The Integral Coefficient Geometric Satake Equivalence in Mixed Characteristic and its Arithmetic Applications

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To my mother, Heping Pei

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ABSTRACT

The first main result of this thesis is the proof of the integral coefficient geometric Satake equivalence in mixed characteristic setting. Our proof can be divided into three parts: the construction of the monoidal structure of the hypercohomology functor on the category of integral coefficient equivariant perverse sheaves on the mixed characteristic affine Grassmannian; a generalized Tannakian formalism; and, the identification of group schemes. In particular, our proof does not employ Scholze's theory of diamonds.

We derive a geometric construction of the Jacquet-Langlands transfer for weighted automorphic forms as an application of the geometric Satake equivalence in the above setting. Our strategy follows the recent work of Xiao-Zhu [XZ17]. We relate the geometry and (ℓ-adic) cohomology of the mod p fibers of the canonical smooth integral models of different Hodge type Shimura varieties, and obtain a Jacquet-Langlands transfer for weighted automorphic forms.

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Chapter 1

INTRODUCTION

The geometric Satake equivalence establishes an equivalence between two symmetric monoidal categories which are of great importance in algebraic geometry, representation theory, and number theory. The first category is $\mathrm{Rep}_\Lambda(\hat{G})$, the category of finitely generated $\hat{G}$-modules over Λ for a connected reductive group G; and the second category is $\mathrm{P}_{L^+G}(Gr_G, \Lambda)$, the category of Λ-coefficient L^+G-equivariant perverse sheaves on the affine Grassmannian Gr_G of G. This equivalence may be regarded as a categorification of the classical Satake isomorphism for connected reductive groups.

The geometric Satake equivalence in equal characteristic setting [Lus], [Gin95], [BD91], [MV07] and in mixed characteristic setting with $\bar{\mathbb{Q}}_\ell$-coefficient [Zhu17] have found many significant applications. For example, V. Lafforgue proved the "automorphic to Galois" direction of the Langlands correspondence over global function fields in his groundbreaking work [Laf18]. The geometric Satake equivalence in the equal characteristic is used to transfer the representations of the Langlands dual group to the perverse sheaves on the moduli of Shtukas. Another noticeable example is a recent work of Xiao-Zhu [XZ17], in which they use the $\bar{\mathbb{Q}}_\ell$-coefficient geometric Satake equivalence in mixed characteristic to prove the "generic" cases of Tate conjecture for the mod p fibers of many Shimura varieties. It is desirable to obtain an integral coefficient version of this equivalence in mixed characteristic setting. In this thesis, we give a new construction of the hypercohomology functor on $\mathrm{P}_{L^+G}(Gr_G, \Lambda)$ which allows us to apply a generalized Tannakian formalism to establish the geometric Satake equivalence in the desired setting.

Let G and G' be two algebraic groups over $\mathbb{Q}$. We assume that they are isomorphic at all finite places of $\mathbb{Q}$, but not necessarily at infinity. Roughly speaking, the Jacquet-Langlands correspondence predicts, in many cases, the following phenomenon: there exists a natural map between the set of automorphic representations of G and that of G' such that if π' is the automorphic representation of G' which corresponds to an automorphic representation π of G, then π_v is isomorphic to π'_v at all finite places v. It is considered as one of the first examples of the Langlands philosophy that maps between L-groups induce maps between automorphic representations.

The classical way of establishing this correspondence is via a comparison of the trace formulas for G and G'. This approach allows us to conclude a map between suitable spaces of automorphic forms for G and G' as abstract representations. However, the resulting map is not canonical, and we, therefore, hope to have a more natural way of understanding this correspondence. Alternative geometric approaches of establishing the Jacquet-Langlands correspondence were first noticed by Ribet [Rib+89] and Serre [SL96], and later followed by Ghitza [Ghi03], [Ghi05], and Helm [Hel+10] [Hel12].

Our second main result in this thesis is a geometric construction of the Jacquet-Langlands transfer for weighted automorphic forms by relating the geometry and (ℓ-adic) cohomology of the mod p fibres of different Hodge type Shimura varieties following the idea of Xiao-Zhu [XZ17]. The integral coefficient geometric Satake equivalence in mixed characteristic plays an indispensable role in this construction.

1.1 The Integral Coefficient Geometric Satake Equivalence in Mixed Characteristic

Main Result

Consider an algebraically closed field k of characteristic $p > 0$ and denote by $W(k)$ its ring of Witt vectors. Let F denote a totally ramified finite extension of $W(k)[1/p]$ and O the ring of integers of F. Let G be a connected reductive group over O and Gr_G be the Witt vector affine Grassmannian defined as in [Zhu17]. In this paper, we consider the category $\mathrm{P}_{L^+G}(Gr_G, \Lambda)$ of L^+G-equivariant perverse sheaves in Λ-coefficient on the affine Grassmannian Gr_G for $\Lambda = \mathbb{F}_\ell$ and $\mathbb{Z}_\ell$, where ℓ is a prime number different from p. We call this category the Satake category and sometimes write it as $\mathrm{Sat}_{G,\Lambda}$ for simplicity. The convolution product of sheaves equips the Satake category with a monoidal structure. Let $\hat{G}_\Lambda$ denote the Langlands dual group of G, i.e. the canonical smooth split reductive group scheme over Λ whose root datum is dual to that of G. Our main theorem is the geometric Satake equivalence in the current setting.

Theorem 1.1.1. *There is an equivalence of monoidal categories between $P_{L^+G}(Gr_G, \Lambda)$ and the category of representations of the Langlands dual group $\hat{G}_\Lambda$ of G on finitely generated Λ-modules.*

We mention that Peter Scholze has announced the same result as part of his work on the local Langlands conjecture for p-adic groups using his beautiful theory of diamonds.

The equal characteristic counterpart of the geometric Satake equivalence was previously achieved by the works of Beilinson-Drinfeld, Ginzburg, Lusztig, and Mirkovic-Vilonen(cf. [BD91], [Gin95], [Lus], [MV07]). Later, Zhu [Zhu17] considered the category of L^+G-equivariant perverse sheaves in $\bar{\mathbb{Q}}_\ell$-coefficient on the mixed characteristic affine Grassmannian Gr_G and established the geometric Satake equivalence in this setting.

In the equal characteristic case, the Beilinson-Drinfeld Grassmannians play a crucial role in establishing the geometric Satake equivalence. In fact, they can be used to construct the monoidal structure of the hypercohomology functor

$$\mathrm{H}^* : P_{L^+G}(Gr_G, \Lambda) \longrightarrow \mathrm{Mod}_\Lambda$$

and the commutativity constraint in the Satake category by interpreting the convolution product as fusion product. In mixed characteristic, Peter Scholze's theory of diamonds allows him to construct an analogue of the Beilinson-Drinfeld Grassmannian and prove the geometric Satake equivalence in this setting in a similar way as in [MV07]. Our approach of constructing the geometric Satake equivalence makes use of some ideas in [Zhu17]. However, our situation is different from *loc.cit* and new difficulties arise. For example, the Satake category in $\bar{\mathbb{Q}}_\ell$-coefficient is semisimple, while, in our case, the semisimplicity of the Satake category fails. In addition, the monoidal structure of the hypercohomology functor was constructed by studying the equivariant cohomology of (convolutions of) irreducible objects in the Satake category in [Zhu17]. Nevertheless, in our situation, the equivariant cohomology may have torsion. Thus the method in *loc.cit* does not apply to our case directly.

Strategy of the Proof

The first key ingredient of the proof is the following proposition.

Proposition 1.1.2. *The hypercohomology functor* $\mathrm{H}^* : \mathrm{P}_{L^+G}(Gr_G, \Lambda) \longrightarrow \mathrm{Mod}_\Lambda$ *is a monoidal functor.*

We study the $\mathbb{G}_m$-action (in fact, we consider the action of the perfection of the group scheme $\mathbb{G}_m$) on the convolution Grassmannian $Gr_G \tilde{\times} Gr_G$. Applying the Mirković-Vilonen theory for mixed characteristic affine Grassmannians established in [Zhu17] and Braden's hyperbolic localization functor, we can decompose the hypercohomology functor $\mathrm{H}^* : \mathrm{P}_{L^+G}(Gr_G \tilde{\times} Gr_G, \Lambda) \to \mathrm{Mod}(\Lambda)$ into a direct sum of compactly supported cohomologies. Each direct summand can be further realized as the tensor product of two compactly supported cohomologies on Gr_G by the Künneth formula. Putting these together completes the proof of Proposition 1.1.2. In particular, the monoidal structure constructed by our approach is compatible with that obtained in [Zhu17].

We further notice that as in the cases discussed in [MV07] and [Zhu17], the hypercohomology functor is representable by projective objects when restricting to full subcategories of the Satake category. In addition, these projective objects are isomorphic to the projective objects studied in [Zhu17] after base change to $\bar{\mathbb{Q}}_\ell$. This, together with Proposition 1.1.2, allows us to directly construct a Λ-algebra $B(\Lambda)$ as in [MV07]. The compatibility of the monoidal structure of H^* and the projective objects constructed in our case with those obtained in [Zhu17] enable us to inherit a commutative multiplication map of $B(\Lambda)$ from that of $B(\bar{\mathbb{Q}}_\ell)$. The commutative multiplication map of $B(\bar{\mathbb{Q}}_\ell)$ comes from the commutativity constraint of $\mathrm{Sat}_{G,\bar{\mathbb{Q}}_\ell}$ constructed in *loc.cit*. In other words, we derive the following proposition.

Proposition 1.1.3. *The Λ-algebra $B(\Lambda)$ admits the structure of a commutative Hopf algebra with an antipode.*

The general Tannakian construction (cf.[MV07]) yields an equivalence of tensor categories

$$P_{L^+G}(Gr_G, \Lambda) \simeq \mathrm{Rep}_\Lambda(\widetilde{G}_\Lambda),$$

where $\widetilde{G}_\Lambda := \mathrm{Spec}B(\Lambda)$ is an affine flat group scheme and $\mathrm{Rep}_\Lambda(\widetilde{G}_\Lambda)$ denotes the category of $\widetilde{G}_\Lambda$-modules which are finitely generated over Λ. We give two approaches identifying $\widetilde{G}_\Lambda$ with $\hat{G}_\Lambda$ and conclude the proof of the theorem by the use of the following result of Prasad-Yu [PY06] on quasi-reductive group schemes.

Theorem 1.1.4. *Let $\mathcal{G}$ be a quasi-reductive group scheme over R. Then*

(1) $\mathcal{G}$ is of finite type over R

(2) $\mathcal{G}_K$ is reductive

(3) $\mathcal{G}_\kappa$ is connected.

In addition, if

(4) the type of $\mathcal{G}_{\bar{K}}$ is of the same type as that of $(\mathcal{G}_{\bar{\kappa}})^\circ_{\mathrm{red}}$,

then $\mathcal{G}$ is reductive.

1.2 A Geometric Jacquet-Langlands Transfer

Main Result

Let (G_1, X_1) and (G_2, X_2) be two Hodge type Shimura data equipped with an isomorphism $\theta : G_{1,\mathbb{A}_f} \simeq G_{2,\mathbb{A}_f}$. Assume that there exists an inner twist $\Psi : G_1 \to G_2$ which is compatible with θ. We assume that $K_i \subset G(\mathbb{A}_f)$ to be sufficiently small such that $\theta K_1 = K_2$. In addition, we assume that p is an unramified prime. Then $K_{1,p}$ (and therefore $K_{2,p}$) is hyperspecial. Let $\underline{G_i}$ be the integral model of $G_{i,\mathbb{Q}_p}$ over $\mathbb{Z}_p$ determined by $K_{i,p}$. Then $\underline{G_1} \simeq \underline{G_2}$, and we can thus identify their Langlands dual groups, which we denote by $\hat{G}_{\mathbb{Q}_\ell}$. We further fix a pinning $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ of $\hat{G}$. Choose an isomorphism $\iota : \mathbb{C} \simeq \bar{\mathbb{Q}}_p$. Let $\{\mu_i\}$ denote the conjugacy class of Hodge cocharacters determined by X_i.

Let $V_i = V_{\mu_i}$ denote the irreducible representation of $\hat{G}_{\mathbb{Q}_\ell}$ of highest weight μ_i. Let $\nu \mid p$ be a place of the compositum of reflex fields E_1E_2 of (G_1, X_1) and (G_2, X_2) determined by our choice of isomorphism ι. Write k_ν for the residue field of E_1E_2 at ν. Results of Kisin [Kis10] and Vasiu [Vas07] state that there exists a smooth canonical integral model of $\mathrm{Sh}_K(G_i, X_i)$ over $O_{E,(\nu)}$. Let $d_i = \dim \mathrm{Sh}_K(G_i, X_i)$ and Sh_{μ_i} denote the mod p fiber of this canonical integral model, base changed to $\bar{k}_\nu$. Our assumption on p implies that the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ factors through some finite quotient $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ for some finite field $\mathbb{F}_{p^n}$ which contains k_ν. Write $\sigma \in \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ for the arithmetic Frobenius. Consider the conjugation action of $\hat{G}_{\mathbb{Q}_\ell}$ on the (non-neutral) component $\hat{G}_{\mathbb{Q}_\ell}\sigma \subset \hat{G}_{\mathbb{Q}_\ell} \rtimes \langle\sigma\rangle$. Denote by $\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$ the abelian category of coherent sheaves on the quotient stack $[\hat{G}_{\mathbb{Q}_\ell}\sigma/\hat{G}_{\mathbb{Q}_\ell}]$.

To each representation W of $G_{\mathbb{Q}_\ell}$, we can attach an ℓ-adic étale local system $\mathcal{L}_{i,W,\mathbb{Q}_\ell}$ on Sh_{μ_i} (see §12.1) by varying the level structure at ℓ. The natural projection $[\hat{G}_{\mathbb{Q}_\ell}\sigma/\hat{G}_{\mathbb{Q}_\ell}] \to \mathbb{B}\hat{G}_{\mathbb{Q}_\ell}$ attaches to each representation V of $\hat{G}_{\mathbb{Q}_\ell}$ a vector bundle $\widetilde{V}$ on $[\hat{G}_{\mathbb{Q}_\ell}\sigma/\hat{G}_{\mathbb{Q}_\ell}]$. Denote the global section of the structure sheaf on the quotient stack $[\hat{G}\sigma/\hat{G}]$ by $\mathcal{J}$, and the prime-to-p Hecke algebra by $\mathcal{H}^p$. Fix a half Tate twist $\mathbb{Q}_\ell(1/2)$.

We state our main theorem.

Theorem 1.2.1. *Let $\mathcal{L}_i := \mathcal{L}_{i,W,\mathbb{Q}_\ell}[d_i](d_i/2)$. Under a mild assumption,* *(1) there exists a map*

$$\mathrm{Spc}:\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V_1},\widetilde{V_2})\to\mathrm{Hom}_{\mathcal{H}^p\otimes\mathcal{J}}(\mathrm{H}_c^*(\mathrm{Sh}_{\mu_1},\mathcal{L}_1),\mathrm{H}_c^*(\mathrm{Sh}_{\mu_2},\mathcal{L}_2)),\tag{1.1}$$

which is compatible with compositions in the source and target.

(2) the ring of endomorphisms $\mathrm{End}_{[\hat{G}\sigma/\hat{G}]}(\mathcal{O}_{[\hat{G}\sigma/\hat{G}]})$ *acts on the compactly supported cohomology* $\mathrm{H}_c^*(\mathrm{Sh}_{\mu_i},\mathcal{L}_i)$ *via* Spc *and this action can be identified with the classical Satake isomorphism if* $\mathrm{Sh}_K(G_1,X_1)=\mathrm{Sh}_K(G_2,X_2)$ *is a Shimura set.*

This is a Jacquet-Langlands transfer for automorphic forms of higher weights which generalizes a previous construction given in [XZ17]. We briefly discuss the proof of Theorem 1.1.

Strategy of the Proof

Let $\Lambda=\mathbb{Z}_\ell,\mathbb{F}_\ell$. The following theorem plays an essential role in the proof of the main result.

Theorem 1.2.2. *For any projective objects* $\Lambda_1,\Lambda_2\in\mathrm{Rep}_\Lambda(\hat{G}_\Lambda)$*, we choose appropriate integers* (m_1,n_1,m_2,n_2) *and a dominant coweight* λ*, and consider the following Hecke correspondence*

$$\mathrm{Sht}_{\Lambda_1}^{\mathrm{loc}(m_1,n_1)}\xleftarrow{h^{\leftarrow}_{\Lambda_1}}\mathrm{Sht}_{\Lambda_1|\Lambda_2}^{\lambda,\mathrm{loc}(m_1,n_1)}\xrightarrow{h^{\rightarrow}_{\Lambda_2}}\mathrm{Sht}_{\Lambda_2}^{\mathrm{loc}(m_2,n_2)}\ .\tag{1.2}$$

Then there exists the following map

$$\mathcal{S}_{\Lambda_1,\Lambda_2}:\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_\Lambda}(\hat{G}_\Lambda\sigma)}(\widetilde{\Lambda_1},\widetilde{\Lambda_2})\longrightarrow\mathrm{Hom}_{\mathrm{D}(\mathrm{Sht}_{\Lambda_1|\Lambda_2}^{\mathrm{loc}(m_1,n_1)})}\Big((h^{\leftarrow}_{\Lambda_1})^*S(\widetilde{\Lambda_1}),(h^{\rightarrow}_{\Lambda_2})^!S(\widetilde{\Lambda_2})\Big),\tag{1.3}$$

which is independent of auxiliary choices.

We remark that the target of $\mathcal{S}_{\Lambda_1,\Lambda_2}$ can be understood as limits of the cohomological correspondence between $(\mathrm{Sht}_{\Lambda_1}^{\mathrm{loc}(m_1,n_1)},S(\widetilde{\Lambda_1}))$ and $(\mathrm{Sht}_{\Lambda_2}^{\mathrm{loc}(m_2,n_2)},S(\widetilde{\Lambda_2}))$ supported on the Hecke correspondence (1.2). In [XZ17], Xiao-Zhu construct the maps $\mathcal{S}_{V,W}$ for $\bar{\mathbb{Q}}_\ell$-representations in a categorical way. In fact, they define the category $\mathrm{P}^{\mathrm{Hk}}(\mathrm{Sht}^{\mathrm{loc}},\bar{\mathbb{Q}}_\ell)$. Its objects are exactly the same as those of $\mathrm{P}(\mathrm{Sht}^{\mathrm{loc}},\bar{\mathbb{Q}}_\ell)$, and its spaces of morphisms are given by (limits of) cohomological correspondences supported on Hecke correspondences of restricted local Shtukas. It receives a canonical functor from the category $\mathrm{P}(\mathrm{Sht}^{\mathrm{loc}}, \bar{\mathbb{Q}}_\ell)$, and this functor has a σ-twisted trace structure. Hence, the universal property of categorical traces asserts that it admits a functor from the σ-twisted categorical trace $\mathrm{Tr}_\sigma(\mathrm{Rep}(\hat{G}_{\bar{\mathbb{Q}}_\ell})) \cong \mathrm{Coh}_{\mathrm{fr}}^{\hat{G}_{\bar{\mathbb{Q}}_\ell}}(\hat{G}_{\bar{\mathbb{Q}}_\ell}\sigma)$ in the sense of [Zhu18]. Here, the latter category is the full subcategory of $\mathrm{Coh}^{\hat{G}_{\bar{\mathbb{Q}}_\ell}}(\hat{G}_{\bar{\mathbb{Q}}_\ell}\sigma)$ generated by objects coming from $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G}_{\bar{\mathbb{Q}}_\ell})$. This idea is made more explicit in *loc.cit*.

The above strategy does not carry over in our situation. To calculate the left σ-twisted categorical trace of $\mathrm{Rep}_{\mathbb{Z}_\ell}(\hat{G}_{\mathbb{Z}_\ell})$, we need to appeal to a more general construction of the tensor product for finitely cocomplete categories. In addition, the correspondence category $\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}, \mathbb{Z}_\ell)$ is not finitely cocomplete any more and the desired maps $\mathcal{S}$ therefore cannot be obtained by the universal property of the categorical trace. One possible way to overcome this difficulty is to upgrade $\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}, \mathbb{Z}_\ell)$ to a higher category.

Instead of pursuing this idea, we take a more concrete approach. We note that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_\Lambda}(\hat{G}_\Lambda\sigma)}(\widetilde{\Lambda_1}, \widetilde{\Lambda_2}) \cong \mathrm{Hom}_{\hat{G}_\Lambda}(\Lambda_1, \mathcal{O}_G \otimes \Lambda_2),$$

where $\mathcal{O}_G$ denotes the regular representation of $\hat{G}_\Lambda$. By the Peter-Weyl theorem, $\mathcal{O}_G$ admits a filtration with associated graded $\oplus_\lambda W \otimes S(\lambda^*)$ where W denotes the Schur module of $\hat{G}_\Lambda$. For each $\mathbf{a} \in \mathrm{Hom}_{\hat{G}}(V, W \otimes S(\lambda^*) \otimes W)$, we use the integral coefficient geometric Satake equivalence discussed in §8 to construct a cohomological correspondence on restricted local Hecke stacks of $G \times G$. The maps $\mathcal{S}_{V,W}$ are constructed by first pulling this cohomological correspondence back to a cohomological correspondence on restricted local Hecke stacks and then pulling it back to a cohomological correspondence on restricted local Shtukas.

1.3 Notations

In this section, we fix notations for later use.

Ring of Witt Vectors

Let F be a mixed characteristic local ring with ring of integers $\mathcal{O}$ and residue field $k = \mathbb{F}_q$. We write σ for the arithmetic Frobenius of $\mathbb{F}_q$. For any k-algebra R, its ring of Witt vectors is denoted by

$$W(R) = \{(r_0, r_1, \cdots) \mid r_i \in R\}.$$

We denote by $W_h(R)$ the ring of truncated Witt vectors of length h. For perfect k-algebra R, we know that $W_h(R) = W(R)/p^h W(R)$. We define the ring of Witt vectors in R with coefficient in O as

$$W_O(R) := W(R)\hat{\otimes}_{W(k)}O := \varprojlim_n W_{O,n}(R), \text{ and } W_{O,n}(R) = W(R) \otimes_{W(k)} O/\varpi^n.$$

We define the (n-th) formal unit disk and formal punctured unit disk to be

$$D_{n,R} := \mathrm{Spec} W_{O,n}(R),\ D_R := \mathrm{Spec} W_O(R),\ D_R^\times := \mathrm{Spec} W_O(R)[1/\varpi],$$

respectively.

Reductive Group Schemes

Let L be the completion of the maximal unramified extension of F. Denote by O_L its ring of integers, and we fix a uniformizer $\varpi \in O_L$. We will assume G to be an unramified reductive group scheme over O. We denote by T the abstract Cartan subgroup of G. Let $S \subset T$ denote the maximal split subtorus. In the case where G is a split reductive group, we will choose a Borel subgroup $B \subset G$ over O and a split maximal torus $T \subset B$. When we need to embed T (or S) into G as a (split) maximal torus, we will state it explicitly.

Let $\mathbb{X}_\bullet$ denote the coweight lattice of T and $\mathbb{X}^\bullet$ the weight lattice. Let $\Delta \subset \mathbb{X}^\bullet$ (resp. $\Delta^\vee \subset \mathbb{X}_\bullet$) the set of roots (resp. coroots). A choice of the Borel subgroup $B \subset G$ determines the semi-group of dominant coweights $\mathbb{X}_\bullet^+ \subset \mathbb{X}_\bullet$ and the set of positive roots $\Delta_+ \subset \Delta$. In fact, $\mathbb{X}_\bullet^+$ and Δ_+ are both independent of the choice of B. The q-power (arithmetic) Frobenius σ acts on $(\mathbb{X}^\bullet, \Delta, \mathbb{X}_\bullet, \Delta^\vee)$ preserving $\mathbb{X}_\bullet^+$. The Langlands dual group of G is denoted by $\hat{G}$.

Let $2\rho \in \mathbb{X}^\bullet$ be the sum of all positive roots. Define the partial order " $\leq$ " on $\mathbb{X}_\bullet$ to be such that $\lambda \leq \mu$ if and only if $\mu - \lambda$ equals a non-negative integral linear combination of positive coroots. For any $\mu \in \mathbb{X}_\bullet$, denote ϖ^μ by the image of μ under the composition of maps

$$\mathbb{G}_m \to T \subset G.$$

Let H be a reductive group over a field K. We write H_{ad} for its adjoint group, H_{der} for its derived group, and H_{sc} for the simply connected cover of H_{ad}.

Let $T \subset H$ be a maximal torus, and we denote by T_{ad} its image in the quotient H_{ad}. We also write T_{sc} for the preimage of T in H_{sc}.

Algebraic Geometry

We denote by $\mathcal{E}^0$ the trivial G-torsor. For any perfect k-algebra R, the arithmetic Frobenius σ induces an automorphism $\sigma \otimes \mathrm{id}$ of the D_R. Let $\mathcal{E}$ be a G-torsor over D_R, and we denote the G-torsor $(\sigma \otimes \mathrm{id})^*\mathcal{E}$ by ${}^{\sigma}\mathcal{E}$.

Let X be an algebraic space over k. We write σ_X for the absolute Frobenius morphism of X. We denote by $X^{p^\infty} := \varprojlim_{\sigma_X} X$ the perfection of X.

Let $\ell \neq p$ be a prime number. Fix a half Tate twist $\mathbb{Z}_\ell(1/2)$. We write $\langle d \rangle := [d](d/2)$. Let X and Y be two algebraic stacks which are perfectly of finitely presentation in the opposite category of perfect k-algebras. For each $f : X \to Y$ being a perfectly smooth morphism of relative dimension d, we write $f^\star := f^*\langle d \rangle$.

Throughout the thesis, we write Λ for $\mathbb{Z}_\ell$ and $\mathbb{F}_\ell$ and E for $\mathbb{Z}_\ell$ unless otherwise stated. For stacks X_1, X_2, and perverse sheaves $\mathcal{F}_i \in \mathrm{P}(X_i, \Lambda)$, we sometimes write the space of cohomological correspondences $\mathrm{Corr}_X((X_1, \mathcal{F}_1), (X_2, \mathcal{F}_2))$ as $\mathrm{Corr}_X(\mathcal{F}_1, \mathcal{F}_2)$ for simplicity.

Let X be a stack, we denote by $\omega_X \in D^c_b(X, E)$ the dualizing sheaf of X in the bounded derived category of sheaves on X. For a perfect pfp algebraic space (cf.[XZ17, A.1.7]) X over k, we denote by $\mathrm{H}^{BM}_i(X_{\bar{k}}) := \mathrm{H}^{-i}(X_{\bar{k}}, \omega_x(-i/2))$ the ith Borel-Moore homology of $X_{\bar{k}}$.

1.4 Summary of the Contents

In §2, we define the affine Grassmannian in mixed characteristic and briefly discuss its geometry.

In §3, we define the Satake category and endow it with a monoidal structure.

In §4 and §5 discuss the semi-infinite orbits in mixed characteristic affine Grassmannians and the weight functors.

Chapter 6 is devoted to construct the monoidal structure of the hypercohomology functor on the Satake category.

In §7, we construct a Λ-coalgebra by studying the cohomology of projective objects in the Satake category and we endow it with the structure of a commutative Hopf algebra with an antipode.

In §8, we identify the group scheme arising from the previous construction with the Langlands dual group of the connected reductive group scheme we start with. This completes the proof of our first main result in this thesis.

Sections §9 and §10 introduce the local Hecke stacks and the moduli of local Shtukas. We discuss their geometry and define the categories of perverse sheaves on them.

In §11, we construct a map from the space of morphisms between two coherent sheaves on the stack of unramified local Langlands parameters to the space of cohomological correspondences supported on the Hecke correspondence of moduli of local Shtukas. This is a key theorem in the proof of our second main theorem.

In §12, we study the cohomological correspondences between the mod p fibres of two different Shimura varieties and prove our second main theorem.

Chapter 2

THE MIXED CHARACTERISTIC AFFINE GRASSMANNIANS

In this chapter, we review the construction of affine Grassmannians in mixed characteristic and summarize their geometric properties which will be used later. Most properties appearing in this section have analogies in the equal characteristic setting, and we refer to [MV07] for a detailed discussion.

2.1 Preliminaries

We start this section by defining p-adic *jet spaces* that are similar to their equal characteristic counterparts. Let $\mathcal{X}$ be a finite type $\mathcal{O}$-scheme. We consider the following two presheaves on the category of affine k-schemes defined as follows

$$L_p^+\mathcal{X}(R) := \mathcal{X}(W_{\mathcal{O}}(R)), \text{ and } L_p^h\mathcal{X}(R) := X(W_{\mathcal{O},h}(R)),$$

which are represented by schemes over k. Their perfections are denoted by

$$L^+\mathcal{X} := (L_p^+\mathcal{X})^{p^{-\infty}}, \text{ and } L^h\mathcal{X} := (L_p^h\mathcal{X})^{p^{-\infty}}$$

respectively, and we call them p-adic *jet spaces*.

Let X be an affine scheme over F. We define the p-adic *loop space* LX of X as the perfect space by assigning a perfect k-algebra R to the set

$$LX(R) = X(W_{\mathcal{O}}(R)[1/p]).$$

2.2 The Mixed Characteristic Affine Grassmannian

Let $\mathcal{X} = G$ be a smooth affine group scheme over $\mathcal{O}$. We write $G^{(0)} = G$ and define the h-th congruence group scheme of G over $\mathcal{O}$, denoted by $G^{(h)}$, as the dilatation of $G^{(h-1)}$ along the unit. The group $L^+G^{(h)}$ can be identified with $\ker(L^+G \to L^hG)$ via the natural map $G^{(h)} \to G$. Then L^+G acts on LG by multiplication on the right. We define the affine Grassmannian Gr_G of G to be the perfect space

$$Gr_G := [LG/L^+G]$$

on the category of perfect k-algebras.

In the work of Bhatt-Scholze [BS17], the functor Gr_G is proved to be representable by an inductive limit of perfections of projective varieties.

We recall the following proposition in [Zhu17] for later use.

Proposition 2.2.1. *Let $\rho : G \to GL_n$ be a linear representation such that GL_n/G is quasi-affine. Then ρ induces a locally closed embedding $Gr_G \to Gr_{GL_n}$. If in addition, GL_n/G is affine, then $Gr_G \to Gr_{GL_n}$ is in fact a closed embedding.*

Explicitly, the affine Grassmannian Gr_G can be described as assigning a perfect k-algebra R the set of pairs (P, ϕ), where P is an L^+G-torsor over Spec$R$ and $\phi : P \to LG$ is an L^+G-equivariant morphism. It is clear from the definition that $LG \to Gr_G$ is an L^+G-torsor and L^+G naturally acts on Gr_G, then we can form the twisted product which we also call the convolution product in the current setting

$$Gr_G \tilde{\times} Gr_G := LG \times^{L^+G} Gr_G := [LG \times Gr_G / L^+G],$$

where L^+G acts on $LG \times Gr_G$ anti-diagonally as $g^+ \cdot ([g_1], [g_2]) := ([g_1(g^+)^{-1}], [g^+ g_2])$.

As in the equal characteristic case, the affine Grassmannians can be interpreted as the moduli stack of G-torsors on the formal unit disk with trivialization away from the origin. More precisely, for each perfect k-algebra R,

$$Gr_G(R) = \left\{ (\mathcal{E}, \phi) \middle| \begin{matrix} \mathcal{E} \to D_R \text{ is a } G\text{-torsor, and} \\ \phi : \mathcal{E}|_{D_R^\times} \simeq \mathcal{E}^0|_{D_R^\times} \end{matrix} \right\}.$$

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two G-torsors over D_R, and let $\beta : \mathcal{E}_1|_{D_R^\times} \simeq \mathcal{E}_2|_{D_R^\times}$ be an isomorphism. One can define the relative position $\mathrm{Inv}(\beta)$ of β as an element in $\mathbb{X}_\bullet^+$ as in [Zhu17].

Definition 2.2.2. For each $\mu \in \mathbb{X}_\bullet^+$, we define

(1) the (spherical) Schubert variety

$$Gr_{\leq \mu} := \{(\mathcal{E}, \beta) \in Gr_G | \mathrm{Inv}(\beta) \leq \mu\},$$

(2) the Schubert cell

$$Gr_\mu := \{(\mathcal{E}, \beta) \in Gr_G | \mathrm{Inv}(\beta) = \mu\}.$$

Proposition 2.2.3. *(1) Let $\mu \in \mathbb{X}_\bullet^+$, and $\varpi^\mu \in Gr_G$ be the corresponding point in the affine Grassmannian. Then the map*

$$i_\mu : L^+G/(L^+G \cap \varpi^\mu L^+G \varpi^{-\mu}) \longrightarrow LG/L^+G, \text{ such that } g \longmapsto g\varpi^\mu$$

induces an isomorphism

$$L^+G/(L^+G \cap \varpi^\mu L^+G \varpi^{-\mu}) \simeq Gr_\mu.$$

(2) *Gr_μ is the perfection of a quasi-projective smooth variety of dimension $(2\rho,\mu)$.*

(3) *$Gr_{\leq\mu}$ is the Zariski closure of Gr_μ in Gr_G, and therefore is perfectly proper of dimension $(2\rho,\mu)$.*

The convolution Grassmannian $Gr_G\tilde{\times}Gr_G$ admits a moduli interpretation as follows

$$Gr_G\tilde{\times}Gr_G(R)=\left\{(\mathcal{E}_1,\mathcal{E}_2,\beta_1,\beta_2)\middle|\begin{matrix}\mathcal{E}_1,\mathcal{E}_2 \text{ are } G-\text{torsors on } D_R, \text{and}\\ \beta_1:\mathcal{E}_1|_{D_R^*}\simeq\mathcal{E}_0|_{D_R^*},\beta_2:\mathcal{E}_2|_{D_R^*}\simeq\mathcal{E}_1|_{D_R^*}\end{matrix}\right\}.$$

Via this interpretation, we define the convolution morphism as in the equal characteristic case

$$m:Gr_G\tilde{\times}Gr_G\longrightarrow Gr_G,$$

such that

$$(\mathcal{E}_1,\mathcal{E}_2,\beta_1,\beta_2)\longmapsto(\mathcal{E}_2,\beta_1\beta_2).$$

Note that there is also the natural projection morphism

$$pr_1:Gr_G\tilde{\times}Gr_G\longrightarrow Gr_G,$$

such that

$$(\mathcal{E}_1,\mathcal{E}_2,\beta_1,\beta_2)\longmapsto(\mathcal{E}_1,\beta_1).$$

It is clear to see that $(pr_1,m):Gr_G\tilde{\times}Gr_G\simeq Gr_G\times Gr_G$ is an isomorphism.

One can define the n-fold convolution Grassmannian $Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G$ in a similar manner as follows

$$Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G:=\left\{(\mathcal{E}_i,\beta_i)\middle|\begin{matrix}\mathcal{E}_i \text{ is a } G\text{-torsor over } D_R, \text{and}\\ \beta_i:\mathcal{E}_i|_{D_R^*}\simeq\mathcal{E}_{i-1}|_{D_R^*}\end{matrix}\right\}.$$

We define the morphism

$$m_i:Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G\longrightarrow Gr_G$$

such that

$$(\mathcal{E}_i,\beta_i)\longmapsto(\mathcal{E}_i,\beta_1\beta_2\cdots\beta_i:\mathcal{E}_i|_{D_R^*}\simeq\mathcal{E}_0|_{D_R^*}).$$

As for the 2-fold convolution Grassmannian, we have an isomorphism

$$(m_1,m_2,\cdots,m_n):Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G\simeq Gr_G\times\cdots\times Gr_G.$$

We also call the map m_n the convolution map.

Given a sequence of dominant coweights $\mu_\bullet = (\mu_1, \cdots, \mu_n)$ of G, we define the following closed subspace of $Gr_G \tilde{\times} \cdots \tilde{\times} Gr_G$,

$$Gr_{\leq\mu_\bullet} := Gr_{\leq\mu_1} \tilde{\times} \cdots \tilde{\times} Gr_{\leq\mu_n} := \{(\mathcal{E}_i, \beta_i) \in Gr_G \tilde{\times} \cdots \tilde{\times} Gr_G | \mathrm{Inv}(\beta_i) \leq \mu_i\}.$$

For a perfect k-algebra R, $Gr_{\leq\mu_\bullet}$ classifies isomorphism classes of modifications of G-torsors over D_R

$$\mathcal{E}_n \overset{\beta_1}{\dashrightarrow} \mathcal{E}_{n-1} \overset{\beta_{n-1}}{\dashrightarrow} \cdots \overset{\beta_1}{\dashrightarrow} \mathcal{E}_0 = \mathcal{E}^0, \tag{2.1}$$

where $\mathrm{Inv}(\beta_i) \leq \mu_i$. We define $Gr^{(\infty)}_{\leq\mu_\bullet}$ as the L^+G-torsor over $Gr_{\leq\mu_\bullet}$ which classifies a point in $Gr_{\leq\mu_\bullet}$ as (2.1) together with an isomorphism $\mathcal{E}_n \simeq \mathcal{E}^0$. For any integer n, we define $Gr^{(n)}_{\leq\mu_\bullet}$ to be the L^nG-torsor over $Gr_{\leq\mu_\bullet}$ which classifies a point in $Gr_{\leq\mu_\bullet}$ together with an isomorphism $\mathcal{E}_n |_{D_{n,R}} \simeq \mathcal{E}^0 |_{D_{n,R}}$. For any $m < n$, there is an isomorphism

$$Gr_{\leq\mu_\bullet} \simeq Gr_{\leq\mu_1,\cdots,\mu_m} \tilde{\times} Gr_{\leq\mu_{m+1},\cdots,\mu_n} := Gr^{(\infty)}_{\leq\mu_1,\cdots,\mu_m} \times^{L^+G} Gr_{\leq\mu_{m+1},\cdots,\mu_n}.$$

Many of the constructions in later sections make use of the following lemma [Zhu17, Lemma 3.1.7]:

Lemma 2.2.4. *For any sequence of dominant coweights $\mu_\bullet = (\mu_1, \mu_2, \cdots, \mu_n)$, there exists a non-negative integer m, such that for any non-negative integer n, the action of $L^{m+n}G$ on $Gr_{\leq\mu_\bullet}$ is trivial. In other words, the natural action of L^+G on $Gr_{\leq\mu_\bullet}$ factors through the finite type quotient $L^{m+n}G$.*

We will call such an integer m a *$\mu_\bullet$-large*. We also call a pair of non-negative integers (m, n) ($m = \infty$ allowed) to be *$\mu_\bullet$-large* if $m - n$ is a $\mu_\bullet$-large integer.

Replacing $Gr_{\leq\mu_i}$ by Gr_{μ_i}, we can similarly define $Gr_{\mu_\bullet} := Gr_{\mu_1} \tilde{\times} \cdots \tilde{\times} Gr_{\mu_n}$. By Proposition 2.3, we have

$$Gr_{\leq\mu_\bullet} = \cup_{\mu'_\bullet \leq \mu_\bullet} Gr_{\mu'_\bullet}, \tag{2.2}$$

where $\mu'_\bullet \leq \mu_\bullet$ means $\mu'_i \leq \mu_i$ for each i. This gives a stratification of $Gr_{\mu_\bullet}$.

As in [Zhu17], we let $|\mu_\bullet| := \sum \mu_i$. Then the convolution map induces the following morphism

$$m : Gr_{\leq\mu_\bullet} \longrightarrow Gr_{\leq|\mu_\bullet|},$$

such that

$$(\mathcal{E}_i, \beta_i) \longmapsto (\mathcal{E}_n, \beta_1 \cdots \beta_n).$$

Let $\nu_\bullet$ be another sequence of dominant coweights. We define the following stack

$$Gr^0_{\mu_\bullet|\nu_\bullet} := Gr_{\leq\mu_\bullet} \times_{m_{\mu_\bullet}, Gr_G, m_{\nu_\bullet}} Gr_{\leq\nu_\bullet}.$$

Write the natural projections from $Gr_{\mu_\bullet} \times_{m_{\mu_\bullet}, Gr_G, m_{\nu_\bullet}}$ to $Gr_{\leq\mu_\bullet}$ and $Gr_{\leq\nu_\bullet}$ as $h^{\leftarrow}_{\mu_\bullet}$ and $h^{\rightarrow}_{\nu_\bullet}$, respectively. We call the following diagram

$$Gr_{\leq\mu_\bullet} \xleftarrow{h^{\leftarrow}_{\mu_\bullet}} \mathrm{Gr}^0_{\mu_\bullet|\nu_\bullet} \xrightarrow{h^{\rightarrow}_{\nu_\bullet}} Gr_{\leq\nu_\bullet} \tag{2.3}$$

the *Satake correspondence*.

Definition 2.2.5. An irreducible component of $\mathrm{Gr}^0_{\mu_\bullet|\nu_\bullet}$ of dimension $(\rho, |\mu_\bullet| + |\nu_\bullet|)$ is called a *Satake cycle*. Denote the set of Satake cycles of $\mathrm{Gr}^0_{\mu_\bullet|\nu_\bullet}$ by $\mathbb{S}_{\mu_\bullet|\nu_\bullet}$. For $\mathbf{a} \in \mathbb{S}_{\mu_\bullet|\nu_\bullet}$, write $\mathrm{Gr}^{0,\mathbf{a}}_{\mu_\bullet|\nu_\bullet}$ for the Satake cycle labelled by $\mathbf{a}$.

It is clear that $\mathrm{Gr}^0_{\mu_\bullet|\nu_\bullet} \cong Gr^0_{\nu_\bullet|\mu_\bullet}$, and we conclude that

$$\mathbb{S}_{\mu_\bullet|\nu_\bullet} = \mathbb{S}_{\nu_\bullet|\mu_\bullet}. \tag{2.4}$$

Chapter 3

THE SATAKE CATEGORY

In this chapter, we first define the Satake category $\mathrm{Sat}_{G,\Lambda}$ as the category of $L^+G \otimes \bar{k}$-equivariant Λ-coefficient perverse sheaves on $Gr_G \otimes \bar{k}$. We then define the convolution map which enables us to equip the Satake category with a monoidal structure.

3.1 The Satake Category as an Abelian Category

We know that $\pi_0(Gr_G \otimes \bar{k}) \simeq \pi_1(G)$ (cf. [Zhu17, Proposition 1.21]). The affine Grassmannian $Gr_G \otimes \bar{k}$ has the decomposition into connected components

$$Gr_G \otimes \bar{k} = \sqcup_{\xi \in \pi_1(G)} Gr_\xi.$$

Recall our discussion in §2.1. We have

$$Gr_\xi = \varinjlim_{\mu \in \xi} Gr_\mu,$$

where $\mu \in \xi$ means that the natural map $\mathbb{X}_\bullet \to \pi_1(G)$ sends μ to ξ. The connecting morphism $i_{\mu,\nu} : Gr_\mu \to Gr_\nu$ is a closed embedding if $\mu \leq \nu$. For $m \leq m'$ be integers such that L^+G acts on Gr_μ through L^mG and $L^{m'}G$, there is a canonical equivalence

$$P_{L^mG\otimes\bar{k}}(Gr_\mu, \Lambda) \cong P_{L^{m'}G\otimes\bar{k}}(Gr_\mu, \Lambda).$$

We define the category of $L^+G \otimes \bar{k}$-equivariant Λ-coefficient perverse sheaves on $Gr_G \otimes \bar{k}$ as

$$\mathrm{P}_{L^+G\otimes\bar{k}}(Gr_G \otimes \bar{k}, \Lambda) := \bigoplus_{\xi \in \pi_1(G)} \mathrm{P}_{L^+G\otimes\bar{k}}(Gr_\xi, \Lambda),$$

$$\mathrm{P}_{L^+G\otimes\bar{k}}(Gr_\xi, \Lambda) := \varinjlim_{(\mu,m)} P_{L^mG\otimes\bar{k}}(Gr_\mu, \Lambda).$$

Here, the limit is taken over the pairs $\{(\mu, m) \mid \mu \in \xi, m \text{ is large enough}\}$ with partial order given by $(\mu, m) \leq (\mu', m)$ if $\mu \leq \mu'$ and $m \leq m'$. The connecting morphism is the composition

$$\mathrm{P}_{L^mG\otimes\bar{k}}(Gr_\mu, \Lambda) \cong \mathrm{P}_{L^{m'}G\otimes\bar{k}}(Gr_\mu, \Lambda) \xrightarrow{i^*_{\mu,\mu'}} \mathrm{P}_{L^{m'}G\otimes\bar{k}}(Gr_{\mu'}, \Lambda)$$

which is a fully faithful embedding. We also call this category the *Satake category* and sometimes denote it by $\mathrm{Sat}_{G,\Lambda}$ for simplicity. We denote by IC_μ for each $\mu \in \mathbb{X}_\bullet^+$ the intersection cohomology sheaf on $Gr_{\leq\mu}$. Its restriction to each open strata Gr_μ is constant and in particular, $\mathrm{IC}_\mu \mid_{Gr_\mu} \simeq \Lambda[(2\rho, \mu)]$.

3.2 The Monoidal Structure of the Satake Category

With the above preparation, we can define the monoidal structure in $\mathrm{Sat}_{G,\Lambda}$ by Lusztig's convolution of sheaves as in the equal characteristic counterpart. Consider the following diagram

$$Gr_G \times Gr_G \xleftarrow{p} LG \times Gr_G \xrightarrow{q} Gr_G \tilde{\times} Gr_G \xrightarrow{m} Gr_G,$$

where p and q are projection maps. We define for any $\mathcal{A}_1, \mathcal{A}_2 \in P_{L^+G}(Gr_G, \Lambda)$,

$$\mathcal{A} \star \mathcal{A}_2 := Rm_!(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2),$$

where $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2 \in P_{L^+G}(Gr_G \tilde{\times} Gr_G, \Lambda)$ is the unique sheaf such that

$$q^*(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2) \simeq p^*({}^p\mathrm{H}^0(\mathcal{A}_1 \boxtimes \mathcal{A}_2)).$$

Unlike the construction in $P_{L^+G}(Gr_G, \bar{\mathbb{Q}}_\ell)$, we emphasize that taking the 0-th perverse cohomology ${}^p\mathrm{H}(\bullet)$ in the above definition is necessary. This is because when we work with $\mathbb{Z}_\ell$-sheaves, the external tensor product $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ may not be perverse. In fact, $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ is perverse if one of $\mathrm{H}^*(\mathcal{A}_i)$ is a flat $\mathbb{Z}_\ell$-module. For more details, we refer to [MV07, Lemma 4.1] for a detailed explanation.

The following proposition is called a "miraculous theorem" of the Satake category in equal characteristic (cf.[BD91]).

Proposition 3.2.1. *For any $\mathcal{A}_1, \mathcal{A}_2 \in P_{L^+G}(Gr_G, \Lambda)$, the convolution product $\mathcal{A}_1 \star \mathcal{A}_2$ is perverse.*

Proof. Note by [Zhu17, Proposition 2.3] that the convolution morphism m is a stratified semi-small morphism with respect to the stratification (2.1). Then the proposition follows from [MV07, Lemma 4.3] □

We can also define the n-fold convolution production in $\mathrm{Sat}_{G,\Lambda}$

$$\mathcal{A}_1 \star \cdots \star \mathcal{A}_n := Rm_!(\mathcal{A}_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathcal{A}_n),$$

where $\mathcal{A}_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathcal{A}_n$ is defined in a similar way as $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$. By considering the following isomorphism:

$$ {}^p\mathrm{H}^0(\mathcal{A}_1 \boxtimes ({}^p\mathrm{H}^0(\mathcal{A}_2 \boxtimes \mathcal{A}_3)) \simeq {}^p\mathrm{H}^0(\mathcal{A}_1 \boxtimes \mathcal{A}_2 \boxtimes \mathcal{A}_3) \simeq {}^p\mathrm{H}^0({}^p\mathrm{H}^0(\mathcal{A}_1 \boxtimes \mathcal{A}_2) \boxtimes \mathcal{A}_3), $$

we conclude that the convolution product is associative:

$$ (\mathcal{A}_1 \star \mathcal{A}_2) \star \mathcal{A}_3 \simeq Rm_!(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2 \tilde{\boxtimes} \mathcal{A}_3) \simeq \mathcal{A}_1 \star (\mathcal{A}_2 \star \mathcal{A}_3). $$

Thus, the category $(\mathrm{Sat}_{G,\Lambda}, \star)$ is a monoidal category.

Chapter 4

SEMI-INFINITE ORBITS AND WEIGHT FUNCTORS

In this section, we review the construction and geometry of semi-infinite orbits of Gr_G. By studying a $\mathbb{G}_m$-action on the affine Grassmannian Gr_G, we realize the semi-infinite orbits as the attracting loci of the $\mathbb{G}_m$-action in the sense of [DG14]. We also define the weight functors and relate them to the hyperbolic localization functors and study their properties.

4.1 The Geometry of Semi-infinite Orbits

We fix embeddings $T \subset B \subset G$ and let U be the unipotent radical of B. Since $U\backslash G$ is quasi-affine, recall Proposition 2.2.1 we know that that $Gr_U \hookrightarrow Gr_G$ is a locally closed embedding. For any $\lambda \in \mathbb{X}_\bullet$, define

$$S_\lambda := LU\varpi^\lambda$$

to be the orbit of ϖ^λ under the LU-action. Then $S_\lambda = \varpi^\lambda Gr_U$, and therefore is locally closed in Gr_G via the embedding $Gr_U \hookrightarrow Gr_G$. By the Iwasawa decomposition for p-adic groups, we know that

$$Gr_G = \cup_{\lambda\in\mathbb{X}_\bullet} S_\lambda.$$

Similarly, consider the opposite Borel B^- and let U^- be its unipotent radical. We also define the opposite semi-infinite orbits

$$S_\lambda^- := LU^-\varpi^\lambda.$$

for $\lambda \in \mathbb{X}_\bullet$

Recall the following closure relations as in [Zhu17, Proposition 2.5] (the equal characteristic analogue of this statement is proved in [MV07, Proposition 3.1].

Proposition 4.1.1. *Let $\lambda \in \mathbb{X}_\bullet$, then $S_{\leq\lambda} := \overline{S_\lambda} = \cup_{\lambda'\leq\lambda} S_{\lambda'}$ and $S^-_{\leq\lambda} := \overline{S^-_\lambda} = \cup_{\lambda'\leq\lambda} S^-_{\lambda'}$.*

Applying the reduction of structure group to the L^+G-torsor $LG \to Gr_G$ to S_μ, we obtain an L^+U-torsor $LU \to S_\mu$. This allows us to construct the convolution of semi-infinite orbits $S_{\mu_1}\tilde{\times}S_{\mu_2}\tilde{\times}\cdots\tilde{\times}S_{\mu_n}$. Let $\mu_\bullet = (\mu_1, \cdots, \mu_n)$ be a sequence of (not necessarily dominant) coweights of G. We define

$$S_{\mu_\bullet} := S_{\mu_1}\tilde{\times}S_{\mu_2}\tilde{\times}\cdots\tilde{\times}S_{\mu_n} \subset Gr_G\tilde{\times}Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G.$$

The morphism

$$m : S_{\mu_\bullet} \longrightarrow S_{\mu_1} \times S_{\mu_1+\mu_2} \times \cdots \times S_{|\mu_\bullet|}$$

given by

$$(\varpi^{\mu_1}x_1, \varpi^{\mu_2}x_2, \cdots \varpi^{\mu_1 n}x_n) \longmapsto (\varpi^{\mu_1}x_1, \varpi^{\mu_1+\mu_2}(\varpi^{-\mu_2}x_1\varpi^{\mu_2}x_2), \cdots, \varpi^{|\mu_\bullet|}(\varpi^{-|\mu_\bullet|+\mu_1}x_1\cdots\varpi^{\mu_n}x_n))$$

is an isomorphism. The morphism m fits into the following commutative diagram

$$\begin{array}{ccc} S_{\leq\mu_0} & \xrightarrow{\quad m \quad} & S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_n} \\ \big\downarrow & & \big\downarrow \\ Gr_G\tilde{\times}Gr_G\tilde{\times}\cdots\tilde{\times}Gr_G & \xrightarrow{(m_1,\ldots,m_n)} & Gr_G \times Gr_G \times \cdots \times Gr_G. \end{array}$$

We also note that there is a canonical isomorphism

$$(S_{\nu_1} \cap Gr_{\leq\mu_1})\tilde{\times}(S_{\nu_2} \cap Gr_{\leq\mu_2})\tilde{\times}\cdots\tilde{\times}(S_{\nu_n} \cap Gr_{\leq\mu_n}) \simeq S_{\nu_\bullet} \cap Gr_{\leq\mu_\bullet}. \tag{4.1}$$

4.2 The Weight Functors

Similar to the equal characteristic situation (cf. [MV07] (3.16), (3.17)), the semi-infinite orbits may be interpreted as the attracting loci of certain torus action which we describe here.

Let $2\rho^\vee$ be the sum of all positive coroots of G with respect to B and regard it as a cocharacter of G. The projection map $L_p^+\mathbb{G}_m \to \mathbb{G}_m$ admits a unique section $\mathbb{G}_m \to L_p^+\mathbb{G}_m$ which identifies $\mathbb{G}_m$ as the maximal torus of $L_p^+\mathbb{G}_m$. This section allows us to define a cocharacter

$$\mathbb{G}_m \longrightarrow L^+\mathbb{G}_m \overset{L^+(2\rho^\vee)}{\longrightarrow} L^+T \subset L^+G.$$

Then the $\mathbb{G}_m$-action on Gr_G is induced by the action of L^+G on Gr_G. Under this action by $\mathbb{G}_m$, the set of fixed points are precisely $R := \{\varpi^\lambda | \lambda \in \mathbb{X}_\bullet\}$. The attracting loci of this action are semi-infinite orbits i.e.

$$S_\lambda = \{g \in Gr_G | \lim_{t\to 0} L^+(2\rho^\vee(t)) \cdot (g) = \varpi^\lambda \text{ for } t \in \mathbb{G}_m\}.$$

The repelling loci are the opposite semi-infinite orbits

$$S_\lambda^- = \{g \in Gr_G | \lim_{t\to\infty} L^+(2\rho^\vee(t)) \cdot (g) = \varpi^\lambda \text{ for } t \in \mathbb{G}_m\}.$$

Recall that if X is a scheme and $i : Y \hookrightarrow X$ is an inclusion of a locally closed subscheme, then for any $\mathcal{F} \in D_c^b(X, \Lambda)$, the local cohomology group is defined as $\mathrm{H}_Y^k(X, \mathcal{F}) := \mathrm{H}^k(Y, i^!\mathcal{F})$.

Proposition 4.2.1. *For any $\mathcal{F} \in P_{L^+G}(Gr_G, \Lambda)$, there is an isomorphism*

$$\mathrm{H}_c^k(S_\mu, \mathcal{F}) \simeq \mathrm{H}_{S_\mu^-}^k(\mathcal{F}),$$

and both sides vanish if $k \neq (2\rho, \mu)$.

Proof. The proof is similar to the equal characteristic case (cf. [MV07, Theorem 3.5]) as the dimension estimation of the intersections of the semi-infinite orbits and Schubert varieties are established in [Zhu17, Corollary 2.8]. Since $\mathcal{F}$ is perverse, then for any $\nu \in \mathbb{X}_\bullet^+$, we know that $\mathcal{F}|_{Gr_\nu} \in D^{\leq -\dim(Gr_\nu)} = D^{\leq -(2\rho,\nu)}$. By [Zhu17, Corollary 2.8], we know that $\mathrm{H}_c^k(S_\mu \cap Gr_{\leq\nu}, \mathcal{F}) = 0$ if $k > 2\dim(S_\mu \cap Gr_{\leq\nu}) = (2\rho, \mu + \nu)$. Filtering Gr_G by $Gr_{\leq\mu}$, we apply a dévissage argument and conclude that

$$\mathrm{H}_c^k(S_\mu, \mathcal{F}) = 0 \text{ if } k > (2\rho, \mu).$$

An analogous argument proves that

$$\mathrm{H}_{S_\mu^-}^k(\mathcal{F}) = 0$$

if $k < (2\rho, \mu)$.

Now by regarding S_μ and S_μ^- as the attracting and repelling loci of the $\mathbb{G}_m$-action, we apply the hyperbolic localization as in [DG14] and obtain

$$\mathrm{H}_c^k(S_\mu, \mathcal{F}) \simeq \mathrm{H}_{S_\mu^-}^k(\mathcal{F}).$$

The proposition is thus proved. □

Let Mod_Λ denote the category of finitely generated Λ-modules and $\mathrm{Mod}(\mathbb{X}_\bullet)$ denote the category of $\mathbb{X}_\bullet$-graded finitely generated Λ-modules.

Definition 4.2.2. For any $\mu \in \mathbb{X}_\bullet$, we define

(1) the weight functor

$$\mathrm{CT}_\mu : P_{L^+G}(Gr_G, \Lambda) \longrightarrow \mathrm{Mod}(\mathbb{X}_\bullet),$$

by

$$\mathrm{CT}_\mu(\mathcal{F}) := \mathrm{H}_c^{(2\rho,\mu)}(S_\mu, \mathcal{F}),$$

(2) the total weight functor

$$\mathrm{CT} := \bigoplus_\mu \mathrm{CT}_\mu : P_{L^+G}(Gr_G, \Lambda) \longrightarrow \mathrm{Mod}(\mathbb{X}_\bullet),$$

by

$$\mathrm{CT}(\mathcal{F}) := \bigoplus_\mu \mathrm{CT}_\mu(\mathcal{F}) := \bigoplus_\mu \mathrm{H}_c^{(2\rho,\mu)}(S_\mu, \mathcal{F}).$$

We denote by F the forgetful functor from $\mathrm{Mod}(\mathbb{X}_\bullet)$ to Mod_Λ.

Proposition 4.2.3. *There is a canonical isomorphism of functors*

$$\mathrm{H}^*(Gr_G, \bullet) \cong F \circ \mathrm{CT} : P_{L^+G}(Gr_G, \Lambda) \longrightarrow \mathrm{Mod}_\Lambda.$$

In addition, both functors are exact and faithful.

Proof. By the definition of the semi-infinite orbits and the Iwasawa decomposition, we obtain two stratifications of Gr_G by $\{S_\mu \mid \mu \in \mathbb{X}_\bullet\}$ and $\{S_\mu^- \mid \mu \in \mathbb{X}_\bullet\}$, respectively. The first stratification induces a spectral sequence with E_1-terms $\mathrm{H}_c^k(S_\mu, \mathcal{F})$ and abutment $\mathrm{H}^*(Gr_G, \mathcal{F})$. This spectral sequence degenerates at the E_1-page by Proposition 4.2.1. Thus, there is a filtration of $\mathrm{H}^*(Gr_G, \mathcal{F})$ indexed by $(\mathbb{X}_\bullet, \leq)$ defined as

$$\mathrm{Fil}_{\geq\mu}\mathrm{H}^*(Gr_G, \mathcal{F}) := \ker(\mathrm{H}^*(Gr_G, \mathcal{F}) \longrightarrow \mathrm{H}^*(S_{<\mu}, \mathcal{F})),$$

where $S_{<\mu} := \cup_{\mu'<\mu} S_{\mu'}$. Direct computation yields that the associated graded of the above filtration is $\bigoplus_\mu \mathrm{H}_c^{(2\rho,\mu)}(S_\mu, \mathcal{F})$.

Consider the second stratification of Gr_G. It also induces a filtration of $\mathrm{H}^*(Gr_G, \mathcal{F})$ as

$$\mathrm{Fil}'_{<\mu}\mathrm{H}^k(Gr_G, \mathcal{F}) := \mathrm{Im}(\mathrm{H}^*_{S^-_{\leq\mu}}(\mathcal{F}) \longrightarrow \mathrm{H}^*(Gr_G, \mathcal{F})),$$

where $S^-_{<\mu} := \cup_{\mu'<\mu} S^-_{\mu'}$.

Now, by Proposition 4.2.1, the two filtrations are complementary to each other and together define the decomposition $\mathrm{H}^*(Gr_G, \bullet) \simeq \bigoplus_\mu \mathrm{H}_c^{(2\rho,\mu)}(S_\mu, \bullet)$.

Next, we prove that the total weight functor CT is exact. To do so, it suffices to show that the weight functor CT_μ is exact for each $\mu \in \mathbb{X}_\bullet$. Let

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

be an exact sequence in $P_{L^+G}(Gr_G, \Lambda)$. It is given by a distinguished triangle

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \xrightarrow{+1}$$

in $D_c^b(Gr_G, \Lambda)$. We thus have a long exact sequence of cohomology

$$\cdots \longrightarrow \mathrm{H}_c^k(S_\mu, \mathcal{F}_1) \longrightarrow \mathrm{H}_c^k(S_\mu, \mathcal{F}_2) \longrightarrow \mathrm{H}_c^k(S_\mu, \mathcal{F}_3) \longrightarrow \mathrm{H}_c^k(S_\mu, \mathcal{F}_3) \longrightarrow \mathrm{H}_c^{k+1}(S_\mu, \mathcal{F}_1) \longrightarrow \cdots.$$

Then Proposition 4.2 gives the desired exact sequence

$$0 \longrightarrow \mathrm{CT}_\mu(\mathcal{F}_1) \longrightarrow \mathrm{CT}_\mu(\mathcal{F}_2) \longrightarrow \mathrm{CT}_\mu(\mathcal{F}_3) \longrightarrow 0.$$

We conclude the proof by showing that CT is faithful. Since CT is exact, it suffices to prove that CT maps non-zero objects to non-zero objects. Let $\mathcal{F} \in \mathrm{Sat}_{G,\Lambda}$ be a nonzero object. Then $\mathrm{supp}(\mathcal{F})$ is a finite union of Schubert cells Gr_ν. Choose ν to be maximal for this property. Then $\mathcal{F}|_{Gr_\nu} \simeq \underline{\Lambda}^{\oplus n}[(2\rho, \nu)]$ for some natural number n and it follows that $\mathrm{CT}_\nu(\mathcal{F}) \neq 0$. Thus the functor H^* is faithful. □

Remark 4.2.4. *The weight functor is in fact independent of the choice of the maximal torus T. The proof for this is analogous to the equal characteristic case (cf. [MV07, Theorem 3.6]).*

We note that the analogue of [Zhu17, Corollary 2.9] also holds in our setting. In particular, $\mathrm{H}^*(\mathrm{IC}_\mu)$ is a free Λ-module for any $\mu \in \mathbb{X}_\bullet^+$.

We end this section by proving a weaker statement of [MV07, Proposition 2.1] which will be used in the process of identification of group schemes in §8.

Lemma 4.2.5. *There is a natural equivalence of tensor categories*

$$\alpha : P_{L^+G}(Gr_G, \Lambda) \cong P_{L^+(G/Z)}(Gr_G, \Lambda),$$

where Z is the center of G.

Proof. We first note that the category $P_{L^+(G/Z)}(Gr_G, \Lambda)$ can be identified as a full subcategory of $P_{L^+G}(Gr_G, \Lambda)$. Let $X \subset Gr_G$ be a finite union of L^+G-orbits. Since

L^+Z acts on Gr_G trivially, the action of L^+G on Gr_G factors through the quotient $L^+(G/Z)$. In other words, the following diagram commutes

$$\begin{array}{ccc} L^+G \times X & \xrightarrow{a_1} & X \\ \Big\downarrow q & \nearrow_{a_2} & \\ L^+(G/Z) \times X & & \end{array}$$

where a_1 and a_2 are the action maps and q is the natural projection map. In addition, the following diagram is clearly commutative.

$$\begin{array}{ccc} L^+G \times X & \xrightarrow{p_1} & X \\ \Big\downarrow q & \nearrow_{p_2} & \\ L^+(G/Z) \times X & & \end{array}$$

It follows that any $\mathcal{F} \in P_{L^+(G/Z)}(Gr_G, \Lambda)$, $\mathcal{F}$ is L^+G-equivariant by checking the definition directly.

Thus it suffices to prove reverse direction. We prove by induction on the number of L^+G-orbits in X as in the proof of [MV07, Proposition A.1]. First, we assume that X contains exactly one L^+G-orbit. Write $X = Gr_\mu$ for some $\mu \in \mathbb{X}_\bullet^+$. Recall Proposition 2.2.3.(1) and [Zhu17, p. 1.4.4]. There is a natural projection with fibres isomorphic to the perfection of affine spaces

$$\pi_\mu : Gr_\mu \simeq L^+G/(L^+G \cap \varpi^\mu L^+G \varpi^{-\mu}) \longrightarrow (\overline{G}/\overline{P}_\mu)^{p^{-\infty}}$$

$$(gt^\mu \operatorname{mod} L^+G) \longmapsto (\bar{g} \operatorname{mod} \overline{P}_\mu^{p^{-\infty}})$$

where P_μ denotes the parabolic subgroup of G generated by the root subgroups U_α of G corresponding to those roots α satisfying $\langle \alpha, \mu \rangle \leq 0$, and $\overline{P}_\mu$ denotes the fibre of P_μ at O/ϖ. Assume that L^+G acts on $(\overline{G}/\overline{P}_\mu)^{p^{-\infty}}$ by a finite type quotient L^nG. Since the stabilizer of this action of L^+G is connected, we have a canonical equivalence of categories (cf. [Zhu17, A.3.4]) $P_{L^+G}((\overline{G}/\overline{P}_\mu)^{p^{-\infty}}, \Lambda) \simeq P_{L^nG}((\overline{G}/\overline{P}_\mu)^{p^{-\infty}}, \Lambda)$. Finally, we note that $P_{L^nG}((\overline{G}/\overline{P}_\mu)^{p^{-\infty}}, \Lambda)$ is equivalent to Mod_Λ, and we conclude that $P_{L^+G}(Gr_\mu, \Lambda) \simeq \mathrm{Mod}_\Lambda(\mathbb{B}L^nG)$. A completely similar argument implies that $P_{L^+(G/Z)}(Gr_\mu, \Lambda) \simeq \mathrm{Mod}_\Lambda(\mathbb{B}L^nG)$ which concludes the proof in the case $X = Gr_\mu$.

Now we treat the general X. Choose $\mu \in \mathbb{X}_\bullet^+$ such that $Gr_\mu \subset X$ is a closed subspace, and let $U := X \backslash Gr_\mu$. By induction hypothesis, we know that $P_{L^+G}(U, \Lambda)$

is equivalent to $P_{L^+(G/Z)}(U,\Lambda)$. Denote by $i : Gr_\mu \hookrightarrow X$ and $j : U \hookrightarrow X$ the closed and open embeddings, respectively. Let $\widetilde{Gr}_\mu := L^+(G/Z)\times Gr_\mu$, $\widetilde{X} := L^+(G/Z)\times X$, and $\widetilde{U} := L^+(G/Z)\times U$. Denote by $\tilde{j} : \widetilde{U} \hookrightarrow \widetilde{X}$ the open embedding. The stratification on X induces a stratification on $\widetilde{X}$ which has strata equal to products of $L^+(G/Z)$ with strata in X. Restricting to $\widetilde{U}$, we get a stratification of $\widetilde{U}$. Considering the action of L^+G on $\widetilde{X}$ and $\widetilde{U}$ by left multiplication on the second factor, we can define categories $P_{L^+G}(\widetilde{X},\Lambda)$ and $P_{L^+G}(\widetilde{U},\Lambda)$. Define the functor

$$\widetilde{\mathrm{CT}}_\mu : P_{L^+G}(\widetilde{X},\Lambda) \longrightarrow \mathrm{Loc}_\Lambda(L^+(G/Z))$$

$$\widetilde{\mathrm{CT}}_\mu(\mathcal{F}) := \mathcal{H}^{(2\rho,\mu)+\dim L^+(G/Z)}(\pi_!\tilde{i}^*(\mathcal{F}))$$

for any $\mathcal{F} \in P_{L^+G}(\widetilde{X},\Lambda)$, where $\tilde{i} : L^+(G/Z)\times(S_\mu\cap X) \hookrightarrow \widetilde{X}$ is the locally closed embedding, $\pi : L^+(G/Z)\times(S_\mu\cap X) \to L^+(G/Z)$ is the natural projection, and $\mathrm{Loc}_\Lambda(L^+(G/Z))$ denotes the category of Λ-local systems on $L^+(G/Z)$. A completely similar argument as in Proposition 4.2.1 shows that $\widetilde{\mathrm{CT}}_\mu$ is an exact functor. Let $\widetilde{F}_1 := \widetilde{\mathrm{CT}}_\mu \circ {}^p\tilde{j}_!, \widetilde{F}_2 : \widetilde{\mathrm{CT}}_\mu \circ {}^p j_* : P_{L^+G}(\widetilde{U},\Lambda) \to \mathrm{Loc}_\Lambda(L^+(G/Z))$. Finally let $\widetilde{T} := \widetilde{\mathrm{CT}}_\mu({}^p\tilde{j}_! \to {}^p\tilde{j}_*)$. Then as in [MV07, Appendix A], we get an equivalence of abelian categories

$$\widetilde{E} : P_{L^+G}(\widetilde{X},\Lambda) \simeq C(\widetilde{F}_1,\widetilde{F}_2,\widetilde{T})$$

where the second category in the above is defined in *loc.cit*. Note that any $\mathcal{F} \in P_{L^+(G/Z)}(\widetilde{X},\Lambda)$ is $\mathbb{G}_m$-equivariant. The same argument in [MV07, Proposition A.1] applies here and gives

$$\widetilde{E}(a_2^*\mathcal{F}) \simeq \widetilde{E}(p_2^*\mathcal{F}).$$

Then we deduce an isomorphism $a_2^*\mathcal{F} \simeq p_2^*\mathcal{F}$ and the lemma is thus proved. □

4.3 Notations

We close this chapter by introducing the following notations. For any $\mu \in \mathbb{X}_\bullet^+$ and $\nu \in \mathbb{X}_\bullet$, the irreducible components of the intersection $Gr_{\leq\mu} \cap S_\nu$ are called the *Mirković-Vilonen cycles*. We denote the set of Mirković-Vilonen cycles by $\mathbb{MV}(\mu)(\nu)$. For each $\mathbf{a} \in \mathbb{MV}_\mu(\nu)$, we write $(Gr_{\leq\mu}\cap S_\nu)^{\mathbf{a}}$ for the irreducible component of $Gr_{\leq\mu}\cap S_\nu$ labelled by $\mathbf{a}$.

Chapter 5

REPRESENTABILITY OF WEIGHT FUNCTORS AND THE STRUCTURE OF REPRESENTING OBJECTS

In chapter 4, we constructed the weight functors and the total weight functor

$$\mathrm{CT}_\mu, \mathrm{CT} : P_{L^+G}(Gr_G, \Lambda) \to \mathrm{Mod}_\Lambda.$$

We will prove in this section that both functors are (pro)representable, so that we can apply the (generalized) Deligne and Milne's Tannakian formalism as in [MV07, §11]. In the following, we will recall the induction functor (cf. [MV07]) to explicitly construct the representing object of each weight functor and use the representability of the total weight functor to prove that the Satake category has enough projective objects. At the end of this section, we give a few propositions of the representing objects which will be used later when we apply the (generalized) Tannakian formalism.

Let $Z \subset Gr_G$ be a closed subspace which is a union of finitely many L^+G-orbits. Choose $n \in \mathbb{Z}$ large enough so that L^+G acts on Z via the quotient L^nG. Let $\nu \in \mathbb{X}_\bullet$. As in [MV07, §9], we consider the following commutative diagram

$$\begin{array}{ccccc} S_\nu^- \cap Z & \longleftarrow & L^nG \times (S_\nu^- \cap Z) & \xrightarrow{\tilde{a}} & Z \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow & & \Big\| \\ Z & \xleftarrow[p]{} & L^nG \times Z & \xrightarrow{a} & Z \end{array}$$

where i is the locally closed embedding, a and $\tilde{a}$ are the action maps, and p is the projection map. Then we define

$$P_Z(\nu, \Lambda) := {}^p\mathrm{H}^0(a_! p^! i_! \underline{\Lambda}_{S_\nu^- \cap Z}[-(2\rho, \nu)]).$$

The following two results are analogues of the equal characteristic counterparts and can be proved exactly in the same manner. We omit the proofs and refer readers to [MV07, Proposition 9.1, Corollary 9.2] for details.

Proposition 5.0.1. *The restriction of the weight functor* CT_ν *to* $P_{L^+G}(Z, \Lambda)$ *is represented by the projective object* $P_Z(\nu, \Lambda)$ *in* $P_{L^+G}(Z, \Lambda)$.

Corollary 5.0.2. *The category $P_{L^+G}(Z,\Lambda)$ has enough projectives.*

Let $P_Z(\Lambda) := \oplus_\nu P_Z(\nu,\Lambda)$. We note the following mixed characteristic analogues of results of the projective objects in the equal characteristic (cf [MV07, Proposition 10.1]) hold in our setting.

Proposition 5.0.3. *(1) Let $Y \subset Z$ be a closed subset which is a union of L^+G-orbits. Then*

$$P_Y(\Lambda) = {}^p\mathrm{H}^0(P_Z(\Lambda)|_Y),$$

and there is a canonical surjective morphism

$$p_Y^Z : P_Z(\Lambda) \longrightarrow P_Y(\Lambda).$$

(2) For each L^+G-orbit Gr_λ, denote by $j_\lambda : Gr_\lambda \hookrightarrow Gr_G$ the inclusion map. The projective object $P_Z(\Lambda)$ has a filtration with associated graded

$$gr(P_Z(\Lambda)) \simeq \bigoplus_{Gr_\lambda \subset Z} \mathrm{CT}({}^p j_{\lambda,*}\underline{\Lambda}_{Gr_\lambda}[(2\rho,\lambda)])^* \otimes {}^p j_{\lambda,!}\underline{\Lambda}_{Gr_\lambda}[(2\rho,\lambda)].$$

In particular, $\mathrm{H}^*(P_Z(\Lambda))$ is a finitely generated free Λ-module.

(3) For $\Lambda = \bar{\mathbb{Q}}_\ell$ and $\mathbb{F}_\ell$, there is a canonical isomorphism

$$P_Z(\Lambda) \simeq P_Z(\mathbb{Z}_\ell) \otimes^L_{\mathbb{Z}_\ell} \Lambda.$$

Again, as the proof in [MV07, Prp.10.1] extends verbatim in our setting, we refer to *loc.cit* for details.

For the rest of this section, we set $\Lambda = \mathbb{Z}_\ell$.

Proposition 5.0.4. *Let $\mathcal{F} \in P_{L^+G}(Z,\Lambda)$ be a projective object. Then $\mathrm{H}^*(\mathcal{F})$ is a projective Λ-module. In particular, $\mathrm{H}^*(\mathcal{F})$ is torsion-free.*

Proof. Since $\mathrm{Hom}(P_Z(\Lambda), \bullet)$ is exact and faithful, the object $P_Z(\Lambda)$ is a projective generator of $P_{L^+G}(Z,\Lambda)$. Then each object in the Satake category admits a resolution by direct sums of $P_Z(\Lambda)$. Choose such a resolution for $\mathcal{F}$

$$P_Z(\Lambda)^{\oplus m} \longrightarrow \mathcal{F} \longrightarrow 0. \tag{5.1}$$

In this way, $\mathcal{F}$ can be realized as a direct summand of $P_Z(\Lambda)^{\oplus m}$. By Proposition 5.0.3 (2), we notice that $\mathrm{H}^*(P_Z(\Lambda)^{\oplus m})$ is a finitely generated free Λ-module. Finally, by the exactness of the global cohomology functor $\mathrm{H}^*(\bullet)$, we conclude that $\mathrm{H}^*(\mathcal{F})$ is a direct summand of $\mathrm{H}^*(P_Z(\Lambda)^{\oplus m})$ and is thus a projective Λ-module. □

Remark 5.0.5. *The results established in Proposition* 5.0.4 *become immediate once the geometric Satake equivalence is established.*

Chapter 6

THE MONOIDAL STRUCTURE OF H^*

In this section, we study the $\mathbb{G}_m$-action on $Gr_G\tilde{\times}Gr_G$ and apply the hyperbolic localization theorem to prove that the hypercohomology functor $\mathrm{H}^* : P_{L^+G}(Gr_G, \Lambda) \to \mathrm{Mod}(\Lambda)$ is a monoidal functor. Then we study the relation between the global weight functor CT and the global cohomology functor H^*. At the end of this section, we prove that the monoidal structure on H^* we constructed is compatible with the one constructed in [Zhu17].

6.1 Weight Functors on the Convolution Grassmannian

Recall the action of $\mathbb{G}_m$ on Gr_G defined in §4, and let $\mathbb{G}_m$ act on $Gr_G \times Gr_G$ diagonally. Then,

$$R \times R := \{(g_1, g_2) \in Gr_G \times Gr_G | L^+(2\rho^\vee(t)) \cdot (g_1, g_2) = (g_1, g_2)\},$$

$$S_{\mu_1} \times S_{\mu_2} = \{(g_1, g_2) \in Gr_G \times Gr_G | \lim_{t\to 0} L^+(2\rho^\vee(t)) \cdot (g_1, g_2) = (\varpi^{\mu_1}, \varpi^{\mu_2})\},$$

and

$$S^-_{\mu_1} \times S^-_{\mu_2} = \{(g_1, g_2) \in Gr_G \times Gr_G | \lim_{t\to \infty} L^+(2\rho^\vee(t)) \cdot (g_1, g_2) = (\varpi^{\mu_1}, \varpi^{\mu_2})\}$$

are the stable, attracting, and repelling loci of the $\mathbb{G}_m$-action, respectively. We write $(g_1\tilde{\times}g_2) \in Gr_G\tilde{\times}Gr_G$ for $(pr_1, m)^{-1}([g_1], [g_1g_2])$. Define the action of $\mathbb{G}_m$ on $Gr_G\tilde{\times}Gr_G$ by $t(g_1\tilde{\times}g_2) := (tg_1\tilde{\times}g_1^{-1}g_2)$ for any $t \in \mathbb{G}_m$. Then the isomorphism $(pr_1, m) : Gr_G\tilde{\times}Gr_G \simeq Gr_G \times Gr_G$ is automatically $\mathbb{G}_m$-equivariant. The stable loci, attracting, and repelling loci of the $\mathbb{G}_m$-action on $Gr_G\tilde{\times}Gr_G$ are

$$R\tilde{\times}R = \{(\varpi^{\mu_1}\tilde{\times}\varpi^{\mu_2-\mu_1}) | \mu_1, \mu_2 \in \mathbb{X}^+_\bullet\},$$

$$S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1} = \{(g_1\tilde{\times}g_2) \in Gr_G\tilde{\times}Gr_G | \lim_{t\to 0} L^+(2\rho^\vee(t))\cdot(g_1\tilde{\times}g_2) = (\varpi^{\mu_1}\tilde{\times}\varpi^{\mu_2-\mu_1})\},$$

and

$$S^-_{\mu_1}\tilde{\times}S^-_{\mu_2-\mu_1} = \{((g_1\tilde{\times}g_2) \in Gr_G\tilde{\times}Gr_G | \lim_{t\to \infty} L^+(2\rho^\vee(t))\cdot(g_1\tilde{\times}g_2) = (\varpi^{\mu_1}\tilde{\times}\varpi^{\mu_2-\mu_1})\},$$

respectively.

Lemma 6.1.1. *For any $\mathcal{F}, \mathcal{G} \in \mathrm{Sat}_{G,\Lambda}$, we have the following isomorphisms*

$$\mathrm{H}^*_{S^-_{\mu_1}\tilde{\times}S^-_{\mu_2-\mu_1}}(Gr_G\tilde{\times}Gr_G, \mathcal{F}\tilde{\boxtimes}\mathcal{G}) \simeq \mathrm{H}^*_c(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}\tilde{\boxtimes}\mathcal{G}) \simeq \mathrm{H}^*_c(S_{\mu_1}, \mathcal{F})\otimes\mathrm{H}^*_c(S_{\mu_2-\mu_1}, \mathcal{G}). \tag{6.1}$$

In addition, the above cohomology groups vanish outside degree $(2\rho, \mu_2)$.

Proof. By our discussion on the $\mathbb{G}_m$-action on $Gr_G\tilde{\times}Gr_G$ above, the first isomorphism can be obtained by applying Braden's hyperbolic localization theorem as in [DG14]. Therefore, we are left to prove the second isomorphism and the vanishing property of the cohomology. We first establish a canonical isomorphism

$$\mathrm{H}^*_c(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}\tilde{\boxtimes}\mathcal{G}) \cong \mathrm{H}^*_c(S_{\mu_1}\times S_{\mu_2-\mu_1}, {}^p\mathrm{H}^0(\mathcal{F}\tilde{\boxtimes}\mathcal{G})).$$

The idea of constructing this isomorphism is completely similar to the one that appears in [Zhu17, Coro.2.17], and we sketch it here.

Assume LU acts on S_{μ_1} via the quotient L^nU for some positive integer n. Denote by $S^{(n)}_{\mu_1}$ the pushout of the L^+U-torsor $LU \to S_{\mu_1}$ along $L^+U \to L^nU$. Then $\pi : S^{(n)}_{\mu_1} \to S_{\mu_1}$ is an L^nU-torsor. Denote by $\pi^*\mathcal{F}$ the pullback of $\mathcal{F}$ along π. Then we have the following projection morphisms

$$S_{\mu_1}\times S_{\mu_2-\mu_1} \xleftarrow{\pi\times\mathrm{id}} S^{(n)}_{\mu_1}\times S_{\mu_2-\mu_1} \xrightarrow{q} S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}.$$

Since L^nU is isomorphic to the perfection of an affine space of dimension $n\dim U$, we have the following canonical isomorprhisms

$$\begin{aligned}
&\mathrm{H}^*_c(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}\tilde{\boxtimes}\mathcal{G})\\
\cong&\mathrm{H}^*_c(S^{(n)}_{\mu_1}\times S_{\mu_2-\mu_1}, q^*(\mathcal{F}\tilde{\boxtimes}\mathcal{G}))\\
\cong&\mathrm{H}^*_c(S^{(n)}_{\mu_1}\times S_{\mu_2-\mu_1}, (\pi\times\mathrm{id})^*({}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G})))\\
\cong&\mathrm{H}^*_c(S_{\mu_1}\times S_{\mu_2-\mu_1}, {}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G})).
\end{aligned}$$

Next, we prove that there is a natural isomorphism

$$\mathrm{H}^*_c(S_{\mu_1}\times S_{\mu_2-\mu_1}, {}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G})) \cong \mathrm{H}^*_c(S_{\mu_1}, \mathcal{F})\otimes\mathrm{H}^*_c(S_{\mu_2-\mu_1}, \mathcal{G}). \tag{6.2}$$

Assume that $\mathcal{G}$ is a projective object in the Satake category. Then by Proposition 5.4 and discussion in §2, we have ${}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G}) = \mathcal{F}\boxtimes\mathcal{G}$ and (6.2) thus holds. Now we come back to the general situation. Note that there is always a map from $\mathrm{H}^*_c(S_{\mu_1}, \mathcal{F})\otimes\mathrm{H}^*_c(S_{\mu_2-\mu_1}, \mathcal{G})$ to $\mathrm{H}^*_c(S_{\mu_1}\times S_{\mu_2-\mu_1}, {}^p\mathrm{H}^0(\mathcal{F}\boxtimes\mathcal{G}))$. In fact, let

$a \in \mathrm{H}_c^m(S_{\mu_1}, \mathcal{F})$ and $b \in \mathrm{H}_c^n(S_{\mu_2-\mu_1}, \mathcal{G})$ be two arbitrary elements in the cohomology groups. Then a and b may be realized as

$$a : \underline{\Lambda}_{S_{\mu_1}} \to \mathcal{F}[m], \text{ and } b : \underline{\Lambda}_{S_{\mu_2-\mu_1}} \to \mathcal{G}[n].$$

These two morphisms together induce a morphism

$$a \boxtimes b : \underline{\Lambda}_{S_{\mu_1} \times S_{\mu_2-\mu_1}} \longrightarrow \mathcal{F} \boxtimes \mathcal{G}[m+n].$$

Since $\mathcal{F} \boxtimes \mathcal{G}$ concentrates in non-positive perverse degrees, we can compose the above morphism with the natural truncation morphism to get the following element

$$a \boxtimes b : \underline{\Lambda}_{S_{\mu_1} \times S_{\mu_2-\mu_1}} \longrightarrow {}^p\mathrm{H}^0(\mathcal{F} \boxtimes G)[m+n]$$

of $\mathrm{H}_c^{m+n}(S_{\mu_1} \times S_{\mu_2-\mu_1}, {}^p\mathrm{H}^0(\mathcal{F} \boxtimes \mathcal{G}))$.

By Corollary 5.2, we can find a projective resolution $\cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ for $\mathcal{F}$. Since the functor ${}^p\mathrm{H}^0(\bullet \boxtimes \mathcal{G})$ is right exact, we get the following exact sequence

$$\cdots \longrightarrow {}^p\mathrm{H}^0(\mathcal{F}_2 \boxtimes \mathcal{G}) \longrightarrow {}^p\mathrm{H}^0(\mathcal{F}_1 \boxtimes \mathcal{G}) \longrightarrow {}^p\mathrm{H}^0(\mathcal{F} \boxtimes \mathcal{G}) \longrightarrow 0. \tag{6.3}$$

Recall the diagonal action of $\mathbb{G}_m$ on $Gr_G \times Gr_G$. We can apply the same argument as in Proposition 4.2.3 to show that

$$\mathrm{H}^*(Gr_G \times Gr_G, \bullet) \simeq \oplus \mathrm{H}_c^*(S_{\mu_1} \times S_{\mu_2-\mu_1}, \bullet)$$

is an exact functor. As a result, the functor

$$\mathrm{H}_c^*(S_{\mu_1} \times S_{\mu_2-\mu_1}, \bullet) : P_{L^+G \times L^+G}(Gr_G \times Gr_G, \Lambda) \longrightarrow \mathrm{Mod}_\Lambda$$

is also exact. Applying this functor to (6.3) gives an exact sequence

$$\cdots \to \mathrm{H}_c^*(S_{\mu_1}, \mathcal{F}_2) \otimes \mathrm{H}_c^*(S_{\mu_2-\mu_1}, \mathcal{G}) \to \mathrm{H}_c^*(S_{\mu_1}, \mathcal{F}_1) \otimes \mathrm{H}_c^*(S_{\mu_2-\mu_1}, \mathcal{G}) \to \mathrm{H}_c^*(S_{\mu_1} \times S_{\mu_2-\mu_1}, {}^p\mathrm{H}^0(\mathcal{F} \boxtimes \mathcal{G})) \to 0.$$

Comparing the above exact sequence with the one obtained from tensoring the following exact sequence

$$\mathrm{H}_c^*(S_{\mu_1}, \mathcal{F}_2) \longrightarrow \mathrm{H}_c^*(S_{\mu_1}, \mathcal{F}_1) \longrightarrow \mathrm{H}_c^*(S_{\mu_1}, \mathcal{F}) \longrightarrow 0$$

with $\mathrm{H}_c^*(S_{\mu_2-\mu_1}, \mathcal{G})$, we complete the proof of (6.2).

Finally, consider Proposition 4.0.2 together with (6.2), and we conclude the proof of the lemma. □

The previous lemma motivates us to study the analogue of the total weight functor

$$\mathrm{CT}' := \bigoplus_{\mu_1,\mu_2\in\mathbb{X}_\bullet} \mathrm{H}_c^*(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1},\bullet) : P_{L^+G}(Gr_G,\Lambda)\times P_{L^+G}(Gr_G,\Lambda) \longrightarrow \mathrm{Mod}(\mathbb{X}_\bullet).$$

Recall that we denote $F : \mathrm{Mod}(\mathbb{X}_\bullet) \to \mathrm{Mod}_\Lambda$ to be the forgetful functor.

Proposition 6.1.2. *There is a canonical isomorphism*

$$\mathrm{H}^*(Gr_G\tilde{\times}Gr_G,\mathcal{F}\tilde{\boxtimes}\mathcal{G}) \cong F\circ\mathrm{CT}'(\mathcal{F}\tilde{\boxtimes}\mathcal{G}) : P_{L^+G}(Gr_G,\Lambda)\times P_{L^+G}(Gr_G,\Lambda) \to \mathrm{Mod}_\Lambda, \tag{6.4}$$

for all $\mathcal{F}, G \in P_{L^+G}(Gr_G,\Lambda)$.

Proof. The convolution Grassmannian $Gr_G\tilde{\times}Gr_G$ admits a stratification by the convolution of semi-infinite orbits

$$\{S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}|\mu_1,\mu_2\in\mathbb{X}_\bullet\}.$$

For any $\mathcal{F},\mathcal{G}\in P_{L^+G}(Gr_G,\Lambda)$, there is a spectral sequence with E_1-terms $\mathrm{H}_c^*(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1},\mathcal{F}\tilde{\boxtimes}\mathcal{G})$ and abutment $\mathrm{H}^*(Gr_G\tilde{\times}Gr_G,\mathcal{F}\tilde{\boxtimes}\mathcal{G})$. By the above lemma, it degenerates at the E_1 page. Hence, there exists a filtration

$$\mathrm{Fil}_{\geq\mu_1,\mu_2}\mathrm{H}^*(\mathcal{F}\tilde{\boxtimes}\mathcal{G}) := \ker(\mathrm{H}^*(\mathcal{F}\tilde{\boxtimes}\mathcal{G}) \to \mathrm{H}^*(S_{<\mu_1,<\mu_2},\mathcal{F}\tilde{\boxtimes}\mathcal{G})),$$

where $S_{<\mu_1,<\mu_2} := \bigcup_{\nu_1<\mu_1,\nu_1+\nu_2<\mu_2} S_{\nu_1}\tilde{\times}S_{\nu_2-\nu_1}$. It is clear that the associated graded of this filtration is $\oplus_{\mu_1,\mu_2\in\mathbb{X}_\bullet}\mathrm{H}_c^*(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1},\mathcal{F}\tilde{\boxtimes}\mathcal{G})$.

Similarly, consider the stratification $\{S^-_{\mu_1}\tilde{\times}S^-_{\mu_2-\mu_1}|,\mu_1,\mu_2\in\mathbb{X}_\bullet\}$ of $Gr_G\tilde{\times}Gr_G$. It also induces a filtration

$$\mathrm{Fil}'_{<\mu_1,\mu_2}\mathrm{H}^*(\mathcal{F}\tilde{\boxtimes}\mathcal{G}) := \mathrm{Im}(\mathrm{H}^*_{T_{<\mu_1,<\mu_2}}(\mathcal{F}\tilde{\boxtimes}\mathcal{G}) \to \mathrm{H}^*(\mathcal{F}\tilde{\boxtimes}\mathcal{G}))$$

on $\mathrm{H}^*(Gr_G\tilde{\times}Gr_G,\mathcal{F}\tilde{\boxtimes}\mathcal{G})$ where $T_{<\mu_1,<\mu_2} := \bigcup_{\nu_1<\mu_1,\nu_1+\nu_2<\mu_2} T_{\nu_1}\tilde{\times}T_{\nu_2-\nu_1}$. The two filtrations are complementary to each other by Lemma 6.1.1 and the proposition is proved. □

6.2 Monoidal Structure of the Hypercohomology Functor

Proposition 6.2.1. *Under the canonical isomorphism*

$$\mathrm{H}^*(Gr_G,\mathcal{F}\star\mathcal{G}) \cong \mathrm{H}^*(Gr_G\tilde{\times}Gr_G,\mathcal{F}\tilde{\boxtimes}\mathcal{G}),$$

the weight functor decomposition of the hypercohomology functor obtained in Proposition 4.2.3 *and the analogous decomposition given by Proposition* 6.1.2 *are compatible. More precisely, for any $\mathcal{F}, \mathcal{G} \in P_{L^+G}(Gr_G, \Lambda)$ and any $\mu_2 \in \mathbb{X}_\bullet$, we have the following isomorphism*

$$\mathrm{H}_c^*(S_{\mu_2}, \mathcal{F} \star \mathcal{G}) \simeq \bigoplus_{\mu_1} \mathrm{H}_c^*(S_{\mu_1} \tilde{\times} S_{\mu_2-\mu_1}, \mathcal{F} \tilde{\boxtimes} \mathcal{G}), \tag{6.5}$$

which identifies both sides as direct summands of the direct sum decomposition of $\mathrm{H}^*(Gr_G, \mathcal{F} \star \mathcal{G})$ *and* $\mathrm{H}^*(Gr_G \tilde{\times} Gr_G, \mathcal{F} \tilde{\boxtimes} \mathcal{G})$*, respectively.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} m^{-1}(S_{\mu_2}) & \xrightarrow{\tilde{f}^+} & Gr_G \tilde{\times} Gr_G \\ \downarrow{\scriptstyle m_1} & & \downarrow{\scriptstyle m} \\ S_{\mu_2} & \xrightarrow{f^+} & Gr_G. \end{array}$$

Here, f and $\tilde{f}^+$ are the natural locally closed embeddings. The morphism m_1 is the convolution morphism m restricted to $m^{-1}(S_{\mu_2})$.

Consider the $\mathbb{G}_m$-equivariant isomorphism $(pr_1, m) : Gr_G \tilde{\times} Gr_G \simeq Gr_G \times Gr_G$. The preimage of S_{μ_2} along m can be described as

$$(pr_1, m) : m^{-1}(S_{\mu_2}) \simeq Gr_G \times S_{\mu_2}.$$

As before, the diagonal action of $\mathbb{G}_m$ on $Gr_G \times S_{\mu_2}$ induces a $\mathbb{G}_m$- action on $m^{-1}(S_{\mu_2})$ with invariant loci $\{(\varpi^{\mu_1}, \varpi^{\mu_2}) | \mu_1 \in \mathbb{X}_\bullet\}$. Via the isomorphism $(pr_1, m)^{-1}$, the attracting and repelling loci for $(\varpi^{\mu_1} \tilde{\times} \varpi^{\mu_2-\mu_1})$ in $m^{-1}(S_{\mu_2})$ are

$$S_{\mu_1} \tilde{\times} S_{\mu_2-\mu_1},$$

and

$$T_{\mu_1,\mu_2} := (pr_1, m)^{-1}(S^-_{\mu_1} \times \{\varpi^{\mu_2}\}),$$

respectively. Applying the hyperbolic localization theorem to $m^{-1}(S_{\mu_2})$, we have the following isomorphism

$$\mathrm{H}_c^*(S_{\mu_1} \tilde{\times} S_{\mu_2-\mu_1}, \mathcal{F} \tilde{\boxtimes} \mathcal{G}) \simeq \mathrm{H}^*_{T_{\mu_1,\mu_2}}(\mathcal{F} \tilde{\boxtimes} \mathcal{G}). \tag{6.6}$$

By Lemma 6.1.1, the above cohomology groups concentrate in a single degree.

Filtering the space $m^{-1}(S_{\mu_2})$ by $\{S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}|\mu_1 \in \mathbb{X}_\bullet\}$, we get a spectral sequence with E_1-terms $\mathrm{H}_c^*(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}\tilde{\boxtimes}\mathcal{G})$. As noticed in Lemma 6.1.1, this spectral sequence degenerates at E_1-page. Then, there exists a filtration

$$\mathrm{Fil}_{\mu_1,\mu_2'} := \mathrm{Ker}(\mathrm{H}^*(m^{-1}(S_{\mu_2}), \mathcal{F}\tilde{\boxtimes}\mathcal{G}) \to \mathrm{H}^*(\cup_{\mu_1'<\mu_1} S_{\mu_1'}\tilde{\times}S_{\mu_2-\mu_1'}, \mathcal{F}\tilde{\boxtimes}\mathcal{G}))$$

with associated graded

$$\bigoplus_{\mu_1} \mathrm{H}_c^*(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}\tilde{\boxtimes}\mathcal{G}).$$

Similarly, filtering $m^{-1}(S_{\mu_2})$ by $\{T_{\mu_1,\mu_2}|\mu_1 \in \mathbb{X}_\bullet\}$, we get an induced spectral sequence with E_1-terms $\mathrm{H}^*_{T_{\mu_1,\mu_2}}(\mathcal{F}\tilde{\boxtimes}\mathcal{G})$. This spectral sequence also degenerates at the E_1-page and there is an induced filtration

$$\mathrm{Fil}'_{\mu_1,\mu_2} := \mathrm{Im}(\mathrm{H}^*_{T_{<\mu_1,\mu_2}}(\mathcal{F}\tilde{\boxtimes}\mathcal{G}) \to \mathrm{H}^*(\mathcal{F}\tilde{\boxtimes}\mathcal{G})),$$

where $T_{<\mu_1,\mu_2} := \cup_{\mu_1'<\mu_1} T_{\mu_1',\mu_2}$. The two filtrations are complementary to each other by (6.6) and together define the decomposition

$$\mathrm{H}_c^*(S_{\mu_2}, \mathcal{F} \star \mathcal{G}) \simeq \bigoplus_{\mu_1} \mathrm{H}_c^*(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}\tilde{\boxtimes}\mathcal{G}).$$

□

By the above proposition, Proposition 6.1.2 induces a monoidal structure of the functor H^*.

Proposition 6.2.2. *The hypercohomology functor* $\mathrm{H}^*(Gr_G, \bullet) : P_{L^+G}(Gr_G, \Lambda) \to \mathrm{Mod}_\Lambda$ *is a monoidal functor. In addition, the obtained monoidal structure is compatible with the weight functor decomposition established in Proposition* 4.2.1

Proof. Recall for $\mathcal{F}, \mathcal{G} \in P_{L^+G}(Gr_G, \Lambda)$, the convolution product $\mathcal{F} \star \mathcal{G}$ is defined as $\mathcal{F} \star \mathcal{G} = Rm_!(\mathcal{F}\tilde{\boxtimes}\mathcal{G})$. Then by Lemma 6.1.1 and Proposition 6.1.2, there are canonical isomorphisms

$$\begin{aligned}
&\mathrm{H}^*(Gr_G, \mathcal{F} \star \mathcal{G}) \\
\cong&\mathrm{H}^*(Gr_G\tilde{\times}Gr_G, \mathcal{F}\tilde{\boxtimes}\mathcal{G}) \\
\cong&\bigoplus_{\mu_1,\mu_2} \mathrm{H}_c^*(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}\tilde{\boxtimes}\mathcal{G}) \\
\cong&\bigoplus_{\mu_1,\mu_2} \Big(\mathrm{H}_c^*(S_{\mu_1}, \mathcal{F}) \otimes \mathrm{H}_c^*(S_{\mu_2-\mu_1}, \mathcal{G})\Big) \\
\cong&\Big(\bigoplus_{\mu_1} \mathrm{H}_c^*(S_{\mu_1}, \mathcal{F})\Big) \otimes \Big(\bigoplus_{\mu_2} \mathrm{H}_c^*(S_{\mu_2}, \mathcal{G})\Big) \\
\cong&\mathrm{H}^*(\mathcal{F}) \otimes \mathrm{H}^*(\mathcal{G}).
\end{aligned}$$

Note that by Proposition 4.2.3, we have the decomposition of the total weight functor into direct sum of weight functors $\mathrm{H}^*(Gr_G, \mathcal{F} \star \mathcal{G}) \simeq \oplus_\lambda \mathrm{H}^*_c(S_\lambda, \mathcal{F} \star \mathcal{G})$. Proposition 6.3 then shows that the monoidal structure obtained above is compatible with the weight functor decomposition. Finally, we need to show that the monoidal structure of H^* is compatible with the associativity constraint. This can be proved by considering the $\mathbb{G}_m$- action on $Gr_G \tilde{\times} Gr_G \tilde{\times} Gr_G$ induced by the diagonal action of $\mathbb{G}_m$ on $Gr_G \times Gr_G \times Gr_G$ via the isomorphism

$$(m_1, m_2, m_3)^{-1} : Gr_G \times Gr_G \times Gr_G \simeq Gr_G \tilde{\times} Gr_G \tilde{\times} Gr_G.$$

Note that in this case we can still split the intersection $(S_{\nu_1} \tilde{\times} S_{\nu_2} \tilde{\times} S_{\nu_3}) \cap (Gr_{\leq \mu_1} \tilde{\times} Gr_{\leq \mu_2} \tilde{\times} Gr_{\leq \mu_3})$ by (4.1). This allows us to apply the hyperbolic localization theorem and a similar spectral sequence argument as before. We obtain the desired compatibility property and the proposition is thus proved.

□

With the monoidal structure of H^* established above, we are now ready to prove the following results.

Proposition 6.2.3. *For any $\mathcal{F} \in \mathrm{Sat}_{G,\Lambda}$, the functors $(\bullet) \star \mathcal{F}$ and $\mathcal{F} \star (\bullet)$ are both right exact. If in addition $\mathcal{F}$ is a projective object, then these functors are exact.*

Proof. Let

$$0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0 \tag{6.7}$$

be an exact sequence in $\mathrm{Sat}_{G,\Lambda}$. By Proposition 4.2.3, taking global cohomology gives an exact sequence

$$\mathrm{H}^*(\mathcal{G}') \longrightarrow \mathrm{H}^*(\mathcal{G}) \longrightarrow \mathrm{H}^*(\mathcal{G}'') \longrightarrow 0. \tag{6.8}$$

Tensoring (6.8) with $\mathrm{H}^*(\mathcal{F})$ gives the exact sequence

$$\mathrm{H}^*(\mathcal{G}') \otimes \mathrm{H}^*(\mathcal{F}) \longrightarrow \mathrm{H}^*(\mathcal{G}) \otimes \mathrm{H}^*(\mathcal{F}) \longrightarrow \mathrm{H}^*(\mathcal{G}'') \otimes \mathrm{H}^*(\mathcal{F}) \longrightarrow 0. \tag{6.9}$$

By proposition 6.3, (6.9) is canonically isomorphic to the following sequence

$$\mathrm{H}^*(\mathcal{G}' \star \mathcal{F}) \longrightarrow \mathrm{H}^*(\mathcal{G} \star \mathcal{F}) \longrightarrow \mathrm{H}^*(\mathcal{G}'' \star \mathcal{F}) \longrightarrow 0. \tag{6.10}$$

Notice that by Proposition 4.2.3, the global cohomology functor $\mathrm{H}^*(\bullet)$ is faithful, then the exactness of (6.9) implies that the sequence

$$\mathcal{G}' \star \mathcal{F} \longrightarrow \mathcal{G} \star \mathcal{F} \longrightarrow \mathcal{G}'' \star \mathcal{F} \longrightarrow 0$$

is also exact. The right exactness for $\mathcal{F} \star (\bullet)$ can be proved similarly.

Now, assume $\mathcal{F}$ to be a projective object in the Satake category. By Proposition 5.4, we know that the functors $(\bullet) \otimes \mathrm{H}^*(\mathcal{F})$ and $\mathrm{H}^*(\mathcal{F}) \otimes (\bullet)$ are both exact. Then arguing as before and using the monoidal structure and the faithfulness of the functor $\mathrm{H}^*(\bullet)$, we conclude the proof. □

We conclude the discussion on the monoidal structure of H^* by identifying it with the one constructed in [Zhu17]. For this purpose, we briefly recall the construction in *loc.cit.*

Let $\mathcal{F}, \mathcal{G} \in P_{L^+G}(Gr_G, \bar{\mathbb{Q}}_\ell)$. Assume that L^+G acts on $\mathrm{supp}(\mathcal{G})$ via the quotient $L^+G \to L^mG$. Define $\mathrm{supp}(\mathcal{F})\tilde{\times}\mathrm{supp}(\mathcal{G}) := \mathrm{supp}(\mathcal{F})^{(m)} \times^{L^mG} \mathrm{supp}(\mathcal{G})$ and denote by π the projection morphism $\mathrm{supp}(\mathcal{F})^{(m)} \to \mathrm{supp}(\mathcal{F})$. Then we have an $L^+G \times L^mG$-equivariant projection morphism

$$p : \mathrm{supp}(\mathcal{F})^{(m)} \times \mathrm{supp}(\mathcal{G}) \longrightarrow \mathrm{supp}(\mathcal{F})\tilde{\times}\mathrm{supp}(\mathcal{G})$$

where L^+G acts on $\mathrm{supp}(\mathcal{F})^{(m)}$ by multiplication on the left and L^mG acts on $\mathrm{supp}(\mathcal{F})^{(m)} \times \mathrm{supp}(\mathcal{G})$ diagonally from the middle. Then p induces a canonical isomorphism of the L^+G-equivariant cohomology (cf. [Zhu17] A.3.5)

$$\mathrm{H}^*_{L^+G}(\mathrm{supp}(\mathcal{F})\tilde{\times}\mathrm{supp}(\mathcal{G}), \mathcal{F}\tilde{\boxtimes}\mathcal{G}) \cong \mathrm{H}^*_{L^+G\times L^mG}(\mathrm{supp}(\mathcal{F})^{(m)} \times \mathrm{supp}(\mathcal{G}), \pi^*\mathcal{F} \boxtimes \mathcal{G}). \tag{6.11}$$

By the equivariant Künneth formula (cf. [Zhu16, A.1.15]), there is a canonical isomorphism

$$\begin{aligned} &\mathrm{H}^*_{L^+G\times L^mG}(\mathrm{supp}(\mathcal{F})^{(m)} \times \mathrm{supp}(\mathcal{G}), \pi^*\mathcal{F} \boxtimes \mathcal{G}) \\ \cong &\mathrm{H}^*_{L^+G\times L^mG}(\mathrm{supp}(\mathcal{F})^{(m)}, \mathcal{F}) \otimes \mathrm{H}^*_{L^+G\times L^mG}(\mathrm{supp}(\mathcal{G}), \mathcal{G}). \end{aligned} \tag{6.12}$$

Combine (6.11) with (6.12), and we conclude a canonical isomorphism

$$\mathrm{H}^*_{L^+G}(\mathrm{supp}(\mathcal{F})\tilde{\times}\mathrm{supp}(\mathcal{G}), \mathcal{F}\tilde{\boxtimes}\mathcal{G}) \cong \mathrm{H}^*_{L^+G}(\mathrm{supp}(\mathcal{F}), \mathcal{F}) \otimes \mathrm{H}^*_{L^+G}(\mathrm{supp}(\mathcal{G}), \mathcal{G}). \tag{6.13}$$

We denote by $\overline{G}_{\bar{\mathbb{Q}}_\ell}$ the base change of $\overline{G}$ to $\bar{\mathbb{Q}}_\ell$. Let $R_{\overline{G},\ell} := \mathrm{Sym}(\mathfrak{g}_{\bar{\mathbb{Q}}_\ell}(-1))^{G_{\bar{\mathbb{Q}}_\ell}}$ denote the algebra of invariant polynomials on the Lie algebra $\mathfrak{g}_{\bar{\mathbb{Q}}_\ell}(-1)$. Then (6.13) induces an isomorphism of $R_{\overline{G},\ell}$-bimodules. In addition, the two $R_{\overline{G},\ell}$-module structures coincide ([Zhu17, Lemma 2.19]) and the base change of (6.13) along the argumentation map $R_{\overline{G},\ell} \to \bar{\mathbb{Q}}_\ell$, the canonical isomorphism

$$\mathrm{H}^*_{L^+G}(\mathcal{F}) \otimes_{R_{\overline{G},\ell}} \bar{\mathbb{Q}}_\ell \simeq \mathrm{H}^*(\mathcal{F}) \tag{6.14}$$

gives the monoidal structure of H^* in the $\bar{\mathbb{Q}}_\ell$-case ([Zhu17], Proposition 2.20).

Then to identify the monoidal structures, it suffices to prove the following proposition.

Proposition 6.2.4. *Let $\mathcal{F}, \mathcal{G} \in P_{L^+G}(Gr_G, \mathbb{Z}_\ell)$ be two projective objects. We denote $\mathcal{F} \otimes \bar{\mathbb{Q}}_\ell$ and $\mathcal{G} \otimes \bar{\mathbb{Q}}_\ell$ by $\mathcal{F}'$ and $\mathcal{G}'$, respectively. Then the following diagram commutes*

$$\begin{array}{ccc}
\mathrm{H}^*_{L^+G}(\mathrm{supp}(\mathcal{F}')\tilde{\times}\mathrm{supp}(\mathcal{G}'), \mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \otimes_{R_{\overline{G},\ell}} \bar{\mathbb{Q}}_\ell & \xrightarrow{(6.14)} & \mathrm{H}^*(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \\
\downarrow (6.13) & & \downarrow \alpha \\
(\mathrm{H}^*_{L^+G}(\mathcal{F}') \otimes_{R_{\overline{G},\ell}} \mathrm{H}^*_{L^+G}(\mathcal{G}')) \otimes_{R_{\overline{G},\ell}} \bar{\mathbb{Q}}_\ell & & \bigoplus_{\mu_1,\mu_2} \mathrm{H}^*_c(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \\
\downarrow \cong & & \downarrow \beta \\
(\mathrm{H}^*_{L^+G}(\mathcal{F}') \otimes_{R_{\overline{G},\ell}} \bar{\mathbb{Q}}_\ell) \otimes_{\bar{\mathbb{Q}}_\ell} (\mathrm{H}^*_{L^+G}(\mathcal{G}') \otimes_{R_{\overline{G},\ell}} \bar{\mathbb{Q}}_\ell) & \xrightarrow{\simeq} & (\bigoplus_{\mu_1} \mathrm{H}^*_c(S_{\mu_1}, \mathcal{F}')) \otimes_{\bar{\mathbb{Q}}_\ell} (\bigoplus_{\mu_2} \mathrm{H}^*_c(S_{\mu_2-\mu_1}, \mathcal{G}'))
\end{array} \tag{6.15}$$

where the morphisms α and β are the base change of isomorphisms (6.4) *and* (6.1) *to $\bar{\mathbb{Q}}_\ell$, respectively.*

Proof. Consider the filtrations

$$\mathrm{Fil}_{\geq \mu_1,\mu_2}\mathrm{H}^*(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}'), \mathrm{Fil}_{\geq\mu}\mathrm{H}^*(\mathcal{F}'), \text{ and } \mathrm{Fil}_{\geq\mu}\mathrm{H}^*(\mathcal{G}')$$

defined as in Proposition 6.1.2 and Proposition 4.2.3. To prove the proposition, it suffices to prove that these filtrations respect (6.13). Then taking the Verdier dual (note now $\mathrm{H}^*(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}')$, $\mathrm{H}^*(\mathcal{F}')$, and $\mathrm{H}^*(\mathcal{G}')$ are all $\bar{\mathbb{Q}}_\ell$-vector spaces) implies that the complementary filtrations $\mathrm{Fil}'_{<\mu_1,\mu_2}$ and $\mathrm{Fil}'_{<\mu}$ also respect (6.13). This will provide the commutativity of (6.15).

The approach we will use is similar to the one given in [Zhu16, Proposition 5.3.14], and we sketch it here. Although the semi-infinite orbit S_μ does not admit an L^+G-action, it is stable under the action of the constant torus $T \subset L^+T \subset L^+G$. Then so is the convolution product of semi-infinite orbits $S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}$. Stratifying $Gr_G\tilde{\times}Gr_G$ by $\{S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1} \mid \mu_1, \mu_2 \in \mathbb{X}_\bullet\}$, we get a spectral sequence with E_1-terms $\mathrm{H}^*_{T,c}(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}'\tilde{\boxtimes}\mathcal{G}')$ which abuts to $\mathrm{H}^*_T(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}')$. By [Zhu17, Proposition 2.7], the spectral sequence degenerates at the E_1-page and the filtration $\mathrm{Fil}_{\geq\mu_1,\mu_2}$ thus lifts to a new filtration of H^*_T

$$\mathrm{Fil}_{\geq\mu_1,\mu_2}\mathrm{H}^*_T(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') := \ker(\mathrm{H}^*_T(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \to \mathrm{H}^*_T(S_{<\mu_1}\tilde{\times}S_{<\mu_2-\mu_1}, \mathcal{F}'\tilde{\boxtimes}\mathcal{G}')).$$

Using a similar argument as in the proof of Proposition 6.1.2, the associated graded of this filtration equals $\bigoplus_{\mu_1,\mu_2} \mathrm{H}^*_{T,c}(S_{\mu_1}\tilde{\times}S_{\mu_2-\mu_1}, \mathcal{F}'\tilde{\boxtimes}\mathcal{G}')$. Note that all the terms in this filtration and the associated graded are in fact free $R_{\overline{T},\ell}$-modules, then base change to $\bar{\mathbb{Q}}_\ell$ along the argumentation map $R_{\overline{T},\ell} \to \bar{\mathbb{Q}}_\ell$ recovers our original filtration $\mathrm{Fil}_{\geq\mu_1,\mu_2}$. Similarly, we can define the filtrations $\mathrm{Fil}_{\geq\mu}\mathrm{H}^*_T(\mathcal{F}')$ and $\mathrm{Fil}_{\geq\mu}\mathrm{H}^*_T(\mathcal{G}')$ which recover the original filtrations $\mathrm{Fil}_{\geq\mu}\mathrm{H}^*(\mathcal{F}')$ and $\mathrm{Fil}_{\geq\mu}\mathrm{H}^*(\mathcal{G}')$ in the same way.

Since

$$\mathrm{H}^*_T(\bullet) \simeq \mathrm{H}^*_{L^+G}(\bullet) \otimes_{R_{\overline{G},\ell}} R_{\overline{T},\ell} : P_{L^+G}(Gr_G, \bar{\mathbb{Q}}_{ell}) \longrightarrow \mathrm{Vect}_{\bar{\mathbb{Q}}_\ell},$$

then (6.13) induces a monoidal structure on the T-equivariant cohomology

$$\mathrm{H}^*_T(\mathcal{F}' \star G') \simeq \mathrm{H}^*_T(\mathcal{F}') \otimes \mathrm{H}^*_T(\mathcal{G}'). \tag{6.16}$$

Then we are left to show that (6.16) is compatible with the filtrations $\mathrm{Fil}_{\geq\mu_1,\mu_2}$ and $\mathrm{Fil}_{\geq\mu}$. It suffices to check the compatibility with filtrations $\mathrm{Fil}_{\geq\mu_1,\mu_2}\mathrm{H}^*_T$ and $\mathrm{Fil}_{\geq\mu}\mathrm{H}^*_T$ over the generic point of $\mathrm{Spec}R_{\overline{T},\ell}$. Denote

$$\mathrm{H}_\lambda := \mathrm{H}^*_T \otimes_{R_{\overline{T},\ell}} Q$$

where Q is the fraction filed of $R_{\overline{T},\ell}$. By the equivariant localization theorem, we have isomorphisms

$$\mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \simeq \bigoplus_{\mu_1,\mu_2} \mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}'|_{(\varpi^{\mu_1}\tilde{\times}\varpi^{\mu_2-\mu_1})})$$

and

$$\mathrm{H}^*_\lambda(S_{<\mu_1,<\mu_2}\,\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \simeq \bigoplus_{\nu_1<\mu_1,\nu_2<\mu_2} \mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}'|_{(\varpi^{\nu_1}\tilde{\times}\varpi^{\nu_2-\nu_1})}).$$

Then it follows that

$$\mathrm{Fil}_{\geq\mu_1,\mu_2}\mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') := \mathrm{Fil}_{\geq\mu_1,\mu_2}\mathrm{H}^*_T(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}')\otimes_{R_{\overline{T},\ell}}Q \simeq \bigoplus_{\nu_1\geq\mu_1,\nu_2\geq\mu_2} \mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}'|_{(\varpi^{\nu_1}\tilde{\times}\varpi^{\nu_2-\nu_1})}).$$

Applying the equivariant localization theorem again gives isomorphisms

$$\mathrm{H}_\lambda(\mathcal{F}') \simeq \bigoplus_{\mu} \mathrm{H}_\lambda(\mathcal{F}'|_{\varpi^\mu})$$

and

$$\mathrm{H}_\lambda(S_{<\mu}, \mathcal{F}') \simeq \bigoplus_{\nu<\mu} \mathrm{H}_\lambda(\mathcal{F}'|_{\varpi^\nu}).$$

Similarly, we get a filtration $\mathrm{Fil}_{\geq\mu}\mathrm{H}_\lambda(\mathcal{F}') \simeq \oplus_{\nu\geq\mu}\mathrm{H}_\lambda(\mathcal{F}'|_{\varpi^\nu})$ induced by $\mathrm{Fil}_{\geq\mu}\mathrm{H}^*_T(\mathcal{F})$.

Notice that as for $\mathrm{H}^*_{L^+G}(\bullet)$, the monoidal structure (6.16) is defined via the composition of the following isomorphisms

$$\begin{aligned}
&\mathrm{Fil}_{\geq \mu_1,\mu_2}\mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}') \\
\simeq &\bigoplus_{\nu_1\geq\mu_1,\nu_2\geq\mu_2} \mathrm{H}_\lambda(\mathcal{F}'\tilde{\boxtimes}\mathcal{G}'|_{(\varpi^{\nu_1}\tilde{\times}\varpi^{\nu_2-\nu_1})}) \\
\simeq &\bigoplus_{\nu_1\geq\mu_1,\nu_2\geq\mu_2} \mathrm{H}_\lambda(\mathcal{F}'|_{\varpi^{\nu_1}})\otimes \mathrm{H}_\lambda(\mathcal{G}'|_{\varpi^{\nu_2-\nu_1}}) \\
\simeq &\bigoplus_{\nu_1\geq\mu_1} \mathrm{H}_\lambda(\mathcal{F}'|_{\varpi^{\nu_1}})\otimes \bigoplus_{\nu_2\geq\mu_2-\mu_1} \mathrm{H}_\lambda(\mathcal{G}'|_{\varpi^{\nu_2}}) \\
\simeq &\mathrm{Fil}_{\geq\mu_1}\mathrm{H}_\lambda(\mathcal{F}')\bigotimes \mathrm{Fil}_{\geq\mu_2-\mu_1}\mathrm{H}_\lambda(\mathcal{G}')
\end{aligned}$$

where the second isomorphism is obtained by an analogue of (6.13) for T-equivariant cohomology and the equivariant Künneth formula. Note that the monoidal structure of the total weight functor CT is compatible with that of the hypercohomology functor H^* by Proposition 6.3. We thus conclude the proof. □

Chapter 7

TANNAKIAN CONSTRUCTION

Let $Z \subset Gr_G$ denote a closed subspace consisting of a finite union of L^+G-orbits. Then any $\mathcal{F} \in P_{L^+G}(Z, \Lambda)$ admits a presentation

$$P_1 \longrightarrow P_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where P_1 and P_0 are finite direct sums of $P_Z(\Lambda)$.

Write $A_Z(\Lambda)$ for $\mathrm{End}_{P_{L^+G}(Z,\Lambda)}(P_Z(\Lambda))^{op}$. By Proposition 5.0.3.(2), $A_Z(\Lambda)$ is a finite free Λ-module, and any finitely generated $A_Z(\Lambda)$-module is also finitely presented. Now we recall the following version of Gabriel and Mitchell's theorem as formulated in [BR18, Theorem 9.1].

Theorem 7.0.1. *Let C be an abelian category. Let P be a projective object and write $A = \mathrm{End}_C(P)^{op}$. Denote $\mathcal{M}$ to be the full subcategory of C consisting of objects M which admits a presentation*

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where P_1 and P_0 are finite direct sums of P. Let $\mathcal{M}'_A$ be the category of finitely presented right A-modules. Then

(1) there is an equivalence of abelian categories $\mathcal{M} \simeq \mathcal{M}'$ induced by the functor $\mathrm{Hom}_C(P, \bullet)$,

(2) there is a canonical isomorphism between the endomorphism ring of the functor $\mathrm{Hom}_C(P, \bullet)$ and A^{op}.

The above theorem and the discussion before it enable us to deduce an equivalence of abelian categories

$$E_Z : P_{L^+G}(Z, \Lambda) \simeq \mathcal{M}'_{A_Z(\Lambda)}.$$

Let $i : Y \hookrightarrow Z$ be an inclusion of closed subsets consisting of L^+G-orbits, then we have the functor $i_* : P_{L^+G}(Y, \Lambda) \to P_{L^+G}(Z, \Lambda)$. In addition, i_* induces a functor $(i^Z_Y)^* : \mathcal{M}'_{A_Y(\Lambda)} \to \mathcal{M}'_{A_Z(\Lambda)}$ which in turn gives a ring homomorphism

$i_Y^Z : A_Z(\Lambda) \to A_Y(\Lambda)$. Note for any $a \in A_Z(\Lambda)$ and $\mathcal{F} \in P_{L^+G}(Y, \Lambda)$, we have canonical isomorphisms

$$\begin{aligned} & a \cdot E_Z(i_*\mathcal{F}) \\ \simeq & a \cdot \mathrm{Hom}(P_Z(\Lambda), i_*\mathcal{F})) \\ \simeq & a \cdot (i_Y^Z)^*(\mathrm{Hom}(P_Z(\Lambda), i_*\mathcal{F})) \\ \simeq & i_Y^Z(a) \cdot \mathrm{Hom}(P_Y(\Lambda), \mathcal{F})) \\ \simeq & i_Y^Z(a) \cdot E_Y(\mathcal{F}). \end{aligned}$$

Define $B_Z(\Lambda) := \mathrm{Hom}(A_Z(\Lambda), \Lambda))$. Since $A_Z(\Lambda)$ is a finite free Λ-module, then so is $B_Z(\Lambda)$ and we have the following canonical equivalence of abelian categories

$$\mathcal{M}'_{A_Z(\Lambda)} \simeq \mathrm{Comod}_{B_Z(\Lambda)}.$$

The dual map of i_Y^Z gives a map $\iota_Z^Y : B_Y(\Lambda) \to B_Z(\Lambda)$. Let $B(\Lambda) = \varinjlim B_Z(\Lambda)$, we conclude that $\mathrm{Sat}_{G,\Lambda} \simeq \mathrm{Comod}_{B(\Lambda)}$ as abelian categories. Moreover, by Proposition 5.0.3.(3) we know that

$$B(\Lambda) \simeq B(\mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \Lambda \tag{7.1}$$

for $\Lambda = \bar{\mathbb{Q}}_\ell$ and $\mathbb{F}_\ell$.

Take any $\mu \in \mathbb{X}_\bullet^+$, and write $A_\mu(\Lambda)$ and $B_\mu(\Lambda)$ for $A_{Gr_{\leq\mu}}(\Lambda)$ and $B_{Gr_{\leq\mu}}(\Lambda)$, respectively. For any $\mu, \nu \in \mathbb{X}_\bullet^+$ such that $\mu \leq \nu$, use notations i_μ^ν and ι_ν^μ for $i_{Gr_{\leq\mu}}^{Gr_{\leq\nu}}$ and $\iota_{Gr_{\leq\nu}}^{Gr_{\leq\mu}}$, respectively. Also, we denote by $P_\theta(\Lambda)$ the projective object $P_{Gr_{\leq\theta}(\Lambda)}$ for any $\theta \in \mathbb{X}_\bullet^+$. Note the following canonoical isomorphism by the monoidal structure of H^* established by Proposition 6.2.2

$$\begin{aligned} & \mathrm{Hom}(P_{\mu+\nu}(\Lambda), P_\mu \star P_\nu) \\ \simeq & \mathrm{H}^*(P_\mu \star P_\nu) \\ \simeq & \mathrm{H}^*(P_\mu) \otimes \mathrm{H}^*(P_\nu) \\ \simeq & \mathrm{Hom}(P_\mu(\Lambda), P_\mu(\Lambda)) \otimes \mathrm{Hom}(P_\nu(\Lambda), P_\nu(\Lambda)). \end{aligned}$$

Then, the element $id_{P_\mu(\Lambda)} \otimes id_{P_\nu(\Lambda)} \in \mathrm{Hom}(P_\mu(\Lambda), P_\mu(\Lambda)) \otimes \mathrm{Hom}(P_\nu(\Lambda), P_\nu(\Lambda))$ gives rise to a morphism

$$f_{\mu,\nu} : P_{\mu+\nu}(\Lambda) \longrightarrow P_\mu(\Lambda) \star P_\nu(\Lambda).$$

Applying the functor H^* and dualizing, we get a morphism

$$g_{\mu,\nu} : B_\mu(\Lambda) \otimes B_\nu(\Lambda) \to B_{\mu+\nu}(\Lambda).$$

We check that the multiplication maps $g_{\bullet,\bullet}$ are compatible with the maps $\iota_\bullet^\bullet$ i.e. for any $\mu \le \mu', \nu \le \nu' \in \mathbb{X}_\bullet^+$, the following diagram commutes

$$\begin{CD} B_\mu(\Lambda) \otimes B_\nu(\Lambda), @>{g_{\mu,\nu}}>> B_{\mu+\nu}(\Lambda) \\ @V{\iota^\mu_{\mu'} \otimes \iota^\nu_{\nu'}}VV @VV{\iota^{\mu+\nu}_{\mu'+\nu'}}V \\ B_{\mu'}(\Lambda) \otimes B_{\nu'}(\Lambda) @>{g_{\mu',\nu'}}>> B_{\mu'+\nu'}(\Lambda). \end{CD} \tag{7.2}$$

By the constructions of g's and ι's, it suffices to check the commutativity for

$$\begin{CD} P_{\mu'+\nu'}(\Lambda) @>{f_{\mu',\nu'}}>> P_{\mu'}(\Lambda) \star P_{\nu'}(\Lambda) \\ @V{p^{\mu'+\nu'}_{\mu+\nu}}VV @VV{p^{\mu'}_\mu \star p^{\nu'}_\nu}V \\ P_{\mu+\nu}(\Lambda) @>{f_{\mu,\nu}}>> P_\mu(\Lambda) \star P_\nu(\Lambda), \end{CD} \tag{7.3}$$

Here, maps $p_\bullet^\bullet$ appearing in the above diagram are the maps $p_\bullet^\bullet$ in Proposition 5.0.3.(1). The construction of f's implies that we are left to show that the following diagram commutes

$$\begin{CD} \mathrm{H}^*(P_{\mu'}(\Lambda) \star P_{\nu'}(\Lambda)) @>{\cong}>> \mathrm{H}^*(P_{\mu'}(\Lambda)) \otimes \mathrm{H}^*(P_{\nu'}(\Lambda)) \\ @V{\mathrm{H}^*(p^{\mu'}_\mu \star p^\nu_{\nu'})}VV @VV{\mathrm{H}^*(p^{\mu'}_\mu) \otimes \mathrm{H}^*(p^{\nu'}_\nu)}V \\ \mathrm{H}^*(P_\mu(\Lambda) \star P_\nu(\Lambda)) @>{\cong}>> \mathrm{H}^*(P_\mu(\Lambda)) \otimes \mathrm{H}^*(P_\nu(\Lambda)) \end{CD}$$

By the monoidal structure of H^*, the above diagram commutes and so is diagram (9.2). Taking direct limit, the morphisms $g_{\mu,\nu}$ give a multiplication map on $B(\Lambda)$ by the above discussion. Our observation at the end of §3 ensures the multiplication on $B(\Lambda)$ is associative.

Note clearly that $B_0(\Lambda) = \Lambda$ and the canonical map $B_0(\Lambda) \to B(\Lambda)$ gives the unit map for $B(\Lambda)$. Now to endow $B(\Lambda)$ with a bialgebra structure in the sense of [DM82, §2], it suffices to prove the multiplication on $B(\Lambda)$ is commutative and $B(\Lambda)$ admits an antipode. The later statement can be proved in a completely similar manner as in [BR18, Proposition 13.4] once the former statement is proved. Thus it suffices to prove the commutativity of the multiplication of $B(\Lambda)$.

First by the compatibility of morphisms $g_{\bullet,\bullet}$ with $\iota_\bullet^\bullet$, it suffices to prove for each $\mu \in \mathbb{X}_\bullet^+$ that the multiplication on $B_\mu(\Lambda)$ is commutative. Consider the following diagram

$$\begin{array}{ccc} B_\mu(\Lambda) \otimes B_\mu(\Lambda) & \xrightarrow{d} & B_{2\mu}(\Lambda) \\ \big\downarrow & & \big\downarrow \\ B_\mu(\bar{\mathbb{Q}}_\ell) \otimes B_\mu(\bar{\mathbb{Q}}_\ell) & \xrightarrow{d'} & B_{2\mu}(\bar{\mathbb{Q}}_\ell). \end{array} \tag{7.4}$$

The vertical arrows are inclusions by noting (7.1) and the fact that $B_\mu(\Lambda)$ is a finite free Λ-module. The map d is defined to map $b_1 \otimes b_2$ to $g_{\mu,\mu}(b_1 \otimes b_2) - g_{\mu,\mu}(b_2 \otimes b_1)$. The map d' is defined similarly. By Proposition 6.2.4 and isomorphism (7.1), diagram (7.4) is commutative. By the construction of the commutativity constraint in $P_{L^+G}(Gr_G, \bar{\mathbb{Q}}_\ell)$ in [Zhu17], we conclude that d is the zero map and the multiplication map in $B(\Lambda)$ is thus commutative. Thus, by a complete similar argument as in [DM82, Proposition 2.16], the category $\mathrm{Comod}_{B(\Lambda)}$ can be equipped with a commutativity constraint. This commutativity constraint then induces that of $\mathrm{Sat}_{G,\Lambda}$. Thus we have endowed $\mathrm{Sat}_{G,\Lambda}$ with a tensor category structure. Then a complete similar argument as in [BR18, Proposition 13.4] shows that $B(\Lambda)$ admits an antipode.

Chapter 8

IDENTIFICATION OF GROUP SCHEMES

With the work in previous chapters, we have constructed the category $P_{L^+G}(Gr_G, \Lambda)$, and equipped it with

(1) the convolution product $\star$ and an associativity constraint,

(2) the hypercohomology functor $\mathrm{H}^* : P_{L^+G}(Gr_G, \Lambda) \to \mathrm{Mod}_\Lambda$ which is Λ-linear, exact, and faithful,

(3) a commutativity constraint which makes $\mathrm{Sat}_{G,\Lambda}$ a tensor category,

(3) a unit object IC_0,

(4) a bialgebra $B(\Lambda)$ such that $\mathrm{Sat}_{G,\Lambda}$ is equivalent to $\mathrm{Comod}_{B(\Lambda)}$ as tensor categories.

Note that by Proposition 5.0.3 (2), $\mathrm{H}^*(P_Z(\mathbb{Z}_\ell))$ is a free $\mathbb{Z}_\ell$-module for any $Z \subset Gr_G$ consisting of a finite union of L^+G-orbits Z. We also know that the representing object $P_Z(\mathbb{Z}_\ell)$ is stable under base change by Proposition 5.0.3 (3). By our discussion in the previous section, we have the following generalized Tannakian construction similar to [MV07, Proposition 11.1]

Proposition 8.0.1. *The category of representations of the group scheme* $\widetilde{G}_{\mathbb{Z}_\ell} := \mathrm{Spec}(B(\mathbb{Z}_\ell))$ *which are finitely generated over* $\mathbb{Z}_\ell$*, is equivalent to* $P_{L^+G}(Gr_G, \mathbb{Z}_\ell)$ *as tensor categories. Furthermore, the coordinate ring of* $\widetilde{G}_{\mathbb{Z}_\ell}$ *is free over* $\mathbb{Z}_\ell$ *and* $\widetilde{G}_{\mathbb{F}_\ell} = \mathrm{Spec}(\mathbb{F}_\ell) \times_{\mathbb{Z}_\ell} \widetilde{G}_{\mathbb{Z}_\ell}$.

We are left to identify the group scheme $\widetilde{G}_{\mathbb{Z}_\ell}$ with the Langlands dual group $\hat{G}_{\mathbb{Z}_\ell}$.

Note that reductive group schemes over $\mathbb{Z}_\ell$ are uniquely determined by their root datum. Then it suffices to prove the followings for our purpose

(1) $\widetilde{G}_{\mathbb{Z}_\ell}$ is smooth over $\mathbb{Z}_\ell$,

(2) the group scheme $\widetilde{G}_{\overline{\mathbb{F}}_\ell}$ is reductive,

(3) the dual split torus $\hat{T}_{\mathbb{Z}_\ell}$ is a maximal torus of $\widetilde{G}_{\mathbb{Z}_\ell}$.

In §7, we showed that $B(\mathbb{Z}_\ell)$ is a free $\mathbb{Z}_\ell$-modules. As a result, the group scheme $\widetilde{G}_{\mathbb{Z}_\ell}$ is affine flat over $\mathbb{Z}_\ell$. Then the affineness of $\widetilde{G}_{\mathbb{Z}_\ell}$ together with the statements (1) and (2) in this paragraph amount to the definition of a reductive group over $\mathbb{Z}_\ell$. Recall in [PY06], a group scheme $\mathcal{G}$ over a discrete valuation ring R with uniformizer π, field of fractions K, and residue field κ is said to be *quasi-reductive* if

(1) $\mathcal{G}$ is affine flat over R,

(2) $\mathcal{G}_K := \mathcal{G} \otimes_R K$ is connected and smooth over K,

(3) $\mathcal{G}_\kappa := \mathcal{G} \otimes_R \kappa$ is of finite type over κ and the neutral component $(\mathcal{G}_{\bar{\kappa}})^\circ_{\text{red}}$ of the reduced geometric fibre is a reductive group of dimension equals $\dim \mathcal{G}_K$.

We will make use of the following theorem for quasi-reductive group schemes proved in *loc.cit*.

Theorem 8.0.2. *Let $\mathcal{G}$ be a quasi-reductive group scheme over R. Then*

(1) $\mathcal{G}$ is of finite type over R

(2) $\mathcal{G}_K$ is reductive

(3) $\mathcal{G}_\kappa$ is connected.

In addition, if

(4) the type of $\mathcal{G}_{\bar{K}}$ is of the same type as that of $(\mathcal{G}_{\bar{\kappa}})^\circ_{\text{red}}$,

then $\mathcal{G}$ is reductive.

As noted above, the requirement (1) of quasi-reductiveness is satisfied by $\widetilde{G}_{\mathbb{Z}_\ell}$. In addition, by [Zhu17], the group scheme $\widetilde{G}_{\mathbb{Q}_\ell}$ is connected reductive with root datum dual to that of G and the condition 2 of quasi-reductiveness is met.

Lemma 8.0.3. *The group scheme $\widetilde{G}_{\bar{\mathbb{F}}_\ell}$ is connected.*

Proof. Note that the same proof as in [MV07, §12] and [BR18, Lemma 9.3] applies in our setting to show that the Satake category $P_{L^+G}(Gr_G, \bar{\mathbb{F}}_\ell)$ has no object $\mathcal{F}$ such that the subcategory $\langle \mathcal{F} \rangle$, which is the strictly full subcategory of $P_{L^+G}(Gr_G, \bar{\mathbb{F}}_\ell)$ whose objects are those isomorphic to a subquotient of $\mathcal{F}^{\star n}$ for some $n \in \mathbb{N}$, is stable under $\star$. This is equivalent to the fact that there does not exist an object $X \in \mathrm{Rep}_{\bar{\mathbb{F}}_\ell}(\widetilde{G}_{\bar{\mathbb{F}}_\ell})$ such that $\langle X \rangle$ is stable under $\bigotimes$ via Proposition 8.1. Then by [BR18, Corollary 2.11.2], we conclude our proof. □

From now on, let $\kappa = \bar{\mathbb{F}}_\ell$. We have proved in Proposition 6.2.4 that the monoidal structure of H^* is compatible with the weight functor decomposition. In other words, we get a monoidal functor

$$\mathrm{CT} : \mathrm{Sat}_{G,\mathbb{Z}_\ell} \longrightarrow \mathrm{Mod}_{\mathbb{Z}_\ell}(\mathbb{X}_\bullet) \simeq \mathrm{Sat}_{T,\mathbb{Z}_\ell}.$$

Base change to κ, the same reasoning yields a monoidal functor

$$\mathrm{CT} : \mathrm{Sat}_{G,\kappa} \longrightarrow \mathrm{Mod}_{\kappa}(\mathbb{X}_\bullet) \simeq \mathrm{Sat}_{T,\kappa}.$$

Applying the construction in §7 to the above two Satake categories, we get a natural homomorphism $\hat{T} \to \widetilde{G}$. Note that by [Zhu17, Corollary 2.8] and Proposition 5.0.3.(2), any $M \in \mathrm{Mod}_\kappa(\mathbb{X}_\bullet)$ can be realized as a subquotient of some projective object in $\mathrm{Sat}_{G,\kappa}$. It then follows from [DM82, Proposition 2.21(b)] that the homomorphism $\hat{T} \to \widetilde{G}$ is in fact a closed embedding, which realizes the dual torus $\hat{T}_\kappa$ as a subtorus of $\widetilde{G}_\kappa$. In addition, since $\widetilde{G}_{\mathbb{Z}_\ell}$ is flat, the same argument in [MV07, §12] applies to give the following dimension estimate

$$\dim G = \dim \widetilde{G}_{\mathbb{Q}_\ell} \geq \dim(\widetilde{G}_\kappa)_{\mathrm{red}}. \tag{8.1}$$

We can write $\widetilde{G}_\kappa = \varprojlim \widetilde{G}^*_\kappa$ where $\widetilde{G}^*_\kappa$ satisfies the following conditions

(1) $\widetilde{G}^*_\kappa$ is of finite type,

(2) the canonical map $\mathrm{Irr}_{\widetilde{G}^*_\kappa} \to \mathrm{Irr}_{\widetilde{G}_\kappa}$ is a bijection, where Irr denotes the set of irreducible representations.

In addition, we require that the transition morphisms are surjective. The first requirement may be satisfied since any group scheme is a projective limit of group schemes of finite type. To ensure that condition (2) can be satisfied, it is enough to choose $\widetilde{G}^*_\kappa$ sufficiently large so that the irreducible representations $L(\eta)$ associated to a finite set of generators η of the semigroup of dominant cocharacters $\mathbb{X}_\bullet^+$, are pull-backs of representations of $\widetilde{G}_\kappa^*$. For any $\mu, \nu \in \mathbb{X}_\bullet^+$, the sheaf $\mathrm{IC}_{\mu+\nu}$ supports on $Gr_{\leq\mu+\nu}$ and hence is a subquotient of $\mathrm{IC}_\mu \star \mathrm{IC}_\nu$. Thus all irreducible representations of $\widetilde{G}_\kappa$ come from $\widetilde{G}_\kappa^*$. By our choice of the finite type quotients, we have $(\widetilde{G}_\kappa)_{\mathrm{red}} = \varprojlim(\widetilde{G}_\kappa^*)_{\mathrm{red}}$. In addition, the composition of maps $\hat{T}_\kappa \to \widetilde{G}_\kappa \to \widetilde{G}_\kappa^*$ is a closed embedding.

We claim that

$$\text{Each finite type quotient } \widetilde{G}_\kappa^* \text{ is connected, reductive, and isomorphic to } \hat{G}_\kappa. \tag{8.2}$$

If (8.2) holds, then the arguments in [MV18] apply and yield $\widetilde{G}_\kappa^* = (\hat{G})_\kappa$. Thus we deduce that condition (3) of the quasi-reductiveness and condition (4) in Theorem 8.2 are satisfied by $\widetilde{G}_\kappa$ and we complete the identification of group schemes by Theorem 8.2. Next, we prove (8.2) following the approach given in [MV07, §12].

Write H for the reductive quotient of $(\widetilde{G}_\kappa^*)_{\mathrm{red}}$, and we have that $\hat{T}_\kappa \to H$ is a closed embedding. Note that any irreducible representation of $(\widetilde{G}_\kappa^*)_{\mathrm{red}}$ is trivial on the unipotent radical. We then have:

$$\text{The canonical map } \mathrm{Irr}_H \to \mathrm{Irr}_{(\widetilde{G}_\kappa^*)_{\mathrm{red}}} \text{ is a bijection.} \tag{8.3}$$

We first note the following lemma.

Lemma 8.0.4. *The subtorus $\hat{T}_\kappa$ is a maximal subtorus of H.*

Proof. Choose a maximal torus T_H for H and denote its Weyl group W_H. Then the irreducible representations of H are parametrized by $\mathbb{X}^\bullet(T_H)/W_H$. On the other hand, write the Weyl group for G by W_G, then Proposition 2.3 implies that $\mathbb{X}_\bullet(T_\kappa)/W_G$ parametrizes Schubert cells in Gr_{G_κ}. The IC-sheaf attached to each Schubert cell is an irreducible object in the Satake category, and thus gives rise to an irreducible representation of $\widetilde{G}_\kappa$. By our choice of $\widetilde{G}_\kappa^*$ and (8.3), we get a bijection $\mathbb{X}^\bullet(T_H)/W_H \simeq \mathbb{X}_\bullet(T)/W_G$. Hence, $T_H/W_H \simeq \hat{T}_\kappa/W_G$. Note that the Weyl group acts faithfully on the maximal torus, and we conclude that $\mathbb{X}^\bullet(T_H) = \mathbb{X}_\bullet(T)$ and $\hat{T}_\kappa$ is a maximal torus in H.

□

From now on, we write W_H for the Weyl group of H with respect to $\hat{T}_\kappa$. Recall that a (co)character of a reductive group is called regular if the cardinality of its orbit under the Weyl group action attains the maximum. Then 2ρ is a regular character in G with respect to T. By the proof of Lemma 8.4, it is a cocharacter in H with respect to $\hat{T}_\kappa$. In addition, the proof of Lemma 8.4 also shows that $W_H \cdot 2\rho = W_G \cdot 2\rho$ and thus the Weyl group orbit $W_H \cdot 2\rho$ has maximal cardinality and it follows that 2ρ is a regular cocharacter in H. Thus 2ρ fixes a Borel B_H which only depends on the Weyl chamber containing 2ρ. It also fixes a set of positive roots.

From the proof of Lemma 8.4, we deduce the followings

$$\text{the (dominant) weights of } (H, B_H, \hat{T}_\kappa) \text{ coincide with (dominant) coweights of } G. \tag{8.4}$$

$$W_H \text{ coincides with } W_G \text{ together with their subsets of simple reflections identified.} \tag{8.5}$$

To show (8.2), we hope to prove the following:

$$\Delta(H, B_H, \hat{T}_\kappa) = \Delta^\vee(G, B, T) \text{ and } \Delta^\vee(H, B_H, \hat{T}_\kappa) = \Delta(G, B, T). \tag{8.6}$$

We first prove a weaker version of (8.6).

Lemma 8.0.5. *Assume G to be semisimple, then statement* (8.6) *holds.*

Proof. Since G is assumed to be semisimple, then $\mathbb{Q} \cdot \mathbb{X}_\bullet^+(T) = \mathbb{Q} \cdot \Delta^\vee(G, B, T)$. Hence,

$$\mathbb{Z}_{\geq 0} \cdot \Delta_s(G, B, T) = \{\alpha \in \mathbb{X}^\bullet(T) | \langle \alpha, \lambda \rangle \geq 0 \text{ for all } \lambda \in \mathbb{X}_\bullet^+(T)\}. \tag{8.7}$$

On the other hand, it follows from (8.5) that W_H and W_G have the same cardinality. Together with (8.4), we conclude that H is also semisimple. Thus,

$$\mathbb{Z}_{\geq 0} \cdot \Delta_s^\vee(H, B_H, \hat{T}_\kappa) = \{\alpha^\vee \in \mathbb{X}_\bullet(\hat{T}_\kappa) | \langle \alpha, \lambda \rangle \geq 0 \text{ for all } \lambda \in \mathbb{X}_+^\bullet(\hat{T}_\kappa)\}. \tag{8.8}$$

Comparing (8.7) and (8.8), we have

$$\mathbb{Z}_{\geq 0} \cdot \Delta_s(G, B, T) = \mathbb{Z}_{\geq 0} \cdot \Delta_s^\vee(H, B_H, \hat{T}_\kappa).$$

Thus, $\Delta_s(G, B, T) = \Delta_s^\vee(H, B_H, \hat{T}_\kappa)$ and we conclude that $\Delta(G, B, T) = \Delta^\vee(H, B_H, \hat{T}_\kappa)$ by noting (8.5). Finally, since for a semisimple reductive group, the coroots are uniquely determined by roots and vice versa, we also conclude that $\Delta^\vee(G, B, T) = \Delta(H, B_H, \hat{T}_\kappa)$. □

In fact, Lemma 8.0.5 may be proved following the idea of [BR18, §14][1] and [MV07, §12] and, we sketch this approach here.

Lemma 8.0.6. *We have the following inclusion of lattices*

$$\mathbb{Z} \cdot \Delta(H, \hat{T}_\kappa) \subseteq \mathbb{Z} \cdot \Delta^\vee(G, T) \tag{8.9}$$

for general G.

Proof. The proof is similar to that for [MV07, (12.21)] and we sketch it here. Note that the Satake category $\mathrm{Sat}_{G,\kappa}$ is equipped with a grading by $\pi_0(Gr_G) \simeq \pi_1(G) = \mathbb{X}_\bullet(T)/\mathbb{Z} \cdot \Delta^\vee(G,T)$ by [Zhu17, Proposition 1.21]. In addition, this grading is compatible with the tensor structure in $\mathrm{Sat}_{G,\kappa}$. Write Z for the center of G, then it can be identified with the group scheme

$$\mathrm{Hom}(\mathbb{X}_\bullet(T)/\mathbb{Z} \cdot \Delta^\vee(G,T), \mathbb{G}_{m,\kappa}). \tag{8.10}$$

Our previous observation implies that the forgetful functor

$$\mathrm{Sat}_{G,\kappa} \simeq \mathrm{Rep}_\kappa(\widetilde{G}_\kappa) \longrightarrow \mathrm{Rep}_\kappa(Z)$$

is compatible with the grading considered above. In this way, Z is realized as a central subgroup of $\widetilde{G}_\kappa$. Since $\hat{T}_\kappa \to H$ is a closed embedding, Z is also contained in the center of H. Finally, note that the center of H can be identified with the group scheme

$$\mathrm{Hom}(\mathbb{X}^\bullet(\hat{T}_\kappa)/\mathbb{Z} \cdot \Delta^\vee(H, \hat{T}_\kappa), \mathbb{G}_{m,\kappa}). \tag{8.11}$$

Our discussion together with (8.10) and (8.11) completes the proof of the lemma. □

Lemma 8.0.7. *The set of dominant weights of $(H, \hat{T}_\kappa)$ is equal to $\mathbb{X}_\bullet^+(T) \subset \mathbb{X}_\bullet(T) = \mathbb{X}^\bullet(\hat{T}_\kappa)$.*

Proof. By our construction, we have a bijection between the set of irreducible representations of $\widetilde{G}_\kappa$ and that of $\widetilde{G}_\kappa^*$. Since irreducible representations restrict trivially to the unipotent radical, we get a bijection between the set of irreducible

[1]We note that the situation considered in [BR18, §14] is slightly different from ours. In the equal characteristic case, the group scheme $\widetilde{G}_\kappa$ is proven to be algebraic by directly exhibiting a tensor generator in the Satake category. Then, there is no need to pass to finite type quotient $\widetilde{G}_\kappa^*$ as we do in this section.

representations of $\widetilde{G}_\kappa$ and that of H. Thus, the dominant weights of $(H, \hat{T}_\kappa)$ equal that of $(\widetilde{G}_\kappa, \hat{T}_\kappa)$.

Let $\lambda \in \mathbb{X}_\bullet(T)$ be a dominant weight of $(\widetilde{G}_\kappa, \hat{T}_\kappa)$ and write the $L^{\widetilde{G}_\kappa}(\lambda)$ for the irreducible representation of $\widetilde{G}_\kappa$ associated to λ. Assume $\mu \in \mathbb{X}_\bullet^+(T)$ be a dominant coweight of G such that the simple perverse sheaf corresponding to $L(\lambda)$ is IC_μ. Note that in the Grothendieck group of $\mathrm{Sat}_{G,\kappa}$, we have

$$[\mathrm{IC}_\mu] = \left[{}^p j_{\mu,*}\underline{\kappa}_{Gr_\mu}[(2\rho,\mu)]\right] + \sum_{\nu\in\mathbb{X}_\bullet^+(T),\nu<\mu} a_\nu^\mu \left[{}^p j_{\nu,!}\underline{\kappa}_{Gr_\nu}[(2\rho,\nu)]\right].$$

Then we conclude that $\lambda = \mu \in \mathbb{X}_\bullet^+(T)$.

On the other hand, if $\mu \in \mathbb{X}_\bullet^+(T)$, then the weights of the $\widetilde{G}_\kappa$-representation which correspond to ${}^p j_{\mu,!}\underline{\kappa}_{Gr_\mu}$ are independent of the coefficient κ by [MV07, Proposition. 8.1]. Hence, they are weights of the irreducible $\hat{G}_{\bar{\mathbb{Q}}_\ell}$-representation of highest weight μ. Thus μ is a dominant weight of $(\widetilde{G}_\kappa, \hat{T}_\kappa)$. □

Lemma 8.0.8. *The Weyl groups W_G and W_H coincide when considered as automorphism groups of $\mathbb{X}_\bullet(T)$, and their subsets of simple reflections S_G and S_H coincide.*

Proof. The proof of this lemma is completely similar to the proof of [BR18, Lemma 14.9] and we sketch it here. For any $\lambda \in \mathbb{X}_\bullet^+(T)$, we consider it as a dominant weight of $(H, \hat{T}_\kappa)$. Then the orbit $W_H \cdot \lambda$ is the set of extremal points of the convex polytope consisting of the convex hull of weights of the irreducible H-representation $L^H(\lambda)$. Since the set of irreducible representations of H are bijective to that of $\widetilde{G}_\kappa$, we conclude that

$$W_H \cdot \lambda = W_G \cdot \lambda. \tag{8.12}$$

For any $\lambda \in \mathbb{X}_\bullet^+(T)$, we call λ regular if it is not orthogonal to any simple root of (G,T). Then for a regular $\lambda \in \mathbb{X}_\bullet^+(T)$, the orbit $S_G \cdot \lambda \subset W \cdot \lambda$ is the subset of $W_G \cdot \lambda$ consisting of elements μ such that the line segment connecting λ and μ is extremal in the convex hull of $W_G \cdot \lambda$. By (8.12), we have the same description for the orbit $S_H \cdot \lambda$. Thus,

$$S_G \cdot \lambda = S_H \cdot \lambda. \tag{8.13}$$

Choose an arbitrary $s_G \in S_G$. For any $\lambda \in \mathbb{X}_\bullet^+(T)$ regular, by (8.13) there exists $s_H \in S_H$ such that $s_G \cdot \lambda = s_H \cdot \lambda$. In addition, the direction of the line segment connecting λ with $s_G \cdot \lambda$ is determined by the line segment joining the coroot of

G associated with s_G with the root of H associated with s_H. Thus for any other $\lambda' \in \mathbb{X}_\bullet^+(T)$ regular, we also have $s_G \cdot \lambda' = s_H \cdot \lambda'$. It follows that $s_G = s_H$ and thus $S_G = S_H$. Thus, we deduce that $W_G = W_H$. □

Lemma 8.0.9. *We have the following inclusion of lattices*

$$\mathbb{Z} \cdot \Delta(G,T) \subseteq \mathbb{Z} \cdot \Delta^\vee(H, \hat{T}_\kappa).$$

Proof. The proof is similar to the one for [BR18, Lemma 14.10] and we sketch it here. Firstly, we observe by Lemma 8.0.7 that

$$\mathbb{Q}_+ \cdot \Delta_s^\vee(H, B_H, \hat{T}_\kappa) = \mathbb{Q}_+ \cdot \Delta_s(G, B, T). \tag{8.14}$$

This is because both sets consist of extremal rays of the rational convex polyhedral cone determined by $\{\lambda \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{X}^\bullet(T) | \text{ for any } \mu \in \mathbb{X}_\bullet^+(T), \langle \lambda, \mu \rangle \geq 0\}$. For $\mu \in \Delta_s(G, B, T)$, it follows from (8.14) that there exists $a \in \mathbb{Q}_+ \backslash \{0\}$ such that $a\mu \in \Delta_s^\vee(H, B_H, \hat{T}_\kappa)$. Lemma 8.8 then implies that

$$\mathrm{id} - \langle \mu^\vee, \bullet \rangle = \mathrm{id} - \langle (a\mu)^\vee, \bullet \rangle (a\mu)$$

as an automorphism of $\mathbb{X}^\bullet(T) = \mathbb{X}_\bullet(\hat{T}_\kappa)$. Thus, $(a\mu)^\vee = \frac{1}{a}\mu^\vee$. Note that Lemma 8.0.6 shows that $a\mu \in \mathbb{Z}{\cdot}\Delta^\vee(G,T)$. Thus, $\frac{1}{a} \in \mathbb{Z}$ and $\mu = \frac{1}{a}(a\mu) \in \mathbb{Z}{\cdot}\Delta^\vee(H, \hat{T}_\kappa)$. □

The arguments above prepare us for a second proof of Lemma 8.0.5 as follows.

Proof. If G is in particular semisimple of adjoint type, then $\mathbb{Z} \cdot \Delta(G,T) = \mathbb{X}^*(T)$. Lemma 8.0.9 then implies that $\mathbb{Z} \cdot \Delta(G,T) = \mathbb{Z} \cdot \Delta^\vee(H, \hat{T}_\kappa)$. Then the arguments in the proof of Lemma 8.0.9 imply that $\Delta_s(G,T) = \Delta_s^\vee(H, \hat{T}_\kappa)$. In addition, $\Delta_s^\vee(G,B,T) = \Delta_s(H, B_H, \hat{T}_\kappa)$, and the canonical bijections between the roots and coroots of H and G coincide. It then follows from Lemma 8.0.8 that $\Delta(H, B_H, \hat{T}_\kappa) = \Delta^\vee(G,T)$ and $\Delta^\vee(H, \hat{T}_\kappa) = \Delta(G,T)$. Thus, the root datum of H with respect to $\hat{T}_\kappa$ is dual to that of (G,T). Then the dimension estimate (8.1) concludes the proof of the lemma in the semisimple of adjoint type case.

Assume G is a general semisimple reductive group scheme. Recall notations in §1.3. We denote by G_{ad} the adjoint quotient of G and by T_{ad} the quotient of the maximal torus T. The construction in §7 goes through and we get the group scheme $(\widetilde{G}_{\mathrm{ad}})_\kappa$. As noted in the proof of Lemma 8.0.6, the Satake category $\mathrm{Sat}_{G_{\mathrm{ad}},\kappa}$ admits a grading by the finite group $\pi_1(G_{\mathrm{ad}})/\pi_1(G)$ which is compatible with the tensor structure of $\mathrm{Sat}_{G_{\mathrm{ad}},\kappa}$. By Lemma 4.6, the category $\mathrm{Sat}_{G,\kappa}$ can be realized as a tensor subcategory of $\mathrm{Sat}_{G_{\mathrm{ad}},\kappa}$ corresponding to the identity coset of $\pi_1(G)$. Thus, we have a surjective quotient

$$(\widetilde{G})_{\mathrm{ad},\kappa} \twoheadrightarrow \widetilde{G}_\kappa$$

with finite central kernel given by $\mathrm{Hom}(\pi_1(G_{\mathrm{ad}})/\pi_1(G), \mathbb{G}_{m,\kappa})$. Hence, $\widetilde{G}_\kappa$ is reductive and in particular semisimple. The result for G being semisimple of adjoint type applies here to complete the proof. □

Lemma 8.0.10. *Let G be a general connected reductive group, then the same result as in Lemma* 8.0.5 *holds.*

Proof. We sketch a proof similar to the arguments for [MV07, §12] and [BR18, Lemma 14.13]. Denote by $Z(G)$ the center of G and let $A = Z(G)^\circ$. Then A is a torus and G/A is semisimple. As in *loc.cit*, the exact sequence

$$1 \to A \to G \to G/A \longrightarrow 1$$

induces maps

$$Gr_A \xrightarrow{i} Gr_G \xrightarrow{\pi} Gr_{G/A}$$

which exhibit Gr_G as a trivial Gr_A-cover over $Gr_{G/A}$. This induces an exact sequence of functors

$$P_{L^+A}(Gr_A, \kappa) \xrightarrow{i_*} P_{L^+G}(Gr_G, \kappa) \xrightarrow{\pi_*} P_{L^+G/A}(Gr_{G/A}, \kappa). \tag{8.15}$$

Note that $(Gr_A)_{\mathrm{red}}$ is a set of discrete points indexed by $\mathbb{X}^+_\bullet(A)$, then taking pushforward along i gives a fully faithful functor $i_* : P_{L^+A}(Gr_A, \kappa) \to P_{L^+G}(Gr_G, \kappa)$. The functor π_* is made sense by Lemma 4.6 and is essentially surjective.

Applying the Tannakian construction as in §7, we get flat affine group schemes $\widetilde{A}_\kappa$ and $\widetilde{(G/A)}_\kappa$. Lemma 8.0.5 implies that $\widetilde{A}_\kappa$ and $\widetilde{(G/A)}_\kappa$ are isomorphic to the dual groups of H and G/A respectively. The same arguments in [MV07, §12] and [BR18, §14] apply here to deduce that the sequence

$$1 \longrightarrow \widetilde{G/A}_\kappa \longrightarrow \widetilde{G}_\kappa \longrightarrow \widetilde{A}_\kappa \longrightarrow 1$$

induced by (8.9) is exact. Then $\widetilde{G}_\kappa$ is identified as the extension of smooth group schemes $\widetilde{A}_\kappa$ and $\widetilde{G/A}_\kappa$, and is thus also smooth. Moreover, the unipotent radical of $\widetilde{G}_\kappa$ has trivial image in the torus $\widetilde{A}_\kappa$. Hence it is included in $\widetilde{G/A}_\kappa$. Since the latter group is semisimple, it follows that $\widetilde{G}_\kappa$ is also reductive. Arguing as in [BR18, Lemma 14.14], we complete the proof of the lemma. □

Thus we identify the group scheme $\widetilde{G}_{\mathbb{Z}_\ell}$ which arises from the general Tannakian construction with the Langlands dual group $\hat{G}_{\mathbb{Z}_\ell}$. We have our main theorem.

Theorem 8.0.11. *There is an equivalence of tensor categories between $P_{L^+G}(Gr_G, \Lambda)$ and the category of Λ-representations of the Langlands dual group $\hat{G}_\Lambda$ of G which are finitely generated over Λ for $\Lambda = \mathbb{F}_\ell$, and $\mathbb{Z}_\ell$.*

Now, we complete the final step of identifying the group schemes.

Lemma 8.0.12. *Let G be a general connected reductive group, then the same result as in Lemma* 8.0.6 *holds.*

Proof. We sketch a proof similar to the arguments for [MV07, §12] and [BR18, Lemma 14.13]. Denote by $Z(G)$ the center of G and let $A = Z(G)^\circ$. Then A is a torus and G/A is semisimple. As in *loc.cit*, the exact sequence

$$1 \to A \to G \to G/A \longrightarrow 1$$

induces maps

$$Gr_A \xrightarrow{i} Gr_G \xrightarrow{\pi} Gr_{G/A}$$

which exhibit Gr_G as a trivial Gr_A-cover over $Gr_{G/A}$. This induces an exact sequence of functors

$$P_{L^+A}(Gr_A, \kappa) \xrightarrow{i_*} P_{L^+G}(Gr_G, \kappa) \xrightarrow{\pi_*} P_{L^+G/A}(Gr_{G/A}, \kappa). \tag{8.16}$$

Note that $(Gr_A)_{\mathrm{red}}$ is a set of discrete points indexed by $\mathbb{X}^+_\bullet(A)$, then taking pushforward along i gives a fully faithful functor $i_* : P_{L^+A}(Gr_A, \kappa) \to P_{L^+G}(Gr_G, \kappa)$. The functor π_* is made sense by Lemma 4.6 and is essentially surjective.

Applying the Tannakian construction as in §7, we get flat affine group schemes $\widetilde{A}_\kappa$ and $\widetilde{(G/A)}_\kappa$. Lemma 8.0.5 implies that $\widetilde{A}_\kappa$ and $\widetilde{(G/A)}_\kappa$ are isomorphic to the dual groups of H and G/A respectively. The same arguments in [MV07, §12] and [BR18, §14] apply here to deduce that the sequence

$$1 \longrightarrow \widetilde{G/A}_\kappa \longrightarrow \widetilde{G}_\kappa \longrightarrow \widetilde{A}_\kappa \longrightarrow 1$$

induced by (8.9) is exact. Then $\widetilde{G}_\kappa$ is identified as the extension of smooth group schemes $\widetilde{A}_\kappa$ and $\widetilde{G/A}_\kappa$, and is thus also smooth. Moreover, the unipotent radical of $\widetilde{G}_\kappa$ has trivial image in the torus $\widetilde{A}_\kappa$. Hence it is included in $\widetilde{G/A}_\kappa$. Since the latter group is semisimple, it follows that $\widetilde{G}_\kappa$ is also reductive. Arguing as in [BR18, Lemma 14.14], we complete the proof of the lemma. □

Thus we identify the group scheme $\widetilde{G}_{\mathbb{Z}_\ell}$ which arises from the general Tannakian construction with the Langlands dual group $\hat{G}_{\mathbb{Z}_\ell}$. We have our main theorem.

Theorem 8.0.13. *The hypercohomology functor* $\mathrm{H}^* : P_{L^+G\otimes\bar{k}}(Gr_G\otimes\bar{k},\Lambda) \to \mathrm{Mod}_\Lambda$ *lifts to a natural equivalence of monoidal categories*

$$\mathrm{H}^* : P_{L^+G\otimes\bar{k}}(Gr_G \otimes \bar{k},\Lambda) \to \mathrm{Rep}_\Lambda(\hat{G}_\Lambda).$$

From now on, we will write the inverse of the geometric Satake equivalence as Sat.

Remark 8.0.14. *As explained in [Zhu16, §5.5], the Galois group* $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ *acts on the Satake category* $\mathrm{Sat}_{G,\Lambda}$ *by tensor auto-equivalences. It, in turn, induces an action of* $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ *on* $\hat{G}$ *which preserves* $(\hat{G},\hat{B},\hat{T})$*. Let* $V \in \mathrm{Rep}_\Lambda(\hat{G})$ *and* $\gamma \in \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$*. We write* ${}^\gamma V$ *for the representation*

$$\hat{G} \xrightarrow{\gamma^{-1}} \hat{G} \to GL_\Lambda(V)$$

of $\hat{G}$*.*

For three sequences of dominant weight $\mu_{1\bullet}$, $\mu_{2\bullet}$, and $\mu_{3\bullet}$, the following lemma is an immediate consequence of Theorem 8.0.13.

Corollary 8.0.15. *We have the following natural isomorphism*

$$\mathrm{Hom}_{\hat{G}}(V_{\mu_{i\bullet}},V_{\mu_{j\bullet}}) \cong \mathrm{Corr}_{Gr^0_{\mu_{i\bullet}|\mu_{j\bullet}}}((Gr_{\le\mu_{i\bullet}},\mathrm{IC}_{\mu_{i\bullet}}),(Gr_{\le\mu_{j\bullet}},\mathrm{IC}_{\mu_{j\bullet}})), \tag{8.17}$$

such that the natural composition on the left hand

$$\mathrm{Hom}_{\hat{G}}(V_{\mu_{1\bullet}},V_{\mu_{2\bullet}}) \otimes \mathrm{Hom}_{\hat{G}}(V_{\mu_{2\bullet}},V_{\mu_{3\bullet}}) \to \mathrm{Hom}_{\hat{G}}(V_{\mu_{1\bullet}},V_{\mu_{3\bullet}})$$

is compatible with the composition of cohomological correspondences on the right hand side

$$\mathrm{Corr}_{Gr^0_{\mu_{1\bullet}|\mu_{2\bullet}}}((Gr_{\le\mu_{1\bullet}},\mathrm{IC}_{\mu_{1\bullet}}),(Gr_{\le\mu_{2\bullet}},\mathrm{IC}_{\mu_{2\bullet}})) \otimes \mathrm{Corr}_{Gr^0_{\mu_{2\bullet}|\mu_{3\bullet}}}((Gr_{\le\mu_{2\bullet}},\mathrm{IC}_{\mu_{2\bullet}}),(Gr_{\le\mu_{3\bullet}},\mathrm{IC}_{\mu_{3\bullet}}))$$
$$\to \mathrm{Corr}_{Gr^0_{\mu_{1\bullet}|\mu_{3\bullet}}}((Gr_{\le\mu_{1\bullet}},\mathrm{IC}_{\mu_{1\bullet}}),(Gr_{\le\mu_{3\bullet}},\mathrm{IC}_{\mu_{3\bullet}}))$$

which is obtained by pushing forward the cohomological correspondences along the map

$$Gr^0_{\mu_{1\bullet}|\mu_{2\bullet}} \times_{Gr_{\mu_{2\bullet}}} Gr^0_{\mu_{2\bullet}|\mu_{3\bullet}} \to Gr^0_{\mu_{1\bullet}|\mu_{3\bullet}}.$$

In addition, there is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{P}(Gr_G)}(m_{\mu_\bullet *}\mathrm{IC}_{\mu_\bullet}, m_{\nu_\bullet *}\mathrm{IC}_{\nu_\bullet}) \cong \mathrm{H}^{\mathrm{BM}}_{(2\rho,|\mu_\bullet|+|\nu_\bullet|)}(Gr^0_{\mu_\bullet|\nu_\bullet}). \tag{8.18}$$

Proof. The lemma can be proved exactly as [Zhu17, Corollary 3.4.4], and we refer to *loc.cit* for details of the proof. □

Chapter 9

LOCAL HECKE STACKS

We review the definition of local Hecke stacks and study their geometric properties in this section. All results are proved in [XZ17], and we refer to *loc.cit* for proofs.

Definition 9.0.1. Let $\mu_\bullet = (\mu_1, \mu_2, \cdots, \mu_n)$ be a sequence of dominant coweights of G. The *local Hecke stack* $\mathrm{Hk}^{\mathrm{loc}}_{\mu_\bullet}$ is defined as the moduli problem which assigns to each perfect k-algebra R the groupoid of chains of modifications of G-torsors

$$\mathcal{E}_n \dashrightarrow \mathcal{E}_{n-1} \dashrightarrow \cdots \dashrightarrow \mathcal{E}_0 \tag{9.1}$$

over D_R of relative positions $\leq \mu_n, \cdots, \leq \mu_1$, respectively.

It may also be understood as the homogeneous space $[L^+G\backslash Gr_{\mu_{\leq \bullet}}]$. Similarly, we define

$$\mathrm{Hk}^{0,\mathrm{loc}}_{\mu_\bullet|\nu_\bullet} := [L^+G\backslash Gr^0_{\mu_\bullet|\nu_\bullet}]$$

as the stack which classifies for each perfect k-algebra R the rectangles of modifications

$$\begin{array}{ccc} \mathcal{E}_n & \dashrightarrow \cdots \dashrightarrow & \mathcal{E}_0 \\ \| & & \| \\ \mathcal{E}'_m & \dashrightarrow \cdots \dashrightarrow & \mathcal{E}'_0, \end{array}$$

of G-torsors over D_R with modifications in the upper (resp. lower) row bounded by $\mu_\bullet$ (resp. $\nu_\bullet$).

Taking quotient of the Satake correspondence (2.3) of affine Grassmannians by L^+G, we get the *Satake correspondence* for local Hecke stacks,

$$\mathrm{Hk}^{\mathrm{loc}}_{\mu_\bullet} \xleftarrow{h^{\leftarrow}_{\mu_\bullet}} \mathrm{Hk}^{0,\mathrm{loc}}_{\mu_\bullet|\nu_\bullet} \xrightarrow{h^{\rightarrow}_{\nu_\bullet}} \mathrm{Hk}^{\mathrm{loc}}_{\nu_\bullet}. \tag{9.2}$$

It is clear from the definition that these stacks are not of finite type, thus we need their finite dimensional quotient to apply the ℓ-adic formalism. We recall the following definition as in [Zhu17].

Definition 9.0.2. For a sequence of dominant coweights $\mu_\bullet = (\mu_1, \mu_2, \cdots, \mu_n)$, choose a $\mu_\bullet$-large integer m, and we define the *m-restricted local Hecke stack* to be the stack

$$\mathrm{Hk}^{\mathrm{loc}(m)}_{\mu_\bullet} := [L^mG\backslash Gr_{\leq \mu}].$$

Similarly, choose m large enough for $\mu_\bullet$ and $\nu_\bullet$, for example, m is taken to be $(\mu_\bullet, \nu_\bullet)$-large, and we define $\mathrm{Hk}^{0,\mathrm{loc}(m)}_{\mu_\bullet|\nu_\bullet} := [L^m G\backslash Gr^0_{\mu_\bullet|\nu_\bullet}]$. We have the Satake correspondence on restricted local Hecke stacks,

$$\mathrm{Hk}^{\mathrm{loc}(m)}_{\mu_\bullet} \xleftarrow{h^{\leftarrow}_{\mu_\bullet}} \mathrm{Hk}^{0,\mathrm{loc}(m)}_{\mu_\bullet|\nu_\bullet} \xrightarrow{h^{\rightarrow}_{\nu_\bullet}} \mathrm{Hk}^{\mathrm{loc}(m)}_{\nu_\bullet}. \tag{9.3}$$

9.1 Torsors over the Local Hecke Stacks

Let $\mathbb{B}L^+G$ (resp. BL^mG for $m \in \mathbb{Z}_{\geq 0}$) denote the moduli stack which classifies for every perfect k-algebra R the groupoid of G-torsors over D_R (resp. $D_{m,R}$). For non-negative integers $m_1 \leq m_2$, the natural quotient maps

$$L^+G \xrightarrow{\mathrm{res}_{m_2} := \mathrm{res}^{\infty}_{m_2}} L^{m_2}G \xrightarrow{\mathrm{res}^{m_2}_{m_1}} L^{m_1}G$$

induce restriction maps between stacks

$$\mathbb{B}L^+G \xrightarrow{\mathrm{res}_{m_2} := \mathrm{res}^{\infty}_{m_2}} \mathbb{B}L^{m_2}G \xrightarrow{\mathrm{res}^{m_2}_{m_1}} \mathbb{B}L^{m_1}G\ . \tag{9.4}$$

Clearly, for any non-negative integers $m_1 \leq m_2 \leq m_3$, we have $\mathrm{res}^{m_2}_{m_1} \circ \mathrm{res}^{m_3}_{m_2} = \mathrm{res}^{m_3}_{m_1}$, where m_3 can be taken to be ∞.

Let $\mu_\bullet = (\mu_1, \mu_2, \cdots, \mu_n)$ be a sequence of dominant coweights. We have natural morphisms

$$t_{\leftarrow}, t_{\rightarrow} : \mathrm{Hk}^{\mathrm{loc}}_{\mu_\bullet} \to \mathbb{B}L^+G$$

which send (9.1) to the torsors $\mathcal{E}_n$ and $\mathcal{E}_0$, respectively.

For restricted local Hecke stacks, we choose a pair of $\mu_\bullet$-large integers (m, n). Then the natural maps

$$\mathrm{Hk}^{\mathrm{loc}(m)}_{\mu_\bullet} \simeq [L^mG\backslash Gr^{(n)}_{\leq\mu_\bullet}/L^nG] \xrightarrow{t_{\leftarrow}\times t_{\rightarrow}} \mathbb{B}L^nG \times \mathbb{B}L^mG$$

induce the L^mG-torsor

$$Gr_{\leq\mu_\bullet} \to \mathrm{Hk}^{\mathrm{loc}(m)}_{\mu_\bullet}, \tag{9.5}$$

and the L^nG-torsor

$$[L^mG\backslash Gr_{\leq\mu_\bullet}] \to [L^{m+n}G\backslash Gr_{\leq\mu_\bullet}]. \tag{9.6}$$

Following the notations in [XZ17], we denote the two torsors by $\mathcal{E}_{\leftarrow}$ and $\mathcal{E}_{\rightarrow}$, respectively. For any pairs of $\mu_\bullet$-large integers (m_1, n_1) and (m_2, n_2) such that

$m_1 \leq m_2$ and $n_1 \leq n_2$, denote the natural restriction map of restricted local Hecke stacks as

$$\mathrm{res}^{m_2}_{m_1} : \mathrm{Hk}^{\mathrm{loc}(m_2)}_{\mu_\bullet} \to \mathrm{Hk}^{\mathrm{loc}(m_1)}_{\mu_\bullet}. \tag{9.7}$$

It is compatible with the restriction maps in (9.4) in the sense that the following diagram is commutative

$$\begin{array}{ccccc} \mathbb{B}L^{n_2}G & \xleftarrow{t_\leftarrow} & \mathrm{Hk}^{\mathrm{loc}(m_2)}_{\mu_\bullet} & \xrightarrow{t_\rightarrow} & \mathbb{B}L^{m_2}G \\ \downarrow{\scriptstyle \mathrm{res}^{n_2}_{n_1}} & & \downarrow{\scriptstyle \mathrm{res}^{m_2}_{m_1}} & & \downarrow{\scriptstyle \mathrm{res}^{m_2}_{m_1}} \\ \mathbb{B}L^{n_1}G & \xleftarrow{t_\leftarrow} & \mathrm{Hk}^{\mathrm{loc}(m_1)}_{\mu_\bullet} & \xrightarrow{t_\rightarrow} & \mathbb{B}L^{m_1}G. \end{array}$$

Let $\mu_\bullet$ and $\nu_\bullet$ be two sequences of dominant coweights. Choose non-negative integers m_1, m_2, n such that (m_1, m_2) is $\mu_\bullet$-large and (m_2, n) is $\nu_\bullet$-large. Then we have the following isomorphism

$$[L^{m_1}G\backslash Gr^{(n)}_{\leq \mu_\bullet,\nu_\bullet}] \cong \mathrm{Hk}^{\mathrm{loc}(m_1)}_{\mu_\bullet,\nu_\bullet} \cong \mathrm{Hk}^{\mathrm{loc}(m_2)}_{\nu_\bullet} \times_{t_\rightarrow, \mathbb{B}L^{m_2}G, \mathrm{res}^{m_1}_{m_2}\circ t_\leftarrow} \mathrm{Hk}^{\mathrm{loc}(m_1)}_{\mu_\bullet}, \tag{9.8}$$

which induces the following perfectly smooth morphisms

$$\mathrm{Hk}^{\mathrm{loc}(m_1)}_{\mu_\bullet,\nu_\bullet} \to \mathrm{Hk}^{\mathrm{loc}(m_2)}_{\nu_\bullet} \times \mathrm{Hk}^{\mathrm{loc}(m_1)}_{\mu_\bullet} \xrightarrow{\mathrm{id}\times \mathrm{res}^{m_1}_{m_1-m_2}} \mathrm{Hk}^{\mathrm{loc}(m_2)}_{\nu_\bullet} \times \mathrm{Hk}^{\mathrm{loc}(m_2-m_1)}_{\mu_\bullet}. \tag{9.9}$$

9.2 Perverse Sheaves on the Moduli of Local Hecke Stacks

Let $m_1 \leq m_2$ be two $\mu_\bullet$-large integers. The natural (twisted) pullback functor

$$\mathrm{Res}^{m_2}_{m_1} := (\mathrm{res}^{m_2}_{m_1})^\star := (\mathrm{res}^{m_2}_{m_1})^*[d](d/2) : \mathrm{P}(\mathrm{Hk}^{\mathrm{loc}(m_1)}_{\mu_\bullet}) \to \mathrm{P}(\mathrm{Hk}^{\mathrm{loc}(m_2)}_{\mu_\bullet})$$

is an equivalence of categories. We define the category of perverse sheaves on the local Hecke stack as

$$\mathrm{P}(\mathrm{Hk}^{\mathrm{loc}}_{\bar{k}}, \Lambda) := \bigoplus_{\xi\in\pi_1(G)} \mathrm{P}(\mathrm{Hk}^{\mathrm{loc}}_{\xi}, \Lambda), \mathrm{P}(\mathrm{Hk}^{\mathrm{loc}}_{\xi}, \Lambda) := \varinjlim_{(\mu,m)\in\xi\times\mathbb{Z}_{\geq 0}} \mathrm{P}(\mathrm{Hk}^{\mathrm{loc}(m)}_{\mu}, \Lambda).$$

Here the connecting morphism in the definition of $\mathrm{P}(\mathrm{Hk}^{\mathrm{loc}}, \Lambda)$ is the fully faithful embedding

$$\mathrm{P}(\mathrm{Hk}^{\mathrm{loc}(m_1)}_{\mu_1}, \Lambda) \xrightarrow{\mathrm{Res}^{m_2}_{m_1}} \mathrm{P}(\mathrm{Hk}^{\mathrm{loc}^{m_1}}_{\mu_1}, \Lambda) \xrightarrow{i_{\mu_1,\mu_2 *}} \mathrm{P}(\mathrm{Hk}^{\mathrm{loc}(m_2)}_{\mu_2}, \Lambda).$$

Finally, via descent, there is a natural equivalence of categories $\mathrm{P}(\mathrm{Hk}^{\mathrm{loc}(m)}_{\mu}, E) \cong \mathrm{P}_{L^mG}(Gr_{\leq\mu}, E)$, which induces an equivalence $\mathrm{P}(\mathrm{Hk}^{\mathrm{loc}}_{\bar{k}}, E) \cong \mathrm{P}_{L^+G\otimes\bar{k}}(Gr_G \otimes \bar{k}, E)$.

Chapter 10

MODULI OF LOCAL SHTUKAS

In this chapter, we define different versions of moduli of local Shtukas and correspondences between them. Using these results, we define the category of E-coefficient perverse sheaves on the moduli of local Shtukas and their cohomological correspondences. In the rest of this thesis, we will make use of the theory of cohomological correspondences between perfect schemes and perfect pfp algebraic spaces. We refer to [XZ17, Appendix A] for reference.

Definition 10.0.1. Let $\mu_\bullet = (\mu_1, \mu_2, \cdots, \mu_n)$ be a sequence of dominant coweights. The *moduli of local Shtukas* $\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet}$ classifies for each perfect k-algebra R sequences of modifications of G-torsors

$$\mathcal{E}_n \dashrightarrow \mathcal{E}_{n-1} \dashrightarrow \cdots \dashrightarrow \mathcal{E}_0 \cong {}^\sigma\mathcal{E}_n$$

over D_R of relative positions $\leq \mu_n, \cdots, \leq \mu_1$ respectively.

It follows from the definition that

$$\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet} \cong \mathrm{Hk}^{\mathrm{loc}}_{\mu_\bullet} \times_{t_\leftarrow \times t_\rightarrow, \mathbb{B}L^+G \times \mathbb{B}L^+G, \mathrm{id}\times\sigma} \mathbb{B}L^+G.$$

There is a natural forgetful map $\psi^{\mathrm{loc}} : \mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet} \to \mathrm{Hk}^{\mathrm{loc}}_{\mu_\bullet}$ which forgets the isomorphisms $\mathcal{E}_0 \cong {}^\sigma\mathcal{E}_n$. One can define the stack

$$\mathrm{Sht}^{0,\mathrm{loc}}_{\mu_\bullet|\nu_\bullet}$$

which classifies for each perfect k-algebra R the following rectangle of modifications

$$\begin{array}{ccc} \mathcal{E}_n \dashrightarrow \cdots \dashrightarrow & \mathcal{E}_0 \cong {}^\sigma\mathcal{E}_n \\ \| & \| \\ \mathcal{E}'_m \dashrightarrow \cdots \dashrightarrow & \mathcal{E}'_0 \cong {}^\sigma\mathcal{E}'_m, \end{array}$$

of G-torsors over D_R with modifications in the upper (resp. lower) row bounded by $\mu_\bullet$ (resp. $\nu_\bullet$). We get the *Satake correspondence* for moduli of local Shtukas

$$\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet} \xleftarrow{s^{\leftarrow}_{\mu_\bullet}} \mathrm{Sht}^{0,\mathrm{loc}}_{\mu_\bullet|\nu_\bullet} \xrightarrow{s^{\rightarrow}_{\mu_\bullet}} \mathrm{Sht}^{\mathrm{loc}}_{\nu_\bullet}.$$

We introduce the partial Frobenius morphism between the moduli of local Shtukas which will play an important role in later constructions.

Definition 10.0.2. Let $\mu_\bullet = (\mu_1, \mu_2, \cdots, \mu_n)$ be a sequence of dominant coweights of G. We define the *partial Frobenius morphism* to be

$$F_{\mu_\bullet} : \mathrm{Sht}^{\mathrm{loc}}_{\mu_1,\cdots,\mu_n} \longrightarrow \mathrm{Sht}^{\mathrm{loc}}_{\sigma(\mu_n),\mu_1,\cdots,\mu_{n-1}} \tag{10.1}$$

$$(\mathcal{E}_n \dashrightarrow \cdots \dashrightarrow \mathcal{E}_0 \cong {}^\sigma\mathcal{E}_n) \mapsto (\mathcal{E}_{n-1} \dashrightarrow \cdots \dashrightarrow {}^\sigma\mathcal{E}_n \dashrightarrow {}^\sigma\mathcal{E}_{n-1}).$$

Definition 10.0.3. Let $\mu_\bullet$ and $\nu_\bullet$ be two sequences of dominant coweights. For each perfect k-algebra R, the prestack $\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet|\nu_\bullet}$ classifies the following commutative diagram of modifications of G-torsors over D_R

$$\begin{array}{ccc} \mathcal{E}_n \dashrightarrow \cdots \dashrightarrow & \mathcal{E}_0 \cong {}^\sigma\mathcal{E}_n \\ \downarrow \beta & \downarrow \beta^\sigma \\ \mathcal{E}'_m \dashrightarrow \cdots \dashrightarrow & \mathcal{E}'_0 \cong {}^\sigma\mathcal{E}'_m, \end{array}$$

where the top (resp. bottom) row defines an R-point of $\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet}$ (resp. $\mathrm{Sht}^{\mathrm{loc}}_{\nu_\bullet}$). Let $\overleftarrow{h}^{\mathrm{loc}}_{\mu_\bullet}$ (reps. $\overrightarrow{h}^{\mathrm{loc}}_{\nu_\bullet}$) denote the morphism which maps the above commutative rectangle to its upper (resp. lower) row. We define the *Hecke correspondence* of local Shtukas to be to following diagram

$$\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet} \xleftarrow{\overleftarrow{h}^{\mathrm{loc}}_{\mu_\bullet}} \mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet|\nu_\bullet} \xrightarrow{\overrightarrow{h}^{\mathrm{loc}}_{\nu_\bullet}} \mathrm{Sht}^{\mathrm{loc}}_{\nu_\bullet}. \tag{10.2}$$

If in addition, the relative position of β is bounded by λ, we get a closed subprestack $\mathrm{Sht}^{\lambda,\mathrm{loc}}_{\mu_\bullet|\nu_\bullet}$. In particular, if $\lambda = 0$, the Hecke correspondence (4.2) reduces to the Satake correspondence.

The Hecke correspondence can be considered as the composition of two Satake correspondences and the cohomological correspondence given by the partial Frobenius morphism. More precisely, we recall [XZ17, Lemma 5.2.14].

Lemma 10.0.4. *Let $\mu_\bullet$ and $\nu_\bullet$ be two sequences of dominant coweights. Choose λ to be a dominant coweight such that $\lambda \geq |\mu_\bullet| + \sigma(\lambda)$ or $\lambda \geq |\mu_\bullet| + \nu$, then we have* *the following commutative diagram of prestacks*

$$
\begin{array}{ccccccc}
 & & & \mathrm{Sht}^{\theta,\mathrm{loc}}_{\mu_\bullet|\nu_\bullet} & & & \\
 & & \swarrow & & \searrow & & \\
 & \mathrm{Sht}^{0,\mathrm{loc}}_{\mu_\bullet|(\sigma(\theta^*),\lambda)} & & & & \mathrm{Sht}^{0,\mathrm{loc}}_{(\lambda,\theta^*)|\nu_\bullet} & \\
 \swarrow{\scriptstyle s^{\leftarrow}_{\mu_\bullet}} & \downarrow{\scriptstyle s^{\rightarrow}_{\sigma(\theta^*),\lambda}} & & & & \downarrow{\scriptstyle s^{\leftarrow}_{\lambda,\theta^*}} & \searrow{\scriptstyle s^{\rightarrow}_{\nu_\bullet}} \\
\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet} & \mathrm{Sht}_{\sigma(\theta^*),\lambda} & & \xrightarrow{F^{-1}_{\lambda,\theta^*}} & & \mathrm{Sht}^{\mathrm{loc}}_{\lambda,\theta^*} & \mathrm{Sht}^{\mathrm{loc}}_{\nu_\bullet}.
\end{array}
$$

In addition, the pentagon in the middle is a Cartesian square when composing $s^{\rightarrow}_{\sigma(\theta^*),\lambda}$ *with* $F^{-1}_{\lambda,\theta^*}$.

10.1 Moduli of Restricted Local Shtukas

Let $\mu_\bullet = 0$ or more generally, a central cocharacter, $\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet} \simeq \mathbb{B}G(O)$ which is not perfectly of finite presentation as a prestack. Thus to apply the ℓ-adic formalism, it is desirable to study the following approximation of $\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet}$.

Definition 10.1.1. Let $\mu_\bullet$ be a sequence of dominant coweights and (m, n) a pair of $\mu_\bullet$-large integers. We define the moduli stack $\mathrm{Sht}^{\mathrm{loc}(m,n)}_{\mu_\bullet}$ of (m, n)-*restricted local iterated shtukas* as the stack that classifies for every perfect k-algebra R,

(1) an R-point of $\mathrm{Hk}^{\mathrm{loc}(m)}_{\mu_\bullet}$,

(2) an isomorphism

$$\Psi : {}^{\sigma}(\mathcal{E}_{\leftarrow} |_{D_{n,R}}) \simeq (\mathcal{E}_{\rightarrow} |_{D_{m,R}}) |_{D_{n,R}}$$

of L^nG-torsos over $\mathrm{Spec}R$, where $\mathcal{E}_{\leftarrow}$ and $\mathcal{E}_{\rightarrow}$ are defined in (9.5) and (9.6), respectively.

The above definition gives a canonical isomorphism

$$\mathrm{Sht}^{\mathrm{loc}(m,n)}_{\mu_\bullet} \cong \mathrm{Hk}^{\mathrm{loc}(m)}_{\mu_\bullet} \times_{t_{\leftarrow}\times\mathrm{res}^m_n\circ t_{\rightarrow},\mathbb{B}L^nG\times\mathbb{B}L^nG,\mathrm{id}\times\sigma} \mathbb{B}L^nG.$$

The natural forgetful morphism $\psi^{\mathrm{loc}(m,n)} : \mathrm{Sht}^{\mathrm{loc}(m,n)}_{\mu_\bullet} \to \mathrm{Hk}^{\mathrm{loc}(m)}_{\mu_\bullet}$ is a perfectly smooth morphism of relative dimension $n \dim G$. For two sequences of dominant coweights $\mu_\bullet$, $\nu_\bullet$, we define $\mathrm{Sht}^{\mathrm{loc}(m,n)}_{\mu_\bullet|\nu_\bullet}$ to be the stack which classifies for each perfect k-algebra R, an R-point of $\mathrm{Hk}^{\mathrm{loc}(m,n)}_{\mu_\bullet|\nu_\bullet}$ together with an isomorphism ${}^{\sigma}(\mathcal{E}_{\leftarrow} |_{D_{n,R}}) \simeq$

$(\mathcal{E}\mid_{D_{m,R}})\mid_{D_{n,R}}$. Let (m_1,n_1) and (m_2,n_2) be two pairs of $\mu_\bullet$-large integers such that $m_1\le m_2$ and $n_1\le n_2$. We define the restriction morphism

$$\mathrm{res}^{m_2,n_2}_{m_1.n_1}:\mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{\mu_\bullet}\to\mathrm{Sht}^{\mathrm{loc}(m_1,n_1)}_{\mu_\bullet} \tag{10.3}$$

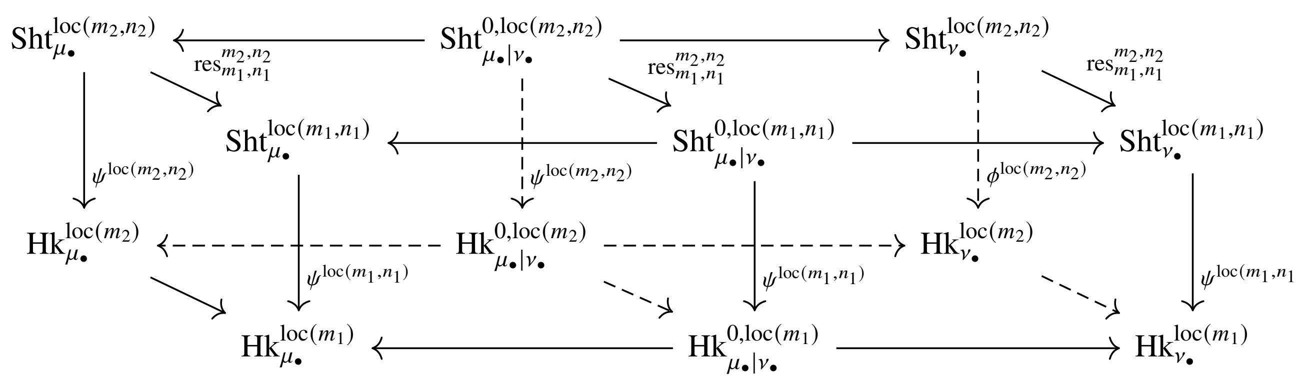
as the composition of the following morphisms

$$\begin{aligned}\mathrm{res}^{m_2,n_2}_{m_1.n_1}:&\mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{\mu_\bullet}\cong\mathrm{Hk}^{\mathrm{loc}(m_2)}_{\mu_\bullet}\times_{t_\leftarrow\times\mathrm{res}^{m_2}_{n_2}\circ t_\rightarrow,\mathbb{B}L^{n_2}G\times\mathbb{B}L^{n_2}G,\mathrm{id}\times\sigma}\mathbb{B}L^{n_2}G\\&\xrightarrow{\mathrm{res}^{m_2}_{m_1}\times\mathrm{res}^{n_2}_{n_1}}\mathrm{Hk}^{\mathrm{loc}(m_1)}_{\mu_\bullet}\times_{t_\leftarrow\times\mathrm{res}^{m_1}_{n_1}\circ t_\rightarrow,\mathbb{B}L^{n_1}G\times\mathbb{B}L^{n_1}G,\mathrm{id}\times\sigma}\mathbb{B}L^{n_1}G\\&\cong\mathrm{Sht}^{\mathrm{loc}(m_1,n_1)}_{\mu_\bullet}.\end{aligned}$$

For $(m_2,n_2)=(\infty,\infty)$, we write $\mathrm{res}^{m_2,n_2}_{m_1.n_1}$ as res_{m_1,n_1} for simplicity. For three pairs of $\mu_\bullet$-large integers (m_i,n_i) such that $m_1\le m_2\le m_3$ and $n_1\le n_2\le n_3$, we have

$$\mathrm{res}^{m_2,n_2}_{m_1,n_1}\circ\mathrm{res}^{m_3,n_3}_{m_2,n_2}=\mathrm{res}^{m_3,n_3}_{m_1,n_1}. \tag{10.4}$$

The Satake correspondences for restricted local Hecke stacks and the Satake correspondences for restricted local Shtukas are related by the restriction morphisms and summarized in the following diagram



(10.5)

where

(1) all rectangles are commutative,

(2) all rectangles are Cartesian except for the two on the left and right side of the cuboid.

Let $\mu_\bullet=(\mu_1,\cdots,\mu_n)$ be a sequence of dominant cocharacters. We call a quadruple of non-negative integers (m_1,n_1,m_2,n_2) $\mu_\bullet$-*acceptable* if

(1) $m_1 - m_2 = n_1 - n_2$ are μ_n-large (or equivalently $\sigma(\mu_n)$-large),

(2) $m_2 - n_1$ is $\mu_\bullet$-large.

We can define the partial Frobenius morphism

$$F_{\mu_\bullet}^{-1} : \mathrm{Sht}^{\mathrm{loc}(m_1,n_1)}_{\sigma(\mu_n),\mu_1,\cdots,\mu_{n-1}} \to \mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{\mu_1,\cdots,\mu_2} \tag{10.6}$$

for restricted local Shtukas. The construction of $F_{\mu_\bullet}^{-1}$ is technical and we refer to [XZ17, Construction 5.3.12] for detailed discussion.

10.2 Perverse Sheaves on the Moduli of Local Shtukas

Let $\mu_\bullet$ be a sequence of dominant coweights and (m_1, n_1), (m_2, n_2) be two pairs of $\mu_\bullet$-large integers such that $m_1 \le m_2$, $n_1 \le n_2$, and $m_2 \ne \infty$. Define the functor

$$\mathrm{Res}^{m_2,n_2}_{m_1,n_1} := (\mathrm{res}^{m_2,n_2}_{m_1.n_1})^\star : \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{\mu_\bullet}, \Lambda) \to \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}(m_1,n_1)}_{\mu_\bullet,\Lambda}). \tag{10.7}$$

Then (10.4) yields

$$\mathrm{Res}^{m_2,n_2}_{m_1,n_1} \circ \mathrm{Res}^{m_3,n_3}_{m_2,n_3} = \mathrm{Res}^{m_3,n_3}_{m_1,n_1}. \tag{10.8}$$

Like Res^m_n, the functor $\mathrm{Res}^{m_i,n_i}_{m_j,n_j}$ is also an equivalence of categories if $m_j > 1$.

We define the category of perverse sheaves on the moduli of local Shtukas as

$$\mathrm{P}(\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}}, \Lambda) := \bigoplus_{\xi\in\pi_1(G)} \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}}_{\xi}, \Lambda),\ \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}}_{\xi}, \Lambda) := \varinjlim_{(m,n,\mu)} \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}(m,n)}_{\mu}, \Lambda) \tag{10.9}$$

where the limit is taken over the triples $\{(m, n, \mu) \in \mathbb{Z}^2 \times \xi \mid (m, n) \text{ is } \mu \text{ large}\}$ with the product partial order. As in [XZ17], we call objects in $\mathrm{P}(\mathrm{Sht}^{\mathrm{loc}}_{\xi}, \Lambda)$ *connected objects*. The connecting morphism is given by the composite of fully faithful functor

$$\mathrm{P}(\mathrm{Sht}^{\mathrm{loc}(m_1,n_1)}_{\mu_1}, \Lambda) \xrightarrow{\mathrm{Res}^{m_2,n_2}_{m_1,n_1}} \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{\mu_1}, \Lambda) \xrightarrow{i_{\mu_1,\mu_1'}} \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{\mu_1'}, \Lambda).$$

For each dominant coweight μ and a pair of μ-large integers (m, n), we define the natural pullback functor

$$\Psi^{\mathrm{loc}(m,n)} := \mathrm{Res}^{m,n}_{m,0} : \mathrm{P}(\mathrm{Hk}^{\mathrm{loc}(m)}_{\mu}, \Lambda) \to \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}(m,n)}_{\mu}, \Lambda). \tag{10.10}$$

We observe that $\Psi^{\mathrm{loc}(m,n)}$ commutes with the connecting morphism in (10.9) by (10.8) and the proper smooth base change. Then we can take the limit and direct sum of $\Psi^{\mathrm{loc}(m,n)}$ and derive the following well-defined functor

$$\Psi^{\mathrm{loc}} : \mathrm{P}(\mathrm{Hk}^{\mathrm{loc}}_{\bar{k}}, \Lambda) \to \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}}, \Lambda). \tag{10.11}$$

Let $\mathcal{F}_i \in \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}}_{\xi_i}, E)$ be connected objects. It is realized as $\mathcal{F}^{(m_i,n_i)}_{i,\mu_i} \in \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}(m_i,n_i)}_{\mu_i}, E)$ for some μ_i and some pair of μ_i-large integers (m_i, n_i). We define the set of cohomological correspondences between $\mathcal{F}_1$ and $\mathcal{F}_2$ as

$$\begin{aligned}&\mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(\mathcal{F}_1, \mathcal{F}_2)\\ :=& \bigoplus_{\xi\in\pi_1(G)} \varinjlim \mathrm{Corr}_{\mathrm{Sht}^{\lambda,\mathrm{loc}(m_1,n_1)}_{\mu_1|\mu_2}}\Big((\mathrm{Sht}^{\mathrm{loc}(m_1,n_1)}_{\mu_1}, \mathcal{F}^{(m_1,n_1)}_{1,\mu_1}), (\mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{\mu_2}, \mathcal{F}^{(m_2,n_2)}_{2,\mu_2})\Big),\end{aligned}$$

where the limit is taken over all partially ordered sextuples $(\mu_1, \mu_2, \lambda, m_1, n_1, m_2, n_2)$ such that

- (m_1, n_1, m_2, n_2) is $(\mu_1 + \lambda, \lambda)$ and $(\mu_2 + \lambda, \lambda)$-acceptable,
- $\mu_i \in \xi_i$, for some $\xi_i \in \pi_1(G)$,
- $\lambda \in \xi$.

Let $(\mu_1, \mu_2, \lambda, m_1, n_1, m_2, n_2) \le (\mu'_1, \mu'_2, \lambda', m'_1, n'_1, m'_2, n'_2)$ be another such sextuple. The connecting morphism between the cohomological correspondences

$$\mathrm{Corr}_{\mathrm{Sht}^{\lambda,\mathrm{loc}(m_1,n_1)}_{\mu_1|\mu_2}}\Big((\mathrm{Sht}^{\mathrm{loc}(m_1,n_1)}_{\mu_1}, \mathcal{F}_{1,\mu_1}), (\mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{\mu_2}, \mathcal{F}_{2,\mu_2})\Big) \tag{10.12}$$

and

$$\mathrm{Corr}_{\mathrm{Sht}^{\lambda',\mathrm{loc}(m'_1,n'_1)}_{\mu'_1|\mu'_2}}\Big((\mathrm{Sht}^{\mathrm{loc}(m'_1,n'_1)}_{\mu'_1}, \mathcal{F}_{1,\mu'_1}), (\mathrm{Sht}^{\mathrm{loc}(m'_2,n'_2)}_{\mu'_2}, \mathcal{F}_{2,\mu'_2})\Big) \tag{10.13}$$

is given by first pulling back (4.13) to the Hecke correspondence

$$\mathrm{Sht}^{\mathrm{loc}(m'_1,n'_1)}_{\mu_1} \longleftarrow \mathrm{Sht}^{\lambda,\mathrm{loc}(m'_1,n'_1)}_{\mu_1|\mu_2} \longrightarrow \mathrm{Sht}^{\mathrm{loc}(m'_2,n'_2)}_{\mu_2},$$

along the restriction morphism, then pushing it forward to the Hecke correspondence

$$\mathrm{Sht}^{\mathrm{loc}(m'_1,n'_1)}_{\mu'_1} \longleftarrow \mathrm{Sht}^{\lambda',\mathrm{loc}(m'_1,n'_1)}_{\mu'_1|\mu'_2} \longrightarrow \mathrm{Sht}^{\mathrm{loc}(m'_2,n'_2)}_{\mu'_2}.$$

The connecting morphism is well-defined and can be composed. We refer to [XZ17, §5.4.1] for more discussions.

Chapter 11

KEY THEOREM FOR CONSTRUCTING THE JACQUET-LANGLANDS TRANSFER

In this chapter, we state and prove the key theorem for our construction of the Jacquet-Langlands transfer. We will make use of the theory of the cohomological correspondences throughout this chapter. Instead of explaining all the details, we refer to [XZ17, Appendix A.2] for a nice discussion.

11.1 Preliminaries

Fix a half Tate twist $\Lambda(1/2)$. Recall notations $\langle d\rangle$ and $f^\star$ introduced in §1.3. Throughout this section, we consider the Langlands dual group scheme $\hat{G}_\Lambda$ over Λ of G and its Λ-representations. The subscripts Λ will be omitted for simplicity. We generalize a few notions introduced in previous sections for the sake of stating the key theorem.

More on Local Hecke Stacks

Let $V_\bullet := V_1 \boxtimes V_2 \boxtimes \cdots \boxtimes V_s \in \mathrm{Rep}(\hat{G}^s)$ and assume that for each i, V_i has the Jordan-Hölder factors $\{V_{\mu_{ij}}\}_j$.

The integral geometric Satake equivalence (Theorem 8.0.14) Sat_{G^s} sends $V_\bullet$ to an $(L^+G \otimes \bar{k})^s$-equivariant perverse sheaf $\mathrm{Sat}_{G^s}(V_\bullet)$ on $(Gr_G \otimes \bar{k})^s$. We write $Gr_{V_\bullet}$ for the support of the external tensor product $\mathrm{Sat}(V_1)\tilde{\boxtimes}\mathrm{Sat}(V_2)\tilde{\boxtimes}\cdots\tilde{\boxtimes}\mathrm{Sat}(V_s)$. Let m be a non-negative integer. We call it V_i*-large* if m is μ_{ij}-large for each j, and we call it $V_\bullet$*-large* if $m = m_1 + m_2 + \cdots + m_s$ such that m_i is V_i-large for each i. For a $V_\bullet$-large integer m, $\mathrm{Sat}_{G^n}(V_\bullet)$ descends to a perverse sheaf supported on $\mathrm{Hk}_{V_\bullet}^{\mathrm{loc}(m)} := [L^mG\backslash Gr_{V_\bullet}]$. We write $S(V)^{\mathrm{loc}(m)}$ for the twist of this perverse sheaf by $\langle m \dim G\rangle$. Note that $S(V_\bullet)^{\mathrm{loc}(m)}$ is isomorphic to the "$\star$"-pullback of $S(V_1)^{\mathrm{loc}(m_1)} \boxtimes S(V_2)^{\mathrm{loc}(m_2)} \boxtimes \cdots \boxtimes S(V_s)^{\mathrm{loc}(m_s)}$ along the perfectly smooth morphism $\mathrm{Hk}_{V_\bullet}^{\mathrm{loc}(m)} \to \prod_i \mathrm{Hk}_{V_i}^{\mathrm{loc}(m_i)}$ constructed in (9.9).

In the case $s = 1$, we have

$$Gr_{V_1} = \cup_j Gr_{\mu_{1j}},\ \mathrm{Hk}_{V_1}^{\mathrm{loc}(m)} = \cup_j \mathrm{Hk}_{\mu_{1j}}^{\mathrm{loc}m}.$$

In general, $\mathrm{Hk}_{V_\bullet}^{\mathrm{loc}(m)}$ is of the form $\cup_{\mu_\bullet}\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m)}$. Via descent, Corollary 8.0.15 gives the following natural isomorphism:

$$\mathrm{Hom}_{\hat{G}}(V_\bullet, W_\bullet) \cong \mathrm{Corr}_{\mathrm{Hk}^{0,\mathrm{loc}(m)}_{V_\bullet|W_\bullet}}\big((\mathrm{Hk}^{\mathrm{loc}(m)}_{V_\bullet}, \mathrm{Sat}^{\mathrm{loc}(m)}_{G}(V_\bullet)), (\mathrm{Hk}^{\mathrm{loc}(m)}_{W_\bullet}, \mathrm{Sat}^{\mathrm{loc}(m)}_{G}(W_\bullet))\big). \tag{11.1}$$

Here and below, we regard $V_\bullet$ and $W_\bullet$ as representations of $\hat{G}$ via the diagonal embedding $\hat{G} \hookrightarrow \hat{G}^s$.

Let $V_\bullet$ and $W_\bullet$ be two representations of $\hat{G}^s$. We can similarly define $Gr^0_{V_\bullet|W_\bullet} := Gr_{V_\bullet} \times_{Gr_G} Gr_{W_\bullet}$ and $\mathrm{Hk}^{0,\mathrm{loc}(m)}_{V_\bullet|W_\bullet} = [L^m G \backslash Gr^0_{V_\bullet|W_\bullet}]$.

More on Moduli of Local Shtukas

Let $V_\bullet \in \mathrm{Rep}(\hat{G}^s)$. For a pair of non-negative integers (m, n), we can generalize the notion of $\mu_\bullet$*-large* and define the notion of $V_\bullet$*-large*. Let (m, n) be a pair of $V_\bullet$ and $W_\bullet$-large integers, we can define the moduli of restricted local Shtukas $\mathrm{Sht}^{\mathrm{loc}(m,n)}_{V_\bullet}$ and $\mathrm{Sht}^{\mathrm{loc}(m,n)}_{V_\bullet|W_\bullet}$. Similar to $\mathrm{Hk}^{\mathrm{loc}}_{V_\bullet}$, the stacks $\mathrm{Sht}^{\mathrm{loc}(m,n)}_{V_\bullet}$ and $\mathrm{Sht}^{\mathrm{loc}(m,n)}_{V_\bullet|W_\bullet}$ can be regarded as unions of $\mathrm{Sht}^{\mathrm{loc}(m,n)}_{\mu_\bullet}$ and unions of $\mathrm{Sht}^{\mathrm{loc}(m,n)}_{\mu_\bullet|\nu_\bullet}$. We have the natural forgetful map

$$\psi^{\mathrm{loc}(m,n)} : \mathrm{Sht}^{\mathrm{loc}(m,n)}_{V_\bullet} \to \mathrm{Hk}^{\mathrm{loc}(m)}_{V_\bullet}. \tag{11.2}$$

Choose a pair of $V_\bullet$-large integers (m, n) such that $n > 0$. Write

$$S(\widetilde{V_\bullet})^{\mathrm{loc}(m,n)} := \Psi^{\mathrm{loc}(m,n)}(\mathrm{Sat}(V_\bullet)^{\mathrm{loc}(m)}) \in \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}(m,n)}, \Lambda)$$

for the pullback of $\mathrm{Sat}(V)^{\mathrm{loc}(m)}$ along the morphism $\psi^{\mathrm{loc}(m,n)}$ (up to a shift and twist). For $s = 1$, $S(\widetilde{V})^{\mathrm{loc}(m,n)}$ represents the perverse sheaf $S(\widetilde{V}) := \Psi(\mathrm{Sat}_G(V)) \in \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}}, \Lambda)$.

Consider the front face of the diagram (10.5). The second and third vertical maps are perfectly smooth. Pulling back the cohomological correspondence on the right hand side of (11.1) to the upper edge and pre-composing it with (11.1), we get the map

$$C^{\mathrm{loc}(m,n)} : \mathrm{Hom}_{\hat{G}}(V_\bullet, W_\bullet) \to \mathrm{Corr}_{\mathrm{Sht}^{0,\mathrm{loc}(m,n)}_{V_\bullet|W_\bullet}}(S(\widetilde{V_\bullet})^{\mathrm{loc}(m,n)}, S(\widetilde{W_\bullet})^{\mathrm{loc}(m,n)}). \tag{11.3}$$

The map $C^{\mathrm{loc}(m,n)}$ is compatible with the compositions at the source and target, and we refer to [XZ17, Lemma 6.1.8] for the proof.

Let $V_\bullet \in \mathrm{Rep}(\hat{G}^s)$ and $W \in \mathrm{Rep}(\hat{G})$. We call a quadruple of non-negative integers (m_1, n_2, m_1, n_1) $V_\bullet \boxtimes W$*-acceptable* if

- $m_1 - m_2 = n_1 - n_2$ is W-large,

- (m_2, n_1) is $V_\bullet$-large.

For a quadruple of $V_\bullet \boxtimes W$-acceptable integers (m_1, n_2, m_1, n_1), we can construct the partial Frobenius morphism

$$F^{-1}_{V_\bullet \boxtimes W} : \mathrm{Sht}^{\mathrm{loc}(m_1,n_1)}_{\sigma W \boxtimes V_\bullet} \to \mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{V_\bullet \boxtimes W} \tag{11.4}$$

similar to (10.1). Here, σW is the Frobenius twist of W as in Remark 8.0.14.

Let $V_1, V_2 \in \mathrm{Rep}(\hat{G})$. For any projective object $W \in \mathrm{Rep}(\hat{G})$, choose a quadruple of $((V_1 \otimes V_2 \otimes W) \boxtimes W^*)$-acceptable integers (m_1, n_1, m_2, n_2). We define the following stack

$$\mathrm{Sht}^{W,\mathrm{loc}(m_1,n_1)}_{V_1|V_2} := \mathrm{Sht}^{\mathrm{loc}(m_1,n_1)}_{V_1|\sigma W^* \boxtimes (\sigma W \otimes V_1)} \times_{\mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{(\sigma W \otimes V_1) \boxtimes W^*}} \mathrm{Sht}^{\mathrm{loc}(m_1.n_1)}_{(\sigma W \otimes V_1) \boxtimes W^* | V_2}. \tag{11.5}$$

The Category $\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$

Recall from Remark 8.0.14 that the Langlands dual group $\hat{G}$ is naturally equipped with an action of the arithmetic Frobenius σ. Consider the σ-twisted conjugation action of $\hat{G}$ on $\hat{G}$. We denote by $\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$ the abelian category of $\hat{G}$-equivariant coherent sheaves on the (non-neutral) component $\hat{G}\sigma \subset \hat{G} \rtimes \sigma$. Equivalently, $\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$ can be regarded as the abelian category of coherent sheaves on the quotient stack $[\hat{G}\sigma/\hat{G}]$ where $\hat{G}$ acts on $\hat{G}\sigma$ by the usual conjugation action.

Let $V \in \mathrm{Rep}(\hat{G})$ be an algebraic representation of $\hat{G}$. There is an associated vector bundle on $\hat{G}\sigma$ with global section $\mathcal{O}_{\hat{G}} \otimes V$. Consider the following action of $\hat{G}$ on $\mathcal{O}_{\hat{G}} \otimes V$. For any $g \in \hat{G}$ and $(f, v) \in \mathcal{O}_{\hat{G}} \otimes V$, $g \cdot (f, v) := (gf\sigma^{-1}(g), gv)$. The associated vector bundle thus gives an object $\widetilde{V} \in \mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$.

11.2 Key Theorem

The following theorem is an analogue of [XZ17, Theorem 6.0.1].

Theorem 11.2.1. *Let $V_1, V_2 \in \mathrm{Rep}(\hat{G})$ be two projective Λ-modules. Then there exists the following map*

$$\mathcal{S}_{V_1,V_2} : \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V_1}, \widetilde{V_2}) \longrightarrow \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{V_1}), S(\widetilde{V_2})), \tag{11.6}$$

which is compatible with the natural composition maps in the source and target.

We prove this theorem in the rest of this section.

We give an explicit construction of $\mathcal{S}_{V_1,V_2}$. Consider the following canonical isomorphisms

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V_1},\widetilde{V_2}) \\ \cong&\mathrm{Hom}_{O_{\hat{G}\sigma}}(O_{\hat{G}\sigma}\otimes V_1, O_{\hat{G}\sigma}\otimes V_2)^{\hat{G}} \\ \cong&\mathrm{Hom}(V_1, O_{\hat{G}\sigma}\otimes V_2)^{\hat{G}} \\ \cong&(V_1^*\otimes O_{\hat{G}\sigma}\otimes V_2)^{\hat{G}}.\end{aligned} \tag{11.7}$$

Let $W \in \mathrm{Rep}_\Lambda(\hat{G}_\Lambda)$ be a projective Λ-module with Λ-basis $\{e_i\}_i$ and dual basis $\{e_i^*\}_i$. We construct the map

$$\Theta_W : \mathrm{Hom}_{\hat{G}_\Lambda}(V_1, \sigma W^*\otimes V_2\otimes W) \cong \mathrm{Hom}_{\hat{G}}(V_1, \mathrm{Hom}(\sigma W\otimes W^*, V_2)) \to \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V_1},\widetilde{V_2}),$$

by sending $\mathbf{a} \in \mathrm{Hom}_{\hat{G}_\Lambda}(V_1, \sigma W^*\otimes V_2\otimes W)$ to the $V_1^*\otimes V_2$-valued function $\Theta_W(\mathbf{a})$ on $\hat{G}\sigma$ defined by

$$(\Theta_W(\mathbf{a})(g))(v_1) := \sum_i (\mathbf{a}(v_1))(ge_i^*\otimes e_i).$$

It suffices to construct the map

$$C_W : \mathrm{Hom}_{\hat{G}}(V_1, \sigma W^*\otimes W\otimes V_2) \to \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{V_1}), S(\widetilde{V_2})).$$

for every $W \in \mathrm{Rep}_\Lambda(\hat{G}_\Lambda)$. Let $\mathbf{a} \in \mathrm{Hom}_{\hat{G}}(V_1, \sigma W^*\otimes W\otimes V_2)$. We have the following coevaluation and evaluation maps:

$$\delta_{\sigma W} : \mathbf{1} \to \sigma W^*\otimes\sigma W, e_W : W\otimes W^* \to \mathbf{1}.$$

Choose a quadruple (m_1, n_1, m_2, n_2) of $(V_1\otimes V_2\otimes W)\boxtimes W^*$-large integers. Then the map $C^{\mathrm{loc}(m_1,n_1)}$ defined in (11.3) sends $\mathbf{a}$ to the cohomological correspondence

$$C^{\mathrm{loc}(m_1,n_1)}(\mathbf{a}) : S(\widetilde{V_1})^{\mathrm{loc}(m_1,n_1)} \longrightarrow S(\sigma\widetilde{W^*}\boxtimes(\widetilde{V_2}\otimes\widetilde{W}))^{\mathrm{loc}(m_1,n_1)}. \tag{11.8}$$

The partial Frobenius morphism (11.4) gives rise to the cohomological correspondence (cf.[XZ17, A.2.3])

$$\mathbb{D}\Gamma^*_{F^{-1}_{(W\otimes V_2)\boxtimes W^*}} : S(\sigma\widetilde{W^*}\boxtimes(\widetilde{V_2}\otimes\widetilde{W}))^{\mathrm{loc}(m_1,n_1)} \longrightarrow S((\widetilde{V_2}\otimes\widetilde{W})\boxtimes\widetilde{W^*})^{\mathrm{loc}(m_2,n_2)}. \tag{11.9}$$

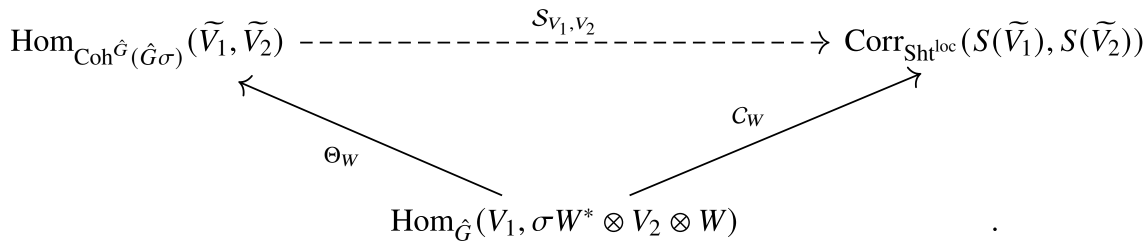
Finally, $C^{\mathrm{loc}(m_2,n_2)}$ sends $\mathrm{id}\otimes e_W$ to the cohomological correspondence

$$C^{\mathrm{loc}(m_2,n_2)}(\mathrm{id}\otimes e_W) : S((\widetilde{V_2}\otimes\widetilde{W})\boxtimes\widetilde{W^*})^{\mathrm{loc}(m_2,n_2)} \longrightarrow S(\widetilde{V_2})^{\mathrm{loc}(m_2,n_2)}. \tag{11.10}$$

The composition of cohomological correspondences (11.8), (11.9), and (11.10) yields a cohomological correspondence

$$C_W(\mathbf{a}) \in \mathrm{Corr}_{\mathrm{Sht}^{W,\mathrm{loc}(m_1,n_1)}_{V_1|V_2}}(S(\widetilde{V_1})^{\mathrm{loc}(m_1,n_1)}, S(\widetilde{V_2})^{\mathrm{loc}(m_2,n_2)}).$$

The construction of the map $\mathcal{S}_{V_1,V_2}$ can be summarized in the following diagram



We prove that the cohomological correspondence constructed in the previous section is well-defined and can be composed.

Let $\mathbf{a}'$ denote the image of $\mathbf{a}$ under the canonical isomorphism $\mathrm{Hom}_{\hat{G}}(V_1, \sigma W^* \otimes V_2 \otimes W) \cong \mathrm{Hom}_{\hat{G}}(\sigma W \otimes V_1 \otimes W^*, V_2)$.

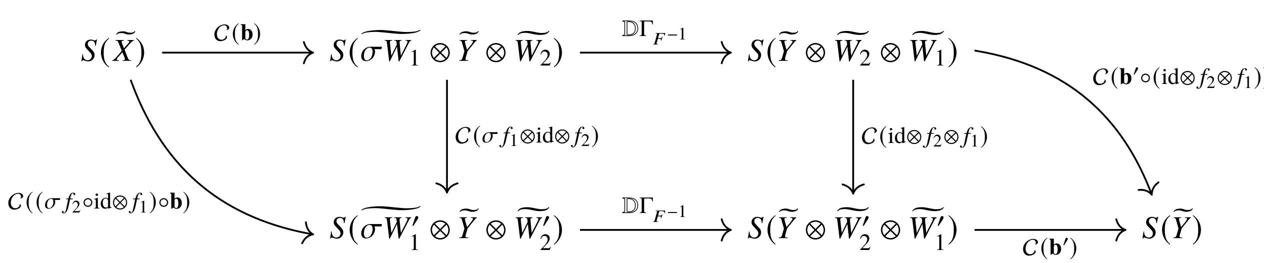
Lemma 11.2.2. *Let $X, Y, W_1, W_2, W_1', W_2'$ be representations of $\hat{G}$, and $f_1 \otimes f_2 : W_1 \otimes W_2 \to W_1' \otimes W_2'$ be a $\hat{G} \times \hat{G}$-module homomorphism. Let $\mathbf{b} \in \mathrm{Hom}_{\hat{G}}(X, \sigma W_1 \otimes Y \otimes W_2)$ and $\mathbf{b}' \in \mathrm{Hom}_{\hat{G}}(Y \otimes W_2' \otimes W_1', Y)$. We omit choosing appropriate integers (m_i, n_i) for simplicity. Then we have*

$$C(\mathbf{b}' \circ (\mathrm{id} \otimes f_2 \otimes f_1)) \circ \mathbb{D}\Gamma^*_{F^{-1}} \circ C(\mathbf{b}) = C(\mathbf{b}') \circ \mathbb{D}\Gamma^*_{F^{-1}} \circ C((\sigma f_1 \circ \mathrm{id} \otimes f_2) \circ \mathbf{b}). \quad (11.11)$$

In particular, the cohomological correspondence $\mathcal{S}_{V_1,V_2}(\mathbf{a})$ equals to the composition of the following cohomological correspondences:

$$C(\delta_{\sigma W} \otimes \mathrm{id}_{V_1}) : S(\widetilde{V_1}) \longrightarrow S(\widetilde{\sigma W^*} \boxtimes (\sigma \widetilde{W} \otimes \widetilde{V_1})),$$
$$\mathbb{D}\Gamma^*_{F^{-1}_{(W \otimes V_1) \boxtimes W^*}} : S(\sigma \widetilde{W^*} \boxtimes (\sigma \widetilde{W} \otimes \widetilde{V_1})) \longrightarrow S((\sigma \widetilde{W} \otimes \widetilde{V_1}) \boxtimes \widetilde{W^*})$$
$$C(\mathbf{a}') : S((\sigma \widetilde{W} \otimes \widetilde{V_1}) \boxtimes \widetilde{W^*}) \longrightarrow S(\widetilde{V_2}).$$

Proof. Consider the following diagram



(11.12)

The bent triangles on the left and right are clearly commutative by Corollary 8.0.15. It suffices to prove that the rectangle in the middle is commutative. But this is a direct consequence of [XZ17, Lemma 6.1.13].

Let $X = V_1, Y = \mathbf{1}$, $W_1 = W_1' = W^*$, $W_2 = \sigma W \otimes V_1, W_2' = W \otimes V_2$. Write $\mathbf{a}''$ for the image of $\mathbf{a}$ under the canonical isomorphism $\mathrm{Hom}(\sigma W \otimes V_1 \otimes W^*, V_2) \cong \mathrm{Hom}(\sigma W \otimes V_1, W \otimes V_2)$. Take $\mathbf{b} = \delta_{\sigma W} \otimes \mathrm{id}$, $f_1 = \mathrm{id}$, and $f_2 = \mathbf{a}''$. Then the second assertion follows from the above commutative diagram. □

Lemma 11.2.3. *For any $\alpha \in \mathrm{Hom}_{\hat{G}}(\widetilde{V_1}, \widetilde{V_2})$, the construction of $\mathcal{S}_{V_1,V_2}$ is independent from the choice of*

(1) projective Λ-modules $W \in \mathrm{Rep}_\Lambda(\hat{G}_\Lambda)$,

(2) $\mathbf{a} \in \mathrm{Hom}_{\hat{G}}(V_1, \sigma W^* \otimes V_2 \otimes W)$, such that $\Theta_W(\mathbf{a}) = \alpha$,

(3) $(V_1 \otimes V_2) \otimes W \boxtimes W^*$-acceptable integers (m_1, n_1, m_2, n_2).

Proof. The proof is completely similar to that of [XZ17, Lemma 6.2.5], and we briefly discuss it here.

We start by proving the independence of (3). Choose another quadruple of $(V_1 \otimes V_2) \otimes W \boxtimes W^*$-acceptable integers $(m_1', n_1', m_2', n_2') \geq (m_1, n_1, m_2, n_2)$. We have the following diagram of Hecke correspondences

$$\begin{array}{ccccc} \mathrm{Sht}_{V_1}^{\mathrm{loc}(m_1',n_1')} & \longleftarrow & \mathrm{Sht}_{V_1|V_2}^{\lambda,\mathrm{loc}(m_1',n_1')} & \longrightarrow & \mathrm{Sht}_{V_2}^{\mathrm{loc}(m_2',n_2')} \\ \downarrow {\scriptstyle \mathrm{res}^{m_1',n_1'}_{m_1,n_1}} & & \downarrow {\scriptstyle \mathrm{res}^{m_1',n_1'}_{m_1,n_1}} & & \downarrow {\scriptstyle \mathrm{res}^{m_2',n_2'}_{m_2,n_2}} \\ \mathrm{Sht}_{V_1}^{\mathrm{loc}(m_1,n_1)} & \longleftarrow & \mathrm{Sht}_{V_1|V_2}^{\lambda,\mathrm{loc}(m_1,n_1)} & \longrightarrow & \mathrm{Sht}_{V_2}^{\mathrm{loc}(m_2,n_2)}. \end{array}$$

This is the upper face of diagram (10.5). As we discussed in §10, all the vertical maps are smooth, the two squares are commutative, and the left square is Cartesian. Then $C_W^{\mathrm{loc}(m_1',n_1')}(\mathbf{a})$ equals the pullback of $C_W^{\mathrm{loc}(m_1,n_1)}(\mathbf{a})$ along the vertical maps.

Next, we prove the independence of (1) and (2) simultaneously. Consider that $\hat{G}$ acts on the filtration of $\mathcal{O}_G$ by right regular representation. Then $\mathcal{O}_G$ is realized as an ind-object in $\mathrm{Rep}_\Lambda(\hat{G})$. Let $X \in \mathrm{Rep}_\Lambda(\hat{G})$ be a projective object and we denote by $\underline{X}$ the underline E-module of X equipped with the trivial $\hat{G}$-action. Consider the following $\hat{G}$-equivariant maps

$$\mathrm{a}_X : X \to \mathcal{O}_G \otimes \underline{X},\ x \mapsto \mathrm{a}_X(x)(g) := gx,$$

$$m_X : \underline{X}^* \otimes X \to O_G,\ (x^*, x) \mapsto m_X(x^*, x)(g) := x^*(gx),$$

where we identify $O_G \otimes \underline{X}$ as the space of $\underline{X}$-valued functions on $\hat{G}$ in the definition of a_X and m_X. Taking $X = W$, we have the following $\hat{G} \times \hat{G}$-module maps

$$\sigma W^* \otimes V_2 \otimes W \xrightarrow{\mathrm{a}_{\sigma W^*}} \underline{W}^* \otimes \sigma O_G \otimes V_2 \otimes W \xrightarrow{m_W} \sigma O_G \otimes V_2 \otimes O_G.$$

The map $\hat{G} \times \hat{G} \to \hat{G}\sigma,\ (g_1, g_2) \mapsto \sigma(g_1)^{-1}\sigma(g_2)\sigma$ induces a natural map $d_\sigma : E[\hat{G}\sigma] \to \sigma O_G \otimes O_G$ which intertwines the σ-twisted conjugation action on $E[\hat{G}\sigma]$ and the diagonal action of $\hat{G}$ on $\sigma O_G \otimes O_G$. For any $\alpha \in \mathrm{Hom}_{\hat{G}}(V_1, O_G \otimes V_2)$, denote by α' the image of α under the following map

$$\mathrm{Hom}_{\hat{G}}(V_1, O_G \otimes V_2) \xrightarrow{d_\sigma} \mathrm{Hom}_{\hat{G}}(V_1, \sigma O_G \otimes V_2 \otimes O_G).$$

Direct computation yields the followings

$$(m_W \circ \mathrm{a}_{\sigma W^*}) \circ \mathbf{a}' = d_\sigma(\alpha') : V_1 \to \sigma O_G \otimes V_2 \otimes O_G,$$

and

$$\mathrm{id}_{V_2} \otimes e_W = \mathrm{ev}_{(1,1)} \circ (m_W \circ a_{W^*}) : V_2 \otimes W \otimes W^* \to V_2,$$

where $\mathrm{ev}_{(1,1)}$ denotes the evaluation at $(1,1) \in \hat{G} \times \hat{G}$. In Lemma 11.2.2, let $W_1 \otimes W_2 := W \otimes W^*$, $W_1' \otimes W_2' := O_G \otimes O_G$, $f_1 \otimes f_2 := m_W \circ a_{W^*}$, $\mathbf{b} := \mathbf{a}'$, and $\mathbf{b}' := \mathrm{ev}_{(1,1)}$. Then we have

$$\begin{aligned} C_W(\mathbf{a}) &= C(\mathrm{id}_{V_2} \otimes e_W) \circ \mathbb{D}\Gamma^*_{F^{-1}_{(V_2 \otimes W) \boxtimes W^*}} \circ C(\mathbf{a}') \\ &= C(\mathrm{ev}_{(1,1)}) \circ \mathbb{D}\Gamma^*_{F^{-1}_{(V_2 \otimes O_G) \boxtimes O_G}} \circ C(d_\sigma(\alpha')). \end{aligned}$$

We see from the last equality in the above that $C_W(\mathbf{a})$ depends only on α and the lemma is thus proved. □

We claim that our construction of $\mathcal{S}_{V_1,V_2}$ is compatible with the composition of morphisms. More precisely, we have the following lemma.

Lemma 11.2.4. *For any representations V_1, V_2, V_3, let $S_1, S_2, S_3 \in \mathrm{Rep}_\Lambda(\hat{G}_\Lambda)$ be projective Λ-modules, and we have the following commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \otimes \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_2, \widetilde{V}_3) & \xrightarrow{\phi} & \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_3) \\ \uparrow & & \uparrow \\ \mathrm{Hom}_{\hat{G}}(\sigma S_1 \otimes V_1 \otimes S_2^*, V_2) \otimes \mathrm{Hom}_{\hat{G}}(\sigma S_2 \otimes V_2 \otimes S_2^*, V_3) & \xrightarrow{\phi'} & \mathrm{Hom}_{\hat{G}}(\sigma S_2 \otimes \sigma S_1 \otimes V_1 \otimes S_1^* \otimes S_2^*, V_3) \\ \downarrow{\scriptstyle C_{S_1} \otimes C_{S_2}} & & \downarrow{\scriptstyle C_{S_1 \otimes S_2}} \\ \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{V}_1), S(\widetilde{V}_2)) \otimes \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{V}_2), S(\widetilde{V}_3)) & \xrightarrow{\phi''} & \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{V}_1), S(\widetilde{V}_3)). \end{array} \tag{11.13}$$

Here

- *the unlabelled vertical arrows are given by the Peter-Weyl theorem*
- ϕ *is the compositions of morphisms in* $\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$
- ϕ'' *is the composition described in* §10.2
- $\phi'(\mathbf{a_1} \otimes \mathbf{a_2})$ *is defined to be the homomorphism*

$$\sigma S_2 \otimes \sigma S_1 \otimes V_1 \otimes S_1^* \otimes S_2^* \xrightarrow{\mathrm{id}_{\sigma S_2} \otimes \mathbf{a_1} \otimes \mathrm{id}_{S_2^*}} \sigma S_2 \otimes V_2 \otimes S_2^* \xrightarrow{\mathbf{a_2}} V_3.$$

Proof. The lemma can be proved following the same idea in the proof of [XZ17, Lemma 6.2.7]. □

We study the endomorphism ring of the unit object in $\mathrm{P}(\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}}, \Lambda)$. This will be used to prove the "$S = T$" theorem for Shimura sets in §12.3.

Let $\delta_{\mathbf{1}}$ denote the intersection cohomology sheaf IC_0 on $\mathrm{Sht}_0^{\mathrm{loc}(m,n)}$. The group theoretic description of the moduli of restricted local Shtukas (cf. [XZ17, §5.3.2]) implies that $\mathrm{Sht}_0^{\mathrm{loc}(m,n)}$ is perfectly smooth. Thus $\delta_{\mathbf{1}}$ may be realized as

$$\delta_{\mathbf{1}}^{m,n} := \Lambda\langle (m-n)\dim G\rangle \in \mathrm{P}(\mathrm{Sht}_0^{\mathrm{loc}(m,n)}, \Lambda)$$

for every $m \geq n$. Fix a square root $q^{1/2}$.

Corollary 11.2.5. *(1) There is a natural isomorphism*

$$\mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(\delta_{\mathbf{1}}, \delta_{\mathbf{1}}) \simeq \mathcal{H}_{G,E}$$

where $\mathcal{H}_{G,E}$ *denotes the Hecke algebra* $C_c^\infty(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}), E)$.

(2) We denote the map

$$\mathcal{S}_{\mathcal{O}_{[\hat{G}\sigma/\hat{G}]}, \mathcal{O}_{[\hat{G}\sigma/\hat{G}]}} : \mathrm{End}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\mathcal{O}_{[\hat{G}\sigma/\hat{G}]}) \to \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(\delta_{\mathbf{1}}, \delta_{\mathbf{1}})$$

by $\mathcal{S}_{\mathcal{O}}$ *for simplicity. Under the isomorphism in* (1), *the map* $\mathcal{S}_{\mathcal{O}} \otimes \mathrm{id}_{E[q^{-1/2}, q^{1/2}]}$ *coincides with the classical Satake isomorphism.*

Proof. Recall the definition of the Borel-Moore homology $\mathrm{H}_i^{\mathrm{BM}}(X)$ for a perfect pfp algebraic space which is defined over an algebraically closed field (cf. [XZ17, A.1.3]). Assume X_1 and X_2 to be perfectly smooth algebraic spaces of pure dimension. Let $X_1 \leftarrow C \rightarrow X_2$ be a correspondence. Then

$$\begin{aligned} &\mathrm{Corr}_C\big((X_1, E\langle d_1\rangle), (X_2, E\langle d_2\rangle)\big) \\ =&\mathrm{Hom}_{D_b^c(C,E)}\big(E\langle d_1\rangle, \omega_C\langle d_2 - 2\dim X_2\rangle\big) \\ =&\mathrm{H}^{\mathrm{BM}}_{2\dim X_2+d_1-d_2}(C). \end{aligned} \tag{11.14}$$

Then if $2\dim C = 2\dim X_2 + d_1 - d_2$, the cohomological correspondences from $(X_1, E\langle d_1\rangle)$ to $(X_2, E\langle d_2\rangle)$ can be identified as the set of irreducible components of C of maximal dimension.

For a perfect pfp algebraic space X of dimension d, define I to be the set of top-dimensional irreducible components of X. Then $\mathrm{H}_d^{\mathrm{BM}}(I)$ is the free E-module generated by the d-dimensional irreducible components of X, and thus can be identified with the space $C(I, E)$ of E-valued functions on I. The map $f \mapsto \sum_{C_i\in I} f(C_i)[C_i]$ establishes a bijection

$$C(I, E) = \mathrm{H}_d^{\mathrm{BM}}(X). \tag{11.15}$$

With the above preparations, we get an isomorphism

$$\mathcal{H}_{G,E} \simeq \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(\delta_{\mathbf{1}}, \delta_{\mathbf{1}}), \tag{11.16}$$

via a similar argument as for [XZ17, Proposition 5.4.4], and we finish the proof of (1).

To prove part (2), we first note that the statement holds for $E = \mathbb{Q}_\ell$ by [XZ17, Theorem 6.0.1(2)]. We sketch the proof here. Let μ be a central minuscule dominant coweight, and ν be a dominant coweight such that $\sigma(\nu) = \nu$. Choose (m_1, n_1, m_2, n_2) to be $(\nu + \mu, \nu)$-acceptable. Take $\mathbf{a} \in \mathrm{Hom}_{\hat{G}}(V_\nu \otimes V_\mu \otimes V_{\nu^*}, V_\mu)$ to be the map induced by the evaluation map $\mathbf{e}_\nu : V_\nu \otimes V_{\nu^*} \longrightarrow \mathbf{1}$. Consider the following diagram

$$\mathrm{pt} \longleftarrow Gr_{\leq\nu^*} \xrightarrow{\ \Delta\ } Gr_{\leq\nu^*} \times Gr_{\leq\nu^*} \xleftarrow{\sigma\times\mathrm{id}} Gr_{\mu^*} \times Gr_{\leq\mu^*} \xleftarrow{\ \Delta\ } Gr_{\leq\nu^*} \longrightarrow \mathrm{pt}.$$

Recall the cohomological correspondences $\delta_{\mathrm{Ic}_{\nu^*}}$ and $e_{\mathrm{Ic}_{\nu^*}}$ defined in [XZ17, §A.2.3.4]. Then $C_{V_\nu}^{\mathrm{loc}(m_1,n_1)}(\mathbf{a}) = \delta_{\mathrm{IC}_{\nu^*}} \circ \Gamma^*_{\sigma\times\mathrm{id}} \circ e_{\mathrm{Ic}_{\nu^*}} \in \mathrm{H}_0^{\mathrm{BM}}(Gr_{\nu^*}(k))$, and the cohomological correspondence $C_{V_\nu}^{\mathrm{loc}(m_1,n_1)}(\mathbf{a})$ can be identified with the function f on $Gr_{\nu^*}(k)$

whose value at $x \in Gr_{\nu^*}(k)$ is given by $\mathrm{tr}(\phi_x \mid \mathrm{Sat}(V_{\nu^*})_{\bar{x}})$. Then up to a choice of $q^{1/2}$, the map $S_{O,O} \otimes_{\mathbb{Q}_\ell} \mathrm{id}_{\mathbb{Q}_\ell[q^{1/2},q^{-1/2}]}$ coincides with the classical Satake isomorphism.

Now we come back to the case $E = \mathbb{Z}_\ell$. Write Q for $\mathbb{Q}_\ell[q^{1/2}, q^{-1/2}]$. The above argument shows that

$$\mathcal{S}_O \otimes Q : \mathrm{End}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\mathcal{O}_{[\hat{G}\sigma/\hat{G}]}) \otimes_{\mathbb{Z}_\ell} Q \to \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(\delta_{\mathbf{1}}, \delta_{\mathbf{1}}) \otimes_{\mathbb{Z}_\ell} Q$$

coincide with the classical Satake isomorphism. Note that

$$\mathrm{End}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\mathcal{O}_{[\hat{G}\sigma/\hat{G}]}) \otimes_{\mathbb{Z}_\ell} Q \simeq \mathbb{Z}_\ell[\hat{G}]^{\hat{G}} \otimes_{\mathbb{Z}_\ell} Q,$$

where $\hat{G}$ acts on $\hat{G}$ by the σ-twisted conjugation. Considering the Satake transfer of the image of $\mathbb{Z}_\ell$-basis of $\mathbb{Z}_\ell[\hat{G}]^{(\hat{G})}$ in $\mathbb{Z}_\ell[\hat{G}]^{(\hat{G})} \otimes_{\mathbb{Z}_\ell} Q$, we conclude the proof of (2). □

Chapter 12

COHOMOLOGICAL CORRESPONDENCES BETWEEN SHIMURA VARIETIES

In this section, we adapt the machinery developed in previous sections and apply it to the study of the cohomological correspondences between different Hodge type Shimura varieties following the idea of [XZ17].

12.1 Preliminaries

Let (G, X) be a Shimura datum and E be its reflex field (cf. [Mil05]). Let $K \subset G(\mathbb{A}_f)$ be a (sufficiently small) open compact subgroup and denote by $\mathrm{Sh}_K(G, X)$ the corresponding Shimura variety defined over E. Fix a prime $p > 2$ such that K_p is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. We write $\underline{G}$ for the reductive group which extends G to $\mathbb{Z}_{(p)}$ and such that $\underline{G}(\mathbb{Z}_p) = K_p$. Choose v to be a place of E lying over p. We write $\mathcal{O}_{E,(v)}$ for the localization of $\mathcal{O}_E$ at v. Results of Kisin [Kis10] and Vasiu [Vas07] state that for any Hodge type Shimura datum (G, X), there is a smooth integral canonical model $\mathcal{S}_K(G, X)$ of $\mathrm{Sh}_K(G, X)$, which is defined over $\mathcal{O}_{E,(v)}$. Let k_v denote the residue field of $\mathcal{O}_{E,v}$ and fix an algebraic closure $\bar{k}_v$ of k_v. We denote by $\mathrm{Sh}_{\mu,K} := (\mathcal{S}_K(G, X) \otimes k_v)^{\mathrm{pf}}$ the perfection of the special fiber of $\mathcal{S}_K(G, X)$. The perfection of mod p fibre of Shimura varieties and moduli of local Shtukas are related by a map $\mathrm{loc}_p : \mathrm{Sh}_{\mu,K} \to \mathrm{Sht}_\mu^{\mathrm{loc}}$. The construction of loc_p is via a $\underline{G}$-torsor over the crystalline cite $(\mathcal{S}_{K,k_v}/\mathcal{O}_{E,v})_{\mathrm{CRIS}}$ and we refer to [XZ17, §7.2.1] for a detailed discussion. In the Siegel case, it may be understood as the perfection of the morphism sending an abelian variety to its underlying p-divisible group. We need the following result of Xiao-Zhu [XZ17, Proposition 7.2.4] for our proof of the main theorem.

Proposition 12.1.1. *Let (m, n) be a pair of μ-large integers. The morphism*

$$\mathrm{loc}_p(m, n) := \mathrm{res}_{m,n} \circ \mathrm{loc}_p : \mathrm{Sh}_\mu \to \mathrm{Sht}_\mu^{\mathrm{loc}(m,n)}$$

is perfectly smooth.

Étale Local Systems on $\mathrm{Sh}_{\mu,K}$

Let $\ell \neq p$ be a prime number. Assume that $\rho : G \to GL_{\mathbb{Q}_\ell}(W)$ is a $\mathbb{Q}_\ell$-representation of G. If $K \subset G(\mathbb{A}_f)$ is sufficiently small, we associate an étale local system $\mathcal{L}_{\ell,W}$ on $\mathrm{Sh}_{\mu,K}$ to W following the idea of [LZ17, §4] and [Mil90, §III.6] as follows.

Write $K = K_\ell K^\ell$ with $K_\ell \subset G(\mathbb{Q}_\ell)$ and $K^\ell \subset G(\mathbb{A}_f^\ell)$. The representation ρ restricts to a representation

$$\rho_{K_\ell} : K(\mathbb{Q}_\ell) \to G(\mathbb{Q}_\ell) \to GL(W_{\mathbb{Q}_\ell}).$$

Note that $K(\mathbb{Q}_\ell)$ is compact, and there exists a lattice $\Lambda_{W,\ell} \subset W_{\mathbb{Q}_\ell}$ fixed by $K(\mathbb{Q}_\ell)$. Now we vary the levels at ℓ. Define

$$K_\ell^{(n)} := K_\ell \cap \rho^{-1}_{K(\mathbb{Q}_\ell)}(\{g \in GL(\Lambda_{W,\ell}) \mid g \equiv 1 \mod \ell^n\}).$$

Then we get a system of open neighborhoods of $1 \in G(\mathbb{Q}_\ell)$. For each n, the construction of $K_\ell^{(n)}$ gives rise to a representation

$$\rho^n_{K_\ell} : K_\ell / K_\ell^{(n)} \to GL(\Lambda_{W,\ell}/\ell^n \Lambda_{W,\ell}).$$

The natural projection map $\mathrm{Sh}_{\mu,K_\ell^{(n)}K^\ell} \to \mathrm{Sh}_{\mu,K_\ell K^\ell}$ is a finite étale cover with the group of deck transformations being $K_\ell/K_\ell^{(n)}$. Then the trivial étale $\mathbb{Z}/\ell^n\mathbb{Z}$-local system $\mathrm{Sh}_{\mu,K_\ell^{(n)}K^\ell} \times \Lambda_{W,\ell}/\ell^n\Lambda_{W,\ell}$ on $\mathrm{Sh}_{\mu,K_\ell^{(n)}K^\ell}$ gives rise to the étale $\mathbb{Z}/\ell^n\mathbb{Z}$-local system

$$\mathcal{L}_{W,\ell,n} := \mathrm{Sh}_{\mu,K_\ell^{(n)}K^\ell} \times^{K_\ell/K_\ell^{(n)}} \Lambda_{W,\ell}/\ell^n\Lambda_{W,\ell}.$$

Let

$$\mathcal{L}_{W,\mathbb{Z}_\ell} := \varprojlim_n \mathcal{L}_{W,\ell,n}. \tag{12.1}$$

This is an étale $\mathbb{Z}_\ell$-local system on $\mathrm{Sh}_{\mu,K}$. It can be checked that $\mathcal{L}_{W,\mathbb{Q}_\ell} := \mathcal{L}_{W,\mathbb{Z}_\ell} \otimes \mathbb{Q}$ is an étale $\mathbb{Q}_\ell$-local system on $\mathrm{Sh}_{\mu,K}$ which is independent of the choice of Λ_ℓ.

12.2 Main Theorem

Let (G_1, X_1) and (G_2, X_2) be two Hodge type Shimura data (cf. [Mil05]) equipped with an isomorphism $\theta : G_{1,\mathbb{A}_f} \simeq G_{2,\mathbb{A}_f}$. Let $\{\mu_i\}$ denote the conjugacy class of Hodge cocharacters determined by X_i and consider them as dominant characters of $\hat{T}$. In particular, μ_1 and μ_2 are both minuscule. Then [XZ17, Corollary 2.1.5] implies that there is a canonical inner twist $\Psi_\mathbb{R} : G_1 \to G_2$ over $\mathbb{C}$. Recall notations in §1.3. We define $\mu_{i,\mathrm{ad}}$ to be the composition of μ_i with the quotient $G \to G_{\mathrm{ad}}$ and consider it as a character of $\hat{T}_{\mathrm{sc}}$. We assume that

$$\mu_{1,\mathrm{ad}} \mid_{Z(\hat{G}_{\mathrm{sc}}^{\Gamma_\mathbb{Q}})} = \mu_{2,\mathrm{ad}} \mid_{Z(\hat{G}_{\mathrm{sc}}^{\Gamma_\mathbb{Q}})} .$$

It follows from [XZ17, Corollary 2.1.6] that $\Psi_{\mathbb{R}}$ comes from a unique global inner twist $\Psi : G_{1\bar{\mathbb{Q}}} \to G_{2\bar{\mathbb{Q}}}$ such that $\Psi = \mathrm{Int}(h) \circ \theta$, for some $\theta : G_{1,\mathbb{A}_f} \simeq G_{2,\mathbb{A}_f}$ and $h \in G_{2,\mathrm{ad}}(\bar{\mathbb{A}}_f)$.

We assume that $K_i \subset G(\mathbb{A}_f)$ to be sufficiently small such that $\theta K_1 = K_2$. Choose a prime p such that $K_{1,p}$ (and therefore $K_{2,p}$) is hyperspecial. Let $\underline{G_i}$ be the integral model of $G_{i,\mathbb{Q}_p}$ over $\mathbb{Z}_p$ determined by $K_{i,p}$. Then $\underline{G_1} \simeq \underline{G_2}$, and we can thus identify their Langlands dual groups $(\hat{G}, \hat{B}, \hat{T})$. Choose an isomorphism $\iota : \mathbb{C} \simeq \bar{\mathbb{Q}}_p$. Let $v \mid p$ be a place of the compositum of reflex fields of (G_i, X_i) determined by our choice of isomorphism ι. We write Sh_{μ_i} for the mod p fibre of the canonical integral model of $\mathrm{Sh}_{K_i}(G_i, X_i)$ base change to k_v. We make the following assumption

$$\mu_1 \mid_{Z(\hat{G}^{\Gamma_{\mathbb{Q}_p}})} = \mu_2 \mid_{Z(\hat{G}^{\Gamma_{\mathbb{Q}_p}})} . \tag{12.2}$$

The assumption guarantees the existence of the ind-scheme $\mathrm{Sh}_{\mu_1|\mu_2}$ which fits into the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Sh}_{\mu_1,K_1} & \xleftarrow{\overleftarrow{h}_{\mu_1}} & \mathrm{Sh}_{\mu_1|\mu_2} & \xrightarrow{\overrightarrow{h}_{\mu_2}} & \mathrm{Sh}_{\mu_2,K_2} \\ \downarrow{\scriptstyle \mathrm{loc}_p} & & \downarrow & & \downarrow{\scriptstyle \mathrm{loc}_p} \\ \mathrm{Sht}^{\mathrm{loc}}_{\mu_1} & \xleftarrow{\overleftarrow{h}^{\mathrm{loc}}_{\mu_1}} & \mathrm{Sht}^{\mathrm{loc}}_{\mu_1|\mu_2} & \xrightarrow{\overrightarrow{h}^{\mathrm{loc}}_{\mu_2}} & \mathrm{Sht}^{\mathrm{loc}}_{\mu_2} \end{array} , \tag{12.3}$$

and makes both squares to be Cartesian.

Remark 12.2.1. *In the case that* $(G_1, X_1) = (G_2, X_2)$*,* $\mathrm{Sh}_{\mu_1|\mu_2}$ *is the perfection of the mod p fibre of a natural integral model of some Hecke correspondence. If* $(G_1, X_1) \neq (G_2, X_2)$*, then* $\mathrm{Sh}_{\mu_1|\mu_2}$ *can be regarded as "exotic Hecke correspondences" between mod p fibres of different Shimura varieties. We refer to [XZ17, §7.3.3, §7.3.4] for a detailed discussion.*

Let (G_i, X_i) $i = 1, 2, 3$ be three Hodge type Shimura data, together with the isomorphisms $\theta_{i,j} : G_{i,\mathbb{A}_f} \simeq G_{j,\mathbb{A}_f}$ satisfying the natural cocycle condition. Choose a common level K using the isomorphism $\theta_{i,j}$. Let p be an unramified prime, such that the assumption (12.2) holds for each pair of $((G_i, X_i), (G_j, X_j))$. Choose a half Tate twist $\mathbb{Q}_\ell(1/2)$.

Let $V_i := V_{\mu_i}$ be the highest weight representation of $\hat{G}_{\mathbb{Q}_\ell}$ of highest weight μ_i. Write $\widetilde{V_i} \in \mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)$ for the vector bundle associated to V_i analogous to

§11.4. Recall from §12.1 that, to each representation W of $G_{\mathbb{Q}_\ell}$, we can attach the étale local system $\mathcal{L}_{W,\mathbb{Q}_\ell}$ on Sh_{μ_i}. Let $d_i = \langle 2\rho, \mu_i \rangle = \dim \mathrm{Sh}_K(G_i, X_i)$. Denote the global section of the structure sheaf on the quotient stack $[\hat{G}\sigma/\hat{G}]$ by $\mathcal{J}$, and the prime-to-p Hecke algebra by $\mathcal{H}^p$.

Theorem 12.2.2. *There exists a map*

$$\mathrm{Spc} : \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V_1}, \widetilde{V_2}) \to \mathrm{Hom}_{\mathcal{H}^p \otimes \mathcal{J}}(\mathrm{H}^*_c(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_1 \rangle), \mathrm{H}^*_c(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_2 \rangle)), \tag{12.4}$$

which is compatible with compositions on the source and target.

Proof. Choose a lattice $\Lambda_i \in \mathrm{Rep}_{\mathbb{Z}_\ell}(\hat{G}_{\mathbb{Z}_\ell})$ in V_i. We denote by $\widetilde{\Lambda_i} \in \mathrm{Coh}^{\hat{G}_{\mathbb{Z}_\ell}}(\hat{G}_{\mathbb{Z}_\ell}\sigma)$ the coherent sheaf which corresponds to Λ_i as in §11.1. Then

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V_1}, \widetilde{V_2}) &\simeq \mathrm{Hom}_{\hat{G}_{\mathbb{Q}_\ell}}(V_1, V_2 \otimes \mathbb{Q}_\ell[\hat{G}]) \\ &\simeq \mathrm{Hom}_{\hat{G}_{\mathbb{Q}_\ell}}(\Lambda_1 \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, (\Lambda_2 \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[\hat{G}]) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \\ &\simeq \mathrm{Hom}_{\hat{G}_{\mathbb{Z}_\ell}}(\Lambda_1, \Lambda_2 \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[\hat{G}]) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \\ &\simeq \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Z}_\ell}}(\hat{G}_{\mathbb{Z}_\ell}\sigma)}(\widetilde{\Lambda_1}, \widetilde{\Lambda_2}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell. \end{aligned} \tag{12.5}$$

By Theorem 11.2.1, we get a map

$$\mathcal{S}_{\Lambda_1,\Lambda_2} : \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Z}_\ell}}(\hat{G}_{\mathbb{Z}_\ell}\sigma)}(\widetilde{\Lambda_1}, \widetilde{\Lambda_2}) \to \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{\Lambda_1}), S(\widetilde{\Lambda_2})). \tag{12.6}$$

Combining (12.5) with (12.6), we get the following map

$$\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V_1}, \widetilde{V_2}) \to \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{\Lambda_1}), S(\widetilde{\Lambda_2})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell. \tag{12.7}$$

Choose a dominant coweight ν and a quadruple (m_1, n_1, m_2, n_2) that is $(\mu_1 + \nu, \nu)$-acceptable and $(\mu_2 + \nu, \nu)$-acceptable. We have the following diagram

$$\begin{array}{ccccc} \mathrm{Sh}_{\mu_1} & \xleftarrow{\overleftarrow{h}_{\mu_1}} & \mathrm{Sh}^{\nu}_{\mu_1|\mu_2} & \xrightarrow{\overrightarrow{h}_{\mu_2}} & \mathrm{Sh}_{\mu_2} \\ \downarrow{\scriptstyle \mathrm{loc}_p} & & \downarrow{\scriptstyle \mathrm{loc}^{\nu}_p} & & \downarrow{\scriptstyle \mathrm{loc}_p} \\ \mathrm{Sht}^{\mathrm{loc}}_{\mu_1} & \xleftarrow{\overleftarrow{h}^{\mathrm{loc}}_{\mu_1}} & \mathrm{Sht}^{\nu,\mathrm{loc}}_{\mu_1|\mu_2} & \xrightarrow{\overrightarrow{h}^{\mathrm{loc}}_{\mu_2}} & \mathrm{Sht}^{\mathrm{loc}}_{\mu_2} \\ \downarrow{\scriptstyle \mathrm{res}_{m_1,n_1}} & & \downarrow{\scriptstyle \mathrm{res}^{\nu}_{m_1,n_1}} & & \downarrow{\scriptstyle \mathrm{res}_{m_2,n_2}} \\ \mathrm{Sht}^{\mathrm{loc}(m_1,n_1)}_{\mu_1} & \xleftarrow{\overleftarrow{h}^{\mathrm{loc}(m_1,n_1)}_{\mu_1}} & \mathrm{Sht}^{\nu,\mathrm{loc}(m_1,n_1)}_{\mu_1|\mu_2} & \xrightarrow{\overrightarrow{h}^{\mathrm{loc}(m_2,n_2)}_{\mu_2}} & \mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{\mu_2} \end{array}, \tag{12.8}$$

where

- all squares are commutative (discussions on diagram (10.5) and diagram (12.3),
- except for the square at the down right corner, and the other three squares are Cartesian (discussions on diagram (12.3) and diagram (12.5),
- the morphism $\overleftarrow{h}_{\mu_1}$ is perfectly proper ([XZ17, Lemma 5.2.12]),
- the morphisms $\mathrm{loc}_p(m_i, n_i)$ are perfectly smooth (Proposition 12.1.1).

Then the morphism $\mathrm{loc}_p^\nu(m_1, n_1) := \mathrm{res}^\nu_{m_1,n_1} \circ \mathrm{loc}_p^\nu$ is also perfectly proper. Thus we can pullback the cohomological correspondences (cf. [XZ17, A.2.11)]) on the right hand side of (12.6) along $\mathrm{loc}_p^\nu(m_1, n_1)$ to obtain a map

$$\mathrm{loc}_p^\nu(m_1,n_1)^\star : \mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{\Lambda_1}), S(\widetilde{\Lambda_2})) \to \mathrm{Corr}_{\mathrm{Sh}^\nu_{\mu|\mu}}(\mathrm{loc}_p(m_1,n_1)^\star S(\widetilde{\Lambda_1}), \mathrm{loc}_p(m_2,n_2)^\star(S(\widetilde{\Lambda_2})).$$

Note that μ_i are minuscule, then the $\star$-pullback of $S(\widetilde{\Lambda}_i)$ along $\mathrm{loc}_p(m_i, n_i)$ equals $\mathbb{Z}_\ell\langle d_i\rangle$. Next, we construct a natural map

$$\mathfrak{C}_W : \mathrm{Corr}_{\mathrm{Sh}^\nu_{\mu_1|\mu_2}}\big((\mathrm{Sh}_{\mu_1}, \mathbb{Z}_\ell\langle d_1\rangle), (\mathrm{Sh}_{\mu_2}, \mathbb{Z}_\ell\langle d_2\rangle)\big) \to \mathrm{Corr}_{\mathrm{Sh}^\nu_{\mu_1|\mu_2}}\big((\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Z}_\ell}\langle d_1\rangle), (\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W,\mathbb{Z}_\ell}\langle d_2\rangle)\big). \tag{12.9}$$

For each $n \in \mathbb{Z}^+$, we note that there exists an ind-scheme $\mathrm{Sh}^{(n)}_{\mu_1|\mu_2}$ which fits into the following commutative diagram such that both squares are Cartesian

$$\begin{array}{ccccc}
\mathrm{Sh}_{\mu_1,K_\ell^{(n)}K^\ell} & \xleftarrow{\overleftarrow{h}^{(n)}_{\mu_1}} & \mathrm{Sh}^{\nu,(n)}_{\mu_1|\mu_2} & \xrightarrow{\overrightarrow{h}^{(n)}_{\mu_2}} & \mathrm{Sh}_{\mu_2,K_\ell^{(n)}K^\ell} \\
\downarrow{\scriptstyle p_1^n} & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle p_2^n} \\
\mathrm{Sh}_{\mu_1} & \xleftarrow{\overleftarrow{h}_{\mu_1}} & \mathrm{Sh}^{\nu}_{\mu_1|\mu_2} & \xrightarrow{\overrightarrow{h}_{\mu_2}} & \mathrm{Sh}_{\mu_2}.
\end{array}$$

Here the three vertical maps are the natural quotients by the finite group K_ℓ/K_ℓ^n and are thus étale.

Let $(f_n)_n : (\overleftarrow{h}_{\mu_1})^*(\mathbb{Z}/\ell^n\mathbb{Z}\langle d_1\rangle)_n \to (\overrightarrow{h}_{\mu_2})^!(\mathbb{Z}/\ell^n\mathbb{Z}\langle d_2\rangle)_n$ be a cohomological correspondence in $\mathrm{Corr}_{\mathrm{Sh}^\nu_{\mu_1|\mu_2}}\big((\mathrm{Sh}_{\mu_1}, \mathbb{Z}_\ell\langle d_1\rangle), (\mathrm{Sh}_{\mu_2}, \mathbb{Z}_\ell\langle d_2\rangle)\big)$. For each $n \in \mathbb{Z}^+$, the shifted pullback (cf. [XZ17, A.2.12]) of f_n gives rise to a cohomological correspondence

$$\tilde{f}_n : (\overleftarrow{h}^{(n)}_{\mu_1})^*(\mathbb{Z}/\ell^n\mathbb{Z}\langle d_1\rangle) \to (\overrightarrow{h}^{(n)}_{\mu_2})^!(\mathbb{Z}/\ell^n\mathbb{Z}\langle d_2\rangle)$$

in $\mathrm{Corr}_{\mathrm{Sh}^{\nu,(n)}_{\mu_1|\mu_2}}\big((\mathrm{Sh}_{\mu_1,K_\ell^{(n)}K^\ell},\mathbb{Z}/\ell^n\mathbb{Z}\langle d_1\rangle),(\mathrm{Sh}_{\mu_2,K_\ell^{(n)}K^\ell},\mathbb{Z}/\ell^n\mathbb{Z}\langle d_2\rangle)\big)$. For any representation W of $G_{\mathbb{Q}_\ell}$, recall the $\mathbb{Z}/\ell^n\mathbb{Z}$ module $\Lambda_{W,\ell}/\ell^n\Lambda_{W,\ell}$ constructed in §12.2. The cohomological correspondence $\tilde{f}_n$ gives rise to a cohomological correspondence

$$\tilde{g}_n \in \mathrm{Corr}_{\mathrm{Sh}^{\nu,(n)}_{\mu_1|\mu_2}}(\mathrm{Sh}_{\mu_1,K_\ell^{(n)}K^\ell}\times\Lambda_{W,\ell}/\ell^n\Lambda_{W,\ell}\langle d_1\rangle,\mathrm{Sh}_{\mu_2,K_\ell^{(n)}K^\ell}\times\Lambda_{W,\ell}/\ell^n\Lambda_{W,\ell}\langle d_2\rangle).$$

In addition, the cohomological correspondence $\tilde{f}_n$ is $K_\ell/K_\ell^{(n)}$-equivariant. Then it follows that the cohomological correspondence $\tilde{g}_n$ is also $K_\ell/K_\ell^{(n)}$-equivariant and descends to a cohomological correspondence

$$g_n \in \mathrm{Corr}_{\mathrm{Sh}^{\nu}_{\mu_1|\mu_2}}((\mathrm{Sh}_{\mu_1},\mathcal{L}_{W,\ell,n}\langle d_1\rangle),(\mathrm{Sh}_{\mu_2},\mathcal{L}_{W,\ell,n}\langle d_2\rangle)).$$

Defining $\mathfrak{C}_W((f_n)_n) := (g_n)_n$ completes the construction of $\mathfrak{C}_W$.

Compose the maps we previously construct,

$$\mathrm{Corr}_{\mathrm{Sht}^{\nu,\mathrm{loc}(m_1,n_1)}_{\mu_1|\mu_2}}\Big((\mathrm{Sht}^{\mathrm{loc}(m_1,n_1)}_{\mu_1},S(\widetilde{\Lambda_1})^{\mathrm{loc}(m_1,n_1)}),(\mathrm{Sht}^{\mathrm{loc}(m_2,n_2)}_{\mu_1},S(\widetilde{\Lambda_2})^{\mathrm{loc}(m_2,n_2)})\Big) \tag{12.10}$$

$$\xrightarrow{\mathrm{loc}^\nu_p(m_1,n_1)^\star}\mathrm{Corr}_{\mathrm{Sh}^{\nu}_{\mu_1|\mu_2}}((\mathrm{Sh}_{\mu_1},\mathbb{Z}_\ell\langle d_1\rangle),(\mathrm{Sh}_{\mu_2},\mathbb{Z}_\ell\langle d_2\rangle))$$
$$\xrightarrow{\mathfrak{C}_W}\mathrm{Corr}_{\mathrm{Sh}^{\nu}_{\mu_1|\mu_2}}((\mathrm{Sh}_{\mu_1},\mathcal{L}_{W,\mathbb{Z}_\ell}\langle d_1\rangle),(\mathrm{Sh}_{\mu_2},\mathcal{L}_{W,\mathbb{Z}_\ell}\langle d_2\rangle))$$
$$\xrightarrow{\mathrm{H}^*_c}\mathrm{Hom}_{\mathcal{H}^p}(\mathrm{H}^*_c(\mathrm{Sh}_{\mu_1},\mathcal{L}_{W,\mathbb{Z}_\ell}\langle d_1\rangle),\mathrm{H}^*_c(\mathrm{Sh}_{\mu_2},\mathcal{L}_{W,\mathbb{Z}_\ell}\langle d_2\rangle)).$$

We justify that the composition of maps in (12.11) factors through $\mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{V_1}),S(\widetilde{V_2}))$. Note that the proof of Lemma 11.2.3.(3) and the definition of $\mathrm{loc}^\nu_p(m_1,n_1)^\star$ imply that for a quadruple (m_1',n_1',m_2',n_2') of $(\mu_1+\nu,\nu)$-acceptable and $(\mu_2+\nu,\nu)$-acceptable integers, the functor $\mathrm{loc}^\nu_p(m_1,n_1)^\star$ commutes with the connecting morphism in (10.12) (with μ_1,μ_2,λ fixed). Let $\nu\le\nu'$ and (m_1',n_1',m_2',n_2') be a quadruple of non-negative integers satisfying appropriate acceptance conditions. The proper smooth base change shows that $\mathrm{loc}^\nu_p(m_1',n_1')^\star$ commutes with enlarging ν to ν'. In addition, the proper smooth base change together with the construction of $\mathfrak{C}_W$ show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Corr}_{\mathrm{Sh}^{\nu}_{\mu_1|\mu_2}}((\mathrm{Sh}_{\mu_1},\mathbb{Z}_\ell\langle d_1\rangle),(\mathrm{Sh}_{\mu_2},\mathbb{Z}_\ell\langle d_2\rangle)) & \xrightarrow{\mathfrak{C}_W} & \mathrm{Corr}_{\mathrm{Sh}^{\nu}_{\mu_1|\mu_2}}((\mathrm{Sh}_{\mu_1},\mathcal{L}_{W,\mathbb{Z}_\ell}\langle d_1\rangle),(\mathrm{Sh}_{\mu_2},\mathcal{L}_{W,\mathbb{Z}_\ell}\langle d_2\rangle)) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
\mathrm{Corr}_{\mathrm{Sh}^{\nu'}_{\mu_1|\mu_2}}((\mathrm{Sh}_{\mu_1},\mathbb{Z}_\ell\langle d_1\rangle),(\mathrm{Sh}_{\mu_2},\mathbb{Z}_\ell\langle d_2\rangle)) & \xrightarrow{\mathfrak{C}_W} & \mathrm{Corr}_{\mathrm{Sh}^{\nu'}_{\mu_1|\mu_2}}((\mathrm{Sh}_{\mu_1},\mathcal{L}_{W,\mathbb{Z}_\ell}\langle d_1\rangle),(\mathrm{Sh}_{\mu_2},\mathcal{L}_{W,\mathbb{Z}_\ell}\langle d_2\rangle)).
\end{array}$$

Thus the map $\mathfrak{C}_W$ is compatible with the enlargement of ν. Finally, by [XZ17, Lemma A.2.8], the composition of maps $\mathrm{H}_c^* \circ \mathfrak{C}_W$ commutes with enlarging ν to ν'. We complete the proof of the statement at the beginning of this paragraph.

Composing (12.7) with (12.11), we get a canonical map

$$\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \to \mathrm{Hom}_{\mathcal{H}^p}(\mathrm{H}_c^*(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_1\rangle), \mathrm{H}_c^*(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_2\rangle)). \tag{12.11}$$

The fact that (12.9) is compatible with the compositions of the source and target can be proved in an analogous way as [XZ17, Lemma 7.3.12], and we omit the details. Then the action of $\mathcal{J}$ naturally translates to the right hand side of (12.9) and upgrades it to our desired map

$$\mathrm{Spc} : \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \to \mathrm{Hom}_{\mathcal{H}^p \otimes \mathcal{J}}(\mathrm{H}_c^*(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_1\rangle), \mathrm{H}_c^*(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_2\rangle)).$$

□

As discussed in *loc.cit*, the action of $\mathcal{J}$ on $\mathrm{H}_c^*(\mathrm{Sh}_{\mu_i}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_i\rangle)$ is expected to coincide with the usual Hecke algebra action, which may be understood as the Shimura variety analogue of V. Lafforgue's "$S = T$" theorem (cf. [Laf18]). We prove this in the case of Shimura sets.

Proposition 12.2.3. *Let* $\mathrm{Sh}_K(G, X)$ *be a zero-dimensional Shimura variety. Then the action of* $\mathcal{J}$ *on* $\mathrm{H}_c^*(\mathrm{Sh}_{\mu_i}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_i\rangle)$ *is given by the classical Satake isomorphism.*

Proof. Let $f \in \mathcal{J}$. Since the Shimura variety we consider is zero-dimensional, it follows from [XZ17, A.2.3(5)] that the cohomological correspondence $\mathrm{loc}_p^\star(\mathcal{S}_O(f))$ can be identified with a $\mathbb{Z}_\ell$-valued function on $\mathrm{Sh}_{\mu|\mu}$. By our construction of the map Spc, this function is given by the pullback of a function f' on $\mathrm{Sht}^{\mathrm{loc}}_{\mu|\mu} = G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$. Corollary 11.2.5(2) thus implies that the function f' is exactly the function $\mathcal{S}_O(f) \in H_{G,E[p^{-1/2},p^{1/2}]}$ which is the image of f under the classical Satake isomorphism.

For any $n \in \mathbb{Z}^+$, take $W = \mathbb{Z}_\ell^n$. Recall our construction of $\mathfrak{C}_W$, the cohomological correspondence $\tilde{f}_n$ is given by a finite direct sum of the function $\mathrm{loc}_p^\star(\mathcal{S}_O(f))$ since the Shimura variety we consider is a set of discrete points. Then the action of $\mathrm{Spc}(f)$ on $\mathrm{H}_c^*(\mathrm{Sh}_{\mu_i}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_i\rangle)$ is given by the classical Satake isomorphism. For general W, we take resolutions of it as in (12.10), and the statement follows from the case $W = \mathbb{Z}_\ell^n$. □

12.3 Non-Vanishing of the Geometric Jacquet-Langlands Transfer

In Theorem 12.2.2, we constructed the geometric Jacquet-Langlands transfer

$$\mathrm{Spc} : \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \to \mathrm{Hom}_{\mathcal{H}^p \otimes \mathcal{J}}(\mathrm{H}^*_c(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_1\rangle), \mathrm{H}^*_c(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_2\rangle).$$

It is natural to ask when this transfer map is nonzero. We discuss this issue in this section. The idea essentially follows from the discussion in [XZ17, §7.4], and we briefly sketch it here.

Assume that $\mathrm{Sh}_{\mu_1,K_1}(G_1, X_1)$ is a zero dimensional Shimura variety. The Jacquet-Langlands transfer map induces the following map

$$\mathrm{JL}_{1,2}(\mathbf{a}) : \mathrm{H}^0_c(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell}) \to \mathrm{H}^*_c(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_2\rangle),$$

for $\mathbf{a} \in \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma_p)}(\widetilde{V}_1, \widetilde{V}_2)$. Let $\mathbf{a}' \in \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma_p)}(\widetilde{V}_2, \widetilde{V}_1)$be the morphism such that the induced map

$$\mathrm{JL}_{2,1}(\mathbf{a}') : \mathrm{H}^*_c(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W,\mathbb{Q}_\ell}\langle d_2\rangle) \to \mathrm{H}^0_c(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell})$$

is dual to $\mathrm{JL}_{1,2}(\mathbf{a})$ when viewing it as a cohomological correspondence (cf. [XZ17, §A.2.18]).

The composition map $\mathrm{JL}_{2,1}(\mathbf{a}') \circ \mathrm{JL}_{1,2}(\mathbf{a})$ gives rise to an endomorphism of $\widetilde{V_{\mu_1}} \in \mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma_p)$. By [XZ17, Theorem 1.4.1], the hom spaces $\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V}_1, \widetilde{V}_2)$ and $\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V}_2, \widetilde{V}_1)$ are both finite projective $\mathcal{J}$-modules. Thus it makes sense to consider the determinant of the pairing

$$\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma_p)}(\widetilde{V}_2, \widetilde{V}_1) \otimes \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma_p)}(\widetilde{V}_1, \widetilde{V}_2) \to \mathcal{J}. \tag{12.12}$$

In particular, this determinant can be regarded as a regular function on the stack $[\hat{G}_{\mathbb{Q}_\ell}\sigma_p/\hat{G}_{\mathbb{Q}_\ell}]$; for a detailed discussion on the pairing (12.12), see [XZ19].

By Theorem 6.1.2 in *loc.cit*, we conclude the following result:

Theorem 12.3.1. *Let π_f be an irreducible $\mathcal{H}_K$-module, and let*

$$\mathrm{H}^0_c(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell})[\pi_f] := \mathrm{Hom}_{\mathcal{H}_K}(\pi_f, \mathrm{H}^0_c(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell})) \otimes \pi_f$$

denote the π_f-isotypical component. Then, the map

$$\mathrm{JL}_{1,2}(\mathbf{a}) : \mathrm{H}^0_c(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell}) \to \mathrm{H}^d_c(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W,\mathbb{Q}_\ell})$$

restricted to $\mathrm{H}^0_c(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W,\mathbb{Q}_\ell})[\pi_f]$ is injective if the Satake parameters of π_f is general with respect to V_{μ_2} in the sense of [XZ17].

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