ARITHMETIZED TRIGONOMETRICAL EXPANSIONS OF DOUBLY PERIODIC FUNCTIONS OF THE THIRD KIND.

Thesis by,

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ARITHMETIZED TRIGONOMETRICAL EXPANSIONS OF DOUBLY PERIODIC FUNCTIONS OF THE THIRD KIND.

We present a method of expanding the members of a certain class of functions, doubly periodic of the third kind, due to Appell.[1] An extension of the details is given, and the method is used to obtain the expansions of those numbers of the class considered whose complexity, in a technical sense, is within an arbitrary chosen range. The details in the expansion of any member of the class are discussed and some information is obtained as to the general form of the result.

-I-

The functions considered are formed from the small theta functions of Jacobi. The following notation[2] is adopted for the theta functions:

$$\vartheta_0(z,q) = \sum_{n=-\infty}^{n=\infty} (-1)^n q^{n^2} e^{2niz}$$

$$\vartheta_1(z,q) = -\sum_{n=-\infty}^{n=\infty} (-1)^{\frac{2n+1}{2}} q^{\left(\frac{2n+1}{2}\right)^2} e^{(2n+1)iz}$$

(1)

$$\vartheta_2(z,q) = \sum_{n=-\infty}^{n=\infty} q^{\left(\frac{2n+1}{2}\right)^2} e^{(2n+1)iz}$$

$$\vartheta_3(z,q) = \sum_{n=-\infty}^{n=\infty} q^{n^2} e^{2niz} \qquad |q|<1$$

(1) Sur les Fonctions Doublement Périodiques de Troisième Espèce. Annales Scientifique de L'Ecole Normale Superieure, Third Series, Vol. 1, 1864, p. 135.
Developpements en Series des Fonctions Doublement Périodiques. Loc. cit. Third Series, Vol. II, 1885, p. 2.

and the functions considered are of the form

$$(2) \qquad F(z) = \vartheta_0^{m_0}(z)\,\vartheta_1^{m_1}(z)\,\vartheta_2^{m_2}(z)\,\vartheta_3^{m_3}(z)$$

where m_0, m_1, m_2, m_3 are integers, including zero, such that $m_0+m_1+m_2+m_3 < 0$. The usual conventions, $\vartheta_i(z) \equiv \vartheta_i(z, q)$ and $\vartheta_i \equiv \vartheta_i(0)$ are used.

A doubly periodic function of the third kind[3] is a uniform function of z which satisfied the equations of definition

$$(3) \qquad \begin{aligned} f(z+\tau_1) &= e^{az+b} f(z) \\ f(z+\tau_2) &= e^{a'z+b'} f(z) \end{aligned}$$

where the quantities τ_1 and τ_2 are called the periods of the function. It is easily verified that $F(z)$ is a doubly periodic function of the third kind. For defining τ by the equation $q = e^{i\pi\tau}$, and replacing $F(z)$ by its value in terms of the ϑ's, we see that $F(z)$ satisfies

$$(4) \qquad \begin{aligned} F(z+\pi) &= (-1)^{m_1+m_2} F(z) \\ F(z+\pi\tau) &= e^{2\mu i z}(-1)^{m_0+m_1} F(z) \end{aligned}$$

μ an integer, >0

which are of the form (3).

The method followed in the present paper, as stated, is due to Appell. In the series of papers cited, Appell solves completely the following problem:

Given a function, $G(z)$, which is uniform and meromorphic and which satisfies the equations

$$(5) \qquad \begin{aligned} G(z+\pi) &= G(z) \\ G(z+\pi\tau) &= e^{-2miz} G(z) \end{aligned}$$

m an integer $\gtrless 0$

Sur les Fonctions Doublement Périodique de Troisième Espèce. Loc. cit. Third Series, Vol. III, 1886, p. 2.
(2) Jacobi, Werke, Vol. I, p. 501.
(3) Krause, Theorie der Doppeltperiodischen Functionen einer Veranderlichen Grösse, Vol. 1, p. 59.

to exhibit $G(z)$ as a sum of simple elements, each element having but one singularity in a period parallelogram, and an integral function of z. Having solved this problem Appell indicates how this method of expressing $G(z)$ leads, in the case $m<0$, to arithmetized trigonometrical expansions by giving the details for certain values of m_0, m_1, m_2 and m_3. In the present paper the general method, or rather the details involved, are given in full but are carried out in a slightly different manner.

It is found, from the point of view of practical application of the theory to obtain specific expansions, that the case in which m of (5) is negative is essentially distinct from the case in which m is positive. For $m>0$, the integral function mentioned above is in general different from zero, and certain constants appear whose determination is given only in terms of certain integrals. Since no general method of evaluating the latter is given, it would appear that Appell's method is of limited use. This question is discussed at some length in a California Institute of Technology dissertation by Mr. M. A. Basoco.

In the case $m<0$, that is for functions of the type of $F(z)$, Appell's theory leads to completely determinate results, since the constants which appear are the coefficients in the principal part of the expansion of $F(z)$ near each of its poles in a cell, and are obtained by elementary means. It is evident from the nature of the theta functions that the integer m which is associated with $F(z)$ represents the excess of the number of zeros of the function over the number of poles in a cell. It is readily shown that this is true of any function of the type .

A short account of those parts of Appell's theory which are needed in this paper is given in the following section:

-II-

Let[4]

$$A_\mu(z, y) = \sum_{n=-\infty}^{n=\infty} e^{2\mu n i y} q^{\mu n(n-1)} \cot(z - y - n\pi\tau) \tag{6}$$

where μ is an integer greater than zero and τ is as above. It is shown that $A_\mu(z, y)$ considered as a function of y

(a) is meromorphic and single valued, and converges for all values of y such that $y \not\equiv z \mod \pi, \pi\tau$,

(b) possesses simple poles at those points for which $y \equiv z \mod \pi, \pi\tau$, and

(c) the residue at $y = z$ is -1 .

The Appell function $A_\mu(z, y)$ as a function of y satisfies

$$\begin{aligned} A_\mu(z, y+\pi) &= A_\mu(z, y) \\ A_\mu(z, y+\pi\tau) &= e^{-2\mu i y} A_\mu(z, y) \end{aligned} \tag{7}$$

the first of which is evident from (6). To establish the second we have from (6), recalling that $q = e^{i\pi\tau}$,

$$A_\mu(z, y+\pi\tau) = \sum_{n=-\infty}^{n=\infty} e^{2\mu n i y} q^{\mu n(n+1)} \cot(z - y - (n+1)\pi\tau)$$

$$= e^{-2\mu i y} \sum_{n=-\infty}^{\infty} e^{2\mu(n+1) i y} q^{\mu n(n+1)} \cot(z - y - (n+1)\pi\tau)$$

which is

$$e^{-2\mu i y} A_\mu(z, y)$$

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(4) Appell defines

$$\chi_\mu(z, y) = \frac{\pi}{2K} \sum_{n=-\infty}^{\infty} e^{\mu n i y \frac{\pi}{K}} [\cot(z - y - 2ni K')] q^{\mu n(n-1)}$$

but it is found that the expression (6) leads to simpler expressions with respect to constants when the notation of (1) is used for the theta functions.

Now let $F(z)$ be any function which is uniform, meromorphic and which satisfies

(8) $$F(z+\pi) = F(z)$$
$$F(z+\pi\tau) = e^{2\mu i z}F(z)$$

where μ, as above, is a positive integer. Let $F(z)$ have in a period parallelogram poles of order l_i at the points $z=\alpha_i$, $i=0,1,\cdots p$, and let the principal part of the expansion of $F(z)$ at $z=\alpha_i$ be given by

$$\sum_{j=1}^{l_i} \frac{R_i^{(j)}}{(z-\alpha_i)^j}$$

Consider

(9) $$\Phi(y) = A_\mu(z,y)\,F(y)$$

From (7) and (8) we see that $\Phi(y)$ is doubly periodic (of the first kind) and hence the sum of the residues in a period parallelogram is zero. The poles of $\Phi(y)$ are at $z, \alpha_0, \alpha_1 \cdots\cdots \alpha_p$. The residue of $A_\mu(z,y)$ being -1 at $y=z$ the residue of $\Phi(y)$ at $y=z$ is $-F(z)$ To obtain the residue of $\Phi(y)$ at $y=\alpha_i$ expand $A_\mu(z,y)$, in a Taylor series about $y=\alpha_i$. This can be written

$$A_\mu(z,y) = A_\mu(z,\alpha_i) + \cdots + \frac{A_\mu^{(k)}(z,\alpha_i)}{k!}(y-\alpha_i)^k \cdots\cdots$$

where $A_\mu^{(k)}(z)$ designates the result of placing $y=\alpha_i$ in the k th derivative of $A_\mu(z,y)$ with respect to y. Recalling that the principal part of the expansion of $F(y)$ about $y=\alpha_i$ was

$$\sum_{j=1}^{l_i} \frac{R_i^{(j)}}{(z-\alpha_i)^j}$$

we see that the residue of $\Phi(y)$ at $y=\alpha_i$ is

$$\sum_{j=1}^{l_i} \frac{A_\mu^{(j-1)}(z,\alpha_i)}{(j-1)!}R_i^{(j)}$$

Hence applying the result that the sum of the residues of $\Phi(y)$ is zero in a period parallelogram we have

(10) $$F(z) = \sum_{i=0}^{p}\sum_{j=1}^{l_i} \frac{A_\mu^{(j-1)}(z,\alpha_i)}{(j-1)!}R_i^{(j)}$$

This result is of fundamental importance: it gives directly the explicit expression for any function of the type of in terms of the Appell function and its derivatives, and the coefficients of the terms of the principal parts of the expansions of the function considered near each of its poles in a period parallelogram.

At this point the treatment in the present paper departs from that given by Appell, not in the general theory which, as has been shown is complete in every respect for the type of function here considered, but in the details of the method used in obtaining trigonometrical series for a given function from its equivalent expression in the form (10). The preceding has made clear that we are concerned with the Appell function and its derivatives at the poles of the function considered. Appell, in the earlier parts of the papers cited, obtains the first derivative of the function by direct differentiation under the summation sign, but it is evident that such a process is not, in general, feasible, owing to the increasing number and complexity of the terms introduced by the differentiation. Later he obtains expressions[5] for $A_\mu(x+\frac{\pi\tau}{2}, a+\frac{\pi\tau}{2})$, $A_\mu(x+\frac{\pi\tau}{2}, a)$, $A_\mu(x, a+\frac{\pi\tau}{2})$ and $A_\mu(x, a)$, which are valid for certain ranges of $|x-a|$, and arithmetizes these results. He does not obtain the general derivatives of these expressions, but from the context it seems possible that such a step was regarded as merely a detail to be carried out and that it offered no further interest

(5) Appell gives, for instance,

$$\frac{K}{\pi}\chi_\mu(x+iK', a+iK') = \frac{1}{2}\cot\frac{\alpha}{2} - \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty} 2\, q^{\mu n^2+2n\nu} \sin(n\beta-\nu\alpha) \qquad |q^2|<|e^{\alpha i}|<|q^{-2}| .$$

where $\alpha = \frac{x-a}{K}\pi$, $\beta = \frac{\mu\pi a}{K}$ and the coefficient 2 in the double sum is replaced by 1 when $\nu = 0$.

from the author's point of view which pertains to the theory of the function involved, and not to the details in its application. It may be remarked in this connection that after the above expressions are found, they are used in several illustfative examples, but that no one of these offers an instance of a pole of higher than the first order. Of greater importance is the fact that the convergence of the derivatives of the Appell function is not discussed; and indeed if the derived series were not all convergent the theory would lose much of its value.

In the next section expressions similar to those above are obtained by a well known device, and from these results expressions are found for the corresponding forms $A_\mu^{(k)}$. These results are shown to be valid in the same regions, respectively, as those belonging to the corresponding modified expressions for $A_\mu^{(0)}$.

-III-

Replacing the trigonometrical functions by their exponential forms, and recalling that $q = e^{i\pi\tau}$ we have the following

$$
\begin{aligned}
\cot(u - k\pi\tau) &= i\left\{1 + 2\sum_{r=1}^{\infty} e^{-2riu} q^{2rk}\right\} \\
\cot(u + k\pi\tau) &= -i\left\{1 + 2\sum_{r=1}^{\infty} e^{2riu} q^{2rk}\right\}
\end{aligned}
\tag{11}
$$

the series converging absolutely for the ranges defined respectively by $|e^{-iu}q^{k}|<1$, $|e^{iu}q^{k}|<1$. Hence both results are simultaneously valid for values of u and k satisfying $|I(u)| < I(k\pi\tau)$, where k is taken to be real and positive. From (6) we have

$$
\begin{aligned}
A_\mu(z + a\pi\tau, y + a\pi\tau) &= \sum_{n=-\infty}^{\infty} e^{2\mu n i(y + a\pi\tau)} q^{\mu n(n-1)} \cot(z - y - n\pi\tau) \\
&= \sum_{n=-\infty}^{\infty} e^{2\mu n i y} q^{\mu n(n-1+2a)} \cot(z - y - n\pi\tau)
\end{aligned}
$$

$$= \cot(z-y) + \sum_{n=1}^{\infty} e^{2\mu n i y} q^{\mu n(n+2a-1)} \cot(z-y-n\pi\tau) +$$

$$+ \sum_{n=1}^{\infty} e^{-2\mu n i y} q^{\mu n(n-2a+1)} \cot(z-y+n\pi\tau) .$$

Using (11) this may be written

$$A_\mu(z+a\pi\tau, y+a\pi\tau) = \cot(z-y) + i\sum_{n=1}^{\infty} e^{2\mu n i y} q^{\mu n(n+2a-1)} - i\sum_{n=1}^{\infty} e^{-2\mu n i y} q^{\mu n(n-2a+1)} +$$

(12)

$$+2i\sum_{n=1}^{\infty}\sum_{r=1}^{\infty} e^{2iy(\mu n+r)} e^{-2riz} q^{\mu n(n+2a-1)+2rn} - 2i\sum_{n=1}^{\infty}\sum_{r=1}^{\infty} e^{-2iy(\mu n+r)} e^{2riz} q^{\mu n(n-2a+1)+2rn} ,$$

the result being valid for $|I(z-y)| < I(\pi\tau)$. Differentiating K times with respect to y we get

$$A_\mu^{(K)}(z+a\pi\tau, y+a\pi\tau) = \frac{d^K \cot(z-y)}{dy^K} + i\sum_{n=1}^{\infty}(2\mu n i)^K e^{2\mu n i y} q^{\mu n(n+2a-1)}$$

(13)

$$-i\sum_{n=1}^{\infty}(-2\mu n i)^K e^{-2\mu n i y} q^{\mu n(n-2a+1)} + 2i\sum_{n=1}\sum_{r=1}[2i(\mu n+r)]^K e^{2iy(\mu n+r)} q^{\mu n(n+2a-1)+2rn} e^{-2riz} +$$

$$-2i\sum_{n=1}\sum_{r=1}[-2i(\mu n+r)]^K e^{-2iy(\mu n+r)} q^{\mu n(n-2a+1)+2rn} e^{2riz} .$$

Of the four series which appear in (13), the first two converge for all values of y. This follows from the usual ratio test and need not be discussed in detail. Since

$$(\mu n+r)^K = \sum_{\ell=0}^{K} \binom{K}{\ell} (\mu n)^{K-\ell} r^\ell$$

we see that each of the last two series can be separated into $K+1$ series, the ℓth of these being respectively

$$\sum_{n=1}^{\infty}(2i)^K \binom{K}{\ell} (\mu n)^{K-\ell} e^{2\mu n i y} q^{\mu n(n+2a-1)} \sum_{r=1}^{\infty} e^{-2ir(z-y)} r^\ell q^{2nr}$$

and

$$\sum_{n=1}^{\infty}(-2i)^K \binom{K}{\ell} (\mu n)^{K-\ell} e^{-2\mu n i y} q^{\mu n(n-2a+1)} \sum_{r=1}^{\infty} e^{-2ir(y-z)} r^\ell q^{2nr} .$$

Now

$$\sum_{r=1}^{\infty} \left| e^{-2ir(z-y)} r^{\ell} q^{2nr} \right| \leqslant \sum_{r=1}^{\infty} \left| e^{-2ir(z-y)} r^{\ell} q^{2r} \right|$$

and

$$\sum_{r=1}^{\infty} \left| e^{-2ir(y-z)} r^{\ell} q^{2nr} \right| \leqslant \sum_{r=1}^{\infty} \left| e^{-2ir(y-z)} r^{\ell} q^{2r} \right| \qquad n = 1, 2 \cdots$$

Further if $|I(z-y)| < I(\pi\tau)$, both of the series on the right are absolutely convergent, this following from the usual ratio test. Hence if their is a region of absolute convergence of the two series

$$\sum_{n=1} (\mu n)^{k-\ell} e^{2\mu n i y} q^{\mu n(n+2a-1)}$$

$$\sum_{n=1} (\mu n)^{k-\ell} e^{-2\mu n i y} q^{\mu n(n-2a+1)}$$

which is common to the region defined by $|I(z-y)| < I(\pi\tau)$ it follows that (13) converges absolutely in this region, excluding $z \equiv y \bmod \pi$. The ratio test shows the two series above converge absolutely for all values of y and z. Hence we have the result: (13) has the same region of absolute convergence as (12).

Now consider $A_\mu(z, y+a\pi\tau)$. From (6) this is

$$A_\mu(z, y+a\pi\tau) = \sum_{n=0}^{\infty} e^{2\mu n i y} q^{\mu n(n+2a-1)} \cot(z - y - (n+a)\pi\tau)$$
$$+ \sum_{n=1}^{\infty} e^{-2\mu n i y} q^{\mu n(n-2a+1)} \cot(z - y + (n-a)\pi\tau).$$

Assume $0 < a < 1$ and substitute from (11). We get

(14)
$$A_\mu(z, y+a\pi\tau) = i\sum_{n=0}^{\infty} e^{2\mu n i y} q^{\mu n(n+2a-1)} - i\sum_{n=1}^{\infty} e^{-2\mu n i y} q^{\mu n(n-2a+1)}$$
$$+2i \sum_{n=0}^{\infty}\sum_{r=1}^{\infty} e^{2iy(\mu n+r)} q^{\mu n(n+2a-1)+2r(n+a)} e^{-2riz}$$
$$-2i \sum_{n=1}^{\infty}\sum_{r=1}^{\infty} e^{-2iy(\mu n+r)} q^{\mu n(n-2a+1)+2r(n-a)} e^{2riz}$$

Differentiating:

$$A_\mu^{(k)}(z, y+a\pi\tau) = i\sum_{n=0}(2\mu ni)^k q^{\mu n(n+2a-1)} e^{2\mu niy} +$$

(15) $$-i\sum_{n=1}(-2\mu niy)^k q^{\mu n(n-2a+1)} e^{-2\mu niy} + 2i\sum_{n=0}\sum_{r=1} e^{2iy(\mu n+r)} [2i(\mu n+r)]^k q^{\mu n(n+2a-1)+2r(n+a)} e^{-2riz} +$$

$$-2i\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}[-2i(\mu n+r)]^k e^{-2iy(\mu n+r)} q^{\mu n(n-2a+1)+2r(n-a)} e^{2riz}.$$

The convergence of (15) is discussed in the same way as that of (12). It is found that (14) and (15) are absolutely convergent in the region defined by $|I(z-y)| < I[(1-a)\pi\tau]$ and $|I(z-y)| < I(a\pi\tau)$.

Similarly we obtain

$$A_\mu^{(k)}[z+(a+\tfrac{1}{2})\pi\tau, y+a\pi\tau] = i\sum_{n=1}^{\infty}(2\mu ni)^k e^{2\mu niy} q^{\mu n(n+2a-1)} +$$

$$-i\sum_{n=0}^{\infty}(-2\mu ni)^k e^{-2\mu niy} q^{\mu n(n-2a+1)} +$$

(16) $$+2i\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}[2i(\mu n+r)]^k e^{2iy(\mu n+r)} q^{\mu n(n+2a-1)+(2n-1)r} e^{-2riz}$$

$$-2i\sum_{n=0}\sum_{r=1}[-2i(\mu n+r)]^k e^{-2iy(\mu n+r)} q^{\mu n(n-2a+1)+(2n+1)r} e^{2riz}.$$

$$|I(z-y)| < I(\tfrac{\pi\tau}{2})$$

$$A_\mu^{(k)}(z+a\pi\tau, y+(a+\tfrac{1}{2})\pi\tau) = i\sum_{n=0}^{\infty}(2\mu ni)^k e^{2\mu niy} q^{\mu n(n+2a)} +$$

$$-i\sum_{n=1}^{\infty}(-2\mu ni)^k e^{-2\mu niy} q^{\mu n(n-2a)} +$$

(17) $$+2i\sum_{n=0}^{\infty}\sum_{r=1}^{\infty}[2i(\mu n+r)]^k e^{2iy(\mu n+r)} q^{\mu n(n+2a)+(2n+1)r} e^{-2riz} +$$

$$-2i\sum_{n=1}\sum_{r=1}[-2i(\mu n+r)]^k e^{-2iy(\mu n+r)} q^{\mu n(n-2a)+(2n-1)r} e^{2riz}.$$

$$|I(z-y)| < I(\tfrac{\pi\tau}{2})$$

$$A_\mu^{(k)}(z, y+\pi\tau) = \frac{d^k}{dy^k}[e^{-2\mu i y}\cot(z-y)] + i\sum_{n=1}^{\infty} e^{2\mu i(n-1)y}[2\mu i(n-1)]^k q^{\mu n(n-1)}$$

$$- i\sum_{n=1}^{\infty}[-2\mu i(n+1)]^k e^{-2\mu i(n+1)y}\, q^{\mu n(n+1)}$$

(18)

$$+2i\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}[2i(\mu n-\mu+r)]^k e^{2iy(\mu n-\mu+r)}\, q^{\mu n(n-1)+2rn}\, e^{-2riz}$$

$$-2i\sum_{n=1}\sum_{r=1}[-2i(\mu n+\mu+r)]^k e^{-2iy(\mu n+\mu+r)}\, q^{\mu n(n+1)+2rn}\, e^{2riz}$$

$$|I(z-y)| < I(\pi\tau), \quad z \not\equiv y \mod \pi$$

These formulae will be used repeatedly. Continued use will also be made of various well known properties of the ϑ functions. For convenience, a tabulation of these formulae and properties is given.

-IV-

As a typical member of the set of functions to be expanded, for which $\mu = 1$, consider[6]

$$F(z) = \frac{\vartheta_0^2(z)e^{-iz}}{\vartheta_1^3(z)}$$

This function satisfies

$$F(z+\pi) = F(z)$$

$$F(z+\pi\tau) = e^{2i(z+\frac{\pi}{2})}F(z)$$

Putting $t = z + \frac{\pi}{2}$ and writing

$$F(z) \equiv \varphi(t) = \frac{\vartheta_0^2(t-\frac{\pi}{2})\,e^{-i(t-\frac{\pi}{2})}}{\vartheta_1^3(t-\frac{\pi}{2})}$$

It is seen that $\varphi(t)$ satisfies

$$\varphi(t+\pi) = \varphi(t)$$

$$\varphi(t+\pi\tau) = e^{2it}\varphi(t)$$

---

(6) The function whose expansion is desired is $\frac{\vartheta_0^2(z)}{\vartheta_1^3(z)}$ but it is clear from what follows that to obtain a function of the form (8) by a linear transformation on z requires the factor $e^{\pm iz}$

which are of the form (8) with $\mu=1$. Further, in a cell $\varphi(t)$ has the sole singularity, a pole of order three, at $t=\frac{\pi}{2}$. To get the coefficients $R_1^{(j)}$ let $t-\frac{\pi}{2}=\varepsilon$, and use the tabulated expansions of the ϑ functions needed. In detail

$$\varphi(\tfrac{\pi}{2}+\varepsilon) = \frac{\vartheta_0^2(\varepsilon)\,e^{-i\varepsilon}}{\vartheta_1^3(\varepsilon)}$$

$$= \frac{\vartheta_0^2\left[1+\frac{\vartheta_0''}{\vartheta_0}\frac{\varepsilon^2}{2}\cdots\right]^2\left[1-i\varepsilon-\frac{\varepsilon^2}{2}\cdots\right]}{\vartheta_1'^3\varepsilon^3\left[1+\frac{\vartheta_1'''}{\vartheta_1'}\frac{\varepsilon^2}{3!}\cdots\right]^3}$$

$$= \frac{\vartheta_0^2}{\vartheta_1'^3\varepsilon^3}\left[1-i\varepsilon+\frac{\varepsilon^2}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\cdots\right]$$

Hence

$$R_1^{(3)} = \frac{\vartheta_0^2}{\vartheta_1'^3}, \quad R_1^{(2)} = \frac{-i\vartheta_0^2}{\vartheta_1'^3}, \quad R_1^{(1)} = \frac{\vartheta_0^2}{2\vartheta_1'^3}\left[2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right]$$

Substituting in (10) and replacing t by $z+\frac{\pi}{2}$ we get

$$\frac{\vartheta_1'^3}{\vartheta_0^2}F(z) = \frac{1}{2}A_1^{(2)}(z+\tfrac{\pi}{2},\tfrac{\pi}{2}) - i\,A_1^{(1)}(z+\tfrac{\pi}{2},\tfrac{\pi}{2})$$

(19)

$$+\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}A_1^{(0)}(z+\tfrac{\pi}{2},\tfrac{\pi}{2})$$

In (13) replace z by $z+\frac{\pi}{2}$, y by $\frac{\pi}{2}$ and a by 0 and use in (19). This gives, after a slight reduction

$$\frac{\vartheta_1'^3\vartheta_0^2(z)\,e^{-iz}}{\vartheta_0^2\vartheta_1^3(z)} = \frac{1}{2}\left\{\frac{2\cos z}{\sin^3 z} + i\sum_{n=1}^{\infty}4n^2(-1)^{n+1}q^{n(n-1)} + \right.$$

$$+i\sum_{n=1}^{\infty}4n^2q^{n(n+1)}(-1)^{n+1} + 2i\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+1}4(n+r)^2q^{n(n-1)+2rn}e^{-2riz} +$$

$$\left.-2i\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}(-1)^{n+1}4(n+r)^2q^{n(n+1)+2rn}e^{2riz}\right\} - i\left\{\frac{1}{\sin^2 z}+\sum_{n=1}^{\infty}2n(-1)^{n+1}q^{n(n-1)}+\right.$$

$$+\sum_{n=1}^{\infty}2n(-1)^nq^{n(n+1)} - 2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}2(n+r)(-1)^nq^{n(n+2r-1)}e^{-2riz}$$

$$\left.-2\sum_{n=1}\sum_{r=0}2(n+r)(-1)^nq^{n(n+2r+1)}e^{2riz}\right\} + \frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\left\{\frac{\cos z}{\sin z}+\right.$$

$$-i + i\sum_{n=1}^{\infty}(-1)^nq^{n(n-1)} + i\sum_{n=0}^{\infty}(-1)^nq^{n(n+1)} + 2i\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^nq^{n(n+2r-1)}e^{-2riz}+$$

$$\left.-2i\sum_{n=1}\sum_{r=0}(-1)^n q^{n(n+2r+1)}e^{2riz}\right\}.$$

In the second series in each bracket replace n by $n-1$. The lower limit will be 1 in each case. In the last series in each bracket replace r by $r-1$. Certain terms cancel and we have

$$\frac{\vartheta_1'^3 \vartheta_0^2(z) e^{-iz}}{\vartheta_0^2 \vartheta_1^3(z)} = \frac{\cos z - i\sin z}{\sin^3 z} + \frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\}\left\{\frac{\cos z - i\sin z}{\sin z}\right\}$$

$$+4i\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+1}\left[(n+r)^2-(n+r)+\tfrac{1}{4}\right] q^{n(n+2r-1)} e^{-2riz} +$$

$$-4i\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+1}\left[(n+r-1)^2+(n+r-1)+\tfrac{1}{4}\right] q^{n(n+2r-1)} e^{2(r-1)iz} +$$

$$+i\left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n=1}\sum_{r=1}(-1)^n q^{n(n+2r-1)}\left[e^{-2riz} - e^{2(r-1)iz}\right]$$

From this follows, on multiplying through by e^{iz},

$$\frac{\vartheta_1'^3 \vartheta_0^2(z)}{\vartheta_0^2 \vartheta_1^3(z)} = \frac{1}{\sin^3 z} + \left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\}\frac{1}{2\sin z} +$$

(20) $$+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+1}\left[2(n+r)-1\right]^2 q^{n(n+2r-1)}\sin(2r-1)z$$

$$+2\left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^n q^{n(n+2r-1)}\sin(2r-1)z$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3 \vartheta_3^2(z)}{\vartheta_0^2 \vartheta_2^3(z)} = \frac{1}{\cos^3 z} + \left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\}\frac{1}{2\cos z}$$

(21) $$+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+r}\left[2(n+r)-1\right]^2 q^{n(n+2r-1)}\cos(2r-1)z +$$

$$+2\left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n=1}\sum_{r=1}(-1)^{n+r-1} q^{n(n+2r-1)}\cos(2r-1)z$$

In (19) replace z by $z+\frac{\pi\tau}{2}$. We get

$$\frac{\vartheta_1'^3\,\vartheta_1^2(z)\,q^{-\frac14}}{i\,\vartheta_0^2\,\vartheta_0^3(z)} = \frac12 A_1^{(2)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2},\frac{\pi}{2}\right) - i\,A_1^{(1)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$+\frac12\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} A_1^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2},\frac{\pi}{2}\right)$$

Using (16) with $a=0$, $y=\frac{\pi}{2}$ and z replaced by $z+\frac{\pi}{2}$, and replacing n by $n-1$ in the summations whose lower limits are $n=0$, gives

$$\frac{\vartheta_1'^3\,\vartheta_1^2(z)}{\vartheta_0^2\,\vartheta_0^3(z)} = \frac12 q^{\frac14}\Bigg\{\sum_{n=1}^{\infty}(-1)^n 4n^2 q^{n(n-1)} + \sum_{n=1}^{\infty}(-1)^n 4(n-1)^2 q^{n(n-1)} +$$

$$+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^n 4(n+r)^2 q^{n(n-1)+(2n-1)r}e^{-2riz} + 2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^n 4(n+r-1)^2 q^{n(n-1)+(2n-1)r}e^{2riz}\Bigg\} +$$

$$+q^{\frac14}\Bigg\{\sum_{n=1}^{\infty}(-1)^{n+1}2n\,q^{n(n-1)} + \sum_{n=1}^{\infty}(-1)^n 2(n-1)\,q^{n(n-1)} + 2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+1}2(n+r)\,q^{n(n-1)+(2n-1)r}e^{-2riz} +$$

$$+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^n 2(n+r-1)\,q^{n(n-1)+(2n-1)r}e^{2riz}\Bigg\} + \frac12 q^{\frac14}\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\Bigg\{2\sum_{n=1}^{\infty}(-1)^{1+n}q^{n(n-1)}$$

$$+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+1}q^{n(n-1)+(2n-1)r}\left[e^{-2riz}+e^{2riz}\right]\Bigg\}$$

Carrying out reductions which are analogous to those used in obtaining (20) we get

$$\frac{\vartheta_1'^3\,\vartheta_1^2(z)}{\vartheta_0^2\,\vartheta_0^3(z)} = \sum_{n=1}^{\infty}(-1)^n(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^n\left[2(n+r)-1\right]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz +$$

(23)

$$+2\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{2\vartheta_1'}\right\}\Bigg\{\sum_{n=1}^{\infty}(-1)^{n+1}q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+1}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\Bigg\}$$

Replacing z by $z-\frac{\pi}{2}$

(24) $$\frac{\vartheta_1'^3\,\vartheta_2^2(z)}{\vartheta_0^2\,\vartheta_3^3(z)} = \sum_{n=1}^{\infty}(-1)^n(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+r}\left[2(n+r)-1\right]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz +$$

$$+\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n=1}^{\infty}(-1)^{n+1}q^{\left(\frac{2n-1}{2}\right)^2}+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+r+1}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

In (20), (21), (23) and (24) replace q by $-q$. There follow

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_3^2(z)}{\vartheta_3^2\vartheta_1^3(z)} &= \frac{1}{\sin^3 z}+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+1}[2(n+r)-1]^2q^{n(n+2r-1)}\sin(2r-1)z\;+\\ &+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{1}{2\sin z}+2\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n}q^{n(n+2r-1)}\sin(2r-1)z\;\Big\}.\end{aligned}\tag{25}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_0^2(z)}{\vartheta_3^2\vartheta_2^3 z} &= \frac{1}{\cos^3 z}+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+r}[2(n+r)-1]^2q^{n(n+2r-1)}\cos(2r-1)z\;+\\ &+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{1}{2\cos z}+2\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+r+1}q^{n(n+2r-1)}\cos(2r-1)z.\end{aligned}\tag{26}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_1^2(z)}{\vartheta_3^2\vartheta_3^3(z)} &= \sum_{n=1}^{\infty}(-1)^{n+1}(2n-1)^2q^{\left(\frac{2n-1}{2}\right)^2}+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+r+1}[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\;+\\ &+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n=1}^{\infty}(-1)^{n}q^{\left(\frac{2n-1}{2}\right)^2}+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+r}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.\end{aligned}\tag{27}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_2^2(z)}{\vartheta_3^2\vartheta_0^3(z)} &= \sum_{n=1}^{\infty}(-1)^{n+1}(2n-1)^2q^{\left(\frac{2n-1}{2}\right)^2}+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n+1}[2(n+r)-1]^2q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\;+\\ &+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n=1}^{\infty}(-1)^{n}q^{\left(\frac{2n-1}{2}\right)^2}+2\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^{n}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.\end{aligned}\tag{28}$$

The arithmetized forms of these will now be developed. Consider the double sums which appear in (20). Each of these can be grouped to form a single series in powers of q. In each of these we need the coefficient of q^N. Only those terms of the double sums of (20) can contribute to the term containing q^N which satisfy $N=n(n+2r-1)$. Let $N=b\beta$ and put $b=n$ and $\beta=n+2r-1$. From this follows

$$n=b \qquad r=\frac{\beta-b+1}{2}$$

$$2r-1=\beta-b \qquad 2(n+r)-1=\beta+b$$

Since r is an integer > 0, and since $0 < b < \beta$ we must have

$$(29) \qquad 0 < b < \sqrt{N} \qquad\qquad \beta - b \equiv 1 \bmod 2$$

Similarly two series of q can be formed in (23), the exponent of q being of the form $\frac{N}{4}$ where $N = (2n-1)(2n+4r-1)$. Here let $N = d\delta$,

$d = 2n-1$, $\delta = 2n+4r-1$. Hence

$$n = \frac{d+1}{2}, \quad r = \frac{\delta - d}{4}, \quad 2(n+r)-1 = \frac{\delta+d}{2},$$

and d, δ must satisfy

$$(30) \qquad 0 < d < \sqrt{N}, \quad d \equiv 1 \bmod 2, \qquad \delta - d \equiv 0 \bmod 4.$$

Hence the expansions given can be written in the second forms

$$(20.1) \qquad \frac{\vartheta_1'^3 \vartheta_0^2(z)}{\vartheta_0^2 \vartheta_1^3(z)} = \frac{1}{\sin^3 z} + 2\sum q^N \left\{ \sum (-1)^{b+1} (\beta+b)^2 \sin(\beta-b)z \right\} + \left\{ 2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1 \right\} \frac{1}{2\sin z} + 2\left\{ 2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} \right\} \sum q^N \left\{ \sum (-1)^b \sin(\beta-b)z \right\}.$$

$$(21.1) \qquad \frac{\vartheta_1'^3 \vartheta_3^2(z)}{\vartheta_0^2 \vartheta_2^3(z)} = \frac{1}{\cos^3 z} + 2\sum q^N \left\{ \sum (-1)^{\frac{\beta+b+1}{2}} (\beta+b)^2 \cos(\beta-b)z \right\} + \left\{ 2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1 \right\} \frac{1}{2\cos z} + 2\left\{ 2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} \right\} \sum q^N \left\{ \sum (-1)^{\frac{\beta+b-1}{2}} \cos(\beta-b)z \right\}.$$

$$(23.1) \qquad \frac{\vartheta_1'^3 \vartheta_1^2(z)}{\vartheta_0^2 \vartheta_0^3(z)} = \sum_{n=1}^{\infty} (-1)^n (2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{d+1}{2}} (\delta+d)^2 \cos\frac{\delta-d}{2} z \right\} + \left\{ 2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} \right\} \left\{ \sum_{n=1}^{\infty} (-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{d-1}{2}} \cos\frac{\delta-d}{2} z \right\} \right\}.$$

$$(24.1) \qquad \frac{\vartheta_1'^3 \vartheta_2^2(z)}{\vartheta_0^2 \vartheta_3^3(z)} = \sum_{n=1}^{\infty} (-1)^n (2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}} \left\{ \sum (-1)^{\frac{\delta+d+2}{4}} (\delta+d)^2 \cos\frac{\delta-d}{2} z \right\} + \left\{ 2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} \right\} \left\{ \sum_{n=1}^{\infty} (-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\delta+d-2}{4}} \cos\frac{\delta-d}{2} z \right\} \right\}.$$

(25.1)
$$\frac{\vartheta_1'^3\vartheta_3^2(z)}{\vartheta_3^2\vartheta_1^3(z)} = \frac{1}{\sin^3 z} + 2\sum q^N\left\{\sum(-1)^{b+1}(\beta+b)^2\sin(\beta-b)z\right\} +$$
$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{1}{2\sin z} + 2\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{\sum(-1)^b\sin(\beta-b)z\right\}$$

(26.1)
$$\frac{\vartheta_1'^3\vartheta_0^2(z)}{\vartheta_3^2\vartheta_2^3(z)} = \frac{1}{\cos^3 z} + 2\sum q^N\left\{\sum(-1)^{\frac{\beta+b+1}{2}}(\beta+b)^2\cos(\beta-b)z\right\} +$$
$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{1}{2\cos z} + 2\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta+b+3}{2}}\cos(\beta-b)z\right\}$$

(27.1)
$$\frac{\vartheta_1'^3\vartheta_1^2(z)}{\vartheta_3^2\vartheta_3^3(z)} = \sum_{n=1}^{\infty}(-1)^{n+1}(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{\sum(-1)^{\frac{\delta+d-2}{4}}(\delta+d)^2\cos\frac{\delta-d}{2}z\right\} +$$
$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n=1}^{\infty}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{\sum(-1)^{\frac{\delta+d+2}{4}}\cos\frac{\delta-d}{2}z\right\}\right\}$$

(28.1)
$$\frac{\vartheta_1'^3\vartheta_2^2(z)}{\vartheta_3^2\vartheta_0^3(z)} = \sum_{n=1}^{\infty}(-1)^{n+1}(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+d)^2\cos\frac{\delta-d}{2}z\right\}$$
$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n=1}^{\infty}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{\sum(-1)^{\frac{d+1}{2}}\cos\frac{\delta-d}{2}z\right\}\right\}$$

it being understood that the coefficient of q^N or $q^{\frac{N}{4}}$ is the sum indicated taken over all pairs of conjugate divisors of N which satisfy (29) or (30) respectively.

- V -

The practical use of the theory has been made clear in the last section. In this section are presented expansions for all functions (2) whose denominators are of the fourth, or lower, total degree in the ϑ's, for which μ is one. Intermediate stages of the calculations are also given.

The exponents of q containing both m and r are all included in

$$m(m+2r) \qquad m(m+2r-1)$$

$$\frac{(2m-1)(2m+4r-3)}{4} \qquad \frac{(2m-1)(2m+4r-1)}{4}$$

Further discussion similar to that of page 15 leads to

$$\begin{array}{llll} N = m(m+2r) = a\alpha & & N = m(m+2r-1) = b\beta & \\ a = m & \alpha = m+2r & b = m & \beta = m+2r-1 \\ m = a & r = \frac{\alpha - a}{2} & m = b & r = \frac{\beta - b + 1}{2} \\ 0 < a < \sqrt{N} & \alpha - a \equiv 0 \bmod 2. & 0 < b < \sqrt{N} & \beta - b \equiv 1 \bmod 2. \end{array}$$

(31)

$$\begin{array}{llll} \frac{N}{4} = \frac{(2m-1)(2m+4r-3)}{4} = \frac{c\gamma}{4} & & \frac{N}{4} = \frac{(2m-1)(4r+2m-1)}{4} = \frac{d\delta}{4} & \\ c = 2m-1 & \gamma = 2m+4r-3 & d = 2m-1 & \delta = 2m+4r-1 \\ m = \frac{c+1}{2} & r = \frac{\gamma - c + 2}{4} & m = \frac{d+1}{2} & r = \frac{\delta - d}{4} \\ c \equiv 1 \bmod 2 & \gamma - c \equiv 2 \bmod 4 & d \equiv 1 \bmod 2 & \delta - d \equiv 0 \bmod 4 \\ 0 < c < \sqrt{N} & & 0 < d < \sqrt{N} & \end{array}$$

The convention is adopted that whenever q^N or $q^{\frac{N}{4}}$ is multiplied by a function of (a,α), (b,β), (c,γ), or (d,δ), this function is summed over all pairs of conjugate divisors of N which satisfy the corresponding conditions of (31). The upper limit of any summation with respect to m or r is ∞, the lower 1, unless written otherwise.

Group I

$$\frac{1}{\vartheta_0(z)} \qquad \frac{1}{\vartheta_1(z)} \qquad \frac{1}{\vartheta_2(z)} \qquad \frac{1}{\vartheta_3(z)}$$

Consider

$$F(z) = \frac{e^{-iz}}{\vartheta_1(z)}$$

The substitutions needed are $t = z + \frac{\pi}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ is of the form (8); its sole singularity in the period parallelogram is a simple pole at $t = \frac{\pi}{2}$. Computing the residue at this pole and using (10) we have

$$(32)\qquad \vartheta_1' F(z) = A_1^{(0)}\left(z+\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

Using (13) this gives after reduction

$$(33)\qquad \frac{\vartheta_1'}{\vartheta_1(z)} = \frac{1}{\sin z} + 4\sum_{n,r}(-1)^n q^{n(n+2r-1)}\sin(2r-1)z$$

Replacing z by $z+\frac{\pi}{2}$ gives

$$(34)\qquad \frac{\vartheta_1'}{\vartheta_2(z)} = \frac{1}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n(n+2r-1)}\cos(2r-1)z$$

In (32) replace z by $z+\frac{\pi\tau}{2}$. This gives

$$\frac{\vartheta_1'}{\vartheta_0(z)} = i\,q^{1/4} A_1^{(0)}\left(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

Using (16) gives

$$(35)\qquad \frac{\vartheta_1'}{\vartheta_0(z)} = 2\sum_{n}(-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2} + 4\sum_{n,r}(-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz$$

Replacing z by $z-\frac{\pi}{2}$

$$(36)\qquad \frac{\vartheta_1'}{\vartheta_3(z)} = 2\sum_{n}(-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2} + 4\sum_{n,r}(-1)^{n+r+1} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz .$$

The arithmetized forms of these are

$$(33.1)\qquad \frac{\vartheta_1'}{\vartheta_1(z)} = \frac{1}{\sin z} + 4\sum q^N\left\{(-1)^b \sin(\beta-b)z\right\}$$

$$(34.1)\qquad \frac{\vartheta_1'}{\vartheta_2(z)} = \frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta+b-1}{2}} \cos(\beta-b)z\right\}$$

$$(35.1)\qquad \frac{\vartheta_1'}{\vartheta_0(z)} = 2\sum_{n}(-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2} + 4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}} \cos\frac{\delta-d}{2} z\right\}$$

$$(36.1)\qquad \frac{\vartheta_1'}{\vartheta_3(z)} = 2\sum_{n}(-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2} + 4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d-2}{4}} \cos\frac{\delta-d}{2} z\right\}.$$

Group IIa.

$$\frac{\vartheta_1(z)}{\vartheta_0^2(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_3^2(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_1^2(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_2^2(z)}$$

$$\frac{\vartheta_1(z)}{\vartheta_3^2(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0^2(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_1^2(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_2^2(z)}$$

Consider

$$F(z) = \frac{\vartheta_1(z)\, e^{-iz}}{\vartheta_0^2(z)}$$

The required substitutions are $t = z + \frac{\pi}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8); its sole singularity in the period parallelogram is a pole of order two at $t = \frac{\pi}{2} + \frac{\pi\tau}{2}$. Making the usual computations and using (10) leads to

$$(37) \qquad \frac{\vartheta_1'^2}{\vartheta_0} F(z) = -i q^{1/4} A_1^{(1)}\left(z + \tfrac{\pi}{2}, \tfrac{\pi}{2} + \tfrac{\pi\tau}{2}\right)$$

Substituting on the right side from (15) and reducing gives

$$(38) \qquad \frac{\vartheta_1'^2 \vartheta_1(z)}{\vartheta_0 \vartheta_0^2(z)} = 4 \sum_{n,r} (-1)^{n+1} 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z .$$

Replacing z by $z + \frac{\pi}{2}$

$$(39) \qquad \frac{\vartheta_1'^2 \vartheta_2(z)}{\vartheta_0 \vartheta_3^2(z)} = 4 \sum_{n,r} (-1)^{n+r} 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z .$$

Replace z by $z + \frac{\pi\tau}{2}$ in (37). We obtain

$$\frac{\vartheta_1'^2 \vartheta_0(z)}{\vartheta_0 \vartheta_1^2(z)} = A_1^{(1)}\left(z + \tfrac{\pi\tau}{2} + \tfrac{\pi}{2}, \tfrac{\pi}{2} + \tfrac{\pi\tau}{2}\right)$$

and expanding this by (13) results in

$$(40) \qquad \frac{\vartheta_1'^2 \vartheta_0(z)}{\vartheta_0 \vartheta_1^2(z)} = \frac{1}{\sin^2 z} + 2\sum_n (-1)^{n+1} 2n\, q^{n^2} + 4 \sum_{n,r} (-1)^{n+1} 2(n+r)\, q^{n^2+2nr} \cos 2rz$$

Replacing z by $z + \frac{\pi}{2}$

$$(41) \qquad \frac{\vartheta_1'^2 \vartheta_3(z)}{\vartheta_0 \vartheta_2^2(z)} = \frac{1}{\cos^2 z} + 2\sum_n (-1)^{n+1} 2n\, q^{n^2} + 4 \sum_{n,r} (-1)^{n+r+1} 2(n+r)\, q^{n^2+2nr} \cos 2rz .$$

Substitute $-q$ for q in the above series. We obtain

(42) $$\frac{\vartheta_1'^2\,\vartheta_1(z)}{\vartheta_3\,\vartheta_3^2(z)} = 4\sum_{n,r}(-1)^{r+1}2(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \;.$$

(43) $$\frac{\vartheta_1'^2\,\vartheta_2(z)}{\vartheta_3\,\vartheta_0^2(z)} = 4\sum_{n,r}2(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \;.$$

(44) $$\frac{\vartheta_1'^2\,\vartheta_3(z)}{\vartheta_3\,\vartheta_1^2(z)} = \frac{1}{\sin^2 z} - 2\sum_n 2n\,q^{n^2} - 4\sum_{n,r}2(n+r)\,q^{n^2+2nr}\cos 2rz \;.$$

(45) $$\frac{\vartheta_1'^2\,\vartheta_0(z)}{\vartheta_3\,\vartheta_2^2(z)} = \frac{1}{\cos^2 z} - 2\sum_n 2n\,q^{n^2} + 4\sum_{n,r}(-1)^{r+1}2(n+r)\,q^{n^2+2nr}\cos 2rz \;.$$

The arithmetized forms of these are

(38.1) $$\frac{\vartheta_1'^2\,\vartheta_1(z)}{\vartheta_0\,\vartheta_0^2(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(c+\gamma)\sin\frac{\gamma-c}{2}z\right\}$$

(39.1) $$\frac{\vartheta_1'^2\,\vartheta_2(z)}{\vartheta_0\,\vartheta_3^2(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}$$

(40.1) $$\frac{\vartheta_1'^2\,\vartheta_0(z)}{\vartheta_0\,\vartheta_1^2(z)} = \frac{1}{\sin^2 z} + 2\sum_n(-1)^{n+1}2n\,q^{n^2} + 4\sum q^N\left\{(-1)^{a+1}(\alpha+a)\cos(\alpha-a)z\right\}$$

(41.1) $$\frac{\vartheta_1'^2\,\vartheta_3(z)}{\vartheta_0\,\vartheta_2^2(z)} = \frac{1}{\cos^2 z} + 2\sum_n(-1)^{n+1}2n\,q^{n^2} + 4\sum q\left\{(-1)^{\frac{\alpha+a+2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}$$

(42.1) $$\frac{\vartheta_1'^2\,\vartheta_1(z)}{\vartheta_3\,\vartheta_3^2(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\}$$

(43.1) $$\frac{\vartheta_1'^2\vartheta_2(z)}{\vartheta_3\vartheta_0^2(z)} = 2\sum q^{\frac{N}{4}}\left\{(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}.$$

(44.1) $$\frac{\vartheta_1'^2\vartheta_3(z)}{\vartheta_3\vartheta_1^2(z)} = \frac{1}{\sin^2 z} - 2\sum_n 2n q^{n^2} - 4\sum q^N\left\{(\alpha+a)\cos(\alpha-a)z\right\}$$

(45.1) $$\frac{\vartheta_1'^2\vartheta_0(z)}{\vartheta_3\vartheta_2^2(z)} = \frac{1}{\cos^2 z} - 2\sum_n 2n q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}.$$

Group II-b

$$\frac{\vartheta_2(z)}{\vartheta_1^2(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_2^2(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_0^2(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_3^2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2(z)e^{-iz}}{\vartheta_1^2(z)}$$

$F(z)$ satisfies (8) and has a pole of order two at $z=0$. Calculating the appropriate $R_i^{(j)}$ and substituting in (10) gives

(46) $$\frac{\vartheta_1'^2}{\vartheta_2}F(z) = A_1^{(1)}(z,0) - iA_1^{(0)}(z,0)$$

Expanding on the right by (13) and simplifying gives

(47) $$\frac{\vartheta_1'^2\vartheta_2(z)}{\vartheta_2\vartheta_1^2(z)} = \frac{\cos z}{\sin^2 z} - 4\sum_{n,r}[2(n+r)-1]\, q^{n(n+2r-1)}\cos(2r-1)z.$$

Replacing z by $z+\frac{\pi}{2}$

(48) $$\frac{\vartheta_1'^2\vartheta_1(z)}{\vartheta_2\vartheta_2^2(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{n(n+2r-1)}\sin(2r-1)z.$$

Substitute $z+\frac{\pi\tau}{2}$ for z in (46). From this follows

$$\frac{\vartheta_1'^2\vartheta_3 z}{\vartheta_2\vartheta_0^2(z)} = -q^{\frac14}A_1^{(1)}(z+\tfrac{\pi\tau}{2},0) + i\,q^{\frac14}A_1^{0}(z+\tfrac{\pi\tau}{2},0).$$

Using (16) and reducing we get

$$(49)\qquad \frac{\vartheta_1'^2\vartheta_3(z)}{\vartheta_2\vartheta_0^2(z)} = 2\sum_n (2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + 4\sum_{n,r}[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \,.$$

Replacing z by $z+\frac{\pi}{2}$

$$(50)\qquad \frac{\vartheta_1'^2\vartheta_0(z)}{\vartheta_2\vartheta_3^2(z)} = 2\sum_n (2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + 4\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \,.$$

From these follow

$$(47.1)\qquad \frac{\vartheta_1'^2\vartheta_2(z)}{\vartheta_2\vartheta_1^2(z)} = \frac{\cos z}{\sin^2 z} - 4\sum q^N\left\{(\beta+b)\cos(\beta-b)z\right\}$$

$$(48.1)\qquad \frac{\vartheta_1'^2\vartheta_1(z)}{\vartheta_2\vartheta_2^2(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)\sin(\beta-b)z\right\}$$

$$(49.1)\qquad \frac{\vartheta_1'^2\vartheta_3(z)}{\vartheta_2\vartheta_0^2(z)} = 2\sum_n (2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(\delta+d)\cos\frac{\delta-d}{2}z\right\}$$

$$(50.1)\qquad \frac{\vartheta_1'^2\vartheta_0(z)}{\vartheta_2\vartheta_3^2(z)} = 2\sum_n (2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)\cos\frac{\delta-d}{2}z\right\}$$

Group III-a

$$\frac{\vartheta_3(z)}{\vartheta_0(z)\vartheta_1(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_2(z)\vartheta_3(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0(z)\vartheta_1(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_2(z)\vartheta_3(z)}$$

$$\frac{\vartheta_0(z)}{\vartheta_1(z)\vartheta_3(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_0(z)\vartheta_2(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_1(z)\vartheta_3(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_0(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_3(z)\, e^{-iz}}{\vartheta_0(z)\vartheta_1(z)}$$

$F(z)$ satisfies (8) and has simple poles at $z = \frac{\pi\tau}{2}$ and $z = 0$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$\vartheta_0 \vartheta_1' F(z) = -q^{-1/4} \vartheta_2 A_1^{(0)}(z, \tfrac{\pi\tau}{2}) + \vartheta_3 A_1^{(0)}(z, 0) \; . \tag{51}$$

Expanding by means of (15) and (13) and reducing gives

$$\begin{aligned} \frac{\vartheta_0\vartheta_1'\vartheta_3(z)}{\vartheta_0(z)\vartheta_1(z)} = {} & \vartheta_3 \left\{ \frac{1}{\sin z} + 4 \sum_{n,r} q^{n(n+2r-1)} \sin(2r-1)z \right\} + \\ & -4\vartheta_2 \sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2 + (2n-1)(r-\frac{1}{2})} \sin(2r-1)z \; . \end{aligned} \tag{52}$$

Replacing z by $z + \frac{\pi}{2}$

$$\begin{aligned} \frac{\vartheta_0\vartheta_1'\vartheta_0(z)}{\vartheta_2(z)\vartheta_3(z)} = {} & \vartheta_3 \left\{ \frac{1}{\cos z} + 4 \sum_{n,r} (-1)^{r+1} q^{n(n+2r-1)} \cos(2r-1)z \right\} + \\ & +4\vartheta_2 \sum_{n,r} (-1)^r q^{\left(\frac{2n-1}{2}\right)^2 + (2n-1)(r-\frac{1}{2})} \cos(2r-1)z \; . \end{aligned} \tag{53}$$

In (51) replace z by $z + \frac{\pi\tau}{2}$. This gives

$$\frac{\vartheta_0\vartheta_1'\vartheta_2(z)}{\vartheta_0(z)\vartheta_1(z)} = \vartheta_2 A_1^{(0)}(z + \tfrac{\pi\tau}{2}, \tfrac{\pi\tau}{2}) - \vartheta_3 q^{1/4} A_1^{(0)}(z + \tfrac{\pi\tau}{2}, 0) \; .$$

Using (13) and (15) leads to

$$\begin{aligned} \frac{\vartheta_0\vartheta_1'\vartheta_2(z)}{\vartheta_0(z)\vartheta_1(z)} = {} & \vartheta_2 \left\{ \frac{\cos z}{\sin z} + 4 \sum_{n,r} q^{n^2+2nr} \sin 2rz \right\} + \\ & -4\vartheta_3 \sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2 + (2n-1)r} \sin 2rz \; . \end{aligned} \tag{54}$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'\vartheta_0\vartheta_1(z)}{\vartheta_2(z)\vartheta_3(z)} = \vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{n^2+2nr}\sin 2rz\right\}$$

(55)

$$+4\vartheta_3\sum_{n,r}(-1)^r q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \quad .$$

In these results, substitute $-q$ for q . We get

$$\frac{\vartheta_1'\vartheta_3\vartheta_0(z)}{\vartheta_1(z)\vartheta_3(z)} = \vartheta_0\left\{\frac{1}{\sin z} + 4\sum_{n,r} q^{n(n+2r-1)}\sin(2r-1)z\right\} +$$

(56)

$$+4\vartheta_2\sum_{n,r}(-1)^{n+r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \quad .$$

$$\frac{\vartheta_3\vartheta_1'\vartheta_3(z)}{\vartheta_0(z)\vartheta_2(z)} = \vartheta_0\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{n(n+2r-1)}\cos(2r-1)z\right\} +$$

(57)

$$+4\vartheta_2\sum_{n,r}(-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \quad .$$

$$\frac{\vartheta_1'\vartheta_3\vartheta_2(z)}{\vartheta_1(z)\vartheta_3(z)} = \vartheta_2\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^n q^{n^2+2nr}\sin 2rz\right\} +$$

(58)

$$+4\vartheta_0\sum_{n,r}(-1)^{r+1} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \quad .$$

$$\frac{\vartheta_1'\vartheta_3\vartheta_1(z)}{\vartheta_0(z)\vartheta_2(z)} = \vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n^2+2nr}\sin 2rz\right\} +$$

(59)

$$+4\vartheta_0\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \quad .$$

These have the arithmetized forms

$$\begin{aligned}\frac{\vartheta_0\vartheta_1'\vartheta_3(z)}{\vartheta_0(z)\vartheta_1(z)} &= \vartheta_3\left\{\frac{1}{\sin z} + 4\sum q^N\left\{\sin(\beta-b)z\right\}\right\} + \\ &\quad -4\vartheta_2\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-c}{2}z\right\}\end{aligned} \tag{52.1}$$

$$\begin{aligned}\frac{\vartheta_0\vartheta_1'\vartheta_0(z)}{\vartheta_2(z)\vartheta_3(z)} &= \vartheta_3\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\}\right\} + \\ &\quad +4\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}\cos\frac{\gamma-c}{2}z\right\}\end{aligned} \tag{53.1}$$

$$\begin{aligned}\frac{\vartheta_0\vartheta_1'\vartheta_2(z)}{\vartheta_0(z)\vartheta_1(z)} &= \vartheta_2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(\alpha-a)z\right\}\right\} + \\ &\quad -4\vartheta_3\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-d}{2}z\right\}\end{aligned} \tag{54.1}$$

$$\begin{aligned}\frac{\vartheta_0\vartheta_1'\vartheta_1(z)}{\vartheta_2(z)\vartheta_3(z)} &= \vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a+2}{2}}\sin(\alpha-a)z\right\}\right\} + \\ &\quad +4\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}\sin\frac{\delta-d}{2}z\right\}\end{aligned} \tag{55.1}$$

$$\begin{aligned}\frac{\vartheta_1'\vartheta_3\vartheta_0(z)}{\vartheta_1(z)\vartheta_3(z)} &= \vartheta_0\left\{\frac{1}{\sin z} + 4\sum q^N\left\{\sin(\beta-b)z\right\}\right\} + \\ &\quad +4\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}\sin\frac{\gamma-c}{2}z\right\}\end{aligned} \tag{56.1}$$

$$\frac{\vartheta_1'\vartheta_3\vartheta_3(z)}{\vartheta_0(z)\vartheta_2(z)} = \vartheta_0\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\}\right\} +$$

(57.1)

$$+ 4\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-c}{2}z\right\} .$$

$$\frac{\vartheta_1'\vartheta_3\vartheta_2(z)}{\vartheta_1(z)\vartheta_3(z)} = \vartheta_2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{(-1)^a\sin(\alpha-a)z\right\}\right\} +$$

(58.1)

$$+ 4\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-4}{4}}\sin\frac{\delta-d}{2}z\right\}$$

$$\frac{\vartheta_1'\vartheta_3\vartheta_1(z)}{\vartheta_0(z)\vartheta_2(z)} = \vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\right\}\right\} +$$

(59.1)

$$+ 4\vartheta_0\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-d}{2}z\right\} .$$

Group III-b

$$\frac{\vartheta_0(z)}{\vartheta_1(z)\vartheta_2(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_1(z)\vartheta_2(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_0(z)\vartheta_3(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0(z)\vartheta_3(z)}$$

Consider

$$F(z) = \frac{\vartheta_0(z)}{\vartheta_1(z)\vartheta_2(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has simple poles at $\frac{\pi\tau}{2}$ and $\frac{\pi\tau}{2} + \frac{\pi}{2}$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

(60) $$\vartheta_1'\vartheta_2 F(z) = \vartheta_0 A_1^{(0)}\left(z + \frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) - \vartheta_3 A_1^{(0)}\left(z + \frac{\pi\tau}{2}, \frac{\pi\tau}{2} + \frac{\pi}{2}\right)$$

From this follows

$$\frac{\vartheta_1'\vartheta_2\vartheta_0(z)}{\vartheta_1(z)\vartheta_2(z)} = \vartheta_0\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{n^2+2nr}\sin 2rz\right\} +$$

(61)

$$+ \vartheta_3\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n^2+2nr}\sin 2rz\right\} .$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'\vartheta_2\vartheta_3(z)}{\vartheta_1(z)\vartheta_2(z)} = \vartheta_0\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{n^2+2nr}\sin 2rz\right\} + \tag{62}$$

$$+\vartheta_3\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^{n} q^{n^2+2nr}\sin 2rz\right\}.$$

In (60) replace z by $z - \frac{\pi\tau}{2}$. There follows

$$\frac{\vartheta_1'\vartheta_2\vartheta_1(z)e^{-iz}}{\vartheta_0(z)\vartheta_3(z)} = q^{\frac{1}{4}}\vartheta_0 A_1^{(0)}\left(z, \frac{\pi\tau}{2}\right) - q^{\frac{1}{4}}\vartheta_3 A_1^{(0)}\left(z, \frac{\pi\tau}{2}+\frac{\pi}{2}\right)$$

Substituting from (15) and simplifying, we have

$$\frac{\vartheta_1'\vartheta_2\vartheta_1(z)}{\vartheta_0(z)\vartheta_3(z)} = 4\vartheta_0\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \;+ \tag{63}$$

$$+4\vartheta_3\sum_{n,r}(-1)^{n+r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'\vartheta_2\vartheta_2(z)}{\vartheta_0(z)\vartheta_3(z)} = 4\vartheta_0\sum_{n,r}(-1)^{r+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \;+ \tag{64}$$

$$+4\vartheta_3\sum_{n,r}(-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z.$$

These give an arithmetization

$$\frac{\vartheta_1'\vartheta_2\vartheta_0(z)}{\vartheta_1(z)\vartheta_2(z)} = \vartheta_0\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(\alpha-a)z\right\}\right\} + \tag{61.1}$$

$$+\vartheta_3\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'\vartheta_2\vartheta_3(z)}{\vartheta_1(z)\vartheta_2(z)} = \vartheta_0\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a+2}{2}}\sin(\alpha-a)z\right\}\right\} + \tag{62.1}$$

$$+\vartheta_3\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{(-1)^{a}\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'\vartheta_2\vartheta(z)}{\vartheta_0(z)\vartheta_3(z)} = 4\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}\sin\frac{\gamma-c}{2}z\right\} + \tag{C 3.1}$$

$$+4\vartheta_0\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-c}{2}z\right\}$$

$$\frac{\vartheta_1'\vartheta_2\vartheta_2(z)}{\vartheta_0(z)\vartheta_3(z)} = 4\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-c}{2}z\right\} + \tag{C 4.1}$$

$$+4\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}\cos\frac{\gamma-c}{2}z\right\}$$

Group IV-a

$$\frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_3^3(z)} \qquad \frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_0^3(z)} \qquad \frac{\vartheta_0(z)\vartheta_3(z)}{\vartheta_2^3(z)} \qquad \frac{\vartheta_0(z)\vartheta_3(z)}{\vartheta_1^3(z)}$$

Consider

$$F(z) = \frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_3^3(z)}$$

Let $t = z+\frac{\pi}{2}+\frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8); it has a pole of order three at $t = \pi+\pi\tau$. Computing the corresponding $R_i^{(j)}$, and using (10) gives

$$\frac{\vartheta_1'^3\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_3^3(z)} = \frac{1}{2}q^{1/4}A_2^{(2)}(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2};\pi\tau) + \tag{G 5}$$

$$+iq^{1/4}A_1^{(1)}(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2},\pi\tau) - \frac{1}{2}q^{1/4}\left\{\frac{\vartheta_2''}{\vartheta_2}+1\right\}A_1^{(0)}(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2},\pi\tau).$$

From this follows, after substitution from (17) and simplifying

$$\frac{\vartheta_1'^3\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_3^3(z)} = 2\sum_{n,r}(-1)^{r+1}[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \tag{G 6}$$

$$+2\frac{\vartheta_2''}{\vartheta_2}\sum_{n,r}(-1)^{r+1}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz$$

Replacing z by $z-\frac{\pi}{2}$

(67)
$$\frac{\vartheta_1'^3\,\vartheta_1(z)\,\vartheta_2(z)}{\vartheta_0\vartheta_3\,\vartheta_0^3(z)} = 2\sum_{n,r}[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \; +$$
$$+2\frac{\vartheta_2''}{\vartheta_2}\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \; .$$

In (65) replace z by $z+\frac{\pi\tau}{2}$, obtaining
$$\frac{e^{iz}\,\vartheta_1'^3\,\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\,\vartheta_3\,\vartheta_2^3(z)} = -\frac{i}{2}A_1^{(2)}(z+\pi\tau+\tfrac{\pi}{2},\pi\tau) + A_1^{(1)}(z+\pi\tau+\tfrac{\pi}{2},\pi\tau)$$
$$+\frac{i}{2}\left\{\frac{\vartheta_2''}{\vartheta_2}+1\right\}A_1^{(0)}(z+\pi\tau+\tfrac{\pi}{2},\pi\tau)$$

From this we get

(68)
$$\frac{\vartheta_1'^3\,\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\,\vartheta_2^3(z)} = \frac{1}{\cos^3 z} - \frac{1}{2\cos z}\left\{\frac{\vartheta_2''}{\vartheta_2}+1\right\} + 2\sum_{n,r}(-1)^r[2(n+r)-1]^2 q^{n(n+2r-1)}\cos(2r-1)z \; +$$
$$+2\frac{\vartheta_2''}{\vartheta_2}\sum_{n,r}(-1)^r q^{n(n+2r-1)}\cos(2r-1)z$$

Replacing z by $z+\frac{\pi}{2}$

(69)
$$\frac{\vartheta_1'^3\,\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\,\vartheta_1^3(z)} = \frac{1}{\sin^3 z} - \frac{1}{2\sin z}\left\{\frac{\vartheta_2''}{\vartheta_2}+1\right\} - 2\sum_{n,r}[2(n+r)-1]^2 q^{n(n+2r-1)}\sin(2r-1)z \; +$$
$$-2\frac{\vartheta_2''}{\vartheta_2}\sum q^{n(n+2r-1)}\sin(2r-1)z$$

Arithmetizing we have

(66.1)
$$\frac{\vartheta_1'^3\,\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_3^3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d+4}{4}}(\delta+d)^2\sin\frac{\delta-d}{2}z\right\} \; +$$
$$+2\frac{\vartheta_2''}{\vartheta_2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d+4}{4}}\sin\frac{\delta-d}{2}z\right\}$$

(67.1)
$$\frac{\vartheta_1'^3\,\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0^3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\delta+d)^2\sin\frac{\delta-d}{2}z\right\} \; +$$
$$+2\frac{\vartheta_2''}{\vartheta_2}\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-d}{2}z\right\} \; .$$

$$\frac{\vartheta_1'^3\,\vartheta_0(z)\,\vartheta_3(z)}{\vartheta_0\,\vartheta_3\,\vartheta_2^3(z)} = \frac{1}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)^2\cos(\beta-b)z\right\} +$$

(68.1)

$$-\left\{\frac{\vartheta_2''}{\vartheta_2}+1\right\}\frac{1}{2\cos z} + 2\frac{\vartheta_2''}{\vartheta_2}\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}\cos(\beta-b)z\right\} .$$

$$\frac{\vartheta_1'^3\,\vartheta_0(z)\,\vartheta_3(z)}{\vartheta_0\,\vartheta_3\,\vartheta_1^3(z)} = \frac{1}{\sin^3 z} - 2\sum q^N\left\{(\beta+b)^2\sin(\beta-b)z\right\} +$$

(69.1)

$$-\left\{\frac{\vartheta_2''}{\vartheta_2}+1\right\}\frac{1}{2\sin z} - 2\frac{\vartheta_2''}{\vartheta_2}\sum q^N\left\{\sin(\beta-b)z\right\} .$$

Group IV-b

$$\frac{\vartheta_1(z)\vartheta_3(z)}{\vartheta_0^3(z)} \qquad \frac{\vartheta_0(z)\vartheta_2(z)}{\vartheta_3^3(z)} \qquad \frac{\vartheta_1(z)\vartheta_3(z)}{\vartheta_2^3(z)} \qquad \frac{\vartheta_0(z)\vartheta_2(z)}{\vartheta_1^3(z)}$$

$$\frac{\vartheta_1(z)\vartheta_0(z)}{\vartheta_3^3(z)} \qquad \frac{\vartheta_3(z)\vartheta_2(z)}{\vartheta_0^3(z)} \qquad \frac{\vartheta_1(z)\vartheta_0(z)}{\vartheta_2^3(z)} \qquad \frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_1^3(z)}$$

Consider

$$F(z) = \frac{\vartheta_1(z)\,\vartheta_3(z)\,e^{-iz}}{\vartheta_0^3(z)}$$

$F(z)$ satisfies (8) and has a pole of order three at $z=\frac{\pi\tau}{2}$. Computing the corresponding $R_i^{(j)}$ and using (10) gives

(70)
$$\frac{\vartheta_1'^3}{\vartheta_0\,\vartheta_2}F(z) = -\frac{1}{2}q^{\frac14}A_1^{(2)}\left(z,\frac{\pi\tau}{2}\right) + \frac{1}{2}q^{\frac14}\frac{\vartheta_3''}{\vartheta_3}A_1^{(0)}\left(z,\frac{\pi\tau}{2}\right)$$

From this follows

$$\frac{\vartheta_1'^3\,\vartheta_1(z)\,\vartheta_3(z)}{\vartheta_0\,\vartheta_2\,\vartheta_0^3(z)} = 2\sum_{n,r}4(n+r-1)^2\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \;+$$

(71)

$$+2\frac{\vartheta_3''}{\vartheta_3}\sum_{n,r}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z .$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\,\vartheta_0(z)\,\vartheta_2(z)}{\vartheta_0\,\vartheta_2\,\vartheta_3^3(z)} = 2\sum_{n,r}(-1)^{r+1}4(n+r-1)^2\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \;+$$

(72)

$$+2\frac{\vartheta_3''}{\vartheta_3}\sum_{n,r}(-1)^{r+1}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z .$$

In (70) replace z by $z+\frac{\pi\tau}{2}$. This gives

$$\frac{\vartheta_1'^3\,\vartheta_0(z)\,\vartheta_2(z)}{\vartheta_0\,\vartheta_2\,\vartheta_1^3(z)} = \frac{1}{2}A_1^{(2)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right) - \frac{1}{2}\frac{\vartheta_3''}{\vartheta_3}A_1^{(0)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right).$$

From this

$$(73)\qquad \begin{aligned} \frac{\vartheta_1'^3\,\vartheta_0(z)\,\vartheta_2(z)}{\vartheta_0\,\vartheta_2\,\vartheta_1^3(z)} &= \frac{\cos z}{\sin^3 z} - 2\sum_{n,r} 4(n+r)^2\, q^{n^2+2nr}\sin 2rz \; + \\ &\quad - \frac{\vartheta_3''}{\vartheta_3}\left\{\frac{\cos z}{2\sin z} + 2\sum_{n,r} q^{n^2+2nr}\sin 2rz\right\}. \end{aligned}$$

Replacing z by $z-\frac{\pi}{2}$

$$(74)\qquad \begin{aligned} \frac{\vartheta_1'^3\,\vartheta_1(z)\,\vartheta_3(z)}{\vartheta_0\,\vartheta_2\,\vartheta_2^3(z)} &= \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^r\, 4(n+r)^2\, q^{n^2+2nr}\sin 2rz \; + \\ &\quad - \frac{\vartheta_3''}{\vartheta_3}\left\{\frac{\sin z}{2\cos z} + 2\sum_{n,r}(-1)^{r+1} q^{n^2+2nr}\sin 2rz\right\}. \end{aligned}$$

Replace q by $-q$ in these results. There follow

$$(75)\qquad \begin{aligned} \frac{\vartheta_1'^3\,\vartheta_0(z)\,\vartheta_1(z)}{\vartheta_2\,\vartheta_3\,\vartheta_3^3(z)} &= 2\sum_{n,r}(-1)^{r+n}\, 4(n+r-1)^2\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; + \\ &\quad + 2\frac{\vartheta_0''}{\vartheta_0}\sum_{n,r}(-1)^{n+r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z. \end{aligned}$$

$$(76)\qquad \begin{aligned} \frac{\vartheta_1'^3\,\vartheta_2(z)\,\vartheta_3(z)}{\vartheta_2\,\vartheta_3\,\vartheta_0^3(z)} &= 2\sum_{n,r}(-1)^{n+1}\, 4(n+r-1)^2\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; + \\ &\quad + 2\frac{\vartheta_0''}{\vartheta_0}\sum_{n,r}(-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z. \end{aligned}$$

$$\frac{\vartheta_1'^3\,\vartheta_2(z)\,\vartheta_3(z)}{\vartheta_2\,\vartheta_3\,\vartheta_1^3(z)} = \frac{\cos z}{\sin^3 z} + 2\sum_{n,r}(-1)^{n+1}4(n+r)^2 q^{n^2+2nr}\sin 2rz \;+$$

(77)

$$-\frac{\vartheta_0''}{\vartheta_0}\left\{\frac{\cos z}{2\sin z} + 2\sum_{n,r}(-1)^n q^{n^2+2nr}\sin 2rz\right\}.$$

$$\frac{\vartheta_1'^3\,\vartheta_0(z)\,\vartheta_1(z)}{\vartheta_2\,\vartheta_3\,\vartheta_2^3(z)} = \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^{n+r}4(n+r)^2 q^{n^2+2nr}\sin 2rz \;+$$

(78)

$$-\frac{\vartheta_0''}{\vartheta_0}\left\{\frac{\sin}{2\cos z} + 2\sum_{n,r}(-1)^{n+r+1} q^{n^2+2nr}\sin 2rz\right\}.$$

Arithmetizing these we get

(71.1) $$\frac{\vartheta_1'^3\,\vartheta_3(z)\,\vartheta_1(z)}{\vartheta_0\vartheta_2\,\vartheta_0^3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\gamma+c)^2\sin\frac{\gamma-c}{2}z\right\} + 2\frac{\vartheta_3''}{\vartheta_3}\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-c}{2}z\right\}.$$

(72.1) $$\frac{\vartheta_1'^3\,\vartheta_0(z)\,\vartheta_2(z)}{\vartheta_0\,\vartheta_2\,\vartheta_3^3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}(\gamma+c)^2\cos\frac{\gamma-c}{2}z\right\} + 2\frac{\vartheta_3''}{\vartheta_3}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}\cos\frac{\gamma-c}{2}z\right\}.$$

(73.1) $$\frac{\vartheta_1'^3\,\vartheta_0(z)\,\vartheta_2(z)}{\vartheta_0\vartheta_2\,\vartheta_1^3(z)} = \frac{\cos z}{\sin^3 z} - 2\sum q^N\left\{(\alpha+a)^2\sin(\alpha-a)z\right\} \;+$$

$$-\frac{\vartheta_3''}{\vartheta_3}\left\{\frac{\cos z}{2\sin z} + 2\sum q^N\left\{\sin(\alpha-a)z\right\}\right\}.$$

(74.1) $$\frac{\vartheta_1'^3\,\vartheta_1(z)\,\vartheta_3(z)}{\vartheta_0\,\vartheta_2\,\vartheta_2^3(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha-a}{2}}(\alpha+a)^2\sin(\alpha-a)z\right\} \;+$$

$$-\frac{\vartheta_3''}{\vartheta_3}\left\{\frac{\sin z}{2\cos z} + 2\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_0(z)\vartheta_1(z)}{\vartheta_2\vartheta_3\vartheta_3^3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}(\gamma+c)^2\sin\frac{\gamma-c}{2}z\right\} +$$

(75.1)

$$+2\frac{\vartheta_0''}{\vartheta_0}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}\sin\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_0^3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+c)^2\cos\frac{\gamma-c}{2}z\right\} +$$

(76.1)

$$+2\frac{\vartheta_0''}{\vartheta_0}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_1^3(z)} = \frac{\cos z}{\sin^3 z} + 2\sum q^N\left\{(-1)^{a+1}(\alpha+a)^2\sin(\alpha-a)z\right\} +$$

(77.1)

$$-\frac{\vartheta_0''}{\vartheta_0}\left\{\frac{\cos z}{2\sin z} + 2\sum q^N\left\{(-1)^a\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_0(z)\vartheta_1(z)}{\vartheta_2\vartheta_3\vartheta_2^3(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha+a}{2}}(\alpha+a)^2\sin(\alpha-a)z\right\} +$$

(78.1)

$$-\frac{\vartheta_0''}{\vartheta_0}\left\{\frac{\sin z}{2\cos z} + 2\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

Group V-a

$$\frac{\vartheta_0^2(z)}{\vartheta_1^3(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_2^3(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_0^3(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_3^3(z)}$$

$$\frac{\vartheta_3^2(z)}{\vartheta_1^3(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_2^3(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_3^3(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_0^3(z)}$$

This group has been obtained in section four. For completeness the results there obtained are given again, the equations being renumbered.

$$(79)\quad \frac{\vartheta_1'^{3}\,\vartheta_0^{2}(z)}{\vartheta_0^{2}\,\vartheta_1^{3}(z)} = \frac{1}{\sin^3 z} + 2\sum_{n,r}(-1)^{n+1}[2(n+r)-1]^2\, q^{n(n+2r-1)}\sin(2r-1)z \; + $$
$$+ \left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\}\frac{1}{2\sin z} + 2\left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n}\, q^{n(n+2r-1)}\sin(2r-1)z \,.$$

$$(80)\quad \frac{\vartheta_1'^{3}\,\vartheta_3^{2}(z)}{\vartheta_0^{2}\,\vartheta_2^{3}(z)} = \frac{1}{\cos^3 z} + 2\sum_{n,r}(-1)^{n+r}[2(n+r)-1]^2\, q^{n(n+2r-1)}\cos(2r-1)z \; + $$
$$+ \left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\}\frac{1}{2\cos z} + 2\left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1}\, q^{n(n+2r-1)}\cos(2r-1)z \,.$$

$$(81)\quad \frac{\vartheta_1'^{3}\,\vartheta_1^{2}(z)}{\vartheta_0^{2}\,\vartheta_0^{3}(z)} = \sum_{n}(2n-1)^2(-1)^n\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n}[2(n+r)-1]^2\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz $$
$$+ \left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n}(-1)^{n+1}\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+1}\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

$$(82)\quad \frac{\vartheta_1'^{3}\,\vartheta_2^{2}(z)}{\vartheta_0^{2}\,\vartheta_3^{3}(z)} = \sum_{n}(-1)^n(2n-1)^2\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+r}[2(n+r)-1]^2\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \; + $$
$$+ \left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n}(-1)^{n+1}\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+r+1}\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

$$(83)\quad \frac{\vartheta_1'^{3}\,\vartheta_3^{2}(z)}{\vartheta_3^{2}\,\vartheta_1^{3}(z)} = \frac{1}{\sin^3 z} + 2\sum_{n,r}(-1)^{n+1}[2(n+r)-1]^2\, q^{n(n+2r-1)}\sin(2r-1)z \; + $$
$$+ \left\{2\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\}\frac{1}{2\sin z} + 2\left\{2\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n}\, q^{n(n+2r-1)}\sin(2r-1)z \,.$$

$$(84)\quad \frac{\vartheta_1'^{3}\,\vartheta_0^{2}(z)}{\vartheta_3^{2}\,\vartheta_2^{3}(z)} = \frac{1}{\cos^3 z} + 2\sum_{n,r}(-1)^{n+r}[2(n+r)-1]^2\, q^{n(n+2r-1)}\cos(2r-1)z \; + $$
$$+ \left\{2\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\}\frac{1}{2\cos z} + 2\left\{2\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1}\, q^{n(n+2r-1)}\cos(2r-1)z \,.$$

$$\frac{\vartheta_1'^3\,\vartheta_1^2(z)}{\vartheta_3^2\,\vartheta_3^3(z)} = \sum_n (-1)^{n+1}(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+r+1}[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \; + \tag{85}$$

$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n (-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+r} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^3\,\vartheta_2^2(z)}{\vartheta_3^2\,\vartheta_0^3(z)} = \sum_n (-1)^{n+1}(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+1}[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \; + \tag{86}$$

$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n (-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^3\,\vartheta_0^2(z)}{\vartheta_0^2\,\vartheta_1^3(z)} = \frac{1}{\sin^3 z} + 2\sum q^N\left\{(-1)^{b+1}(\beta+b)^2\sin(\beta-b)z\right\} \; + \tag{79.1}$$

$$+\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\frac{1}{2\sin z} + 2\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^b\sin(\beta-b)z\right\}.$$

$$\frac{\vartheta_1'^3\,\vartheta_3^2(z)}{\vartheta_0^2\,\vartheta_2^3(z)} = \frac{1}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\beta+b+1}{2}}(\beta+b)^2\cos(\beta-b)z\right\} \; + \tag{80.1}$$

$$+\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{1}{2\cos z} + 2\sum q^N\left\{(-1)^{\frac{\beta+b-1}{2}}\cos(\beta-b)z\right\}\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}$$

$$\frac{\vartheta_1'^3\,\vartheta_1^2(z)}{\vartheta_0^2\,\vartheta_0^3(z)} = \sum_n (-1)^n(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}(\delta+d)^2\cos\frac{\delta-d}{2}z\right\} \; + \tag{81.1}$$

$$+\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n (-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

(82.1)
$$\frac{\vartheta_1'^3\vartheta_2^2(z)}{\vartheta_0^3\vartheta_3^3 z} = \sum_n (-1)^n(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d+2}{4}}(\delta+d)^2\cos\frac{\delta-d}{2}z\right\} +$$
$$+\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d-2}{4}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

(83.1)
$$\frac{\vartheta_1'^3\vartheta_3^2(z)}{\vartheta_3^2\vartheta_1^3(z)} = \frac{1}{\sin^3 z} + 2\sum q^N\left\{(-1)^{b+1}(\beta+b)^2\sin(\beta-b)z\right\} +$$
$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{1}{2\sin z} + 2\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^b\sin(\beta-b)z\right\}.$$

(84.1)
$$\frac{\vartheta_1'^3\vartheta_0^2(z)}{\vartheta_3^2\vartheta_2^3(z)} = \frac{1}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\beta+b+1}{2}}(\beta+b)^2\cos(\beta-b)z\right\} +$$
$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\frac{1}{2\cos z} + 2\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta+b-1}{2}}\cos(\beta-b)z\right\}.$$

(85.1)
$$\frac{\vartheta_1'^3\vartheta_1^2(z)}{\vartheta_3^2\vartheta_3^3(z)} = \sum_n(-1)^{n+1}(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d-2}{4}}(\delta+d)^2\cos\frac{\delta-d}{2}z\right\} +$$
$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d+2}{4}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

(86.1)
$$\frac{\vartheta_1'^3\vartheta_2^2(z)}{\vartheta_3^2\vartheta_0^3(z)} = \sum_n(-1)^{n+1}(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+d)^2\cos\frac{\delta-d}{2}z\right\} +$$
$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

Group V-b

$$\frac{\vartheta_3^2(z)}{\vartheta_0^3(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_3^3(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_2^3(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_1^3(z)}$$

Consider

$$F(z) = \frac{\vartheta_3^2}{\vartheta_0^3(z)}$$

Let $t = z + \frac{\pi\tau}{2} + \frac{\pi}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a triple pole at $t = \frac{\pi}{2} + \pi\tau$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$\frac{\vartheta_1'^3}{\vartheta_2^2} F(z) = \frac{i}{2} q^{\frac{1}{4}} A_1^{(2)}\left(z + \frac{\pi\tau}{2} + \frac{\pi}{2}, \pi\tau + \frac{\pi}{2}\right) - q^{\frac{1}{4}} A_1^{(1)}\left(z + \frac{\pi}{2} + \frac{\pi\tau}{2}, \pi\tau + \frac{\pi}{2}\right)$$

(87)

$$+ \frac{i}{2} q^{\frac{1}{4}} \left\{ 2\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'} - 1 \right\} A_1^{(0)}\left(z + \frac{\pi\tau}{2} + \frac{\pi}{2}, \pi\tau + \frac{\pi}{2}\right) .$$

From this follows

$$\frac{\vartheta_1'^3 \vartheta_3^2(z)}{\vartheta_2^2 \vartheta_0^3(z)} = \sum_n (-1)^{n+1} (2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r} (-1)^{n+1} [2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2 + (2n-1)r} \cos 2rz +$$

(88)

$$+ \left\{ 2\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'} \right\} \left\{ \sum_n (-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r} (-1)^n q^{\left(\frac{2n-1}{2}\right)^2 + (2n-1)r} \cos 2rz \right\} .$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'^3 \vartheta_0^2(z)}{\vartheta_2^2 \vartheta_3^3(z)} = \sum_n (-1)^{n+1} (2n-1) q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r} (-1)^{n+r+1} [2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2 + (2n-1)r} \cos 2rz +$$

(89)

$$+ \left\{ 2\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'} \right\} \left\{ \sum_n (-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r} (-1)^{n+r} q^{\left(\frac{2n-1}{2}\right)^2 + (2n-1)r} \cos 2rz \right\} .$$

In (87) replace z by $z - \frac{\pi\tau}{2} - \frac{\pi}{2}$. We get

$$-\frac{\vartheta_1'^3 \vartheta_1^2(z) e^{-iz} q^{\frac{1}{4}}}{\vartheta_2^2 \vartheta_2^3(z)} = \frac{i}{2} q^{\frac{1}{4}} A_1^{(2)}\left(z, \pi\tau + \frac{\pi}{2}\right) - q^{\frac{1}{4}} A_1^{(1)}\left(z, \pi\tau + \frac{\pi}{2}\right)$$

$$+ q^{\frac{1}{4}} \frac{i}{2} \left\{ 2\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'} \right\} A_1^{(0)}\left(z, \pi\tau + \frac{\pi}{2}\right)$$

From this follows

$$\frac{\vartheta_1'^3\,\vartheta_1^2(z)}{\vartheta_2^2\,\vartheta_2^3(z)} = \frac{1}{\cos^3 z} + 2\sum_{n,r}(-1)^{n+r}\left[2(n+r)-1\right]^2 q^{n(n+2r-1)}\cos(2r-1)z \; +$$

(90)

$$+\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{1}{2\cos z} + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1} q^{n(n+2r-1)}\cos(2r-1)z \; .$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\,\vartheta_2^2(z)}{\vartheta_2^2\,\vartheta_1^3(z)} = \frac{1}{\sin^3 z} + 2\sum_{n,r}(-1)^{n+1}\left[2(n+r)-1\right]^2 q^{n(n+2r-1)}\sin(2r-1)z \; +$$

(91)

$$+\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{1}{2\sin z} + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n} q^{n(n+2r-1)}\sin(2r-1)z \; .$$

From these we have

$$\frac{\vartheta_1'^3\,\vartheta_3^2(z)}{\vartheta_2^2\,\vartheta_0^3(z)} = \sum_{n}(-1)^{n+1}(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+d)^2\cos\frac{\delta-d}{2}z\right\} +$$

(88.1)

$$+\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n}(-1)^{n} q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-d}{2}z\right\}\right\}$$

$$\frac{\vartheta_1'^3\,\vartheta_0^2(z)}{\vartheta_2^2\,\vartheta_3^3(z)} = \sum_{n}(-1)^{n+1}(2n-1)^2 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d-2}{4}}(\delta+d)^2\cos\frac{\delta-d}{2}z\right\} +$$

(89.1)

$$+\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n}(-1)^{n} q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d+2}{4}}\cos\frac{\delta-d}{2}z\right\}\right\} .$$

$$\frac{\vartheta_1'^3\,\vartheta_1^2(z)}{\vartheta_2^2\,\vartheta_2^3(z)} = \frac{1}{\cos^3 z} + 2\sum q^{N}\left\{(-1)^{\frac{\beta+b+1}{2}}(\beta+b)^2\cos(\beta-b)z\right\} +$$

(90.1)

$$+\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{1}{2\cos z} + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{N}\left\{(-1)^{\frac{\beta+b-1}{2}}\cos(\beta-b)z\right\} .$$

$$\frac{\vartheta_1'^3\,\vartheta_2^2(z)}{\vartheta_2^2\,\vartheta_1^3(z)} = \frac{1}{\sin^3 z} + 2\sum q^{N}\left\{(-1)^{\beta}(\beta+b)^2\sin(\beta-b)z\right\} +$$

(91.1)

$$+\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{1}{2\sin z} + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{N}\left\{(-1)^{\beta+1}\sin(\beta-b)z\right\} .$$

Group VI-a

$$\frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_0^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_1(z)\vartheta_0(z)}{\vartheta_3^2(z)\vartheta_2(z)} \qquad \frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_0(z)} \qquad \frac{\vartheta_0(z)\vartheta_1(z)}{\vartheta_2^2(z)\vartheta_3(z)}$$

$$\frac{\vartheta_2(z)\vartheta_0(z)}{\vartheta_3^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_1(z)\vartheta_3(z)}{\vartheta_0^2(z)\vartheta_2(z)} \qquad \frac{\vartheta_2(z)\vartheta_0(z)}{\vartheta_1^2(z)\vartheta_3(z)} \qquad \frac{\vartheta_3(z)\vartheta_1(z)}{\vartheta_2^2(z)\vartheta_0(z)}$$

Consider

$$F(z) = \frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_0^2(z)\vartheta_1(z)}$$

Let $t = z + \frac{\pi\tau}{2} + \frac{\pi}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders two and one at $\pi\tau + \frac{\pi}{2}$ and $\frac{\pi\tau}{2} + \frac{\pi}{2}$ respectively. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$\frac{\vartheta_1'^2\vartheta_0}{\vartheta_2\vartheta_3}F(z) = i\,q^{\frac{1}{4}}A_1^{(1)}\left(z + \tfrac{\pi\tau}{2} + \tfrac{\pi}{2}, \pi\tau + \tfrac{\pi}{2}\right) + \tag{92}$$

$$- q^{\frac{1}{4}}A_1^{(0)}\left(z + \tfrac{\pi\tau}{2} + \tfrac{\pi}{2}, \pi\tau + \tfrac{\pi}{2}\right) + \vartheta_2\vartheta_3\, A_1^{(0)}\left(z + \tfrac{\pi\tau}{2} + \tfrac{\pi}{2}, \tfrac{\pi\tau}{2} + \tfrac{\pi}{2}\right).$$

From this follows

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_0^2(z)\vartheta_1(z)} = 4\sum_{n,r}(-1)^n[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \tag{93}$$

$$+\vartheta_2\vartheta_3\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^n q^{n^2+2nr}\sin 2rz\right\}.$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_1(z)\vartheta_0(z)}{\vartheta_2\vartheta_3\vartheta_3^2(z)\vartheta_2(z)} = 4\sum_{n,r}(-1)^{n+r+1}[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \tag{94}$$

$$+\vartheta_2\vartheta_3\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n^2+2nr}\sin 2rz\right\}.$$

In (92) replace z by $z - \frac{\pi\tau}{2}$. We get

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2(z)\vartheta_3(z)e^{-iz}}{\vartheta_2\vartheta_3\vartheta_1^2(z)\vartheta_0(z)} = -A_1^{(1)}\left(z+\tfrac{\pi}{2}, \pi\tau+\tfrac{\pi}{2}\right) - i\,A_1^{(0)}\left(z+\tfrac{\pi}{2}, \pi\tau+\tfrac{\pi}{2}\right)$$

$$+\, i\,q^{-\frac{1}{4}}\vartheta_2\vartheta_3\,A_1^{(0)}\left(z+\tfrac{\pi}{2}, \tfrac{\pi\tau}{2}+\tfrac{\pi}{2}\right).$$

Expanding on the right gives

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_1^2(z)\vartheta_0(z)} = \frac{\cos z}{\sin^2 z} + 4\sum_{n,r}(-1)^{n+1}[2(n+r)-1]\,q^{n(n+2r-1)}\cos(2r-1)z \;+ \tag{95}$$

$$+\;4\vartheta_2\vartheta_3\sum_{n,r}(-1)^n\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; .$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_0(z)\vartheta_1(z)}{\vartheta_2\vartheta_3\vartheta_2^2(z)\vartheta_3(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^{n+r}[2(n+r)-1]\,q^{n(2r+n-1)}\sin(2r-1)z \tag{96}$$

$$+\;4\vartheta_2\vartheta_3\sum_{n,r}(-1)^{n+r+1}\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; .$$

In these results replace q by $-q$, This gives

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_0(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_3^2(z)\vartheta_1(z)} = 4\sum_{n,r}(-1)^{n+r}[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \tag{97}$$

$$+\;\vartheta_2\vartheta_0\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{n^2+2nr}\sin 2rz\right\} .$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_0^2(z)\vartheta_2(z)} = 4\sum_{n,r}(-1)^{n+1}[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \;+ \tag{98}$$

$$+\;\vartheta_0\vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{n^2+2nr}\sin 2rz\right\} .$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_0(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_1^2(z)\vartheta_3(z)} = \frac{\cos z}{\sin^2 z} + 4\sum_{n,r}(-1)^{n+1}[2(n+r)-1]\,q^{n(n+2r-1)}\cos(2r-1)z \;+ \tag{99}$$

$$+\;4\vartheta_0\vartheta_2\sum_{n,r}(-1)^{r+1}\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; .$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_2^2(z)\vartheta_0(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^{n+r}[2(n+r)-1]\,q^{n(n+2r-1)}\sin(2r-1)z \;+ \tag{100}$$

$$+\;4\vartheta_0\vartheta_2\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; .$$

From these follow

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_0^2(z)\vartheta_1(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} +$$

(93.1)

$$+\vartheta_2\vartheta_3\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{(-1)^a\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_0(z)\vartheta_1(z)}{\vartheta_2\vartheta_3\vartheta_3^2(z)\vartheta_2(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d-2}{4}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} +$$

(94.1)

$$+\vartheta_2\vartheta_3\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_1^2(z)\vartheta_0(z)} = \frac{\cos z}{\sin^2 z} + 4\sum q^N\left\{(-1)^{b+1}(\beta+b)\cos(\beta-b)z\right\} +$$

(95.1)

$$+4\vartheta_2\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_1(z)\vartheta_0(z)}{\vartheta_2\vartheta_3\vartheta_2^2(z)\vartheta_3(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta+b+1}{2}}(\beta+b)\sin(\beta-b)z\right\} +$$

(96.1)

$$+4\vartheta_2\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}\sin\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_0(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_3^2(z)\vartheta_1(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d+2}{4}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} +$$

(97.1)

$$+\vartheta_0\vartheta_2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_0^2(z)\vartheta_2(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} +$$

(98.1)

$$+\vartheta_0\vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a+2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

(99.1)
$$\frac{\vartheta_1'^2\vartheta_3\,\vartheta_0(z)\,\vartheta_2(z)}{\vartheta_0\vartheta_2\,\vartheta_1^2(z)\,\vartheta_3(z)} = \frac{\cos z}{\sin^2 z} + 4\sum q^N\left\{(-1)^{b+1}(\beta+b)\cos(\beta-b)z\right\}$$
$$+ 4\vartheta_0\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}\cos\frac{\gamma-c}{2}z\right\}.$$

(100.1)
$$\frac{\vartheta_1'^2\vartheta_3\,\vartheta_1(z)\,\vartheta_3(z)}{\vartheta_0\vartheta_2\,\vartheta_2^2(z)\,\vartheta_0(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta+b+1}{2}}(\beta+b)\sin(\beta-b)z\right\} +$$
$$+ 4\vartheta_0\vartheta_2\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-c}{2}z\right\}.$$

Group VI-b

$$\frac{\vartheta_1(z)\,\vartheta_2(z)}{\vartheta_0^2(z)\,\vartheta_3(z)} \qquad \frac{\vartheta_1(z)\,\vartheta_2(z)}{\vartheta_3^2(z)\,\vartheta_0(z)} \qquad \frac{\vartheta_0(z)\,\vartheta_3(z)}{\vartheta_1^2(z)\,\vartheta_2(z)} \qquad \frac{\vartheta_0(z)\,\vartheta_3(z)}{\vartheta_2^2(z)\,\vartheta_1(z)}$$

Consider
$$F(z) = \frac{\vartheta_1(z)\,\vartheta_2(z)}{\vartheta_0^2(z)\,\vartheta_3(z)}$$

Let $t = z+\frac{\pi}{2}+\frac{\pi\tau}{2}$. $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders one and two at $\pi\tau$ and $\pi\tau+\frac{\pi}{2}$ respectively. Calculating the corresponding $R_i^{(i)}$ and using (10) gives

(101)
$$\frac{\vartheta_1'^2\vartheta_2}{\vartheta_0\vartheta_3}F(z) = -iq^{\frac{1}{4}}A_{(1)}^{(1)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2},\ \pi\tau+\frac{\pi}{2}\right) +$$
$$+ q^{\frac{1}{4}}A_1^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2},\ \pi\tau+\frac{\pi}{2}\right) - q^{\frac{1}{4}}\vartheta_0\vartheta_3\,A_1^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2},\ \pi\tau\right)$$

From this follows

(102)
$$\frac{\vartheta_1'^2\vartheta_2\,\vartheta_1(z)\,\vartheta_2(z)}{\vartheta_0\vartheta_3\,\vartheta_0^2(z)\,\vartheta_3(z)} = 4\sum_{n,r}(-1)^{n+1}[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \ +$$
$$+ 4\vartheta_0\vartheta_3\sum_{n,r}(-1)^{r+1}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz$$

Replacing z by $z+\frac{\pi}{2}$

$$(103)\qquad \frac{\vartheta_1'^2\vartheta_2\vartheta_0(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0(z)\vartheta_3^2(z)} = 4\sum_{n,r}(-1)^{n+r}[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + 4\vartheta_0\vartheta_3\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz$$

In (100) replace z by $z+\frac{\pi}{2}$. There follows

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0(z)\vartheta_3(z)e^{iz}}{\vartheta_0\vartheta_3\vartheta_1^2(z)\vartheta_2(z)} = A_1^{(1)}(z+\pi\tau+\tfrac{\pi}{2},\ \pi\tau+\tfrac{\pi}{2}) + i\,A_1^{(0)}(z+\pi\tau+\tfrac{\pi}{2},\ \pi\tau+\tfrac{\pi}{2}) - i\,\vartheta_0\vartheta_3 A_1^{(0)}(z+\pi\tau+\tfrac{\pi}{2},\ \pi\tau)$$

and from this we have

$$(104)\qquad \frac{\vartheta_1'^2\vartheta_2\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_1^2(z)\vartheta_2(z)} = \frac{\cos z}{\sin^2 z} + 4\sum_{n,r}(-1)^{n+1}[2(n+r)-1]\,q^{n(n+2r-1)}\cos(2r-1)z + \vartheta_0\vartheta_3\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{r+1}q^{n(n+2r-1)}\cos(2r-1)z\right\}$$

Replacing z by $z-\frac{\pi}{2}$

$$(105)\qquad \frac{\vartheta_1'^2\vartheta_2\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_2^2(z)\vartheta_1(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^{n+r}[2(n+r)-1]\,q^{n(n+2r-1)}\sin(2r-1)z + \vartheta_0\vartheta_3\left\{\frac{1}{\sin z} + 4\sum_{n,r} q^{n(n+2r-1)}\sin(2r-1)z\right\}.$$

From these follow

$$(102.1)\qquad \frac{\vartheta_1'^2\vartheta_2\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0^2(z)\vartheta_3(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} + 4\vartheta_0\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-4}{4}}\sin\frac{\delta-d}{2}z\right\}.$$

$$(103.1)\qquad \frac{\vartheta_1'^2\vartheta_2\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_3^2(z)\vartheta_0(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d+2}{4}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} + 4\vartheta_0\vartheta_3\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-d}{2}z\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_1^2(z)\vartheta_2(z)} = \frac{\cos z}{\sin^2 z} + 4\sum q^N\left\{(-1)^{b+1}(\beta+b)\cos(\beta-b)z\right\} +$$

(104.1)

$$+\vartheta_0\vartheta_3\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_2^2(z)\vartheta_1(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta+b+1}{2}}(\beta+b)\sin(\beta-b)z\right\} +$$

(105.1)

$$+\vartheta_0\vartheta_3\left\{\frac{1}{\sin z} + 4\sum q^N\left\{\sin(\beta-b)z\right\}\right\}.$$

Group VII-a

$$\frac{\vartheta_2^2(z)}{\vartheta_0^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_3^2(z)\vartheta_2(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_1^2(z)\vartheta_0(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_2^2(z)\vartheta_3(z)}$$

$$\frac{\vartheta_2^2(z)}{\vartheta_3^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_0^2(z)\vartheta_2(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_1^2(z)\vartheta_3(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_2^2(z)\vartheta_0(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^2(z)e^{-iz}}{\vartheta_0^2(z)\vartheta_1(z)}$$

Let $t = z+\frac{\pi}{2}$, $F(z)\equiv\varphi(t)$. $\varphi(t)$ satisfies (8) and has poles or orders two and one at $\frac{\pi}{2}+\frac{\pi\tau}{2}$ and $\frac{\pi}{2}$ respectively. Calculate the corresponding $A_i^{(j)}$ and use (10). There follows

(106)
$$\frac{\vartheta_0\vartheta_1'^2}{\vartheta_2^2}F(z) = i\,q^{-1/4}\frac{\vartheta_3^2}{\vartheta_2^2}A_1^{(1)}\left(z+\tfrac{\pi}{2},\tfrac{\pi\tau}{2}+\tfrac{\pi}{2}\right) + \vartheta_2\vartheta_3\,A_1^{(0)}\left(z+\tfrac{\pi}{2},\tfrac{\pi}{2}\right)$$

Hence

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2^2(z)}{\vartheta_2^2\vartheta_0^2(z)\vartheta_1(z)} = 2\frac{\vartheta_3^2}{\vartheta_2^2}\sum_{n,r}(-1)^n 4(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z +$$

(107)

$$+\vartheta_2\vartheta_3\left\{\frac{1}{\sin z} + 4\sum_{n,r}(-1)^n q^{n(n+2r-1)}\sin(2r-1)z\right\}.$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_1^2(z)}{\vartheta_2^2\vartheta_3^2(z)\vartheta_2(z)} = 2\frac{\vartheta_3^2}{\vartheta_2^2}\sum_{n,r}(-1)^{n+r+1}4(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z +$$

(108)

$$+\vartheta_2\vartheta_3\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{n+r+1}q^{n(n+2r-1)}\cos(2r-1)\cdot z\right\}.$$

In (106) replace z by $z+\frac{\pi\tau}{2}$. This gives

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3^2(z)}{\vartheta_2^2\vartheta_1^2(z)\vartheta_0(z)} = \frac{\vartheta_3^2}{\vartheta_2^2}A_1^{(1)}\left(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2},\tfrac{\pi\tau}{2}+\tfrac{\pi}{2}\right)-i\vartheta_2\vartheta_3A_1^{(0)}\left(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2},\tfrac{\pi}{2}\right)q^{-\frac{1}{4}}$$

Hence

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3^2(z)}{\vartheta_2^2\vartheta_1^2(z)\vartheta_0(z)} = \frac{\vartheta_3^2}{\vartheta_2^2}\left\{\frac{1}{\sin^2 z}+4\sum_n(-1)^{n+1}n\,q^{n^2}+4\sum_{n,r}(-1)^{n+1}2(n+r)\,q^{n^2+2nr}\cos 2rz\right\}+$$

(109)

$$+2\vartheta_2\vartheta_3\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2}+2\sum_{n,r}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_0^2(z)}{\vartheta_2^2(z)\vartheta_3(z)\vartheta_2^2} = \frac{\vartheta_3^2}{\vartheta_2^2}\left\{\frac{1}{\cos^2 z}+4\sum_n(-1)^{n+1}n\,q^{n^2}+4\sum_{n,r}(-1)^{n+r+1}2(n+r)\,q^{n^2+2nr}\cos 2rz\right\}+$$

(110)

$$+2\vartheta_2\vartheta_3\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2}+2\sum_{n,r}(-1)^{n+r}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

In these results replace q by $-q$. There results

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_2^2(z)}{\vartheta_2^2\vartheta_3^2(z)\vartheta_1(z)} = 2\frac{\vartheta_0^2}{\vartheta_2^2}\sum_{n,r}(-1)^r4(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z +$$

(111)

$$+\vartheta_0\vartheta_2\left\{\frac{1}{\sin z}+4\sum_{n,r}(-1)^n q^{n(n+2r-1)}\sin(2r-1)z\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_1^2(z)}{\vartheta_2^2\vartheta_0^2(z)\vartheta_2(z)} = -2\frac{\vartheta_0^2}{\vartheta_2^2}\sum_{n,r}4(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z +$$

(112)

$$+\vartheta_0\vartheta_2\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{n+r+1}q^{n(n+2r-1)}\cos(2r-1)z\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_0^2(z)}{\vartheta_2^2 \vartheta_1^2(z) \vartheta_3(z)} = \frac{\vartheta_0^2}{\vartheta_2^2}\left\{\frac{1}{\sin^2 z} - 4\sum_n n q^{n^2} - 4\sum_{n,r} 2(n+r) q^{n^2+2nr} \cos 2rz\right\} +$$

(113)

$$+2\vartheta_0\vartheta_2\left\{\sum_n (-1)^{n+1} q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+r+1} q^{\frac{(2n-1)^2}{2}+(2n-1)r} \cos 2rz\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_3^2(z)}{\vartheta_2^2 \vartheta_2^2(z) \vartheta_0(z)} = \frac{\vartheta_0^2}{\vartheta_2^2}\left\{\frac{1}{\cos^2 z} - 4\sum_n n q^{n^2} + 4\sum_{n,r}(-1)^{r+1} 2(n+r) q^{n^2+2nr} \cos 2rz\right\} +$$

(114)

$$+2\vartheta_0\vartheta_2\left\{\sum_n (-1)^{n+1} q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+1} q^{\frac{(2n-1)^2}{2}+(2n-1)r} \cos 2rz\right\}.$$

These give

$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_2^2(z)}{\vartheta_2^2 \vartheta_0^2(z) \vartheta_1(z)} = 2\frac{\vartheta_3^2}{\vartheta_2^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\} +$$

(107.1)

$$+\vartheta_2\vartheta_3\left\{\frac{1}{\sin z} + 4\sum q^N\left\{(-1)^b \sin(\beta-b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_1^2(z)}{\vartheta_2^2 \vartheta_3^2(z) \vartheta_2(z)} = 2\frac{\vartheta_3^2}{\vartheta_2^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}(\gamma+c)\cos\frac{\gamma-c}{2}z\right\} +$$

(108.1)

$$+\vartheta_2\vartheta_3\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta+b-1}{2}}\cos(\beta-b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_3^2(z)}{\vartheta_2^2 \vartheta_1^2(z) \vartheta_0(z)} = \frac{\vartheta_3^2}{\vartheta_2^2}\left\{\frac{1}{\sin^2 z} + 4\sum_n (-1)^{n+1} n q^{n^2} + 4\sum q^N\left\{(-1)^{a+1}(\alpha+a)\cos(\alpha-a)z\right\}\right\} +$$

(109.1)

$$+2\vartheta_2\vartheta_3\left\{\sum_n (-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_0^2(z)}{\vartheta_2^2 \vartheta_2^2(z) \vartheta_3(z)} = \frac{\vartheta_3^2}{\vartheta_2^2}\left\{\frac{1}{\cos^2 z} + 4\sum_n (-1)^{n+1} n q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\} +$$

(110.1)

$$+2\vartheta_2\vartheta_3\left\{\sum_n (-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d+2}{4}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_2^2(z)}{\vartheta_2^2\vartheta_3^2(z)\vartheta_1(z)} = 2\frac{\vartheta_0^2}{\vartheta_2^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\} +$$

(111.1)

$$+\vartheta_0\vartheta_2\left\{\frac{1}{\sin z} + 4\sum q^{N}\left\{(-1)^{b}\sin(\beta-b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_1^2(z)}{\vartheta_2^2\vartheta_0^2(z)\vartheta_2(z)} = -2\frac{\vartheta_0^2}{\vartheta_2^2}\sum q^{\frac{N}{4}}\left\{(\gamma+c)\cos\frac{\gamma-c}{2}z\right\} +$$

(112.1)

$$+\vartheta_0\vartheta_2\left\{\frac{1}{\cos z} + 4\sum q^{N}\left\{(-1)^{\frac{\beta+b-1}{2}}\cos(\beta-b)z\right\}\right\}$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_0^2(z)}{\vartheta_2^2\vartheta_1^2(z)\vartheta_3(z)} = \frac{\vartheta_0^2}{\vartheta_2^2}\left\{\frac{1}{\sin^2 z} - 4\sum_n n q^{n^2} - 4\sum q^{N}\left\{(\alpha+a)\cos(\alpha-a)z\right\}\right\} +$$

(113.1)

$$+2\vartheta_0\vartheta_2\left\{\sum_n(-1)^{n+1}q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d-2}{4}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_3^2(z)}{\vartheta_2^2\vartheta_2^2(z)\vartheta_0(z)} = \frac{\vartheta_0^2}{\vartheta_2^2}\left\{\frac{1}{\cos^2 z} - 4\sum_n n q^{n^2} + 4\sum q^{N}\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\} +$$

(114.1)

$$+2\vartheta_0\vartheta_2\left\{\sum_n(-1)^{n+1}q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}\cos\frac{\delta-d}{2}z\right\}\right\}$$

Group VII-b

$$\frac{\vartheta_3^2(z)}{\vartheta_0^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_3^2(z)\vartheta_2(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_1^2(z)\vartheta_0(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_2^2(z)\vartheta_3(z)}$$

$$\frac{\vartheta_0^2(z)}{\vartheta_3^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_0^2(z)\vartheta_2(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_1^2(z)\vartheta_3(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_2^2(z)\vartheta_0(z)}$$

Consider

$$F(z) = \frac{\vartheta_3^2(z)\,e^{-iz}}{\vartheta_0^2(z)\,\vartheta_1(z)}$$

Let $t = z+\frac{\pi}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders two and one at $t = \frac{\pi\tau}{2}+\frac{\pi}{2}$ and $t = \frac{\pi}{2}$ respectively. The corresponding $R_i^{(\nu)}$ in (10) gives

$$(115)\qquad \frac{\vartheta_1'^2\vartheta_0}{\vartheta_3^2}F(z) = i\,q^{-1/4}\frac{\vartheta_2^2}{\vartheta_3^2}A_1^{(1)}\left(z+\tfrac{\pi}{2},\ \tfrac{\pi\tau}{2}+\tfrac{\pi}{2}\right) + \vartheta_2\vartheta_3 A_1^{(0)}\left(z+\tfrac{\pi}{2},\ \tfrac{\pi}{2}\right) .$$

This differs from (106) only in the constants. Hence we have

$$(116)\qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_0\vartheta_3^2(z)}{\vartheta_3^2\vartheta_0^2(z)\vartheta_1(z)} &= 2\frac{\vartheta_2^2}{\vartheta_3^2}\sum_{n,r}(-1)^n 4(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \;+ \\ &+\vartheta_2\vartheta_3\left\{\frac{1}{\sin z}+4\sum_{n,r}(-1)^n q^{n(n+2r-1)}\sin(2r-1)z\right\} . \end{aligned}$$

$$(117)\qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_0\vartheta_0^2(z)}{\vartheta_3^2\vartheta_3^2(z)\vartheta_2(z)} &= 2\frac{\vartheta_2^2}{\vartheta_3^2}\sum_{n,r}(-1)^{n+r+1} 4(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \;+ \\ &+\vartheta_2\vartheta_3\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{n+r+1} q^{n(n+2r-1)}\cos(2r-1)z\right\} . \end{aligned}$$

$$(118)\qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_0\vartheta_2^2(z)}{\vartheta_3^2\vartheta_1^2(z)\vartheta_0(z)} &= \frac{\vartheta_2^2}{\vartheta_3^2}\left\{\frac{1}{\sin^2 z}+4\sum_{n}(-1)^{n+1} n\,q^{n^2}+4\sum_{n,r}(-1)^{n+1}2(n+r)\,q^{n^2+2nr}\cos 2rz\right\} + \\ &+2\vartheta_2\vartheta_3\left\{\sum_{n}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2}+2\sum_{n,r}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} . \end{aligned}$$

$$(119)\qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_0\vartheta_1^2(z)}{\vartheta_3^2\vartheta_2^2(z)\vartheta_3(z)} &= \frac{\vartheta_2^2}{\vartheta_3^2}\left\{\frac{1}{\cos^2 z}+4\sum_{n}(-1)^{n+1} n\,q^{n^2}+4\sum_{n,r}(-1)^{n+r+1}2(n+r)\,q^{n^2+2nr}\cos 2rz\right\} + \\ &+2\vartheta_2\vartheta_3\left\{\sum_{n}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2}+2\sum_{n,r}(-1)^{n+r} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} \end{aligned}$$

$$(120)\qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_3\vartheta_0^2(z)}{\vartheta_0^2\vartheta_3^2(z)\vartheta_1(z)} &= 2\frac{\vartheta_2^2}{\vartheta_0^2}\sum_{n,r}(-1)^{r+1} 4(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \;+ \\ &+\vartheta_0\vartheta_2\left\{\frac{1}{\sin z}+4\sum_{n,r}(-1)^n q^{n(n+2r-1)}\sin(2r-1)z\right\} . \end{aligned}$$

$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_3^2(z)}{\vartheta_0^2 \vartheta_0^2(z)\vartheta_2(z)} = 2\frac{\vartheta_2^2}{\vartheta_0^2}\sum_{n,r} 4(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z \; +$$

(121)

$$+\,\vartheta_0\vartheta_2\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{n+r+1} q^{n(n+2r-1)}\cos(2r-1)z\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_2^2(z)}{\vartheta_0^2\vartheta_1^2(z)\vartheta_3(z)} = \frac{\vartheta_2^2}{\vartheta_0^2}\left\{\frac{1}{\sin^2 z} - 4\sum_n n\, q^{n^2} - 4\sum_{n,r} 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\} +$$

(122)

$$+\,2\vartheta_0\vartheta_2\left\{\sum_n (-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+r} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_1^2(z)}{\vartheta_0^2\vartheta_2^2(z)\vartheta_0(z)} = \frac{\vartheta_2^2}{\vartheta_0^2}\left\{\frac{1}{\cos^2 z} - 4\sum_n n\, q^{n^2} + 4\sum_{n,r}(-1)^{r+1}\, 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\} +$$

(123)

$$+\,2\vartheta_0\vartheta_2\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

From these follow

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3^2(z)}{\vartheta_3^2\vartheta_0^2(z)\vartheta_1(z)} = 2\frac{\vartheta_2^2}{\vartheta_3^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\} +$$

(116.1)

$$+\,\vartheta_2\vartheta_3\left\{\frac{1}{\sin z}+4\sum q^N\left\{(-1)^b\sin(\beta-b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_0^2(z)}{\vartheta_3^2\vartheta_3^2(z)\vartheta_2(z)} = 2\frac{\vartheta_2^2}{\vartheta_3^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}(\gamma+c)\cos\frac{\gamma-c}{2}z\right\} +$$

(117.1)

$$+\,\vartheta_2\vartheta_3\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta+b-1}{2}}\cos(\beta-b)z\right\}\right\},$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2^2(z)}{\vartheta_3^2\vartheta_1^2(z)\vartheta_0(z)} = \frac{\vartheta_2^2}{\vartheta_3^2}\left\{\frac{1}{\sin^2 z}+4\sum_n(-1)^{n+1} n\, q^{n^2}+4\sum q^N\left\{(-1)^{a+1}(\alpha+a)\cos(\alpha-a)z\right\}\right\}+$$

(118.1)

$$+\,2\vartheta_2\vartheta_3\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2}+2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_1^2(z)}{\vartheta_3^2\vartheta_2^2(z)\vartheta_3(z)} = \frac{\vartheta_2^2}{\vartheta_3^2}\left\{\frac{1}{\cos^2 z} + 4\sum_n (-1)^{n+1} n q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a-2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\}+$$

(119.1)

$$+2\vartheta_2\vartheta_3\left\{\sum_n (-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d+2}{4}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_0^2(z)}{\vartheta_0^2\vartheta_3^2(z)\vartheta_1(z)} = 2\frac{\vartheta_2^2}{\vartheta_0^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\}+$$

(120.1)

$$+\vartheta_0\vartheta_2\left\{\frac{1}{\sin z} + 4\sum q^N\left\{(-1)^b\sin(\beta-b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_3^2(z)}{\vartheta_0^2\vartheta_0^2(z)\vartheta_2(z)} = 2\frac{\vartheta_2^{+2}}{\vartheta_0^2}\sum q^{\frac{N}{4}}\left\{(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}+$$

(121.1)

$$+\vartheta_0\vartheta_2\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta+b-1}{2}}\cos(\beta-b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_2^2(z)}{\vartheta_0^2\vartheta_1^2(z)\vartheta_3(z)} = \frac{\vartheta_2^2}{\vartheta_0^2}\left\{\frac{1}{\sin^2 z} - 4\sum_n n q^{n^2} - 4\sum q^N\left\{(\alpha+a)\cos(\alpha-a)z\right\}\right\}+$$

(122.1)

$$+2\vartheta_0\vartheta_2\left\{\sum_n (-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d+2}{4}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_1^2(z)}{\vartheta_0^2\vartheta_2^2(z)\vartheta_0(z)} = \frac{\vartheta_2^2}{\vartheta_0^2}\left\{\frac{1}{\cos^2 z} - 4\sum_n n q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\}+$$

(123.1)

$$+2\vartheta_0\vartheta_2\left\{\sum_n (-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

Group VII-c

$$\frac{\vartheta_1^2(z)}{\vartheta_0^2(z)\,\vartheta_3(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_3^2(z)\,\vartheta_0(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_1^2(z)\,\vartheta_2(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_2^2(z)\,\vartheta_1(z)}$$

$$\frac{\vartheta_1^2(z)}{\vartheta_3^2(z)\,\vartheta_0(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_0^2(z)\,\vartheta_3(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_1^2(z)\,\vartheta_2(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_2^2(z)\,\vartheta_1(z)}$$

Consider

$$F(z) = \frac{\vartheta_1^2(z)}{\vartheta_0^2(z)\,\vartheta_3(z)}$$

Let $t = z+\frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8). Calculating the $A_i^{(j)}$ associated with $\varphi(t)$ and using (10) gives

$$\frac{\vartheta_1'^2\vartheta_2}{\vartheta_3^2}F(z) = q^{\frac14}\frac{\vartheta_0^2}{\vartheta_3^2}\left\{A_1^{(1)}\left(z+\tfrac{\pi\tau}{2}, \pi\tau\right) + i\,A_1^{(0)}\left(z+\tfrac{\pi\tau}{2}, \pi\tau\right)\right\}$$

(124)

$$-\,i\,q^{\frac14}\vartheta_0\vartheta_3\,A_1^{(0)}\left(z+\tfrac{\pi\tau}{2}, \pi\tau+\tfrac{\pi}{2}\right) .$$

This gives

$$-\frac{\vartheta_1'^2\vartheta_2\vartheta_1^2(z)}{\vartheta_3^2\vartheta_0^2(z)\vartheta_3(z)} = 2\frac{\vartheta_0^2}{\vartheta_3^2}\left\{\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}+$$

(125)

$$+2\vartheta_0\vartheta_3\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+r}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} .$$

Replacing z by $z-\frac{\pi}{2}$

$$-\frac{\vartheta_1'^2\vartheta_2\vartheta_2^2(z)}{\vartheta_3^2\vartheta_3^2(z)\vartheta_0(z)} = 2\left\{\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^r[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}\frac{\vartheta_0^2}{\vartheta_3^2}+$$

(126)

$$+2\vartheta_0\vartheta_3\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} .$$

In (124) replace z by $z-\frac{\pi\tau}{2}$. We get

$$\frac{\vartheta_2\vartheta_1'^2\vartheta_0^2(z)e^{-iz}}{\vartheta_3^2\vartheta_1^2(z)\vartheta_2(z)} = \frac{\vartheta_0^2}{\vartheta_3^2}\left\{A_1^{(1)}(z,\pi\tau)+iA_1^{(0)}(z,\pi\tau)\right\} - i\,\vartheta_0\vartheta_3\,A_1^{(0)}\left(z,\pi\tau+\tfrac{\pi}{2}\right)$$

From this follows

$$(127)\quad \frac{\vartheta_1'^2\vartheta_2\vartheta_0^2(z)}{\vartheta_3^2\vartheta_1^2(z)\vartheta_2(z)} = \frac{\vartheta_0^2}{\vartheta_3^2}\left\{\frac{\cos z}{\sin^2 z} - 4\sum_{n,r}[2(n+r)-1]\, q^{n(n+2r-1)}\cos(2r-1)z\right\} + \\ + \vartheta_0\vartheta_3\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n(n+2r-1)}\cos(2r-1)z\right\}.$$

Replacing z by $z-\frac{\pi}{2}$

$$(128)\quad \frac{\vartheta_1'^2\vartheta_2\vartheta_3^2(z)}{\vartheta_3^2\vartheta_2^2(z)\vartheta_1(z)} = \frac{\vartheta_0^2}{\vartheta_3^2}\left\{\frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{n(n+2r-1)}\sin(2r-1)z\right\} \\ + \vartheta_0\vartheta_3\left\{\frac{1}{\sin z} + 4\sum_{n,r}(-1)^n q^{n(n+2r-1)}\sin(2r-1)z\right\}.$$

Changing q into $-q$ in these gives

$$(129)\quad \frac{\vartheta_1'^2\vartheta_2\vartheta_3^2(z)}{\vartheta_0^2\vartheta_3^2(z)\vartheta_0(z)} = 2\frac{\vartheta_3^2}{\vartheta_0^2}\left\{\sum_n(2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} + \\ +2\vartheta_0\vartheta_3\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

$$(130)\quad \frac{\vartheta_1'^2\vartheta_2\vartheta_2^2(z)}{\vartheta_0^2\vartheta_0^2(z)\vartheta_3(z)} = 2\frac{\vartheta_3^2}{\vartheta_0^2}\left\{\sum_n(2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} + \\ +2\vartheta_0\vartheta_3\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+r} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

$$(131)\quad \frac{\vartheta_1'^2\vartheta_2\vartheta_3^2(z)}{\vartheta_0^2\vartheta_1^2(z)\vartheta_2(z)} = \frac{\vartheta_3^2}{\vartheta_0^2}\left\{\frac{\cos z}{\sin^2 z} - 4\sum_{n,r}[2(n+r)-1]\, q^{n(n+2r-1)}\cos(2r-1)z\right\} + \\ + \vartheta_0\vartheta_3\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n(n+2r-1)}\cos(2r-1)z\right\}.$$

$$(132)\quad \frac{\vartheta_1'^2\vartheta_2\vartheta_0^2(z)}{\vartheta_0^2\vartheta_2^2(z)\vartheta_1(z)} = \frac{\vartheta_3^2}{\vartheta_0^2}\left\{\frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{n(n+2r-1)}\sin(2r-1)z\right\} + \\ + \vartheta_0\vartheta_3\left\{\frac{1}{\sin z} + 4\sum_{n,r}(-1)^n q^{n(n+2r-1)}\sin(2r-1)z\right\}$$

From these follow

$$-\frac{\vartheta_1'^2\vartheta_2\vartheta_1^2(z)}{\vartheta_3^2\vartheta_0^2(z)\vartheta_3(z)} = 2\frac{\vartheta_0^2}{\vartheta_3^2}\left\{\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(\delta+d)\cos\frac{\delta-d}{2}\right\}\right\} +$$

(125.1)

$$+2\vartheta_0\vartheta_3\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d+2}{4}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$-\frac{\vartheta_1'^2\vartheta_2\vartheta_2^2(z)}{\vartheta_3^2\vartheta_3^2(z)\vartheta_0(z)} = 2\frac{\vartheta_0^2}{\vartheta_3^2}\left\{\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\} +$$

(126.1)

$$+2\vartheta_0\vartheta_3\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0^2(z)}{\vartheta_3^2\vartheta_1^2(z)\vartheta_2(z)} = \frac{\vartheta_0^2}{\vartheta_3^2}\left\{\frac{\cos z}{\sin^2 z} - 4\sum q^N\left\{(\beta+b)\cos(\beta-b)z\right\}\right\} +$$

(127.1)

$$+\vartheta_0\vartheta_3\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta+b-1}{2}}\cos(\beta-b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_3^2(z)}{\vartheta_3^2\vartheta_2^2(z)\vartheta_1(z)} = \frac{\vartheta_0^2}{\vartheta_3^2}\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)\sin(\beta-b)z\right\}\right\} +$$

(128.1)

$$+\vartheta_0\vartheta_3\left\{\frac{1}{\sin z} + 4\sum q^N\left\{(-1)^b\sin(\beta-b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_1^2(z)}{\vartheta_0^2\vartheta_3^2(z)\vartheta_0(z)} = 2\frac{\vartheta_3^2}{\vartheta_0^2}\left\{\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\} +$$

(129.1)

$$+2\vartheta_0\vartheta_3\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_2^2(z)}{\vartheta_0^2\vartheta_0^2(z)\vartheta_3(z)} = 2\frac{\vartheta_3^2}{\vartheta_0^2}\left\{\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\} +$$

(130.1)

$$+2\vartheta_0\vartheta_3\left\{\sum_n(-1)^n q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d+2}{4}}\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_3^2(z)}{\vartheta_0^2\vartheta_1^2(z)\vartheta_2(z)} = \frac{\vartheta_3^2}{\vartheta_0^2}\left\{\frac{\cos z}{\sin^2 z} - 4\sum q^N\{(\beta+b)\cos(\beta-b)z\}\right\} +$$

(131.1)

$$+\vartheta_0\vartheta_3\left\{\frac{1}{\cos z} + 4\sum q^N\{(-1)^{\frac{\beta+b-1}{2}}\cos(\beta-b)z\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0^2(z)}{\vartheta_0^2\vartheta_2^2(z)\vartheta_1(z)} = \frac{\vartheta_3^2}{\vartheta_0^2}\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)\sin(\beta-b)z\}\right\} +$$

(132.1)

$$+\vartheta_0\vartheta_3\left\{\frac{1}{\sin z} + 4\sum q^N\{(-1)^b\sin(\beta-b)z\}\right\}.$$

Group VIII

$$\frac{\vartheta_3^2(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)}$$

Consider

$$F(z) = \frac{\vartheta_3^2(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)}$$

Let $t = z+\frac{\pi\tau}{2}$, $F(z)\equiv\varphi(t)$. $\varphi(t)$ satisfies (8). Calculating the appropriate $R_i^{(i)}$ and using (10) gives

$$\vartheta_1'^2F(z) = -q^{1/4}\vartheta_2^3A_1^{(0)}(z+\tfrac{\pi\tau}{2},\pi\tau) + \vartheta_3^3A_1^{(0)}(z+\tfrac{\pi\tau}{2},\tfrac{\pi\tau}{2}) +$$

(133)

$$-\vartheta_0^3A_1^{(0)}(z+\tfrac{\pi\tau}{2},\tfrac{\pi\tau}{2}+\tfrac{\pi}{2})$$

From this follows

$$\frac{\vartheta_1'^2\vartheta_3^2(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)} = -4\vartheta_2^3\sum_{n,r}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz +$$

(134)

$$+\vartheta_3^3\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}q^{n^2+2nr}\sin 2rz\right\} + \vartheta_0^3\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1}q^{n^2+2nr}\sin 2rz\right\}.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0^2(z)}{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)} = 4\vartheta_2^3\sum_{n,r}(-1)^rq^{n^2+2nr}\sin 2rz +$$

(135)

$$+\vartheta_3^3\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{n^2+2nr}\sin 2rz\right\} + \vartheta_0^3\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^nq^{n^2+2nr}\sin 2rz\right\}.$$

In (133) replace z by $z-\frac{\pi\tau}{2}$. We find

$$\frac{\vartheta_1'^2\,\vartheta_2^2(z)\,e^{-iz}}{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)} = \vartheta_2^3 A_1^{(0)}(z,\pi\tau) - q^{-1/4}\vartheta_3^3 A_1^{(0)}\left(z,\tfrac{\pi\tau}{2}\right) + \vartheta_0^3 q^{-1/4} A_1^{(0)}\left(z,\tfrac{\pi\tau}{2}+\tfrac{\pi}{2}\right).$$

Hence

$$(136)\qquad \frac{\vartheta_1'^2\,\vartheta_2^2(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)} = 4\vartheta_0^3\sum_{n,r}(-1)^{n+r+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z + \vartheta_2^3\left\{\frac{1}{\sin z}+4\sum_{n,r} q^{n(n+2r-1)}\sin(2r-1)z\right\} - 4\vartheta_3^3\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z.$$

Replacing z by $z+\frac{\pi}{2}$

$$(137)\qquad \frac{\vartheta_1'^2\,\vartheta_1^2(z)}{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)} = 4\vartheta_0^3\sum_{n,r}(-1)^{n} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + \vartheta_2^3\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{r+1} q^{n(n+2r-1)}\cos(2r-1)z\right\} + 4\vartheta_3^3\sum_{n,r}(-1)^r q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z.$$

From these follow

$$(134.1)\qquad \frac{\vartheta_1'^2\,\vartheta_3^2(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)} = -4\vartheta_2^3\sum q^{\frac{N}{4}}\left\{\sin\tfrac{\delta-d}{2}z\right\} + \vartheta_3^3\left\{\frac{\cos z}{\sin z}+4\sum q^N\{\sin(\alpha-a)z\}\right\} + \vartheta_0^3\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

$$(135.1)\qquad \frac{\vartheta_1'^2\,\vartheta_0^2(z)}{\vartheta_3(z)\vartheta_1(z)\vartheta_2(z)} = 4\vartheta_2^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}\sin\tfrac{\delta-d}{2}z\right\} + \vartheta_3^3\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\right\}\right\} + \vartheta_0^3\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^a\sin(\alpha-a)z\right\}\right\}.$$

$$(136.1)\qquad \frac{\vartheta_1'^2\,\vartheta_2^2(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)} = 4\vartheta_0^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}\sin\tfrac{\gamma-c}{2}z\right\} - 4\vartheta_3^3\sum q^{\frac{N}{4}}\left\{\sin\tfrac{\gamma-c}{2}z\right\} + \vartheta_2^3\left\{\frac{1}{\sin z}+4\sum q^N\{\sin(\beta-b)z\}\right\}.$$

$$(137.1)\qquad \frac{\vartheta_1'^2\,\vartheta_1^2(z)}{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)} = 4\vartheta_0^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\tfrac{\gamma-c}{2}z\right\} + 4\vartheta_3^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}\cos\tfrac{\gamma-c}{2}z\right\} + \vartheta_2^3\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\}\right\}.$$

Group IX-a

$$\frac{\vartheta_3^3(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_0^3(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_2^3(z)}{\vartheta_3^4(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_0^4 z}$$

$$\frac{\vartheta_0^3(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_3^3(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_2^3(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_3^4 z}$$

Consider

$$F(z) = \frac{\vartheta_3^3(z)}{\vartheta_2^4(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order four at $t = \frac{\pi}{2} + \frac{\pi\tau}{2}$. Calculating the corresponding $R_i^{(j)}$ we find, using (10)

$$\frac{\vartheta_1'^4 \vartheta_3^3(z)}{\vartheta_0^3 \vartheta_2^4(z)} = \frac{1}{6} A_1^{(3)}\left(z + \tfrac{\pi\tau}{2}, \tfrac{\pi\tau}{2} + \tfrac{\pi}{2}\right) +$$

(138)

$$+ \frac{1}{6}\left\{9\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\} A_1^{(1)}\left(z + \tfrac{\pi\tau}{2}, \tfrac{\pi\tau}{2} + \tfrac{\pi}{2}\right).$$

From this follows

$$\frac{\vartheta_1'^4 \vartheta_3^3(z)}{\vartheta_0^3 \vartheta_2^4(z)} = \frac{1}{\cos^4 z} - \frac{2}{3\cos^2 z} + \frac{1}{3}\sum_n (-1)^n (2n)^3 q^{n^2} + \frac{2}{3}\sum_{n,r} (-1)^{n+r} [2(n+r)]^3 q^{n^2+2nr} \cos 2rz +$$

(139)

$$+ \frac{1}{6}\left\{9\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z} + 4\sum_n (-1)^{n+1} n q^{n^2} + 4\sum (-1)^{n+r+1} 2(n+r) q^{n^2+2nr} \cos 2rz\right\},$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'^4 \vartheta_0^3(z)}{\vartheta_0^3 \vartheta_1^4(z)} = \frac{1}{\sin^4 z} - \frac{2}{3\sin^2 z} + \frac{1}{3}\sum_n (-1)^n (2n)^3 q^{n^2} + \frac{2}{3}\sum_{n,r} (-1)^n [2(n+r)]^3 q^{n^2+2nr} \cos 2rz +$$

(140)

$$+ \frac{1}{6}\left\{9\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z} + 4\sum_n (-1)^{n+1} n q^{n^2} + 4\sum_{n,r} (-1)^{n+1} 2(n+r) q^{n^2+2nr} \cos 2rz\right\}.$$

Replacing z by $z - \frac{\pi\tau}{2}$ in (138) gives

$$\frac{\vartheta_1'^4 \vartheta_2^3(z) e^{-iz}}{\vartheta_0^3 \vartheta_3^4(z)} = \frac{1}{6} q^{-\frac{1}{4}} A_1^{(3)}\left(z, \tfrac{\pi\tau}{2} + \tfrac{\pi}{2}\right) + \frac{1}{6} q^{-\frac{1}{4}}\left\{9\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\} A_1^{(1)}\left(z, \tfrac{\pi\tau}{2} + \tfrac{\pi}{2}\right).$$

Hence

$$\frac{\vartheta_1'^4\,\vartheta_2^3(z)}{\vartheta_0^3\,\vartheta_3^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+r+1}\left[2(n+r-1)\right]^3 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(141)

$$+\;\frac{4}{3}\left\{9\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r}(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \,.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\,\vartheta_1^3(z)}{\vartheta_0^3\,\vartheta_0^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n}\left[2(n+r-1)\right]^3 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(142)

$$+\;\frac{4}{3}\left\{9\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+1}(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \,.$$

Replacing q by $-q$ in these results in

$$\frac{\vartheta_1'^4\,\vartheta_0^3(z)}{\vartheta_3^3\,\vartheta_2^4(z)} = \frac{1}{\cos^4 z}-\frac{2}{3\cos^2 z}+\frac{1}{3}\sum_{n}(2n)^3 q^{n^2}+\frac{2}{3}\sum_{n,r}(-1)^r\left[2(n+r)\right]^3 q^{n^2+2nr}\cos 2rz \; +$$

(143)

$$+\;\frac{1}{6}\left\{9\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z}-4\sum_{n} n\, q^{n^2}+4\sum_{n,r}(-1)^{r+1}2(n+r)\, q^{n^2+2nr}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_3^3(z)}{\vartheta_3^3\,\vartheta_1^4(z)} = \frac{1}{\sin^4 z}-\frac{2}{3\sin^2 z}+\frac{1}{3}\sum_{n}(2n)^3 q^{n^2}+\frac{2}{3}\sum_{n,r}\left[2(n+r)\right]^3 q^{n^2+2nr}\cos 2rz \; +$$

(144)

$$+\;\frac{1}{6}\left\{9\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z}-4\sum_{n} n\, q^{n^2}-4\sum_{n,r}2(n+r)\, q^{n^2+2nr}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_2^3(z)}{\vartheta_3^3\,\vartheta_0^4(z)} = \frac{2}{3}\sum_{n,r}\left[2(n+r-1)\right]^3 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(145)

$$-\;\frac{4}{3}\left\{9\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \,.$$

$$\frac{\vartheta_1'^4\,\vartheta_1^3(z)}{\vartheta_3^3\,\vartheta_3^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{r+1}\left[2(n+r-1)\right]^3 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(146)

$$+\;\frac{4}{3}\left\{9\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r}(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \,.$$

From these we get

(139.1)
$$\frac{\vartheta_1'^{4}\vartheta_3^{3}(z)}{\vartheta_0^{3}\vartheta_2^{4}(z)} = \frac{1}{\cos^4 z} - \frac{2}{3\cos^2 z} + \frac{1}{3}\sum_n (-1)^n (2n)^3 q^{n^2} + \frac{2}{3}\sum q^N\left\{(-1)^{\frac{\alpha+a}{2}}(\alpha+a)^3 \cos(\alpha-a)z\right\} +$$
$$+\frac{1}{6}\left\{9\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z} + 2\sum_n (-1)^{n+1} 2n\, q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

(140.1)
$$\frac{\vartheta_1'^{4}\vartheta_0^{3}(z)}{\vartheta_0^{3}\vartheta_1^{4}(z)} = \frac{1}{\sin^4 z} - \frac{2}{3\sin^2 z} + \frac{1}{3}\sum_n (-1)^n (2n)^3 q^{n^2} + \frac{2}{3}\sum q^N\left\{(-1)^{a}(\alpha+a)^3 \cos(\alpha-a)z\right\} +$$
$$+\frac{1}{6}\left\{9\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z} + 2\sum_n (-1)^{n+1} 2n\, q^{n^2} + 4\sum q^N\left\{(-1)^{a+1}(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

(141.1)
$$\frac{\vartheta_1'^{4}\vartheta_2^{3}(z)}{\vartheta_0^{3}\vartheta_3^{4}(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}(\gamma+c)^3 \cos\frac{\gamma-c}{2}z\right\} +$$
$$+\frac{1}{3}\left\{9\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}.$$

(142.1)
$$\frac{\vartheta_1'^{4}\vartheta_1^{3}(z)}{\vartheta_0^{3}\vartheta_0^{4}(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+c)^3 \sin\frac{\gamma-c}{2}z\right\} +$$
$$+\frac{1}{3}\left\{9\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\}.$$

(143.1)
$$\frac{\vartheta_1'^{4}\vartheta_0^{3}(z)}{\vartheta_3^{3}\vartheta_2^{4}(z)} = \frac{1}{\cos^4 z} - \frac{2}{3\cos^2 z} + \frac{1}{3}\sum_n (2n)^3 q^{n^2} + \frac{2}{3}\sum q^N\left\{(-1)^{\frac{\alpha-a}{2}}(\alpha+a)^3 \cos(\alpha-a)z\right\} +$$
$$+\frac{1}{6}\left\{9\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z} - 4\sum_n n\, q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

(144.1)
$$\frac{\vartheta_1'^{4}\vartheta_3^{3}(z)}{\vartheta_3^{3}\vartheta_1^{4}(z)} = \frac{1}{\sin^4 z} - \frac{2}{3\sin^2 z} + \frac{1}{3}\sum_n (2n)^3 q^{n^2} + \frac{2}{3}\sum q^N\left\{(\alpha+a)^3 \cos(\alpha-a)z\right\} +$$
$$+\frac{1}{6}\left\{9\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z} - 2\sum_n (2n)\, q^{n^2} - 4\sum q^N\left\{(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_1^3(z)}{\vartheta_3^3\,\vartheta_3^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}(\gamma+c)^3 \sin\frac{\gamma-c}{2}z\right\} +$$

(146.1)

$$+\frac{1}{3}\left\{9\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\}.$$

Group IX-b

$$\frac{\vartheta_3^3(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_0^3(z)}{\vartheta_3^4(z)} \qquad \frac{\vartheta_2^3(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_2^4(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^3 e^{-iz}}{\vartheta_1^4(z)}$$

$F(z)$ satisfies (8) and has a pole of order four at $z=0$. Calculating the corresponding $A_i^{(j)}$ and using (10) gives

$$\frac{\vartheta_1'^4}{\vartheta_2^3}F(z) = \frac{1}{6}A_1^{(3)}(z,0) - \frac{i}{2}A_1^{(2)}(z,0)$$

(147)

$$+\frac{1}{6}\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-3\right\}A_1^{(1)}(z,0) - \frac{i}{6}\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}A_1^{(0)}(z,0).$$

This gives

$$\frac{\vartheta_1'^4\,\vartheta_2^3(z)}{\vartheta_2^3\,\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z} + \frac{2}{3}\sum_{n,r}[2(n+r)-1]^3 q^{n(n+2r-1)}\cos(2r-1)z +$$

(148)

$$+\frac{1}{6}\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{\cos z}{\sin^2 z} - \frac{2}{3}\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}[2(n+r)-1]q^{n(n+2r-1)}\cos(2r-1)z.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\,\vartheta_1^3(z)}{\vartheta_2^3\,\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z} + \frac{2}{3}\sum_{n,r}(-1)^{r+1}[2(n+r)-1]^3 q^{n(n+2r-1)}\sin(2r-1)z +$$

(149)

$$+\frac{1}{6}\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{\sin z}{\cos^2 z} + \frac{2}{3}\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^r[2(n+r)-1]q^{n(n+2r-1)}\sin(2r-1)z.$$

In (147) replace z by $z+\frac{\pi\tau}{2}$. Hence

$$\frac{\vartheta_1'^4\vartheta_3^3(z)}{\vartheta_2^3\vartheta_0^4(z)}=\frac{1}{6}q^{\frac14}A_1^{(3)}(z+\tfrac{\pi\tau}{2},0)-\frac{i}{2}q^{\frac14}A_1^{(2)}(z+\tfrac{\pi\tau}{2},0)+$$

$$+\frac{1}{6}q^{\frac14}\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-3\right\}A_1^{(1)}(z+\tfrac{\pi\tau}{2},0)-\frac{i}{6}q^{\frac14}\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}A_1^{(0)}(z+\tfrac{\pi\tau}{2},0)$$

From this follows

$$\frac{\vartheta_1'^4\vartheta_3^3(z)}{\vartheta_2^3\vartheta_0^4(z)}=\frac13\sum_n(2n-1)^3q^{\left(\frac{2n-1}{2}\right)^2}+\frac23\sum_{n,r}[2(n+r)-1]q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz+$$

(150)

$$-\frac13\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2}+2\sum_{n,r}[2(n+r)-1]q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\vartheta_0^3(z)}{\vartheta_2^3\vartheta_3^4(z)}=\frac13\sum_n(2n-1)^3q^{\left(\frac{2n-1}{2}\right)^2}+\frac23\sum_{n,r}(-1)^r[2(n+r)-1]^3q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz+$$

(151)

$$-\frac13\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2}+2\sum_{n,r}(-1)^r[2(n+r)-1]q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

From these follow

$$\frac{\vartheta_1'^4\vartheta_2^3(z)}{\vartheta_2^3\vartheta_1^4(z)}=\frac{\cos z}{\sin^4 z}+\frac23\sum q^N\left\{(\beta+b)^3\cos(\beta-b)z\right\}+$$

(148.1)

$$+\frac16\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{\cos z}{\sin^2 z}-\frac23\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(\beta+b)\cos(\beta-b)z\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_1^3(z)}{\vartheta_2^3\vartheta_2^4(z)}=\frac{\sin z}{\cos^4 z}+\frac23\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}(\beta+b)^3\sin(\beta-b)z\right.$$

(149.1)

$$\left.+\frac16\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{\sin z}{\cos^2 z}+\frac23\left\{9\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)\sin(\beta-b)z\right\}.$$

$$\frac{\vartheta_1'^4 \vartheta_3^3(z)}{\vartheta_2^3 \vartheta_0^4(z)} = \frac{1}{3}\sum_n (2n-1)^3 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{12}\sum q^{\frac{N}{4}}\left\{ (\delta+d)^3 \cos\frac{\delta-d}{2} z\right\} +$$

(150.1)

$$-\frac{1}{3}\left\{ 9\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n (2n-1) q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{N}\left\{ (\delta+d)\cos\frac{\delta-d}{2} z\right\}\right\}.$$

$$\frac{\vartheta_1'^4 \vartheta_0^3(z)}{\vartheta_2^3 \vartheta_3^4(z)} = \frac{1}{3}\sum_n (2n-1)^3 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}} (\delta+d)^3 \cos\frac{\delta-d}{2} z\right\} +$$

(151.1)

$$-\frac{1}{3}\left\{ 9\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n (2n-1) q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)\cos\frac{\delta-d}{2} z\right\}\right\}.$$

Group X-a

$$\frac{\vartheta_0^2(z)\vartheta_2(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_3^2(z)\vartheta_1(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_2^2(z)\vartheta_0(z)}{\vartheta_3^4(z)}$$

$$\frac{\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_0^2(z)\vartheta_1(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_1^2(z)\vartheta_0(z)}{\vartheta_3^4(z)} \qquad \frac{\vartheta_2^2(z)\vartheta_3(z)}{\vartheta_0^4(z)}$$

Consider

$$F(z) = \frac{\vartheta_0^2(z)\vartheta_2(z)\, e^{-iz}}{\vartheta_1^4(z)}$$

$F(z)$ satisfies (8) and has a pole of order four at $z=0$. Calculating the corresponding $H_i^{(j)}$ we find

$$\frac{\vartheta_1'^4}{\vartheta_0^2\vartheta_2}F(z) = \frac{1}{6}H_1^{(3)}(z,0) - \frac{i}{2}H_1^{(2)}(z,0) + \frac{1}{6}\left\{6\frac{\vartheta_0''}{\vartheta_0} + 3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 3\right\}H_1^{(1)}(z,0) +$$

(152)

$$-\frac{i}{6}\left\{6\frac{\vartheta_0''}{\vartheta_0} + 3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 1\right\}H_1^{(0)}(z,0)$$

From this follows

$$\frac{\vartheta_1'^4\vartheta_0^2(z)\vartheta_2(z)}{\vartheta_0^2\vartheta_2\,\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z} + \frac{1}{6}\left\{6\frac{\vartheta_0''}{\vartheta_0} + 3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 1\right\}\frac{\cos z}{\sin^2 z} +$$

(153)

$$+\frac{2}{3}\sum[2(n+r)-1]^3 q^{n(n+2r-1)}\cos(2r-1)z - \frac{2}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0} + 3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum[2(n+r)-1]\, q^{n(n+2r-1)}\cos(2r-1)z\,.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\vartheta_3^2(z)\vartheta_1(z)}{\vartheta_0^2\vartheta_2\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z} + \frac{2}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^r[2(n+r)-1]\,q^{n(n+2r-1)}\sin(2r-1)z +$$

(154)

$$+\frac{1}{6}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{\sin z}{\cos^2 z} + \frac{2}{3}\sum_{n,r}(-1)^{r+1}[2(n+r)-1]^3\,q^{n(n+2r-1)}\sin(2r-1)z \,.$$

In (152) replace z by $z+\frac{\pi\tau}{2}$. We get

$$-\frac{\vartheta_1'^4\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_2\vartheta_0^2\vartheta_0^4(z)} = \frac{1}{6}q^{\frac14}A_1^{(3)}(z+\tfrac{\pi\tau}{2},0) - \frac{i}{2}q^{\frac14}A_1^{(2)}(z+\tfrac{\pi\tau}{2},0) +$$

$$+\frac{1}{6}q^{\frac14}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-3\right\}A_1^{(1)}(z+\tfrac{\pi\tau}{2},0) - \frac{i}{6}q^{\frac14}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}A_1^{(0)}(z+\tfrac{\pi\tau}{2},0) \,.$$

From this follows

$$\frac{\vartheta_1'^4\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_0^2\vartheta_2\vartheta_0^4(z)} = -\frac{1}{3}\sum_{n}(2n-1)^3 q^{\left(\frac{2n-1}{2}\right)^2} - \frac{2}{3}\sum_{n,r}[2(n+r)-1]^3 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \;+$$

(155)

$$+\frac{1}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n}(2n-1)q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}[2(n+r)-1]q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} \,.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\vartheta_2^2(z)\vartheta_0(z)}{\vartheta_0^2\vartheta_2\vartheta_3^4(z)} = -\frac{1}{3}\sum_{n}(2n-1)^3 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{2}{3}\sum_{n,r}(-1)^{r+1}[2(n+r)-1]^3 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \;+$$

(156)

$$+\frac{1}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n}(2n-1)q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^r[2(n+r)-1]q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} .$$

Replacing q by $-q$ in these results in

$$\frac{\vartheta_1'^4\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_3^2\vartheta_2\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z} - \frac{2}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}[2(n+r)-1]\,q^{n(n+2r-1)}\cos(2r-1)z \;+$$

(157)

$$+\frac{1}{6}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{\cos z}{\sin^2 z} + \frac{2}{3}\sum_{n,r}[2(n+r)-1]^3\,q^{n(n+2r-1)}\cos(2r-1)z \,.$$

$$\frac{\vartheta_1'^4\vartheta_0^2(z)\vartheta_1(z)}{\vartheta_3^2\vartheta_2\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z} + \frac{2}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^r[2(n+r)-1]\,q^{n(n+2r-1)}\sin(2r-1)z \;+$$

(158)

$$+\frac{1}{6}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{\sin z}{\cos^2 z} + \frac{2}{3}\sum_{n,r}(-1)^{r+1}[2(n+r)-1]^3\,q^{n(n+2r-1)}\sin(2r-1)z \,.$$

(159)
$$\frac{\vartheta_1'^4\,\vartheta_1^2(z)\,\vartheta_0(z)}{\vartheta_3^2\,\vartheta_2\,\vartheta_3^4(z)} = \frac{1}{3}\sum_n (2n-1)^3 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{2}{3}\sum_{n,r}(-1)^r[2(n+r)-1]^3 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \;+$$
$$-\frac{1}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n (2n-1) q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

(160)
$$\frac{\vartheta_1'^4\,\vartheta_2^2(z)\,\vartheta_3(z)}{\vartheta_3^2\,\vartheta_2\,\vartheta_0^4(z)} = \frac{1}{3}\sum_n (2n-1) q^{\left(\frac{2n-1}{2}\right)^2} + \frac{2}{3}\sum_{n,r}[2(n+r)-1]^3 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \;+$$
$$-\frac{1}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n (2n-1) q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

These give

(153.1)
$$\frac{\vartheta_1'^4\,\vartheta_0^2(z)\,\vartheta_2(z)}{\vartheta_0^2\,\vartheta_2\,\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z} + \frac{1}{6}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{\cos z}{\sin^2 z} \;+$$
$$+\frac{2}{3}\sum q^N\left\{(\beta+b)^3\cos(\beta-b)z\right\} - \frac{2}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(\beta+b)\cos(\beta-b)z\right\}.$$

(154.1)
$$\frac{\vartheta_1'^4\,\vartheta_3^2(z)\,\vartheta_1(z)}{\vartheta_0^2\,\vartheta_2\,\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z} + \frac{1}{6}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}\frac{\sin z}{\cos^2 z} \;+$$
$$+\frac{2}{3}\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}(\beta+b)^3\sin(\beta-b)z\right\} + \frac{2}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)\sin(\beta-b)z\right\},$$

(155.1)
$$\frac{\vartheta_1'^4\,\vartheta_1^2(z)\,\vartheta_3(z)}{\vartheta_0^2\,\vartheta_2\,\vartheta_0^4(z)} = -\frac{1}{3}\sum_n (2n-1)^3 q^{\left(\frac{2n-1}{2}\right)^2} - \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(\delta+d)^3\cos\frac{\delta-d}{2}z\right\} \;+$$
$$+\frac{1}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n (2n-1) q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\},$$

(156.1)
$$\frac{\vartheta_1'^4\,\vartheta_2^2(z)\,\vartheta_0(z)}{\vartheta_0^2\,\vartheta_2\,\vartheta_3^4(z)} = -\frac{1}{3}\sum_n (2n-1)^3 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-4}{4}}(\delta+d)^3\cos\frac{\delta-d}{2}z\right\} \;+$$
$$+\frac{1}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n (2n-1) q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_3^2\vartheta_2\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z} + \frac{\cos z}{6\sin^2 z}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} +$$

(157.1)

$$+\frac{2}{3}\sum q^N\left\{(\beta+b)^3\cos(\beta-b)z\right\} - \frac{2}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(\beta+b)\cos(\beta-b)z\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_0^2(z)\vartheta(z)}{\vartheta_3^2\vartheta_2\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z} + \frac{\sin z}{6\cos z}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} +$$

(158.1)

$$+\frac{2}{3}\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}(\beta+b)^3\sin(\beta-b)z\right\} + \frac{2}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)\sin(\beta-b)z\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_1^2(z)\vartheta_0(z)}{\vartheta_3^2\vartheta_2\vartheta_3^4(z)} = \frac{1}{3}\sum_n(2n-1)^3 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)^3\cos\frac{\delta-d}{2}z\right\} +$$

(159.1)

$$-\frac{1}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_2^2(z)\vartheta_3(z)}{\vartheta_3^2\vartheta_2\vartheta_0^4(z)} = \frac{1}{3}\sum_n(2n-1)^3 q^{\left(\frac{2n-1}{2}\right)^2} + \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(\delta+d)^3\cos\frac{\delta-d}{2}z\right\} +$$

(160.1)

$$-\frac{1}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\}.$$

Group X-b

$$\frac{\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_3^4(z)}$$

$$\frac{\vartheta_0(z)\vartheta_3^2(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_3^4(z)} \qquad \frac{\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_0^4(z)}$$

Consider

$$F(z) = \frac{\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_1^4(z)}$$

Let $t = z + \frac{\pi i}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order four at $t = \frac{\pi i}{2}$. Calculating the corresponding $A_i^{(\nu)}$ and using (10) gives

$$(161)\qquad \frac{\vartheta_1'^4}{\vartheta_0^2\vartheta_3}F(z) = \frac{1}{6}A_1^{(3)}\left(z+\frac{\pi i}{2}, \frac{\pi i}{2}\right) + \frac{1}{6}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}A_1^{(1)}\left(z+\frac{\pi i}{2}, \frac{\pi i}{2}\right).$$

Hence

$$(162)\qquad \begin{aligned} \frac{\vartheta_1'^4\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_0^2\vartheta_2\vartheta_1^4(z)} &= \frac{1}{\sin^4 z} - \frac{2}{3\sin^2 z} + \frac{1}{3}\sum_n (2n)^3 q^{n^2} + \frac{2}{3}\sum_{n,r}[2(n+r)]^3 q^{n^2+2nr}\cos 2rz + \\ &+ \frac{1}{6}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z} - 2\sum_n 2n q^{n^2} - 4\sum_{n,r}2(n+r)q^{n^2+2nr}\cos 2rz\right\}. \end{aligned}$$

Replacing z by $z - \frac{\pi}{2}$

$$(163)\qquad \begin{aligned} \frac{\vartheta_1'^4\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_0^2\vartheta_3\vartheta_2^4(z)} &= \frac{1}{\cos^4 z} - \frac{2}{3\cos^2 z} + \frac{1}{3}\sum_n (2n)^3 q^{n^2} + \frac{2}{3}\sum_{n,r}(-1)^r[2(n+r)]^3 q^{n^2+2nr}\cos 2rz + \\ &+ \frac{1}{6}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z} - 4\sum_n n q^{n^2} + 4\sum_{n,r}(-1)^{r+1}2(n+r)q^{n^2+2nr}\cos 2rz\right\}. \end{aligned}$$

In (161) replace z by $z - \frac{\pi i}{2}$. We get

$$-\frac{\vartheta_1'^4\vartheta_1^2(z)\vartheta_2(z)e^{-iz}}{\vartheta_0^2\vartheta_3\vartheta_0^4(z)} = \frac{1}{6}q^{-1/4}A_1^{(3)}\left(z, \frac{\pi i}{2}\right) + \frac{1}{6}q^{-\frac{1}{4}}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}A_1^{(1)}\left(z, \frac{\pi i}{2}\right).$$

From this follows

$$(164)\qquad \begin{aligned} \frac{\vartheta_1'^4\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_0^2\vartheta_3\vartheta_0^4(z)} &= -\frac{2}{3}\sum_{n,r}[2(n+r-1)]^3 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + \\ &+ \frac{2}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}[2(n+r)-2]\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \end{aligned}$$

Replacing z by $z - \frac{\pi}{2}$

$$(165)\qquad \begin{aligned} \frac{\vartheta_1'^4\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_0^2\vartheta_3\vartheta_3^4(z)} &= \frac{2}{3}\sum_{n,r}(-1)^r[2(n+r-1)]^3 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z + \\ &+ \frac{2}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1}2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z. \end{aligned}$$

In these results replace q by $-q$. There follow

$$\frac{\vartheta_1'^4\,\vartheta_3^2(z)\,\vartheta_0(z)}{\vartheta_3^2\,\vartheta_0\,\vartheta_1^4(z)} = \frac{1}{\sin^4 z} - \frac{2}{3\sin^2 z} + \frac{1}{3}\sum_n (-1)^n (2n)^3 q^{n^2} + \frac{2}{3}\sum_{n,r}(-1)^n[2(n+r)]^3 q^{n^2+2nr}\cos 2rz +$$

(166)

$$+\frac{1}{6}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z}+4\sum_n(-1)^{n+1} n\, q^{n^2}+4\sum_{n,r}(-1)^{n+1}2(n+r)\,q^{n^2+2nr}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_0^2(z)\,\vartheta_3(z)}{\vartheta_3^2\,\vartheta_0\,\vartheta_2^4(z)} = \frac{1}{\cos^4 z} - \frac{2}{3\cos^2 z} + \frac{1}{3}\sum_n (-1)^n (2n)^3 q^{n^2} + \frac{2}{3}\sum_{n,r}(-1)^{n+r}[2(n+r)]^3 q^{n^2+2nr}\cos 2rz +$$

(167)

$$+\frac{1}{6}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z}+4\sum_n(-1)^{n+1} n\, q^{n^2}+4\sum_{n,r}(-1)^{n+r+1}2(n+r)\,q^{n^2+2nr}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_1^2(z)\,\vartheta_2(z)}{\vartheta_3^2\,\vartheta_0\,\vartheta_3^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+r}[2(n+r-1)]^3 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \;+$$

(168)

$$+\frac{2}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1}2(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z .$$

$$\frac{\vartheta_1'^4\,\vartheta_2^2(z)\,\vartheta_1(z)}{\vartheta_3^2\,\vartheta_0\,\vartheta_0^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+1}[2(n+r-1)]^3 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \;+$$

(169)

$$+\frac{2}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n}2(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z .$$

From the above results we get

$$\frac{\vartheta_1'^4\,\vartheta_0^2(z)\,\vartheta_3(z)}{\vartheta_0^2\,\vartheta_3\,\vartheta_1^4(z)} = \frac{1}{\sin^4 z} - \frac{2}{3\sin^2 z} + \frac{1}{3}\sum_n (2n)^3 q^{n^2} + \frac{2}{3}\sum q^N\{(\alpha+a)^3\cos(\alpha-a)z\} +$$

(162.1)

$$+\frac{1}{6}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z}-2\sum_n 2n\,q^{n^2}-4\sum q^N\{(\alpha+a)\cos(\alpha-a)z\}\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_0(z)\,\vartheta_3^2(z)}{\vartheta_0^2\,\vartheta_3\,\vartheta_2^4(z)} = \frac{1}{\cos^4 z} - \frac{2}{3\cos^2 z} + \frac{1}{3}\sum_n (2n)^3 q^{n^2} + \frac{2}{3}\sum q^N\left\{(-1)^{\frac{\alpha-a}{2}}(\alpha+a)^3\cos(\alpha-a)z\right\} +$$

(163.1)

$$+\frac{1}{6}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z}-2\sum_n 2n\,q^{n^2}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_1^2(z)\,\vartheta_2(z)}{\vartheta_0^2\,\vartheta_3\,\vartheta_0^4(z)} = -\frac{1}{12}\sum q^{\frac{N}{4}}\left\{(\gamma+c)^3\cos\frac{\gamma-c}{2}z\right\} + \tag{164.1}$$

$$+\frac{1}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_2^2(z)\,\vartheta_1(z)}{\vartheta_0^2\,\vartheta_3\,\vartheta_3^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}(\gamma+c)^3\sin\frac{\gamma-c}{2}z\right\} + \tag{165.1}$$

$$+\frac{1}{3}\left\{6\frac{\vartheta_0''}{\vartheta_0}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_3^2(z)\,\vartheta_0(z)}{\vartheta_3^2\,\vartheta_0\,\vartheta_1^4(z)} = \frac{1}{\sin^4 z}-\frac{2}{3\sin^2 z}+\frac{1}{3}\sum_n(-1)^n(2n)^3q^{n^2}+\frac{2}{3}\sum q^N\left\{(-1)^a(\alpha+a)^3\cos(\alpha-a)z\right\}+ \tag{166.1}$$

$$+\frac{1}{6}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z}+4\sum(-1)^{n+1}n\,q^{n^2}+4\sum q^N\left\{(-1)^{a+1}(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_0^2(z)\,\vartheta_3(z)}{\vartheta_3^2\,\vartheta_0\,\vartheta_2^4(z)} = \frac{1}{\cos^4 z}-\frac{2}{3\cos^2 z}+\frac{1}{3}\sum_n(-1)^n(2n)^3q^{n^2}+\frac{2}{3}\sum q^N\left\{(-1)^{\frac{\alpha+a}{2}}(\alpha+a)^3\cos(\alpha-a)z\right\}+ \tag{167.1}$$

$$+\frac{1}{6}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z}+4\sum_n(-1)^{n+1}n\,q^{n^2}+4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_1^2(z)\,\vartheta_2(z)}{\vartheta_3^2\,\vartheta_0\,\vartheta_3^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}(\gamma+c)^3\cos\frac{\gamma-c}{2}z\right\} + \tag{168.1}$$

$$+\frac{1}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_2^2(z)\,\vartheta_1(z)}{\vartheta_3^2\,\vartheta_0\,\vartheta_0^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+c)^3\sin\frac{\gamma-c}{2}z\right\} + \tag{169.1}$$

$$+\frac{1}{3}\left\{6\frac{\vartheta_3''}{\vartheta_3}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\}.$$

Group X-c

$$\frac{\vartheta_2^2(z)\,\vartheta_3(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_1^2(z)\,\vartheta_0(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_3^2(z)\,\vartheta_2(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_0^2(z)\,\vartheta_1(z)}{\vartheta_3^4(z)}$$

$$\frac{\vartheta_2^2(z)\,\vartheta_0(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_1^2(z)\,\vartheta_3(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_0^2(z)\,\vartheta_2(z)}{\vartheta_3^4(z)} \qquad \frac{\vartheta_3^2(z)\,\vartheta_1(z)}{\vartheta_0^4(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^2(z)\,\vartheta_3(z)}{\vartheta_1^4(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order four at $t = \frac{\pi\tau}{2}$. Calculating the $A_i^{(j)}$ and proceeding as before gives

$$(170) \qquad \frac{\vartheta_1'^4}{\vartheta_2^2\vartheta_3} F(z) = \frac{1}{6} A_1^{(3)}\left(z+\tfrac{\pi\tau}{2}, \tfrac{\pi\tau}{2}\right) + \frac{1}{6}\left\{6\frac{\vartheta_2''}{\vartheta_2} + 3\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\} A_1^{(1)}\left(z+\tfrac{\pi\tau}{2}, \tfrac{\pi\tau}{2}\right)$$

From this follows

$$(171) \qquad \begin{aligned} \frac{\vartheta_1'^4\,\vartheta_2^2(z)\,\vartheta_3(z)}{\vartheta_2^2\vartheta_3\,\vartheta_1^4(z)} &= \frac{1}{\sin^4 z} - \frac{2}{3\sin^2 z} + \frac{1}{3}\sum_n (2n)^3 q^{n^2} + \frac{2}{3}\sum_{n,r}[2(n+r)]^3 q^{n^2+2nr}\cos 2rz \\ &+ \frac{1}{6}\left\{6\frac{\vartheta_2''}{\vartheta_2} + 3\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z} - 4\sum_n n q^{n^2} - 4\sum_{n,r} 2(n+r) q^{n^2+2nr}\cos 2rz\right\}. \end{aligned}$$

Replacing z by $z - \frac{\pi}{2}$

$$(172) \qquad \begin{aligned} \frac{\vartheta_1'^4\,\vartheta_1^2(z)\,\vartheta_0(z)}{\vartheta_2^2\vartheta_3\,\vartheta_2^4(z)} &= \frac{1}{\cos^4 z} - \frac{2}{3\cos^2 z} + \frac{1}{3}\sum_n (2n)^3 q^{n^2} + \frac{2}{3}\sum_{n,r}(-1)^r[2(n+r)]^3 q^{n^2+2nr}\cos 2rz \; + \\ &+ \frac{1}{6}\left\{6\frac{\vartheta_2''}{\vartheta_2} + 3\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z} - 4\sum_n n q^{n^2} + 4\sum_{n,r}(-1)^{r+1} 2(n+r) q^{n^2+2nr}\cos 2rz\right\} \end{aligned}$$

In (170) replace z by $z - \frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^4\,\vartheta_3^2(z)\,\vartheta_2(z)\,e^{-iz}}{\vartheta_2^2\vartheta_3\,\vartheta_0^4(z)} = \frac{1}{6} q^{-\frac{1}{4}} A_1^{(3)}\left(z, \tfrac{\pi\tau}{2}\right) + \frac{1}{6} q^{-1/4}\left\{6\frac{\vartheta_2''}{\vartheta_2} + 3\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\} A_1^{(1)}\left(z, \tfrac{\pi\tau}{2}\right).$$

From this follows

$$(173) \qquad \begin{aligned} \frac{\vartheta_1'^4\,\vartheta_3^2(z)\,\vartheta_2(z)}{\vartheta_2^2\vartheta_3\,\vartheta_0^4(z)} &= \frac{2}{3}\sum_{n,r}[2(n+r-1)]^3 q^{\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; + \\ &- \frac{2}{3}\left\{6\frac{\vartheta_2''}{\vartheta_2} + 3\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r} 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \end{aligned}$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\vartheta_0^2(z)\vartheta_1(z)}{\vartheta_2^2\vartheta_3\vartheta_3^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{r+1}[2(n+r-1)]^3\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(174)

$$+\,\frac{2}{3}\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^r\, 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

The change of q into $-q$ gives

$$\frac{\vartheta_1'^4\vartheta_2^2(z)\vartheta_0(z)}{\vartheta_2^2\vartheta_0\vartheta_1^4(z)} = \frac{1}{\sin^4 z}-\frac{2}{3\sin^2 z}+\frac{1}{3}\sum_n(-1)^n(2n)^3q^{n^2}+\frac{2}{3}\sum_{n,r}(-1)^n[2(n+r)]^3q^{n^2+2nr}\cos 2rz \; +$$

(175)

$$+\,\frac{1}{6}\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z}+4\sum_n(-1)^{n+1}nq^{n^2}+4\sum_{n,r}(-1)^{n+1}2(n+r)q^{n^2+2nr}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_2^2\vartheta_0\vartheta_2^4(z)} = \frac{1}{\cos^4 z}-\frac{2}{3\cos^2 z}+\frac{1}{3}\sum_n(-1)^n(2n)^3q^{n^2}+\frac{2}{3}\sum_{n,r}(-1)^{n+r}[2(n+r)]^3q^{n^2+2nr}\cos 2rz \; +$$

(176)

$$+\,\frac{1}{6}\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z}+4\sum_n(-1)^{n+1}nq^{n^2}+4\sum_{n,r}(-1)^{n+1+r}2(n+r)q^{n^2+2nr}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_0^2(z)\vartheta_2(z)}{\vartheta_2^2\vartheta_0\vartheta_3^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+r}[2(n+r-1)]^3\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(177)

$$+\,\frac{2}{3}\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1}\,2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z.$$

$$\frac{\vartheta_1'^4\vartheta_3^2(z)\vartheta_1(z)}{\vartheta_2^2\vartheta_0\vartheta_0^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+1}[2(n+r-1)]^3\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(178)

$$+\,\frac{2}{3}\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^n\,2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z.$$

From these follow

$$\frac{\vartheta_1'^4\vartheta_1^2(z)\vartheta_0(z)}{\vartheta_2^2\vartheta_3\vartheta_2^4(z)} = \frac{1}{\cos^4 z}-\frac{2}{3\cos^2 z}+\frac{1}{3}\sum_n(2n)^3q^{n^2}+\frac{2}{3}\sum q^N\left\{(-1)^{\frac{\alpha-a}{2}}(\alpha+a)^3\cos(\alpha-a)z\right\} +$$

(172.1)

$$+\,\frac{1}{6}\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z}-4\sum_n nq^{n^2}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\},$$

(171.1)
$$\frac{\vartheta_1'^4\,\vartheta_2^2(z)\,\vartheta_3(z)}{\vartheta_2^2\vartheta_3\,\vartheta_1^4(z)} = \frac{1}{\sin^4 z} - \frac{2}{3\sin^2 z} + \frac13\sum_n (2n)^3 q^{n^2} + \frac23\sum q^N\left\{(\alpha+a)^3\cos(\alpha-a)z\right\} +$$
$$+\frac16\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z} - 4\sum n q^{n^2} - 4\sum q^N\left\{(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

(173.1)
$$\frac{\vartheta_1'^4\,\vartheta_3^2(z)\,\vartheta_2(z)}{\vartheta_2^2\vartheta_3\,\vartheta_0^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(\gamma+c)^3\cos\frac{\gamma-c}{2}z\right\} +$$
$$-\frac13\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}.$$

(174.1)
$$\frac{\vartheta_1'^4\,\vartheta_0^2(z)\,\vartheta_1(z)}{\vartheta_2^2\vartheta_3\,\vartheta_3^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}(\gamma+c)^3\sin\frac{\gamma-c}{2}z\right\} +$$
$$+\frac13\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\}.$$

(175.1)
$$\frac{\vartheta_1'^4\,\vartheta_2^2(z)\,\vartheta_0(z)}{\vartheta_2^2\vartheta_0\,\vartheta_1^4(z)} = \frac{1}{\sin^4 z} - \frac{2}{3\sin^2 z} + \frac13\sum_n (-1)^n(2n)^3 q^{n^2} + \frac23\sum q^N\left\{(-1)^a(\alpha+a)^3\cos(\alpha-a)z\right\} +$$
$$+\frac16\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\sin^2 z} + 4\sum_n (-1)^{n+1} n q^{n^2} + 4\sum q^N\left\{(-1)^{a+1}(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

(176.1)
$$\frac{\vartheta_1'^4\,\vartheta_1^2(z)\,\vartheta_3(z)}{\vartheta_2^2\vartheta_0\,\vartheta_2^4(z)} = \frac{1}{\cos^4 z} - \frac{2}{3\cos^2 z} + \frac13\sum_n (-1)^n(2n)^3 q^{n^2} + \frac23\sum q^N\left\{(-1)^{\frac{\alpha+a}{2}}(\alpha+a)^3\cos(\alpha-a)z\right\} +$$
$$+\frac16\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{1}{\cos^2 z} + 4\sum_n (-1)^{n+1} n q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

(177.1)
$$\frac{\vartheta_1'^4\,\vartheta_0^2(z)\,\vartheta_2(z)}{\vartheta_2^2\vartheta_0\,\vartheta_3^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}(\gamma+c)^3\cos\frac{\gamma-c}{2}z\right\} +$$
$$+\frac13\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}.$$

(178.1)
$$\frac{\vartheta_1'^4\,\vartheta_3^2(z)\,\vartheta_1(z)}{\vartheta_2^2\vartheta_0\,\vartheta_0^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+c)^3\sin\frac{\gamma-c}{2}z\right\} +$$
$$+\frac13\left\{6\frac{\vartheta_2''}{\vartheta_2}+3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\}$$

Group XI

$$\frac{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)}{\vartheta_3^4(z)}$$

Consider

$$F(z) = \frac{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)}{\vartheta_1^4(z)}\, e^{-iz}$$

Let $z+\frac{\pi}{2}=t$, $F(z)\equiv\varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of the order four at $t=\frac{\pi}{2}$. Calculating the corresponding $A_i^{(j)}$ and using (10) gives

(179) $$\vartheta_1'^3 F(z) = \frac{1}{6}A_1^{(3)}(z+\tfrac{\pi}{2},\tfrac{\pi}{2}) - \frac{i}{2}A_1^{(2)}(z+\tfrac{\pi}{2},\tfrac{\pi}{2}) - \frac{1}{6}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\right\}A_1^{(1)}(z+\tfrac{\pi}{2},\tfrac{\pi}{2}) + \frac{i}{6}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+1\right\}A_1^{(0)}(z+\tfrac{\pi}{2},\tfrac{\pi}{2}).$$

From this follows

(180) $$\frac{\vartheta_1'^3\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)}{\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z} - \frac{\cos z}{6\sin^2 z}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+1\right\} + \frac{2}{3}\sum_{n,r}(-1)^n[2(n+r)-1]^3\, q^{n(n+2r-1)}\cos(2r-1)z \; +$$
$$+ \frac{2}{3}\frac{\vartheta_1'''}{\vartheta_1'}\sum_{n,r}(-1)^n[2(n+r)-1]\, q^{n(n+2r-1)}\cos(2r-1)z$$

(181) $$\frac{\vartheta_1'^3\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)}{\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z} - \frac{\sin z}{6\cos^2 z}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+1\right\} + \frac{2}{3}\sum_{n,r}(-1)^{n+r+1}[2(n+r)-1]^3\, q^{n(n+2r-1)}\sin(2r-1)z \; +$$
$$+ \frac{2}{3}\frac{\vartheta_1'''}{\vartheta_1'}\sum_{n,r}(-1)^{n+r+1}[2(n+r)-1]\, q^{n(n+2r-1)}\sin(2r-1)z$$

Replace z by $z+\frac{\pi\tau}{2}$ in (179). We get

$$\frac{\vartheta_1'^3\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)}{\vartheta_0^4(z)} = -\frac{i}{6}q^{1/4}A_1^{(3)}(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2},\tfrac{\pi}{2}) - \frac{1}{2}q^{1/4}A_1^{(2)}(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2},\tfrac{\pi}{2}) \; +$$
$$+\frac{i}{6}q^{1/4}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\right\}A_1^{(1)}(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2},\tfrac{\pi}{2}) + \frac{1}{6}q^{1/4}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+1\right\}A_1^{(0)}(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2},\tfrac{\pi}{2}) .$$

There follows

(182) $$\frac{\vartheta_1'^3\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)}{\vartheta_0^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+1}[2(n+r)-1]^3\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz$$
$$+ \frac{2\vartheta_1'''}{3\vartheta_1'}\sum_{n,r}(-1)^{n+1}[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz$$

Replacing z by $z-\frac{\pi}{2}$

(183)
$$\frac{\vartheta_1'^3\,\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)}{\vartheta_3^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+r}[2(n+r)-1]^3\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r} \sin 2rz \; +$$
$$+\;\frac{2\vartheta_1'''}{3\vartheta_1'}\sum_{n,r}(-1)^{n+r}[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r} \sin 2rz \;.$$

From these follow

(180.1)
$$\frac{\vartheta_1'^3\,\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)}{\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z} - \frac{\cos z}{6\sin^2 z}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+1\right\} + \frac{2}{3}\sum q^N\left\{(-1)^b(\beta+b)^3\cos(\beta-b)z\right\} +$$
$$+\;\frac{2}{3}\frac{\vartheta_1'''}{\vartheta_1'}\sum q^N\left\{(-1)^b(\beta+b)\cos(\beta-b)z\right\} .$$

(181.1)
$$\frac{\vartheta_1'^3\,\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)}{\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z} - \frac{\sin z}{6\cos^2 z} + \frac{2}{3}\sum q^N\left\{(-1)^{\frac{\beta+b-1}{2}}(\beta+b)^3\sin(\beta-b)z\right\} +$$
$$+\;\frac{2\vartheta_1'''}{3\vartheta_1'}\sum q^N\left\{(-1)^{\frac{\beta+b-1}{2}}(\beta+b)\sin(\beta-b)z\right\} .$$

(182.1)
$$\frac{\vartheta_1'^3\,\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)}{\vartheta_0^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+d)^3\sin\frac{\delta-d}{2}z\right\} +$$
$$+\;\frac{1\vartheta_1'''}{3\vartheta_1'}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} .$$

(183.1)
$$\frac{\vartheta_1'^3\,\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)}{\vartheta_3^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d+2}{4}}(\delta+d)^3\sin\frac{\delta-d}{2}z\right\} +$$
$$+\;\frac{1\vartheta_1'''}{3\vartheta_1'}\sum q^{N}\left\{(-1)^{\frac{\delta+d+2}{4}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} .$$

Group XII-a

$$\frac{\vartheta_2^3(z)}{\vartheta_1^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_2^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_3^3(z)}{\vartheta_0^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_0^3(z)}{\vartheta_3^3(z)\vartheta_2(z)}$$

$$\frac{\vartheta_2^3(z)}{\vartheta_1^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_2^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_0^3(z)}{\vartheta_3^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_3^3(z)}{\vartheta_0^3(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^3(z)}{\vartheta_1^3(z)\vartheta_0(z)}$$

Let $z + \frac{\pi\tau}{2} = t$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders three and one at $t = \frac{\pi\tau}{2}$, and $t = \pi\tau$ respectively. Finding the corresponding $R_i^{(j)}$ and using (10) gives

$$\frac{\vartheta_1'^3\vartheta_0 F(z)}{\vartheta_2^3} = \frac{1}{2} A_1^{(2)}\left(z + \frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right)$$

(184)

$$+\frac{1}{6}\left\{9\frac{\vartheta_2''}{\vartheta_2} - 3\frac{\vartheta_0''}{\vartheta_0} - 3\frac{\vartheta_1'''}{\vartheta_1'}\right\} A_1^{(0)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) + q^{\frac{1}{4}}\frac{\vartheta_3^5}{\vartheta_2} A_1^{(0)}\left(z + \frac{\pi\tau}{2}, \pi\tau\right) .$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^3(z)}{\vartheta_2^3\vartheta_1^3(z)\vartheta_0(z)} = \frac{\cos z}{\sin^3 z} - 2\sum_{n,r}[2(n+r)]^2 q^{n^2+2nr}\sin 2rz + 4\frac{\vartheta_3^5}{\vartheta_2}\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz +$$

(185)

$$+\frac{1}{2}\left\{3\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{n^2+2nr}\sin 2rz\right\} .$$

Replacing z by $z - \frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1^3(z)}{\vartheta_2^3\vartheta_2^3(z)\,\vartheta_3(z)} = \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^r[2(n+r)]^2 q^{n^2+2nr}\sin 2rz + 4\frac{\vartheta_3^5}{\vartheta_2}\sum_{n,r}(-1)^{r+1} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz +$$

(186)

$$+\frac{1}{2}\left\{3\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{n^2+2nr}\sin 2rz\right\} .$$

Replacing z by $z - \frac{\pi\tau}{2}$ in (184) gives

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^3(z)e^{-iz}}{\vartheta_2^3\vartheta_0^3(z)\vartheta_1(z)} = \frac{q^{-\frac{1}{4}}}{2}A_1^{(2)}\left(z,\frac{\pi\tau}{2}\right) + \frac{q^{-\frac{1}{4}}}{2}\left\{3\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}A_1^{(0)}\left(z,\frac{\pi\tau}{2}\right) + \frac{\vartheta_3^5}{\vartheta_2}A_1^{(0)}(z,\pi\tau)$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^3(z)}{\vartheta_2^3\vartheta_0^3(z)\vartheta_1(z)} = -2\sum_{n,r}[2(n+r-1)]^2 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z +$$

(187)

$$+2\left\{3\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z + \frac{\vartheta_3^5}{\vartheta_2}\left\{\frac{1}{\sin z} + 4\sum_{n,r} q^{n(n+2r-1)}\sin(2r-1)z\right\} .$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0^3(z)}{\vartheta_2^3\vartheta_3^3(z)\vartheta_2(z)} = 2\sum_{n,r}(-1)^r[2(n+r-1)]^2 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(188)

$$+2\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + \frac{\vartheta_3^5}{\vartheta_0}\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{n(n+2r-1)}\cos(2r-1)z\right\}.$$

Replacing q **by** $-q$ **in these gives**

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2^3(z)}{\vartheta_2^3\vartheta_1^3(z)\vartheta_3(z)} = \frac{\cos z}{\sin^3 z} + 2\sum_{n,r}(-1)^{n+1}[2(n+r)]^2 q^{n^2+2nr}\sin 2rz + 4\frac{\vartheta_0^5}{\vartheta_2}\sum_{n,r}(-1)^r q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \; +$$

(189)

$$+\frac{1}{2}\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^n q^{n^2+2nr}\sin 2rz\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1^3(z)}{\vartheta_2^3\vartheta_2^3(z)\vartheta_0(z)} = \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^{n+r}[2(n+r)]^2 q^{n^2+2nr}\sin 2rz - 4\frac{\vartheta_0^5}{\vartheta_2}\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \; +$$

(190)

$$+\frac{1}{2}\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1} q^{n^2+2nr}\sin 2rz\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0^3(z)}{\vartheta_2^3\vartheta_3^3(z)\vartheta_1(z)} = 2\sum_{n,r}(-1)^{n+r}[2(n+r-1)]^2 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(191)

$$+2\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z + \frac{\vartheta_0^5}{\vartheta_2}\left\{\frac{1}{\sin z}+4\sum q^{n(n+2r-1)}\sin(2r-1)z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3^3(z)}{\vartheta_2^3\vartheta_0^3(z)\vartheta_2(z)} = 2\sum_{n,r}(-1)^{n+1}[2(n+r-1)]^2 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(192)

$$+2\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + \frac{\vartheta_3^5}{\vartheta_2}\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{n(n+2r-1)}\cos(2r-1)z\right\}.$$

From these follow

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^3(z)}{\vartheta_2^3\vartheta_1^3(z)\vartheta_0(z)} = \frac{\cos z}{\sin^3 z} - 2\sum q^N\left\{(\alpha+a)^2\sin(\alpha-a)z\right\} + 4\frac{\vartheta_3^5}{\vartheta_2}\sum q^{\frac{N}{4}}\left\{\sin(\alpha-a)z\right\} \; +$$

(185.1)

$$+\frac{1}{2}\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{\sin(\alpha-a)z\right\}\right\}.$$

(186.1)

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1^3(z)}{\vartheta_2^3\vartheta_2^3(z)\vartheta_3(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha-a}{2}}(\alpha+a)^2\sin(\alpha-a)z\right\} + 4\frac{\vartheta_3^5}{\vartheta_2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-4}{4}}\sin\frac{\delta-d}{2}z\right\} +$$

$$+\frac{1}{2}\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

(187.1)

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^3(z)}{\vartheta_2^3\vartheta_0^3(z)\vartheta_1(z)} = -\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\gamma+c)^2\sin\frac{\gamma-c}{2}z\right\} +$$

$$+2\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-c}{2}z\right\} + \frac{\vartheta_3^5}{\vartheta_2}\left\{\frac{1}{\sin z}+4\sum q^N\left\{\sin(\beta-b)z\right\}\right\}.$$

(188.1)

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0^3(z)}{\vartheta_2^3\vartheta_3^3(z)\vartheta_2(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}(\gamma+c)^2\cos\frac{\gamma-c}{2}z\right\} +$$

$$+2\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}\cos\frac{\gamma-c}{2}z\right\} + \frac{\vartheta_3^5}{\vartheta_2}\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\}\right\}.$$

(189.1)

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2^3(z)}{\vartheta_2^3\vartheta_1^3(z)\vartheta_3(z)} = \frac{\cos z}{\sin^3 z} + 2\sum q^N\left\{(-1)^{a+1}(\alpha+a)^2\sin(\alpha-a)z\right\} + 4\frac{\vartheta_0^5}{\vartheta_2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}\sin\frac{\delta-d}{2}z\right\} +$$

$$+\frac{1}{2}\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^{a}\sin(\alpha-a)z\right\}\right\}.$$

(190.1)

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1^3(z)}{\vartheta_2^3\vartheta_1^3(z)\vartheta_0(z)} = \frac{\sin z}{\cos^3(z)} + 2\sum q^N\left\{(-1)^{\frac{\alpha+a}{2}}(\alpha+a)^2\sin(\alpha-a)z\right\} - 4\frac{\vartheta_0^5}{\vartheta_2}\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-d}{2}z\right\} +$$

$$+\frac{1}{2}\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha+a-2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

(191.1)

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0^3(z)}{\vartheta_2^3\vartheta_3^3(z)\vartheta_1(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}(\gamma+c)^2\sin\frac{\gamma-c}{2}z\right\} +$$

$$+2\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}\sin\frac{\gamma-c}{2}z\right\} + \frac{\vartheta_0^5}{\vartheta_2}\left\{\frac{1}{\sin z}+4\sum q^N\left\{\sin(\beta-b)z\right\}\right\}.$$

(192.1)

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3^3(z)}{\vartheta_2^3\vartheta_0^3(z)\vartheta_2(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+c)^2\cos\frac{\gamma-c}{2}z\right\} +$$

$$+2\left\{3\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-c}{2}z\right\} + \frac{\vartheta_0^5}{\vartheta_2}\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\}\right\}.$$

Group XII-b

$$\frac{\vartheta_3^3(z)}{\vartheta_1^3(z)\,\vartheta_0(z)} \qquad \frac{\vartheta_0^3(z)}{\vartheta_2^3(z)\,\vartheta_3(z)} \qquad \frac{\vartheta_2^3(z)}{\vartheta_0^3(z)\,\vartheta_1(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_3^3(z)\,\vartheta_2(z)}$$

$$\frac{\vartheta_0^3(z)}{\vartheta_1^3(z)\,\vartheta_3(z)} \qquad \frac{\vartheta_3^3(z)}{\vartheta_2^3(z)\,\vartheta_0(z)} \qquad \frac{\vartheta_2^3(z)}{\vartheta_3^3(z)\,\vartheta_1(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_0^3(z)\,\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_3^3(z)\, e^{-iz}}{\vartheta_1^3(z)\,\vartheta_0(z)}$$

$F(z)$ satisfies (8) and has poles of orders three and one at $z=0$ and $\frac{\pi\tau}{2}$ respectively. Calculating the corresponding $A_i^{(j)}$ we find

$$(193) \qquad \frac{\vartheta_0\vartheta_1'^3}{\vartheta_3^3} F(z) = \frac{1}{2} A_1^{(2)}(z,0) - i A_1^{(1)}(z,0) + \frac{1}{2}\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} A_1^{(0)}(z,0) + q^{-\frac{1}{4}}\frac{\vartheta_2^5}{\vartheta_3} A_1^{(0)}\left(z,\frac{\pi\tau}{2}\right).$$

From this follows

$$(194) \qquad \begin{aligned} \frac{\vartheta_1'^3\vartheta_0\vartheta_3^3(z)}{\vartheta_3^3\vartheta_1^3(z)\vartheta_0(z)} &= \frac{1}{\sin^3 z} + \frac{1}{2\sin z}\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} - 2\sum_{n,r}[2(n+r)-1]^2 q^{n(n+2r-1)}\sin(2r-1)z \;+ \\ &+ 2\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r} q^{n(n+2r-1)}\sin(2r-1)z + 4\frac{\vartheta_2^5}{\vartheta_3}\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \,. \end{aligned}$$

Replacing z by $z+\frac{\pi}{2}$

$$(195) \qquad \begin{aligned} \frac{\vartheta_1'^3\vartheta_0\vartheta_0^3(z)}{\vartheta_3^3\vartheta_2^3(z)\vartheta_3(z)} &= \frac{1}{\cos^3 z} + \frac{1}{2\cos z}\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} + 2\sum_{n,r}(-1)^r[2(n+r)-1]^2 q^{n(n+2r-1)}\cos(2r-1)z \;+ \\ &+ 2\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1} q^{n(n+2r-1)}\cos(2r-1)z + 4\frac{\vartheta_2^5}{\vartheta_3}\sum_{n,r}(-1)^{r+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \,. \end{aligned}$$

In (193) replace z by $z+\frac{\pi\tau}{2}$. This gives

$$\begin{aligned} \frac{\vartheta_1'^3\vartheta_0\vartheta_2^3(z)}{\vartheta_3^3\vartheta_0^3(z)\vartheta_1(z)} &= \frac{q^{\frac{1}{4}}}{2} A_1^{(2)}\left(z+\frac{\pi\tau}{2},0\right) - i q^{\frac{1}{4}} A_1^{(1)}\left(z+\frac{\pi\tau}{2},0\right) \;+ \\ &+ \frac{1}{2} q^{\frac{1}{4}}\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} A_1^{(0)}\left(z+\frac{\pi\tau}{2},0\right) + \frac{\vartheta_2^5}{\vartheta_3} A_1^{(0)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right) . \end{aligned}$$

and from this follows

$$(196)\quad \frac{\vartheta_1'^3\vartheta_0\vartheta_2^3(z)}{\vartheta_3^3\vartheta_0^3(z)\vartheta_1(z)} = -2\sum_{n,r}[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \; + $$

$$+2\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \frac{\vartheta_2^5}{\vartheta_3}\left\{\frac{\cos z}{\sin z}+4\sum q^{n^2+2nr}\sin 2rz\right\}.$$

Replacing z **by** $z-\frac{\pi}{2}$

$$(197)\quad \frac{\vartheta_1'^3\vartheta_0\vartheta_1^3(z)}{\vartheta_3^3\vartheta_3^3(z)\vartheta_2(z)} = 2\sum_{n,r}(-1)^r[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \; + $$

$$+2\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \frac{\vartheta_2^5}{\vartheta_3}\left\{\frac{\sin z}{\cos z}+4\sum(-1)^{r+1} q^{n^2+2nr}\sin 2rz\right\}.$$

In these results replace q **by** $-q$ **. We get**

$$(198)\quad \frac{\vartheta_1'^3\vartheta_3\vartheta_0^3(z)}{\vartheta_0^3\vartheta_1^3(z)\vartheta_3(z)} = \frac{1}{\sin^3 z}+\frac{1}{2\sin z}\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}-2\sum_{n,r}[2(n+r)-1]^2 q^{n(n+2r-1)}\sin(2r-1)z \; + $$

$$+2\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{n(n+2r-1)}\sin(2r-1)z + 4\frac{\vartheta_2^5}{\vartheta_0}\sum_{n,r}(-1)^{n+r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z.$$

$$(199)\quad \frac{\vartheta_1'^3\vartheta_3\vartheta_3^3(z)}{\vartheta_0^3\vartheta_2^3(z)\vartheta_0(z)} = \frac{1}{\cos^3 z}+\frac{1}{2\cos z}\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}+2\sum_{n,r}(-1)^r[2(n+r)-1]^2 q^{n(n+2r-1)}\cos(2r-1)z \; + $$

$$+2\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1} q^{n(n+2r-1)}\cos(2r-1)z + 4\frac{\vartheta_2^5}{\vartheta_0}\sum_{n,r}(-1)^{n+r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z.$$

$$(200)\quad \frac{\vartheta_1'^3\vartheta_3\vartheta_2^3(z)}{\vartheta_0^3\vartheta_3^3(z)\vartheta_1(z)} = 2\sum_{n,r}(-1)^r[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \; + $$

$$+2\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \frac{\vartheta_2^5}{\vartheta_0}\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^n q^{n^2+2nr}\sin 2rz\right\}.$$

$$(201)\quad \frac{\vartheta_1'^3\vartheta_3\vartheta_1^3(z)}{\vartheta_0^3\vartheta_0^3(z)\vartheta_2(z)} = -2\sum_{n,r}[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \; + $$

$$+2\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \frac{\vartheta_2^5}{\vartheta_0}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1} q^{n^2+2nr}\sin 2rz\right\}$$

From these follow

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^3(z)}{\vartheta_3^3\vartheta_1^3(z)\vartheta_0(z)} = \frac{1}{\sin^3 z} + \frac{1}{2\sin z}\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} - 2\sum q^N\left\{(\beta+b)^2\sin(\beta-b)z\right\} +$$

(194.1)

$$+2\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{\sin(\beta-b)z\right\} + 4\frac{\vartheta_2^5}{\vartheta_3}\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0^3(z)}{\vartheta_3^3\vartheta_2^3(z)\vartheta_3(z)} = \frac{1}{\cos^3 z} + \frac{1}{2\cos z}\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} + 2\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)^2\cos(\beta-b)z\right\} +$$

(195.1)

$$+2\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\} + 4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}\cos\frac{\gamma-c}{2}z\right\}\frac{\vartheta_2^5}{\vartheta_3}.$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^3(z)}{\vartheta_3^3\vartheta_0^3(z)\vartheta_1(z)} = -\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\delta+d)^2\sin\frac{\delta-d}{2}z\right\} + 2\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-d}{2}z\right\} +$$

(196.1)

$$+\frac{\vartheta_2^5}{\vartheta_3}\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1^3(z)}{\vartheta_3^3\vartheta_3^3(z)\vartheta_2(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)^2\sin\frac{\delta-d}{2}z\right\} + 2\left\{3\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-4}{4}}\sin\frac{\delta-d}{2}z\right\} +$$

(197.1)

$$+\frac{\vartheta_2^5}{\vartheta_3}\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0^3(z)}{\vartheta_0^3\vartheta_1^3(z)\vartheta_3(z)} = \frac{1}{\sin^3 z} + \frac{1}{2\sin z}\left\{3\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} - 2\sum q^N\left\{(\beta+b)^2\sin(\beta-b)z\right\} +$$

(198.1)

$$+2\left\{3\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{\sin(\beta-b)z\right\} + 4\frac{\vartheta_2^5}{\vartheta_0}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}\sin\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3^3(z)}{\vartheta_0^3\vartheta_2^3(z)\vartheta_0(z)} = \frac{1}{\cos^3 z} + \frac{1}{2\cos z}\left\{3\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} + 2\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)^2\cos(\beta-b)z\right\} +$$

(199.1)

$$+2\left\{3\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\} + 4\frac{\vartheta_2^5}{\vartheta_0}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2^3(z)}{\vartheta_0^3\vartheta_3^3(z)\vartheta_1(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)^2\sin\frac{\delta-d}{2}z\right\} + 2\left\{3\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-4}{4}}\sin\frac{\delta-d}{2}z\right\} +$$

(200.1)

$$+\frac{\vartheta_2^5}{\vartheta_0}\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{(-1)^a\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1^3(z)}{\vartheta_0\vartheta_0^3(z)\vartheta_2(z)} = -\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\delta+d)^2\sin\frac{\delta-d}{2}z\right\} + 2\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-d}{2}z\right\} +$$

(201.1)

$$+\frac{\vartheta_2^5}{\vartheta_0}\left\{\frac{\sin z}{\cos z} + 4\sum q^{N}\left\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

Group XII-c

$$\frac{\vartheta_0^3(z)}{\vartheta_1^3(z)\vartheta_2(z)} \qquad \frac{\vartheta_3^3(z)}{\vartheta_2^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_0^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_2^3(z)}{\vartheta_3^3(z)\vartheta_0(z)}$$

$$\frac{\vartheta_3^3(z)}{\vartheta_1^3(z)\vartheta_2(z)} \qquad \frac{\vartheta_0^3(z)}{\vartheta_2^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_3^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_2^3(z)}{\vartheta_0^3(z)\vartheta_3(z)}$$

Consider

$$F(z) = \frac{\vartheta_0^3(z)}{\vartheta_1^3(z)\vartheta_2(z)}$$

Let $t = z+\frac{\pi\tau}{2}$, $F(z)\equiv\varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders three and one at $t=\frac{\pi\tau}{2}$ and $t=\frac{\pi\tau}{2}+\frac{\pi}{2}$ respectively. Calculating the corresponding $R_i^{(j)}$ we find

$$\frac{\vartheta_1'^3\vartheta_2}{\vartheta_0^3}F(z) = \frac{1}{2}A_1^{(2)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right) + \frac{1}{2}\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}A_1^{(0)}\left(z+\frac{\pi\tau}{2},\frac{\pi}{2}\right)$$

(202)

$$-\frac{\vartheta_3^5}{\vartheta_0}A_1^{(0)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}+\frac{\pi}{2}\right)$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0^3(z)}{\vartheta_0^3\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} + \frac{\cos z}{2\sin z}\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\} - 2\sum_{n,r}[2(n+r)]^2 q^{n^2+2nr}\sin 2rz \quad +$$

(203)

$$+2\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}q^{n^2+2nr}\sin 2rz \quad + \frac{\vartheta_3^5}{\vartheta_0}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1}q^{n^2+2nr}\sin 2rz\right\}.$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3^3(z)}{\vartheta_0^3\vartheta_2^3(z)\vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + \frac{\sin z}{2\cos z}\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\} + 2\sum_{n,r}(-1)^r[2(n+r)]^2 q^{n^2+2nr}\sin 2rz \quad +$$

(204)

$$+2\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1}q^{n^2+2nr}\sin 2rz \quad + \frac{\vartheta_3^5}{\vartheta_0}\left\{\frac{\cos z}{\sin z}+4\sum(-1)^n q^{n^2+2nr}\sin 2rz\right\}.$$

In (202) replace z by $z-\frac{\pi\tau}{2}$. There results

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1^3(z)e^{-iz}}{\vartheta_0^3\vartheta_0^3(z)\vartheta_3(z)} = \frac{1}{2}q^{-1/4}A_1^{(2)}\left(z,\frac{\pi\tau}{2}\right) + \frac{1}{2}q^{-1/4}\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}A_1^{(0)}\left(z,\frac{\pi\tau}{2}\right) +$$

$$- q^{-1/4}\frac{\vartheta_3^5}{\vartheta_0}A_1^{(0)}\left(z,\frac{\pi\tau}{2}+\frac{\pi}{2}\right)$$

This gives

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_2\vartheta_1^3(z)}{\vartheta_0^3\vartheta_0^3(z)\vartheta_3(z)} &= -2\sum_{n,r}[2(n+r-1)]^2 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; + \\ &+2\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; + 4\frac{\vartheta_3^5}{\vartheta_0}\sum(-1)^{n+r}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \,.\end{aligned} \tag{205}$$

Replacing z by $z+\frac{\pi}{2}$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_2\vartheta_2^3(z)}{\vartheta_0^3\vartheta_3^3(z)\vartheta_0(z)} &= 2\sum_{n,r}(-1)^r[2(n+r-1)]^2 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; + \\ &+2\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; + 4\frac{\vartheta_3^5}{\vartheta_0}\sum_{n,r}(-1)^{n+1}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \,.\end{aligned} \tag{206}$$

Replacing q by $-q$ in these results in

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_2\vartheta_3^3(z)}{\vartheta_3^3\vartheta_1^3(z)\vartheta_2(z)} &= \frac{\cos z}{\sin^3 z} + \frac{\cos z}{2\sin z}\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\} + 2\sum_{n,r}(-1)^{n+1}[2(n+r)]^2 q^{n^2+2nr}\sin 2rz \; + \\ &+2\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^n q^{n^2+2nr}\sin 2rz \; + \frac{\vartheta_0^5}{\vartheta_3}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{n^2+2nr}\sin 2rz\right\} ,\end{aligned} \tag{207}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_2\vartheta_0^3(z)}{\vartheta_3^3\vartheta_2^3(z)\vartheta_1(z)} &= \frac{\sin z}{\cos^3 z} + \frac{\sin z}{2\cos z}\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\} + 2\sum_{n,r}(-1)^{n+r}[2(n+r)]^2 q^{n^2+2nr}\sin 2rz \; + \\ &+2\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1}q^{n^2+2nr}\sin 2rz \; + \frac{\vartheta_0^5}{\vartheta_3}\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}q^{n^2+2nr}\sin 2rz\right\} .\end{aligned} \tag{208}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_2\vartheta_1^3(z)}{\vartheta_3^3\vartheta_3^3(z)\vartheta_0(z)} &= 2\sum_{n,r}(-1)^{n+r}[2(n+r-1)]^2 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; + \\ &+2\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; - 4\frac{\vartheta_0^5}{\vartheta_3}\sum q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \,.\end{aligned} \tag{209}$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_2^3(z)}{\vartheta_3^3\vartheta_0^3(z)\vartheta_3(z)} = 2\sum_{n,r}(-1)^{n+1}[2(n+r-1)]^2\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

$$+2\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; + 4\frac{\vartheta_0^5}{\vartheta_3}\sum_{n,r}(-1)^r q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \;. \tag{210}$$

There follows

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0^3(z)}{\vartheta_0^3\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} - 2\sum q^N\left\{(\alpha+a)^2\sin(\alpha-a)z\right\} \; +$$

$$+\frac12\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{\sin(\alpha-a)z\right\}\right\} + \frac{\vartheta_3^5}{\vartheta_0}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\right\}\right\}. \tag{203.1}$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3^3(z)}{\vartheta_0^3\vartheta_2^3(z)\vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha-a}{2}}(\alpha+a)^2\sin(\alpha-a)z\right\} \; +$$

$$+\frac12\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\right\}\right\} + \frac{\vartheta_3^5}{\vartheta_0}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^a\sin(\alpha-a)z\right\}\right\} \tag{204.1}$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1^3(z)}{\vartheta_0^3\vartheta_0^3(z)\vartheta_3(z)} = -\frac12\sum q^{\frac N4}\left\{(\gamma+c)^2\sin\frac{\gamma-c}{2}z\right\} \; +$$

$$+2\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac N4}\left\{\sin\frac{\gamma-c}{2}z\right\} + 4\frac{\vartheta_3^5}{\vartheta_0}\sum q^{\frac N4}\left\{(-1)^{\frac{\gamma+c+4}{4}}\sin\frac{\gamma-c}{2}z\right\}. \tag{205.1}$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_2^3(z)}{\vartheta_0^3\vartheta_3^3(z)\vartheta_0(z)} = \frac12\sum q^{\frac N4}\left\{(-1)^{\frac{\gamma-c+2}{4}}(\gamma+c)^2\cos\frac{\gamma-c}{2}z\right\} \; +$$

$$+2\left\{3\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac N4}\left\{(-1)^{\frac{\gamma-c-2}{4}}\cos\frac{\gamma-c}{2}z\right\} + 4\frac{\vartheta_3^5}{\vartheta_0}\sum q^{\frac N4}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-c}{2}z\right\}. \tag{206.1}$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3^3(z)}{\vartheta_3^3\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} + 2\sum q^N\left\{(-1)^{a+1}(\alpha+a)^2\sin(\alpha-a)z\right\} \; +$$

$$+\frac12\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^a\sin(\alpha-a)z\right\}\right\} + \frac{\vartheta_0^5}{\vartheta_3}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\right\}\right\}. \tag{207.1}$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0^3(z)}{\vartheta_3^3\vartheta_2^3(z)\vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha+a}{2}}(\alpha+a)^2\sin(\alpha-a)z\right\} \; +$$

$$+\frac12\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha+a-2}{2}}\sin(\alpha-a)z\right\}\right\} + \frac{\vartheta_0^5}{\vartheta_3}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{\sin(\alpha-a)z\right\}\right\}. \tag{208.1}$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1^3(z)}{\vartheta_3^3\vartheta_3^3(z)\vartheta_0(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}(\gamma+c)^2\sin\frac{\gamma-c}{2}z\right\} +$$

(209.1)

$$+2\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}\sin\frac{\gamma-c}{2}z\right\} - 4\frac{\vartheta_0^5}{\vartheta_3}\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_2^3(z)}{\vartheta_3^3\vartheta_0^3(z)\vartheta_3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+c)^2\cos\frac{\gamma-c}{2}z\right\}$$

(210.1)

$$+2\left\{3\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-c}{2}z\right\} + 4\frac{\vartheta_0^5}{\vartheta_3}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+2-c}{4}}\cos\frac{\gamma-c}{2}z\right\}.$$

Group XIII-a

$$\frac{\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_1^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_0^2(z)\vartheta_1(z)}{\vartheta_2^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_2^2(z)\vartheta_3(z)}{\vartheta_0^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_1^2(z)\vartheta_0(z)}{\vartheta_3^3(z)\vartheta_2(z)}$$

$$\frac{\vartheta_0^2(z)\vartheta_2(z)}{\vartheta_1^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_3^2(z)\vartheta_1(z)}{\vartheta_2^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_2^2(z)\vartheta_0(z)}{\vartheta_3^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_0^3(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_1^3(z)\vartheta_0(z)}$$

Let $t = z+\frac{\pi\tau}{2}$, $F(z)\equiv\varphi(t)$. $\varphi(t)$ satisfies (8) and possesses poles of orders three and one at $t=\frac{\pi\tau}{2}$ and $t=\pi\tau$ respectively. Calculating the corresponding $A_i^{(j)}$ and using (10) gives

$$(211)\qquad \frac{\vartheta_1'^3\vartheta_0}{\vartheta_3^2\vartheta_2}F(z) = \frac{1}{2}A_1^{(2)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right) + \frac{1}{2}\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\}A_1^{(0)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right) + q^{\frac{1}{4}}\vartheta_3\vartheta_2^3A_1^{(0)}\left(z+\frac{\pi\tau}{2},\pi\tau\right)$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_3^2\vartheta_2\vartheta_1^3(z)\vartheta_0(z)} = \frac{\cos z}{\sin^3 z} - 2\sum_{n,r}[2(n+r)]^2q^{n^2+2nr}\sin 2rz + \frac{1}{2}\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\frac{\cos z}{\sin z} +$$

(212)

$$+2\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{n^2+2nr}\sin 2rz + 4\vartheta_2^3\vartheta_3\sum_{n,r}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0^2(z)\vartheta_1(z)}{\vartheta_3^2\vartheta_2\vartheta_2^3(z)\vartheta_3(z)} = \frac{\sin z}{\cos^3 z} + \frac{\sin z}{2\cos z}\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\} + 2\sum_{n,r}(-1)^r[2(n+r)]^2q^{n^2+2nr}\sin 2rz +$$

(213)

$$+2\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r}(-1)^{r+1}q^{n^2+2nr}\sin 2rz + 4\vartheta_2^3\vartheta_3\sum(-1)^{r+1}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz.$$

In (211) replace z by $z-\frac{\pi\tau}{2}$. This gives

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^2(z)\vartheta_3(z)e^{-iz}}{\vartheta_3^2\vartheta_2\vartheta_0^3(z)\vartheta_1(z)}q^{\frac14}=\frac12 A_1^{(2)}(z,\tfrac{\pi\tau}{2})+\frac12\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\}A_1^{(0)}(z,\tfrac{\pi\tau}{2})+q^{\frac14}\vartheta_2^3\vartheta_3A_1^{(0)}(z,\pi\tau),$$

and from this

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^2(z)\vartheta_3(z)}{\vartheta_3^2\vartheta_2\vartheta_0^3(z)\vartheta_1(z)}=-2\sum_{n,r}[2(n+r-1)]^2q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z+$$

(214)

$$+2\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z+\vartheta_2^3\vartheta_3\left\{\frac{1}{\sin z}+4\sum_{n,r}q^{n(n+2r-1)}\sin(2r-1)z\right\}.$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1^2(z)\vartheta_0(z)}{\vartheta_3^2\vartheta_2\vartheta_3^3(z)\vartheta_2(z)}=2\sum_{n,r}(-1)^r[2(n+r-1)]^2q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z+$$

(215)

$$+2\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r}(-1)^{r+1}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z+\vartheta_2^3\vartheta_3\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{n(n+2r-1)}\cos(2r-1)z\right\}.$$

In these results replace q by $-q$. Hence

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0^2(z)\vartheta_2(z)}{\vartheta_0^2\vartheta_2\vartheta_1^3(z)\vartheta_3(z)}=\frac{\cos z}{\sin^3 z}+2\sum_{n,r}(-1)^{n+1}[2(n+r)]^2q^{n^2+2nr}\sin 2rz+$$

(216)

$$+\frac12\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^nq^{n^2+2nr}\sin 2rz\right\}+4\vartheta_0\vartheta_2^3\sum_{n,r}(-1)^{r+1}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3^2(z)\vartheta_1(z)}{\vartheta_0^2\vartheta_2\vartheta_2^3(z)\vartheta_0(z)}=\frac{\sin z}{\cos^3 z}+2\sum_{n,r}(-1)^{n+r}[2(n+r)]^2q^{n^2+2nr}\sin 2rz+$$

(217)

$$+\frac12\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1}q^{n^2+2nr}\sin 2rz\right\}+4\vartheta_2^3\vartheta_0\sum_{n,r}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2^2(z)\vartheta_0(z)}{\vartheta_0^2\vartheta_2\vartheta_3^3(z)\vartheta_1(z)}=2\sum_{n,r}(-1)^{n+r+1}[2(n+r-1)]^2q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z+$$

(218)

$$+2\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^{n+r}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z+\vartheta_2^3\vartheta_0\left\{\frac{1}{\sin z}+4\sum_{n,r}q^{n(n+2r-1)}\sin(2r-1)z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_0^2\vartheta_2\vartheta_0^3(z)\vartheta_2(z)}=2\sum_{n,r}(-1)^n[2(n+r-1)]^2q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z+$$

(219)

$$+2\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^{n+1}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z+\vartheta_0\vartheta_2^3\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{n(n+2r-1)}\cos(2r-1)z\right\}.$$

These give

(212.1)
$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_3^2\vartheta_2\vartheta_1^3(z)\vartheta_0(z)} = \frac{\cos z}{\sin^3 z} - 2\sum q^N\left\{(\alpha+a)^2\sin(\alpha-a)z\right\} + \frac{1}{2}\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{\sin(\alpha-a)z\right\}\right\} + 4\vartheta_2^3\vartheta_3\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-d}{2}z\right\}.$$

(213.1)
$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0^2(z)\vartheta_1(z)}{\vartheta_3^2\vartheta_2\vartheta_2^3(z)\vartheta_3(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha-a}{2}}(\alpha+a)^2\sin(\alpha-a)z\right\} + \frac{1}{2}\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\right\}\right\} + 4\vartheta_2^3\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-4}{4}}\sin\frac{\delta-d}{2}z\right\}.$$

(214.1)
$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^2(z)\vartheta_3(z)}{\vartheta_3^2\vartheta_2\vartheta_0^3(z)\vartheta_1(z)} = -\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\gamma+c)^2\sin\frac{\gamma-c}{2}z\right\} + 2\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-c}{2}z\right\} + \vartheta_2^3\vartheta_3\left\{\frac{1}{\sin z}+4\sum q^N\left\{\sin(\beta-b)z\right\}\right\}.$$

(215.1)
$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1^2(z)\vartheta_0(z)}{\vartheta_3^2\vartheta_2\vartheta_3^3(z)\vartheta_2(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}(\gamma+c)^2\cos\frac{\gamma-c}{2}z\right\} + 2\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}\cos\frac{\gamma-c}{2}z\right\} + \vartheta_2^3\vartheta_3\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\}\right\}.$$

(216.1)
$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0^2(z)\vartheta_2(z)}{\vartheta_0^2\vartheta_2\vartheta_1^3(z)\vartheta_3(z)} = \frac{\cos z}{\sin^3 z} + 2\sum q^N\left\{(-1)^{a+1}(\alpha+a)^2\sin(\alpha-a)z\right\} + \frac{1}{2}\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^{a}\sin(\alpha-a)z\right\}\right\} + 4\vartheta_2^3\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-4}{4}}\sin\frac{\delta-d}{2}z\right\}.$$

(217.1)
$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3^2(z)\vartheta_1(z)}{\vartheta_0^2\vartheta_2\vartheta_2^3(z)\vartheta_0(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha+a}{2}}(\alpha+a)^2\sin(\alpha-a)z\right\} + \frac{1}{2}\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\right\}\right\} + 4\vartheta_2^3\vartheta_0\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-d}{2}z\right\}.$$

(218.1)
$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2^2(z)\vartheta_0(z)}{\vartheta_0^2\vartheta_2\vartheta_3^3(z)\vartheta_1(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}(\gamma+c)^2\sin\frac{\gamma-c}{2}z\right\} + 2\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}\sin\frac{\gamma-c}{2}z\right\} + \vartheta_2^3\vartheta_0\left\{\frac{1}{\sin z}+4\sum q^N\left\{\sin(\beta-b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_0^2\vartheta_2\vartheta_0^3(z)\vartheta_2(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+c)^2\cos\frac{\gamma-c}{2}z\right\} +$$

(219.1)

$$+2\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-c}{2}z\right\} + \vartheta_2^3\vartheta_0\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\}\right\}.$$

Group XIII-b

$$\frac{\vartheta_2^2(z)\vartheta_3(z)}{\vartheta_1^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_1^2(z)\vartheta_0(z)}{\vartheta_2^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_0^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_0^2(z)\vartheta_1(z)}{\vartheta_3^3(z)\vartheta_2(z)}$$

$$\frac{\vartheta_2^2(z)\vartheta_0(z)}{\vartheta_1^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_2^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_0^2(z)\vartheta_2(z)}{\vartheta_3^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_3^2(z)\vartheta_1(z)}{\vartheta_0^3(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^2(z)\vartheta_3(z)e^{-iz}}{\vartheta_1^3(z)\vartheta_0(z)}$$

$F(z)$ satisfies (8) and has poles of orders three and one at $z=0$ and $z=\frac{\pi\tau}{2}$ respectively. From the corresponding we find

(220) $$\frac{\vartheta_1'^3\vartheta_0}{\vartheta_2^2\vartheta_3}F(z) = \frac{1}{2}A_1^{(2)}(z,0) - iA_1^{(1)}(z,0) + \frac{1}{2}\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}-1\right\}A_1^{(0)}(z,0) + q^{-\frac{1}{4}}\vartheta_3^3\vartheta_2 A_1^{(0)}\left(z,\frac{\pi\tau}{2}\right)$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^2(z)\vartheta_3(z)}{\vartheta_2^2\vartheta_3\vartheta_1^3(z)\vartheta_0(z)} = \frac{1}{\sin^3 z} + \frac{1}{2\sin z}\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}-1\right\} - 2\sum_{n,r}[2(n+r)-1]^2 q^{n(n+2r-1)}\sin(2r-1)z +$$

(221)

$$+2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r} q^{n(n+2r-1)}\sin(2r-1)z + 4\vartheta_3^3\vartheta_2\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z.$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1^2(z)\vartheta_0(z)}{\vartheta_2^2\vartheta_3\vartheta_2^3(z)\vartheta_3(z)} = \frac{1}{\cos^3 z} + \frac{1}{2\cos z}\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}-1\right\} + 2\sum_{n,r}(-1)^r[2(n+r)-1]^2 q^{n(n+2r-1)}\cos(2r-1)z +$$

(222)

$$+2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r}(-1)^{r+1} q^{n(n+2r-1)}\cos(2r-1)z + 4\vartheta_3^3\vartheta_2\sum_{n,r}(-1)^{r+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z.$$

In (220) replace z by $z+\frac{\pi\tau}{2}$. This gives

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_2^2\vartheta_3\vartheta_0^3(z)\vartheta_1(z)}q^{-\frac{1}{4}} = \frac{1}{2}A_1^{(2)}\left(z+\frac{\pi\tau}{2},0\right) - iA_1^{(1)}\left(z+\frac{\pi\tau}{2},0\right) +$$

$$+\frac{1}{2}\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}-1\right\}A_1^{(0)}\left(z+\frac{\pi\tau}{2},0\right) + q^{-\frac{1}{4}}\vartheta_3^3\vartheta_2 A_1^{(0)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right)$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_2^2\vartheta_3\vartheta_0^3(z)\vartheta_1(z)} = -2\sum_{n,r}[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz +$$

(223)

$$+2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \vartheta_3^3\vartheta_2\left\{\frac{\cos z}{\sin z}+4\sum_{n,r} q^{n^2+2nr}\sin 2rz\right\}.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0^2(z)\vartheta_1(z)}{\vartheta_2^2\vartheta_3\vartheta_3^3(z)\vartheta_2(z)} = 2\sum_{n,r}(-1)^r[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz$$

(224)

$$+2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r}(-1)^{r+1} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \vartheta_3^3\vartheta_2\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{r+1} q^{n^2+2nr}\sin 2rz\right\}.$$

These give , on replacing q by $-q$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2^2(z)\vartheta_0(z)}{\vartheta_2^2\vartheta_0\vartheta_1^3(z)\vartheta_3(z)} = \frac{1}{\sin^3 z}+\frac{1}{2\sin z}\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_3''}{\vartheta_3}-1\right\}-2\sum_{n,r}[2(n+r)-1]^2 q^{n(n+2r-1)}\sin(2r-1)z +$$

(225)

$$+2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r} q^{n(n+2r-1)}\sin(2r-1)z + 4\vartheta_0^3\vartheta_2\sum_{n,r}(-1)^{n+r+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_2^2\vartheta_0\vartheta_2^3(z)\vartheta_0(z)} = \frac{1}{\cos^3 z}+\frac{1}{2\cos z}\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_3''}{\vartheta_3}-1\right\}+2\sum_{n,r}(-1)^r[2(n+r)-1]^2 q^{n(n+2r-1)}\cos(2r-1)z +$$

(226)

$$+2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^{r+1} q^{n(n+2r-1)}\cos(2r-1)z + 4\vartheta_0^3\vartheta_2\sum_{n,r}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0^2(z)\vartheta_2(z)}{\vartheta_2^2\vartheta_0\vartheta_3^3(z)\vartheta_1(z)} = 2\sum_{n,r}(-1)^{r+1}[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz +$$

(227)

$$+2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^r q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \vartheta_0^3\vartheta_2\left\{\frac{\cos z}{\sin z}+4\sum(-1)^n q^{n^2+2nr}\sin 2rz\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3^2(z)\vartheta_1(z)}{\vartheta_2^2\vartheta_0\vartheta_0^3(z)\vartheta_2(z)} = 2\sum_{n,r}[2(n+r)-1]^2 q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz$$

(228)

$$-2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \vartheta_0^3\vartheta_2\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1} q^{n^2+2nr}\sin 2rz\right\}.$$

From these follow

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^2(z)\vartheta_3(z)}{\vartheta_2^2\vartheta_3\vartheta_1^3(z)\vartheta_0(z)} = \frac{1}{\sin^3 z}+\frac{1}{2\sin z}\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}-1\right\}-2\sum q^N\left\{(\beta+b)^2\sin(\beta-b)z\right\} +$$

(221.1)

$$+2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^N\left\{\sin(\beta-b)z\right\} + 4\vartheta_3^3\vartheta_2\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-c}{2}z\right\}.$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_0\vartheta_1^2(z)\vartheta_0(z)}{\vartheta_2^2\vartheta_3\vartheta_2^3(z)\vartheta_3(z)} &= \frac{1}{\cos^3 z} + \frac{1}{2\cos z}\left\{\frac{\vartheta_2''}{\vartheta_2} - 2\frac{\vartheta_0''}{\vartheta_0} - 1\right\} + 2\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)^2\cos(\beta-b)z\right\} + \\ &+ 2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\} + 4\vartheta_3^3\vartheta_2\sum q^{\frac N4}\left\{(-1)^{\frac{\gamma-c-2}{4}}\cos\frac{\gamma-c}{2}z\right\}.\end{aligned} \tag{222.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_0\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_2^2\vartheta_3\vartheta_0^3(z)\vartheta_1(z)} &= -\frac12\sum q^{\frac N4}\left\{(\delta+d)^2\sin\frac{\delta-d}{2}z\right\} + \\ &+ 2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{\frac N4}\left\{\sin\frac{\delta-d}{2}z\right\} + \vartheta_3^3\vartheta_2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(\alpha-a)z\right\}\right\}.\end{aligned} \tag{223.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_0\vartheta_0^2(z)\vartheta_1(z)}{\vartheta_2^2\vartheta_3\vartheta_3^3(z)\vartheta_2(z)} &= \frac12\sum q^{\frac N4}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)^2\sin\frac{\delta-d}{2}z\right\} + \\ &+ 2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{\frac N4}\left\{(-1)^{\frac{\delta-d-4}{4}}\sin\frac{\delta-d}{2}z\right\} + \vartheta_3^3\vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\right\}\right\}.\end{aligned} \tag{224.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_3\vartheta_2^2(z)\vartheta_0(z)}{\vartheta_2^2\vartheta_0\vartheta_1^3(z)\vartheta_3(z)} &= \frac{1}{\sin^3 z} + \frac{1}{2\sin z}\left\{\frac{\vartheta_2''}{\vartheta_2} - 2\frac{\vartheta_3''}{\vartheta_3} - 1\right\} - 2\sum q^N\left\{(\beta+b)^2\sin(\beta-b)z\right\} + \\ &+ 2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^N\left\{\sin(\beta-b)z\right\} + 4\vartheta_0^3\vartheta_2\sum q^{\frac N4}\left\{(-1)^{\frac{\gamma+c}{4}}\sin\frac{\gamma-c}{2}z\right\}.\end{aligned} \tag{225.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_3\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_2^2\vartheta_0\vartheta_2^3(z)\vartheta_0(z)} &= \frac{1}{\cos^3 z} + \frac{1}{2\cos z}\left\{\frac{\vartheta_2''}{\vartheta_2} - 2\frac{\vartheta_3''}{\vartheta_3} - 1\right\} + 2\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)^2\cos(\beta-b)z\right\} + \\ &+ 2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\} + 4\vartheta_0^3\vartheta_2\sum q^{\frac N4}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-c}{2}z\right\}.\end{aligned} \tag{226.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_3\vartheta_0^2(z)\vartheta_2(z)}{\vartheta_2^2\vartheta_0\vartheta_3^3(z)\vartheta_1(z)} &= \frac12\sum q^{\frac N4}\left\{(-1)^{\frac{\delta-d-4}{4}}(\delta+d)^2\sin\frac{\delta-d}{2}z\right\} \\ &+ 2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac N4}\left\{(-1)^{\frac{\delta-d}{4}}\sin\frac{\delta-d}{2}z\right\} + \vartheta_0^3\vartheta_2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{(-1)^a\sin(\alpha-a)z\right\}\right\}.\end{aligned} \tag{227.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_3\vartheta_3^2(z)\vartheta_1(z)}{\vartheta_2^2\vartheta_0\vartheta_0^3(z)\vartheta_2(z)} &= \frac12\sum q^{\frac N4}\left\{(\delta+d)^2\sin\frac{\delta-d}{2}z\right\} + \\ &- 2\left\{\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac N4}\left\{\sin\frac{\delta-d}{2}z\right\} + \vartheta_0^3\vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\right\}\right\}.\end{aligned} \tag{228.1}$$

$$\frac{\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_1^3(z)\vartheta_2(z)} \qquad \frac{\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_2^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_0^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_3^3(z)\vartheta_0(z)}$$

$$\frac{\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_1^3(z)\vartheta_2(z)} \qquad \frac{\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_2^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_3^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_0^3(z)\vartheta_3(z)}$$

Consider
$$F(z) = \frac{\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_1^3(z)\vartheta_2(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders three and one at $t = \frac{\pi\tau}{2}$ and $t = \frac{\pi\tau}{2} + \frac{\pi}{2}$ respectively. Calculating the corresponding $R_i^{(i)}$ and using (10) gives

$$\frac{\vartheta_1'^3\vartheta_2}{\vartheta_3^2\vartheta_0}F(z) = \frac{1}{2}A_1^{(2)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) + \frac{1}{2}\left\{\frac{\vartheta_3''}{\vartheta_3} - 2\frac{\vartheta_2''}{\vartheta_2}\right\}A_1^{(0)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) - \vartheta_0^3\vartheta_3 A_1^{(0)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}+\frac{\pi}{2}\right) \tag{229}$$

from which follows

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_2\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_3^2\vartheta_0\vartheta_1^3(z)\vartheta_2(z)} &= \frac{\cos z}{\sin^3 z} - 2\sum_{n,r}[2(n+r)]^2 q^{n^2+2nr}\sin 2rz \;+ \\ &+ \frac{1}{2}\left\{\frac{\vartheta_3''}{\vartheta_3} - 2\frac{\vartheta_2''}{\vartheta_2}\right\}\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{n^2+2nr}\sin 2rz\right\} + \vartheta_0^3\vartheta_3\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n^2+2nr}\sin 2rz\right\}\end{aligned} \tag{230}$$

Replacing z by $z + \frac{\pi}{2}$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_2\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_3^2\vartheta_0\vartheta_2^3(z)\vartheta_1(z)} &= \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^r[2(n+r)]^2 q^{n^2+2nr}\sin 2rz \;+ \\ &+ \frac{1}{2}\left\{\frac{\vartheta_3''}{\vartheta_3} - 2\frac{\vartheta_2''}{\vartheta_2}\right\}\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{n^2+2nr}\sin 2rz\right\} + \vartheta_0^3\vartheta_3\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^n q^{n^2+2nr}\sin 2rz\right\}\end{aligned} \tag{231}$$

In (229) replace z by $z - \frac{\pi\tau}{2}$, obtaining

$$-\frac{\vartheta_1'^3\vartheta_2\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_3^2\vartheta_0\vartheta_0^3(z)\vartheta_3(z)}e^{-iz}q^{\frac{1}{4}} = \frac{1}{2}A_1^{(2)}\left(z, \frac{\pi\tau}{2}\right) + \frac{1}{2}\left\{\frac{\vartheta_3''}{\vartheta_3} - 2\frac{\vartheta_2''}{\vartheta_2}\right\}A_1^{(0)}\left(z, \frac{\pi\tau}{2}\right) - \vartheta_0^3\vartheta_3 A_1^{(0)}\left(z, \frac{\pi\tau}{2}+\frac{\pi}{2}\right).$$

Which gives

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_2\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_3^2\vartheta_0\vartheta_0^3(z)\vartheta_3(z)} &= 2\sum_{n,r}[2(n+r-1)]^2 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \;+ \\ &-2\left\{\frac{\vartheta_3''}{\vartheta_3} - 2\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z + 4\vartheta_0^3\vartheta_3\sum_{n,r}(-1)^{n+r+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z\end{aligned} \tag{232}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_3^2\vartheta_0\vartheta_3^3(z)\vartheta_0(z)} = 2\sum_{n,r}(-1)^{r+1}[2(n+r-1)]^2 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(233)

$$+2\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r}(-1)^r q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + 4\vartheta_0^3\vartheta_3\sum_{n,r}(-1)^n q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z .$$

In these results replace q by $-q$. Hence

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_0^2\vartheta_3\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} + 2\sum_{n,r}(-1)^{n+1}[2(n+r)]^2 q^{n^2+2nr}\sin 2rz \; +$$

(234)

$$+\frac{1}{2}\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^n q^{n^2+2nr}\sin 2rz\right\}+\vartheta_3^3\vartheta_0\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{n^2+2nr}\sin 2rz\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_0^2\vartheta_3\vartheta_2^3(z)\vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^{n+r}[2(n+r)]^2 q^{n^2+2nr}\sin 2rz \; +$$

(235)

$$+\frac{1}{2}\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1}q^{n^2+2nr}\sin 2rz\right\}+\vartheta_3^3\vartheta_0\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}q^{n^2+2nr}\sin 2rz\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_0^2\vartheta_3\vartheta_3^3(z)\vartheta_0(z)} = 2\sum_{n,r}(-1)^{r+n+1}[2(n+r-1)]^2 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(236)

$$+2\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r}(-1)^{r+n} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z + 4\vartheta_3^3\vartheta_0\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z ;$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_0^2\vartheta_3\vartheta_0^3(z)\vartheta_3(z)} = 2\sum_{n,r}(-1)^n[2(n+r-1)]^2 q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(237)

$$+2\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r}(-1)^{n+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + 4\vartheta_3^3\vartheta_0\sum_{n,r}(-1)^{r+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z .$$

From these follow

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_3^2\vartheta_0\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} - 2\sum q^N\{(\alpha+a)\sin(\alpha-a)z\} \; +$$

(230.1)

$$+\frac{1}{2}\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\{\sin(\alpha-a)z\}\right\}+\vartheta_0^3\vartheta_3\left\{\frac{\sin z}{\cos z}+4\sum q^N\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_3^2\vartheta_0\vartheta_2^3(z)\vartheta_1(z)}=\frac{\sin z}{\cos^3 z}+2\sum q^N\left\{(-1)^{\frac{\alpha-a}{2}}(\alpha+a)^2\sin(\alpha-a)z\right\}+$$

(231.1)

$$+\frac{1}{2}\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\right\}\right\}+\vartheta_0^3\vartheta_3\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^a\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_3^2\vartheta_0\vartheta_0^3(z)\vartheta_3(z)}=\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\gamma+c)^2\sin\frac{\gamma-c}{2}z\right\}+$$

(232.1)

$$-2\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^{\frac{N}{4}}\left\{\sin(\gamma-c)z\right\}+4\vartheta_0^3\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}\sin\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_3^2\vartheta_0\vartheta_3^3(z)\vartheta_0(z)}=\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}(\gamma+c)^2\cos\frac{\gamma-c}{2}z\right\}+$$

(233.1)

$$+2\left\{\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}\cos\frac{\gamma-c}{2}z\right\}+4\vartheta_0^3\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_0^2\vartheta_3\vartheta_1^3(z)\vartheta_2(z)}=\frac{\cos z}{\sin^3 z}+2\sum q^N\left\{(-1)^{a+1}(\alpha+a)^2\sin(\alpha-a)z\right\}+$$

(234.1)

$$+\frac{1}{2}\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^a\sin(\alpha-a)z\right\}\right\}+\vartheta_3^3\vartheta_0\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_0^2\vartheta_3\vartheta_2^3(z)\vartheta_1(z)}=\frac{\sin z}{\cos^3 z}+2\sum q^N\left\{(-1)^{\frac{\alpha+a}{2}}(\alpha+a)^2\sin(\alpha-a)z\right\}+$$

(235.1)

$$+\frac{1}{2}\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\right\}\right\}+\vartheta_3^3\vartheta_0\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{\sin(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_0^2\vartheta_3\vartheta_0(z)\vartheta_3^3(z)}=\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}(\gamma+c)^2\sin\frac{\gamma-c}{2}z\right\}+$$

(236.1)

$$+2\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}\sin\frac{\gamma-c}{2}z\right\}+4\vartheta_3^3\vartheta_0\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_0^2\vartheta_3\vartheta_0^3(z)\vartheta_3(z)}=\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+c)^2\cos\frac{\gamma-c}{2}z\right\}+$$

(237.1)

$$+2\left\{\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-c}{2}z\right\}+4\vartheta_3^3\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}\cos\frac{\gamma-c}{2}z\right\}.$$

Group XIV-a

$$\frac{\vartheta_2^3(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_3^3(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} \qquad \frac{\vartheta_0^3(z)}{\vartheta_2^2(z)\vartheta_3^2(z)}$$

$$\frac{\vartheta_2^3(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_2^2(z)\vartheta_0^2(z)} \qquad \frac{\vartheta_0^3(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_3^3(z)}{\vartheta_2^2(z)\vartheta_0^2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^2(z)e^{-iz}}{\vartheta_1^2(z)\vartheta_3^2(z)}$$

$F(z)$ satisfies (8) and has poles of order two at $z=0$ and $z=\frac{\pi\tau}{2}$
Calculating the corresponding $A_i^{(j)}$ and using (10) gives

$$\vartheta_1'^2\vartheta_0^2 F(z) = \vartheta_2^3\left\{A_1^{(1)}(z,0) - i A_1^{(0)}(z,0)\right\} + \vartheta_3^3 q^{-1/4} A_1^{(1)}\left(z, \frac{\pi\tau}{2}\right) \tag{238}$$

From this follows

$$\begin{aligned}\frac{\vartheta_1'^2\vartheta_0^2\vartheta_2^3(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} &= \vartheta_2^3\left\{\frac{\cos z}{\sin^2 z} - 4\sum_{n,r}[2(n+r)-1]\, q^{n(n+2r-1)}\cos(2r-1)z\right\} + \\ &\quad -4\vartheta_3^3\sum_{n,r} 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z .\end{aligned} \tag{239}$$

Replacing z by $z-\frac{\pi}{2}$

$$\begin{aligned}\frac{\vartheta_1'^2\vartheta_0^2\vartheta_1^3(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} &= \vartheta_2^3\left\{\frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{n(n+2r-1)}\sin(2r-1)z\right\} + \\ &\quad +4\vartheta_3^3\sum_{n,r}(-1)^r 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z .\end{aligned} \tag{240}$$

In (238) replace z by $z+\frac{\pi\tau}{2}$. We get

$$\frac{\vartheta_0^2\vartheta_1'^2\vartheta_3^3(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} = \vartheta_3^3 A_1^{(1)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) + \vartheta_2^3 q^{\frac{1}{4}}\left\{A_1^{(1)}\left(z+\frac{\pi\tau}{2},0\right) - i A_1^{(0)}\left(z+\frac{\pi\tau}{2},0\right)\right\}$$

Hence

$$\begin{aligned}\frac{\vartheta_1'^2\vartheta_0^2\vartheta_3^3(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} &= -2\vartheta_2^3\left\{\sum_n(2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} + \\ &\quad +\vartheta_3^3\left\{\frac{1}{\sin^2 z} - 4\sum_n n\, q^{n^2} - 4\sum 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\} .\end{aligned} \tag{241}$$

Replacing z by $z+\frac{\pi}{2}$

(242)

$$\frac{\vartheta_0^2\vartheta_1'^2\vartheta_0^3(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} = 2\vartheta_2^3\left\{-\sum_n (2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{r+1}[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} +$$

$$+\vartheta_3^3\left\{\frac{1}{\cos^2 z} - 4\sum_n n\, q^{n^2} + 4\sum_{n,r}(-1)^{r+1}\,2(n+r)\, q^{n^2+2nr}\cos 2rz\right\}.$$

In these results replace q by $-q$, obtaining

(243)

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_2^3(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} = \vartheta_2^3\left\{\frac{\cos z}{\sin^2 z} - 4\sum_{n,r}[2(n+r)-1]\, q^{n(n+2r-1)}\cos(2r-1)z\right\} +$$

$$+4\vartheta_0^3\sum_{n,r}(-1)^{n+r+1}\,2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z.$$

(244)

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_1^3(z)}{\vartheta_2^2(z)\vartheta_0^2(z)} = \vartheta_2^3\left\{\frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{n(n+2r-1)}\sin(2r-1)z\right\} +$$

$$+4\vartheta_0^3\sum_{n,r}(-1)^n\,2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

(245)

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_0^3(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} = 2\vartheta_2^3\left\{\sum_n(2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} +$$

$$+\vartheta_0^3\left\{\frac{1}{\sin^2 z} + 4\sum_n(-1)^{n+1} n\, q^{n^2} + 4\sum_{n,r}(-1)^{n+1}\,2(n+r)\, q^{n^2+2nr}\cos 2rz\right\},$$

(246)

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_3^3(z)}{\vartheta_2^2(z)\vartheta_0^2(z)} = 2\vartheta_2^3\left\{\sum_n(2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}$$

$$+\vartheta_0^3\left\{\frac{1}{\cos^2 z} + 4\sum_n(-1)^{n+1} n\, q^{n^2} + 4\sum_{n,r}(-1)^{n+r+1}\,2(n+r)\, q^{n^2+2nr}\cos 2rz\right\}$$

From these follow

(239.1)

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_2^3(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} = \vartheta_2^3\left\{\frac{\cos z}{\sin^2 z} - 4\sum q^N\left\{(\beta+b)\cos(\beta-b)z\right\}\right\} +$$

$$-2\vartheta_3^3\sum q^{\frac{N}{4}}\left\{(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}.$$

(240.1)

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_1^3(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} = \vartheta_2^3\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}\sin(\beta-b)z\right\}\right\}$$

$$+2\vartheta_3^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_0^2 \vartheta_3^3(z)}{\vartheta_1^2(z)\,\vartheta_0^2(z)} = -2\vartheta_2^3\left\{\sum_n (2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\}$$

(241.1)

$$+\vartheta_3^3\left\{\frac{1}{\sin^2 z} - 4\sum_n n\, q^{n^2} - 4\sum q^N\left\{(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_0^2 \vartheta_0^3(z)}{\vartheta_2^2(z)\,\vartheta_3^2(z)} = -2\vartheta_2^3\left\{\sum_n (2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\}+$$

(242.1)

$$+\vartheta_3^3\left\{\frac{1}{\cos^2 z} - 4\sum_n n\, q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_3^2 \vartheta_2^3(z)}{\vartheta_1^2(z)\,\vartheta_3^2(z)} = \vartheta_2^3\left\{\frac{\cos z}{\sin^2 z} - 4\sum q^N\left\{(\beta+b)\cos(\beta-b)z\right\}\right\} +$$

(243.1)

$$+2\vartheta_0^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_3^2 \vartheta_1^3(z)}{\vartheta_0^2(z)\,\vartheta_2^2(z)} = \vartheta_2^3\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)\sin(\beta-b)z\right\}\right\} +$$

(244.1)

$$+2\vartheta_0^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_3^2 \vartheta_0^3(z)}{\vartheta_1^2(z)\,\vartheta_3^2(z)} = 2\vartheta_2^3\left\{\sum_n (2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\}+$$

(245.1)

$$+\vartheta_0^3\left\{\frac{1}{\sin^2 z} + 4\sum_n (-1)^{n+1} n\, q^{n^2} + 4\sum q^N\left\{(-1)^{a+1}(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_3^2 \vartheta_3^3(z)}{\vartheta_2^2(z)\,\vartheta_0^2(z)} = 2\vartheta_2^3\left\{\sum_n (2n-1)\, q^{\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\}+$$

(246.1)

$$+\vartheta_0^3\left\{\frac{1}{\cos^2 z} + 4\sum_n (-1)^{n+1} n\, q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\}.$$

Group XIV-b

$$\frac{\vartheta_1^3(z)}{\vartheta_0^2(z)\,\vartheta_3^2(z)} \qquad \frac{\vartheta_2^3(z)}{\vartheta_0^2(z)\,\vartheta_3^2(z)} \qquad \frac{\vartheta_0^3(z)}{\vartheta_1^2(z)\,\vartheta_2^2(z)} \qquad \frac{\vartheta_3^3(z)}{\vartheta_1^2(z)\,\vartheta_2^2(z)}$$

Consider

$$F(z) = \frac{\vartheta_1^3(z)\,e^{-iz}}{\vartheta_0^2(z)\,\vartheta_3^2(z)}$$

Let $z+\frac{\pi}{2}=t$, $F(z)\equiv\varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of order two at $t=\frac{\pi\tau}{2}+\frac{\pi}{2}$ and $t=\frac{\pi\tau}{2}$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$\vartheta_1'^2\vartheta_2^2 F(z) = i q^{-1/4}\vartheta_0^3 A_1^{(1)}\left(z+\frac{\pi}{2}, \frac{\pi\tau}{2}+\frac{\pi}{2}\right) + i q^{-1/4}\vartheta_3^3 A_1^{(1)}\left(z+\frac{\pi}{2}, \frac{\pi\tau}{2}\right) \tag{247}$$

Hence

$$\begin{aligned}\frac{\vartheta_1'^2\vartheta_2^2\vartheta_1^3(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} &= 4\vartheta_0^3\sum_{n,r}(-1)^n 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \;+ \\ &+ 4\vartheta_3^3\sum_{n,r}(-1)^{n+r+1} 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z\end{aligned} \tag{248}$$

Replacing z by $z+\frac{\pi}{2}$

$$\begin{aligned}\frac{\vartheta_1'^2\vartheta_2^2\vartheta_2^3(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} &= 4\vartheta_0^3\sum_{n,r}(-1)^{n+r+1} 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \;+ \\ &+ 4\vartheta_3^3\sum_{n,r} 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; .\end{aligned} \tag{249}$$

In (247) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0^3(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0^3 A_1^{(1)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2}, \frac{\pi\tau}{2}+\frac{\pi}{2}\right) + \vartheta_3^3 A_1^{(1)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2}, \frac{\pi\tau}{2}\right)$$

From this follows

$$\begin{aligned}\frac{\vartheta_1'^2\vartheta_2^2\vartheta_0^3(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} &= \vartheta_0^3\left\{\frac{1}{\sin^2 z} + 4\sum_n(-1)^{n+1} n q^{n^2} + 4\sum_{n,r}(-1)^{n+1} 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\} + \\ &+ \vartheta_3^3\left\{\frac{1}{\cos^2 z} - 4\sum_n n q^{n^2} + 4\sum_{n,r}(-1)^{r+1} 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\} .\end{aligned} \tag{250}$$

Replacing z by $z+\frac{\pi}{2}$

$$\begin{aligned}\frac{\vartheta_1'^2\vartheta_2^2\vartheta_3^3(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} &= \vartheta_0^3\left\{\frac{1}{\cos^2 z} + 4\sum_n(-1)^{n+1} n q^{n^2} + 4\sum(-1)^{n+r+1} 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\} + \\ &+ \vartheta_3^3\left\{\frac{1}{\sin^2 z} - 4\sum_n n q^{n^2} - 4\sum_{n,r} 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\} ,\end{aligned} \tag{251}$$

These results give

$$\frac{\vartheta_1'^2\,\vartheta_2^2\,\vartheta_1^3(z)}{\vartheta_0^2(z)\,\vartheta_3^2(z)} = 2\,\vartheta_0^3 \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\} +$$

(248.1)
$$+2\,\vartheta_3^3 \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-2}{4}}(\gamma-c)\sin\frac{\gamma-c}{2}z\right\} .$$

$$\frac{\vartheta_1'^2\,\vartheta_2^2\,\vartheta_2^3(z)}{\vartheta_0^2(z)\,\vartheta_3^2(z)} = 2\,\vartheta_0^3 \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}(\gamma+c)\cos\frac{\gamma-c}{2}z\right\} +$$

(249.1)
$$+2\,\vartheta_3^3 \sum q^{\frac{N}{4}}\left\{(\gamma+c)\cos\frac{\gamma-c}{2}z\right\} .$$

$$\frac{\vartheta_1'^2\,\vartheta_2^2\,\vartheta_0^3(z)}{\vartheta_1^2(z)\,\vartheta_2^2(z)} = \vartheta_0^3\left\{\frac{1}{\sin^2 z} + 4\sum_n(-1)^{n+1} n\, q^{n^2} + 4\sum q^N\left\{(-1)^{a+1}(\alpha+a)\cos(\alpha-a)z\right\}\right\} +$$

(250.1)
$$+\vartheta_3^3\left\{\frac{1}{\cos^2 z} - 4\sum_n n\, q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\} .$$

$$\frac{\vartheta_1'^2\,\vartheta_2^2\,\vartheta_3^3(z)}{\vartheta_1^2(z)\,\vartheta_2^2(z)} = \vartheta_0^3\left\{\frac{1}{\cos^2 z} + 4\sum_n(-1)^{n+1} n\, q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\} +$$

(251.1)
$$+\vartheta_3^3\left\{\frac{1}{\sin^2 z} - 4\sum_n n\, q^{n^2} - 4\sum q^N\left\{(\alpha+a)\cos(\alpha-a)z\right\}\right\} .$$

Group XV-a

$$\frac{\vartheta_2^2(z)\,\vartheta_3(z)}{\vartheta_1^2(z)\,\vartheta_2^2(z)} \qquad \frac{\vartheta_1^2(z)\,\vartheta_0(z)}{\vartheta_2^2(z)\,\vartheta_3^2(z)} \qquad \frac{\vartheta_3^2(z)\,\vartheta_2(z)}{\vartheta_1^2(z)\,\vartheta_0^2(z)} \qquad \frac{\vartheta_0^2(z)\,\vartheta_1(z)}{\vartheta_2^2(z)\,\vartheta_3^2(z)}$$

$$\frac{\vartheta_2^2(z)\,\vartheta_0(z)}{\vartheta_1^2(z)\,\vartheta_2^2(z)} \qquad \frac{\vartheta_1^2(z)\,\vartheta_3(z)}{\vartheta_2^2(z)\,\vartheta_0^2(z)} \qquad \frac{\vartheta_0^2(z)\,\vartheta_2(z)}{\vartheta_1^2(z)\,\vartheta_3^2(z)} \qquad \frac{\vartheta_3^2(z)\,\vartheta_1(z)}{\vartheta_2^2(z)\,\vartheta_0^2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^2(z)\,\vartheta_3(z)}{\vartheta_0^2(z)\,\vartheta_1^2(z)}$$

Let $t = z+\frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of order two at $t=\frac{\pi\tau}{2}$ and $t=\pi\tau$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

(252)
$$\frac{\vartheta_1'^2\,\vartheta_0^2}{\vartheta_2\,\vartheta_3}F(z) = \vartheta_2\,A_1^{(1)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right) + \vartheta_3\,q^{\frac{1}{4}}\left\{A_1^{(1)}\left(z+\frac{\pi\tau}{2},\pi\tau\right) + i\,A_1^{(0)}\left(z+\frac{\pi\tau}{2},\pi\tau\right)\right.$$

From this we have

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_2^2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_1^2(z)\vartheta_0^2(z)} = \vartheta_2\left\{\frac{1}{\sin^2 z} - 4\sum_n n q^{n^2} - 4\sum_{n,r} 2(n+r) q^{2nr+n^2}\cos 2rz\right\} + \tag{253}$$

$$-2\vartheta_3\left\{\sum_n (2n-1) q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_1^2(z)\vartheta_0(z)}{\vartheta_2\vartheta_3\vartheta_2^2(z)\vartheta_3^2(z)} = \vartheta_2\left\{\frac{1}{\cos^2 z} - 4\sum_n n q^{n^2} + 4\sum_{n,r}(-1)^{r+1} 2(n+r) q^{n^2+2nr}\cos 2rz\right\} + \tag{254}$$

$$-2\vartheta_3\left\{\sum_n (2n-1) q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

In (252) replace z by $z-\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_3^2(z)\vartheta_2(z)e^{-iz}}{\vartheta_2\vartheta_3\vartheta_0^2(z)\vartheta_1^2(z)}q^{1/4} = \vartheta_2 A_1^{(1)}\left(z,\frac{\pi\tau}{2}\right) + q^{1/4}\vartheta_3\left\{A_1^{(1)}(z,\pi\tau) + i A_0^{(0)}(z,\pi\tau)\right\}.$$

This gives

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_2\vartheta_3\vartheta_0^2(z)\vartheta_1^2(z)} = -4\vartheta_2\sum_{n,r} 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \tag{255}$$

$$+\vartheta_3\left\{\frac{\cos z}{\sin^2 z} - 4\sum_{n,r}[2(n+r)-1]\, q^{n(n+2r-1)}\cos(2r-1)z\right\}.$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_0^2(z)\vartheta_1(z)}{\vartheta_2\vartheta_3\vartheta_3^2(z)\vartheta_2^2(z)} = 4\vartheta_2\sum_{n,r}(-1)^r 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \tag{256}$$

$$+\vartheta_3\left\{\frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{n(n+2r-1)}\sin(2r-1)z\right\}.$$

In the above results replace q by $-q$. There follows

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_2^2(z)\vartheta_0(z)}{\vartheta_0\vartheta_2\vartheta_1^2(z)\vartheta_3^2(z)} = \vartheta_2\left\{\frac{1}{\sin^2 z} + 4\sum_n(-1)^{n+1} n q^{n} + 4\sum_{n,r}(-1)^{n+1} 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\} + \tag{257}$$

$$-2\vartheta_0\left\{\sum_n (2n-1) q^{\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^r[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_2^2(z)\vartheta_0^2(z)} = \vartheta_2\left\{\frac{1}{\cos^2 z}+4\sum_n(-1)^{n+1}n\,q^{n^2}+4\sum_{n,r}(-1)^{n+r+1}2(n+r)\,q^{n^2+2nr}\cos 2rz\right\}+ \tag{258}$$

$$+2\vartheta_0\left\{-\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2}-2\sum_{n,r}[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_0^2(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_1^2(z)\vartheta_3^2(z)} = 4\vartheta_2\sum_{n,r}(-1)^{n+r}2(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \quad + \tag{259}$$

$$+\vartheta_0\left\{\frac{\cos z}{\sin^2 z}-4\sum_{n,r}[2(n+r)-1]\,q^{n(n+2r-1)}\cos(2r-1)z\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_1(z)\vartheta_3^2(z)}{\vartheta_2\vartheta_0\vartheta_0^2(z)\vartheta_2^2(z)} = 4\vartheta_2\sum_{n,r}(-1)^{n+1}2(n+r-1)\,q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \quad + \tag{260}$$

$$+\vartheta_0\left\{\frac{\sin z}{\cos^2 z}+4\sum_{n,r}(-1)^r[2(n+r)-1]\,q^{n(n+2r-1)}\sin(2r-1)z\right\}$$

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_2^2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_1^2(z)\vartheta_0^2(z)} = \vartheta_2\left\{\frac{1}{\sin^2 z}-4\sum_n q^{n^2}n-4\sum q^N\{(\alpha+a)\cos(\alpha-a)z\}\right\}+ \tag{253.1}$$

$$-2\left\{\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2}+\sum q^{\frac{N}{4}}\left\{(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\}\vartheta_3.$$

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_1^2(z)\vartheta_0(z)}{\vartheta_2\vartheta_3\vartheta_2^2(z)\vartheta_3^2(z)} = \vartheta_2\left\{\frac{1}{\cos^2 z}-4\sum_n nq^{n^2}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\}+ \tag{254.1}$$

$$+2\left\{-\sum_n(2n-1)q^{\left(\frac{2n-1}{2}\right)^2}+\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-4}{4}}(\delta+d)\cos\frac{\delta-d}{2}\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_3^2(z)\vartheta_2(z)}{\vartheta_2\vartheta_3\vartheta_0^2(z)\vartheta_1^2(z)} = \vartheta_3\left\{\frac{\cos z}{\sin^2 z}-4\sum q^N\{(\beta+b)\cos(\beta-b)z\}\right\}+ \tag{255.1}$$

$$-2\vartheta_2\left\{\sum q^{\frac{N}{4}}\left\{(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}\right.$$

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_0^2(z)\vartheta_1(z)}{\vartheta_2\vartheta_3\vartheta_2^2(z)\vartheta_3^2(z)} = \vartheta_3\left\{\frac{\sin z}{\cos^2 z}+4\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)\sin(\beta-b)z\right\}\right\}+ \tag{256.1}$$

$$+2\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_2^2(z)\vartheta_0(z)}{\vartheta_2\vartheta_0\vartheta_1^2(z)\vartheta_3^2(z)} = \vartheta_2\left\{\frac{1}{\sin^2 z} + 4\sum_n(-1)^{n+1} n\, q^{n^2} + 4\sum q^N\left\{(-1)^{a+1}(\alpha+a)\cos(\alpha-a)z\right\}\right\} +$$

(257.1)
$$-2\vartheta_0\left\{\sum_n(2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_1^2(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_0^2(z)\vartheta_2^2(z)} = \vartheta_2\left\{\frac{1}{\cos^2 z} + 4\sum_n(-1)^{n+1} n\, q^{n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a+a}{2}}(\alpha+a)\cos(\alpha-a)z\right\}\right\} +$$

(258.1)
$$-2\vartheta_0\left\{\sum_n(2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(\delta+d)\cos\frac{\delta-d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_0^2(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_1^2(z)\vartheta_3^2(z)} = 2\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+9}{4}}(\gamma+c)\cos\frac{\gamma-c}{2}z\right\} +$$

(259.1)
$$+\vartheta_0\left\{\frac{\cos z}{\sin^2 z} - 4\sum q^N\left\{(\beta+b)\cos(\beta-b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_3^2(z)\vartheta_1(z)}{\vartheta_0\vartheta_2\vartheta_0^2(z)\vartheta_2^2(z)} = 2\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\} +$$

(260.1)
$$+\vartheta_0\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+b)\sin(\beta-b)z\right\}\right\}.$$

Group XV-b

$$\frac{\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} \qquad \frac{\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_1^2(z)\vartheta_2^2(z)}$$

Consider

$$F(z) = \frac{\vartheta_1^2(z)\vartheta_2(z)e^{-iz}}{\vartheta_0^2(z)\vartheta_3^2(z)}$$

$F(z)$ satisfies (8) and has poles of order two at $z=\frac{\pi\tau}{2}$ and $z=\frac{\pi\tau}{2}+\frac{\pi}{2}$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

(261) $$\frac{\vartheta_1'^2\vartheta_2^2}{\vartheta_0\vartheta_3}F(z) = \vartheta_0\, q^{-\frac{1}{4}}A_1^{(1)}\left(z,\frac{\pi\tau}{2}\right) + \vartheta_3\, q^{-\frac{1}{4}}A_1^{(1)}\left(z,\frac{\pi\tau}{2}+\frac{\pi}{2}\right)$$

From this follows

$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0^2(z)\vartheta_3^2(z)} = -4\vartheta_0\sum_{n,r} 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(262)

$$+\,4\vartheta_3\sum_{n,r}(-1)^{n+r}\,2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; .$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_0\vartheta_3\vartheta_0^2(z)\vartheta_3^2(z)} = 4\vartheta_0\sum_{n,r}(-1)^r\, 2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(263)

$$+\,4\vartheta_3\sum_{n,r}(-1)^{n+1}\,2(n+r-1)\, q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

In (261) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0 A_1^{(1)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) + \vartheta_3 A_1^{(1)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}+\frac{\pi}{2}\right)$$

There follows

$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0\left\{\frac{1}{\sin^2 z} - 4\sum_n n q^{n^2} - 4\sum_{n,r} 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\} +$$

(264)

$$+\,\vartheta_3\left\{\frac{1}{\cos^2 z} + 4\sum_n (-1)^{n+1} n q^{n^2} + 4\sum_{n,r}(-1)^{n+r+1}\, 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\} .$$

Replacing by

$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_0\vartheta_3\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0\left\{\frac{1}{\cos^2 z} - 4\sum_n n q^{n^2} + 4\sum_{n,r}(-1)^{r+1}\, 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\} +$$

(265)

$$+\,\vartheta_3\left\{\frac{1}{\sin^2 z} + 4\sum_n (-1)^{n+1} n q^{n^2} + 4\sum_{n,r}(-1)^{n+1}\, 2(n+r)\, q^{n^2+2nr}\cos 2rz\right\}$$

These give

$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_1^2(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0^2(z)\vartheta_3^2(z)} = -2\vartheta_0\sum q^{\frac{N}{4}}\left\{(\gamma+c)\cos\frac{\gamma-c}{2}z\right\} \; +$$

(262.1)

$$+\,2\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c+4}{4}}(\gamma+c)\cos\frac{\gamma-c}{2}z\right\}$$

$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_2^2(z)\vartheta_1(z)}{\vartheta_0\vartheta_3\vartheta_0^2(z)\vartheta_3^2(z)} = 2\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\} \; +$$

(263.1)

$$+\,2\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+c)\sin\frac{\gamma-c}{2}z\right\} .$$

$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_0^2(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0\left\{\frac{1}{\sin^2 z} - 4\sum_n n q^{n^2} - 4\sum q^N\{(\alpha+a)\cos(\alpha-a)z\}\right\} +$$

$$+\vartheta_3\left\{\frac{1}{\cos^2 z} + 4\sum_n (-1)^{n+1} n q^{n^2} + 4\sum q^N\{(-1)^{\frac{\alpha+a+2}{2}}(\alpha+a)\cos(\alpha-a)z\}\right\}. \tag{264.1}$$

$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_3^2(z)\vartheta_0(z)}{\vartheta_0\vartheta_3\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0\left\{\frac{1}{\cos^2 z} - 4\sum_n n q^{n^2} + 4\sum q^N\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+a)\cos(\alpha-a)z\}\right\} +$$

$$+\vartheta_3\left\{\frac{1}{\sin^2 z} + 4\sum_n (-1)^{n+1} n q^{n^2} + 4\sum q^N\{(-1)^{a+1}(\alpha+a)\cos(\alpha-a)z\}\right\}. \tag{265.1}$$

Group XVI-a

$$\frac{\vartheta_3^3(z)}{\vartheta_0^2(z)\vartheta_1(z)\vartheta_2(z)} \qquad \frac{\vartheta_0^3(z)}{\vartheta_3^2(z)\vartheta_1(z)\vartheta_2(z)} \qquad \frac{\vartheta_2^3(z)}{\vartheta_1^2(z)\vartheta_0(z)\vartheta_3(z)} \qquad \frac{\vartheta_1^3(z)}{\vartheta_2^2(z)\vartheta_0(z)\vartheta_3(z)}$$

Consider

$$F(z) = \frac{\vartheta_3^3(z)}{\vartheta_0^2(z)\vartheta_1(z)\vartheta_2(z)}$$

Let $t = z + \frac{\pi\tau}{2} + \frac{\pi}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8), has a double pole at $t = \pi\tau + \frac{\pi}{2}$ and simple poles at $t = \frac{\pi\tau}{2} + \frac{\pi}{2}$ and $t = \frac{\pi\tau}{2}$. Calculating the corresponding $A_i^{(j)}$ and using (10) we get

$$\vartheta_1'^3 F(z) = i q^{1/4}\vartheta_2^4\left\{A_1^{(1)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2}, \pi\tau+\frac{\pi}{2}\right) + i A_1^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2}, \frac{\pi}{2}+\pi\tau\right)\right\} +$$

$$+ \vartheta_3^5\vartheta_2 A_1^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2}, \frac{\pi\tau}{2}+\frac{\pi}{2}\right) - \vartheta_0^5\vartheta_2 A_1^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{2}, \frac{\pi\tau}{2}\right). \tag{266}$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_3^3(z)}{\vartheta_0^2(z)\vartheta_1(z)\vartheta_2(z)} = 4\vartheta_2^4\sum_{n,r}(-1)^n[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \;+$$

$$+\vartheta_3^5\vartheta_2\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^n q^{n^2+2nr}\sin 2rz\right\} + \vartheta_0^5\vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{n^2+2nr}\sin 2rz\right\} \tag{267}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0^3(z)}{\vartheta_3^2(z)\vartheta_1(z)\vartheta_2(z)} = 4\vartheta_2^4\sum_{n,r}(-1)^{n+r+1}[2(n+r)-1]\, q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz$$

$$+\vartheta_3^5\vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n^2+2nr}\sin 2rz\right\} + \vartheta_0^5\vartheta_2\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{n^2+2nr}\sin 2rz\right\} \tag{268}$$

In (266) replace $z+\frac{\pi\tau}{2}+\frac{\pi}{2}$ by z, obtaining

$$\frac{\vartheta_1'^3\,\vartheta_1^3(z)\,e^{-iz}}{\vartheta_2^2(z)\,\vartheta_0(z)\,\vartheta_3(z)} = i\,\vartheta_2^4\left\{A_1^{(1)}(z,\pi\tau+\tfrac{\pi}{2}) + i\,A_1^{(0)}(z,\pi\tau+\tfrac{\pi}{2})\right. \left. + \vartheta_3^5\vartheta_2\, q^{-1/4} A_1^{(0)}(z,\tfrac{\pi\tau}{2}+\tfrac{\pi}{2}) - \vartheta_0^5\vartheta_2\, q^{-1/4} A_1^{(0)}(z,\tfrac{\pi\tau}{2})\right\}.$$

This gives

(269)
$$\frac{\vartheta_1'^3\,\vartheta_1^3(z)}{\vartheta_2^2(z)\,\vartheta_0(z)\,\vartheta_3(z)} = \vartheta_2^4\left\{\frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^{n+r}[2(n+r)-1]\, q^{n(n+2r-1)}\sin(2r-1)z\right\} +$$
$$+4\vartheta_3^5\vartheta_2\sum_{n,r}(-1)^{n+r+1} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \;-\; 4\vartheta_0^5\vartheta_2\sum_{n,r} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

Replacing z by $z+\frac{\pi}{2}$

(270)
$$\frac{\vartheta_1'^3\,\vartheta_2^3(z)}{\vartheta_1^2(z)\,\vartheta_0(z)\,\vartheta_3(z)} = \vartheta_2^4\left\{\frac{\cos z}{\sin^2 z} + 4\sum_{n,r}(-1)^{n+1}[2(n+r)-1]\, q^{n(n+2r-1)}\cos(2r-1)z\right\} +$$
$$+4\vartheta_3^5\vartheta_2\sum_{n,r}(-1)^{n} q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \;+\; 4\vartheta_0^5\vartheta_2\sum_{n,r}(-1)^r q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

From these follow

(267.1)
$$\frac{\vartheta_1'^3\,\vartheta_3^3(z)}{\vartheta_0^2(z)\,\vartheta_1(z)\,\vartheta_2(z)} = 2\vartheta_2^4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}(\delta+d)\sin\tfrac{\delta-d}{2}z\right\} +$$
$$+\vartheta_3^5\vartheta_2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\{(-1)^a\sin(\alpha-a)z\}\right\} + \vartheta_0^5\vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\}\right\},$$

(268.1)
$$\frac{\vartheta_1'^3\,\vartheta_0^3(z)}{\vartheta_3^2(z)\,\vartheta_1(z)\,\vartheta_2(z)} = 2\vartheta_2^4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d-2}{4}}(\delta+d)\sin\tfrac{\delta-d}{2}z\right\} +$$
$$+\vartheta_3^5\vartheta_2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\{(-1)^{\frac{d+a+2}{2}}\sin(d-a)z\}\right\} + \vartheta_0^5\vartheta_2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\{\sin(\alpha-a)z\}\right\},$$

(269.1)
$$\frac{\vartheta_1'^3\,\vartheta_1^3(z)}{\vartheta_2^2(z)\,\vartheta_0(z)\,\vartheta_3(z)} = \vartheta_2^4\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\{(-1)^{\frac{\beta+b+1}{2}}(\beta+b)\sin(\beta-b)z\}\right\} +$$
$$+4\vartheta_3^5\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}\sin\tfrac{\gamma-c}{2}z\right\} - 4\vartheta_0^5\vartheta_2\sum q^{\frac{N}{4}}\left\{\sin\tfrac{\gamma-c}{2}z\right\}.$$

(270.1)
$$\frac{\vartheta_1'^3\,\vartheta_2^3(z)}{\vartheta_1^2(z)\,\vartheta_0(z)\,\vartheta_3(z)} = \vartheta_2^5\left\{\frac{\cos z}{\sin^2 z} + 4\sum q^N\{(-1)^{b+1}(\beta+b)\cos(\beta-b)z\}\right\} +$$
$$+4\vartheta_3^5\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\tfrac{\gamma-c}{2}z\right\} + 4\vartheta_0^5\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}\cos\tfrac{\gamma-c}{2}z\right\}.$$

Group XVI-b

$$\frac{\vartheta_2^3(z)}{\vartheta_0^2(z)\vartheta_1(z)\vartheta_3(z)} \quad \frac{\vartheta_1^3(z)}{\vartheta_3^2(z)\vartheta_0(z)\vartheta_2(z)} \quad \frac{\vartheta_3^3(z)}{\vartheta_1^2(z)\vartheta_0(z)\vartheta_2(z)} \quad \frac{\vartheta_0^3(z)}{\vartheta_2^2(z)\vartheta_1(z)\vartheta_3(z)}$$

$$\frac{\vartheta_2^3(z)}{\vartheta_3^2(z)\vartheta_1(z)\vartheta_0(z)} \quad \frac{\vartheta_1^3(z)}{\vartheta_0^2(z)\vartheta_3(z)\vartheta_2(z)} \quad \frac{\vartheta_0^3(z)}{\vartheta_1^2(z)\vartheta_3(z)\vartheta_2(z)} \quad \frac{\vartheta_3^3(z)}{\vartheta_2^2(z)\vartheta_1(z)\vartheta_0(z)}$$

Consider

$$F(z) = \frac{\vartheta_3^3(z)\, e^{-iz}}{\vartheta_0(z)\vartheta_2(z)\vartheta_1^2(z)}$$

Let $t = z + \frac{\pi}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8), has a double pole at $t = \frac{\pi}{2}$, and simple poles at $t = \frac{\pi\tau}{2} + \frac{\pi}{2}$ and $t = 0$. Calculating the corresponding $R_0^{(i)}$ we get from (10)

$$\vartheta_1'^3 F(z) = \vartheta_3^4 \left\{ A_1^{(1)}(z+\tfrac{\pi}{2}, \tfrac{\pi}{2}) - i A_1^{(0)}(z+\tfrac{\pi}{2}, \tfrac{\pi}{2}) \right\} +$$

(271)

$$i q^{-1/4} \vartheta_2^5 \vartheta_3 A_1^{(0)}(z+\tfrac{\pi}{2}, \tfrac{\pi\tau}{2}+\tfrac{\pi}{2}) + i \vartheta_0^5 \vartheta_3 A_1^{(0)}(z+\tfrac{\pi}{2}, 0)$$

This results in

$$\frac{\vartheta_1'^3 \vartheta_3^3(z)}{\vartheta_1^2(z)\vartheta_0(z)\vartheta_2(z)} = \vartheta_3^4 \left\{ \frac{\cos z}{\sin^2 z} + 4\sum_{n,r} (-1)^{n+1} [2(n+r)-1]\, q^{n(n+2r-1)} \cos(2r-1)z \right\} +$$

(272)

$$+ 4\vartheta_2^5 \vartheta_3 \sum_{n,r} (-1)^n q^{\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z + \vartheta_0^5 \vartheta_3 \left\{ \frac{1}{\cos z} + 4\sum_{n,r} (-1)^{r+1} q^{n(n+2r-1)} \cos(2r-1)z \right\}$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'^3 \vartheta_0^3(z)}{\vartheta_2^2(z)\vartheta_3(z)\vartheta_1(z)} = \vartheta_3^4 \left\{ \frac{\sin z}{\cos^2 z} + 4\sum_{n,r} (-1)^{n+r} [2(n+r)-1]\, q^{n(n+2r-1)} \sin(2r-1)z \right\} +$$

(273)

$$+ 4\vartheta_2^5 \vartheta_3 \sum_{n,r} (-1)^{n+r+1} q^{\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z + \vartheta_0^5 \vartheta_3 \left\{ \frac{1}{\sin z} + 4\sum q^{n(n+2r-1)} \sin(2r-1)z \right\}$$

In (271) replace z by $z + \frac{\pi\tau}{2}$ obtaining

$$\frac{\vartheta_1'^3 \vartheta_2^3(z)}{\vartheta_0^2(z)\vartheta_1(z)\vartheta_3(z)} = -i \vartheta_3^4 q^{1/4} \left\{ A_1^{(1)}(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2}, \tfrac{\pi}{2}) - i A_1^{(0)}(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2}, \tfrac{\pi}{2}) \right\} +$$

$$+ \vartheta_2^5 \vartheta_3 A_1^{(0)}(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2}, \tfrac{\pi\tau}{2}+\tfrac{\pi}{2}) + q^{1/4} \vartheta_0^5 \vartheta_3 A_1^{(0)}(z+\tfrac{\pi\tau}{2}+\tfrac{\pi}{2}, 0)$$

This gives

$$\frac{\vartheta_1'^3 \vartheta_2^3(z)}{\vartheta_0^2(z)\vartheta_1(z)\vartheta_3(z)} = 4\vartheta_3^4 \sum_{n,r} (-1)^n q^{\left(\frac{2n-1}{2}\right)^2 + (2n-1)r} [2(n+r)-1] \sin 2rz \quad +$$

(274)

$$+ 4\vartheta_0^5 \vartheta_3 \sum_{n,r} (-1)^r q^{\left(\frac{2n-1}{2}\right)^2 + (2n-1)r} \sin 2rz + \vartheta_2^5 \vartheta_3 \left\{ \frac{\cos z}{\sin z} + 4\sum_{n,r} (-1)^n q^{n^2+2nr} \sin 2rz \right\}$$

Replacing z by $z+\frac{\pi}{2}$

(275)
$$\frac{\vartheta_1'^3\vartheta_1^3(z)}{\vartheta_3^2(z)\vartheta_0(z)\vartheta_2(z)} = 4\vartheta_3^4\sum_{n,r}(-1)^{n+r+1}[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \; +$$
$$-4\vartheta_0^5\vartheta_3\sum_{n,r}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \vartheta_2^5\vartheta_3\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1}q^{n^2+2nr}\sin 2rz\right\}.$$

Replacing q by $-q$ in these results gives

(276)
$$\frac{\vartheta_1'^3\vartheta_0^3(z)}{\vartheta_1^2(z)\vartheta_2(z)\vartheta_3(z)} = \vartheta_0^4\left\{\frac{\cos z}{\sin^2 z}+4\sum_{n,r}(-1)^{n+1}[2(n+r)-1]\,q^{n(n+2r-1)}\cos(2r-1)z\right\} +$$
$$+4\vartheta_2^5\vartheta_0\sum_{n,r}(-1)^r q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + \vartheta_3^5\vartheta_0\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{n(n+2r-1)}\cos(2r-1)z\right\}.$$

(277)
$$\frac{\vartheta_1'^3\vartheta_3^3(z)}{\vartheta_2^2(z)\vartheta_1(z)\vartheta_0(z)} = \vartheta_0^4\left\{\frac{\sin z}{\cos^2 z}+4\sum_{n,r}(-1)^{n+r}[2(n+r)-1]\,q^{n(n+2r-1)}\sin(2r-1)z\right\} +$$
$$-4\vartheta_2^5\vartheta_0\sum_{n,r}q^{\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z + \vartheta_3^5\vartheta_0\left\{\frac{1}{\sin z}+4\sum_{n,r}q^{n(n+2r-1)}\sin(2r-1)z\right\}$$

(278)
$$\frac{\vartheta_1'^3\vartheta_2^3(z)}{\vartheta_3^2(z)\vartheta_0(z)\vartheta_1(z)} = 4\vartheta_0^4\sum_{n,r}(-1)^{n+r+1}[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \; +$$
$$-4\vartheta_3^5\vartheta_0\sum_{n,r}q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \vartheta_2^5\vartheta_0\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}q^{n^2+2nr}\sin 2rz\right\}.$$

(279)
$$\frac{\vartheta_1'^3\vartheta_1^3(z)}{\vartheta_0^2(z)\vartheta_2(z)\vartheta_3(z)} = 4\vartheta_0^4\sum_{n,r}(-1)^n[2(n+r)-1]\,q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \; +$$
$$+4\vartheta_3^5\vartheta_0\sum_{n,r}(-1)^r q^{\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \vartheta_2^5\vartheta_0\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{n^2+2nr}\sin 2rz\right\}$$

From these follow

(272.1)
$$\frac{\vartheta_1'^3\vartheta_3^3(z)}{\vartheta_1^2(z)\vartheta_0(z)\vartheta_2(z)} = \vartheta_3^4\left\{\frac{\cos z}{\sin^2 z}+4\sum q^N\left\{(-1)^{b+1}(\beta+b)\cos(\beta-b)z\right\}\right\} +$$
$$+4\vartheta_2^5\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-c}{2}z\right\} + \vartheta_0^5\vartheta_3\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\}\right\}$$

(273.1)
$$\frac{\vartheta_1'^3\vartheta_0^3(z)}{\vartheta_2^2(z)\vartheta_1(z)\vartheta_3(z)} = \vartheta_3^4\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta+b+1}{2}}(\beta+b)\sin(\beta-b)z\right\}\right\} +$$
$$+4\vartheta_2^5\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+c}{4}}\sin\frac{\gamma-c}{2}z\right\} + \vartheta_0^5\vartheta_3\left\{\frac{1}{\sin z} + 4\sum q^N\left\{\sin(\beta-b)z\right\}\right\}$$

(274.1)
$$\frac{\vartheta_1'^3\vartheta_2^3(z)}{\vartheta_0^2(z)\vartheta_1(z)\vartheta_3(z)} = 2\vartheta_3^4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} +$$
$$+4\vartheta_0^5\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}\sin\frac{\delta-d}{2}z\right\} + \vartheta_2^5\vartheta_3\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{(-1)^a\sin(\alpha-a)z\right\}\right\}.$$

(275.1)
$$\frac{\vartheta_1'^3\vartheta_1^3(z)}{\vartheta_3^2(z)\vartheta_2(z)\vartheta_0(z)} = 2\vartheta_3^4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d-2}{4}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} +$$
$$-4\vartheta_0^5\vartheta_3\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-d}{2}z\right\} + \vartheta_2^5\vartheta_3\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha+a+2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

(276.1)
$$\frac{\vartheta_1'^3\vartheta_0^3(z)}{\vartheta_1^2(z)\vartheta_2(z)\vartheta_3(z)} = \vartheta_0^4\left\{\frac{\cos z}{\sin^2 z} + 4\sum q^N\left\{(-1)^{b+1}(\beta+b)\cos(\beta-b)z\right\}\right\} +$$
$$+4\vartheta_2^5\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+2}{4}}\cos\frac{\gamma-c}{2}z\right\} + \vartheta_3^5\vartheta_0\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-b)z\right\}\right\}.$$

(277.1)
$$\frac{\vartheta_1'^3\vartheta_3^3(z)}{\vartheta_2^2(z)\vartheta_0(z)\vartheta_1(z)} = \vartheta_0^4\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta+b+1}{2}}(\beta+b)\sin(\beta-b)z\right\}\right\} +$$
$$-4\vartheta_2^5\vartheta_0\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-c}{2}z\right\} + \vartheta_3^5\vartheta_0\left\{\frac{1}{\sin z} + 4\sum q^N\left\{\sin(\beta-b)z\right\}\right\}.$$

(278.1)
$$\frac{\vartheta_1'^3\vartheta_2^3(z)}{\vartheta_3^2(z)\vartheta_0(z)\vartheta_1(z)} = 2\vartheta_0^4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+d-2}{4}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} +$$
$$-4\vartheta_3^5\vartheta_0\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-d}{2}z\right\} + \vartheta_2^5\vartheta_0\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(\alpha-a)z\right\}\right\}.$$

(279.1)
$$\frac{\vartheta_1'^3\vartheta_1^3(z)}{\vartheta_0^2(z)\vartheta_2(z)\vartheta_3(z)} = 2\vartheta_0^4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}(\delta+d)\sin\frac{\delta-d}{2}z\right\} +$$
$$+4\vartheta_3^5\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d}{4}}\sin\frac{\delta-d}{2}z\right\} + \vartheta_2^5\vartheta_0\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-a)z\right\}\right\}.$$

The expansions of this section are those for which μ equals two whose denominators are of the fourth or lower degree in the ϑ's . The following table replaces the corresponding one of the previous section, and is understood throughout this section.

$$N = m(2m+2r) = a\alpha$$
$$a = m \qquad \alpha = 2m+2r$$
$$m = a \qquad r = \frac{\alpha-2a}{2}$$
$$m+r = \frac{\alpha}{2} \qquad 2m+r = \frac{\alpha+2a}{2}$$
$$0 \equiv \alpha \bmod 2 \qquad 0 < a < \sqrt{\frac{N}{2}}$$

$$N = m(2m+2r-1) = b\beta$$
$$b = m \qquad \beta = 2m+2r-1$$
$$m = b \qquad r = \frac{\beta-2b+1}{2}$$
$$m+r = \frac{\beta+1}{2} \qquad 2r-1 = \beta-2b$$
$$2(2m+r)-1 = \beta+2b$$
$$\beta \equiv 1 \bmod 2 \qquad 0 < b < \sqrt{\frac{N}{2}}$$

$$\frac{N}{4} = \frac{(2m-1)^2}{2} + \frac{(2m-1)(2r-1)}{2} = \frac{c\gamma}{4}$$
$$c = 2m-1 \qquad \gamma = 4m+4r-4$$
$$m = \frac{c+1}{2} \qquad r = \frac{\gamma-2c+2}{4}$$
$$2r-1 = \frac{\gamma-2c}{2} \qquad m+r = \frac{\gamma+4}{4}$$
$$2(2m+r)-3 = \frac{\gamma+2c}{2}$$
$$c \equiv 1 \bmod 2 \qquad \gamma \equiv 0 \bmod 4$$
$$0 < c < \sqrt{\frac{N}{2}}$$

$$\frac{N}{4} = \frac{(2m-1)^2}{2} + (2m-1)r = \frac{d\delta}{4}$$
$$d = 2m+1 \qquad \delta = 2(2m+2r-1)$$
$$m = \frac{d+1}{2} \qquad r = \frac{\delta-2d}{4}$$
$$m+r = \frac{\delta-2}{4} \qquad 2m+r-1 = \frac{\delta+2d}{4}$$
$$d \equiv 1 \bmod 2 \qquad \delta \equiv 2 \bmod 4$$
$$0 < d < \sqrt{\frac{N}{2}}$$

In reducing the expressions given on using (10) it may happen that every term except one is odd, or is even. Hence we infer that the excepted term must vanish. This implies an identity in the g's . The identities which arise in this way are used to simplify the form of the results.

Group I

$$\frac{1}{\vartheta_1^2(z)} \qquad \frac{1}{\vartheta_2^2(z)} \qquad \frac{1}{\vartheta_0^2(z)} \qquad \frac{1}{\vartheta_3^2(z)}$$

Consider

$$F(z) = \frac{1}{\vartheta_1^2(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order two at $t = \frac{\pi\tau}{2}$. Calculating the corresponding $R_i^{(N)}$ and using (10) gives

(280) $$\vartheta_1'^3 F(z) = A_2^{(1)}\left(z + \tfrac{\pi\tau}{2}, \tfrac{\pi\tau}{2}\right)$$

From this follows

(281) $$\frac{\vartheta_1'^2}{\vartheta_1^2(z)} = \frac{1}{\sin^2 z} - 4\sum_n 2n\, q^{2n^2} - 4\sum_{n,r} 2(2n+r)\, q^{2n^2+2nr} \cos 2rz .$$

Replacing z by $z + \frac{\pi}{2}$

(282) $$\frac{\vartheta_1'^2}{\vartheta_2^2(z)} = \frac{1}{\cos^2 z} - 4\sum_n 2n\, q^{2n^2} + 4\sum 2(2n+r)(-1)^{r+1} q^{2n^2+2nr} \cos 2rz .$$

In (280) replace z by $z - \frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2 e^{-2iz}}{\vartheta_0^2(z)} = -q^{-1/2} A_2^{(1)}\left(z, \tfrac{\pi\tau}{2}\right)$$

which gives

(283) $$\frac{\vartheta_1'^2}{\vartheta_0^2(z)} = 4\sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + 4\sum_{n,r} 2(2n+r-1)\, q^{2\left(\frac{2n-1}{2}\right)^2 + (2n-1)r} \cos 2rz .$$

Replacing z by $z + \frac{\pi}{2}$

(284) $$\frac{\vartheta_1'^2}{\vartheta_3^2(z)} = 4\sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + 4\sum_{n,r} (-1)^r 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2} + (2n-1)r} \cos 2rz .$$

From these we have

(281.1) $$\frac{\vartheta_1'^2}{\vartheta_1^2(z)} = \frac{1}{\sin^2 z} - 4\sum 2n\, q^{2n^2} - 4\sum q^N \left\{(d + 2a)\cos(d - 2a) z\right\}$$

$$(282.1) \qquad \frac{\vartheta_1'^{\,2}}{\vartheta_2^2(z)} = \frac{1}{\cos^2 z} - 4\sum_n 2n\, q^{2n^2} + 4\sum q^N \left\{ (-1)^{\frac{\alpha-2a+2}{2}} (\alpha+2a)\cos(\alpha-2a)z \right\}.$$

$$(283.1) \qquad \frac{\vartheta_1'^{\,2}}{\vartheta_0^2(z)} = 4\sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}} \left\{ (\delta+2d)\cos\frac{\delta-2d}{2} z \right\}$$

$$(284.1) \qquad \frac{\vartheta_1'^{\,2}}{\vartheta_3^2(z)} = 4\sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\delta-2d}{4}} (\delta+2d)\cos\frac{\delta-2d}{2} z \right\}.$$

Group II-a

$$\frac{1}{\vartheta_1(z)\vartheta_2(z)} \qquad \frac{1}{\vartheta_0(z)\vartheta_3(z)}$$

Consider

$$F(z) = \frac{1}{\vartheta_1(z)\vartheta_2(z)}$$

Let $t = z + \frac{\pi\tau}{2} + \frac{\pi}{4}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has simple poles at $t = \frac{\pi\tau}{2} + \frac{\pi}{4}$ and $t = \frac{\pi\tau}{2} + \frac{3\pi}{4}$. Calculating the corresponding $R_i^{(0)}$ and using (10) gives

$$(285) \qquad \frac{\vartheta_1'\vartheta_2}{\vartheta_1(z)\vartheta_2(z)} = A_2^{(0)}\left(z + \frac{\pi\tau}{2} + \frac{\pi}{4}, \frac{\pi\tau}{2} + \frac{\pi}{4}\right) - A_2^{(0)}\left(z + \frac{\pi\tau}{2} + \frac{\pi}{2}, \frac{\pi\tau}{2} + \frac{3\pi}{4}\right)$$

and from this

$$(286) \qquad \frac{\vartheta_1'\vartheta_2}{\vartheta_1(z)\vartheta_2(z)} = \frac{2}{\sin 2z} + 4\sum_{n,r} 2(-1)^n q^{2n^2+2n(2r-1)} \sin 2(2r-1)z$$

In (285) replace z by $z - \frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'\vartheta_2 e^{-2iz}}{\vartheta_0(z)\vartheta_3(z)} q^{1/2} = -i\, A_2^{(0)}\left(z + \frac{\pi}{4}, \frac{\pi\tau}{2} + \frac{\pi}{4}\right) + i\, A_2^{(0)}\left(z + \frac{\pi}{4}, \frac{\pi\tau}{2} + \frac{3\pi}{4}\right)$$

which gives

$$(287) \qquad \frac{\vartheta_1'\vartheta_2}{\vartheta_0(z)\vartheta_3(z)} = 4\sum_{n,0} (-1)^{1+n} q^{\frac{(2n-1)^2}{2}} + 8\sum_{n,r} (-1)^{n+1} q^{\frac{(2n-1)^2}{2} + 2r(2n-1)} \cos 4rz$$

From these follow

$$(286.1)\quad \frac{\vartheta_1'\vartheta_2}{\vartheta_1(z)\vartheta_2(z)} = \frac{2}{\sin 2z} + 8\sum q^N\{(-1)^a \sin(\alpha-2a)z\}$$

Where the additional condition $\alpha - 2a \equiv 2 \bmod 4$ must hold, and

$$(287.1)\quad \frac{\vartheta_1'\vartheta_2}{\vartheta_0(z)\vartheta_3(z)} = 4\sum_n (-1)^{n+1} q^{\frac{(2n-1)^2}{2}} + 8\sum q^{\frac{N}{4}}\{(-1)^{\frac{d-1}{2}} \cos\frac{\delta-2d}{2}z\}$$

where the additional condition $\delta - 2d \equiv 0 \bmod 8$ must hold.

Group II-b

$$\frac{1}{\vartheta_1(z)\vartheta_3(z)} \qquad \frac{1}{\vartheta_2(z)\vartheta_0(z)} \qquad \frac{1}{\vartheta_1(z)\vartheta_0(z)} \qquad \frac{1}{\vartheta_2(z)\vartheta_3(z)}$$

Consider

$$F(z) = \frac{e^{-iz}}{\vartheta_1(z)\vartheta_0(z)}$$

Let $t = z + \frac{\pi\tau}{4}$, $F(z) = \varphi(t)$. $\varphi(t)$ satisfies (8) and has simple poles at $t = \frac{\pi\tau}{4}$ and at $t = \frac{3\pi\tau}{4}$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$\frac{\vartheta_1'\vartheta_0 e^{-iz}}{\vartheta_1(z)\vartheta_0(z)} = A_2^{(0)}(z+\tfrac{\pi\tau}{4}, \tfrac{\pi\tau}{4}) - A_2^{(0)}(z+\tfrac{\pi\tau}{4}, \tfrac{3\pi\tau}{4})$$

From this follows

$$(288)\quad \frac{\vartheta_1'\vartheta_0}{\vartheta_1(z)\vartheta_0(z)} = \frac{1}{\sin z} + 4\sum_{n,r} q^{n(2n+2r-1)} \sin(2r-1)z - 4\sum_{n,r} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z .$$

Replacing z by $z+\frac{\pi}{2}$

$$(289)\quad \frac{\vartheta_1'\vartheta_0}{\vartheta_2(z)\vartheta_3(z)} = \frac{1}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{n(2n+2r-1)} \cos(2r-1)z + 4\sum_{n,r}(-1)^r q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z$$

In the last two results replace q by $-q$ obtaining

(290) $$\frac{\vartheta_1'\vartheta_3}{\vartheta_1(z)\vartheta_3(z)} = \frac{1}{\sin z} + 4\sum_{n,r}(-1)^n q^{n(2n+2r-1)}\sin(2r-1)z + 4\sum_{n,r}(-1)^{n+r} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

(291) $$\frac{\vartheta_1'\vartheta_3}{\vartheta_0(z)\vartheta_2(z)} = \frac{1}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n(2n+2r-1)}\cos(2r-1) + 4\sum_{n,r}(-1)^{n+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

From these follow

(288.1) $$\frac{\vartheta_1'\vartheta_0}{\vartheta_1(z)\vartheta_0(z)} = \frac{1}{\sin z} - 4\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-2c}{2}z\right\} + 4\sum q^N\left\{\sin(\beta-2b)z\right\}.$$

(289.1) $$\frac{\vartheta_1'\vartheta_0}{\vartheta_2(z)\vartheta_3(z)} = \frac{1}{\cos z} + 4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c+2}{4}}\cos\frac{\gamma-2c}{2}z\right\} + 4\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}\cos(\beta-2b)z\right\}.$$

(290.1) $$\frac{\vartheta_1'\vartheta_3}{\vartheta_1(z)\vartheta_3(z)} = \frac{1}{\sin z} + 4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+4}{4}}\sin\frac{\gamma-2c}{2}z\right\} + 4\sum q^N\left\{(-1)^b\sin(\beta-2b)z\right\}$$

(291.1) $$\frac{\vartheta_1'\vartheta_3}{\vartheta_0(z)\vartheta_2(z)} = \frac{1}{\cos z} + 4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-2c}{2}z\right\} + 4\sum q^N\left\{(-1)^{\frac{\beta-1}{2}}\cos(\beta-2b)z\right\}.$$

Group III-a

$$\frac{\vartheta_0(z)}{\vartheta_1^3(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_2^3(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_0^3(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_3^3(z)}$$

$$\frac{\vartheta_3(z)}{\vartheta_1^3(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_2^3(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_3^3(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0^3(z)}$$

Consider

$$F(z) = \frac{\vartheta_0(z)\,e^{-iz}}{\vartheta_1^3(z)}$$

Let $t = z + \frac{\pi I}{4}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order three at $t = \frac{\pi I}{4}$. Calculating the corresponding $A_i^{(j)}$ and using (10) gives

(292) $$\frac{\vartheta_1'^3\vartheta_0(z)e^{-iz}}{\vartheta_0\vartheta_1^3(z)} = \frac{1}{2}A_2^{(2)}\left(z+\frac{\pi I}{4},\frac{\pi I}{4}\right) - i\,A_2^{(1)}\left(z+\frac{\pi I}{4},\frac{\pi I}{4}\right) + \frac{1}{2}\left\{\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\}A_2^{(0)}\left(z+\frac{\pi I}{4},\frac{\pi I}{4}\right)$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_0(z)}{\vartheta_0\vartheta_1^3(z)} = \frac{1}{\sin^3 z} - 2\sum_{n,r}[2(2n+r)-1]^2 q^{n(2n+2r-1)}\sin(2r-1)z \; +$$

(293)

$$+\frac{1}{2\sin z}\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} + 2\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{n(2n+2r-1)}\sin(2r-1)z$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'\vartheta_3(z)}{\vartheta_0\vartheta_2^3(z)} = \frac{1}{\cos^3 z} + 2\sum_{n,r}(-1)^r[2(2n+r)-1]^2 q^{n(2n+2r-1)}\cos(2r-1)z \; +$$

(294)

$$+\frac{1}{2\cos z}\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} + 2\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1} q^{n(2n+2r-1)}\cos(2r-1)z$$

In (292) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$-\frac{\vartheta_1'^3\vartheta_1(z)e^{iz}}{\vartheta_0\vartheta_0^3(z)} = \frac{1}{2}A_2^{(2)}\left(z+\frac{3\pi\tau}{4},\frac{\pi\tau}{4}\right) - i\,A_2^{(1)}\left(z+\frac{3\pi\tau}{4},\frac{\pi\tau}{4}\right) \; +$$

$$+\frac{1}{2}\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}A_2^{(0)}\left(z+\frac{3\pi\tau}{4},\frac{\pi\tau}{4}\right)$$

which gives

(295) $$\frac{\vartheta_1'^3\vartheta_1(z)}{\vartheta_0\vartheta_0^3(z)} = 2\sum_{n,r}[2(2n+r)-3]^2 q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z - 2\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z .$$

Replacing z by $z+\frac{\pi}{2}$

(296) $$\frac{\vartheta_1'^3\vartheta_2(z)}{\vartheta_0\vartheta_3^3(z)} = 2\sum_{n,r}(-1)^{r+1}[2(2n+r)-3]^2 q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + 2\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^r q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z .$$

In these results replace q by $-q$. This gives

$$\frac{\vartheta_1'^3\vartheta_3(z)}{\vartheta_3\vartheta_1^3(z)} = \frac{1}{\sin^3 z} + 2\sum_{n,r}(-1)^{n+1}[2(2n+r)-1]^2 q^{n(2n+2r-1)}\sin(2r-1)z \; +$$

(297)

$$+\frac{1}{2\sin z}\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} + 2\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^n q^{n(2n+2r-1)}\sin(2r-1)z$$

(298)
$$\frac{\vartheta_1'^3\,\vartheta_0(z)}{\vartheta_3\,\vartheta_2^3(z)} = \frac{1}{\cos^3 z} + 2\sum_{n,r}(-1)^{n+r}\left[2(2n+r)-1\right]^2 q^{n(2n+2r-1)}\cos(2r-1)z \; +$$
$$+\frac{1}{2\cos z}\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} + 2\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1} q^{n(2n+2r-1)}\cos(2r-1)z \, .$$

(299)
$$\frac{\vartheta_1'^3\,\vartheta_1(z)}{\vartheta_3\,\vartheta_3^3(z)} = 2\sum_{n,r}(-1)^{n+r}\left[2(2n+r)-3\right]^2 q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z + 2\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

(300)
$$\frac{\vartheta_1'^3\,\vartheta_2(z)}{\vartheta_3\,\vartheta_0^3(z)} = 2\sum_{n,r}(-1)^{n+1}\left[2(2n+r)-3\right]^2 q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + 2\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

From these follow

(293.1)
$$\frac{\vartheta_1'^3\,\vartheta_0(z)}{\vartheta_0\,\vartheta_1^3(z)} = \frac{1}{\sin^3 z} - 2\sum q^N\left\{(\beta+2b)^2\sin(\beta-2b)z\right\} \; +$$
$$+\frac{1}{2\sin z}\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} + 2\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{\sin(\beta-2b)z\right\} .$$

(294.1)
$$\frac{\vartheta_1'^3\,\vartheta_3(z)}{\vartheta_0\,\vartheta_2^3(z)} = \frac{1}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\beta-2b+1}{2}}(\beta+2b)^2\cos(\beta-2b)z\right\} \; +$$
$$+\frac{1}{2\cos z}\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\} + 2\left\{\frac{\vartheta_0''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}\cos(\beta-2b)z\right\}$$

(295.1)
$$\frac{\vartheta_1'^3\,\vartheta_1(z)}{\vartheta_0\,\vartheta_0^3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\gamma+2c)^2\sin\frac{\gamma-2c}{2}z\right\} - 2\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-2c}{2}z\right\}$$

(296.1)
$$\frac{\vartheta_1'^3\,\vartheta_2(z)}{\vartheta_0\,\vartheta_3^3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c-2}{4}}(\gamma+2c)^2\cos\frac{\gamma-2c}{2}z\right\} + 2\left\{\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c+2}{4}}\cos\frac{\gamma-2c}{2}z\right\} .$$

$$\frac{\vartheta_1'^3\vartheta_3(z)}{\vartheta_3\vartheta_1^3(z)} = \frac{1}{\sin^3 z} + 2\sum q^N\left\{(-1)^{b+1}(\beta+2b)^2\sin(\beta-2b)z\right\} +$$

(297.1)

$$+\frac{1}{2\sin z}\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} + 2\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^b\sin(\beta-2b)z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_0(z)}{\vartheta_3\vartheta_2^3(z)} = \frac{1}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\beta+1}{2}}(\beta+2b)^2\cos(\beta-2b)z\right\} +$$

(298.1)

$$+\frac{1}{2\cos z}\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} + 2\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta-1}{2}}\cos(\beta-2b)z\right\}.$$

(299.1) $$\frac{\vartheta_1'^3\vartheta_1(z)}{\vartheta_3\vartheta_3^3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+4}{4}}(\gamma+2c)^2\sin\frac{\gamma-2c}{2}z\right\} + 2\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma}{4}}\sin\frac{\gamma-2c}{2}z\right\}.$$

(300.1) $$\frac{\vartheta_1'^3\vartheta_2(z)}{\vartheta_3\vartheta_0^3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+2c)^2\cos\frac{\gamma-2c}{2}z\right\} + 2\left\{\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-2c}{2}z\right\}.$$

Group III-b

$$\frac{\vartheta_2(z)}{\vartheta_1^3(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_2^3(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_3^3(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_0^3(z)}$$

Consider

$$F(z) = \frac{\vartheta_2(z)}{\vartheta_1^3(z)}$$

Let $t = z + \frac{\pi\tau}{2}+\frac{\pi}{4}$, $F(z)\equiv\varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order three at $t = \frac{\pi\tau}{2}+\frac{\pi}{4}$. Calculating the corresponding $R_i^{(\nu)}$ and using (10) we get

(301) $$\frac{\vartheta_1'^3}{\vartheta_2}F(z) = \frac{1}{2}A_2^{(2)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{4},\ \frac{\pi\tau}{2}+\frac{\pi}{4}\right) +$$

$$+\frac{1}{2}\left\{\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}A_2^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{4},\ \frac{\pi\tau}{2}+\frac{\pi}{4}\right)$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_2(z)}{\vartheta_2\vartheta_1^3(z)} = \frac{\cos z}{\sin^3 z} + 2\sum_{n,r}(-1)^{n+1}[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz +$$

(302)

$$+\frac{1}{2}\left\{\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^n q^{2n^2+2nr}\sin 2rz\right\}.$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_1(z)}{\vartheta_2\vartheta_2^3(z)} = \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^{n+r}[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz +$$

(303)

$$+\frac{1}{2}\left\{\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1} q^{2n^2+2nr}\sin 2rz\right\}.$$

In (301) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^3\vartheta_3(z)e^{2iz}}{\vartheta_2\vartheta_0^3(z)}q^{1/2} = -\frac{i}{2}A_2^{(2)}\left(z+\pi\tau+\frac{\pi}{4},\frac{\pi\tau}{2}+\frac{\pi}{4}\right)-\frac{i}{2}\left\{\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}A_2^{(0)}\left(z+\frac{\pi}{4}+\pi\tau,\frac{\pi\tau}{2}+\frac{\pi}{4}\right).$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_3(z)}{\vartheta_2\vartheta_0^3(z)} = \sum_{n}(-1)^{n+1}[2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+1}[2(2n+r-1)]^2 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz +$$

(304)

$$+\left\{\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n}(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^n q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}.$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0(z)}{\vartheta_2\vartheta_3^3(z)} = \sum_{n}(-1)^{n+1}[2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+r+1}[2(2n+r-1)]^2 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz$$

(305)

$$+\left\{\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}.$$

From these we get

$$\frac{\vartheta_1'^3\vartheta_2(z)}{\vartheta_2\vartheta_1^3(z)} = \frac{\cos z}{\sin^3 z} + 2\sum q^N\left\{(-1)^{a+1}(\alpha+2a)^2\sin(\alpha-2a)z\right\} +$$

(302.1)

$$+\frac{1}{2}\left\{\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^a\sin(\alpha-2a)z\right\}\right\}$$

$$\frac{\vartheta_1'^3 \vartheta_1(z)}{\vartheta_2 \vartheta_2^3(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N \left\{ (-1)^{\frac{\alpha}{2}} (\alpha + 2a)^2 \sin(\alpha - 2a) z \right\} +$$

(303.1)

$$+ \frac{1}{2}\left\{ \frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'} \right\} \left\{ \frac{\sin z}{\cos z} + 4\sum q^N \left\{ (-1)^{\frac{\alpha+2}{2}} \sin(\alpha - a) z \right\} \right\} .$$

$$\frac{\vartheta_1'^3 \vartheta_3(z)}{\vartheta_2 \vartheta_0^3(z)} = \sum_n (-1)^{n+1} [2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + \frac{1}{2}\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{d-1}{2}} (\delta + 2d)^2 \cos \frac{\delta - 2d}{2} z \right\} +$$

(304.1)

$$+ \left\{ \frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'} \right\} \left\{ \sum_n (-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{d+1}{2}} \cos \frac{\delta - 2d}{2} z \right\} \right\} .$$

$$\frac{\vartheta_1'^3 \vartheta_0(z)}{\vartheta_2 \vartheta_3^3(z)} = \sum_n (-1)^{n+1} [2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + \frac{1}{2}\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\delta+2}{4}} (\delta + 2d)^2 \cos \frac{\delta - 2d}{2} z \right\} +$$

(305.1)

$$+ \left\{ \frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'} \right\} \left\{ \sum_n (-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\delta-2}{4}} \cos \frac{\delta - 2d}{2} z \right\} \right\}$$

Group IV-a

$$\frac{\vartheta_2(z)}{\vartheta_1^2(z)\vartheta_0(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_2^2(z)\vartheta_3(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_0^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_3^2(z)\vartheta_2(z)}$$

$$\frac{\vartheta_2(z)}{\vartheta_1^2(z)\vartheta_3(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_2^2(z)\vartheta_0(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_3^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_0^2(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2(z)\, e^{-iz}}{\vartheta_1^2(z)\vartheta_0(z)}$$

Let $t = z + \frac{\pi\tau}{4} + \frac{\pi}{4}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders two and one at $t = \frac{\pi\tau}{4} + \frac{\pi}{4}$ and $t = \frac{3\pi\tau}{4} + \frac{\pi}{4}$ respectively. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$\frac{\vartheta_1'^2 \vartheta_0}{\vartheta_2(z)} F(z) = A_2^{(1)}\left(z + \frac{\pi\tau}{4} + \frac{\pi}{4}, \frac{\pi\tau}{4} + \frac{\pi}{4}\right) - i\, A_2^{(0)}\left(z + \frac{\pi\tau}{4} + \frac{\pi}{4}, \frac{\pi}{4} + \frac{\pi\tau}{4}\right) +$$

(306)

$$+ i\,\vartheta_3^2\, A_2^{(0)}\left(z + \frac{\pi\tau}{4} + \frac{\pi}{4}, \frac{\pi}{4} + \frac{3\pi\tau}{4}\right)$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2(z)}{\vartheta_2\vartheta_1^2(z)\vartheta_0(z)} = \frac{\cos z}{\sin^2 z} + 4\sum_{n,r}(-1)^{n+1}[2(2n-r)-1]\, q^{n(2n+2r-1)}\cos(2r-1)z \; +$$

(307)

$$+\, 4\vartheta_3^2\sum_{n,r}(-1)^n q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \quad .$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_1(z)}{\vartheta_2\vartheta_2^2(z)\vartheta_3(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^{n+r}[2(2n+r)-1]\, q^{n(2n+2r-1)}\sin(2r-1)z \; +$$

(308)

$$+\, 4\vartheta_3^2\sum_{n,r}(-1)^{n+r+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

In (306) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3(z)e^{iz}}{\vartheta_2\vartheta_0^2(z)\vartheta_1(z)} = -i\, A_2^{(1)}\left(z+\tfrac{\pi}{4}+\tfrac{3\pi\tau}{4}, \tfrac{\pi}{4}+\tfrac{\pi\tau}{4}\right) \; +$$

$$+\, \vartheta_3^2 A_2^{(0)}\left(z+\tfrac{\pi}{4}+\tfrac{3\pi\tau}{4}, \tfrac{\pi}{4}+\tfrac{3\pi\tau}{4}\right) - A_2^{(0)}\left(z+\tfrac{\pi}{4}+\tfrac{3\pi\tau}{4}, \tfrac{\pi}{4}+\tfrac{\pi\tau}{4}\right)$$

From this follows

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3(z)}{\vartheta_2\vartheta_0^2(z)\vartheta_1(z)} = 4\sum_{n,r}(-1)^n[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin 2rz \; +$$

(309)

$$+\, \vartheta_3^2\left\{\frac{1}{\sin z} + 4\sum_{n,r}(-1)^n q^{n(2n+2r-1)}\sin(2r-1)z\right\} .$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_0(z)}{\vartheta_2\vartheta_3^2(z)\vartheta_2(z)} = 4\sum_{n,r}(-1)^{n+r+1}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(310)

$$+\, \vartheta_3^2\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n(2n+2r-1)}\cos(2r-1)z\right\}$$

In these results, replace q by $-q$. There follow

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_2(z)}{\vartheta_2\vartheta_1^2(z)\vartheta_3(z)} = \frac{\cos z}{\sin^2 z} - 4\sum_{n,r}[2(2n+r)-1]\, q^{n(2n+2r-1)}\cos(2r-1)z$$

(311)

$$+\, 4\vartheta_0^2\sum_{n,r}(-1)^{r+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \quad .$$

$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_1(z)}{\vartheta_2 \vartheta_2^2(z) \vartheta_0(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r} (-1)^r [2(2n+r)-1]\, q^{n(2n+2r-1)} \sin(2r-1)z \tag{312}$$
$$+ 4\vartheta_0^2 \sum q^{\frac{(2n-1)^2}{2} + \frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z$$

$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_0(z)}{\vartheta_2 \vartheta_3^2(z) \vartheta_1(z)} = 4\sum_{n,r} (-1)^{r+1} [2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2} + \frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z \; + \tag{313}$$
$$+ \vartheta_0^2 \left\{ \frac{1}{\sin z} + 4\sum_{n,r} q^{n(2n+2r-1)} \sin(2r-1)z \right\} .$$

$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_3(z)}{\vartheta_2 \vartheta_0^2(z) \vartheta_2(z)} = 4\sum_{n,r} [2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2} + \frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z \; + \tag{314}$$
$$+ \vartheta_0^2 \left\{ \frac{1}{\cos z} + 4\sum_{n,r} (-1)^{r+1} q^{n(2n+2r-1)} \cos(2r-1)z \right\}$$

These give

$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_2(z)}{\vartheta_2 \vartheta_1^2(z) \vartheta_0(z)} = \frac{\cos z}{\sin^2 z} + 4\sum q^N \left\{ (-1)^{b+1} (\beta+2b) \cos(\beta-2b)z \right\} + 4\vartheta_3^2 \sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{c+1}{2}} \cos\frac{\gamma-2c}{2} z \right\} . \tag{307.1}$$

$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_1(z)}{\vartheta_2 \vartheta_2^2(z) \vartheta_3(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N \left\{ (-1)^{\frac{\beta+1}{2}} (\beta+2b) \sin(\beta-2b)z \right\} + 4\vartheta_3^2 \sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\gamma}{4}} \sin\frac{\gamma-2c}{2} z \right\} . \tag{308.1}$$

$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_3(z)}{\vartheta_2 \vartheta_0^2(z) \vartheta_1(z)} = 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{c+1}{2}} (\gamma+2c) \sin\frac{\gamma-2c}{2} z \right\} + \vartheta_3^2 \left\{ \frac{1}{\sin z} + 4\sum q^N \left\{ (-1)^b \sin(\beta-2b)z \right\} \right\} . \tag{309.1}$$

$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_0(z)}{\vartheta_2 \vartheta_3^2(z) \vartheta_2(z)} = 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\gamma}{4}} (\gamma+2c) \cos\frac{\gamma-2c}{2} z \right\} + \vartheta_3^2 \left\{ \frac{1}{\cos z} + 4\sum q^N \left\{ (-1)^{\frac{\beta-1}{2}} \cos(\beta-2b)z \right\} \right\} . \tag{310.1}$$

$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_2(z)}{\vartheta_2 \vartheta_1^2(z) \vartheta_3(z)} = \frac{\cos z}{\sin^2 z} - 4\sum q^N \left\{ (\beta+2b) \cos(\beta-2b)z \right\} + 4\vartheta_0^2 \sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\gamma-2c-2}{4}} \cos\frac{\gamma-2c}{2} z \right\} . \tag{311.1}$$

(312.1) $$\frac{\vartheta_1'^2\vartheta_3\vartheta_1(z)}{\vartheta_2\vartheta_2^2(z)\vartheta_0(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta-2b+1}{2}}(\beta+2b)\sin(\beta-2b)z\right\} + 4\vartheta_0^2\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-2c}{2}z\right\}$$

(313.1) $$\frac{\vartheta_1'^2\vartheta_3\vartheta_0(z)}{\vartheta_2\vartheta_3^2(z)\vartheta_1(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c-2}{4}}(\gamma+2c)\sin\frac{\gamma-2c}{2}z\right\} + \vartheta_0^2\left\{\frac{1}{\sin z} + 4\sum q^N\left\{\sin(\beta-2b)z\right\}\right\}.$$

(314.1) $$\frac{\vartheta_1'^2\vartheta_3\vartheta_3(z)}{\vartheta_2\vartheta_0^2(z)\vartheta_2(z)} = 2\sum q^{\frac{N}{4}}\left\{(\gamma+2c)\cos\frac{\gamma-2c}{2}z\right\} + \vartheta_0^2\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}\cos(\beta-2b)z\right\}\right\}.$$

Group IV-5

$$\frac{\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_0(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_2^2(z)\vartheta_3(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_3^2(z)\vartheta_2(z)}$$

$$\frac{\vartheta_0(z)}{\vartheta_1^2(z)\vartheta_3(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_2^2(z)\vartheta_0(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_3^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_0^2(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_0(z)}$$

Let $t = z + \frac{\pi\tau}{2} + \frac{\pi}{4}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders two and one at $t = \frac{\pi\tau}{2} + \frac{\pi}{4}$ and $t = \frac{\pi}{4} + \pi\tau$ respectively. Calculating the corresponding $A_c^{(j)}$ and using (10) gives

(315) $$\frac{\vartheta_1'^2\vartheta_0}{\vartheta_3}F(z) = A_2^{(1)}\left(z + \frac{\pi\tau}{2} + \frac{\pi}{4}, \frac{\pi\tau}{2} + \frac{\pi}{4}\right) + i\,q^{\frac{1}{2}}\vartheta_2^2 A_2^{(0)}\left(z + \frac{\pi\tau}{2} + \frac{\pi}{4}, \pi\tau + \frac{\pi}{4}\right)$$

From this follows

(316) $$\frac{\vartheta_1'^2\vartheta_0\vartheta_3(z)}{\vartheta_3\vartheta_1^2(z)\vartheta_0(z)} = \frac{1}{\sin^2 z} + 2\sum_n (-1)^{n+1}4n\,q^{2n^2} + 4\sum_{n,r}(-1)^{n+1}2(2n+r)\,q^{2n^2+2nr}\cos 2rz + $$
$$+ 2\vartheta_2^2\left\{\sum_n (-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^n q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}.$$

Replacing z by $z + \frac{\pi}{2}$

(317) $$\frac{\vartheta_1'^2\vartheta_0\vartheta_0(z)}{\vartheta_3\vartheta_2^2(z)\vartheta_3(z)} = \frac{1}{\cos^2 z} + 2\sum_n (-1)^{n+1}4n\,q^{2n^2} + 4\sum_{n,r}(-1)^{n+r+1}2(2n+r)\,q^{2n^2+2nr}\cos 2rz + $$
$$+ 2\vartheta_2^2\left\{\sum_n (-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}.$$

In (315) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2(z)e^{2iz}}{\vartheta_3\vartheta_0^2(z)\vartheta_1(z)} = -iq^{-1/2}A_2^{(1)}(z+\pi\tau+\tfrac{\pi}{4},\tfrac{\pi\tau}{2}+\tfrac{\pi}{4})+\vartheta_2^2A_2^{(0)}(z+\pi\tau+\tfrac{\pi}{4},\tfrac{\pi}{4}+\pi\tau),$$

This gives

(318)
$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2(z)}{\vartheta_3\vartheta_0^2(z)\vartheta_1(z)} = 4\sum_{n,r}(-1)^n 2(2n+r-1)q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + $$
$$+\vartheta_2^2\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^n q^{2n^2+2nr}\sin 2rz\right\}$$

Replacing z by $z+\frac{\pi}{2}$

(319)
$$\frac{\vartheta_1'^2\vartheta_0\vartheta_1(z)}{\vartheta_3\vartheta_3^2(z)\vartheta_2(z)} = 4\sum_{n,r}(-1)^{n+r+1}2(2n+r-1)q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + $$
$$+\vartheta_2^2\left\{\left(\frac{\cos z}{\sin z}\right)^{-1}+4\sum_{n,r}(-1)^{n+r+1}q^{2n^2+2nr}\sin 2rz\right\}$$

In these results replace q by $-q$. We get

(320)
$$\frac{\vartheta_1'^2\vartheta_3\vartheta_0(z)}{\vartheta_0\vartheta_1^2(z)\vartheta_3(z)} = \frac{1}{\sin^2 z}+2\sum_n(-1)^{n+1}4nq^{2n^2}+4\sum_{n,r}(-1)^{n+1}2(2n+r)q^{2n^2+2nr}\cos 2rz + $$
$$+2\vartheta_2\left\{\sum_n(-1)^{n+1}q^{\frac{(2n-1)^2}{2}}+2\sum_{n,r}(-1)^{n+r+1}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

(321)
$$\frac{\vartheta_1'^2\vartheta_3\vartheta_3(z)}{\vartheta_0\vartheta_2^2(z)\vartheta_0(z)} = \frac{1}{\cos^2 z}+2\sum_n(-1)^{n+1}4nq^{2n^2}+4\sum_{n,r}(-1)^{n+r+1}2(2n+r)q^{2n^2+2nr}\cos 2rz$$
$$+2\vartheta_2^2\left\{\sum_n(-1)^{n+1}q^{\frac{(2n-1)^2}{2}}+2\sum_{n,r}(-1)^{n+1}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

(322)
$$\frac{\vartheta_1'^2\vartheta_3\vartheta_2(z)}{\vartheta_0\vartheta_3^2(z)\vartheta_1(z)} = 4\sum_{n,r}(-1)^{n+r}2(2n+r-1)q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + $$
$$+\vartheta_2^2\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^n q^{2n^2+2nr}\sin 2rz\right\}$$

(323)
$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_1(z)}{\vartheta_0 \vartheta_0^2(z) \vartheta_2(z)} = 4\sum_{n,r} (-1)^{n+1} 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r} \sin 2rz \; +$$
$$+ \vartheta_2^2 \left\{ \frac{\sin z}{\cos z} + 4\sum_{n,r} (-1)^{n+r+1} q^{2n^2+2nr} \sin 2rz \right\}.$$

(316.1)
$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_3(z)}{\vartheta_3 \vartheta_1^2(z) \vartheta_0(z)} = \frac{1}{\sin^2 z} + 4\sum_n (-1)^{n+1} 2n\, q^{2n^2} + 4\sum q^N \left\{ (-1)^{a+1} (\alpha+2a) \cos(\alpha-2a) z \right\} +$$
$$+ 2\vartheta_2^2 \left\{ \sum_n (-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{d+1}{2}} \cos \frac{\delta-2d}{2} z \right\} \right\}.$$

(317.1)
$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_0(z)}{\vartheta_3 \vartheta_2^2(z) \vartheta_3(z)} = \frac{1}{\cos^2 z} + 4\sum_n (-1)^{n+1} 2n\, q^{2n^2} + 4\sum q^N \left\{ (-1)^{\frac{\alpha+2}{2}} (\alpha+2a) \cos(\alpha-2a) z \right\} +$$
$$+ 2\vartheta_2^2 \left\{ \sum_n (-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\delta-2}{4}} \cos \frac{\delta-2d}{2} z \right\} \right\}.$$

(318.1)
$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_2(z)}{\vartheta_3 \vartheta_0^2(z) \vartheta_1(z)} = 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{d+1}{2}} (\delta+2d) \sin \frac{\delta-2d}{2} z \right\} +$$
$$+ \vartheta_2^2 \left\{ \frac{\cos z}{\sin z} + 4\sum q^N \left\{ (-1)^a \sin(\alpha-2a) z \right\} \right\}.$$

(319.1)
$$\frac{\vartheta_1'^2 \vartheta_0 \vartheta_1(z)}{\vartheta_3 \vartheta_3^2(z) \vartheta_2(z)} = 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\delta+2}{4}} (\delta+2d) \sin \frac{\delta-2d}{2} z \right\} +$$
$$+ \vartheta_2^2 \left\{ \frac{\sin z}{\cos z} + 4\sum q^N \left\{ (-1)^{\frac{\alpha+2}{2}} \sin(\alpha-2a) z \right\} \right\}.$$

(320.1)
$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_0(z)}{\vartheta_0 \vartheta_1^2(z) \vartheta_3(z)} = \frac{1}{\sin^2 z} + 2\sum_n (-1)^{n+1} 4n\, q^{2n^2} + 4\sum_n q^N \left\{ (-1)^{a+1} (\alpha+2a) \cos(\alpha-2a) z \right\} +$$
$$+ 2\vartheta_2^2 \left\{ \sum_n (-1)^{n+1} q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\delta+2}{4}} \cos \frac{\delta-2d}{2} z \right\} \right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_3(z)}{\vartheta_0 \vartheta_2^2(z) \vartheta_0(z)} = \frac{1}{\cos^2 z} + 2\sum_n (-1)^{n+1} 4n\, q^{2n^2} + 4\sum q^N \left\{(-1)^{\frac{\alpha+2}{2}} (\alpha+2a) \cos(\alpha-2a) z\right\} +$$

(321.1)

$$+ 2\vartheta_2^2 \left\{ \sum_n (-1)^{n+1} q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}} \left\{(-1)^{\frac{d-1}{2}} \cos \frac{\delta-2d}{2} z\right\}\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_2(z)}{\vartheta_0 \vartheta_3^2(z) \vartheta_1(z)} = 2\sum q^{\frac{N}{4}} \left\{(-1)^{\frac{\delta-2}{4}} (\delta+2d) \sin \frac{\delta-2d}{2} z\right\} +$$

(322.1)

$$+ \vartheta_2^2 \left\{ \frac{\cos z}{\sin z} + 4\sum q^N \left\{(-1)^a \sin(\alpha-2a) z\right\}\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_3 \vartheta_1(z)}{\vartheta_0 \vartheta_0^2(z) \vartheta_2(z)} = 2\sum q^{\frac{N}{4}} \left\{(-1)^{\frac{d-1}{2}} (\delta+2d) \sin \frac{\delta-2d}{2} z\right\} +$$

(323.1)

$$+ \vartheta_2^2 \left\{ \frac{\sin z}{\cos z} + 4\sum q^N \left\{(-1)^{\frac{\alpha+2}{2}} \sin(\alpha-2a) z\right\}\right\}$$

Group IV-c

$$\frac{\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_2(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_2^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0^2(z)\vartheta_3(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_3^2(z)\vartheta_0(z)}$$

$$\frac{\vartheta_0(z)}{\vartheta_1^2(z)\vartheta_2(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_2^2(z)\vartheta_1(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_3^2(z)\vartheta_0(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_0^2(z)\vartheta_3(z)}$$

Consider

$$F(z) = \frac{\vartheta_3(z)\, e^{-iz}}{\vartheta_1^2(z)\vartheta_2(z)}$$

Let $t = z + \frac{\pi\tau}{4}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders two and one at $t = \frac{\pi\tau}{4}$ and $t = \frac{\pi\tau}{4} + \frac{\pi}{2}$, respectively. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

(324) $$\frac{\vartheta_1'^2 \vartheta_2}{\vartheta_3} F(z) = A_2^{(1)}\left(z+\tfrac{\pi\tau}{4}, \tfrac{\pi\tau}{4}\right) - i\, A_2^{(0)}\left(z+\tfrac{\pi\tau}{4}, \tfrac{\pi\tau}{4}\right) + i\,\vartheta_0^2 A_2^{(0)}\left(z+\tfrac{\pi\tau}{4}, \tfrac{\pi\tau}{4}+\tfrac{\pi}{2}\right)$$

From this follows

$$\frac{\vartheta_1'^2 \vartheta_2 \vartheta_3(z)}{\vartheta_3 \vartheta_1^2(z)\vartheta_2(z)} = \frac{\cos z}{\sin^2 z} - 4\sum_{n,r}[2(2n+r)-1]\, q^{n(2n+2r-1)} \cos(2r-1)z \; +$$

(325)

$$+\, \vartheta_0^2\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{n(2n+2r-1)} \cos(2r-1)z\right\}$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^2 \vartheta_2 \vartheta_0(z)}{\vartheta_3 \vartheta_2^2(z)\vartheta_1(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^r[2(2n+r)-1]\, q^{n(2n+2r-1)} \sin(2r-1)z \; +$$

(326)

$$+\, \vartheta_0^2\left\{\frac{1}{\sin z} + 4\sum_{n,r} q^{n(2n+2r-1)} \sin(2r-1)z\right\}.$$

In (324) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2 \vartheta_2 \vartheta_2(z)e^{iz}}{\vartheta_3 \vartheta_0^2(z)\vartheta_3(z)} = -A_2^{(1)}\left(z+\frac{3\pi\tau}{4},\frac{\pi\tau}{4}\right) + iA_2^{(0)}\left(z+\frac{3\pi\tau}{4},\frac{\pi\tau}{4}\right) - i\vartheta_0^2 A_2^{(0)}\left(z+\frac{3\pi\tau}{4},\frac{\pi\tau}{4}+\frac{\pi}{2}\right).$$

which gives

$$\frac{\vartheta_1'^2 \vartheta_2 \vartheta_2(z)}{\vartheta_3 \vartheta_0^2(z)\vartheta_3(z)} = 4\sum_{n,r}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z \; +$$

(327)

$$+\, 4\vartheta_0^2\sum_{n,r}(-1)^{r+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z\,.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^2 \vartheta_2 \vartheta_1(z)}{\vartheta_3 \vartheta_3^2(z)\vartheta_0(z)} = 4\sum_{n,r}(-1)^{r+1}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z \; +$$

(328)

$$+\, 4\vartheta_0^2\sum_{n,r} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z\,.$$

In these results replace q by $-q$. This gives

$$\frac{\vartheta_1'^2 \vartheta_2 \vartheta_0(z)}{\vartheta_0 \vartheta_1^2(z)\vartheta_2(z)} = \frac{\cos z}{\sin^2 z} + 4\sum_{n,r}(-1)^{n+1}[2(2n+r)-1]\, q^{2(2n+2r-1)} \cos(2r-1)z \; +$$

(329)

$$+\, \vartheta_3^2\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n(2n+2r-1)} \cos(2r-1)z\right\}$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_3(z)}{\vartheta_0\vartheta_2^2(z)\vartheta_1(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^{r+n}[2(2n+r)-1]\, q^{n(2n+2r-1)}\sin(2r-1)z + \tag{330}$$

$$+\vartheta_3^2\left\{\frac{1}{\sin z} + 4\sum_{n,r}(-1)^n q^{n(2n+2r-1)}\sin(2r-1)z\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3\vartheta_2(z)}{\vartheta_0\vartheta_3^2(z)\vartheta_0(z)} = 4\sum_{n,r}(-1)^{n+r}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + \tag{331}$$

$$+4\vartheta_3^2\sum_{n,r}(-1)^{n+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_1(z)}{\vartheta_0\vartheta_0^2(z)\vartheta_3(z)} = 4\sum_{n,r}(-1)^{n+1}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z + \tag{332}$$

$$+4\vartheta_3^2\sum_{n,r}(-1)^{n+r} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z.$$

From these follow

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_3(z)}{\vartheta_3\vartheta_1^2(z)\vartheta_2(z)} = \frac{\cos z}{\sin^2 z} - 4\sum q^N\{(\beta+2b)\cos(\beta-2b)z\} + \tag{325.1}$$

$$+\vartheta_0^2\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}\cos(\beta-2b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0(z)}{\vartheta_3\vartheta_2^2(z)\vartheta_1(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta-2b+1}{2}}(\beta+2b)\sin(\beta-2b)z\right\} + \tag{326.1}$$

$$+\vartheta_0^2\left\{\frac{1}{\sin z} + 4\sum q^N\{\sin(\beta-2b)z\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_2(z)}{\vartheta_3\vartheta_0^2(z)\vartheta_3(z)} = 2\sum q^{\frac{N}{4}}\left\{(\gamma+2c)\cos\frac{\gamma-2c}{2}z\right\} + 4\vartheta_0^2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c-2}{4}}\cos\frac{\gamma-2c}{2}z\right\}. \tag{327.1}$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_1(z)}{\vartheta_3\vartheta_3^2(z)\vartheta_0(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c-2}{4}}(\gamma+2c)\sin\frac{\gamma-2c}{2}z\right\} + 4\vartheta_0^2\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-2c}{2}z\right\}. \tag{328.1}$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0(z)}{\vartheta_0\vartheta_1^2(z)\vartheta_2(z)} = \frac{\cos z}{\sin^2 z} + 4\sum q^N\left\{(-1)^{b+1}(\beta+2b)\cos(\beta-2b)z\right\} + \tag{329.1}$$

$$+\vartheta_3^2\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-1}{2}}\cos(\beta-2b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_3(z)}{\vartheta_0\vartheta_2^2(z)\vartheta_1(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta+1}{2}}(\beta+2b)\sin(\beta-2b)z\right\} + \tag{330.1}$$

$$+\vartheta_3^2\left\{\frac{1}{\sin z} + 4\sum q^N\left\{(-1)^{b}\sin(\beta-2b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_2(z)}{\vartheta_0\vartheta_3^2(z)\vartheta_0(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+4}{4}}(\gamma+2c)\cos\frac{\gamma-2c}{2}z\right\} + 4\vartheta_3^2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-2c}{2}z\right\}. \tag{331.1}$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_1(z)}{\vartheta_0\vartheta_0^2(z)\vartheta_3(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+2c)\sin\frac{\gamma-2c}{2}z\right\} + 4\vartheta_3^2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+4}{4}}\sin\frac{\gamma-2c}{2}z\right\}. \tag{332.1}$$

Group V

$$\frac{\vartheta_0(z)}{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)}$$

Consider

$$F(z) = \frac{\vartheta_0(z)}{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has simple poles at $t = \frac{\pi\tau}{2}$, $t = \frac{\pi\tau}{2} + \frac{\pi}{2}$ and $t = \pi\tau + \frac{\pi}{2}$. Calculating the corresponding values of $A_i^{(j)}$ and using (10) gives

$$\vartheta_1'^2 F(z) = \vartheta_0^2 A_2^{(0)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) - \vartheta_3^2 A_2^{(0)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}+\frac{\pi}{2}\right) + \vartheta_2^2 q^{\frac{1}{2}} A_2^{(0)}\left(z+\frac{\pi\tau}{2}, \pi\tau+\frac{\pi}{2}\right) \tag{333}$$

From this follows

$$\frac{\vartheta_1'^2\vartheta_0(z)}{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)} = \vartheta_0^2\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{2n^2+2nr}\sin 2rz\right\} \tag{334}$$

$$+\vartheta_3^2\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{2n^2+2nr}\sin 2rz\right\} + 4\vartheta_2^2\sum_{n,r}(-1)^r q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz.$$

Replacing z by $z-\frac{\pi}{2}$

$$(335)\quad \frac{\vartheta_1'^2\vartheta_3(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)} = \vartheta_0^2\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{2n^2+2nr}\sin 2rz\right\} + $$
$$+\vartheta_3^2\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{2n^2+2nr}\sin 2rz\right\} - 4\vartheta_2^2\sum_{n,r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz$$

In (333) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2\vartheta_1(z)e^{2iz}q^{1/2}}{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)} = \vartheta_0^2 A_2^{(0)}(z+\pi\tau, \tfrac{\pi\tau}{2}) - \vartheta_3^2 A_2^{(0)}(z+\pi\tau, \tfrac{\pi\tau}{2}+\tfrac{\pi}{2}) + \vartheta_2^2 q^{1/2} A_2^{(0)}(z+\pi\tau, \pi\tau+\tfrac{\pi}{2}).$$

There follows

$$(336)\quad \frac{\vartheta_1'^2\vartheta_1(z)}{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)} = 4\vartheta_0^2\sum_{n,r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + 4\vartheta_3^2\sum_{n,r}(-1)^r q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + $$
$$+\vartheta_2^2\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{2n^2+2nr}\sin 2rz\right\}.$$

Replacing z by $z-\frac{\pi}{2}$

$$(337)\quad \frac{\vartheta_1'^2\vartheta_2(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)} = 4\vartheta_0^2\sum_{n,r}(-1)^{r+1} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz - 4\vartheta_3^2\sum_{n,r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + $$
$$+\vartheta_2^2\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{2n^2+2nr}\sin 2rz\right\}$$

From these we have

$$(334.1)\quad \frac{\vartheta_1'^2\vartheta_0(z)}{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)} = \vartheta_0^2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\{\sin(\alpha-2a)z\}\right\} + $$
$$+\vartheta_3^2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\{(-1)^{\frac{\alpha-2a-2}{2}}\sin(\alpha-2a)z\}\right\} + 4\vartheta_2^2\sum q^{\frac{N}{4}}\{(-1)^{\frac{\delta-2d}{4}}\sin\tfrac{\delta-2d}{2}z\}.$$

$$(335.1)\quad \frac{\vartheta_1'^2\vartheta_3(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)} = \vartheta_0^2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\{(-1)^{\frac{\alpha-2a-2}{2}}\sin(\alpha-2a)z\}\right\} + $$
$$+\vartheta_3^2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\{\sin(\alpha-2a)z\}\right\} - 4\vartheta_2^2\sum q^{\frac{N}{4}}\{\sin\tfrac{\delta-2d}{2}z\}.$$

$$\frac{\vartheta_1'^2\,\vartheta_1(z)}{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)} = 4\vartheta_0^2\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-2d}{2}z\right\} + 4\vartheta_3^2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}\sin\frac{\delta-2d}{2}z\right\} + \tag{336.1}$$

$$+\vartheta_2^2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{d-2a-2}{2}}\sin(d-2a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\,\vartheta_2(z)}{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)} = 4\vartheta_0^2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d-4}{4}}\sin\frac{\delta-2d}{2}z\right\} - 4\vartheta_3^2\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-2d}{2}z\right\} + \tag{337.1}$$

$$+\vartheta_2^2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(d-2a)z\right\}\right\}.$$

Group VI-a

$$\frac{\vartheta_0^2(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_3^4(z)}$$

$$\frac{\vartheta_3^2(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_3^4(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_0^4(z)}$$

Consider

$$F(z) = \frac{\vartheta_0^2(z)}{\vartheta_1^4(z)}$$

Let $t = z+\frac{\pi}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order four at $t=\frac{\pi}{2}$. Calculating the corresponding $A_i^{(i)}$ and using (10) gives

$$\frac{\vartheta_1'^4}{\vartheta_0^2}F(z) = \frac{1}{6}A_2^{(3)}\left(z+\frac{\pi}{2},\frac{\pi}{2}\right) + \frac{1}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 2\frac{\vartheta_1'''}{\vartheta_1'}\right\}A_2^{(1)}\left(z+\frac{\pi}{2},\frac{\pi}{2}\right) \tag{338}$$

From this follows

$$\frac{\vartheta_1'^4\vartheta_0^2(z)}{\vartheta_0^2\vartheta_1^4(z)} = \frac{1}{\sin^4 z} + \frac{1}{3\sin^2 z}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 2\frac{\vartheta_1'''}{\vartheta_1'} - 2\right\} + \frac{1}{3}\sum_n (4n)^3 q^{2n^2} + \frac{2}{3}\sum_{n,r}[2(2n+r)]^3 q^{2n^2+2nr}\cos 2rz + \tag{339}$$

$$-\frac{4}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2n\, q^{2n^2} + \sum_{n,r} 2(2n+r)\, q^{2n^2+2nr}\cos 2rz\right\}.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\vartheta_3^2(z)}{\vartheta_0^2\vartheta_2^4(z)} = \frac{1}{\cos^4 z} + \frac{1}{3\cos^2 z}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 2\frac{\vartheta_1'''}{\vartheta_1'} - 2\right\} + \frac{1}{3}\sum_n (4n)^3 q^{2n^2} + \frac{2}{3}\sum_{n,r}(-1)^r[2(2n+r)]^3 q^{2n^2+2nr}\cos 2rz + \tag{340}$$

$$-\frac{4}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2n\, q^{2n^2} + \sum_{n,r}(-1)^r 2(2n+r)\, q^{2n^2+2nr}\cos 2rz\right\}.$$

In (338) replace by , obtaining

$$q^{\frac{1}{4}}\frac{\vartheta_1'^4\vartheta_1^2(z)e^{-2iz}}{\vartheta_0^2\vartheta_0^4(z)} = -\frac{1}{6}A_2^{(3)}(z,\tfrac{\pi I}{2}) - \frac{1}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 2\frac{\vartheta_1'''}{\vartheta_1'}\right\}A_2'''(z,\tfrac{\pi I}{2})$$

From this follows

(341)
$$\frac{\vartheta_1'^4\vartheta_1^2(z)}{\vartheta_0^2\vartheta_0^4(z)} = -\frac{1}{3}\sum_n[2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} - \frac{2}{3}\sum_{n,r}[2(2n+r-1)]^3 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz \; +$$
$$+\frac{2}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2(2n-1)q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}2(2n+r-1)q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}.$$

Replacing z by $z-\frac{\pi}{2}$

(342)
$$\frac{\vartheta_1'^4\vartheta_2^2(z)}{\vartheta_0^2\vartheta_3^4(z)} = -\frac{1}{3}\sum_n[2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} + \frac{2}{3}\sum_{n,r}(-1)^{r+1}[2(2n+r-1)]^3 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz$$
$$+\frac{2}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2(2n-1)q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^r 2(2n+r-1)q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}.$$

In these results replace q by $-q$. This gives

(343)
$$\frac{\vartheta_1'^4\vartheta_3^2(z)}{\vartheta_3^2\vartheta_1^4(z)} = \frac{1}{\sin^4 z} + \frac{1}{3\sin^2 z}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 2\frac{\vartheta_1'''}{\vartheta_1'} - 2\right\} + \frac{1}{3}\sum_n(4n)^3 q^{2n^2} + \frac{2}{3}\sum_{n,r}[2(2n+r)]^3 q^{2n^2+2nr}\cos 2rz \; +$$
$$-\frac{4}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 2\frac{\vartheta_0'''}{\vartheta_1'}\right\}\left\{\sum_n 2n\, q^{2n^2} + \sum_{n,r}2(2n+r)\, q^{2n^2+2nr}\cos 2rz\right\}$$

(344)
$$\frac{\vartheta_1'^4\vartheta_0^2(z)}{\vartheta_3^2\vartheta_2^4(z)} = \frac{1}{\cos^4 z} + \frac{1}{3\cos^2 z}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 2\frac{\vartheta_1'''}{\vartheta_1'} - 2\right\} + \frac{1}{3}\sum_n(4n)^3 q^{2n^2} + \frac{2}{3}\sum_{n,r}(-1)^r[2(2n+r)]^3 q^{2n^2+2nr}\cos 2rz \; +$$
$$-\frac{4}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 2\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\sum_n 2n\, q^{2n^2} + \sum_{n,r}2(2n+r)(-1)^r q^{2n^2+2nr}\cos 2rz\right\}.$$

(345)
$$\frac{\vartheta_1'^4\vartheta_1^2(z)}{\vartheta_3^2\vartheta_3^4(z)} = \frac{1}{3}\sum_n[2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^r[2(2n+r-1)]^3 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz \; +$$
$$-\frac{2}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2(2n-1)q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^r 2(2n+r-1)q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

(346)
$$\frac{\vartheta_1'^4\vartheta_2^2(z)}{\vartheta_3^2\vartheta_0^4(z)} = \frac{1}{3}\sum_n [2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} + \frac{2}{3}\sum_{n,r}[2(2n+r-1)]^3 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz \;+$$
$$-\frac{2}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2(2n-1)q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}2(2n+r-1)q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}.$$

From these follow

(339.1)
$$\frac{\vartheta_1'^4\vartheta_0^2(z)}{\vartheta_0^2\vartheta_1^4(z)} = \frac{1}{\sin^4 z} + \frac{1}{3\sin^2 z}\left\{3\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_1'''}{\vartheta_1'}-2\right\} + \frac{1}{3}\sum_n (4n)^3 q^{2n^2} + \frac{2}{3}\sum q^N\{(\alpha+2a)^3\cos(\alpha-2a)z\} \;+$$
$$-\frac{4}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2n\, q^{2n^2} + \sum q^N\{(\alpha+2a)\cos(\alpha-2a)z\}\right\}.$$

(340.1)
$$\frac{\vartheta_1'^4\vartheta_3^2(z)}{\vartheta_0^2\vartheta_2^4(z)} = \frac{1}{\cos^4 z} + \frac{1}{3\cos^2 z}\left\{3\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_1'''}{\vartheta_1'}-2\right\} + \frac{1}{3}\sum_n (4n)^3 q^{2n^2} + \frac{2}{3}\sum q^N\left\{(-1)^{\frac{\alpha-2a}{2}}(\alpha+2a)^3\cos(\alpha-2a)z\right\} \;+$$
$$-\frac{4}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2n\, q^{2n^2} + \sum q^N\left\{(-1)^{\frac{\alpha-2a}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\}\right\}.$$

(341.1)
$$\frac{\vartheta_1'^4\vartheta_1^2(z)}{\vartheta_0^2\vartheta_0^4(z)} = -\frac{1}{3}\left\{\sum_n [2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} + \frac{1}{4}\sum q^{\frac{N}{4}}\left\{(\delta+2d)^3\cos\frac{\delta-2d}{2}z\right\}\right\} \;+$$
$$+\frac{2}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2(2n-1)q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}.$$

(342.1)
$$\frac{\vartheta_1'^4\vartheta_2^2(z)}{\vartheta_0^2\vartheta_3^4(z)} = -\frac{1}{3}\left\{\sum_n [2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} + \frac{1}{4}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)^3\cos\frac{\delta-2d}{2}z\right\}\right\}$$
$$+\frac{2}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2(2n-1)q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}.$$

(343.1)
$$\frac{\vartheta_1'^4\vartheta_3^2(z)}{\vartheta_3^2\vartheta_1^4 z} = \frac{1}{\sin^4 z} + \frac{1}{3\sin^2 z}\left\{3\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_1'''}{\vartheta_1'}-2\right\} + \frac{1}{3}\sum_n (4n)^3 q^{2n^2} + \frac{2}{3}\sum q^N\{(\alpha+2a)^3\cos(\alpha-2a)z\} \;+$$
$$-\frac{4}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2n\, q^{n^2} + \sum q^N\{(\alpha+2a)\cos(\alpha-2a)z\}\right\}.$$

(344.1)
$$\frac{\vartheta_1'^4\vartheta_0^2(z)}{\vartheta_3^2\vartheta_2^4(z)} = \frac{1}{\cos^4 z} + \frac{1}{3\cos^2 z}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 2\frac{\vartheta_1'''}{\vartheta_1'} - 2\right\} + \frac{1}{3}\sum (4n)^3 q^{2n^2} + \frac{2}{3}\sum q^N \left\{(-1)^{\frac{\alpha-2a}{2}}(\alpha+2a)^3\cos(\alpha-2a)z\right\} +$$
$$-\frac{4}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2n\, q^{2n^2} + \sum q^N \left\{(-1)^{\frac{\alpha-2a}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\}\right\}.$$

(346.1)
$$\frac{\vartheta_1'^4\vartheta_2^2(z)}{\vartheta_3^2\vartheta_0^4(z)} = \frac{1}{3}\sum_n [2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} + \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(\delta+2d)^3\cos\frac{\delta-2d}{2}z\right\} +$$
$$-\frac{2}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2(2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}.$$

(345.1)
$$\frac{\vartheta_1'^4\vartheta_1^2(z)}{\vartheta_3^2\vartheta_3^4(z)} = \frac{1}{3}\sum_n [2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} + \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)^3\cos\frac{\delta-2d}{2}z\right\} +$$
$$-\frac{2}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2(2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}.$$

Group VI-b

$$\frac{\vartheta_2^2(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_3^4(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^2(z)}{\vartheta_1^4(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order four at $t = \frac{\pi\tau}{2}$. Calculating the corresponding value of $R_i^{(j)}$ and using (10) gives

(347)
$$\frac{\vartheta_1'^4}{\vartheta_2^2}F(z) = \frac{1}{6}A_2^{(3)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right) + \frac{1}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}A_2^{(1)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right)$$

There follows

(348)
$$\frac{\vartheta_1'^4\vartheta_2^2(z)}{\vartheta_2^2\vartheta_1^4(z)} = \frac{1}{\sin^4 z} + \frac{1}{3\sin^2 z}\left\{3\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_1'''}{\vartheta_1'}-2\right\} + \frac{1}{3}\sum_n (4n)^3 q^{2n^2} + \frac{2}{3}\sum_{n,r}[2(2n+r)]^3 q^{2n^2+2nr}\cos 2rz$$
$$-\frac{4}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2n\, q^{2n^2} + \sum_{n,r}2(2n+r)\, q^{2n^2+2nr}\cos 2rz\right\}.$$

Replace z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\vartheta_1^2(z)}{\vartheta_2^2\vartheta_2^4(z)} = \frac{1}{\cos^4 z} + \frac{1}{3\cos^2 z}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 2\frac{\vartheta_1'''}{\vartheta_1'} - 2\right\} + \frac{1}{3}\sum_n (4n)^3 q^{2n^2} + \frac{2}{3}\sum_{n,r}(-1)^r q^{2n^2+2nr}[2(2n+r)]^3\cos 2rz +$$

(349)
$$-\frac{4}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2n\, q^{2n^2} + \sum_{n,r}(-1)^r\, 2(2n+r)\, q^{2n^2+2nr}\cos 2rz\right\}.$$

In (347) replace z by $z-\frac{\pi\tau}{2}$, obtaining

$$q^{\frac{1}{2}}\frac{\vartheta_1'^4\vartheta_3^2(z)e^{-2iz}}{\vartheta_2^2\vartheta_0^4(z)} = \frac{1}{6}A_2^{(3)}\left(z,\frac{\pi\tau}{2}\right) + \frac{1}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}A_2^{(1)}\left(z,\frac{\pi\tau}{2}\right)$$

From which we find

$$\frac{\vartheta_1'^4\vartheta_3^2(z)}{\vartheta_2^2\vartheta_0^4(z)} = \frac{1}{3}\sum_n[2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}[2(2n+r-1)]^3 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz +$$

(350)
$$-\frac{2}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{2\sum_n(2n-1)\,q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}2(2n+r-1)\,q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}.$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\vartheta_0^2(z)}{\vartheta_2^2\vartheta_3^4(z)} = \frac{1}{3}\sum_n[2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} + \frac{2}{3}\sum_{n,r}(-1)^r[2(2n+r-1)]^3 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz +$$

(351)
$$-\frac{2}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2(2n-1)\,q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}2(2n+r-1)(-1)^r q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}.$$

From these follow

$$\frac{\vartheta_1'^4\vartheta_2^2(z)}{\vartheta_2^2\vartheta_1^4(z)} = \frac{1}{\sin^4 z} + \frac{1}{3\sin^2 z}\left\{3\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_1'''}{\vartheta_1'}-2\right\} + \frac{1}{3}\sum_n(4n)^3 q^{2n^2} + \frac{2}{3}\sum q^N\left\{(\alpha+2a)^3\cos(\alpha-2a)z\right\} +$$

(348.1)
$$-\frac{4}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{2\sum_n n\,q^{2n^2} + \sum q^N\left\{(\alpha+2a)\cos(\alpha-2a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_1^2(z)}{\vartheta_2^2\vartheta_2^4(z)} = \frac{1}{\cos^4 z} + \frac{1}{3\cos^2 z}\left\{3\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_1'''}{\vartheta_1'}-2\right\} + \frac{1}{3}\sum_n(4n)^3 q^{2n^2} + \frac{2}{3}\sum q^N\left\{(-1)^{\frac{\alpha-2a}{2}}(\alpha+2a)^3\cos(\alpha-2a)z\right\}$$

(349.1)
$$-\frac{4}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 2n\,q^{2n^2} + \sum q^N\left\{(-1)^{\frac{\alpha-2a}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^4 \vartheta_3^2(z)}{\vartheta_2^2 \vartheta_0^4(z)} = \frac{1}{3}\sum_n [2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} + \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(\delta+2d)^3 \cos\frac{\delta-2d}{2} z\right\} +$$

(350.1)

$$-\frac{2}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n (2n-1)2\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(\delta+2d)\cos\frac{\delta-2d}{2} z\right\}\right\}.$$

$$\frac{\vartheta_1'^4 \vartheta_0^2(z)}{\vartheta_2^2 \vartheta_3^4(z)} = \frac{1}{3}\sum_n [2(2n-1)]^3 q^{\frac{(2n-1)^2}{2}} + \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)^3 \cos\frac{\delta-2d}{2} z\right\} +$$

(351.1)

$$-\frac{2}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 2\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum 2(2n-1) q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{2}}(\delta+2d)\cos\frac{\delta-2d}{2} z\right\}\right\}.$$

Group VII-a

$$\frac{\vartheta_0(z)\vartheta_2(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_1(z)\vartheta_3(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_1(z)\vartheta_3(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_0(z)\vartheta_2(z)}{\vartheta_3^4(z)}$$

$$\frac{\vartheta_3(z)\vartheta_2(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_1(z)\vartheta_0(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_1(z)\vartheta_0(z)}{\vartheta_3^4(z)} \qquad \frac{\vartheta_3(z)\vartheta_2(z)}{\vartheta_0^4(z)}$$

Consider

$$F(z) = \frac{\vartheta_0(z)\vartheta_2(z)}{\vartheta_1^4(z)} e^{-iz}$$

Let $t = z + \frac{\pi\tau}{4} + \frac{\pi}{4}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order four at $t = \frac{\pi\tau}{4} + \frac{\pi}{4}$. Calculating the corresponding $P_i^{(i)}$ and using (10) gives

$$\frac{\vartheta_1'^4}{\vartheta_0\vartheta_2} F(z) = \frac{1}{6} H_2^{(3)}\left(z + \frac{\pi\tau}{4} + \frac{\pi}{4}, \frac{\pi\tau}{4} + \frac{\pi}{4}\right) - \frac{i}{2} H_2^{(2)}\left(z + \frac{\pi\tau}{4} + \frac{\pi}{4}, \frac{\pi\tau}{4} + \frac{\pi}{4}\right) +$$

(352)

$$-\frac{1}{6}\left\{3\frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_1'''}{\vartheta_1'} + 3\right\} H_2^{(1)}\left(z + \frac{\pi\tau}{4} + \frac{\pi}{4}, \frac{\pi\tau}{4} + \frac{\pi}{4}\right) + \frac{i}{6}\left\{3\frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_1'''}{\vartheta_1'} + 1\right\} H_2^{(0)}\left(z + \frac{\pi\tau}{4} + \frac{\pi}{4}, \frac{\pi\tau}{4} + \frac{\pi}{4}\right),$$

There follows

$$\frac{\vartheta_1'^4 \vartheta_0(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z} - \frac{\cos z}{6\sin^2 z}\left\{3\frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_1'''}{\vartheta_1'} + 1\right\} + \frac{2}{3}\sum_{n,r}(-1)^n [2(2n+r)-1]^3 q^{n(2n+2r-1)} \cos(2r-1)z +$$

(353)

$$+\frac{2}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^n [2(2n+r)-1]\, q^{n(2n+2r-1)} \cos(2r-1)z.$$

Replacing z by $z - \frac{\pi}{2}$

$$\frac{\vartheta_1'^4\,\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z} - \frac{\sin z}{6\cos^2 z}\left\{3\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_1'''}{\vartheta_1'}+1\right\} + \frac{2}{3}\sum_{n,r}(-1)^{n+r+1}[2(2n+r)-1]^3\, q^{n(2n+2r-1)}\sin(2r-1)z \; +$$

$$+\frac{2}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1}[2(2n+r)-1]\, q^{n(2n+2r-1)}\sin(2r-1)z\,. \tag{354}$$

In (352) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{i e^{iz}\vartheta_1(z)\vartheta_3(z)\vartheta_1'^4}{\vartheta_0\vartheta_2\vartheta_0^4(z)} = \frac{1}{6}A_2^{(3)}\left(z+\tfrac{3\pi\tau}{4}+\tfrac{\pi}{4},\tfrac{\pi\tau}{4}+\tfrac{\pi}{4}\right) - \frac{i}{2}A_2^{(2)}\left(z+\tfrac{3\pi\tau}{4}+\tfrac{\pi}{4},\tfrac{\pi\tau}{4}+\tfrac{\pi}{4}\right) \; +$$

$$-\frac{1}{6}\left\{3\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_1'''}{\vartheta_1'}+3\right\}A_2^{(1)}\left(z+\tfrac{3\pi\tau}{4}+\tfrac{\pi}{4},\tfrac{\pi\tau}{4}+\tfrac{\pi}{4}\right) + \frac{i}{6}\left\{3\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_1'''}{\vartheta_1'}+1\right\}A_2^{(0)}\left(z+\tfrac{3\pi\tau}{4}+\tfrac{\pi}{4},\tfrac{\pi\tau}{4}+\tfrac{\pi}{4}\right).$$

From this follows

$$\frac{\vartheta_1'^4\,\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_0^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+1}[2(2n+r)-3]^3\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

$$+\frac{2}{3}\left\{3\frac{\vartheta_0''}{\vartheta_3}+\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+1}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z\,. \tag{355}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\,\vartheta_0(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_3^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+r}[2(2n+r)-3]^3\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

$$+\frac{2}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z\,. \tag{356}$$

In these results replace q by $-q$. We get

$$\frac{\vartheta_1'^4\,\vartheta_2(z)\vartheta_3(z)}{\vartheta_3\vartheta_2\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z} - \frac{\cos z}{6\sin^2 z}\left\{3\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_1'''}{\vartheta_1'}+1\right\} + \frac{2}{3}\sum_{n,r}[2(2n+r)-1]^3\, q^{n(2n+2r-1)}\cos(2r-1)z \; +$$

$$+\frac{2}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}[2(2n+r)-1]\, q^{n(2n+2r-1)}\cos(2r-1)z\,. \tag{357}$$

$$\frac{\vartheta_1'^4\,\vartheta_0(z)\vartheta_1(z)}{\vartheta_3\vartheta_2\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z} - \frac{\sin z}{6\cos^2 z}\left\{3\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_1'''}{\vartheta_1'}+1\right\} + \frac{2}{3}\sum_{n,r}(-1)^{r+1}[2(2n+r)-1]^3\, q^{n(2n+2r-1)}\sin(2r-1)z \; +$$

$$+\frac{2}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}[2(2n+r)-1](-1)^{r+1}\, q^{n(2n+2r-1)}\sin(2r-1)z\,. \tag{358}$$

$$\frac{\vartheta_1'^4\,\vartheta_1(z)\vartheta_0(z)}{\vartheta_2\vartheta_3\vartheta_3^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{r+1}[2(2n+r)-3]^3\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(359)

$$+\frac{2}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \, .$$

$$\frac{\vartheta_1'^4\,\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_0^4(z)} = \frac{2}{3}\sum[2(2n+r)-3]^3\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(360)

$$+\frac{2}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

From these follow

$$\frac{\vartheta_1'^4\,\vartheta_0(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z}-\frac{\cos z}{6\sin^2 z}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_3''}{\vartheta_3}+1\right\}+\frac{2}{3}\sum q^N\left\{(-1)^b(\beta+2b)^3\cos(\beta-2b)z\right\}+$$

(353.1)

$$+\frac{2}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^N\left\{(-1)^b(\beta+2b)\cos(\beta-2b)z\right\}$$

$$\frac{\vartheta_1'^4\,\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z}-\frac{\sin z}{6\cos^2 z}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_3''}{\vartheta_3}+1\right\}+\frac{2}{3}\sum q^N\left\{(-1)^{\frac{\beta-1}{2}}(\beta+2b)^3\sin(\beta-2b)z\right\}+$$

(354.1)

$$+\frac{2}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^N\left\{(-1)^{\frac{\beta-1}{2}}(\beta+2b)\sin(\beta-2b)z\right\} .$$

$$\frac{\vartheta_1'^4\,\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_0^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+2c)^3\sin\frac{\gamma-2c}{2}z\right\} \; +$$

(355.1)

$$+\frac{1}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+2c)\sin\frac{\gamma-2c}{2}z\right\} .$$

$$\frac{\vartheta_1'^4\,\vartheta_0(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_3^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+4}{4}}(\gamma+2c)^3\cos\frac{\gamma-2c}{2}z\right\} \; +$$

(356.1)

$$+\frac{1}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+4}{4}}(\gamma+2c)\cos\frac{\gamma-2c}{2}z\right\} .$$

(357.1)
$$\frac{\vartheta_1'^4\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z} - \frac{\cos z}{6\sin^2 z}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_0''}{\vartheta_0}+1\right\} + \frac{2}{3}\sum q^N\left\{(\beta+2b)^3\cos(\beta-2b)z\right\} +$$
$$+\frac{2}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^N\left\{(\beta+2b)\cos(\beta-2b)z\right\}.$$

(358.1)
$$\frac{\vartheta_1'^4\vartheta_0(z)\vartheta_1(z)}{\vartheta_2\vartheta_3\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z} - \frac{\sin z}{6\cos^2 z}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_0''}{\vartheta_0}+1\right\} + \frac{2}{3}\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}(\beta+2b)^3\sin(\beta-2b)z\right\} +$$
$$+\frac{2}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}(\beta+2b)\sin(\beta-2b)z\right\}.$$

(359.1)
$$\frac{\vartheta_1'^4\vartheta_0(z)\vartheta_1(z)}{\vartheta_2\vartheta_3\vartheta_3^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c-2}{4}}(\gamma+2c)^3\sin\frac{\gamma-2c}{2}z\right\} +$$
$$+\frac{1}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c-2}{4}}(\gamma+2c)\sin\frac{\gamma-2c}{2}z\right\}.$$

(360.1)
$$\frac{\vartheta_1'^4\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_0^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(\gamma+2c)^3\cos\frac{\gamma-2c}{2}z\right\} +$$
$$+\frac{1}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{\frac{N}{4}}\left\{(\gamma+2c)\cos\frac{\gamma-2c}{2}z\right\}$$

Group VII-b

$$\frac{\vartheta_0(z)\vartheta_3(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_0(z)\vartheta_3(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_3^4(z)}$$

Consider
$$F(z) = \frac{\vartheta_0(z)\vartheta_3(z)}{\vartheta_1^4(z)}$$

Let $t = z+\frac{\pi\tau}{2}+\frac{\pi}{4}$, $F(z)=\varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order four at $t=\frac{\pi\tau}{2}+\frac{\pi}{4}$. Calculating the corresponding $R_i^{(j)}$ and using (10) we find

(361)
$$\frac{\vartheta_1'^4}{\vartheta_0\vartheta_3}F(z) = \frac{1}{6}A_2^{(3)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{4},\frac{\pi\tau}{2}+\frac{\pi}{4}\right) - \frac{1}{6}\left\{3\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_1'''}{\vartheta_1'}\right\}A_2^{(1)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{4},\frac{\pi\tau}{2}+\frac{\pi}{4}\right)$$

There follows

$$\frac{\vartheta_1'^4\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_1^4(z)} = \frac{1}{\sin^4 z} - \frac{1}{6\sin^2 z}\left\{\frac{\vartheta_1'''}{\vartheta_1'} + 3\frac{\vartheta_2''}{\vartheta_2} + 4\right\} + \frac{1}{3}\sum(-1)^n(4n)^3 q^{2n^2} +$$

(362)
$$+\frac{2}{3}\sum_{n,r}(-1)^n[2(2n+r)]^3 q^{2n^2+2nr}\cos 2rz + \frac{1}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^n 4n q^{2n^2} + 2\sum_{n,r}(-1)^n 2(2n+r) q^{2n^2+2nr}\cos 2rz\right\}$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_2^4(z)} = \frac{1}{\cos^4 z} - \frac{1}{6\cos^2 z}\left\{\frac{\vartheta_1'''}{\vartheta_1'} + 3\frac{\vartheta_2''}{\vartheta_2} + 4\right\} + \frac{1}{3}\sum(-1)^n(4n)^3 q^{2n^2} +$$

(363)
$$+\frac{2}{3}\sum_{n,r}(-1)^{n+r}[2(2n+r)]^3 q^{2n^2+2nr}\cos 2rz + \frac{1}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^n 4n q^{2n^2} + 2\sum_{n,r}(-1)^{n+r} 2(2n+r) q^{2n^2+2nr}\cos 2rz\right\}.$$

In (361) replace z by $z-\frac{\pi\tau}{2}$, obtaining

$$q^{\frac{1}{2}}\frac{\vartheta_1'^4\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0^4(z)}e^{-2iz} = \frac{i}{6}A_2^{(3)}\left(z+\frac{\pi}{4}, \frac{\pi\tau}{2}+\frac{\pi}{4}\right) - \frac{i}{6}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_2''}{\vartheta_2}\right\}A_2^{(1)}\left(z+\frac{\pi}{4}, \frac{\pi\tau}{2}+\frac{\pi}{4}\right)$$

There follows

$$\frac{\vartheta_1'^4\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+r}[2(2n+r-1)]^3 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz +$$

(364)
$$+\frac{2}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r}(-1)^{n+r}2(2n+r-1)q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz$$

$$\frac{\vartheta_1'^4\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_3^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+r}[2(2n+r-1)]^3 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz$$

(365)
$$+\frac{2}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r}(-1)^{n+r}2(2n+r-1)q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz.$$

From these results we find

$$\frac{\vartheta_1'^4\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_1^4(z)} = \frac{1}{\sin^4 z} - \frac{1}{6\sin^2 z}\left\{\frac{\vartheta_1'''}{\vartheta_1'} + 3\frac{\vartheta_2''}{\vartheta_2} + 4\right\} + \frac{1}{3}\sum_n(-1)^n(4n)^3 q^{2n^2} +$$

(362.1)
$$+\frac{2}{3}\sum q^N\{(-1)^a(\alpha+2a)^3\cos(\alpha-2a)z\} + \frac{1}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'}+3\frac{\vartheta_2''}{\vartheta_2}\right\}\left\{\sum_n(-1)^n 4n q^{2n^2} + 2\sum q^N\{(-1)^a(\alpha+2a)\cos(\alpha-2a)z\}\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_2^4(z)} = \frac{1}{\cos^4 z} - \frac{1}{6\cos^2 z}\left\{\frac{\vartheta_1'''}{\vartheta_1'} + 3\frac{\vartheta_2''}{\vartheta_2} + 4\right\} + \frac{1}{3}\sum_n (-1)^n (4n)^3 q^{2n^2} +$$

(363.1)

$$+ \frac{2}{3}\sum q^N\left\{(-1)^{\frac{\alpha}{2}}(\alpha+2a)^3\cos(\alpha-2a)z\right\} + \frac{1}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'} + 3\frac{\vartheta_2''}{\vartheta_2}\right\}\left\{\sum_n (-1)^n 4n\, q^{2n^2} + 2\sum q^N\left\{(-1)^{\frac{\alpha}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+2d)^3\sin\frac{\delta-2d}{2}z\right\} +$$

(364.1)

$$+ \frac{1}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'} + 3\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+2d)\sin\frac{\delta-2d}{2}z\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_3^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2}{4}}(\delta+2d)^3\sin\frac{\delta-2d}{2}z\right\} +$$

(365.1)

$$+ \frac{1}{3}\left\{\frac{\vartheta_1'''}{\vartheta_1'} + 3\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2}{4}}(\delta+2d)\sin\frac{\delta-2d}{2}z\right\}$$

Group VIII-a

$$\frac{\vartheta_2^2(z)}{\vartheta_1^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_2^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_0^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_3^3(z)\vartheta_2(z)}$$

$$\frac{\vartheta_2^2(z)}{\vartheta_1^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_2^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_3^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_0^3(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^2(z)\, e^{-iz}}{\vartheta_1^3(z)\vartheta_0(z)}$$

Let $t = z + \frac{\pi\tau}{4}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders three and one at $t = \frac{\pi\tau}{4}$ and $t = \frac{3\pi\tau}{4}$ respectively. Calculating the corresponding $A_i^{(j)}$ and using (10) gives

$$\frac{\vartheta_1'^3\vartheta_0}{\vartheta_2^2}F(z) = \frac{1}{2}A_2^{(2)}\left(z+\frac{\pi\tau}{4}, \frac{\pi\tau}{4}\right) - i\,A_2^{(1)}\left(z+\frac{\pi\tau}{4}, \frac{\pi\tau}{4}\right) +$$

(366)

$$+ \frac{1}{2}\left\{2\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\}A_2^{(0)}\left(z+\frac{\pi\tau}{4}, \frac{\pi\tau}{4}\right) + \vartheta_3^4 A_2^{(0)}\left(z+\frac{\pi\tau}{4}, \frac{3\pi\tau}{4}\right)$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^2(z)}{\vartheta_2^2\vartheta_1^3(z)\vartheta_0(z)} = \frac{1}{\sin^3 z} + \frac{1}{2\sin z}\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\} + 4\vartheta_3^4\sum_{n,r} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z +$$

(367)

$$-2\sum_{n,r}[2(2n+r)-1]^2 q^{n(2n+2r-1)}\sin(2r-1)z + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r} q^{n(2n+2r-1)}\sin(2r-1)z$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1^2(z)}{\vartheta_2^2\vartheta_2^3(z)\vartheta_3(z)} = \frac{1}{\cos^3 z} + \frac{1}{2\cos z}\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\} + 4\vartheta_3^4\sum_{n,r}(-1)^{r+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z +$$

(368)

$$+2\sum_{n,r}(-1)^r[2(2n+r)-1]^2 q^{n(2n+2r-1)}\cos(2r-1)z + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1} q^{n(2n+2r-1)}\cos(2r-1)z .$$

In (366) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^2(z)e^{iz}}{\vartheta_2^2\vartheta_0^3(z)\vartheta_1(z)} = \frac{1}{2}A_2^{(2)}\left(z+\frac{3\pi\tau}{4},\frac{\pi\tau}{4}\right) - i A_2^{(1)}\left(z+\frac{3\pi\tau}{4},\frac{\pi\tau}{4}\right) +$$

$$+\frac{1}{2}\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}A_2^{(0)}\left(z+\frac{3\pi\tau}{4},\frac{\pi\tau}{4}\right) + \vartheta_3^4 A_2^{(0)}\left(z+\frac{3\pi\tau}{4},\frac{3\pi\tau}{4}\right) .$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^2(z)}{\vartheta_2^2\vartheta_0^3(z)\vartheta_1(z)} = \vartheta_3^4\left\{\frac{1}{\sin z} + 4\sum q^{n(2n+2r-1)}\sin(2r-1)z\right\} - 2\sum_{n,r}[2(2n+r)-3]^2 q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

(369)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0^2(z)}{\vartheta_2^2\vartheta_3^3(z)\vartheta_2(z)} = \vartheta_3^4\left\{\frac{1}{\cos z} + 4\sum q^{n(2n+2r-1)}(-1)^{r+1}\cos(2r-1)z\right\} + 2\sum_{n,r}(-1)^r[2(2n+r)-3]^2 q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z +$$

(370)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

In these results, replace q by $-q$. This gives

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2^2(z)}{\vartheta_2^2\vartheta_1^3(z)\vartheta_3(z)} = \frac{1}{\sin^3 z} + \frac{1}{2\sin z}\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} + 4\vartheta_0^4\sum(-1)^{n+r+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z +$$

(371)

$$+2\sum_{n,r}(-1)^{n+1}[2(2n+r)-1]^2 q^{n(2n+2r-1)}\sin(2r-1)z + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^n q^{n(2n+2r-1)}\sin(2r-1)z$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1^2(z)}{\vartheta_2^2\vartheta_2^3(z)\vartheta_0(z)} = \frac{1}{\cos^3 z} + \frac{1}{2\cos z}\left\{2\frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'} - \frac{\vartheta_3''}{\vartheta_3} - 1\right\} + 4\vartheta_0^4 \sum_{n,r} (-1)^n q^{\frac{(2n-1)^2}{2} + \frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z \; +$$

(372)

$$+2\sum_{n,r}(-1)^{n+r}[2(2n+r)-1]^2 q^{n(2n+2r-1)}\cos(2r-1)z + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^{n+r+1}q^{n(2n+2r-1)}\cos(2r-1)z \, .$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0^2(z)}{\vartheta_2^2\vartheta_3^3(z)\vartheta_1(z)} = \vartheta_0^4\left\{\frac{1}{\sin z} + 4\sum q^{n(2n+2r-1)}(-1)^n \sin(2r-1)z\right\} + 2\sum_{n,r}(-1)^{n+r}[2(2n+r)-3]^2 q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z \; +$$

(373)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^{n+r+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z \, .$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3^2(z)}{\vartheta_2^2\vartheta_0^3(z)\vartheta_2(z)} = \vartheta_0^4\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n(2n+2r-1)}\cos(2r-1)z\right\} + 2\sum_{n,r}(-1)^{n+1}[2(2n+r)-3]^2 q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(374)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^n q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \, .$$

From these follow

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^2(z)}{\vartheta_2^2\vartheta_1^3(z)\vartheta_0(z)} = \frac{1}{\sin^3 z} + \frac{1}{2\sin z}\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_0''}{\vartheta_0}-1\right\} + 4\vartheta_3^4\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-2c}{2}z\right\} \; +$$

(367.1)

$$-2\sum q^N\left\{(\beta+2b)^2\sin(\beta-2b)z\right\} + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^N\left\{\sin(\beta-2b)z\right\} .$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1^2(z)}{\vartheta_2^2\vartheta_2^3(z)\vartheta_3(z)} = \frac{1}{\cos^3 z} + \frac{1}{2\cos z}\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\} + 4\vartheta_3^4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c-2}{4}}\cos\frac{\gamma-2c}{2}z\right\} \; +$$

(368.1)

$$+2\sum q^N\left\{(-1)^{\frac{\beta-2b+1}{2}}(\beta+2b)\cos(\beta-2b)z\right\} + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}\cos(\beta-2b)z\right\} .$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^2(z)}{\vartheta_2^2\vartheta_0^3(z)\vartheta_1(z)} = \vartheta_3^4\left\{\frac{1}{\sin z} + 4\sum q^N\left\{\sin(\beta-2b)z\right\}\right\} - \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\gamma+2c)^2\sin\frac{\gamma-2c}{2}z\right\} \; +$$

(369.1)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-2c}{2}z\right\} .$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0^2(z)}{\vartheta_2^2\vartheta_3^3(z)\vartheta_2(z)} = \vartheta_3^4\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}\cos(\beta-2b)z\right\}\right\} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c+2}{4}}(\gamma+2c)^2\cos\frac{\gamma-2c}{2}z\right\} +$$

(370.1)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c-2}{4}}\cos\frac{\gamma-2c}{2}z\right\}$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^2(z)}{\vartheta_2^2\vartheta_1^3(z)\vartheta_3(z)} = \frac{1}{\sin^3 z} + \frac{1}{2\sin z}\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}-1\right\} + 4\vartheta_0^4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma}{4}}\sin\frac{\gamma-2c}{2}z\right\} +$$

(371.1)

$$+2\sum q^N\left\{(-1)^{b+1}(\beta+2b)^2\sin(\beta-2b)z\right\} + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^N\left\{(-1)^b\sin(\beta-2b)z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1^2(z)}{\vartheta_2^2\vartheta_2^3(z)\vartheta_0(z)} = \frac{1}{\cos^3 z} + \frac{1}{2\cos z}\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}-1\right\} + 4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-2c}{2}z\right\}\vartheta_0^4$$

(372.1)

$$+2\sum q^N\left\{(-1)^{\frac{\beta+1}{2}}(\beta+2b)^2\cos(\beta-2b)z\right\} + 2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^N\left\{(-1)^{\frac{\beta-1}{2}}\cos(\beta-2b)z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0^2(z)}{\vartheta_2^2\vartheta_3^3(z)\vartheta_1(z)} = \vartheta_0^4\left\{\frac{1}{\sin z} + 4\sum q^N\left\{(-1)^b\sin(\beta-2b)z\right\}\right\} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+4}{4}}(\gamma+2c)^2\sin\frac{\gamma-2c}{2}z\right\} +$$

(373.1)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma}{4}}\sin\frac{\gamma-2c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3^2(z)}{\vartheta_2^2\vartheta_0^3(z)\vartheta_2(z)} = \vartheta_0^4\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-1}{2}}\cos(\beta-2b)z\right\}\right\} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+2c)^2\cos\frac{\gamma-2c}{2}z\right\} +$$

(374.1)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-2c}{2}z\right\}.$$

Group VIII-b

$$\frac{\vartheta_3^2(z)}{\vartheta_1^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_2^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_0^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_3^3(z)\vartheta_2(z)}$$

$$\frac{\vartheta_0^2(z)}{\vartheta_1^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_2^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_3^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_0^3(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^2(z)\, e^{-iz}}{\vartheta_1^3(z)\,\vartheta_0(z)}$$

Let $t = z + \frac{\pi\tau}{4}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders three and one at $t = \frac{\pi\tau}{4}$ and $t = \frac{3\pi\tau}{4}$ respectively. Calculating the corresponding values of $R_i^{(j)}$ and using (10) gives

$$\frac{\vartheta_1'^3\vartheta_0}{\vartheta_3^2} F(z) = \frac{1}{2} A_2^{(2)}\left(z+\frac{\pi\tau}{4}, \frac{\pi\tau}{4}\right) - i\, A_2^{(1)}\left(z+\frac{\pi\tau}{4}, \frac{\pi\tau}{4}\right) +$$

(375)

$$+ \frac{1}{2}\left\{2\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} A_2^{(0)}\left(z+\frac{\pi\tau}{4}, \frac{\pi\tau}{4}\right) + \vartheta_2^4 A_2^{(0)}\left(z+\frac{\pi\tau}{4}, \frac{3\pi\tau}{4}\right)$$

There follows

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^2(z)}{\vartheta_3^2\vartheta_1^3(z)\vartheta_0(z)} = \frac{1}{\sin^3 z} + \frac{1}{2\sin z}\left\{2\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'} - \frac{\vartheta_0''}{\vartheta_0} - 1\right\} + 4\vartheta_2^4 \sum_{n,r} q^{\frac{(2n-1)^2}{2} + \frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z \; +$$

(376)

$$-2\sum_{n,r}[2(2n+r)-1]^2 q^{n(2n+2r-1)} \sin(2r-1)z + 2\left\{2\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'} - \frac{\vartheta_0''}{\vartheta_0}\right\} \sum_{n,r} q^{n(2n+2r-1)} \sin(2r-1)z$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0^2(z)}{\vartheta_3^2\vartheta_2^3(z)\vartheta_3(z)} = \frac{1}{\cos^3 z} + \frac{1}{2\cos z}\left\{2\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} + 4\vartheta_2^4 \sum_{n,r} (-1)^{r+1} q^{\frac{(2n-1)^2}{2} + \frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z \; +$$

(377)

$$+2\sum_{n,r}(-1)^r[2(2n+r)-1]^2 q^{n(2n+2r-1)} \cos(2r-1)z + 2\left\{2\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\} \sum_{n,r}(-1)^{r+1} q^{n(2n+2r-1)} \cos(2r-1)z \, .$$

In (375) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^2(z)\, e^{iz}}{\vartheta_3^2\vartheta_0^3(z)\vartheta_1(z)} = \frac{1}{2} A_2^{(2)}\left(z+\frac{3\pi\tau}{4}, \frac{\pi\tau}{4}\right) - i\, A_2^{(1)}\left(z+\frac{3\pi\tau}{4}, \frac{\pi\tau}{4}\right) +$$

$$+ \frac{1}{2}\left\{2\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_1'''}{\vartheta_1'} - \frac{\vartheta_0''}{\vartheta_0} - 1\right\} A_2^{(0)}\left(z+\frac{3\pi\tau}{4}, \frac{\pi\tau}{4}\right) + \vartheta_2^4 A_2^{(0)}\left(z+\frac{3\pi\tau}{4}, \frac{3\pi\tau}{4}\right).$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2^2(z)}{\vartheta_3^2\vartheta_0^3(z)\vartheta_1(z)} = \vartheta_2^4\left\{\frac{1}{\sin z} + 4\sum_{n,r} q^{n(2n+2r-1)} \sin(2r-1)z\right\} - 2\sum_{n,r}[2(2n+r)-3]^2 q^{\frac{(2n-1)^2}{2} + \frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z \; +$$

(378)

$$+2\left\{2\frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_1'''}{\vartheta_1'}\right\} \sum_{n,r} q^{\frac{(2n-1)^2}{2} + \frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1^2(z)}{\vartheta_3^2\vartheta_3^3(z)\vartheta_2(z)} = \vartheta_2^4\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{n(2n+2r-1)}\cos(2r-1)z\right\}+2\sum_{n,r}[2(2n+r)-3]^2(-1)^r q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z\ +$$

(379)

$$+2\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r+1}q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

In these results replace q by $-q$. We get

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0^2(z)}{\vartheta_0^2\vartheta_1^3(z)\vartheta_3(z)} = \frac{1}{\sin^3 z}+\frac{1}{2\sin z}\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}+4\vartheta_2^4\sum_{n,r}(-1)^{n+r}q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z\ +$$

(380)

$$+2\sum_{n,r}(-1)^{n+1}q^{n(2n+2r-1)}[2(2n+r)-1]^2\sin(2r-1)z+2\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^n q^{n(2n+2r-1)}\sin(2r-1)z\,.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3^2(z)}{\vartheta_0^2\vartheta_2^3(z)\vartheta_0(z)} = \frac{1}{\cos^3 z}+\frac{1}{2\cos z}\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}+4\vartheta_2^4\sum(-1)^{n+1}q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z\ +$$

(381)

$$+2\sum_{n,r}(-1)^{n+r}[2(2n+r)-1]^2q^{n(2n+2r-1)}\cos(2r-1)z+2\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1}q^{n(2n+2r-1)}\cos(2r-1)z\,.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2^2(z)}{\vartheta_0^2\vartheta_3^3(z)\vartheta_1(z)} = \vartheta_2^4\left\{\frac{1}{\sin z}+4\sum_{n,r}(-1)^n q^{n(2n+2r-1)}\sin(2r-1)z\right\}+2\sum_{n,r}(-1)^{n+r+1}[2(2n+r)-3]^2q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z\ +$$

(382)

$$+2\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^{n+r}q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z\,.$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1^2(z)}{\vartheta_0^2\vartheta_0^3(z)\vartheta_2(z)} = \vartheta_2^4\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{n+r+1}q^{n(2n+2r-1)}\cos(2r-1)z\right\}+2\sum_{n,r}(-1)^n[2(2n+r)-3]^2q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

(383)

$$+2\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^{n+1}q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

From these follow

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3^2(z)}{\vartheta_3^2\vartheta_1^3(z)\vartheta_0(z)} = \frac{1}{\sin^3 z}+\frac{1}{2\sin z}\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}+4\vartheta_2^4\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-2c}{2}z\right\}\ +$$

(376.1)

$$-2\sum q^N\left\{(\beta+2b)^2\sin(\beta-2b)z\right\}+2\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{\sin(\beta-2b)z\right\}.$$

$$
\begin{aligned}
\frac{\vartheta_1'^3\vartheta_0\vartheta_0^2(z)}{\vartheta_3^2\vartheta_2^3(z)\vartheta_3(z)} &= \frac{1}{\cos^3 z}+\frac{1}{2\cos z}\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}+4\vartheta_2^4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c-2}{4}}\cos\frac{\gamma-2c}{2}z\right\} \\
&+2\sum q^N\left\{(-1)^{\frac{\beta-2b+1}{2}}(\beta+2b)^2\cos(\beta-2b)z\right\}+2\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}\cos(\beta-2b)z\right\}.
\end{aligned}
\tag{377.1}
$$

$$
\begin{aligned}
\frac{\vartheta_1'^3\vartheta_0\vartheta_2^2(z)}{\vartheta_3^2\vartheta_0^3(z)\vartheta_1(z)} &= \vartheta_2^4\left\{\frac{1}{\sin z}+4\sum q^N\left\{\sin(\beta-2b)z\right\}\right\}-\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\gamma+2c)^2\sin\frac{\gamma-2c}{2}z\right\}+ \\
&+2\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-2c}{2}z\right\}.
\end{aligned}
\tag{378.1}
$$

$$
\begin{aligned}
\frac{\vartheta_1'^3\vartheta_0\vartheta_1^2(z)}{\vartheta_3^2\vartheta_3^3(z)\vartheta_2(z)} &= \vartheta_2^4\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}\cos(\beta-2b)z\right\}\right\}+\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c+2}{4}}(\gamma+2c)^2\cos\frac{\gamma-2c}{2}z\right\}+ \\
&+2\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c-2}{4}}\cos\frac{\gamma-2c}{2}z\right\}.
\end{aligned}
\tag{379.1}
$$

$$
\begin{aligned}
\frac{\vartheta_1'^3\vartheta_3\vartheta_0^2(z)}{\vartheta_0^2\vartheta_1^3(z)\vartheta_3(z)} &= \frac{1}{\sin^3 z}+\frac{1}{2\sin z}\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}+4\vartheta_2^4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+4}{4}}\sin\frac{\gamma-2c}{2}z\right\}+ \\
&+2\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^b\sin(\beta-2b)z\right\}+2\sum q^N\left\{(-1)^{b+1}(\beta+2b)^2\sin(\beta-2b)z\right\}.
\end{aligned}
\tag{380.1}
$$

$$
\begin{aligned}
\frac{\vartheta_1'^3\vartheta_3\vartheta_3^2(z)}{\vartheta_0^2\vartheta_2^3(z)\vartheta_0(z)} &= \frac{1}{\cos^3 z}+\frac{1}{2\cos z}\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}+4\vartheta_2^4\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-2c}{2}z\right\}+ \\
&+2\sum q^N\left\{(-1)^{\frac{\beta+1}{2}}(\beta+2b)^2\cos(\beta-2b)z\right\}+2\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta-1}{2}}\cos(\beta-2b)z\right\}.
\end{aligned}
\tag{381.1}
$$

$$
\begin{aligned}
\frac{\vartheta_1'^3\vartheta_3\vartheta_2^2(z)}{\vartheta_0^2\vartheta_3^3(z)\vartheta_1(z)} &= \vartheta_2^4\left\{\frac{1}{\sin z}+4\sum q^N\left\{(-1)^b\sin(\beta-2b)z\right\}\right\}+\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma}{4}}(\gamma+2c)^2\sin\frac{\gamma-2c}{2}z\right\}+ \\
&+2\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma+4}{4}}\sin\frac{\gamma-2c}{2}z\right\}.
\end{aligned}
\tag{382.1}
$$

$$
\begin{aligned}
\frac{\vartheta_1'^3\vartheta_3\vartheta_1^2(z)}{\vartheta_0^2\vartheta_0^3(z)\vartheta_2(z)} &= \vartheta_2^4\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-1}{2}}\cos(\beta-2b)z\right\}\right\}+\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+2c)^2\cos\frac{\gamma-2c}{2}z\right\}+ \\
&+2\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-2c}{2}z\right\}
\end{aligned}
\tag{383.1}
$$

Group VIII-c

$$\frac{\vartheta_0^2(z)}{\vartheta_1^3(z)\vartheta_2(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_2^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_0^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_3^3(z)\vartheta_0(z)}$$

$$\frac{\vartheta_3^2(z)}{\vartheta_1^3(z)\vartheta_2(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_2^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_3^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_0^3(z)\vartheta_3(z)}$$

Consider

$$F(z) = \frac{\vartheta_0^2(z)}{\vartheta_1^3(z)\vartheta_2(z)}$$

Let $t = z + \frac{\pi\tau}{2} + \frac{\pi}{4}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders three and one at $t = \frac{\pi\tau}{2} + \frac{\pi}{4}$ and $t = \frac{\pi\tau}{2} + \frac{3\pi}{4}$ respectively. Calculating the corresponding values of $A_i^{(k)}$ and using (10) gives

$$F(z)\frac{\vartheta_1'^3\vartheta_2}{\vartheta_0^2(z)} = \frac{1}{2}A_2^{(2)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{4}, \frac{\pi\tau}{2}+\frac{\pi}{4}\right) - \vartheta_3^4 A_2^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{4}, \frac{\pi\tau}{2}+\frac{3\pi}{4}\right)$$

(384)

$$+\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'}\right\} A_2^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{4}, \frac{\pi\tau}{2}+\frac{\pi}{4}\right)$$

From this follows

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0^2(z)}{\vartheta_0^2\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} + 2\sum_{n,r}(-1)^{n+1}[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz + \vartheta_3^4\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{2n^2+2nr}\sin 2rz\right\} +$$

(385)

$$+\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^n q^{2n^2+2nr}\sin 2rz\right\}.$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3^2(z)}{\vartheta_0^2\vartheta_2^3(z)\vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^{n+r}[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz + \vartheta_3^4\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^n q^{2n^2+2nr}\sin 2rz\right\} +$$

(386)

$$+\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{2n^2+2nr}\sin 2rz\right\}.$$

In (384) replace z by $z - \frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1^2(z)e^{-2iz}}{\vartheta_0^2\vartheta_0^3(z)\vartheta_3(z)} = -\frac{i}{2}q^{-1/2}A_2^{(2)}\left(z+\frac{\pi}{4}, \frac{\pi\tau}{2}+\frac{\pi}{4}\right)$$

$$-\frac{i}{2}q^{-1/2}\left\{2\frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1'''}{\vartheta_1'}\right\}A_2^{(0)}\left(z+\frac{\pi}{4}, \frac{\pi\tau}{2}+\frac{\pi}{4}\right) + i q^{-1/2}\vartheta_3^4 A_2^{(0)}\left(z+\frac{\pi}{4}, \frac{\pi\tau}{2}+\frac{3\pi}{4}\right)$$

This gives

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1^2(z)}{\vartheta_0^2\vartheta_0^3(z)\vartheta_3(z)} = \sum_n (-1)^n[2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^n[2(2n+r-1)]^2 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz \;+$$

(387) $$+\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^{n+1}q^{\frac{(2n-1)^2}{2}}+2\sum_{n,r}(-1)^{n+1}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}+$$

$$+2\vartheta_0^4\left\{\sum_n(-1)^{n+1}q^{\frac{(2n-1)^2}{2}}+2\sum_{n,r}(-1)^{n+r+1}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}.$$

Replacing z **by** $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_2^2(z)}{\vartheta_0^2\vartheta_3^3(z)\vartheta_0(z)} = \sum_n (-1)^n[2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+r}[2(2n+r-1)]^2 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz \;+$$

(388) $$+\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^{n+1}q^{\frac{(2n-1)^2}{2}}+2\sum_{n,r}(-1)^{n+r+1}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}+$$

$$+2\vartheta_3^4\left\{\sum_n(-1)^{n+1}q^{\frac{(2n-1)^2}{2}}+2\sum_{n,r}(-1)^{n+1}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

In these results replace q **by** $-q$ **. There follow**

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3^2(z)}{\vartheta_3^2\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} + 2\sum_{n,r}(-1)^{n+1}[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz + \vartheta_0^4\left\{\frac{\sin z}{\cos z}+4\sum(-1)^{n+r+1}q^{2n^2+2nr}\sin 2rz\right\}+$$

(389) $$+\frac{1}{2}\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^n q^{2n^2+2nr}\sin 2rz\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0^2(z)}{\vartheta_3^2\vartheta_2^3(z)\vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^{n+r}[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz + \vartheta_0^4\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^n q^{2n^2+2nr}\sin 2rz\right\}+$$

(390) $$+\frac{1}{2}\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1}q^{2n^2+2nr}\sin 2rz\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1^2(z)}{\vartheta_3^2\vartheta_3^3(z)\vartheta_0(z)} = \sum_n (-1)^{n+1}[2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+r+1}[2(2n+r-1)]^2 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz \;+$$

(391) $$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}}+2\sum_{n,r}(-1)^{n+r}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

$$+2\vartheta_0^4\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}}+\sum_{n,r}(-1)^n q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}.$$

$$\frac{\vartheta_1'^3\,\vartheta_2\,\vartheta_2^2(z)}{\vartheta_3^2\,\vartheta_0^3(z)\,\vartheta_3(z)} = \sum_n (-1)^{n+1}[2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+1}[2(2n+r)-2]^2 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos(2r-1)z \; +$$

(392)
$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^n q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

$$+2\vartheta_0^4\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

From these follow

$$\frac{\vartheta_1'^3\,\vartheta_2\,\vartheta_0^2(z)}{\vartheta_0^2\,\vartheta_1^3(z)\,\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} + 2\sum q^N\left\{(-1)^{a+1}(\alpha+2a)^2\sin(\alpha-2a)z\right\} + \vartheta_3^4\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha+2}{2}}\sin(\alpha-2a)z\right\}\right\}+$$

(385.1)
$$+\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^a\sin(\alpha-2a)z\right\}\right\}$$

$$\frac{\vartheta_1'^3\,\vartheta_2\,\vartheta_3^2(z)}{\vartheta_0^2\,\vartheta_2^3(z)\,\vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha}{2}}(\alpha+2a)^2\sin(\alpha-2a)z\right\} + \vartheta_3^4\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^a\sin(\alpha-2a)z\right\}\right\}+$$

(386.1)
$$+\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha+2}{2}}\sin(\alpha-2a)z\right\}\right\}$$

$$\frac{\vartheta_1'^3\,\vartheta_2\,\vartheta_1^2(z)}{\vartheta_0^2\,\vartheta_0^3(z)\,\vartheta_3(z)} = \sum_n(-1)^n[2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}(\delta+2d)^2\cos\frac{\delta-2d}{2}z\right\} \; +$$

(387.1)
$$+\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^{n+1} q^{\frac{(2n-1)^2}{2}} + 2\sum q^N\left\{(-1)^{\frac{d-1}{2}}\cos\frac{\delta-2d}{2}z\right\}\right\}+$$

$$+2\vartheta_3^4\left\{\sum_n(-1)^{n+1} q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+2}{4}}\cos\frac{\delta-2d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\,\vartheta_2\,\vartheta_2^2(z)}{\vartheta_0^2\,\vartheta_3^3(z)\,\vartheta_0(z)} = \sum_n(-1)^n[2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2}{4}}(\delta+2d)^2\cos\frac{\delta-2d}{2}z\right\} \; +$$

(388.1)
$$+\left\{2\frac{\vartheta_0''}{\vartheta_0}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^{n+1} q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+2}{4}}\cos\frac{\delta-2d}{2}z\right\}\right\}+$$

$$+2\vartheta_3^4\left\{\sum_n(-1)^{n+1} q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}\cos\frac{\delta-2d}{2}z\right\}\right\}$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3^2(z)}{\vartheta_3^2\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} + 2\sum q^N\left\{(-1)^{a+1}(d+2a)^2\sin(d-2a)z\right\} + \vartheta_0^4\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{d+2}{2}}\sin(d-2a)z\right\}\right\} +$$

(389.1)

$$+\frac{1}{2}\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\cos z}{\sin z}+\sum q^N\left\{(-1)^a\sin(d-2a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0^2(z)}{\vartheta_3^2\vartheta_2^3(z)\vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{d}{2}}(d+2a)^2\sin(d-2a)z\right\} + \vartheta_0^4\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{(-1)^a\sin(d-2a)z\right\}\right\} +$$

(390.1)

$$+\frac{1}{2}\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{d+2}{2}}\sin(d-2a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1^2(z)}{\vartheta_3^2\vartheta_3^3(z)\vartheta_0(z)} = \sum_n(-1)^{n+1}[2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+2}{4}}(\delta+2d)^2\cos\frac{\delta-2d}{2}z\right\} +$$

(391.1)

$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2}{4}}\cos\frac{\delta-2d}{2}z\right\}\right\} +$$

$$+2\vartheta_0^4\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-2d}{2}z\right\}\right\}$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_2^2(z)}{\vartheta_3^2\vartheta_0^3(z)\vartheta_3(z)} = \sum_n(-1)^{n+1}[2(2n-1)]^2 q^{\frac{(2n-1)^2}{2}} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+2d)^2\cos\frac{\delta-2d}{2}z\right\} +$$

(392.1)

$$+\left\{2\frac{\vartheta_3''}{\vartheta_3}-\frac{\vartheta_2''}{\vartheta_2}-\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-2d}{2}z\right\}\right\} +$$

$$+2\vartheta_0^4\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2}{4}}\cos\frac{\delta-2d}{2}z\right\}\right\}$$

Group IX-a

$$\frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_1^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_1(z)\vartheta_0(z)}{\vartheta_2^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_0^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_1(z)\vartheta_0(z)}{\vartheta_3^3(z)\vartheta_2(z)}$$

$$\frac{\vartheta_2(z)\vartheta_0(z)}{\vartheta_1^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_1(z)\vartheta_3(z)}{\vartheta_2^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_2(z)\vartheta_0(z)}{\vartheta_3^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_1(z)\vartheta_3(z)}{\vartheta_0^3(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_1^3(z)\vartheta_0(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders three and one at $t = \frac{\pi\tau}{2}$ and $t = \pi\tau$ respectively. Calculating the corresponding values of $R_i^{(k)}$ and using (10) gives

$$\frac{\vartheta_1'^3\vartheta_0}{\vartheta_2\vartheta_3}F(z) = \frac{1}{2}A_2^{(2)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right) - \frac{\vartheta_0''}{\vartheta_0}A_2^{(0)}\left(z+\frac{\pi\tau}{2},\frac{\pi\tau}{2}\right) + q^{1/2}\vartheta_2^2\vartheta_3^2A_2^{(0)}\left(z+\frac{\pi\tau}{2},\pi\tau\right) \tag{393}$$

There follows

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_0\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_1^3(z)\vartheta_0(z)} &= \frac{\cos z}{\sin^3 z} - 2\sum_{n,r}[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz \; + \\ &- \frac{\vartheta_0''}{\vartheta_0}\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{2n^2+2nr}\sin 2rz\right\} + 4\vartheta_2^2\vartheta_3^2\sum_{n,r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz \,.\end{aligned} \tag{394}$$

Replacing z by $z+\frac{\pi}{2}$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_0\vartheta_1(z)\vartheta_0(z)}{\vartheta_2\vartheta_3\vartheta_2^3(z)\vartheta_3(z)} &= \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^r[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz \; + \\ &- \frac{\vartheta_0''}{\vartheta_0}\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{2n^2+2nr}\sin 2rz\right\} + 4\vartheta_2^2\vartheta_3^2\sum_{n,r}(-1)^{r+1} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz \,.\end{aligned} \tag{395}$$

In (393) replace z by $z - \frac{\pi\tau}{2}$, obtaining

$$q^{1/2}\frac{\vartheta_1'^3\vartheta_0\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_0^3(z)\vartheta_1(z)}e^{-2iz} = \frac{1}{2}A_2^{(2)}\left(z,\frac{\pi\tau}{2}\right) - \frac{\vartheta_0''}{\vartheta_0}A_2^{(0)}\left(z,\frac{\pi\tau}{2}\right) + q^{1/2}\vartheta_2^2\vartheta_3^2A_2^{(0)}(z,\pi\tau).$$

This gives

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_0\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_0^3(z)\vartheta_1(z)} &= \vartheta_2^2\vartheta_3^2\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{2n^2+2nr}\sin 2rz\right\} + \\ &- 2\sum_{n,r}[2(2n+r-1)]^2 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz - 4\frac{\vartheta_0''}{\vartheta_0}\sum_{n,r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz\end{aligned} \tag{396}$$

Replacing z by $z+\frac{\pi}{2}$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_0\vartheta_1(z)\vartheta_0(z)}{\vartheta_2\vartheta_3\vartheta_3^3(z)\vartheta_2(z)} &= \vartheta_3^2\vartheta_2^2\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{2n^2+2nr}\sin 2rz\right\} + \\ &+ 2\sum_{n,r}(-1)^r[2(2n+r-1)]^2 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + 4\frac{\vartheta_0''}{\vartheta_0}\sum_{n,r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}(-1)^r\sin 2rz\end{aligned} \tag{397}$$

In these results replace q by $-q$. We get

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2(z)\vartheta_0(z)}{\vartheta_0\vartheta_2\vartheta_1^3(z)\vartheta_3(z)} = \frac{\cos z}{\sin^3 z} - 2\sum_{n,r}[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz \; + \tag{398}$$

$$-\frac{\vartheta_3''}{\vartheta_3}\left\{\frac{\cos z}{\sin z}+4\sum_{n,r} q^{2n^2+2nr}\sin 2rz\right\}+4\vartheta_0^2\vartheta_2^2\sum_{n,r}(-1)^{r+1}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_2^3(z)\vartheta_0(z)} = \frac{\sin z}{\cos^3 z}+2\sum_{n,r}(-1)^r[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz \; + \tag{399}$$

$$-\frac{\vartheta_3''}{\vartheta_3}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{2n^2+2nr}\sin 2rz\right\}+4\vartheta_0^2\vartheta_2^2\sum_{n,r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2(z)\vartheta_0(z)}{\vartheta_0\vartheta_2\vartheta_3^3(z)\vartheta_1(z)} = \vartheta_2^2\vartheta_0^2\left\{\frac{\cos z}{\sin z}+4\sum_{n,r} q^{2n^2+2nr}\sin 2rz\right\} + \tag{400}$$

$$+2\sum_{n,r}(-1)^{r+1}[2(2n+r-1)]^2 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + 4\frac{\vartheta_3''}{\vartheta_3}\sum_{n,r}(-1)^{r+1}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz .$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_0^3(z)\vartheta_2} = \vartheta_0^2\vartheta_2^2\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{2n^2+2nr}\sin 2rz\right\} + \tag{401}$$

$$+2\sum_{n,r}[2(2n+r-1)]^2 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + 4\frac{\vartheta_3''}{\vartheta_3}\sum_{n,r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz$$

From these follow

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_1^3(z)\vartheta_0(z)} = \frac{\cos z}{\sin^3 z} - 2\sum q^N\left\{(\alpha+2a)^2\sin(\alpha-2a)z\right\} + \tag{394.1}$$

$$-\frac{\vartheta_0''}{\vartheta_0}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{\sin(\alpha-2a)z\right\}\right\}+4\vartheta_2^2\vartheta_3^2\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-2d}{2}z\right\} .$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0(z)\vartheta_1(z)}{\vartheta_2\vartheta_3\vartheta_2^3(z)\vartheta_3(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha-2a}{2}}(\alpha+2a)^2\sin(\alpha-2a)z\right\} + \tag{395.1}$$

$$-\frac{\vartheta_0''}{\vartheta_0}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-2a-2}{2}}\sin(\alpha-2a)z\right\}\right\}+4\vartheta_2^2\vartheta_3^2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d+4}{4}}\sin\frac{\delta-2d}{2}z\right\}$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_0^3(z)\vartheta_1(z)} = \vartheta_2^2\vartheta_3^2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(d-2a)z\right\}\right\} +$$

(396.1)

$$-\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\delta+2d)^2\sin\frac{\delta-2d}{2}z\right\} - 4\frac{\vartheta_0''}{\vartheta_0}\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-2d}{2}z\right\}$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0(z)\vartheta_1(z)}{\vartheta_2\vartheta_3\vartheta_3^3(z)\vartheta_2(z)} = \vartheta_2^2\vartheta_3^2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{d-2a-2}{2}}\sin(d-2a)z\right\}\right\} +$$

(397.1)

$$+\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)^2\sin\frac{\delta-2d}{2}z\right\} + 4\frac{\vartheta_0''}{\vartheta_0}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}\sin\frac{\delta-2d}{2}z\right\}$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_1^3(z)\vartheta_3(z)} = \frac{\cos z}{\sin^3 z} - 2\sum q^N\left\{(d+2a)^2\sin(d-2a)z\right\} +$$

(398.1)

$$-\frac{\vartheta_3''}{\vartheta_3}\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(d-2a)z\right\}\right\} + 4\vartheta_0^2\vartheta_2^2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d-4}{4}}\sin\frac{\delta-2d}{2}z\right\}$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3(z)\vartheta_1(z)}{\vartheta_0\vartheta_2\vartheta_2^3(z)\vartheta_0(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{d-2a}{2}}(d+2a)^2\sin(d-2a)z\right\} +$$

(399.1)

$$-\frac{\vartheta_3''}{\vartheta_3}\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{d-2a-2}{2}}\sin(d-2a)z\right\}\right\} + 4\vartheta_0^2\vartheta_2^2\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-2d}{2}z\right\}$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2(z)\vartheta_0(z)}{\vartheta_0\vartheta_2\vartheta_3^3(z)\vartheta_1(z)} = \vartheta_0^2\vartheta_2^2\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(d-2a)z\right\}\right\} +$$

(400.1)

$$+\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d-4}{4}}(\delta+2d)^2\sin\frac{\delta-2d}{2}z\right\} + 4\frac{\vartheta_3''}{\vartheta_3}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d-4}{4}}\sin\frac{\delta-2d}{2}z\right\}$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_0^3(z)\vartheta_2(z)} = \vartheta_0^2\vartheta_2^2\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{d-2a-2}{2}}\sin(d-2a)z\right\}\right\} +$$

(401.1)

$$+\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\delta+2d)^2\sin\frac{\delta-2d}{2}z\right\} + 4\frac{\vartheta_3''}{\vartheta_3}\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-2d}{2}z\right\}$$

Group IX-b

$$\frac{\vartheta_0(z)\vartheta_3(z)}{\vartheta_1^3(z)\vartheta_2(z)} \qquad \frac{\vartheta_0(z)\vartheta_3(z)}{\vartheta_2^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_0^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_3^3(z)\vartheta_0(z)}$$

Consider

$$F(z) = \frac{\vartheta_0(z)\vartheta_3(z)}{\vartheta_1^3(z)\vartheta_2(z)}$$

Let $t = z+\frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders three and one at $t=\frac{\pi\tau}{2}$ and $t=\frac{\pi\tau}{2}+\frac{\pi}{2}$ respectively. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$(402) \qquad \frac{\vartheta_1'^3\vartheta_2}{\vartheta_0\vartheta_3}F(z) = \frac{1}{2}A_2^{(2)}(z+\tfrac{\pi\tau}{2},\tfrac{\pi\tau}{2}) - \frac{\vartheta_2''}{\vartheta_2}A_2^{(0)}(z+\tfrac{\pi\tau}{2},\tfrac{\pi\tau}{2}) - \vartheta_0^2\vartheta_3^2A_2^{(0)}(z+\tfrac{\pi\tau}{2},\tfrac{\pi\tau}{2}+\tfrac{\pi}{2})$$

There follows

$$(403) \qquad \begin{aligned}\frac{\vartheta_1'^3\vartheta_2\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_1^3(z)\vartheta_2(z)} &= \frac{\cos z}{\sin^3 z} - 2\sum_{n,r}[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz + \\ &-\frac{\vartheta_2''}{\vartheta_2}\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}q^{2n^2+2nr}\sin 2rz\right\} - \vartheta_0^2\vartheta_3^2\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^r q^{2n^2+2nr}\sin 2rz\right\}.\end{aligned}$$

Replacing z by $z+\frac{\pi}{2}$

$$(404) \qquad \begin{aligned}\frac{\vartheta_1'^3\vartheta_2\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_2^3(z)\vartheta_1(z)} &= \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^r[2(2n+r)]^2 q^{2n^2+2nr}\sin 2rz + \\ &-\frac{\vartheta_2''}{\vartheta_2}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{2n^2+2nr}\sin 2rz\right\} - \vartheta_0^2\vartheta_3^2\left\{\frac{\cos z}{\sin z}-4\sum q^{2n^2+2nr}\sin 2rz\right\}\end{aligned}$$

In (402) replace z by $z-\frac{\pi\tau}{2}$, obtaining

$$q^{\frac{1}{2}}\frac{\vartheta_1'^3\vartheta_2\vartheta_1(z)\vartheta_2(z)e^{-2iz}}{\vartheta_0\vartheta_3\vartheta_0^3(z)\vartheta_3(z)} = -\frac{1}{2}A_2^{(2)}(z,\tfrac{\pi\tau}{2}) + \frac{\vartheta_2''}{\vartheta_2}A_2^{(0)}(z,\tfrac{\pi\tau}{2}) + \vartheta_0^2\vartheta_3^2A_2^{(0)}(z,\tfrac{\pi\tau}{2}+\tfrac{\pi}{2}$$

This gives

$$(405) \qquad \begin{aligned}\frac{\vartheta_1'^3\vartheta_2\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0^3(z)\vartheta_3(z)} &= 2\sum_{n,r}[2(2n+r-1)]^2 q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz \quad + \\ &+4\frac{\vartheta_2''}{\vartheta_2}\sum_{n,r}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + 4\vartheta_0^2\vartheta_3^2\sum_{n,r}(-1)^{r+1}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz\end{aligned}$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_3^3(z)\vartheta_0(z)} = 2\sum_{n,r}(-1)^{r+1}[2(2n+r-1)]^2\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz \; +$$

(406)
$$+\,4\frac{\vartheta_2''}{\vartheta_2}\sum_{n,r}(-1)^{r+1}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + 4\vartheta_0^2\vartheta_3^2\sum_{n,r}q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz\,.$$

From these follow

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} - 2\sum q^N\left\{(\alpha+2a)^2\sin(\alpha-2a)z\right\} \; +$$

(403.1)
$$-\,\frac{\vartheta_2''}{\vartheta_2}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{\sin(\alpha-2a)z\right\}\right\} - \vartheta_0^2\vartheta_3^2\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-2a}{2}}\sin(\alpha-2a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_2^3(z)\vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha-2a}{2}}(\alpha+2a)^2\sin(\alpha-2a)z\right\} \; +$$

(404.1)
$$-\,\frac{\vartheta_2''}{\vartheta_2}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-2a-2}{2}}\sin(\alpha-2a)z\right\}\right\} - \vartheta_0^2\vartheta_3^2\left\{\frac{\cos z}{\sin z}-4\sum q^N\left\{\sin(\alpha-2a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0^3(z)\vartheta_3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\delta+2d)^2\sin\frac{\delta-2d}{2}z\right\} \; +$$

(405.1)
$$+\,4\frac{\vartheta_2''}{\vartheta_2}\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-2d}{2}z\right\} + 4\vartheta_0^2\vartheta_3^2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d-4}{4}}\sin\frac{\delta-2d}{2}z\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_3^3(z)\vartheta_0(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d-4}{4}}(\delta+2d)^2\sin\frac{\delta-2d}{2}z\right\} \; +$$

(406.1)
$$+\,4\vartheta_0^2\vartheta_3^2\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-2d}{2}z\right\} + 4\frac{\vartheta_2''}{\vartheta_2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d-4}{4}}\sin(\delta-2d)\frac{z}{2}\right\}$$

Group X-a

$$\frac{\vartheta_2^2(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_2^2(z)\vartheta_3^2(z)}$$

$$\frac{\vartheta_2^2(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_2^2(z)\vartheta_0^2(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_2^2(z)\vartheta_0^2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^2(z)}{\vartheta_1^2(z)\vartheta_0^2(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of order two at $t = \frac{\pi\tau}{2}$ and $t = \pi\tau$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$(407) \qquad \vartheta_1'^2 \vartheta_0^2 F(z) = \vartheta_2^2 A_2^{(1)}\left(z+\tfrac{\pi\tau}{2}, \tfrac{\pi\tau}{2}\right) + q^{1/2}\vartheta_3^2\left\{ A_2^{(1)}\left(z+\tfrac{\pi\tau}{2}, \pi\tau\right) + 2i\, A_2^{(0)}\left(z+\tfrac{\pi\tau}{2}, \pi\tau\right)\right\}$$

There follows

$$(408) \qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_0^2\vartheta_2^2(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} &= \vartheta_2^2\left\{ \frac{1}{\sin^2 z} - 2\sum_n 4n\, q^{2n^2} - 4\sum 2(2n+r)\, q^{2n^2+2nr}\cos 2rz \right\} + \\ &\quad -4\vartheta_3^2\left\{ \sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum_{n,r} 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz \right\} \end{aligned}$$

Replacing z by $z+\frac{\pi}{2}$

$$(409) \qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_0^2\vartheta_1^2(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} &= \vartheta_2^2\left\{ \frac{1}{\cos^2 z} - 2\sum_n 4n\, q^{2n^2} + 4\sum_{n,r} (-1)^{r+1} 2(2n+r)\, q^{2n^2+2nr}\cos 2rz \right\} + \\ &\quad -4\vartheta_3^2\left\{ \sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum_{n,r} (-1)^r 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz \right\} \end{aligned}$$

In (407) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$q^{1/2}\frac{\vartheta_1'^2\vartheta_0^2\vartheta_3^2(z)\, e^{2iz}}{\vartheta_0^2(z)\vartheta_1^2(z)} = \vartheta_2^2 A_2^{(1)}\left(z+\pi\tau, \tfrac{\pi\tau}{2}\right) + \vartheta_3^2 q^{1/2}\left\{ A_2^{(1)}(z+\pi\tau, \pi\tau) + 2i A_2^{(0)}(z+\pi\tau, \pi\tau)\right\}$$

There follows

$$(410) \qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_0^2\vartheta_3^2(z)}{\vartheta_0^2(z)\vartheta_1^2(z)} &= \vartheta_3^2\left\{ \frac{1}{\sin^2 z} - 2\sum_n 4n\, q^{2n^2} - 4\sum_{n,r} 2(2n+r)\, q^{2n^2+2nr}\cos 2rz \right\} + \\ &\quad -4\vartheta_2^2\left\{ \sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum_{n,r} 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz \right\} \end{aligned}$$

Replacing z by $z+\frac{\pi}{2}$

$$(411) \qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_0^2\vartheta_0^2(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} &= \vartheta_3^2\left\{ \frac{1}{\cos^2 z} - 2\sum_n 4n\, q^{2n^2} + 4\sum_{n,r} (-1)^{r+1} 2(2n+r)\, q^{2n^2+2nr}\cos 2rz \right\} + \\ &\quad -4\vartheta_2^2\left\{ \sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum (-1)^r 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz \right\} \end{aligned}$$

In these results replace q by $-q$. We get

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_2^2(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} = \vartheta_2^2\left\{\frac{1}{\sin^2 z} - 2\sum_n 4n\, q^{2n^2} - 4\sum_{n,r} 2(2n+r)\, q^{2n^2+2nr}\cos 2rz\right\} + \tag{412}$$

$$-4\vartheta_0^2\left\{\sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum_{n,r}(-1)^r 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_1^2(z)}{\vartheta_2^2(z)\vartheta_0^2(z)} = \vartheta_2^2\left\{\frac{1}{\cos^2 z} - 2\sum_n 4n\, q^{2n^2} + 4\sum_{n,r}(-1)^{r+1} 2(2n+r)\, q^{2n^2+2nr}\cos 2rz\right\} \tag{413}$$

$$-4\vartheta_0^2\left\{\sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum_{n,r} 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_0^2(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} = \vartheta_0^2\left\{\frac{1}{\sin^2 z} - 2\sum_n 4n\, q^{2n^2} - 4\sum_{n,r} 2(2n+r)\, q^{2n^2+2nr}\cos 2rz\right\} + \tag{414}$$

$$+4\vartheta_2^2\left\{\sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum_{n,r}(-1)^r 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_3^2(z)}{\vartheta_2^2(z)\vartheta_0^2(z)} = \vartheta_0^2\left\{\frac{1}{\cos^2 z} - 2\sum_n 4n\, q^{2n^2} + 4\sum_{n,r}(-1)^{r+1} q^{2n^2+2nr}\, 2(2n+r)\cos 2rz\right\} + \tag{415}$$

$$+4\vartheta_2^2\left\{\sum_n (2n-1)^2\, q^{\frac{(2n-1)^2}{2}} + \sum_{n,r} 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

From these results follow

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_2^2(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} = \vartheta_2^2\left\{\frac{1}{\sin^2 z} - 2\sum_n 4n\, q^{2n^2} - 4\sum q^N\left\{(\alpha+2a)\cos(\alpha-2a)z\right\}\right\} + \tag{408.1}$$

$$-2\vartheta_3^2\left\{2\sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_1^2(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} = \vartheta_2^2\left\{\frac{1}{\cos^2 z} - 2\sum_n 4n\, q^{2n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-2a-2}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\}\right\} + \tag{409.1}$$

$$-2\vartheta_3^2\left\{2\sum_n (2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}$$

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_3^2(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} = \vartheta_3^2\left\{\frac{1}{\sin^2 z} - 2\sum_n 4n\, q^{2n^2} - 4\sum q^N\left\{(\alpha+2a)\cos(\alpha-2a)z\right\}\right\} + \tag{410.1}$$

$$-2\vartheta_2^2\left\{\sum 2(2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_0^2(z)\vartheta_0^2}{\vartheta_2^2(z)\vartheta_3^2(z)} = \vartheta_3^2\left\{\frac{1}{\cos^2 z} - 2\sum_n 4n\,q^{2n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-2a-2}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\}\right\} +$$

(411.1)

$$-2\vartheta_2^2\left\{2\sum_n(2n-1)\,q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_2^2(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} = \vartheta_2^2\left\{\frac{1}{\sin^2 z} - 2\sum 4n\,q^{2n^2} - 4\sum q^N\left\{(\alpha+2a)\cos(\alpha-2a)z\right\}\right\} +$$

(412.1)

$$-2\vartheta_0^2\left\{2\sum_n(2n-1)\,q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_1^2(z)}{\vartheta_0^2(z)\vartheta_2^2(z)} = \vartheta_2^2\left\{\frac{-1}{\cos^2 z} - 2\sum_n 4n\,q^{2n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-2a-2}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\}\right\} +$$

(413.1)

$$-2\vartheta_0^2\left\{2\sum(2n-1)\,q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_3^2(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} = \vartheta_0^2\left\{\frac{1}{\sin^2 z} - 2\sum_n 4n\,q^{2n^2} - 4\sum q^N\left\{(\alpha+2a)\cos(\alpha-2a)z\right\}\right\} +$$

(414.1)

$$+2\vartheta_2^2\left\{2\sum_n(2n-1)\,q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_3^2(z)}{\vartheta_0^2(z)\vartheta_2^2(z)} = \vartheta_0^2\left\{\frac{1}{\cos^2 z} - 2\sum_n 4n\,q^{2n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-2a-2}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\}\right\} +$$

(415.1)

$$+2\vartheta_2^2\left\{2\sum_n(2n-1)\,q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}.$$

Group X-b

$$\frac{\vartheta_0^2(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_0^2(z)\vartheta_3^2(z)}$$

Consider

$$F(z) = \frac{\vartheta_0^2(z)}{\vartheta_1^2(z)\,\vartheta_2^2(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of order two at $t = \frac{\pi\tau}{2}$ and $t = \frac{\pi\tau}{2} + \frac{\pi}{2}$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$\vartheta_1'^2 \vartheta_2^2 F(z) = \vartheta_0^2 A_2^{(1)}\left(z + \frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) + \vartheta_3^2 A_2^{(1)}\left(z + \frac{\pi\tau}{2}, \frac{\pi\tau}{2} + \frac{\pi}{2}\right). \tag{416}$$

There follows

$$\begin{aligned} \frac{\vartheta_1'^2 \vartheta_2^2 \vartheta_0^2(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} &= \vartheta_0^2 \left\{ \frac{1}{\sin^2 z} - 2\sum_n 4n q^{2n^2} - 4\sum 2(2n+r) q^{2n^2+2nr} \cos 2rz \right\} \\ &+ \vartheta_3^2 \left\{ \frac{1}{\cos^2 z} - 2\sum_n 4n q^{2n^2} + 4\sum_{n,r} (-1)^{r+1} 2(2n+r) q^{2n^2+2nr} \cos 2rz \right\} \end{aligned} \tag{417}$$

Replacing z by $z + \frac{\pi}{2}$

$$\begin{aligned} \frac{\vartheta_1'^2 \vartheta_2^2 \vartheta_3^2(z)}{\vartheta_1^2(z)\vartheta_2^2 z} &= \vartheta_0^2 \left\{ \frac{1}{\cos^2 z} - 2\sum_n 4n q^{2n^2} + 4\sum_{n,r} (-1)^{r+1} 2(2n+r) q^{2n^2+2nr} \cos 2rz \right\} + \\ &+ \vartheta_3^2 \left\{ \frac{1}{\sin^2 z} - 2\sum_n 4n q^{2n^2} - 4\sum_{n,r} 2(2n+r) q^{2n^2+2nr} \cos 2rz \right\} \end{aligned} \tag{418}$$

In (416) replace z by $z - \frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2 \vartheta_2^2 \vartheta_1^2(z) e^{-2iz}}{\vartheta_0^2(z)\vartheta_3^2(z)} = q^{-1/2} \vartheta_0^2 A_2^{(1)}\left(z, \frac{\pi\tau}{2}\right) + \vartheta_3^2 q^{-1/2} A_2^{(1)}\left(z, \frac{\pi\tau}{2} + \frac{\pi}{2}\right).$$

which gives

$$\begin{aligned} \frac{\vartheta_1'^2 \vartheta_2^2 \vartheta_1^2(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} &= 4\vartheta_3^2 \left\{ \sum_n (2n-1) q^{\frac{(2n-1)^2}{2}} + \sum_{n,r} 2(2n+r-1)(-1)^r q^{\frac{(2n-1)^2}{2}+(2n-1)r} \cos 2rz \right\} + \\ &- 4\vartheta_0^2 \left\{ \sum_n (2n-1) q^{\frac{(2n-1)^2}{2}} + \sum_{n,r} 2(2n+r-1) q^{\frac{(2n-1)^2}{2}+(2n-1)r} \cos 2rz \right\} \end{aligned} \tag{419}$$

Replacing z by $z + \frac{\pi}{2}$

$$\begin{aligned} \frac{\vartheta_1'^2 \vartheta_2^2 \vartheta_2^2(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} &= 4\vartheta_3^2 \left\{ \sum_n (2n-1) q^{\frac{(2n-1)^2}{2}} + \sum_{n,r} 2(2n+r-1) q^{\frac{(2n-1)^2}{2}+(2n-1)r} \cos 2rz \right\} \\ &- 4\vartheta_0^2 \left\{ \sum_n (2n-1) q^{\frac{(2n-1)^2}{2}} + \sum_{n,r} (-1)^r 2(2n+r-1) q^{\frac{(2n-1)^2}{2}+(2n-1)r} \cos 2rz \right\} \end{aligned} \tag{420}$$

We get from these

$$\frac{\vartheta_1'^2 \vartheta_2^2 \vartheta_0^2(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0^2\left\{\frac{1}{\sin^2 z} - 2\sum_n 4n\, q^{2n^2} - 4\sum q^N\left\{(\alpha+2a)\cos(\alpha-2a)z\right\}\right\} + \tag{417.1}$$
$$+\ \vartheta_3^2\left\{\frac{1}{\cos^2 z} - 2\sum_n 4n\, q^{2n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-2a-2}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_2^2 \vartheta_3^2(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0^2\left\{\frac{1}{\cos^2 z} - 2\sum_n 4n\, q^{2n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-2a-2}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\}\right\} + \tag{418.1}$$
$$+\ \vartheta_3^2\left\{\frac{1}{\sin^2 z} - 2\sum_n 4n\, q^{2n^2} - 4\sum q^N\left\{(\alpha+2a)\cos(\alpha-2a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_2^2 \vartheta_1^2(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} = 2\,\vartheta_3^2\left\{\sum_n 2(2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\} + \tag{419.1}$$
$$-\,2\,\vartheta_0^2\left\{\sum_n 2(2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2 \vartheta_2^2 \vartheta_2^2(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} = 2\,\vartheta_3^2\left\{\sum_n 2(2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\} + \tag{420.1}$$
$$-\,2\,\vartheta_0^2\left\{\sum_n 2(2n-1)\, q^{\frac{(2n-1)^2}{2}} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2d}{4}}(\delta+2d)\cos\frac{\delta-2d}{2}z\right\}\right\}$$

Group XI-a

$$\frac{\vartheta_2(z)\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} \qquad \frac{\vartheta_1(z)\vartheta_0(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_2(z)\vartheta_0(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_1(z)\vartheta_3(z)}{\vartheta_2^2(z)\vartheta_3^2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2(z)\vartheta_3(z)e^{-iz}}{\vartheta_1^2(z)\vartheta_0^2(z)}$$

Let $t = z + \frac{\pi\tau}{4}$ $F(z) = \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of order two at $t = \frac{\pi\tau}{4}$ and $t = \frac{3\pi\tau}{4}$. Calculating the corresponding $R_i^{(\nu)}$ and using (10) gives

$$\frac{\vartheta_1'^2 \vartheta_0^2}{\vartheta_2\vartheta_3}F(z) = A_2^{(1)}\left(z+\tfrac{\pi\tau}{4}, \tfrac{\pi\tau}{4}\right) - i\,A_2^{(0)}\left(z+\tfrac{\pi\tau}{4}, \tfrac{\pi\tau}{4}\right) + A_2^{(1)}\left(z+\tfrac{\pi\tau}{4}, \tfrac{3\pi\tau}{4}\right) + \tag{421}$$
$$+\ i\,A_2^{(0)}\left(z+\tfrac{\pi\tau}{4}, \tfrac{3\pi\tau}{4}\right).$$

There follows

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_1^2(z)\vartheta_0^2(z)} = \frac{\cos z}{\sin^2 z} - 4\sum_{n,r}[2(2n+r)-1]\, q^{n(2n+2r-1)}\cos(2r-1)z \; +$$

$$(422) \qquad -4\sum_{n,r}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \, .$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_1(z)\vartheta_0(z)}{\vartheta_2\vartheta_3\vartheta_2^2(z)\vartheta_3^2(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^r[2(2n+r)-1]\, q^{n(2n+2r-1)}\sin(2r-1)z \; +$$

$$(423) \qquad +4\sum_{n,r}(-1)^r[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

In the above replace q by $-q$, obtaining

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_0^2(z)\vartheta_2^2(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^{n+r}[2(2n+r)-1]\, q^{n(2n+2r-1)}\sin(2r-1)z \; +$$

$$(425) \qquad +4\sum_{n,r}(-1)^{n+1}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

$$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_0(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_1^2(z)\vartheta_3^2(z)} = \frac{\cos z}{\sin^2 z} + 4\sum_{n,r}(-1)^{n+1}[2(2n+r)-1]\, q^{n(2n+2r-1)}\cos(2r-1)z \; +$$

$$(424) \qquad +4\sum_{n,r}(-1)^{n+r}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

These give

$$(422.1) \qquad \frac{\vartheta_1'^2\vartheta_0^2\vartheta_2(z)\vartheta_3(z)}{\vartheta_2\vartheta_3\vartheta_1^2(z)\vartheta_0^2(z)} = \frac{\cos z}{\sin^2 z} - 4\sum q^N\{(\beta+2b)\cos(\beta-2b)z\} - 2\sum q^{\frac{N}{4}}\{(\gamma+2c)\cos\tfrac{\gamma-2c}{2}z\}$$

$$(423.1) \qquad \frac{\vartheta_1'^2\vartheta_0^2\vartheta_1(z)\vartheta_0(z)}{\vartheta_2\vartheta_3\vartheta_2^2(z)\vartheta_3^2(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\{(-1)^{\frac{\beta-2b+1}{2}}(\beta+2b)\sin(\beta-2b)z\} + 2\sum q^{\frac{N}{4}}\{(-1)^{\frac{\gamma-2c+2}{2}}(\gamma+2c)\sin\tfrac{\gamma-2c}{2}z\}$$

$$(424.1) \qquad \frac{\vartheta_1'^2\vartheta_3^2\vartheta_0(z)\vartheta_2(z)}{\vartheta_0\vartheta_2\vartheta_1^2(z)\vartheta_3^2(z)} = \frac{\cos z}{\sin^2 z} + 4\sum q^N\{(-1)^{b+1}(\beta+2b)\cos(\beta-2b)z\} + 2\sum q^{\frac{N}{4}}\{(-1)^{\frac{\gamma+4}{4}}(\gamma+2c)\cos\tfrac{\gamma-2c}{2}z\}$$

$$(425.1) \qquad \frac{\vartheta_1'^2\vartheta_3^2\vartheta_1(z)\vartheta_3(z)}{\vartheta_0\vartheta_2\vartheta_0^2(z)\vartheta_2^2(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\{(-1)^{\frac{\beta+1}{2}}(\beta+2b)\sin(\beta-2b)z\} + 2\sum q^{\frac{N}{4}}\{(-1)^{\frac{c-1}{2}}(\gamma+2c)\sin\tfrac{\gamma-2c}{2}z\}$$

Group XI-b

$$\frac{\vartheta_0(z)\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} \qquad \frac{\vartheta_1(z)\vartheta_2(z)}{\vartheta_0^2(z)\vartheta_3^2(z)}$$

Consider

$$F(z) = \frac{\vartheta_0(z)\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_2^2(z)}$$

Let $t = z + \frac{\pi\tau}{2} + \frac{\pi}{4}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of order two at $t = \frac{\pi\tau}{2} + \frac{\pi}{4}$ and $t = \frac{\pi\tau}{2} + \frac{3\pi}{4}$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$(426) \qquad \frac{\vartheta_1'^2\vartheta_2^2}{\vartheta_0\vartheta_3} F(z) = A_2^{(1)}\left(z + \frac{\pi\tau}{2} + \frac{\pi}{4}, \frac{\pi\tau}{2} + \frac{\pi}{4}\right) + A_2^{(1)}\left(z + \frac{\pi\tau}{2} + \frac{\pi}{4}, \frac{\pi\tau}{2} + \frac{3\pi}{4}\right)$$

There follows

$$(427) \qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_2^2\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_1^2(z)\vartheta_2^2(z)} &= \frac{1}{\sin^2 z} + 2\sum_{n}(-1)^{n+1} 4n\, q^{2n^2} + 4\sum_{n,r}(-1)^{n+1} 2(2n+r)\, q^{2n^2+2nr}\cos 2rz + \\ &+ \frac{1}{\cos^2 z} + 2\sum_{n}(-1)^{n+1} 4n\, q^{2n^2} + 4\sum_{n,r}(-1)^{n+r+1} 2(2n+r)\, q^{2n^2+2nr}\cos 2rz \end{aligned}$$

In (426) replace z by $z - \frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_1(z)\vartheta_2(z)e^{-2iz}}{\vartheta_0\vartheta_3\vartheta_0^2(z)\vartheta_3^2(z)} = -i\, q^{-1/2} A_2^{(1)}\left(z + \frac{\pi}{4}, \frac{\pi\tau}{2} + \frac{\pi}{4}\right) - i A_2^{(1)}\left(z + \frac{\pi}{4}, \frac{\pi\tau}{2} + \frac{3\pi}{4}\right)$$

which gives

$$(428) \qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_2^2\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0^2(z)\vartheta_3^2(z)} &= 4\sum_{n,r}(-1)^{n+1}[2(2n+r-1)]\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + \\ &+ 4\sum_{n,r}(-1)^{n+r} 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz \end{aligned}$$

From these follow

$$(427.1) \qquad \begin{aligned} \frac{\vartheta_1'^2\vartheta_2^2\vartheta_0(z)\vartheta_3(z)}{\vartheta_0\vartheta_3\vartheta_1^2(z)\vartheta_2^2(z)} &= \frac{1}{\sin^2 z} + 2\sum(-1)^{n+1} 4n\, q^{2n^2} + 4\sum q^N\left\{(-1)^{a+1}(\alpha+2a)\cos(\alpha-2a)z\right\} + \\ &+ \frac{1}{\cos^2 z} + 2\sum(-1)^{n+1} 4n\, q^{2n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha+2}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\} \end{aligned}$$

$$(428.1) \qquad \frac{\vartheta_1'^2\vartheta_2^2\vartheta_1(z)\vartheta_2(z)}{\vartheta_0\vartheta_3\vartheta_0^2(z)\vartheta_3^2(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+2d)\sin\frac{\delta-2d}{2}z\right\} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2}{4}}(\delta+2d)\sin\frac{\delta-2d}{2}z\right\}$$

Group XII-a

$$\frac{\vartheta_0^2(z)}{\vartheta_1^2(z)\vartheta_2(z)\vartheta_3(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_2^2(z)\vartheta_1(z)\vartheta_0(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_0^2(z)\vartheta_2(z)\vartheta_3(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_3^2(z)\vartheta_1(z)\vartheta_0(z)}$$

$$\frac{\vartheta_3^2(z)}{\vartheta_1^2(z)\vartheta_2(z)\vartheta_0(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_2^2(z)\vartheta_1(z)\vartheta_3(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_3^2(z)\vartheta_2(z)\vartheta_0(z)} \qquad \frac{\vartheta_2^2(z)}{\vartheta_0^2(z)\vartheta_1(z)\vartheta_3(z)}$$

Consider

$$F(z) = \frac{\vartheta_0^2(z)\, e^{-iz}}{\vartheta_1^2(z)\vartheta_2(z)\vartheta_3(z)}$$

Let $t = z + \frac{\pi\tau}{4}$, $F(z) \equiv \phi(t)$. $\phi(t)$ satisfies (8) and has poles of orders two, and one, at $t = \frac{\pi\tau}{4}$; $t = \frac{\pi\tau}{4} + \frac{\pi}{2}$ and $t = \frac{3\pi\tau}{4} + \frac{\pi}{2}$ respectively. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$(429) \quad \frac{\vartheta_1'^2\vartheta_2\vartheta_3}{\vartheta_0^2} F(z) = A_2^{(1)}\left(z + \frac{\pi\tau}{4}, \frac{\pi\tau}{4}\right) - i\, A_2^{(0)}\left(z + \frac{\pi\tau}{4}, \frac{\pi\tau}{4}\right) + i\,\frac{\vartheta_3^4}{\vartheta_0^2} A_2^{(0)}\left(z + \frac{\pi\tau}{4}, \frac{\pi\tau}{4} + \frac{\pi}{2}\right) - i\,\frac{\vartheta_2^4}{\vartheta_0^2} A_2^{(0)}\left(z + \frac{\pi\tau}{4}, \frac{3\pi\tau}{4} + \frac{\pi}{2}\right)$$

There follows

$$(430) \quad \frac{\vartheta_1'^2\vartheta_2\vartheta_3\vartheta_0^2(z)}{\vartheta_0^2\vartheta_1^2(z)\vartheta_2(z)\vartheta_3(z)} = \frac{\cos z}{\sin^2 z} - 4\sum_{n,r}[2(2n+r)-1]\, q^{n(2n+2r-1)} \cos(2r-1)z + \frac{\vartheta_3^4}{\vartheta_0^2}\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{n(2n+2r-1)}\cos(2r-1)z\right\} + 4\frac{\vartheta_2^4}{\vartheta_0^2}\sum_{n,r}(-1)^r q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

Replacing z by $z - \frac{\pi}{2}$

$$(431) \quad \frac{\vartheta_1'^2\vartheta_2\vartheta_3\vartheta_3^2(z)}{\vartheta_0^2\vartheta_2^2(z)\vartheta_1(z)\vartheta_0(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^r[2(2n+r)-1]\, q^{n(2n+2r-1)}\sin(2r-1)z + \frac{\vartheta_3^4}{\vartheta_0^2}\left\{\frac{1}{\sin z} + 4\sum_{n,r} q^{n(2n+2r-1)}\sin(2r-1)z\right\} - 4\frac{\vartheta_2^4}{\vartheta_0^2}\sum_{n,r} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

In (429) replace z by $z + \frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_3\vartheta_1^2(z)e^{iz}}{\vartheta_0^2\vartheta_0^2(z)\vartheta_2(z)\vartheta_3(z)} = A_2^{(1)}\left(z + \frac{\pi\tau}{4}, \frac{3\pi\tau}{4}\right) - i\, A_2^{(0)}\left(z + \frac{3\pi\tau}{4}, \frac{\pi\tau}{4}\right) - i\,\frac{\vartheta_2^4}{\vartheta_0^2} A_2^{(0)}\left(z + \frac{3\pi\tau}{4}, \frac{3\pi\tau}{4} + \frac{\pi}{2}\right) + i\,\frac{\vartheta_3^4}{\vartheta_0^2} A_2^{(0)}\left(z + \frac{3\pi\tau}{4}, \frac{\pi\tau}{4} + \frac{\pi}{2}\right)$$

This gives

$$(432) \quad \frac{\vartheta_1'^2\vartheta_2\vartheta_3\vartheta_1^2(z)}{\vartheta_0^2\vartheta_0^2(z)\vartheta_2(z)\vartheta_3(z)} = -4\sum_{n,r}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + \frac{\vartheta_2^4}{\vartheta_0^2}\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{n(2n+2r-1)}\cos(2r-1)z\right\} + 4\frac{\vartheta_3^4}{\vartheta_0^2}\sum_{n,r}(-1)^r q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

Replace z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_3\vartheta_2^2(z)}{\vartheta_0^2\vartheta_3^2(z)\vartheta_1(z)\vartheta_0(z)} = 4\sum_{n,r}(-1)^r[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(433)

$$+\frac{\vartheta_2^4}{\vartheta_0^2}\left\{\frac{1}{\sin z}+4\sum q^{n(2n+2r-1)}\sin(2r-1)z\right\} - 4\frac{\vartheta_3^4}{\vartheta_0^2}\sum q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z .$$

In these results replace q by $-q$, obtaining

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2\vartheta_3^2(z)}{\vartheta_3^2\vartheta_1^2(z)\vartheta_0(z)\vartheta_2(z)} = \frac{\cos z}{\sin^2 z} + 4\sum_{n,r}(-1)^{n+1}[2(2n+r)-1]\, q^{n(2n+2r-1)}\cos(2r-1)z \; +$$

(434)

$$+\frac{\vartheta_0^4}{\vartheta_3^2}\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{n+r+1} q^{n(2n+2r-1)}\cos(2r-1)z\right\} + 4\frac{\vartheta_2^4}{\vartheta_3^2}\sum_{n,r}(-1)^n q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2\vartheta_0^2(z)}{\vartheta_3^2\vartheta_2^2(z)\vartheta_1(z)\vartheta_3(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^{n+r}[2(2n+r)-1]\, q^{n(2n+2r-1)}\sin(2r-1)z \; +$$

(435)

$$+\frac{\vartheta_0^4}{\vartheta_3^2}\left\{\frac{1}{\sin z}+4\sum_{n,r}(-1)^n q^{n(2n+2r-1)}\sin(2r-1)z\right\} + 4\frac{\vartheta_2^4}{\vartheta_3^2}\sum_{n,r}(-1)^{n+r+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2\vartheta_1^2(z)}{\vartheta_3^2\vartheta_3^2(z)\vartheta_0(z)\vartheta_2(z)} = 4\sum_{n,r}(-1)^{n+r+1}[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(436)

$$+\frac{\vartheta_2^4}{\vartheta_3^2}\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{n+r+1} q^{n(2n+2r-1)}\cos(2r-1)z\right\} + 4\frac{\vartheta_0^4}{\vartheta_3^2}\sum_{n,r}(-1)^n q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2\vartheta_2^2(z)}{\vartheta_3^2\vartheta_0^2(z)\vartheta_1(z)\vartheta_3(z)} = 4\sum_{n,r}(-1)^n[2(2n+r)-3]\, q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(437)

$$+\frac{\vartheta_2^4}{\vartheta_3^2}\left\{\frac{1}{\sin z}+4\sum_{n,r}(-1)^n q^{(2n+2r-1)n}\sin(2r-1)z\right\} + 4\frac{\vartheta_0^4}{\vartheta_3^2}\sum_{n,r}(-1)^{n+r+1} q^{\frac{(2n-1)^2}{2}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

From these follow

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_3\vartheta_0^2(z)}{\vartheta_0^2\vartheta_1^2(z)\vartheta_2(z)\vartheta_3(z)} = \frac{\cos z}{\sin^2 z} - 4\sum q^N\{(\beta+2b)\cos(\beta-2b)z\} \; +$$

(430.1)

$$+\frac{\vartheta_3^4}{\vartheta_0^2}\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}\cos(\beta-2b)z\right\}\right\} + 4\frac{\vartheta_2^4}{\vartheta_3^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c+2}{4}}\cos\frac{\gamma-2c}{2}z\right\}$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_3\vartheta_3^2(z)}{\vartheta_0^2\vartheta_2^2(z)\vartheta_1(z)\vartheta_0(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta-2b+1}{2}}(\beta+2b)\sin(\beta-2b)z\right\} +$$

(431.1)

$$+\frac{\vartheta_3^4}{\vartheta_0^2}\left\{\frac{1}{\sin z}+4\sum q^N\left\{\sin(\beta-2b)z\right\}\right\} - 4\frac{\vartheta_2^4}{\vartheta_0^2}\sum q^{\frac{N}{4}}\left\{\quad \sin\frac{\gamma-2c}{2}z\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_3\vartheta_1^2(z)}{\vartheta_0^2\vartheta_0^2(z)\vartheta_2(z)\vartheta_3(z)} = -2\sum q^{\frac{N}{4}}\left\{(\gamma+2c)\cos\frac{\gamma-2c}{2}z\right\} + 4\frac{\vartheta_3^4}{\vartheta_0^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c+2}{4}}\cos\frac{\gamma-2c}{2}z\right\}$$

(432.1)

$$+\frac{\vartheta_2^4}{\vartheta_0^2}\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-2b-1}{2}}\cos(\beta-2b)z\right\}\right\}$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_3\vartheta_2^2(z)}{\vartheta_0^2\vartheta_3^2(z)\vartheta_1(z)\vartheta_0(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-2c+2}{4}}(\gamma+2c)\sin\frac{\gamma-2c}{2}z\right\} - 4\frac{\vartheta_3^4}{\vartheta_0^2}\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-2c}{2}z\right\} +$$

(433.1)

$$+\frac{\vartheta_2^4}{\vartheta_0^2}\left\{\frac{1}{\sin z}+4\sum q^N\left\{\sin(\beta-2b)z\right\}\right\}$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0\vartheta_3^2(z)}{\vartheta_3^2\vartheta_1^2(z)\vartheta_2(z)\vartheta_0(z)} = \frac{\cos z}{\sin^2 z} + 4\sum q^N\left\{(-1)^{b+1}(\beta+2b)\cos(\beta-2b)z\right\} +$$

(434.1)

$$+\frac{\vartheta_0^4}{\vartheta_3^2}\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-1}{2}}\cos(\beta-2b)z\right\}\right\} + 4\frac{\vartheta_2^4}{\vartheta_3^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-2c}{2}z\right\}$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0\vartheta_0^2(z)}{\vartheta_3^2\vartheta_2^2(z)\vartheta_1(z)\vartheta_3(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta+1}{2}}(\beta+2b)\sin(\beta-2b)z\right\} +$$

(435.1)

$$+\frac{\vartheta_0^4}{\vartheta_3^2}\left\{\frac{1}{\sin z}+4\sum q^N\left\{(-1)^b\sin(\beta-2b)z\right\}\right\} + 4\frac{\vartheta_2^4}{\vartheta_3^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma}{4}}\sin\frac{\gamma-2c}{2}z\right\}$$

$$\frac{\vartheta_1'^2\vartheta_2\vartheta_0\vartheta_1^2(z)}{\vartheta_3^2\vartheta_3^2(z)\vartheta_2(z)\vartheta_0(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma}{4}}(\gamma+2c)\cos\frac{\gamma-2c}{2}z\right\} + 4\frac{\vartheta_0^4}{\vartheta_3^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-2c}{2}z\right\} +$$

(436.1)

$$+\frac{\vartheta_2^4}{\vartheta_3^4}\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-1}{2}}\cos(\beta-2b)z\right\}\right\}$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_2\vartheta_2^2(z)}{\vartheta_3^2\vartheta_0^2(z)\vartheta_1(z)\vartheta_3(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+2c)\sin\frac{\gamma-2c}{2}z\right\} +$$

(437.1)

$$+4\frac{\vartheta_0^4}{\vartheta_3^2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma}{4}}\sin\frac{\gamma-2c}{2}z\right\} + \frac{\vartheta_2^4}{\vartheta_3^2}\left\{\frac{1}{\sin z}+4\sum q^N\left\{(-1)^b\sin(\beta-2b)z\right\}\right\}.$$

Group XII-b

$$\frac{\vartheta_2^2(z)}{\vartheta_1^2(z)\vartheta_0(z)\vartheta_3(z)} \qquad \frac{\vartheta_1^2(z)}{\vartheta_2^2(z)\vartheta_0(z)\vartheta_3(z)} \qquad \frac{\vartheta_3^2(z)}{\vartheta_0^2(z)\vartheta_1(z)\vartheta_2(z)} \qquad \frac{\vartheta_0^2(z)}{\vartheta_3^2(z)\vartheta_1(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2^2(z)}{\vartheta_1^2(z)\vartheta_0(z)\vartheta_3(z)}$$

Let $t = z+\frac{\pi\tau}{2}+\frac{\pi}{4}$ $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisifes (8) and has poles of orders two, and one at $t=\frac{\pi\tau}{2}+\frac{\pi}{4}$, and $t=\pi\tau+\frac{\pi}{4}$, $t=\pi\tau+\frac{3\pi}{4}$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3}{\vartheta_2^2}F(z) = A_2^{(1)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{4}, \frac{\pi\tau}{2}+\frac{\pi}{4}\right) + i q^{\frac{1}{2}}\frac{\vartheta_3^4}{\vartheta_2^2}A_2^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{4}, \pi\tau+\frac{\pi}{4}\right)+$$

(438)

$$+ i q^{\frac{1}{2}}\frac{\vartheta_0^4}{\vartheta_2^2}A_2^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{4}, \pi\tau+\frac{3\pi}{4}\right)$$

There follows

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3\vartheta_2^2(z)}{\vartheta_2^2\vartheta_1^2(z)\vartheta_0(z)\vartheta_3(z)} = \frac{1}{\sin^2 z} + 2\sum_n(-1)^{n+1}4n\,q^{2n^2} + 4\sum_{n,r}(-1)^{n+1}2(2n+r)\,q^{2n^2+2nr}\cos 2rz +$$

(439)

$$+2\frac{\vartheta_3^4}{\vartheta_2^2}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^n q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\} +$$

$$+2\frac{\vartheta_0^4}{\vartheta_2^2}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3\vartheta_1^2(z)}{\vartheta_2^2\vartheta_2^2(z)\vartheta_0(z)\vartheta_3(z)} = \frac{1}{\cos^2 z} + 2\sum_n(-1)^{n+1}4n\,q^{2n^2} + 4\sum_{n,r}(-1)^{n+r+1}2(2n+r)\,q^{2n^2+2nr}\cos 2rz +$$

(440)

$$+2\frac{\vartheta_3^4}{\vartheta_2^2}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^{n+r} q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\} +$$

$$+2\frac{\vartheta_0^4}{\vartheta_2^2}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum_{n,r}(-1)^n q^{\frac{(2n-1)^2}{2}+(2n-1)r}\cos 2rz\right\}$$

In (438) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3\vartheta_3^2(z)e^{2iz}}{\vartheta_2^2\vartheta_0^2(z)\vartheta_1(z)\vartheta_2(z)} = -i\, q^{-1/2} A_2^{(1)}\left(z+\pi\tau+\tfrac{\pi}{4}, \tfrac{\pi\tau}{2}+\tfrac{\pi}{4}\right) + \frac{\vartheta_3^4}{\vartheta_2^2} A_2^{(0)}\left(z+\pi\tau+\tfrac{\pi}{4}, \pi\tau+\tfrac{\pi}{4}\right) + \frac{\vartheta_0^4}{\vartheta_2^2} A_2^{(0)}\left(z+\pi\tau+\tfrac{\pi}{4}, \pi\tau+\tfrac{3\pi}{4}\right).$$

There follows

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3\vartheta_3^2(z)}{\vartheta_2^2\vartheta_0^2(z)\vartheta_1(z)\vartheta_2(z)} = 4\sum_{n,r}(-1)^n 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + \frac{\vartheta_3^4}{\vartheta_2^2}\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^n q^{2n^2+2nr}\sin 2rz\right\} + \frac{\vartheta_0^4}{\vartheta_2^2}\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{2n^2+2nr}\sin 2rz\right\}. \tag{441}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3\vartheta_0^2(z)}{\vartheta_2^2\vartheta_3^2(z)\vartheta_1(z)\vartheta_2(z)} = 4\sum_{n,r}(-1)^{n+r+1} 2(2n+r-1)\, q^{\frac{(2n-1)^2}{2}+(2n-1)r}\sin 2rz + \frac{\vartheta_3^4}{\vartheta_2^2}\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{2n^2+2nr}\sin 2rz\right\} + \frac{\vartheta_0^4}{\vartheta_2^2}\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^n q^{2n^2+2nr}\sin 2rz\right\}. \tag{442}$$

These give

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3\vartheta_2^2(z)}{\vartheta_2^2\vartheta_1^2(z)\vartheta_0(z)\vartheta_3(z)} = \frac{1}{\sin^2 z} + 2\sum_n(-1)^{n+1}4n\, q^{2n^2} + 4\sum q^N\left\{(-1)^{a+1}(\alpha+2a)\cos(\alpha-2a)z\right\} + 2\frac{\vartheta_3^4}{\vartheta_2^2}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-2d}{2}z\right\}\right\} + 2\frac{\vartheta_0^4}{\vartheta_2^2}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2}{4}}\cos\frac{\delta-2d}{2}z\right\}\right\}. \tag{439.1}$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3\vartheta_1^2(z)}{\vartheta_2^2\vartheta_2^2(z)\vartheta_0(z)\vartheta_3(z)} = \frac{1}{\cos^2 z} + 2\sum_n(-1)^{n+1}4n\, q^{2n^2} + 4\sum q^N\left\{(-1)^{\frac{d+2}{2}}(\alpha+2a)\cos(\alpha-2a)z\right\} + 2\frac{\vartheta_3^4}{\vartheta_2^2}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-2}{4}}\cos\frac{\delta-2d}{2}z\right\}\right\} + 2\frac{\vartheta_0^4}{\vartheta_2^2}\left\{\sum_n(-1)^n q^{\frac{(2n-1)^2}{2}} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-2d}{2}z\right\}\right\} \tag{440.1}$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3\vartheta_3^2(z)}{\vartheta_2^2\vartheta_0^2(z)\vartheta_1(z)\vartheta_2(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}(\delta+2d)\sin\frac{\delta-2d}{2}z\right\} +$$

$$(441.1) \qquad + \frac{\vartheta_3^4}{\vartheta_2^2}\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{(-1)^a \sin(\alpha-2a)z\right\}\right\} + \frac{\vartheta_0^4}{\vartheta_2^2}\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha+2}{2}}\sin(\alpha-2a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_0\vartheta_3\vartheta_0^2(z)}{\vartheta_2^2\vartheta_3^2(z)\vartheta_1(z)\vartheta_2(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta+2}{4}}(\delta+2d)\sin\frac{\delta-2d}{2}z\right\} +$$

$$(442.1) \qquad + \frac{\vartheta_3^4}{\vartheta_2^2}\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha+2}{2}}\sin(\alpha-2a)z\right\}\right\} + \frac{\vartheta_0^4}{\vartheta_2^2}\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{(-1)^a \sin(\alpha-2a)z\right\}\right\}$$

VII

The introductory remarks of the last section apply here with the exceptions that here μ equals three and that the table below replaces the previous one.

$$\begin{array}{ll}
N = n(3n+r) = a\alpha & \\
a = n & \alpha = 3n+r \\
n = a & 2r = \alpha - 3a \\
2(n+r) = \alpha - a & \\
2(3n+r) = \alpha + 3a & \\
\alpha \equiv a \bmod 2 & \\
0 < a < \sqrt{\frac{N}{3}} &
\end{array}$$

$$\begin{array}{ll}
N = n(3n+2r-1) = b\beta & \\
b = n & \beta = 3n+2r-1 \\
n = b & 2r = \beta - 3b + 1 \\
2(n+r) = \beta - b + 1 & \\
2(3n+r) - 1 = \beta + 3b & \\
\beta - b \equiv 1 \bmod 2 & \\
0 < b < \sqrt{\frac{N}{3}} &
\end{array}$$

$$\begin{array}{ll}
\frac{N}{4} = 3\left(\frac{2n-1}{2}\right)^2 + (2n-1)r = \frac{\delta d}{4} & \\
d = 2n-1 & \delta = 6n+4r-3 \\
2n = d+1 & 4r = \delta - 3d \\
4(n+r) = \delta - d + 2 & \\
2[2(3n+r)-3] = \delta + 3d & \\
\delta \equiv 3d \bmod 4, \quad 0 < d < \sqrt{\frac{N}{3}} & \\
d \equiv 1 \bmod 2 &
\end{array}$$

$$\begin{array}{ll}
\frac{N}{4} = 3\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2} = \frac{c\gamma}{4} & \\
c = 2n-1 & \gamma = 6n+4r-5 \\
2n = c+1 & 4r = \gamma - 3c + 2 \\
2(2r-1) = \gamma - 3c & \\
4(n+r) = \gamma - c + 4 & \\
4(3n+r-2) = \gamma + 3c & \\
\gamma \equiv 3c+2 \bmod \bmod 4 & \\
c \equiv 1 \bmod 2 & 0 < c < \sqrt{\frac{N}{3}}
\end{array}$$

Group I-a

$$\frac{1}{\vartheta_1^3(z)} \qquad \frac{1}{\vartheta_2^3(z)} \qquad \frac{1}{\vartheta_3^3(z)} \qquad \frac{1}{\vartheta_0^3(z)}$$

Consider

$$F(z) = \frac{e^{-iz}}{\vartheta_1^3(z)}$$

Let $t = z + \frac{\pi\tau}{3} + \frac{\pi}{6}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order three at $t = \frac{\pi\tau}{3} + \frac{\pi}{6}$. Calculating the corresponding $R_i^{(i)}$ and using (10) gives

$$\vartheta_1'^3 F(z) = \frac{i}{2} A_3^{(2)}\left(z+\tfrac{\pi\tau}{3}+\tfrac{\pi}{6}, \tfrac{\pi\tau}{3}+\tfrac{\pi}{6}\right) - i A_3^{(1)}\left(z+\tfrac{\pi\tau}{3}+\tfrac{\pi}{6}, \tfrac{\pi\tau}{3}+\tfrac{\pi}{6}\right) - \frac{1}{2}\left\{1+\frac{\vartheta_1'''}{\vartheta_1'}\right\} A_3^{(0)}\left(z+\tfrac{\pi\tau}{3}+\tfrac{\pi}{6}, \tfrac{\pi\tau}{3}+\tfrac{\pi}{6}\right) \tag{443}$$

There follows

$$\frac{\vartheta_1'^3}{\vartheta_1^3(z)} = \frac{1}{\sin^3 z} - \frac{1}{2\sin z}\left\{1+\frac{\vartheta_1'''}{\vartheta_1'}\right\} + 2\sum_{n,r}(-1)^{n+1}[2(3n+r)-1]^2 q^{n(3n+2r-1)} \sin(2r-1)z + 2\frac{\vartheta_1'''}{\vartheta_1'}\sum_{n,r}(-1)^{n+1} q^{n(3n+2r-1)}\sin(2r-1)z \tag{444}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3}{\vartheta_2^3(z)} = \frac{1}{\cos^3 z} - \frac{1}{2\cos z}\left\{1+\frac{\vartheta_1'''}{\vartheta_1'}\right\} + 2\sum_{n,r}(-1)^{n+r}[2(3n+r)-1]^2 q^{n(3n+2r-1)} \cos(2r-1)z + 2\frac{\vartheta_1'''}{\vartheta_1'}\sum_{n,r}(-1)^{n+r} q^{n(3n+2r-1)}\cos(2r-1)z \tag{445}$$

In (443) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^3 e^{2iz}}{\vartheta_0^3(z)} = -\frac{i}{2} q^{-1/4} A_3^{(2)}\left(z+\tfrac{5\pi\tau}{6}+\tfrac{\pi}{6}, \tfrac{\pi\tau}{3}+\tfrac{\pi}{6}\right) - q^{-1/4} A_3^{(1)}\left(z+\tfrac{5\pi\tau}{6}+\tfrac{\pi}{6}, \tfrac{\pi\tau}{3}+\tfrac{\pi}{6}\right) + \frac{i}{2} q^{-1/4}\left\{1+\frac{\vartheta_1'''}{\vartheta_1'}\right\} A_3^{(0)}\left(z+\tfrac{5\pi\tau}{6}+\tfrac{\pi}{6}, \tfrac{\pi\tau}{3}+\tfrac{\pi}{6}\right)$$

This gives

$$\frac{\vartheta_1'^3}{\vartheta_0^3(z)} = \sum_n(-1)^{n+1}[3(2n-1)]^2 q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+1}[2(3n+r)-3]^2 q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz + \frac{\vartheta_1'''}{\vartheta_1'}\left\{\sum_n(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} \tag{446}$$

Replacing z by $z-\frac{\pi}{2}$

$$\frac{\vartheta_1'^3}{\vartheta_3^3(z)} = \sum_n(-1)^{n+1}[3(2n-1)]^2 q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+r+1}[2(3n+r)-3]^2 q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz + \frac{\vartheta_1'''}{\vartheta_1'}\left\{\sum_n(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+r+1} q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} \tag{447}$$

From these follow

$$(444.1)\quad \frac{\vartheta_1'^3}{\vartheta_1^3(z)} = \frac{1}{\sin^3 z} - \frac{1}{2\sin z}\left\{1 + \frac{\vartheta_1'''}{\vartheta_1'}\right\} + 2\sum q^N\left\{(-1)^{b+1}(\beta+3b)^2 \sin(\beta-3b)z\right\} + 2\frac{\vartheta_1'''}{\vartheta_1'}\sum q^N\left\{(-1)^{b+1}\sin(\beta-3b)z\right\}.$$

$$(445.1)\quad \frac{\vartheta_1'^3}{\vartheta_2^3(z)} = \frac{1}{\cos^3 z} - \frac{1}{2\cos z}\left\{1 + \frac{\vartheta_1'''}{\vartheta_1'}\right\} + 2\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+3b)^2 \cos(\beta-3b)z\right\} + 2\frac{\vartheta_1'''}{\vartheta_1'}\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}\cos(\beta-3b)z\right\}.$$

$$(446.1)\quad \frac{\vartheta_1'^3}{\vartheta_0^3(z)} = \sum_m (-1)^{m+1}[3(2m-1)]^2 q^{3\left(\frac{2m-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+3d)^2\cos\frac{\delta-3d}{2}z\right\} + \frac{\vartheta_1'''}{\vartheta_1'}\left\{\sum_m (-1)^{m+1} q^{3\left(\frac{2m-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}\cos\frac{\delta-3d}{2}z\right\}\right\}$$

$$(447.1)\quad \frac{\vartheta_1'^3}{\vartheta_3^3(z)} = \sum_m (-1)^{m+1}[3(2m-1)]^2 q^{3\left(\frac{2m-1}{2}\right)^2} + \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-2}{4}}(\delta+3d)^2\cos\frac{\delta-3d}{2}z\right\} + \frac{\vartheta_1'''}{\vartheta_1'}\left\{\sum_m (-1)^{m+1} q^{3\left(\frac{2m-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-2}{4}}\cos\frac{\delta-3d}{2}z\right\}\right\}$$

Group II-a

$$\frac{1}{\vartheta_1^2(z)\vartheta_0(z)} \qquad \frac{1}{\vartheta_2^2(z)\vartheta_3(z)} \qquad \frac{1}{\vartheta_0^2(z)\vartheta_1(z)} \qquad \frac{1}{\vartheta_3^2(z)\vartheta_2(z)}$$

$$\frac{1}{\vartheta_1^2(z)\vartheta_3(z)} \qquad \frac{1}{\vartheta_2^2(z)\vartheta_0(z)} \qquad \frac{1}{\vartheta_3^2(z)\vartheta_1(z)} \qquad \frac{1}{\vartheta_0^2(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{1}{\vartheta_1^2(z)\vartheta_0(z)}$$

Let $t = z + \frac{\pi\tau}{2} + \frac{\pi}{6}$ $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders two and one at $t = \frac{\pi\tau}{2} + \frac{\pi}{6}$ and $t = \pi\tau + \frac{\pi}{6}$ respectively. Calculating thе corresponding and using (10) gives

$$\vartheta_1'^2\vartheta_0 F(z) = A_3^{(1)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{6}, \frac{\pi\tau}{2}+\frac{\pi}{6}\right) + i\, q^{3/4}\vartheta_2\vartheta_3 A_3^{(0)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{6}, \pi\tau+\frac{\pi}{6}\right) \tag{448}$$

There follows

$$\begin{aligned}\frac{\vartheta_1'^2\vartheta_0}{\vartheta_1^2(z)\vartheta_0(z)} &= \frac{1}{\sin^2 z} + 2\sum_{n}(-1)^{n+1}6n\,q^{3n^2} + 4\sum_{n,r}(-1)^{n+1}2(3n+r)\,q^{3n^2+2nr}\cos 2rz \;+ \\ &\quad + 2\vartheta_2\vartheta_3\left\{\sum_{n}(-1)^n q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^n q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}\end{aligned} \tag{449}$$

Replacing z by $z+\frac{\pi}{2}$

$$\begin{aligned}\frac{\vartheta_1'^2\vartheta_0}{\vartheta_2^2(z)\vartheta_3(z)} &= \frac{1}{\cos^2 z} + 2\sum_{n}(-1)^{n+1}6n\,q^{3n^2} + 4\sum_{n,r}(-1)^{n+r+1}2(3n+r)\,q^{3n^2+2nr}\cos 2rz \;+ \\ &\quad + 2\vartheta_2\vartheta_3\left\{\sum_{n}(-1)^n q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+r} q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}\end{aligned} \tag{450}$$

In (448) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2\vartheta_0 e^{3iz}}{\vartheta_0^2(z)\vartheta_1(z)} = -i\,q^{-3/4}A_3^{(1)}\left(z+\pi\tau+\frac{\pi}{6}, \frac{\pi\tau}{2}+\frac{\pi}{6}\right) + \vartheta_2\vartheta_3 A_3^{(0)}\left(z+\pi\tau+\frac{\pi}{6}, \pi\tau+\frac{\pi}{6}\right)$$

This gives

$$\frac{\vartheta_1'^2\vartheta_0}{\vartheta_0^2(z)\vartheta_1(z)} = 4\sum_{n,r}(-1)^n 2(3n+r-2)\,q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z + \vartheta_2\vartheta_3\left\{\frac{1}{\sin z} + 4\sum_{n}(-1)^n q^{n(3n+2r-1)}\sin(2r-1)z\right\}. \tag{451}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^2\vartheta_0}{\vartheta_3^2(z)\vartheta_2(z)} = 4\sum_{n,r}(-1)^{n+r+1}2(3n+r-2)\,q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z + \vartheta_2\vartheta_3\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n(3n+2r-1)}\cos(2r-1)z\right\}. \tag{452}$$

In these results replace by . We get

$$\begin{aligned}\frac{\vartheta_1'^2\vartheta_3}{\vartheta_1^2(z)\vartheta_3(z)} &= \frac{1}{\sin^2 z} - 2\sum_{n}6n\,q^{3n^2} - 4\sum_{n,r}2(3n+r)\,q^{3n^2+2nr}\cos 2rz \;+ \\ &\quad - 2\vartheta_0\vartheta_2\left\{\sum_{n}(-1)^n q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+r} q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}\end{aligned} \tag{454}$$

$$
\frac{\vartheta_1'^2\vartheta_3}{\vartheta_2^2(z)\vartheta_0(z)} = \frac{1}{\cos^2 z} - 2\sum_n 6n\, q^{3n^2} + 4\sum_{n,r}(-1)^{r+1}2(3n+r)\, q^{3n^2+2nr}\cos 2rz \; +
$$

(455)

$$
+ 2\vartheta_2\vartheta_0\left\{\sum_n(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}.
$$

(456) $$\frac{\vartheta_1'^2\vartheta_3}{\vartheta_3^2(z)\vartheta_1(z)} = 4\sum_{n,r}(-1)^{r+1}2(3n+r-2)\, q^{\frac{3(2n-1)^2}{4}+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; + \vartheta_0\vartheta_2\left\{\frac{1}{\sin z} + 4\sum_{n,r}(-1)^n q^{n(3n+2r-1)}\sin(2r-1)z\right\}$$

(457) $$\frac{\vartheta_1'^2\vartheta_3}{\vartheta_0^2(z)\vartheta_2(z)} = 4\sum_{n,r}2(3n+r-2)\, q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; + \vartheta_0\vartheta_2\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n(3n+2r-1)}\cos(2r-1)z\right\}$$

From these follow

$$
\frac{\vartheta_1'^2\vartheta_0}{\vartheta_1^2(z)\vartheta_0(z)} = \frac{1}{\sin^2 z} + 2\sum_n(-1)^{n+1}6n\, q^{3n^2} + 4\sum q^N\left\{(-1)^{a+1}(\alpha+3a)\cos(\alpha-3a)z\right\} \; +
$$

(449.1)

$$
+ 2\vartheta_2\vartheta_3\left\{\sum_n(-1)^n q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}\cos\frac{\delta-3d}{2}z\right\}\right\}
$$

$$
\frac{\vartheta_1'^2\vartheta_0}{\vartheta_2^2(z)\vartheta_3(z)} = \frac{1}{\cos^2 z} + 2\sum_n(-1)^{n+1}6n\, q^{3n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+3a)\cos(\alpha-3a)z\right\} \; +
$$

(450.1)

$$
+ 2\vartheta_2\vartheta_3\left\{\sum_n(-1)^n q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d+2}{4}}\cos\frac{\delta-3d}{2}z\right\}\right\}.
$$

(451.1) $$\frac{\vartheta_1'^2\vartheta_0}{\vartheta_0^2(z)\vartheta_1(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+3c)\sin\frac{\gamma-3c}{2}z\right\} + \vartheta_2\vartheta_3\left\{\frac{1}{\sin z} + 4\sum q^N\left\{(-1)^b\cos(\beta-3b)z\right\}\right\}.$$

(452.1) $$\frac{\vartheta_1'^2\vartheta_0}{\vartheta_3^2(z)\vartheta_2(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c}{4}}(\gamma+3c)\cos\frac{\gamma-3c}{2}z\right\} + \vartheta_2\vartheta_3\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-3b)z\right\}\right\}$$

$$\frac{\vartheta_1'^2\vartheta_3}{\vartheta_1^2(z)\vartheta_3(z)} = \frac{1}{\sin^2 z} - 2\sum_n 6n\, q^{3n^2} - 4\sum q^N\left\{(\alpha+3a)\cos(\alpha-3a)z\right\} +$$

(454.1)

$$+ 2\vartheta_0\vartheta_2\left\{\sum_n (-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-2}{2}}\cos\frac{\delta-3d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^2\vartheta_3}{\vartheta_2^2(z)\vartheta_0(z)} = \frac{1}{\cos^2 z} - 2\sum_n 6n\, q^{3n^2} + 4\sum q^N\left\{(-1)^{\frac{d-3a-2}{2}}(\alpha+3a)\cos(\alpha-3a)z\right\} +$$

(455.1)

$$+ 2\vartheta_0\vartheta_2\left\{\sum_n (-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}\cos\frac{\delta-3d}{2}z\right\}\right\}$$

(456.1) $$\frac{\vartheta_1'^2\vartheta_3}{\vartheta_3^2(z)\vartheta_1(z)} = 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c-2}{4}}(\gamma+3c)\sin\frac{\gamma-3c}{2}z\right\} + \vartheta_0\vartheta_2\left\{\frac{1}{\sin z} + 4\sum q^N\left\{(-1)^b\sin(\beta-3b)z\right\}\right\}.$$

(457.1) $$\frac{\vartheta_1'^3\vartheta_3}{\vartheta_0^2(z)\vartheta_2(z)} = 2\sum q^{\frac{N}{4}}\left\{(\gamma+3c)\cos\frac{\gamma-3c}{2}z\right\} + \vartheta_0\vartheta_2\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-3b)z\right\}\right\}$$

Group II-b

$$\frac{1}{\vartheta_1^2(z)\vartheta_2(z)} \qquad \frac{1}{\vartheta_2^2(z)\vartheta_1(z)} \qquad \frac{1}{\vartheta_0^2(z)\vartheta_3(z)} \qquad \frac{1}{\vartheta_3^2(z)\vartheta_0(z)}$$

Consider

$$F(z) = \frac{e^{-iz}}{\vartheta_1^2(z)\vartheta_2(z)}$$

Let $t = z + \frac{\pi\tau}{3}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders two and one at $t = \frac{\pi\tau}{3}$ and $t = \frac{\pi\tau}{3} + \frac{\pi}{3}$ respectively. Calculating the corresponding and using (10) gives

(458) $$\vartheta_1'^2\vartheta_2 F(z) = A_3^{(1)}\left(z + \frac{\pi\tau}{3}, \frac{\pi\tau}{3}\right) - i\, A_3^{(0)}\left(z + \frac{\pi\tau}{3}, \frac{\pi\tau}{3}\right) + i\,\vartheta_0\vartheta_3\, A_3^{(c)}\left(z + \frac{\pi\tau}{3}, \frac{\pi\tau}{3} + \frac{\pi}{3}\right)$$

There follows

$$(459)\qquad \frac{\vartheta_1'^2\vartheta_2}{\vartheta_1^2(z)\vartheta_2(z)} = \frac{\cos z}{\sin^2 z} - 2\sum_{n,r} 2[2(3n+r)-1]\, q^{n(3n+2r-1)}\cos(2r-1)z + \vartheta_0\vartheta_3\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{n(3n+2r-1)}\cos(2r-1)z\right\}.$$

Replacing z **by** $z+\frac{\pi}{2}$

$$(460)\qquad \frac{\vartheta_1'^2\vartheta_2}{\vartheta_2^2(z)\vartheta_1(z)} = \frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^r[2(3n+r)-1]\, q^{n(3n+2r-1)}\sin(2r-1)z + \vartheta_0\vartheta_3\left\{\frac{1}{\sin z} + 4\sum_{n,r}(-1)^n q^{n(3n+2r-1)}\sin(2r-1)z\right\}$$

In (458) replace z **by** $z+\frac{\pi\tau}{2}$ **, obtaining**

$$q^{\frac14}\frac{\vartheta_1'^2\vartheta_2 e^{2iz}}{\vartheta_0^2(z)\vartheta_3(z)} = -A_3^{(1)}\left(z+\tfrac{5\pi\tau}{6},\tfrac{\pi\tau}{3}\right) + i\,A_3^{(0)}\left(z+\tfrac{5\pi\tau}{6},\tfrac{\pi\tau}{3}\right) - i\,\vartheta_3\vartheta_0 A_3^{(0)}\left(z+\tfrac{5\pi\tau}{6},\tfrac{\pi\tau}{3}+\tfrac{\pi}{6}\right)$$

There follows

$$(461)\qquad \frac{\vartheta_1'^2\vartheta_2(z)}{\vartheta_0^2(z)\vartheta_3(z)} = 2\sum_n 3(2n-1)\, q^{3\left(\frac{2n-1}{2}\right)} + 4\sum_{n,r}[2(3n+r)-3]\, q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \; + $$
$$+\,\vartheta_0\vartheta_3\left\{2\sum_n(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2} + 4\sum_{n,r}(-1)^{n+r+1} q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}$$

Replacing z **by** $z+\frac{\pi}{2}$

$$(462)\qquad \frac{\vartheta_1'^2\vartheta_2}{\vartheta_3^2(z)\vartheta_0(z)} = 2\sum_n 3(2n-1)\, q^{3\left(\frac{2n-1}{2}\right)^2} + 4\sum_{n,r}(-1)^r[2(3n+r)-3]\, q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \; +$$
$$+\,\vartheta_0\vartheta_3\left\{2\sum_n(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2} + 4\sum_{n,r}(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}$$

These give

$$(459.1)\qquad \frac{\vartheta_1'^2\vartheta_2}{\vartheta_1^2(z)\vartheta_2(z)} = \frac{\cos z}{\sin^2 z} - 4\sum q^N\left\{(\beta+3b)\cos(\beta-3b)\right\} + \vartheta_0\vartheta_3\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b-1}{2}}\cos(\beta-3b)z\right\}\right\}.$$

$$(460.1)\qquad \frac{\vartheta_1'^2\vartheta_2}{\vartheta_2^2(z)\vartheta_1(z)} = \frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta-3b+1}{2}}(\beta+3b)\sin(\beta-3b)z\right\} + \vartheta_0\vartheta_3\left\{\frac{1}{\sin z} + 4\sum q^N\left\{(-1)^b\sin(\beta-3b)z\right\}\right\}$$

$$(461.1)\qquad \frac{\vartheta_1'^2\vartheta_2(z)}{\vartheta_0^2(z)\vartheta_3(z)} = 2\sum_n 3(2n-1)\, q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(\delta+3d)\cos\frac{\delta-3d}{2}z\right\}$$
$$+\,2\vartheta_0\vartheta_3\left\{\sum_n(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-2}{4}}\cos\frac{\delta-3d}{2}z\right\}\right\}$$

(462.1)
$$\frac{\vartheta_1'^2\vartheta_2}{\vartheta_3^2(z)\vartheta_0(z)} = 2\sum_n 3(2n-1)q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d}{4}}(\delta+3d)\cos\frac{\delta-3d}{2}z\right\} +$$
$$+2\vartheta_0\vartheta_3\left\{\sum_n(-1)^{n+1}q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}\cos\frac{\delta-3d}{2}z\right\}\right\}$$

Group III

$$\frac{1}{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)} \qquad \frac{1}{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)} \qquad \frac{1}{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)} \qquad \frac{1}{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)}$$

Consider
$$F(z) = \frac{1}{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)}$$

Let $t = z + \frac{\pi\tau}{2}$ $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has simple poles at $t = \pi\tau$, $t = \frac{\pi\tau}{2}$ and $t = \frac{\pi\tau}{2} + \frac{\pi}{2}$. Calculating the corresponding $R_i^{(1)}$ and using (10) gives

(463)
$$\vartheta_1'^2 F(z) = -\vartheta_2 q^{\frac{3}{4}} A_3^{(0)}(z+\tfrac{\pi\tau}{2}, \pi\tau) + \vartheta_3 A_3^{(0)}(z+\tfrac{\pi\tau}{2}, \tfrac{\pi\tau}{2}) - \vartheta_0 A_3^{(0)}(z+\tfrac{\pi\tau}{2}, \tfrac{\pi\tau}{2}+\tfrac{\pi}{2})$$

There follows

(464)
$$\frac{\vartheta_1'^2}{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)} = -4\vartheta_2\sum_{n,r} q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \vartheta_3\left\{\frac{\cos z}{\sin z} + 4\sum q^{3n^2+2nr}\sin 2rz\right\} +$$
$$+\vartheta_0\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1}q^{3n^2+2nr}\sin 2rz\right\}$$

Replacing z by $z+\frac{\pi}{2}$

(465)
$$\frac{\vartheta_1'^2}{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)} = 4\vartheta_2\sum_{n,r}(-1)^r q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz + \vartheta_3\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1}q^{3n^2+2nr}\sin 2rz\right\}$$
$$+\vartheta_0\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^n q^{3n^2+2nr}\sin 2rz\right\}.$$

In (463) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^2 e^{3iz}}{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)} = \vartheta_2 A_3^{(0)}(z+\pi\tau, \pi\tau) - \vartheta_3 q^{-\frac{3}{4}} A_3^{(0)}(z+\pi\tau, \tfrac{\pi\tau}{2}) + \vartheta_0 q^{-\frac{3}{4}} A_3^{(0)}(z+\pi\tau, \tfrac{\pi\tau}{2}+\tfrac{\pi}{2})$$

This gives

$$\frac{\vartheta_1'^2}{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)} = 4\vartheta_0\sum_{n,r}(-1)^{n+r} q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z \quad +$$

(466)
$$-4\vartheta_3\sum_{n,r} q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z \quad + \vartheta_2\left\{\frac{1}{\sin z}+4\sum_{n,r} q^{n(3n+2r-1)}\sin(2r-1)z\right\}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^2}{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)} = 4\vartheta_0\sum_{n,r}(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z \quad +$$

(467)
$$+4\vartheta_3\sum_{n,r}(-1)^{r} q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z \quad + \vartheta_2\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{r+1} q^{n(3n+2r-1)}\cos(2r-1)z\right\}$$

There follow

$$\frac{\vartheta_1'}{\vartheta_0(z)\vartheta_1(z)\vartheta_2(z)} = -4\vartheta_2\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-3d}{2}z\right\} + \vartheta_3\left\{\frac{\cos z}{\sin z}+4\sum q^{N}\left\{\sin(\alpha-3a)z\right\}\right\} +$$

(464.1)
$$+\vartheta_0\left\{\frac{\sin z}{\cos z}+4\sum q^{N}\left\{(-1)^{\frac{\alpha-a+2}{2}}\sin(\alpha-3a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2}{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)} = 4\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d}{4}}\sin\frac{\delta-3d}{2}z\right\} + \vartheta_3\left\{\frac{\sin z}{\cos z}+4\sum q^{N}\left\{(-1)^{\frac{\alpha-3a-2}{2}}\sin(\alpha-3a)z\right\}\right\} +$$

(465.1)
$$+\vartheta_0\left\{\frac{\cos z}{\sin z}+4\sum q^{N}\left\{(-1)^{a}\sin(\alpha-3a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2}{\vartheta_0(z)\vartheta_1(z)\vartheta_3(z)} = 4\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+4}{4}}\sin\frac{\gamma-3c}{2}z\right\} \quad +$$

(466.1)
$$-4\vartheta_3\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-3c}{2}z\right\} + \vartheta_2\left\{\frac{1}{\sin z}+4\sum q^{N}\left\{\sin(\beta-3b)z\right\}\right\}.$$

$$\frac{\vartheta_1'^2}{\vartheta_0(z)\vartheta_2(z)\vartheta_3(z)} = 4\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-3c}{2}z\right\} \quad +$$

(467.1)
$$+4\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c+2}{4}}\cos\frac{\gamma-3c}{2}z\right\} + \vartheta_2\left\{\frac{1}{\cos z}+4\sum q^{N}\left\{(-1)^{\frac{\beta-3b-1}{2}}\cos(\beta-3b)z\right\}\right\}$$

Group IV-a

$$\frac{\vartheta_0(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_3^4(z)}$$

$$\frac{\vartheta_1(z)}{\vartheta_3^4(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_1^4(z)}$$

Consider

$$F(z) = \frac{\vartheta_0(z)}{\vartheta_1^4(z)}$$

Let $t = z + \frac{\pi\tau}{2} + \frac{\pi}{6}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order four at $t = \frac{\pi\tau}{2} + \frac{\pi}{6}$. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$\frac{\vartheta_1'^4}{\vartheta_0} F(z) = \frac{1}{6} H_3^{(3)}\left(z + \frac{\pi\tau}{2} + \frac{\pi}{6}, \frac{\pi\tau}{2} + \frac{\pi}{6}\right) + \frac{1}{2}\left\{\frac{\vartheta_0''}{\vartheta_0} - \frac{4}{3}\frac{\vartheta_1'''}{\vartheta_1'}\right\} H_3^{(1)}\left(z + \frac{\pi\tau}{2} + \frac{\pi}{6}, \frac{\pi\tau}{2} + \frac{\pi}{6}\right) \tag{468}$$

There follows

$$\frac{\vartheta_1'^4 \vartheta_0(z)}{\vartheta_0 \vartheta_1^4(z)} = \frac{1}{\sin^4 z} + \frac{1}{6\sin^2 z}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 4\right\} + \frac{1}{3}\sum_{n}(-1)^n (6n)^3 q^{3n^2} + \frac{2}{3}\sum_{n,r}(-1)^n [2(3n+r)]^3 q^{3n^2+2nr} \cos 2rz \; +$$

$$+ \frac{1}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_{n}(-1)^{n+1} 6n\, q^{3n^2} + 2\sum_{n,r}(-1)^{n+1} 2(3n+r)\, q^{3n^2+2nr} \cos 2rz\right\}. \tag{469}$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'^4 \vartheta_3(z)}{\vartheta_0 \vartheta_2^4(z)} = \frac{1}{\cos^4 z} + \frac{1}{6\cos^2 z}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 4\right\} + \frac{1}{3}\sum_{n}(-1)^n (6n)^3 q^{3n^2} + \frac{2}{3}\sum_{n,r}(-1)^{n+r} [2(3n+r)]^3 q^{3n^2+2nr} \cos 2rz \; +$$

$$+ \frac{1}{3}\left\{4\frac{\vartheta_1'''}{\vartheta_1'} - 3\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\sum_{n}(-1)^n 6n\, q^{3n^2} + 2\sum_{n,r}(-1)^{n+r} 2(3n+r)\, q^{3n^2+2nr} \cos 2rz\right\} \tag{470}$$

In (468) replace z by $z - \frac{\pi\tau}{2}$, obtaining

$$-i\frac{q^{3/4} e^{-3iz} \vartheta_1'^4 \vartheta_1(z)}{\vartheta_0 \vartheta_0^4(z)} = \frac{1}{6} H_3^{(3)}\left(z + \frac{\pi}{6}, \frac{\pi\tau}{2} + \frac{\pi}{6}\right) + \frac{1}{2} H_3^{(1)}\left(z + \frac{\pi}{6}, \frac{\pi\tau}{2} + \frac{\pi}{6}\right)\left\{\frac{\vartheta_0''}{\vartheta_0} - \frac{4}{3}\frac{\vartheta_1'''}{\vartheta_1'}\right\}$$

There follows

$$\frac{\vartheta_1'^4 \vartheta_1(z)}{\vartheta_0 \vartheta_0^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+1} [2(3n+r-2)]^3 q^{3\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z$$

$$+ \frac{2}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^n q^{3\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}}\, 2(3n+r-2) \sin(2r-1)z \tag{471}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\vartheta_2(z)}{\vartheta_0\vartheta_3^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{n+r}[2(3n+r-2)]^3 q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(472)

$$+\frac{2}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{n+r+1}\,2(3n+r-2)\,q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

In these results replace q by $-q$. There follow

$$\frac{\vartheta_1'^4\vartheta_3(z)}{\vartheta_3\vartheta_1^4(z)} = \frac{1}{\sin^4 z}+\frac{1}{6\sin^2 z}\left\{3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}-4\right\}+\frac{1}{3}\sum_n(6n)^3q^{3n^2}+\frac{2}{3}\sum_{n,r}[2(3n+r)]^3q^{3n^2+2nr}\cos 2rz \; +$$

(473)

$$-\frac{1}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 6n\,q^{3n^2}+2\sum_{n,r}2(3n+r)\,q^{3n^2+2nr}\cos 2rz\right\}$$

$$\frac{\vartheta_1'^4\vartheta_0(z)}{\vartheta_3\vartheta_2^4(z)} = \frac{1}{\cos^4 z}+\frac{1}{6\cos^2 z}\left\{3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}-4\right\}+\frac{1}{3}\sum_n(6n)^3q^{3n^2}+\frac{2}{3}\sum_{n,r}(-1)^r[2(3n+r)]^3q^{3n^2+2nr}\cos 2rz \; +$$

(474)

$$-\frac{1}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 6n\,q^{3n^2}+2\sum_{n,r}(-1)^r\,2(3n+r)\,q^{3n^2+2nr}\cos 2rz\right\}$$

$$\frac{\vartheta_1'^4\vartheta_1(z)}{\vartheta_3\vartheta_3^4(z)} = \frac{2}{3}\sum_{n,r}(-1)^{r+1}[2(3n+r-2)]^3 q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(475)

$$+\frac{2}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^r\,2(3n+r-2)\,q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

$$\frac{\vartheta_1'^4\vartheta_2(z)}{\vartheta_3\vartheta_0^4(z)} = \frac{2}{3}\sum_{n,r}[2(3n+r-2)]^3 q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(476)

$$-\frac{2}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}2(3n+r-2)\,q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

These give

$$\frac{\vartheta_1'^4\vartheta_0(z)}{\vartheta_0\vartheta_1^4(z)} = \frac{1}{\sin^4 z}+\frac{1}{6\sin^2 z}\left\{3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}-4\right\}+\frac{1}{3}\sum_n(-1)^n(6n)^3q^{3n^2}+\frac{2}{3}\sum q^N\left\{(-1)^a(\alpha+3a)^3\cos(\alpha-3a)z\right\} +$$

(469.1)

$$+\frac{1}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n(-1)^{n+1}6n\,q^{3n^2}+2\sum q^N\left\{(-1)^{a+1}(\alpha+3a)\cos(\alpha-3a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_3(z)}{\vartheta_0\,\vartheta_2^4(z)} = \frac{1}{\cos^4 z} + \frac{1}{6\cos^2 z}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 4\right\} + \frac{1}{3}\sum_n (-1)^n (6n)^3 q^{3n^2} + \frac{2}{3}\sum q^N \left\{(-1)^{\frac{\alpha-a}{2}} (\alpha+3a)^3 \cos(\alpha-3a)z\right\} +$$

(470.1)

$$+\frac{1}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 6n(-1)^n q^{3n^2} + 2\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+3a)\cos(\alpha-3a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_1(z)}{\vartheta_0\vartheta_0^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+3c)^3 \sin\frac{\gamma-3c}{2}z\right\} +$$

(471.1)

$$+\frac{1}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}(\gamma+3c)\sin\frac{\gamma-3c}{2}z\right\}\right\}$$

$$\frac{\vartheta_1'^4\,\vartheta_2(z)}{\vartheta_0\,\vartheta_3^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+4}{4}}(\gamma+3c)^3\cos\frac{\gamma-3c}{2}z\right\} +$$

(472.1)

$$+\frac{1}{3}\left\{3\frac{\vartheta_0''}{\vartheta_0} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c}{4}}(\gamma+3c)\cos\frac{\gamma-3c}{2}z\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_3(z)}{\vartheta_3\,\vartheta_1^4(z)} = \frac{1}{\sin^4 z} + \frac{1}{6\sin^2 z}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 4\right\} + \frac{1}{3}\sum_n (6n)^3 q^{3n^2} + \frac{2}{3}\sum q^N\left\{(\alpha+3a)^3\cos(\alpha-3a)z\right\} +$$

(473.1)

$$-\frac{1}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 6n\,q^{3n^2} + 2\sum q^N\left\{(\alpha+3a)\cos(\alpha-3a)z\right\}\right\}$$

$$\frac{\vartheta_1'^4\,\vartheta_0(z)}{\vartheta_3\,\vartheta_2^4(z)} = \frac{1}{\cos^4 z} + \frac{1}{6\cos^2 z}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 4\right\} + \frac{1}{3}\sum_n (6n)^3 q^{3n^2} + \frac{2}{3}\sum q^N\left\{(-1)^{\frac{\alpha-3a}{2}}(\alpha+3a)^3\cos(\alpha-3a)z\right\} +$$

(474.1)

$$-\frac{1}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 6n\,q^{3n^2} + 2\sum q^N\left\{(-1)^{\frac{\alpha-3a}{2}}(\alpha+3a)\cos(\alpha-3a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_1(z)}{\vartheta_3\,\vartheta_3^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c-2}{4}}(\gamma+3c)^3\sin\frac{\gamma-3c}{2}z\right\} +$$

(475.1)

$$+\frac{1}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c+2}{4}}(\gamma+3c)\sin\frac{\gamma-3c}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\,\vartheta_2(z)}{\vartheta_3\,\vartheta_0^4(z)} = \frac{1}{12}\sum q^{\frac{N}{4}}\left\{(\gamma+3c)^3\cos\frac{\gamma-3c}{2}z\right\} +$$

(476.1)

$$-\frac{1}{3}\left\{3\frac{\vartheta_3''}{\vartheta_3} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^{\frac{N}{4}}\left\{(\gamma+3c)\cos\frac{\gamma-3c}{2}z\right\}$$

Group IV-b

$$\frac{\vartheta_2(z)}{\vartheta_1^4(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_2^4(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_0^4(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_3^4(z)}$$

Consider

$$F(z) = \frac{\vartheta_2(z)\, e^{-iz}}{\vartheta_1^4(z)}$$

Let $t = z + \frac{\pi\tau}{3}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has a pole of order four at $t = \frac{\pi\tau}{3}$. Calculating the corresponding $R_i^{(i)}$ and using (10) we get

$$\frac{\vartheta_1'^4}{\vartheta_2} F(z) = \frac{1}{6} A_3^{(3)}\left(z+\frac{\pi\tau}{3}, \frac{\pi\tau}{3}\right) - \frac{i}{2} A_3^{(2)}\left(z+\frac{\pi\tau}{3}, \frac{\pi\tau}{3}\right) +$$

(477)

$$+\frac{1}{6}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 3\right\} A_3^{(1)}\left(z+\frac{\pi\tau}{3}, \frac{\pi\tau}{3}\right) - \frac{i}{6}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} A_3^{(0)}\left(z+\frac{\pi\tau}{3}, \frac{\pi\tau}{3}\right)$$

There follows

$$\frac{\vartheta_1'^4 \vartheta_2(z)}{\vartheta_2 \vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z} + \frac{\cos z}{6\sin^2 z}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} + \frac{2}{3}\sum_{n,r}[2(3n+r)-1]^3 q^{n(3n+2r-1)} \cos(2r-1)z \; +$$

(478)

$$-\frac{2}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}[2(3n+r)-1]\, q^{n(3n+2r-1)} \cos(2r-1)z$$

Replacing z by $z+\frac{\pi}{2}$,

$$\frac{\vartheta_1'^4 \vartheta_1(z)}{\vartheta_2 \vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z} + \frac{\sin z}{6\cos^2 z}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} + \frac{2}{3}\sum_{n,r}(-1)^{r+1}[2(3n+r)-1]^3 q^{n(3n+2r-1)} \sin(2r-1)z \; +$$

(479)

$$+\frac{2}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum_{n,r}(-1)^{r}[2(3n+r)-1]\, q^{n(3n+2r-1)} \sin(2r-1)z$$

In (477) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^4 \vartheta_3(z)}{\vartheta_2 \vartheta_0^4(z)} e^{2iz} q^{1/4} = \frac{1}{6} A_3^{(3)}\left(z+\frac{5\pi\tau}{6}, \frac{\pi\tau}{3}\right) - \frac{i}{2} A_3^{(2)}\left(z+\frac{5\pi\tau}{6}, \frac{\pi\tau}{3}\right) +$$

$$+\frac{1}{6}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 3\right\} A_3^{(1)}\left(z+\frac{5\pi\tau}{6}, \frac{\pi\tau}{3}\right) - \frac{i}{6}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'} - 1\right\} A_3^{(0)}\left(z+\frac{5\pi\tau}{6}, \frac{\pi\tau}{3}\right)$$

This gives

$$\frac{\vartheta_1'^4 \vartheta_3(z)}{\vartheta_2 \vartheta_0^4(z)} = \frac{1}{3}\sum_{n}[3(2n-1)]^3 q^{3\left(\frac{2n-1}{2}\right)^2} + \frac{2}{3}\sum_{n,r}[2(3n+r)-3]^3 q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r} \cos 2rz \; +$$

(480)

$$-\frac{1}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2} - 4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum 3(2n-1)\, q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}[2(3n+r)-3]\, q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r} \cos 2rz\right\}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^4\vartheta_0(z)}{\vartheta_2\vartheta_3^4(z)} = \frac{1}{3}\sum_n [3(2n-1)]^3 q^{3\left(\frac{2n-1}{2}\right)^2} + \frac{2}{3}\sum_{n,r}(-1)^r[2(3n+r)-3]^3 q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz \; +$$

(481)

$$-\frac{1}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 3(2n-1)\, q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^r[2(3n+r)-3]\, q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\}$$

From these follow

$$\frac{\vartheta_1'^4\vartheta_2(z)}{\vartheta_2\vartheta_1^4(z)} = \frac{\cos z}{\sin^4 z}+\frac{\cos z}{6\sin^2 z}\left\{3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}+\frac{2}{3}\sum q^N\left\{(\beta+3b)^3\cos(\beta-3b)z\right\} \; +$$

(478.1)

$$-\frac{2}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(\beta+3b)\cos(\beta-3b)z\right\}$$

$$\frac{\vartheta_1'^4\vartheta_1(z)}{\vartheta_2\vartheta_2^4(z)} = \frac{\sin z}{\cos^4 z}+\frac{\sin z}{6\cos^2 z}\left\{3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}-1\right\}+\frac{2}{3}\sum q^N\left\{(-1)^{\frac{\beta-3b-1}{2}}(\beta+3b)^3\sin(\beta-3b)z\right\} \; +$$

(479.1)

$$+\frac{2}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\sum q^N\left\{(-1)^{\frac{\beta-3b+1}{2}}(\beta+3b)\sin(\beta-3b)z\right\}$$

$$\frac{\vartheta_1'^4\vartheta_3(z)}{\vartheta_2\vartheta_0^4(z)} = \frac{1}{3}\sum_n[3(2n-1)]^3 q^{3\left(\frac{2n-1}{2}\right)^2}+\frac{1}{12}\sum q^{\frac{N}{4}}\left\{(\delta+3d)^3\cos\frac{\delta-3d}{2}z\right\} \; +$$

(480.1)

$$-\frac{1}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 3(2n-1)\, q^{3\left(\frac{2n-1}{2}\right)^2}+\sum q^{\frac{N}{4}}\left\{(\delta+3d)\cos\frac{\delta-3d}{2}z\right\}\right\}.$$

$$\frac{\vartheta_1'^4\vartheta_0(z)}{\vartheta_2\vartheta_3^4(z)} = \frac{1}{3}\sum_n[3(2n-1)]^3 q^{3\left(\frac{2n-1}{2}\right)^2}+\frac{1}{12}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d}{4}}(\delta+3d)^3\cos\frac{\delta-3d}{2}z\right\}$$

(481.1)

$$-\frac{1}{3}\left\{3\frac{\vartheta_2''}{\vartheta_2}-4\frac{\vartheta_1'''}{\vartheta_1'}\right\}\left\{\sum_n 3(2n-1)\, q^{3\left(\frac{2n-1}{2}\right)^2}+\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d}{4}}(\delta+3d)\cos\frac{\delta-3d}{2}z\right\}\right\}.$$

Group V-a

$$\frac{\vartheta_2(z)}{\vartheta_1^3(z)\vartheta_0(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_3(z)\vartheta_2^3(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_0^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_3^3(z)\vartheta_2(z)}$$

$$\frac{\vartheta_2(z)}{\vartheta_1^3(z)\vartheta_3(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_0(z)\vartheta_2^3(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_3^3(z)\vartheta_1(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_0^3(z)\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2(z)}{\vartheta_1^3(z)\,\vartheta_0(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \psi(t)$. $\psi(t)$ satisfies (8) and has poles of orders three and one at $t = \frac{\pi\tau}{2}$ and $t = \pi\tau$ respectively. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$(482) \qquad \frac{\vartheta_1'^3 \vartheta_0}{\vartheta_2} F(z) = \frac{1}{2} A_3^{(2)}\left(z + \frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) - \frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0} + \frac{\vartheta_3''}{\vartheta_3}\right\} A_3^{(0)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) + q^{\frac{3}{4}} \vartheta_2 \vartheta_3^3 A_3^{(0)}\left(z+\frac{\pi\tau}{2}, \pi\tau\right)$$

There follows

$$(483) \qquad \frac{\vartheta_1'^3 \vartheta_0 \vartheta_2(z)}{\vartheta_1^3(z)\vartheta_0(z)\vartheta_2} = \frac{\cos z}{\sin^3 z} - 2\sum_{n,r}[2(3n+r)]^2 q^{3n^2+2nr} \sin 2rz \; + $$
$$-\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0} + \frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\frac{\cos z}{\sin z} + 4\sum q^{3n^2+2nr} \sin 2rz\right\} + 4\vartheta_2\vartheta_3^3 \sum_{n,r} q^{3\left(\frac{2n-1}{2}\right)^2 + (2n-1)r} \sin 2rz .$$

Replacing z by $z+\frac{\pi}{2}$

$$(484) \qquad \frac{\vartheta_1'^3 \vartheta_0 \vartheta_1(z)}{\vartheta_2 \vartheta_2^3(z) \vartheta_3(z)} = \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^r[2(3n+r)]^2 q^{3n^2+2nr} \sin 2rz \; +$$
$$-\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0} + \frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{3n^2+2nr} \sin 2rz\right\} + 4\vartheta_2\vartheta_3^3 \sum_{n,r}(-1)^{r+1} q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r} \sin 2rz$$

In (482) replace z by $z - \frac{\pi\tau}{2}$, obtaining

$$q^{\frac{3}{4}} \frac{\vartheta_1'^3 \vartheta_0 \vartheta_3(z) e^{-3iz}}{\vartheta_2 \vartheta_0^3(z) \vartheta_1(z)} = \frac{1}{2} A_3^{(2)}\left(z, \frac{\pi\tau}{2}\right) - \frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0} + \frac{\vartheta_3''}{\vartheta_3}\right\} A_3^{(0)}\left(z, \frac{\pi\tau}{2}\right) + q^{\frac{3}{4}} \vartheta_2 \vartheta_3^3 A_3^{(0)}(z, \pi\tau)$$

From this

$$(485) \qquad \frac{\vartheta_1'^3 \vartheta_0 \vartheta_3(z)}{\vartheta_2 \vartheta_0^3(z) \vartheta_1(z)} = -2\sum_{n,r}[2(3n+r-2)]^2 q^{3\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z \; +$$
$$-2\left\{2\frac{\vartheta_0''}{\vartheta_0} + \frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r} q^{3\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z + \vartheta_2\vartheta_3^3\left\{\frac{1}{\sin z} + 4\sum_{n,r} q^{n(3n+2r-1)} \sin(2r-1)z\right\}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0(z)}{\vartheta_2\vartheta_3^3(z)\vartheta_2(z)} = 2\sum_{n,r}(-1)^r[2(3n+r-2)]^2\, q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(486)

$$+2\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^r q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \;+\vartheta_2\vartheta_3^3\left\{\frac{1}{\cos z}+4\sum_{n,r} q^{n(3n+2r-1)}(-1)^{r+1}\cos(2r-1)z\right\}.$$

In these results replace q by $-q$. This gives

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2(z)}{\vartheta_2\vartheta_1^3(z)\vartheta_3(z)} = \frac{\cos z}{\sin^3 z}+2\sum_{n,r}(-1)^{n+1}[2(3n+r)]^2\, q^{3n^2+2nr}\sin 2rz \; +$$

(487)

$$-\frac{1}{2}\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^n q^{3n^2+2nr}\sin 2rz\right\}+4\vartheta_2\vartheta_0^3\sum_{n,r}(-1)^{r+1} q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz .$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1(z)}{\vartheta_2\vartheta_2^3(z)\vartheta_0(z)} = \frac{\sin z}{\cos^3 z}+2\sum_{n,r}(-1)^{n+r}[2(3n+r)]^2\, q^{3n^2+2nr}\sin 2rz \; +$$

(488)

$$-\frac{1}{2}\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1} q^{3n^2+2nr}\sin 2rz\right\}+4\vartheta_2\vartheta_0^3\sum_{n,r} q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz .$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0(z)}{\vartheta_2\vartheta_3^3(z)\vartheta_1(z)} = 2\sum_{n,r}(-1)^{n+r}[2(3n+r-2)]^2\, q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(489)

$$+2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r}(-1)^{n+r} q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \;+\vartheta_2\vartheta_0^3\left\{\frac{1}{\sin z}+4\sum_{n,r} q^{n(3n+2r-1)}\sin(2r-1)z\right\}$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3(z)}{\vartheta_2\vartheta_0^3(z)\vartheta_2(z)} = 2\sum_{n,r}(-1)^{n+1}[2(3n+r-2)]^2\, q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(490)

$$+2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r}(-1)^{n+1} q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \;+\vartheta_2\vartheta_0^3\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{r+1} q^{n(3n+2r-1)}\cos(2r-1)z\right\}$$

From these follow

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2(z)}{\vartheta_2\vartheta_1^3(z)\vartheta_0(z)} = \frac{\cos z}{\sin^3 z} - 2\sum q^N\left\{(d+3a)^2\sin(d-3a)z\right\} \; +$$

(483.1)

$$-\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\sum q^N 4\left\{\sin(d-3a)z\right\}+\frac{\cos z}{\sin z}\right\}+4\vartheta_2\vartheta_3^3\sum q^{\frac{N}{4}}\left\{\sin\frac{a-3d}{2}z\right\}$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1(z)}{\vartheta_2\vartheta_2^3(z)\vartheta_3(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha-3a}{2}}(\alpha+3a)^2\sin(\alpha-3a)z\right\} + 4\vartheta_2\vartheta_3^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d-4}{4}}\sin\frac{\delta-3d}{2}z\right\} +$$

$$-\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-3a-2}{2}}\sin(\alpha-3a)z\right\}\right\} \tag{484.1}$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3(z)}{\vartheta_2\vartheta_0^3(z)\vartheta_1(z)} = -\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\gamma+3c)^2\sin\frac{\gamma-3c}{2}z\right\} - 2\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-3c}{2}z\right\} +$$

$$+\vartheta_2\vartheta_3^3\left\{\frac{1}{\sin z}+4\sum q^N\left\{\sin(\beta-3b)z\right\}\right\} \tag{485.1}$$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_0(z)}{\vartheta_2\vartheta_3^3(z)\vartheta_2(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c+2}{4}}(\gamma+3c)^2\cos\frac{\gamma-3c}{2}z\right\} + 2\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c+2}{4}}\cos\frac{\gamma-3c}{2}z\right\} +$$

$$+\vartheta_2\vartheta_3^2\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-3b-1}{2}}\cos(\beta-3b)z\right\}\right\} \tag{486.1}$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2(z)}{\vartheta_2\vartheta_1^3(z)\vartheta_3(z)} = \frac{\cos z}{\sin^3 z} + 2\sum q^N\left\{(-1)^{a+1}(\alpha+3a)^2\sin(\alpha-3a)z\right\} + 4\vartheta_2\vartheta_0^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d-4}{4}}\sin\frac{\delta-3d}{2}z\right\} +$$

$$-\frac{1}{2}\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^{a}\sin(\alpha-3a)z\right\}\right\}. \tag{487.1}$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1(z)}{\vartheta_2\vartheta_2^3(z)\vartheta_0(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha-a}{2}}(\alpha+3a)^2\sin(\alpha-3a)z\right\} + 4\vartheta_2\vartheta_0^3\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-3d}{2}z\right\} +$$

$$-\frac{1}{2}\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-3a)z\right\}\right\} \tag{488.1}$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0(z)}{\vartheta_2\vartheta_3^3(z)\vartheta_1(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+4}{4}}(\gamma+3c)^2\sin\frac{\gamma-3c}{2}z\right\} + 2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+4}{4}}\sin\frac{\gamma-3c}{4}z\right\} +$$

$$+\vartheta_2\vartheta_0^3\left\{\frac{1}{\sin z}+4\sum q^N\left\{\sin(\beta-3b)z\right\}\right\}. \tag{489.1}$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3(z)}{\vartheta_2\vartheta_0^3(z)\vartheta_2(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+3c)^2\cos\frac{\gamma-3c}{2}z\right\} + 2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-3c}{2}z\right\} +$$

$$+\vartheta_2\vartheta_0^3\left\{\frac{1}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\beta-3b-1}{2}}\cos(\beta-3b)z\right\}\right\} \tag{490.1}$$

Group V-b

$$\frac{\vartheta_3(z)}{\vartheta_1^3(z)\,\vartheta_0(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_2^3(z)\,\vartheta_3(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0^3(z)\,\vartheta_1(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_3^3(z)\,\vartheta_2(z)}$$

$$\frac{\vartheta_0(z)}{\vartheta_1^3(z)\,\vartheta_3(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_2^3(z)\,\vartheta_0(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_3^3(z)\,\vartheta_1(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_0^3(z)\,\vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_3(z)\,e^{-iz}}{\vartheta_1^3(z)\,\vartheta_0(z)}$$

Let $z + \frac{\pi\tau}{3} = t$, $F(z) \equiv \psi(t)$. $\psi(t)$ satisfies (8) and has poles of orders three and one at $t = \frac{\pi\tau}{3}$ and $t = \frac{5\pi\tau}{6}$ respectively. Calculating the corresponding $R_i^{(j)}$ and using (10) we get

$$\frac{\vartheta_1'^3\vartheta_0}{\vartheta_3}F(z) = \frac{1}{2}A_3^{(2)}\left(z+\frac{\pi\tau}{3},\frac{\pi\tau}{3}\right) - i A_3^{(1)}\left(z+\frac{\pi\tau}{3},\frac{\pi\tau}{3}\right)$$

(491)

$$-\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}+1\right\}A_3^{(0)}\left(z+\frac{\pi\tau}{3},\frac{\pi\tau}{3}\right) + q^{\frac{1}{4}}\vartheta_2^3\vartheta_3\,A_3^{(0)}\left(z+\frac{\pi\tau}{3},\frac{5\pi\tau}{6}\right)$$

There follows

$$\frac{\vartheta_1'^3\vartheta_0\,\vartheta_3(z)}{\vartheta_3\,\vartheta_1^3(z)\,\vartheta_0(z)} = \frac{1}{\sin^3 z} - \frac{1}{2\sin z}\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}+1\right\} - 2\sum_{n,r}[2(3n+r)-1]^2 q^{n(3n+2r-1)}\sin(2r-1)z \;+$$

(492)

$$-2\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r}q^{n(3n+2r-1)}\sin(2r-1)z + 4\vartheta_2^3\vartheta_3\sum_{n,r}q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin(2r-1)z$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\,\vartheta_0(z)}{\vartheta_3\,\vartheta_2^3(z)\,\vartheta_3(z)} = \frac{1}{\cos^3 z} - \frac{1}{2\cos z}\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}+1\right\} + 2\sum_{n,r}(-1)^r[2(3n+r)-1]^2 q^{n(3n+2r-1)}\cos(2r-1)z \;+$$

(493)

$$+2\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r}(-1)^r q^{n(3n+2r-1)}\cos(2r-1)z + 4\vartheta_2^3\vartheta_3\sum_{n,r}(-1)^{r+1}q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

In (491) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^3\vartheta_0\,\vartheta_2(z)\,e^{2iz}}{\vartheta_2\,\vartheta_0^3(z)\,\vartheta_1(z)}q^{\frac{1}{4}} = \frac{1}{2}A_3^{(2)}\left(z+\frac{5\pi\tau}{6},\frac{\pi\tau}{3}\right) - i A_3^{(1)}\left(z+\frac{5\pi\tau}{6},\frac{\pi\tau}{3}\right) \;+$$

$$-\frac{1}{2}\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}+1\right\}A_3^{(0)}\left(z+\frac{5\pi\tau}{6},\frac{\pi\tau}{3}\right) + q^{\frac{1}{4}}\vartheta_2^3\vartheta_3\,A_3^{(0)}\left(z+\frac{5\pi\tau}{6},\frac{5\pi\tau}{6}\right)$$

This gives

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_2(z)}{\vartheta_3\vartheta_0^3(z)\vartheta_1(z)} = -2\sum_{n,r}[2(3n+r)-3]^2 q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz \; +$$

(494)

$$-2\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r} q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz + \vartheta_2^3\vartheta_3\left\{\frac{\cos z}{\sin z}+4\sum_{n,r} q^{3n^2+2nr}\sin 2rz\right\}$$

Replacing z **by** $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_1(z)}{\vartheta_3\vartheta_3^3(z)\vartheta_2(z)} = 2\sum_{n,r}(-1)^r[2(3n+r)-3]^2 q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz \; +$$

(495)

$$+2\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r}(-1)^r q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz + \vartheta_2^3\vartheta_3\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{r+1} q^{3n^2+2nr}\sin 2rz\right\}$$

In these results replace q **by** $-q$ **. This gives**

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_0(z)}{\vartheta_0\vartheta_1^3(z)\vartheta_3(z)} = \frac{1}{\sin^3 z}-\frac{1}{2\sin z}\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}+1\right\} - 2\sum_{n,r}[2(3n+r)-1]^2 q^{n(3n+2r-1)}\sin(2r-1)z \; +$$

(496)

$$-2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r} q^{n(3n+2r-1)}\sin(2r-1)z + 4\vartheta_2^3\vartheta_0\sum_{n,r}(-1)^{n+r} q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_3(z)}{\vartheta_0\vartheta_2^3(z)\vartheta_0(z)} = \frac{1}{\cos^3 z}-\frac{1}{2\cos z}\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}+1\right\} + 2\sum(-1)^r[2(3n+r)-1]^2 q^{n(3n+2r-1)}\cos(2r-1)z$$

(497)

$$+2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r}(-1)^r q^{n(3n+2r-1)}\cos(2r-1)z + 4\vartheta_2^3\vartheta_0\sum_{n,r}(-1)^{n+1} q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_2(z)}{\vartheta_0\vartheta_3^3(z)\vartheta_1(z)} = 2\sum_{n,r}(-1)^{r+1}[2(3n+r)-3]^2 q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz$$

(498)

$$+2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r}(-1)^{r+1} q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz + \vartheta_2^3\vartheta_0\left\{\frac{\cos z}{\sin z}+4\sum(-1)^n q^{3n^2+2nr}\sin 2rz\right\}$$

$$\frac{\vartheta_1'^3\vartheta_3\vartheta_1(z)}{\vartheta_0\vartheta_0^3(z)\vartheta_2(z)} = 2\sum_{n,r}[2(3n+r)-3]^2 q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz \; +$$

(499)

$$+2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum_{n,r} q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz + \vartheta_2^3\vartheta_0\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1} q^{3n^2+2nr}\sin 2rz\right\}$$

These give

$$\frac{\vartheta_1'^3\vartheta_0\vartheta_3(z)}{\vartheta_3\vartheta_1^3(z)\vartheta_0(z)} = \frac{1}{\sin^3 z}-\frac{1}{2\sin z}\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}+1\right\} - 2\sum q^N\left\{(\beta+3b)^2\sin(\beta-3b)z\right\} \; +$$

(492,1)

$$+4\vartheta_2^3\vartheta_3\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-3c}{2}z\right\} - 2\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^N\left\{\sin(\beta-3b)z\right\}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_0\vartheta_0(z)}{\vartheta_3\vartheta_2^3(z)\vartheta_3(z)} &= \frac{1}{\cos^3 z} - \frac{1}{2\cos z}\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}+1\right\} + 2\sum q^N\left\{(-1)^{\frac{\beta-3b+1}{2}}(\beta+3b)^2\cos(\beta-3b)z\right\} + \\ &+ 2\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^N\left\{(-1)^{\frac{\beta-3b+1}{2}}\cos(\beta-3b)z\right\} + 4\vartheta_2^3\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c-2}{4}}\sin\frac{\gamma-3c}{2}z\right\}.\end{aligned}\tag{493.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_0\vartheta_2(z)}{\vartheta_3\vartheta_0^3(z)\vartheta_1(z)} &= -\frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\delta+3d)^2\sin\frac{\delta-3d}{2}z\right\} - 2\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-3d}{2}z\right\} + \\ &+ i\vartheta_2^3\vartheta_3\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(\alpha-3a)z\right\}\right\}.\end{aligned}\tag{494.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_0\vartheta_1(z)}{\vartheta_3\vartheta_3^3(z)\vartheta_2(z)} &= \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d}{4}}(\delta+3d)^2\sin\frac{\delta-3d}{2}z\right\} + \\ &+ 2\left\{2\frac{\vartheta_0''}{\vartheta_0}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d}{4}}\sin\frac{\delta-3d}{2}z\right\} + \vartheta_2^3\vartheta_3\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha-3a-2}{2}}\sin(\alpha-3a)z\right\}\right\}.\end{aligned}\tag{495.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_3\vartheta_0(z)}{\vartheta_0\vartheta_1^3(z)\vartheta_3(z)} &= \frac{1}{\sin^3 z} - \frac{1}{2\sin z}\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}+1\right\} - 2\sum q^N\left\{(\beta+3b)^2\sin(\beta-3b)z\right\} + \\ &- 2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^N\left\{\sin(\beta-3b)z\right\} + 4\vartheta_2^3\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+4}{4}}\sin\frac{\gamma-3c}{2}z\right\}\end{aligned}\tag{496.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_3\vartheta_3(z)}{\vartheta_0\vartheta_2^3(z)\vartheta_0(z)} &= \frac{1}{\cos^3 z} - \frac{1}{2\cos z}\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}+1\right\} + 2\sum q^N\left\{(-1)^{\frac{\beta-3b+1}{2}}(\beta+3b)^2\cos(\beta-3b)z\right\} + \\ &+ 2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^N\left\{(-1)^{\frac{\beta-3b+1}{2}}\cos(\beta-3b)z\right\} + 4\vartheta_2^3\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-3c}{2}z\right\}.\end{aligned}\tag{497.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_3\vartheta_2(z)}{\vartheta_0\vartheta_3^3(z)\vartheta_1(z)} &= \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d-4}{4}}(\delta+3d)^2\sin\frac{\delta-3d}{2}z\right\} + \\ &+ 2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d-4}{4}}\sin\frac{\delta-3d}{2}z\right\} + \vartheta_0\vartheta_2^3\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{(-1)^a\sin(\alpha-3a)z\right\}\right\}\end{aligned}\tag{498.1}$$

$$\begin{aligned}\frac{\vartheta_1'^3\vartheta_3\vartheta_1(z)}{\vartheta_0\vartheta_0^3(z)\vartheta_2(z)} &= \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\delta+3d)^2\sin\frac{\delta-3d}{2}z\right\} + \\ &+ 2\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_2''}{\vartheta_2}\right\}\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-3d}{2}z\right\} + \vartheta_2^3\vartheta_0\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-3a)z\right\}\right\}.\end{aligned}\tag{499.1}$$

Group V-c

$$\frac{\vartheta_0(z)}{\vartheta_1^3(z)\,\vartheta_2(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_2^3(z)\,\vartheta_1(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_0^3(z)\,\vartheta_3(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_3^3(z)\,\vartheta_0(z)}$$

$$\frac{\vartheta_3(z)}{\vartheta_1^3(z)\,\vartheta_2(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_2^3(z)\,\vartheta_1(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_3^3(z)\,\vartheta_0(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0^3(z)\,\vartheta_3(z)}$$

Consider

$$F(z) = \frac{\vartheta_0(z)}{\vartheta_1^3(z)\,\vartheta_2(z)}$$

Let $t = z + \frac{\pi\tau}{2}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders three and one at $t = \frac{\pi\tau}{2}$ and $t = \frac{\pi\tau}{2} + \frac{\pi}{2}$ respectively. Calculating the corresponding values of $R_i^{(j)}$ and using (10) gives

$$(500) \qquad \frac{\vartheta_1'^3\vartheta_2}{\vartheta_0} F(z) = \frac{1}{2} A_3^{(2)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) - \frac{1}{2}\left\{2\frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3}\right\} A_3^{(0)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}\right) - \vartheta_0\vartheta_3^3 A_3^{(0)}\left(z+\frac{\pi\tau}{2}, \frac{\pi\tau}{2}+\frac{\pi}{2}\right)$$

There follows

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0(z)}{\vartheta_0\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} - 2\sum_{n,r}[2(3n+r)]^2 q^{3n^2+2nr}\sin 2rz \; +$$

(501)

$$-\frac{1}{2}\left\{2\frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{3n^2+2nr}\sin 2rz\right\} + \vartheta_0\vartheta_3^3\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{3n^2+2nr}\sin 2rz\right\}.$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3(z)}{\vartheta_0\vartheta_2^3(z)\vartheta_1(z)} = \frac{\sin z}{\cos z} + 2\sum_{n,r}(-1)^r[2(3n+r)]^2 q^{3n^2+2nr}\sin 2rz \; +$$

(502)

$$-\frac{1}{2}\left\{2\frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{3n^2+2nr}\sin 2rz\right\} + \vartheta_0\vartheta_3^3\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^{n} q^{3n^2+2nr}\sin 2rz\right\}$$

In (500) replace z by $z - \frac{\pi\tau}{2}$, obtaining

$$-\frac{\vartheta_1'^3\vartheta_2\vartheta_1(z)q^{3/4}}{\vartheta_0\vartheta_0^3(z)\vartheta_3(z)} e^{-3iz} = \frac{1}{2}A_3^{(2)}\left(z, \frac{\pi\tau}{2}\right) - \frac{1}{2}\left\{2\frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3}\right\}A_3^{(0)}\left(z, \frac{\pi\tau}{2}\right) - \vartheta_0\vartheta_3^3 A_3^{(0)}\left(z, \frac{\pi\tau}{2}+\frac{\pi}{2}\right)$$

This gives

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1(z)}{\vartheta_0\vartheta_0^3(z)\vartheta_3(z)} = 2\sum_{n,r}[2(3n+r-2)]^2 q^{3\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(503)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r} q^{3\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z + 4\vartheta_0\vartheta_3^3\sum_{n,r}(-1)^{n+r} q^{3\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_2(z)}{\vartheta_0\vartheta_3^3(z)\vartheta_0(z)} = 2\sum_{n,r}(-1)^{r+1}[2(3n+r-2)]^2 q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(503)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_3''}{\vartheta_3}\right\}\sum_{n,r}(-1)^{r+1} q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; + 4\vartheta_0\vartheta_3^3\sum_{n,r}(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

In these results replace q by $-q$. We get

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0(z)}{\vartheta_3\vartheta_2^3(z)\vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + 2\sum_{n,r}(-1)^{n+r}[2(3n+r)]^2 q^{3n^2+2nr}\sin 2rz$$

(505)

$$-\frac{1}{2}\left\{2\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{n+r+1} q^{3n^2+2nr}\sin 2rz\right\} + \vartheta_3\vartheta_0^3\left\{\frac{\cos z}{\sin z}+4\sum_{n,r} q^{3n^2+2nr}\sin 2rz\right\}$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3(z)}{\vartheta_2\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} + 2\sum_{n,r}(-1)^{n+1}[2(3n+r)]^2 q^{3n^2+2nr}\sin 2rz \; +$$

(506)

$$-\frac{1}{2}\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^{n} q^{3n^2+2nr}\sin 2rz\right\} + \vartheta_3\vartheta_0^3\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{r+1} q^{3n^2+2nr}\sin 2rz\right\}.$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_1(z)}{\vartheta_3\vartheta_3^3(z)\vartheta_0(z)} = 2\sum_{n,r}(-1)^{n+r}[2(3n+r-2)]^2 q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; +$$

(507)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r}(-1)^{n+r} q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \; + 4\vartheta_3\vartheta_0^3\sum_{n,r} q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_2(z)}{\vartheta_3\vartheta_0^3(z)\vartheta_3(z)} = 2\sum_{n,r}(-1)^{n+1}[2(3n+r-2)]^2 q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; +$$

(508)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_0''}{\vartheta_0}\right\}\sum_{n,r}(-1)^{n+1} q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \; + 4\vartheta_3\vartheta_0^3\sum_{n,r}(-1)^{r} q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

From these follow

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_0(z)}{\vartheta_0\vartheta_1^3(z)\vartheta_2(z)} = \frac{\cos z}{\sin^3 z} - 2\sum q^N\left\{(\alpha+3a)^2\sin(\alpha-3a)z\right\} \; +$$

(501.1)

$$-\frac{1}{2}\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{\sin(\alpha-3a)z\right\}\right\} + \vartheta_0\vartheta_3^3\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-3a)z\right\}\right\},$$

$$\frac{\vartheta_1'^3\vartheta_2\vartheta_3(z)}{\vartheta_0\vartheta_2^3(z)\vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^N\left\{(-1)^{\frac{\alpha-3a}{2}}(\alpha+3a)^2\sin(\alpha-3a)z\right\} \; +$$

(502.1)

$$-\frac{1}{2}\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_3''}{\vartheta_3}\right\}\left\{\frac{\sin z}{\cos z}+4\sum q^N\left\{(-1)^{\frac{\alpha-3a-2}{2}}\sin(\alpha-3a)z\right\}\right\} + \vartheta_0\vartheta_3^3\left\{\frac{\cos z}{\sin z}+4\sum q^N\left\{(-1)^{a}\sin(\alpha-3a)z\right\}\right\}$$

$$\frac{\vartheta_1'^3 \vartheta_2 \vartheta_1(z)}{\vartheta_0 \vartheta_0^3(z) \vartheta_3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(\gamma+3c)^2 \sin\frac{\gamma-3c}{2}z\right\} +$$

(503.1)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-3c}{2}z\right\} + 4\vartheta_0\vartheta_3^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+4}{4}}\sin\frac{\gamma-3c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3 \vartheta_2 \vartheta_2(z)}{\vartheta_0 \vartheta_3^3(z) \vartheta_0(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c-2}{4}}(\gamma+3c)^2 \cos\frac{\gamma-3c}{2}z\right\} +$$

(504.1)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_3''}{\vartheta_3}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c-2}{4}}\cos\frac{\gamma-3c}{2}z\right\} + 4\vartheta_0\vartheta_3^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-3c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3 \vartheta_2 \vartheta_3(z)}{\vartheta_3 \vartheta_1^3(z) \vartheta_2(z)} = \frac{\cos z}{\sin^3 z} + 2\sum q^{N}\left\{(-1)^{a+1}(\alpha+3a)^2 \sin(\alpha-3a)z\right\} +$$

(506.1)

$$-\frac{1}{2}\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\frac{\cos z}{\sin z} + 4\sum q^{N}\left\{(-1)^{a}\sin(\alpha-3a)z\right\}\right\} + \vartheta_3\vartheta_0^3\left\{\frac{\sin z}{\cos z} + 4\sum q^{N}\left\{(-1)^{\frac{\alpha-3a-2}{2}}\sin(\alpha-3a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3 \vartheta_2 \vartheta_0(z)}{\vartheta_3 \vartheta_2^3(z) \vartheta_1(z)} = \frac{\sin z}{\cos^3 z} + 2\sum q^{N}\left\{(-1)^{\frac{\alpha-a}{2}}(\alpha+3a)^2 \sin(\alpha-3a)z\right\} +$$

(505.1)

$$-\frac{1}{2}\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_0''}{\vartheta_0}\right\}\left\{\frac{\sin z}{\cos z} + 4\sum q^{N}\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-3a)z\right\}\right\} + \vartheta_3\vartheta_0^3\left\{\frac{\cos z}{\sin z} + 4\sum q^{N}\left\{\sin(\alpha-3a)z\right\}\right\}$$

$$\frac{\vartheta_1'^3 \vartheta_2 \vartheta_1(z)}{\vartheta_3 \vartheta_3^3(z) \vartheta_0(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-4}{4}}(\gamma+3c)^2 \sin\frac{\gamma-3c}{2}z\right\} +$$

(507.1)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c-4}{4}}\sin\frac{\gamma-3c}{2}z\right\} + 4\vartheta_0^3\vartheta_3\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-3c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3 \vartheta_2 \vartheta_2(z)}{\vartheta_3 \vartheta_0^3(z) \vartheta_3(z)} = \frac{1}{2}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+3c)^2 \cos\frac{\gamma-3c}{2}z\right\} +$$

(508.1)

$$+2\left\{2\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_0''}{\vartheta_0}\right\}\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}\cos\frac{\gamma-3c}{2}z\right\} + 4\vartheta_0^3\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c+4}{4}}\cos\frac{\gamma-3c}{2}z\right\}.$$

Group VI-a

$$\frac{\vartheta_2(z)}{\vartheta_1^2(z)\,\vartheta_0^2(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_2^2(z)\,\vartheta_3^2(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_1^2(z)\,\vartheta_0^2(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_2^2(z)\,\vartheta_3^2(z)}$$

$$\frac{\vartheta_2(z)}{\vartheta_1^2(z)\,\vartheta_3^2(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_2^2(z)\,\vartheta_0^2 z} \qquad \frac{\vartheta_0(z)}{\vartheta_1^2(z)\,\vartheta_3^2(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_2^2(z)\,\vartheta_0^2(z)}$$

Consider

$$F(z) = \frac{\vartheta_2(z)\, e^{-iz}}{\vartheta_1^2(z)\,\vartheta_0^2(z)}$$

Let $t = z + \frac{\pi\tau}{3}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of order two at $t = \frac{\pi\tau}{3}$ and $t = \frac{\pi\tau}{3} + \frac{\pi\tau}{2}$. Calculating the corresponding $R_i^{(j)}$ and using (10) we find

$$\vartheta_1'^2 \vartheta_0^2 F(z) = \vartheta_2 \left\{ A_3^{(1)}\left(z + \tfrac{\pi\tau}{3}, \tfrac{\pi\tau}{3}\right) - i A_3^{(0)}\left(z + \tfrac{\pi\tau}{3}, \tfrac{\pi\tau}{3}\right) \right\} +$$

(509)

$$+ q^{\frac14} \vartheta_3 \left\{ A_3^{(1)}\left(z + \tfrac{\pi\tau}{3}, \tfrac{5\pi\tau}{6}\right) + 2i A_3^{(0)}\left(z + \tfrac{\pi\tau}{3}, \tfrac{5\pi\tau}{6}\right) \right\}$$

There follows

(510) $$\frac{\vartheta_1'^2 \vartheta_0^2 \vartheta_2(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} = \vartheta_2 \left\{ \frac{\cos z}{\sin^2 z} - 4\sum_{n,r} [2(3n+r)-1]\, q^{n(3n+2r-1)} \cos(2r-1)z \right\} - 4\vartheta_3 \sum_{n,r} 2(3n+r-2)\, q^{3\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}} \cos(2r-1)z .$$

Replacing z by $z - \frac{\pi}{2}$

(511) $$\frac{\vartheta_1'^2 \vartheta_0^2 \vartheta_1(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} = \vartheta_2 \left\{ \frac{\sin z}{\cos z} + 4\sum_{n,r} (-1)^r [2(3n+r)-1]\, q^{n(3n+2r-1)} \sin(2r-1)z \right\} + 4\vartheta_3 \sum_{n,r} (-1)^r\, 2(3n+r-2)\, q^{3\left(\frac{2n-1}{2}\right)^2 + \frac{(2n-1)(2r-1)}{2}} \sin(2r-1)z .$$

In (509) replace z by $z + \frac{\pi\tau}{2}$, obtaining

$$e^{2iz} \frac{\vartheta_1'^2 \vartheta_0^2 \vartheta_3(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} = \vartheta_2\, q^{-\frac14} \left\{ A_3^{(1)}\left(z + \tfrac{5\pi\tau}{6}, \tfrac{\pi\tau}{3}\right) - i A_3^{(0)}\left(z + \tfrac{5\pi\tau}{6}, \tfrac{\pi\tau}{3}\right) \right\} +$$

$$+ \vartheta_3 \left\{ A_3^{(1)}\left(z + \tfrac{5\pi\tau}{6}, \tfrac{5\pi\tau}{6}\right) + 2i A_3^{(0)}\left(z + \tfrac{5\pi\tau}{6}, \tfrac{5\pi\tau}{6}\right) \right\}$$

which gives

$$\frac{\vartheta_1'^2 \vartheta_0^2 \vartheta_3(z)}{\vartheta_1^2(z)\vartheta_0^2(z)} = -2\vartheta_2 \left\{ \sum_n 3(2n-1)\, q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r} [2(3n+r)-3]\, q^{3\left(\frac{2n-1}{2}\right)^2 + (2n-1)r} \cos 2rz \right\} +$$

(512)

$$+ \vartheta_3 \left\{ \frac{1}{\sin^2 z} - 2\sum_n 6n\, q^{3n^2} - 4\sum_{n,r} 2(3n+r)\, q^{3n^2+2nr} \cos 2rz \right\}$$

Replacing z by $z + \frac{\pi}{2}$

$$\frac{\vartheta_1'^2 \vartheta_0^2 \vartheta_0(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} = -2\vartheta_2 \left\{ \sum_n 3(2n-1)\, q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r} (-1)^r [2(3n+r)-3]\, q^{3\left(\frac{2n-1}{2}\right)^2 + (2n-1)r} \cos 2rz \right\}$$

(513)

$$+ \vartheta_3 \left\{ \frac{1}{\cos^2 z} - 2\sum_n 6n\, q^{3n^2} + 4\sum_{n,r} (-1)^{r+1}\, 2(3n+r)\, q^{3n^2+2nr} \cos 2rz \right\}$$

In these results replace q by $-q$. This gives

(514) $$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_2(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} = \vartheta_2\left\{\frac{\cos z}{\sin^2 z} - 4\sum_{n,r}[2(3n+r)-1]\,q^{n(3n+2r-1)}\cos(2r-1)z\right\} + 4\vartheta_0\sum_{n,r}(-1)^{n+r}2(3n+r-2)\,q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

(515) $$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_1(z)}{\vartheta_0^2(z)\vartheta_2^2(z)} = \vartheta_2\left\{\frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)[2(3n+r)-1]\,q^{n(3n+2r-1)}\sin(2r-1)z\right\} + 4\vartheta_0\sum_{n,r}(-1)^{n+1}2(3n+r-2)\,q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

(516) $$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_0(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} = 2\vartheta_2\left\{\sum 3(2n-1)\,q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}(-1)^r[2(3n+r)-3]\,q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right\} +$$
$$+\vartheta_0\left\{\frac{1}{\sin^2 z} + 2\sum_{n}(-1)^{n+1}6n\,q^{3n^2} + 4\sum_{n,r}(-1)^{n+1}2(3n+r)\,q^{3n^2+2nr}\cos 2rz\right\}.$$

(517) $$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_3(z)}{\vartheta_0^2(z)\vartheta_2^2(z)} = 2\vartheta_2\left\{\sum_n 3(2n-1)\,q^{3\left(\frac{2n-1}{2}\right)^2} + 2\sum_{n,r}[2(3n+r)-3]\,q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\cos 2rz\right.$$
$$\left.+\vartheta_0\left\{\frac{1}{\cos^2 z} + 2\sum_n(-1)^{n+1}6n\,q^{3n^2} + 4\sum_{n,r}(-1)^{n+r+1}(3n+r)2\,q^{3n^2+2nr}\cos 2rz\right\}\right.$$

From these follow

(510.1) $$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_2(z)}{\vartheta_0^2(z)\vartheta_1^2(z)} = \vartheta_2\left\{\frac{\cos z}{\sin^2 z} - 4\sum q^N\{(\beta+3b)\cos(\beta-3b)z\}\right\} - 2\vartheta_3\sum q^{\frac{N}{4}}\left\{(\gamma+3c)\cos\frac{\gamma-3c}{2}z\right\}.$$

(511.1) $$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_1(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} = \vartheta_2\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\{(-1)^{\frac{\beta-3b+1}{2}}(\beta+3b)\sin(\beta-3b)z\}\right\} + 2\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c+2}{4}}(\gamma+3c)\sin\frac{\gamma-3c}{2}z\right\}$$

(512.1) $$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_3(z)}{\vartheta_0^2(z)\vartheta_1^2(z)} = -2\vartheta_2\left\{\sum_n 3(2n-1)\,q^{3\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(\delta+3d)\cos\frac{\delta-3d}{2}z\right\}\right\} +$$
$$+\vartheta_3\left\{\frac{1}{\sin^2 z} - 2\sum_n 6n\,q^{3n^2} - 4\sum q^N\{(\alpha+3a)\cos(\alpha-3a)z\}\right\}.$$

(513.1) $$\frac{\vartheta_1'^2\vartheta_0^2\vartheta_0(z)}{\vartheta_2^2(z)\vartheta_3^2(z)} = -2\vartheta_2\left\{\sum_n 3(2n-1)\,q^{3\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d}{4}}(\delta+3d)\cos\frac{\delta-3d}{2}z\right\}\right\} +$$
$$+\vartheta_3\left\{\frac{1}{\cos^2 z} - 2\sum_n 6n\,q^{3n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-3a-2}{2}}(\alpha-3a)\cos(\alpha-3a)z\right\}\right\}$$

(514.1) $$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_2(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} = \vartheta_2\left\{\frac{\cos z}{\sin^2 z} - 4\sum q^N\{(\beta+3b)\cos(\beta-3b)z\}\right\} + 2\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c-1}{4}}(\gamma+3c)\cos\frac{\gamma-3c}{2}\right\}.$$

(515.1) $$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_1(z)}{\vartheta_2^2(z)\vartheta_0^2(z)} = \vartheta_2\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\{(-1)^{\frac{\beta-3b+1}{2}}(\beta+3b)\sin(\beta-3b)z\}\right\} + 2\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+3c)\sin\frac{\gamma-3c}{2}z\right\}.$$

(516.1) $$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_0(z)}{\vartheta_1^2(z)\vartheta_3^2(z)} = 2\vartheta_2\left\{\sum_n 3(2n-1)q^{3\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d}{4}}(\delta+3d)\cos\frac{\delta-3d}{2}z\right\}\right\} +$$
$$+\vartheta_0\left\{\frac{1}{\sin^2 z} + 2\sum_n(-1)^{n+1}6n\,q^{3n^2} + 4\sum q^N\{(-1)^{a+1}(\alpha+3a)\cos(\alpha-3a)z\}\right\}.$$

(517.1) $$\frac{\vartheta_1'^2\vartheta_3^2\vartheta_3(z)}{\vartheta_0^2(z)\vartheta_2^2(z)} = 2\vartheta_2\left\{\sum_n 3(2n-1)q^{3\left(\frac{2n-1}{2}\right)^2} + \sum q^{\frac{N}{4}}\left\{(\delta+3d)\cos\frac{\delta-3d}{2}z\right\}\right\} +$$
$$+\vartheta_0\left\{\frac{1}{\cos^2 z} + 2\sum_n(-1)^{n+1}6n\,q^{3n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+3a)\cos(\alpha-3a)z\right\}\right\}$$

Group VI-b

$$\frac{\vartheta_0(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0^2(z)\vartheta_3^2(z)}$$

Consider
$$F(z) = \frac{\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_2^2(z)}$$

Let $z = \frac{\pi\tau}{2} + \frac{\pi}{6}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of order two at $t = \frac{\pi\tau}{2} + \frac{\pi}{6}$ and $t = \frac{\pi\tau}{2} + \frac{4\pi}{6}$. Calculating the corresponding $R_i^{(1)}$ and using (10) gives

(518) $$\vartheta_1'^2\vartheta_2^2 F(z) = \vartheta_0\,\mathfrak{Z}_3^{(1)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{6}, \frac{\pi\tau}{2}+\frac{\pi}{6}\right) + \vartheta_3\,\mathfrak{Z}_3^{(1)}\left(z+\frac{\pi\tau}{2}+\frac{\pi}{6}, \frac{\pi\tau}{2}+\frac{4\pi}{6}\right)$$

There follows

(519) $$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_0(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0\left\{\frac{1}{\sin^2 z} + 2\sum_n(-1)^{n+1}6n\,q^{3n^2} + 4\sum_{n,r}(-1)^{n+1}2(3n+r)\,q^{3n^2+2nr}\cos 2rz\right\} +$$
$$+\vartheta_3\left\{\frac{1}{\cos^2 z} - 2\sum_n 6n\,q^{3n^2} + 4\sum_{n,r}(-1)^{r+1}2(3n+r)\,q^{3n^2+2nr}\cos 2rz\right\}$$

Replacing z by $z + \frac{\pi}{2}$

(520)
$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0\left\{\frac{1}{\cos^2 z} + 2\sum_n(-1)^{n+1}6n\,q^{3n^2} + 4\sum_{n,r}(-1)^{n+r+1}2(3n+r)q^{3n^2+2nr}\cos 2rz\right\} +$$
$$+\vartheta_3\left\{\frac{1}{\sin^2 z} - 2\sum_n 6n\,q^{3n^2} - 4\sum_{n,r}2(3n+r)\,q^{3n^2+2nr}\cos 2rz\right\}.$$

In (518) replace z by $z-\frac{\pi\tau}{2}$, obtaining

$$\frac{i\,q^{3/4}\vartheta_1'^2\vartheta_2^2\vartheta_1(z)}{\vartheta_0^2(z)\vartheta_3^2(z)}e^{-3iz} = \vartheta_0 A_3^{(1)}\left(z+\tfrac{\pi}{6}, \tfrac{\pi\tau}{2}+\tfrac{\pi}{6}\right) + \vartheta_3 A_3^{(1)}\left(z+\tfrac{\pi}{6}, \tfrac{\pi\tau}{2}+\tfrac{4\pi}{6}\right)$$

This gives

(521)
$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_1(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} = 4\vartheta_0\sum_{n,r}(-1)^{n+1}2(3n+r-2)\,q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z +$$
$$+4\vartheta_3\sum_{n,r}(-1)^{r+1}3(2n+r-2)\,q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z,$$

Replacing z by $z+\frac{\pi}{2}$

(522)
$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_2(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} = 4\vartheta_0\sum_{n,r}(-1)^{n+r}2(3n+r-2)\,q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z +$$
$$+4\vartheta_3\sum_{n,r}2(3n+r-2)\,q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z.$$

From these follow

(519.1)
$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_0(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0\left\{\frac{1}{\sin^2 z} + 2\sum_n(-1)^{n+1}6n\,q^{3n^2} + 4\sum q^N\left\{(-1)^{a+1}(\alpha+3a)\cos(\alpha-3a)z\right\}\right\} +$$
$$+\vartheta_3\left\{\frac{1}{\cos^2 z} - 2\sum_n 6n\,q^{3n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-3a-2}{2}}(\alpha+3a)\cos(\alpha-3a)z\right\}\right\}.$$

(520.1)
$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_2^2(z)} = \vartheta_0\left\{\frac{1}{\cos^2 z} + 2\sum_n(-1)^{n+1}6n\,q^{3n^2} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}(\alpha+3a)\cos(\alpha-3a)z\right\}\right\} +$$
$$+\vartheta_3\left\{\frac{1}{\sin^2 z} - 2\sum_n 6n\,q^{3n^2} - 4\sum q^N\left\{(\alpha+3a)\cos(\alpha-3a)z\right\}\right\}.$$

(521.1)
$$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_1(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} = 2\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c-1}{2}}(\gamma+3c)\sin\frac{\gamma-3c}{2}z\right\} + 2\vartheta_3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c-2}{4}}(\gamma+3c)\sin\frac{\gamma-3c}{2}z\right\}.$$

(522.1) $$\frac{\vartheta_1'^2\vartheta_2^2\vartheta_2(z)}{\vartheta_0^2(z)\vartheta_3^2(z)} = 2\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{r-c+4}{4}}(r+3c)\cos\frac{r-3c}{2}z\right\} + 2\vartheta_3\sum q^{\frac{N}{4}}\left\{(r+3c)\cos\frac{r-3c}{2}z\right\}$$

Group VII-a

$$\frac{\vartheta_0(z)}{\vartheta_1^2(z)\vartheta_2(z)\vartheta_3(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_2^2(z)\vartheta_1(z)\vartheta_0(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_1^2(z)\vartheta_2(z)\vartheta_3(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_3^2(z)\vartheta_0(z)\vartheta_1(z)}$$

$$\frac{\vartheta_3(z)}{\vartheta_1^2(z)\vartheta_2(z)\vartheta_0(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_2^2(z)\vartheta_1(z)\vartheta_3(z)} \qquad \frac{\vartheta_1(z)}{\vartheta_3^2(z)\vartheta_2(z)\vartheta_0(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_0^2(z)\vartheta_3(z)\vartheta_1(z)}$$

Consider

$$F(z) = \frac{\vartheta_3(z)e^{-iz}}{\vartheta_2^2(z)\vartheta_1(z)\vartheta_0(z)}$$

Let $t = z + \frac{\pi\tau}{3}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders two, and one, at $t = \frac{\pi\tau}{3} + \frac{\pi}{2}$ and $t = \frac{\pi\tau}{3}, \frac{5\pi\tau}{6}$ respectively. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

(523) $$\frac{\vartheta_1'^3F(z)}{\vartheta_0} = -i\vartheta_0\left\{A_3^{(1)}\left(z+\tfrac{\pi\tau}{3}, \tfrac{\pi\tau}{3}+\tfrac{\pi}{2}\right) - i\,A_3^{(0)}\left(z+\tfrac{\pi\tau}{3}, \tfrac{\pi\tau}{3}+\tfrac{\pi}{2}\right)\right\} + \vartheta_3^3A_3^{(0)}\left(z+\tfrac{\pi\tau}{3}, \tfrac{\pi\tau}{3}\right) - \vartheta_2^3q^{\frac{1}{4}}A_3^{(0)}\left(z+\tfrac{\pi\tau}{3}, \tfrac{5\pi\tau}{6}\right)$$

There follows

(524) $$\frac{\vartheta_1'^3\vartheta_3(z)}{\vartheta_0\vartheta_2^2(z)\vartheta_1(z)\vartheta_0(z)} = \vartheta_0\left\{\frac{\sin z}{\cos^2 z} + 4\sum_{n,r}[2(3n+r)-1](-1)^{n+r}q^{n(3n+2r-1)}\sin(2r-1)z\right\} + \vartheta_3^3\left\{\frac{1}{\sin z} + 4\sum_{n,r}q^{n(3n+2r-1)}\sin(2r-1)z\right\} - 4\vartheta_2^3\sum_{n,r}q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

Replacing z by $z+\frac{\pi}{2}$

(525) $$\frac{\vartheta_1'^3\vartheta_0(z)}{\vartheta_0\vartheta_1^2(z)\vartheta_2(z)\vartheta_3(z)} = \vartheta_0\left\{\frac{\cos z}{\sin^2 z} + 4\sum_{n,r}(-1)^{n+1}[2(3n+r)-1]q^{n(3n+2r-1)}\cos(2r-1)z\right\} + \vartheta_3^3\left\{\frac{1}{\cos z} + 4\sum_{n,r}(-1)^{r+1}q^{n(3n+2r-1)}\cos(2r-1)z\right\} + 4\vartheta_2^3\sum_{n,r}(-1)^r q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

In (523) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^3 q^{\frac14}\vartheta_2(z)e^{2iz}}{\vartheta_0\vartheta_3^2(z)\vartheta_1(z)\vartheta_0(z)} = i\vartheta_0\left\{A_3^{(1)}\left(z+\tfrac{5\pi\tau}{6},\tfrac{\pi\tau}{3}+\tfrac{\pi}{2}\right)-iA_3^{(0)}\left(z+\tfrac{5\pi\tau}{6},\ \tfrac{\pi\tau}{3}+\tfrac{\pi}{2}\right)\right\}$$

$$-\vartheta_3^3 A_3^{(0)}\left(z+\tfrac{5\pi\tau}{6},\tfrac{\pi\tau}{3}\right)+\vartheta_2^3 q^{\frac14}A_3^{(0)}\left(z+\tfrac{6\pi\tau}{6},\tfrac{5\pi\tau}{6}\right)$$

This gives

(526)
$$\frac{\vartheta_1'^3\vartheta_2(z)}{\vartheta_0\vartheta_3^2(z)\vartheta_1(z)\vartheta_0(z)} = 4\vartheta_0\sum_{n,r}(-1)^{n+r}[2(3n+r)-3]\,q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \;+$$
$$-4\vartheta_3^3\sum_{n,r}q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz+\vartheta_2^3\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}q^{3n^2+2nr}\sin 2rz\right\}.$$

Replacing z by $z+\frac{\pi}{2}$

(527)
$$\frac{\vartheta_1'^3\vartheta_1(z)}{\vartheta_0\vartheta_0^3(z)\vartheta_2(z)\vartheta_3(z)} = 4\vartheta_0\sum_{n,r}(-1)^{n+1}[2(3n+r)-3]\,q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \;+$$
$$+4\vartheta_3^3\sum_{n,r}(-1)^r q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz+\vartheta_2\left\{\frac{\sin z}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{3n^2+2nr}\sin 2rz\right\}.$$

In these results replace q by $-q$. There follows

(528)
$$\frac{\vartheta_1'^3\vartheta_0(z)}{\vartheta_3\vartheta_2^2(z)\vartheta_1(z)\vartheta_3(z)} = \vartheta_3\left\{\frac{\sin z}{\cos^2 z}+4\sum_{n,r}(-1)^{n+r}[2(3n+r)-1]\,q^{n(3n+r-1)}\sin(2r-1)z\right\}+$$
$$+\vartheta_0^3\left\{\frac{1}{\sin z}+4\sum_{n,r}q^{n(3n+2r-1)}\sin(2r-1)z\right\}+4\vartheta_2^3\sum_{n,r}(-1)^{n+r+1}q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z$$

(529)
$$\frac{\vartheta_1'^3\vartheta_3(z)}{\vartheta_3\vartheta_1^2(z)\vartheta_2(z)\vartheta_0(z)} = \vartheta_3\left\{\frac{\cos z}{\sin^2 z}+4\sum_{n,r}(-1)^{n+1}[2(3n+r)-1]\,q^{n(3n+2r-1)}\cos(2r-1)z\right\}+$$
$$+\vartheta_0^3\left\{\frac{1}{\cos z}+4\sum_{n,r}(-1)^{r+1}q^{n(3n+2r-1)}\cos(2r-1)z\right\}+4\vartheta_2^3\sum_{n,r}(-1)^n q^{3\left(\frac{2n-1}{2}\right)^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z$$

(530)
$$\frac{\vartheta_1'^3\vartheta_2(z)}{\vartheta_3\vartheta_0^2(z)\vartheta_1(z)\vartheta_3(z)} = 4\vartheta_3\sum_{n,r}(-1)^n[2(3n+r)-3]\,q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \;+$$
$$+4\vartheta_0^3\sum_{n,r}(-1)^{r+1}q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz+\vartheta_2^3\left\{\frac{\cos z}{\sin z}+4\sum_{n,r}(-1)^n q^{3n^2+2nr}\sin 2rz\right\}.$$

$$\frac{\vartheta_1'^3\,\vartheta_1(z)}{\vartheta_3\,\vartheta_3^2(z)\,\vartheta_2(z)\,\vartheta_0(z)} = 4\vartheta_3\sum_{n,r}(-1)^{n+r+1}[2(3n+r)-3]\,q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \;+ \tag{531}$$

$$+\,4\vartheta_0^3\sum_{n,r} q^{3\left(\frac{2n-1}{2}\right)^2+(2n-1)r}\sin 2rz \;+\; \vartheta_2^3\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{3n^2+2nr}\sin 2rz\right\}$$

These give

$$\frac{\vartheta_1'^3\,\vartheta_3(z)}{\vartheta_0\,\vartheta_2^2(z)\,\vartheta_1(z)\,\vartheta_0(z)} = \vartheta_0\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+3b)\sin(\beta-3b)z\right\}\right\} + \tag{524.1}$$

$$+\,\vartheta_3^3\left\{\frac{1}{\sin z} + 4\sum q^N\left\{\sin(\beta-3b)z\right\}\right\} - 4\vartheta_2^3\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-3c}{2}z\right\}\,,$$

$$\frac{\vartheta_1'^3\,\vartheta_0(z)}{\vartheta_0\,\vartheta_1^2(z)\,\vartheta_2(z)\,\vartheta_3(z)} = \vartheta_0\left\{\frac{\cos z}{\sin^2 z} + 4\sum q^N\left\{(-1)^{b+1}(\beta+3b)\cos(\beta-3b)z\right\}\right\} + \tag{525.1}$$

$$+\,\vartheta_3^3\left\{\frac{1}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\beta-3b-1}{2}}\cos(\beta-3b)z\right\}\right\} + 4\vartheta_2^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c+2}{4}}\cos\frac{\gamma-3c}{2}z\right\}$$

$$\frac{\vartheta_1'^3\,\vartheta_2(z)}{\vartheta_0\,\vartheta_3^2(z)\,\vartheta_1(z)\,\vartheta_0(z)} = 2\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d+2}{4}}(\delta+3d)\sin\frac{\delta-3d}{2}z\right\} + \tag{526.1}$$

$$-\,4\vartheta_3^3\sum q^{\frac{N}{4}}\left\{\sin\frac{\delta-3d}{2}z\right\} + \vartheta_2^3\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(\alpha-3a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\,\vartheta_1(z)}{\vartheta_0\,\vartheta_0^2(z)\,\vartheta_2(z)\,\vartheta_3(z)} = 2\vartheta_0\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d-1}{2}}(\delta+3d)\sin\frac{\delta-3d}{2}z\right\} + \tag{527.1}$$

$$+\,4\vartheta_3^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-3d}{4}}\sin\frac{\delta-3d}{2}z\right\} + \vartheta_2^3\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha-3a-2}{2}}\sin(\alpha-3a)z\right\}\right\}.$$

$$\frac{\vartheta_1'^3\,\vartheta_0(z)}{\vartheta_3\,\vartheta_2^2(z)\,\vartheta_1(z)\,\vartheta_3(z)} = \vartheta_3\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+3b)\sin(\beta-3b)z\right\}\right\} + \tag{528.1}$$

$$+\,\vartheta_0^3\left\{\frac{1}{\sin z} + 4\sum q^N\left\{\sin(\beta-3b)z\right\}\right\} + 4\vartheta_2^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c}{4}}\sin\frac{\gamma-3c}{2}z\right\}.$$

$$\frac{\vartheta_1'^3 \vartheta_3(z)}{\vartheta_3 \vartheta_1^2(z)\, \vartheta_0(z)\, \vartheta_2(z)} = \vartheta_3 \left\{ \frac{\cos z}{\sin^2 z} + 4 \sum q^N \left\{ (-1)^{b+1} (\beta + 3b) \cos (\beta - 3b) z \right\} \right\} +$$

(529.1)

$$+ \vartheta_0^3 \left\{ \frac{1}{\cos z} + 4 \sum q^N \left\{ (-1)^{\frac{\beta - 3b - 1}{2}} \cos (\beta - 3b) z \right\} \right\} + 4 \vartheta_2^3 \sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{c+1}{2}} \cos \frac{\gamma - 3c}{2} z \right\} .$$

$$\frac{\vartheta_1'^3 \vartheta_2(z)}{\vartheta_3 \vartheta_0^2(z)\, \vartheta_1(z)\, \vartheta_3(z)} = 2 \vartheta_3 \sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{d+1}{2}} (\delta + 3d) \sin \frac{\delta - 3d}{2} z \right\} +$$

(530.1)

$$+ 4 \vartheta_0^3 \sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\delta - 3d - 4}{4}} \sin \frac{\delta - 3d}{2} z \right\} + \vartheta_2^3 \left\{ \frac{\cos z}{\sin z} + 4 \sum q^N \left\{ (-1)^a \sin (\alpha - 3a) z \right\} \right\} .$$

$$\frac{\vartheta_1'^3 \vartheta_1(z)}{\vartheta_3 \vartheta_3^2(z)\, \vartheta_2(z)\, \vartheta_0(z)} = 2 \vartheta_3 \sum q^{\frac{N}{4}} \left\{ (-1)^{\frac{\delta - d - 2}{4}} (\delta + 3d) \sin \frac{\delta - 3d}{2} z \right\} +$$

(531.1)

$$+ 4 \vartheta_0^3 \sum q^{\frac{N}{4}} \left\{ \sin \frac{\delta - 3d}{2} z \right\} + \vartheta_2^3 \left\{ \frac{\sin z}{\cos z} + 4 \sum q^N \left\{ (-1)^{\frac{\alpha - a - 2}{2}} \sin (\alpha - 3a) z \right\} \right\}$$

Group VII-b

$$\frac{\vartheta_1(z)}{\vartheta_2^2(z)\, \vartheta_0(z)\, \vartheta_3(z)} \qquad \frac{\vartheta_2(z)}{\vartheta_1^2(z)\, \vartheta_0(z)\, \vartheta_3(z)} \qquad \frac{\vartheta_0(z)}{\vartheta_3^2(z)\, \vartheta_1(z)\, \vartheta_2(z)} \qquad \frac{\vartheta_3(z)}{\vartheta_0^2(z)\, \vartheta_1(z)\, \vartheta_2(z)}$$

Consider

$$F(z) = \frac{\vartheta_1(z)\, e^{-iz}}{\vartheta_2^2(z)\, \vartheta_0(z)\, \vartheta_3(z)}$$

Let $t = z + \frac{\pi\tau}{3}$, $F(z) \equiv \varphi(t)$. $\varphi(t)$ satisfies (8) and has poles of orders two, and one at $t = \frac{\pi\tau}{3} + \frac{\pi}{2}$, and at $t = \frac{5\pi\tau}{6}$, $\frac{\pi}{2} + \frac{5\pi\tau}{6}$ respectively. Calculating the corresponding $R_i^{(j)}$ and using (10) gives

$$\frac{\vartheta_1'^3}{\vartheta_2} F(z) = -i \vartheta_2 \left\{ A_3^{(1)}\left(z + \frac{\pi\tau}{3}, \frac{\pi\tau}{3} + \frac{\pi}{2}\right) - i A_3^{(0)}\left(z + \frac{\pi\tau}{3}, \frac{\pi\tau}{3} + \frac{\pi}{2}\right) \right\} +$$

(532)

$$+ q^{\frac{1}{4}} \vartheta_0^3 A_3^{(0)}\left(z + \frac{\pi\tau}{3}, \frac{5\pi\tau}{6}\right) + q^{\frac{1}{4}} \vartheta_3^3 A_3^{(0)}\left(z + \frac{5\pi\tau}{6}, \frac{5\pi\tau}{6} + \frac{\pi}{2}\right)$$

There follows

$$\frac{\vartheta_1'^3\vartheta_1(z)}{\vartheta_2\vartheta_2^2(z)\vartheta_0(z)\vartheta_3(z)} = \vartheta_2\left\{\frac{\sin z}{\cos^2 z} + 4\sum_{n,r}(-1)^{n+r}[2(3n+r)-1]\,q^{n(3n+2r-1)}\sin(2r-1)z\right\} +$$

(533)

$$+ 4\vartheta_0^3\sum_{n,r} q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \quad + 4\vartheta_3^3\sum_{n,r}(-1)^{n+r+1} q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\sin(2r-1)z \, .$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_2(z)}{\vartheta_2\vartheta_1^2(z)\vartheta_0(z)\vartheta_3(z)} = \vartheta_2\left\{\frac{\cos z}{\sin^2 z} + 4\sum_{n,r}(-1)^{n+1}[2(3n+r)-1]\,q^{n(3n+2r-1)}\cos(2r-1)z\right\} +$$

(534)

$$+ 4\vartheta_0^3\sum_{n,r}(-1)^{r+1} q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \quad + 4\vartheta_3^3\sum(-1)^n q^{3(\frac{2n-1}{2})^2+\frac{(2n-1)(2r-1)}{2}}\cos(2r-1)z \, .$$

In (532) replace z by $z+\frac{\pi\tau}{2}$, obtaining

$$\frac{\vartheta_1'^3\vartheta_3(z)e^{2iz}q^{1/4}}{\vartheta_2\vartheta_0^2(z)\vartheta_1(z)\vartheta_2(z)} = -i\vartheta_2\left\{A_3^{(1)}(z+\tfrac{5\pi\tau}{6},\tfrac{\pi\tau}{3}+\tfrac{\pi}{2}) - iA_3^{(0)}(z+\tfrac{5\pi\tau}{6},\tfrac{\pi\tau}{3}+\tfrac{\pi}{2})\right\} +$$

$$+ q^{1/4}\vartheta_0^3 A_3^{(0)}(z+\tfrac{5\pi\tau}{6},\tfrac{5\pi\tau}{6}) \quad + q^{1/4}\vartheta_3^3 A_3^{(0)}(z+\tfrac{5\pi\tau}{6},\tfrac{5\pi\tau}{6}+\tfrac{\pi}{2})$$

This gives

$$\frac{\vartheta_1'^3\vartheta_3(z)}{\vartheta_2\vartheta_0^2(z)\vartheta_1(z)\vartheta_2(z)} = 4\vartheta_2\sum_{n,r}(-1)^{n+r+1}[2(3n+r)-3]\,q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz \quad +$$

(535)

$$+\vartheta_0^3\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r} q^{3n^2+2nr}\sin 2rz\right\} \quad + \vartheta_3^3\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{n+r+1} q^{3n^2+2nr}\right\}$$

Replacing z by $z+\frac{\pi}{2}$

$$\frac{\vartheta_1'^3\vartheta_0(z)}{\vartheta_2\vartheta_3^2(z)\vartheta_1(z)\vartheta_2(z)} = 4\vartheta_2\sum_{n,r}(-1)^n[2(3n+r)-3]\,q^{3(\frac{2n-1}{2})^2+(2n-1)r}\sin 2rz \quad +$$

(536)

$$+\vartheta_0^3\left\{\frac{\sin z}{\cos z} + 4\sum_{n,r}(-1)^{r+1} q^{3n^2+2nr}\sin 2rz\right\} + \vartheta_3^3\left\{\frac{\cos z}{\sin z} + 4\sum_{n,r}(-1)^n q^{3n^2+2nr}\sin 2rz\right\} .$$

From these follow

$$\frac{\vartheta_1'^3\vartheta_1(z)}{\vartheta_2\vartheta_2^2(z)\vartheta_0(z)\vartheta_3(z)} = \vartheta_2\left\{\frac{\sin z}{\cos^2 z} + 4\sum q^N\left\{(-1)^{\frac{\beta-b+1}{2}}(\beta+3b)\sin(\beta-3b)z\right\}\right\} +$$

(533.1)

$$+ 4\vartheta_0^3\sum q^{\frac{N}{4}}\left\{\sin\frac{\gamma-3c}{2}z\right\} + \ 4\vartheta_3^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-c}{4}}\sin\frac{\gamma-3c}{2}z\right\} .$$

$$\frac{\vartheta_1'^{3}\vartheta_2(z)}{\vartheta_2\vartheta_1^2(z)\vartheta_0(z)\vartheta_3(z)} = \vartheta_2\left\{\frac{\cos z}{\sin^2 z} + 4\sum q^N\left\{(-1)^{b+1}(\beta+3b)\cos(\beta-3b)z\right\}\right\} +$$

(534.1)

$$+ 4\vartheta_0^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\gamma-3c-2}{4}}\cos\frac{\gamma-3c}{2}z\right\} + 4\vartheta_3^3\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{c+1}{2}}\cos\frac{\gamma-3c}{2}z\right\},$$

$$\frac{\vartheta_1'^{3}\vartheta_3(z)}{\vartheta_2\vartheta_0^2(z)\vartheta_1(z)\vartheta_2(z)} = 2\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{\delta-d-2}{4}}(\delta+3d)\sin\frac{\delta-3d}{2}z\right\} +$$

(535.1)

$$+\vartheta_0^3\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{\sin(\alpha-3a)z\right\}\right\} + \vartheta_3^3\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha-a-2}{2}}\sin(\alpha-3a)z\right\}\right\},$$

$$\frac{\vartheta_1'^{3}\vartheta_0(z)}{\vartheta_2\vartheta_3^2(z)\vartheta_1(z)\vartheta_2(z)} = 2\vartheta_2\sum q^{\frac{N}{4}}\left\{(-1)^{\frac{d+1}{2}}(\delta+3d)\sin\frac{\delta-3d}{2}z\right\} +$$

(536.1)

$$+\vartheta_0^3\left\{\frac{\sin z}{\cos z} + 4\sum q^N\left\{(-1)^{\frac{\alpha-3a-2}{2}}\sin(\alpha-3a)z\right\}\right\} + \vartheta_3^3\left\{\frac{\cos z}{\sin z} + 4\sum q^N\left\{(-1)^a\sin(\alpha-3a)z\right\}\right\}.$$

VIII

The expansions obtained in the previous pages exhibit a general uniformnity of structure.Every expansion consists of a sum of single terms,which show explicitly the singularities corresponding to real values of z, and of series of the types

$$\sum q^N\left\{(\alpha+\mu a)^{\ell}\,{\cos \atop \sin}(\alpha-\mu a)z\right\}$$

$$\sum q^N\left\{(\beta+\mu b)^{\ell}\,{\cos \atop \sin}(\beta-\mu b)z\right\}$$

$$\sum q^{\frac{N}{4}}\left\{(\gamma+\mu c)^{l}\,{\cos \atop \sin}\,\frac{\gamma-\mu c}{2}z\right\}$$

$$\sum q^{\frac{N}{4}}\left\{(\delta+\mu d)^{l}\,{\cos \atop \sin}\,\frac{\delta-\mu d}{2}z\right\}$$

where the notation is as before, and (α, a), ...satisfy the conditions corresponding to the values $1, 2, 3$ of μ.

Each pole of order k in the function expanded gives rise to corresponding series in which l takes precisely the values $k-1$, $k-3$, $k-5$..., ending in 0 or 1 according as k is odd or is even. The factors whidh multiply these series are all simple functions of the corresponding $R_i^{(j)}$. It seems probable that this general structure of expansions of the type of function here considered is independent of the value of μ. It is planned to investigate this more general question in a future paper.

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Pasadena, California.

$$\vartheta_0(z,q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz \qquad \vartheta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z$$

$$\vartheta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz \qquad \vartheta_2(z,q) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos(2n+1)z$$

$$\vartheta_0\left(\frac{\pi\tau}{2}\right) = \vartheta_1(0) = \vartheta_2\left(\frac{\pi}{2}\right) = \vartheta_3\left(\frac{\pi\tau}{2}+\frac{\pi}{2}\right) = 0 \qquad q = e^{i\pi\tau}$$

$$\vartheta_0(z,-q) = \vartheta_3(z,q) \qquad \vartheta_1(z,-q) = (-1)^{\frac{1}{4}}\vartheta_1(z,q)$$

$$\vartheta_3(z,-q) = \vartheta_0(z,q) \qquad \vartheta_2(z,-q) = (-1)^{\frac{1}{4}}\vartheta_2(z,q)$$

$$\vartheta_0\left(z+\frac{\pi}{2}\right) = \vartheta_3(z) \qquad \vartheta_0\left(z-\frac{\pi}{2}\right) = \vartheta_3(z) \qquad \vartheta_0\left(z \pm \frac{\pi\tau}{2}\right) = \pm i e^{\mp i(z \pm \frac{\pi\tau}{4})}\vartheta_1(z)$$

$$\vartheta_1\left(z+\frac{\pi}{2}\right) = \vartheta_2(z) \qquad \vartheta_1\left(z-\frac{\pi}{2}\right) = -\vartheta_2(z) \qquad \vartheta_1\left(z \pm \frac{\pi\tau}{2}\right) = \pm i e^{\mp i(z \pm \frac{\pi\tau}{4})}\vartheta_0(z)$$

$$\vartheta_2\left(z+\frac{\pi}{2}\right) = -\vartheta_1(z) \qquad \vartheta_2\left(z-\frac{\pi}{2}\right) = \vartheta_1(z) \qquad \vartheta_2\left(z \pm \frac{\pi\tau}{2}\right) = e^{\mp i(z \pm \frac{\pi\tau}{4})}\vartheta_3(z)$$

$$\vartheta_3\left(z+\frac{\pi}{2}\right) = \vartheta_0(z) \qquad \vartheta_3\left(z-\frac{\pi}{2}\right) = \vartheta_0(z) \qquad \vartheta_3\left(z \pm \frac{\pi\tau}{2}\right) = e^{\mp i(z \pm \frac{\pi\tau}{4})}\vartheta_2(z)$$

$$\vartheta_0(z+\pi) = \vartheta_0(z) \qquad \vartheta_0(z+\pi\tau) = -e^{-2i(z+\frac{\pi\tau}{2})}\vartheta_0(z)$$

$$\vartheta_1(z+\pi) = -\vartheta_1(z) \qquad \vartheta_1(z+\pi\tau) = -e^{-2i(z+\frac{\pi\tau}{2})}\vartheta_1(z)$$

$$\vartheta_2(z+\pi) = -\vartheta_2(z) \qquad \vartheta_2(z+\pi\tau) = e^{-2i(z+\frac{\pi\tau}{2})}\vartheta_2(z)$$

$$\vartheta_3(z+\pi) = \vartheta_3(z) \qquad \vartheta_3(z+\pi\tau) = e^{-2i(z+\frac{\pi\tau}{2})}\vartheta_3(z)$$

$$\vartheta_1' = \vartheta_0\vartheta_2\vartheta_3 \qquad \frac{\vartheta_1'''}{\vartheta_1'} = \frac{\vartheta_0''}{\vartheta_0} + \frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3} \qquad \vartheta_3^4 = \vartheta_0^4 + \vartheta_2^4$$

$$\vartheta_0^4 = \frac{\vartheta_3''}{\vartheta_3} - \frac{\vartheta_2''}{\vartheta_2} \qquad \vartheta_2^4 = \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_3''}{\vartheta_3} \qquad \vartheta_3^4 = \frac{\vartheta_0''}{\vartheta_0} - \frac{\vartheta_2''}{\vartheta_2}$$

$$\vartheta_0(\varepsilon) = \vartheta_0 + \frac{\vartheta_0}{2}\varepsilon^2 + \frac{\vartheta_0^{IV}}{4!}\varepsilon^4 + \cdots$$

$$\vartheta_1(\varepsilon) = \vartheta_1'\varepsilon + \frac{\vartheta_1'''}{3!}\varepsilon^3 + \frac{\vartheta_1^{V}}{5!}\varepsilon^5 \cdots$$

$$\vartheta_2(\varepsilon) = \vartheta_2 + \frac{\vartheta_2''}{2}\varepsilon^2 + \vartheta_2^{IV}\frac{\varepsilon^4}{4!} + \cdots$$

$$\vartheta_3(\varepsilon) = \vartheta_3 + \frac{\vartheta_3''}{2}\varepsilon^2 + \frac{\vartheta_3^{IV}}{4!}\varepsilon^4 + \cdots$$

$$\vartheta_0\left(\frac{\pi}{2}+\varepsilon\right) = \vartheta_3 + \frac{\vartheta_3''}{2}\varepsilon^2 + \frac{\vartheta_3^{IV}}{4!}\varepsilon^4 \cdots$$

$$\vartheta_1\left(\frac{\pi}{2}+\varepsilon\right) = \vartheta_2 + \frac{\vartheta_2''}{2}\varepsilon^2 + \frac{\vartheta_2^{IV}}{4!}\varepsilon^4 + \cdots$$

$$\vartheta_2\left(\frac{\pi}{2}+\varepsilon\right) = -\vartheta_1'\varepsilon - \frac{\vartheta_1'''}{3!}\varepsilon^3 - \frac{\vartheta_1^{V}}{5!}\varepsilon^5 - \cdots$$

$$\vartheta_3\left(\frac{\pi}{2}+\varepsilon\right) = \vartheta_0 + \frac{\vartheta_0''}{2}\varepsilon^2 + \frac{\vartheta_0^{IV}}{4!}\varepsilon^4 + \cdots$$

$$\vartheta_0\left(\frac{\pi\tau}{2}+\varepsilon\right) = i\,e^{-\frac{i\pi\tau}{4}}\left\{\varepsilon\vartheta_1' - i\vartheta_1'\varepsilon^2 + (\vartheta_1''' - 3\vartheta_1')\frac{\varepsilon^3}{3!} - i(\vartheta_1''' - \vartheta_1')\frac{\varepsilon^4}{3!}\right\}$$

$$\vartheta_1\left(\frac{\pi\tau}{2}+\varepsilon\right) = i\,e^{-\frac{i\pi\tau}{4}}\left\{\vartheta_0 - i\vartheta_0\varepsilon + (\vartheta_0'' - \vartheta_0)\frac{\varepsilon^2}{2} - i(3\vartheta_0'' - \vartheta_0)\frac{\varepsilon^3}{3!} + (\vartheta_0^{IV} - 6\vartheta_0'' + \vartheta_0)\frac{\varepsilon^4}{4!}\cdots\right\}$$

$$\vartheta_2\left(\frac{\pi\tau}{2}+\varepsilon\right) = e^{-\frac{i\pi\tau}{4}}\left\{\vartheta_3 - i\vartheta_3\varepsilon + (\vartheta_3'' - \vartheta_3)\frac{\varepsilon^2}{2} - i(3\vartheta_3'' - \vartheta_3)\frac{\varepsilon^3}{3!} + (\vartheta_3^{IV} - 6\vartheta_3'' + \vartheta_3)\frac{\varepsilon^4}{4!}\cdots\right\}$$

$$\vartheta_3\left(\frac{\pi\tau}{2}+\varepsilon\right) = e^{-\frac{i\pi\tau}{4}}\left\{\vartheta_2 - i\vartheta_2\varepsilon + (\vartheta_2'' - \vartheta_2)\frac{\varepsilon^2}{2} - i(3\vartheta_2'' - \vartheta_2)\frac{\varepsilon^3}{3!} + (\vartheta_3^{IV} - 6\vartheta_3'' + \vartheta_3)\frac{\varepsilon^4}{4!}\cdots\right\}$$

$$\vartheta_0\left(\frac{\pi}{2}+\frac{\pi\tau}{2}+\varepsilon\right) = e^{-\frac{i\pi\tau}{4}}\left\{\vartheta_2 - i\vartheta_2\varepsilon + (\vartheta_2'' - \vartheta_2)\frac{\varepsilon^2}{2} - i(3\vartheta_2'' - \vartheta_2)\frac{\varepsilon^3}{3!} + (\vartheta_2^{IV} - 6\vartheta_2'' + \vartheta_2)\frac{\varepsilon^4}{4!}\cdots\right\}$$

$$\vartheta_1\left(\frac{\pi}{2}+\frac{\pi\tau}{2}+\varepsilon\right) = e^{-\frac{i\pi\tau}{4}}\left\{\vartheta_3 - i\vartheta_3\varepsilon + (\vartheta_3'' - \vartheta_3)\frac{\varepsilon^2}{2} - i(3\vartheta_3'' - \vartheta_3)\frac{\varepsilon^3}{3!} + (\vartheta_3^{IV} - 6\vartheta_3'' + \vartheta_3)\frac{\varepsilon^4}{4!}\cdots\right\}$$

$$\vartheta_2\left(\frac{\pi}{2}+\frac{\pi\tau}{2}+\varepsilon\right) = -i\, e^{-\frac{i\pi\tau}{4}}\left\{ \vartheta_0 - i\vartheta_0\varepsilon + (\vartheta_0'' - \vartheta_0)\frac{\varepsilon^2}{2} - i(3\vartheta_0'' - \vartheta_0)\frac{\varepsilon^3}{3!} + (\vartheta_0^{iv} - 6\vartheta_0'' + \vartheta_0)\frac{\varepsilon^4}{4!} \right\}$$

$$\vartheta_3\left(\frac{\pi}{2}+\frac{\pi\tau}{2}+\varepsilon\right) = i\, e^{-\frac{i\pi\tau}{4}}\left\{ \vartheta_1'\varepsilon - i\vartheta_1'\varepsilon^2 + (\vartheta_1''' - 3\vartheta_1' \,\frac{\varepsilon^3}{3!} - i(\vartheta_1''' - \vartheta_1')\frac{\varepsilon^4}{3!} \right\}$$

$$\frac{\vartheta_0''(-q)}{\vartheta_0(-q)} = \frac{\vartheta_3''(q)}{\vartheta_3(q)} \qquad \frac{\vartheta_1'''(-q)}{\vartheta_1'(-q)} = \frac{\vartheta_1'''(q)}{\vartheta_1'(q)} \qquad \frac{\vartheta_2''(-q)}{\vartheta_2(-q)} = \frac{\vartheta_2''(q)}{\vartheta_2(q)} \qquad \frac{\vartheta_3(-q)}{\vartheta_3(-q)} = \frac{\vartheta_0''(q)}{\vartheta_0(q)}$$

$$\frac{\vartheta_0''}{\vartheta_0} = 8\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2} = 8\sum q^N\Big\{\sum_{N=d\delta}\delta\Big\} \quad d\equiv 1 \bmod 2$$

$$\frac{\vartheta_1'''}{\vartheta_1'} = 24\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} - 1 = -1 + 24\sum q^N\Big\{\sum_{N=d\delta}\delta\Big\}, \ d \text{ even}$$

$$\frac{\vartheta_2''}{\vartheta_2} = -1 - 8\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2} = -1 + 8\sum q^N\Big\{\sum_{N=d\delta}(-1)^\delta\delta\Big\}, \ d \text{ even}$$

$$\frac{\vartheta_3''}{\vartheta_3} = -8\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1+q^{2n-1})^2} = 8\sum q^N\Big\{\sum_{N=d\delta}(-1)^\delta\delta\Big\} \quad d\equiv 1 \bmod 2$$

$$A_\mu^{(k)}(z+a\pi\tau,\ y+a\pi\tau) = \frac{d^k}{dy^k}\cot(z-y) \quad + i\sum_{n=1}(2\mu n i)^k e^{2\mu n i y} q^{\mu n(n+2a-1)} \; +$$

$$-i\sum_{n=1}(-2\mu n i)^k e^{-2\mu n i y} q^{\mu n(n-2a+1)} + 2i\sum_{n=1}\sum_{r=1}[2i(\mu n+r)]^k e^{2iy(\mu n+r)} q^{\mu n(n+2a-1)+2nr} e^{-2riz} \; +$$

$$-2i\sum_{n=1}\sum_{r=1}[-2i(\mu n+r)]^k e^{-2iy(\mu n+r)} q^{\mu n(n-2a+1)+2rn} e^{2riz} \quad .$$

$$A_\mu^{(k)}(z,\ y+a\pi\tau) = i\sum_{n=0}(2\mu n i)^k e^{2\mu n i y} q^{\mu n(n+2a-1)} - i\sum_{n=1}(-\mu 2 n i)^k e^{-2\mu n i y} q^{\mu n(n-2a+1)} \; +$$

$$+2i\sum_{n=0}\sum_{r=1}[2i(\mu n+r)]^k e^{2iy(\mu n+r)-2riz} q^{\mu n(n+2a-1)+2r(n+a)} \; +$$

$$-2i\sum_{n=1}\sum_{r=1}[-2i(\mu n+r)]^k e^{-2iy(\mu n+r)+2riz} q^{\mu n(n-2a+1)+2r(n-a)} \qquad 0<a<1 \quad .$$

$$A_\mu^{(k)}(z+a\pi\tau,\ y+(a+\tfrac{1}{2})\pi\tau) = i\sum_{n=0}(2\mu n i)^k e^{2\mu n i y} q^{\mu n(n+2a)} - i\sum_{n=1}(-2\mu n i)^k e^{-2\mu n i y} q^{\mu n(n-2a)} \; +$$

$$+2i\sum_{n=0}\sum_{r=1}[2i(\mu n+r)]^k e^{2iy(\mu n+r)} q^{\mu n(n+2a)+(2n+1)r} e^{-2riz} \; +$$

$$-2i\sum_{n=1}\sum_{r=1}[-2i(\mu n+r)]^k e^{-2iy(\mu n+r)} q^{\mu n(n-2a)+(2n-1)r} e^{2riz} \quad .$$

$$A_\mu^{(k)}(z+(a+\tfrac{1}{2})\pi\tau,\ y+a\pi\tau) = i\sum_{n=1}(2\mu n i)^k e^{2\mu n i y} q^{\mu n(n+2a-1)} - i\sum_{n=0}(2\mu n i)^k e^{-2\mu n i y} q^{\mu n(n-2a+1)} \; +$$

$$+2i\sum_{n=1}\sum_{r=1}[2i(\mu n+r)]^k e^{2iy(\mu n+r)} q^{\mu n(n+2a-1)+(2n-1)r} e^{-2riz} \; +$$

$$-2i\sum_{n=0}\sum_{r=1}[-2i(\mu n+r)]^k e^{-2iy(\mu n+r)} q^{\mu n(n-2a+1)+(2n+1)r} e^{2riz} \quad .$$