# Generalizations of a Theorem of Hecke

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– Charles Dickens, *A Tale of Two Cities*

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# ABSTRACT

Let $p > 3$ be an odd prime, $p \equiv 3 \bmod 4$ and let $\pi^+, \pi^-$ be the pair of cuspidal representations of $SL_2(\mathbb{F}_p)$. It is well known by Hecke that the difference $m_{\pi^+} - m_{\pi^-}$ in the multiplicities of these two irreducible representations occurring in the space of weight 2 cusps forms with respect to the principal congruence subgroup $\Gamma(p)$, equals the class number $h(-p)$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

This thesis consists of two main parts. In the first part, we extend Hecke's result to *all fundamental discriminants* of imaginary quadratic fields, including the even case. The proof is geometric in nature and uses the holomorphic Lefschetz number.

In the second part, we consider generalizations to groups with higher $\mathbb{Q}$-rank. In particular, we focus on the rank 2 special unitary group $SU(2,2)$. On the representation theory side, we prove the regular unipotent classes have positive contribution to an alternating sum of multiplicities of certain irreducible cuspidal representations of $SU(2,2)$ over the finite field of $p$ elements. We also show that the semisimple classes have zero contribution, which is again a direct generalization of the $SL_2$ case. To obtain these two results, we make use of the Deligne-Lusztig theory and the connection of the traces to the Gelfand-Graev representations.

# PUBLISHED CONTENT AND CONTRIBUTIONS

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Panda, Corina B conducted all of the research and authored the manuscript. With minor modifications, Chapter 2 is solely based on this paper.

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*Chapter 1*

# INTRODUCTION

## 1.1 Motivation

Let $p > 3$ be odd prime. There is, up to twist equivalence, a unique irreducible cuspidal representation $\pi$ of $GL_2(\mathbb{F}_p)$, which, when restricted to $SL_2(\mathbb{F}_p)$, splits into a pair of irreducible representations $\pi^+$, $\pi^-$ of the same dimension. The group $SL_2(\mathbb{F}_p)$ acts naturally on the space $\mathcal{S}_2(\Gamma(p))$ of weight 2 cusp forms with respect to the principal congruence subgroup $\Gamma(p) = \ker\big(Sl_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/p\mathbb{Z})\big)$. One might think that $\pi^+$, $\pi^-$ would occur with the same multiplicity in $\mathcal{S}_2(\Gamma(p))$. Indeed, this holds true when $p \equiv 1 \bmod 4$. However, as Hecke showed in [14], [15], the two cuspidal irreducible representations $\pi^+, \pi^-$ of $SL_2(\mathbb{F}_p)$ have different multiplicities when $p \equiv 3 \bmod 4$. One could say that this was a precursor to the modern theory of $L$-indistinguishability [23]. Furthermore, Hecke showed (in *loc. cit.*) that in this case, the difference in multiplicities $m_{\pi^+} - m_{\pi^-}$, is exactly $h(-p)$, the class number of $\mathbb{Q}(\sqrt{-p})$. Note that there is exactly one more, up to twist equivalence, irreducible representation $\tau$ of $GL_2(\mathbb{F}_p)$ that also splits into two irreducible representations $\tau^+, \tau^-$ of $SL_2(\mathbb{F}_p)$, upon restriction. In this case, $\tau$ is in the principal series and $m_{\tau^+} = m_{\tau^-}$ for all odd $p$.

The main results of this thesis are motivated by Hecke's work for $SL_2$. Let $G$ be a semisimple connected algebraic group over $\mathbb{Q}$ and $K$ a maximal compact subgroup of $G(\mathbb{R})$. We are interested in the cases where the associated symmetric space $D = G(\mathbb{R})/K$ is Hermitian of non-compact type, that is, a Hermitian symmetric domain. Note that there exists a holomorphic diffeomorphism of $D$ onto a bounded symmetric domain. Suppose G has a $\mathbb{Z}$-structure. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup, say $\Gamma = G(\mathbb{Z})$, and denote by $Y_\Gamma$ the locally symmetric variety $\Gamma\backslash D$. When $Y_\Gamma$ is non-compact, let $Y_\Gamma^*$ be its Baily-Borel-Satake compactification (see [2]). However, $Y_\Gamma^*$ is usually singular, thus instead we shall consider its smooth toroidal resolution (see [1]), which we denote by $X_\Gamma$.

Let $\Gamma(N)$ be a principal congruence subgroup of level $N$ defined as $\ker\big(G(\mathbb{Z}) \twoheadrightarrow G(\mathbb{Z}/N\mathbb{Z})\big)$. Assume $\Gamma(N)$ is neat, in particular torsion-free, which holds for $N$

large, with $Y_\Gamma$ smooth. Then $X_{\Gamma(N)} \to X_\Gamma$ is a (ramified) covering and we get a natural action of $G(\mathbb{Z}/N\mathbb{Z})$ on the space $X_{\Gamma(N)}$. This induces an action on the sheaf cohomology groups $H^0(X_{\Gamma(N)}, \Omega^j)$, where $\Omega^j$ is the sheaf of holomorphic differentials of degree $j$ on $X_{\Gamma(N)}$. Note $H^0(X_{\Gamma(N)}, \Omega^j)$ is finite dimensional and corresponds to holomorphic cusp forms. This generalizes the case of $SL_2$, where the cohomology group $H^0(X_{\Gamma(p)}, \Omega^1)$ is isomorphic to the space $\mathcal{S}_2(\Gamma(p))$ of cusp forms of weight 2 with respect to the principal congruence subgroup $\Gamma(p)$. For this particular case, $X_{\Gamma(p)}$ is the compactified modular curve of level $p$.

Let $\widetilde{G}$ be a reductive group over $\mathbb{Q}$ with derived subgroup $G$. While some of the results obtained apply more generally or are interesting on their own, the goal of this thesis is to focus on the following questions:

($Q_1$) Find irreducible representations $\pi$ of $\widetilde{G}(\mathbb{Z}/N\mathbb{Z})$, which are cuspidal when $N$ is prime, that decompose upon restriction to $G(\mathbb{Z}/N\mathbb{Z})$, that is $\pi|_{G(\mathbb{Z}/N\mathbb{Z})}$ can be written as $\pi_1 + \cdots + \pi_{d_0}$ with $\pi_i$ irreducible for $i \in \{1, \cdots, d_0\}$, $d_0 \in \mathbb{Z}_{>1}$. Of particular interest is the case when $d_0 = 2$.

($Q_2$) Denote the representation of $G(\mathbb{Z}/N\mathbb{Z})$ on the group $H^0(X_{\Gamma(N)}, \Omega^j)$ by $(\rho, H^0(X_{\Gamma(N)}, \Omega^j))$. Let $m_\pi$ be the multiplicity of an irreducible representation $\pi$ of $G(\mathbb{Z}/N\mathbb{Z})$ in $\rho$ and consider the alternating sum of multiplicities

$$\Delta M_\pi = \sum_{i=1}^{d_0} \xi_{d_0}^{i-1} m_{\pi_i}, \tag{1.1}$$

where $\xi_{d_0}$ is a primitive $d_0^{\text{th}}$ root of unity.
Determine cases when $\Delta M_\pi$ is nonzero. We call such a phenomenon multiplicity defect.

($Q_3$) Relate the multiplicity defect $\Delta M_\pi$ to some arithmetic invariants.

In the first part of the thesis, we extend Hecke's result to all fundamental discriminants $-D$ of imaginary quadratic fields $\mathbb{Q}(\sqrt{-D})$ (Theorem 2.1.1, Chapter 2). Our main contribution is twofold, one dealing with the delicate situation when $N$ is even and the other addressing ($Q_3$) above by using a geometric argument involving the holomorphic Lefschetz number. Note that the $\mathbb{Q}$-rank of

$SL_2$ is 1 and the minimal Baily-Borel-Satake compactification is smooth, which allows us to compute the holomorphic Lefschetz numbers on $X_{\Gamma(D)} = Y^*_{\Gamma(D)}$.

We want to remark that there are various analogues of Hecke's result in several contexts, using different methods. Alongside weight 2 cusp forms, similar results hold for cusp forms of higher weight as investigated by Hecke and Feldman (c.f. [11]). There is an analogue for Maass cusp forms in an article by J. Stopple (see [38]). On the other hand, Lee and Weintraub extend Hecke's work to the Siegel space of degree 2 by considering the case of $Sp_4$ in [25].

The second part of our thesis focuses on tackling the above questions for another group of $\mathbb{Q}$-rank 2, $SU(2,2)$. To put it in perspective, recall the classification of irreducible Hermitian symmetric spaces of non-compact type via work of E. Cartan (see Chapter X, p. 518 in [16]). The exhaustive list of the associated groups $G$ consists of $SU(p,q)$, $SO^*(2n)$ for $n > 2$, $SO_0(p,2)$ for $p > 2$, $Sp_n(\mathbb{R})$ and some exceptional groups. Here $SO^*(2n)$ is the group of complex matrices $g$ such that $\overline{g}^t J_n g = J_n$ and $g^t g = I_{2n}$, where $J_n$ is the skew-hermitian form given by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. $SO_0(p,2)$ denotes the identity component of $SO(p,2)$. Note we have the following special isomorphisms in small dimensions: $\mathfrak{so}(2,2) \cong \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{so}^*(6) \cong \mathfrak{su}(3,1)$ and $\mathfrak{so}^*(8) = \mathfrak{so}(6,2)$. As a result, among the $\mathbb{Q}$-rank 2 groups, $SU(2,2)$ is the simplest. Note that as $\mathfrak{su}(1,1)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, the case of $SU(2,2)$ can also be viewed as a direct generalization of Hecke's treatment of $SL_2$, see Chapter 3.

The reason we focus on groups of $\mathbb{Q}$-rank 2 is because they provide the first examples in the theory of toroidal compactifications where the rational boundary components are not all zero-dimensional. Note the minimal Baily-Borel-Satake compactification is not smooth. Thus, answering question $(Q_3)$ on the geometric side becomes increasingly nontrivial.

Indeed, the case of $SU(2,2)$ is more subtle than the $Sp_4$ case, which has been already studied in the literature ([25]). The representation theory of $Sp_4(\mathbb{F}_p)$ is fairly known via the work of Srinivasan (c.f. [37]). The main results of the second half of the thesis focus on answering questions $(Q_1)$ and $(Q_2)$ on the representation theory side for $SU(2,2)$. In particular, we find desired cuspidal regular irreducible representations of the quasi-split group $U(2,2)$ over a finite field, that decompose upon restriction to $SU(2,2)$. To get the quasi-split special unitary group over $\mathbb{F}_p$ as a reductive quotient $G(\mathbb{Z}/p\mathbb{Z})$, it is necessary and sufficient to consider the case when $p$ is inert in the imaginary quadratic field $E$ over which the Hermitian form for $SU(2,2)$ is defined. In the case of $p$ ramified, we have $G(\mathbb{Z}/p\mathbb{Z}) \cong SO(2,2)$, which is not as interesting given the special isomorphism $\mathfrak{so}(2,2) \cong \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$. On the other hand, in the case of $p$ split we have $G(\mathbb{Z}/p\mathbb{Z}) \cong SL_4(\mathbb{F}_p)$, about whose representations a great deal is known. Above all, we note that this case can be dealt with by using the same theory developed in Chapter 6 since both groups are different rational forms of the same group, $SL_4$, as a connected semisimple algebraic group over $\overline{\mathbb{F}}_p$. Moreover, when working out the character theory, the $SL_4(\mathbb{F}_p)$ case, just as it happens in the case of $Sp_4(\mathbb{F}_p)$, gets simplified as the group is split.

In order to tackle $(Q_2)$, rewrite $\Delta M_\pi$ as

$$\Delta M_\pi = \frac{1}{|G(\mathbb{Z}/N\mathbb{Z})|} \sum_{g \in G(\mathbb{Z}/N\mathbb{Z})} \Big( \sum_{i=1}^{d_0} \xi_{d_0}^{i-1} \pi_i(g) \Big) \overline{\chi_\rho(g)},$$

where $\chi_\rho$ is the character of $\rho$ and by abuse of notation we denote by $\pi_i(g)$ the characters of the irreducible components $\pi_i$ on the element $g$ of $G(\mathbb{Z}/N\mathbb{Z})$. Note that a first step in proving multiplicity defect is finding conjugacy classes $g$ on which the alternating sum of characters

$$\sum_{i=1}^{d_0} \xi_{d_0}^{i-1} \pi_i \tag{1.2}$$

is nonzero.

We prove this alternating sum is zero on the semisimple classes. The key step in our work is showing that, when $\pi$ comes from a Deligne-Lusztig representation, the alternating sum is nonzero on the regular unipotent classes. Thus the regular unipotent classes have positive contribution to the multiplicity defect $\Delta M_\pi$.

The proofs use the duality operator on cuspidal representations and Gelfand-Graev characters to write the characters of the irreducible representations $\pi_i$ on regular unipotent classes in terms of certain trigonometric sums. A key computation in Chapter 6 evaluates these trigonometric sums in terms of Gauss sums. Towards answering $(Q_3)$, this suggests links with possible interpretations of $\Delta M_\pi$ as certain arithmetic invariants, analogues of the class number appearing in Hecke's original theorem. The results hold for any $G$, see Section 1.2; in particular, in Chapter 7 we give a formula for the contribution to the multiplicity defect coming from the regular unipotent classes of $SU(2,2)$. We note the reason the character theory of the finite group of Lie type $SU(2,2)$ defined over a finite field is arithmetical in nature is because the underlying group $SL_4(\overline{\mathbb{F}}_p)$ has non-connected center.

As a remark, note that one has to also consider the values of the alternating sum above on conjugacy classes that are neither semisimple, nor regular unipotent; this problem has not been addressed in this thesis and it will be subject of future work.

We do not solve $(Q_3)$ in this thesis. However, we propose the following strategy. One needs to work on the geometric side by constructing the toroidal compactification $X_{\Gamma(p)}$ and applying some Lefschetz type formula, such as the Atiyah-Singer holomorphic Lefschetz formula, to compute the characters $\chi_\rho(g)$. In the case of $Sp_4(\mathbb{R})$, the toroidal compactification, first developed by Igusa in [19], has been worked out explicitly. The analysis in Chapter 3 focuses on the lower $\mathbb{Q}$-rank case of $SU(1,1)$ from the geometric perspective, thus informing the steps required in constructing the picture for the higher $\mathbb{Q}$-rank case. Note that the same derivations carried in the $SU(2,2)$ case give us two different rational boundary components, a zero-dimensional one and a one-dimensional one isomorphic to $SU(1,1)$. Thus one has to also treat the higher dimensional cusps, when constructing the smooth toroidal compactification.

## 1.2 Overview of the results

In the first part of the thesis, we extend Hecke's result for $SL_2(\mathbb{F}_p)$ to all fundamental discriminants $-D, D > 3$ of imaginary quadratic fields $\mathbb{Q}(\sqrt{-D})$ and provide an alternate geometric proof even for the case when $D$ is a prime $p \equiv 3 \bmod 4$. We consider certain distinctive irreducible representations of $G = SL_2(\mathbb{Z}/D\mathbb{Z})$ described in Section 2.2, that are (up to isomorphism) partitioned along tuples of the shape $(\epsilon_0, \cdots, \epsilon_t|e)$, where $\epsilon_i \in \{\pm\}$ and $e \in \{\pm 1\}$. Note that these distinctive representations agree with Hecke's representations in the case $D = p > 3$. Let $m_\pi$ be the multiplicity of a distinctive representation $\pi$ in the space of weight 2 cusp forms for the principal congruent subgroup $\Gamma(D)$, and consider the alternating sum of multiplicities over distinctive $G$-representations of type $(\epsilon_0, \cdots, \epsilon_t|e)$ defined in (2.1)

$$\Delta M_{t,e} = \sum_{(\epsilon_0,\cdots,\epsilon_t|e)} \prod_{i=0}^{t} \epsilon_i \sum_{\pi\in(\epsilon_0,\cdots,\epsilon_t|e)} m_\pi.$$

The main result we prove is given by Theorem 2.1.1 in Chapter 2:

**Theorem A.** *For $D > 3$, let $G = SL_2(\mathbb{Z}/D\mathbb{Z})$, where $-D$ is a fundamental discriminant associated to the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$. We may write $D$ as $D_0 p_1 \cdots p_t$, with $D_0 \in \{p_0, 4, 8\}$, $t \geq 0$. Consider the expression $\Delta M_{t,e}$ introduced above and let $\Delta M_t = \Delta M_{t,1}$. Then the following identity relating $\Delta M_t$ and the class number $h(-D)$ of $\mathbb{Q}(\sqrt{-D})$ holds*

$$\Delta M_t = \begin{cases} 0, & \text{if } D_0 = 4, t = 0 \\ \operatorname{sgn}_{D_0,t} 2^t \big[h(-D) + h(-D/2)\big], & \text{if } D_0 = 8, t = 0 \\ \operatorname{sgn}_{D_0,t} 2^t h(-D), & \text{if } D_0 = p_0; D_0 = 4, t \geq 1; \\ & D_0 = 8, t \geq 1, \\ & p_1 \cdots p_t \equiv 3 \bmod 4 \\ \operatorname{sgn}_{D_0,t} 2^t \big[h(-D) + 2h(-D/2)\big], & \text{if } D_0 = 8, t \geq 1, \\ & p_1 \cdots p_t \equiv 1 \bmod 4, \end{cases}$$

*where* $\operatorname{sgn}_{D_0,t} \in \{\pm 1\}$ *is given by*

$$\operatorname{sgn}_{D_0,t} = \begin{cases} 1, & \text{if } D_0 = p_0 t = 0; D_0 = 4, t = 1; \\ & D_0 = 8, t \in \{0, 1\} \\ \prod\limits_{0 \leq i < j \leq t} (-1)^{\frac{(p_i - 1)(p_j - 1)}{4}}, & \text{if } D_0 = p_0, t \geq 1 \\ \prod\limits_{1 \leq i < j \leq t} (-1)^{\frac{(p_i - 1)(p_j - 1)}{4}}, & \text{if } D_0 \in \{4, 8\}, t > 1. \end{cases}$$

The second part of the thesis focuses on answering $(Q_1)$, $(Q_2)$, with a particular focus on $SU(2,2)$. However, the results hold in higher generality, as we shall outline below. The derivations use Deligne-Lusztig theory, Gelfand-Graev representations and related notions for finite groups of Lie type.

Let us fix notations. When $N$ is a prime $p$, let $G(\mathbb{Z}/N\mathbb{Z})$ be an arbitrary finite group of Lie type. It can be realized as the fixed points under a Frobenius map $F$ on a connected semisimple group $G$ over $\overline{\mathbb{F}}_p$. The group $G$ can be embedded in a reductive group $\widetilde{G}$ over $\overline{\mathbb{F}}_p$ with connected center and compatible $F$-structure. As a result, $G^F$ will be naturally identified with $G(\mathbb{Z}/p\mathbb{Z})$, while $\widetilde{G}^F = \widetilde{G}(\mathbb{Z}/p\mathbb{Z})$. This notation is consistent throughout the thesis, starting with Chapter 4.

We answer $(Q_1)$ for the case when $G^F = SU(2,2)$ and $\widetilde{G}^F = U(2,2)$, specifically, we find (cuspidal) irreducible representations $\pi$ of $\widetilde{G}^F$ that split into several irreducibles upon restriction to $G^F$, e.g. $\pi_{|_{G^F}} = \pi_1 + \cdots + \pi_{d_0}$, where $\pi_i$ are irreducible representations of $G^F$, $i \in \{1, \cdots, d_0\}$. Let $\Gamma$ be the character of the Gelfand-Graev representation of $\widetilde{G}^F$; refer to Chapter 4 for an introduction of Gelfand-Graev representations and related notions. We call an irreducible representation of $\widetilde{G}^F$ regular if it is a component of the Gelfand-Graev representation. Note that if $\langle \pi, \Gamma \rangle_{\widetilde{G}^F} = 0$, it is known that $\pi_i(u) = 0$ on all regular unipotent classes $u \in G^F$ ([26]), so we restrict ourselves to the case where $\pi$ is a regular cuspidal irreducible representation.

Let the group $G^F = SU(2,2)$ be as defined in Section 5.1 and let $\widetilde{T}_w$ be the torus of type $w$ with respect to the maximally split $F$-stable torus of diagonal matrices $\widetilde{T}$, where $w \in W(\widetilde{T}) \cong S_4$ is given by $(1,2,3,4) \mapsto (4,3,2,1)$. Let $Z$ be the non-connected center of $G$ and denote by $d$ the order of the cohomology group $H^1(F, Z)$. The result in Theorem 5.1.7 gives us the desired regular cuspidal irreducible representations:

**Proposition B.** *Let $\widetilde{T}_w$ be the maximal $F$-stable torus defined above, $\widetilde{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$ a character of $\widetilde{T}_w^F$ in general position. The character $\epsilon_{\widetilde{G}} \epsilon_{\widetilde{T}_w} R_{\widetilde{T}_w}^{\widetilde{G}}(\widetilde{\theta})$ is an irreducible cuspidal regular representation of $\widetilde{G}^F$.*

*Assuming $\pi = \epsilon_{\widetilde{G}} \epsilon_{\widetilde{T}_w} R_{\widetilde{T}_w}^{\widetilde{G}}(\widetilde{\theta})$ does not stay irreducible when restricted to $G^F$, its splitting behaviour is as follows:*

1. *$\pi_{|_{G^F}} = \pi_1 + \pi_2$ when $\widetilde{\theta}$ is given by $(\theta_1, \alpha\theta_1, \theta_3, \alpha\theta_3)$, for $\alpha$ the unique nontrivial quadratic character of $U(1)$ and $\theta_1, \theta_3$ irreducible characters of $U(1)$ such that $\theta_1/\theta_3 \neq 1, \alpha$. Note that $\widetilde{\theta}$ is a quadratic character.*

2. *$\pi_{|_{G^F}} = \pi_1 + \pi_2 + \pi_3 + \pi_4$ when $\widetilde{\theta}$ is, up to a twist by a character of $U(1)$, given by $(1, \alpha, \alpha^2, \alpha^3)$, for $\alpha$ a quartic character of $U(1)$. Note that this splitting can happen only when $d = 4$; in this case $\widetilde{\theta}$ is a quartic character.*

*Conversely, given any datum of one of these two types for $\widetilde{\theta}$, the character $\pi = \epsilon_{\widetilde{G}} \epsilon_{\widetilde{T}_w} R_{\widetilde{T}_w}^{\widetilde{G}}(\widetilde{\theta})$ splits upon restriction to $G^F$ in the corresponding manner described above.*

Here $U(1) \cong (\mathbb{F}_{p^2}^\times)^1$ is the cyclic group of order $p+1$ consisting of elements of norm 1 in $\mathbb{F}_{p^2}^\times$. In general, for a connected reductive group $\widetilde{G}$ over an algebraically closed field of prime characteristic, the group $\widetilde{G}^F$ always has a cuspidal complex representation given up to a sign by a Deligne-Lusztig character (see [22]). More specifically, let $\widetilde{T}$ be an $F$-stable maximal torus that lies in no proper $F$-stable parabolic subgroup of $\widetilde{G}$ and let $\widetilde{\theta}$ be a character of $\widetilde{T}^F$ in general position; in this case $T$ is anisotropic. Then $\pi = \pm R_{\widetilde{T}}^{\widetilde{G}}(\widetilde{\theta})$ is an irreducible cuspidal representation of $\widetilde{G}^F$. Also remark that generically, such an irreducible representation $\pi$ of $\widetilde{G}^F$ will stay irreducible upon restriction to $G^F$; when it splits, the splitting behaviour can be determined in general, using the same ideas as for the $SU(2,2)$ case.

The remaining results focus on answering $(Q_2)$. In Theorem 5.2.4 we show the semisimple conjugacy classes have zero contribution to $\Delta M_\pi$:

**Theorem C.** *Let $\pi$ be an irreducible representation of $\widetilde{G}^F$ that splits into $d_0$ components upon restriction to $G^F$, that is $\pi_{|_{G^F}} = \pi_1 + \cdots + \pi_{d_0}$, where $d_0|d$. The alternating sum $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i$ introduced in equation* (1.2) *is zero on the semisimple elements of $G^F$. Recall $\xi_{d_0}$ is a primitive $d_0^{th}$ root of unity.*

This result is true for arbitrary $G$ semisimple connected and simply-connected and holds for any irreducible representation of $\widetilde{G}^F$ that splits upon restriction to $G^F$. It is a direct generalization of Hecke's $SL_2$ case, where $\pi^+ - \pi^-$ is zero on the semisimple elements.

More interestingly, we prove there is positive contribution to $\Delta M_\pi$ on the regular unipotent classes. First, we write the characters of the irreducible regular cuspidal representations $\pi_i$ on regular unipotent classes in terms of certain trigonometric sums $\sigma_z$ indexed by the cohomology group $H^1(F, Z)$. For example, we give such a character formula for the case when $d_0 = |H^1(F, Z)|$ in Theorem 6.2.10; similar results can be derived for any possible $d_0$ by using the same type of reasoning. The formula holds true in general for $G$ semisimple connected in good characteristic and $p$ sufficiently large to ensure that all maximal tori of $G^F$ are nondegenerate. Next, results in [8] allow us to reformulate an expression for the so called "Mellin transforms" of $\sigma_z$ in terms of Gauss sums over finite fields; see Theorem 6.3.5. As a consequence, we show the alternating sum of characters $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i$ evaluated on regular unipotent classes can be written as non-zero multiples of Gauss sums. Note this result holds under the same conditions above, for $G$ semisimple connected in good characteristic, with $p$ sufficiently large. In particular, when specializing to the case of $G^F = SU(2,2)$ we get the following result (Theorem 7.0.8), where recall that $d$ is the order of $H^1(F,Z)$:

**Theorem D.** *Let $\pi$ be an irreducible cuspidal regular representation of $\widetilde{G}^F = U(2,2)$ that splits into $d_0|d$ irreducible components upon restriction to $G^F = SU(2,2)$, $\pi_{|_{G^F}} = \pi_1 + \cdots + \pi_{d_0}$. Let $\zeta_2$ be the unique non-trivial quadratic character of $\mathbb{F}_{p^2}^\times$ and $\zeta_4$ be the quartic character of $\mathbb{F}_{p^2}^\times$ given by $g^{4r-i} \mapsto \xi_4^{3i}$, for $i \in \{0,1,2,3\}, r \in \mathbb{Z}$ and $g$ a generator for $\mathbb{F}_{p^2}^\times$. Let $G$ be the Gauss sum introduced in Definition 6.1.1.*

*Then the alternating sum $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i$ takes the following values on regular unipotent classes:*

$$\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i(u) = \begin{cases} -G(\zeta_2) & \text{if } d_0 = 2,\ d \in \{2,4\}, \\ i\sqrt{p}G(\zeta_4) & \text{if } d_0 = 4,\ d = 4, \end{cases}$$

*for any $u \in U_{z_1}$, where $z_1$ is the identity element in $H^1(F,Z)$.*

Note that it is sufficient to compute the values on $u \in U_{z_1}$; the values on the other unipotent classes are the same up to sign if $d_0 = 2$ or up to multiplication by a $4^{\text{th}}$ root of unity if $d_0 = 4$.

Again, this mimics the case of $SL_2$, where the non-zero contribution to $\pi^+ - \pi^-$ comes from the unipotent conjugacy classes. Remark that the case of $SU(2,2)$ is different from that of $SL_2$ in that we have a new type of conjugacy classes of mixed Jordan decomposition.

## 1.3 Structure of the thesis

The structure of the thesis is as follows. In Chapter 2 we extend Hecke's result for $SL_2(\mathbb{F}_p)$ to all fundamental discriminants of imaginary quadratic fields. In Chapter 3, we investigate the geometric side for the case of $SU(1,1)$, to both reframe the case of $SL_2(\mathbb{F}_p)$ and inform the picture of the higher $\mathbb{Q}$-rank case of $SU(2,2)$; the latter is needed when answering $Q_3$.

Chapter 4 contains background material on Deligne-Lusztig theory, Gelfand-Graev characters and related notions, and character values on regular unipotent classes. Chapters 5, 6 and 7 contain our main results for the second part of the thesis. In Chapter 5 we find regular cuspidal irreducible representations of $\widetilde{G}^F$ that split when restricted to $G^F$ for the case of $SU(2,2)$. We also prove the semisimple conjugacy classes have zero contribution to $\Delta M_\pi$. Chapter 6 focuses on the general case and gives a formula for the multiplicity defect at regular unipotent elements. In Chapter 7 we specialize the theory previously developed to the case of $SU(2,2)$.

## 1.4 Concluding remarks

To summarize, we study generalizations of Hecke's result in two different directions.

First, we extend it to higher levels in the case of $SL_2$ by using a geometric argument, and provide a complete result in Chapter 2.

Second, we focus on groups of higher $\mathbb{Q}$-rank, in particular we consider the case of $SU(2,2)$ of $\mathbb{Q}$-rank 2. Let $p$ be a prime that is inert in the imaginary quadratic field $E$ over which $SU(2,2)$ is defined. Let $\pi$ be an irreducible regular cuspidal representation of the finite group of Lie type $\widetilde{G}^F = U(2,2)(\mathbb{F}_p)$ that splits upon restriction to $G^F = SU(2,2)(\mathbb{F}_p)$, that is $\pi_{|_{G^F}} = \pi_1 + \cdots + \pi_{d_0}$, with $\pi_i$ irreducible for $i \in \{1, \cdots, d_0\}$, $d_0 \in \{2,4\}$. Let $\rho$ denote the representation of $G^F$ on the holomorphic part of the middle cohomology of $X_{\Gamma(p)}$. Denote by $m_{\pi_i}$ the multiplicity of $\pi_i$ in $\rho$. Then

$$\Delta M_\pi = \frac{1}{|G^F|} \sum_{g \in G^F} \Big( \sum_{i=1}^{d_0} \xi_{d_0}^{i-1} \pi_i(g) \Big) \overline{\chi_\rho(g)},$$

where $\chi_\rho$ is the character of $\rho$.

In particular, we have

$$\Delta M_\pi = \frac{1}{|G^F|} \Big( \Delta M_{ss} + \Delta M_{int} + \Delta M_{ru} \Big),$$

where $\Delta M_{ss}$ is the sum over the semisimple classes, $\Delta M_{ru}$ is the sum over the regular unipotent classes, while $\Delta M_{int}$ is the sum over classes of mixed Jordan decomposition $g = su$, with $s$ semisimple and $u$ unipotent.

$\Delta M_{ss} = 0$ by Theorem 5.2.4. As already noted, this in fact holds for general $G$ semisimple, connected and simply-connected.

Results in Chapter 7 give a precise formula for $\Delta M_{ru}$. This is derived from the character formula for $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1} \pi_i$ in Theorem 7.0.8. This part generalizes as well.

Assume, for simplicity, that $d_0 = 2$. Denoting $\pi_1$ by $\pi^+$ and $\pi_2$ by $\pi^-$, we get

$$\Delta M_\pi = m_{\pi^+} - m_{\pi^-}.$$

Let $U_{z_i}$ be the conjugacy class of regular unipotent elements indexed by $z_i = \omega^{i-1} I_4$ of $H^1(F, Z)$ for $i \in \{1, \cdots, 4\}$, where $\omega$ is a primitive $4^{\text{th}}$ root of unity. The order of $H^1(F, Z)$, denoted by $d$, is either 2 or 4, both of which are relevant to us. When $|H^1(F, Z)| = 2$, we have $z_1 \sim z_3, z_2 \sim z_4$ in $H^1(F, Z)$. Then

$$\Delta M_{ru} = \begin{cases} -G(\zeta_2)\Big( \sum_{u \in U_{z_1}} \overline{\chi_\rho(u)} - \sum_{u \in U_{z_2}} \overline{\chi_\rho(u)} \Big) & \text{if } d = 2, \\ -G(\zeta_2)\Big( \sum_{u \in U_{z_1} \cup U_{z_3}} \overline{\chi_\rho(u)} - \sum_{u \in U_{z_2} \cup U_{z_4}} \overline{\chi_\rho(u)} \Big) & \text{if } d = 4. \end{cases}$$

Note this is a direct analogue of Hecke's result for the case of $SL_2(\mathbb{F}_p)$. In order to completely determine $\Delta M_{ru}$, we need to find the value of the characters $\chi_\rho$ on regular unipotents, which amounts to working on the geometric side by applying the holomorphic Lefschetz trace formula on the toroidal compactification. Also, note that here we have the intermediary term $\Delta M_{int}$, which does not appear in the $SL_2$ case. We expect to be able to either relate each separate term $\Delta M_{int}$ and $\Delta M_{ru}$, on its own, or perhaps their sum, to an arithmetic invariant such as the class number in Hecke's case.

*Chapter 2*

# A GENERALIZATION OF HECKE'S THEOREM FOR $SL_2(\mathbb{F}_p)$ TO FUNDAMENTAL DISCRIMINANTS

The goal of this chapter is to extend Hecke's result for $SL_2(\mathbb{F}_p)$ to all fundamental discriminants $-D$ of imaginary quadratic fields $\mathbb{Q}(\sqrt{-D})$ and prove that an alternating sum of multiplicities of certain irreducibles of $SL_2(\mathbb{Z}/D\mathbb{Z})$ is an explicit multiple, up to a sign and a power of 2, of either the class number $h(-D)$ or of the sums $h(-D)+h(-D/2)$, $h(-D)+2h(-D/2)$; the last two possibilities occur in some of the cases when $D \equiv 0 \bmod 8$. The proof uses the holomorphic Lefschetz number.

## 2.1 Preliminaries

The purpose of this chapter is to extend Hecke's result to all fundamental discriminants $-D$, $D > 3$ of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$ and to provide an alternate geometric proof even for the case when $D$ is a prime $p \equiv 3 \bmod 4$. We write the fundamental discriminant $-D$ of $K$ as $-D_0 p_1 \cdots p_t$ with $t \geq 0$ (the second product being 1 if $t = 0$), $D_0 \in \{p_0, 4, 8\}$ and $p_0, p_1, \cdots, p_t$ distinct odd primes such that the typical fundamental discriminant congruences are satisfied. We consider certain distinctive irreducible representations of $G = SL_2(\mathbb{Z}/D\mathbb{Z})$ described in Section 2.2; for the moment, it suffices to say that these representations are (up to isomorphism) partitioned along tuples of the shape $(\epsilon_0, \cdots, \epsilon_t | e)$, where $\epsilon_i \in \{\pm\}$ and $e \in \{\pm 1\}$. Note that these distinctive representations agree with Hecke's representations in the case $D = p > 3$. Let $\mathcal{S}_2(\Gamma(D))$ be the space of weight 2 cusp forms for the principal congruence subgroup $\Gamma(D)$. The natural action of $G$ on $\mathcal{S}_2(\Gamma(D))$ gives a $G$-representation, which we denote by $(\rho, \mathcal{S}_2(\Gamma(D)))$. Let $m_\pi$ be the multiplicity of a distinctive representation $\pi$ in $\rho$ and consider the alternating sum of multiplicities over distinctive $G$-representations of type $(\epsilon_0, \cdots, \epsilon_t | e)$

$$\Delta M_{t,e} = \sum_{(\epsilon_0, \cdots, \epsilon_t | e)} \prod_{i=0}^{t} \epsilon_i \sum_{\pi \in (\epsilon_0, \cdots, \epsilon_t | e)} m_\pi. \tag{2.1}$$

Note that by $\prod_{i=0}^{t} \epsilon_i$, we clearly mean the product of $\pm 1$ when $\epsilon_i$ takes values in $\{\pm\}$. The main result we prove is as follows:

**Theorem 2.1.1.** *For $D > 3$, let $G = SL_2(\mathbb{Z}/D\mathbb{Z})$, where $-D$ is a fundamental discriminant associated to the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$. We may write $D$ as $D_0 p_1 \cdots p_t$, with $D_0 \in \{p_0, 4, 8\}$, $t \geq 0$. Consider the expression $\Delta M_{t,e}$ introduced in (2.1) above and let $\Delta M_t = \Delta M_{t,1}$. Then the following identity relating $\Delta M_t$ and the class number $h(-D)$ of $\mathbb{Q}(\sqrt{-D})$ holds*

$$\Delta M_t = \begin{cases} 0, & \text{if } D_0 = 4, t = 0 \\ \operatorname{sgn}_{D_0,t} 2^t \big[h(-D) + h(-D/2)\big], & \text{if } D_0 = 8, t = 0 \\ \operatorname{sgn}_{D_0,t} 2^t h(-D), & \text{if } D_0 = p_0; D_0 = 4, t \geq 1; \\ & D_0 = 8, t \geq 1, \\ & p_1 \cdots p_t \equiv 3 \bmod 4 \\ \operatorname{sgn}_{D_0,t} 2^t \big[h(-D) + 2h(-D/2)\big], & \text{if } D_0 = 8, t \geq 1, \\ & p_1 \cdots p_t \equiv 1 \bmod 4, \end{cases}$$

*where $\operatorname{sgn}_{D_0,t} \in \{\pm 1\}$ is given by*

$$\operatorname{sgn}_{D_0,t} = \begin{cases} 1, & \text{if } D_0 = p_0 t = 0; D_0 = 4, t = 1; \\ & D_0 = 8, t \in \{0, 1\} \\ \prod_{0 \leq i < j \leq t} (-1)^{\frac{(p_i - 1)(p_j - 1)}{4}}, & \text{if } D_0 = p_0, t \geq 1 \\ \prod_{1 \leq i < j \leq t} (-1)^{\frac{(p_i - 1)(p_j - 1)}{4}}, & \text{if } D_0 \in \{4, 8\}, t > 1. \end{cases}$$

Note that in the case when $D > 3$ is an odd prime $p \equiv 3 \bmod 4$, we have $\Delta M_0 = h(-p)$, so our result matches Hecke's original theorem. Towards the end of our work, we came to learn that this extension of Hecke's result has already been proved *up to a sign* for the case of *odd* discriminants $D$ in a paper of McQuillan [29], though by a different method. We hope our result is still of some interest for two reasons. First, the even case is more subtle. Second, our method also makes explicit the sign $\operatorname{sgn}_{D_0,t}$, which was previously unknown even in the odd case.

We prove the main theorem by a geometric argument using the holomorphic Lefschetz number. The structure of this chapter is as follows. The second section sets the necessary notation in introducing the desired distinctive irreducible representations of $G = SL_2(\mathbb{Z}/D\mathbb{Z})$ we are considering for the alternating sum $\Delta M_t$ of multiplicities of these representations into the space $\mathcal{S}_2(\Gamma(D))$. The general idea of the proof consists of computing the characters $\Delta\chi_{t,e}$, $\chi_{\mathcal{S}_2(\Gamma(D))}$ and then comparing the resulting expression for $\Delta M_t$ with an analytic formula for the class number. Moving to Section 2.3, we find the values of the virtual character $\Delta\chi_{t,e}$. Since $G$ acts on the modular curve $X(D) = \Gamma(D)\backslash\mathcal{H}^*$, we view $g : X(D) \to X(D)$ as a map on a one-dimensional compact complex manifold for all $g \in G$. We compute the fixed points of $g$ on $X(D)$ in Section 2.4, which allows us to compute the holomorphic Lefschetz number of the map g in Section 2.5. As the Lefschetz numbers give us the characters $\chi_{\mathcal{S}_2(\Gamma(D))}(g)$, we have all the ingredients to compute the alternating sum $\Delta M_t$, which is done in the final two sections of the paper for both $D_0$ odd and even.

### 2.2 Ingredients of the main theorem

Let $G = SL_2(\mathbb{Z}/D\mathbb{Z})$, where $-D$ is a fundamental discriminant associated to the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$. We have the following possibilities for $D$:

$$D = \begin{cases} p_0 \prod_{i=1}^{t} p_i, & \text{with } p_0, p_i \text{ distinct odd primes s.t. } p_0 \prod_{i=1}^{t} p_i \equiv 3 \bmod 4, t \geq 0 \\ 4 \prod_{i=1}^{t} p_i, & \text{with } p_i \text{ distinct odd primes s.t. } \prod_{i=1}^{t} p_i \equiv 1 \bmod 4, t \geq 0 \\ 8 \prod_{i=1}^{t} p_i, & \text{with } p_i \text{ distinct odd primes, } t \geq 0. \end{cases}$$

Thus, we can let $D = D_0 \prod_{i=1}^{t} p_i$, where $D_0 \in \{p_0, 4, 8\}$, $t \geq 0$ and the primes satisfy the above congruences. The object of interest of the paper is an expression for $\Delta M_t$ in terms of the class number $h(-D)$ of $K$. In the following, we first introduce the distinctive $G$-representations of type $(\epsilon_0, \cdots, \epsilon_t|e)$ that appear in the alternating sum $\Delta M_{t,e}$, as seen in (2.1).

Since $G \cong SL_2(\mathbb{Z}/D_0\mathbb{Z}) \times SL_2(\mathbb{F}_{p_1}) \times \cdots \times SL_2(\mathbb{F}_{p_t})$, all complex irreducible representations of $G$ arise from the irreducible representations of $SL_2(\mathbb{Z}/D_0\mathbb{Z})$ and $SL_2(\mathbb{F}_{p_i})$, $i \in [1, t]$ an integer. Thus an irreducible representation of $G$ can be written as $\pi = \otimes_i \pi_i = (\pi_0, \pi_1, \cdots \pi_t)$, where $\pi_0$ is an irreducible of

$SL_2(\mathbb{Z}/D_0\mathbb{Z})$ and $\pi_i$ is an irreducible of $SL_2(\mathbb{F}_{p_i})$ for $i \in [1,t]$. If we denote by $\chi_\pi$ the character of $\pi$, we have $\chi_\pi = \chi_0 \prod_{i=1}^{t} \chi_i$, where $\chi_i$ is the character of $\pi_i$.

In order to describe a distinctive $G$-representation $\pi = (\pi_0, \pi_1, \cdots \pi_t)$, we need to introduce the types of representations $\pi_i$ that compose it. We are interested in irreducible representations of $GL_2(\mathbb{Z}/D_0\mathbb{Z})$, respectively $GL_2(\mathbb{F}_p)$ for $p$ odd prime, that split into two irreducibles when restricted to $SL_2(\mathbb{Z}/D_0\mathbb{Z})$, respectively $SL_2(\mathbb{F}_p)$. Let $\pi_0$, respectively $\pi_i$, be one of the two irreducible representations of $SL_2(\mathbb{Z}/D_0\mathbb{Z})$, respectively $SL_2(\mathbb{F}_{p_i})$ that appear as constituents of this restriction from $GL_2$ to $SL_2$. We then call such a representation $\pi = (\pi_0, \pi_1, \cdots, \pi_t)$ a *distinctive representation* of $G$. As we will see later, there are $4 \cdot 2^{2t}$ such distinctive representations if $D_0 \in \{p_0, 4\}$ and $20 \cdot 2^{2t}$ of them in the case $D_0 = 8$.

In view of the product representation of $SL_2(\mathbb{Z}/D\mathbb{Z})$, it suffices to describe what representations $\pi_i$ can occur in a distinctive $G$-representation for the two basic cases, namely $SL_2(F_p)$, when $p$ is an odd prime, and $SL_2(\mathbb{Z}/D_0\mathbb{Z})$, when $D_0 \in \{4, 8\}$; we accomplish this in the following subsections.

**p odd prime case**

The case of $p$ odd is well known, see [34], Chapters 1, 2, p. $1-48$ or [12], Chapter 5, Section 5.2, p. $67-73$, for example. There are two types of representations that appear, the ones *induced* from the Borel subgroup of upper triangular matrices $B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, c \in \mathbb{F}_p^\times, b \in \mathbb{F}_p \right\}$ and the *cuspidal* ones. For the first type, if $\theta$, $\phi$ are two characters of $\mathbb{F}_p^\times$, then we can define a character of $B$ by $\tau \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \theta(a)\phi(c)$. The induced representation to $GL_2(\mathbb{F}_p)$ $\tau_{\theta,\phi} = \mathrm{Ind}_B^{GL_2(\mathbb{F}_p)} \tau$ will be irreducible of dimension $p+1$ iff $\theta \neq \phi$; we have $\tau_{\theta,\phi} \cong \tau_{\phi,\theta}$. Thus there are $\frac{1}{2}(p-1)(p-2)$ such representations. For $\alpha$ a character of $\mathbb{F}_p^\times$, we consider the characters of $GL_2(\mathbb{F}_p)$ given by the determinant function, $\chi_\alpha(g) = \alpha(\det g)$, which are trivial when restricted to $SL_2(\mathbb{F}_p)$. Since $\tau_{\theta,\phi} \otimes \chi_\alpha \cong \tau_{\theta\alpha,\phi\alpha}$, the induced representations above can be considered up to a twist equivalence. There is, up to twist equivalence, a unique irreducible induced representation $\tau_{\theta,1}$ that when restricted to $SL_2(\mathbb{F}_p)$ splits into two irreducibles $\tau^+, \tau^-$of the same dimension; the representation $\tau_{\theta,1}$ is given by the unique nontrivial *quadratic* character $\theta$ of $\mathbb{F}_p^\times$. We refer to the pair of representations $\tau^+, \tau^-$ as irreducibles of $SL_2(\mathbb{F}_p)$ induced from the Borel subgroup.

On the other hand, the cuspidal representations of $GL_2(\mathbb{F}_p)$ are those that do not appear in a representation induced from the Borel subgroup. They are associated to characters $\lambda$ of the cyclic group $\mathbb{F}_{p^2}^\times$ that do not come from characters of $\mathbb{F}_p^\times$, that is characters $\lambda$ for which there exists no character $\mu$ of $\mathbb{F}_p^\times$ such that $\lambda(x) = \mu(N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x))$ for all $x \in \mathbb{F}_{p^2}^\times$. For each such character $\lambda$, there is a corresponding irreducible cuspidal representation $\pi_\lambda$ such that $\pi_\lambda \cong \pi_{\lambda'}$ iff $\lambda' = \lambda$ or $\lambda' = \lambda^p$. There are $\frac{1}{2}p(p-1)$ such irreducibles. Any character $\alpha$ of $\mathbb{F}_p^\times$ can be extended to a character $\lambda_\alpha$ of $\mathbb{F}_{p^2}^\times$ by $\lambda_\alpha(x) = \alpha(N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x))$. We then have $\pi_\lambda \otimes \lambda_\alpha \cong \pi_{\lambda\lambda_\alpha}$, so we may partition the cuspidal representations $\pi_\lambda$ according to twist equivalence. The restriction of $\pi_\lambda$ to $SL_2(\mathbb{F}_p)$ depends only on the restriction of $\lambda$ to the cyclic subgroup of order $p+1$ containing elements in $\mathbb{F}_{p^2}^\times$ of norm 1. There is, up to twist equivalence, a unique irreducible cuspidal representation $\pi_\lambda$ that when restricted to $SL_2(\mathbb{F}_p)$ splits into a pair of two irreducibles $\pi^+$, $\pi^-$ of the same dimension; the representation $\pi_\lambda$ is given by the unique nontrivial *quadratic* character $\lambda$ of order $p+1$. We refer to the pair of representations $\pi^+, \pi^-$ as cuspidal irreducibles of $SL_2(\mathbb{F}_p)$.

We denote the above irreducible representations that can appear as components of a distinctive $G$-representation $\pi = (\pi_0, \pi_1, \cdots \pi_t)$ by $\pi_\delta^\epsilon$, where $\delta = +1$ if the representation is induced, $\delta = -1$ if it is cuspidal, $\epsilon \in \{\pm\}$. The characters of these representations take the value $(\delta + \epsilon G_p)/2$ on $u_1$, where $u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, as seen for example in [18], Section 7, p. 30. Clearly, $\epsilon G_p = \pm G_p$ depending on whether $\epsilon$ is $+$ or $-$.

Table 2.1: Characters of distinctive $SL_2(\mathbb{F}_p)$-representations

| | $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | $\pm\begin{pmatrix} 1 & \eta_p \\ 0 & 1 \end{pmatrix}$ | $a^l$ | $b^m$ |
|---|---|---|---|---|---|
| $\chi_{-1}^+$ | $\lambda_p(\pm1)\frac{p-1}{2}$ | $\lambda_p(\pm1)\frac{-1+G_p}{2}$ | $\lambda_p(\pm1)\frac{-1-G_p}{2}$ | 0 | $-\lambda_p(b)^m$ |
| $\chi_{-1}^-$ | $\lambda_p(\pm1)\frac{p-1}{2}$ | $\lambda_p(\pm1)\frac{-1-G_p}{2}$ | $\lambda_p(\pm1)\frac{-1+G_p}{2}$ | 0 | $-\lambda_p(b)^m$ |
| $\chi_{+1}^+$ | $\theta_p(\pm1)\frac{p+1}{2}$ | $\theta_p(\pm1)\frac{1+G_p}{2}$ | $\theta_p(\pm1)\frac{1-G_p}{2}$ | $\theta_p(\nu)^l$ | 0 |
| $\chi_{+1}^-$ | $\theta_p(\pm1)\frac{p+1}{2}$ | $\theta_p(\pm1)\frac{1-G_p}{2}$ | $\theta_p(\pm1)\frac{1+G_p}{2}$ | $\theta_p(\nu)^l$ | 0 |

Here $\eta_p$ is a non-square mod $p$, $a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$, where $\nu$ generates the multiplicative group of $\mathbb{F}_p$, $b$ is an element of order $p+1$ which is not diagonalizable over $\mathbb{F}_p$, $l \in [1, (p-3)/2]$, $m \in [1, (p-1)/2]$ integers. Also $\lambda_p, \theta_p$ are the unique nontrivial quadratic characters of cyclic groups of order $p+1$, respectively $p-1$, and $G_p$ is the Gauss sum given by $\sum_{x \in \mathbb{F}_p^\times} \theta_p(x)\xi^x$, where $\xi = \exp(2\pi i/p)$. It is well known that $G_p = \sqrt{\theta_p(-1)p}$. In terms of notation, if $D_0 = p_0$, $\pi_0$ will be of the type $\pi_{0,\delta_0}^{\epsilon_0}$ above, while $\pi_i$ will be of type $\pi_{i,\delta_i}^{\epsilon_i}$, where $\epsilon_0, \epsilon_i \in \{\pm\}$, $\delta_0, \delta_i \in \{\pm 1\}$ and $i \in [1, t]$ an integer.

$D_0 = 4$ **case**

Recall that for the case $D_0$ even, we look at representations of $GL_2(\mathbb{Z}/D_0\mathbb{Z})$ that split into two irreducibles of the same dimension when restricted to $SL_2(\mathbb{Z}/D_0\mathbb{Z})$. In the case $D_0 = 4$, there are two such representations of $GL_2(\mathbb{Z}/D_0\mathbb{Z})$. Therefore, there are two pairs of representations that appear in a distinctive $G$-representation, 2 one-dimensional ones and 2 of dimension 3. We denote them by $\pi_{0,\delta_0}^{\epsilon_0}$, with $\epsilon_0 \in \{\pm\}$, $\delta_0 \in \{1, 3\}$, their characters appearing in the following table:

Table 2.2: Characters of distinctive $SL_2(\mathbb{Z}/4\mathbb{Z})$-representations

| | 1<br>$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | 6<br>$\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | 3<br>$\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ | 6<br>$\pm\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$ | 8<br>$\pm\begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$ |
|---|---|---|---|---|---|
| $\chi_1^+$ | $\pm 1$ | $\pm i$ | $\pm(-1)$ | $\pm i$ | $\pm(-1)$ |
| $\chi_1^-$ | $\pm 1$ | $\pm(-i)$ | $\pm(-1)$ | $\pm(-i)$ | $\pm(-1)$ |
| $\chi_3^+$ | $\pm 3$ | $\pm i$ | $\pm 1$ | $\pm(-i)$ | 0 |
| $\chi_3^-$ | $\pm 3$ | $\pm(-i)$ | $\pm 1$ | $\pm i$ | 0 |

$D_0 = 8$ **case**

When $D_0 = 8$, the situation is as follows, where $\alpha = \xi_8 + \xi_8^3$, $\beta = \xi_8 - \xi_8^3$ with $\xi_8 = \exp(2\pi i/8)$, and $\gamma = 1 + 2i$:

Table 2.3: Characters of distinctive $SL_2(\mathbb{Z}/8\mathbb{Z})$-representations on the conjugacy classes of interest

| | 1 | 12 | 12 | 12 | 12 | 24 | 24 | 6 | 6 |
|---|---|---|---|---|---|---|---|---|---|
| | $\pm u_0$ | $\pm u_1$ | $\pm u_3$ | $\pm u_5$ | $\pm u_7$ | $\pm a_0$ | $\pm a_4$ | $\pm u_2$ | $\pm u_6$ |
| $\chi_{1,1}^{+}$ | $\pm 1$ | $\pm i$ | $\pm(-i)$ | $\pm i$ | $\pm(-i)$ | $\pm(-i)$ | $\pm(-i)$ | $\pm(-1)$ | $\pm(-1)$ |
| $\chi_{1,1}^{-}$ | $\pm 1$ | $\pm(-i)$ | $\pm i$ | $\pm(-i)$ | $\pm i$ | $\pm i$ | $\pm i$ | $\pm(-1)$ | $\pm(-1)$ |
| $\chi_{2,1}^{+}$ | $2$ | $\alpha$ | $\alpha$ | $-\alpha$ | $-\alpha$ | $0$ | $0$ | $0$ | $0$ |
| $\chi_{2,1}^{-}$ | $2$ | $-\alpha$ | $-\alpha$ | $\alpha$ | $\alpha$ | $0$ | $0$ | $0$ | $0$ |
| $\chi_{2,2}^{+}$ | $\pm 2$ | $\pm\beta$ | $\pm(-\beta)$ | $\pm(-\beta)$ | $\pm\beta$ | $0$ | $0$ | $0$ | $0$ |
| $\chi_{2,2}^{-}$ | $\pm 2$ | $\pm(-\beta)$ | $\pm\beta$ | $\pm\beta$ | $\pm(-\beta)$ | $0$ | $0$ | $0$ | $0$ |
| $\chi_{3,1}^{+}$ | $\pm 3$ | $\pm i$ | $\pm(-i)$ | $\pm i$ | $\pm(-i)$ | $\pm i$ | $\pm i$ | $\pm 1$ | $\pm 1$ |
| $\chi_{3,1}^{-}$ | $\pm 3$ | $\pm(-i)$ | $\pm i$ | $\pm(-i)$ | $\pm i$ | $\pm(-i)$ | $\pm(-i)$ | $\pm 1$ | $\pm 1$ |
| $\chi_{3,2}^{+}$ | $3$ | $i$ | $-i$ | $i$ | $-i$ | $-1$ | $1$ | $-\overline{\gamma}$ | $-\gamma$ |
| $\chi_{3,2}^{-}$ | $3$ | $-i$ | $i$ | $-i$ | $i$ | $-1$ | $1$ | $-\gamma$ | $-\overline{\gamma}$ |
| $\chi_{3,3}^{+}$ | $3$ | $i$ | $-i$ | $i$ | $-i$ | $1$ | $-1$ | $-\gamma$ | $-\overline{\gamma}$ |
| $\chi_{3,3}^{-}$ | $3$ | $-i$ | $i$ | $-i$ | $i$ | $1$ | $-1$ | $-\overline{\gamma}$ | $-\gamma$ |
| $\chi_{3,4}^{+}$ | $\pm 3$ | $\pm(-1)$ | $\pm(-1)$ | $\pm(-1)$ | $\pm(-1)$ | $\pm(-i)$ | $\pm i$ | $\pm\gamma$ | $\pm\overline{\gamma}$ |
| $\chi_{3,4}^{-}$ | $\pm 3$ | $\pm(-1)$ | $\pm(-1)$ | $\pm(-1)$ | $\pm(-1)$ | $\pm i$ | $\pm(-i)$ | $\pm\overline{\gamma}$ | $\pm\gamma$ |
| $\chi_{3,5}^{+}$ | $\pm 3$ | $\pm 1$ | $\pm 1$ | $\pm 1$ | $\pm 1$ | $\pm(-i)$ | $\pm i$ | $\pm\overline{\gamma}$ | $\pm\gamma$ |
| $\chi_{3,5}^{-}$ | $\pm 3$ | $\pm 1$ | $\pm 1$ | $\pm 1$ | $\pm 1$ | $\pm i$ | $\pm(-i)$ | $\pm\gamma$ | $\pm\overline{\gamma}$ |
| $\chi_{6,1}^{+}$ | $6$ | $\beta$ | $-\beta$ | $-\beta$ | $\beta$ | $0$ | $0$ | $0$ | $0$ |
| $\chi_{6,1}^{-}$ | $6$ | $-\beta$ | $\beta$ | $\beta$ | $-\beta$ | $0$ | $0$ | $0$ | $0$ |
| $\chi_{6,2}^{+}$ | $\pm 6$ | $\pm\alpha$ | $\pm\alpha$ | $\pm(-\alpha)$ | $\pm(-\alpha)$ | $0$ | $0$ | $0$ | $0$ |
| $\chi_{6,2}^{-}$ | $\pm 6$ | $\pm(-\alpha)$ | $\pm(-\alpha)$ | $\pm\alpha$ | $\pm\alpha$ | $0$ | $0$ | $0$ | $0$ |

In this case, there are 10 pairs of representations that appear in a distinctive $G$-representation. There is a pair of dimension 1, 2 pairs of dimension 2, 5 of dimension 3, and 2 of dimension 6. We denote them by $\pi_{0,\delta_0}^{\epsilon_0}$ with $\epsilon_0 \in \{\pm\}$, $\delta_0 \in \{(1,1),(2,1),(2,2),(3,1),(3,2),\cdots,(3,5),(6,1),(6,2)\}$. As we shall see in Lemma 2.3.2 below, we are only interested in the values of these characters on the conjugacy classes that take different values on $\pi_{0,\delta_0}^{+}$

and $\pi^-_{0,\delta_0}$. As a result, the conjugacy classes that are of interest are represented by $\pm u_x$ with $x \in \{0,1,3,5,7,2,6\}$, where $u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, and by $\pm \begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix}$ and $\pm \begin{pmatrix} 0 & 1 \\ 7 & 4 \end{pmatrix}$; we denoted the last 4 representatives by $\pm a_0$, respectively $\pm a_4$. The characters of the representations appearing in a distinctive $G$-representation on these conjugacy classes are as in Table 2.3 above.

Going back to the general setting, let $\pi$ be a distinctive $G$-representation given by $(\pi^{\epsilon_0}_{0,\delta_0}, \cdots, \pi^{\epsilon_t}_{t,\delta_t})$; note that $\pi^{\epsilon_i}_{i,\delta_i}$ can be either cuspidal or induced for $i \in [1,t]$ an integer, while the possible candidates for $\pi^{\epsilon_0}_{0,\delta_0}$ are those whose characters are given in the above tables. We call such a representation $\pi = (\pi^{\epsilon_0}_{0,\delta_0}, \cdots, \pi^{\epsilon_t}_{t,\delta_t})$ of type $(\epsilon_0, \cdots, \epsilon_t)$. Since the action of $-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\mathrm{Id}$ depends on $\chi^\epsilon_\delta(-\mathrm{Id}) = (-1)^{(p-\delta)/2}(p+\delta)/2$, for $\pi^\epsilon_\delta$ either cuspidal or induced irreducible of $SL_2(\mathbb{F}_p)$, we get that -Id acts as $\mathrm{sgn}(\chi_0(-\mathrm{Id}))\prod_{i=1}^{t}(-1)^{(p_i-\delta_i)/2}{\cdot}\mathrm{Id}$. We say a distinctive representation of $G$ is of type $(\epsilon_0, \cdots, \epsilon_t|e)$ if it is of type $(\epsilon_0, \cdots, \epsilon_t)$ and $\mathrm{sgn}(\chi_0(-\mathrm{Id}))\prod_{i=1}^{t}(-1)^{(p_i-\delta_i)/2} = e$, where $e \in \{\pm 1\}$.

As we saw in (2.1), we consider the following alternating sum over distinctive $G$-representations of type $(\epsilon_0, \cdots, \epsilon_t|e)$:

$$\Delta M_{t,e} = \sum_{(\epsilon_0,\cdots,\epsilon_t|e)} \prod_{i=0}^{t} \epsilon_i \sum_{\pi \in (\epsilon_0,\cdots,\epsilon_t|e)} m_\pi,$$

where $m_\pi$ is the multiplicity of the representation $\pi$ of type $(\epsilon_0, \cdots, \epsilon_t|e)$ in the $G$-representation $\rho$ on the space $\mathcal{S}_2(\Gamma(D))$ of weight 2 cusp forms for the principal congruence subgroup $\Gamma(D)$. If we let $\chi_{\mathcal{S}_2(\Gamma(D))}$ to be the character of $(\rho, \mathcal{S}_2(\Gamma(D)))$, we can rewrite $\Delta M_{t,e}$ as

$$\begin{aligned} \Delta M_{t,e} &= \sum_{(\epsilon_0,\cdots,\epsilon_t|e)} \prod_{i=0}^{t} \epsilon_i \sum_{\pi \in (\epsilon_0,\cdots,\epsilon_t|e)} \frac{1}{|G|} \sum_{g \in G} \chi_\pi(g)\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \Delta\chi_{t,e}(g)\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)}, \end{aligned} \tag{2.2}$$

where the alternating sum of characters $\sum_{(\epsilon_0,\cdots,\epsilon_t|e)}\prod_{i=0}^{t}\epsilon_i\sum_{\pi\in(\epsilon_0,\cdots,\epsilon_t|e)}\chi_\pi$ was denoted by $\Delta\chi_{t,e}$.

After computing the values of $\Delta\chi_{t,e}$ and $\chi_{\mathcal{S}_2(\Gamma(D))}$ on the conjugacy classes of $G$, which will be done in Sections 2.3, 2.5 respectively, we get an analytic expression for $\Delta M_t$. The goal is to rewrite this expression as a multiple involving $h(-D)$, which will be done by using the following modified version of the Dirichlet class number formula:

**Lemma 2.2.1.** *Let $D > 4$ such that $-D$ is the fundamental discriminant associated to the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$. Then the class number of $K$ is given by*

$$h(-D) = \begin{cases} -\frac{1}{D}\sum_{n=1}^{D-1} n\left(\frac{n}{D}\right), & \text{if } D_0 \text{ is odd} \\ -\frac{2}{D}\sum_{\substack{n=1,\\ n\equiv p_1\cdots p_t \bmod 4}}^{D-1} n\left(\frac{n}{D}\right), & \text{if } D_0 \text{ is even} \end{cases}$$

*where $D$ is written as $D_0p_1\cdots p_t$, with $D_0 \in \{p_0, 4, 8\}$, $t \geq 0$ and $p_0, p_i$ are distinct odd primes, $i \in [1, t]$ an integer.*

*Proof.* By the Dirichlet class number formula (c.f. [6], Chapter 6, p. 49 – 50)

$$h(-D) = \frac{w}{2\pi}\sqrt{D}L(\chi, 1),$$

where $w$ is the number of roots of unity in $\mathbb{Q}(\sqrt{-D})$, and $\chi$ is the quadratic character of $\mathbb{Q}(\sqrt{-D})$, $\chi : \mathbb{Z}^+ \to \mathbb{C}^\times$, $\chi(m) = \left(\frac{-D}{m}\right)$. Since $-D < -4, w = 2$.

For a nonzero integer $m$, let $m'$ denote its odd part, that is $m = 2^s m'$ with $(m', 2^s) = 1$. By the quadratic reciprocity of the Kronecker symbol we then have $\left(\frac{-D}{m}\right) = (-1)^{\frac{(m'-1)(-D'-1)}{4}}\left(\frac{m}{D}\right)$, where $D'$ is the odd part of $D$. Moreover,

$$L(\chi, 1) = -\frac{\pi}{D\sqrt{D}}\sum_{n=1}^{D-1} n\chi(n),$$

so if $D_0$ is even we have

$$h(-D) = -\frac{1}{D}\sum_{n=1}^{D-1}(-1)^{\frac{(n'-1)(-D'-1)}{4}} n\left(\frac{n}{D}\right)$$

$$
\begin{aligned}
&= -\frac{1}{D}\sum_{n=1}^{D-1}(-1)^{\frac{(n-1)(-p_1\cdots p_t-1)}{4}} n\left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right)\\
&= -\frac{1}{D}\sum_{n\equiv p_1\cdots p_t} n\left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right)\\
&\qquad\qquad + \left(\frac{-1}{p_1\cdots p_t}\right)\frac{1}{D}\sum_{n\equiv 3p_1\cdots p_t} n\left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right)\\
&= -\frac{2}{D}\sum_{n\equiv p_1\cdots p_t} n\left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right),
\end{aligned}
$$

where the congruences are taken mod 4 and the summation is over integers $n \in [1, D-1]$.

On the other hand, for odd $D_0$ we get

$$
\begin{aligned}
h(-D) &= -\frac{1}{D}\sum_{n=1}^{D-1}(-1)^{\frac{(n'-1)(-D-1)}{4}} n\left(\frac{n}{D}\right)\\
&= -\frac{1}{D}\sum_{n=1}^{D-1} n\left(\frac{n}{D}\right).
\end{aligned}
$$

□

As a side remark, note that we are only interested in finding an expression for $\Delta M_t = \Delta M_{t,1}$ as $\Delta M_{t,-1}$ always vanishes. As we shall see in the following section, this happens because the weight of cusp forms is even and thus forces the action of $-g$ on $\mathcal{S}_2(\Gamma(D))$ to be the same as that of $g$.

## 2.3 A key virtual character

Consider the alternating sum over irreducibles of $G$ of type $(\epsilon_0, \cdots, \epsilon_t|e)$ as introduced above:

$$
\Delta\chi_{t,e} = \sum_{(\epsilon_0,\cdots,\epsilon_t|e)} \prod_{i=0}^{t}\epsilon_i \sum_{\pi\in(\epsilon_0,\cdots,\epsilon_t|e)} \chi_\pi.
$$

As seen in (2.2), the values $\Delta\chi_{t,e}$ takes on the conjugacy classes of $G$ appear in the expression $\Delta M_t$. We obtain two results, see Lemmas 2.3.1, 2.3.2 below.

**Lemma 2.3.1.** *Let $g = (g_0, \cdots, g_t)$ represent a conjugacy class of $G$, where $g_i \in SL_2(\mathbb{F}_{p_i})$, for all $i \in [1, t]$ an integer and $g_0 \in SL_2(\mathbb{Z}/D_0\mathbb{Z})$. Then*

$$\Delta\chi_{t,e}(g) = \begin{cases} 2^{t-1}\Big[\Delta\chi_{0,e}(g_0) + \Delta\chi_{0,-e}(g_0)\Big] \prod_{i=1}^{t} \left(\dfrac{x_i}{p_i}\right) G_{p_i}, \\ \qquad\qquad\qquad\qquad \textit{if } g = (g_0, u_{x_1}, \cdots, u_{x_t}) \\ 2^{t-1}\Big[\Delta\chi_{0,e}(g_0) - \Delta\chi_{0,-e}(g_0)\Big] \prod_{i=1}^{t} \left(\dfrac{x_i}{p_i}\right) G_{p_i}, \\ \qquad\qquad\qquad\qquad \textit{if } g = (g_0, -u_{x_1}, \cdots, -u_{x_t}) \\ 0, \ \textit{otherwise} \end{cases}$$

*for* $t \geq 1$*, where* $x_i \in \{1, \eta_{p_i}\}$ *with* $\eta_{p_i}$ *a non-square* $\bmod p_i$ *for all* $i \in [1,t]$ *an integer.*

*Proof.* We have

$$\begin{aligned}
&\Delta\chi_{t,e}(g_0, \cdots, g_t) \\
&= \sum_{(\epsilon_0,\cdots,\epsilon_t|e)} \prod_{i=0}^{t} \epsilon_i \sum_{\pi\in(\epsilon_0,\cdots,\epsilon_t|e)} \chi_\pi(g_0, \cdots, g_t) \\
&= \sum_{(\epsilon_0,\cdots,\epsilon_t|e)} \prod_{i=0}^{t-1} \epsilon_i\epsilon_t \sum_{\delta_t} \sum_{\pi\in(\epsilon_0,\cdots,\epsilon_{t-1}|e(-1)^{(p_t-\delta_t)/2})} \chi_\pi(g_0, \cdots, g_{t-1})\chi^{\epsilon_t}_{t,\delta_t}(g_t) \\
&= \sum_{\delta_t} \sum_{(\epsilon_0,\cdots,\epsilon_{t-1}|e(-1)^{(p_t-\delta_t)/2})} \prod_{i=0}^{t-1} \epsilon_i \sum_{\pi\in(\epsilon_0,\cdots,\epsilon_{t-1}|e(-1)^{(p_t-\delta_t)/2}))} \chi_\pi(g_0, \cdots, g_{t-1}) \\
&\qquad\qquad \times \Big[\chi^+_{t,\delta_t}(g_t) - \chi^-_{t,\delta_t}(g_t)\Big] \\
&= \sum_{\delta_t} \Delta\chi_{t-1,e(-1)^{(p_t-\delta_t)/2}}(g_0, \cdots, g_{t-1})\Big[\chi^+_{t,\delta_t}(g_t) - \chi^-_{t,\delta_t}(g_t)\Big],
\end{aligned}$$

where $\chi^{\epsilon_t}_{t,\delta_t}$ denotes the character of the component $\pi^{\epsilon_t}_{t,\delta_t}$ in the $G$-representation $\pi = (\pi^{\epsilon_0}_{0,\delta_0}, \cdots, \pi^{\epsilon_t}_{t,\delta_t})$ of type $(\epsilon_0, \cdots, \epsilon_t|e)$.

For odd $p$, we get

$$\chi^+_\delta(g) - \chi^-_\delta(g) = \begin{cases} \left(\frac{x}{p}\right) G_p, & \text{if } g = u_x \text{ for } x \in \{1, \eta\} \\ (-1)^{(p-\delta)/2} \left(\frac{x}{p}\right) G_p, & \text{if } g = -u_x \text{ for } x \in \{1, \eta\} \\ 0, & \text{otherwise} \end{cases}$$

where $\eta$ is a non-square $\bmod p$. Thus, we must have $\Delta\chi_{t,e}(g_0, \cdots, g_t) = 0$ for all $g_i \neq \pm u_{x_i}$, $i \in [1,t]$ an integer. On the other hand,

$$
\begin{aligned}
&\Delta\chi_{t,e}(g_0,\cdots,g_{t-1},\pm u_{x_t})\\
&=\sum_{\delta_t}\Delta\chi_{t-1,e(-1)^{(p_t-\delta_t)/2}}(g_0,\cdots,g_{t-1})][\chi^{+}_{t,\delta_t}(\pm u_{x_t})-\chi^{-}_{t,\delta_t}(\pm u_{x_t})]\\
&=\Big[\Delta\chi_{t-1,e}(g_0,\cdots,g_{t-1})\pm\Delta\chi_{t-1,-e}(g_0,\cdots,g_{t-1})\Big]\left(\frac{x_t}{p_t}\right)G_{p_t}, \qquad (2.3)
\end{aligned}
$$

so

$$\Delta\chi_{t,e}(g_0,\cdots,g_{t-1},\pm u_{x_t})=\pm\Delta\chi_{t,-e}(g_0,\cdots,g_{t-1},\pm u_{x_t}). \qquad (2.4)$$

We claim that if there exists $i,j\in\{1,\cdots,t\}$ such that $g_i=u_{x_i}$ and $g_j=-u_{x_j}$ then we must have $\Delta\chi_{t,e}(g_0,\cdots,g_t)=0$ for $t\geq 2$. Clearly, if for some $s\leq t$ we have $\Delta\chi_{s,e}(g_0,\cdots,g_s)=0$, then by (2.3) and (2.4), we get $\Delta\chi_{t,e}(g_0,\cdots,g_s,\cdots,g_t)=0$. Thus we can assume WLOG that $i=t,j=t-1$, the case $i=t-1,j=t$ being exactly the same. Since we know $\Delta\chi_{t-1,e}(g_0,\cdots,g_{t-2},-u_{x_{t-1}})=-\Delta\chi_{t-1,-e}(g_0,\cdots,g_{t-2},-u_{x_{t-1}})$, we get that $\Delta\chi_{t,e}(g_0,\cdots,-u_{x_{t-1}},u_{x_t})=0$. As a result $\Delta\chi_{t,e}(g_0,\cdots,g_t)$ is zero outside the conjugacy classes of type $(g_0,u_{x_1},\cdots,u_{x_t})$ or $(g_0,-u_{x_1},\cdots,-u_{x_t})$.

Then

$$
\begin{aligned}
\Delta\chi_{t,e}(g_0,u_{x_1},\cdots,u_{x_t})&=2\Delta\chi_{t-1,e}(g_0,u_{x_1},\cdots,u_{x_{t-1}})\left(\frac{x_t}{p_t}\right)G_{p_t}\\
&=2^{t-1}\Big[\Delta\chi_{0,e}(g_0)+\Delta\chi_{0,-e}(g_0)\Big]\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i},
\end{aligned}
$$

and similarly,

$$\Delta\chi_{t,e}(g_0,-u_{x_1},\cdots,-u_{x_t})=2^{t-1}[\Delta\chi_{0,e}(g_0)-\Delta\chi_{0,-e}(g_0)]\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i}.$$

□

On the other hand, since

$$\Delta\chi_{0,e}(g_0)=\sum_{(\epsilon_0|e)}\epsilon_0\sum_{\pi\in(\epsilon_0|e)}\chi_\pi(g_0),$$

we can compute $\Delta\chi_{0,1}(g_0)+\Delta\chi_{0,-1}(g_0)$ for the different values of $D_0$ as follows:

| $D_0$ | $\Delta\chi_{0,1}(g_0)+\Delta\chi_{0,-1}(g_0)$ |
|---|---|
| $p_0$ | $\begin{cases} 2\left(\frac{x_0}{p_0}\right)G_{p_0}, & \text{if } g_0 = u_{x_0} \text{ for } x_0 \in \{1, \eta_{p_0}\} \\ 0, & \text{otherwise} \end{cases}$ |
| 4 | $\begin{cases} 4\xi_4^{x_0}, & \text{if } g_0 = u_{x_0} \text{ for } x_0 \in \{\pm 1\} \\ 0, & \text{otherwise} \end{cases}$ |
| 8 | $\begin{cases} 8\xi_8^{x_0} + 8\xi_4^{x_0}, & \text{if } g_0 = u_{x_0} \text{ for } x_0 \in \{\pm 1, \pm 3\} \\ \pm(-4\xi_4), & \text{if } g_0 = \pm a_0 \\ \pm 4\xi_4, & \text{if } g_0 = \pm a_4 \\ 0, & \text{otherwise.} \end{cases}$ |

The values for $\Delta\chi_{0,1}(g_0)-\Delta\chi_{0,-1}(g_0)$ are computed in a similar fashion. Note that we used the fact that $-u_1$ and $u_{-1}$ are in the same conjugacy class of $SL_2(\mathbb{Z}/4\mathbb{Z})$. Also, $\xi_{D_0} = \exp(2\pi i/D_0)$ for $D_0 \in \{4, 8\}$.

Thus, we can state the following result:

**Lemma 2.3.2.** *For all possible $D_0$, $\Delta\chi_{t,e}$ takes the following values on conjugacy classes $g = (g_0, \cdots, g_t)$:*

- *If $D_0 = p_0$,*

$$\Delta\chi_{t,e}(g) = \begin{cases} 2^t\left(\frac{x_0}{p_0}\right)G_{p_0}\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i}, & \text{if } g = (u_{x_0}, \cdots, u_{x_t}) \\ e2^t\left(\frac{x_0}{p_0}\right)G_{p_0}\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i}, & \text{if } g = (-u_{x_0}, \cdots, -u_{x_t}) \\ 0, & \text{otherwise} \end{cases}$$

  *for all $t \geq 0$.*

- *If* $D_0 = 4$,

$$\Delta\chi_{t,e}(g) = \begin{cases} 2^{t+1}\xi_4^{x_0}\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i}, & \text{if } g = (u_{x_0}, \cdots, u_{x_t}), \\ & x_0 \in \{\pm 1\} \\ e2^{t+1}\xi_4^{x_0}\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i}, & \text{if } g = (-u_{x_0}, \cdots, -u_{x_t}), \\ & x_0 \in \{\pm 1\} \\ 0, & \text{otherwise} \end{cases}$$

*for all* $t \geq 1$. *Also,* $\Delta\chi_{0,-1}(g) = \begin{cases} 2^2\xi_4^{x_0}, & \text{if } g = u_{x_0}, x_0 \in \{\pm 1\} \\ 0, & \text{otherwise,} \end{cases}$ *and* $\Delta\chi_{0,1}(g) = 0$.

- *If* $D_0 = 8$,

$$\Delta\chi_{t,e}(g) = \begin{cases} 2^{t+2}(\xi_8^{x_0} + \xi_4^{x_0})\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i}, & \text{if } g = (u_{x_0}, \cdots, u_{x_t}), \\ & x_0 \in \{\pm 1, \pm 3\} \\ \mp 2^{t+1}\xi_4\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i}, & \text{if } g = (\pm a_0, \cdots, u_{x_t}) \\ \pm 2^{t+1}\xi_4\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i}, & \text{if } g = (\pm a_4, \cdots, u_{x_t}) \\ e2^{t+2}(\xi_8^{x_0} + \xi_4^{x_0})\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i}, & \text{if } g = (-u_{x_0}, \cdots, -u_{x_t}), \\ & x_0 \in \{\pm 1, \pm 3\} \\ \mp e2^{t+1}\xi_4\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i}, & \text{if } g = (\mp a_0, \cdots, -u_{x_t}) \\ \pm e2^{t+1}\xi_4\prod_{i=1}^{t}\left(\frac{x_i}{p_i}\right)G_{p_i}, & \text{if } g = (\mp a_4, \cdots, -u_{x_t}) \\ 0, & \text{otherwise} \end{cases}$$

*for all* $t \geq 1$.

*Also,* $\Delta\chi_{0,1}(g) = \begin{cases} 2^2(\xi_8^{x_0} + \xi_4^{x_0}), & \text{if } g = \pm u_{x_0}, x_0 \in \{\pm 1, \pm 3\} \\ 0, & \text{otherwise} \end{cases}$

$$\textit{and } \Delta\chi_{0,-1}(g) = \begin{cases} \pm 2^2(\xi_8^{x_0} + \xi_4^{x_0}), & \textit{if } g = \pm u_{x_0}, x_0 \in \{\pm 1, \pm 3\} \\ \mp 2^2\xi_4, & \textit{if } g = \pm a_0 \\ \pm 2^2\xi_4, & \textit{if } g = \pm a_4 \\ 0, & \textit{otherwise.} \end{cases}$$

*When not specified, $x_i$ above takes values in $\{1, \eta_{p_i}\}$, where $\eta_{p_i}$ is a non-square* $\bmod p_i$ *for $i \in [0, t]$ an integer. Also, $\xi_{D_0} = \exp(2\pi i/D_0)$ for $D_0 \in \{4, 8\}$.*

Using the above result in (2.2), the alternating sum $\Delta M_{t,e}$ can be rewritten as follows:

$$\begin{aligned} &\Delta M_{t,e} \\ &= \frac{1}{|G|}\sum_{g\in G}\Delta\chi_{t,e}(g)\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)} \\ &= \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|}\prod_{i=1}^{t}\frac{1}{(p_i^2-1)p_i} \\ &\qquad \times \Bigg[\sum_{(g_0,\cdots,u_{x_t})} c_0\prod_{i=1}^{t}\frac{p_i^2-1}{2}\Delta\chi_{t,e}(g_0,\cdots,u_{x_t})\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g_0,\cdots,u_x)} \\ &\qquad + \sum_{(-g_0,\cdots,-u_{x_t})} c_0\prod_{i=1}^{t}\frac{p_i^2-1}{2}\Delta\chi_{t,e}(-g_0,\cdots,-u_{x_t})\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(-g_0,\cdots,-u_{x_t})}\Bigg] \\ &= \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|}\prod_{i=1}^{t}\frac{1}{2p_i} \\ &\qquad \times \sum_{(g_0,\cdots,u_{x_t})} c_0(1+e)\Delta\chi_{t,e}(g_0,\cdots,u_{x_t})\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g_0,\cdots,u_{x_t})}, \end{aligned}$$

so

$$\begin{aligned} \Delta M_{t,e} = \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|}\prod_{i=1}^{t}\frac{1}{2p_i} \\ \times \sum_{(g_0,\cdots,u_{x_t})} c_0(1+e)\Delta\chi_{t,e}(g_0,\cdots,u_{x_t})\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g_0,\cdots,u_{x_t})}, \end{aligned} \tag{2.5}$$

where $c_0$ is the size of the conjugacy class of $g_0$. The last equality follows since $\Delta\chi_{t,e}(-g) = e\Delta\chi_{t,e}(g)$ by the result of Lemma 2.3.2. Clearly $\Delta M_{t,-1} = 0$, as was previously mentioned.

## 2.4 Fixed points on the modular curve

Let $M$ be the modular curve $X(D) = \Gamma(D)\backslash\mathcal{H}^*$. $M$ will be a one-dimensional compact complex manifold. $G$ acts on $M$ and $g : M \to M$ is a holomorphic endomorphism. If we consider $\tilde{g}$ to be a lift of $g$ to $SL_2(\mathbb{Z})$, then the map $g : M \to M$ is given by $g\pi(z) = \pi(\tilde{g}z)$, where $\pi$ is the natural projection $\mathcal{H}^* \to M$. We are in the situation where we look at maps $g : M \to M$ whose fixed points, if they exist, are isolated and non-degenerate. Using the holomorphic Lefschetz number, one can compute $\chi_{\mathcal{S}_2(\Gamma(D))}(g)$ by knowing the fixed points of $g$ on $M$, as we shall see in the next section. In the following, we find the fixed points of maps of the form $g = (g_0, \cdots, u_{x_t})$ for which $\Delta\chi_{t,1}(g) \neq 0$, where $g_0$ depends on $D_0$ as seen in Lemma 2.3.2.

**Lemma 2.4.1.** *For $D > 3$, the map $g = (g_0, \cdots, u_{x_t})$ has no fixed points on $Y(D) = \Gamma(D)\backslash\mathcal{H}$; all the possible fixed points happen at the cusps of $\Gamma(D)$.*

*Proof.* If $\pi(z)$ is a fixed point on $\Gamma(D)\backslash\mathcal{H}^*$, $z \in \mathcal{H}^*$, then there exists $\eta \in \Gamma(D)$ such that $\tilde{g}z = \eta z$, so we need to look at the fixed points of $\eta^{-1}\tilde{g}$ on $\mathcal{H}^*$. Since $\mathrm{Tr}(\eta^{-1}\tilde{g}) \equiv \mathrm{Tr}\,\tilde{g} \bmod D$, we get $\mathrm{Tr}(\eta^{-1}\tilde{g}) \equiv 2 \bmod p_i$ for $i \in [1, t]$ an integer, so if $t \geq 1$ and $p_i > 3$ for some $i$, we must have $|\mathrm{Tr}(\eta^{-1}\tilde{g})| \geq 2$ . As $D > 3$, we also have $|\mathrm{Tr}(\eta^{-1}\tilde{g})| \geq 2$ for the case $D_0 = p_0$. If $D_0 = 4$, we get $\mathrm{Tr}(\eta^{-1}\tilde{g}) \equiv 2 \bmod 4$, which gives us $|\mathrm{Tr}(\eta^{-1}\tilde{g})| \geq 2$ as well. If $D = 8$, $\mathrm{Tr}(\eta^{-1}\tilde{g}) \equiv 2 \bmod 8$. The last case to consider is $D = 8 * 3$. If $g_0 = u_{x_0}$, $x_0 \in \{\pm 1, \pm 3\}$ or $g_0 = \pm a_4$ then $\mathrm{Tr}(\eta^{-1}\tilde{g})$ is either 2 or 4 mod 8. If $(g_0, u_{x_1}) = (a_0, u_1)$, a choice for $\tilde{g}$ is the matrix $\begin{pmatrix} -8 & 1 \\ 63 & -8 \end{pmatrix}$, while if $(g_0, u_{x_1}) = (a_0, u_{-1})$, we can choose a lift $\tilde{g} = \begin{pmatrix} -8 & -7 \\ -9 & -8 \end{pmatrix}$. Thus $\mathrm{Tr}(\eta^{-1}\tilde{g}) \equiv -16 \bmod 24$; similarly, one gets the same result if $g_0 = -a_0$. Therefore, $|\mathrm{Tr}(\eta^{-1}\tilde{g})| \geq 2$ holds for all possible values of $D > 3$, so $\eta^{-1}\tilde{g}$ is either parabolic or hyperbolic and thus it has either one or two fixed points on $\mathbb{R} \cup \{\infty\}$. □

For the following, assume $D > 4$. Recall that two cusps $\frac{a}{b}$ and $\frac{c}{d}$ of $\Gamma(D)$ with integers $a, b, c, d$ such that $(a, b) = 1, (c, d) = 1$ are $\Gamma(D)$-equivalent iff $\begin{pmatrix} a \\ b \end{pmatrix} \equiv \pm \begin{pmatrix} c \\ d \end{pmatrix} \bmod D$ ([36], Chapter 1, Section 1.6, Lemma 1.42, p. 23). Now, if the cusp $\frac{a}{b}$ with $a, b \in \mathbb{Z}, (a, b) = 1$ is a fixed point of $g$, then $\frac{a}{b}$ and

$\tilde{g}_{\frac{a}{b}}$ are $\Gamma(D)$-equivalent. Depending on the values of $D_0$, we get the following cases:

- If $D_0 = p_0$, then $g_0 = u_{x_0}$ with $x_0 \in \{1, \eta_{p_0}\}$, where $\eta_{p_0}$ is a non-square mod $p_0$. We have

$$a + bx_i \equiv \pm a \bmod p_i,$$
$$b \equiv \pm b \bmod p_i,$$

for all $i \in [0, t]$ an integer. As $(a, b) = 1$, we must be in the case

$$a + bx_i \equiv a \bmod p_i,$$
$$b \equiv b \bmod p_i,$$

for all $i \in [0, t]$ an integer, so $b \equiv 0 \bmod D$.

- If $D_0 = 4$, then $g_0 = u_{x_0}$, with $x_0 \in \{\pm 1\}$. Since $D > 4$, $t \geq 1$, so by the same reasoning as above we must have $b \equiv 0 \bmod \prod_{i=1}^{t} p_i$ for all $i \in [1, t]$ an integer. Moreover, we must be in the case

$$a + bx_0 \equiv a \bmod 4,$$
$$b \equiv b \bmod 4,$$

so $b \equiv 0 \bmod 4$ and thus $b \equiv 0 \bmod D$.

- If $D_0 = 8$, first consider the case when $g_0 = u_{x_0}$, with $x_0 \in \{\pm 1, \pm 3\}$. If

$$a + bx_0 \equiv -a \bmod 8,$$
$$b \equiv -b \bmod 8,$$

then $b \equiv 0 \bmod 4$ and $2a + bx_0 \equiv 0 \bmod 8$, so $a$ must be even, contradiction. Thus, we must be in the case

$$a + bx_0 \equiv a \bmod 8,$$
$$b \equiv b \bmod 8,$$

so $b \equiv 0 \bmod 8$. If $t \geq 1$, by the same reasoning as above, we must have $b \equiv 0 \bmod \prod_{i=1}^{t} p_i$ and thus $b \equiv 0 \bmod D$.

If $g_0 = a_0$, then we have

$$\begin{aligned} b &\equiv \pm a \bmod 8, \\ -a &\equiv \pm b \bmod 8, \end{aligned}$$

which forces both $a, b$ to be even, so there are no fixed points in this case. Similarly, there are no fixed points for the cases $g_0 = -a_0$ and $g_0 = \pm a_4$.

Notice that $\Delta\chi_{t,1}(g) = 0$ when $D = 4$. Therefore, we can state the following result:

**Lemma 2.4.2.** *Let $D > 3$ and $g = (g_0, \cdots, u_{x_t})$ an element of $G$ such that $\Delta\chi_{t,1}(g) \neq 0$ and $g_0$ depending on $D_0$ as seen in Lemma 2.3.2. Then the fixed points of $g$ on $M$ are as follows:*

- *If $D_0 \in \{p_0, 4\}$, $g_0 = u_{x_0}$ and $g$ has fixed points $\frac{l}{D}$ with $(l, D) = 1$, $l \in [1, D/2]$ an integer.*
- *If $D_0 = 8$ and $g_0 = u_{x_0}$, then $g$ has fixed points $\frac{l}{D}$ with $(l, D) = 1$, $l \in [1, D/2]$ an integer and there are no fixed points when $g_0 \in \{\pm a_0, \pm a_4\}$.*

*In the above, we have $x_i \in \{1, \eta_{p_i}\}$, with $\eta_{p_i}$ a non-square $\bmod p_i$ for all $i \in [1, t]$ an integer and*

$$x_0 \in \begin{cases} \{1, \eta_{p_0}\}, \textit{with } \eta_{p_0} \textit{ a non-square } \bmod p_0, & \textit{if } D_0 = p_0 \\ (\mathbb{Z}/D_0\mathbb{Z})^\times, & \textit{if } D_0 \in \{4, 8\}. \end{cases}$$

## 2.5 The holomorphic Lefschetz number

For $G$ acting on the one-dimensional compact complex manifold $M$, we identify any $g \in G$ with a map $g : M \to M$. Suppose the fixed points of $g$ are isolated and non-degenerate. The holomorphic Lefschetz number of the map $g$ relative to the holomorphic line bundle defined by the structure sheaf $\mathcal{O}$ is given by (c.f. [13], Chapter 3, Section 4, p. $422 - 426$)

$$L(g, \mathcal{O}) = \sum_q (-1)^q \operatorname{Tr}(g^* | H^{0,q}_{\bar{\partial}}(M)).$$

Let $dg_\kappa : T_\kappa(M) \to T_\kappa(M)$ be the differential induced by the map $g$ on the holomorphic tangent space at the fixed point $\kappa$ . By the holomorphic Lefschetz fixed-point formula we have

$$L(g, \mathcal{O}) = \sum_{g(\kappa)=\kappa} \frac{1}{\det(I - dg_\kappa)},$$

where, by abuse of notation, by $dg_\kappa$ we mean the above differential evaluated at the fixed point. The goal of this section is to compute the characters $\chi_{\mathcal{S}_2(\Gamma(D))}(g)$ which appear in the expression of $\Delta M_t$ in (2.2). We compute the Lefschetz numbers by using the fixed points in Lemma 2.4.2, which in turn give us the characters $\chi_{\mathcal{S}_2(\Gamma(D))}$.

We have $H^{0,q}_{\bar\partial}(M) \cong H^q(M, \mathcal{O})$. As one knows, $H^q(M, \mathcal{O})$ vanishes for $q > 1$ and $H^0(M, \mathcal{O}) \cong \mathbb{C}$. Let $\Omega^i$ define the sheaf of holomorphic differentials of degree $i$ on $M$, so that $\Omega^0 = \mathcal{O}$. By Hodge theory $H^1(M, \mathcal{O}) \cong \overline{H^0(M, \Omega^1)}$, where the space $H^0(M, \Omega^1)$ is exactly the space $\mathcal{S}_2(\Gamma(D))$ of weight 2 cusp forms for the principal congruence subgroup $\Gamma(D)$.

As a result,

$$L(g, \mathcal{O}) = \mathrm{Tr}(g^*|\mathbb{C}) - \mathrm{Tr}(g^*|\overline{\mathcal{S}_2(\Gamma(D))}).$$

But $\mathrm{Tr}(g^*|\mathbb{C}) = 1$, since the action of $g^*$ on $H^0(M, \mathcal{O})$ is trivial and we have $\mathrm{Tr}(g^*|\overline{\mathcal{S}_2(\Gamma(D))}) = \overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)}$. Thus

$$\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)} = 1 - L(g, \mathcal{O}). \tag{2.6}$$

Moreover, if $g$ has no fixed points, the Lefschetz number is zero and we get $\chi_{\mathcal{S}_2(\Gamma(D))}(g) = 1$.

Next step is to compute the differentials $dg_\kappa$. As seen in Lemma 2.4.2, we are interested in the cases when $g = (g_0, \cdots, u_{x_t})$, with $g_0$ of the form $u_{x_0}$. The fixed points of $g$ are given by $\frac{l}{D}$, with $(l, D) = 1$, $l \in [1, D/2]$ an integer. Note that the cusp $\frac{1}{D}$ is equivalent to infinity.

**Lemma 2.5.1.** *Let $D > 3$ and $g = (g_0, \cdots, u_{x_t})$ an element of $G$ having fixed points on $M$ such that $\Delta\chi_{t,1}(g) \neq 0$. We must have $g_0 = u_{x_0}$ with the values of $x_0$ depending on $D_0$ as seen in Lemma 2.4.2. The differential $dg_{\frac{l}{D}}$ at the cusp $\frac{l}{D}$ with $(l, D) = 1, l \in [1, D/2]$ an integer, is given by*

$$dg_{\frac{l}{D}} = \xi^{\lambda l^{-2}},$$

*where $\xi = \exp(2\pi i/D)$, $\lambda \in \mathbb{Z}$ such that $\lambda \equiv x_0 \bmod D_0$, $\lambda \equiv x_i \bmod p_i$, for all $i \in [1, t]$ an integer.*

*Proof.* The idea is to translate the cusp $\frac{l}{D}$ to $\infty$ and compute the differential there. Say the fixed points of $g$ are at the cusps $\kappa$, so the complex structure on $M$ is given locally by homeomorphisms into open sets of $\mathbb{C}$ through the map

$$\pi(z) \mapsto \exp(2\pi i \rho(z)/D),$$

where $\pi : \mathcal{H}^* \to M$ is the natural projection, $\rho \in SL_2(\mathbb{R})$ such that $\rho(\kappa) = \infty$.

For the cusp $\frac{l}{D}$, let $\gamma_l \in SL_2(\mathbb{Z})$ such that $\gamma_l(\frac{l}{D}) = \infty$. There exists an induced differential $d\gamma_l : T_{\frac{l}{D}}(M) \to T_\infty(M)$ such that the map $dg_{\frac{l}{D}}$ translated to $\infty$ is given by

$$d\gamma_l g \gamma_l^{-1} : T_\infty(M) \to T_\infty(M),$$

where the map $\gamma_l g \gamma_l^{-1}$ on $M$ is given by $\pi(z) \mapsto \pi(\gamma_l \tilde{g} \gamma_l^{-1} z)$, with $\tilde{g}$ a lift of $g$ to $SL_2(\mathbb{Z})$. For the cusp $\frac{l}{D}$ with $(l, D) = 1$, $\gamma_l$ will be given by the matrix $\begin{pmatrix} b & a \\ -D & l \end{pmatrix}$, where $a, b \in \mathbb{Z}$ such that $aD + bl = 1$.

As

$$\begin{pmatrix} b & a \\ -D & l \end{pmatrix} \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} l & -a \\ D & b \end{pmatrix} \equiv \begin{pmatrix} 1 + Dbx_0 & b^2 x_0 \\ -D^2 x_0 & 1 - Dbx_0 \end{pmatrix} \bmod D_0,$$

we get that $\gamma_l \tilde{g} \gamma_l^{-1} \equiv \begin{pmatrix} 1 & b^2 x_0 \\ 0 & 1 \end{pmatrix} \bmod D_0$. Similarly, we have $\gamma_l \tilde{g} \gamma_l^{-1} \equiv \begin{pmatrix} 1 & b^2 x_i \\ 0 & 1 \end{pmatrix} \bmod p_i$ for all $i \in [1, t]$ an integer. If $\lambda = \lambda_{x_0, \cdots, x_t} \in \mathbb{Z}$ such that $\lambda \equiv x_0 \bmod D_0$ and $\lambda \equiv x_i \bmod p_i$ for all $i \in [1, t]$ an integer, the action of $\gamma_l g \gamma_l^{-1}$ on $M$ will be a translation by $b^2 \lambda$. Thus, if $\exp(2\pi i z/D)$ is the local coordinate for $\infty$ on $M$, then $\exp(2\pi i(z + b^2\lambda)/D)$ will be the local coordinate for $\gamma_l g \gamma_l^{-1}(\infty)$. So

$$d\gamma_l g \gamma_l^{-1} = \frac{d \exp(2\pi i(z + b^2 \lambda)/D)}{d \exp(2\pi i z/D)},$$

and thus $dg_{\frac{l}{D}} = \xi^{\lambda b^2}$. Since $aD + bl = 1$, we have $b^2 \equiv l^{-2} \bmod D$.

□

Under the setting of Lemma 2.5.1, we get

$$L(g, \mathcal{O}) = \sum_{\substack{l=1, \\ (l,D)=1}}^{\lfloor D/2 \rfloor} \frac{1}{1 - \xi^{\lambda l^{-2}}} = \frac{1}{2} \sum_{\substack{l=1, \\ (l,D)=1}}^{D-1} \frac{1}{1 - \xi^{\lambda l^2}}$$

$$= 2^t n(D_0) \sum_{\substack{l\in[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2,\\ \left(\frac{l}{p_i}\right)=1}} \frac{1}{1-\xi^{\lambda l}} = 2^t n(D_0) \sum_{\substack{l\in\lambda[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2,\\ \left(\frac{l}{p_i}\right)=\left(\frac{\lambda}{p_i}\right)}} \frac{1}{1-\xi^{l}}$$

$$= 2^t n(D_0) \sum_{\substack{l\in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2,\\ \left(\frac{l}{p_i}\right)=\left(\frac{x_i}{p_i}\right)}} \frac{1}{1-\xi^{l}},$$

where $n(D_0) = \begin{cases} 1, \text{ if } D_0 \in \{p_0, 4\} \\ 2, \text{ if } D_0 = 8 \end{cases}$ and the summation is over $l \in [1, D-1]$ an integer, $(l, D) = 1$.

Under the same conditions, from (2.6) we get

$$\overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g)} = 1 - 2^t n(D_0) \sum_{\substack{l\in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2,\\ \left(\frac{l}{p_i}\right)=\left(\frac{x_i}{p_i}\right)}} \frac{1}{1-\xi^{l}}.$$

Thus the expression in (2.5) gives us

$$\begin{aligned}
&\Delta M_t \\
&= \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{1}{2p_i} \sum_{(g_0,\cdots,u_{x_t})} 2c_0 \Delta\chi_{t,1}(g_0,\cdots,u_{x_t}) \overline{\chi_{\mathcal{S}_2(\Gamma(D))}(g_0,\cdots,u_{x_t})} \\
&= \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{1}{2p_i} \sum_{(g_0,\cdots,u_{x_t})} 2c_0 \Delta\chi_{t,1}(g_0,\cdots,u_{x_t}) \\
&\quad - \frac{1}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{1}{2p_i} \sum_{(g_0,\cdots,u_{x_t})} 2c_0 \Delta\chi_{t,1}(g_0,\cdots,u_{x_t}) L((g_0,\cdots u_{x_t}),\mathcal{O}) \\
&= -\frac{2c_0}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{1}{2p_i} \\
&\quad \times \sum_{(u_{x_0},\cdots,u_{x_t})} \Delta(D_0,x_0) \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) G_{p_i} 2^t n(D_0) \sum_{\substack{l\in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2,\\ \left(\frac{l}{p_i}\right)=\left(\frac{x_i}{p_i}\right)}} \frac{1}{1-\xi^{l}} \\
&= -\frac{2n(D_0)c_0}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{G_{p_i}}{p_i} \sum_{(u_{x_0},\cdots,u_{x_t})} \Delta(D_0,x_0) \prod_{i=1}^{t} \left(\frac{x_i}{p_i}\right) \sum_{\substack{l\in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2,\\ \left(\frac{l}{p_i}\right)=\left(\frac{x_i}{p_i}\right)}} \frac{1}{1-\xi^{l}},
\end{aligned}$$

so

$$\Delta M_t = -\frac{2n(D_0)c_0}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|}\prod_{i=1}^{t}\frac{G_{p_i}}{p_i}\sum_{x_0}\Delta(D_0,x_0)\sum_{l\in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2}\left(\frac{l}{p_1\cdots p_t}\right)\frac{1}{1-\xi^l}, \tag{2.7}$$

with $c_0 = \begin{cases} \frac{p_0^2-1}{2}, & \text{if } D_0 = p_0 \\ 6, & \text{if } D_0 = 4 \\ 12, & \text{if } D_0 = 8, \end{cases}$ $\quad n(D_0) = \begin{cases} 1, & \text{if } D_0 \in \{p_0, 4\} \\ 2, & \text{if } D_0 = 8 \end{cases}$

and $\Delta(D_0,x_0) = \begin{cases} 2^t\left(\frac{x_0}{p_0}\right)G_{p_0}, & \text{if } D_0 = p_0, x_0 \in \{1,\eta_{p_0}\} \\ 2^{t+1}\xi_4^{x_0}, & \text{if } D_0 = 4, x_0 \in (\mathbb{Z}/4\mathbb{Z})^\times \\ 2^{t+2}(\xi_8^{x_0}+\xi_4^{x_0}), & \text{if } D_0 = 8, x_0 \in (\mathbb{Z}/8\mathbb{Z})^\times, \end{cases}$

where $\eta_{p_0}$ is a non-square in $\mathbb{F}_{p_0}^\times$ and the summation is over $l \in [1, D-1]$ an integer. Note that the above result works for $D > 4$ and if $D = 4$ we have $\Delta M_0 = 0$. So for the rest of the paper we work with $D > 4$, unless mentioned otherwise.

## 2.6 Some useful lemmas

The following results provide key steps in bringing the expression for $\Delta M_t$ in (2.7) in the form of the analytical formula for $h(-D)$ appearing in Lemma 2.2.1.

**Lemma 2.6.1.** *If $D \in \mathbb{Z}_{>1}$, $\xi = \exp(2\pi i/D)$, then*

$$\frac{1}{1-\xi^l} = \frac{1}{D}\sum_{n=0}^{D-1} n\xi^{-l(n+1)},$$

*for all $l \in [1, D-1]$ an integer.*

*Proof.* Let $\theta_D$ be the polynomial $\theta_D(x) = \sum_{n=0}^{D-1} x^n = \prod_{n=1}^{D-1}(x-\xi^n)$.Then

$$\theta_D(x)' = \sum_{n=0}^{D-1} nx^{n-1} = \sum_{n=1}^{D-1}\prod_{j\neq n}(x-\xi^j).$$

Evaluating at $\xi^l$ we get $\sum_{n=0}^{D-1} n\xi^{l(n-1)} = \prod_{j\neq l}(\xi^l - \xi^j) = \xi^{l(D-2)}\prod_{j\neq l}(1-\xi^{j-l})$. Thus

$$\sum_{n=0}^{D-1} n\xi^{l(n-1)} = \xi^{-2l}\prod_{n\neq D-l}(1-\xi^n),$$

so

$$\frac{1}{1-\xi^l} = \frac{1}{D}\prod_{n\neq l}(1-\xi^n) = \frac{1}{D}\sum_{n=0}^{D-1} n\xi^{-l(n+1)}.$$

□

**Lemma 2.6.2.** *Let $D \in \mathbb{Z}_{>1}$, $D = D_* p_1 \cdots p_t$, with $p_i$ distinct odd primes, $(D_*, p_i) = 1$ for all $i \in [1, t]$ an integer, $x \in \mathbb{Z}$. Then*

$$\sum_{l\equiv x}\left(\frac{l}{p_1\cdots p_t}\right)\frac{1}{1-\xi^l}$$
$$= \frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\sum_{n=1}^{D-1}(n-1)\left(\frac{n}{p_1\cdots p_t}\right)\sum_{l\equiv -nx}\left(\frac{l}{p_1\cdots p_t}\right)\xi^l + \eta_t,$$

*where $\xi = \exp(2\pi i/D)$, $\eta_t = \begin{cases}\frac{D-1}{D}, & \text{if } t=0\\ 0, & \text{otherwise,}\end{cases}$ the summation is over $l \in [1, D-1]$ an integer and the congruences are* mod $D_*$.

*Proof.* If we denote $\sum_{l\equiv x \bmod D_*}\left(\frac{l}{p_1\cdots p_t}\right)\frac{1}{1-\xi^l}$ by $E$, then from Lemma 2.6.1, we have

$$\begin{aligned}
E &= \frac{1}{D}\sum_{l\equiv x \bmod D_*}\left(\frac{l}{p_1\cdots p_t}\right)\sum_{n=0}^{D-1} n\xi^{-l(n+1)}\\
&= \frac{1}{D}\sum_{n=0}^{D-1} n \sum_{l\equiv x \bmod D_*}\left(\frac{l}{p_1\cdots p_t}\right)\xi^{-l(n+1)}\\
&= \frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\sum_{n=0}^{D-2} n\left(\frac{n+1}{p_1\cdots p_t}\right)\sum_{l\equiv x \bmod D_*}\left(\frac{-l(n+1)}{p_1\cdots p_t}\right)\xi^{-l(n+1)}\\
&\qquad + \frac{1}{D}(D-1)\sum_{l\equiv x \bmod D_*}\left(\frac{l}{p_1\cdots p_t}\right)\\
&= \frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\sum_{n=1}^{D-1}(n-1)\left(\frac{n}{p_1\cdots p_t}\right)\sum_{l\equiv x \bmod D_*}\left(\frac{-ln}{p_1\cdots p_t}\right)\xi^{-ln} + \eta_t\\
&= \frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\sum_{n=1}^{D-1}(n-1)\left(\frac{n}{p_1\cdots p_t}\right)\sum_{l\equiv -nx \bmod D_*}\left(\frac{l}{p_1\cdots p_t}\right)\xi^{l} + \eta_t.
\end{aligned}$$

□

**Lemma 2.6.3.** *Let $p_i$ be distinct odd primes, $i \in [1,t]$ an integer, $t \geq 1$. Then*

$$\sum_{l=1}^{p_1\cdots p_t-1} \left(\frac{l}{p_1\cdots p_t}\right) \xi_{p_1\cdots p_t}^l = e_t \prod_{i=0}^{t} G_{p_i},$$

*where* $\xi_{p_1\cdots p_t} = \exp(2\pi i/p_1\cdots p_t)$, $e_t = \begin{cases} \prod_{1\leq i<j\leq t} (-1)^{\frac{(p_i-1)(p_j-1)}{4}} & \text{if } t>1 \\ 1, & \text{if } t=1. \end{cases}$

*Proof.* For an odd prime $p$ we know $\sum_{l=1}^{p-1} \left(\frac{l}{p}\right) \xi_p^l = G_p$, where $\xi_p = \exp(2\pi i/p)$. If $p, q$ distinct odd primes,

$$\begin{aligned} \sum_{i=1}^{p-1} \left(\frac{i}{p}\right) \xi_p^i \sum_{j=1}^{q-1} \left(\frac{j}{q}\right) \xi_q^j &= \sum_{i,j} \left(\frac{i}{p}\right) \left(\frac{j}{q}\right) \xi_{pq}^{iq+jp} \\ &= \sum_{i,j} \left(\frac{iq+jp}{pq}\right) \left(\frac{p}{q}\right) \left(\frac{q}{p}\right) \xi_{pq}^{iq+jp} \\ &= (-1)^{\frac{(p-1)(q-1)}{4}} \sum_{l=1}^{pq-1} \left(\frac{l}{pq}\right) \xi_{pq}^l, \end{aligned}$$

and the result follows by induction.

□

**Lemma 2.6.4.** *Let $D = D_0 p_1 \cdots p_t$, with $D_0 \in \mathbb{Z}_{>1}$ and $p_i$ distinct odd primes such that $(D_0, p_i) = 1$ for all integers $i \in [1,t]$. Let $S \subset \mathbb{Z}$ finite set, $n, c \in \mathbb{Z}$. Then*

$$\sum_{x\in S} \xi_{D_0}^{cx} \sum_{\substack{l=1, \\ l\equiv -nx(p_1\cdots p_t)^2 \bmod D_0}}^{D-1} \left(\frac{l}{p_1\cdots p_t}\right) \xi^l = \left(\frac{D_0}{p_1\cdots p_t}\right) e_t \prod_{i=1}^{t} G_{p_i} \sum_{x\in S} \xi_{D_0}^{x(c-np_1\cdots p_t)},$$

*where* $\xi = \exp(2\pi i/D)$, $\xi_{D_0} = \exp(2\pi i/D_0)$

*and* $e_t = \begin{cases} \prod_{1\leq i<j\leq t} (-1)^{\frac{(p_i-1)(p_j-1)}{4}}, & \text{if } t>1 \\ 1, & \text{if } t\in\{0,1\}. \end{cases}$

*Proof.* There is nothing to prove if $t = 0$. For $t \geq 1$, by Lemma 6.1.1, we have

$$\sum_{j=1}^{p_1\cdots p_t-1} \left(\frac{j}{p_1\cdots p_t}\right) \xi_{p_1\cdots p_t}^j = e_t \prod_{i=1}^{t} G_{p_i}.$$

Then

$$\left(\sum_{x\in S}\xi_{D_0}^{cx}\xi_{D_0}^{-nxp_1\cdots p_t}\right)\left[\sum_{j=1}^{p_1\cdots p_t-1}\left(\frac{j}{p_1\cdots p_t}\right)\xi_{p_1\cdots p_t}^{j}\right]$$

$$= \sum_{x\in S}\xi_{D_0}^{cx}\sum_{j=1}^{p_1\cdots p_t-1}\left(\frac{j}{p_1\cdots p_t}\right)\xi^{jD_0-nx(p_1\cdots p_t)^2}$$

$$= \left(\frac{D_0}{p_1\cdots p_t}\right)\sum_{x\in S}\xi_{D_0}^{cx}\sum_{j=1}^{p_1\cdots p_t-1}\left(\frac{jD_0}{p_1\cdots p_t}\right)\xi^{jD_0-nx(p_1\cdots p_t)^2}$$

$$= \left(\frac{D_0}{p_1\cdots p_t}\right)\sum_{x\in S}\xi_{D_0}^{cx}\sum_{\substack{l=1,\\ l\equiv -nx(p_1\ldots p_t)^2 \bmod D_0}}^{D-1}\left(\frac{l}{p_1\cdots p_t}\right)\xi^{l}.$$

□

## 2.7 Proof of the main theorem

We prove the main result in both the odd and even cases, by using the key lemmas from the previous section in the expression (2.7) for $\Delta M_t$.

**The odd case** $D_0 = p_0$

From (2.7) we have

$$\Delta M_t$$

$$= -\frac{2n(D_0)c_0}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|}\prod_{i=1}^{t}\frac{G_{p_i}}{p_i}\sum_{x\in\{1,\eta_{p_0}\}}\Delta(D_0,x_0)\sum_{l\in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2}\left(\frac{l}{p_1\cdots p_t}\right)\frac{1}{1-\xi^l},$$

$$= -2^t\frac{G_{p_0}}{p_0}\prod_{i=1}^{t}\frac{G_{p_i}}{p_i}\sum_{l=1}^{D-1}\left(\frac{l}{D}\right)\frac{1}{1-\xi^l}.$$

We have $D = p_0\prod_{i=1}^{t}p_i$, with $-D\equiv 1 \bmod 4$ and let's first assume $t\geq 1$. We need to compute $\sum_{l=1}^{D-1}\left(\frac{l}{D}\right)\frac{1}{1-\xi^l}$, which we denote by $\Delta_{p_0}$. From Lemma 2.6.2 for $D_* = 1$ we get

$$\sum_{l=1}^{D-1}\left(\frac{l}{D}\right)\frac{1}{1-\xi^l} = \frac{1}{D}\left(\frac{-1}{D}\right)\sum_{n=1}^{D-1}(n-1)\left(\frac{n}{D}\right)\sum_{l=1}^{D-1}\left(\frac{l}{D}\right)\xi^l,$$

and using the results of Lemmas 6.1.1 and 2.2.1 we have

$$\begin{aligned}
\Delta_{p_0} &= \prod_{0\le i<j\le t} (-1)^{\frac{(p_i-1)(p_j-1)}{4}} \frac{1}{D} \left(\frac{-1}{D}\right) G_{p_0} \prod_{i=1}^{t} G_{p_i} \left[ \sum_{n=1}^{D-1} n \left(\frac{n}{D}\right) - \sum_{n=1}^{D-1} \left(\frac{n}{D}\right) \right] \\
&= \prod_{0\le i<j\le t} (-1)^{\frac{(p_i-1)(p_j-1)}{4}} \frac{1}{D} \left(\frac{-1}{D}\right) G_{p_0} \prod_{i=1}^{t} G_{p_i} \sum_{n=1}^{D-1} n \left(\frac{n}{D}\right) \\
&= - \prod_{0\le i<j\le t} (-1)^{\frac{(p_i-1)(p_j-1)}{4}} \left(\frac{-1}{D}\right) G_{p_0} \prod_{i=1}^{t} G_{p_i} h(-D).
\end{aligned}$$

Thus for $t \geq 1$ we have

$$\begin{aligned}
\Delta M_t &= -2^t \prod_{i=0}^{t} \frac{G_{p_i}}{p_i} \sum_{l=1}^{D-1} \left(\frac{l}{D}\right) \frac{1}{1-\xi^l} \\
&= \prod_{0\le i<j\le t} (-1)^{\frac{(p_i-1)(p_j-1)}{4}} 2^t \prod_{i=0}^{t} \frac{G_{p_i}^2}{p_i} \left(\frac{-1}{D}\right) h(-D) \\
&= \prod_{0\le i<j\le t} (-1)^{\frac{(p_i-1)(p_j-1)}{4}} 2^t h(-D),
\end{aligned}$$

since $G_p = \sqrt{(-1)^{\frac{p-1}{2}} p} = \sqrt{\left(\frac{-1}{p}\right) p}$.

The case $t = 0$ works similarly and we get $\Delta M_0 = h(-p)$, when $p \equiv 3 \bmod 4$, which is Hecke's initial result.

**The even case** $D_0 \in \{4, 8\}$

From (2.7) we get

$$\begin{aligned}
\Delta M_t = &- \frac{2n(D_0)c_0}{|SL_2(\mathbb{Z}/D_0\mathbb{Z})|} \prod_{i=1}^{t} \frac{G_{p_i}}{p_i} \\
&\times \sum_{x_0 \in (\mathbb{Z}/D_0\mathbb{Z})^\times} \Delta(D_0, x_0) \sum_{l \in x_0[(\mathbb{Z}/D_0\mathbb{Z})^\times]^2} \left(\frac{l}{p_1 \cdots p_t}\right) \frac{1}{1-\xi^l} \\
= &-2^{t-1} \prod_{i=1}^{t} \frac{G_{p_i}}{p_i} \Delta_{D_0},
\end{aligned}$$

where

$$\Delta_{D_0} = \begin{cases} \displaystyle\sum_{x_0 \in (\mathbb{Z}/D_0\mathbb{Z})^\times} \xi_{D_0}^{x_0} \sum_{\substack{l=1, \\ l \equiv x_0 \bmod D_0}}^{D-1} \left(\frac{l}{p_1 \cdots p_t}\right) \frac{1}{1-\xi^l}, & \text{if } D_0 = 4 \\ \displaystyle\sum_{x_0 \in (\mathbb{Z}/D_0\mathbb{Z})^\times} (\xi_{D_0}^{x_0} + \xi_{D_0}^{2x_0}) \sum_{\substack{l=1, \\ l \equiv x_0 \bmod D_0}}^{D-1} \left(\frac{l}{p_1 \cdots p_t}\right) \frac{1}{1-\xi^l}, & \text{if } D_0 = 8. \end{cases}$$

Note that for $D_0 \in \{4, 8\}$, we have $(p_1 \cdots p_t)^2 \equiv 1 \bmod D_0$. Thus, using Lemma 2.6.2 for $D_* = D_0$ and Lemma 2.6.4 for $S = (\mathbb{Z}/D_0\mathbb{Z})^\times$, the expression $E := \displaystyle\sum_{x_0 \in (\mathbb{Z}/D_0\mathbb{Z})^\times} \xi_{D_0}^{cx_0} \sum_{\substack{l=1, \\ l \equiv x_0 \bmod D_0}}^{D-1} \left(\frac{l}{p_1 \cdots p_t}\right) \frac{1}{1-\xi^l}$, where $c = 1$ when $D_0 = 4$ and $c \in \{1, 2\}$ when $D_0 = 8$, can be rewritten as

$$\begin{aligned} E = &\sum_{x_0 \in (\mathbb{Z}/D_0\mathbb{Z})^\times} \xi_{D_0}^{cx_0} \Bigg[ \frac{1}{D} \left(\frac{-1}{p_1 \cdots p_t}\right) \\ &\times \sum_{n=1}^{D-1} (n-1) \left(\frac{n}{p_1 \cdots p_t}\right) \sum_{\substack{l \equiv -nx_0 \\ \bmod D_0}} \left(\frac{l}{p_1 \cdots p_t}\right) \xi^l + \eta_t \Bigg] \\ = &\frac{1}{D} \left(\frac{-1}{p_1 \cdots p_t}\right) \sum_{x_0 \in (\mathbb{Z}/D_0\mathbb{Z})^\times} \xi_{D_0}^{cx_0} \sum_{n=1}^{D-1} (n-1) \left(\frac{n}{p_1 \cdots p_t}\right) \sum_{\substack{l \equiv -nx_0 \\ \bmod D_0}} \left(\frac{l}{p_1 \cdots p_t}\right) \xi^l \\ = &\frac{1}{D} \left(\frac{-1}{p_1 \cdots p_t}\right) \sum_{n=1}^{D-1} (n-1) \left(\frac{n}{p_1 \cdots p_t}\right) \sum_{x_0 \in (\mathbb{Z}/D_0\mathbb{Z})^\times} \xi_{D_0}^{cx_0} \sum_{\substack{l \equiv -nx_0 \\ \bmod D_0}} \left(\frac{l}{p_1 \cdots p_t}\right) \xi^l \\ = &\frac{1}{D} \left(\frac{-1}{p_1 \cdots p_t}\right) \left(\frac{D_0}{p_1 \cdots p_t}\right) e_t \prod_{i=1}^{t} G_{p_i} \\ &\times \sum_{n=1}^{D-1} (n-1) \left(\frac{n}{p_1 \cdots p_t}\right) \sum_{x_0 \in (\mathbb{Z}/D_0\mathbb{Z})^\times} \xi_{D_0}^{x_0(c - np_1 \cdots p_t)}, \end{aligned}$$

where $e_t = \begin{cases} \displaystyle\prod_{1 \le i < j \le t} (-1)^{\frac{(p_i-1)(p_j-1)}{4}}, & \text{if } t > 1 \\ 1, & \text{if } t \in \{0, 1\}. \end{cases}$

An easy computation gives us the following:

$$\sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times} \xi_{D_0}^{x_0(1-np_1\cdots p_t)} = \begin{cases} \pm\frac{D_0}{2}, & \text{if } n\equiv p_1\cdots p_t, -p_1\cdots p_t \bmod D_0, \\ & \hfill \text{when } D_0=4 \\ \pm\frac{D_0}{2}, & \text{if } n\equiv p_1\cdots p_t, -3p_1\cdots p_t \bmod D_0, \\ & \hfill \text{when } D_0=8 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sum_{x_0\in(\mathbb{Z}/D_0\mathbb{Z})^\times} \xi_{D_0}^{x_0(2-np_1\cdots p_t)} = \begin{cases} \pm\frac{D_0}{2}, & \text{if } n\equiv 2p_1\cdots p_t, -2p_1\cdots p_t \bmod D_0, \\ & \hfill \text{when } D_0=8 \\ 0, & \text{otherwise.} \end{cases}$$

As a result, we get

$$\Delta_{D_0} = \frac{1}{D}\left(\frac{-1}{p_1\cdots p_t}\right)\left(\frac{D_0}{p_1\cdots p_t}\right) e_t \prod_{i=0}^{t} G_{p_i} \frac{D_0}{2}\Delta_{D_0}^*,$$

where

$$\Delta_{D_0}^* = \begin{cases} \displaystyle\sum_{n\equiv p_1\cdots p_t \bmod 4} n\left(\frac{n}{p_1\cdots p_t}\right) - \sum_{n\equiv -p_1\cdots p_t \bmod 4} n\left(\frac{n}{p_1\cdots p_t}\right), \\ \hfill \text{if } D_0=4 \\ \displaystyle\sum_{\substack{n\equiv p_1\cdots p_t \bmod 8,\\ n\equiv 2p_1\cdots p_t \bmod 8}} n\left(\frac{n}{p_1\cdots p_t}\right) - \sum_{\substack{n\equiv -3p_1\cdots p_t \bmod 8,\\ n\equiv -2p_1\cdots p_t \bmod 8}} n\left(\frac{n}{p_1\cdots p_t}\right), \\ \hfill \text{if } D_0=8. \end{cases}$$

For $D_0=8$, we have

$$\sum_{\substack{n=1,\\ n\equiv 2p_1\cdots p_t \bmod 8}}^{D-1} n\left(\frac{n}{p_1\cdots p_t}\right) - \sum_{\substack{n=1,\\ n\equiv -2p_1\cdots p_t \bmod 8}}^{D-1} n\left(\frac{n}{p_1\cdots p_t}\right)$$
$$= \begin{cases} 2\left(\frac{2}{p_1\cdots p_t}\right)\Delta_{\frac{D_0}{2}}^*, & \text{if } \left(\frac{-1}{p_1\cdots p_t}\right)=1 \\ 0, & \text{otherwise;} \end{cases}$$

we denote this difference by $\delta_8^*$.

A trivial check gives us

$$\Delta_{D_0}^* = \begin{cases} 2\left(\frac{D_0}{p_1\cdots p_t}\right) \displaystyle\sum_{n\equiv p_1\cdots p_t \bmod 4} n\left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right), & \text{if } D_0=4, t\geq 1 \\ \left(\frac{D_0}{p_1\cdots p_t}\right) \displaystyle\sum_{n\equiv p_1\cdots p_t \bmod 4} n\left(\frac{n}{D_0}\right)\left(\frac{n}{p_1\cdots p_t}\right) + \delta_8^*, & \text{if } D_0=8 \end{cases}$$

where the summations are over $n \in [1, D-1]$ an integer.

Using the result of Lemma 2.2.1, we have

$$\Delta_{D_0} = \begin{cases} -\left(\frac{-1}{p_1\cdots p_t}\right) e_t 2 \prod_{i=1}^{t} G_{p_i} h(-D), & \text{if } D_0 = 4, t \geq 1 \\ -e_t 2\Big[h(-D) + h(-D/2)\Big], & \text{if } D_0 = 8, t = 0 \\ -\left(\frac{-1}{p_1\cdots p_t}\right) e_t 2 \prod_{i=1}^{t} G_{p_i} \Big[h(-D) + 2h(-D/2)\Big], & \text{if } D_0 = 8, t \geq 1, \\ & p_1 \cdots p_t \equiv 1 \bmod 4 \\ -\left(\frac{-1}{p_1\cdots p_t}\right) e_t 2 \prod_{i=1}^{t} G_{p_i} h(-D), & \text{if } D_0 = 8, t \geq 1, \\ & p_1 \cdots p_t \equiv 3 \bmod 4, \end{cases}$$

so we get

$$\Delta M_t = \begin{cases} e_t 2^t h(-D), & \text{if } D_0 = 4, t \geq 1 \\ e_t 2^t \big[h(-D) + h(-D/2)\big], & \text{if } D_0 = 8, t = 0 \\ e_t 2^t \big[h(-D) + 2h(-D/2)\big], & \text{if } D_0 = 8, t \geq 1, p_1 \cdots p_t \equiv 1 \bmod 4 \\ e_t 2^t h(-D), & \text{if } D_0 = 8, t \geq 1, p_1 \cdots p_t \equiv 3 \bmod 4, \end{cases}$$

which is what we want for the case $D > 4$ even. This concludes the proof of the main theorem for all cases $D_0 \in \{p_0, 4, 8\}$.

*Remark.* Note that when $D_0 = 8$, there is a multiple of $h(-D/2)$ appearing in the expression for $\Delta M_t$. Morally, this term comes from the distinctive $G$-representations whose $SL_2(\mathbb{Z}/8\mathbb{Z})$ part can be factored through $SL_2(\mathbb{Z}/4\mathbb{Z})$. There are two such pairs of irreducibles of $SL_2(\mathbb{Z}/8\mathbb{Z})$ that can appear in a distinctive $G$-representation, that is $\pi^+_{1,1}, \pi^-_{1,1}$, respectively $\pi^+_{3,1}, \pi^-_{3,1}$. Interchanging $\pi^+$ and $\pi^-$ for some of the irreducibles appearing in the $SL_2(\mathbb{Z}/8\mathbb{Z})$ part and discarding those above that factor through $SL_2(\mathbb{Z}/4\mathbb{Z})$ will give us $\Delta M_t = \operatorname{sgn}_{D_0,t} 2^t h(-D)$ for all cases $D_0 = 8, t \geq 1$; here $\operatorname{sgn}_{D_0,t}$ is as given in the statement of the main theorem. For example, in order to get such a result, one can interchange $\pi^+$ and $\pi^-$ for $\pi_{3,3}$ and $\pi_{3,4}$ in the $SL_2(\mathbb{Z}/8\mathbb{Z})$ part of a distinctive $G$-representation.

*Chapter 3*

# STUDY CASE $SU(1,1)$: REFRAMING HECKE'S ORIGINAL SETTING FOR $SL_2$

In this chapter we shall consider the case when $G = SU(1,1)$. First of all, since we have the exceptional isomorphism of $\mathfrak{su}_2(\mathbb{R}) \cong \mathfrak{su}(1,1)$, we can reframe the case of $SL_2(\mathbb{F}_p)$ studied in Chapter 2 in the language of the special unitary group $SU(1,1)$. On the other hand, we can use the $SU(1,1)$ case to inform the steps needed for the higher rank case of $SU(2,2)$ on the geometry side. We mention that since the $\mathbb{Q}$-rank of $SU(2,2)$ is not 1, the minimal Baily-Borel compactification will not be smooth, and thus one needs to go to the toroidal compactification in this case - thus, the $SU(1,1)$ case below does not complete the picture for $SU(2,2)$ on the geometry side.

**Preliminaries**

Let $E$ be an imaginary quadratic field with discriminant $D > 0$, ring of integers $\mathcal{O}_E$ and Galois automorphism given by complex conjugation. That is, $E = \mathbb{Q}(\beta)$, where $\beta = \sqrt{-D}$ and $\overline{\beta} = -\beta$. Let $V = E^2$ be the 2-dimensional vector space over $E$ with standard basis, and $L \subset V$ the standard $\mathcal{O}_E$-lattice in $V$. Choose $J : V \times V \to E$ to be a nondegenerate hermitian form on $V$ with $J(au, bv) = \overline{a}b\overline{J(v,u)}$, which has signature $(1,1)$ on $V_{\mathbb{R}} = V \otimes_E \mathbb{R}$ and whose matrix in the basis for $V$ is given by

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where by abuse of notation we define the matrix of $J$ by $J$ as well. Note that $J$ is $\mathcal{O}_E$-valued on $L$.

Let $G = SU(1,1)$ be the special unitary group of signature $(1,1)$ defined by $J$, viewed as a semisimple connected algebraic group over $\mathbb{Q}$. Then for any $\mathbb{Q}$-algebra $A$,

$$G(A) = \{g \in SL(V \otimes_{\mathbb{Q}} A) | \overline{g}^t Jg = J\}.$$

Similarly, for any $\mathbb{Z}$-algebra $A'$, define

$$G(A') = \{g \in SL(L \otimes_{\mathbb{Z}} A') | \overline{g}^t Jg = J\}.$$

Thus, $G(\mathbb{Z}) = SL_4(\mathcal{O}_E) \cap G(\mathbb{Q})$ is the group of matrices in $G(\mathbb{Q})$ that preserve the lattice $L$.

We are interested in irreducible cuspidal representations of $\widetilde{G}(\mathbb{Z}/p\mathbb{Z})$, for prime $p > 3$, that split upon restriction to $G(\mathbb{Z}/p\mathbb{Z})$. As we shall see later on, $G(\mathbb{Z}/p\mathbb{Z})$ is either $SL_2(\mathbb{F}_p)$, when $p$ splits in $E$, $SU((1,1), \mathbb{F}_{p^2})$, when $p$ is inert or $SO((1,1), \mathbb{F}_p)$ when p is ramified in $E$. Note that by $SU((1,1), \mathbb{F}_{p^2})$ we mean the special unitary group of signature $(1,1)$ over the field of $p$ elements; note the elements in $SU((1,1), \mathbb{F}_{p^2})$ have entries in $\mathbb{F}_{p^2}$, that is, they are elements $g \in SL_2(\mathbb{F}_{p^2})$ such that $\overline{g}^t J g = J$, for $J$ the nondegenerate hermitian form defined above. Here complex conjugation is given by entrywise raising to the $p^{\text{th}}$-power map. Similarly, $SO((1,1), \mathbb{F}_p)$ is a special unitary group of signature $(1,1)$ over the field of $p$ elements. Thus, if $p$ inert or split, $G(\mathbb{Z}/p\mathbb{Z})$ is a finite group of Lie type, which is a rational form of the connected semisimple reductive group $SL_2$ defined over the algebraically closed field $K = \overline{\mathbb{F}}_p$. Now, since we know that any two nondegenerate Hermitian (or skew Hermitian) forms of the same dimension over a finite field are equivalent, when working with the finite group of Lie type $G(\mathbb{Z}/p\mathbb{Z})$, we can consider $J$ being given by the matrix $\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$ instead. This is exactly the form we shall use for the case of $SU(2,2)$, as it makes the derivations cleaner. For details on hermitian forms and their classification up to isometry check [27], [31].

### 3.1 The associated symmetric domain $G/K$

In the following section we shall analyze the hermitian symmetric domain associated to the semisimple connected group $G = SU(1,1)$ defined over $\mathbb{Q}$. The goal is to realize it as a bounded symmetric domain $D$ and view the action of $G$ on $D$ as being given by the usual action of $SL_2(\mathbb{R})$ on the upper half plane. As a result, we can reinterpret the case of $SL_2$ done in Chapter 2 in the frame of the special unitary groups of signature $(1,1)$. The scope of this analysis is to help inform the steps needed to complete the higher rank cases, so we shall focus on explaining in detail only some of the aspects of this particular case, while mentioning the general picture otherwise. The presentation below mainly follows the theory in Chapter 7, Section 9 in [20]. For the general frame of symmetric domains and compactifications of locally symmetric varieties we refer to in Chapter 3 of [1], while for general results on hermitian symmetric domains and locally symmetric varieties , we refer to Chapters $1, 3$ and $4$ in

[30] or [16].

We have $G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \middle| a\overline{a} - b\overline{b} = 1 \right\}$ with maximal compact $K = \left\{ \begin{pmatrix} c & 0 \\ 0 & \overline{c} \end{pmatrix} \right\}$ with $c\overline{c} = 1$. Note that $a, b, c \in E \otimes_{\mathbb{Q}} \mathbb{R}$. In order to put a complex structure on the symmetric space $G/K$, choose an embedding $E \hookrightarrow \mathbb{C}$ and identify $E \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}$. Now $G/K$ can be identified with the disc $\{z \in \mathbb{C} | z\overline{z} < 1\}$ by $gK \mapsto b/\overline{a}$.

$G$ is a semisimple and $G/K$ is a hermitian symmetric space of noncompact type, that is a hermitian symmetric domain. As a result, it admits a complex manifold structure on which $G$ acts by holomorphic transformations.

**Lemma 3.1.1.** *The action of* $G$ *on the space* $G/K$ *is given by linear fractional transformations*

$$(g', z) \mapsto \frac{a'z + b'}{\overline{b'}z + \overline{a'}}$$

*where* $g' = \begin{pmatrix} a' & b' \\ \overline{b'} & \overline{a'} \end{pmatrix}$.

*Proof.* From the identification of $G/K$ with the disc, we have $gK \mapsto b/\overline{a}$, for $g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$. Since $g'g = \begin{pmatrix} a'a + b'\overline{b} & a'b + b'\overline{a} \\ \overline{b'}a + \overline{a'b} & \overline{b'}b + \overline{a'a} \end{pmatrix}$, we get that

$$g'(gK) \mapsto \frac{a'b + b'\overline{a}}{\overline{b'}b + \overline{a'a}} = \frac{a'b/\overline{a} + b'}{\overline{b'}b/\overline{a} + \overline{a'}},$$

which ends the proof. □

Note that for every hermitian symmetric space of noncompact type, there exists a bounded symmetric domain that is diffeomorphic to it. For $G/K$ above, the unit disc is a bounded symmetric domain.

### Realization of $G/K$ as a bounded domain inside $P_+$

In this section we present the general theory of realizing $G/K$ as a bounded domain, in particular, we will see how the disc realization fits into the theory of the Harish-Chandra decomposition. Let $\mathfrak{p}$ be the $-1$ eigenspace in the Cartan decomposition of the Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2$ corresponding to the complexification $G_{\mathbb{C}}$ and let $P_+$ be its corresponding Lie group, which will be a complex subgroup of $G_{\mathbb{C}}$.

By the Harish-Chandra decomposition, we know there exists a bounded open subset $D \subsetneq P_+$ such that the map $G \to D$ given by

$$g \mapsto (P_+ \text{ component of } g)$$

induces a diffeormorphism between $G/K$ and $D$. Let us quickly check that $K$ maps to the identity in $P_+$.

**Lemma 3.1.2.** *For $G_{\mathbb{C}} = SL_2(\mathbb{C})$, the decomposition of an open subset of $G_{\mathbb{C}}$ as $P_+ \times K_{\mathbb{C}} \times P_-$ is given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix},$$

*which is valid whenever $d$ is nonzero.*

A direct result of the lemma shows that $K$, which consists of diagonal elements, has trivial $P_+$ component.

Moreover, we have that the set $D$ is given by

$$D = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{C}) | 1 - \overline{z}z > 0 \right\},$$

which is exactly the unit disc, as we saw earlier. $G$ acts holomorphically on $D$ by

$$g(\omega) = (P_+ \text{ component of } g\omega).$$

This action is in fact the action given by fractional linear transformation that we saw earlier. Indeed, if $\omega = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $g\omega = \begin{pmatrix} a & az+b \\ c & cz+d \end{pmatrix}$. Thus the $P_+$ component of $g\omega$ is given by

$$\begin{pmatrix} 1 & \frac{az+b}{cz+d} \\ 0 & 1 \end{pmatrix},$$

so $G$ acts on $D$ by linear fractional transformations.

**The action of $G$ on $D$ seen as the action of $SL_2(\mathbb{R})$ on the upper half plane**

Let $G' = SL_2(\mathbb{R})$, $u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ and $D' = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} | \operatorname{Im}(z) > 0 \right\}$.

**Lemma 3.1.3.** *The Cayley transform $u$ conjugates $G$ into $G'$, that is $uGu^{-1} = G'$. Also, $uGB = G'uB = D'K_{\mathbb{C}}P_-$. $D'$ can be identified with $G/K$. Moreover $G'$ acts on $D'$ by the usual action of $SL_2(\mathbb{R})$ on the upper half plane.*

*Proof.* The first check is trivial. Since we know that $GB = DK_{\mathbb{C}}P_-$, we have $D'$ to be the $P_+$ part of $uD$. Indeed, if $\omega = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in D$, we have $u\omega = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & z+i \\ i & zi+1 \end{pmatrix}$ with $P_+(u\omega) = \begin{pmatrix} 1 & \frac{z+i}{zi+1} \\ 0 & 1 \end{pmatrix}$. One can easily check that $P_+(u\omega)$ is in the upper half plane.

We know the action of $G$ on $D$ is given by

$$g(\omega) = P_+(g\omega)$$

for $g \in G, \omega \in D$. We have the identificaiton $D \cong D'$ given by

$$\omega \mapsto P_+(u\omega).$$

Then action of $G$ on $D$ induces an action of $G'$ on $D'$ as follows:

$$\begin{array}{ccc} \omega & \longrightarrow & P_+(u\omega) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle ugu^{-1}} \\ P_+(g\omega) & \longrightarrow & P_+(ug\omega). \end{array}$$

As we have seen before, $G$ acts on $D$ by fractional linear transformations. One can easily check that the induced action of $D'$ on $D'$ is the usual action of $SL_2(\mathbb{R})$ on the upper half plane.

□

Note that in this case we have $SL_2(\mathbb{R}) = SU(1,1,J_r)(\mathbb{R})$, where $J_r = uJu^{-1}$. By the notation $SU(1,1,J_r)$ we mean the semisimple connected group $SU(1,1)$ defined over $\mathbb{Q}$ by the hermitian form $J_r$. Notice that $J_r$ is given by the matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, which is equivalent to using the form given by $\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$.

Since the action of $G$ on $D$ is the same as the action of $SL_2(\mathbb{R})$ on the upper half plane, and $D$ can be identified with the upper half plane, one expects the smooth toroidal compactification of the locally symmetric variety $\Gamma\backslash D$ to be given by the actual Baily-Borel compactification of adding points at the cusps. As usual, $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup. Therefore, we would be able to translate Hecke's problem for $SL_2$ in Chapter 2 in the frameset of the group $SU(1,1)$. The next shows that, as the $\mathbb{Q}$-rank of $SU(1,1)$ is one, the boundary components of $D$ are indeed zero dimensional points.

### 3.2 Rational boundary components

In the following, we shall prove the proper rational boundary components of $D$ are zero dimensional via the general theory exhibited in Chapter 3 of [1]. Most of the results and notations used in the proofs below, unless cited otherwise, are from [20]. The main idea is to start with a maximally compact $\theta$-stable Cartan subalgebra and switch to a maximally noncompact $\theta$-stable Cartan subalgebra by using Cayley transforms. This switch establishes the system of restricted roots, which are then used to determine the rational boundary components.

#### Cartan decomposition

The Lie algebra of $G$ is given by $\mathfrak{g} = \mathfrak{su}(1,1)$ comprising of elements $g \in M_2(\mathbb{C})$ such that $(\overline{1+\epsilon g})^t J(1+\epsilon g) = J$ and $\det(1+\epsilon g) = 1$. Thus we have

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ \overline{b} & -a \end{pmatrix} \Big| a, b \in \mathbb{C} \text{ with } a = -\overline{a} \right\}.$$

Considering the negative conjugate transpose as the Cartan involution, we have $\theta : g \mapsto -\overline{g}^t$. We get the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$, $\mathfrak{p}$ are the 1 and $-1$ eigenspaces under $\theta$. Thus

$$\mathfrak{k} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \Big| a \in \mathbb{C}) \text{ with } a = -\overline{a} \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & b \\ \overline{b} & 0 \end{pmatrix} \Big| b \in \mathbb{C} \right\}.$$

#### Root decomposition of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$

Let $\mathfrak{t} \subset \mathfrak{k}$ be a maximal abelian subspace of $\mathfrak{k}$. Then $\mathfrak{h} = Z_{\mathfrak{g}}(\mathfrak{t})$ is a $\theta$ stable Cartan subalgebra of $\mathfrak{g}$ of the form $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}_0$, with $\mathfrak{a}_0 \subset \mathfrak{p}$ by Proposition 6.60, p. 328. We know from Theorem 7.117, p. 439, that $G/K$ is Hermitian if and only if $Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$, where $\mathfrak{c}$ is the center of $\mathfrak{k}$. As a result, $\mathfrak{c} \subset \mathfrak{t}$, so $Z_{\mathfrak{g}}(\mathfrak{t}) \subset Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$. Thus $\mathfrak{a}_0 = 0$, so $\mathfrak{t}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}$.

Let $\Delta$ be the roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Notice that $(\operatorname{ad} X)^* = -\operatorname{ad} \theta X$ by Lemma 6.27, p. 304, so $\operatorname{ad} H$ is skew adjoint for $H \in \mathfrak{k}$ and every root in $\Delta$ is imaginary, hence compact or non-compact.

In our case $\mathfrak{t}$ consists of the diagonal matrices $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, with $\lambda_i \in i\mathbb{R}$ and $\lambda_1 + \lambda_2 = 0$, so $\mathfrak{t} = \mathfrak{l}$. The complexification $\mathfrak{g}_{\mathbb{C}}$ is $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{t}_{\mathbb{C}}$ consists of the diagonal matrices. Defining a member $e_i$ of the dual space $\mathfrak{t}_{\mathbb{C}}^*$ by

$$e_i \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \lambda_i$$

for $i \in \{1, 2\}$, we know the root space $\Delta$ is given by the $(e_i - e_j)$'s for $i \neq j$ with eigenspaces $\mathfrak{g}_{e_i - e_j}$ generated by $E_{ij}$. Here by $\mathfrak{g}_{e_i - e_j}$ we mean $(\mathfrak{g}_{\mathbb{C}})_{e_i - e_j}$.

As we noted above, all the roots are imaginary since $\lambda_i$ is purely imaginary, thus either compact or noncompact. By Lemma 7.127, p. 441, we know a root $\alpha$ is compact if and only if it vanishes on the center $\mathfrak{c}_{\mathbb{C}}$ of $\mathfrak{k}_{\mathbb{C}}$. It is not necessary to figure out compactness of the roots this way since for that it's enough to check whether the corresponding eigenvectors are in $\mathfrak{k}_{\mathbb{C}}$ or not. However, computing the center is needed for establishing a good ordering for $i\mathfrak{t}$, that is a system of positivity in which every noncompact positive root is larger than every compact root. To do this, we can for example use a lexicographic order that is build from a basis of $i\mathfrak{c}$ following a basis of its orthogonal complement in $i\mathfrak{t}$.

In our case, since $\mathfrak{l} = \mathfrak{t}$ is abelian, $\mathfrak{c} = \mathfrak{l}$ is given by matrices $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $\lambda_i \in i\mathbb{R}$ and zero trace. Thus the roots that vanish on the complexification $\mathfrak{c}_{\mathbb{C}}$, and thus there are no compact roots. We fix a good ordering on $i\mathfrak{t}^*$ by considering the spanning vector $H$ for $i\mathfrak{t}$, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We then have $\alpha > 0$ if there exists an index $k$ such that $\alpha(H) > 0$. A quick computation gives us a valid ordering where the positive roots are given by the noncompact one $\Delta_n^+ = \{e_1 - e_2\}$. Now, we can define $\mathfrak{p}_{\mathbb{C}}^+ = \oplus_{\alpha\in\Delta_n^+}\mathfrak{g}_\alpha$ and $\mathfrak{p}_{\mathbb{C}}^- = \oplus_{\alpha\in\Delta_n^+}\mathfrak{g}_{-\alpha}$ and it's easy to see that in our case we have

$$\mathfrak{p}_{\mathbb{C}}^+ = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \mathfrak{p}_{\mathbb{C}}^- = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}.$$

**Cayley transforms given by the noncompact imaginary roots**

In order to switch from the maximally compact Cartan subalgebra $\mathfrak{t}$ to the maximally noncompact one, we need a maximally strongly orthogonal set of noncompact imaginary roots, such that after applying the Cayley transforms we're left with no noncompact imaginary roots. By Harish-Chandra in Lemma 7.143, p. 447, we know a maximal set of strongly orthogonal roots is given by $\gamma_1', \cdots, \gamma_r'$, where $\gamma_i'$ is the smallest element of $\Delta_n^+$ that is strongly orthogonal to $\gamma_1', \cdots \gamma_{i-1}'$. Moreover, we know that if $E_{\gamma_i'}$ is the nonzero root vector for $\gamma_i'$, then the space given by

$$\mathfrak{a} = \oplus_{i=1}^r \mathbb{R}(E_{\gamma_i'} + \overline{E_{\gamma_i'}})$$

is a maximal abelian subspace of $\mathfrak{p}$. Note that since the roots are imaginary, we have $\overline{E_{\gamma_i'}} \in \mathfrak{g}_{-\gamma_i'}$.

In our case, we have a single noncompact positive root $\gamma' = e_1 - e_2$, so we already have a maximal set of strongly orthogonal roots. An easy computation gives us that the corresponding maximal abelian subalgebra of $\mathfrak{p}$ is given by matrices

$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$$

with $\lambda \in \mathbb{R}$. Notice that the real rank of $\mathfrak{a}$ is 1, so the noncompact dimension of a maximal noncompact subalgebra of $\mathfrak{g}$ is 1. Here, we have $E_{\gamma'} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{-\gamma'} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ so $H_{\gamma'} = [E_{\gamma'}, E_{-\gamma'}] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Notice that we can normalize $E_{\gamma'}, H_{\gamma'}$ such that we have the bracket relations

$$[H, E] = 2E, \quad [H, \overline{E}] = -2\overline{E}, \quad [E, \overline{E}] = H$$

which define a copy of $\mathfrak{sl}_2(\mathbb{C})$. Notice that $E + \overline{E}, i(E - \overline{E})$ are fixed by bar and thus are in $\mathfrak{g}$. Moreover, $H$ must be in $i\mathfrak{g}$ and also since $[E, \overline{E}] \subset \mathfrak{t}_{\mathbb{C}}$, we must have $H \in i\mathfrak{t}$. One can check that $E + \overline{E}, i(E - \overline{E}), iH$ generate over $\mathbb{R}$ a copy of $\mathfrak{sl}_2(\mathbb{R})$. Furthermore, we can easily check that $2\frac{<\gamma', \psi>}{<\gamma', \gamma'>} = \psi(H)$ for all $\psi \in \Delta$ , so the notation is consistent with that of [1].

Having a root $\beta$ imaginary noncompact, the Cayley transform defined by it is given by

$$c_\beta = \mathrm{Ad}(\exp \frac{\pi}{4}(\overline{E_\beta} - E_\beta)).$$

Notice that $c_\beta$ sends the Cartan subalgebra $\mathfrak{h}$ to

$$\mathfrak{h}' = \mathfrak{g} \cap c_\beta(\mathfrak{h}_\mathbb{C}) = \ker(\beta|_\mathfrak{h}) \oplus \mathbb{R}(E_\beta + \overline{E}_\beta).$$

Note that the Cayley transform leaves fixed the part of the Cartan subalgebra that is orthogonal to the embedded copy of $\mathfrak{sl}_2(\mathbb{C})$, that is, if $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}_0$, then everything in $\mathfrak{h}_\mathbb{C}$ that is orthogonal to $H_\beta$ remains fixed under $c_\beta$. That's because if $B(H, H_\beta) = 0$, where $B$ is the Killing form, then $\beta(H) = 0$, so $H \in \ker(\beta|_{\mathfrak{h}_\mathbb{C}})$. On the other hand, we also know that

$$\begin{aligned} c_\beta(H_\beta) &= E_\beta + \overline{E_\beta} \\ c_\beta(E_\beta - \overline{E_\beta}) &= E_\beta - \overline{E_\beta} \\ c_\beta(E_\beta + \overline{E_\beta}) &= -H_\beta. \end{aligned}$$

Thus we have $c_\beta(iH_\beta) = i(E_\beta + \overline{E_\beta}) \notin \mathfrak{g}$, so since $iH_\beta \in \mathfrak{t}$, the compact dimension decreases by 1, while since $c_\beta(H_\beta) = E_\beta + \overline{E_\beta} \subset \mathfrak{p} \cap \mathfrak{g}$, the noncompact dimension increases by 1. In our case, applying the Cayley transforms for $\gamma'$, we switch from $\mathfrak{t}$ to a maximally noncompact Cartan subalgebra whose noncompact part is given by $\mathbb{R}(E_{\gamma'} + \overline{E_{\gamma'}}) = \mathfrak{a} \subset \mathfrak{p}$.

On the other hand, if we let $\mathfrak{a}' = \mathbb{R}H_{\gamma'} \in i\mathfrak{t}$, it's clear that $\mathfrak{a}'$ is send to $\mathfrak{a}$ by the composition of Cayley transforms. Moreover, note that within the embedded $\mathfrak{sl}_2(\mathbb{C})$ copy in $\mathfrak{g}_\mathbb{C}$, we have the following correspondences:

$$H_\beta = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad E_\beta = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad \overline{E_\beta} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},$$

so $c_\beta$ corresponds to $\mathrm{Ad}(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix})$ within the embedded copy $SL_2(\mathbb{R})$ in $G$, which is consistent with the theory of [1].

**Restricted root space decomposition of $\mathfrak{g}$**

Given $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace, we consider the decomposition of $\mathfrak{g}$ with respect to the action of $\mathrm{ad}\,\mathfrak{a}$. Since by Lemma 6.27, p. 304, $(\mathrm{ad}\,X)^* = -\mathrm{ad}\,\theta X$, ad is self-adjoint, so the eigenvalues $\lambda \in \mathfrak{a}^*$ will be real. Denoting the corresponding eigenspace by $\mathfrak{g}_\lambda$, $\lambda$ is a restricted root of $\mathfrak{g}$ if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$. We then have the restricted root space decomposition

$$\mathfrak{g} = Z(\mathfrak{a}) \oplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda,$$

where $\Sigma$ is the set of restricted roots.

For our case, let $f \in \mathfrak{a}^*$ take value $\lambda$ on matrices in $\mathfrak{a}$. Then the restricted roots are linear combinations of the $f$'s so the restricted root space is given by $\Sigma = \{\pm 2f\}$.

**Equivalence of the restricted root space decomposition with the theory exhibitied by Ash-Mumford-Rapoport-Tai in [1]**

We have a root space decomposition with respect to $\mathfrak{t}$ as

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus_{\alpha \in \Delta} (\mathfrak{g}_\mathbb{C})_\alpha.$$

We can now clump together the root spaces corresponding to equivalent eigenvalues in $(\mathfrak{a}')^*$, where a linear map $\lambda' : \mathfrak{a}' \to \mathbb{R}$ is given by restricting the roots in $\Delta$ to $\mathfrak{a}'$; call these roots $\Delta'_\mathbb{R}$. So we get

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus_{\substack{\alpha \in \Delta \\ \alpha|_{\mathfrak{a}'} = 0}} (\mathfrak{g}_\mathbb{C})_\alpha \oplus_{\lambda' \in \Delta'_\mathbb{R}} (\mathfrak{g}_\mathbb{C})_{\lambda'},$$

where $\mathfrak{t}_\mathbb{C} \oplus_{\substack{\alpha \in \Delta \\ \alpha|_{\mathfrak{a}'} = 0}} (\mathfrak{g}_\mathbb{C})_\alpha = Z_{\mathfrak{g}_\mathbb{C}}(\mathfrak{a}')$ and $(\mathfrak{g}_\mathbb{C})'_\lambda = \oplus_{\substack{\alpha \in \Delta \\ \alpha|_{\mathfrak{a}'} = \lambda}} (\mathfrak{g}_\mathbb{C})_\alpha$ is the eigenspace in $\mathfrak{g}_\mathbb{C}$ where $\operatorname{ad} \mathfrak{a}'$ is given by the character $\lambda' \in \Delta'_\mathbb{R}$. Now, let $c$ denote the composition of the Cayley transforms corresponding to a maximal set of strongly orthogonal roots. In our case $c$ corresponding to the noncompact imaginary root $\gamma'$.

We know that $c$ sends $\mathfrak{a}'$ to $\mathfrak{a}$, but also $c(\mathfrak{a}) = \mathfrak{a}'$. Thus, the map $c$ induces a set of linear maps $\mathfrak{a} \to \mathbb{R}$ coming from $\Delta'_\mathbb{R}$. Thus, if we denote $c^*(\Delta'_\mathbb{R})$ by $\Delta_\mathbb{R}$, we have

$$\mathfrak{g}_\mathbb{C} = Z_{\mathfrak{g}_\mathbb{C}}(\mathfrak{a}) \oplus_{\lambda \in \Delta_\mathbb{R}} (\mathfrak{g}_\mathbb{C})_\lambda,$$

where $(\mathfrak{g}_\mathbb{C})_\lambda$ is the eigenspace of $\mathfrak{g}_\mathbb{C}$ where the $\operatorname{ad} \mathfrak{a}$ is given by the character $\lambda \in \Delta_\mathbb{R}$. However, $\mathfrak{a} \in \mathfrak{g}$, so we get a decomposition

$$\mathfrak{g} = Z(\mathfrak{a}) \oplus_{\lambda \in \Delta_\mathbb{R}} \mathfrak{g}_\lambda,$$

which is exactly the restricted root decomposition of $\mathfrak{g}$. As a result $\Delta_\mathbb{R}$ coincides with $\sum$.

We make one more remark regarding obtaining the restricted roots from successful applications of the Cayley transforms. Notice that the restricted roots are obtained by projecting all of the roots in $\Delta$ on the linear span of the maximal set of strongly orthogonal noncompact imaginary roots. Thus, the restricted roots are gotten from $\Delta$ by means of an equivalence relation where $\phi \sim \psi$ if and only if $\phi - \psi$ is orthogonal to the maximal set of strongly orthogonal noncompact imaginary roots. But then $< \phi - \psi, \gamma_i' >= 0$ if and only if $(\phi - \psi)H_i = 0$, so $\phi - \psi$ is equivalent to zero if and only if it vanishes on all $H_i$'s, and thus $(\phi - \psi)|_{a'} = 0$. As a result, projecting on the linear space of the maximal strongly orthogonal set is equivalent to the above description of finding the restricted roots. Note that except when we project the roots, we actually first find $\Delta_{\mathbb{R}}'$ as before and then we need to switch to $\Delta_{\mathbb{R}}$ using the Cayley transforms.

For our case, it is trivial that when we restrict the roots $\alpha \in \Delta$ to $\mathfrak{a}'$ we get

$$\Delta_{\mathbb{R}}' = \{\pm\gamma'\}.$$

Here we have $\gamma'$ takes the value $2\lambda$ on elements of $\mathfrak{a}'$, where $\mathfrak{a}'$ consists of matrices of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \in i\mathfrak{t}$$

with $\lambda \in \mathbb{R}$. As a result, the induced set of roots $\Delta_{\mathbb{R}}$ is given by

$$\Delta_{\mathbb{R}} = \{\pm\gamma\},$$

where $\gamma$ takes value $2\lambda$ on elements in $\mathfrak{a}$. Notice that this is exactly the set $\sum$ of restricted roots we found above.

**Description of the rational boundary components**

We know the rational boundary components $F$ of $D = G/K$ are parametrized in terms of the subsets $S \subset \{1, \cdots, r\}$, where $r$ is the rank of $\mathfrak{g}$. They are given by subgroups $G_h(F_S)$ whose Lie algebra is given by

$$\mathfrak{l}_S = \sum_{\substack{\lambda \in \Sigma \\ \lambda = \sum_j a_j \gamma_i, j \notin S}} (\mathfrak{g}_\lambda + [\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}]).$$

We thus have one type of component corresponding to the subset $\{1\}$. Clearly, when $S = \{1\}$, the Lie algebra is trivial, so we get a trivial component.

As noted before, in the case of $\mathbb{Q}$-rank 1, the toroidal compactification of $\Gamma\backslash D$ reduces to the Baily-Borel mimimal compactification of adding a finite number of points to the quotient $\Gamma\backslash D$. Note that this is because we have no higher dimensional cusps as rational boundary components. However, in the case of $SU(2,2)$ the $\mathbb{Q}$-rank is 2, so we shall get two types of rational boundary components, a zero dimensional one and a one dimensional one. As a result, one has to treat the higher dimensional cusps when constructing the smooth toroidal compactification.

*Remark.* We want to make another remark regarding the rational boundary components. Note the the toroidal compactification for the $Sp_4$ case is the same as the Igusa compactification. The boundary components in this case correspond to based lines and based planes in $V = (\mathbb{Z}/p\mathbb{Z})^4$ and are glued via the Tits building for $Sp_4(\mathbb{Z}/p\mathbb{Z})$ as seen in [25], [24], [32]. Similarly, one can try to describe the rational boundary components for the $SU(1,1)$ case from the Tits building perspective as well, following the general theory outlined in [39], for example.

### 3.3 Locally symmetric varieties and the group $G(\mathbb{Z}/p\mathbb{Z})$

Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. In particular, say $\Gamma = \Gamma(p)$ is the principal congruence subgroup of level $p > 3$ defined by the exact sequence

$$1 \to \Gamma(p) \to G(\mathbb{Z}) \to G(\mathbb{Z}/p\mathbb{Z}) \to 1.$$

We have $G(\mathbb{Z}) = SU((1,1), \mathcal{O}_E)$ and $G(\mathbb{Z}/p\mathbb{Z}) = SU((1,1), \mathcal{O}_E/p\mathcal{O}_E)$. Recall that by $SU((1,1), K)$ we mean the special unitary group of signature $(1,1)$ whose matrices have entries in $K$. Note that $G(\mathbb{Z}/p\mathbb{Z})$ acts on the locally symmetric variety $\Gamma(p)\backslash D$ and we have already seen $\Gamma(p)\backslash D$ has a smooth compactification given by the Baily-Borel compactification. Since we are interested in irreducible cuspidal representations of $\widetilde{G}(\mathbb{Z}/p\mathbb{Z})$ that split upon restriction to $G(\mathbb{Z}/p\mathbb{Z})$, let us first analyze the group $G(\mathbb{Z}/p\mathbb{Z})$.

We have three cases: $p$ splits in $E$, $p$ ramified, and $p$ inert. For general references on the splitting on primes in extension fields, see Chapter 1 in [33] and [35].

#### The case of $p$ split

We have $p = \mathfrak{p}\overline{\mathfrak{p}}$, so the inertia degree is 1; that is, the residue field $\mathcal{O}_E/\mathfrak{p}\mathcal{O}_E$ equals $\mathbb{F}_p$, so $\mathcal{O}_E/p\mathcal{O}_E \cong \mathbb{F}_p \times \mathbb{F}_p$. As a result, we have $SU((1,1), \mathcal{O}_E/p\mathcal{O}_E) \cong$

$SU((1,1),\mathbb{F}_p\times\mathbb{F}_p)$.

**Lemma 3.3.1.** *We have* $SU((1,1),\mathbb{F}_p\times\mathbb{F}_p)\cong SL_2(\mathbb{F}_p)$.

*Proof.* We have $SU((1,1),\mathbb{F}_p\times\mathbb{F}_p)=\{g\in SL_2(\mathbb{F}_p\times\mathbb{F}_p)|\overline{g}^tJg=J\}$. If we let $g=(A,B)\in\mathbb{F}_p\times\mathbb{F}_p$,then we get $B=(J^t)^{-1}(A^t)^{-1}J^t$. As a result, the map $g\mapsto A$ gives the desired isomorphism. □

**The case of $p$ inert**

If $p$ inert, we have $\mathcal{O}_E/p\mathcal{O}_E=\mathbb{F}_{p^2}$ as the inertia degree is 2.

**Lemma 3.3.2.** *We have* $SU((1,1),\mathbb{F}_{p^2})\cong SL_2(\mathbb{F}_p)$.

*Proof.* The proof idea is the same as the one in Lemma 3.1.3. As $\mathcal{O}_E/p\mathcal{O}_E\equiv\mathbb{F}_{p^2}$, we have $\mathbb{F}_{p^2}\cong\mathbb{F}_p(\beta)$.

Consider the Cayley transform $u=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$. Then

$$uJu^{-1}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}=J_r,$$

where recall $J$ is the hermitian form of signature $(1,1)$ defining $SU(1,1)$.

For $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{F}_{p^2})$ we have

$$J_rg^{-1}J_r^{-1}=\begin{pmatrix}a & c\\ b & d\end{pmatrix}.$$

So $\overline{g}^tJ_rg=J_r$ iff $g^t=\overline{g}^t$ iff $g=\overline{g}$ iff $g$ has entries in $\mathbb{F}_p$. As a result we have

$$SU((1,1),\mathbb{F}_{p^2},J_r)=SL_2(\mathbb{F}_p),$$

where $SU((1,1),\mathbb{F}_{p^2},J_r)$ is the special unitary subgroup group of signature $(1,1)$ given by $J_r$ over $\mathbb{F}_p$. It is trivial to check that $SU((1,1),\mathbb{F}_{p^2},J_r)=uSU((1,1),\mathbb{F}_{p^2})u^{-1}$. □

**The case of $p$ ramified**

If $p$ ramified, $p=\mathfrak{p}^2$, the inertia degree is 1, ramification index is 2 and $\mathcal{O}_E/p\mathcal{O}_E=\mathcal{O}_E/\mathfrak{p}^2\mathcal{O}_E$. The conjugation acts as the identity on $\mathcal{O}_E/p\mathcal{O}_E$, so $SU((1,1),\mathcal{O}_E/p\mathcal{O}_E)$ is given by $SO(1,1)$ over the finite field of $p$ elements.

To sum up, for odd prime $p > 3$, we have an induced action of the group $SU((1,1), \mathcal{O}_E/p\mathcal{O}_E)$ on the space $\Gamma(p)\backslash D$. In the light of the results of Lemmas 3.3.2 and 3.1.3, when $p$ inert or split, this action can be seen as the usual linear fractional transformations induced action of $SL(2, \mathbb{F}_p)$ on the modular curve with respect to the principal congruent subgroup in $SL_2(\mathbb{Z})$. As a result, we are in the original case of Hecke for $SL_2$. However, we note that $\Gamma(p)\backslash D$ and the modular curve with respect to the principal congruent subgroup of level $p$ in $SL_2(\mathbb{Z})$ are related to Shimura varieties that have different moduli space interpretations. One knows the modular curve of level $p$ is the moduli space of isomorphisms classes of complex elliptic curves and $p$-torsion data. On the other hand, $\Gamma(p)\backslash D$ related to the moduli space of principally polarized abelian surfaces with an $\mathcal{O}_E$-action and additional level $p$ structure.

*Chapter 4*

# PRELIMINARIES ON DELIGNE-LUSZTIG THEORY AND CHARACTER VALUES ON REGULAR UNIPOTENT CLASSES

The goal of this chapter is to provide a short introduction into the necessary background material needed in the subsequent chapters. The first two sections shall focus on general results on finite groups of Lie type and Deligne-Lusztig theory. We try to keep the notation consistent throughout the following chapters, so unless noted otherwise, one should assume the notation introduced below. The last section gives a brief introduction into Gelfand-Graev representations and related notions, with the ultimate goal of stating a formula for character values of irreducible representations on regular unipotent elements. This will be useful later on in proving a formula for multiplicity defect for $SU(2,2)$ in Chapter 7.

## 4.1 Basic notions on algebraic groups and maximal tori

In the following, we give a brief overview of notions and results on algebraic groups and maximal tori. For proofs and more details one can refer to Chapters $0, 1, 2$ and $3$ in [9] or Chapters $1, 2, 3$ and $4$ in [4]. More generally, [3] and [17] provide great introductions into some of the notions presented below.

### Jordan-Chevalley decomposition

We know that in a linear algebraic group $G$, each element $g \in G$ has a unique decomposition $g = g_s g_u = g_u g_s$, with $g_s$ semisimple and $g_u$ unipotent. In a realization of $G$ as a closed subgroup of $GL_n$, the matrix $g_s$ is diagonabilizable, while $g_u$ has all eigenvalues equal to 1. If $G$ is defined over an algebraically closed field of characteristic $p$, then the semisimple elements have order $p'$ coprime to $p$, while the unipotent elements have order a power of $p$.

### Roots, coroots and the Weyl group

Let $G$ be a connected group over an algebraically closed field $K$ and $T$ a maximal torus of $G$. By the rigidity of tori, if $T$ is a torus in an algebraic group $G$, then $N_G(T)^\circ = C_G(T)^\circ$. Since $G$ is a connected algebraic group, the centralizer $C_G(T)$ of any torus is connected, and as a result $N_G(T)^\circ = C_G(T)$. Moreover, as T maximal, $C_G(T) = T$, so the Weyl group of $T$ is given by $W(T) = N_G(T)/T$.

We denote the character group $\mathrm{Hom}(T, \mathbb{G}_m)$ by $X$ and the group 1-parameter subgroups of $T$, $\mathrm{Hom}(\mathbb{G}_m, T)$ by $Y$. The nondegenerate exact pairing $X \times Y \to \mathbb{Z}$ puts $X$ and $Y$ in duality and is given by $(\chi, \nu) \mapsto \langle \chi, \nu \rangle$, where $\langle \chi, \nu \rangle = n$ if $(\chi \circ \nu)(a) = a^n$ for all $a \in \mathbb{G}_m$. The Weyl group acts on both $X$ and $Y$ as follows:

$$^w\chi(t) = \chi(t^w) \quad \text{for } \chi \in X,\, t \in T,$$

$$\nu^w(a) = \nu(a)^w \quad \text{for } \nu \in Y, a \in \mathbb{G}_m.$$

If $B$ is a Borel subgroup of $G$ containing $T$, we have the semidirect product decomposition $B = TU$. There is a unique Borel subgroup $B^- = TU^-$ such that $B \cap B^- = T$. The groups $U, U^-$ are maximal unipotent subgroups of $G$ normalized by $T$. They give rise to minimal proper subgroups of $U, U^-$ normalized by $T$, which are connected unipotent subgroups of dimension 1 and thus isomorphic to the additive group $\mathbb{G}_a$. The action by conjugation of $T$ on each of these subgroups gives a homomorphism $T \to \mathrm{Aut}\, \mathbb{G}_a$ and thus gives rise to an element of $\mathrm{Hom}(T, \mathbb{G}_m) = X$. All the elements of $X$ arising in this way are called the roots and we denote by $\Phi$ the set of roots with respect to $T$. For each root $\alpha \in \Phi$, we denote the 1-dimensional unipotent subgroup associated to it by $U_\alpha$, where $U_\alpha$ is also called a root subgroup. Note that $U = \prod_{\alpha \in \Phi^+} U_\alpha$ and $G = \langle T, U_\alpha, \alpha \in \Phi \rangle$.

For each root $\alpha \in \Phi$, let $\alpha^\vee$ be its corresponding coroot. $\alpha^\vee$ is an element of $Y$ satisfying $\langle \alpha, \alpha^\vee \rangle = 2$. We denote by $\Phi^\vee \subseteq Y$ the group of coroots. For each root $\alpha$, there exists an element $w_\alpha \in \langle U_\alpha, U_{-\alpha} \rangle$ that belongs to the Weyl group. $w_\alpha$ acts on $X$ by

$$w_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha \quad \chi \in X,$$

while $w_{\alpha^\vee}$ acts on $Y$ by

$$w_{\alpha^\vee}(\nu) = \nu - \langle \alpha, \nu \rangle \alpha^\vee \quad \nu \in Y,$$

with $w_\alpha(\Phi) = \Phi$ and $w_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$. We have $w_\alpha = w_{-\alpha}$ and $w_\alpha^2 = 1$ and by abuse of notation we can denote $w_{\alpha^\vee}$ by $w_\alpha$. Also, we know the Weyl group

$W$ is generated by $w_\alpha$ for $\alpha \in \Phi$. With the properties above, the quadruple $(X, \Phi, Y, \Phi^\vee)$ is called a root datum.

While $\Phi$ is the set of roots, we denote by $\Phi^+$, $\Phi^-$ the sets of positive, respectively negative roots. The set of simple roots generating $\Phi^+$ will be denoted by $\Pi$.

**Finite groups of Lie type and $G^F$-classes of maximal tori**

Let $G$ be a linear algebraic group over an algebraically closed field $K = \overline{\mathbb{F}}_p$ of characteristic $p$. A homomorphism is called a standard Frobenius map if it is given by the raising to the $q^{\text{th}}$ power map $F_q : (a_{ij}) \mapsto (a_{ij}^q)$, where we consider $G$ as embedded in $GL_n(K)$. Here $q$ is a power of $p$. A homomorphism $F : G \to G$ is called a Frobenius map if some power of $F$ is a standard Frobenius map. The finite groups $G^F$ arising as the fixed points of a Frobenius map $F : G \to G$ on a connected reductive group $G$ are called the finite groups of Lie type.

A maximal torus of $G^F$ is defined to be a subgroup of the form $T^F$, where $T$ is an $F$-stable maximal torus of $G$. Note that although every maximal torus of $G$ lies in a Borel subgroup of $G$, it need not be true that every $F$-stable maximal torus of $G$ lies in an $F$-stable Borel. As a result, it can happen that there is a maximal torus of $G^F$ that does not lie in a Borel subgroup of $G^F$. We call an $F$-stable maximal torus that lies in an $F$-stable Borel subgroup of $G$ maximally split. Terminology wise, rational means $F$-stable. Moreover, the Frobenius map defines an $\mathbb{F}_q$-structure on $G$, which is equivalent to saying $G$ is defined over $\mathbb{F}_q$ with $G(\mathbb{F}_q) = G^F$.

Let $B$ be an $F$-stable Borel subgroup of $G$ and $T$ and $F$-stable maximal torus of $G$ contained in $B$. The action of $F$ on the character and cocharacter groups of the maximally split torus $T$ is given by

$$(F(\chi))(t) = \chi(F(t)) \quad \text{for } \chi \in X,$$

$$(F(\nu))(a) = F(\nu(a)) \quad \text{for } \nu \in Y, a \in K^\times.$$

Let us make some remarks regarding maximal tori in connected reductive groups. We know that any two such maximal tori in $G$ are conjugate. Also, we know that there exists an $F$-stable Borel subgroup and thus there exists $F$-stable maximal tori of $G$. In fact, there exists $F$-stable maximal tori of $G$

which lie in $F$-stable Borel subgroups of $G$. These are the maximally split tori. Moreover, any two maximally split $F$-stable maximal tori of $G$ are conjugate by an element of $G^F$. In general, not every $F$-stable maximal torus will be maximally split, so we need to know how the set of $F$-stable maximal tori of $G$ falls into conjugacy classes under the action of $G^F$.

Let $T_0$ be a maximally split $F$-stable torus. Let $W$ be the Weyl group of $G$ with respect to the maximal torus $T_0$, that is $W = W(T_0) = N_G(T_0)/T_0$. The map $\pi : N_G(T_0) \to W$ is the natural projection. Moreover, we say that $w, w'$ are $F$-conjugate if there exists $x \in W$ such that $w' = xwF(x)^{-1}$. Note that this is the same notion of $F$-conjugacy given by the usual action of a group $M$ on itself defined for any $m_0 \in M$ by $m \mapsto m_0 m F(m_0)^{-1}$. We have the following result based on a Lang-Steinberg theorem presented later:

**Proposition 4.1.1.** *The map ${}^gT_0 \to \pi(g^{-1}F(g))$ determines a bijection between the $G^F$-classes of $F$-stable maximal tori of $G$ and the $F$-conjugacy classes of $W$.*

If $T$ is an $F$-stable maximal torus of $G$ for which the corresponding $F$-conjugacy class of $W$ contains $w$, we say that $T$ is obtained from the maximally split torus $T_0$ by twisting with $w$. That is, if $T$ is such an $F$-stable maximal torus, then $T =^g T_0$, where $\pi(g^{-1}F(g)) = w$. Then $T$ is of type $w$ with respect to $T_0$. By conjugation by $g^{-1}$, the torus $T$ with the action of $F$ can be identified with the torus $T_0$ endowed with the action of $wF$. Note that we can define $G^F$-conjugacy and type with respect to a rational maximal torus that does not have to be maximally split.

### More results on tori

Let $G$ be a connected reductive group over an algebraically closed field $K$ of characteristic $p$, that is $K = \overline{\mathbb{F}}_p$, and let $F$ be a Frobenius map on $G$.

We first present a result about the structure of the group $K^\times$. Let $\Omega$ be the group of alll complex roots of unity and $\Omega_{p'}$ a subgroup of $\Omega$ given by

$$\Omega_{p'} = \{z \in \Omega | z^n = 1 \text{ for some } n \text{ not divisible by } p\}.$$

Let $\mathbb{Q}_{p'}$ be the additive group of rational numbers $r/s$, where $r, s \in \mathbb{Z}$ and $s$ is not divisible by $p$. We then have the following:

**Proposition 4.1.2.** *The following groups are isomorphic:*

1. $K^\times$
2. $\Omega_{p'}$
3. $\mathbb{Q}_{p'}/\mathbb{Z}$.

Let $T$ be a torus over $K$ with $X, Y$ its character and cocharacter groups. The following propositions will be useful in proving results later on:

**Proposition 4.1.3.** *1.* $Y \otimes K^\times \cong \mathrm{Hom}(X, K^\times)$ *as abelian groups.*

*2.* $X \otimes K^\times \cong \mathrm{Hom}(Y, K^\times)$.

*3.* $\mathrm{Hom}(X, K^\times) \cong T$ *as abelian groups.*

*4.* $Y \otimes K^\times \cong T$.

We choose and isomorphism between $\mathbb{Q}_{p'}/\mathbb{Z}$ and $K^\times$. The following isomorphisms depend on this choice and thus are not canonical:

**Proposition 4.1.4.** $T^F$ *is isomorphic to* $Y/(F-1)Y$.

Let us consider $(T^F)^\wedge = \mathrm{Hom}(T^F, \mathbb{C}^\times)$. As $T^F$ is a finite group of order prime to $p$, we have $(T^F)^\wedge = \mathrm{Hom}(T^F, \Omega_{p'})$ and thus $(T^F)^\wedge \cong \mathrm{Hom}(T^F, \mathbb{Q}_{p'}/\mathbb{Z})$. We then have:

**Proposition 4.1.5.** $(T^F)^\wedge$ *is isomorphic to* $X/(F-1)X$.

**Duality of connected reductive groups**

Let $G$ be a connected reductive group with maximal torus $T$ and associated root datum $(X, \Phi, Y, \Phi^\vee)$. By introducing the concept of isomorphic root data, we note that since all maximal tori in $G$ are conjugate, the group $G$ determines a root datum uniquely up to isomorphism. Moreover, two connected reductive groups are isomorphic if and only if their root data is isomorphic.

On the other hand, one can introduce the notion of duality. Two connected reductive groups $G$, $G^*$ are said to be dual if their root data are dual. One can prove that each connected reductive group $G$ has a dual group $G^*$ that is unique up to isomorphism. We are interested in duality of connected reductive groups with a Frobenius map. The pairs $(G, F)$ and $(G^*, F^*)$ are in duality if the conditions in the following propositions hold for tori $T, T^*$ maximally split in $G, G^*$ respectively:

**Proposition 4.1.6.** *Let $T$ be an $F$-stable maximal torus of $G$ and $X, Y$ be its character and cocharacter groups. Let $T^*$ be an $F^*$-stable maximal torus of $G^*$ and $X^*, Y^*$ be its character and cocharacter groups. Then the following two conditions are equivalent:*

1. *There exists an isomorphism $\delta : X \to Y^*$ such that:*
   a) $\delta(\Phi) = (\Phi^*)^\vee$.
   b) $\langle \chi, \alpha^\vee \rangle = \langle \alpha^*, \delta(\chi) \rangle$ *for all* $\chi \in X, \alpha \in \Phi$ *where* $\delta(\alpha) = (\alpha^*)^\vee$.
   c) $\delta(F(\chi)) = F^*(\delta(\chi))$ *for all* $\chi \in X$.

2. *There exists an isomorphism $\epsilon : Y \to X^*$ such that:*
   a) $\epsilon(\Phi^\vee) = \Phi^*$.
   b) $\langle \alpha, \nu \rangle = \langle \epsilon(\nu), \epsilon(\alpha^\vee)^\vee \rangle$ *for all* $\nu \in Y, \alpha \in \Phi$.
   c) $\epsilon(F(\nu)) = F^*(\epsilon(\nu))$ *for all* $\nu \in Y$.

Note that if $(G, F)$ and $(G^*, F^*)$ are in duality, then for any $F$-stable maximal torus $T$, there is an $F^*$-stable maximal torus $T^*$ of $G^*$ such that the conditions of the above proposition hold. Also, we note that there is an isomorphism $\delta : W \to W^*$ given by the map mapping $w_\alpha$ to $w_{\delta(\alpha)}$.

Now suppose the pairs $(G, F)$ and $(G^*, F^*)$ are in duality. We have

**Proposition 4.1.7.** *Let $T$ be an $F$-stable maximal torus of $G$ and $T^*$ an $F^*$-stable maximal torus of $G^*$ such that $T, T^*$ are in duality in the sense of Proposition 4.1.6. Then the duality map $\delta : X \to Y^*$ gives rise to an isomorphism between $T^{*F^*}$ and the character group $(T^F)^\wedge$.*

Let us make a small remark on the idea of the proof of this proposition. The result follows from the following isomorphisms:

$$(T^F)^\wedge \cong X/(F-1)X \cong Y^*/(F^*-1)Y^* \cong T^{*F^*},$$

where the middle map is induced by $\delta : X \to Y^*$.

## 4.2 Deligne-Lusztig representations

The goal of this section is to briefly define the Deligne-Lusztig representations and provide a short introduction into the subject, enough to outline the tools necessarily to carry on the subsequent computations of proving the non-vanishing of certain alternating sums of characters. We closely follow the presentation of Chapter 7 in [4]. For more details on Deligne-Lustig theory refer to the original work of Deligne and Lusztig in [7] or Chapters $4, 6, 11, 12$ and 13 in [9].

### The generalized characters $R_T^G(\theta)$

One can define the generalized Deligne-Lusztig characters $R_T^G(\theta)$ for each $F$-stable maximal torus $T$ of $G$ and each character $\theta \in (T^F)^\wedge$. In order to define $R_T^G(\theta)$, we shall assume knowledge of $l$-adic cohomology groups of an algebraic variety over an algebraically closed field of characteristic $p$, where $l$ is a prime different from $p$.

We first introduce a generalization of a theorem of Lang that was proven by Steinberg:

**Proposition 4.2.1** (Lang-Steinberg)**.** *If $G$ is a connected group over an algebraically closed field $K$ of characteristic $p$ and if $F$ is any surjective homomorphism $F : G \to G$ such that $G^F$ is finite, then the map $\mathcal{L} : G \to G$ given by $\mathcal{L}(g) = g^{-1}F(g)$ is surjective.*

In particular, the above result holds if $F$ is a Frobenius map.

Let $G$ be a connected reductive group with $T$ be an $F$-stable maximal torus. Let $B$ be a Borel containing $T$, which may not be $F$-stable. We then have $B = TU$. Then $\widetilde{X} = \mathcal{L}^{-1}(U)$ is an affine algebraic variety over $K = \overline{\mathbb{F}}_p$. The $i^{\text{th}}$ $l$-adic cohomology group of $\widetilde{X}$ with compact support $H_c^i(\widetilde{X}, \overline{\mathbb{Q}}_l)$ is a left $G^F$-module and a right $T^F$-module such that

$$(gv)t = g(vt)g \in G^F, t \in T^F, v \in H_c^i(\widetilde{X}, \overline{\mathbb{Q}}_l).$$

We can now define the Deligne-Lusztig generalized characters $R_T^G(\theta)$, where $(T^F)^\wedge$ is an irreducible character of $T^F$. As $\overline{\mathbb{Q}}_l$ contains the field of algebraic numbers and the values of $\theta$ as an irreducible character of $T^F$ will all be algebraic integers, we have

$$(T^F)^\wedge = \mathrm{Hom}(T^F, \mathbb{C}^\times) = \mathrm{Hom}(T^F, \overline{\mathbb{Q}}_l^\times).$$

Let $H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_l)_\theta$ be the $T^F$-submodule of elements on which $T^F$ acts by the character $\theta$. We claim $H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_l)_\theta$ is a left $G^F$-module and a right $T^F$-module.

**Definition 4.2.2.** For all $g \in G^F$, define $R^G_T(\theta) : G^F \to \overline{\mathbb{Q}}_l$ by

$$R^G_T(\theta)(g) = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(g, H^i_c(\widetilde{X}, \overline{\mathbb{Q}}_l)_\theta).$$

Note that since the values $R^G_T(\theta)(g)$ are again algebraic integers, $R^G_T(\theta)$ will take values in $\mathbb{C}$, so it can be viewed as a generalized complex character of $G^F$. Moreover, one can show $R^G_T(\theta)$ is independent of the choice of $B$.

*Remark.* Let us make some remarks regarding the construction of $R^G_T(\theta)$. In the case when $T$ is maximally split and thus contained in an $F$-stable Borel $B$, the generalized character $R^G_T(\theta)$ will be exactly the induction from $B^F$ to $G^F$ of a character lifting $\theta$ from $T^F$ to $B^F$. The characters $R^G_T(\theta)$ were introduced precisely to give an analogue of this induction from $B^F$ to $G^F$ when $T$ is not maximally split.

Indeed, one can define the Harish-Chandra induction $R^G_L$ for $P$ a rational parabolic and $L$ a rational Levi subgroup of $P$ where $P = LU$, as the functor $R^G_L : E \to \mathbb{C}[G^F/U^F] \otimes_{\mathbb{C}[L^F]} E$ from the category of left $\mathbb{C}[L^F]$-modules to left $\mathbb{C}[G^F]$-modules. Note $\mathbb{C}[G^F/U^F]$ is a left $G^F$-module and a right $L^F$-module. The adjoint functor ${}^*R^G_L$ is the Harish-Chandra restriction. As a remark, as in the case of the Deligne-Lusztig construction above, the parabolic subgroup $P$ does not appear in the notation since one can prove $R^G_L$, ${}^*R^G_L$ do not depend on the parabolic subgroup used in their construction. When $L = T$ is a maximal torus, $R^G_T(\theta)$, for $\theta$ an irreducible character of $T^F$, is exactly the induction from $B^F$ to $G^F$, mentioned above, of a character of $B^F$ induced by $\theta$.

Note that the Harish-Chandra induction gives us a functor $R^G_L$ when $L$ is a rational Levi of a rational parabolic $P$ of $G$. The Deligne-Lusztig induction generalizes the Harish-Chandra induction to the case of non-rational parabolic $P$ being defined as the generalized induction functor associated to the $G^F$-module-$L^F$ afforded by $H^*_c(\widetilde{X}, \overline{\mathbb{Q}}_l)$. When $P$ is rational, $P = LU$, $U$ is also rational, so one can prove $H^*_c(\widetilde{X}, \overline{\mathbb{Q}}_l) \cong \overline{\mathbb{Q}}_l[G^F/U^F]$ as $G^F$-modules-$L^F$, so the Deligne-Lusztig induction is the Harish-Chandra induction in this case. Now, as seen above, when $L = T$ is a rational maximal torus, $R^G_T(\theta)$, for $\theta$ an irreducible of $T^F$, is called a Deligne-Lusztig character.

**Some remarks on cuspidal representations and the duality functor**

We have already introduced the Harish-Chandra induction as the functor $R_L^G$, where $L$ is a rational Levi of a rational parabolic subgroup of $G$, and we saw the Deligne-Lusztig induction generalizes it to non-rational parabolics. In the following we give a brief explanation of cuspidal representations as detailed in Chapter 6 of [9]. Furthermore, we introduce the notion of the duality functor.

First, let us define a partial order on the set of pairs $(L, \delta)$, where $L$ is a rational Levi subgroup of a rational parabolic subgroup of $G$ and $\delta$ is an irreducible representation of $L^F$, by letting $(L', \delta') \leq (L, \delta)$ if $L' \subset L$ and $\langle \delta, R_{L'}^L(\delta') \rangle_{L^F} \neq 0$.

**Definition 4.2.3.** The representation $\delta$ of $L^F$ is said to be cuspidal if the following equivalent properties hold:

1. The pair $(L, \delta)$ is minimal for the partial order we defined above.
2. For any rational Levi subgroup $L'$ of a rational parabolic subgroup of $L$, we have ${}^*R_{L'}^L(\delta) = 0$

Let us make some remarks on cuspidal representations. The set of irreducible components of $R_L^G(\delta)$ is called the Harish-Chandra series associated to $(L, \delta)$. When $L$ fixed and $\delta$ runs through the set of cuspidal representations of $L^F$, this series is called the Harish-Chandra series associated to $L$. If $L$ is a rational maximal torus, all irreducible representations of $L^F$ must be cuspidal; also, all such rational maximal tori are conjugate in $G^F$. As a result, the series associated to any rational maximal torus is called the principal series. On the other hand the set of cuspidal representation of $G^F$ is called the discrete series associated to $G$.

For the rest of this subsection we will define the duality functor. We first need to introduce some notions of rank. The rank of a connected reductive group $G$ is the dimension of its maximum torus. We also need to introduce the notion of relative rank, or $\mathbb{F}_q$-rank, where $G$ is connected reductive over an algebraically closed field $K$ of characteristic $p$. Note we are in the case of $G$ connected reductive with an $\mathbb{F}_q$-structure given by the Frobenius map, and thus $G$ is defined over $\mathbb{F}_q$.

Recall from Subsection 4.1 that a maximally split torus is a rational maximal torus that is contained in some rational Borel subgroup; note that another terminology for a maximally split torus is that of quasi-split.

**Definition 4.2.4.**

1. The $\mathbb{F}_q$-rank of a torus defined over $\mathbb{F}_q$, is the rank of its maximum split subtorus.
2. We call the $\mathbb{F}_q$-rank of an algebraic group $G$, defined over $\mathbb{F}_q$, the $\mathbb{F}_q$-rank of a maximally split torus of $G$.

**Definition 4.2.5.** We define

$$\epsilon_G = (-1)^{\mathbb{F}_q\text{-}\operatorname{rank}(G)}.$$

The following property will be useful later on:

**Lemma 4.2.6.** *With the above notations, we have*

$$\mathbb{F}_q - \operatorname{rank}(G) = \mathbb{F}_q - \operatorname{rank}(G/R(G)) + \mathbb{F}_q - \operatorname{rank}(R(G)),$$

*where $R(G)$ is the radical of the reductive group $G$, $G/R(G)$ is its semisimple part.*

**Definition 4.2.7.** We call the semisimple $\mathbb{F}_q$-rank of $G$, denoted by $r(G)$, the $\mathbb{F}_q$-rank of $G/R(G)$.

We shall now introduce the duality operator on the characters of $G^F$, which will appear later on in the formula computing the character of an irreducible representation of $G^F$ at regular unipotent elements:

**Definition 4.2.8.** Let $B$ be a rational Borel subgroup of a connected reductive group $G$ defined over $\mathbb{F}_p$. By duality we mean the operator $D_G$ defined on the class functions of $G^F$ given by

$$D_G = \sum_{P \supseteq B} (-1)^{r(P)} R_L^G \circ^* R_L^G,$$

where the summation is taken over the rational parabolics of $G$ that contain $B$ and where $L$ is a rational Levi subgroup of $P$.

**More on Deligne-Lusztig characters**

We have seen so far that $R_T^G(\theta)$ for $T$ rational maximal torus and $\theta$ an irreducible character of $T^F$, is a generalized character of $G^F$. We shall first give a formula for the scalar product $\langle R_T^G(\theta), R_{T'}^G(\theta')\rangle_{G^F}$ which will tell us, up to a sign, when $R_T^G(\theta)$ is an irreducible character of $G^F$.

If $T, T'$ are two $F$-stable maximal tori, we define $N(T,T') = \{g \in G | {}^gT = T'\}$ and $W(T,T') = \{Tg | g \in N(T,T')\}$. We then have

$$\langle R_T^G(\theta), R_{T'}^G(\theta')\rangle_{G^F} = |\{\omega \in W(T,T')^F | {}^{\omega}\theta' = \theta\}|. \tag{4.1}$$

*Remark.* We note that $R_T^G(\theta)$ are parametrized by $G^F$-conjugacy classes of pairs $(T,\theta)$ as $R_T^G(\theta) = R_{T'}^G(\theta')$ when ${}^g(T,\theta) = (T',\theta')$ for some $g \in G^F$. Moreover, one can prove that if $T, T'$ are $F$-stable maximal tori of $G$ that are not $G^F$-conjugate, then $\langle R_T^G(\theta), R_{T'}^G(\theta')\rangle_{G^F} = 0$. Thus, $R_T^G, R_{T'}^G$ are orthogonal, but they may have some common irreducible component. In fact, in order for $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ to have no irreducible component in common, we need $(T,\theta)$ and $(T,\theta)$ to not be geometrically conjugate.

Thus, we saw that $R_T^G(\theta)$ are parametrized by $G^F$-conjugacy classes of pairs $(T,\theta)$. One can give another parametrization using the notion of the dual group presented in Subsection 4.1, as follows:

**Proposition 4.2.9.** *[Proposition* 13.13 *in [9]] The $G^F$-conjugacy classes of pairs $(T,\theta)$, where $T$ is a rational maximal torus of $G$ and $\theta$ is an irreducible representation of $T^F$ are in one-to-one correspondence with the ${G^*}^{F^*}$-conjugacy classes of pairs $(T^*, s)$ where $s$ is a semisimple element of ${G^*}^{F^*}$ and $T^*$ is a rational maximal torus containing $s$.*

In the light of the above proposition, we shall sometimes use the notation $R_{T^*}^G(s)$ for $R_T^G(\theta)$.

Next, let us see when the generalized characters $R_T^G(\theta)$ are irreducible and cuspidal.

**Definition 4.2.10.** $\theta \in (T^F)^\wedge$ is said to be in general position if no non-identity element in $W(T)^F = (N(T)/T)^F$ fixes $\theta$.

As a result, the following holds:

**Proposition 4.2.11.** *If $\theta$ is in general position, then $\pm R_T^G(\theta)$ is an irreducible character of $G^F$. Moreover, one can prove $\epsilon_G \epsilon_T R_T^G(\theta)$ is in fact irreducible.*

Lastly, recall the definition of cuspidal representations above. The following proposition describes when the irreducible representation $\epsilon_G \epsilon_T R_T^G(\theta)$ is cuspidal:

**Proposition 4.2.12.** *Let $T$ be an $F$-stable maximal torus of $G$ and let $\theta$ be an irreducible character of $T^F$ in general position. Then the irreducible character $\epsilon_G \epsilon_T R_T^G(\theta)$ of $G^F$ is cuspidal if and only if $T$ lies in no proper $F$-stable parabolic subgroup of $G$.*

## 4.3 Character values on regular unipotent classes

The aim of this section is to give a brief introduction into the results needed to state a theorem that gives the value of irreducible representations $\chi$ of $G^F$ on the regular unipotent classes. The presentation revolves around introducing the Gelfand-Graev representations and related notions, and roughly follows the material in Chapter 14 in [9] or Chapters 1, 2 and 3 in [8].

### The Galois cohomology group $H^1(F, Z)$

The goal of this subsection is to give some explicit realizations of the first Galois cohomology group $H^1(F, Z)$, which plays a paramount role in computing the characters of irreducible representations of $G^F$ on regular unipotent elements.

Using the surjectivity of the Lang map in Proposition 4.2.1, we have $(F-1)Z = Z$ and thus the Galois cohomology group $H^1(F, Z) = Z/(F-1)Z$ is trivial when the center $Z$ is connected. One can think of $H^1(F, Z)$ as the collection of orbits in $Z$ for the action of $Z$ on $Z$ induced by $F$, that is the $F$-conjugacy classes of $Z$.

**Lemma 4.3.1.** *Let $T$ be an $F$-stable maximal torus which contains the center $Z$. Then*

$$H^1(F, Z) \cong \mathcal{L}_T^{-1}(Z)/ZT^F,$$

*where $\mathcal{L}_T$ is the Lang map $\mathcal{L}_T : T \to T$.*

Note that in our case $G$ is a reductive group and we know that if $T$ is a maximal torus of $G$, then $C_G(T) = T$, so $Z \subseteq T$.

**Lemma 4.3.2.** *With the above notations, we have a canonical isomorphism*

$$H^1(F, Z) \cong Z/\mathcal{L}_Z(Z).$$

**Regular unipotent elements**

We start with defining the regular unipotent elements. For more information on the regular unipotent elements, Chapter 5, Section 5.1 in [4] offers a more detailed description.

**Definition 4.3.3.** An element $g$ of an algebraic group $G$ is said to be regular if the dimension of its centralizer is minimal.

Moreover, it was proved by Steinberg that if $G$ is connected and reductive, then $\dim C_G(g) \geq \mathrm{rank}(G)$, where the rank of $G$ is defined to be the dimension of the maximal tori of $G$. As a result, the regular elements are those such that $\dim C_G(g) = \mathrm{rank}(G)$. It is also well-known that every connected reductive group $G$ contains regular unipotent elements and any two are conjugate in $G$. Recall $U = \prod_{\alpha\in\Phi^+} U_\alpha$, so every unipotent element $u \in U$ will be of the form $\prod_{\alpha\in\Phi^+} x_\alpha(a_\alpha)$.

**Proposition 4.3.4.** *Let $G$ be a connected reductive group and $u \in G$ be unipotent. Then the following conditions on $u$ are equivalent:*

1. *$u$ is regular.*
2. *$u$ lies in a unique Borel subgroup of $G$.*
3. *$u$ is conjugate to an element of the form $\prod_{\alpha\in\Phi^+} x_\alpha(a_\alpha)$ with $a_\alpha \neq 0$ for all $\alpha \in \Pi$.*

We can now consider the regular unipotent elements $G^F$ given by the Frobenius map $F : G \to G$. The finite group $G^F$ contains regular unipotent elements. As in the case of semisimple conjugacy classes, we are interested in the case when the unique regular unipotent class of $G$ splits into several $G^F$-conjugacy classes of rational regular unipotent elements upon restriction to $G^F$.

The following result on the classes of regular unipotent elements in $G^F$ involves the notion of good characteristic for $G$. A prime is said to be bad for a simple group $G$ if $p$ divides the coefficient of some root $\alpha$ when expressed as a combination of simple roots $\alpha = \sum_i n_i \alpha_i$. It is well known that there is no bad characteristic for root systems of type $A_n$ and thus, since in our applications $G = SL_4$, we will always be in good characteristic.

It is well known that if $Z(G)$ is connected and the characteristic of $K$ is a good prime for $G$, then any two regular unipotent elements in $G^F$ are conjugate in $G^F$. However, if the center is not connected this is not the case, which is the crux of being able to prove that the alternating sums introduced in (1.2) do not vanish on the regular unipotent elements. The following result follows from Theorem 5.2.1:

**Proposition 4.3.5.** *If the characteristic is good for $G$, the $G^F$-conjugacy classes of rational regular unipotent elements are parametrizes by the $F$-conjugacy classes of $Z(G)/Z(G)^\circ$.*

As a result, once we choose a rational class of regular unipotent elements, the above parametrization is complete. We shall fix such an element $u_0 \in U^F$ which will be explicitly given later on.

**Regular characters**

Let $B$ be a fixed $F$-stable Borel subgroup of $G$ that contains a rational maximal torus $T$, and let $U$ be the unipotent radical of $B$ such that $B = TU$. Let $\Phi, \Phi^+$ and $\Pi$ be the root system of $G$ with respect to $T$, respectively the positive roots and the simple roots corresponding to the order on $\Phi$ determined by $B$.

The action of $F$ on $X(T)$ induces a permutation $\tau$ on the simple roots. Indeed, as $B$ is $F$-stable, $U$ will be $F$-stable, so the $U_\alpha$'s for $\alpha \in \Phi^+$ will be permuted by $F$. There will therefore be a permutation $\tau$ on the positive roots such that $F(U_\alpha) = U_{\tau(\alpha)}$. Moreover, the set of simple roots $\Pi$ is stable under $\tau$, so $\tau$ gives a permutation on $\Pi$. More specifically, $\tau$ is given as the permutation on $\Pi$ induced by the action of the element $a \mapsto a^q$ of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on $X(T)$ as follows:

$$(\tau\alpha)(F(t)) = \alpha(t)^q. \tag{4.2}$$

As $\mathbb{G}_a \cong U_\alpha$ through $a \mapsto x_\alpha(a)$, the following diagram commutes:

$$\begin{array}{ccc} U_\alpha & \xrightarrow{F} & U_{\tau\alpha} \\ \simeq\uparrow & & \uparrow\simeq \\ \mathbb{G}_a & \xrightarrow[a\mapsto a^q]{} & \mathbb{G}_a. \end{array}$$

Thus, the isomorphism $U_{\tau\alpha} \cong \mathbb{G}_a$ on the right gives us $F(x_\alpha(a)) = x_{\tau\alpha}(a^q)$, which is the same action as the one described in the above equation. Indeed, $T$ acts on $U_\alpha$ as $tx_\alpha(a)t^{-1} = x_\alpha(\alpha(t)a)$ for all $t \in T$, so the action of $T$ on $U_{\tau\alpha}$ will be given by $F(t)F(x_\alpha(a))F(t^{-1}) = F(x_\alpha(\alpha(t)a))$, so $F(t)x_{\tau\alpha}(a^q)F(t)^{-1} = x_{\tau\alpha}(\alpha(t)^q a^q)$ and thus $(\tau\alpha)(F(t)) = \alpha(t)^q$.

We have $U = \prod_{\alpha\in\Phi^+} U_\alpha$ and let $U^* = \prod_{\alpha\in\Phi^+\setminus\Pi} U_\alpha$. Then $U^*$ is the derived group of $U$ and it is connected. Moreover we have $U/U^*$ abelian and $U/U^* \cong \prod_{\alpha\in\Pi} U_\alpha$. For any orbit $I \in \Pi/\tau$ of $\tau$ in $\Pi$, denote by $U_I$ the image of $\prod_{\alpha\in I} U_\alpha$ in $U/U^*$. Using $U_\alpha \cong \mathbb{G}_a$, it can be shown that the group of rational points $U_I^F$ is isomorphic to the additive group $\mathbb{F}_{q^{|I|}}^+$, where $|I|$ is the size of the orbit $I$. Explicitly, the isomorphism $x_I : \mathbb{F}_{q^{|I|}}^+ \to U_I^F$ is given by

$$x_I(a) = x_{\alpha_I}(a)x_{\tau\alpha_I}(a^q)\cdots x_{\tau^{|I|-1}\alpha_I}(a^{q^{|I|-1}})$$

for $a \in \mathbb{F}_{q^{|I|}}^+$ and $\alpha_I$ is a representative of the orbit $I \in \Pi/\tau$. Moreover, the following result holds:

$$U^F/U^{*F} \cong (U/U^*)^F \cong \prod_{I\in\Pi/\tau} U_I^F,$$

where $\Pi/\tau$ is the set of orbits of $\tau$ on $\Pi$.

**Definition 4.3.6.** A linear irreducible character $\psi$ of $U^F$ is regular if $\psi$ is trivial on $U^{*F}$ and its restriction to $U_I^F$ is not trivial for any orbit $I$ of $\tau$ in $\Pi$.

In the following, we will state some results on the regular characters of $U^F$ which will be useful later.

**Theorem 4.3.7** (Theorem 2.4 in [8]). *There is a natural regular permutation action of $H^1(F, Z)$ on the set of $T^F$-orbits of regular characters of $U^F$.*

As a result, since $H^1(F, Z)$ is canonically isomorphic to $\mathcal{L}_T^{-1}(Z)/ZT^F$ as we have seen in Lemma 4.3.1, it follows that $\mathcal{L}_T^{-1}(Z)/Z$ acts regularly on the regular characters of $U^F$.

Thus, the regular characters of $U^F$ are parametrized by $\mathcal{L}_T^{-1}/Z$. As in the case of regular unipotent elements, this parametrization is well-defined as soon as we choose a regular character. Let us fix such a character $\psi_0$. In the following, we'll make this choice of $\psi_0$ explicit.

First, let $N$ be a fixed integer which is a multiple of $|I|$ for any orbit $I$. Choose $\chi_0$ to be an additive character of $\mathbb{F}_{q^N}$ such that the restriction of $\chi_0$ to $\mathbb{F}_q$ is non-trivial.

Then any linear character $\psi_I$ of $U_I^F$ is given by

$$\psi_I(x_I(a)) = \chi_0(\lambda_I a)$$

for $a \in \mathbb{F}_{q^{|I|}}$ and some $\lambda_I \in \mathbb{F}_{q^{|I|}}$. Moreover, any linear character $\psi$ of $(U/U^*)^F$ is of the form $\psi = \prod_{I \in \Pi/\tau} \psi_I$, where the $\psi_I$ as defined as above. In particular, there is an isomorphism $((U/U^*)^F)^\wedge \to \prod_{I \in \Pi/\tau} \mathbb{F}_{q^{|I|}}$ defined by the map

$$\psi \mapsto (\lambda_{I_1}, \lambda_{I_2}, \cdots, \lambda_{I_r}),$$

where $r = |\Pi/\tau|$. Notice $\psi$ is regular iff none of the $\psi_I$'s are trivial, which happens precisely when no $\lambda_I$ is zero.

Let $\Psi$ be the set of regular characters of $U^F$. Since $\psi_0$ is regular, it is trivial on $U^{*F}$, so it is a character of $U^F/U^{*F}$. Given the choice of $\chi_0$ above, we can set $\psi_0 \in \Psi$ to be the distinguished regular character that corresponds to taking $\lambda_I = 1$ for all $I \in \Pi/\tau$. As a result, $\psi_0(\prod_{I \in \Pi/\tau} x_I(a_I)) = \prod_{I \in \Pi/\tau} \psi_{0,I}(x_I(a_I)) = \prod_{I \in \Pi/\tau} \chi_0(a_I)$ and thus

$$\psi_0\Big(\prod_{I \in \Pi/\tau} x_I(a_I)\Big) = \chi_0\Big(\sum_{I \in \Pi/\tau} a_I\Big). \tag{4.3}$$

Note that $\sum_{I \in \Pi/\tau} a_I \in \mathbb{F}_{q^N}$ as $a_I \in \mathbb{F}_{q^{|I|}}$.

Lastly, as $T$ acts on $U$, it also acts on $U/U^* \cong \prod_{\alpha \in \Pi} U_\alpha$ by

$$t\Big(\prod_{\alpha \in \Pi} x_\alpha(a_\alpha)\Big)t^{-1} = \prod_{\alpha \in \Pi} x_\alpha(\alpha(t)a_\alpha), \tag{4.4}$$

where $t \in T, a_\alpha \in \overline{\mathbb{F}}_q$.

On the other hand, one can prove $\mathcal{L}_T^{-1}(Z)$ acts on the set of characters of $U^F$ ( and $U^F/U^{*F}$) as follows:

$$ {}^t\psi(u) = \psi(t^{-1}ut), \tag{4.5} $$

where $t \in \mathcal{L}_T^{-1}(Z), u \in U^F$ and $\psi$ is a character of $U^F$. Using (4.4), one can prove that regular characters are taken to regular characters by this action, which is used in proving the result of Theorem 4.3.7.

**Gelfand-Graev representations**

Theorem 4.3.7 gives us an explicit parametrization of the $T^F$-orbits on the set of regular characters $\Psi$ of $U^F$. That is, for each $z \in H^1(F, Z) = \mathcal{L}_T^{-1}(Z)/ZT^F$, let $t_z \in \mathcal{L}_T^{-1}(Z)$ be a representative for $z$ and define

$$ \psi_z =^{t_z} \psi_0. $$

We then define $\Psi_z$ to be the $T^F$-orbit of $\psi_z$ in $\Psi$.

**Definition 4.3.8.** For $z$ in $H^1(F, Z)$, define the Gelfand-Graev representation $\Gamma_z$ by $\Gamma_z = \mathrm{Ind}_{U^F}^{G^F}(\psi_z)$.

Note that if the center is connected, there exists only one unique Gelfand-Graev representation.

It is a well known result of Steinberg that the Gelfand-Graev representations are multiplicity free. We devote the rest of this section to study the irreducible components of Gelfand-Graev representations, which will be of use when trying to see what irreducible representations of $G^F$ do not vanish on regular unipotent elements.

We shall now define a class function $\chi_{(s)}$ of $G^F$ associated to a semisimple element $s \in G^{*F^*}$ and a rational maximal torus $T^*$ containing $s$. Here $(G, F)$ and $(G^*, F^*)$ are in duality.

First, it is well known the connected component of the centralizer $C_{G^*}(s)^\circ$ is reductive, with Weyl group $W^{*\circ}(s)$ generated by reflexions $w_{\alpha^*}$ with $\alpha^* \in \Phi^*$ such that $\alpha^*(s) = 1$. It is a normal subgroup of the Weyl group of $C_{G^*}(s)$, which is $W^*(s) = \{w \in W^*(T^*) | s^w = s\}$.

Recall the notation $R_{T^*}^G(s)$ introduced after Proposition 4.2.9. The class function $\chi_{(s)}$ is defined as follows:

**Definition 4.3.9.** If $s$ is a semisimple element of $G^{*F^*}$ and if $T^*$ is a rational maximal torus containing $s$, we define a class function $\chi_{(s)}$ on $G^F$ by

$$\chi_{(s)} = |W^{*\circ}(s)|^{-1} \sum_{w \in W^{*\circ}(s)} \epsilon_G \epsilon_{T^*_w} R^G_{T^*_w}(s),$$

where $T^*_w$ is a torus of $G^*$ of type $w$ with respect to $T^*$.

Note that the above definition does not depend on the choice of the maximal torus $T^*$; by choosing another maximal torus containing $s$, we only make a translation on the types inside $W^{*\circ}(s)$. Also, $\chi_{(s)}$ is constant on the rational semisimple conjugacy class of $s$ in $G^{*F^*}$, that is, we can consider $\chi_{(s)}$ as being in fact defined for the conjugacy class of $(s)$ in $G^{*F^*}$.

Notice $\chi_{(s)}$ contains irreducible components of $R^G_{T^*}(s)$, where $(s)$ is a semisimple class of $G^{*F^*}$. We define the rational series of characters, denoted by $\mathcal{E}(G^F, (s)_{G^{*F^*}})$, as the set of irreducible components of $R^G_{T^*}(s)$, where $(s)_{G^{*F^*}}$ is the semisimple class of $s \in G^{*F^*}$. Thus the irreducible components of $\chi_{(s)}$ are in the rational series. Note that the rational series are a subset of the geometric series $\mathcal{E}(G^F, (s))$. When the centre of $G$ is connected, the rational series and the geometric series are the same. Moreover, the rational series of characters form a partition of the irreducible characters of $G^F$.

One can prove $\chi_{(s)}$ is in fact a proper character for any $s \in {G^*}^{F^*}$. Moreover, the characters $\chi_{(s)}$ give the decomposition of the Gelfand-Graev representation in general.

**Proposition 4.3.10.** *For any $z \in H^1(F, Z)$, we have $\langle \chi_{(s)}, \Gamma_z \rangle_{G^F} = 1$.*

**Proposition 4.3.11.** *For any $z \in H^1(F, Z)$ and any rational semisimple conjugacy class $(s)$ of $G^{*F^*}$, there is exactly one irreducible common component of $\chi_{(s)}$ and $\Gamma_z$. Let us call the common component $\chi_{s,z}$. We then have*

$$\Gamma_z = \sum_{(s)} \chi_{s,z},$$

*where the sum runs over the semisimple classes of $G^{*F^*}$.*

**Proposition 4.3.12.** *We have*

$$|Z^F|/|Z^{\circ F}| \sum_{(s)} \left| (W^*(s)/W^{*\circ}(s))^{F^*} \right|^{-1} \chi_{(s)} = \sum_{z \in H^1(F,Z)} \Gamma_z,$$

*where the sum on the left-hand side runs over the rational semisimple conjugacy classes* $(s)$ *of* $G^{*F^*}$.

There are some important remarks to be made here. Notice that each rational semisimple conjugacy class $(s)$ of $G^{*F^*}$ has only one irreducible component that is in $\Gamma_z$, for all $z \in H^1(F, Z(G))$. More than that, every irreducible component of $\chi_{(s)}$ must be in some $\Gamma_z$. However, it is possible that one irreducible component of $\chi_{(s)}$ is in both $\Gamma_z$ and $\Gamma_{z'}$ for $z \neq z'$, that is $\chi_{s,z} = \chi_{s,z'}$. For example, when $s = 1$ one can check that $\chi_{(1)} = St_G$, and thus an irreducible character. As a result, $\chi_{1,z} = \chi_{(1)}$ for all $z \in H^1(F, Z)$.

**Definition 4.3.13.** An irreducible character of $G^F$ is regular if it is a component of some Gelfand-Graev character.

As a result, the set of regular characters is given by

$$\{\chi_{s,z} | (s) \text{ is a semisimple conjugacy class of } G^{*F^*}, z \in H^1(F, Z)\}. \tag{4.6}$$

We shall see later there is another set of characters of $G^F$ that we are interested in, that is, the ones that take non-zero values on regular unipotent classes. In good characteristic, these characters will be precisely those whose dual is, up to a sign, a regular character, as will be clear by the main result of this section on the values of an irreducible character $\chi$ of $G^F$ on regular unipotent elements.

**A formula for character values on regular unipotent elements**

Using the Gelfand-Graev representations, we shall state a result that gives the value of an irreducible character of $G^F$ on regular unipotent elements.

We first start with explicitly defining the fixed regular unipotent element $u_0$ of $U^F$. Specifically, let us define $u_0$ as the element of $U$ having its projection to $U/U^* \cong \prod_{\alpha \in \Pi} U_\alpha$ be given by

$$\overline{u}_0 = \prod_{\alpha \in \Pi} x_\alpha(1). \tag{4.7}$$

Note that as $u_0 \in U^F$, $\overline{u}_0$ must belong to $U^F/U^{*F}$, which is trivial to see.

For each $z \in H^1(F, Z)$, let $t \in \mathcal{L}_T^{-1}$ be a representative for $z$. We then define the set $U_z = \{x \in G^F | x \sim_{G^F} {}^t(u_0 U^{*F})\}$. Note this definition does not depend on the representative $t$. One can prove that the sets $U_z$ form a partition on the regular unipotent elements of $G^F$ as $z$ goes through $H^1(F, Z)$. Moreover, if the characteristic is good for $G$, then the $U_z$ are precisely the regular unipotent classes in $G^F$.

In order to state the main result, we need to introduce the following:

**Definition 4.3.14.** Let $z \in H^1(F, Z)$. We define the complex number $\sigma_z$ by

$$\sigma_z = \sum_{\psi \in \Psi_{z^{-1}}} \psi(u_0).$$

**Theorem 4.3.15.** *[Theorem 3.5 in [8], Theorem 14.35 in [9] ] Let $\chi$ be the character of an irreducible representation of $G^F$ and $z \in H^1(F, Z)$. Then*

$$|U_z|^{-1} \sum_{u \in U_z} \chi(u) = \sum_{z' \in H^1(F,Z)} \sigma_{zz'^{-1}} \langle (-1)^{|\Pi/\tau|} D_G(\chi), \Gamma_{z'} \rangle_{G^F},$$

*where $D_G(\chi)$ is the dual representation.*

Note that if the characteristic is good for $G$, the set $U_z$ is one conjugacy class, so we have

$$\chi(u) = \sum_{z' \in H^1(F,Z)} \sigma_{zz'^{-1}} \langle (-1)^{|\Pi/\tau|} D_G(\chi), \Gamma_{z'} \rangle_{G^F}$$

for any $u \in U_z$. There are thus two pieces in figuring out the value of a certain irreducible character of $G^F$ on regular unipotent elements. First is figuring out the effect of $D_G$ on irreducibles and the inner product of $D_G(\chi)$ with a Gelfand-Graev representation. Secondly, one has to evaluate $\sigma_z$. We shall carry out both of these tasks in the following chapters, focusing on the case of interest where $\chi$ is a cuspidal representation.

*Chapter 5*

# STUDY CASE $SU(2,2)$, PART I: ON FINDING DESIRED CUSPIDAL IRREDUCIBLES AND PROVING SEMISIMPLE CLASSES HAVE ZERO CONTRIBUTION TO $\Delta M_\pi$

The aim of this chapter is to introduce the case of the finite group of Lie type defined by $SU(2,2)$ over a finite field, with the goal of treating the first two questions $(Q1)$,$(Q_2)$ posed in the introduction of the thesis. Specifically, we first find regular cuspidal irreducible representations of $U(2,2)$ that split upon restriction to $SU(2,2)$ in Theorem 5.1.7. Secondly, Theorem 5.2.4 proves the semisimple conjugacy classes have zero contribution to the alternating sum of multiplicities $\Delta M_\pi$ defined in (1.1).

## 5.1 On regular cuspidal irreducible representations

Let $E$ be an imaginary quadratic field with discriminant $D > 0$, ring of integers $\mathcal{O}_E$ and Galois automorphism given by complex conjugation. That is, $E = \mathbb{Q}(\beta)$, where $\beta = \sqrt{-D}$ and $\overline{\beta} = -\beta$. Let $V = E^4$ be the 4-dimensional vector space over $E$ with standard basis, and $L \subset V$ the standard $\mathcal{O}_E$-lattice in $V$. Choose $J : V \times V \to E$ to be a nondegenerate hermitian form on $V$ with $J(au, bv) = \overline{a}b\overline{J(v,u)}$, which has signature $(2,2)$ on $V_\mathbb{R} = V \otimes_E \mathbb{R}$ and whose matrix in the basis for $V$ is given by

$$J = \begin{pmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & \beta & 0 \\ 0 & -\beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \end{pmatrix},$$

where by abuse of notation we define the matrix of $J$ by $J$ as well. Note that $J$ is $\mathcal{O}_E$-valued on $L$. As $J(u,v) = \overline{u}^t Jv$ for all $u, v \in V$, the hermitian form $J$ is given by $-\beta\overline{z}_4 z_1 - \beta\overline{z}_3 z_2 + \beta\overline{z}_2 z_3 + \beta\overline{z}_1 z_4$.

Let $G' = SU(2,2)$ be the special unitary group of signature $(2,2)$ defined by $J$, viewed as a semisimple connected algebraic group over $\mathbb{Q}$. Then for any $\mathbb{Q}$-algebra $A$,

$$G'(A) = \{g \in SL(V \otimes_\mathbb{Q} A) | \overline{g}^t J g = J\}.$$

Moreover, the group $G'(A)$ can be thought of the group of matrices that preserve the hermtitian form, that is $g \in SL(V \otimes_{\mathbb{Q}} A)$ such that $J(gu, gv) = J(u, v)$ for all $u, v \in V \otimes_{\mathbb{Q}} A$. Similarly, for any $\mathbb{Z}$-algebra $A'$, define

$$G'(A') = \{g \in SL(L \otimes_{\mathbb{Z}} A') | \overline{g}^t J g = J\}.$$

Thus, $G'(\mathbb{Z}) = SL_4(\mathcal{O}_E) \cap G'(\mathbb{Q})$ is the group of matrices in $G'(\mathbb{Q})$ that preserve the lattice $L$.

Now, let $D$ be the hermitian symmetric domain associated to $G' = SU(2,2)$ and let $\Gamma \in G'(\mathbb{Q})$ be an arithmetic subgroup. In particular, let $\Gamma = \Gamma(p)$ be the principal congruence subgroup of level $p$ odd prime, defined by the exact sequence

$$1 \to \Gamma(p) \to G'(\mathbb{Z}) \to G'(\mathbb{Z}/p\mathbb{Z}) \to 1,$$

where $G'(\mathbb{Z}/p\mathbb{Z}) = SU((2,2,), \mathcal{O}_E/p\mathcal{O}_E)$ has entries in $\mathcal{O}_E/p\mathcal{O}_E$. Note that one can think of $\Gamma(p)$ as $\ker\big(G'(\mathbb{Z}) \twoheadrightarrow SL_4(\mathcal{O}_E/p\mathcal{O}_E)\big)$. As in the case of $SU(1,1)$ in Chapter 3, if $p$ is inert in $E$, $G'(\mathbb{Z}/p\mathbb{Z}) = SU((2,2), \mathbb{F}_{p^2})$, the special unitary group of signature $(2,2)$ over $\mathbb{F}_p$. When $p$ ramified, the conjugation acts as the identity on $\mathcal{O}_E/p\mathcal{O}_E$, so $G'(\mathbb{Z}/p\mathbb{Z})$ is $SO(2,2)$ over the finite field $\mathbb{F}_p$. The case of $p$ split reduces to studying irreducible cuspidal representations of $GL_4(\mathbb{F}_p)$ that split when restricted to $SL_4(\mathbb{F}_p)$. As a great deal is known about the explicit character table of $SL_n(\mathbb{F}_p)$ and given the special isomorphism $\mathfrak{so}(2,2) \cong \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$, we shall focus on the case when $G'(\mathbb{Z}/p\mathbb{Z})$ is the special unitary group over $\mathbb{F}_p$. However, note that both $SL_4$ and $SU(2,2)$ over the finite field of $p$ elements are rational forms of $SL_4(\overline{\mathbb{F}}_p)$ given by different Frobenius maps. As a result, one can carry the same derivations we do for the unitary group in the case of $SL_4$ as well.

The finite group of Lie type $SU(2,2)$ can be realized as the fixed points under a Frobenius map $F: G \to G$ on a connected reductive group $G$ defined over the algebraic closure $\overline{\mathbb{F}}_p$.

Let $G = SL_4(\overline{\mathbb{F}}_p)$ and $F: G \to G$ be given by

$$(a_{ij}) \mapsto J^{-1}((a_{ij}^p)^t)^{-1} J,$$

where $J$ is the nondegenerate hermitian form defining $U(2,2)$. It is easy to check that this homomorphism is a Frobenius map since $F^2$ is given by $(a_{ij}) \mapsto J^{-1}(((J^{-1}((a_{ij}^p)^t)^{-1}J)^p)^t)^{-1}J = J^{-1}((J^{-1}((a_{ij}^{p^2})^t)^{-1}J)^t)^{-1}J$, that is

$(a_{ij}) \mapsto J^{-1}(J(a_{ij}^{p^2})^{-1}J^{-1})^{-1}J = a_{ij}^{p^2}$. Thus $F$ is a power of the standard Frobenius map.

Then the finite subgroup

$$G^F = \{g \in G | F(g) = g\}$$

is given by matrices $g \in G$ such that $g = J^{-1}(\overline{g}^t)^{-1}J$, so $\overline{g}^t J g = J$. Moreover, $F^2(g) = g$, so the entries of $g$ are in $\mathbb{F}_{p^2}$ and thus $G^F$ is $SU(2,2)$ over the finite field with $p$ elements.

**Embedding of $G$ into a reductive group with connected center $\widetilde{G}$**

Recall we are interested in irreducible cuspidal representations of $U(2,2)$ that split when restricted to $SU(2,2)$. Note that $U(2,2)$ is a connected reductive group with connected center that contains $SU(2,2)$, while $SU(2,2)$ is a rational form of $G = SL_4(\overline{\mathbb{F}}_p)$.

The group $G$ does not have connected center, but it can be embedded via a Deligne-Lusztig construction in a reductive group $\widetilde{G}$ with connected centre and compatible $F$-structure (see Section 5 in [7]). As a result, one can study the characters of $G^F$ via the characters of $\widetilde{G}^F$, about which a little more is known as the centre of $\widetilde{G}$ is connected. Note that the character theory in the non-connected case is more arithmetical than in the case of connected center; as a result, we shall see in Chapter 7 that the values of the alternating sums on the regular unipotent elements are given in terms of Gauss sums over finite fields, involving the arithmetic in the field $\mathbb{F}_p$.

Let us first construct $\widetilde{G}$. In general, let $G$ is a connected semisimple group defined over an algebraically closed field $K = \overline{\mathbb{F}}_q$ of positive characteristic and let $Z = Z(G)$ be its center. Let $Z \to \widetilde{Z}$ be an embedding of $Z$ into a torus that is defined over $\mathbb{F}_q$. Then $\widetilde{G}$ is the pushout of the diagram

$$\begin{array}{ccc} G & \dashrightarrow & \widetilde{G} \\ \uparrow & & \uparrow \\ Z & \longrightarrow & \widetilde{Z}, \end{array}$$

so $\widetilde{G} = G \times \widetilde{Z}/\{(z, z^{-1}) | z \in Z\}$. As $\widetilde{Z}$ is $F$-stable, we have that the maps $G \to \widetilde{G}$, $Z \to \widetilde{Z}$ are $F$-equivariant and $F$ is extended to $\widetilde{G}$ in the obvious way.

Note that $\widetilde{Z} = Z \cap \widetilde{G}$ is the center of $\widetilde{G}$ and any $F$-stable maximal torus $T$ of $G$ is contained in an $F$-stable maximal torus $\widetilde{T}$ of $\widetilde{G}$ such that $T = G \cap \widetilde{T}$.

In the case of $G = SL_4$, $Z \cong \mu_4$, where $\mu_4$ are the $4^{\text{th}}$ roots of unity in $\overline{\mathbb{F}}_p$. Let us take $\widetilde{Z} \cong \overline{\mathbb{F}}_p^\times$ to be given by a scalars times the identity, where clearly $\mu_4 \subset \overline{\mathbb{F}}_p^\times$. Then we have $\widetilde{G} = SL_4 \times_Z \overline{\mathbb{F}}_p^\times = GL_4(\overline{\mathbb{F}}_p)$, so $\widetilde{G}^F \cong U(2,2)$.

**Splitting of characters of $\widetilde{G}^F$ upon restriction to $G^F$**

We are concerned with the following: given an irreducible representation of $\tilde{G}^F = U(2,2)$, how does it split when restricted to $G^F = SU(2,2)$. The results below are consequences of Frobenius reciprocity and Clifford theory and can be found in Section 2 of [26], for example.

Let us introduce Clifford's theorem for completion, see Chapter 14 in [10]. Note it is stated in the language of modules.

**Theorem 5.1.1** (Clifford's Theorem)**.** *Let $k$ be any field, $V$ an irreducible $kG$-module, $N$ a normal subgroup of $G$. For $g \in G$, $W$ a $kN$-submodule of $V$, we denote $gW = \{gw | w \in W\} \subseteq V$.*

1. *If $0 \neq W$ is a $kN$-submodule of $V$, then $V = \sum_{g \in G} gW$. If $W$ is irreducible, then so is every $gW$, proving that $V_N$ is a completely reducible $kN$-module.*

2. *Let $W_1, \cdots W_m$ be representatives of the isomorphism classes of irreducible $kN$ -submodules of $V$, and denote by $V_i$ the sum of all $kN$-submodules of $V$ which are isomorphic to $W_i$. Then $V = V_1 \oplus \cdots \oplus V_m$. (The $V_i$ are the homogeneous components of $V$).*

3. *If $g \in G$, then each $gV_i$ is some $V_j$. $G$ is a transitive permutation group on $\{V_1, \cdots, V_m\}$.*

4. *If $H_1 = \{g \in G | gV_1 = V_1\}$, then $V_1$ is irreducible as a $kH_1$-module and $V \cong V_1^G = kG \otimes_{kH_1} V_1$.*

5. *For some integer $e$, $V_N \cong e(W_1 \oplus \cdots \oplus W_m)$ (that is, the direct sum of $e$ copies of $W_1 \oplus \cdots \oplus W_m$).*

6. *Let $V$ afford the character $\theta$ of $G$, $W_i$ the character $\chi_i$ of $N$. Then*

$$\theta_N = e(\chi_1 + \cdots + \chi_m),$$

*where* $W_i \cong g_i W_1$*, some* $g_i \in G$ *and* $\chi_i = \chi_1^{g_i}$*;* $\chi_1^{g_i}$ *is defined by* $\chi_1^{g_i}(x) = \chi_1(x^{g_i})$*, for all* $x \in N$*.*

In terms of notations, we have to mention that by $V_N$ we mean the restriction of $V$ to $N$, that is, we view $V$ as a $kN$-module. Also, $V_1^G = kG \otimes_{kH_1} V_1$ is the induction from $H_1$ to $G$, while by $x^{g_i}$ we mean $g_i^{-1} x g_i$. In our case $G = U(2,2)$ over $\mathbb{F}_q$ and $N = SU(2,2)$. Thus, every irreducible representation $V$, when restricted to $SU(2,2)$, either stays irreducible, or decomposes into a sum of irreducible representations of $SU(2,2)$ that each appear with the same multiplicity. Moreover, $U(2,2)$ acts transitively on the set of these irreducibles, that is, they are all conjugate under $U(2,2)$.

Let $\pi$ be an irreducible representation of $\widetilde{G}^F$. We have two results that we shall use:

**Proposition 5.1.2.** *If the center* $Z$ *of* $G$ *is cyclic, then* $\pi_{|_{G^F}}$ *is multiplicity free for any irreducible character* $\pi$ *of* $\widetilde{G}^F$*.*

**Proposition 5.1.3.** *The group* $A(\pi) = \{\alpha \in (\widetilde{G}^F/G^F)^\wedge | \alpha\pi = \pi\}$ *is isomorphic to a subgroup of* $Z/(F-1)Z$ *and the number of irreducible components of* $\pi_{|_{G^F}}$ *divides* $d = |Z/(F-1)Z|$*. Moreover, if* $\pi_{|_{G^F}}$ *is multiplicity free, then* $A(\pi)$ *acts regularly on the irreducible components of* $\pi_{|_{G^F}}$*.*

In our case, note that $H^1(F, Z) = Z/(F-1)Z \cong \mu_4/\mu_4^{p+1} \cong \mu_d$ where $d = (4, p+1)$. Thus if the restriction $\pi$ of $\widetilde{G}^F$ is not irreducible, it must split into either 2 or 4 irreducibles when $p \equiv 3 \bmod 4$ and 2 irreducibles when $p \equiv 1 \bmod 4$.

*Remark.* We remark that for regular characters $\pi$, that is, irreducible components of the Gelfand-Graev representation of $\widetilde{G}^F$, the restriction $\pi_{|_{G^F}}$ is multiplicity free. In this case we do not require $Z$ to be cyclic and as we shall see later these are the types of characters we work with.

**Regular cuspidal irreducible representations of $\widetilde{G}^F$**

Let $\Gamma$ be the Gelfand-Graev character of $\widetilde{G}^F$. We have $\pi_{|_{G^F}} = \pi_1 + \cdots + \pi_{d_0}$, where $\pi_i$ are irreducible representations of $G^F$, $i \in \{1, \cdots, d_0\}$. Recall that we wish to compute the alternating sum on the components $\pi_i$, when $\pi$ is cuspidal. Following the direct generalization of the case of $SL_2$, we wish this alternating sum to be nonzero on the regular unipotent classes, so we require

$\pi$ to be regular as well. The reason is that if $\langle \pi, \Gamma \rangle_{\widetilde{G}^F} = 0$, then $\pi_i(u) = 0$ for all regular unipotent classes $u \in G^F$, as follows from Theorems $4.1, 4.2$, p. 86 in [26]. As a result, we can restrict ourselves to the case where $\pi$ is a regular cuspidal representation.

The following theorem characterizes the regular cuspidal irreducible representations of $G^F$ when $G$ has connected center:

**Theorem 5.1.4** (Theorem 3.7 in [26])**.** *Suppose the center of $G$ is connected. If $\pi$ is cuspidal, irreducible and a component of $\Gamma$, the Gelfand-Graev representation of $G^F$, then $\pi$ is of the form $\pi = \pm R_T^G(\theta)$, where $T$ is an $F$-stable maximal torus of $G$ and $\theta$ is a nonsingular character of $T^F$.*

This theorem clearly applies to $\widetilde{G}^F$. As a result, $\pi$ is an irreducible Deligne-Lusztig character $\pi = \epsilon_{\widetilde{G}} \epsilon_{\widetilde{T}} R_{\widetilde{T}}^{\widetilde{G}}(\widetilde{\theta})$. It is well known that for a connected reductive group $\widetilde{G}$ over $K$, the group $\widetilde{G}^F$ has a cuspidal complex representation given by a Deligne-Lusztig character, for case base proof on classical simple adjoint groups, see [22]. More specifically, $\widetilde{\theta}$ must be in general position and the $F$-stable maximal torus $\widetilde{T}$ must lie in no proper $F$-stable parabolic subgroup of $\widetilde{G}$. We have the following results that give us such cuspidal irreducible representations:

**Lemma 5.1.5.** *Let $\widetilde{B}$ the subgroup of upper triangular matrices in $\widetilde{G}$. Then $\widetilde{B}$ is $F$-stable and it contains the maximally split torus $F$-stable $\widetilde{T}$ of diagonal matrices. Let $w$ be given by the $(1, 2, 3, 4) \mapsto (4, 3, 2, 1)$ symmetry in $W(\widetilde{T}) \cong S_4$ and let $\widetilde{T}_w$ be a torus of type $w$ with respect to $\widetilde{T}$. Then $\widetilde{T}_w$ is an $F$-stable maximal torus not contained in any proper $F$-stable parabolic subgroup of $\widetilde{G}$.*

*Proof.* It is easy to see that if $t = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix}$ with $t_i \in \overline{\mathbb{F}}_p$, then we have $F(t) = \begin{pmatrix} t_4^{-p} & 0 & 0 & 0 \\ 0 & t_3^{-p} & 0 & 0 \\ 0 & 0 & t_2^{-p} & 0 \\ 0 & 0 & 0 & t_1^{-p} \end{pmatrix}$. As a result, $\widetilde{T}^F$ consists of matrices of the form $\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-p} & 0 \\ 0 & 0 & 0 & t_1^{-p} \end{pmatrix}$ with $t_1, t_2 \in \mathbb{F}_{p^2}$.

On the other hand, $\widetilde{T}_w^F \cong \widetilde{T}^{wF} = \left\{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix} \middle| t_i \in (\mathbb{F}_{p^2})^1 \right\}$, where $(\mathbb{F}_{p^2})^1$ consists of elements in $\mathbb{F}_{p^2}$ of norm 1. Note that $\widetilde{T}_w^F \cong U(1) \times U(1) \times U(1) \times U(1)$. We claim $\widetilde{T}_w$ is $\mathbb{F}_p$-anisotropic. In order to see that, we need to prove $X(\widetilde{T}_w)_{\mathbb{F}_p} = 0$, that is, there are no nontrivial $\mathbb{F}_p$-homomorphisms $\widetilde{T}_w \to \mathbb{G}_m$. This follows as $\widetilde{T}_w^F$ is $\mathbb{F}_p$-anisotropic in $\widetilde{G}^F$.

But as $\widetilde{T}_w$ is $\mathbb{F}_p$-anisotropic, using the dynamic description of parabolic subgroups (see Section 6 in [5]), it is clear that $\widetilde{T}_w$ is not contained in any proper $F$-stable parabolic of $\widetilde{G}$. Indeed, if $\widetilde{T}_w \subset P$ for $P$ a proper $\mathbb{F}_p$-parabolic subgroup of $\widetilde{G}$, then $\widetilde{T}_w$ must have a non-trivial one parameter $\mathbb{F}_p$-subgroup and thus $\widetilde{T}_w$ cannot be $\mathbb{F}_p$-anisotropic. This is a contradiction, thus $\widetilde{T}_w$ is an $F$-stable maximal torus not contained in any proper $F$-stable parabolic subgroup of $\widetilde{G}$.

□

**Lemma 5.1.6.** *Assume the notations above. A character $\widetilde{\theta}$ of $\widetilde{T}_w^F \cong U(1) \times U(1) \times U(1) \times U(1)$ is of the form $\widetilde{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$, where $\theta_i$ is an irreducible character of $U(1)$. $\widetilde{\theta} \in (\widetilde{T}_w^F)^\wedge$ is in general position precisely when all of $\theta_i$ are distinct.*

*Proof.* The first part of the lemma is an elementary fact. For the second part, we know $\widetilde{\theta}$ is in general position if no non-identity element in $W(\widetilde{T}_w)^F$ fixes $\widetilde{\theta}$. We claim that the elements of $W(\widetilde{T}_w)^F$ act by permutation on $\widetilde{\theta}$, that is $(\theta_1, \theta_2, \theta_3, \theta_4)^{w_0} = (\theta_{\tau(1)}, \theta_{\tau(2)}, \theta_{\tau(3)}, \theta_{\tau(4)})$ for $w_0 \in W(\widetilde{T}_w)^F$ and $\tau$ the permutation in $S_4$ given by $w_0$.

As $\widetilde{G} = GL_4$, we know $W(\widetilde{T})$ is given by the group of symmetries $S_4$ generated by $w_1, w_2, w_3$ corresponding to the set of simple roots given by $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ with $\alpha_i(t) = t_i t_{i+1}^{-1}$ for a typical element $t \in \widetilde{T}$. $\widetilde{T}_w$ is obtained from $\widetilde{T}$ by twisting with $w$, that is $\widetilde{T}_w =^g \widetilde{T}$ for some $g \in \widetilde{G}$ such that $\pi(g^{-1}F(g)) = w$.

As a result, it is easy to see that the Weyl group of ${}^g\widetilde{T}$ is the conjugate under $g$ of $W(\widetilde{T})$. Moreover, by conjugation with $g^{-1}$, the Weyl group $W({}^g\widetilde{T})$ with an action of $F$ can be identified with the Weyl group $W(\widetilde{T})$ with the action of $wF$. So $W(\widetilde{T}_w)^F \cong W(\widetilde{T})^{wF}$. It is now easy to check that all generators $w_i$ are fixed under $wF$, so $W(\widetilde{T})^{wF} \cong S_4$. As a result the claim is proved.

The statement of the lemma follows easily: if any two characters $\theta_i$, $\theta_j$ for $i \neq j$ are equal, we'll have $\theta^{w_{ij}} = \theta$ for $w_{ij} \in W(\widetilde{T}_w)^F$ given by the transposition $(ij)$ in $S_4$. The converse is trivial.

□

**Theorem 5.1.7.** *Let $\widetilde{T}_w$ be the maximal $F$-stable torus defined above, $\widetilde{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$ a character of $\widetilde{T}_w^F$ in general position. The character $\epsilon_{\widetilde{G}}\epsilon_{\widetilde{T}_w} R_{\widetilde{T}_w}^{\widetilde{G}}(\widetilde{\theta})$ is an irreducible cuspidal regular representation of $\widetilde{G}^F$.*

*Assuming $\pi = \epsilon_{\widetilde{G}}\epsilon_{\widetilde{T}_w} R_{\widetilde{T}_w}^{\widetilde{G}}(\widetilde{\theta})$ does not stay irreducible when restricted to $G^F$, its splitting behaviour is as follows:*

1. *$\pi_{|_{G^F}} = \pi_1 + \pi_2$ when $\widetilde{\theta}$ is given by $(\theta_1, \alpha\theta_1, \theta_3, \alpha\theta_3)$, for $\alpha$ the unique nontrivial quadratic character of $U(1)$ and $\theta_1, \theta_3$ irreducible characters of $U(1)$ such that $\theta_1/\theta_3 \neq 1, \alpha$. Note that $\widetilde{\theta}$ is a quadratic character.*

2. *$\pi_{|_{G^F}} = \pi_1 + \pi_2 + \pi_3 + \pi_4$ when $\widetilde{\theta}$ is, up to a twist by a character of $U(1)$, given by $(1, \alpha, \alpha^2, \alpha^3)$, for $\alpha$ a quartic character of $U(1)$. Note that this splitting can happen only when $d = 4$; in this case $\widetilde{\theta}$ is a quartic character.*

*Conversely, given any datum of one of these two types for $\widetilde{\theta}$, the character $\pi = \epsilon_{\widetilde{G}}\epsilon_{\widetilde{T}_w} R_{\widetilde{T}_w}^{\widetilde{G}}(\widetilde{\theta})$ splits upon restriction to $G^F$ in the corresponding manner described above.*

*Proof.* Let us first prove the direct implication. We are in the setting of Proposition 5.1.3, so the subgroup $A(\pi) \subseteq Z/(F-1)Z$ acts regularly on the irreducible components of $\pi_{|_{G^F}}$. As a result, $\pi$ splits only when it admits a self-twist, that is $\pi \cong \alpha\pi$ for some nontrivial $\alpha \in (\widetilde{G}^F/G^F)^\wedge$. Here $\alpha\pi = \alpha \otimes \pi$. Now, one can prove $\widetilde{G}^F/G^F \cong \widetilde{T}_w^F/T_w^F$, where $T_w = \widetilde{T}_w \cap G$ is a maximal $F$-stable torus in $G$. Thus, the character $\alpha \in (\widetilde{G}^F/G^F)^\wedge$ can be viewed as an element of $(\widetilde{T}_w^F/T_w^F)^\wedge$. Let us denote by $\alpha$ its restriction to $\widetilde{T}_w^F$. By Corrolary 1.27, p. 116 in [7], we have

$$R_{\widetilde{T}_w}^{\widetilde{G}}(\alpha\widetilde{\theta}) = \alpha \otimes R_{\widetilde{T}_w}^{\widetilde{G}}(\widetilde{\theta}).$$

Thus, as $\alpha \otimes \pi \cong \pi$, the characters $\widetilde{\theta}$ and $\alpha\widetilde{\theta}$ must give the same Deligne-Lusztig characters. From (4.1), we get that $\alpha\widetilde{\theta} =^{w_0} \widetilde{\theta}$ for some $w_0 \in W(\widetilde{T}_w)^F$. We have already seen that $w_0$ must be a permutation in $S_4$, so $\alpha(\theta_1, \theta_2, \theta_3, \theta_4)$ and $(\theta_1, \theta_2, \theta_3, \theta_4)$ must be the same up to permutation. We have two cases depending on whether $d = 2$ or $d = 4$, where $d = |Z/(F-1)Z|$.

If $d = 2$, we must have $A(\pi)$ generated by the quadratic character $\alpha$ of $\widetilde{G}^F/G^F \cong U(1)$. Note that $U(1) \cong (\mathbb{F}_{p^2}^\times)^1$ is cyclic, so there is a unique nontrivial quadratic character $\alpha$. WLOG say $\theta_1 = \alpha\theta_2$, so $\theta_2 = \alpha\theta_1$. Moreover, we must have $\theta_3 = \alpha\theta_4$ and $\theta_4 = \alpha\theta_3$. Thus, $\widetilde{\theta}$ is given by $(\theta_1, \alpha\theta_1, \theta_3, \alpha\theta_3)$ for $\theta_1, \theta_3$ irreducibles of $U(1)$ and $\theta_1/\theta_3 \neq 1, \alpha$.

If $d = 4$, then $A(\pi)$ is either cyclic of order 2 generated by the unique nontrivial quadratic character of $U(1)$ or cyclic of order 4 generated by a character of $U(1)$ or order 4. The case when $A(\pi)$ of order 2 was already discussed above. For the case when $A(\pi)$ cyclic of order 4, as the order of $U(1) \cong (\mathbb{F}_{p^2})^1$ is $p+1$, a quartic character of $U(1)$ exists iff $p \equiv 3 \bmod 4$. However, $d = 4$ iff $(p+1, 4) = 4$, so when $d = 4$ one can always find such a character $\alpha$ of order 4. Now, WLOG say $\theta_2 = \alpha\theta_1$. Then $\alpha\theta_2$ must be different from $\theta_1, \theta_2$, so say $\alpha\theta_2 = \theta_3$ and thus we get $\widetilde{\theta}$ is given by $(\theta_1, \alpha\theta_1, \alpha^2\theta_1, \alpha^3\theta_1)$.

The converse follows trivially from the above reasoning. □

*Remark.* One important remark to make is that the irreducible representations $\pi_i$ above are also cuspidal. Let us see why this is the case. It is well known that $\pi$ being cuspidal representation of $\widetilde{G}^F$ means $\langle \pi, \operatorname{Ind}_{U^F}^{\widetilde{G}^F} 1\rangle_{\widetilde{G}^F} = 0$ for each unipotent radical $U$ of an $F$-stable proper parabolic subgroup of $\widetilde{G}$. Note that a unipotent radical of an $F$-stable parabolic of $\widetilde{G}$ is contained in $G$. By Frobenius reciprocity, $\langle \operatorname{Ind}_{U^F}^{G^F} 1, \operatorname{Res}_{G^F}^{\widetilde{G}^F} \pi\rangle_{\widetilde{G}^F} = \langle \pi, \operatorname{Ind}_{G^F}^{\widetilde{G}^F} \operatorname{Ind}_{U^F}^{G^F} 1\rangle_{G^F}$. The second inner product is 0 as $\operatorname{Ind}_{G^F}^{\widetilde{G}^F} \operatorname{Ind}_{U^F}^{G^F} 1 = \operatorname{Ind}_{U^F}^{\widetilde{G}^F} 1$, so all components $\pi_i$ of $\pi_{|_{G^F}}$ must be cuspidal.

## 5.2 Zero contribution for $\Delta M_\pi$ on the semisimple conjugacy classes of $G^F$

Extrapolating from the case of $SL_2$, one might expect the alternating sum of characters $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i$ in (1.2) to be zero on the semisimple conjugacy classes and nonzero on the regular unipotent classes. The goal of this section is to indeed prove there is no contribution on the semisimple classes of $G^F$. Note that the proof holds for a general connected semisimple simply-connected group $G$ over an algebraically closed field $K$ of positive characteristic. The case of regular unipotent classes will be the object of the next chapters.

We now introduce the main tool we are going to use, which is an application of the Lang-Steinberg theorem:

**Theorem 5.2.1** (Application 3.25 in [9])**.** *The $G^F$-conjugacy classes of rational elements conjugate to some fixed $x \in G^F$ under $G$ are parametrized by the $F$-conjugacy classes of $C_G(x)/C_G(x)^\circ$.*

Here, a rational element is an element that belongs to $G^F$. Also, by $F$-conjugation in a group $M$ we mean the action of $M$ on itself defined for any $m_0 \in M$ by $m \mapsto m_0 m F(m_0)^{-1}$. We can easily check that if $g \in C_G(x)$, then $kgF(k)^{-1} \in C_G(x)$ for some $k \in C_G(x)$ since $F(k) \in C_G(x)$. The last assertion holds true since $kx = x$ implies $F(k)F(x) = F(x)$, but as $x \in G^F$ we have $F(x) = x$.

We also need the following result concerning the centralizer of semisimple elements in $G$ due to Steinberg:

**Theorem 5.2.2** (Theorem 3.5.6 in [4])**.** *Let $G$ be a connected reductive group whose derived group $G'$ is simply-connected. Let $s$ be a semisimple element of $G$. Then $C_G(s)$ is connected. In fact, if $G$ is semisimple connected and simply-connected, then the centralizer of any semisimple element of $G$ is connected.*

For example, $\widetilde{G} = GL_4(\overline{\mathbb{F}}_p)$ is a connected reductive group with derived group $G = SL_4(\overline{\mathbb{F}}_p)$ simply-connected, so $C_{\widetilde{G}}(s)$, for $s \in \widetilde{G}$ semisimple, is connected. We now have all the tools to prove that the semisimple conjugacy classes of $\widetilde{G}^F = U(2,2)$ do not split when restricted to $G^F = SU(2,2)$.

**Lemma 5.2.3.** *The semisimple conjugacy classes of $\widetilde{G}^F$ do not split when restricted to $G^F$.*

*Proof.* Let $x, y \in G^F$ be two semisimple elements that are conjugate over $\widetilde{G}^F$. Then there exists $g \in \widetilde{G}^F$ such that $x = gyg^{-1}$. Since $g \in \widetilde{G}$, we get that there exists $g' \in G$ such that $x = g'yg'^{-1}$. Note that two elements in $G^F$ conjugate under $\widetilde{G}$ are said to be geometrically conjugate, thus $x$ and $y$ are geometrically conjugate.

Since $G = SL_4(\overline{\mathbb{F}}_p)$ is semisimple connected and simply connected, the centralizer $C_G(x)$ is connected by Theorem 5.2.2. As a result, since $C_G(x)/C_G(x)^\circ$ is trivial, Theorem 5.2.1 tells us that there is only one $G^F$-conjugacy class in the intersection of the geometric conjugacy class of $x$ with $G^F$. As a result $x$ and $y$ are in the same $G^F$-conjugacy class, which ends the proof. □

**Theorem 5.2.4.** *Let $\pi$ be an irreducible representation of $\widetilde{G}^F$ that splits into $d_0$ components upon restriction to $G^F$, that is $\pi_{|_{G^F}} = \pi_1 + \cdots + \pi_{d_0}$, where $d_0|d$. The alternating sum $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i$ introduced in equation* (1.2) *is zero on the semisimple elements of $G^F$. Recall $\xi_{d_0}$ is a primitive $d_0^{th}$ root of unity.*

*Proof.* By Clifford's Theorem 5.1.1, we have that $\pi_i = \pi_1^g$ for some $g \in \widetilde{G}^F$, that is the two irreducibles of $G^F$ are conjugate under $U(2,2)$. As a result, we have $\pi_i(s) = \pi_1^g(s) = \pi_1(g^{-1}sg)$. But from Lemma 5.2.3 above, we know that since $s$ and $g^{-1}sg$ are in the same semisimple conjugacy class of $\widetilde{G}^F$, they will belong to the same semisimple conjugacy class of $G^F$ and thus $\pi_1(g^{-1}sg) = \pi_1(s)$. As a result, $\pi_i(s) = \pi_1(s) = \pi_{|_{G^F}}(s)/d_0$, so $\sum_{i=0}^{d-1} \xi_{d_0}^i \pi_i^i(s) = 0$ for $s$ semisimple in $G^F$.

□

*Chapter 6*

# TOWARDS A FORMULA FOR MULTIPLICITY DEFECT FOR $G^F$

In the light of the semisimple classes having zero contribution to the alternating sum of multiplicities $\Delta M_\pi$, we want to show the regular unipotent classes are non-zero on the alternating sum $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i$ introduced in (1.2). The scope of this chapter is to carry out the two steps needed to compute characters of irreducible representations of $G^F$ on regular unipotent elements. First, we shall write these characters in terms of the complex numbers $\sigma_z$; this is done in Theorem 6.2.10 when $d_0 = d$. Secondly, we give a formula, in terms of Gauss sums, for the Mellin transforms $\sigma_\zeta$ in Theorem 6.3.5, which is equivalent to having a formula for evaluating $\sigma_z$. The following analysis is done in full generality for connected semisimple groups $G$ defined in Section 6.2 below.

## 6.1 Some basic results on Gauss sums

This section is meant to recall basic results on Gauss sums that will be instrumental in computing $\sigma_\zeta$. We loosely follow the exposition of Chapter 2, Section 2.1 on Gauss sums in [21].

**Definition 6.1.1.** Let $\phi \in (\mathbb{F}_{q^{|I|}}^\times)^\wedge$ and $\chi_0$ the additive character of $\mathbb{F}_{q^N}$ introduced in Subsection 4.3. Define the Gauss sum $G(\phi) := G(\phi, \chi_0)$ to be given by

$$G(\phi) = \sum_{x \in \mathbb{F}_{q^{|I|}}^\times} \phi(x)\chi_0(x).$$

Note that in the above definition, $\phi$ is a multiplicative character of $\mathbb{F}_{q^{|I|}}^\times$, while $\chi_0$ is an additive character of $\mathbb{F}_{q^N}$ viewed as a character of $\mathbb{F}_{q^{|I|}}$. As a remark, Gauss sums of the form above can be defined over any finite field of prime characteristic.

**Lemma 6.1.2.** *Let* $1$ *be the trivial character of* $\mathbb{F}_{q^{|I|}}^\times$. *We then have* $G(1) = -1$.

*Proof.* The result follows trivially as $G(1) = \sum_{x \in \mathbb{F}_{q^{|I|}}^\times} \chi_0(x)$ is a sum over $\mathbb{F}_{q^{|I|}}^\times$ and $\chi_0$ is viewed as a non-trivial additive character of $\mathbb{F}_{q^{|I|}}$. □

**Proposition 6.1.3** (Proposition 2.4 in [21])**.** *Let $\mathbb{F}_q$ be a finite field with $q$ elements, $\chi$ a non-trivial additive character and $\phi$ a non-trivial multiplicative character of $\mathbb{F}_q$. Then we have*

$$|G(\phi, \chi)| = \sqrt{q}.$$

Note that in our case $\chi_0$ is non-trivial, so we have $G(\phi) := G(\phi, \chi_0)$ of modulus $\sqrt{q^{|I|}}$ if $\phi$ is a multiplicative character of $\mathbb{F}_{p^{|I|}}^\times$. In particular $G(\phi)$ is nonzero, which is a key observation in proving the multiplicity defect.

## 6.2 Step I: On a final form of character values of regular cuspidal irreducibles at regular unipotent elements

The goal of this section is to carry out the first step in computing the value of a character of an irreducible representation $\chi$ of $G^F$ at regular unipotent elements. As outlined by the result of Theorem 4.3.15, we shall see what is the value of the inner product of the dual $D_G(\chi)$ with a Gelfand-Graev representation. As motivated by the introduction of our problem of interest in Section 1.1, we restrict ourselves to certain representations $\chi$, and thus we shall focus on proving results for this particular setting. We note that the characters $\chi_{(s)}$ appear naturally in the case we consider and are a key ingredient in our proofs.

Let $G$ be a connected semisimple algebraic group over an algebraically closed field $K$ of prime characteristic, that is $K = \overline{\mathbb{F}}_p$. Recall from Section 1.1 that we are interested in irreducible cuspidal representations $\pi$ of $\widetilde{G}^F$ that split when restricted to $G^F$. Moreover, we ask that $\langle \pi, \Gamma \rangle_{\widetilde{G}^F} = 1$, where $\Gamma$ is the Gelfand-Graev character of the group $\widetilde{G}^F$, that is, we want $\pi$ to be a regular character. Note $\Gamma$ is unique since $\widetilde{G}^F$ has connected center.

We shall also restrict ourselves to the case where $G$ has good characteristic and $q$ is sufficiently large. Note that for the particular case of $G = SL_4$ we are interested in, we are always in good characteristic, while the condition on $q$ is to ensure that all maximal tori of $G^F$ are nondegenerate. As $\pi$ is an irreducible component of the Gelfand-Graev character $\Gamma$ of $\widetilde{G}^F$, the restriction $\pi_{|_{G^F}}$ is multiplicity free by Theorem 3.1, p.83 in [26] and that the number of irreducible components divides $d = |H^1(F, Z)|$ by the result of Proposition 5.1.3. We shall carry out the derivations in the case where we get exactly $d$ irreducible components upon restricting $\pi$ to $G^F$. As a result, we have

$$\pi_{|_{G^F}} = \pi_1 + \cdots + \pi_d, \tag{6.1}$$

where $\pi_i$ are irreducible representations of $G^F$, $i \in \{1, \cdots, d\}$.

*Remark.* Note that in the particular case of $\widetilde{G}^F = U(2,2)$ we are interested in, if $d = 4$, we also have the case where $\pi$ splits into two irreducible representations when restricted to $G^F$. While the following analysis will not deal with this situation in detail, the exact same ideas of the proof apply, with minor modifications that have to deal with the fact that the irreducible components of a certain character $\chi_{(s)}$ will be appear in more than one Gelfand-Graev representation $\Gamma_z$ of $G^F$, $z \in H^1(F, Z)$. The crux in the analysis is figuring out for what $z, z'$ we have $\chi_{s,z} = \chi_{s,z'}$. A formula for $\pi_1, \pi_2$ on the regular unipotent elements in this particular case, will be given in the proof of the main result of Chapter 7.

**Preliminaries**

Since $\pi$ is an irreducible cuspidal representation of $\widetilde{G}^F$ which is a component of the Gelfand-Graev representation $\Gamma$, we saw by the result of Theorem 5.1.4 that $\pi$ must be of the form $\pi = \pm R_{\widetilde{T}}^{\widetilde{G}}(\widetilde{\theta})$, where $R_{\widetilde{T}}^{\widetilde{G}}(\widetilde{\theta})$ is the generalized Deligne-Lusztig character of $G^F$ corresponding to the $F$-stable maximal torus $\widetilde{T}$ of $\widetilde{G}$ and the irreducible character $\widetilde{\theta}$ of $\widetilde{T}^F$ in general position. Moreover, by Proposition 4.2.11, we must have $\pi = \epsilon_{\widetilde{G}}\epsilon_{\widetilde{T}}R_{\widetilde{T}}^{\widetilde{G}}(\widetilde{\theta})$. Also, recall that since $\pi$ is cuspidal, $\widetilde{T}$ is not contained in any proper $F$-stable parabolic subgroup of $\widetilde{G}$.

In the particular case we are interested in $\widetilde{G} = GL_4(\overline{\mathbb{F}}_p)$, $G = SL_4(\overline{\mathbb{F}}_p)$ and $F$ is the Frobenius map defined in Section 5.1 such that $\widetilde{G}^F$, $G^F$ are the groups $U(2,2)$, respectively $SU(2,2)$ over the finite field of $p$ elements. As we saw in Lemma 5.1.5, such an $F$-stable maximal torus $\widetilde{T}$ that is not contained in any proper $F$-stable parabolic of $\widetilde{G}$ is given by the torus of type $\omega$ with respect to the maximally split $F$-stable torus of diagonal matrices, where $\omega$ is the symmetry $(1,2,3,4) \mapsto (4,3,2,1)$ in the Weyl group. Then $\widetilde{T}^F \cong U(1) \times U(1) \times U(1) \times U(1)$. As seen in Lemma 5.1.6, an irreducible character $\widetilde{\theta} \in (\widetilde{T}^F)^\wedge$ in general position is given by $(\theta_1, \theta_2, \theta_3, \theta_4)$ with all $\theta_i$ distinct, where $\theta_i$ is an irreducible character of $U(1)$.

For more details on the general results presented below, one can refer to Section 3 in [26] or Chapters 13 and 14 of [9], as well as Section 5 in [7] . A useful overview of some of the background ingredients used in our proofs is also outlined in Chapters $1, 3$ and 4 of [4] and Chapter 2 in [17].

The following lemma follows immediately from general properties of the Deligne-Lusztig induction:

**Lemma 6.2.1.** *The restriction of $R_{\widetilde{T}}^{\widetilde{G}}(\widetilde{\theta})$ to $G^F$ is given by $R_T^G(\theta)$, where $T = \widetilde{T} \cap G$ and $\theta$ is the restriction of $\widetilde{\theta} \in (\widetilde{T}^F)^\wedge$ to $T^F$.*

As a result, we must have $\mathrm{Res}_{G^F}^{\widetilde{G}^F}\left(\epsilon_{\widetilde{G}}\epsilon_{\widetilde{T}}R_{\widetilde{T}}^{\widetilde{G}}(\widetilde{\theta})\right) = \epsilon_G\epsilon_T R_T^G(\theta)$. Indeed, by construction, $\widetilde{G}$ is a connected reductive group with the same derived group as $G$. Since $G$ is semisimple, we have $\widetilde{G} = G\widetilde{Z}^\circ$, where the connected center $\widetilde{Z}^\circ = \widetilde{Z}$ of $\widetilde{G}$ coincides with the radical of $\widetilde{G}$ since $\widetilde{G}$ is reductive. Now by the property given by Lemma 4.2.6, we get that $\epsilon_{\widetilde{G}} = \epsilon_G\epsilon_{\widetilde{Z}}$. On the other hand, $T$ is a maximal $F$-stable torus in $G$ contained in a maximal $F$-stable torus $\widetilde{T}$ of $\widetilde{G}$ such that $\widetilde{T} = T\widetilde{Z}^0$ and by using the same ideas as in the proof of the result in Lemma 4.2.6, we get $\epsilon_{\widetilde{T}} = \epsilon_T\epsilon_{\widetilde{Z}}$. As a result, $\epsilon_{\widetilde{G}}\epsilon_{\widetilde{T}} = \epsilon_G\epsilon_T$.

Thus, $\pi_{|_{G^F}} = \epsilon_G\epsilon_T R_T^G(\theta)$ in the above notation. Since we know that the value of any Deligne-Lusztig character at a regular unipotent element is 1, in the light of wanting to compute the value of $\pi_i$, $i \in \{1, \cdots, d\}$, on a regular unipotent, it is worth mentioning $\pi_{|_{G^F}}$ takes values $\pm 1$ on all regular unipotent elements.

Now, $\widetilde{\theta}$ is an irreducible character of $\widetilde{T}^F$ in general position. We say a character of $T^F$ is nonsingular if it is not orthogonal to any coroot. It is a fact that in a group with a connected center, $\widetilde{\theta}$ is nonsingular if and only if it is in general position. The following holds true in general:

**Proposition 6.2.2** (Corrollary 5.18 in [7])**.** *For any $G$, if $\theta$ is in general position, then $\theta$ is nonsingular.*

Let us outline the idea of the proof. We embed $G$ in a group with connected center $\widetilde{G}$ as seen before. Now $T$ is contained in an $F$-stable maximal torus $\widetilde{T}$ of $\widetilde{G}$ and $\theta$ is the restriction to $T^F$ of some character $\widetilde{\theta}$ of $\widetilde{T}^F$. It is a fact that since $\theta$ is in general position, $\widetilde{\theta}$ must, a forteriori, be in general position as well. Since $\widetilde{G}$ has connected center, $\widetilde{\theta}$ is nonsingular. Moreover, a character $\widetilde{\theta}$ of $\widetilde{T}^F$ is nonsingular if and only if its restriction to $T^F$ is nonsingular, hence the result follows.

In particular, we see that since $\widetilde{\theta}$ is in general position and thus nonsingular, then its restriction to $T^F$, $\theta$, will be nonsingular. Now, recall by Proposition 4.2.9, that there exists a one-to-one correspondence between $G^F$-conjugacy classes of pairs $(T, \theta)$, where $T$ is a rational maximal torus of $G$ and $\theta$ is an irreducible representation of $T^F$, and $G^{*F^*}$-conjugacy classes of pairs $(T^*, s)$, where $s$ is a semisimple element of $G^{*F^*}$ and $T^*$ is a rational maximal torus containing $s$. As a result, the $G^F$-class of $(T, \theta)$ corresponds to the $G^{*F^*}$-conjugacy class of a pair $(T^*, s)$. The semisimple element $s$ of $G^{*F^*}$ defines a character $\chi_{(s)}$ of $G^F$ as described in Definition 4.3.9. The following subsection focuses on the character $\chi_{(s)}$ when, as in this case, the corresponding $\theta$ is nonsingular.

Let us go back to the notion of a regular element in an algebraic group $G$ in Definition 4.3.3. While regular unipotent elements were introduced in Section 4.3, the following proposition in Chapter 2, Section 2.3, p. 27 in [17] gives us a number of characterizations of regular semisimple elements:

**Proposition 6.2.3.** *Let $s \in G$ be semisimple. Then the following conditions are equivalent:*

1. *$s$ is regular.*
2. $\dim C_G(s) = \operatorname{rank}(G)$.
3. *For any maximal torus $T$ of $G$ containing $s$ and any root $\alpha$ relative to $T$, $\alpha(s) \neq 1$.*
4. *$C_G(s)^\circ$ is a maximal torus of $G$.*
5. *$s$ lies in a unique maximal torus of $G$.*
6. *$C_G(s)$ consists of semisimple elements.*

We have one last remark regarding the $F$-stable maximal torus $T$ in $G$, specifically we want $T^F$ to be nondegenerate, that is, $T$ will be the only maximal torus of $G$ containing $T^F$. By Proposition 3.6.1, Chapter 3, p. 96 in [4] , $T^F$ is nondegerate if and only if no root of $G$ with respect to $T$ satisfies $\alpha(t) = 1$ for all $t \in T^F$. Moreover, one can prove that all the maximal tori of $G^F$ are nondegenerate provided $q$ is sufficiently large, which is an assumption we made when setting the problem.

**On $\chi_{(s)}$ when $s$ corresponds to a nonsingular irreducible $\theta$ of $T^F$**

The goal of this subsection is to prove that when $\theta$ is nonsingular in $(T, \theta)$, then in the corresponding $G^{*F^*}$-class $(T^*, s)$, the semisimple element $s$ is regular. We shall see that this means the restriction of the cuspidal representation $\pi$ to $G^F$ will be given by $\chi_{(s)}$.

First, we saw that $\theta$ is nonsingular, so as a result $\theta$ is not orthogonal to any coroot. Let $(X, \Phi, Y, \Phi^\vee)$ be the root data of $G$ and $W$ its Weyl group and let $\alpha^\vee$ be a coroot. Note that from Proposition 4.1.5 we know $(T^F)^\wedge \cong X/(F-1)X$, so we can view $\theta$ as an element of $X$ and as a result the notation $\langle\theta, \alpha^\vee\rangle$ makes sense. Recall the action of $W$ on $X$ is given by

$$w_\alpha(\chi) = \chi - \langle\chi, \alpha^\vee\rangle\alpha,$$

for $\chi \in X$. As a result, we trivially get that $w_\alpha$ fixes $\theta$ iff $\theta$ is orthogonal to $\alpha^\vee$.

Before moving onto the main result, we need the following lemma concerning semisimple elements in $G$:

**Lemma 6.2.4.** *Let $s$ be a semisimple element of $G$ contained in a maximal torus $T$. If the Weyl group is generated by $w_\alpha$, for $\alpha \in \Phi$, then we have*

$$s^{w_\alpha} = s \Leftrightarrow \alpha(s) = 1.$$

*Proof.* This lemma appears to be a standard result about root systems. Moreover, we only need the converse implication later on, so in the following we give a proof for that.

First, notice that as $(-\alpha)(s) = \alpha(s)^{-1}$, we have $\alpha(s) = 1$ iff $(-\alpha)(s) = 1$. Since $w_\alpha = w_{-\alpha}$, we can assume WLOG that $\alpha \in \Phi^+$.

Recall from Subsection 4.1 that each root subgroup $U_\alpha$ is isomorphic to the additive group $\mathbb{G}_a$. We have an isomorphism $a \mapsto x_\alpha(a)$ of $\mathbb{G}_a$ into $U_\alpha$ such that the action of $T$ on the subspace $U_\alpha$ is given by

$$tx_\alpha(a)t^{-1} = x_\alpha(\alpha(t)a),$$

that is, $t$ acts on $U_\alpha$ by $\alpha(t)$. First, we claim $\alpha(s) = 1$ iff $U_\alpha \subseteq C_G(s)$. Indeed, an element in $U_\alpha$ is of the form $x_\alpha(a)$, and $x_\alpha(a) \in C_G(s)$ iff $x_\alpha(a)^{-1}sx_\alpha(a) = s$. As a result, $x_\alpha(a) \in C_G(s)$ iff

$$sx_\alpha(a)s^{-1} = x_\alpha(a),$$

and thus $U_\alpha$ lies in $C_G(s)$ iff $\alpha(s) = 1$. Moreover, using the remark above, $U_\alpha$ and $U_{-\alpha}$ both lie in $C_G(s)$ iff $\alpha(s) = 1$.

Now, recall from Subsection 4.1 that $w_\alpha \in \langle U_\alpha, U_{-\alpha}\rangle$. So if $\alpha(s) = 1$, we have $U_\alpha, U_{-\alpha} \subseteq C_G(s)$ and thus $w_\alpha$ must belong to $C_G(s)$ as well, so the converse implication is proved. □

Let us now go back to the result of Proposition 4.2.9. Recall that the construction of the bijection relies on the fact that if $T$ is an $F$-stable maximal torus that puts $(G, F)$ and $(G^*, F^*)$ in duality, then we have an isomorphism between ${T^*}^{F^*}$ and the character group $(T^F)^\wedge$ as seen in Proposition 4.1.7. The following lemmas build up towards the main result of this section:

**Lemma 6.2.5.** *Let $T$ as above, an $F$-stable maximal torus in $G$ such that $T^F$ is nondegenerate. By abuse of notation, denote by $\delta$ the isomorphism $X/(F-1)X \cong Y^*/(F^*-1)Y^*$ induced by $\delta : X \to Y^*$. If $\theta \in X$ such that $\langle \theta, \alpha^\vee\rangle \neq 0$ for $\alpha^\vee \in \Phi^\vee$, then $\delta(\theta)$ and $\delta(\theta)^{w_{\delta(\alpha)}}$ are distinct elements in the quotient $Y^*/(F^*-1)Y^*$ for all $\delta(\alpha) \in \Phi^{*\vee}$.*

*Proof.* The first step in the proof is the following claim:

**Claim.** *Let $\theta \in X$. Then $\langle \theta, \alpha^\vee\rangle \neq 0$ for $\alpha^\vee \in \Phi^\vee$ if and only if $\theta$ and ${}^{w_\alpha}\theta$ have different images in the quotient $X/(F-1)X$.*

*Proof of Claim.* Let $\overline{\theta}$ to be the image of $\theta$ in $X/(F-1)X$. It is clear that $\overline{\theta} \neq \overline{{}^{w_\alpha}\theta}$ implies $\theta \neq^{w_\alpha} \theta$. For the converse, assume $\theta -^{w_\alpha} \theta \in (F-1)\chi$ for some $\chi \in X$. By the action of $w_\alpha$ on $X$, we have ${}^{w_\alpha}\theta = \theta - \langle \theta, \alpha^\vee\rangle\alpha$ in the notation of Subsection 4.1. Thus $(F-1)\chi = \langle \theta, \alpha^\vee\rangle\alpha$.

Now, $(F-1)\chi(t) = \chi(F(t)t^{-1})$ for all $t \in T$, so $(F-1)\chi$ is trivial on $T^F$. As a result, $\alpha$ must take the same value on all of $T^F$. But as $\alpha$ is trivial on $Z^F$, it must thus be trivial on all of $T^F$. As a result, $\alpha(t) = 1$ for all $t \in T^F$, which is a contradiction with the fact that $T^F$ is nondegenerate. Thus our initial assumption is false and thus $\theta$ and ${}^{w_\alpha}\theta$ have different images in the quotient $X/(F-1)X$.

We thus have $\theta \neq^{w_\alpha} \theta$ in $X/(F-1)X$. As a result, $\delta(\theta) \neq \delta({}^{w_\alpha}\theta)$ in $Y^*/(F^*-1)Y^*$. We also know the isomorphism $\delta$ transforms the map $w_\delta : X \to X$ into the map $w_{\delta(\alpha)} : Y^* \to Y^*$ such that

$$w_{\delta(\alpha)}(\delta(\chi)) = \delta(\chi) - \langle \alpha^*, \delta(\chi)\rangle\delta(\alpha),$$

with $\delta(\alpha) = \alpha^{*\vee}$ and $\langle \chi, \alpha^\vee \rangle = \langle \alpha^*, \delta(\chi) \rangle$. So $\delta(\theta)^{w_{\delta(\alpha)}} = \delta({}^{w_\alpha}\theta)$, and thus $\delta(\theta) \neq \delta(\theta)^{w_{\delta(\theta)}}$ in $Y^*/(F^* - 1)Y^*$. $\square$

**Lemma 6.2.6.** *Let $T$ be a torus that puts $(G, F)$ and $(G^*, F^*)$ in duality as above, with $T^F$ nondegenerate. If $\nu \in Y^*$ such that $\nu^{w_{\delta(\alpha)}} \neq \nu$ in $Y^*/(F^*-1)Y^*$ for all $\delta(\alpha) \in \Phi^{*\vee}$, then $s^{w_{\delta(\alpha)}} \neq s$ for all $\delta(\alpha) \in \Phi^{*\vee}$, where $s \in T^{*F^*}$ is the image of $\nu$ through the isomorphism $Y^*/(F^* - 1)Y^* \cong T^{*F^*}$.*

*Proof.* The isomorphism $Y^*/(F^* - 1)Y^* \cong T^{*F^*}$ in Proposition 4.1.4 comes from the following commutative diagram, as seen in Proposition 13.7, Chapter 13, p. 102 in [9]:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y^* & \xrightarrow{F^{*n}-1} & Y^* & \longrightarrow & T^{*F^{*n}} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle N_{F^{*n}/F^*}} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle N_{F^{*n}/F^*}} & & \\
0 & \longrightarrow & Y^* & \xrightarrow{F^{*}-1} & Y^* & \longrightarrow & T^{*F^{*}} & \longrightarrow & 1.
\end{array}$$

Here $n$ is such that $T^{*F^{*n}}$ is split over $\mathbb{F}_{q^n}$ and $N_{F^{*n}/F^*} : F^{*n} \to T^{*F^*}$ is the norm map defined by

$$t \mapsto tF^*(t)(F^*)^2(t) \cdots (F^*)^{n-1}(t).$$

The top right map $Y^* \to T^{*F^{*n}}$ sends $\nu$ to $\nu(\zeta)$, where $\zeta$ is the $(q^n - 1)^{\text{th}}$ root of 1 in $\overline{\mathbb{F}}_q^\times$. As a result, the isomorphism $Y^*/(F^* - 1)Y^* \cong T^{*F^*}$ is given by $\nu \mapsto N_{F^{*n}/F^*}(\nu(\zeta))$.

If we denote $N_{F^{*n}/F}(t) \in T^{*F^*}$ by $s$, the following claim finishes the proof of the lemma:

**Claim.** *If $\nu^{w_{\delta(\alpha)}} \neq \nu$ for all $\delta(\alpha) \in \Phi^{*\vee}$ then $N_{F^{*n}/F^*}(t)^{w_{\delta(\alpha)}} \neq N_{F^{*n}/F^*}(t)$ for all $\delta(\alpha) \in \Phi^{*\vee}$, where $t = \nu(\zeta)$.*

*Proof of Claim.* As $\nu \neq \nu^{w_{\delta(\alpha)}}$ for all $\delta(\alpha) \in \Phi^{*\vee}$, we have $\langle \alpha^*, \nu \rangle \neq 0$ for all $\alpha^* \in \Phi^*$. Since $F^*(\alpha^*) \in \Phi^*$ and $\langle F^*(\alpha^*), \nu \rangle = \langle \alpha^*, F^*(\nu) \rangle$, we must have $\langle (1 + F^* + \cdots + (F^*)^{n-1})(\alpha^*), \nu \rangle = \langle \alpha^*, (1 + F^* + \cdots + (F^*)^{n-1})(\nu) \rangle$ and thus $(1 + F^* + \cdots + (F^*)^{n-1})(\nu)$ has nonzero inner product with all roots in $\Phi^*$. Thus $(1 + F^* + \cdots + (F^*)^{n-1})(\nu)^{w_{\delta(\alpha)}} \neq (1 + F^* + \cdots + (F^*)^{n-1})(\nu)$. Now, by the same reasoning as that of Lemma 6.2.5, as $T^F$ is nondegenerate, we get that $(1 + F^* + \cdots + (F^*)^{n-1})(\nu)^{w_{\delta(\alpha)}}$ and $(1 + F^* + \cdots + (F^*)^{n-1})(\nu)$ must be distinct in the quotient $Y^*/(F^* - 1)Y^*$. Since $(F^*(\nu))(\zeta) = F^*(t)$, we get $N_{F^{*n}/F^*}(t)^{w_{\delta(\alpha)}} \neq N_{F^{*n}/F^*}(t)$ for all $\delta(\alpha) \in \Phi^{*\vee}$.

□

**Lemma 6.2.7.** *Let $T$ is an $F$-stable maximal torus of $G$ and $T^*$ an $F^*$ stable maximal torus of $G^*$ such that $T$ and $T^*$ are in duality. Assume $T^F$ is nondegenerate. Then a nonsingular element $\theta \in (T^F)^\wedge$ is sent to a regular semisimple element $s \in {T^*}^{F^*}$ through the isomorphism given by the duality map $\delta : X \to Y^*$.*

*Proof.* Recall that the proof of Proposition 4.1.7 mapping the characters of a torus to elements of a dual torus is based on the following isomorphisms:

$$(T^F)^\wedge \cong X/(F-1)X \cong Y^*/(F^*-1)Y^* \cong {T^*}^{F^*},$$

where the middle isomorphism is given by the duality map $\delta$. Since $\theta$ is nonsingular, it means that $\langle \theta, \alpha^\vee \rangle \neq 0$ for all coroots $\alpha \in \Phi^\vee$.

As we've already seen in the proof of Lemma 6.2.5, we know that the Weyl group $W^*$ is generated by $w_{\delta(\alpha)}$ for all $\delta(\alpha) \in \Phi^{*\vee}$, where $w_{\delta(\alpha)}$ acts on $Y^*$ as

$$w_{\delta(\alpha)}(\delta(\chi)) = \delta(\chi) - \langle \alpha^*, \delta(\chi) \rangle \delta(\alpha),$$

with $\delta(\alpha) = \alpha^{*\vee}$ and $\langle \chi, \alpha^\vee \rangle = \langle \alpha^*, \delta(\chi) \rangle$. Consequently, $\langle \alpha^*, \delta(\theta) \rangle \neq 0$ is equivalent to $w_{\delta(\alpha)}(\delta(\theta)) \neq \delta(\theta)$. By the result of the same lemma, we then have $\delta(\theta)^{w_{\delta(\alpha)}} \neq \delta(\theta)$ in $Y^*/(F^*-1)Y^*$ for all $\delta(\alpha) \in \Phi^{*\vee}$. Furthermore, Lemma 6.2.6 gives us that $s^{w_{\delta(\alpha)}} \neq s$ for all $\delta(\alpha) \in \Phi^{*\vee}$, where $s = (\delta(\theta))(\zeta)$ in the notation of said lemma.

Now, note the Weyl group $W^*$ is the group of transformations of $Y^*$ generated by the $w_{\delta(\alpha)}$ for all $\delta(\alpha) \in \Phi^{*\vee}$, which is the same as the group of transformations of $X^*$ generated by the $w_{\alpha^*}$ for all $\alpha^* \in \Phi^*$. As a result, we get $s^{w_{\alpha^*}} \neq s$ for all $\alpha^* \in \Phi^*$, so by Lemma 6.2.4, we get that $\alpha^*(s) \neq 1$ for all roots $\alpha^* \in \Phi^*$. But by Proposition 6.2.3, this means that $s$ is regular semisimple, which ends the proof.

□

As a result, we can state the following:

**Proposition 6.2.8.** *Let $T$ be an $F$-stable maximal torus in $G$, $\theta$ a nonsingular irreducible character of $T^F$. Assume $G$ has good characteristic and $q$ is sufficiently large such that $T^F$ is nondegenerate and let $(G^*, F^*)$ be the dual* *pair to $(G, F)$. Recall the bijection given in Proposition 4.2.9. Then the $G^F$-conjugacy class of the pair $(T, \theta)$ corresponds to the $G^{*F^*}$-conjugacy class of the pair $(T^*, s)$, where $s$ is a regular semisimple element of $G^{*F^*}$ and $T^*$ is a rational maximal torus containing $s$.*

As a result, recall that since the conjugacy class $(T, \theta)$ determines the Deligne-Lusztig representation $R_T^G(\theta)$, we can use the notation $R_{T^*}^G(s)$ for $R_T^G(\theta)$. Let us look at the character $\chi_{(s)}$ of $G^F$ determined by the semisimple regular element $s \in G^{*F^*}$. As $s$ is regular, the Weyl group $W^{*\circ}(s) = \{w_{\alpha^*} | \alpha^* \in \Phi^*, \alpha^*(s) = 1\}$ of $C_{G^*}^{\circ}(s)$, is trivial. Thus, there is only one maximal torus containing $s$ and we have

$$\chi_{(s)} = \epsilon_G \epsilon_{T^*} R_{T^*}^G(s).$$

Thus, since by Lemma 6.2.1 the restriction of the irreducible cuspidal representation $\pi$ of $\widetilde{G}^F$ introduced in the beginning of Section 6.2, is given by $\pi_{|_{G^F}} = \epsilon_G \epsilon_T R_T^G(\theta)$, we have $\pi_{|_{G^F}} = \chi_{(s)}$. Indeed, the only argument to make is $\epsilon_T = \epsilon_{T^*}$, which follows easily as $T, T^*$ are in duality. In the following subsections, we shall use properties of $\chi_{(s)}$ to evaluate the inner product of $D_G(\chi_{(s)})$ with a Gelfand-Graev representation.

*Remark.* Let us look at the embedding $G \to \widetilde{G}$ and reason at the level of the group $\widetilde{G}$, which has connected center. $\theta$ is the restriction to $T^F$ of the character $\widetilde{\theta}$ of $\widetilde{T}^F$ that is in general position, and thus nonsingular as $\widetilde{G}$ has connected center. By the same reasoning as above, there is a bijection between the $\widetilde{G}^F$-conjugacy classes of pairs $(\widetilde{T}, \widetilde{\theta})$ and the $\widetilde{G}^{*F^*}$-conjugacy classes of pairs $(\widetilde{T}^*, \widetilde{s})$ with $\widetilde{s}$ is a semisimple element of $\widetilde{G}^{*F^*}$ and $\widetilde{T}^*$ is a rational maximal torus containing $\widetilde{s}$, where $(\widetilde{G}, F)$ and $(\widetilde{G}^*, F^*)$ are in duality. Now $\widetilde{s}$ is regular semisimple by the same reasoning above and it is the semisimple class in $\widetilde{G}^{*F^*}$ sitting above $s$. We thus have $\pi = \chi_{(\widetilde{s})}$ and by general properties of the Deligne-Lusztig induction $\mathrm{Res}_{G^F}^{\widetilde{G}^F}(\chi_{(\widetilde{s})}) = \chi_{(s)}$, so we reach the same conclusion that $\pi_{|_{G^F}} = \chi_{(s)}$, where $s$ is regular semisimple in $G^{*F^*}$.

**Character formula**

In the following, we shall finalize the first step in computing the values of the irreducible components of $\pi_{|_{G^F}}$ on regular unipotent elements, as outlined in the discussion following Theorem 4.3.15. From the previous subsection we have $\pi_{|_{G^F}} = \chi_{(s)}$ for $s \in G^{*F^*}$ regular semisimple element belonging to a rational maximal torus $T^*$.

First, we shall prove the dual $D_G$ sends a cuspidal representation $\chi$ to $\pm\chi$:

**Lemma 6.2.9.** *Let $\chi$ be a cuspidal representation of $G^F$. Then $D_G(\chi) = (-1)^{r(G)}\chi$, where $r(G)$ is the semisimple $\mathbb{F}_q$-rank of $G$.*

*Proof.* Recall that by definition, we have

$$D_G(\chi) = \sum_{P \supseteq B} (-1)^{r(P)} R_L^G \circ {}^* R_L^G(\chi),$$

where the sum is taken over the rational parabolics of $G$ that contain the rational Borel subgroup $B$ and where $L$ is a rational Levi subgroup of $P$. Since $\chi$ is a cuspidal representation of $G^F$, the pair $(G, \chi)$ is minimal in the sense of Definition 4.2.3, thus for any rational Levi $L$ of a rational parabolic subgroup of $G$ we have ${}^*R_L^G(\chi) = 0$. As a result,

$$D_G(\chi) = (-1)^{r(G)}\chi,$$

which ends the proof of the lemma. □

Note we are in the same setting as stated in Section 6.2, where we assume good characteristic for $G$, $q$ sufficiently large and let $\pi$ be a regular cuspidal irreducible representation of $\widetilde{G}^F$, such that $\pi_{|_{G^F}} = \pi_1 + \cdots + \pi_d$ as in (6.1), where $d = |H^1(F, Z)|$. Since $\pi_i$ is cuspidal as well, by the lemma above we have $D_G(\pi_i) = (-1)^{r(G)}\pi_i$, so $\langle D_G(\pi_i), \Gamma_z\rangle_{G^F} = (-1)^{r(G)}\langle \pi_i, \Gamma_z\rangle_{G^F}$ for $i \in \{1, \cdots, d\}$. As seen in Proposition 4.3.11, $\pi_{|_{G^F}} = \chi_{(s)}$ and $\Gamma_z$ have exactly one irreducible common component $\chi_{s,z}$, thus we may assume WLOG that $\pi_i = \chi_{s,z_i}$ for each $i \in \{1, \cdots, d\}$, where the $z_i$'s run through the elements of $H^1(F, Z)$.

Using the formula of Theorem 4.3.15 we can state the following result:

**Theorem 6.2.10.** *Let $\pi_i$ for $i \in \{1, \cdots, d\}$ be an irreducible cuspidal representation of $G^F$ as seen above. The values of the character of $\pi_i$ on regular unipotent elements are given by*

$$\pi_i(u) = (-1)^{|\Pi/\tau| + r(G)} \sigma_{zz_i^{-1}}$$

*for any $u \in U_z$.*

Note that once we evaluate $\sigma_z$, which we shall do in the following chapter, we will know the exact values the characters $\pi_i$ take on each regular unipotent class.

Let us make another small remark. Recall from (4.6) that the set of regular characters is given by

$$\{\chi_{s,z} | (s) \text{ is a semisimple conjugacy class of } G^{*F^*}, z \in H^1(F,Z)\}.$$

Clearly $\pi_i$ is a regular character. Moreover, we also have semisimple characters, which are irreducible characters whose dual is (up to a sign) a regular irreducible character. We shall see later there is another set of characters of $G^F$ that we are interested in, that is, the ones that take non-zero values on regular unipotent classes. As a result, $\pi_i$ is a regular semisimple character for all $i \in \{1, \cdots, d\}$. One can actually prove that the regular semisimple characters are

$$\{\chi_{s,z} | (s) \text{ is a regular semisimple conjugacy class of } G^{*F^*}, z \in H^1(F,Z)\}.$$

## 6.3 Step II: Evaluating $\sigma_z$ in terms of Gauss sums

The goal of this section is to evaluate $\sigma_z$ in terms Gauss sums, along the ideas developed in Chapter 4 of [8]. We shall introduce the definitions and mention the results for the general case closely following [8], with sketches of proofs done in more detail only where that is needed for further computations in the $SU(2,2)$ case.

We have to mention however, that [8] contains several small, but fixable errors. They are not of big consequence for the paper itself, but are crucial for our computations. As a result, the below exposition is meant to fix the errors and give a clear frame of how the computations have to be carried out.

We have $\sigma_z = \sum_{\psi \in \Psi_{z^{-1}}} \psi(u_0)$. Recall from the observation following Theorem 4.3.7, that $\mathcal{L}_T^{-1}(Z)/Z$ acts regularly on the regular characters of $U^F$. As a result, for each $\psi \in \Psi$, there exists a unique $t_\psi \in \mathcal{L}_T^{-1}(Z)/Z$ such that $\psi = {}^{t_\psi} \psi_0$. Also, recall we have the canonical isomorphism $\overline{\mathcal{L}} : H^1(F,Z) = \mathcal{L}_T^{-1}(Z)/ZT^F \to Z/\mathcal{L}_Z(Z)$ given by $\overline{\mathcal{L}}(tZT^F) = \mathcal{L}(t)\mathcal{L}(Z)$. So if $t_\psi$ a representative for $z^{-1}$ in $\mathcal{L}_T^{-1}(Z)/Z$, then $\psi = \psi_{z^{-1}}$. Since $\Psi_{z^{-1}}$ is the $T^F$-orbit of $\psi_{z^{-1}}$ in $\Psi$, we get

$$\sigma_z = \sum_t {}^t\psi_0(u_0), \tag{6.2}$$

where the sum is over the set $\{t \in \mathcal{L}_T^{-1}(Z)/Z | \overline{\mathcal{L}}(t) = z^{-1}\}$. Here, by abuse of notation, we denoted by $\overline{\mathcal{L}}$ the induced map on the quotient $\mathcal{L}_T^{-1}(Z)/Z$.

Recall we fixed $u_0$ as a regular unipotent element in $U^F$, and from (4.7) we have that $u_0$ is an element of $U^F$ having its projection to $U^F/U^{*F}$ be given by

$$\overline{u}_0 = \prod_{\alpha \in \Pi} x_\alpha(1).$$

Now, as $\psi_0 \in \Psi$ is a regular character of $U^F$, is it trivial on $U^{*F}$, so it can be considered as a character of $U^F/U^{*F}$. Thus from (4.5), we have ${}^t\psi_0(u_0) = \psi_0(t^{-1}u_0t)$. As $t^{-1}(\prod_{\alpha\in\Pi} x_\alpha(1))t = \prod_{\alpha\in\Pi} x_\alpha(\alpha(t^{-1}))$, the projection of $t^{-1}u_0t$ to $U/U^*$ is given by $\prod_{\alpha\in\Pi} x_\alpha(\alpha(t^{-1}))$. But as $u_0 \in U^F$, $\prod_{\alpha\in\Pi} x_\alpha(\alpha(t^{-1}))$ must be fixed by $F$. Recall $F(x_\alpha(a)) = x_{\tau\alpha}(a^q)$, so we must have

$$\prod_{\alpha\in\Pi} x_\alpha(\alpha(t^{-1})) = \prod_{I\in\Pi/\tau} x_{\alpha_I}(\alpha_I(t^{-1}))x_{\tau\alpha_I}(\alpha_I(t^{-1})^q)\cdots x_{\tau^{|I|-1}\alpha_I}(\alpha_I(t^{-1})^{q^{|I|-1}}),$$

where $\alpha_I$ is a representative of the $\tau$-orbit $I$ on $\Pi$ and $\alpha_I(t^{-1}) \in \mathbb{F}_{q^{|I|}}^\times$. We then have ${}^t\psi_0(u_0) = \psi_0\Big(\prod_{I\in\Pi/\tau} x_I(\alpha_I(t^{-1}))\Big)$ and using (4.3) and (6.2) we get

$$\sigma_z = \sum_t \chi_0\Big(\sum_{I\in\Pi/\tau} \alpha_I(t)\Big), \tag{6.3}$$

where the sum is over $\{t \in \mathcal{L}_T^{-1}(Z)/Z | \overline{\mathcal{L}}(t) = z\}$.

**Description of the isomorphism $\overline{\gamma}$**

As $H^1(F, Z)$ is a finite abelian group, the structure theorem gives us an isomorphism

$$\overline{\gamma} : H^1(F, Z) \to \prod_{k=1}^r \mu_{d_k},$$

where $\mu_d$ is the group of $d^{\text{th}}$ roots of unity in $\overline{\mathbb{F}}_q$. The goal of the following lemmas is to describe the isomorphism $\overline{\gamma}$ such as to rewrite equation (6.3) in terms of Gauss sums. While we shall state the results in full generality, for the sake of clarity, we will give proofs or sketches of proofs only for the case where the group $G$ is simply connected, more precisely $G = SL_4$, as it is in the case of $SU(2, 2)$ we are concerned with. Also, note the proofs only need minor modifications to work in general.

The canonical isomorphism $\overline{\mathcal{L}} : H^1(F, Z) \to Z/\mathcal{L}_Z(Z)$ induces an isomorphism on the character groups $\hat{\mathcal{L}} : (Z/\mathcal{L}_Z(Z))^\wedge \to (H^1(F, Z))^\wedge$. As $(Z/\mathcal{L}_Z(Z))^\wedge \cong \{\phi \in Z^\wedge | \mathcal{L}(Z) \subseteq \ker \phi\} = (Z/Z^\circ)^{\wedge F} \cong X(Z/Z^\circ)^F \subseteq X(Z)$ as $Z/Z^\circ$ is finite. If we let $\overline{\gamma}_k$ to be the $k^{th}$ component of $\overline{\gamma}$, then the $\overline{\gamma}_k$'s can be thought of as elements of $X(Z)$. Moreover, using the exact sequence

$$0 \to X(T/Z) \to X(T) \to X(Z) \to 0,$$

we can lift the characters $\overline{\gamma}_k$'s to characters $\gamma_k \in X(T)$.

**Lemma 6.3.1.** *Let $\gamma_1, \cdots, \gamma_r \in X(T)$ be arbitrary lifts of $\overline{\gamma}_1, \cdots, \overline{\gamma}_r$. Then $t \in \mathcal{L}_T^{-1}(Z)/Z$ satisfies $\overline{\mathcal{L}}(t) = z$ if and only if $\overline{\gamma}_k(z) = ((F-1)\gamma_k)(t)$ for $k = 1, \cdots, r$.*

*Proof.* First of all, notice that $((F-1)\gamma_k)(t)$ is well defined since $\gamma_k|_Z = \overline{\gamma}_k \in X(Z)^F$, so $\gamma_k \circ (F-1) = (F-1)\gamma_k$ is trivial on $Z$. Now, $\overline{\mathcal{L}}(t) = z$ if and only if all characters of $Z/(F-1)Z = H^1(F, Z)$ take the same values on $z$ and $\overline{\mathcal{L}}(t)$. As a result, $\overline{\mathcal{L}}(t) = z$ if and only if $\overline{\gamma}_k(z) = \overline{\gamma}_k(\overline{\mathcal{L}}(t))$ for all $k \in \{1, \cdots, r\}$. But the last equality is equivalent to $\overline{\gamma}_k(z) = \gamma_k((F-1)t) = ((F-1)\gamma_k)(t)$, which ends the proof. □

Notice that the lifts $\gamma_k$ of $\overline{\gamma}_k$ are only unique modulo $X(T/Z)$. The following lemma gives us a description of the group $X(T/Z)$ that will be useful in choosing lifts of $\overline{\gamma}_k$ in such a manner that will make rewriting equation (6.3) in terms of Gauss sums possible:

**Lemma 6.3.2.** *Let $T$ be the maximal torus of $G$ relative to which the root system $\Phi$ has been defined. We have the short exact sequence*

$$0 \to \mathbb{Z}\Phi \to X(T/Z) \to P \to 0,$$

*where $P$ is the $p$-torsion group of $X/\mathbb{Z}\Phi$, $X = X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and $Z$ is the center of $G$.*

*Proof.* The reasoning below uses ideas presented in Chapter 1, Section 1.12, p. 26 in [4]. Let $T$ be the maximal torus of $G$ such that $X = X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$. For each closed subgroup $S$ of $T$, we define a subgroup $S^\perp$ of $X$ by $S^\perp = \{\chi \in X | \chi(s) = 1 \text{ for all } s \in S\}$. Now let $A$ be a subgroup of $X$. We define a subgroup $A^\perp$ of $T$ by $A^\perp = \{t \in T | \chi(t) = 1 \text{ for all } \chi \in A\}$. Then $A^\perp$

is a closed subgroup of $T$. We can then consider $A^{\perp\perp} \subseteq X$ for each subgroup $A$ of $X$. It is well-known that if $K$ has characteristic $p$, then $A \subseteq A^{\perp\perp}$ and $A^{\perp\perp}/A$ is the $p$-torsion subgroup of $X/A$.

Let us now take $A$ to be the root lattice $\mathbb{Z}\Phi$. Then

$$\mathbb{Z}\Phi^{\perp} = \{t \in T | \chi(t) = 1 \text{ for all } \chi \in \mathbb{Z}\Phi\}.$$

It is enough to find $t \in T$ such that $\alpha(t) = 1$ for all $\alpha \in \Phi$. We know $G = \langle T, U_\alpha, \alpha \in \Phi\rangle$. We have $Z \subseteq T$ and $t \in T$ acts on $U_\alpha$ by $tx_\alpha(a)t^{-1} = x_\alpha(\alpha(t)a)$. Thus, $t \in Z$ if and only if $\alpha(t) = 1$ for all $\alpha \in \Phi$. As a result, $\mathbb{Z}\Phi^{\perp} = Z$.

Consequently,

$$\mathbb{Z}\Phi^{\perp\perp} = \{\chi \in X | \chi(z) = 1 \text{ for all } z \in Z\},$$

so $\mathbb{Z}\Phi^{\perp\perp} = X(T/Z)$. Since $\mathbb{Z}\Phi \subseteq \mathbb{Z}\Phi^{\perp\perp}$, we have a short exact sequence

$$0 \to \mathbb{Z}\Phi \to \mathbb{Z}\Phi^{\perp\perp} \to \mathbb{Z}\Phi^{\perp\perp}/\mathbb{Z}\Phi \to 0,$$

where $\mathbb{Z}\Phi^{\perp\perp} = X(T/Z)$ and $\mathbb{Z}\Phi^{\perp\perp}/\mathbb{Z}\Phi$ is the $p$-torsion subgroup of $X/\mathbb{Z}\Phi$, q.e.d. □

**Lemma 6.3.3.** *The lifts* $\{\gamma_k | k \in \{1, \cdots, r\}\}$ *above can be chosen so that*

$$(F-1)\gamma_k = \sum_{I \in \Pi/\tau} \frac{c_{I,k}}{d_k}(q^{|I|} - 1)\alpha_I,$$

*where* $\alpha_I$ *as in (6.3),* $c_{I,k} \in \mathbb{Z}$ *and if* $c_{I,k} \neq 0$*, then* $c_{I,k}(q^{|I|} - 1)$ *is divisible by* $d_k$*.*

*Proof.* We shall do the proof in the case when $G$ is simply connected, in particular $G = SL_4$. If we let $(X, \Phi, Y, \Phi^\vee)$ be the root system of $G$, $\mathbb{Z}\Phi^\vee \subseteq Y$ will be the coroot lattice. Since $X = X(T)$ is canonically isomorphic to $\mathrm{Hom}(Y, \mathbb{Z})$, we have a restriction map $\mathrm{Hom}(Y, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z}) = \Omega$ which is injective. $\Omega$ is the lattice of weights and we have $\Omega \supseteq X \supseteq \mathbb{Z}\Phi$. Since $G$ is simply connected, we have $X = \Omega$ and as a result, the above restriction map must be an isomorphism. It is also well-known that since $G$ is simply connected, its center $Z$ will be isomorphic to the finite group $\mathrm{Hom}(\Omega/\mathbb{Z}\Phi, \mathbb{G}_m)$. As a result, the finite group $X/\mathbb{Z}\Phi$ will be isomorphic to $Z$. But $Z$ is cyclic of order 4 in this case, so it has no $p$-torsion and thus Lemma 6.3.2 gives us $X(T/Z) \cong \mathbb{Z}\Phi$.

Now, since $\overline{\gamma}_k$ has order $d_k$, for any lift $\gamma_k$ we will have $d_k\gamma_k$ trivial on $Z$. Thus $d_k\gamma_k \in X(T/Z) = \mathbb{Z}\Phi$, so $\gamma_k = \sum_{\alpha\in\Pi} \frac{c_{\alpha,k}}{d_k}\alpha$ for certain integers $c_{\alpha,k}$. As observed before in Lemma 6.3.1, $(F-1)\gamma_k$ must be trivial on $Z$ as well, so we must have $(F-1)\gamma_k \in \mathbb{Z}\Phi$. Therefore, since we know from (4.2) in Subsection 4.3, that $\alpha(F(t)) = (q\tau^{-1}\alpha)(t)$ we have

$$(F-1)\gamma_k = \sum_{\alpha\in\Pi} \frac{c_{\alpha,k}}{d_k} q\tau^{-1}\alpha - \sum_{\alpha\in\Pi} \frac{c_{\alpha,k}}{d_k}\alpha \in \mathbb{Z}\Phi$$

and as a result

$$\frac{qc_{\tau\alpha,k} - c_{\alpha,k}}{d_k} \in \mathbb{Z}. \tag{6.4}$$

Now, let us fix $I \in \Pi/\tau$ and define $c'_{\tau^{-j}\alpha_I,k} = q^j c_{\alpha_I,k}$ for $j = 0, 1, \cdots, |I|-1$, where $\alpha_I$ is a representative of the orbit $I$ as seen before. Let $\gamma'_k = \sum_{\alpha\in\Pi} \frac{c'_{\alpha,k}}{d_k}\alpha$. We shall prove $\gamma'_k$ is another lift for $\overline{\gamma}_k$. Indeed, we have

$$\begin{aligned}
\gamma'_k - \gamma_k &= \sum_{\alpha\in\Pi} \frac{c'_{\alpha,k} - c_{\alpha,k}}{d_k}\alpha \\
&= \sum_{I\in\Pi/\tau} \sum_{j=0}^{|I|-1} \frac{c'_{\tau^{-j}\alpha_I,k} - c_{\tau^{-j}\alpha_I,k}}{d_k} \tau^{-j}\alpha_I \\
&= \sum_{I\in\Pi/\tau} \sum_{j=0}^{|I|-1} \frac{q^j c_{\alpha_I,k} - c_{\tau^{-j}\alpha_I,k}}{d_k} \tau^{-j}\alpha_I.
\end{aligned}$$

But from (6.4), we know that $\frac{q^j c_{\alpha,k} - c_{\tau^{-j}\alpha,k}}{d_k} \in \mathbb{Z}$ for $j = 1, \cdots, |I|-1$ and as a result $\gamma'_k - \gamma_k \in \mathbb{Z}\Phi = X(T/Z)$. Thus $\gamma_k$ is indeed another lift of $\overline{\gamma}_k$, where

$$\gamma'_k = \sum_{I\in\Pi/\tau} \sum_{j=0}^{|I|-1} \frac{c'_{\tau^{-j}\alpha_I,k}}{d_k} \tau^{-j}\alpha_I = \sum_{I\in\Pi/\tau} \sum_{j=0}^{|I|-1} \frac{q^j c_{\alpha_I,k}}{d_k} \tau^{-j}\alpha_I.$$

Thus

$$\begin{aligned}
(F-1)\gamma'_k &= \sum_{I\in\Pi/\tau} \sum_{j=0}^{|I|-1} \frac{q^j q c_{\alpha_I,k}}{d_k} \tau^{-j-1}\alpha_I - \sum_{I\in\Pi/\tau} \sum_{j=0}^{|I|-1} \frac{q^j c_{\alpha_I,k}}{d_k} \tau^{-j}\alpha_I \\
&= \sum_{I\in\Pi/\tau} \frac{c_{\alpha_I,k}}{d_k} (q^{|I|}-1)\alpha_I,
\end{aligned}$$

so $\gamma'_k$ is a lift of $\overline{\gamma}_k$ satisfying the desired property, where $c_{I,k} = c_{\alpha_I,k}$.

□

Now, let $\gamma_k$ be lifts as in Lemma 6.3.3. We then have

$$(F-1)\gamma_k(t) = \prod_{I \in \Pi/\tau} \alpha_I(t)^{c_{I,k}(q^{|I|}-1)/d_k}$$

for $t \in T$. As a result, by Lemma 6.3.1, if $z \in H^1(F, Z)$ such that $\overline{\mathcal{L}}(t) = z$, $\overline{\gamma}_k(z) = \prod_{I \in \Pi/\tau} \alpha_I(t)^{c_{I,k}(q^{|I|}-1)/d_k}$. On the other hand, from equation (6.3) we have

$$\sigma_z = \sum_t \prod_{I \in \Pi/\tau} \chi_0(\alpha_I(t)),$$

where the sum is over $\{t \in \mathcal{L}_Z^{-1}(Z)/Z | \overline{\mathcal{L}}(t) = z\}$. As a result, we have

$$\sigma_z = \sum_{(s_I)_{I \in \Pi/\tau}} \prod_{I \in \Pi/\tau} \chi_0(s_I), \tag{6.5}$$

where $(s_I) \in \prod_{I \in \Pi/\tau} \mathbb{F}^\times_{p^{|I|}}$ such that $\overline{\gamma}_k(z) = \prod_{I \in \Pi/\tau} s_I^{c_{I,k}(q^{|I|}-1)/d_k}$.

### Computation of $\sigma_z$ in terms of Gauss sums

The next step in evaluating $\sigma_z$ above in terms of Gauss sums is to introduce the "Mellin transforms" of the $\sigma_z$:

**Definition 6.3.4.** For $\zeta \in (H^1(F, Z))^\wedge$, define $\sigma_\zeta = \sum_{z \in H^1(F,Z)} \zeta(z)\sigma_z$.

Since $\zeta(z_0^{-1})\sigma_\zeta = \sum_{z \in H^1(F,Z)} \zeta(zz_0^{-1})\sigma_z$, we get

$$\sum_{\zeta \in (H^1(F,Z))^\wedge} \zeta(z_0^{-1})\sigma_\zeta = \sum_{z \in H^1(F,Z)} \sum_{\zeta \in (H^1(F,Z))^\wedge} \zeta(zz_0^{-1})\sigma_z.$$

We know all sums $\sum_{\zeta \in (H^1(F,Z))^\wedge} \zeta(x)$ vanish on $x \neq 1$, and thus we have an inversion formula by

$$\sigma_z = |H^1(F, Z)|^{-1} \sum_{\zeta \in (H^1(F,Z))^\wedge} \zeta(z^{-1})\sigma_\zeta.$$

As a result computing $\sigma_z$ is equivalent to computing $\sigma_\zeta$. The main result evaluating $\sigma_\zeta$ in terms of the Gauss sums defined in Definition 6.1.1 is as follows:

**Theorem 6.3.5.** *With the above notations, we have*

$$\sigma_\zeta = \prod_{I\in\Pi/\tau} G\Big(\prod_k \zeta_{I,k}^{c_{I,k}(q^{|I|}-1,d_k)/d_k}\Big),$$

*where* $\zeta_{I,k}$ *is a multiplicative character of* $\mathbb{F}_{q^{|I|}}^\times$ *defined by* $\zeta_{I,k}(x)=\zeta(z)$*. Here* $z\in\mu_{d_k}\subseteq H^1(F,Z)$ *is defined by* $\overline{\gamma}_k(z) = x^{(q^{|I|}-1)/(q^{|I|}-1,d_k)}$ *for each* $k$*.*

*Proof.* First, recall from Lemma 6.3.3 that $c_{I,k}(q^{|I|}-1)/d_k\in\mathbb{Z}$, so $c_{I,k}(q^{|I|}-1,d_k)/d_k\in\mathbb{Z}$ as $d_k/(q^{|I|}-1,d_k)$ and $(q^{|I|}-1)/(q^{|I|}-1,d_k)$ are coprime. Thus $\zeta_{I,k}^{c_{I,k}(q^{|I|}-1,d_k)/d_k}$ in the statement of the theorem makes sense.

Using equation (6.5) in definition 6.3.4, we have

$$\sigma_\zeta = \sum_{z\in H^1(F,Z)} \zeta(z) \sum_{(s_I)_{I\in\Pi/\tau}} \prod_{I\in\Pi/\tau} \chi_0(s_I),$$

where the inner sum is over $(s_I)\in\prod_{I\in\Pi/\tau}\mathbb{F}_{p^{|I|}}^\times$ such as in equation (6.5). Now, for each $k$, $\overline{\gamma}_k(z)=\prod_{I\in\Pi/\tau} s_I^{c_{I,k}(q^{|I|}-1)/d_k}$ implies $\prod_{I\in\Pi/\tau}\zeta_{I,k}(s_I)^{c_{I,k}(q^{|I|}-1,d_k)/d_k}=\zeta(z)$. This is not hard to see, as if we consider $z_I$ such that $\overline{\gamma}_k(z_I)=s_I^{c_{I,k}(q^{|I|}-1)/d_k}$, then we have $\zeta_{I,k}(s_I)^{c_{I,k}(q^{|I|}-1,d_k)/d_k}=\zeta(z_I)$. Moreover, one can prove $z$ must be of the form $\prod_{I\in\Pi/\tau} z_I$ so then $\zeta(z)=\prod_{I\in\Pi/\tau}\zeta(z_I)=\prod_{I\in\Pi/\tau}\zeta_{I,k}(s_I)^{c_{I,k}(q^{|I|}-1,d_k)/d_k}$ for $z\in\mu_{d_k}\subseteq H^1(F,Z)$. We note there does not need to be a unique way to write $z$ as a product of $z_I$'s for $I\in\Pi/\tau$. Moreover as $z$ runs through $H^1(F,Z)$, $(s_I)_{I\in\Pi/\tau}$ runs through $\prod_{I\in\Pi/\tau}\mathbb{F}_{p^{|I|}}^\times$.

We then have

$$\begin{aligned}
\sigma_\zeta &= \sum_{(s_I)_{I\in\Pi/\tau}} \Big(\prod_k\prod_{I\in\Pi/\tau} \zeta_{I,k}(s_I)^{c_{I,k}(q^{|I|}-1,d_k)/d_k}\Big)\Big(\prod_{I\in\Pi/\tau}\chi_0(s_I)\Big)\\
&= \sum_{(s_I)_{I\in\Pi/\tau}} \Big(\prod_{I\in\Pi/\tau}\prod_k \zeta_{I,k}(s_I)^{c_{I,k}(q^{|I|}-1,d_k)/d_k}\Big)\Big(\prod_{I\in\Pi/\tau}\chi_0(s_I)\Big)\\
&= \prod_{I\in\Pi/\tau}\Big(\sum_{s_I\in\mathbb{F}_{p^{|I|}}^\times}\Big(\prod_k \zeta_{I,k}^{c_{I,k}(q^{|I|}-1,d_k)/d_k}\Big)(s_I)\chi_0(s_I)\Big),
\end{aligned}$$

so $\sigma_\zeta=\prod_{I\in\Pi/\tau} G\Big(\prod_k \zeta_{I,k}^{c_{I,k}(q^{|I|}-1,d_k)/d_k}\Big)$, which is what we wanted to prove.

∎

*Chapter 7*

# STUDY CASE $SU(2,2)$, PART II: A FORMULA FOR MULTIPLICITY DEFECT

The goal of this chapter is to give a formula for multiplicity defect in the $SU(2,2)$ case, which is done in Theorem 7.0.8. In particular, we show the alternating sum $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i$ on regular unipotent classes is given in terms of Gauss sums over $\mathbb{F}_{p^2}^{\times}$. Besides proving the desired multiplicity defect, this suggests possible interpretations of $\Delta M_\pi$ for $SU(2,2)$ in terms of certain arithmetic invariants that would be analogues of the class number appearing in Hecke's original problem for $SL_2$. The explicit derivations in this chapter are fully based on the theory developed in the previous chapter.

**Preliminaries**

Recall $G = SL_4(\overline{\mathbb{F}}_p)$. As introduced in Section 5.1, the Frobenius map $F$ is defined by $(a_{i,j}) \mapsto J^{-1}((a_{i,j}^p)^t)^{-1}J$, where $J$ is given by the hermitian form $\begin{pmatrix} 0 & 0 & 0 & \beta \\ 0 & 0 & \beta & 0 \\ 0 & -\beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \end{pmatrix}$, with $\overline{\beta} = -\beta$. The elements of $g \in G$ that are fixed by $F$ are such that $\overline{g}^t Jg = J$, where complex conjugation is given by entrywise raising to the p$^{th}$-power map. Therefore, $G^F$ is $SU(2,2)$ over the finite field of $p$ elements.

We are in the case where $Z \cong \mu_4$ is cyclic and $H^1(F,Z) = Z/(F-1)Z \cong \mu_4/\mu_4^{p+1} \cong \mu_d$, where $d = (4, p+1)$. Thus $\overline{\gamma} : H^1(F,Z) \to \mu_d$ is an isomorphism and we want to choose a lift $\gamma \in X(T)$ in the form presented in Lemma 6.3.3.

First, let us describe $\Phi, \Pi$ and the orbits $I \in \Pi/\tau$ of the induced action of $F$ on $\Pi$. Let $B$ be the subgroup of upper triangular matrices. It is easy to check $B$ is $F$-stable. Let $T \subseteq B$ be the $F$-stable maximal torus consisting of diagonal matrices and $U$ the subgroup of unipotent upper triangular matrices. It is then clear that the usual computations of the roots of $SL_4$ with respect to $B$ and $T$ go through, so we have $\Phi$ such that the set of simple roots is given by $\Pi = (\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_i(t) = t_i t_{i+1}^{-1}$ for a typical element $t \in T$,

$$t = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix} \text{ with } \prod_{i=1}^{4} t_i = 1.$$

Recall that the action of $F$ on $X(T)$ defines a permutation $\tau$ on the set of simple roots $\Pi$. We have the following result:

**Lemma 7.0.1.** *In the above notation, we have two orbits of $\tau$ in $\Pi$, namely $I_1 = (\alpha_1, \alpha_3)$, $I_2 = (\alpha_2)$.*

*Proof.* The checks are trivial. Recall from (4.2) in Subsection 4.3, that the induced action of $\tau$ on $\Pi$ is given by $F(\alpha) = q\tau^{-1}\alpha$, so we have

$$\alpha(F(t)) = (\tau^{-1}\alpha)(t)^p.$$

As a result $\alpha_1(F(t)) = \alpha_1 \begin{pmatrix} t_4^{-p} & 0 & 0 & 0 \\ 0 & t_3^{-p} & 0 & 0 \\ 0 & 0 & t_2^{-p} & 0 \\ 0 & 0 & 0 & t_1^{-p} \end{pmatrix} = t_4^{-p}t_3^p = \alpha_3(t)^p$. On the other hand, $\alpha_2(F(t)) = t_3^{-p}t_2^p = \alpha_2(t)^p$. □

**Lemma 7.0.2.** *Maintaining the above notation, a lift of the type presented in Lemma 6.3.3 for the isomorphism $\overline{\gamma} : H^1(F, Z) \to \mu_d$ is given by*

$$\gamma = \frac{1}{d}(\alpha_1 + 2\alpha_2 + p\alpha_3),$$

*where $d = (4, p + 1)$.*

*Proof.* Per the comment before Lemma 6.3.1, we first want to check $\overline{\gamma} \in X(Z/Z^\circ)^F$. We have

$$\gamma(t) = (t_1t_2^{-1}(t_2t_3^{-1})^2(t_3t_4^{-1})^p)^{1/d} = (t_1^{p+1}t_2^{p+1}t_3^{2p-2})^{1/d}$$

and as both $p + 1$ and $2(p - 1)$ are divisible by $d$, we have $\gamma \in X(T)$. Since $Z \cong \mu_4$, we also have $\overline{\gamma} \in X(Z)^F \cong X(Z/Z^\circ)^F$. As a result, $(F - 1)\gamma$ must be trivial on $Z$. But it is easy to see, via the same computations as in the proof of Lemma 6.3.3, that $(F - 1)\gamma = \frac{1}{d}((p^2 - 1)\alpha_1 + 2(p - 1)\alpha_2)$. As $p^2 - 1$ and $2(p - 1)$ are divisible by $d$, we have $(F - 1)\gamma \in \mathbb{Z}\Phi \cong X(T/Z)$ and thus $(F - 1)\gamma$ is trivial on $Z$. As a result, we checked $\gamma$ is a lift in $X(T)$ such that restriction $\overline{\gamma}$ to $X(Z)$ is indeed an element of $X(Z/Z^{\circ})^{F}$. Moreover, we checked $\gamma$ is of the desired form of Lemma 6.3.3.

We are left to checking $\overline{\gamma}$ is an isomorphism between $H^1(F,Z)$ and $\mu_d$. As $\overline{\gamma} = \gamma|_Z$, we have $\overline{\gamma}(z_\omega) = \omega^{4p/d}$, where $\omega$ is a primitive $4^{\text{th}}$ root of unity and $z_\omega$ is the scalar matrix $\omega I_4$. It is trivial to check that $z_\omega$ is a generator for $H^1(F,Z)$, so we must check $\overline{\gamma}(z_\omega)$ generates $\mu_d$. Clearly, $\omega^{4p/d}$ is a generator for $\mu_d$, for both when $d=2$ and $d=4$, $\overline{\gamma}$ is indeed a generator for $(H^1(F,Z))^{\wedge}$. $\square$

*Remark.* Since we are in the case of $G$ simply connected, we have $X(T)=\Omega$, so $\gamma$ is in fact an element of the lattice of weights. As a sanity check, one can compute the fundamental weights of $G$ with respect to $B,T$ above, as being given by $\lambda_1,\lambda_2,\lambda_3$, where $\lambda_i = t_1\cdots t_i$. One can check $\lambda_2 = 2\lambda_1-\alpha_1, \lambda_3 = 3\lambda_1-2\alpha_1-\alpha_2$. Clearly these weights form a basis for $\Omega$ and one can check that $\Omega/\mathbb{Z}\Phi$ is indeed cyclic of order 4 generated by $\lambda_1$, so isomorphic to $Z$, as stated in the proof of Lemma 6.3.3. Now notice that $\gamma(t) = (t_1^{p+1}t_2^{p+1}t_3^{2p-2})^{1/d} = (t_1t_2t_3)^{2(p-1)/d}(t_1t_2)^{(3-p)/d}$, so $\gamma = 2(p-1)/d\lambda_3 + (3-p)/d\lambda_2 \in \Omega$ as $d = (p+1,4)$. For more details on fundamental weights, one can refer to Chapter 15 in [28].

Thus, we have two orbits, $I_1 = (\alpha_1,\alpha_3)$, $I_2 = (\alpha_2)$, and note that $c_{I_1} = 1$, $c_{I_2} = 2$. We need to find the characters $\zeta_{I_1}$ of $\mathbb{F}_{p^2}^{\times}$ and $\zeta_{I_2}$ of $\mathbb{F}_p^{\times}$ in order to compute $\sigma_\zeta$, which is will be done in the following subsections, depending on whether $(4,p+1)$ is 2 or 4.

**I. Case of $p \equiv 1 \bmod 4$**

We have $d=2$, so there are only two characters $\zeta$ of $H^1(F,Z)^{\wedge}$. Clearly, if $\zeta$ is trivial, $\zeta_{I_1},\zeta_{I_2}$ will be both trivial characters and a result

$$\sigma_1 = G(1)^2 = 1.$$

Let us consider the case $\zeta$ is nontrivial. We have $H^1(F,Z)\cong\mu_2$ and it is easy to see $z_1$ and $z_\omega$ can be taken as representatives of the $F$-conjugacy classes in $H^1(F,Z)$. Here $\omega$ is a primitive $4^{\text{th}}$ root of unity and $z_{\omega^i}$ for $i\in\{1,2,3,4\}$ is given by scalar matrices $\omega^i I_4$. Indeed, $(F-1)(z_{\omega^i}) = z_{\omega^{-i(p+1)}} = z_{\omega^{2i}} \in \{I_4,-I_4\}$ and thus $z_1, z_{\omega^2}$ are in $F$-conjugate, while $z_\omega, z_{\omega^3}$ are $F$-conjugate as well.

Recall $\zeta_I(x) = \zeta(z)$, where $z\in H^1(F,Z)$ is such that $\overline{\gamma}(z) = x^{(p^{|I|}-1)/2}$. Given the canonical isormorphism $\overline{\mathcal{L}} : H^1(F,Z)\to Z/\mathcal{L}(Z)$, recall from Lemma 6.3.1

that $\overline{\gamma}(z) = ((F-1)\gamma)(t)$ if and only if $t \in \mathcal{L}_T^{-1}(Z)/Z$ such that $\overline{\mathcal{L}}(t) = z$. Moreover, following the result of Lemma 6.3.3, our choice of $\gamma$ from Lemma 7.0.2 gives us

$$((F-1)\gamma)(t) = \prod_{I \in \Pi/\tau} \alpha_I(t)^{(p^{|I|}-1)c_I/2}.$$

In particular, $\alpha_{I_1} = \alpha_1$, $\alpha_{I_2} = \alpha_2$, so we get

$$\overline{\gamma}(z) = \alpha_1(t)^{(p^2-1)/2}\alpha_2(t)^{p-1},$$

where $t \in \mathcal{L}_T^{-1}(Z)/Z$ such that $\overline{\mathcal{L}}(t) = z$.

As a result, in order to determine the characters $\zeta_I$, we need to find all $t \in \mathcal{L}_T^{-1}(Z)/Z$ such that $\overline{\mathcal{L}}(t) = z$, which is what the following lemma gives us:

**Lemma 7.0.3.** *For each $z \in H^1(F,Z) = \mathcal{L}_T^{-1}(Z)/ZT^F$, let $t_z \in \mathcal{L}_T^{-1}(Z)/Z$ be a representative, where $t_z \in T$ is of the form $\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix}$ with $\prod_i t_i = 1$, $t_i \in \overline{\mathbb{F}}_p$. Then $t_z$ is determined by $t_1^{-1}t_4^{-p} = t_2^{-1}t_3^{-p} = t_3^{-1}t_2^{-p} = t_4^{-1}t_1^{-p} = \omega^i$, where $z = z_{\omega^i}$.*

*Proof.* It is easy to see that

$$\mathcal{L}(t) = t^{-1}F(t) = \begin{pmatrix} t_1^{-1}t_4^{-p} & 0 & 0 & 0 \\ 0 & t_2^{-1}t_3^{-p} & 0 & 0 \\ 0 & 0 & t_3^{-1}t_2^{-p} & 0 \\ 0 & 0 & 0 & t_4^{-1}t_1^{-p} \end{pmatrix},$$

and thus the result follows trivially. □

Note that for the case of $p \equiv 1 \bmod 4$, we have $i \in \{0,1\}$ as $z_1 \sim z_{\omega^2}$, $z_\omega \sim z_{\omega^3}$.

**Lemma 7.0.4.** *Let $\zeta \in H^1(F,Z)^\wedge$ be the nontrivial character in the above notation, in the case when $p \equiv 1 \bmod 4$. Then the multiplicative characters $\zeta_{I_1}$ and $\zeta_{I_2}$ introduced in Theorem 6.3.5 are the unique character of degree two of the cyclic group $\mathbb{F}_{p^2}^\times$, respectively a character of $\mathbb{F}_p^\times$ that takes the value 1 on the squares $\bmod p$.*

*Proof.* As seen above

$$\overline{\gamma}(z) = \alpha_1(t)^{(p^2-1)/2}\alpha_2(t)^{p-1},$$

where $t \in \mathcal{L}_T^{-1}(Z)/Z$ such that $\overline{\mathcal{L}}(t) = z$. Also $\zeta_I(x) = \zeta(z)$, where $z \in H^1(F, Z)$ is defined by $\overline{\gamma}(z) = x^{(p^{|I|}-1)/2}$. We need to find $z_I$, as in the proof of Lemma 6.3.5, such that $\overline{\gamma}(z_I) = \alpha_I(t_I)^{c_I(p^{|I|}-1)/2}$.

Let us first determine $\zeta_{I_1}$. Choose $z_{I_1}$ such that $\alpha_{I_2}(t_{I_1}) = 1$, that is, $t_{I_1} = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & t^{-1}\lambda^{-2} \end{pmatrix}$ will be a representative for $z_{I_1}$, with $\lambda, t \in \overline{\mathbb{F}}_p^\times$. Following the result of Lemma 7.0.3, $t_{I_1}$ is determined by the following equations

$$t^{p-1}\lambda^{2p} = \omega^i \tag{7.1}$$

$$\lambda^{-1-p} = \omega^i \tag{7.2}$$

$$t^{1-p}\lambda^2 = \omega^i, \tag{7.3}$$

where $z_{I_1} = z_{\omega^i}$. First, it is easy to see that $\lambda, t \in \mathbb{F}_{p^2}^\times$. Indeed, from (7.3), $t^{p^2-p}\lambda^{-2p} = \omega^{-pi}$, so multiplying this equation with (7.1), we get $t^{p^2-1} = \omega^{(1-p)i} = 1$ since $p \equiv 1 \bmod 4$. So $t \in \mathbb{F}_{p^2}^\times$ and the proof that $\lambda \in \mathbb{F}_{p^2}^\times$ follows from equation (7.2). As a result, WLOG, let us take $g$ a generator of $\mathbb{F}_{p^2}^\times$ such that $g^{(p^2-1)/4} = \omega$. We thus have $t = g^x, \lambda = g^y$ for some $x, y \in \mathbb{Z}/(p^2-1)\mathbb{Z}$.

From (7.2), we have $-(1+p)y = (p^2-1)i/4 + (p^2-1)k_1$ for $k_1 \in \mathbb{Z}$. Thus

$$y = \frac{1-p}{4}i + (1-p)k_1.$$

On the other hand, from (7.1), we have $(p-1)x + 2py = (p^2-1)i/4 + (p^2-1)k_2$ for $k_2 \in \mathbb{Z}$, so $(p-1)x = (p^2-1)i/4 + (p^2-1)k_2 + 2p(p-1)i/4 + 2p(p-1)k_1$. Then

$$x = \frac{1+3p}{4}i + 2pk_1 + (1+p)k_2.$$

From (7.3), we have $(1-p)x + 2y = (1-p)(1+3p)i/4 + (1-p)2pk_1 + (1-p)(1+p)k_2 + 2(1-p)i/4 + 2(1-p)k_1 = 3(1-p^2)i/4 + 2(1-p^2)k_1 + (1-p^2)k_2$ and thus $t^{1-p}\lambda^2 = g^{-3(p^2-1)i/4} = \omega^i$. Thus (7.3) is verified.

We then have

$$x - y = pi + (3p-1)k_1 + (1+p)k_2$$

$$= -i + (1+p)(i+k_2) + (3p-1)k_1.$$

As a result $\alpha_1(t_{I_1})^{(p^2-1)/2} = g^{(x-y)(p^2-1)/2}$. But $(1+p, 3p-1) = 2$ as $p \equiv 1 \bmod 4$, so there exists $a, b \in \mathbb{Z}$ such that $(1+p)a + (3p-1)b = 2$. Thus, one can choose $k_1, k_2 \in \mathbb{Z}$ such that

$$\alpha_1(t_{I_1})^{(p^2-1)/2} = (g^{-i+2r})^{(p^2-1)/2},$$

where $r \in \mathbb{Z}$.

It is now easy to compute $\zeta_{I_1}$. If $i = 0$ or $i = 2$, then $z_{I_1} = z_1$ or $z_{I_1} = z_{\omega^2}$, and we have $\overline{\gamma}(z_{I_1}) = (g^{-i+2r})^{(p^2-1)/2}$, so $\zeta_{I_1}(g^{2r}) = \zeta_{I_1}(g^{-2+2r}) = 1$ for all $r \in \mathbb{Z}$. This holds because $z_1 \sim z_{\omega^2}$. Now, if $i = 1$ or $i = 3$, that is $z_{I_1} = z_\omega$ or $z_{I_1} = z_{\omega^3}$, then $\overline{\gamma}(z_{I_1}) = (g^{-i+2r})^{(p^2-1)/2}$, so $\zeta_{I_1}(g^{-1+2r}) = \zeta_{I_1}(g^{-3+2r}) = -1$. The last step is true since $\zeta(z_\omega) = \zeta(z_{\omega^3}) = 1$ as $\zeta$ is the nontrivial character of $\mu_2$. Thus $\zeta_{I_1}$ is the multiplicative character of $\mathbb{F}_{p^2}^\times$ which takes the value 1 on the squares in $\mathbb{F}_{p^2}^\times$ and the value $-1$ on the non-squares.

One can compute $\zeta_{I_2}$ as a character of $\mathbb{F}_p^\times$ in a similar fashion. Choose $z_{I_2}$ such that $\alpha_{I_1}(t_{I_2}) = 1$, that is, $t_{I_2} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t^{-1}\lambda^{-2} \end{pmatrix}$ will be a representative for $z_{I_2}$, with $\lambda, t \in \overline{\mathbb{F}}_p^\times$. As before, $t_{I_2}$ is determined by

$$\lambda^{2p-1}t^p = \omega^i \tag{7.4}$$

$$\lambda^{-1}t^{-p} = \omega^i \tag{7.5}$$

$$t^{-1}\lambda^{-p} = \omega^i \tag{7.6}$$

$$t\lambda^{2-p} = \omega^i, \tag{7.7}$$

where $z_{I_2} = z_{\omega^i}$. First, it is easy to see that $\lambda, t \in \mathbb{F}_{p^2}^\times$. Indeed, from (7.6), $t^p\lambda^{p^2} = \omega^{-pi}$, so multiplying this equation with (7.5), we get $\lambda^{p^2-1} = \omega^{(1-p)i} = 1$ since $p \equiv 1 \bmod 4$. So $\lambda \in \mathbb{F}_{p^2}^\times$ and the proof that $t \in \mathbb{F}_{p^2}^\times$ follows exactly the same. As before, let $g$ be a generator of $\mathbb{F}_{p^2}^\times$ such that $g^{(p^2-1)/4} = \omega$, and let $t = g^x, \lambda = g^y$ for some $x, y \in \mathbb{Z}/(p^2-1)\mathbb{Z}$.

From (7.5), we have $-y - px = (p^2-1)i/4 + (p^2-1)k_1$ for $k_1 \in \mathbb{Z}$. Thus

$$y = \frac{1-p^2}{4}i + (1-p^2)k_1 - px.$$

On the other hand, from (7.4), we have $(2p-1)y+px = (p^2-1)i/4+(p^2-1)k_2$ for $k_2 \in \mathbb{Z}$, so $2p(1-p)x = (p^2-1)i/4 + (p^2-1)k_2 + (2p-1)(p^2-1)i/4 + (2p-1)(p^2-1)k_1$. Thus $2px = -(1+p)2pi/4 - 2p(1+p)k_1 + (1+p)(k_1-k_2)$. As a result, $p$ must divide $k_1 - k_2$, so $k_1 - k_2 = pk_3$ for some $k_3 \in \mathbb{Z}$. Then

$$x = -\frac{1+p}{4}i - (1+p)k_1 + \frac{1+p}{2}k_3.$$

From (7.6), we have $-x - py = p(p^2-1)i/4 + p(p^2-1)k_1 + (p^2-1)x = p(p^2-1)i/4+p(p^2-1)k_1-(p^2-1)(1+p)i/4-(p^2-1)(1+p)k_1+(p^2-1)(1+p)k_3/2$, so $t^{-1}\lambda^{-p} = g^{-(p^2-1)i/4} = \omega^{-i}$. Thus we must have $\omega^{-i} = \omega^i$, which implies $i \in \{0,2\}$. Thus $z_{I_2}$ must be $z_1 \sim z_{\omega^2}$.

On the other hand, $x+(2-p)y = (p-1)^2x+(2-p)(1-p^2)i/4+(2-p)(1-p^2)k_1 = -(p-1)^2(1+p)i/4 - (p-1)^2(1+p)k_1 + (p-1)^2(1+p)k_3/2 + (2-p)(1-p^2)i/4+(2-p)(1-p^2)k_1$. Thus $t\lambda^{2-p} = g^{-(p^2-1)i/4} = \omega^{-i} = \omega^i$, so (7.4) is also verified.

We then have

$$\begin{aligned} y - x &= \frac{1+p}{2}i + 2(1+p)k_1 - \frac{(1+p)^2}{2}k_3 \\ &= (1+p)\left(\frac{i}{2} + 2k_1 - \frac{p+1}{2}k_3\right). \end{aligned}$$

As a result $\alpha_2(t_{I_2})^{p-1} = g^{(y-x)(p-1)}$. But as $(2,(1+p)/2) = 1$, there exists $a, b \in \mathbb{Z}$ such that $2a + (1+p)b/2 = 1$. Thus, one can choose $k_1, k_3 \in \mathbb{Z}$ such that

$$\alpha_2(t_{I_2})^{p-1} = \left(g^{(1+p)(i/2+r)}\right)^{p-1},$$

where $r$ is any integer.

It is now easy to compute $\zeta_{I_2}$. Note that since $g$ is a generator for $\mathbb{F}_{p^2}^\times$, $g' := g^{1+p}$ will be a generator for $\mathbb{F}_p^\times$. If $z_{I_2} = z_1$, that is, $i = 0$, we have $\overline{\gamma}(z_{I_2}) = (g'^{2r})^{(p-1)/2}$, so $\zeta_{I_2}(g'^{2r}) = 1$ for all $r \in \mathbb{Z}$. Let us check the case $i = 2$ gives the same result. Indeed, if $z_{I_2} = z_{\omega^2}$, then $\overline{\gamma}(z_{I_2}) = (g'^{2+2r})^{(p-1)/2}$, so $\zeta_{I_2}(g'^{2+2r}) = \zeta(z_{\omega^2}) = 1$. Thus $\zeta_{I_2}$ takes the value 1 on the squares in $\mathbb{F}_p^\times$.

□

We can now state the main result of this subsection:

**Proposition 7.0.5.** *With the above notations, we have*

$$\sigma_\zeta = \begin{cases} 1 & \text{for } \zeta \text{ the trivial character of } H^1(F,Z) \cong \mu_2 \\ -G(\zeta_{I_1}) & \text{for } \zeta \text{ the non-trivial character of } H^1(F,Z) \cong \mu_2 \end{cases},$$

*where $G(\zeta_{I_1}) = G(\zeta_{I_1}, \chi_0)$ is the Gauss sum defined by the unique non-trivial quadratic character $\zeta_{I_1} \in (\mathbb{F}_{p^2}^{\times})^{\wedge}$. In particular, $\sigma_\zeta$ is always non-zero; moreover when $\zeta$ is non-trivial, we have $|\sigma_\zeta| = p$.*

*Proof.* We have already seen the result for the case when $\zeta$ is the trivial character. When $\zeta$ is non-trivial, by Theorem 6.3.5 we have

$$\sigma_\zeta = G(\zeta_{I_1})G(\zeta_{I_2}^2)$$

and using the result of Lemma 7.0.4 above, we know that $\zeta_{I_2}^2$ is a trivial character of $\mathbb{F}_p^{\times}$. As a result $G(\zeta_{I_2}^2) = -1$. The rest of the result follows trivially from Proposition 6.1.3.

□

**II. Case of $p \equiv 3 \bmod 4$**

We have $d = 4$ and thus we have four characters $\zeta$ of $H^1(F, Z)^{\wedge}$, let us denote them $\zeta_j$ for $i \in \{0, 1, 2, 3\}$. Note the $\zeta_j$'s are given by

$$\omega \mapsto \omega^j,$$

where $\omega$ is a primitive $4^{\text{th}}$ root of unity. We think of $H^1(F, Z) \cong \mu_4$ as a cyclic multiplicative group of order 4 generated by $\omega$. Clearly, $\zeta_0$ is trivial.

As $H^1(F, Z) \cong \mu_4$, the elements of $H^1(F, Z)$ are given by $z_{\omega^i}$ for $i \in \{0, 1, 2, 3\}$, where $z_{\omega^i}$ is given by scalar matrices $\omega^i I_4$, as seen in the previous case.

We follow the same reasoning as in the case of $p \equiv 1 \bmod 4$. Thus we again have

$$\overline{\gamma}(z) = \alpha_1(t)^{(p^2-1)/4}\alpha_2(t)^{(p-1)/2},$$

where $t \in \mathcal{L}_T^{-1}(Z)/Z$ such that $\overline{\mathcal{L}}(t) = z$. The result of Lemma 7.0.3 for all $i \in \{0, 1, 2, 3\}$ gives us all $t \in \mathcal{L}_T^{-1}(Z)/Z$ such that $\overline{\mathcal{L}}(t) = z$.

We now have all the ingredients to compute the multiplicative characters $\zeta_{I_1}, \zeta_{I_2}$ as follows:

**Lemma 7.0.6.** *Let $\zeta_j \in H^1(F, Z)^{\wedge}$ be a character in the above notation of the case when $p \equiv 3 \bmod 4$, for $j \in \{0, 1, 2, 3\}$. Then the multiplicative characters $\zeta_{jI_1}$ of $\mathbb{F}_{p^2}^{\times}$ and $\zeta_{jI_2}$ of $\mathbb{F}_p^{\times}$ introduced in Theorem 6.3.5 are as follows:*

1. *$\zeta_{jI_1}(g'^{4r-i}) = \omega^{ij}$ for $i, j \in \{0, 1, 2, 3\}$, $r \in \mathbb{Z}$, so the $\zeta_{jI_1}$'s are the four characters of $\mathbb{F}_{p^2}^{\times}$ of order dividing 4,*

2. *$\zeta_{jI_2}(g''^{2r+i/2}) = \omega^{ij}$ for $i \in \{0,2\}, j \in \{0,1,2,3\}$, $r \in \mathbb{Z}$, so the $\zeta_{jI_2}$'s are the unique character of $\mathbb{F}_p^\times$ of order 2 given by the Legendre symbol if $j$ is odd, and the trivial character if $j$ is even,*

*where $g', g''$ are generators for $\mathbb{F}_{p^2}^\times$, $\mathbb{F}_p^\times$ respectively and $\omega$ is a primitive $4^{th}$ root of unity.*

*Proof.* As seen above

$$\overline{\gamma}(z) = \alpha_1(t)^{(p^2-1)/4}\alpha_2(t)^{(p-1)/2},$$

where $t \in \mathcal{L}_T^{-1}(Z)/Z$ such that $\overline{\mathcal{L}}(t) = z$. Also $\zeta_I(x) = \zeta(z)$, where $z \in H^1(F,Z)$ is defined by $\overline{\gamma}(z) = x^{(p^{|I|}-1)/4}$. Just as in the case of $p \equiv 1 \bmod 4$, we need to find $z_I$ such that $\overline{\gamma}(z_I) = \alpha_I(t_I)^{c_I(p^{|I|}-1)/2}$.

Let us first determine $\zeta_{I_1}$. Choose $z_{I_1}$ such that $\alpha_{I_2}(t_{I_1}) = 1$, that is, $t_{I_1} = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & t^{-1}\lambda^{-2} \end{pmatrix}$ will be a representative for $z_{I_1}$, with $\lambda, t \in \overline{\mathbb{F}}_p^\times$. As in the proof of Lemma 7.0.4, $t_{I_1}$ is determined by the following equations

$$t^{p-1}\lambda^{2p} = \omega^i \tag{7.8}$$

$$\lambda^{-1-p} = \omega^i \tag{7.9}$$

$$t^{1-p}\lambda^2 = \omega^i, \tag{7.10}$$

where $z_{I_1} = z_{\omega^i}$. First, it is easy to see that $\lambda, t \in \mathbb{F}_{p^4}^\times$. Indeed, from (7.10), $t^{p^2-p}\lambda^{-2p} = \omega^{-pi}$, so multiplying this equation with (7.8), we get $t^{p^2-1} = \omega^{(1-p)i} = -1$ since $p \equiv 3 \bmod 4$. Thus $t^{p^4-1} = 1$, so $t \in \mathbb{F}_{p^2}^\times$ and the proof that $\lambda \in \mathbb{F}_{p^4}^\times$ follows from equation (7.9). As a result, WLOG, let us take $g$ a generator of $\mathbb{F}_{p^4}^\times$ such that $g^{(p^4-1)/4} = \omega$. We thus have $t = g^x, \lambda = g^y$ for some $x, y \in \mathbb{Z}/(p^4-1)\mathbb{Z}$.

From (7.9), we have $-(1+p)y = (p^4-1)i/4 + (p^4-1)k_1$ for $k_1 \in \mathbb{Z}$. Thus

$$y = \frac{(1-p)(1+p^2)}{4}i + (1-p)(1+p^2)k_1.$$

On the other hand, from (7.8), we have $(p-1)x + 2py = (p^4-1)i/4 + (p^4-1)k_2$ for $k_2 \in \mathbb{Z}$, so $(p-1)x = (p^4-1)i/4 + (p^4-1)k_2 + 2p(p-1)(1+p^2)i/4 + 2p(p-1)(1+p^2)k_1$. Then

$$x = \frac{(1+3p)(1+p^2)}{4}i + 2p(1+p^2)k_1 + (1+p)(1+p^2)k_2.$$

From (7.10), we have $(1-p)x+2y = (1-p)(1+3p)(1+p^2)i/4 + (1-p)2p(1+p^2)k_1 + (1-p)(1+p)(1+p^2)k_2 + 2(1-p)(1+p^2)i/4 + 2(1-p)(1+p^2)k_1 = 3(1-p^4)i/4 + 2(1-p^4)k_1 + (1-p^4)k_2$ and thus $t^{1-p}\lambda^2 = g^{-3(p^4-1)i/4} = \omega^i$. Thus (7.10) is verified.

We then have

$$\begin{aligned} x-y &= (1+p^2)pi + (1+p^2)(3p-1)k_1 + (1+p^2)(1+p)k_2) \\ &= (1+p^2)\Big(-i + (1+p)(i+k_2) + (3p-1)k_1\Big). \end{aligned}$$

As a result $\alpha_1(t_{I_1})^{(p^2-1)/4} = g^{(x-y)(p^2-1)/4}$. But $(1+p, 3p-1) = 4$ as $p \equiv 3 \bmod 4$, so there exists $a, b \in \mathbb{Z}$ such that $(1+p)a + (3p-1)b = 4$. Thus, one can choose $k_1, k_2 \in \mathbb{Z}$ such that

$$\alpha_1(t_{I_1})^{(p^2-1)/4} = \left(g^{(1+p^2)(-i+4r)}\right)^{(p^2-1)/4},$$

where $r \in \mathbb{Z}$.

It is now easy to compute $\zeta_{I_1}$. Note that since $g$ is a generator of $\mathbb{F}_{p^4}^\times$, $g' := g^{1+p^2}$ will be a generator for $\mathbb{F}_{p^2}^\times$. If $z_{I_1} = z_{\omega^i}$, then $\overline{\gamma}(z_{I_1}) = (g'^{-i+4r})^{(p^2-1)/4}$, so $\zeta_{I_1}(g'^{-i+4r}) = \zeta(\omega^i)$. Note that we used the clear identification of $\zeta_{\omega_i}$ with $\omega^i$, which gives us $\zeta(z_{\omega^i}) = \zeta(\omega^i)$. Thus, if $\zeta = \zeta_j$, we have

$$\zeta_{jI_1}(g'^{-i+4r}) = \zeta_j(\omega^i) = \omega^{ij}.$$

As a result, $\zeta_{jI_1}$ is a multiplicative character of $\mathbb{F}_{p^2}^\times$ of order dividing 4.

Computing $\zeta_{I_2}$ as a character of $\mathbb{F}_p^\times$ can be done in a similar fashion. Choose $z_{I_2}$ such that $\alpha_{I_1}(t_{I_2}) = 1$, that is, $t_{I_2} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t^{-1}\lambda^{-2} \end{pmatrix}$ will be a representative for $z_{I_2}$, with $\lambda, t \in \overline{\mathbb{F}}_p^\times$. As before, $t_{I_2}$ is determined by

$$\lambda^{2p-1}t^p = \omega^i \tag{7.11}$$

$$\lambda^{-1}t^{-p} = \omega^i \tag{7.12}$$

$$t^{-1}\lambda^{-p} = \omega^i \tag{7.13}$$

$$t\lambda^{2-p} = \omega^i, \tag{7.14}$$

where $z_{I_2} = z_{\omega^i}$. First, it is easy to see that $\lambda, t \in \mathbb{F}_{p^4}^\times$. Indeed, from (7.13), $t^p\lambda^{p^2} = \omega^{-pi}$, so multiplying this equation with (7.12), we get $\lambda^{p^2-1} = \omega^{(1-p)i} = -1$ since $p \equiv 1 \bmod 4$. So $\lambda \in \mathbb{F}_{p^4}^\times$ and the proof that $t \in \mathbb{F}_{p^4}^\times$ follows exactly the same. As before, let $g$ be a generator of $\mathbb{F}_{p^4}^\times$ such that $g^{(p^4-1)/4} = \omega$, and let $t = g^x, \lambda = g^y$ for some $x, y \in \mathbb{Z}/(p^4-1)\mathbb{Z}$.

From (7.12), we have $-y - px = (p^4-1)i/4 + (p^4-1)k_1$ for $k_1 \in \mathbb{Z}$. Thus

$$y = \frac{1-p^4}{4}i + (1-p^4)k_1 - px.$$

On the other hand, from (7.11), we have $(2p-1)y+px = (p^4-1)i/4+(p^4-1)k_2$ for $k_2 \in \mathbb{Z}$, so $2p(1-p)x = (p^4-1)i/4 + (p^4-1)k_2 + (2p-1)(p^4-1)i/4 + (2p-1)(p^4-1)k_1$. Thus $2px = -(1+p)(1+p^2)2pi/4 - 2p(1+p)(1+p^2)k_1 + (1+p)(1+p^2)(k_1-k_2)$. As a result, $p$ must divide $k_1 - k_2$, so $k_1 - k_2 = pk_3$ for some $k_3 \in \mathbb{Z}$. Then

$$x = -\frac{(1+p)(1+p^2)}{4}i - (1+p)(1+p^2)k_1 + \frac{(1+p)(1+p^2)}{2}k_3.$$

From (7.13), we have $-x - py = p(p^4-1)i/4 + p(p^4-1)k_1 + (p^2-1)x = p(p^4-1)i/4 + p(p^4-1)k_1 - (p^2-1)(1+p)(1+p^2)i/4 - (p^2-1)(1+p)(1+p^2)k_1 + (p^2-1)(1+p)(1+p^2)k_3/2$, so $t^{-1}\lambda^{-p} = g^{-(p^4-1)i/4} = \omega^{-i}$. Thus we must have $\omega^{-i} = \omega^i$, which implies $i \in \{0, 2\}$. Thus $z_{I_2}$ must be $z_1$ or $z_{\omega^2}$.

On the other hand, $x+(2-p)y = (p-1)^2x+(2-p)(1-p^4)i/4+(2-p)(1-p^4)k_1 = -(p-1)^2(1+p)(1+p^2)i/4-(p-1)^2(1+p)(1+p^2)k_1+(p-1)^2(1+p)(1+p^2)k_3/2+(2-p)(1-p^4)i/4+(2-p)(1-p^4)k_1$. Thus $t\lambda^{2-p} = g^{-(p^4-1)i/4} = \omega^{-i} = \omega^i$, so (7.14) is also verified.

We then have

$$\begin{aligned} y - x &= \frac{(1+p)(1+p^2)}{2}i + 2(1+p)(1+p^2)k_1 - \frac{(1+p)^2(1+p^2)}{2}k_3 \\ &= (1+p)(1+p^2)\Big(\frac{i}{2} + 2k_1 - \frac{1+p}{2}k_3\Big). \end{aligned}$$

As a result $\alpha_2(t_{I_2})^{(p-1)/2} = g^{(y-x)(p-1)/2}$. But as $(2, (1+p)/2) = 2$, there exists $a, b \in \mathbb{Z}$ such that $2a + (1+p)b/2 = 2$. Thus, one can choose $k_1, k_3 \in \mathbb{Z}$ such that

$$\alpha_2(t_{I_2})^{(p-1)/2} = \Big(g^{(1+p)(1+p^2)(i/2+2r)}\Big)^{(p-1)/2},$$

where $r$ is any integer.

It is now easy to compute $\zeta_{I_2}$. Note that since $g$ is a generator for $\mathbb{F}_{p^4}^{\times}$, $g'' := g^{(1+p)(1+p^2)}$ will be a generator for $\mathbb{F}_p^{\times}$. If $z_{I_2} = z_{\omega^i}$, we have $\overline{\gamma}(z_{I_2}) = (g''^{i/2+2r})^{(p-1)/2}$, so $\zeta_{I_2}(g''^{i/2+2r}) = \zeta(\omega^i)$ for all $r \in \mathbb{Z}$, $i \in \{0, 2\}$. Thus

$$\zeta_{jI_2}(g''^{i/2+2r}) = \begin{cases} 1 & \text{for } i = 0 \\ \omega^{2j} & \text{for } i = 2 \end{cases}$$

and as a result the characters $\zeta_{jI_2}$ are trivial for $j \in \{0, 2\}$, while for $j \in \{1, 3\}$, $\zeta_{jI_2}$ is indeed the unique character of order 2 of $\mathbb{F}_p^{\times}$ given by the Legendre symbol. □

We can now compute $\sigma_\zeta$ as follows:

**Proposition 7.0.7.** *With the above notations, we have*

$$\sigma_{\zeta_j} = \begin{cases} -G(\zeta_{jI_1}) & \text{if } j \in \{0, 2\} \\ i\sqrt{p}G(\zeta_{jI_1}) & \text{if } j \in \{1, 3\} \end{cases},$$

*where the $\zeta_j$'s are the characters of $H^1(F, Z) \cong \mu_4$ given by $\omega \mapsto \omega^j$ for $j \in \{0, 1, 2, 3\}$ and $\omega$ a primitive $4^{th}$ root of unity. As defined above, $G(\zeta_I) = G(\zeta_I, \chi_0)$ are the Gauss sums defined by the characters $\zeta_I \in (\mathbb{F}_{p^{|I|}}^{\times})^{\wedge}$. The character $\zeta_{jI_1}$ of $\mathbb{F}_{p^2}^{\times}$ of order dividing 4, is given by $g^{4r-i} \mapsto \omega^{ij}$, for $i \in \{0, 1, 2, 3\}, r \in \mathbb{Z}$ and $g$ a generator for $\mathbb{F}_{p^2}^{\times}$. In particular, $\sigma_{\zeta_j}$ is always non-zero; moreover $\sigma_{\zeta_0} = 1$, while $|\sigma_{\zeta_2}| = p$ and $|\sigma_{\zeta_j}| = p\sqrt{p}$ for $j \in \{1, 3\}$.*

*Proof.* By Theorem 6.3.5 we have

$$\sigma_\zeta = G(\zeta_{I_1})G(\zeta_{I_2})$$

and using the result of Lemma 7.0.6 above, the first part of the lemma follows immediately, while the non-vanishing part of the result follows trivially from Proposition 6.1.3. □

**A formula for multiplicity defect for $SU(2, 2)$**

We have $\pi$ an irreducible regular cuspidal representation of $\widetilde{G}^F$, such that $\pi_{|_{G^F}} = \pi_1 + \cdots + \pi_{d_0}$, with $d_0 \neq 1$ a divisor of $d = |H^1(F, Z)|$. So far we have showed that the alternating sum $\displaystyle\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i$ is zero on semisimple classes, which means the semisimple classes have no contribution in the alternating sum of multiplicities $\Delta M_\pi$ introduced in (1.1). In the following we shall show that $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i$ is non-zero on the regular unipotent classes of $G^F$. Moreover, the values this alternating sum takes are given in terms of Gauss sums, suggesting links with possible interpretations of $\Delta M_\pi$ in terms of certain arithmetic invariants that generalize Hecke's theorem for the $SL_2$.

Given $\sigma_z$, we can find formulas for $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i(u)$ for any $u \in U_z$ using Theorem 6.2.10 and similar arguments for the case when $d_0 \neq d$. We specialize the results to the study case of $G^F = SU(2,2)$ and obtain:

**Theorem 7.0.8.** *Let $\pi$ be an irreducible cuspidal regular representation of $\widetilde{G}^F = U(2,2)$ that splits into $d_0|d$ irreducible components upon restriction to $G^F = SU(2,2)$, $\pi_{|_{G^F}} = \pi_1 + \cdots + \pi_{d_0}$. Let $\zeta_2$ be the unique non-trivial quadratic character of $\mathbb{F}_{p^2}^\times$ and $\zeta_4$ be the quartic character of $\mathbb{F}_{p^2}^\times$ given by $g^{4r-i} \mapsto \xi_4^{3i}$, for $i \in \{0,1,2,3\}, r \in \mathbb{Z}$ and $g$ a generator for $\mathbb{F}_{p^2}^\times$. Let $G$ be the Gauss sum introduced in Definition 6.1.1.*

*Then the alternating sum $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i$ takes the following values on regular unipotent classes:*

$$\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i(u) = \begin{cases} -G(\zeta_2) & \text{if } d_0 = 2,\ d \in \{2,4\}, \\ i\sqrt{p}G(\zeta_4) & \text{if } d_0 = 4,\ d = 4, \end{cases}$$

*for any $u \in U_{z_1}$, where $z_1$ is the identity element in $H^1(F, Z)$.*

*Proof.* The result follows directly from Theorem 6.2.10 when $d_0 = d$, noting that $|\Pi/\tau| = 2$ and $r(G) = 2$. The $\mathbb{F}_p$-rank of $SL_4$ is 2 as one can check the maximum split subtorus in $T^F$, where $T$ is the usual maximally split torus of diagonal matrices, is given by $\mathbb{F}_{p^2}^\times \times \mathbb{F}_{p^2}^\times$. Let $z_i = \omega^{i-1}I_4$, so $z_1$ is the trivial element, $\omega$ is a primitive $4^{\text{th}}$ root of unity. The same proposition makes it clear that is it enough to compute this alternating sum on $u \in U_{z_1}$.

If $d_0 = d = 2$, we get $\pi_1(u) - \pi_2(u) = \sigma_{z_1^{-1}} - \sigma_{z_2^{-1}} = -G(\zeta_{2,2})$ for $u \in U_{z_1}$, where $\zeta_{2,2}$ is the unique non-trivial quadratic character of $\mathbb{F}_{p^2}^\times$. The last equality follows from Proposition 7.0.5.

If $d_0 = d = 4$, we get $\pi_1(u) + \xi\pi_2(u) + \xi^2\pi_3(u) + \xi^3\pi_4(u) = \sigma_{z_1} + \xi\sigma_{z_4} + \xi^2\sigma_{z_3} + \xi^3\sigma_{z_2}$ for $u \in U_{z_1}$, where $\xi = \omega$. From Proposition 7.0.7, we get that

$\sigma_{z_1} + \xi\sigma_{z_4} + \xi^2\sigma_{z_3} + \xi^3\sigma_{z_2} = i\sqrt{p}G(\zeta_{4,4})$, where $\zeta_{4,4}$ is a quartic character of $\mathbb{F}_{p^2}^\times$ given by $g^{4r-i} \mapsto \omega^{3i}$, for $i \in \{0,1,2,3\}, r \in \mathbb{Z}$ and $g$ a generator for $\mathbb{F}_{p^2}^\times$.

Lastly, we shall deal with the case $d_0 = 2, d = 4$. We have $\pi_{|_{G^F}} = \pi_1 + \pi_2$, so similar to the reasoning before Theorem 6.2.10 we have exactly two irreducible components $\chi_{s,z}$ of $\chi_{(s)} = \pi_{|_{G^F}}$.

Let us recall the result of part (ii) of Proposition 3.12 in [8]: We have $\chi_{s,z} = \chi_{s',z'}$ if and only if $(s) = (s')$ and $\phi_z = \phi_{z'}$, where $\phi_z$ is a character of $(C_{G^*}(s)/C_{G^*}(s)^\circ)^{F^*}$.

Let $A$ be the fundamental group of the derived group of $G^*$, that is, its weight lattice modulo its root lattice. The character $\phi_z$ is defined by the isomorphism

$$\omega : H^1(F, Z) \cong (A^{F^*})^\wedge,$$

so $\omega : z \mapsto \phi_z$. One can embed $C_{G^*}(s)/C_{G^*}(s)^\circ$ in $A$, so as a result $\phi_z$ can be thought of as a character of $(C_{G^*}(s)/C_{G^*}(s)^\circ)^{F^*}$.

In our case, $z_1, z_3$ are the trivial element and the element of order 2 in $H^1(F, Z) \cong \mu_4$, while $z_2, z_4$ are the elements of order 4. There are exactly two irreducible components $\chi_{s,z}$ of $\chi_{(s)}$, where $z \in \{z_1, z_2, z_3, z_4\}$. Thus there must be two distinct characters of $(C_{G^*}(s)/C_{G^*}(s)^\circ)^{F^*}$ among the four characters $\phi_z$ of $A^{F^*}$. As a result $((C_{G^*}(s)/C_{G^*}(s)^\circ)^{F^*})^\wedge$ will be cyclic of degree two and because of order reasons we must have $\chi_{s,z_1} = \chi_{s,z_3}$ and $\chi_{s,z_2} = \chi_{s,z_4}$. Consequently, the formulas in Theorem 6.2.10 become

$$\pi_1(u) = (-1)^{|\Pi/\tau| + r(G)}(\sigma_{zz_1^{-1}} + \sigma_{zz_3^{-1}})$$
$$\pi_2(u) = (-1)^{|\Pi/\tau| + r(G)}(\sigma_{zz_2^{-1}} + \sigma_{zz_4^{-1}})$$

for any $u \in U_z$. Thus, $\pi_1(u) - \pi_2(u) = \sigma_{z_1} + \sigma_{z_3} - \sigma_{z_2} - \sigma_{z_4}$ for $u \in U_{z_1}$, so by Proposition 7.0.7 we have $\pi_1(u) - \pi_2(u) = -G(\zeta_{2,4})$, where $\zeta_{2,4}$ is the non-trivial quadratic character of $\mathbb{F}_{p^2}^\times$. □

*Remark.* Note that it is sufficient to compute the values on $u \in U_{z_1}$; the values on the other unipotent classes are the same up to sign if $d_0 = 2$ or up to multiplication by a $4^{\text{th}}$ root of unity if $d_0 = 4$.

Also, remark that the formulas above are up to a sign and choice of primitive root. Obviously, if we change the order in which we consider the irreducibles $\pi_i$, the formulas might change slightly. However, the alternating sum $\sum_{i=1}^{d_0} \xi_{d_0}^{i-1}\pi_i$

will always be, up to a sign, of the form of a Mellin transform $\sigma_\zeta$ defined in Definition 6.3.4.

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