Essays on Market Design and Industrial Organization

Thesis by

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Para Maricruz

“Sabes lo que yo ignoro
y me dices las cosas que no me digo.
Me aprendo en ti más que en mi mismo”.
—Jaime Sabines (1926–1999)

Unless we destroy ourselves utterly, the future belongs to those societies that, while not ignoring the reptilian and mammalian paths of our being, enable the characteristically human components of our nature to flourish; to those societies that encourage diversity rather than conformity; to those societies willing to invest resources in a variety of social, political, economic and cultural experiments, and prepared to sacrifice short-term advantage for long-term benefit; to those societies that treat new ideas as delicate, fragile and immensely valuable pathways to the future.

Carl Sagan, *The Dragons of Eden* (1977)

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ABSTRACT

This dissertation contains three essays. They offer contributions to the study of matching in foster care (Chapters 1 and 2), and to the study of the effect of product market competition on managerial incentives (Chapter 3).

Chapter 1 presents an empirical framework to study the assignment of children into foster homes and its implications on placement outcomes. The empirical application uses a novel dataset of confidential foster care records from Los Angeles County, California. The estimates of the empirical model are used to examine policy interventions aimed at improving placement outcomes by increasing market thickness. If placements were assigned across all the administrative regions of the county, the model predicts that (i) the average number of foster homes children go through before exiting foster care would decrease by 8% and (ii) the distance between foster homes and children's schools would be reduced by 54%.

Chapter 2 proposes and studies a dynamic model of centralized matching in foster care. The optimal matching policy is characterized by minimizing the number of children who remain unmatched in every period. The main finding is that the optimal matching policy gives priority to younger children. The model captures several dynamic trade-offs, most notably between children's ages and the heterogeneity in the expected duration of placements. I also analyze federal data from the Adoption and Foster Care Analysis and Reporting System (AFCARS). I find that, in Los Angeles County, placements and their durations are strongly correlated with the race of children and their foster parents.

Chapter 3, co-authored with Kaniṣka Dam, develops an incentive contracting model under oligopolistic competition to study how incumbent firms adjust managerial incentives following deregulation policies that enhance competition. We show that firms elicit higher managerial effort by offering stronger incentives as an optimal response to entry, as long as incumbent firms act as production leaders. Our model draws a link between an industry-specific feature, the time needed to build production capacity, and the effect that product market competition has on executive compensation. We offer new testable implications regarding how this industry-specific feature shapes the incentive structure of executive pay.

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INTRODUCTION

Many economic transactions are not governed by standard prices, such as the ones used in traditional commodity markets. Two prominent examples are the transactions that take place in (i) matching markets and (ii) within the boundaries of organizations. For various moral and legal concerns, prices are often not used to determine who is matched to whom in matching markets. For example, in modern-day societies people typically do not pay one another when agreeing to get married. Likewise, when deciding how to allocate children to public schools or organs to patients who need a transplant, it is usually illegal for parties to partake in monetary transactions. Economic transactions that take place within organizations are, by nature, not governed by traditional prices. In some sense, organizations such as firms and governments exist precisely to allocate goods outside of traditional markets. A good commonly allocated within organizations is *managerial effort*. In the presence of moral hazard, the optimal allocation of effort requires state-contingent contracts rather than fixed wages or prices.

This dissertation contains three essays. Two of them contribute to the study of matching markets (Chapters 1 and 2) and one to the study of the provision of incentives within organizations (Chapter 3). Chapters 1 and 2 focus on the study of matching in foster care. Chapter 1 presents an empirical framework to study the assignment of children into foster homes and its implication on placement outcomes. Chapter 2 complements Chapter 1 by focusing on the dynamic aspect of matching in foster care. Chapter 3, co-authored with Kaniṣka Dam, studies the effect that product market competition has on the provision of managerial incentives within a firm.

The study of foster care as a matching market is a fairly recent area of research. The empirical application in Chapter 1 uses a novel dataset of confidential foster care records from Los Angeles County, California.[1] The main methodological contribution of Chapter 1 is to formulate an empirical framework to study matching in foster care that accounts for unobservable heterogeneity in the distribution of placement outcomes. In terms of policy, the main contribution of the chapter resides

[1] I obtained IRB approval and a limited waiver of confidentiality from the Juvenile Division of the Superior Court of California to analyze confidential records of the Department of Children and Family Services of Los Angeles County. The analyses and interpretations of all the data used in this thesis are my sole responsibility. The aforementioned institutions and their agents or employees bear no responsibility for the analyses and interpretations presented here.

on the analysis of policies aimed at improving placement outcomes. Specifically, I use the estimates of the empirical model to examine the effect that market thickness has on placement outcomes. Notably, the estimates of the model show that, if placements were assigned across all the administrative regions of Los Angeles County, (i) the average placement disruption probability across all placements in the system would reduce by 4.2 percentage points (which is equivalent to a reduction of 8% in the expected number of foster homes children go through before exiting foster care), and (ii) the distance between foster homes and children's schools would be reduced by 54%.

In Chapter 2, I propose a model to study matching in foster care in a dynamic environment. The optimal matching policy in the model is characterized by minimizing the number of children who remain unmatched on every period. The main finding of this chapter is that the optimal matching policy gives priority to younger children. The model captures several dynamic trade-offs, notably between children's ages and the heterogeneity in the expected duration of placements. I also analyze federal data from the Adoption and Foster Care Analysis and Reporting System (AFCARS).[2] I find that, in Los Angeles County, CA, placements and their durations are strongly correlated with the race of children and their foster parents.

In Chapter 3, co-authored with Kaniṣka Dam, we develop an incentive contracting model under oligopolistic competition to study how incumbent firms adjust managerial incentives following deregulation policies that enhance competition. We show that firms elicit higher managerial effort by offering stronger incentives as an optimal response to entry, as long as incumbent firms act as production leaders. Our model draws a link between an industry-specific feature, the time needed to build production capacity, and the effect that product market competition has on executive compensation. We offer new testable implications regarding how this industry-specific feature shapes the incentive structure of executive pay.

[2]The AFCARS data were made available by the National Data Archive on Child Abuse and Neglect (NDACAN), Cornell University, Ithaca, NY, and were originally collected by the Children's Bureau with funding from the U.S. Department of Health and Human Services. The analyses and interpretations of all the data used in this study are my sole responsibility. The aforementioned institutions and their agents or employees bear no responsibility for the analyses or interpretations presented here.

Chapter 1

WHO GETS PLACED WHERE AND WHY? AN EMPIRICAL FRAMEWORK FOR FOSTER CARE PLACEMENT

1.1 Introduction

The assignment of scarce resources is at the heart of economics. In this chapter, I study one particular assignment setting that has been largely absent in the economics literature—the placement of children into foster homes. I develop an empirical framework that captures how social workers match children and foster homes in the field. The analysis centers on the relationship between placement assignments and outcomes.

I estimate an econometric model using a novel dataset of confidential county records at the micro-level from the largest foster care system in the United States, the one in Los Angeles County, California. Motivated by the literature on children welfare studies (and anecdotal evidence from conversations with social workers), my definition of *placement outcomes* includes both the duration of placements and whether they are disrupted (in which case children are moved from one foster home to another) or terminate because children exit foster care.

I use the estimates of the model to examine various policy interventions aimed at improving placement outcomes. I find that thicker markets generate better outcomes in the sense that they result in lower disruption rates, but the effects are different along different dimensions. Specifically, the model predicts that the gains from assigning placements across geographic regions in the county are greater than those generated by delaying assignments. Counterfactual exercises show that pooling the assignments across all the regional offices in the county would decrease the expected number of placements each child goes through before exiting foster care by 8%. I also quantify the system-wide effects of specific types of foster homes. I find that increasing the share of placements involving children's relatives (also known as "kinship care") would lead to lower placement disruption rates and longer placements. By contrast, the model predicts mixed effects from increasing the share of foster homes that are recruited and trained by non-profit agencies.

The model is designed to capture the co-dependence between placement assignments and outcomes. On the one hand, the model captures how the assignments of placements are driven by their expected outcomes. On the other hand, it also recognizes that the outcomes observed in the data are selected through such assignment. The interplay between assignments and outcomes causes an endogeneity problem. Since the matching mechanism determines which placement outcomes are observable, the observed distribution of placement outcomes is biased inasmuch as placement assignments are driven by unobservables correlated with outcomes. To identify the true distribution of outcomes, the model exploits the exogenous variation across the dates and geographic regions in which children enter foster care. I study matching markets at the daily level across the nineteen administrative regions defined by the Los Angeles County Department of Children and Family Services.

It is widely recognized that stable foster care placements are essential for the social, emotional, and cognitive development of children (UC Davis, 2008). Social workers in the field also strive to assign long-lasting placements to minimize future workloads. Nonetheless, it is fairly common that children go through multiple foster homes while they are in foster care.[1] Understanding how children are assigned to foster homes allows one to analyze how the matching mechanism used in the field translates into outcomes via placement characteristics. For example, the estimates of the model show that the gains from thicker markets come largely from being able to assign children to foster homes that are closer to their schools. The model predicts that if the assignments of placements were determined at the county-level (and not within geographic regions), the average distance between children's schools and their foster homes would be cut by 54%.

I model the assignment of children into foster homes as a centralized matching problem, and I model placement outcomes with a mixed competing risks duration model. The matching problem allows for idiosyncratic variation in the preferences of children over foster home characteristics, and vice versa. At the same time, it takes into account that placements are assigned on the basis of their expected outcomes. I model unobservable heterogeneity through frailty terms in the outcome distribution. To account for possible selection bias (i.e., that placements may be assigned because of unobservables correlated with outcomes), I assume that the decision-maker choosing the matching between children and foster homes observes

[1]For example, of all the children who exited foster care in the U.S. during 2015, 56.1% of them went through at least two placements, and the average number of placements per child was 2.56 (NDACAN, 2015). It has also been shown that the time children spend in foster care, as well as the number of placement disruptions they experience, are associated in adult life with emotional and behavioral difficulties, increased criminal convictions, and higher depression and smoking rates (Dregan and Gulliford, 2012).

such frailty terms. Thus, the distribution of outcomes generated by the model is conditional on the assignment chosen and incorporates unobservable heterogeneity.

The estimates of the matching model allow me to quantify the trade-offs that social workers incur when assigning placements. For instance, at first sight, it seems intuitive that social workers aim to assign the placements that are expected to have the longest durations in order to avoid placement disruptions. However, this reasoning ignores the intimate co-dependence between a placement's duration and its termination reason. Indeed, according to the model estimates, social workers' assignments reflect a dislike for duration conditional on a specific termination reason. That is, if a placement were known to be disrupted, the model estimates indicate that social workers would prefer for it to be disrupted sooner rather than later. Similarly, if it were known that a placement will terminate because the child will exit foster care to a permanent placement, social workers would prefer for this to happen as soon as possible. At the same time, the estimates show that social workers prioritize minimizing disruptions over placement duration. That is, regardless of a placement's duration, the model predicts that social workers would always prefer for placements *not* to be disrupted.

The rest of the chapter is organized as follows. I review the related literature in what remains of the introduction. In Section 1.2, I provide an institutional background of foster care, and describe the data. Section 1.3 presents the econometric model. In Sections 1.4 and 1.5, I discuss the identification of the model and the estimation technique. Section 1.6 reports the estimation results. Section 1.7 shows the results of the counterfactual exercises, and Section 1.8 concludes.

Related Literature.—The main contribution of this chapter is to develop an empirical framework to study (i) how children are assigned into foster homes, and (ii) how the matching mechanism underlying such assignment translates into placement outcomes. Slaugh, Akan, Kesten, and Ünver (2016) is the only other paper in the literature that applies tools from matching and market design to a question related to foster care. They analyze the Pennsylvania Adoption Exchange program, whose main aim is to facilitate the adoption of foster children through a computerized recommendation system. They analyze the effect that improvements to the system—in terms of enhancing the capacity of social workers to match children and prospective adoptive parents—have on the rate of successful adoptions.

Baccara, Collard-Wexler, Felli, and Yariv (2014) analyze data from an online platform that seeks to facilitate adoptions. Although they are distinct in fundamental ways, adoption and foster care are closely related. Parents who are seeking to adopt often become foster parents beforehand, and, in many cases, foster children are adopted by their foster parents. Baccara et al. (2014) focus on the preferences that prospective adoptive parents show for children. They find a favorable preference for girls, and a preference against African Americans.

Overall, the economics literature analyzing questions related to foster care is slim. In a series of papers, Doyle Jr. (2007, 2008, 2013) evaluates the impact of foster care on long-term outcomes. Their approach exploits that, in many cases, social workers are assigned randomly to investigate reports of abuse and neglect. This random assignment allows them to identify the "treatment effect" of foster care on schooling, employment, and criminality. Doyle Jr. and Peters (2007) use variation in the subsidies offered to foster parents to estimate the supply curve of foster homes. Analyzing data from the late 1980s to the early 1990s, they estimate that, in states with shortages of foster homes, an increase in subsidies by 10% increases the quantity supplied by 3%.[2]

In broader terms, this chapter belongs to the empirical matching and market design literature (Roth, 2016). The common denominator in this literature is the formulation and estimation of structural models that incorporate key institutional aspects of the market being studied. In a seminal contribution, Choo and Siow (2006) study the marriage market in a transferable utility (TU) environment. Their setup is based on the Assignment Game developed by Shapley and Shubik (1971). See Graham (2011, 2013), Chiappori, Oreffice, and Quintana-Domeque (2012), and Galichon and Salanié (2015) for extensions and generalizations of their approach. Choo (2015) further extends the analysis to a dynamic setting. More generally, Fox (2018) studies nonparametric identification and estimation of TU matching markets. Buchholz (2019) and Fréchette, Lizzeri, and Salz (2019) study matching models in the market for taxis.[3]

In a non-TU environment, Agarwal (2015) formulates and estimates a matching model of the medical match (NRMP). Agarwal and Somaini (2018) study the strategic incentives of different mechanisms in the assignment of children to public schools. For other recent contributions to the empirical study of school choice, see Narita (2016), Hwang (2016), Calsamiglia, Fu, and Güell (2017), and Abdulka-

[2]See Doyle Jr. and Aizer (2018) for an excellent literature review on the current state of empirical work in economics on child maltreatment and its relation to foster care and intimate partner violence.

[3]Market-clearing transfers need not only be monetary prices (e.g., passengers waiting for taxis "pay" in waiting-time units), see Galichon and Hsieh (2017).

diroğlu, Agarwal, and Pathak (2017). There is also a growing literature analyzing kidney exchange (e.g., Agarwal, Ashlagi, Azevedo, Featherstone, and Karaduman, 2019), waiting-list mechanisms for organ donation (Agarwal, Ashlagi, Rees, Somaini, and Waldinger, 2019), and public housing allocation (Waldinger, 2019).

All the studies cited in the previous two paragraphs model assignments according to specific matching mechanisms. The TU literature generally assumes that the market is cleared via equilibrium transfers. In non-TU environments, the assignment usually results from predetermined matching algorithms.[4] The main differences from previous studies and this chapter is that the assignment mechanism underlying foster care neither involves equilibrium transfers nor makes use of a systematic matching algorithm. Beyond being centralized, the matching between children and foster homes is the consequence of both (i) specific regulations and (ii) discretionary choices made by social workers in the field.

The insights from this chapter are also relevant for the growing literature on dynamic matching. One of the main objectives of this literature is to study the dynamic tradeoffs between waiting time, thickness, incentives, and match quality. For notable examples, see Baccara, Lee, and Yariv (forthcoming), Ünver (2010), Akbarpour, Li, and Gharan (2020), Doval (2018), and Ashlagi, Jaillet, and Manshadi (2013). Specifically, this chapter provides an example in which increasing market thickness by delaying placements does not have sizable effect on outcomes.

1.2 Institutional Background and Data

Foster Care in the U.S. and Los Angeles County

Every year more than a half million children go through foster care in the United States. Foster children are a particularly vulnerable population: most of them are in foster care because they were abused, neglected, or abandoned (NDACAN, 2015). The main goal of foster care is to provide temporary care for children until permanent placements can be arranged for them. When a child is moved from a foster home to a permanent placement, it is said that she exits foster care to "permanency." Children who exit to permanency usually go back to live with their birth families, or, if this

[4]The study of matching algorithms dates back to Gale and Shapley (1962), who formulated the well-known Deferred Acceptance (DA) algorithm. Roth (1984) documents the history of the medical match and, more specifically, how it came to employ the DA algorithm before the findings of Gale and Shapley. Given the attractive features of DA (stability and strategy-proofness), it has been proposed as a mechanism to match children to schools (Abdulkadiroğlu and Sönmez, 2003). A significant portion of the school choice literature compares the DA algorithm with the so-called Boston algorithm (e.g., Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005).

is not possible, are adopted or assigned guardians. When permanent placements cannot be arranged for children, they stay in foster care until they become of age, and are emancipated from the system (also known as "aging out").[5]

The administrative management of the foster care system is at the county level in the United States.[6] The child protection agency of Los Angeles County is the Department of Children and Family Services (DCFS).[7] As other child protection agencies, DCFS is responsible of processing and investigating reports of child abuse, taking cases to court, and implementing court resolutions. After receiving a report, county social workers conduct an investigation to determine if children need to be removed from home. The decision whether a child should be removed or not needs to be approved by a judge. The procedures regarding the investigation and removal decision are independent from placement assignment procedures. Foster care placements are assigned and managed within nineteen regions across the county of Los Angeles. When a child enters foster care, her case is handled by the regional office corresponding to the region where the child's birth mother lives. Social workers from that regional office are responsible for finding a suitable placement for the child, and overseeing her case while she remains in foster care.

Placement Assignment in Foster Care

By law, there are a few factors that social workers must consider when assigning placements: (1) whether a child has relatives who are available to take care of them, in which case children must be placed with their relatives; (2) the location of the foster home: social workers must make efforts to place children in foster homes that are near their schools and their family homes (from where they were removed), and (3) whether a child has siblings who are also in foster care, in which case efforts should be made to place siblings together.[8] However, the law does not provide a systematic way in which these factors are to be waged against one another. The

[5]Foster care is inherently different from adoption. In general, adoptive parents have the same rights and obligations over their children as biological parents. By contrast, foster parents have very limited say in the placement of foster children. Whether a child is removed from home, placed in or exits foster care, is a decision made by the courts, which rely heavily on the input of social workers.

[6]In some cases, there is a single child protection agency for all the counties covering the same urban area (e.g., there is a single agency for the five boroughs of New York City).

[7]Specific foster care regulations vary at the state and county level. In California, the main regulations of the foster care system are provided in the Welfare and Institutions Code (WIC, 2019), and the Family Code (FAM, 2019). In Los Angeles County, foster care regulations are provided in the Child Welfare Policy Manual of DCFS (2019). For a history of the foster care system in the United States, see Rymph (2017).

[8]See DCFS (2019, Sec. 0100-510.60); FAM (2019, Div. 12, Part 6, Sec. 7950), and WIC (2019, Div. 9, Part 4, Ch. 1, Sec. 16002).

law also gives social workers the discretion to assign placements that bypass these guidelines if they consider it is in a child's best interest. Likewise, children who are 10 years or older also have the right to make a brief statement in court regarding the placement decision.

In the field, social workers aim to find placements that fulfill all the requirements stated in the law, and are also suitable for children in more practical ways. For example, when evaluating prospective foster homes, they may take into account scheduling and transportations considerations, the family environment of the foster home (e.g., the age and gender of the family's biological children), and other idiosyncratic factors such as the experience of the foster parents and the history of a child in the system. The reason for taking into account each of these factors is because a main concern of social workers is for placements to be disrupted. Placements are usually disrupted because the foster family and the foster child are not able to establish a harmonious and stable relationship (e.g., the child presents behavioral problems the family is not prepared to deal with, the situation of the family changes, or problems emerge between the foster child and the family's biological children). When placements are disrupted, children need to be moved to new foster homes. In LA County, on average, foster children go through 2.1 foster homes before exiting to permanency.

I gathered the above observations through informal conversations with a handful of social workers with experience in the field. Overall, my impression from these conversations is that apart from the guidelines embedded in the law, social workers work on a case-by-case basis. They treat each case differently, and wage all of the involved factors in a case to find the best possible placement. Another common observation is that, in many cases, ideal placements are just not possible because of the shortage of foster homes. As children enter foster care, social workers within each regional office come together and do their best to find placements that are suitable for the children.

Another characteristic feature of how children are assigned placements in the field is that the process is done as quickly as possible. In most cases, children must be placed on very short notice. Furthermore, even if a social worker knows that a child will be removed in the near future (usually not more than a few days), a placement cannot be assigned until the child has been removed. The reason for this is precisely because foster homes are scarce and there are children in need of placements constantly. Therefore, social workers cannot hold placements and wait for children to be removed from home. It would mean that other children are not being placed, which social workers try to avoid as best they can.[9]

Data Description and Summary Statistics

The data used in this study comes from the confidential county records of DCFS. The database used for the analysis includes the record of every child that was placed in a foster home at any point between January 1, 2011, and February 28, 2011, in LA County.[10] During this period, 2,087 children where assigned to a foster home at least once in LA County, and 2,358 placements were assigned in total across the nineteen regional offices in LA County. On average, roughly 40 placements are assigned every day throughout the county. Table 1.1 contains summary statistics of the placements in the dataset.

Modeling Strategy

In what follows, I develop an econometric model with the objective of analyzing the determinants underlying placement assignment. The main focus is on placements that were assigned on the same day in the same regional office. That is, the model aims to explain what drives the matching between children and foster homes in cases in which two or more placements where assigned in the same day in the same regional office. For this purpose, I slice the data of placements into markets accordingly. The division of the data into markets also incorporates placements with relatives. That is, if a child was placed with a relative, I form an independent market consisting of a single child and a single home in which the assignment problem is trivial. The reason I keep "singleton" markets (i.e., with a single child and single home) is to study their outcomes.

The way I model placement assignment is by considering a single matchmaker that assigns placements in terms of their expected outcomes. That is, when there are several ways in which children and foster homes can be matched, the matchmaker is assumed to consider the expected outcomes of all prospective placements, and weigh them according to a specific utility function. I rationalize the observed

[9]Children who enter foster care at times when there are no placements available are usually placed in Emergency-Foster Care or Emergency Shelter Care while a non-emergency placement can be found (usually in a few days at most). Emergency placements are available 24/7, but are not suitable for stays lasting more than a few days.

[10]The confidentiality waiver needed to access the data granted access to a larger time period. However, I restrict the sample period to a two-month period for computational considerations. As it shall be seen in the coming sections, the econometric framework I develop in this chapter is computationally intensive.

Table 1.1: Summary statistics

	(1) n	(2) mean	(3) sd	(4) p5	(5) p25	(6) p50	(7) p75	(8) p95
Termination Reasons								
Disruption	2358	0.5093	0.5	0	0	1	1	1
Exit	2358	0.4237	0.4942	0	0	0	1	1
Emancipation	2358	0.05174	0.2215	0	0	0	0	1
Censored	2358	0.01527	0.1226	0	0	0	0	0
Duration								
Duration (days)	2358	255.4	343.9	5	35	131.5	339	898.4
Duration|Disrup	1201	164.6	242.7	4	22	74	186	623
Duration|Exit	999	304.1	304.8	5.45	66	223	439.2	879
Duration|Emanc	122	394.7	437.4	8.6	95	232	502	1400
Duration|Cens	36	1461	850.5	25.1	344.5	1969	1988	2002
Children Characteristics								
Time Since Removal (days)	2358	387.7	937.6	0	0	32	292	2184
Placement # In Spell	2358	2.75	2.582	1	1	2	3	8
Spell # in Child	2358	1.194	0.4626	1	1	1	1	2
Zero Waiting Time	2358	0.8562	0.3509	0	1	1	1	1
Waiting Time (days)	2358	0.9326	3.148	0	0	0	0	10.6
Age	2358	8.694	5.967	0.2037	2.916	8.485	14.54	17.35
Male	2358	0.4576	0.4983	0	0	0	1	1
Black	2358	0.3138	0.4641	0	0	0	1	1
Hispanic	2358	0.5424	0.4983	0	0	1	1	1
White	2358	0.1175	0.3221	0	0	0	0	1
Other Race	2358	0.02629	0.16	0	0	0	0	0
English	2358	0.8223	0.3823	0	1	1	1	1
Spanish	2358	0.1773	0.382	0	0	0	0	1
Other Language	2358	0.0004241	0.02059	0	0	0	0	0
Absence/Incapacitation	2358	0.2693	0.4437	0	0	0	1	1
Abuse/Severe Neglect	2358	0.2498	0.433	0	0	0	0	1
General Neglect	2358	0.4597	0.4985	0	0	0	1	1
Other Removal Reason	2358	0.0212	0.1441	0	0	0	0	0
Foster Homes Characteristics								
County Foster Home	2358	0.08567	0.2799	0	0	0	0	1
Agency Foster Home	2358	0.4258	0.4946	0	0	0	1	1
Group Home	2358	0.1158	0.32	0	0	0	0	1
Relative Home	2358	0.3728	0.4836	0	0	0	1	1
Distance Plac-Office (mi.)	2358	22.93	21.27	2.22	7.716	16.05	30.69	71.15
Distance Plac-School (mi.)	2358	18.13	23.77	0	0	7.983	26.9	72.73
No School	2358	0.2472	0.4315	0	0	0	0	1

Note: Summary statistics of placement outcomes and characteristics. The distance measures are at the zip-code level (foster home and school). They were computed using the Google Maps API (accessible through https://cloud.google.com/maps-platform/). No School refers to children for which the dataset includes no school zip-code (presumably because the child does not go to school or the data is missing). sd = standard deviation; p# refers to the #th percentile.

matching by considering it as the optimal matching from the matchmaker's perspective. Apart from considering the expected outcomes of prospective placements, the matchmaker's problem also allows for children and foster homes to have idiosyncratic tastes for the type of foster home and child with whom they are matched. The model is designed to include the most prominent institutional features of foster care placement. That being said, the only feature I abstract away from is the placement of siblings. I ignore the existence of siblings in the system, and focus on one-to-one matchings. The analysis of placement assignment with siblings is ripe ground for future research.

1.3 Model

Market of Foster Care Services

A market is a tuple $(C, H, \mathbf{X}, \mathbf{Y})$, where C is the set of available children, H is the set of available foster homes, $\mathbf{X} = (\mathbf{x}_c)_{c \in C}$ is the matrix of children's (observable) characteristics, i.e., $\mathbf{x}_c \in \mathcal{X} \subseteq \mathbb{R}^{\dim(\mathbf{x})}$ is the vector of characteristics of child $c \in C$, and $\mathbf{Y} = (\mathbf{y}_h)_{h \in H}$ is the matrix of the (observable) characteristics of available homes, i.e., $\mathbf{y}_h \in \mathcal{Y} \subseteq \mathbb{R}^{\dim(\mathbf{y})}$ is the vector of characteristics of home $h \in H$. In order to incorporate idiosyncratic preferences over children's and foster home's characteristics, I define *types* as a coarsening of characteristics. Let $X = \{x\}$ and $Y = \{y\}$ be the sets of child- and home-types; formally, they are finite partitions of $\mathcal{X}$ and $\mathcal{Y}$. Similarly, let $x_c \in X$ and $y_h \in Y$ denote the types of $c \in C$ and $h \in H$, respectively.

A one-to-one matching between children and foster homes is an indicator function $M : C \times H \to \{0, 1\}$ such that $\sum_{h \in H} M(c, h) \leq 1$ for all $c \in C$, and $\sum_{c \in C} M(c, h) \leq 1$ for all $h \in H$. That is, $M(c, h) = 1$ if child c is matched with home h, and 0 otherwise. For simplicity, I also write $(c, h) \in M$ if $M(c, h) = 1$. Let $\mathbb{M}(C, H)$ denote the set of feasible one-to-one matchings between C and H.

Matching a child and a foster home forms a placement. The outcome of a placement is given by $(T, R) \in \mathbb{R}_+ \times \mathcal{R}$, where T denotes the placement's duration and R its termination reason. A placement may terminate because it is disrupted (d), the child exits to permanency (ex), or is emancipated (em). The set of termination reasons is thus $\mathcal{R} \equiv \{d, ex, em\}$. It is convenient to differentiate emancipation from the other termination reasons because the time to emancipation, denoted by T_{em}, is not random (i.e., known ex-ante). I define the set of termination reasons with non-degenerate duration as $\mathcal{R}_0 = \{d, ex\}$.

Children are matched to foster homes on a daily basis within regional offices throughout the county. The unit of observation is a market, indexed by $i = 1, \ldots, n$. Markets correspond to office-days, and also incorporate the restriction that children need to be matched with their relatives whenever possible. That is, children for whom relatives are available as prospective foster parents have their own markets (consisting of a single child and a single foster home). The data consists on (1) a sample of markets, $(C_i, H_i, \mathbf{X}_i, \mathbf{Y}_i)_{i=1}^n$; (2) the matching chosen in each market, $(M_i)_{i=1}^n$, where $M_i \in \mathbb{M}(C_i, H_i)$ for $i = 1, \ldots, n$, and (3) the outcomes of the assigned placements, $(\mathbf{T}_i, \mathbf{R}_i)_{i=1}^n$, where $\mathbf{T}_i = (T_{ch})_{(c,h)\in M_i}$, and $\mathbf{R}_i = (R_{ch})_{(c,h)\in M_i}$.

I take the data of markets, $(C_i, H_i, \mathbf{X}_i, \mathbf{Y}_i)_{i=1}^n$, as given (i.e., as exogenous variables). The observed matching and the realized outcomes, $(M_i, \mathbf{T}_i, \mathbf{R}_i)_{i=1}^n$, are the outcome (or endogenous) variables of the model. Note that this implies that there are no spillovers across office-days. Every day, in every office, a matching is assigned between the available children and foster homes taking the market as given. I outline the data generating process of the endogenous variables, $(M, \mathbf{T}, \mathbf{R})$, in the following sections.

Placement Assignment

Placements are assigned by a single (or representative) utilitarian matchmaker, who has preferences over realized outcomes $(T, R) \in \mathbb{R}_+ \times \mathcal{R}$. The matchmaker's preferences are represented by the utility function:

$$u(T, R; T_{em}) = \mu_R + \varphi_R \log T + \bar{\varphi}_R \log T_{em}, \tag{1.1}$$

where $\mu_R, \varphi_R, \bar{\varphi}_R \in \mathbb{R}$ are unknown parameters for $R \in \mathcal{R}$. The parameter μ_R measures the preference over termination reason $R \in \mathcal{R}$, regardless of duration; φ_R is the marginal utility of duration conditional on terminating due to $R \in \mathcal{R}$. The utility function also includes the time to emancipation in its third term to control for the fact that placements involving younger children may have ex-ante longer durations. For example, if $\bar{\varphi}_R = -\varphi_R$, the matchmaker cares about duration relative to the time to emancipation. More generally, one can see that the sign of the marginal rate of substitution between duration and age, conditional on termination reason $R \in \mathcal{R}$, is equal to the sign of $\varphi_R / \bar{\varphi}_R$.

Consider a prospective placement $(c, h) \in C \times H$. Let $\mathcal{I}_{ch}$ denote the information that the matchmaker has on its outcome distribution. The total payoff of placing child $c \in C$ in home $h \in H$ to the matchmaker is given by:

$$V(c, h) = \pi(c, h) + \varepsilon_{cy_h} + \eta_{x_c h}, \tag{1.2}$$

where $\pi(c,h) := \mathbb{E}\left[u(\tilde{T},\tilde{R};T_{em})\,|\,\mathcal{I}_{ch}\right]$ captures the preferences and information available to the matchmaker about the placement's outcome. I specify the distribution of $(\tilde{T},\tilde{R})\,|\,\mathcal{I}_{ch}$ in the next section.[11] The remaining two terms in (1.2), ε_{cy_h} and $\eta_{x_c h}$, capture idiosyncratic taste variation across children and foster homes (which is unobservable to the econometrician). Specifically, ε_{cy} captures the payoff of matching child c with a home of type $y \in Y$, and η_{xh} that of matching home h with a child of type $x \in X$. In this sense, the model incorporates the preferences of children over being placed in specific types of homes and those of homes over taking care of particular types of children. More generally, the taste variation terms are aimed to capture type-specific idiosyncratic unobservables that affect placement assignment (e.g., the matchmaker may also have preferences over forming certain types of placements, regardless of their outcomes).

The matchmaker chooses the matching $M \in \mathbb{M}(C,H)$ that maximizes its aggregate payoff. Since $V(c,h)$ is observable to the matchmaker for all $(c,h) \in C \times H$, the observed matching is the solution to the following linear programming problem:

$$\max\left\{\sum_{c\in C,h\in H} M(c,h)V(c,h) : M \in \mathbb{M}(C,H)\right\}. \tag{1.3}$$

I restrict attention to matchings in which no child is left unmatched while there is an unmatched home. That is, besides incorporating the natural constraints that every child can be matched with at most one home (and vice versa), the set of feasible matchings $\mathbb{M}(C,H)$ satisfies:

$$M \in \mathbb{M}(C,H) \quad \Leftrightarrow \quad \sum_{c\in C,h\in H} M(c,h) = \min\{|C|,|H|\}. \tag{1.4}$$

Placement Outcomes

Prospective placements are indexed by $(c,h) \in C \times H$. For simplicity, consider a generic placement and omit such index in this section. The full vector of characteristics of a placement is given by $\mathcal{I} = (\mathbf{x},\mathbf{y},\boldsymbol{\omega})$, where $(\mathbf{x},\mathbf{y}) \in \mathcal{X}\times\mathcal{Y}$ are the observable child- and home-characteristics, and $\boldsymbol{\omega} \in \mathbb{R}^{\dim(\omega)}$ is a vector of characteristics *not* observed by the econometrician. The distribution of a placement's outcome, $(\tilde{T},\tilde{R})$, depends on its full vector of characteristics, $\mathcal{I}$.

I model placement outcomes as the result of mixed competing risks. Consider a generic placement with characteristics $\mathcal{I} = (\mathbf{x},\mathbf{y},\boldsymbol{\omega})$. Let $\tilde{T}_R$ be the latent duration

[11] I differentiate random variables that are observable to the econometrician from their realized values with a tilde; $(\tilde{T},\tilde{R})$ denotes the unrealized (random) placement outcome, while $(T,R) \in \mathbb{R}_+ \times \mathcal{R}$ denotes its realization.

associated to the "risk" of terminating due to reason $R \in \mathcal{R}_0$. Up to censoring, due to the sample period or emancipation, a placement's outcome is determined by the least latent duration. Denote the time to the end of the sample period by T_{cen}, and indicate censored placements by $R = cen$. To simplify notation, I define $\tilde{T}_{em} = T_{em}$ and $\tilde{T}_{cen} = T_{cen}$ as the degenerate latent durations corresponding to the time to emancipation and the end of the sample period, respectively. Formally, the outcome of a placement is given by:

$$\tilde{T} = \min\left\{\tilde{T}_R : R \in \mathcal{R} \cup \{cen\}\right\}, \text{ and } \tilde{R} = \arg\min\left\{\tilde{T}_R : R \in \mathcal{R} \cup \{cen\}\right\}. \quad (1.5)$$

Under the above specification, a placement is emancipated (or censored) if and only if it has not been disrupted or has exited to permanency by its emancipation date (or the end of the sample period). Note that each placement in the data is subject to either emancipation or censoring due to the sample period, depending on which of T_{em} and T_{cen} is lower. Both types of censoring, due to emancipation and the end of the sample period, are equivalent in terms of the likelihood of the latent durations. However, they are not equivalent from the matchmaker perspective, who has a preference over the emancipation likelihood and the time to emancipation. Censoring due to the sample period is only statistical in nature.

Assumption 1 (Unobserved heterogeneity) *The unobservable characteristics of a placement are given by the vector* $\boldsymbol{\omega} = (\omega_R)_{R\in\mathcal{R}_0}$. *Furthermore,*

$$\boldsymbol{\omega} \sim N(0, \boldsymbol{\Sigma}_\omega), \quad (1.6)$$

where $\boldsymbol{\Sigma}_\omega$ *is a positive semidefinite and symmetric matrix of size* $|\mathcal{R}_0| \times |\mathcal{R}_0|$.

Assumption 2 (Burr hazards) *Conditional on a placement's characteristics,* $\mathcal{I}$, *the latent durations,* $\{\tilde{T}_R : R \in \mathcal{R}_0\}$, *are independent. Furthermore, the conditional distribution of* $\tilde{T}_R$ *is determined by the following Burr hazard rate*[12],

$$\lambda_R(T \mid \mathcal{I}) = \frac{k_R(\mathcal{I})\alpha_R T^{\alpha_R - 1}}{1 + \gamma_R^2 k_R(\mathcal{I}) T^{\alpha_R}}, \quad R \in \mathcal{R}_0, \quad (1.7)$$

where $k_R(\mathcal{I}) \equiv \exp\{\omega_R + g(\mathbf{x}, \mathbf{y})\beta_R\}$ *with* $\beta_R \in \mathbb{R}^{\dim(\beta)}$, $g : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^{\dim(\beta)}$, $\alpha_R > 0$, *and* $\gamma_R \geq 0$.

[12]The hazard rate of the random variable $\tilde{T}$ is the function defined by $\lambda(T) = f(T)/\bar{F}(T)$, where f denotes the probability density function of $\tilde{T}$, and $\bar{F}$ its survivor function. The survivor function is defined by $\bar{F}(T) \equiv 1 - F(T)$, where F denotes the variable's cumulative distribution function.

Assumption 2 specifies the distribution of placement outcomes from the perspective of the matchmaker, $(\tilde{T}, \tilde{R}) \mid \mathcal{I}$. The matchmaker's additional information, $\boldsymbol{\omega}$, consists of unobservable frailty terms, $(\omega_R)_{R \in \mathcal{R}_0}$, which shift the hazard rate associated to each "risk" (termination reason) upwards or downwards. Since such frailty terms are not observable to the econometrician, the distribution $(\tilde{T}, \tilde{R}) \mid \mathcal{I}$ is not observed directly in the data. The outcome distribution is "mixed" by the distribution of the unobservable frailty terms; one must integrate out $\boldsymbol{\omega}$ to recover the distribution of outcomes in the data. However, note that the distribution of $\boldsymbol{\omega}$ across the placements observed in the data is *not* equal to the unconditional distribution specified in Assumption 1. The distribution of $\boldsymbol{\omega}$ across the placements in the data is conditional on being matched, i.e., to that of $\boldsymbol{\omega}_{ch} \mid M(c, h) = 1$.

The Burr specification in Assumption 2 is a standard parametric assumption used in duration models (e.g., Lancaster, 1990; Wooldridge, 2010).[13] The Burr distribution has the main advantage of being flexible yet tractable. It generalizes other well-known duration distributions, such as the Exponential ($\gamma_R = 0, \alpha_R = 1$), Weibull ($\gamma_R = 0$), and Log-Logistic ($\gamma_R = 1$). A convenient feature of this distribution is that its integrated hazard rate has a closed form, and hence, also its survivor function and likelihood. The parameters α_R and γ_R govern the duration-dependence of the hazard function, which may be flat, monotonic (positive or negative), or have an inverse-U shape. The function g is a shorthand for the covariates used in the model, all of which are derived from observable characteristics. Besides including stand-alone covariates, $g(\mathbf{x}, \mathbf{y})$ may include interactions between variables in $\mathbf{x}$ and $\mathbf{y}$, and other non-trivial transformations, such as distance measures. The effect of the covariates on each hazard rate is controlled by the coefficients in β_R. Since the function λ_R is monotonic in k_R, the sign of the coefficients in β_R indicate the direction in which the covariates shift the hazard rates. A higher hazard rate, say λ_R, implies that a placement is more likely to terminate sooner and due to termination reason $R \in \mathcal{R}_0$.

Assumption 1 specifies the joint distribution of $\boldsymbol{\omega} = (\omega_R)_{R \in \mathcal{R}_0}$ up to the unknown covariance matrix $\boldsymbol{\Sigma}_\omega$. Assuming that $\boldsymbol{\omega}$ has zero mean is without loss of generality, as long as the covariates in the hazard function include a constant. Intuitively, the covariance matrix $\boldsymbol{\Sigma}_\omega$ captures the extent of the variation in the observed outcomes not captured through placement characteristics. Moreover, the correlation between the individual frailty terms introduces dependence among the latent durations. Such correlation captures, for example, if (a) children who are less likely to reach perma-

[13]Another common application of the Burr distribution, also known as the Singh-Maddala distribution, is to model the distribution of income (Singh and Maddala, 1976).

nency are also more likely to experience disruptions (because, say, they experienced worse conditions during their upbringing, and this has an impact on their current behavior), or (b) children who are more likely to exit the system sooner are also more likely to experience disruptions (because, say, foster parents are less invested in nurturing long and stable relationships with children who will leave their households sooner).

Collect the parameters of the hazard rates in $\boldsymbol{\alpha} = (\alpha_R)_{R\in\mathcal{R}_0}$, $\boldsymbol{\gamma} = (\gamma_R)_{R\in\mathcal{R}_0}$ and $\boldsymbol{\beta} = (\beta_R)_{R\in\mathcal{R}_0}$. The conditional outcome distribution, $(\tilde{T}, \tilde{R}) \,|\, \boldsymbol{\mathcal{I}}$, is fully specified in Assumption 2 up to the unknown vector of parameters

$$\boldsymbol{\theta}_T \equiv (\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta}).$$

Observed Matching

In this section, I consider a generic market $(C, H, \mathbf{X}, \mathbf{Y})$, and omit its index $i = 1, \ldots, n$ for simplicity. The problem of the matchmaker in (1.3) is a deterministic problem over matchings. However, from the econometrician's perspective, the observed matching is the realization of a random variable since $V(c, h)$ is not fully observable. Specifically, an econometrician does not observe the frailty terms $(\boldsymbol{\omega}_{ch})_{(c,h)\in C\times H}$, or the taste variation terms, $\varepsilon_c = (\varepsilon_{cy})_{y\in Y}$ for every $c \in C$, and $\eta_h = (\eta_{xh})_{x\in X}$ for every $h \in H$.

Assumption 3 (Multinomial Probit) *The taste variation terms are independent multivariate normal random vectors. Namely,*

$$\varepsilon_c \sim N(0, \boldsymbol{\Sigma}_\varepsilon), \quad \textit{and} \quad \eta_h \sim N(0, \boldsymbol{\Sigma}_\eta), \tag{1.8}$$

where $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{\Sigma}_\eta$ are positive semidefinite and symmetric matrices. Their sizes are $|Y| \times |Y|$ and $|X| \times |X|$, respectively. Furthermore, $\varepsilon_c \perp \varepsilon_{c'}$ for all $c, c' \in C$, and $\eta_h \perp \eta_{h'}$ for all $h, h' \in H$. Also, ε_c, η_h, and $\boldsymbol{\omega}_{c'h'}$ are mutually independent for all $(c, h), (c', h') \in C \times H$.

Under Assumption 3, the observed matching is a realization of the following random variable:

$$\tilde{M}(C, H, \mathbf{X}, \mathbf{Y}) = \arg\max \left\{ \sum_{c\in C, h\in H} M(c, h)\pi(c, h) + \upsilon_M : M \in \mathbb{M}(C, H) \right\}, \tag{1.9}$$

where υ_M is the composite error term given by

$$\upsilon_M \equiv \sum_{c\in C, h\in H} M(c, h)[\varepsilon_{cy_h} + \eta_{x_c h}]. \tag{1.10}$$

Since the composite error term, υ_M, follows a multivariate normal distribution, the matching problem takes the form of a mixed multinomial probit. Below, I show that the distribution of the individual taste variation parameters, ε_c and η_y, can be backed out from the distribution of the composite error term υ_M. Therefore, the distribution of the taste variation parameters can be obtained directly from the matching data.

Assumption 3 also includes several independence assumptions. First, the unobservable taste variation components are independent across parties, i.e., $\varepsilon_c \perp \varepsilon_{c'}$, $\eta_h \perp \eta_{h'}$, and $\varepsilon_c \perp \eta_h$. This assumption rules out unobservable interdependencies among placement assignments by considering preferences over types as independent across children and foster homes. Second, the unobservable frailty terms are independent across placements, i.e., $\boldsymbol{\omega}_{ch} \perp \boldsymbol{\omega}_{c'h'}$. This assumption rules out unobservable interdependencies among placement outcomes. Conditional on being matched, the outcome of (c, h) is independent of that of (c', h'). Third, the taste variation terms, ε_c and η_h, are independent of the frailty terms in $\boldsymbol{\omega}_{ch}$. This assumption separates the unobservables affecting placement assignments into two groups. On the one hand, $\boldsymbol{\omega}_{ch}$ contains unobservables that affect placement assignments through their expected outcomes (i.e., outcome-relevant unobservables). On the other hand, ε_c and η_h capture the rest of the unobservables which affect the matching, but are independent of outcomes.

Collect the preference parameters in $\boldsymbol{\mu} = (\mu_R)_{R\in\mathcal{R}_0}$, $\boldsymbol{\varphi} = (\varphi_R)_{R\in\mathcal{R}_0}$, $\bar{\boldsymbol{\varphi}} = (\bar{\varphi}_R)_{R\in\mathcal{R}_0}$, $\boldsymbol{\Sigma} = (\boldsymbol{\Sigma}_\varepsilon, \boldsymbol{\Sigma}_\eta)$, and define

$$\boldsymbol{\theta}_M \equiv (\boldsymbol{\mu}, \boldsymbol{\varphi}, \bar{\boldsymbol{\varphi}}, \boldsymbol{\Sigma}).$$

Two-by-two Example

In this section, I consider a simple example to illustrate how the model allows for the matching observed in the data to depend on distinct factors. Consider a market with two children and two homes, let $C = \{c_1, c_2\}$ and $H = \{h_1, h_2\}$. Let x_1 and x_2 denote the types of children c_1 and c_2, respectively, and y_1 and y_2, those of homes h_1 and h_2. Let $\varepsilon_1 = (\varepsilon_{11}, \varepsilon_{12})$ and $\varepsilon_2 = (\varepsilon_{21}, \varepsilon_{22})$ be the unobservable tastes of child c_1 and c_2 for home-types y_1 and y_2, respectively, where I take the liberty of writing $\varepsilon_{kj} \equiv \varepsilon_{c_k y_j}$. Similarly, let $\eta_1 = (\eta_{11}, \eta_{21})$ and $\eta_2 = (\eta_{12}, \eta_{22})$ be the unobservable tastes of homes h_1 and h_2 for child-types x_1 and x_2, respectively, with $\eta_{kj} \equiv \eta_{x_k h_j}$. Finally, let $(\boldsymbol{\omega}_{11}, \boldsymbol{\omega}_{12}, \boldsymbol{\omega}_{21}, \boldsymbol{\omega}_{22})$ be the unobservable vectors of frailty terms of each prospective placement, i.e., $\boldsymbol{\omega}_{kj} \equiv (\omega_{c_k h_j, R})_{R\in\mathcal{R}_0}$ for $j, k = 1, 2$.

Let $\pi_{kj} \equiv \pi(c_k, h_j)$ denote the payoff of assigning each prospective placement, which is a function of ω_{kj}, for $j, k = 1, 2$. The set of feasible matchings $\mathbb{M}(C, H)$ contains two matchings: M, which assigns placements (c_1, h_1) and (c_2, h_2), and M', which assigns placements (c_1, h_2) and (c_2, h_1). Let $\mathcal{V}$ and $\mathcal{V}'$ denote their respective aggregate payoffs, i.e.,

$$\mathcal{V} = V(c_1, h_1) + V(c_2, h_2) = (\pi_{11} + \varepsilon_{11} + \eta_{11}) + (\pi_{22} + \varepsilon_{22} + \eta_{22}) \tag{1.11}$$

$$\mathcal{V}' = V(c_1, h_2) + V(c_2, h_1) = (\pi_{12} + \varepsilon_{12} + \eta_{12}) + (\pi_{21} + \varepsilon_{21} + \eta_{21}) . \tag{1.12}$$

Matching M is chosen over M' if and only if $\mathcal{V} \geq \mathcal{V}'$ (the event $\mathcal{V} = \mathcal{V}'$ has zero probability). In principle, all the terms in (1.11) and (1.12) might differ, implying that observing matching M over M' might result for numerous reasons, e.g., the expected outcome of placement (c_1, h_2) or (c_2, h_1) is unfavorable relative to that of (c_1, h_1) or (c_2, h_2) (i.e., π_{12} or π_{21} are low relative to π_{11} or π_{22}) . Alternatively, child c_k might have a higher than usual preference for being matched with a home of type y_k (i.e., ε_{11} or ε_{22} are particularly high), or home h_j might have a higher than usual preference for being matched with a child of type x_j (i.e., η_{11} or η_{22} are high relative to η_{12} or η_{21}).

Now consider the case in which $y_1 = y_2$, so that $\varepsilon_{11} = \varepsilon_{12}$ and $\varepsilon_{21} = \varepsilon_{22}$. In such case, matching M is chosen over M' if and only if

$$(\pi_{11} + \eta_{11}) + (\pi_{22} + \eta_{22}) \geq (\pi_{12} + \eta_{12}) + (\pi_{21} + \eta_{21}) . \tag{1.13}$$

In this case, even though the unobservable taste terms of both children may differ, i.e., $\varepsilon_{11} \neq \varepsilon_{21}$, the preferences of children over home-types play no role in the determination of the optimal matching. Similarly, if the children are also of the same type, $x_1 = x_2$, then matching M is chosen over M' if and only if

$$\pi_{11} + \pi_{22} \geq \pi_{12} + \pi_{21} . \tag{1.14}$$

In this case, the optimal matching is determined only on the basis of expected outcomes. Importantly, the event $\mathcal{V} \geq \mathcal{V}'$ is still random from the econometrician's perspective, since (1.14) depends on the unobservable frailty terms, ω_{kj} for $j, k = 1, 2$.

Expected Placement Outcomes

In this section, I show in more detail how the payoff function depends on a placement's expected outcomes. Using (1.1) and the definition of $\pi(c,h)$, we obtain

$$\pi(c,h) = \sum_{R\in\mathcal{R}} \mathbb{P}(\tilde{R} = R \,|\, \mathcal{I}_{ch}) \left\{\mu_R + \varphi_R \mathbb{E}\left[\log \tilde{T} \,|\, \tilde{R} = R, \mathcal{I}_{ch}\right] + \bar{\varphi}_R \log T_{em,c}\right\}. \quad (1.15)$$

Therefore, the expected placement outcomes that are relevant for the matchmaker's payoff are the termination probability, $\mathbb{P}(\tilde{R} = R \mid \mathcal{I}_{ch})$, and the conditional expected log-duration, $\mathbb{E}\left[\log \tilde{T} \,|\, \tilde{R} = R, \mathcal{I}_{ch}\right]$, of each termination reason $R \in \mathcal{R}$. The expected placement outcomes can be computed using standard results in survival analysis (e.g., Kalbfleisch and Prentice, 2002; Lancaster, 1990).[14] Namely, for $R \in \mathcal{R}_0$,

$$\mathbb{P}(\tilde{R} = R \,|\, \mathcal{I}_{ch}) = \int_0^{T_{em,c}} \bar{F}(T \,|\, \mathcal{I}_{ch}) \lambda_R(T \,|\, \mathcal{I}_{ch}) dT \quad (1.16)$$

$$\mathbb{E}\left[\log \tilde{T} \,|\, \tilde{R} = R, \mathcal{I}_{ch}\right] = \int_0^{T_{em,c}} \log T \left[\frac{\bar{F}(T \,|\, \mathcal{I}_{ch}) \lambda_R(T \,|\, \mathcal{I}_{ch})}{\mathbb{P}(\tilde{R} = R \,|\, \mathcal{I}_{ch})}\right] dT, \quad (1.17)$$

where $\bar{F}(T \,|\, \mathcal{I}_{ch})$ denotes the conditional survival function of $\tilde{T}$, given by

$$\bar{F}(T \,|\, \mathcal{I}_{ch}) = \exp\left\{- \sum_{R\in\mathcal{R}_0} \gamma_R^{-2} \log\left[1 + \gamma_R^2 k_R(\mathcal{I}_{ch}) T^{\alpha_R}\right]\right\}. \quad (1.18)$$

Simple calculations show that the resulting integrals in (1.16) and (1.17) have no closed-form.[15] Therefore, to compute the payoff function of placement $(c,h) \in C \times H$, one needs to compute the integrals in (1.16) and (1.17) numerically at $\mathcal{I}_{ch} = (\mathbf{x}_c, \mathbf{y}_h, \boldsymbol{\omega}_{ch})$, obtain the expected placements outcomes, and replace the respective values in (1.15).

[14]To observe why (1.16) holds, it suffices to note that $\bar{F}(T \,|\, \mathcal{I}_{ch}) \lambda_R(T \,|\, \mathcal{I}_{ch})$ is the likelihood of the placement having duration T and terminating due to $R \in \mathcal{R}_0$. The probability of terminating due to $R \in \mathcal{R}_0$ is the integral of this likelihood over the support of $\tilde{T}$, $[0, T_{em,c}]$. Similarly, to observe why (1.17) holds, it suffices to note that the quotient in brackets in (1.17) is the probability density function (pdf) of $\tilde{T} \mid \tilde{R} = R, \mathcal{I}_{ch}$. To see this, note that the likelihood of the event $(\tilde{T}, \tilde{R}) = (T, R)$ may also be written as $\mathbb{P}(\tilde{R} = R \,|\, \mathcal{I}_{ch}) f(T \,|\, \tilde{R} = R, \mathcal{I}_{ch})$, where $f(T \,|\, \tilde{R} = R, \mathcal{I}_{ch})$ denotes the pdf of $\tilde{T} \,|\, \tilde{R} = R, \mathcal{I}_{ch}$. Expression (1.18) also follows from standard results. Namely, the survivor function of the duration in a competing risks model is given by $\bar{F}(T) = \exp\left\{- \sum_{R\in\mathcal{R}_0} \int_0^T \lambda_R(S) dS\right\}$.

[15]The fact that these integrals have no closed-form is a common feature among most commonly used duration distributions. A notable exception, perhaps the only one, is the competing risks model with symmetric Weibull hazards (all hazards have the same shape parameter). In our case, this corresponds to the case with $\gamma_R = 0$ and $\alpha_R = \alpha$ for all $R \in \mathcal{R}_0$. In such case, the termination probabilities have the same form as the choice probabilities of the multinomial logit, and are constant across time. As shall be seen in next sections, this specification, although attractive for its computational tractability, is too restrictive for the present case.

In order to observe how the aggregate payoff of matching $M \in \mathbb{M}(C, H)$ depends on the expected placement outcomes of the assigned placements, note that

$$\sum_{c,h} M(c,h)\pi(c,h) = \sum_{R\in\mathcal{R}} \left\{ \left[\sum_{c,h} M(c,h)\mathbb{P}(\tilde{R} = R \,|\, \mathcal{I}_{ch}) \right] \mu_R \right.$$
$$+ \left[\sum_{c,h} M(c,h)\mathbb{P}(\tilde{R} = R \,|\, \mathcal{I}_{ch})\mathbb{E}\left(\log \tilde{T} \,|\, \tilde{R} = R, \mathcal{I}_{ch}\right) \right] \varphi_R$$
$$\left. + \left[\sum_{c,h} M(c,h)\mathbb{P}(\tilde{R} = R \,|\, \mathcal{I}_{ch}) \log T_{em,c} \right] \bar{\varphi}_R \right\}, \qquad (1.19)$$

where the sums are over $c \in C, h \in H$. Hence, conditional on the matchmaker's information on every prospective placement, $(\mathcal{I}_{ch})_{(c,h)\in C\times H}$, the problem of the matchmaker in (1.9) takes the form of a multinomial probit. The "systematic" or "observed" portion of the aggregate payoff of matching $M \in \mathbb{M}(C, H)$, given in (1.19), is a linear index on the parameters of the matchmaker's utility function, $(\boldsymbol{\mu}, \boldsymbol{\varphi}, \bar{\boldsymbol{\varphi}})$. The "covariates" of such linear index are sums of the expected outcomes of all the assigned placements under M, which, in essence, are non-linear transformations of the covariates of the assigned placements, $\{g(\mathbf{x}_c, \mathbf{y}_h) : M(c,h) = 1\}$. The unconditional problem of the matchmaker takes the form of a mixed multinomial probit since one must integrate out the unobservable part of $(\mathcal{I}_{ch})_{(c,h)\in C\times H}$, i.e., $(\boldsymbol{\omega}_{ch})_{(c,h)\in C\times H}$.

1.4 Identification

Outcome Distribution

Absent matching, the data on observed outcomes is sufficient to identify the parameters of the distribution of outcomes, $(\boldsymbol{\Sigma}_\omega, \boldsymbol{\theta}_T)$. This observation follows from Heckman and Honoré (1989), who show that the joint distribution of the latent durations in a competing risks model is non-parametrically identified as long as (1) the model includes covariates; (2) the hazard rates of the latent durations have at least one common covariate with a different coefficient in each hazard rate; (3) such covariate is continuous and unbounded, and (4) the mixing distribution is sufficiently smooth (and satisfies certain regularity conditions at the limit). All of these conditions are met given Assumptions 1 and 2. The continuous and unbounded covariates are distance measures, e.g., the distance between children's schools and foster homes, which has a termination-specific coefficient.

Once we take into account the matching part of the model, one must recognize that the distribution of ω across the placements observed in the data, in general, differs from the unconditional distribution specified in Assumption 1. The distribution of ω across the placements observed in the data is given by $\omega_{ch} \,|\, \tilde{M}(c,h) = 1$, where $\tilde{M}$ is the random variable defined in (1.9). Hence, the distribution of ω in the data depends on all the variables involved in the matchmaker's problem. In order to identify this distribution, the model relies on the random variation on the exogenous variables $(C, H, \mathbf{X}, \mathbf{Y})$. The simplest way to see why this variation is sufficient to identify the parameters in the unconditional distribution of ω is to consider placements in singleton markets. Note that the distribution of ω for placements assigned in markets with $|C| = |H| = 1$ is the same as its unconditional distribution. That is, if $|C| = |H| = 1$, the matchmaker's problem is trivial, which implies that the event $\{M(c,h) = 1\}$ is uninformative, and the likelihood of such placement's outcome is the same as its unconditional one. More generally, in non-singleton markets, exogenous variation in $(C, H, \mathbf{X}, \mathbf{Y})$ identifies the unconditional mixing distribution in a similar way in which instruments are used in standard sample selection models (e.g., Heckman, 1979). One needs exogenous variation that affects the likelihood of being "selected" (i.e., of having an observable outcome) that is independent of the outcome itself.

Another aspect that differs from the standard competing risks framework is that the matching may induce endogeneity, which leads to bias when estimating the coefficients of the covariates in the hazard functions. This observation was first noted in the literature by Ackerberg and Botticini (2002) in a setting of contract choice. Their setup is different to the one here, but the underlying intuition is the same. In a reduced-form setting, they show that when the outcome of a match (in their case, a joint sharecropping contract) depends on the characteristics of both parties involved in the match, the presence of unobservables correlated with the matching and the outcome lead to endogeneity. The matching affects the joint distribution of a match's characteristics, causing them to become correlated with the error term in a regression. To see this in our case, write the latent duration as follows[16]

$$\log \tilde{T}_R = K_R - g(\mathbf{x}, \mathbf{y})\beta_R/\alpha_R - \omega_R/\alpha_R + error_R, \tag{1.20}$$

where $error_R \equiv \log \tilde{T}_R - \mathbb{E}\left[\log \tilde{T}_R \,|\, \boldsymbol{I}\right]$ is an exogenous error term, and

$$K_R \equiv \alpha_R^{-1}\left[\psi(1) - \psi(\gamma_R^{-2}) + \log \gamma_R^{-2}\right] \tag{1.21}$$

[16]Expression (1.20) is a well-known feature of the Burr distribution (Lancaster, 1990). Indeed, the fact that the log-duration can be written in the form of a linear regression is a characteristic feature of all accelerated failure time models, the Burr duration model included.

is a constant (ψ denotes the digamma function, $\psi(x) \equiv d \log \Gamma(x)/dx$, where Γ is the gamma function). In (1.20), one can see how the covariates affect the latent log-durations in an analogous way to a linear regression. At first glance, the covariates in $g(\mathbf{x}, \mathbf{y})$ seem to be exogenous. The unconditional distributions of ω_R and $error_R$ are independent of $(\mathbf{x}, \mathbf{y})$. However, the joint distribution of $(\mathbf{x}, \mathbf{y})$ across the assigned placements, $(\mathbf{x}_c, \mathbf{y}_h) \,|\, \tilde{M}(c, h) = 1$, is determined by the matching. Note that $\mathbf{y}_h = \sum_{h' \in H} \tilde{M}(c, h')\mathbf{y}_{h'}$. Hence, the covariates derived from $\mathbf{y}$ are no longer independent from the error term $-\omega_R/\alpha_R + error_R$ in (1.20). A symmetric argument shows that the same holds for the covariates derived from $\mathbf{x}$.

To fix this endogeneity problem, Ackerberg and Botticini (2002) suggest using instrumental variables that affect the matching, but are independent of outcomes. In the present case, this exogenous variation comes through $(C, H, \mathbf{X}, \mathbf{Y})$. Two placements that are observationally equivalent, say (c, h) and (c', h') with $(\mathbf{x}_c, \mathbf{y}_h) = (\mathbf{x}_{c'}, \mathbf{y}_{h'})$, will not have the same mixing distribution if they are assigned in distinct markets. If (say) the first placement is assigned in market $(C, H, \mathbf{X}, \mathbf{Y})$ and the second one in $(C', H', \mathbf{X}', \mathbf{Y}')$, then the distribution of $\boldsymbol{\omega} \mid \tilde{M}(c, h) = 1$, in general, will be distinct to that of $\boldsymbol{\omega} \,|\, \tilde{M}'(c', h') = 1$. The matchings chosen in both markets, $\tilde{M}$ and $\tilde{M}'$, are independent random variables with distinct distributions. This identification strategy has been used in the contracting literature since the seminal contribution of Ackerberg and Botticini (e.g., Sørensen, 2007; Ewens, Gorbenko, and Korteweg, 2019).

Matching Distribution

The identification of the parameters in the matchmaker's utility function, $(\boldsymbol{\mu}, \boldsymbol{\varphi}, \bar{\boldsymbol{\varphi}})$, is straightforward once the mixing distribution is identified, and one sets $\varphi_{em} = 0$. Setting $\varphi_{em} = 0$ is necessary since the time to emancipation appears twice in $u(T, R; T_{em})$ for $R = em$, see (1.1). As mentioned above, see (1.19), the matching problem is a multinomial probit with index linear on $(\boldsymbol{\mu}, \boldsymbol{\varphi}, \bar{\boldsymbol{\varphi}})$.

Finally, I discuss the identification of the covariance matrices of the taste variation terms, $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{\Sigma}_\eta$. Let $\sigma_\varepsilon(y, y')$ be the (y, y')-th entry of $\boldsymbol{\Sigma}_\varepsilon$, i.e., $\sigma_\varepsilon(y, y') = \operatorname{cov}(\varepsilon_{cy}, \varepsilon_{cy'})$. Similarly, let $\sigma_\eta(x, x') = \operatorname{cov}(\eta_{xh}, \eta_{x'h})$. From (1.10), note that the vector of composite error terms, $\boldsymbol{\upsilon} \equiv (\upsilon_M)_{M \in \mathbb{M}(C,H)}$, follows a zero-mean multivariate normal distribution with covariance structure given by (a detailed proof is given in Appendix A.2):

$$\operatorname{cov}(\upsilon_M, \upsilon_{M'}) = \sum_{c \in C} \sigma_\varepsilon(y_{M(c)}, y_{M'(c)}) + \sum_{h \in H} \sigma_\eta(x_{M(h)}, x_{M'(h)}), \tag{1.22}$$

where I write $M(c) = h \Leftrightarrow M(h) = c \Leftrightarrow M(c, h) = 1$. To deal with unmatched children in (1.22), set $\sigma_\varepsilon(y_{M(c)}, y_{M'(c)}) \equiv 0$ if c is unmatched in either M or M'. Standard results in discrete choice models (e.g., Train, 2009) show that the covariance matrix of $\boldsymbol{\upsilon}$ is identified up to location and scale normalizations.

Assumption 4 (Covariance Normalization) *There exists $x_0 \in X$ and $y_0 \in Y$ such that $\sigma_\eta(x_0, x) = 0$ for every $x \in X$, and $\sigma_\varepsilon(y_0, y_0) = 1$.*

Assumption 4 imposes the necessary normalizations to identify the covariance matrices $\mathbf{\Sigma}_\varepsilon$ and $\mathbf{\Sigma}_\eta$. First, it imposes a location normalization by assuming that there exists a child-type, x_0, for which the taste variation term of every home equals to zero. Second, a scale normalization is assumed by assuming there exists a home-type, y_0, for which the variance of the corresponding taste variation term equals one for every child.

Proposition 1 *Under Assumption 4, the covariance matrices $\mathbf{\Sigma}_\varepsilon$ and $\mathbf{\Sigma}_\eta$ are identified.*

The proof of Proposition 1 is provided in Appendix A.2. The proof exploits that the distribution of the taste variation terms is the same regardless of the types of the other available children and homes in the market. The proof relies on analyzing the identified elements of the covariance matrix of $\boldsymbol{\upsilon}$ in specific markets with particular types of children and homes, and use the normalization in Assumption 4 and the covariance structure in (1.22) to back out the covariance matrices $\mathbf{\Sigma}_\varepsilon$ and $\mathbf{\Sigma}_\eta$.

Collect all the parameters of the model in $\boldsymbol{\theta} = [\mathbf{\Sigma}_\omega, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M]$. Let $\mathbf{\Theta} \in \mathbb{R}^{\dim(\boldsymbol{\theta})}$ be the parameter space. That is, $\mathbf{\Theta}$ is the subset of $\mathbb{R}^{\dim(\boldsymbol{\theta})}$ that incorporates the following parameter restrictions: $\alpha_R > 0, \gamma_R \geq 0$ for every $R \in \mathcal{R}_0$, $\varphi_{em} = 0$, $\mathbf{\Sigma}_\eta$ such that $\sigma_\eta(x_0, x) = 0$ for every $x \in X$, $\mathbf{\Sigma}_\varepsilon$ such that $\sigma_\varepsilon(y_0, y_0) = 1$, and $\mathbf{\Sigma}_\omega$, $\mathbf{\Sigma}_\varepsilon$ and $\mathbf{\Sigma}_\eta$ are positive semidefinite and symmetric matrices.

1.5 Estimation

In this section, I explain how to obtain a consistent, efficient, and asymptotically normal estimator of $\boldsymbol{\theta}$. The estimation consists in maximizing the simulated log-likelihood of the model. To simplify notation, let $\mathbf{z}_{ch} = (\mathbf{x}_c, \mathbf{y}_h)$ denote the observable characteristics of placement $(c, h) \in C_i \times H_i$, and group all the observable characteristics of market i in $\mathbf{Z}_i = (\mathbf{X}_i, \mathbf{Y}_i)$. Similarly, let $\mathbf{\Omega}_i = (\omega_{ch})_{(c,h)\in C_i \times H_i}$.

Fix $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Consider an arbitrary market i. The likelihood of observing $(M_i, \mathbf{T}_i, \mathbf{R}_i)$, conditional on $(\boldsymbol{\Omega}_i, \mathbf{Z}_i)$, is given by:

$$\mathcal{L}(M_i, \mathbf{T}_i, \mathbf{R}_i \,|\, \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M) = \mathcal{L}_M(M_i \,|\, \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_M, \boldsymbol{\theta}_T) \times \cdots$$
$$\cdots \;\mathcal{L}_{\mathbf{T},\mathbf{R}}(\mathbf{T}_i, \mathbf{R}_i \,|\, M_i, \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T) \tag{1.23}$$

where $\mathcal{L}_M(M_i \mid \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_M, \boldsymbol{\theta}_T)$ denotes the conditional matching likelihood, and $\mathcal{L}_{\mathbf{T},\mathbf{R}}(\mathbf{T}_i, \mathbf{R}_i \,|\, M_i, \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T)$ denotes the conditional outcome likelihood. Both likelihood functions are conditional on both unobservable and observable characteristics, $\boldsymbol{\Omega}_i$ and $\mathbf{Z}_i$, respectively. In the next two sections, I spell out both conditional likelihood functions. Then, I show how to compute the simulated log-likelihood of the data, which basically amounts to integrating out $\boldsymbol{\Omega}_i$ from (1.23).

Conditional Matching Likelihood

Write the payoff function $\pi(\cdot)$ as a function of placement characteristics and parameters, i.e., $\pi(\boldsymbol{\omega}_{ch}, \mathbf{z}_{ch} \,|\, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M) = \pi(c, h)$. Also, let $\mathbb{M}_i \equiv \mathbb{M}(C_i, H_i)$ denote the set of feasible matchings in market i. The conditional matching likelihood is given by the Probit choice probability:

$$\mathcal{L}_M(M_i \,|\, \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M) = \int 1_{\mathcal{A}(M_i | \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M)}(\boldsymbol{\upsilon}) dF(\boldsymbol{\upsilon}), \tag{1.24}$$

where $\boldsymbol{\upsilon} = (\upsilon_M)_{M \in \mathbb{M}_i}$ is the vector of matching composite errors, $1_{\mathcal{A}}(\boldsymbol{\upsilon})$ denotes the indicator function of set $\mathcal{A}$ (it takes $\boldsymbol{\upsilon}$ as argument), and the set $\mathcal{A}(M_i \mid \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M)$ is the set of $\boldsymbol{\upsilon}$'s for which the matching M_i is optimal, i.e.,

$$\left\{ \boldsymbol{\upsilon} : \upsilon_M - \upsilon_{M_i} \leq \sum_{c,h} \left[M_i(c,h) - M(c,h) \right] \pi(\boldsymbol{\omega}_{ch}, \mathbf{z}_{ch} \,|\, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M) \,\forall M \in \mathbb{M}_i \right\}. \tag{1.25}$$

Conditional Outcomes Likelihood

Let $\mathcal{L}_{T,R}(T, R \mid \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\theta}_T)$ denote the conditional likelihood of a single placement outcome, given by the Burr competing risks likelihood:

$$\mathcal{L}_{T,R}(T, R \,|\, \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\theta}_T) = \bar{F}(T \,|\, \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\theta}_T) \lambda_R(T \,|\, \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\theta}_T)^{1_{R \notin \{em, cen\}}} \tag{1.26}$$

where $\bar{F}(T \mid \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\theta}_T)$ is the survivor function given in (1.18), and $\lambda_R(T \mid \boldsymbol{\omega}, \mathbf{z}, \boldsymbol{\theta}_T)$ the termination specific hazard-rate in Assumption 2. The conditional outcome likelihood of all the placements in market i is given by:

$$\mathcal{L}_{\mathbf{T},\mathbf{R}}(\mathbf{T}_i, \mathbf{R}_i \,|\, M_i, \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T) = \prod_{(c,h) \in M_i} \mathcal{L}_{T,R}(T_{ch}, R_{ch} \,|\, \boldsymbol{\omega}_{ch}, \mathbf{z}_{ch}, \boldsymbol{\theta}_T). \tag{1.27}$$

(Simulated) Log-likelihood

Let $\mathcal{G}$ denote the joint distribution of $\mathbf{\Omega}_i$, i.e., $\mathcal{G} = \times_{c,h} G_{ch}$, where $G_{ch} \equiv N(0, \mathbf{\Sigma}_\omega)$. The conditional likelihood of the market-level data $(M_i, \mathbf{T}_i, \mathbf{R}_i)$ is:

$$\mathcal{L}(M_i, \mathbf{T}_i, \mathbf{R}_i \,|\, \mathbf{Z}_i, \boldsymbol{\theta}) = \int \mathcal{L}_M(M_i \,|\, \mathbf{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M) \times \cdots$$
$$\cdots \mathcal{L}_{\mathbf{T},\mathbf{R}}(\mathbf{T}_i, \mathbf{R}_i \,|\, M_i, \mathbf{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T) \mathcal{G}(d\mathbf{\Omega}_i \,|\, \mathbf{\Sigma}_\omega). \quad (1.28)$$

The log-likelihood of the data is

$$\ell_n(\boldsymbol{\theta} \,|\, \mathbf{Z}) = \sum_{i=1}^{n} \log \mathcal{L}(M_i, \mathbf{T}_i, \mathbf{R}_i \,|\, \mathbf{Z}_i, \boldsymbol{\theta}). \quad (1.29)$$

To estimate $\boldsymbol{\theta}$, I compute the simulated counterpart of $\ell_n(\boldsymbol{\theta} \mid \mathbf{Z})$. There are two multi-dimensional integrals within (1.29) that need to be simulated. The first one is the integral over $\boldsymbol{\upsilon}$ in the conditional matching likelihood, see (1.24). To compute this integral, I draw a sample of S_υ independent draws of the taste variation terms, ε_c and η_h. The sample is drawn independently of the model parameters, in order to keep the simulation draws fixed during the estimation. I use a logit-kernel to smooth the choice probabilities in (1.24). It is well known (e.g., Train, 2009) that such smoothing is computationally convenient when estimating multinomial probit models, especially in cases with a large number of alternatives, as in this case. Let $\zeta > 0$ denote the smoothing parameter of the logit-kernel. The second integral that needs to be computed through simulation is the one over $\mathbf{\Omega}_i$ in (1.28). To compute this integral, I draw a random sample of S_ω independent draws of each $\boldsymbol{\omega}_{ch} = (\omega_{R,ch})_{R \in \mathcal{R}_0}$, for $(c, h) \in C_i \times H_i$, $i = 1, \ldots, n$. Likewise, this sample is drawn independently of the model parameters. Let $\ell_n^{S_\omega, S_\upsilon, \zeta}(\boldsymbol{\theta} \mid \mathbf{Z})$ denote the simulated counterpart of the log-likelihood of the data in (1.29). (See Appendix A.1 for more details on the estimation.) The estimator of $\boldsymbol{\theta}$ is given by:

$$\hat{\boldsymbol{\theta}}_{SMLE} = \arg\max_{\boldsymbol{\theta} \in \mathbf{\Theta}} \ell_n^{S_\omega, S_\upsilon, \zeta}(\boldsymbol{\theta} \,|\, \mathbf{Z}). \quad (1.30)$$

Standard results (e.g., Gourieroux and Monfort, 1997) imply that $\hat{\boldsymbol{\theta}}_{SMLE}$ is a consistent, efficient, and asymptotically normal estimator for $\boldsymbol{\theta}$, as $n, S_\omega, S_\upsilon \to \infty$ with $\min\{S_\omega, S_\upsilon\}/\sqrt{n} \to \infty$.

1.6 Estimation Results

Empirical Specification

In this section, I present the results of the estimation. Due to computational considerations, I consider a small version of the model in terms of the number of covariates I include. The estimates presented below correspond to a model that includes the following placement characteristics: age, type of foster home (relative, county, agency, or group home), and distance to school. I also include a dummy for children for which the school's zip-code is missing (who presumably do not go to school), and interactions between age and the type of foster home.

I define children and home-types (used to specify the taste variation terms) as follows. The set of child-types, X, contains two elements differentiating children who are younger, or older, than 8 years old. The set of home-types, Y, includes one type for each type of foster home other than relatives. It is not necessary to define a home-type for relative foster homes since all of them are in singleton markets.

The dataset used in the estimation contains 1,467 markets and 2,358 assigned placements. This specification of the model has 39 parameters.

Parameter Estimates

In this section, I discuss the simulated maximum likelihood estimates of the model parameters. Table 1.2 presents the parameter estimates of the outcome distribution, $\hat{\boldsymbol{\Sigma}}_\omega$ and $\hat{\boldsymbol{\theta}}_T$. The first two rows of the table present the estimated covariance matrix of $\boldsymbol{\omega}$. The estimated variance of ω_d is higher than that of ω_{ex}, implying that the variance not captured by the covariates is higher for disruption than for exit. The model also captures a positive correlation between both hazard rates: placements which the matchmaker considers as having a higher hazard for disruption, are also considered as having a higher hazard for exiting the system.

The next rows of Table 1.2 report the estimated coefficients of each of the covariates in $g(\mathbf{x}, \mathbf{y})$ for each hazard rate. A larger coefficient of (say) age on the disruption hazard implies that placements with older children are more likely to be disrupted (and sooner) than placements with younger children. The coefficients indicate that older children have higher disruption hazards in all types of foster homes, other than group homes. By contrast, age is found to have a minor effect in the hazard for exiting to permanency in foster homes other than group homes.

Table 1.3 reports average partial effects of placement characteristics on placement outcomes. Partial effects are computed for every placement assigned in the data

Table 1.2: Estimated parameters of outcome distribution ($\boldsymbol{\Sigma}_{\omega}, \boldsymbol{\theta}_T$)

	(1) Disruption	(2) Exit
Var(ω_R)	0.873***	0.02955
	(0.2912)	(0.02867)
Cov(ω_d, ω_{ex})	0.1573*	0.1573*
	(0.08908)	(0.08908)
Age At Placement	0.09872***	-0.01615
	(0.01767)	(0.01047)
County-FH	2.217***	-0.02375
	(0.332)	(0.2101)
Agency-FH	2.983***	0.4547***
	(0.2556)	(0.1237)
Group Home	-2.077**	-1.987***
	(0.9188)	(0.5642)
Age At Plac. × *County-FH*	-0.02272	0.01804
	(0.0261)	(0.01636)
Age At Plac. × *Agency-FH*	-0.07878***	-0.01007
	(0.0194)	(0.0124)
Age At Plac. × *Group Home*	0.2569***	0.1419***
	(0.06179)	(0.03894)
Distance To School (zip)	0.02052***	-0.006059***
	(0.002471)	(0.001724)
Missing Dist. To School	0.9007***	0.1222
	(0.1603)	(0.08942)
Constant	-8.996***	-6.082***
	(0.5408)	(0.2132)
Alpha (α_R)	1.091***	0.9665***
	(0.07551)	(0.03427)
Gamma (γ_R)	0.9527***	0.2222
	(0.1183)	(0.2361)
Number of markets (n)	1467	
$SMLL$	-17005.86	

Note: Estimated parameters of unobserved heterogeneity ($\boldsymbol{\Sigma}_{\omega}$) and conditional hazard rates ($\boldsymbol{\theta}_T$). Standard errors in parenthesis. Significance level of parameters: ***p<0.01, **p<0.05, *p<0.01.

using expressions (1.16) and (1.17). Here, one can see that, on average, placements with older children are more likely to be disrupted. The marginal effect of one year of age on the disruption probability is, on average, 1.4%. Also, placements with older children are more likely to be disrupted sooner when they do so. Indeed, placements with older children tend to have lower durations overall, regardless of the termination reason. Placements with relatives are more stable, they have lower disruption probabilities than every other type of foster home. They also last less than every other type of placement except for group homes. Placements in county and agency foster homes have similar expected outcomes. Both of them are around 30% more likely to be disrupted than placements with relatives. The distance between a foster home and the child's school increases the odds of disruption and overall diminishes a placement's expected duration.

Table 1.3: Average partial effects (APEs)

	(1) $\mathbb{P}$(Disrup)	(2) $\mathbb{P}$(Exit)	(4) $\mathbb{E}(\log T \mid \text{Disrup})$	(5) $\mathbb{E}(\log T \mid \text{Exit})$	(6) $\mathbb{E}(\log T)$
Age At Placement	0.01393	-0.01146	-0.04059	-0.0218	-0.04014
County-FH	0.3168	-0.2661	-0.9689	-0.6275	-0.9266
Agency-FH	0.32	-0.2716	-1.221	-0.8743	-1.174
Group Home	0.1652	-0.1575	0.2872	0.4496	0.3393
Distance To School (zip)	0.004013	-0.003757	-0.007978	-0.003091	-0.007359
Missing Dist. To School	0.1136	-0.09686	-0.5244	-0.3653	-0.5212
Number of placements			2358		

Note: Average partial effects of placement characteristics on expected outcomes. Averages taken across the sample of assigned placements in the data. The partial effects with respect to continuous variables is taken by considering a marginal change of one unit.

Table 1.4: Goodness of fit and estimation parameters

	(1) Predicted	(2) Sample
$\mathbb{P}$*(Disruption)*	0.514	0.5093
$\mathbb{P}$*(Exit)*	0.4303	0.4237
$\mathbb{P}$*(Emanc/Cens)*	0.05568	0.06701
$\mathbb{E}(\log T \mid \textit{Disruption})$	4.482	4.141
$\mathbb{E}(\log T \mid \textit{Exit})$	4.721	4.994
$\mathbb{E}(\log T \mid \textit{Emanc/Cens})$	7.19	5.534
$\mathbb{E}(\log T)$	4.615	4.596
Number of markets (n)	1467	
Number of assigned placements	2358	
Number of prospective placements	8900	
$SMLL$	-17005.86	
S_ω	50	
S_υ	50	
ζ	1e-01	
dim($\boldsymbol{\theta}$)	39	

Note: Average predicted outcomes and sample average outcomes. Averages taken across the sample of assigned placements in the data. The number of assigned placements in the data is equal to $\sum_i \sum_{c,h} M_i(c,h)$. The number of prospective placements is equal to $\sum_i \sum_{c,h} |C_i| \times |H_i|$. $SMLL$ gives the value of the simulated log-likelihood at the estimated vector of parameters. S_ω, S_υ, and ψ are the parameters of the simulated log-likelihood; dim($\boldsymbol{\theta}$) refers to the number of parameters estimated.

Table 1.4 reports goodness of fit measures and the parameters used in the estimation. Overall, the model does good job on matching the average outcomes observed in the data. Note that when computing average expected outcomes for goodness of fit purposes, one must take into account censored placements (those for which the outcome is not observable due to the sample period). This is done by replacing T_{em} for $\min\{T_{em}, T_{cen}\}$ in expressions (1.16) and (1.17). Also, note that the (average) expected log-duration conditional on emancipation/censoring predicted by the model is much higher than the emancipation/censoring times observed in the data. This reflects that the placements that are more likely to be emancipated or censored are precisely the ones that have lower times to emancipation or are closer to the end of the sample period.

Table 1.5 reports the estimated parameters of the matchmaker's utility function. Overall, the matchmaker has a higher payoff from placements that exit to permanency. The least desirable termination reason is disruption. The marginal utility of duration is negative, regardless of termination reason. The magnitude of the parameters show that the matchmaker is *not* willing to trade-off a placement exiting to permanency for it being disrupted, regardless of the time to reach permanency and the time spent in a disruptive placement. To see this, note that if a placement is to be disrupted, the matchmaker prefers for it to be disrupted as soon as possible. However, even if a placement is disrupted right away, $T = 1$, the payoff to the matchmaker is lower than if the child exits to permanency, regardless of the time the child needs to wait before exiting.

The marginal utility of the time to emancipation is positive conditional on duration, but negative conditional on exiting to permanency. This captures that the valuation of the matchmaker for age differs depending on the termination reason. An interpretation of this preference is that the time to disruption and permanency (i.e., the time that it takes for a placement to be disrupted or exit to permanency, conditional on that being its termination reason) is valued differently depending on the age of children. The sign of the coefficients indicate that the matchmaker's preference against children spending time in placements that will be disrupted is stronger for younger children than for older ones. By contrast, the matchmaker's tolerance for children waiting to exit to permanency is higher for younger children than for older ones. The magnitude of the coefficients allow to compute the marginal rate of substitution between duration and age. For instance, consider a child of average age, 8.7 years old, who is in a placement known to be disrupted. Set the disruption time

Table 1.5: Estimated parameters of matching utility ($\boldsymbol{\theta}_M$)

	(1) Disruption	(2) Exit	(3) Emancipation
μ_R — *MgU. Term. Reason*	-2.908***	2.449**	-2.057***
	(0.6972)	(1.091)	(0.7183)
φ_R — *MgU. Duration*	-0.3549***	-0.5265***	0†
	(0.1005)	(0.167)	(0)
$\bar{\varphi}_R$ — *MgU. Emanc. Time*	0.3093***	-0.1179	0.009985
	(0.06172)	(0.09607)	(0.01364)
Number of markets (n)		1467	
$SMLL$		-17005.86	

Note: Estimated parameters of matching utility function ($\boldsymbol{\theta}_M$), where $u = \mu_R + \varphi_R \log T + \bar{\varphi}_R \log T_{em}$. Standard errors in parenthesis. Significance level of parameters: ***p<0.01, **p<0.05, *p<0.01. † indicates fixed parameter (i.e., not estimated).

at its conditional average, 5.4 months (165 days). A placement that is know to be disrupted, but has a child who is younger by one year, generates a higher payoff for the matchmaker as long as its duration is less than 5.9 months (180 days), 9.31% more. If the placement is known to be terminated because the child will exit to permanency, the opposite obtains. Again, consider a placement with a child who is 8.7 years old and, who is known, will exit to permanency in the average time, 10 months (304 days). A placement with a child who is also known will exit to permanency, but who is one year *older*, generates a higher payoff to the matchmaker, as long as the child exits to permanency in no more than 10.2 months (312 days), 2.6% more.

Table 1.6 reports the estimated covariance matrices of the taste variation terms. Overall, the estimates show no significance variance in the taste variation parameters. Intuitively, this reflects that, given the current specification, the expected outcomes of placements seem to be sufficient in order to predict placement assignments.[17]

1.7 Counterfactual Exercises

Counterfactual I: Market Thickness

In this section, I analyze the effect of policies aimed at improving outcomes by increasing market thickness. Market thickness may be increased along two dimensions. First, I consider the case in which placements are assigned every $D > 1$ days, instead of daily as is done in the field ($D = 1$). I consider policies with $D \leq 15$. Second, I consider the case in which non-relative placements are assigned

[17] A caveat of the estimates in Table 1.6 is that the normalizations implemented in this specification do not correspond to the ones in Assumption 4, which is key in proving Proposition 1. The estimates in Table 1.6 may be close to zero because the normalizations are not doing a good job in identifying the parameters.

Table 1.6: Estimates of the covariance matrix of the taste variation shocks ($\boldsymbol{\Sigma}$)

$$\hat{\boldsymbol{\Sigma}}_{\varepsilon} = \begin{pmatrix} 0\dagger & 0\dagger & 0\dagger \\ (0) & (0) & (0) \\ 0\dagger & 0.0002013 & -0.001219 \\ (0) & (0.0009768) & (0.003017) \\ 0\dagger & -0.001219 & 0.01181 \\ (0) & (0.003017) & (0.01172) \end{pmatrix}_{|Y|\times|Y|} \qquad \hat{\boldsymbol{\Sigma}}_{\eta} = \begin{pmatrix} 0\dagger & 0\dagger \\ (0) & (0) \\ 0\dagger & 0.0001188 \\ (0) & (0.000899) \end{pmatrix}_{|X|\times|X|}$$

Note: Estimated parameters of the covariance matrices of taste variation shock of children over home types, $\varepsilon_c = (\varepsilon_{cy})_{y\in Y} \sim N(0, \boldsymbol{\Sigma}_{\varepsilon})$, and of the covariance matrix of the taste variation shock of homes over children types, $\eta_h = (\eta_{xh})_{x\in X} \sim N(0, \boldsymbol{\Sigma}_{\eta})$. Standard errors in parenthesis. Significance level of parameters: ***p<0.01, **p<0.05, *p<0.01. † indicates fixed parameter (i.e., not estimated).

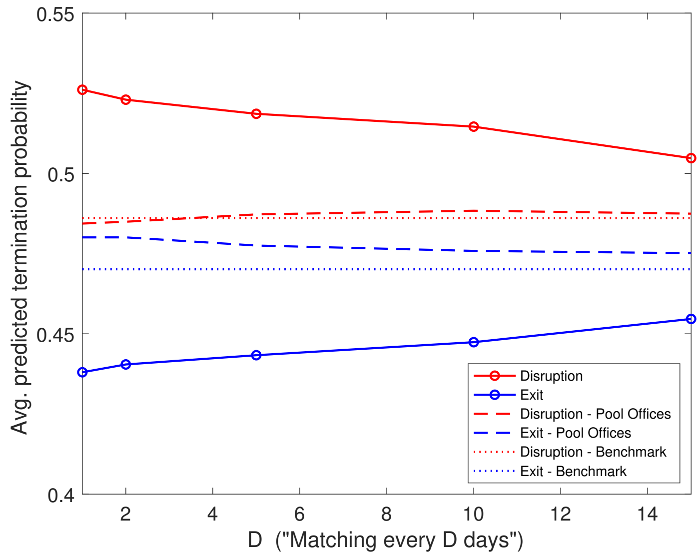
across all regional offices together, instead of within them as is done in the field. I also consider the two types of policies together, i.e., assigning placements every $D > 1$ days and pooling the children and foster homes from all regional offices into a county-wide market.

By design, the aggregate payoff of the matchmaker is higher when markets are thicker. The reason is because the original matching is always feasible when the market is thicker. The effect on the expected outcomes of placements is controlled by the matchmaker's payoff function, which determines which placements are assigned in the counterfactual markets.

Figure 1.1 plots the average predicted termination probabilities across the counterfactual markets. The value of D is plotted in the x-axis. The solid lines correspond to the termination probabilities in the case in which markets are formed within offices. The dashed lines to the case in which markets are pooled across regional offices. The plots also include a dotted line, which is constant across D. The dotted line corresponds to the "benchmark" case in which all placements are assigned at once, $D = \infty$, and regional offices are pooled together. The average predicted outcomes in the benchmark case correspond to the ones of the best placements (from the matchmaker's perspective) that can be formed in the full dataset.

The baseline values of the termination probabilities are the values at $D = 1$, i.e., these values correspond to the predicted probabilities of the model with the assignment observed in the data. From the top panel of Figure 1.1, one can see that in thicker markets the average disruption probability is lower, and that of exiting to permanency or disruption is higher. When the pools of available children and foster homes are larger, the matchmaker is able to assign placements with lower disruption probabilities. However, note that the gains from thicker markets come almost exclu-

Figure 1.1: Counterfactual I: Average predicted termination probabilities



Note: Plot of the average predicted termination probabilities. Averages taken across all assigned placements in each counterfactual. The x-axis plots the value of D, the number of non-matching days in a matching period. Solid lines correspond to counterfactuals in which markets are defined by regional offices; dashed lines to ones in which regional offices are pooled together into the same markets. The benchmark case (dotted line) corresponds to the case in which regional offices are pooled together into markets and $D = \infty$.

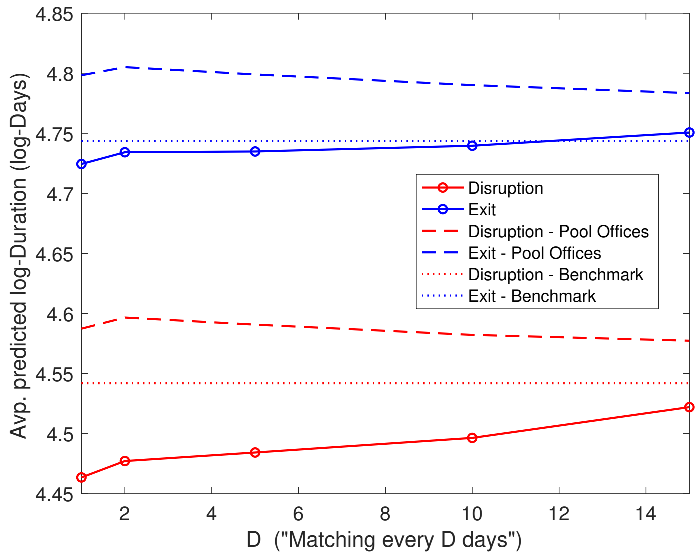
sively from pooling regional offices together. When matchings are assigned daily ($D = 1$) but regional offices are pooled together, the disruption rates diminishes from 52.61% to 48.43%. In terms of expected number of placements per child, this is equivalent from going from 2.11 to 1.94.[18]

Figure 1.2 is analogous to Figure 1.1, but it plots the average predicted conditional durations of placements. Here, one can see that the average duration of placements may be higher or lower than the baseline in thicker markets. Interestingly, when offices are pooled together and placements are assigned daily, the matchmaker assigns placements with higher expected durations than both the baseline and benchmark cases. The reason is because the matchmaker is willing to trade-off duration (which it dislikes) with better termination probabilities. The same can be seen in Figure 1.3, which plots the average expected duration.

The top panel of Figure 1.4 plots the average distance to school across placements in thicker markets. The average distance between foster homes and children's schools is cut in 54% when offices are pooled together into county wide-markets. The

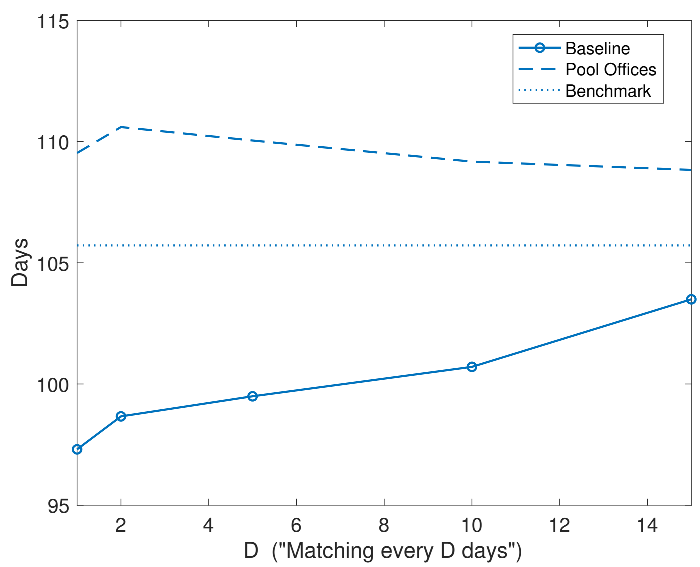
[18]The average disruption probability can be seen as the probability of a "failure" in a series of discrete dichotomic random draws. In this case, the number of placements per child follows a geometric distribution ("number of trials needed to get one success"). If p_d denotes the disruption probability, the expected number of placements per child is $1/(1 - p_d)$.

Figure 1.2: Counterfactual I: Average predicted conditional expected duration



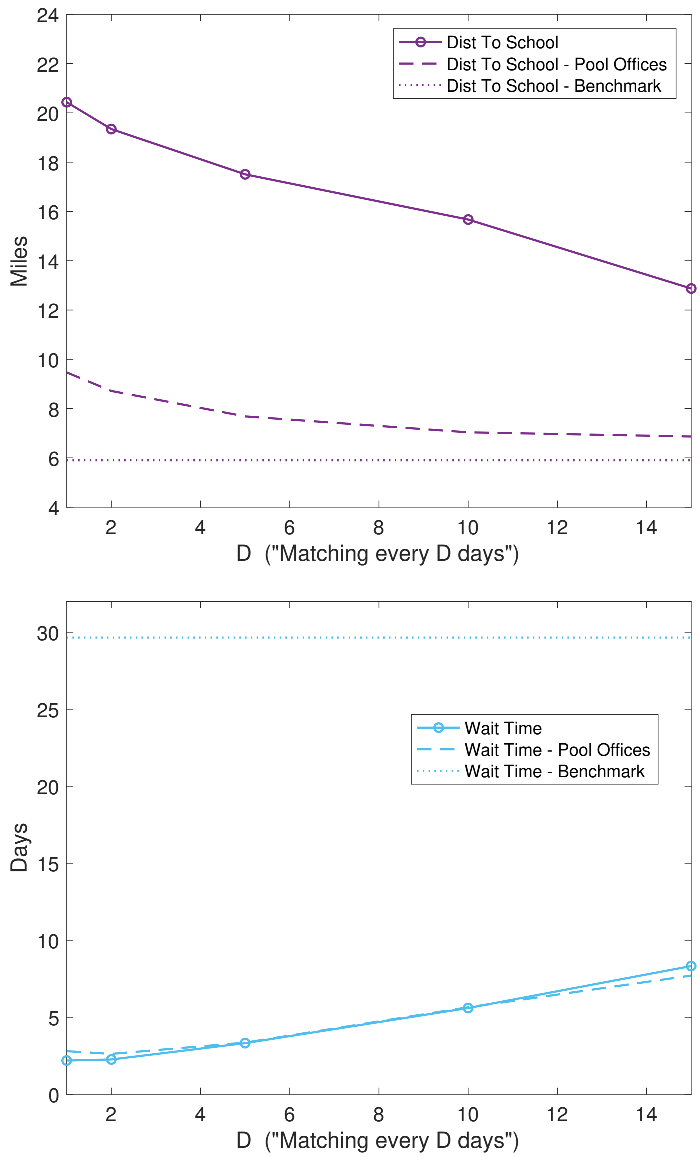
Note: Plot of the average predicted conditional expected durations. Averages taken across all assigned placements in each counterfactual. The x-axis plots the value of D, the number of non-matching days in a matching period. Solid lines correspond to counterfactuals in which markets are defined by regional offices; dashed lines to ones in which regional offices are pooled together into the same markets. The benchmark case (dotted line) corresponds to the case in which regional offices are pooled together into markets and $D = \infty$.

Figure 1.3: Counterfactual I: Average predicted expected duration



Note: Plot of the average predicted expected duration. Averages taken across all assigned placements in each counterfactual. The x-axis plots the value of D, the number of non-matching days in a matching period. Solid lines correspond to counterfactuals in which markets are defined by regional offices; dashed lines to ones in which regional offices are pooled together into the same markets. The benchmark case (dotted line) corresponds to the case in which regional offices are pooled together into markets and $D = \infty$.

Figure 1.4: Counterfactual I: Average distance to school and waiting time



Note: Plots of the average distance to school and waiting time. Averages taken across all assigned placements in each counterfactual. The x-axis plots the value of D, the number of non-matching days in a matching period. Solid lines correspond to counterfactuals in which markets are defined by regional offices; dashed lines to ones in which regional offices are pooled together into the same markets. The benchmark case (dotted line) corresponds to the case in which regional offices are pooled together into markets and $D = \infty$.

average distance goes from 20.43 to 9.5 miles. From the plot, one can see that the gains resulting from lower disruption probabilities follows the same patters as the distance to school: the gains from pooling offices together outweighs those obtained from delaying placement assignments. Finally, the bottom plot of Figure 1.4 shows the average time that children wait before being assigned placements. As expected, delaying placements increases this figure monotonically.

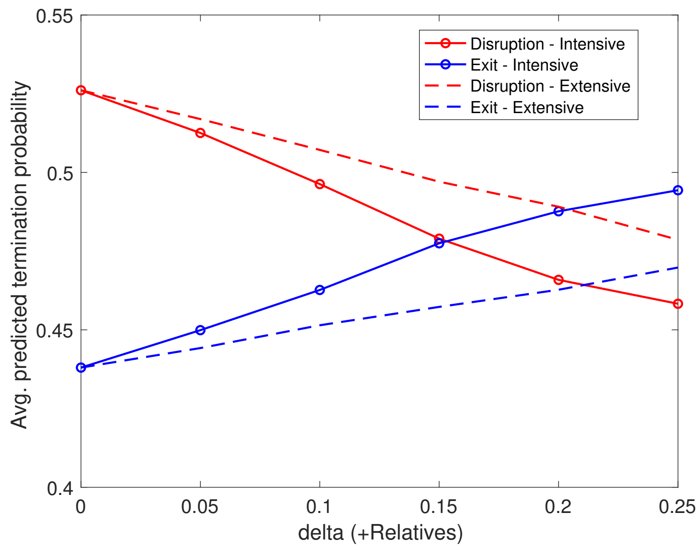
Counterfactual IIa: Relative Foster Homes

In this section, I analyze the effect that relatives have on average expected outcomes. Specifically, I consider an increase in the share of the foster homes that are relatives across all markets. I analyze both an increase in the intensive and extensive margins. Let $\delta_{rel} \in (0, 1)$ be the increase in the share of foster homes that are relatives. I consider policies with $\delta_{rel} \leq 0.25$; $\delta_{rel} = 0$ corresponds to the baseline case, in which the supply of foster homes is the same as the one observed in the data.

I increase the share of relative homes as follows. First, I estimate a binary logit model that predicts whether a child has a relative home or not, as a function of its characteristics. Let $n^*_{rel} = \lfloor \delta_{rel} * n_{rel} \rfloor$, where n_{rel} denotes the number of placements with relatives in the data. Then, from the population of non-relative placements in the data, I select n^*_{rel} at random, weighing them by the predicted probability that each of them had a relative available. That is, I select children who did not had a relative placement, but had a higher likelihood of having it, with higher probability. In the case of the intensive margin, I convert the foster homes of the selected placements into relative homes (leaving all other placement characteristics fixed), and assign them to new singleton markets with the corresponding child. In the extensive margin case, I create a duplicate of the foster homes of the selected children. Then, I convert the duplicated home to a relative home (leaving all the other placement characteristics fixed), and assign it with the corresponding child to a new singleton market. The difference between the intensive and extensive margins is that the set of available foster homes for the rest of the children in the market remains unchanged in the extensive margin, while it is reduced by one home in the intensive margin.

Figure 1.5 reports the predicted average termination probabilities in the distinct counterfactuals. The parameter δ_{rel} is on the x-axis. One can observe that a higher share of relative homes, in both the intensive and extensive margins, has a sizable effect on termination probabilities. Overall, the disruption probability diminishes and the one of exiting to permanency increases. The adjustment is more gradual in the extensive margin. In the intensive margin, the disruption probability goes from 52.6% at $\delta_{rel} = 0$ to 45.82% at $\delta_{rel} = 0.25$ (equivalent to going from an average of 2.1 placement per child to 1.84). In the extensive margin, the change is from 52.6% to 47.8% (equivalent to going from an average of 2.1 placement per child to 1.91). The difference between both margins has to do with how the rest of the children are being placed in the non-relative placements. Figure 1.6 shows the analogous plot

Figure 1.5: Counterfactual IIa-Relatives: Average predicted termination probabilities



Note: Plot of the average predicted termination probabilities. Averages taken across all assigned placements in each counterfactual. The x-axis plots the value of δ_{rel}, the factor by which the supply of Relative Foster Homes is adjusted. Solid lines correspond to counterfactuals in which the supply of Relative Foster Homes is increased in the intensive margin; dashed lines to ones in which the supply of Relative Foster Homes is increased in the extensive margin.

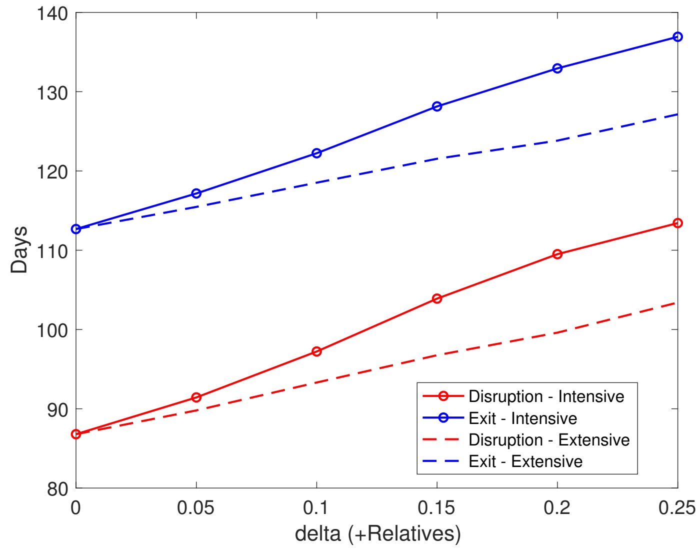
for conditional durations. Overall, placements tend to last longer when the share of relative foster homes in the system is increased.

Counterfactual IIb: Agency Foster Homes

In this section, I analyze the effect that agency foster homes have on average expected outcomes. Specifically, I consider an increase in the share of foster homes that come through non-profit agencies across all markets. I analyze an increase in both the intensive and extensive margins. Let $\delta_{ah} \in (0,1)$ be the increase in the share of foster homes that are agency homes. I consider policies with $\delta_{ah} \leq 0.25$; $\delta_{ah} = 0$ corresponds to the baseline case, in which the supply of foster homes is the same as the one observed in the data.

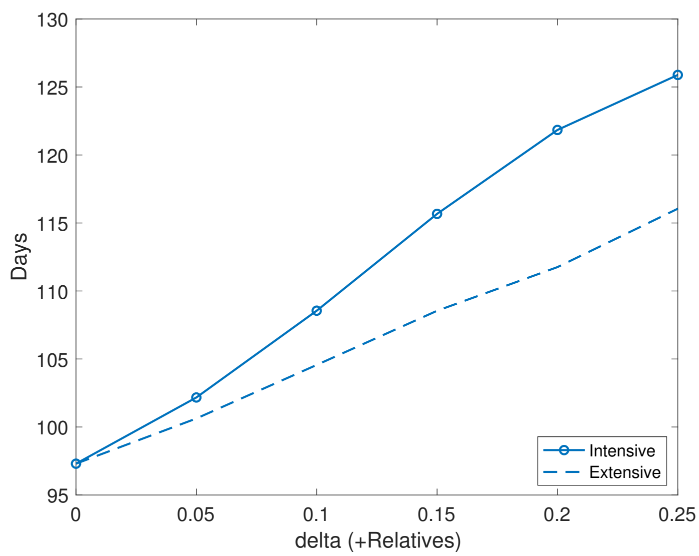
I increase the share of agency homes as follows. Let $n^*_{ah} = \lfloor \delta_{ah} * n_{ah} \rfloor$, where n_{ah} denotes the number of placements with agency homes in the data. Then, from the population of non-agency placements in the data, I select n^*_{ah} uniformly at random, keeping the relative share of the other types of non-agency placements fixed. In the case of the intensive margin, I convert the foster homes of the selected placements into agency homes (leaving all other placement characteristics fixed). In the extensive margin case, I create a duplicate of the foster homes of the selected

Figure 1.6: Counterfactual IIa-Relatives: Average predicted conditional expected duration



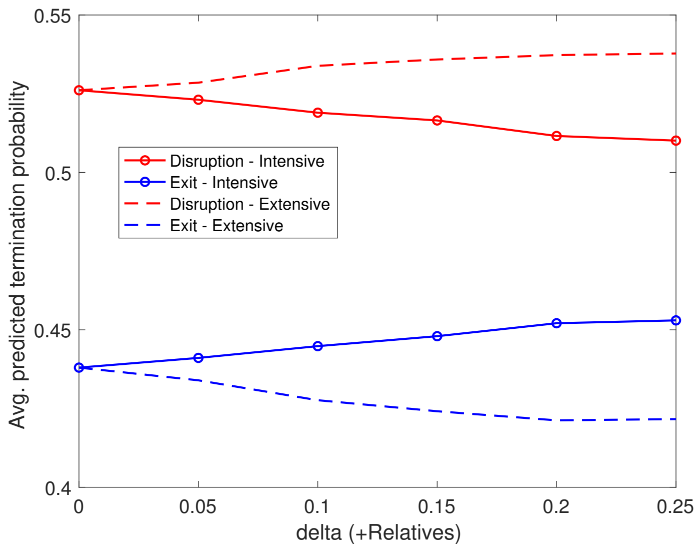
Note: Plots of the average predicted conditional expected durations. Averages taken across all assigned placements in each counterfactual. The x-axis plots the value of δ, the factor by which the supply of Relative Foster Homes is adjusted. Solid lines correspond to counterfactuals in which the supply of Relative Foster Homes is increased in the intensive margin; dashed lines to ones in which the supply of Relative Foster Homes is increased in the extensive margin.

Figure 1.7: Counterfactual IIa-Relatives: Average predicted expected duration



Note: Plot of the average predicted expected duration. Averages taken across all assigned placements in each counterfactual. The x-axis plots the value of δ, the factor by which the supply of Relative Foster Homes is adjusted. Solid lines correspond to counterfactuals in which the supply of Relative Foster Homes is increased in the intensive margin; dashed lines to ones in which the supply of Relative Foster Homes is increased in the extensive margin.

Figure 1.8: Counterfactual IIb-Agency-FH: Average predicted termination probabilities

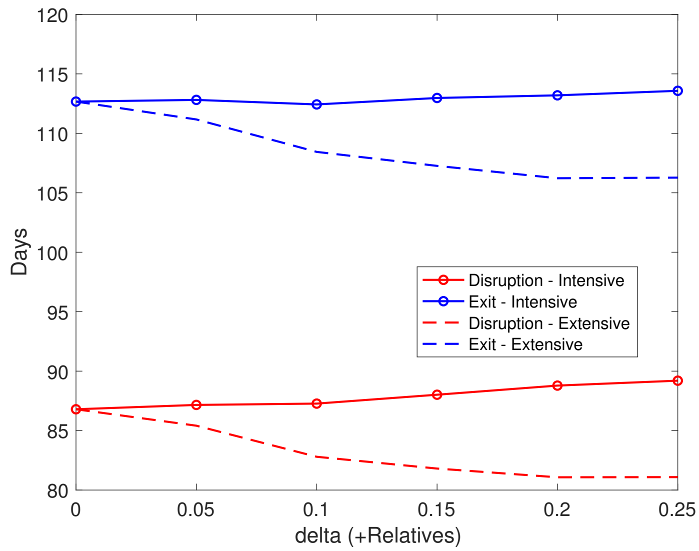


Note: Plot of the average predicted termination probabilities. Averages taken across all assigned placements in each counterfactual. The x-axis plots the value of δ, the factor by which the supply of Agency Foster Homes is adjusted. Solid lines correspond to counterfactuals in which the supply of Relative Foster Homes is increased in the intensive margin; dashed lines to ones in which the supply of Agency Foster Homes is increased in the extensive margin.

children. Then, I convert each duplicated home to a agency home (leaving all the other placement characteristics fixed). The difference between the intensive and extensive margins is that the set of available foster homes in the market has the same number of homes in the intensive margin (with one converted into an agency home), and in the extensive margin it has an extra agency home.

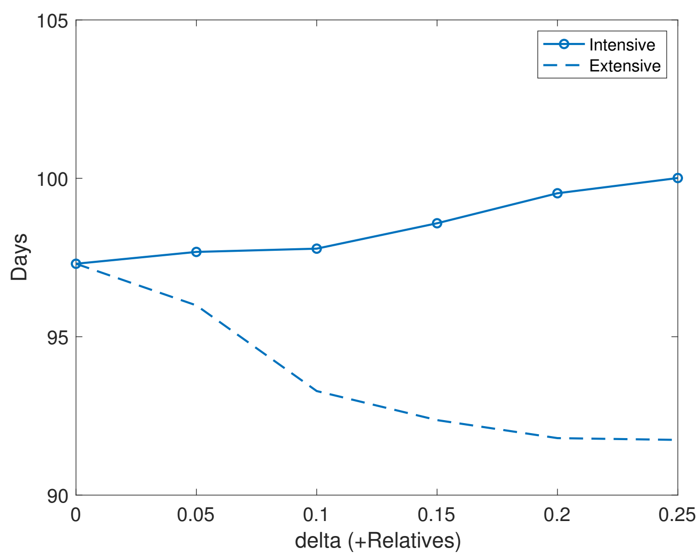
Figure 1.8 reports the predicted average termination probabilities in the distinct counterfactuals. The parameter δ_{ah} is on the x-axis. One can observe that a higher share of agency homes, in both the intensive and extensive margins, has a minor effect on termination probabilities. Interestingly, the effects point in opposite directions in the intensive and the extensive margins. In the intensive margin, the disruption probability diminishes and the one of exiting to permanency increases. The disruption probability goes from 52.6% at $\delta_{ah} = 0$ to 51% at $\delta_{ah} = 0.25$ (equivalent to going from an average of 2.1 placement per child to 2). In the extensive margin, the average disruption probability increases, it goes from 52.6% to 53.8% (equivalent to going from an average of 2.1 placement per child to 2.2). Figure 1.6 shows the analogous plots for conditional durations.

Figure 1.9: Counterfactual IIb-Agency-FH: Average predicted conditional expected duration



Note: Plots of the average predicted conditional expected durations. Averages taken across all assigned placements in each counterfactual. The x-axis plots the value of δ, the factor by which the supply of Relative Foster Homes is adjusted. Solid lines correspond to counterfactuals in which the supply of Agency Foster Homes is increased in the intensive margin; dashed lines to ones in which the supply of Agency Foster Homes is increased in the extensive margin.

Figure 1.10: Counterfactual IIb-Agency-FH: Average predicted expected duration



Note: Plot of the average predicted expected duration. Averages taken across all assigned placements in each counterfactual. The x-axis plots the value of δ, the factor by which the supply of Agency Foster Homes is adjusted. Solid lines correspond to counterfactuals in which the supply of Agency Foster Homes is increased in the intensive margin; dashed lines to ones in which the supply of Relative Foster Homes is increased in the extensive margin.

1.8 Conclusion

This chapter presents a framework to study how placements are assigned in foster care. The model aims to capture how social workers assign placements in the field. The model incorporates key institutional features of placement assignment in foster care: (1) children need to be placed with relatives whenever possible; (2) social workers need to prioritize the location of prospective foster homes in relation to the children's schools, and (3) social workers have discretion in how to weigh all the factors that contribute to successful placements.

A key aspect of the model is that it incorporates the endogeneity arising from placement assignments being affected by unobservables correlated with outcomes. The main identification strategy is to rely on the exogenous variation across the dates and geographic regions at which children enter foster care. The empirical exercise uses a novel dataset of confidential foster care record from Los Angeles County, California. The parameter estimates of the model show that expected outcomes are significant factors when assigning placements. Overall, social workers assign the placements that are less likely to be disrupted and in which it is more likely that the children exit to permanency. Another key variable when determining assignments is the conditional expected duration of prospective placements. Social workers aim to assign placements that, conditional on their termination reason, will have the lowest possible durations.

Through counterfactual exercises, I show the effect of market thickness and the presence of different types of foster homes on the distribution of outcomes. A key contribution of this chapter is to quantify the gains, in terms of better placement outcomes, resulting from thicker markets in foster care. It is shown that the gains due to market thickness are greater when thickness is increased geographically (by assigning placements throughout the county) than time-wise (by delaying placements). Specifically, the model predicts that if placements were assigned in county-wide markets, the expected number of placements children would experience in foster care would diminish by 8%, and the average distance from foster homes to children's schools would be reduces by 54%.

Often, the allocation of resources is the result of individual choices made within exogenously-designed institutions. The findings of this chapter support the view that social workers have a good understanding of which placements are less likely to be disrupted and seem to do a good job when it comes to assigning them. They assign the placements that are more likely to work. However, at the system level, the model shows that the current state of the system does not facilitate the coordination between the distinct regional offices. The empirical analysis shows that by being better at coordinating with one another, regional offices would be able to assign better placements for children and foster parents.

Chapter 2

MATCHING IN FOSTER CARE: A DYNAMIC AND CENTRALIZED APPROACH

2.1 Introduction

The main objective of the foster care (FC) system is to find suitable placements for children who have been temporarily removed from home by child-protective services. The main focus of this chapter is to study the determinants of the duration of assignments (matches) in the FC system. All matches are temporary because all children in FC *exit* the system eventually. They are either reunited with their biological parents, adopted, or emancipated (exit the system when turning 18 or 21 years old, depending on the state). Furthermore, foster care placements may also be *unsuccessful* and terminate before children exit the system.[1] In such cases, children need to be rematched with new foster families.

The assignment of children to foster families gives rise to a matching market. The demand side is comprised by children in need of placements and the supply side by foster families or, more generally, placement providers. The main difficulty of studying the duration of matches is that they are not assigned randomly. County social workers are responsible for finding and assigning placements for children in FC. Furthermore, the pool of children who need a placement and the one of available foster homes evolves dynamically over time. Importantly, the stochastic process governing these dynamics is shaped by the matching policy used by county officials. That is, if "bad" matches are assigned (ie., ones that will be disrupted), then more children will need a placement in the future. The main goal of this chapter is to model the matching process dynamically. Given the lack of data on the exact matching procedure, I model the observed matching as if it were generated by minimizing the number of children who remain unmatched in any given period.

[1]In reality, a high fraction of children go through unsuccessful matches. In 2013, the average number of placements during the last removal was 2.8 across all children who where in FC in the United States. A match may be terminated prematurely by the foster family, a county social worker, the involved private agency (if any), or simply because the child runs away.

First, I propose a general model of centralized one-to-one matching in a dynamic environment.[2] In every period, a single matchmaker decides matches among unmatched children and available families. The duration of each match is random, and its distribution depends on characteristics of both the child and the family. A match may terminate for one of two reasons: the child exits the system or the match is unsuccessful, in which case the child needs to be rematched. At the start of every period, the set of children in the market is comprised by new arrivals, unmatched children from previous periods, and those coming from unsuccessful matches. For simplicity, I assume that unmatched parents exit the market in every period. The matchmaker lives infinitely and seeks to minimize the expected discounted sum of the children left unmatched in every period. The setting gives rise to a dynamic programming problem. When deciding which children to match and to which families, the matchmaker must take into account the expected duration of the prospective matches and, more importantly, their probability of being unsuccessful before the children exit the system.

Second, I conduct a series of analytic and computational exercises. I consider a specification of the general model that highlights the tension between matching younger children with lower expected durations of matches and older children who have higher expected duration of matches, but are closer to the emancipation age. The parameters of the model are such that there is a scarcity of foster parents in the market. The difference between the two types of exercises is the time horizon. Given the complexity of the general model, the analytic exercises consider a finite horizon of at most three periods. By contrast, in the computational exercise, I solve for the matchmaker's optimal matching policy by iterating over the Bellman equation. I describe the optimal matching policy for different specifications of the model's parameters. The main finding is that the optimal matching policy gives priority to younger children in terms of the likelihood of getting matched and the *quality* of the match in terms of its expected duration. The model captures different trade-offs between age and heterogeneity in the expected duration of matches. Moreover, solving the model in infinite horizon allows me to compute its stationary state. I report the expectation of several variables of interest in the stationary state for different specifications of the model's parameters.

[2]In reality, a family may take care of more than one child, so the matching may be many-to-one. However, there is no available data of the number of children cared by the same family, so I restrict the analysis to one-to-one matching.

Third, I present an empirical description of the FC system in Los Angeles County, California. The main data source is the Adoption and Foster Care Analysis & Report System (AFCARS) of the U.S. Children's Bureau. I describe the dataset and present summary statistics of relevant variables. I also describe sorting patterns in the FC in terms of sex, age, and race, and analyze the correlation between the duration of matches and several children characteristics. The key aspects of the data are related to its longitudinal characteristics. Even though the AFCARS databases are cross-sections, they include information and dates of the history of every child in FC, such as the number of removals from home and number of settings during the last or current removal.

The chapter is organized as follows. I review the related literature in what remains of the introduction. I present a general model of centralized matching in a dynamic environment in Section 2.2. The analytic and computational exercises of the model are in Section 2.3. Section 2.4 presents a description of the data, and Section 2.5 concludes by addressing the future challenges and objectives of this research agenda.

Related Literature.—The economic analysis of the foster care system is mostly absent in the economics literature. For some exceptions, see Doyle Jr. (2007, 2008) and Doyle Jr. and Peters (2007), who analyze social workers' removal decisions using a treatment effects framework. The only paper that has analyzed a market design aspect of the foster care is Slaugh, Akan, Kesten, and Ünver (2016). Their main focus is on children who get adopted from foster care. In particular, they analyze an existing government program in Pennsylvania which main objective is to find adoptive families for children in foster care. A related study on adoption markets is Baccara, Collard-Wexler, Felli, and Yariv (2014). The theoretical portion of this chapter is related to the literature on dynamic matching. The approach is novel in that it considers a matching market with reversible matches. In contrast with much of the existing literature, the main focus is on the determinants of the duration of matches in a market in which all matches are temporary. Furthermore, the approach is fully centralized in the sense that a single matchmaker decides which matches are formed. Individuals do not search or otherwise decide for themselves. Hence, I do not impose incentive compatibility or individual rationality constraints. For examples that focus on distinct aspects of dynamic matching markets, such as the relation between preferences, stability, efficiency and sorting patterns, see Baccara, Lee, and Yariv (forthcoming), Doval (2018), and Ünver (2010).

The empirical part of the chapter is related to the literature on structural estimation of matching markets. For examples of the marriage market and transferable utility matching markets in general, see Choo and Siow (2006), Fox (2018), and Galichon and Salanié (2015). For treatments with non-transferable utility and strategic considerations, see Agarwal (2015) and Agarwal and Somaini (2018). More generally, the empirical strategy that this research aims to use falls in the wide empirical literature on dynamic discrete choice models. For notable contributions, see Hotz and Miller (1993), Magnac and Thesmar (2002), Pakes (1986), Rust (1987, 1994), and Wolpin (1984), and the references therein.

2.2 Model

The FC system lasts for $t = 0, 1, \ldots, T \leq \infty$ periods. The sequence of every period t is as follows:

1. Children of age are emancipated.
2. The sets of available children and parents to be matched, and the set of surviving matches from previous periods, are (randomly) determined.
3. The matchmaker decides matches among the available children and parents.
4. All children's ages increase by one.

Specifically, the set of available children in period t consists of the children who remained unmatched in the previous period, new arrivals, and children from broken matches, of which the last two are randomly determined. The set of available parents is composed solely by new arrivals, which are random. I assume that parents who remain unmatched at the end of a period and those from broken matches exit the system permanently. Every existing match at the end of period $t - 1$ may exit the system in period t with some probability, e.g., because the child is adopted or reunited with her biological parents. The matches who do not exit the system may break up, in which case the child returns to the market to be rematched, or remain unbroken and survive period t. Exits and breakups are randomly determined. The goal of the matchmaker is to minimize the amount of children who remain unmatched in every period.

Sketch of the Model

Suppose all parents and children are otherwise homogeneous. The model has three state variables:

1. m^t = number of matches from previous periods that survive period t;
2. c^t = number of available children to be matched in period t;
3. p^t = number of available parents to be matched in period t.

The matchmaker decides a^t, the number of matches formed in period t. Her choice set is given by:

$$a^t \in \Phi\left(c^t, p^t\right) \equiv \min\left\{c^t, p^t\right\}. \tag{2.1}$$

That is, the matchmaker may form as many matches as there are available children or parents. In the next period, each existing match exits the system with probability $e \in (0, 1)$, and those who do not exit, are broken with probability $b \in (0, 1)$. New arrivals of children follow a binomial distribution with parameters (n^c, μ), and those of parents with (n^p, λ). Thus, the transition of the system is governed by:

$$p^{t+1} = \mathrm{B}(n^p, \lambda) \tag{2.2}$$

$$m^{t+1} = \left(m^t + a^t\right) - e^{t+1} - \mathrm{B}\Big(m^t + a^t - e^{t+1}, b\Big) \tag{2.3}$$

$$c^{t+1} = \left(c^t - a^t\right) + \mathrm{B}(n^c, \mu) + \Big(m^t + a^t - e^{t+1} - m^{t+1}\Big), \tag{2.4}$$

where $e^{t+1} = \mathrm{B}\left(m^t + a^t, e\right)$ is the number of matches that exit the system at the start of period $t+1$, and $\mathrm{B}(n, q)$ denotes the realization of a binomial random variable with parameters (n, q). (2.2) formalizes the fact that the set of available parents is simply composed by new arrivals in each period. The first term in (2.3) is the number of existing matches at the end of period t, the ones from previous periods that have survived until period t and the newly formed. The second and third terms in (2.3) correspond to the random numbers of matches that exit the system and are broken at the start of $t + 1$. Thus, the set of available children in $t + 1$ is composed by 1) the children who remained unmatched in t, 2) new arrivals, and 3) children from broken matches. Each corresponds to the respective term in (2.4). For completeness, set

$$c^{-1} = 0 \quad \& \quad m^{-1} = 0, \tag{2.5}$$

so that in period $t = 0$ the market is comprised solely by new arrivals and there are no matches from previous periods. The number of children who remain unmatched in each period is $c_t - a_t$, so the problem of the matchmaker is given by:

$$\begin{aligned} \max_{a=(a_t)_{t=0}^T} \quad & -\mathbb{E} \sum_{t=0}^{T} \delta^t \left(c^t - a^t\right) \\ \text{subject to} \quad & (2.1) - (2.5) \quad \forall\, t = -1, 0, 1, \ldots, T, \end{aligned} \tag{$*$}$$

where $\delta \in (0,1)$ is the matchmaker's discount factor. The model is thus far incomplete because 1) the size of the market may be unbounded, depending on the arrival and exit probabilities, and 2) there is no heterogeneity across agents in the economy, so the matchmaker would simply match as many children as possible in each period. In order to bound the size of the market, I introduce the emancipation age. That is, each child grows one period old as periods go by, and exits the market permanently at a predetermined emancipation age. Furthermore, I enrich the model by assuming that there are different types of children and parents, leading to distinct exit and breakup probabilities.

Complete Model

Let $\bar{g}$ be the emancipation age, and denote by G the set of possible ages for children in the system $\{0, 1, \ldots, \bar{g} - 1\}$. Assume that each child has a type $x \in X$, where X is a finite set of child-types. Similarly, each parent has type $y \in Y$ with Y finite. Define the following:

- $m^t_{g,x,y}$ = number of surviving matches from previous periods composed by a child of age g and type x, and a parent of type y.
- $m^t = \left(m^t_{g,x,y}\right)_{(g,x,y)\in G\times X\times Y}$
- $c^t_{g,x}$ = number of available children of age g and type x in period t.
- $c^t = \left(c^t_{g,x}\right)_{(g,x)\in G\times X}$
- p^t_y = number of available parents of type y in period t.
- $p^t = \left(p^t_y\right)_{y\in Y}$

The matchmaker decides $a^t = \left(a^t_{g,x,y}\right)_{(g,x,y)\in G\times X\times Y}$, where $a^t_{g,x,y}$ indicates the number of matches formed in period t between a child of age g and type x, and a parent of type y. The choice set of the matchmaker in period t is given by:

$$a^t \in \Phi\left(c^t, p^t\right) \equiv \Bigg\{ a \in \mathbb{N}^{|G|\times|X|\times|Y|} : \forall (g, x) \in G \times X,\ 0 \leq \sum_{y\in Y} a_{g,x,y} \leq c^t_{g,x},$$

$$\& \quad \forall y \in Y,\ 0 \leq \sum_{g\in G} \sum_{x\in X} a_{g,x,y} \leq p^t_y \Bigg\}. \quad (2.6)$$

The first set of constraints in (2.6) are on the number of matches per children age and type, whereas the second set of constraints are on the number of matches per type of parent. A match between a child of type x and a parent of type y exits the system (for other reason than emancipation) with probability $e(x, y)$. If a match does not exit the system, it breaks up with probability $b(x, y)$. Hence, a match between a type-x child and type-y parent survives a period with probability $(1 - e(x, y)) \cdot (1 - b(x, y))$. Let $\left(n^c_{g,x}, \mu_{g,x}\right)_{(g,x)\in G\times X}$ describe the arrival distributions of new children by age and type, and $\left(n^p_y, \lambda_y\right)_{y\in Y}$ that of new parents by type. Thus, for every $(x, y) \in X \times Y$ and $g \in G \setminus \{0\}$, the transition of the system is governed by:

$$p^{t+1}_y = \mathrm{B}\left(n^p_y, \lambda_y\right) \quad (2.7)$$

$$m^{t+1}_{0,x,y} = 0 \quad (2.8)$$

$$m^{t+1}_{g,x,y} = m^t_{g-1,x,y} + a^t_{g-1,x,y} - e^{t+1}_{g,x,y} - \mathrm{B}\left(m^t_{g-1,x,y} + a^t_{g-1,x,y} - e^{t+1}_{g,x,y}, b(x, y)\right) \quad (2.9)$$

$$c^{t+1}_{0,x} = \mathrm{B}\left(n^c_{0,x}, \mu_{0,x}\right) \quad (2.10)$$

$$c^{t+1}_{g,x} = \left(c^t_{g-1,x} - \sum_{y\in Y} a^t_{g-1,x,y}\right) + \mathrm{B}\left(n^c_{g,x}, \mu_{g,x}\right) + \sum_{y\in Y} \left[m^t_{g-1,x,y} + a^t_{g-1,x,y} - e^{t+1}_{g,x,y} - m^{t+1}_{g,x,y}\right], \quad (2.11)$$

where

$$e^{t+1}_{g,x,y} = \mathrm{B}\left(m^t_{g-1,x,y} + a^t_{g-1,x,y}, e(x, y)\right)$$

is the number of matches that exit the system at the start of $t + 1$ per age and type. See that (2.7) – (2.11) are the analogues of (2.2) – (2.4) with the appropriate adjustments. First, note that newborns, age $g = 0$, may only be new arrivals and cannot come from broken matches or previous periods. Second, children who reach the emancipation age, $g = \bar{g}$, exit the system by construction. Finally, the matchmaker has any freedom to form matches across types. The number of children who remain unmatched in period t is given by $\sum_{g\in G} \sum_{x\in X} \left(c^t_{g,x} - \sum_{y\in Y} a^t_{g,x,y}\right)$, so the problem of the matchmaker is

$$\begin{aligned} \max_{a=(a_t)_{t=0}^{T}} \quad & -\mathbb{E}\sum_{t=0}^{T}\delta^t \sum_{g\in G}\sum_{x\in X}\left(c_{g,x}^t - \sum_{y\in Y} a_{g,x,y}^t\right) \\ \text{subject to} \quad & (2.6)-(2.11) \quad \forall\, t=-1,0,1,\ldots,T; \\ & c^{-1}=0 \quad \& \quad m^{-1}=0. \end{aligned} \tag{$\star$}$$

The state variables of the problem are m^t, c^t, and p^t. Each has dimension $|G|\times|X|\times|Y|$, $|G|\times|X|$, and $|Y|$, respectively. In any given period, the number of children in the system, unmatched and matched, is bounded since the arrival distribution of children has finite support. Namely, for every $g\in G$ and $x\in X$,

$$c_{g,x}^t + \sum_{y\in Y} m_{g,x,y}^t \leq \sum_{\tilde g=0}^{g} n_{\tilde g,x}^c. \tag{2.12}$$

The number of children of age g and type x in the system in period t, matched or unmatched, is bounded above by the maximum possible number of arrivals of type-x children who are g periods old in period t. The state space of the matchmaker's problem is a proper subset of $\mathbb{N}^{|G|\times|X|\times|Y|+|G|\times|X|+|Y|}$.[3]

2.3 Analytic and Computational Exercises

In this section, I consider a simple specification of the complete model to illustrate the dynamic trade-offs faced by the matchmaker. In particular, I assume there are two types of children and parents whose arrivals are Bernoulli distributed. The main focus is to explore the trade-offs due to the different types and ages of children. First, I consider three examples in finite horizon to illustrate three important observations. 1) It is not optimal to leave children unmatched while parents are available. This is unsurprising since parents exit the market when they are unmatched. 2) The matchmaker gives priority to children who have higher probability of exiting the market if matched, but lower priority to children who are closer to emancipation

[3]Specifically, the dimension of the state space is $\left[\prod_{g\in G\setminus\{0\},x\in X}\sum_{n=0}^{N_{g,x}}\binom{n+|Y|}{n}\right]\cdot\left[\prod_{x\in X}\left(n_{0,x}^c+1\right)\right]\cdot\left[\prod_{y\in Y}\left(n_y^p+1\right)\right]$, where $N_{g,x}=\sum_{\tilde g=0}^{g} n_{\tilde g,x}^c$ for every $g\in G\setminus\{0\}$ and $x\in X$. To see this formally, note that the second and third terms in brackets are the possible configurations of newborn children and parent arrivals. For each $(g,x)\in(G\setminus\{0\})\times X$, the sum $\sum_{n=0}^{N_{g,x}}\binom{n+|Y|}{n}$ is the possible number of configurations of $c_{g,x}^t$ and $\left(m_{g,x,y}^t\right)_{y\in Y}$ satisfying (2.12). Note that there are $\binom{n+|Y|}{n}$ possible values of $c_{g,x}^t$ and $\left(m_{g,x,y}^t\right)_{y\in Y}$ that satisfy (2.12) with equality and n in the right-hand side.

age. 3) The interplay between children ages and types depends on the specification of the breakup probabilities. Rather than achieving great generality, the objective of these three examples is to illustrate the workings of the model and the trade-offs it captures.

Second, I perform a computational exercise in an infinite horizon setup. I write the Bellman equation of the matchmaker's dynamic programming problem and solve it using iteration. The problem is high-dimensional by design, so the optimal policy is hard to interpret as such. I describe it qualitatively by evaluating four observations regarding the optimal decision in a set of predetermined states. The observations highlight the connection between the finite and infinite horizon models. I also perform discrete comparative statics by analyzing how the optimal policy changes in the model parameters. Finally, I compute and analyze the steady state of the system under the optimal policy. The state of the system is a Markov chain under the optimal matching policy. I compute its transition matrix and obtain its invariant distribution. This yields a distribution on the state space, so it allows to describe the market and the matchmaker's decisions in expected terms.

Analytic Examples in Finite Horizon

Let X and Y be binary sets, say $\{0, 1\}$. Assume that arrivals of children and parents are Bernoulli distributed and homogeneous across types. Moreover, assume that only newborns arrive to the market. Set $n^c_{0,x} = n^p_y = 1$ and $n^c_{g,x} = 0$ for all $g \in G \setminus \{0\}$ and $(x, y) \in X \times Y$, and let $\mu_{0,x} = \mu$ and $\lambda_y = \lambda$ for $x, y \in \{0, 1\}$. I consider three examples.

Leaving Children Unmatched

In a simple two-period example, I illustrate the rationale of why it is not optimal to leave children unmatched while parents are available. Simply put, matching a child increases the immediate payoff *and* the expected value of the following periods. This is because, all else equal, it is better for the matchmaker if there are fewer children in the market in each period. Let $T = 1$. Assume there is one child and one parent available in $t = 0$. For simplicity, I omit specifying the age of children. Let V^t_c be the expected optimal value of period t conditional on there being c available children who arrived in previous periods. If the (only) child remains unmatched in $t = 0$, the optimal expected value of the matchmaker's objective function in $t = 1$ is given by:

$$\begin{aligned}V_1^1 &= (1-\lambda)^2\left[\mu^2(-3)+2\mu(1-\mu)(-2)+(1-\mu)^2(-1)\right]\\&+2\lambda(1-\lambda)\left[\mu^2(-2)+2\mu(1-\mu)(-1)+(1-\mu)^2(0)\right]\\&+\lambda^2\left[\mu^2(-1)+2\mu(1-\mu)(0)+(1-\mu)^2(0)\right]\\&=-(1-\lambda)^2(2\mu+1)-4\lambda(1-\lambda)\mu-\lambda^2\mu^2.\end{aligned}\tag{2.13}$$

The value V_1^1 in (2.13) encompasses the optimal decision of the matchmaker in $t = 1$. Note that all children are homogeneous from the matchmaker's perspective. In the final period, she only wants to match as many children as possible. For example, with probability $(1 - \lambda)^2$ no parent arrives in $t = 1$, so at least one child is left unmatched, the one who arrived in $t = 0$. Two more children arrive in $t = 1$ with probability μ^2, in which case three children are left unmatched. With probability $2\mu(1-\mu)$, only one child arrives, so two children are left unmatched. The probability of there being just one child left unmatched is $(1 - \mu)^2$, which is the probability that no child arrives in the second period. Similarly, the optimal expected value in $t = 1$, conditional on there being no children from previous periods in the market, is given by:

$$V_0^1 = -2(1-\lambda)^2\mu - 2\lambda(1-\lambda)^2\mu^2. \tag{2.14}$$

Naturally, the matchmaker prefers that there are less children in the market in $t = 1$ since $V_0^1 > V_1^1$. For this reason, the matchmaker prefers to match the available child in $t = 0$ regardless of the breakup probability. To see this formally, note that the expected optimal value in $t = 0$ of not matching the child is $-1 + \delta V_1^1$, whereas the one of matching her is $-0 + \delta\left[bV_1^1 + (1-b)V_0^1\right]$, where b is the breakup probability of the match. Matching the child in $t = 0$ not only yields a higher immediate payoff, but also a higher expected value from the following periods.

Exit Probability and Emancipation Age

Using the same setup as above, I illustrate why it is optimal for the matchmaker to give priority to children who are more likely to exit the system if matched, e.g., because of getting adopted. Consider a case with two children of different types, x and x', but equal breakup probability. Assume there is a single parent available in $t = 0$. Let e and e' be the exit probabilities of each child if matched, and denote by b their common breakup probability. Assume $e > e'$. The expected value of the matchmaker in period $t = 0$ from matching child x is given by:

$$V_x^0 = -1 + \delta\left[eV_1^1 + (1-e)\left(bV_2^1 + (1-b)V_1^1\right)\right]. \tag{2.15}$$

The expected value on (2.15) encompasses the fact that x' will remain unmatched in $t = 0$, and one of three things may happen in the next period: 1) x exits the system, in which case there will be one child in $t = 1$ from previous periods, 2) x does not exit the system and her match breaks up, or 3) she does not exit, but her match does not break up. It is straightforward to compute V_2^1 and verify $V_1^1 > V_2^1$. The expected value of matching x' instead is the same as in (2.15), but with the exit probability e' in place of e. Since $V_1^1 > V_2^1$ and $e > e'$, (2.15) implies it is optimal to match child x in period $t = 0$. The reason is because a child that is more likely to exit the system permanently when matched is also less likely to come back to the market in future periods because of an unsuccessful match.

Consider instead two children with null exit probability and same breakup probability, but different ages. In particular, assume that one of them, say x', is to be emancipated in the next period. Denote the breakup probability of both children by b. Matching the child who will be emancipated in the next period yields an expected value in $t = 0$ given by $V_{x'}^0 = -1 + \delta V_1^1$, whereas matching x, the child who will not be emancipated, yields

$$V_x^0 = -1 + \delta \left[b V_1^1 + (1 - b) V_0^1 \right] . \tag{2.16}$$

The key thing to note is that there will be no children from previous periods in $t = 1$ if the match of x does not break. This is because x' will be emancipated regardless of having stayed unmatched. Hence, it is optimal to match x and leave x' unmatched.

Types and Emancipation Age

The previous example emphasizes why it is optimal for the matchmaker to give lower priority to children closer to the emancipation age. This observation is particularly strong when looking at the last period of a child previous to emancipation. A child's type is unimportant in the period prior to emancipation because her type only matters to the matchmaker through the breakup probability, which is irrelevant if the child is to be emancipated in the following period. With this observation in mind, in this section I consider an example to explore the tension between age and type. In particular, I consider a situation in which there are two children and one parent available in the market. One of the children has the same type as the parent, so a lower breakup probability if matched, but is also closer to the emancipation age. In order for the type of the older child to be taken into account by the matchmaker, let $T = 2$.

Figure 2.1: Breakup probability $b(x, y)$

		y = 0	y = 1
x	0	$(1-r)/s$	$1/(s \cdot w)$
	1	$1/s$	$(1-r)/(s \cdot w)$

Specify the breakup probability $b(x, y)$ as in the Figure 2.1. The breakup probability depends on three parameters: $r \in [0, 1]$ indicates the same-type bias, $s \in [1, \infty)$ measures how likely are matches to survive in this system overall, and $w \in [1, \infty)$ is the "bonus" of type-1 parents. High values of r indicate that same-type matches are less likely to breakup. Formally, the survival probability $1 - b(x, y)$ is supermodular if and only if $r > 0$. High values of s imply that all matches are more likely to survive, and high values of w that, all else constant, matches involving a type-1 parent are less likely to break up.

The state is fixed in period $t = 0$ at $p_1^0 = 1$, $c_{00}^0 = 1$, $c_{11}^0 = 1$, and all other state variables equal to zero. There is a type-1 parent, a type-0 child who is 0 periods old, and a type-1 child who is one period old. The emancipation age is set to $\bar{g} = 2$, so the type-0 child will be in the system for the three periods, but the type-1 child will be emancipated in period $t = 2$. In every subsequent period, a parent of each type may arrive with probability λ, and a newborn child of each type may arrive with probability μ. I characterize the parameter values for which it is optimal to match the type-0 child, and not the type-1 child, in period $t = 0$.

Proceed by backward induction. In the last period, $t = 2$, the matchmaker matches as many children as possible, regardless of their types. It is straightforward to verify

$$V_0^2 > V_1^2 > V_2^2 > V_3^2. \tag{2.17}$$

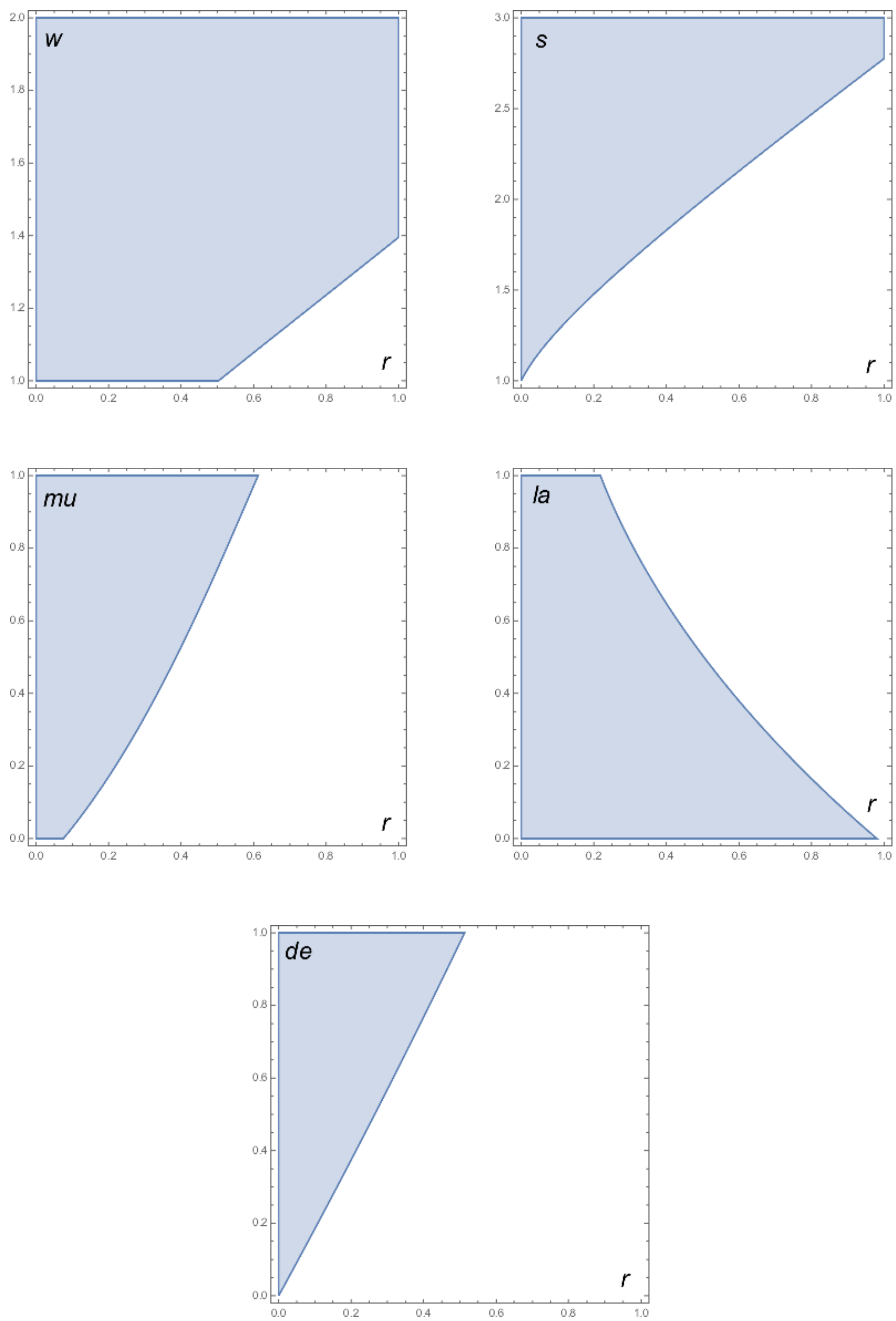
A maximum number of three children may arrive to period $t = 2$ from previous periods: the type-0 child of period $t = 0$, and potentially the two arrivals of period $t = 1$. In period $t = 1$, there are six state variables: c_{10}^1,c_{21}^1,c_{00}^1,c_{01}^1, p_0^1, and p_1^1. The last four variables correspond to new arrivals and are randomly determined: c_{00}^1, $c_{01}^1 \sim$ Bernoulli(μ), and p_0^1, $p_1^1 \sim$ Bernoulli(λ). If the matchmaker matches the type-0 child in $t = 0$, then $c_{21}^1 = 1$, and $c_{10}^1 \sim$ Bernoulli($b(0, 1)$), the older type-1 child remains unmatched and the first-period match breaks up in the next period with probability $b(0, 1) = 1/sw$. If instead the matchmaker matches the type-1 child in period $t = 0$, then $c_{10}^1 = 1$ and $c_{21}^1 \sim \text{Bernoulli}(b(1, 1))$. Hence, there are $5^2 = 32$ possible states in $t = 1$ for each decision of the matchmaker in $t = 0$.

Let $V^1(a^0 = x)$ be the expected optimal value in $t = 1$ if the type-x child is matched in $t = 0$. See Appendix 1 for details. The matchmaker's expected value of matching the type-x child in $t = 0$ is given by

$$-1 + \delta V^1 \left(a^0 = x\right), \tag{2.18}$$

so it is optimal to match the type-0 child if and only if $V^1(a^0 = 0) > V^1(a^0 = 1)$. To illustrate how this inequality changes in the parameters, I use the following benchmark: $w = 1$, $s = 2$, $\mu = 0.75$, $\lambda = 0.5$, and $\delta = 0.98$. Figure 2.2 illustrates the regions of the parameters for which it is optimal to match the type-0 child in period $t = 0$. The x-axis corresponds to the same-type bias r in every plot. I change one parameter at a time from the benchmark values in the y-axis. In general, it is not optimal to match the type-0 child for high values of r. Intuitively, if the same-type bias is sufficiently strong, it is optimal to match the type-1 child even though she is closer to the emancipation age. Moreover, note that if type-1 parents are sufficiently "better" (high values of w), then it is always optimal to match the type-0 child (top-left). This is because a high w "washes out" the same-type bias, so that a type-1 parent is better for either type of child. The same holds for high values of s (top-right). If all children have a low breakup probability, the same-type bias stops playing a role and it is always optimal to match the younger type-0 child. The main disadvantage of matching the older type-1 child is that the type-0 child will stay in the system unmatched for more periods. This is crucial if the arrival probability of children, μ, is high or the arrival probability of parents, λ, is low. The middle panels of Figure 2.2 illustrate this observation. For high values of μ or low values of λ, a stronger same-type bias is needed if matching the type-1 child is to be optimal. Finally, note that the discount factor is also relevant in this case. The bottom panel of Figure 2.2 shows that a lower discount factor, δ, implies that it is optimal to match the type-1 child.

Figure 2.2: Parameter regions in which it is optimal to match type-0 child



Note: Parameter regions in which $V^1(a^0 = 0) > V^1(a^0 = 1)$. Benchmark parameter values: $w = 1$, $s = 2$, $\mu = 0.75$, $\lambda = 0.5$, and $\delta = 0.98$. The x-axis varies r in $[0, 1]$ in all figures. Other parameters are varied from the benchmark as follows: $w \in [1, 2]$ (top-left), $s \in [1, 3]$ (top-right), $\mu \in [0, 1]$ (middle-left), $\lambda \in [0, 1]$ (middle-right), and $\delta \in [0, 1]$ (bottom).

Computational Example with Infinite Horizon

Consider the case with infinite horizon, $T = \infty$. The problem in ($\star$) is stationary, so I focus on the optimal matching policy per period $a = \left(a_{g,x,y}\right)_{(g,x,y)\in G\times X\times Y}$, given the state of the system (m, c, p). In particular, the optimal matching policy satisfies the following Bellman Equation:

$$V(m, c, p) = \max_{a\in\Phi(c,p)} \left\{ -\sum_{g\in G}\sum_{x\in X}\left(c_{g,x} - \sum_{y\in Y} a_{g,x,y} \right) + \delta\mathbb{E}\left[V(\tilde{m}, \tilde{c}, \tilde{p})|m, c, a\right] \right\}, \tag{2.19}$$

where V is the value function of ($\star$) and the state in the next period $(\tilde{m}, \tilde{c}, \tilde{p})$ evolves according to:

$$\begin{aligned} \tilde{p}_y &\sim \mathrm{B}\left(n_y^p, \lambda_y\right) \\ \tilde{m}_{0,x,y} &= 0 \\ \tilde{m}_{g,x,y} &\sim \mathrm{B}\left(m_{g-1,x,y} + a_{g-1,x,y} - \tilde{e}_{g,x,y}, 1 - b(x, y)\right) \\ \tilde{c}_{0,x} &\sim \mathrm{B}\left(n_{0,x}^c, \mu_{0,x}\right) \\ \tilde{c}_{g,x} &= c_{g-1,x} + \mathrm{B}\left(n_{g,x}^c, \mu_{g,x}\right) + \sum_{y\in Y}\left[m_{g-1,x,y} - \tilde{e}_{g,x,y} - \tilde{m}_{g,x,y}\right] \end{aligned} \tag{2.20}$$

where $\tilde{e}_{g,x,y} = \mathrm{B}\left(m_{g-1,x,y} + a_{g-1,x,y}, e(x, y)\right)$, for each $(x, y) \in X\times Y$ and $g \in G\setminus\{0\}$.

Consider the same specification as in the previous section: there are two types of children and parents, only newborn children arrive to the market, and arrivals are Bernoulli distributed and homogeneous across types. Furthermore, set the emancipation age at $\bar{g} = 3$ periods, and assume a zero exit probability, $e(x, y) = 0$, for all $(x, y) \in X \times Y$. Specify the breakup probability $b(x, y)$ as in Figure 2.1.

Given the structure of arrivals, the maximum number of children per age and type in the system (matched and unmatched) is 1, and so is the maximum number of new arrivals of parents per type. To compute the value function, I use a grid that includes every possible state, so its dimension is $4^4 \cdot 2^2 \cdot 2^2 = 4,065$, see footnote 3. Accordingly, the decision variable of the problem, a, has dimension $|G| \times |X| \times |Y|$ $= 3\times2\times2 = 12$. After solving for the optimal policy, I proceed in three steps. First, I analyze four observations in a benchmark of the model's parameters to describe the optimal matching policy qualitatively. Second, I analyze how these observations change in the parameters of the model. Third, I analyze the stationary state of the system under the optimal policy and how it changes for different parameter specifications.

The Optimal Policy: A Benchmark Specification

Consider the following benchmark specification of the model's parameters: $r = 0.5$, $w = 1$, $s = 2$, $\mu = 0.75$, $\lambda = 0.5$, and $\delta = 0.98$. Under the benchmark specification, there is same-type bias ($r > 0$), no type-1 bonus ($w = 1$), and foster parents are scarce in the market ($\mu > \lambda$). (Note that it is the same specification as in Section 2.3.) The following four observations hold true in the optimal policy under the benchmark.

Observation 1 *No children remain unmatched while parents are available.*

Observation 1 goes in line with the reasoning on Section 2.3. Since parents exit the system permanently when they remain unmatched, there is no reason to leave a child unmatched while a parent is available. In the next section, we will see that this observation is robust to different specifications of the model's parameters.

Observation 2 *No child is left unmatched while a same-type older child gets matched.*

It is optimal for the matchmaker to give priority to younger children who are further from the emancipation age. The intuition for this observation is straightforward: children who are closer to the emancipation age imply a lower future cost to the matchmaker if left unmatched because they will be in the market for a fewer number of periods. In the next section, we will see that this observation is robust to different specifications of the model's parameters.

Observation 3 *Assume there is only one available parent and is type-1. If there are two children: one type-1, and one type-0 but younger than the type-1 child, the younger type-0 child is matched only if the older type-1 child is one period prior to emancipation.*

Observation 3 is the opposite to the case analyzed in Section 2.3. Under the benchmark specification, there is a relatively strong same-type bias, $r = 0.5$, so it is optimal to give priority to type over age. Namely, the breakup probability of the type-0 young child if matched is 0.5, whereas that of the older same-type child is 0.25. However, as noted in Section 2.3, this observation naturally depends on the specification of the breakup probabilities. I analyze this issue in the next section.

Observation 4 *Assume there are two available parents, one of each type. If there are two type-0 children and, at the most, one type-1 child, older than the two type-0 children, the only child matched to a same-type parent is the youngest type-0 child.*

Observation 4 explores if age and type are complements or substitutes. Observation 2 highlights the fact that younger children are given priority over older children when only one parent is available. Observation 4 addresses the question of which of the two children, the young or the old one, is matched with the same-type parent if there are two parents available. Under the benchmark specification, the optimal policy implies that age and type are complements because the younger child is matched with the same-type parent, whereas the older one is matched with a different-type parent. Hence, younger children are given priority in getting matched *and* in getting matched with a "better" parent in terms of a lower breakup probability. In the next section, I will see that this observation is robust to different specifications of the parameters with a few exceptions.

Discrete Comparative Statics

In this Section, I evaluate if the previous observations hold for different specifications of the parameters. Specifically, I vary the same-type bias $r \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$, and each of the other parameters one at a time from the benchmark specification: $w \in \{1, 1.2, 1.4, 1.6, 1.8, 2\}$, $s \in \{1, 1.4, 1.8, 2.2, 2.6, 3\}$, and μ, λ, $\delta \in \{0.01, 0.2, 0.4, 0.6, 0.8, 0.99\}$. The total number of specifications computed is 180.

1. Observations 1 and 2 hold in all the specifications. No child is left unmatched while a parent is available or an older same-type child gets matched.

2. I present the parameter values for which Observation 3 holds marked as black dots in Figure 2.3. Observation 3 holds when the same-type bias is sufficiently large. In particular, note that it does not hold in the absence of same-type bias, $r = 0$. The intuition for the cases in which a young type-0 is left unmatched while the older type-1 child is matched with a type-1 parent is the same as in section 2.3. Compare Figures 2.2 and 2.3. The shaded regions in Figure 2.2 correspond to the parameter values under which the type-0 child is matched in a finite horizon setup. Observation 3 in an infinite horizon setup is slightly more general since it also considers cases when the type-1 child is older but one period prior to emancipation age. Roughly, the shaded regions in Figure

2.2 correspond to the white dots in Figure 2.3, which represent the cases in which Observation 3 fails.

3. I present the parameter values for which Observation 4 holds in Figure B.1 in Appendix 2. In general, Observation 4 holds in all the specifications except for high values of w. The reason is because high values of w imply that the same-type bias is not very important for type-0 matches. For large values of w, matches with a type-1 parent have less breakup probability regardless of the child's type. Figure B.1 shows that younger children receive priority, whether it is due to same-type bias or parent-1 bonus. Younger children tend to be in the matches with lower breakup probabilities.

The Stationary State

In this section, I analyze distinct features of the system in the stationary state. I report the expected number of available children, formed matches, children left unmatched and breakups per age and type, and the expected duration of matches for five distinct specifications. Formally, the stationary state is obtained through the invariant (stationary) distribution of the system's transition matrix under the optimal policy. Let $\sigma = (m, c, p)$ index a state. Denote the state space by Σ. For a fixed policy, $a : \Sigma \to \Phi(\sigma)$, the transition equations in (2.20) determine a transition matrix P_a, where

$$P_a(\sigma, \tilde{\sigma}) = \mathbb{P}[\tilde{\sigma}|\sigma, a]. \tag{2.21}$$

That is, the $(\sigma, \tilde{\sigma})$-th entry in P_a is the probability that the state transitions from σ to $\tilde{\sigma}$ if policy a is implemented. Denote the optimal policy by $a^*(\sigma)$. The optimal policy determines an optimal transition matrix P^* with $(\sigma, \tilde{\sigma})$-th entry given by:

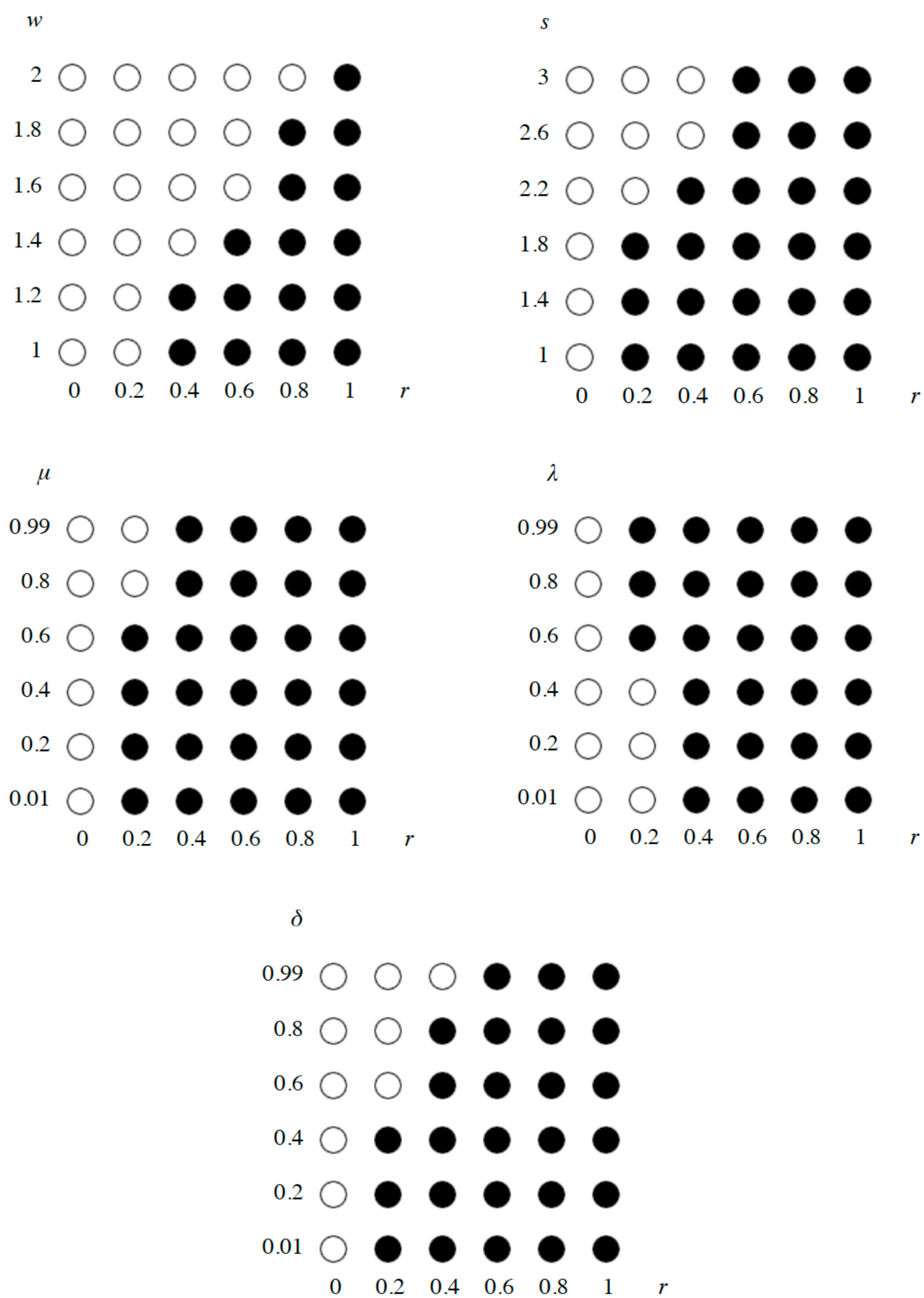
$$P^*(\sigma, \tilde{\sigma}) = P_{a^*(\sigma)}(\sigma, \tilde{\sigma}). \tag{2.22}$$

The invariant distribution of the Markov Chain determined by P^* is given by $\pi \in \Delta(\Sigma)$ such that

$$\pi P^* = \pi. \tag{2.23}$$

The stationary state of the system σ^* is distributed according to π. Once I compute π, I can compute the expected value of any variable of interest in the stationary state. For example, the expected number of available children in the market of age g and type x is given by $c^*_{g,x} = \mathbb{E}_\pi c_{g,x}(\sigma) = \sum_{\sigma \in \Sigma} c_{g,x}(\sigma) \cdot \pi(\sigma)$. Table 2.1 reports the expectation of several variables of interest in the stationary state.

Figure 2.3: Parameter regions in which Observation 3 holds



Note: The black dots indicate the parameter values in which Observation 3 holds with $T = \infty$. Observation 3: assume there is only one type-1 parent, if there are two children, one type-1 and one type-0 younger than the type-1 child, the type-0 child is matched only if the older type-1 child is one period prior to emancipation. Benchmark parameter values: $w = 1$, $s = 2$, $\mu = 0.75$, $\lambda = 0.5$, and $\delta = 0.98$. The x-axis varies r in all figures. Other parameters are varied from the benchmark one at a time: w (top-left), s (top-right), μ (middle-left), λ (middle-right), and δ (bottom).

Table 2.1: Stationary state

Age & Type	*Variable*	*Benchmark*	$w=2$, $r=0$	$w=2$, $r=0.5$	$s=3$, $r=0$	$s=3$, $r=0.5$
	Expected number of available children					
$g=1, x=0$	$c^*_{1,0}$	0.4560	0.3836	0.4336	0.3750	0.4063
$g=1, x=1$	$c^*_{1,1}$	0.4560	0.5774	0.4086	0.5625	0.4063
$g=2, x=0$	$c^*_{2,0}$	0.4826	0.4740	0.4610	0.4648	0.4333
$g=2, x=1$	$c^*_{2,1}$	0.4826	0.6121	0.4217	0.5931	0.4333
	Expected number of matches					
$g=0, x=0, y=0$	$a^*_{0,0,0}$	0.3750	0.2217	0.3750	0.3750	0.3750
$g=0, x=0, y=1$	$a^*_{0,0,1}$	0.0255	0.3408	0.0469	0.1875	0.0469
$g=0, x=1, y=0$	$a^*_{0,1,0}$	0.0255	0.1533	0.0266	0.0938	0.0469
$g=0, x=1, y=1$	$a^*_{0,1,1}$	0.3750	0.1280	0.3750	0.1875	0.3750
$g=1, x=0, y=0$	$a^*_{1,0,0}$	0.0570	0.0461	0.0542	0.0117	0.0317
$g=1, x=0, y=1$	$a^*_{1,0,1}$	0.0148	0.0079	0.0147	0.0410	0.0134
$g=1, x=1, y=0$	$a^*_{1,1,0}$	0.0148	0.0343	0.0132	0.0098	0.0134
$g=1, x=1, y=1$	$a^*_{1,1,1}$	0.0570	0.0139	0.0319	0.0381	0.0317
$g=2, x=0, y=0$	$a^*_{2,0,0}$	0.0119	0.0193	0.0128	0.0040	0.0129
$g=2, x=0, y=1$	$a^*_{2,0,1}$	0.0101	0.0020	0.0109	0.0174	0.0107
$g=2, x=1, y=0$	$a^*_{2,1,0}$	0.0066	0.0127	0.0063	0.0028	0.0070
$g=2, x=1, y=1$	$a^*_{2,1,1}$	0.0062	0.0027	0.0061	0.0129	0.0068
	*Expected number of unmatched children**					
$g=0, x=0$	$u^*_{0,0}$	0.3495	0.1875	0.3281	0.1875	0.3281
$g=0, x=1$	$u^*_{0,1}$	0.3495	0.4687	0.3484	0.4687	0.3281
$g=1, x=0$	$u^*_{1,0}$	0.3842	0.3296	0.3647	0.3223	0.3611
$g=1, x=1$	$u^*_{1,1}$	0.3842	0.5291	0.3635	0.5146	0.3611
$g=2, x=0$	$u^*_{2,0}$	0.4606	0.4526	0.4373	0.4434	0.4097
$g=2, x=1$	$u^*_{2,1}$	0.4698	0.5967	0.4094	0.5774	0.4196
	*Expected number of breakups**					
$g=0, x=0, y=0$	$B^*_{0,0,0}$	0.0938	0.1109	0.0937	0.1250	0.0625
$g=0, x=0, y=1$	$B^*_{0,0,1}$	0.0127	0.0852	0.0117	0.0625	0.0156
$g=0, x=1, y=0$	$B^*_{0,1,0}$	0.0128	0.0766	0.0133	0.0312	0.0156
$g=0, x=1, y=1$	$B^*_{0,1,1}$	0.0937	0.0320	0.0469	0.0625	0.0625
$g=1, x=0, y=0$	$B^*_{1,0,0}$	0.0846	0.0785	0.0839	0.0872	0.0574
$g=1, x=0, y=1$	$B^*_{1,0,1}$	0.0138	0.0659	0.0125	0.0553	0.0149
$g=1, x=1, y=0$	$B^*_{1,1,0}$	0.0846	0.0785	0.0839	0.0872	0.0574
$g=1, x=1, y=1$	$B^*_{1,1,1}$	0.0846	0.0275	0.0450	0.0544	0.0574
	*Expected duration**					
All	l^*	3.7225	2.9556	5.4543	2.9444	5.4140

Note: The number of children who remain unmatched is the difference between the available and matched. For every $g \in \{0,1,2\}$ and $x \in \{0,1\}$, $u_{g,x} = c_{g,x} - (a_{g,x,0} + a_{g,x,1})$. The number of broken matches between two periods is the difference between the matches at the end of the period and the matches that survive the next period. For every $g \in \{0,1\}$ and $x, y \in \{0,1\}$, it is $m_{g-1,x,y} + a_{g-1,x,y} - m'_{g,x,y}$, where $m'_{g,x,y}$ refers to the matches that survive next period. Thus, the expected number of broken matches on any state is $B_{g,x,y} = m_{g-1,x,y} + a_{g-1,x,y} - \mathbb{E}\,\tilde{m}_{g,x,y}$. The expected duration of a (x,y)-match is $1/b(x,y)$. Note that the duration of matches follows a geometric distribution. The expected duration of matches in the stationary state, l^*, is the expected duration of the expected matches.

The first thing to notice from the tables is that the stationary state is very rich, offering a wide range of variables of interest that vary with the parameters of the model. The first panel of Table 2.1 reports the expected number of available children in the stationary state. The values for ages $g = 0$ are not reported because they equal the arrival rates of newborns to the market ($\mu = 0.75$). Note that the number of children available in the market is lower if there is same-type bias ($r = 0.5$) and, either $w = 2$ or $s = 3$, than in the benchmark case. The reason is because a higher w or s imply lower breakup probabilities. Moreover, when there is parent-1 bonus and same-type bias (column 5), the number of available children is greater for type $x = 0$. This is because the matching is positively assortative across types, but type-0 matches have greater breakup probability.

The second panel of Table 2.1 reports the expected matching policy in the stationary state. The first thing to note is that the matching is positively assortative when there is same-type bias ($r = 0.5$), even in the presence of parent-1 bonus. Secondly, the number of expected matches is decreasing in age. The intuition is the one behind Observation 2: children who are closer to the emancipation age represent a lower expected future cost if they remain unmatched. Lastly, when there are both same-type bias and parent-1 bonus (column 5), the number of different-type matches is greater for matches with s type-1 parent. Intuitively, the number of (0, 1)-matches is the least since they have the greatest breakup probability.

The third panel of Table 2.1 presents the expected number of children who remain unmatched. For every $g \in \{0, 1\}$ and $x \in \{0, 1\}$, $u^*_{g,x} = c^*_{g,x} - (a^*_{g,x,0} + a^*_{g,x,1})$. Unsurprisingly, the likelihood of remaining unmatched increases with age. An important remark is that the asymmetry between same-age type-0 and type-1 children who remain unmatched in the absence of same-type bias (columns 4 and 6) is not due to any particular aspect of the model, but to the fact that all children are homogeneous in the absence of same-type bias. Thus, the asymmetry is due to the arbitrary tie-breaking rule of the matchmaker. That is, if there is a single available parent and two children of different type but with same breakup probability, the optimal policy is not unique, so matching any type is optimal.

The fourth panel of Table 2.1 presents the expected number of broken matches. This is computed as the difference between the number of total matches at the end of the period $m_{g-1,x,y} + a_{g-1,x,y}$ and the expected number of matches that survive the next period $\mathbb{E}\,\tilde{m}_{g,x,y}$. In the presence of same-type bias, there is a greater number of broken matches that are same-type than mixed-type. The reason is because there are more same-type matches in absolute terms, so the number of them that break is greater even though they have lower breakup probability. Finally, the bottom panel of Table 2.1 shows the expected duration across all the matches in the system. The calculation does not take into account the emancipation age. By design, all matches last three periods at the most because of the emancipation age. Nonetheless, I compute the expected duration of the matches as the inverse of their breakup probability. The duration of a match follows a geometric distribution. The expected duration of matches in the system may be seen as a measure of the *quality* of the matches from the matchmaker's perspective. Unsurprisingly, the expected duration of matches in the system is greater when there is same-type bias ($r = 0.5$) and parent-1 bonus ($w = 2$, column 5) or higher survival rate ($s = 3$, column 7).

2.4 Data

The main data source is the Adoption and Foster Care Analysis & Report System (AFCARS) Foster Care File Database, distributed by the National Data Archive on Child Abuse and Neglect (NDACAN). Under federal mandate, all the states in the U.S. are required to provide information to the AFCARS of all the children in FC who are under the responsibility of State welfare agencies. NDACAN publishes yearly databases containing information of all children in FC.[4] The data contain characteristics of each child in FC and their current placement setting, information regarding the history of each child in FC, and characteristics of each case, including reason(s) for removal and some characteristics of the biological and foster parents. Each child has a unique identifier across all databases, so children may be tracked across years. The AFCARS Foster Care File Database published by NDACAN contains information of every state since 2001. Even though there is available data for every state, I limit the analysis to Los Angeles County because the administration of state child welfare agencies is usually at the county level. Furthermore, the database does not contain information regarding specific regulations that pertain to single states or counties. For example, it does not contain trustworthy information regarding the payments to foster families. First, I present a description of the dataset and some summary statistics for the fiscal year (FY) 2013.[5] Second, I report results of several regressions in order to describe the correlation among several variables in the dataset. It is important to note that the regression analysis is not causal since it does not account for data selection or omitted variables. I focus the analysis on

[4]For more details, see the AFCARS User's Guide (NDACAN, 2013b).

[5]The AFCARS database is published according to fiscal years. The fiscal year 2013 runs from October 1st, 2012 to September 30th, 2013.

sorting patterns between children and parents in terms of race, sex, and age, and in the association between the duration of matches and other variables.

Description of the Data

Each observation in the dataset corresponds to a child who was in FC for at least 24 hours during the year. There are 29,873 observations in LA County.[6] Reported characteristics of the child include: sex, race, disabilities, and month and year of birth. See Table 2.2. Children in FC may have been *removed* more than one time from home. Furthermore, during each *removal*, a child may go through different placements, indicating that past matches were unsuccessful. Table 2.3 presents summary statistics on the number of (lifetime) removals, placements in the last removal, and lengths of stay in the last removal and placement. It also includes summary statistics on the reasons for removal. It is worth noting that on average children have been in FC for around two years during the last removal, but have been on their last placement only for a year approximately. Accordingly, children have been on 2.45 placements on average during their last removal.

Most of the children in FC (86.15%) are in private homes, 34.81% being cared by relatives. The remaining 13.85% are placed in one of multiple placement settings provided by the state. The database also includes some characteristics on the case of each child. See Table 2.4. Many of the children in FC have a stated "Case Plan Goal." A high fraction of the children have *reunification* listed as a case plan goal (36.86%). Adoption is the second most common case plan goal (17.66%). The biological/principal parents of around 10% of the children in FC have lost their parental rights. However, the case plan goal has not been established or is unknown in around 20% of the cases. Table 2.4 also lists the discharge reason for all the children discharged during the FY 2013. The majority of children (56.62%) left FC to get reunified with their principal/biological caretakers. Note also that a significant fraction of children leave FC because they get adopted (13.81%). Out of the 29,873 children in the database, 9,773 (32.71%) left FC during the FY 2013, i.e., there were no longer in FC on September 30th, 2013.

The data also includes characteristics of the principal/biological and foster families of each child. However, the fraction of missing values is particularly high for these variables in LA County. For example, the family structure of the principal caretaker

[6] LA County is the county in the U.S. with the most children in FC, and California is the state with the most children in FC in the U.S. 34.88% of the children in FC in California are in LA County, and California comprises 13.36% of all children in FC in the country.

family is known only in roughly 30% cases. That being said, single female is the most common family structure within the children for which I observe the structure of the principal caretaker family. Single female is also the most common family structure within the foster families for which the family structure is known. As it may be seen in Table 2.2, even though the database contains the race of the foster family for each child, it is unknown in the majority of the cases.

The main variables of interest for the empirical application are those related to the history of the children in FC. In particular, the database contains entry dates for the first, last, and previous to last removals. This allows to partially construct the duration of each placement for every child. In principle, the data may be completed by tracking the previous matches in previous years. However, if a child was removed more than one time during the same year, some data is not observed. Furthermore, the data faces a censoring problem because many children are still in FC when the data is collected, so we only observe a lower bound for the duration of the current matches. Table 2.6 reports how many children have the same date recorded for particular events in LA County. For example, on average 27.21 children were removed from home for the first time every day between October 1st, 2012, and September 30th, 2013. Similarly, on average 54.9 children were placed on the same date in their current placement. This figure includes first-time removals, but also all the children who are re-placed within a removal. These figures are not surprising since LA County is the county in the U.S. with the largest population in FC. Figure 2.4 presents histograms of the dates at which five distinct events happened in a child's case: 1) first removal, 2) discharge from FC, 3) latest removal, 4) beginning of current placement, and 5) discharge of previous removal. The main thing to notice is that the dates appear to be evenly distributed across the year. This suggests that removals and placements are distributed uniformly across time, so the selection due to the cross-section nature of the dataset is random. The histograms of the begin date of the current placement (middle-right) and the discharge date from the previous removal (bottom) are skewed to the right and left, respectively, because they adjust over time as children are re-placed or discharged from more recent removals.

Regression Analysis

I run two sets of linear regressions to explore the correlation between several variables in the data. First, I analyze the sorting patterns in terms of race. Table 2.7 reports the estimated coefficients. The first thing to note is that only a small fraction of the sample is used in these regressions. This is because the majority of observations do not include the race of the foster family, see Table 2.2. Nonetheless, the results of the regressions suggest that there are strong sorting patterns in the data within the observations for which the race of the foster parents is known. In particular, African American, Hispanic, and White children are significantly more likely to be assigned to foster parents of the same race. The coefficient of Age At Start (age on October 30th, 2012) is estimated precisely, but it is very close to zero. Similarly, I find that the sex of children is not correlated with the race of foster parents. A key finding is the high R-squared of all the regressions, specially the first one. This suggests that race is an important factor in the assignment of children to foster families.

The second set of regressions regards the number of removals and placements, as well as the length of stay during the last and previous placements. These regressions only include characteristics of the children as independent variables, so they use almost all the observations in the data. The results are reported in Table 2.8. I find that African American children have been removed slightly more times from home than White children have, whereas Hispanics slightly less. In contrast, Asian children appear to be the less likely of being removed from home. Similarly, African American children have been through more placements (Settings) than White children during their last removal, and Hispanic children through less. Unsurprisingly, older children have been through more placements during their last removal. On average, a child who is one year older is roughly 10% more likely to have been through one more placement during her last removal. Similar patterns emerge when looking at the average length of stay in the last/current placement. African American children have been for more time at their current/last placement on average than White children, but Hispanic children have been less time. Note that this regression does not account for the termination reason. This regression pools children who were discharged, emancipated, adopted, or were still in FC at the end of the FY. The last regression (4) analyzes the average length of stay in the previous placements. In contrast with regression (3), this is the duration of unsuccessful matches since children got re-assigned after the placement ended. The size of the sample in this regression is smaller since not all children have been through more than one placement. Nonetheless, the results are qualitatively similar: older children were on average more days in their past placements, as well as African American and men (slightly), whereas Hispanic and Asian children were on average shorter periods of time in their previous placements.

2.5 Conclusion and Further Research

The main objective of this chapter is to study the determinants of the duration of matches in the FC system and to recognize the importance of accounting for the underlying dynamic matching mechanism. The first step in the analysis was to develop a general centralized matching model in a dynamic environment. The model aims to capture the fact that matches are not randomly determined in reality. The main assumption of the model is that a single matchmaker forms matches in order to minimize the number of children who remain unmatched. The motivation for this assumption is that the aim of the FC system is to find placements for children who are removed from home. The model is flexible enough to capture several dynamic trade-offs faced by the matchmaker. Through a series of analytic and computational exercises, I illustrated four features of the matchmaker's optimal matching policy: 1) it is not optimal to leave children unmatched while parents are available; 2) the optimal matching policy gives priority to younger children since it does not leave a child unmatched while an older child with the same type gets matched; 3) the matchmaker faces a trade-off between matching children and parents of the same type and matching younger children who are further from the emancipation age, but are of different type; the optimal solution of this trade-off depends on the model's parameters; 4) the optimal policy gives priority to younger children in terms of match *quality* in the sense that younger children tend to be in the matches with lower breakup probabilities.

An important feature of the model is that it allows me to compute and study its stationary state. I present moments of relevant variables for different specifications of the model's parameters. This is crucial for the empirical application since the moments in the stationary state may be seen as the empirical implications of the model. The behavior of the stationary state is fairly intuitive with respect to the model's parameters. For example, I find that the optimal matching policy in the stationary state is positively assortative when the survival probability is supermodular in the types of children and parents ($r > 0$). Finally, I described the main data source for the empirical application of the model. In particular, I report summary statistics and analyze the correlation among several variables of interest for the FC in Los Angeles County on 2013.

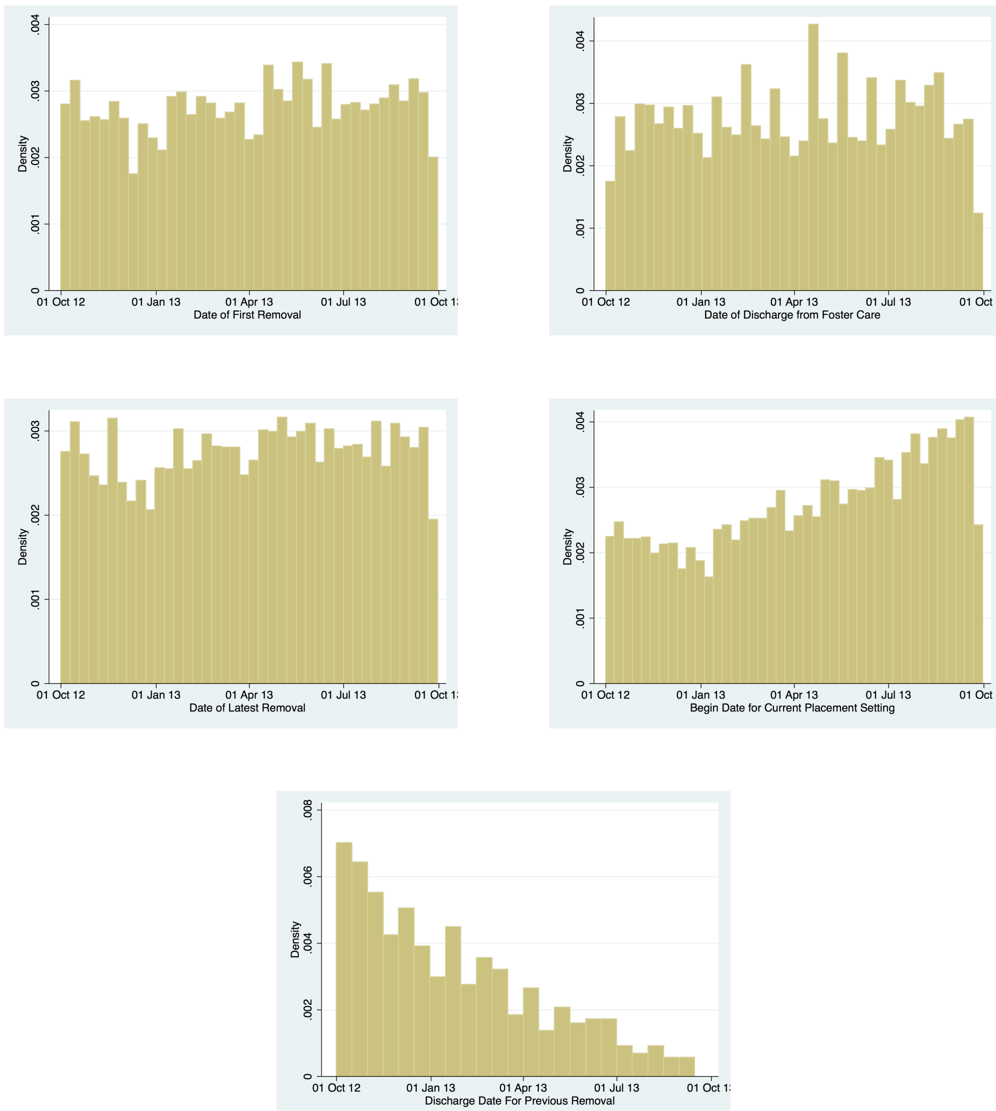
The study of foster care as a dynamic matching market is a new area of study in the economics literature. This and the previous chapter aimed to provide an introduction to the subject and raise key questions for further research in this field. I conclude by listing three key challenges and objectives of this research agenda.

(1) Study further the meaning of "match-quality" in this market. The present chapter focused mainly on the breakup probability of different matches. The next step along this research is to recognize that some children may exit quicker when matched to specific types of families (e.g., because they are adopted).

(2) Study different objectives that social workers may have on the field. While matching as many children as possible is one of the key objectives of the system, the empirical portion of this chapter notes that there are strong sorting patterns across key variables, such as race. This observation raises the question: are children and foster families sorted according to race because this leads to higher "match-quality" (ie., fewer breakups and quicker exits) or is it motivated by other factors?

(3) Study if parents have dynamic objectives. The model presented in this chapter assumes that foster parents simply exited the system after one period. Nonetheless, in reality foster parents may have their own objectives regarding which children to take care of. In principle, they could "strategically wait" for a child of their preferences. While discriminating on the basis of gender or race is illegal in the foster care system, social workers may include some of these considerations in their calculus due to foster parents being more or less likely to adopt a foster child on the basis of demographic characteristics.

Figure 2.4: Histograms of relevant dates



Note: I consider only the cases for which each event happened during the FY 2013. Date of first removal (top-left), discharge from FC (top-right), last removal (middle-left), start of current placement (middle-right), discharge from previous removal (bottom).

Table 2.2: Descriptive statistics: Children and foster parents

Variable	(1) Mean	(2) Std. Dv.	(3) Obs.	(4) Miss. (%)
Children Age				
Age at first removal	6.4273	5.6954	29,849	0.08
Age at end	9.1135	6.0705	29,859	0.04
Children Sex, Race, and Disability				
Man	.5181	.4997	29,873	0
Am. Indian/AK Native	.0048	.0554	29,759	0.4
Asian	.0169	.1206	”	”
African American	.2768	.4369	”	”
White	.1222	.3031	”	”
Hispanic	.6202	.4661	27,573	7.7
Some Disability	.5954	.4908	29,873	0
Foster Parents Race				
Am. Indian/AK Native	.0018	.0397	7,841	67.38
Asian	.0186	.1326	”	”
African American	.2978	.4461	”	”
White	.1292	.3227	”	”
Hispanic	.2378	.2476	15,868	34

Note: The fifth column reports the percentage of observations for which the variable is missing. Races are *proportional* since an individual can identify more than one. *Age at end = min{Age at exit, Age at the end of FY}. Some disability = mental retardation, visually/hearing impaired, physical disability, emotionally disturbed, or "other diagnosed condition."
Source: AFCARS Foster Care File FY 2013, NDACAN, 2013a.

Table 2.3: Descriptive statistics: Removals, placements, and removal reason

Variable	(1) Mean	(2) Std. Dv.	(3) Obs.	(4) Miss. (%)
Removals and Placements				
(Lifetime) Removals	1.3164	.6332	29,873	0
Placements in last removal	2.4518	2.5203	29,871	0
LOS in last removal (years)	2.0753	2.9975	29,864	0.03
LOS in last placement (years)	1.0102	1.7367	29,558	1.05
Removal Reason				
Physical Abuse	.0997	.2997	29,595	0.9
Sexual Abuse	.0235	.1515	"	"
Neglect	.6885	.4631	"	"
Parent Alcohol Abuse	.0146	.1201	"	"
Parent Drug Abuse	.0552	.2285	"	"
Child Alcohol Abuse	.0003	.0184	"	"
Child Drug Abuse	.0151	.1218	"	"
Child Disability	.0023	.0489	"	"
Child Behavior Problem	.1041	.3054	"	"
Parent Death	.0027	.0519	"	"
Parent Incarceration	.0205	.1419	"	"
Caretaker Inability to Cope	.3034	.4597	"	"
Abandonment	.0090	.0942	"	"
Relinquishment	.0057	.0756	"	"
Inadequate Hosing	.0278	.1644	"	"

Note: The fifth column reports the percentage of observations for which the variable is missing. Removals = 1 (75.68%), = 2 (18.54%), ≤ 3 (98.76%). Placements in last removal = 1 (44.00%), = 2 (27.09%), ≤ 7 (95.73%). LOS ≡ Length of Stay. If the child is still in FC at the end of the FY year, *last* removal and placement refer to *current* removal and placement. One child may have more than one removal reason.
Source: AFCARS Foster Care File FY FY 2013, NDACAN, 2013a.

Table 2.4: Descriptive statistics: Placement and case

	(1) Observations	(2) Percent (%)	(3) Cumulative (%)
Current Placement Setting			
Pre-adoptive home	2,098	7.02	7.02
Relative Foster Home	10,398	34.81	41.83
Non-Relative Foster Home	11,545	38.65	80.48
Group home	1,064	3.56	84.04
Institution	1,591	5.33	89.36
Supervised independent living	518	1.73	91.10
Runaway	965	3.23	94.33
Trial home visit	1,694	5.67	100.00
Total	29,873	100.00	
Most Recent Case Plan Goal			
Reunification	11,010	36.86	36.86
Live with other relative(s)	508	1.70	38.56
Adoption	5,276	17.66	56.22
Long-term foster care	1,501	5.02	61.24
Emancipation	1,493	5.00	66.24
Guardianship	3,549	11.88	78.12
Not established	6,536	21.88	100.00
Total	29,873	100.00	
Terminated Parental Rights			
TPR	3,052	10.22	10.22
Discharge Reason			
Reunification	5,533	56.62	56.62
Adoption	1,350	13.81	70.43
Emancipation	1,368	14.00	84.43
Guardianship	1,003	10.26	94.69
Transfer to another agency	514	5.26	99.95
Death of child	5	0.05	100.00
Total	9,773	100.00	

Note: TPR = 1 if both biological/principal parents have terminated their parental rights. **Out of the 29,873 total number of observations in the dataset, 9,773 (32.71%) correspond to children who were discharged from FC during the fiscal year 2013.
Source: AFCARS Foster Care File FY 2013, NDACAN, 2013a.

Table 2.5: Descriptive statistics: Family structure

	(1) Observations	(2) Percent (%)	(3) Cumulative (%)
Principal Caretaker Family Structure			
Married couple	867	2.90	2.90
Unmarried couple	1,269	4.25	7.15
Single female	5,940	19.88	27.03
Single male	821	2.75	29.78
Missing	20,976	70.22	100.00
Total	29,873	100.00	
Foster Family Structure			
Married couple	5,156	21.45	21.45
Unmarried couple	2,306	9.59	31.04
Single female	9,723	40.44	7 1.48
Single cale	2,181	9.07	80.55
Missing	4,675	19.45	100.00
Total	24,041	100.00	

Note: The total of observations with foster family structure, including missing values, is less than the number of observations because not all children are placed in a foster home (see Table 2.4).
Source: AFCARS Foster Care File FY 2013, NDACAN, 2013a.

Table 2.6: Descriptive statistics: Observations with same date on FY 2013

Variable	(1) Mean	(2) Std. Dv.	(3) Obs.	(4) Min. – Max.
First Removal	27.21	6.5849	9,363	8 – 47
Last FC Discharge	3.17	1.5403	573	1 – 8
Last Removal	34.1	7.1160	11,892	12 – 56
Current Placement	54.9	13.613	18,414	25 – 86
FC Discharge	33.52	22.9684	9,625	5 – 110

Note: If the child is still in FC at the end of the FY year, *last* removal and placement refer to *current* removal and placement. The number of observations vary because I only consider observations with the corresponding date between October 1st, 2012, and September 13th, 2013. There are no missing values. All observations have the corresponding dates.
Source: AFCARS Foster Care File FY FY 2013, NDACAN, 2013a.

Table 2.7: Regression analysis: Racial sorting patterns

	(1) FP-Afr. Am.	(2) FP-Hispanic	(3) FP-White
Age At Start	0.0061***	0.0000	-0.0071***
	(0.0007)	(0.0003)	(0.0006)
Man	0.0044	0.0000	0.0007
	(0.0069)	(0.0032)	(0.0070)
African Am.	0.7090***	-0.0680***	-0.4090***
	(0.0146)	(0.0058)	(0.0178)
Hispanic	-0.0530***	0.2740***	-0.3920***
	(0.0126)	(0.0059)	(0.0174)
Am. Indian	0.0080	0.0281	-0.4979***
/AK Native	(0.0719)	(0.0279)	(0.0583)
Asian	0.0255	-0.0073	-0.3320***
	(0.0338)	(0.0138)	(0.0383)
Constant	0.0765***	0.0960***	0.5200***
	(0.0124)	(0.0058)	(0.0172)
Observations	7,250	14,690	7,250
R-squared	0.548	0.369	0.161

Note: The dependent variable is the proportional race of the foster parent: (1) African American, (2) Hispanic, (3) White, and the independent variables are characteristics of the children. Races are *proportional* since an individual can identify with more than one. Age At Start is the age (years) of the child on October 1st, 2013. OLS Estimates. Standard errors (robust to arbitrary heteroskedasticity) in parentheses. *** p<0.01, ** p<0.05, * p<0.1

Table 2.8: Regression analysis: Removals and placements

	(1) Removals	(2) Settings	(3) LOS-Setting	(4) Avg. LOS -Prev. Settings
Removals		-0.156***	-76.94***	-54.29***
		(0.0289)	(6.535)	(4.631)
Settings	-0.00981***		-9.586***	-12.87***
	(0.00184)		(1.504)	(1.062)
Age At Start	0.0300***	0.117***	23.72***	16.51***
	(0.000646)	(0.00289)	(0.917)	(0.757)
Man	-0.00347	-0.191***	17.66**	11.69*
	(0.00720)	(0.0290)	(7.666)	(6.285)
African Am.	0.0410***	0.458***	138.3***	53.05***
	(0.0142)	(0.0592)	(16.02)	(13.07)
Hispanic	-0.0525***	-0.257***	-31.36**	-28.08***
	(0.0124)	(0.0486)	(12.36)	(10.75)
Am. Indian	0.109	0.765***	32.46	-6.179
/AK Native	(0.0707)	(0.279)	(60.47)	(42.59)
Asian	-0.204***	-0.318**	-111.9***	-70.95**
	(0.0233)	(0.131)	(22.26)	(29.88)
Constant	1.115***	1.845***	299.4***	221.6***
	(0.0121)	(0.0559)	(13.11)	(12.27)
Observations	27,571	27,571	25,498	15,426
R-squared	0.088	0.094	0.058	0.061

Note: Removals = lifetime removals. Settings = placements during the last removal. LOS-Setting = Length of stay (days) in the last/current placement. Avg. LOS-Prev. Settings = Average length of stay (days) in the previous settings of the last removal. Races are *proportional* since an individual can identify with more than one. Age At Start is the age (years) of the child on October 1st, 2013. OLS Estimates. Standard errors (robust to arbitrary heteroskedasticity) in parentheses. *** p<0.01, ** p<0.05, * p<0.1

Chapter 3

EXECUTIVE COMPENSATION AND COMPETITIVE PRESSURE IN THE PRODUCT MARKET: HOW DOES FIRM ENTRY SHAPE MANAGERIAL INCENTIVES?

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3.1 Introduction

There is a plethora of empirical evidence that supports the Hicksian view (Hicks, 1935) that executive compensation tends to be more performance-sensitive in more competitive environments (e.g., Nickell, 1996; Van Reenen, 2011). A series of empirical studies have used industry-specific regulatory reforms to analyze the effect of competition on executive pay (Crawford, Ezzell, and Miles, 1995; Cuñat and Guadalupe, 2009a; Dasgupta, Li, and Wang, 2017; Hubbard and Palia, 1995; Kole and Lehn, 1999; Palia, 2000). These studies focus on how deregulation policies that increase competition in the product market affect the structure of managerial incentive contracts. The main takeaway from this literature is that, following a deregulation policy that intensifies product market competition, firms reduce managerial slack by increasing executive compensation and strengthening its pay-performance sensitivity.

Our objective in this paper is to explain the nature of the aforementioned empirical regularity, and to offer new insights into how executive pay is shaped by industry-specific features. First, we provide a simple model of oligopolistic competition with firm entry that shows why incumbent firms find it optimal to reduce managerial slack when competition rises due to deregulation. Then, we use our model to derive novel empirical implications regarding the *time to build production capacity* in an industry. Our model shows that this industry-specific feature is a crucial factor when analyzing the effect that firm entry has on executive compensation. According to our model, the relationship observed in the empirical studies obtains in industries in which the time to build capacity is such that incumbents act as production leaders and entrants as followers. This result goes in line with the empirical literature given that existing studies focus on industries in which it takes time to build production capacity, such as banking, manufacturing, and the airline industry.

The question of how product market competition shapes managerial incentives is far from being new in the literature.[1] Notwithstanding, our approach is novel in that we analyze it explicitly in a framework of firm entry. Because incumbent firms anticipate (and accommodate) future entry with relaxing regulation, we use a standard model of sequential quantity-setting oligopoly, in which entrant firms choose their managerial contracts and quantities after observing those of the incumbents. Our focus is on the strategic response of incumbents regarding managerial incentive pay as they foresee the entry of new firms. In line with the empirical literature, our main finding is that it is optimal for incumbents to strengthen incentive pay and reduce managerial slack when they foresee the entry of new firms into the product market. Moreover, we show that the strength of the managerial incentives offered by incumbents is increasing in the number of entrants—higher competitive pressure leads to steeper incentives and lower managerial slack.

Our model incorporates managerial incentive contracts into the Stackelberg quantity competition framework proposed by Daughety (1990). There is a fixed number of incumbents and a set of potential entrants with more entrants meaning greater competitive pressure on the incumbent firms. Both incumbents (in the pre-entry stage) and entrants (in the post-entry stage) play Cournot games among themselves; entrants take the aggregate output of incumbents as given. All firms are initially inefficient and each hires a risk neutral manager whose principal task is to exert non-verifiable R&D effort to bring down the constant marginal cost of production, what is often termed "process innovation." We assume that the final realizations of marginal costs are private information among firms, and that incentive contracts are publicly observable. Hence, even though the marginal costs of rival firms are unknown, each firm observes a signal of how likely every other firm is to reduce its marginal cost.

The crux of our model is that managerial effort is beneficial to incumbents in two ways. First, steeper incentives that induce each manager to exert higher effort directly increase the likelihood of cost reduction (value-of-cost-reduction effect). Second, they also alter the beliefs of the rival firms about the true cost realization of a given firm (marginal-profitability-of-effort effect). Even if a manager fails to

[1] The notion that monopoly, and market power in general, are detrimental to managerial efficiency dates back to Smith (1776, Book 1, Chapter 11), and has a long tradition in the literature (Hart, 1983; Leibenstein, 1966; Scharfstein, 1988).

achieve the cost target, her effort is profitable in as much as it makes the rivals believe that a cost reduction has actually been attained. More intensified product market competition affects each of these two effects through the market size and the effective size of cost reduction. As the entrants' optimal contracting and production decisions are negatively affected by the aggregate incumbent output, the entry of new firms implies an increase in both market size and the effective size of cost reduction for incumbents. In turn, this implies both a higher expected value of cost reduction and expected marginal profitability of effort, which makes it optimal for the incumbents to elicit higher managerial effort by strengthening incentives. It is worth noting that, even in the absence of the marginal-profitability-of-effort effect, a growing number of entrants strengthens the value-of-cost-reduction effect. Such case arises, for example, when marginal costs are public information and managerial effort is unprofitable beyond cost reduction.

The key to our main result is that incumbent firms are able to strategically pre-commit to managerial contracts, which in turn determine technological efficiency endogenously. The general intuition goes in line with the seminal works of Fudenberg and Tirole (1984) and Bulow, Geanakoplos, and Klemperer (1985). In a standard entry model, when an incumbent and an entrant compete in quantities (strategic substitutes), lowering the marginal cost of the incumbent decreases the entrant's total profits (since the incumbent's optimal output increases). Hence, when costs are endogenously determined, incumbents find it optimal to behave more aggressively in cost-reduction activities. In our framework, this corresponds to incumbents offering stronger managerial incentives which are observed by the entrant firms. Thus, by making a commitment to be more aggressive, the incumbents push the entrants into a more passive posture. This is an example of the "top-dog" strategy, according to the terminology proposed by Fudenberg and Tirole (1984). This sort of aggressive or accommodating behavior on behalf of the incumbent firms does not emerge under simultaneous competition because the incumbents fail to reap such benefits due to the lack of pre-commitment to any investment strategy. By contrast, under strategic complementarity, e.g., price competition, the aforementioned result is reversed because the incumbent firms would commit to a strategy of "underinvestment" (weakened managerial incentives) after which the entrants would optimally respond by lowering their prices. Fudenberg and Tirole (ibid.) call such underinvestment strategy to avoid stoking competition "puppy-dog ploy."

The paper is organized as follows. In Section 3.2, we review the related literature. In the next section, we outline the model. In Section 3.4, we solve for the equilibrium and present our main results. In Section 3.5, we present testable implications of our model. In Section 3.6, we analyze two extensions, hierarchical entry and price competition. We conclude in Section 3.7. All proofs are relegated to Appendix C.2, most of which follow from Result 1 in Appendix C.1.

3.2 Related Literature and Our Contribution

The astounding rise in both the level and incentive component of executive compensation packages over the past three decades is often attributed to changes in industry configurations. The idea is based on the Darwinian view of organizations, which states that, in order to survive and perform well, firms must solve governance problems by adapting their structure of managerial incentive contracts as product market competition rises. As mentioned in the previous section, several studies have exploited regulatory reforms to analyze how product market competition shapes the incentive structure of the executive compensation packages. Kole and Lehn (1999), and Palia (2000) study how the introduction of the Airline Deregulation Act in 1978 has altered the structure of the incentive contracts offered to CEOs in the U.S. airline industry. Crawford, Ezzell, and Miles (1995), Hubbard and Palia (1995), and Cuñat and Guadalupe (2009a), analyze the changes in executive pay in the U.S. banking sector following an important regulatory reform that permitted interstate banking during the 1980s. In the context of international trade, Cuñat and Guadalupe (2009b) study the effect of changes in foreign competition on executive pay in the U.S. firms. Dasgupta, Li, and Wang (2017) analyze the effect of industry-level tariff cuts on CEOs pay-performance sensitivity in the U.S. manufacturing sector. Overall, these studies confirm the view that one of the ways in which firms react to intensifying product market competition is by increasing the pay-performance sensitivity of their executive compensation packages.[2]

We build on Daughety's (1990) Stackelberg leadership model by endogenizing firm technology via managerial incentive contracts.[3] However, Daughety (ibid.) does not consider the possibility of incumbents using (endogenous) cost-reducing R&D

[2]In a related study, Karuna (2007) also finds a positive relationship between the degree of product substitutability and stock options granted to CEOs.

[3]Both the Stackelberg and Cournot settings of our model can be seen as special cases of a slightly more general model, which we refer to as the "base model." The base model may be of independent theoretical interest as it provides a simple method for analyzing comparative statics on the number of firms in Stackelberg and Cournot models under cost uncertainty. See Appendix C.1 for further details.

investments as a pre-commitment device for product-deterrence, as in Fudenberg and Tirole (1984) and Bulow, Geanakoplos, and Klemperer (1985). In our model, since the incumbents are able to pre-commit to strategic managerial incentive contracts, incumbent output is increasing in the number of entrants. Namely, a higher number of entrants implies a higher *expected marginal profitability of effort*. By contrast, in Daughety (1990), incumbent output is independent of the number of entrants.[4]

A couple of other papers also analyze the interaction between entry and R&D incentives in oligopoly with sequential moves. Etro (2004) considers a model of patent race where a monopolist leader faces a fringe of entrants. In the Stackelberg equilibrium under free entry, the incumbent monopolist innovates more aggressively because any profitable innovative opportunity would be reaped by new entrants until entry dissipates profit. Ishida, Matsumura, and Matsushima (2011) analyze a two-stage Cournot competition with ex-ante cost asymmetry, whereas we consider ex-ante symmetry. As in the present paper, investment by one firm in process innovation is an instrument for pre-commitment to expand output in order to deter the output of its rivals. However, none of the aforementioned papers considers the possibility of endogenous production technology in *managerial* firms via optimal incentive contracts; instead, they focus on the direct effect of increased competition on R&D investment.

In agency theoretic models relating product market competition to managerial incentives, competing against more firms invariably reduces equilibrium output and profits.[5] In turn, this lowers the value of attaining a cost reduction and thus makes it optimal to offer weaker managerial incentives (the so-called *scale* or *output* effect). In a framework of hidden information (about the realization of marginal costs), Martin (1993) assumes that the marginal productivity of managerial effort decreases in the number of active firms in a Cournot market, and hence, the equilibrium state-contingent contracts provide weaker incentives as the number of firms grows. Golan, Parlour, and Rajan (2015) also analyze managerial incentives in a Cournot oligopoly. As the expected product market profit of each firm depends on the likelihood of achieving a low marginal cost in the rival firms, the observed profit as a signal of managerial effort becomes noisier, and hence, the cost of incentive provision magnifies in a more competitive environment. This effect points in the same direction as the standard scale effect implying a negative association between competition and incentives.

[4]We owe this observation to an anonymous referee.

[5]See Legros and Newman (2014) for an excellent survey of the extant literature.

In order to counteract the negative effect of competition on managerial incentives due to lower product market profits, one thus requires to identify additional countervailing effects of product market competition on managerial incentives. The effect of competition on executive pay-performance sensitivity may be, in theory, non-monotonic. Hermalin (1992) models CEOs as receiving a fraction of the shareholder income. Because more intense competition erodes this income, managers tend to consume fewer "agency goods," i.e., expend more effort, as agency goods are assumed to be normal goods. Hermalin (1994) assumes that more firms in a Cournot market implies an exogenous decrease in the slope of the inverse market demand (with the intercept remaining constant), and hence, an exogenous increase in the market size of each firm is identified as a countervailing *business stealing effect*, apart from the standard value-of-cost-reduction effect. Schmidt (1997) shows that if a firm is more likely to go bankrupt in a more competitive environment, the manager tends to work harder to avoid liquidation of the firm's assets as liquidation implies a loss of reputation. The value-of-cost-reduction effect and the threat-of-liquidation effect do not often point in the same direction. Piccolo, D'Amato, and Martina (2008) build on Martin (1993), and identify an *agency effect*. In their model, profit-sharing contracts improve productive efficiency, which points in the direction opposite to the standard scale effect. Thus, they obtain an inverted-U relationship between competition and managerial effort. Raith (2003) analyzes a managerial incentive problem in a price-setting oligopoly with horizontal differentiation and privately realized marginal costs. He establishes a positive association between competition and managerial incentives by showing that in a free-entry equilibrium, managerial incentives increase due to a higher degree of product substitutability, market size, or lower cost of entry. Wu (2017) analyzes the interaction between product and labor markets in a model that assigns worker talent to heterogeneous firms. Greater product market competition, as measured by demand elasticity, results in a reallocation of more talented managers from smaller to larger firms, and hence, an increase in the value of managerial efforts in such firms. Consequently, firms strengthen managerial incentives, and the resulting wage distribution becomes more right-skewed.

Our approach is novel because we analyze a new mechanism through which product market competition affects executive pay-performance sensitivity. In particular, we study how incumbent firms adjust their managerial contracts optimally when new firms are about to enter the market. As mentioned earlier, a model of sequential quantity competition is appropriate to analyze the effect of increased competition following a regulatory reform. In line with the empirical evidence, we find a positive relationship—as competition rises, incumbents find it optimal to strengthen executive pay-performance sensitivity in order to reduce managerial slack. Furthermore, we also contribute to the literature by noting that the time to build production capacity in an industry is a key factor in studying how competition affects managerial incentives. In particular, it allows us to relate our model to the earlier literature that finds a negative association between competition and managerial incentives. Our analysis builds on previous literature and conforms to empirical findings.

Our paper is also related to a well-known strand of literature in which incentive contracts are assumed to be linear combination of profit and revenue (e.g., Fershtman and Judd, 1987; Sklivas, 1987; Vickers, 1985). In these models, managers choose output (or price, depending on whether firms compete in quantities or prices) to maximize the incentive scheme. Wang and Wang (2009) extend this framework to sequential managerial delegation and obtain results similar to Daughety (1990), i.e., a more equal distribution on leaders and followers results in higher industry output, lower price, and higher welfare. The main difference of our approach is that, in our case, managers receive state-contingent contracts and choose cost-reducing R&D efforts instead of quantity or price. As a consequence, firms' cost parameters become endogenous, which results in ex-post asymmetry.

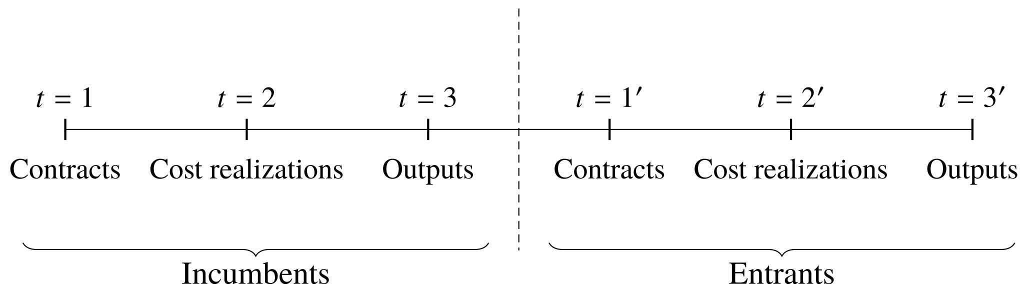
3.3 The Model

Specifications

The economy consists of two classes of risk neutral agents, $n + m$ ex-ante identical firms who compete in quantities in a market for a homogeneous good, and $n + m$ ex-ante identical managers. The firms are divided in two groups—namely, a subset I of $n \geq 1$ incumbents and a subset J of $m \geq 0$ entrants, with $I \cap J = \varnothing$. Our main objective is to analyze the effect of increased competition, i.e., an increase in $|J| = m$, on the optimal managerial contracts in the firms that belong to I. Until section 3.4, where we analyze cross-sectional variation in the number of incumbents, we consider I as a fixed collection of incumbent firms. A typical incumbent firm is denoted by i, and a typical entrant, by j. Often, for convenience, we will denote a generic firm (incumbent or entrant) by $k \in I \cup J$ with $|I \cup J| = n + m$.

Let q_k denote the production of firm k. The inverse market demand is given by $P = 1 - Q$, where Q denotes the aggregate industry output, and P the market price. Each firm k operates on a constant-returns-to-scale production technology with

Figure 3.1: Timing of events in the sequential quantity-setting oligopoly



marginal cost $c_k \in \{0, c\}$ where $0 < c < 1$. Initially, all firms have the inefficient technology, i.e., $c_k = c$ for all k. Each firm hires a manager whose principal task is to exert non-verifiable R&D effort in order to mark down the marginal cost to 0. The probability that the marginal cost is reduced is given by e_k, which is the effort exerted by the manager of firm k. Each firm k offers its manager a take-it-or-leave-it contract $(w_k(0),\ w_k(c))$ which is contingent on the realized marginal cost $c_k \in \{0, c\}$. Contracts are subject to limited liability of the managers. Managerial contracts are publicly observable, but the realized marginal costs remain private information of the firms. Every manager has the same effort cost function $\psi(e) = e^2/2$, and her outside option is normalized to 0.

Timing of Events

The timing of events, which is divided into two phases, is described in Figure 3.1. At date 1, the incumbents hire a manager apiece by offering publicly observable contingent contracts. At $t = 2$, the manager at each incumbent firm exerts non-verifiable effort, and the marginal cost of each incumbent is privately realized. At $t = 3$, the incumbents simultaneously set quantities. After observing the aggregate quantities set by the incumbents, the entrants repeat the timing at dates $t = 1', 2', 3'$. Finally, after date $t = 3'$, the market price is set, and the profits of all firms (incumbents and entrants) are realized.

Managerial Contract and Effort

Each manager k chooses her effort e_k optimally, given the contracts $w_k(0)$ and $w_k(c)$ at firm k. Because the realizations of marginal costs are independent, managerial contracts at each firm k are independent of the realizations of marginal costs at the rival firms. The optimal effort at firm k is given by:

$$e_k = \underset{\hat{e}_k}{\operatorname{argmax}} \left\{ \hat{e}_k w_k(0) + (1 - \hat{e}_k) w_k(c) - \frac{1}{2}\, \hat{e}_k^2 \right\} = w_k \equiv w_k(0) - w_k(c). \tag{IC}$$

The above is the *incentive compatibility constraint* of the manager at firm k in which w_k represents the incentive component of the managerial contract. Therefore, we will refer to a higher (lower) value of w_k, or equivalently, of e_k as 'stronger (weaker) managerial incentives.' We assume limited liability (non-negative income for the manager at each state of nature), i.e.,

$$w_k(c) \geq 0, \quad \text{and} \quad w_k(0) \geq 0. \tag{LL}$$

Finally, the expected utility of the manager at each firm k must be at least as high as her outside option 0, i.e., the *participation constraint* of the manager is given by:

$$u_k \equiv e_k w_k(0) + (1 - e_k) w_k(c) - \frac{1}{2}\, e_k^2 \geq 0. \tag{PC}$$

Quantity Competition

We follow Daughety, 1990, which is a generalization of the standard notion of Stackelberg competition, to model market competition in the present context. After managers have exerted effort, each incumbent i learns its marginal cost c_i privately. Then, the incumbent firms (the "leaders") choose quantities $(q_1, \ldots, q_n)$ simultaneously to maximize expected profit. After observing the aggregate incumbent quantity, $Q_I \equiv \sum_{i \in I} q_i$, the entrants choose managerial contracts simultaneously, taking Q_I as given. Following the choice of managerial effort, e_j, each entrant firm j learns its marginal cost c_j privately. Finally, the entrant firms (the "followers") choose quantities $(q_1, \ldots, q_m)$ in a Cournot fashion to maximize expected profit. We assume that in equilibrium all m entrants decide to enter, i.e., regardless of their own and the incumbents' cost realizations, each entrant finds it optimal to produce a positive output in equilibrium. This rules out the possibility that the incumbents may deter entry. The incumbents are also assumed to produce a positive output in equilibrium regardless of their realized marginal cost. This implies a restriction of the parameter space—namely, an upper bound on c. This is an innocuous but conservative assumption as the incentives to attain a low marginal cost would have been stronger otherwise. We solve for the equilibrium by backward induction, and show that it is unique and symmetric for incumbents and entrants.

3.4 Managerial Incentives in Sequential Oligopoly

Choice of Quantities and Managerial Efforts by the Entrants

Let $Q_J = \sum_{j \in J} q_j$ be the aggregate entrant output, and $q_{-j} \equiv Q_J - q_j = \sum_{k \in J\setminus\{j\}} q_k$, the aggregate output of the rival entrants. Further, let the managerial effort and bonus vectors be denoted by (e_i, e_j) and (w_i, w_j), respectively for $i \in I$ and $j \in J$. At the quantity setting stage, $t = 3'$, each entrant j takes Q_I and q_{-j} as given to solve

$$\max_{q_j} \; q_j(1 - Q_I - q_j - \mathbb{E}q_{-j} - c_j).$$

The subgame played by the entrants at the quantity setting stage, $t = 3'$, is simply a Cournot game among m firms with a residual demand $P = 1 - Q_I - \sum_{j \in J} q_j$. The quantity of each rival entrant is a random variable because its realized marginal cost is unknown to entrant firm j. The expected cost of firm j is $\mathbb{E}c_j = c(1 - e_j)$, where e_j is the incentive compatible level of managerial effort chosen at date $t = 2'$. Because the managerial contracts of all entrant firms are publicly observable, every firm j knows the expected cost of every rival firm. Further, let $e_{-j} \equiv \sum_{k \in J\setminus\{j\}} e_k$. The quantity and expected profit of each entrant firm in the subgame perfect equilibrium are respectively given by:

$$q_j(c_j, e_j, e_{-j}, Q_I) = \frac{2(1 - Q_I) - (m+1)c_j + (m-1)c(1 + e_j) - 2ce_{-j}}{2(m+1)},$$
$$\pi_j(c_j, e_j, e_{-j}, Q_I) = \left\{\frac{2(1 - Q_I) - (m+1)c_j + (m-1)c(1 + e_j) - 2ce_{-j}}{2(m+1)}\right\}^2.$$

Note that $\pi_j(c_j, e_j, e_{-j}, Q_I)$ is the expected market profit of each entrant firm j conditional on its realized cost, c_j. It depends on e_j even when conditioning on c_j because the effort exerted by the manager at firm j pins down the beliefs of the rival entrants about c_j. These beliefs affect the rivals' output decisions in the same way as e_{-j} affects those of firm j, so the effort exerted by the manager at firm j is profitable beyond its cost realization. If the realized marginal costs were publicly observable, the product market profits would not depend on managerial efforts; instead, they would depend on the observed numbers of high- and low-cost firms (cf. Golan, Parlour, and Rajan, 2015), and managerial effort would not be profitable beyond the value of cost reduction.

The optimal contracting problem at $t = 1'$ at each entrant firm j is solved in two stages (e.g., Grossman and Hart, 1983). First, firm j minimizes the expected incentive costs in order to implement a given level of effort subject to the constraints described in Section 3.3, i.e.,

$$C_j(e_j) = \min_{\{w_j(0),\, w_j(c)\}} e_j w_j(0) + (1 - e_j) w_j(c), \qquad (\text{Min}_j)$$

$$\text{subject to} \quad \text{(IC), (LL) and (PC).}$$

The value function, called the 'incentive cost function,' of the above minimization problem is given by:

$$C_j(e_j) = C(e_j) = e_j^2 \quad \text{for all} \quad j \in J.$$

In the second stage, firm j chooses the effort level e_j in order to maximize the expected profits

$$\Pi_j(e_j, e_{-j}, Q_I) \equiv e_j \pi_j(0, e_j, e_{-j}, Q_I) + (1 - e_j)\pi_j(c, e_j, e_{-j}, Q_I)$$

net of its incentive costs $C(e_j)$, i.e.,

$$\max_{e_j} \; \Pi_j(e_j, e_{-j}, Q_I) - C(e_j). \qquad (\text{Max}_j)$$

Let the equilibrium managerial effort in the entrant firms be denoted by $e_J(Q_I, m)$, which is derived from the first-order condition of the maximization problem (Max$_j$). It is analyzed in the following lemma.

Lemma 1 *Given the aggregate output Q_I of the incumbent firms, the equilibrium managerial effort in the entrant firms is unique, symmetric, and is given by:*

$$e_J(Q_I, m) = \frac{c[8m(1 - Q_I) + c(m^2 - 6m + 1)]}{2[4(m + 1)^2 + c^2(m - 1)^2]} \quad \textit{for all} \;\; j \in J. \qquad (\text{EE})$$

The higher the aggregate output of the incumbents, Q_I, the lower is the managerial effort in each entrant firm. This is because when the aggregate output of the incumbents expands, the entrants face a shrunken residual demand, and hence, it is optimal for each of them to offer weaker incentives to its manager, which elicit lower effort.

Quantity Choice of the Incumbents

To set output levels at date $t = 3$, the incumbents solve the following profit maximization:

$$\max_{q_i} \; \pi_i^q(q_i, Q_J) \equiv q_i(1 - q_i - \mathbb{E}q_{-i} - Q_J - c_i). \qquad (\text{Max}_q)$$

In setting quantities, the incumbents take into account the best response of the entrant firms and anticipate their managerial efforts. Let $q_j(c_j,\, e,\, Q_I)$ denote the quantity of an entrant firm j in the subgame perfect equilibrium for a common level of effort e (among the entrants), i.e., with $e_j = e$ for all $j \in J$. Then, the expected aggregate output of the entrants is given by:

$$\begin{aligned} Q_J(Q_I,\, m) &= \sum_{j \in J} \mathbb{E} q_j(c_j, e_J(Q_I, m), Q_I) \\ &= \kappa(m)(1 - C_\kappa Q_I), \end{aligned}$$

where

$$\kappa(m) \equiv \frac{(4 + c^2)m(m+1)}{4(m+1)^2 + c^2(m-1)^2}, \quad \text{and} \quad C_\kappa \equiv 1 - \frac{c(8 + c^2)}{2(4 + c^2)} \in (0,\, 1).$$

It is easily verified that $\kappa'(m) > 0$. Hence, the aggregate best response $Q_J(Q_I, m)$ is linear in the aggregate incumbent quantity Q_I, and it shifts upward as m grows. Importantly, $\partial^2 Q_J(Q_I,\, m)/\partial m \partial Q_I = -C_\kappa \kappa'(m) < 0$. This means that the incumbent output softens the impact of firm entry on the market price, or, equivalently, that more entrants make incumbent output more effective in deterring entrant output.[6] Therefore, (Max$_q$) takes the following form:

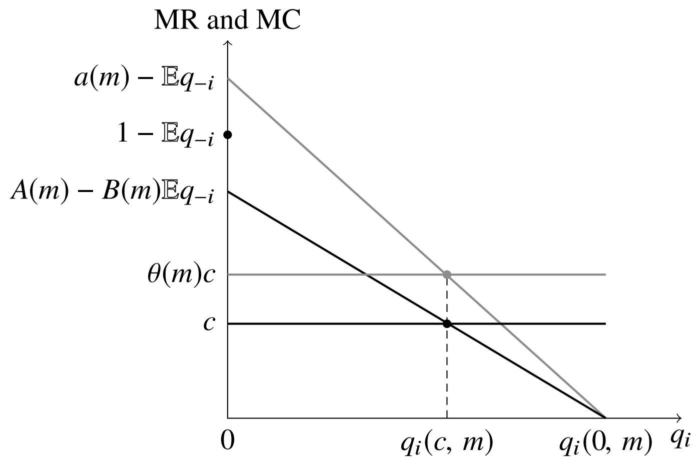
$$\begin{aligned} &\max_{q_i}\; q_i(1 - q_i - \mathbb{E}q_{-i} - Q_J(q_i + \mathbb{E}q_{-i},\, m) - c_i) \\ \iff \quad &\max_{q_i}\; q_i(A(m) - B(m)(q_i + \mathbb{E}q_{-i}) - c_i), \qquad (3.1) \end{aligned}$$

where $A(m) \equiv 1 - C_\kappa \kappa(m)$ and $B(m) \equiv 1 - \kappa(m)$. From the incumbents' perspective, entry of new firms implies two countervailing effects. On the one hand, more firms imply a lower market price, i.e., $A(m) < 1$. However, as the aggregate incumbent output diminishes the optimal effort and output of the entrants, it also implies that the price is less responsive to the incumbents output, i.e., $B(m) < 1$. This gives them more leeway; they can increase output without reducing the equilibrium price too much. For reasons that will become clear below, it is convenient to consider these effects in a different but equivalent way. Note that the solution to (3.1) is equivalent to the solution of the following 'normalized' problem:

$$\max_{q_i}\; q_i(a(m) - (q_i + \mathbb{E}q_{-i}) - \theta(m)c_i),$$

[6]This effect is similar to the one in Daughety (1990), except that, in our model, there is an extra strategic device to achieve this product-deterrence effect—namely, increasing the strength of managerial incentives to reduce marginal costs.

Figure 3.2: Optimal output of incumbents



Note: The optimal output of a representative incumbent firm for a given number of entrants under the actual (black line) and normalized (gray line) marginal revenue and cost functions.

where

$$a(m) \equiv \frac{A(m)}{B(m)} = \frac{1 - C_\kappa \kappa(m)}{1 - \kappa(m)}, \quad \theta(m) \equiv \frac{1}{B(m)} = \frac{1}{1 - \kappa(m)} \quad \text{with} \quad a'(m), \theta'(m) > 0.$$

That is, from the perspective of each incumbent i, the entry of new firms is equivalent to an increase in the market size, $a(m) > 1$, and the size of cost reduction, $\theta(m)c > c$. This means that, even though entrants reduce the market price, the market size increases from the incumbents' perspective as the price is less responsive to their output, which also equates to a higher size of cost reduction.

We depict the equivalence mentioned above graphically in Figure 3.2 by means of the marginal revenue (derived from the residual demand faced by i) and marginal cost curves. The black downward-sloping line is the marginal revenue function derived from the residual demand $[A(m) - B(m)\mathbb{E}q_{-i}] - B(m)q_i$ of incumbent i for $m > 0$. This marginal revenue function has a slope equal to $-2B(m)$. The maximum price is represented by the point $A(m) - B(m)\mathbb{E}q_{-i}$, and the market size is represented by $a(m) - \mathbb{E}q_{-i}$. Hence, the horizontal intercept of the marginal revenue function is given by $[a(m) - \mathbb{E}q_{-i}]/2$. If there were no entrants, we would have $A(0) = B(0) = 1$. Following the entry of at least one firm, we have $A(m) < 1$ and $B(m) < 1$. The black horizontal line is the marginal cost of a high-cost incumbent i. The equilibrium quantity $q_i(c, m)$ is determined by the intersection of the marginal revenue and marginal cost of the high-cost incumbent i for a given number of entrants m. The normalized marginal revenue function that is derived from the normalized residual demand $a(m) - \mathbb{E}q_{-i} - q_i$ with $a(m) > 1$, and the normalized marginal cost curve, $\theta(m)c$, are shown by the gray lines. The normalized marginal revenue curve is steeper than the actual marginal revenue curve because it has a slope equal to -2. These two normalized functions intersect at the same equilibrium output level $q_i(c, m)$ of each high-cost incumbent. For each low-cost incumbent, the equilibrium quantity is given by $q_i(0, m) = [a(m) - \mathbb{E}q_{-i}]/2$ because for such a firm i, $c_i = \theta(m)c_i = 0$.

Let $e_{-i} = \sum_{k \in I \setminus \{i\}} e_k$ be the aggregate managerial efforts of the rival incumbents. The equilibrium output and profit of incumbents are described in the following lemma.

Lemma 2 *Given the number of entrants, m, the privately realized marginal costs* $\{c_1, \ldots, c_n\}$*, and the managerial efforts* $\{e_1, \ldots, e_n\}$ *of the incumbent firms, the equilibrium quantity and profit of each incumbent firm are respectively given by:*

$$q_i(c_i, e_i, e_{-i}, m) = \frac{2a(m) - (n+1)\theta(m)c_i + (n-1)\theta(m)c(1+e_i) - 2\theta(m)ce_{-i}}{2(n+1)},$$

$$\pi_i(c_i, e_i, e_{-i}, m) = \frac{1}{\theta(m)} \left\{ \frac{2a(m) - (n+1)\theta(m)c_i + (n-1)\theta(m)c(1+e_i) - 2\theta(m)ce_{-i}}{2(n+1)} \right\}^2.$$

Although the equilibrium quantity and profit of each entrant j depend on the aggregate incumbent quantity Q_I, those of each incumbent firm i do not depend on the entrant quantity because the incumbents act as Stackelberg leaders in the product market. But they do depend on the number of entrants via the market size $a(m)$ and the size of cost reduction $\theta(m)c$ for the incumbent firms.

Equilibrium Managerial Efforts and Incentives in the Incumbent Firms

In the contracting stage at date 1, each incumbent firm i solves a maximization problem similar to (Max$_j$) (replace j by i everywhere, and drop Q_I from the profit function). Define by $\Delta\pi_i(e_i, e_{-i}, m) \equiv \pi_i(0, e_i, e_{-i}, m) - \pi_i(c, e_i, e_{-i}, m)$ the expected value of cost reduction of each incumbent firm i. The first-order condition for the contracting problem of each incumbent i is given by:

$$\begin{aligned} \frac{\partial \Pi_i(e_i, e_{-i}, m)}{\partial e_i} \equiv \Delta\pi_i(e_i, e_{-i}, m) \Bigg[e_i \frac{\partial \pi_i(0, e_i, e_{-i}, m)}{\partial e_i} \cdots \\ \cdots + (1-e_i) \frac{\partial \pi_i(c, e_i, e_{-i}, m)}{\partial e_i} \Bigg] = 2e_i. \qquad (\mathrm{FOC}_i) \end{aligned}$$

At the optimal managerial effort, the marginal benefit of effort is equalized with the marginal incentive cost. The left-hand-side of (FOC$_i$) is the marginal benefit of effort which comprises of two terms—namely, the *expected value of cost reduction*, given by $\Delta\pi_i(e_i, e_{-i}, m)$, and the *expected marginal profitability of effort*, given by $\mathbb{E}[\partial\pi_i(c_i, e_i, e_{-i}, m)/\partial e_i]$. On the right-hand-side of the above equation is the marginal incentive cost, $C'(e_i)$. Let the equilibrium managerial effort and incentives of incumbents be denoted by $e_I(m)$ and $w_I(m)$, respectively, which are determined from (FOC$_i$) and (IC). Note also that the manager's utility, i.e., the net level of compensation of the manager in each incumbent firm is given by:

$$u_I(m) \equiv e_I(m)w_I(m) - \frac{1}{2}(e_I(m))^2 = \frac{1}{2}(w_I(m))^2. \tag{3.2}$$

The following proposition describes the equilibrium managerial effort, incentives, and the level of executive compensation in the incumbent firms.

Proposition 1 *The equilibrium managerial effort and incentives of the incumbent firms are unique, symmetric, and given by:*

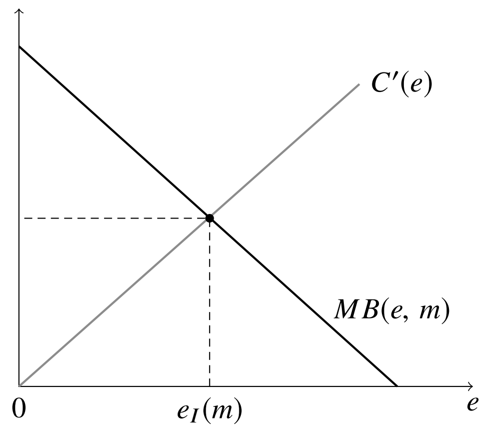
$$e_I(m) = w_I(m) = \frac{c[8a(m)n + \theta(m)c(n^2 - 6n + 1)]}{2[4(n+1)^2 + \theta(m)c^2(n-1)^2]} \in (0,\ 1). \tag{EI}$$

The equilibrium utility accrued to each manager at the incumbent firms is given by $u_I(m)$*, as in* (3.2)*. Moreover, for fixed* $n \geq 1$ *and* $m \geq 0$*, there exists* $\hat{c} \in (0,\ 1)$ *such that every firm (incumbent or entrant) produces a positive output in equilibrium regardless of its realized cost, provided that* $c \in (0,\ \hat{c})$*.*

Note that the first-order condition (FOC$_i$) defines implicitly the best reply in effort at firm i as a function of the aggregate effort at the rival incumbent firms, e_{-i}, which is linear and downward sloping (see proof of Result 1-(a) in Appendix C.1 for more details). Managerial efforts and incentives are strategic substitutes. As a result, the symmetric equilibrium effort $e_I(m)$ is the unique equilibrium outcome. Now, in order to determine the equilibrium managerial effort, we evaluate the first-order condition (FOC$_i$) at a common effort level e. The marginal benefit of effort, denoted by $MB(e, m)$, is strictly decreasing in e as shown by the downward sloping line in Figure 3.3. The upward sloping line, labeled $C'(e)$, is the marginal incentive cost as a function of e. The intersection of $MB(e, m)$ and $C'(e)$ yields the unique equilibrium managerial effort $e_I(m)$.

To find the upper bound $\hat{c}$ on the high marginal cost, note that the firm that produces the least in equilibrium is a high-cost entrant in a market in which all incumbents are low-cost. Let $q_i(c_i, e, m)$ denote the equilibrium output of an incumbent firm i at marginal cost c_i and a common effort level e (among the incumbents), which is

Figure 3.3: Equilibrium managerial effort of incumbents



obtained from Lemma 2. Let $\hat{Q}_I \equiv \sum_{i \in I} q_i(0, e_I(m), m)$ be the aggregate incumbents output at equilibrium when all of them are low-cost. The upper bound $\hat{c}$ is implicitly defined by $q_j(c, e_J(Q_I, m), Q_I) = 0$. For more details, see the proof of Proposition 1 in Appendix C.2.

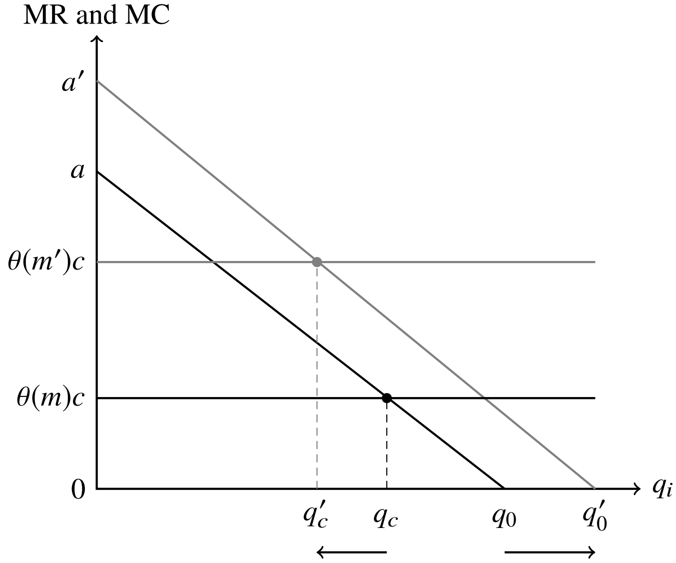
Competition and Managerial Incentives in the Incumbent Firms

Our objective is to analyze how increased competition due to the entry of new firms into the market affects the provision of managerial incentives at the incumbent firms. The following proposition states our main result.

Proposition 2 *Let $m' > m \geq 0$. Given any number of incumbents $n \geq 1$, entry of new firms induces each incumbent firm to elicit higher managerial effort, i.e., $e_I(m') > e_I(m)$, by providing stronger incentives, i.e., $w_I(m') > w_I(m)$, and higher compensation, i.e., $u_I(m') > u_I(m)$.*

The above proposition implies two sorts of effects of competition on managerial incentives. The first one is an *extensive margin* effect. The equilibrium managerial effort, incentives, and compensations, are lower in the incumbent firms in the absence of any entrant firm. Even the entry of only one firm which sets quantity as a Stackelberg follower induces the incumbents to elicit higher managerial effort by offering stronger incentives and compensation. This is a consequence of the fact that both $e_I(m)$ and $w_I(m)$ are strictly increasing in m. The second is an *intensive margin* effect. As the competitive pressure intensifies, each incumbent firm elicits higher managerial effort and offers stronger incentives and compensations.

Figure 3.4: Effect of entrants on incumbents output



Note: Effect of an increase in the number of entrants from m to m' on the equilibrium outputs of low- and high-cost incumbents. Let $a \equiv a(m) - \mathbb{E}q_{-i}$, $a' \equiv a(m') - \mathbb{E}q_{-i}$, $q_{c_i} \equiv q_i(c_i, e, m)$ and $q'_{c_i} \equiv q_i(c_i, e, m')$ for $c_i \in \{0, c\}$. Following an increase in m, q_0 increases to q'_0, but q_c decreases to q'_c.

The effect of an increase in the number of entrants on the equilibrium output of both low- and high-cost incumbents is shown in Figure 3.4. Entry of new firms induces the low-cost incumbents to produce more because both their market size and size of cost reduction increase. As $a(m)$ and $\theta(m)$ are both increasing functions of m, entry benefits the low-cost incumbents implying that $q_i(0, e, m)$ is strictly increasing in m. The same does not obtain for high-cost incumbents. The direction of the change in $q_i(c, e, m)$ following an increase in the number of entrants is *a priori* ambiguous because both $a(m)$ and $\theta(m)$ are increasing in m. From Figure 3.4, it is immediate to see that $q_i(c, e, m)$ is decreasing in m if and only if $a'(m) - \theta'(m)c < 0$, which turns out to be the case, i.e., the loss to the high-cost incumbents due to an increase in the size of cost reduction outweighs the gain from an increase in market size.[7]

In order to see why entry of new firms induces the incumbents to elicit higher managerial effort, we analyze how an increased number of entrants affects the expected value of cost reduction and the expected marginal profitability of effort of incumbents, i.e., the two terms in the left-hand-side of (FOC$_i$) evaluated at a

[7]Note that

$$a'(m) - \theta'(m)c = -\frac{c^3\kappa'(m)}{(4+c^2)[1-\kappa(m)]^2} < 0$$

because $\kappa'(m) > 0$. See Appendix C.2 for details.

common effort level e of the incumbent firms. The expected value of cost reduction is given by:

$$\Delta\pi_i(e, m) = \frac{c\,[4a(m) + \theta(m)(n-3)c - 2\theta(m)(n-1)ce]}{4(n+1)}.$$

Note that if all incumbents increase their effort level, e, the value of the cost reduction diminishes, since it is more likely for all incumbents to lower their marginal cost.[8] Importantly, this effect is amplified if there are more entrants. Hence, it can easily be shown that the 'value-of-cost-reduction effect' is positive, i.e., it is strengthened by the entry of new firms, if the common effort level is sufficiently low:

$$\frac{d\Delta\pi_i(e, m)}{dm} > 0 \quad \Longleftrightarrow \quad e < F(c) \equiv \frac{2(1 - C_\kappa)}{(n-1)c} + \frac{n-3}{2(n-1)}.$$

The expected marginal profit of effort is given by:

$$MPE_i(e, m) \equiv \mathbb{E}\left[\frac{\partial\pi_i(c_i, e, m)}{\partial e_i}\right] = \frac{c(n-1)\,[a(m) - \theta(m)c + \theta(m)ce]}{(n+1)^2}.$$

Note that the expected marginal profit of effort, $MPE_i(e, m)$, is increasing in the common effort level, e. That is, if all the firms believe that all of them are more likely to reduce the cost, then it is more profitable to exert effort. Intuitively, this effect is dampened by the number of incumbents n; however, it is amplified by the number of entrants, m. One can easily show that the 'marginal-profitability-of-effort-effect' is positive, i.e., it is strengthened by the entry of new firms, if the common effort level is sufficiently high:

$$\frac{dMPE_i(e, m)}{dm} > 0 \quad \Longleftrightarrow \quad e > G(c) \equiv \frac{c^2}{2(4 + c^2)}.$$

Overall, a higher number of entrants, m, steepens the expected value of a cost reduction and the marginal profit of effort. Recall that the marginal benefit of effort is the sum of the two effects, i.e.,

$$MB(e, m) = \Delta\pi_i(e, m) + MPE_i(e, m).$$

Therefore, a sufficient condition for the equilibrium effort $e_I(m)$ to be increasing in the number of entrants is for both effects to be positive:

$$G(c) < e_I(m) < F(c) \quad \Longrightarrow \quad \frac{\partial MB_i(e_I(m), m)}{\partial m} > 0$$
$$\Longleftrightarrow \quad \frac{de_I(m)}{dm} > 0.$$

[8]Note that this is not the case if a single firm increases its managerial effort. In such case, such firm would see its value of cost reduction rise while that of the other firms would diminish.

One way of proving Proposition 2 is to show that the inequality $G(c) < e_I(m) < F(c)$ holds for every $c \in (0, \hat{c})$, which is, indeed, the case.[9] Nonetheless, showing these inequalities directly is not the most suitable way of proving Proposition 2. The difficulty lies in that all three, $G(c)$, $e_I(m)$, and $F(c)$, are increasing functions of c. Because the upper bound $\hat{c}$ does not have a closed form solution (see Appendix C.2), it is simpler to verify that the inequality holds numerically, by doing an extensive search in the parameter space.[10] Therefore, the proof of Proposition 2 shows directly that $e_I(m)$ is an increasing function of m. Equivalently, one can show that it is an increasing function of $\kappa(m)$.

Equilibrium Firm Value and Market Profits

In this section, we focus on the effect that firm entry has on the equilibrium "firm value" and market profits of incumbents. The firm value is given by a firm's expected product market profits net of incentive costs. In equilibrium, the expected firm value corresponds to the value function of problem (Max$_j$). Let $V_i(m)$ and $\Pi_i(m)$ denote the expected firm value and expected market profits of incumbent $i \in I$ in equilibrium, respectively. Then,

$$V_i(m) \equiv \Pi_i(m) - C_i(e_i),$$

where $C_i(e_i) = e_i^2$ is the incentive cost function, defined in (Min$_j$), and e_i denotes the equilibrium effort of each incumbent firm i. The expected equilibrium profits are given by:

$$\Pi_i(m) = e_I(m)\pi_i(0, e_I(m), m) + (1 - e_I(m))\pi_i(c, e_I(m), m),$$

where $\pi_i(c_i, e_I(m), m)$ denotes the incumbent expected market profits, conditional on having realized cost c_i, on the equilibrium path (see Lemma 2). Recognizing that the firm value is the value function of a maximization problem, which depends itself on other value functions (the equilibrium profit functions) yields the following result.

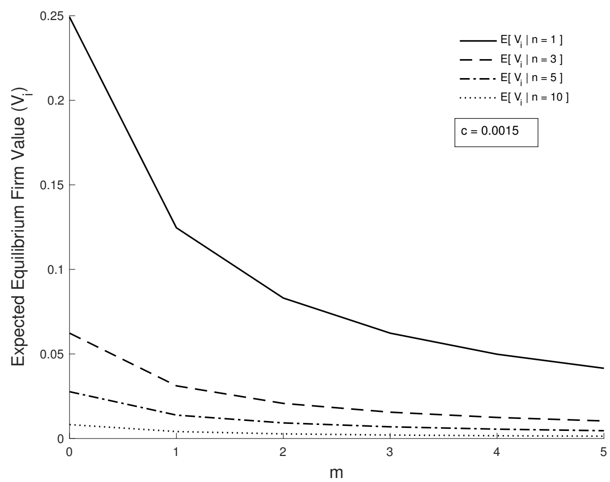
Proposition 3 *The equilibrium firm value of the incumbent firms is decreasing in the number of entrants.*

[9]It is easy to check that $F(c) > G(c)$ if and only if $n + 1 > 0$.

[10]Despite the fact that we prove this claim numerically, the proof of Proposition 2 is fully analytical. For more details on this and the other claims below that we show numerically, see Appendix C.3.

The intuition behind Proposition 3 is straightforward. The objective function in (Max$_j$) depends on m only through the equilibrium expected profit functions at realized marginal costs 0 and c, $\pi_i(0,\ e_I(m),\ m)$ and $\pi_i(c,\ e_I(m),\ m)$, respectively. Hence, it follows from the Envelope theorem that, if both of the expected profit functions are decreasing in m, then $V_i(m)$ is also decreasing in m. This is indeed the case because the objective function of the profit maximization problem, $\pi_i^q(q_i, Q_J)$ in (Max$_q$), only depends on m through Q_J. Since $Q_J = Q_J(Q_I,\ m)$ is strictly increasing in m, due to $\kappa'(m) > 0$, by the Envelope theorem, $\partial \pi_i^q(q_i,\ Q_J)/\partial Q_J < 0$ implies $\partial \pi_i(c_i,\ e_I(m),\ m)/\partial m < 0$ for $c_i \in \{0, c\}$.[11] Figure 3.5 shows graphically the result in Proposition 3.

Figure 3.5: Equilibrium expected incumbent firm value



Cross-Sectional Variation in the Number of Incumbents

Until now, we have maintained the number of incumbents fixed. To emphasize the significance of Proposition 2, we analyze how the equilibrium managerial effort varies with the number of incumbents, n, for a fixed number of entrants. This corresponds to comparing the managerial contracts offered at firms in two markets that face the same amount of competitive pressure (same number of entrants), but one of which is initially more competitive than the other (has more incumbents). Let $e_I(n,\ m) \equiv e_I(m)$, and $w_I(n,\ m) \equiv w_I(m)$, as defined in (EI), and $u_I(n,\ m) \equiv u_I(m)$, as defined in (3.2). (In this section we take the liberty of using notation defined previously, but change m for n to highlight the comparative statics in n.)

[11] We owe this observation to an anonymous referee.

Proposition 4 *Let $n' > n \geq 1$. Given any fixed number of entrants $m \geq 0$, incumbents in more competitive markets, i.e., in ones with more incumbents, elicit lower managerial effort, i.e., $e_I(n', m) < e_I(n, m)$, by providing weaker incentives, i.e., $w_I(n', m) < w_I(n, m)$, and lower compensation, i.e., $u_I(n', m) < u_I(n, m)$.*

The marginal benefit of effort, the left-hand-side of (FOC$_i$), differs in two ways in markets that are initially more competitive. The first one is through the standard 'output channel.' More incumbents means greater aggregate production by rivals, which implies that each firm optimally reduces its output at any realization of marginal cost as quantities are strategic substitutes. The expected value of cost reduction is lower as the effect of the number of incumbents on the optimal output level does not depend on the realized cost. That is, because $\partial q_i(0, e, n)/\partial n = \partial q_i(c, e, n)/\partial n < 0$, one obtains

$$\frac{d\Delta\pi_i(e, n)}{dn} = \frac{2}{\theta(m)} [q_i(0, e, n) - q_i(c, e, n)] \cdot \frac{\partial q_i(c_i, e, n)}{\partial n} < 0.$$

Notably, this 'value-of-cost-reduction effect' would work under the same logic if the realizations of marginal costs would have been public knowledge.

Due to the presence of privately realized marginal costs, a higher number of incumbents also changes the marginal benefit of managerial effort through the 'marginal-profitability-of-effort' effect. By eliciting a higher managerial effort, each incumbent i induces its rivals to believe that it has attained a low marginal cost, and hence, the aggregate rival quantity is lower in expectation. This raises the expected market price, and hence, the expected profits of firm i, i.e., $\partial\pi_i(c_i, e, n)/\partial e_i > 0$ at any realization of marginal cost. The marginal profitability of effort is greater if rivals believe that a given firm has attained cost reduction. In a more concentrated market (less firms), it is easier to influence rivals by affecting their beliefs, and hence, the marginal profitability of effort is increasing in the number of incumbents. In a market with many firms, by contrast, it is harder that more rivals are so influenced (as there are more firms). Thus, the marginal profitability of effort is decreasing in the number of incumbents in competitive markets. Formally,

$$\frac{d\mathbb{E}[\partial\pi_i(c_i, e, n)/\partial e_i]}{dn} = -\frac{c(n-3)[a(m) - \theta(m)c(1-e)]}{(n+1)^3},$$

which is strictly positive (negative) for $n < (>)\, 3$ as $c < \hat{c} < a(m)/\theta(m)$. (To see why this last inequality holds, note that the equilibrium output of a high-cost incumbent would be negative otherwise. See the proof of Proposition 1 in Appendix C.2 for more details.). Thus, the effect of an increase in n on the marginal profitability of effort may be positive or negative depending on the number of incumbents. Nonetheless, in either case, the aggregate effect of a higher number of incumbents on the marginal benefit of effort turns out to be always negative, i.e., $MB(e, n)$ in Figure 3.3 shifts down as n increases with $C'(e)$ remaining unaltered, and hence, $e_I(n, m)$ is decreasing in n. To see this formally, it suffices to note that

$$\begin{aligned}\frac{d\,MB(e,\,n)}{dn} &= \frac{d\Delta\pi_i(e,n)}{dn} + \frac{d\mathbb{E}[\partial\pi_i(c_i,\,e,n)/\partial e_i]}{dn}\\ &= -\frac{2c(n-1)[a(m)-\theta(m)c(1-e)]}{(n+1)^3} < 0.\end{aligned}$$

The crucial difference between varying the number of entrants and incumbents is that the entry of new firms affects the incumbents' output decision by altering the effective market size and the size of cost reduction. If there are more incumbents to start with, this alters directly the number of firms incumbents are competing against, and leaves the market size and the size of cost reduction unaffected. The juxtaposition of Propositions 2 and 4 conveys the main message of our paper. The fact that incumbents find it optimal to elicit higher managerial effort by offering steeper incentive contracts when they foresee the entry of new firms to the market, is due to incumbents being able to affect the entrants' output decisions by committing to an output level before they start producing.

Managerial Incentives in Simultaneous Oligopoly

The objective of this section is to analyze the effect of entry on managerial efforts and incentives in the incumbent firms when the m entrant firms are allowed set quantities simultaneously along with the n incumbents. The simultaneous setting is nothing but a Cournot market with $n+m$ symmetric firms and privately realized marginal costs $(c_1,\,\ldots,\,c_n,\,c_1,\,\ldots,\,c_m)$. The equilibrium managerial effort in each firm (incumbent or entrant) can be obtained directly from the expression (EI) as follows. As the entrants are treated equally as the incumbents, remove the entrants by setting $m=0$, and replace the number of incumbents, n, by $n+m$. In this case, $a(m)=\theta(m)=1$.

Let the symmetric equilibrium managerial effort and incentives in each firm (incumbent or entrant) be denoted by $e^{sim}(n+m)$ and $w^{sim}(n+m)$, respectively, and note that a manager's equilibrium utility is given by:

$$u^{sim}(n+m) \equiv e^{sim}(n+m)w^{sim}(n+m) - \frac{1}{2}\left(e^{sim}(n+m)\right)^2 = \frac{1}{2}\left(w^{sim}(n+m)\right)^2.$$

The effect of an entrant in an incumbent's optimal managerial effort and contract in this setting is analogous to considering a market that has one more incumbent (in this setting, entrants and incumbents are symmetric). Hence, we obtain the following corollary directly from Proposition 4.

Corollary 1 *Let $m' > m \geq 0$. In a simultaneous quantity-setting oligopoly in which m entrants set quantities and managerial contracts along $n \geq 1$ incumbents, entry of new firms implies that each incumbent elicits lower managerial effort, i.e., $e^{sim}(n+m') < e^{sim}(n+m)$, by providing weaker incentives to its manager, i.e., $w^{sim}(n+m') < w^{sim}(n+m)$, and lower compensation, i.e., $u^{sim}(n+m') < u^{sim}(n+m)$.*

The result in Corollary 1 is not new in the literature (see Golan, Parlour, and Rajan, 2015; Hermalin, 1994; Martin, 1993). The intuition behind it goes in the same line as the one underlying Proposition 4. The entrants affect the marginal benefit of effort of the incumbents through the 'value-of-cost-reduction' and 'marginal-profitability-of-effort' effects. As noted in section 3.4 above, in this case, entry implies a lower expected value of cost reduction for the incumbents, and also a lower expected marginal profit of effort as long as the market is already sufficiently competitive or, equivalently, the number of entrants is sufficiently high, i.e., as long as $n + m > 3$. Notably, as highlighted in the extant literature, the result in Corollary 1 does not depend on marginal costs being privately realized. On the contrary, we show that the negative effect of competition on managerial incentives in this setting is reinforced with privately realized marginal costs if the market is sufficiently competitive.

3.5 Testable Implications

Nature of Industry Competition and Time to Build Production Capacity

A key insight of our stylized model is the juxtaposition of Proposition 2 with Corollary 1. If entrants set quantities as Stackelberg followers, incumbent firms offer stronger managerial incentives as the number of entrants grows, whereas the opposite is obtained if they set quantities simultaneously, along with the incumbents.

Allen, Deneckere, Faith, and Kovenock (2000) examine the role of capacity precommitment as an instrument to deter production in a Bertrand-Edgeworth model of price competition. In particular, they analyze a three-stage game where an incumbent firm chooses its capacity first. Having observed the incumbent's capacity level, the entrant then chooses capacity. Finally at stage 3, the firms simultaneously set prices. The authors show that the outcome of this game coincides with that of a Stackelberg quantity competition. The crux of Allen, Deneckere, Faith, and

Kovenock's (2000) analysis is that production capacity cannot be adjusted instantaneously in the post-entry game, i.e., for a potential entrant, capacity requires *time to build*. This is in contrast with the Bertrand-Edgeworth model analyzed by Kreps and Scheinkman (1983), in which firms are able to adjust capacity instantaneously prior to engaging in simultaneous price competition. In this sense, the outcome of Kreps and Scheinkman (ibid.), which coincides with that of Cournot competition, corresponds to industries in which capacity requires *no* time build. This disparity in the time required to build production capacity leads to the following implication.

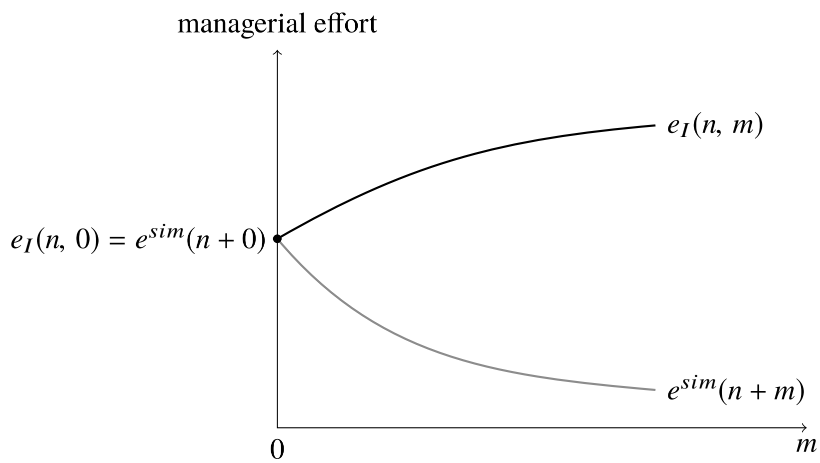
Implication 1 (i) *In an industry where production capacity requires 'time to build,' the incumbent firms offer higher managerial compensation and stronger incentive pay following an increase in the market competition induced by the entry of new firms.* (ii) *By contrast, if the production capacity can be adjusted instantaneously, entry of new firms implies that incumbents would provide lower compensation and weaker incentives to their managers following entry.*

The Stackelberg outcome is more plausible in an industry in which sequential capacity choices are followed by simultaneous price competition. In such industries, such as the airline or banking industries, the sluggishness of capacity adjustment gives rise to an output-deterrence effect due to capacity pre-commitment. By contrast, in industries in which production capacities can be built almost instantaneously, such as services and technology, building capacity does not have a pre-commitment value. Our results imply that this industry-specific feature is key when analyzing the effect of market product competition on executive compensation.

It is worth emphasizing that Implication 1 applies both at the *extensive* and the *intensive margins*. When entrants set quantity as Stackelberg followers, (i) firm entry increases incumbents' managerial effort, i.e., at the extensive margin, and (ii) a higher number of entrants increases each incumbent's managerial effort by a larger magnitude, i.e., at the intensive margin. Figure 3.6 depicts the juxtaposition of Proposition 2 with Corollary 1. From (EI) it follows that $e_I(n, 0) = e^{sim}(n + 0)$. In the absence of any entrant firm ($m = 0$), the equilibrium efforts coincide because it makes no difference whether entrants set managerial contracts and quantities after or along with the incumbents. Because $e_I(n, m)$ is strictly increasing in m, and $e^{sim}(n + m)$ is strictly decreasing in m, the equilibrium managerial incentives are not only higher when time is required to build capacity, but also their differences magnify as the number of entrants grows. Therefore, even a monopolist incumbent ($n = 1$)

would respond more aggressively to an increase in the threat of competition under time-to-build-capacity, whereas she would provide weaker managerial incentives if the time to build capacity were negligible.

Figure 3.6: Equilibrium managerial effort on the number of entrants



Note: Equilibrium managerial effort as a function of the number of entrants m under simultaneous and sequential quantity-setting oligopolies for a given number of incumbents n.

Equilibrium Social Welfare

In this section, we focus on the welfare analysis. We simplify the analysis by analyzing the equilibrium welfare numerically. We use a granular grid of the model's parameters to validate Implication 2 below (see Appendix C.3). The total welfare in the industry consists of three components: (i) consumer surplus (CS), (ii) total producer surplus of incumbents (PS_I), and (iii) total producer surplus of entrants (PS_J). We sketch how to compute each of these components in turn. We provide full details in Appendix C.4.

The consumer surplus is directly obtained from the demand function. Conditional on the total industry output, $Q = Q_I + Q_J$, the consumer surplus can be readily computed as $CS = Q^2/2$. Hence, the expected consumer surplus is given by $\mathbb{E}CS = 0.5 * \mathbb{E}Q^2$. To compute the producer surplus of the incumbents and the entrants, define the producer surplus of a generic firm $k \in K$, as the sum of its firm value and the utility of its manager. Since managerial wages are simply a transfer between a firm and its manager, the producer surplus of a firm is equal to its market profits net of its manager's effort cost. That is, the expected producer surplus of firm $k \in K$, denoted by PS_k is given by:

$$PS_k(m) = \Pi_k(m) - \psi(e_k),$$

where $\psi(e) = e^2/2$ is the managerial effort cost function, and e_k denotes the equilibrium effort of firm $k \in K$. Note that the producer surplus differs from the firm value in that, instead of accounting for the *incentive* costs of effort provision, it accounts for the effort costs from an *efficiency* point of view. The total producer surplus of incumbents is given by $PS_I = \sum_{i \in I} PS_i(m)$, and the total producer surplus of entrants is given by $PS_J = \sum_{j \in J} PS_j(m)$. The total welfare of the industry is defined by:

$$W = CS + PS_I + PS_J.$$

As a measure of social welfare, we use the expected total welfare at equilibrium, $\mathbb{E}W$. As in the previous subsections, define the analog welfare measures for the case in which entrants produce along the incumbents simultaneously, by W^{sim}, CS^{sim}, PS_I^{sim}, and PS_J^{sim}.

Implication 2 *Entry of new firms affects the consumer surplus, the producer surplus, and the social welfare in the following ways:*

(i) *The expected consumer surplus both under sequential (Stackelberg) and simultaneous (Cournot) competition, $\mathbb{E}\,CS$ and $\mathbb{E}\,CS^{sim}$ are increasing in the number of entrants. Moreover, the expected consumer surplus under sequential competition is higher than that under simultaneous competition, i.e., $\mathbb{E}\,CS > \mathbb{E}\,CS^{sim}$ for every $m \geq 1$ and $n \geq 1$;*

(ii) *The aggregate expected producer surplus both under sequential and simultaneous competition, $\mathbb{E}(PS_I + PS_J)$ and $\mathbb{E}(PS_I^{sim} + PS_J^{sim})$ are decreasing in the number of entrants. Moreover, the aggregate expected producer surplus under sequential competition is lower than that under simultaneous competition, i.e., $\mathbb{E}(PS_I + PS_J) < \mathbb{E}(PS_I^{sim} + PS_J^{sim})$ for every $m \geq 1$ and $n \geq 1$;*

(iii) *The expected social welfare both under sequential and simultaneous competition, $\mathbb{E}\,W$ and $\mathbb{E}\,W^{sim}$ are increasing in the number of entrants. Moreover, the expected social welfare under sequential competition is higher than that under simultaneous competition, i.e., $\mathbb{E}\,W > \mathbb{E}\,W^{sim}$ for every $m \geq 1$ and $n \geq 1$.*

According to Implication 2, regardless of the nature of competition, i.e., whether entrants produce after or along with the incumbents, social welfare increases with firm entry. Moreover, for any combination of parameter values, it also obtains that social welfare is higher when the incumbents are able to set quantities before the entrants. This can be seen in Figure 3.7 (top panel), which shows how social welfare is higher under sequential competition (blue curves) than under simultaneous competition (red curves). In the central and bottom panels of Figure 3.7, we plot the consumer surplus and the aggregate producer surplus. Notably, while the consumer surplus is always increasing in the number of entrants, the producer surplus is decreasing. Qualitatively, consumer surplus behaves in a similar fashion as social welfare with respect to entry. However, the producer surplus is higher in the simultaneous case than with sequential competition. All of these findings point to the conclusion that the effect of increased competition via firm entry is fiercer when incumbents behave as Stackelberg leaders. It is worth noting that the above finding that social welfare is higher under Stackelberg quantity competition is similar to what Daughety (1990), and Wang and Wang (2009) find, except that in our case the production technology is endogenized through managerial incentive contracts.

3.6 Extensions

Effect of Hierarchical Entry on Managerial Incentives

Entry of firms seldom takes place simultaneously. The Airline Deregulation Act of 1978 stipulated a transition period of three years over which several small carriers entered the U.S. airline industry sequentially. Even in the absence of entry barriers, some firms are quicker than others to learn about market conditions. Prescott and Visscher (1977) argue that "some entrants become aware of a profitable market before others or require longer periods of time in which to *tool up* [our italics]." In what follows, we analyze an entry game where firms enter sequentially. In particular, following the quantity choice of the incumbents, firms enter in a hierarchical fashion (as in Boyer and Moreaux, 1986). For simplicity, we consider only two entrants, i.e., $J = \{1, 2\}$. There are two consecutive periods of entry, entrant 1 enters in period 1, and entrant 2 enters the market in period 2. We show that hierarchical competition reinforces the effect of entry on managerial effort relative to the case when the post-entry quantity competition is simultaneous.

Note that when the two entrants compete simultaneously by setting quantities, the symmetric equilibrium managerial effort of the incumbent firms are given by $e_I(2)$ which is obtained by substituting $m = 2$ in the expression (EI) in Proposition 1.

Denote by $e_I^h(2)$ the symmetric equilibrium effort elicited by the incumbents under hierarchical entry of firms 1 and 2.

Proposition 5 *Incumbents elicit greater managerial effort under hierarchical entry than that under simultaneous entry, i.e.,* $e_I^h(2) \geq e_I(2)$ *for any number of incumbents,* $n \geq 1$.

Recall that the key determining factors of managerial efforts are the sizes of the market and cost reduction for the incumbent firms. The key to proving Proposition 5 is that, under hierarchical entry, both the market size and the size of the cost reduction for incumbents are higher relative to simultaneous entry. Hierarchical entry implies more intense competition because each predecessor produces more aggressively in order to deter the production of subsequent entrants. As a result, the incumbents, being the first movers, provide stronger incentives (relative to the case of simultaneous entry) in order to reduce managerial slack.

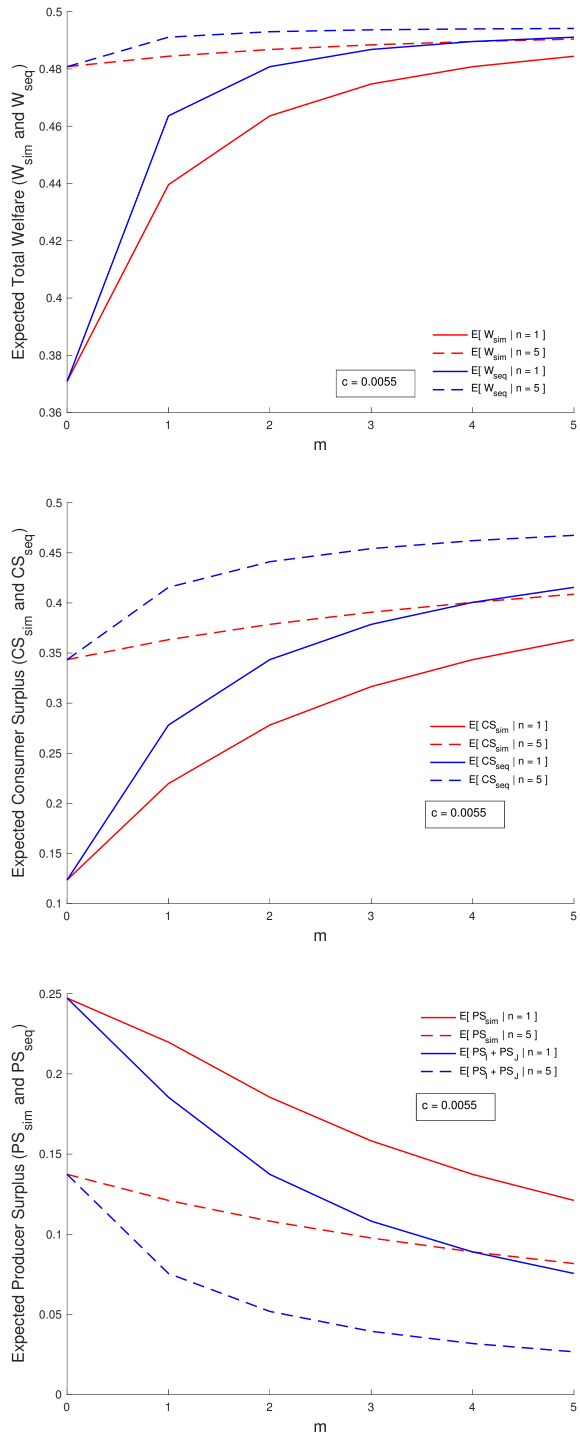
Effect of Price Competition on Managerial Incentives

In this section, we analyze the effect of price competition on managerial incentives. Consider the setting described in Section 3.3 with the only difference that firms set prices instead of quantities. Competition is à la Bertrand, i.e., all firms produce a single homogeneous good, and there is no capacity constraint. The timing of the game is analogous to the one described in Figure 3.1. Incumbents post prices at date $t = 3$, which the entrants observe, and then entrants set prices at date $t' = 3$.

Let p_k be the price set by firm $k \in I \cup J$, and let $P_I = \min\{p_i : i \in I\}$ be the lowest price among those set by the incumbents. Consider the sub-games played by the entrants and the incumbents once their respective marginal costs have been privately realized. Standard arguments show that these sub-games do not have equilibria in pure strategies.[12] We analyze symmetric equilibria with atom-less mixed strategies in the price-setting stages. Also, to simplify the analysis, we restrict attention to the symmetric equilibrium in the choice of managerial effort.

[12]First, note that the only case in which an entrant can obtain positive profits is if $P_I > 0$. Also, see that $P_I \leq c$ in equilibrium (since incumbents would obtain negative profits otherwise). To note that there is no equilibrium in pure strategies, see that (i) setting $p_j > 0$ is not an equilibrium since any other entrant could undercut this price and obtain all the demand, and (ii) setting $p_j = 0$ is also not an equilibrium since there exists the possibility that j is the only entrant with a low cost, in which case it would be profitable to increase the price marginally. Analogous arguments apply to the sub-game in which incumbents set prices.

Figure 3.7: Social welfare



Note: Social welfare (top), consumer surplus (center), and producer surplus (bottom) in the simultaneous and sequential oligopolies.

Proposition 6 *Under price competition,*

(a) *the symmetric equilibrium managerial effort of the entrants is given by $e_J^B(m, P_I)$ which solves*

$$\frac{e_J}{(1-e_J)^{m-1}} = \frac{P_I(1-P_I)}{2}. \tag{3.3}$$

(b) *Similarly, the symmetric equilibrium effort elicited by the incumbents is given by $e_I^B(m)$ which solves*

$$\frac{e_I}{(1-e_I)^{n-1}} = \frac{c(1-c)\left(1-e_J^B(m, c)\right)^m}{2}. \tag{3.4}$$

(c) *Both $e_J^B(m, P_I)$ and $e_I^B(m)$ are decreasing in the number of entrants m, i.e., $e_J^B(m', P_I) < e_J^B(m, P_I)$ and $e_I^B(m') < e_I^B(m)$ for $m' > m \geq 0$.*

Proposition 6 establishes that the equilibrium effort of incumbents is decreasing in the number of entrants m in a price-setting environment. The intuition behind this result lies in the expected equilibrium profits of a low-cost firm who sets its price according to an equilibrium in mixed strategies. Two observations that follow from the above proposition are worth noting.

- First, the expected profits in a Bertrand game with privately realized costs are the same as in the game with publicly observed ones.[13] In our case, this implies that the expected profits of a low-cost entrant are given by:

$$\pi_j(0, e_{-j}, P_I) = P_I(1-P_I)(1-e)^{m-1}, \tag{3.5}$$

  where $e_k = e$ for every $k \in J \setminus \{j\}$. Note that the expected profits in (3.5) correspond to the case with publicly observable costs. In this case, entrant j obtains non-negative profits if and only if it is the only entrant that attains cost reduction by setting price equal to P_I and serving the entire market demand (we assume that entrants have priority over incumbents if they set the same price since they can always undercut any positive price set by an incumbent marginally). From (3.5), one can easily see why the equilibrium effort of the entrants is decreasing in m. The likelihood of being the only entrant who attains cost reduction is decreasing in the number of entrants, which diminishes the marginal profitability of managerial effort.

[13]For a formal proof of this statement, see the proof of Proposition 6.

- Second, the equilibrium managerial effort elicited by the incumbent firms is decreasing in the number of firms when both incumbents and entrants set prices simultaneously. From (3.4), it is immediate to see that the left-hand-side is strictly increasing in both e_I and n, whereas the right-hand-side is constant with respect to n for a given value of m. Thus, as n increases, the left-hand-side of (3.4) shifts up, which implies that e_I^B decreases with n.

The expected profits of a low-cost incumbent are given by:

$$\pi_i(0,\ e_{-i}) = c(1-c)(1-e_J^B(m,\ c))^m(1-e)^{n-1}, \tag{3.6}$$

where $e_k = e$ for every $k \in I \setminus \{i\}$. The expected profit of a low-cost incumbent is also the same as that with known marginal costs. In this case, an incumbent obtains non-negative profits if and only if it is the only firm in the industry which succeeds in attaining cost reduction. Therefore, the only channel through which the number of entrants affects the incumbents' expected profits is the probability that all entrants fail in reducing marginal cost, which is given by $(1-e_J^B(m,\ c))^m$. One can easily see from (3.3) that this probability is decreasing in m. Even though each entrant is more likely to fail to attain the cost reduction individually as m increases, there are more of them, so the probability that all of them fail to attain it decreases in m. This, in turn, drives the profits of every incumbent to zero, which is sufficient to counteract the fact that the entrants themselves provide less effort when there are more of them. Therefore, in a price-setting environment, fiercer competition among the entrants themselves causes them to offer weaker managerial incentives, and, in turn, makes it profitable for the incumbents to weaken managerial incentives.

3.7 Conclusion

Motivated by empirical evidence, in this paper we investigate how firms adjust executive compensation packages following deregulation policies that intensify product market competition by allowing the entry of new firms. Using a standard incentive contracting model under quantity-setting oligopoly, we show that incumbent firms find it optimal to elicit higher managerial effort by offering stronger incentive contracts when they foresee entry of new firms into the product market. Our model allows us to tease out in detail the channels through which product market competition affects managerial incentives in a setting with firm entry. In our model, the key features that link the number of entrants with an incumbent's contracting problem are the market size and the size of cost reduction, both of which affect the marginal benefit of effort, through the expected value of cost reduction, and the expected marginal profitability of effort. By showing that firm entry increases both the market size and the size of cost reduction for incumbents, and analyzing, in turn, how these two affect the expected value of cost reduction and the expected marginal profitability of effort, we show that incumbents find it optimal to offer stronger managerial incentives when new firms enter the market. Furthermore, we also show that the magnitude in which incumbents strengthen managerial incentives is increasing in the number of entrants—a greater competitive pressure triggers a starker reaction by the incumbents.

Beyond conforming to the empirical regularities, our model also sheds light on how the nature of competition in product market affects managerial incentives. Namely, we explore the connection between the time to build production capacity in an industry and the effect that product market competition has on managerial incentives. We find that firm entry increases the pay-performance sensitivity of managerial contracts in markets in which production capacity takes time to build. In other words, the key driver of our result is that entrants act as Stackelberg followers in the product market by taking the aggregate output of incumbents as given. In the opposite case in which production capacity may be obtained instantaneously, i.e., entrants are symmetric to incumbents and set contracts and output simultaneously along with them, the association is negative—incumbents find it optimal to offer weaker managerial incentives as more firms enter the market.

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APPENDIX TO CHAPTER 1

A.1 Estimation Details

In this section, I show the steps to compute the simulated log-likelihood $\ell_n^{S_\omega,S_\upsilon,\psi}(\boldsymbol{\theta}\,|\,\mathbf{Z})$ in detail, see (1.30).

1. Simulate the conditional matching likelihood, using a logit-kernel:

   - Write the surplus of matching $M \in \mathbb{M}_i$ as:

     $$V_{s_\upsilon}(M \mid \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M) = \sum_{c,h} M(c,h)\left[\pi(\omega_{ch}, \mathbf{z}_{ch}\,|\,\boldsymbol{\theta}_T, \boldsymbol{\theta}_M) + \varepsilon_{cy_h}^{s_\upsilon} + \eta_{x_c h}^{s_\upsilon}\right],$$

     where $\varepsilon_c^{s_\upsilon} = \left(\varepsilon_{cy}^{s_\upsilon}\right)_{y\in Y}$ and $\eta_h^{s_\upsilon} = \left(\eta_{xh}^{s_\upsilon}\right)_{x\in X}$ are simulated structural errors.
   - To simulate $\varepsilon_c^{s_\upsilon}$ and $\eta_h^{s_\upsilon}$, let $\boldsymbol{\Gamma}_\varepsilon$ and $\boldsymbol{\Gamma}_\eta$ be the Cholesky factors of $\boldsymbol{\Sigma}_\varepsilon$ and $\boldsymbol{\Sigma}_\eta$, respectively. Draw fixed simulated values

     $$\begin{aligned}\tilde{\varepsilon}_c^{s_\upsilon} &\sim \text{iid } N(0, I_{|Y|}), \quad s_\upsilon = 1, \ldots, S_\upsilon,\\ \tilde{\eta}_h^{s_\upsilon} &\sim \text{iid } N(0, I_{|X|}), \quad s_\upsilon = 1, \ldots, S_\upsilon,\end{aligned}$$

     for every $c \in C_i$ and $h \in H_i$. Set $\varepsilon_c^{s_\upsilon} = \boldsymbol{\Gamma}_\varepsilon \tilde{\varepsilon}_c^{s_\upsilon}$, and $\eta_h^{s_\upsilon} = \boldsymbol{\Gamma}_\eta \tilde{\eta}_h^{s_\upsilon}$.
   - Define the simulated counterpart of the conditional matching likelihood $\mathcal{L}_M(M_i\,|\,\boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M)$ as

     $$\mathcal{L}_M^{s_\upsilon,\psi}(M_i\,|\,\boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M) = \frac{\exp\left\{V_{s_\upsilon}(M_i \mid \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M)/\psi\right\}}{\sum_{M\in\mathbb{M}_i}\exp\left\{V_{s_\upsilon}(M \mid \boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M)/\psi\right\}},$$

     where $\psi > 0$ is the smoothing constant of the logit-kernel.
   - Note that $\mathcal{L}_M^{s_\upsilon,\psi} = 1$ for all $M_i \in \mathbb{M}_i$ if $|\mathbb{M}_i| = 1$. Markets with a single prospective placement do not contribute to the matching likelihood.
   - As $\psi \to 0$, $\mathcal{L}_M^{s_\upsilon,\psi}$ tends to the indicator function over the choice set, given the simulated errors. Formally,

     $$\lim_{\psi\to 0}\mathcal{L}_M^{s_\upsilon,\psi}(M_i\,|\,\boldsymbol{\Omega}_i, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M) = 1_{\mathcal{A}(M_i|\boldsymbol{\Omega}_i,\mathbf{Z}_i,\boldsymbol{\theta}_T,\boldsymbol{\theta}_M)}(\boldsymbol{\upsilon}^{s_\upsilon}),$$

     where $\boldsymbol{\upsilon}^{s_\upsilon} = (\upsilon_M^{s_\upsilon})_{M\in\mathbb{M}_i}$ with $\upsilon_M^{s_\upsilon} = \sum_{c,h} M(c,h)\left[\varepsilon_{cy_h}^{s_\upsilon} + \eta_{x_c h}^{s_\upsilon}\right]$.

2. Integrate over $\mathbf{\Omega}_i$.

   - To simulate $\boldsymbol{\omega}_{ch}$, let $\mathbf{\Gamma}_\omega$ be the Cholesky factor of $\mathbf{\Sigma}_\omega$. Draw fixed simulated values

$$\tilde{\omega}_{ch}^{s_\omega} \sim \text{iid } N(0, I_{|\mathcal{R}_0|}), \quad s_\omega = 1, \ldots, S_\omega,$$

   for every $(c,h) \in C_i \times H_i$. Set $\boldsymbol{\omega}_{ch}^{s_\omega} = \mathbf{\Gamma}_\omega \tilde{\boldsymbol{\omega}}_{ch}^{s_\omega}$, and $\mathbf{\Omega}_i^{s_\omega} = \left(\boldsymbol{\omega}_{ch}^{s_\omega}\right)_{(c,h)\in C_i \times H_i}$.

   - The conditional outcome likelihood $\mathcal{L}_{\mathbf{T},\mathbf{R}}(\mathbf{T}_i, \mathbf{R}_i \mid M_i, \mathbf{\Omega}_i^{s_\omega}, \mathbf{Z}_i, \boldsymbol{\theta}_T)$ has a closed-form.

   - Define the simulated counterpart of the market-level likelihood $\mathcal{L}(M_i, \mathbf{T}_i, \mathbf{R}_i \mid \mathbf{Z}_i, \boldsymbol{\theta})$ as

$$\mathcal{L}^{S_\omega, S_\upsilon, \psi}(M_i, \mathbf{T}_i, \mathbf{R}_i \,|\, \mathbf{Z}_i, \boldsymbol{\theta}) = \frac{1}{S_\omega S_\upsilon} \sum_{s_\omega=1}^{S_\omega} \sum_{s_\upsilon=1}^{S_\upsilon} \mathcal{L}_M^{s_\upsilon,\psi}(M_i \,|\, \mathbf{\Omega}_i^{s_\omega}, \mathbf{Z}_i, \boldsymbol{\theta}_T, \boldsymbol{\theta}_M) \cdots$$
$$\cdots \times \mathcal{L}_{\mathbf{T},\mathbf{R}}(\mathbf{T}_i, \mathbf{R}_i \,|\, M_i, \mathbf{\Omega}_i^{s_\omega}, \mathbf{Z}_i, \boldsymbol{\theta}_T).$$

3. Add over markets and take logs:

   - Finally, define:

$$\ell_n^{S_\omega, S_\upsilon, \psi}(\boldsymbol{\theta} \,|\, \mathbf{Z}) = \sum_{i=1}^{n} \log \mathcal{L}^{S_\omega, S_\upsilon, \psi}(M_i, \mathbf{T}_i, \mathbf{R}_i \,|\, \mathbf{Z}_i, \boldsymbol{\theta}).$$

A.2 Matching Covariance

Claim 1 *The covariance matrix of the composite error $\boldsymbol{\upsilon} = (\upsilon_M)_{M \in \mathbb{M}(C,H)}$ is given by:*

$$cov(\upsilon_M, \upsilon_{M'}) = \sum_{c \in C} \sigma_\varepsilon(y_{M(c)}, y_{M'(c)}) + \sum_{h \in H} \sigma_\eta(x_{M(h)}, x_{M'(h)}). \tag{A.1}$$

Proof of Claim 1: Define $A(c,h) \equiv \varepsilon_{cy_h} + \eta_{x_c h}$. Note that

$$\begin{aligned} A(c,h)A(c',h') &= [\varepsilon_{cy_h} + \eta_{x_c h}][\varepsilon_{c'y_{h'}} + \eta_{x_{c'} h'}] \\ &= \varepsilon_{cy_h}\varepsilon_{c'y_{h'}} + \varepsilon_{cy_h}\eta_{x_{c'} h'} + \eta_{x_c h}\varepsilon_{c'y_{h'}} + \eta_{x_c h}\eta_{x_{c'} h'}. \end{aligned}$$

From Assumption 3, it follows

$$\begin{aligned} \mathbb{E}A(c,h)A(c',h') &= 1\{c = c'\}\mathbb{E}\varepsilon_{cy_h}\varepsilon_{c'y_{h'}} + 1\{h = h'\}\mathbb{E}\eta_{x_c h}\eta_{x_{c'} h'} \\ &= 1\{c = c'\}\sigma_\varepsilon(y_h, y_{h'}) + 1\{h = h'\}\sigma_\eta(x_c, x_{c'}). \end{aligned}$$

Since $\mathbb{E}\upsilon_M = \mathbb{E}\upsilon_{M'} = 0$,

$$\begin{aligned}
\text{cov}(\upsilon_M, \epsilon_{M'}) &= \mathbb{E}\upsilon_M \upsilon_{M'} \\
&= \mathbb{E}\left[\sum_{c,h} M(c,h)A(c,h)\right]\left[\sum_{c',h'} M'(c',h')A(c',h')\right] \\
&= \sum_{c,h}\sum_{c',h'} M(c,h)M'(c',h')\mathbb{E}A(c,h)A(c',h') \\
&= \sum_{c}\sum_{h,h'} M(c,h)M'(c,h')\sigma_\varepsilon(y_h, y_{h'}) \\
&+ \sum_{h}\sum_{c,c'} M(c,h)M'(c',h)\sigma_\eta(x_c, x_{c'}).
\end{aligned}$$

Note that $\sum_{h,h'} M(c,h)M'(c,h')\sigma_\varepsilon(y_h, y_{h'}) = \sigma_\varepsilon(y_h, y_{h'})$ for h and h' such that $M(c,h) = M'(c,h') = 1$, which is equivalent to $\sigma_\varepsilon(y_{M(c)}, y_{M'(c)})$. Using a symmetric argument in the second term yields the desired expression:

$$\text{cov}(\upsilon_M, \epsilon_{M'}) = \sum_{c\in C} \sigma_\varepsilon(y_{M(c)}, y_{M'(c)}) + \sum_{h\in H} \sigma_\eta(x_{M(h)}, x_{M'(h)}).$$

□

Proof of Proposition 1: For an arbitrary market with choice set $\mathbb{M}(C,H)$, let

$$\tilde{\upsilon}_M = \upsilon_M - \upsilon_{M_0} \quad \forall\, M \in \mathbb{M}(C,H) \setminus \{M_0\}, \tag{A.2}$$

for some fixed $M_0 \in \mathbb{M}(C,H)$. Standard results (e.g., Train, 2009) show that the covariance matrix of $\tilde{\boldsymbol{\upsilon}} \equiv (\tilde{\upsilon}_M)_{M\in\mathbb{M}(C,H)\setminus\{M_0\}}$ is identified up to a scale normalization. From (1.22), one can write the elements in the covariance matrix of $\tilde{\boldsymbol{\upsilon}}$ as follows:

$$\begin{aligned}
\text{cov}(\tilde{\upsilon}_{M'}, \tilde{\upsilon}_{M''}) &= \text{cov}(\upsilon_{M'} - \upsilon_{M_0}, \upsilon_{M''} - \upsilon_{M_0}) \\
&= \text{cov}(\upsilon_{M'}, \upsilon_{M''}) + \text{var}(\upsilon_{M_0}) - \text{cov}(\upsilon_{M'}, \upsilon_{M_0}) - \text{cov}(\upsilon_{M''}, \upsilon_{M_0}) \\
&= \sum_c \sigma_\varepsilon(y_{M'(c)}, y_{M''(c)}) + \sum_h \sigma_\eta(x_{M'(h)}, x_{M''(h)}) \\
&+ \sum_c \sigma_\varepsilon(y_{M_0(c)}) + \sum_h \sigma_\eta(x_{M_0(h)}) \\
&- \left[\sum_c \sigma_\varepsilon(y_{M_0(c)}, y_{M'(c)}) + \sum_h \sigma_\eta(x_{M_0(h)}, x_{M'(h)})\right] \\
&- \left[\sum_c \sigma_\varepsilon(y_{M_0(c)}, y_{M''(c)}) + \sum_h \sigma_\eta(x_{{}_0M(h)}, x_{M''(h)})\right],
\end{aligned} \tag{A.3}$$

$$\begin{aligned}\mathrm{var}(\tilde{\upsilon}_{M'}) &= \sum_c \sigma_\varepsilon(y_{M'(c)}) + \sum_h \sigma_\eta(x_{M'(h)}) \\ &+ \sum_c \sigma_\varepsilon(y_{M_0(c)}) + \sum_h \sigma_\eta(x_{M_0(h)}) \\ &- 2\left[\sum_c \sigma_\varepsilon(y_{M_0(c)}, y_{M'(c)}) + \sum_h \sigma_\eta(x_{M_0(h)}, x_{M'(h)})\right],\end{aligned} \tag{A.4}$$

where I write $\sigma_\eta(x) \equiv \sigma_\eta(x, x)$ and $\sigma_\varepsilon(y) \equiv \sigma_\varepsilon(y, y)$ to simplify notation.

First, I show how to identify the elements of the covariance matrix $\mathbf{\Sigma}_\eta$. Consider a market with three children whose types are given by $x, x', x'' \in X$ and a single home whose type is $y \in Y$. The set of feasible matchings in this market contains three matchings: $M_0 = (x, y)$, $M_1 = (x', y)$, and $M_2 = (x'', y)$, where I abuse notation and define the matching over the types of the children and homes. Using (A.3) and (A.4), one may see that the identified elements in the covariance matrix of $\tilde{\boldsymbol{\upsilon}}$ in this market are given by:

$$\sigma_1^* \equiv \frac{\mathrm{cov}(\tilde{\upsilon}_{M_1}, \tilde{\upsilon}_{M_2})}{\mathrm{var}(\tilde{\upsilon}_{M_1})} = \frac{\sigma_\eta(x', x'') + \sigma_\epsilon(y) + \sigma_\eta(x) - \sigma_\eta(x, x') - \sigma_\eta(x, x')}{2\sigma_\varepsilon(y) + \sigma_\eta(x) + \sigma_\eta(x') - 2\sigma_\eta(x, x')} \tag{A.5}$$

$$\sigma_2^* \equiv \frac{\mathrm{var}(\tilde{\upsilon}_{M_2})}{\mathrm{var}(\tilde{\upsilon}_{M_1})} = \frac{2\sigma_\varepsilon(y) + \sigma_\eta(x) + \sigma_\eta(x'') - 2\sigma_\eta(x, x'')}{2\sigma_\varepsilon(y) + \sigma_\eta(x) + \sigma_\eta(x') - \sigma_\eta(x, x')}. \tag{A.6}$$

Let $x = x_0$ and $y = y_0$, so

$$\sigma_1^* = \frac{\sigma_\eta(x', x'') + 1}{2 + \sigma_\eta(x')}. \tag{A.7}$$

Since $\sigma_\eta(x', x'') = \sigma_\eta(x')$ for $x'' = x'$, (A.7) identifies $\sigma_\eta(x')$ for an arbitrary $x' \in X$. Note that this implies that (A.7) also identifies $\sigma_\eta(x', x'')$ for arbitrary $x', x'' \in X$.

Second, I show how to identify the covariance matrix $\mathbf{\Sigma}_\varepsilon$. Consider a market with three children whose types are given by $x, x', x'' \in X$ and two homes whose types are $y, y' \in Y$. In this market, $\mathbb{M}(C, H)$ contains six matchings. Let M_0 be the matching that assigns placements (x, y) and (x', y'); M_1 the one that assigns (x, y) and (x'', y'), and M_2 the one assigning (x'', y) and (x', y'). Using (A.3), compute the following covariance:

$$\begin{aligned}\mathrm{cov}(\tilde{\upsilon}_{M_1}, \tilde{\upsilon}_{M_2}) &= \sigma_\varepsilon(y, y') + \sigma_\eta(x, x'') + \sigma_\eta(x', x'') \\ &+ \sigma_\varepsilon(y) + \sigma_\varepsilon(y') + \sigma_\eta(x) + \sigma_\eta(x') \\ &- \left[\sigma_e(y) + \sigma_\eta(x) + \sigma_\eta(x', x'')\right] \\ &- \left[\sigma_\varepsilon(y') + \sigma_\eta(x, x'') + \sigma_\eta(x')\right] \\ &= \sigma_\varepsilon(y, y').\end{aligned}$$

Let M_3 be the matching assigning the placements (x', y) and (x, y'), and note that:

$$\text{cov}(\tilde{\upsilon}_{M_1}, \tilde{\upsilon}_{M_3}) = \sigma_\varepsilon(y') + \sigma_\varepsilon(y, y') + \sigma_\eta(x') + \sigma_\eta(x', x'') - 2\sigma_\eta(x, x'').$$

Hence, two elements of the covariance matrix of $\tilde{\boldsymbol{\upsilon}}$ in this market are given by:

$$\sigma_3^* = \frac{\text{cov}(\tilde{\upsilon}_{M_1}, \tilde{\upsilon}_{M_2})}{\text{var}(\tilde{\upsilon}_{M_1})} = \frac{\sigma_\varepsilon(y, y')}{2\sigma_\varepsilon(y') + \sigma_\eta(x') + \sigma_\eta(x'') - 2\sigma_\eta(x', x'')}$$
$$\sigma_4^* = \frac{\text{cov}(\tilde{\upsilon}_{M_1}, \tilde{\upsilon}_{M_3})}{\text{var}(\tilde{\upsilon}_{M_1})} = \frac{\sigma_\varepsilon(y') + \sigma_\varepsilon(y, y') + \sigma_\eta(x') + \sigma_\eta(x', x'') - 2\sigma_\eta(x, x'')}{2\sigma_\varepsilon(y') + \sigma_\eta(x') + \sigma_\eta(x'') - 2\sigma_\eta(x', x'')}.$$

Since $\boldsymbol{\Sigma}_\eta$ is identified, the previous two equations define the 2-by-2 system of equations:

$$\sigma_3^* = \frac{\sigma_\varepsilon(y, y')}{2\sigma_\varepsilon(y') + H}$$
$$\sigma_4^* = \frac{\sigma_\varepsilon(y') + \sigma_\varepsilon(y, y') + K}{2\sigma_\varepsilon(y') + H},$$

where H and K are known constants. Identification of $\boldsymbol{\Sigma}_\varepsilon$ follows from noting that the above system of equations has a unique solution for $\sigma_\varepsilon(y')$ and $\sigma_\varepsilon(y, y')$, in terms of identified quantities. $\square$

Appendix B

APPENDIX TO CHAPTER 2

B.1 Details on Example

In this appendix, I provide further details for the example in Section 2.3. In order to compute the expected optimal value on $t = 1$, given that policy a^0 was chosen on the first period, I condition on the different possible cases. For example, assume that the younger type-0 child is matched in the first period, $a^0 = 0$, and that a single type-0 parent arrives on period $t = 1$. There are eight possible cases for the state variables $(c_{10}^1, c_{21}^1, c_{00}^1, c_{01}^1)$. In all cases, $c_{21}^1 = 1$ since the type-1 child remained unmatched in the first period. The cases are determined by each of the other state variables; each may equal 0 or 1 with $c_{10}^1 \sim \text{Bernoulli}(b(0,1))$ and $c_{00}^1, c_{01}^1 \sim \text{Bernoulli}(\mu)$. Let

$$V^1\left(P_0^1 = 1, P_1^1 = 0, c_{10}^1, c_{21}^1, c_{00}^1, c_{01}^1\right)$$

denote the expected optimal value of this case. For example, if the first-period match is broken and two children arrive in the second period, $(c_{10}^1, c_{21}^1, c_{00}^1, c_{01}^1) = (1,1,1,1)$, it is optimal to match the youngest type-0 child who just arrived on $t = 2$, leading to:

$$V^1\left(P_0^1 = 1, P_1^1 = 0, c_{10}^1 = 1, c_{21}^1 = 1, c_{00}^1 = 1, c_{01}^1 = 1\right) \cdots$$
$$\cdots = -3 + \delta\left(b(0,0)\cdot V_3^2 + (1 - b(0,0))\cdot V_2^2\right).$$

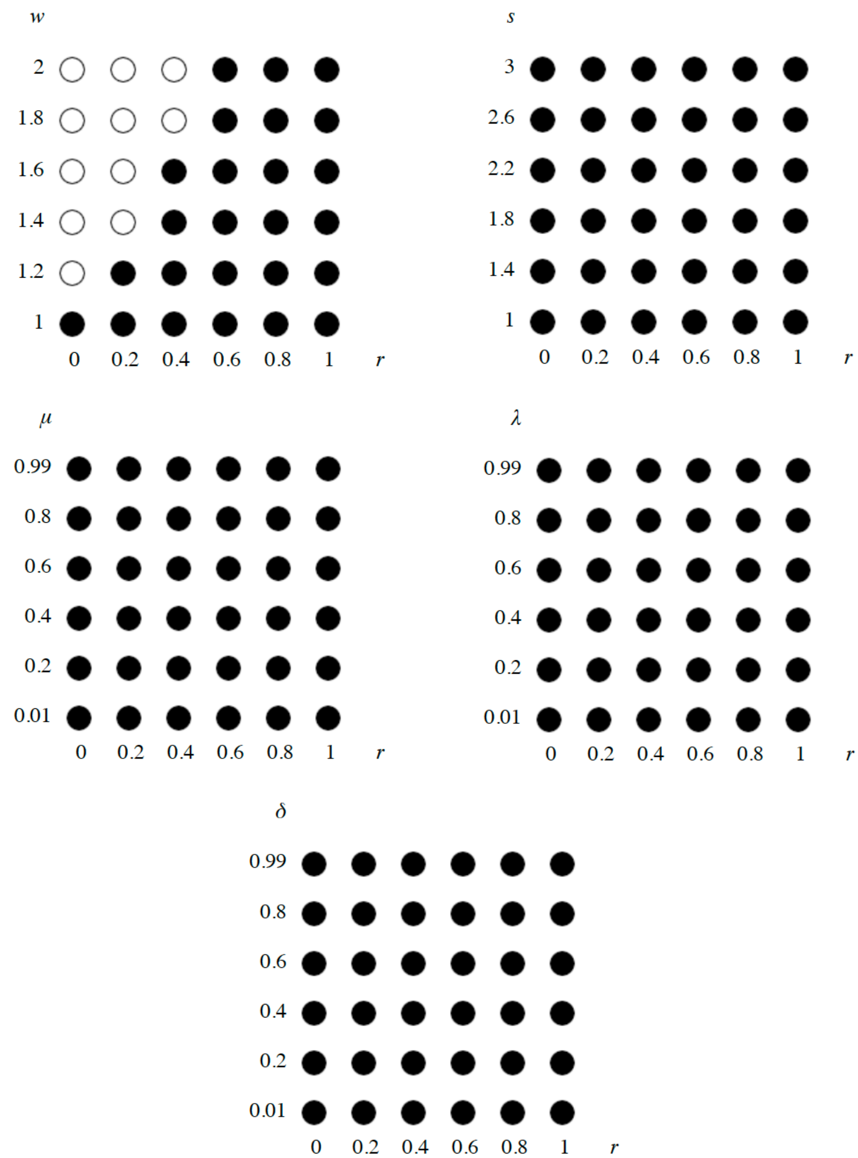
Three children are left unmatched in the market, and one of them is emancipated in the next period. In period $t = 2$, there are three available children from previous periods with probability $b(0,0)$ (if the second-period match breaks up), and two with probability $1 - b(0,0)$ (if the match does not break up). One may compute $V^1\left(P_0^1, P_1^1, c_{10}^1, c_{21}^1, c_{00}^1, c_{01}^1\right)$ in a similar way for each of the corresponding eight cases with $(P_0^1 = 1, P_1^1 = 0)$. Taking the expectation over the eight cases yields the optimal expected value $V^1\left(P_0^1 = 1, P_1^1 = 0; a^0 = 0\right)$. In order to cover all the remaining cases, the same must be done for each of the four possible cases of (P_0^1, P_1^1), where $P_0^1, P_1^1 \sim \text{Bernoulli}(\lambda)$. Thus, one obtains $V^1(a^0 = 0)$ by taking the expectation:

$$
\begin{aligned}
V^1\left(a^0 = 0\right) = {} & (1-\lambda)^2 \cdot V^1\left(P_0^1 = 0, P_1^1 = 0; a^0 = 0\right) \\
& + \lambda(1-\lambda) \cdot V^1\left(P_0^1 = 1, P_1^1 = 0; a^0 = 0\right) \\
& + (1-\lambda)\lambda \cdot V^1\left(P_0^1 = 0, P_1^1 = 1; a^0 = 0\right) \\
& + \lambda^2 \cdot V^1\left(P_0^1 = 1, P_1^1 = 1; a^0 = 0\right).
\end{aligned}
$$

The optimal expected value on $t = 1$ conditional on matching the older type-1 child on the first period, $V^1\left(a^0 = 1\right)$, may be obtained in an analogous way.

B.2 Plots of Observation 4

Figure B.1: Parameter regions in which Observation 4 holds



Note: The black dots indicate the parameter values in which Observation 4 holds with $T = \infty$. Observation 4: assume there are two parents available, one of each type, if there are two type-0 children and one type-1 child, older than the two type-0 children, the only child matched to a same-type parent is the youngest type-0 child. Benchmark parameter values: $w = 1$, $s = 2$, $\mu = 0.75$, $\lambda = 0.5$, and $\delta = 0.98$. The x-axis varies r in all figures. Other parameters are varied from the benchmark one at a time: w (top-left), s (top-right), μ (middle-left), λ (middle-right), and δ (bottom).

Appendix C

APPENDIX TO CHAPTER 3

C.1 The Base Model

The subgames played by the entrants and the incumbents have the same underlying structure. In this section, we analyze a more general version of the simultaneous quantity competition model with a fixed number N of firms, called the *base model*, which yields most of the results in Section 3.4. Let K be the set of firms, with $|K| = N \geq 1$, and index firms with k. Let $P = A - BQ$ be the inverse market demand with A, $B > 0$. The marginal cost of a representative firm k is given by c_k, with $c_k \in \{0, c\}$ with $c \in (0, \bar{c})$. The upper bound $\bar{c}$ is such that all firms produce a positive output in equilibrium regardless of their realized marginal costs. We will prove that such bound exists. In what follows, we describe the main results of the base model.

Result 1 *Consider the base model.*

(a) *The equilibrium effort is symmetric and unique across all firms. It is given by:*

$$e(N) = \frac{c[8AN + c(N^2 - 6N + 1)]}{2[4B(N+1)^2 + c^2(N-1)^2]}\,. \tag{EC}$$

Moreover, the second order condition associated with the individual firm maximization problem in a Bayesian Cournot equilibrium is satisfied for every firm if all of them produce a positive quantity in equilibrium.

(b) *If A is independent of c, then there is $\bar{c} \in (0, A)$ such that, if $c \in (0, \bar{c})$, every firm produces a strictly positive quantity of output and elicits strictly positive level of managerial effort in a symmetric equilibrium, regardless of its realized marginal cost.*

(c) *If $c \in (0, \bar{c})$, the equilibrium effort in* (EC) *is decreasing in the number of firms, i.e., for every $N \geq 1$, we have $e'(N) < 0$.*

Proof Let q_k denote the production of any firm $k \in I \cup J$.

(a) Once all the contracts are observed and marginal costs are privately realized, each firm k solves

$$\max_{q_k} \quad q_k[A - B(q_k + \mathbb{E}q_{-k}) - c_k].$$

The first-order condition of the above maximization problem is given by:

$$\begin{aligned}
& A - 2Bq_k - B\mathbb{E}q_{-k} - c_k = 0 \\
\iff\ & 2Bq_k = A - c_k - B\mathbb{E}q_{-k} \\
\iff\ & q_k(c_k,\, \mathbb{E}q_{-k}) = \frac{A - c_k}{2B} - \frac{1}{2}\,\mathbb{E}q_{-k}.
\end{aligned} \tag{C.1}$$

Taking expectation in the above equation, we get

$$\mathbb{E}q_k = \frac{A - \mathbb{E}c_k}{2B} - \frac{1}{2}\,\mathbb{E}q_{-k} = \frac{A - c(1 - e_k)}{2B} - \frac{1}{2}\,\mathbb{E}q_{-k}. \tag{C.2}$$

Summing the above over k, we get

$$\begin{aligned}
& \sum_{k=1}^{N} \mathbb{E}q_k = \frac{N(A - c)}{2B} + \frac{c}{2B}\sum_{k=1}^{N} e_k - \frac{N-1}{2}\sum_{k=1}^{N} \mathbb{E}q_k \\
\iff\ & \sum_{k=1}^{N} \mathbb{E}q_k = \frac{1}{B(N+1)}\left[N(A - c) + c\sum_{k=1}^{N} e_k\right].
\end{aligned} \tag{C.3}$$

On the other hand, (C.2) can be written as

$$\begin{aligned}
& \mathbb{E}q_k = \frac{A - c(1 - e_k)}{2B} - \frac{1}{2}\left(\sum_{l=1}^{N} \mathbb{E}q_l - \mathbb{E}q_k\right) \\
\iff\ & \frac{1}{2}\mathbb{E}q_k = \frac{A - c(1 - e_k)}{2B} - \frac{1}{2B(N+1)} \cdot \left[N(A - c) + c\sum_{k=1}^{N} e_k\right] \\
\iff\ & \mathbb{E}q_k = \frac{A - c + c(Ne_k - e_{-k})}{B(N+1)},
\end{aligned} \tag{C.4}$$

where $e_{-k} = \sum_{l \in K\setminus\{k\}} e_l$. Thus, using the fact that $\mathbb{E}q_{-k} = \sum_{l=1}^{N} \mathbb{E}q_l - \mathbb{E}q_k$, and substituting for $\sum_{l=1}^{N} \mathbb{E}q_l$ and $\mathbb{E}q_k$ from (C.3) and (C.4), from (C.1) we obtain the quantity and profit of each firm in the Bayesian Cournot equilibrium, which are respectively given by:

$$q_k(c_k,\, e_k,\, e_{-k}) = \frac{2A - (N+1)c_k + (N-1)c(1 + e_k) - 2ce_{-k}}{2B(N+1)} \tag{C.5}$$

$$\pi_k(c_k,\, e_k,\, e_{-k}) = B\left(\frac{2A - (N+1)c_k + (N-1)c(1 + e_k) - 2ce_{-k}}{2B(N+1)}\right)^2. \tag{C.6}$$

At date 1, each firm k chooses the optimal managerial incentives to solve

$$\max_{e_k} \; e_k \pi_k(0, e_k, e_{-k}) + (1 - e_k)\pi_k(c, e_k, e_{-k}) - e_k^2. \tag{C.7}$$

The expected value of cost reduction is defined as $\Delta\pi_k(e_k, e_{-k}) := \pi_k(0, e_k, e_{-k}) - \pi_k(c, e_k, e_{-k})$. For firm k, it is given by:

$$\Delta\pi_k(e_k, e_{-k}) = \frac{c[4A + (N-3)c + 2(N-1)ce_k - 4ce_{-k}]}{4B(N+1)}. \tag{C.8}$$

Also, note that

$$\begin{aligned} e_k \cdot \frac{\partial \pi_k(0, e_k, e_{-k})}{\partial e_k} + (1 - e_k) \cdot \frac{\partial \pi_k(c, e_k, e_{-k})}{\partial e_k} \cdots \\ \cdots = \frac{(N-1)c[A - c + c(Ne_k - e_{-k})]}{B(N+1)^2}. \end{aligned} \tag{C.9}$$

Using the expressions (C.8) and (C.9), the first-order condition of the maximization problem in (C.7) is given by:

$$\begin{aligned} & \Delta\pi_k(e_k, e_{-k}) + e_k \cdot \frac{\partial \pi_k(0, e_k, e_{-k})}{\partial e_k} + (1 - e_k) \cdot \frac{\partial \pi_k(c, e_k, e_{-k})}{\partial e_k} = 2e_k \\ \Longleftrightarrow \quad & \frac{c[8AN + (N^2 - 6N + 1)c + 2(N-1)(3N+1)ce_k - 8Nce_{-k}]}{4B(N+1)^2} = 2e_k. \end{aligned} \tag{FOC$'_k$}$$

Condition (FOC$'_k$) defines the best response (in effort) $e_k(e_{-k})$ of the manager at firm k, which is given by:

$$\begin{aligned} e_k(e_{-k}) = \underbrace{\frac{c[8AN + c(N^2 - 6N + 1)]}{2[4B(N+1)^2 - c^2(N-1)(3N+1)]}}_{\alpha \equiv \alpha(N, A, B)} \cdots \\ \cdots - \underbrace{\left(\frac{4c^2N}{4B(N+1)^2 - c^2(N-1)(3N+1)}\right)}_{\beta \equiv \beta(N, A, B)} e_{-k}. \end{aligned} \tag{BR$'_k$}$$

The best response is linear and downward sloping. Let $e_K = \sum_{k \in K} e_k$. Summing over all k, in equilibrium: $e_K = N\alpha - \beta \sum_k e_{-k}$. Thus,

$$e_K = \frac{N\alpha}{1 + \beta(N-1)},$$

where we use $\sum_k e_{-k} = (N-1)e_K$. As $e_K = e_{-k} + e_k$, the equilibrium effort is given by:

$$e_k = \frac{\alpha}{1 + \beta(N-1)}.$$

Replacing the values for the constants α and β yields the equilibrium effort given in (EC). Because effort choices are strategic substitutes with linear best response functions, there exists a unique symmetric equilibrium.

Next, we show that the second order condition is satisfied for every firm if all of them produce a positive output in equilibrium. Note that the second-order condition of firm k's maximization problem (C.7) is given by:

$$
\begin{aligned}
& 2\left(\frac{\partial \Delta\pi_k}{\partial e_k}\right) + e_k \cdot \frac{\partial^2 \Delta\pi_k}{\partial e_k^2} + \frac{\partial^2 \pi_k(c,\,\cdot)}{\partial e_k^2} - 2 \le 0 \\
\iff \quad & \frac{c^2(N-1)}{B(N+1)} + \frac{c^2(N-1)^2}{2B(N+1)^2} - 2 \le 0. \qquad (\mathrm{SOC}_k)
\end{aligned}
$$

Note that (SOC_k) is strict for $N = 1$ and it is equivalent to

$$
\frac{1}{B} \le \frac{4(N+1)^2}{c^2(N-1)(3N+1)} \quad \text{for} \quad N \ge 2. \qquad (\mathrm{SOC}'_k)
$$

Let $q_k(c_k) \equiv q_k(c_k,\, e_k,\, e_{-k})$ with $e_k = e(N)$ for every $k \in I \cup J$. Note that $q_k(0) - q_k(c) = c/2B$, so $q_k(c) > 0$ for all k implies

$$
\frac{1}{B} < \frac{2}{cN} \cdot \sum_k q_k(0). \qquad (\mathrm{C.10})
$$

The upper bound on $1/B$ in (C.10) is lower than the one in (SOC'_k) as, by construction, $\sum_k q_k(0) < 1$ (otherwise the equilibrium price would be negative), and $4(N+1)^2/(N-1)(3N+1) > 1$ for each $N > 0$.

(b) We prove the existence of $\bar{c} \in (0,\, A)$ such that $c \in (0, \bar{c})$ implies $q_k(c_k) > 0$ in equilibrium for every $k \in K$ and $c_k \in \{0, c\}$. Fix $N \ge 1$. Write $e(N,\, c) \equiv e(N)$. From (C.5), see that the symmetric equilibrium production of a high-cost firm is lower than that of a low-cost firm and satisfies:

$$
\begin{aligned}
q_k(c) = \frac{2(A-c) - (N-1)ce(N,\, c)}{2B(N+1)} > 0 & \\
\iff \quad f(N,\, c) \equiv \frac{2(A-c)}{c(N-1)} - e(N,\, c) > 0. & \qquad (\mathrm{C.11})
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \lim_{c \to 0} f(N,\, c) = \infty, \\
& f(N,\, A) = 0 - \frac{A^2(N+1)^2}{2[4B(N+1)^2 + A^2(N-1)^2]} < 0.
\end{aligned}
$$

Therefore, by Intermediate Value Theorem, there is $c_0 \in (0, A)$ such that $f(N, c_0) = 0$. If c_0 is unique, then take $\bar{c} = c_0$. Otherwise, take $\bar{c} = \min\{c_0\}$. Next, we prove that $e(N, c) > 0$ for $c \in (0, \bar{c})$, which is equivalent to the following:

$$8AN + c(N^2 - 6N + 1) > 0. \tag{C.12}$$

Given that $A > c$, we have

$$8AN + c(N^2 - 6N + 1) > 8cN + c(N^2 - 6N + 1) = c(N + 1)^2 > 0,$$

which proves (C.12) for all $N > 0$.

(c) Fix $N \geq 1$. Differentiating (EC) with respect to N, we obtain

$$e'(N) = -\frac{2c(N^2 - 1)[8B(A - c) + c^2(2A - c)]}{[4B(N + 1)^2 + c^2(N - 1)^2]^2}.$$

The above expression is negative for $A > c$ and $N \geq 1$. It is immediate to see that $e(N)$ is strictly increasing in A and strictly decreasing in B for all $N \geq 1$.

This completes the proof. □

C.2 Proofs

Most of the following proofs follow directly from the analysis of the base model, see Result 1 in Appendix C.1.

Proof of Lemma 1

The proof directly follows from Result 1-(a) with $A = 1 - Q_I$, $B = 1$, and $N = m$.□

Proof of Lemma 2

The proof directly follows from the proof of Result 1 with $A = A(m)$, $B = B(m)$, $a(m) \equiv A(m)/B(m)$, $\theta(m) \equiv 1/B(m)$, and $N = n$; see equations (C.5) and (C.6). □

Proof of Proposition 1

The maximization problem of each incumbent i is given by:

$$\begin{aligned} &\max_{q_i} \; q_i(1 - q_i - \mathbb{E}q_{-i} - Q_J(q_i + \mathbb{E}q_{-i}) - c_i) \\ \Longleftrightarrow \; &\max_{q_i} \; q_i[\underbrace{(1 - C_\kappa \kappa(m))}_{A(m)} - \underbrace{(1 - \kappa(m))}_{B(m)}(q_i + \mathbb{E}q_{-i}) - c_i]. \end{aligned}$$

Therefore, setting $a \equiv a(m) = A(m)/B(m)$, $\theta \equiv \theta(m) = 1/B(m)$, and $N = n$, it follows from Result 1-(a) that

$$e_I(m) = \frac{c[8a(m)n + \theta(m)c(n^2 - 6n + 1)]}{2[4(n+1)^2 + \theta(m)c^2(n-1)^2]}\,.$$

Recall that the subgame played by the entrants is equivalent to the base model with $B = 1$, $a = A/B = 1 - Q_I$, and $\theta = 1/B = 1$. We cannot apply the bound in Result 1-(b) directly as a depends on c (Q_I is an equilibrium object that depends on the model's parameters). Obtain the equilibrium output of a high-cost entrant by replacing $a = 1 - Q_I$ and $\theta = 1$ in (C.5):

$$q_j(c, Q_I) = \frac{2(1 - Q_I - c) - c(m-1)e_J(m,\, Q_I)}{2(m+1)}, \tag{C.13}$$

where $e_J(m,\, Q_I)$ is the optimal effort of the entrants given in (EE). Because low-cost entrants produce more than high-cost ones in equilibrium, the interior solution condition is equivalent to $q_j(c, Q_I^*) > 0$, where Q_I^* is the total output of the incumbents in the symmetric equilibrium. Note that $q_j(c, Q_I)$ is decreasing in Q_I as

$$\begin{aligned} &\frac{\partial q_j(c, Q_I)}{\partial Q_I} < 0 \\ \Longleftrightarrow\ & 2 - \frac{4c^2 m(m-1)}{4(m+1)^2 + c^2(m-1)^2} > 0 \\ \Longleftrightarrow\ & m^2(4 - c^2) + m(8 - c^2) + 4 + c^2 > 0. \end{aligned}$$

Hence, a high-cost entrant produces the least when all incumbents have low costs. Let $q_i(0)$ be the optimal output of a low-cost incumbent, so the interior solution for each entrant j requires $q_j\left(c, \sum_{i \in I} q_i(0)\right) > 0$. By (C.13), this is equivalent to

$$\sum_i q_i(0) < 1 - c - \frac{c(m-1) \cdot e_J\left(m,\, \sum_i q_i(0)\right)}{2}. \tag{C.14}$$

From (C.5), the equilibrium output of a low-cost incumbent is given by:

$$q_i(0) = \frac{2a(m) + \theta(m)c(n-1)(1 - e_I(m))}{2(n+1)}. \tag{C.15}$$

Note that $a(m) \to 1$, $\theta(m) \to m + 1$, and $e_I(m) \to 0$ as $c \to 0$, and hence, we have

$$\lim_{c \to 0} \sum_i q_i(0) = \frac{n}{n+1} < 1 = \lim_{c \to 0} 1 - c - \frac{c(m-1) \cdot e_J\left(m,\, \sum_i q_i(0)\right)}{2}.$$

Therefore, from (C.14), there exists $\bar{c}_J > 0$ such that every entrant produces a positive output in equilibrium, provided $c \in (0,\, \bar{c}_J)$. Furthermore, $\bar{c}_J < 1 - Q_I^*$ as

(C.14) also implies

$$c < c + \frac{c(m-1) \cdot e_J\left(\sum_i q_i(0)\right)}{2} < 1 - \sum_i q_i(0) \leq 1 - Q_I^*.$$

Finally, we characterize the interior solution condition of the incumbents. From (C.5), the interior solution condition for incumbents, $q_i(c) > 0$, is equivalent to

$$e_I(m) < \frac{2(A(m) - c)}{c(n-1)} = \frac{2(a(m) - \theta(m)c)}{\theta(m)c(n-1)}. \tag{C.16}$$

Because $e_I(m) \to 0$ as $c \to 0$, there is $\bar{c}_I > 0$ such that all incumbents produce a positive output in equilibrium provided that $c \in (0,\ \bar{c}_I)$, although the right-hand-side of (C.16) tends to ∞. Moreover, $\bar{c}_I < A(m) = a(m)/\theta(m)$ because (C.16) does not hold if $c > a(m)/\theta(m)$. Define $\hat{c} = \min\{\bar{c}_J,\ \bar{c}_I\}$ to obtain the appropriate bound. By Result 1-(a), every firm's second order condition of the optimal contracting problem is satisfied if $c \in (0,\ \hat{c})$. □

Proof of Proposition 2

We first establish that $\kappa(m)$ is strictly increasing in m. Note that

$$\kappa'(m) = \frac{(4+c^2)[(4+c^2)(m+1)^2 - 4c^2m^2]}{(4(m+1)^2 + c^2(m-1)^2)^2}.$$

The numerator of the above expression is strictly positive if and only if

$$\underbrace{\frac{4+c^2}{4c^2}}_{h(c)} > \left(\frac{m}{m+1}\right)^2.$$

Note that $h(c)$ is strictly decreasing on $[0,\ 1]$ with $\min\{h(c)\} = h(1) = 5/4 > 1$. The right-hand-side of the above inequality is always strictly less than 1 for $m \geq 1$. Hence, $\kappa'(m) > 0$. Next,

$$\begin{aligned} A(m) &= 1 - C_\kappa \kappa(m) &\implies \quad & A'(m) = -C_\kappa \kappa'(m), \\ B(m) &= 1 - \kappa(m) &\implies \quad & B'(m) = -\kappa'(m). \end{aligned}$$

Because

$$e_I(m) = \frac{c[8nA(m) + c(n^2 - 6n + 1)]}{2[4(n+1)^2B(m) + c^2(n-1)^2]},$$

we have

$$e_I'(m) = \frac{8\kappa'(m)[(n+1)^2 e_I(m) - C_\kappa cn]}{2[4(n+1)^2B(m) + c^2(n-1)^2]}.$$

Thus, $e_I'(m) > 0$ if and only if

$$e_I(m) > \frac{nC_\kappa}{(n+1)^2} \quad \iff \quad \frac{8nA(m) + c(n^2 - 6n + 1)}{2[4(n+1)^2 B(m) + c^2(n-1)^2]} > \frac{nC_\kappa}{(n+1)^2}. \tag{C.17}$$

We prove the following condition:

$$\begin{aligned} & \frac{8nA(m) + c(n^2 - 6n + 1)}{2[4(n+1)^2 B(m) + c^2(n-1)^2]} > \frac{n}{(n+1)^2} \\ \iff & \underbrace{A(m) - B(m)}_{\kappa(m)(1-C_\kappa)} > \frac{c[2cn(n-1)^2 - (n+1)^2(n^2 - 6n + 1)]}{8n(n+1)^2}. \end{aligned} \tag{C.18}$$

which implies (C.17) because $C_\kappa < 1$. Note that $A(m) - B(m)$ is a strictly increasing function of m as $A'(m) - B'(m) = \kappa'(m)(1 - C_\kappa) > 0$. So, in order to prove condition (C.18), it suffices to show that the inequality holds for $m = 1$. Note that

$$A(1) - B(1) = \kappa(1)(1 - C_\kappa) = \frac{4 + c^2}{8} \cdot \frac{c(8 + c^2)}{2(4 + c^2)} = \frac{c(8 + c^2)}{16}.$$

Hence, for $m = 1$, (C.18) boils down to:

$$8 + c^2 > \underbrace{\frac{2[2cn(n-1)^2 - (n+1)^2(n^2 - 6n + 1)]}{n(n+1)^2}}_{H(n;\,c)}. \tag{C.19}$$

It is easy to show that $H(n;\, c)$ is strictly decreasing in n for all $c \in (0,\, 1)$. Thus, $H(n;\, c)$ achieves a maximum at $n = 1$, which is equal to $H(1;\, c) = 8 < 8 + c^2$, and hence, the proposition. □

Proof of Proposition 3

The proof is in the text. It follows directly from applying the envelope theorem to the profit maximization problem, and then to the contracting problem. The key is to first note that Q_J is increasing in m, then that the profit functions $\pi_i(c_i,\, e,\, m)$ are decreasing in Q_J and only depend on m through Q_J (by the envelope theorem). Hence, $\pi_i(c_i,\, e,\, m)$ is decreasing in m for $c_i \in \{0,\, c\}$. Lastly, note that by the envelope theorem, the firm value $V_i(m)$ only depends on m through the profit functions. Since $V_i(m)$ is increasing in both profit functions, it is decreasing in the number of entrants m. □

Proof of Proposition 4

The proof directly follows from Result 1-(c) with $A = A(m)$, $B = B(m)$, and $N = n$. □

Proof of Corollary 1

The corollary follows directly from Proposition 4. Because $e_I(n, m)$ is strictly decreasing in n by Proposition 4 and $e^{sim}(n+m) = e(n+m, 0)$, $e^{sim}(n+m') < e^{sim}(n+m)$ for every $m' > m$. $\square$

Proof of Proposition 5

A direct but more mechanical way to prove Proposition 2 is to show that $e_I(m)$ is strictly increasing in $\kappa(m)$. Note that

$$e_I(m) = \frac{c[8n(1 - C_\kappa \kappa(m)) + c(n^2 - 6n + 1)]}{2[4(n+1)^2(1-\kappa(m)) + c^2(n-1)^2]} \equiv \hat{e}_I(\kappa(m))\,.$$

It is easy to show that

$$\text{sign}[\hat{e}_I'(\kappa)] = \text{sign}\,\underbrace{[2c(n+1)^2(8n + c(n^2 - 6n + 1) - 4ncC_\kappa(4(n+1)^2 + c^2(n-1)^2)]}_{h(n,c)}\,.$$

For all $n > 0$ and $c \in (0,\ 1)$, $h(n,\ c) > 0$, and hence, $\hat{e}_I(\kappa)$ is strictly increasing in κ.

To show that the managerial effort elicited by the incumbents is higher under hierarchical entry than that under simultaneous entry, the only thing we require to show is that, under hierarchical entry, $\kappa(2)$ is higher than that under simultaneous entry. First, consider the case of simultaneous entry. Note that

$$\kappa(2) = \frac{6(4 + c^2)}{36 + c^2}\,.$$

Next, consider hierarchical entry. The last mover, entrant 2 solves

$$\max_{q_2}\ q_2\,(1 - Q_I - q_1 - q_2 - c_2)\,,$$

where Q_I is the aggregate incumbent output, and q_1 is the production of entrant 1. The optimal output and profit of entrant 2 are respectively given by:

$$q_2(c_2) = \frac{1 - (Q_I + q_1) - c_2}{2},$$
$$\pi_2(c_2) = \frac{(1 - (Q_I + q_1) - c_2)^2}{4}.$$

The optimal managerial effort of entrant 2 is given by:

$$e_2 = \frac{\pi_2(0) - \pi_2(c)}{2} = \frac{c(2 - 2(Q_I + q_1) - c)}{8}.$$

Thus, the expected output of entrant 2 is given by:

$$\mathbb{E}q_2(Q_I+q_1)=e_2q_2(0)+(1-e_2)q_2(c)=\underbrace{\frac{c^2(2-c)+8(1-c)}{16}}_{G_2(c)}-\underbrace{\frac{4+c^2}{8}}_{K_2(c)}(Q_I+q_1).$$

In previous stage of entry, entrant 1 solves

$$\max_{q_1}\ q_1\left(1-Q_I-q_1-\mathbb{E}q_2(Q_I+q_1)-c_1\right).$$

Following the same procedure as in the case of entrant 2, we obtain

$$\mathbb{E}q_1(Q_I)=\underbrace{\frac{32-32c+c^3(12-2c+c^2)}{4(4-c^2)^2}}_{G_1(c)}-\underbrace{\frac{4+c^2}{8-2c^2}}_{K_1(c)}\cdot Q_I.$$

Using the recursive formulation, we thus get

$$\begin{aligned}Q_J(Q_I)=\mathbb{E}q_1+\mathbb{E}q_2&=G_1(c)-K_1(c)Q_I+G_2(c)-K_2(c)\underbrace{(Q_I+G_1(c)-K_1(c)Q_I)}_{Q_I+\mathbb{E}q_1}\\&=G_2(c)+G_1(c)(1-K_2(c))-(K_2(c)+K_1(c)(1-K_2(c)))Q_I,\end{aligned}$$

i.e., the aggregate best reply of the entrants is linear in Q_I. Each incumbent i thus solves

$$\begin{aligned}&\max_{q_i}\ q_i\left(1-(q_i+\mathbb{E}q_{-i})-Q_J(q_i+\mathbb{E}q_{-i})-c_i\right)\\\iff&\max_{q_i}\ q_i\left(a^h(2)-(q_i+\mathbb{E}q_{-i})-\theta^h(2)c_i\right),\end{aligned}$$

where

$$\begin{aligned}a^h(2)&=\frac{1-[G_2(c)+G_1(c)(1-K_2(c))]}{1-[K_2(c)+K_1(c)(1-K_2(c))]}=\frac{4}{4-c^2}+c\left(\frac{12}{4-3c^2}-\frac{c}{4+2c}\right),\\\theta^h(2)&=\frac{1}{1-[K_2(c)+K_1(c)(1-K_2(c))]}=\frac{16}{4-3c^2}\end{aligned}$$

are respectively the market size and the size of cost reduction of the incumbents under hierarchical entry. Note that

$$\theta^h(2)=\frac{16}{4-3c^2}=\frac{1}{1-\kappa^h(2)}\quad\iff\quad\kappa^h(2)=\frac{12+3c^2}{16}.$$

It is immediate to see that $\kappa^h(2)>\kappa(2)$for any $c\in(0,\ 1)$, and hence, the proposition follows. $\square$

Proof of Proposition 6

To derive the equilibrium managerial efforts, we proceed by backward induction. Consider the problem of an entrant firm. Let $c_j \in \{0, c\}$ be its realized cost, which is private information. Entrants share the following public beliefs about their marginal costs, Pr.$\{c_j = 0\} = e_j$. Let $P_I \in [0, c]$ be the minimum price chosen by the incumbents. In the price-setting stage of entrants, consider a symmetric mixed strategy equilibrium, where $p_j(c_j = c) = c$ and $p_j(c_j = 0) \sim F_J[\underline{p}_J, P_I]$ for every j. That is, high-cost entrants set a price equal to their marginal cost and obtain zero profits, and low-cost entrants randomize their price according to distribution F_J, which has support on $[\underline{p}_J, P_I]$, where $\underline{p}_J \geq 0$. We focus on the equilibrium in which the cdf F_J is a smooth function, i.e., the distribution has no atoms. As explained in the text, there is no equilibrium in pure strategies. Also, note that the low-cost incumbents would obtain zero profit by setting their prices above P_I.

To derive the equilibrium mixed strategy F_J, we exploit the fact that an entrant must be indifferent between setting any price in the support of F_J when all other entrants are playing the equilibrium mixed strategy. Let $p_j \in [\underline{p}_J, P_I]$, and set $e_k = e$ for every $k \in J \setminus \{j\}$. Under this strategy profile, a low-cost entrant obtains positive profits if and only if its price is the lowest among all the prices set by the rival entrants. The probability of this event is given by:

$$\text{Pr.}\{p_j \leq p_k \text{ for all } k \in J \setminus \{j\}\} = \left(1 - eF_J(p_j)\right)^{m-1}. \tag{C.20}$$

Hence, for a low-cost entrant, the expected profits of setting price p_j are given by:

$$\mathbb{E}\left(\pi_j \mid c_j = 0, e_{-j}, P_I\right) = p_j(1 - p_j)\left(1 - eF_J(p_j)\right)^{m-1}. \tag{C.21}$$

The indifference condition implies that (C.21) is a constant function of p_j for the equilibrium strategy F_J, i.e., there exists a constant K_J such that $K_J = \mathbb{E}\big(\pi_j \mid c_j = 0, e_{-j}, P_I\big)$ for every $p_j \in [\underline{p}_J, P_I]$. Using this indifference condition, one obtains

$$F_J(p_j) = \frac{1}{e}\left[1 - \left(\frac{K_J}{p_j(1 - p_j)}\right)^{\frac{1}{m-1}}\right]. \tag{C.22}$$

To find the value of K_J, note that $F(P_I) = 1$, which results in $K_J = P_I(1 - P_I)(1 - e)^{m-1}$. Plugging this value of K_J in (C.22) results in the cdf of the equilibrium mixed strategy, which can easily be shown to be a smooth and increasing function in p_j. Similarly, one may find the value of $\underline{p}_J$ by using the fact that $F(\underline{p}_J) = 0$. It is easily shown that $\underline{p}_J \in (0, P_I)$.

The value of K_J gives the expected profits of a low-cost entrant prior to its own cost realization. That is, if $e_k = e$ for every $k \in J \setminus \{j\}$, a low-cost entrant has expected profits given by:

$$\pi_j(0,\, e_{-j},\, P_I) = P_I(1 - P_I)(1 - e)^{m-1}. \tag{C.23}$$

Two key remarks follow from expression (C.23). First, the expected equilibrium profits do not depend on an entrant's own managerial effort. As opposed to the quantity-setting game, managerial effort has no value beyond the true cost realization, i.e., the marginal profitability of effort is null in a price-setting environment. Second, the expected equilibrium profits are the same as if marginal costs were public information. Note that, if marginal costs were known, (i) entrants with high-marginal cost would have zero profits as well, and (ii) the only case in which a low-cost entrant can have a positive profit is that for every other entrant to have high marginal cost (in which case the entrant would set the price at $p_j = P_I$ and capture the entire market demand).

Therefore, at the contracting stage, the problem of entrant j is given by:

$$\max_{e_j} e_j \pi_j(0,\, e_{-j},\, P_I) - e_j^2, \tag{C.24}$$

which yields the following best-response for effort choice among entrants:

$$e_j(e) = \frac{P_I(1 - P_I)(1 - e)^{m-1}}{2}, \tag{C.25}$$

which implies that effort choices are strategic substitutes, i.e., $e_j'(e) < 0$. Expression (C.25) yields the equilibrium effort of entrants in the symmetric equilibrium, $e_J^B(m,\, P_I)$, defined implicitly in (3.3).

Now, consider the problem of an incumbent firm. Using similar arguments as above, one can see that an incumbent firm realizes positive profits if it attains low marginal cost, sets a price lower than every other incumbent, and every entrant realizes high cost. Also, it can easily be seen that there is no equilibrium in pure strategies in the price-setting stage of the incumbents. Therefore, conditional on setting price p_i, an incumbent with low cost will have expected profits equal to

$$\mathbb{E}(\pi_i \mid c_i = 0,\, p_i) = (1 - e_J)^m \,(1 - eF_I(p_i))^{n-1}\, p_i(1 - p_i), \tag{C.26}$$

where $(1 - e_J)^m$ is the probability that all entrants have high cost, e is the symmetric managerial effort elicited by each rival incumbent, and F_I is the symmetric mixed strategy equilibrium in price choices among the incumbents. Note that conditional on the event that i has the lowest price among incumbents implies $p_i = P_I$, and hence, from (3.3) we obtain

$$(1 - e_J)^m = \frac{2e_J(1 - e_J)}{p_i(1 - p_i)}. \tag{C.27}$$

Therefore, we can write

$$\mathbb{E}(\pi_i \mid c_i = 0,\, p_i) = 2e_J(p_i)(1 - e_J(p_i))\,[1 - eF_I(p_i)]^{n-1}\,. \tag{C.28}$$

To have an equilibrium in mixed strategies, $\mathbb{E}(\pi_i \mid c_i = 0,\, p_i)$ must be constant for all p_i. Set $\mathbb{E}(\pi_i \mid c_i = 0,\, p_i) = K_I$ and solve for F_I to obtain

$$F_I(p_i) = \frac{1}{e}\left[1 - \left(\frac{K_I}{2e_J(p_i)(1 - e_J(p_i))}\right)^{n-1}\right]. \tag{C.29}$$

The support of F_I is given by $[\underline{p}_I,\, c]$ with $\underline{p}_I > 0$. Find the value of K_I by solving $F_I(c) = 1$, which yields

$$K_I = 2e_J(c)(1 - e_J(c))(1 - e)^{n-1}. \tag{C.30}$$

Following the same steps as above, one can verify that the equilibrium mixed strategy F_I is well-defined, i.e., smooth and increasing, and that $\underline{p}_I \in (0,\, c)$.

As with entrants, the value of K_I gives the expected profits of low-cost incumbents, i.e., $\pi_i(0,\, e_{-i}) = K_I$ if $e_k = e$ for every $k \in I \setminus \{i\}$. Solving a problem analogous to (C.24), and using (3.3), one obtains the best-response for the effort choice of incumbents:

$$e_i(e) = \frac{c(1 - c)(1 - e_J(c))^m(1 - e)^{n-1}}{2}. \tag{C.31}$$

Note that the effort choices among incumbents are also strategic substitutes. Using the best-response in (C.31) results in the equilibrium effort of incumbents, provided in (3.4).

Now, we prove that both equilibrium managerial efforts (of the entrants and the incumbents) are decreasing in m. First, from (3.3) note that the equilibrium effort level of the entrants is decreasing in m: the left-hand side of (3.3) is increasing as a function of both e_J and m, hence e_J is decreasing as a function of m. Second, from (3.4), note that the left-hand side is increasing as a function of e_I. Therefore, the sign of the derivative of $e_I^B(m)$ with respect to m is equal to the sign of the derivative of $[1 - e_J^B(m,\, c)]^m$ with respect to m. In what follows, we prove that $[1 - e_J^B(m,\, c)]^m$ is decreasing in m, and conclude the proof.

From (3.3), note that $(1-e_J^B(m,\ c))^m$ is decreasing in m if and only if $e_J^B(m,\ c)(1-e_J^B(m,\ c))$ is also decreasing in m. Because $e_J^B(m,\ c)$ is decreasing in m, $e_J^B(m,\ c)(1-e_J^B(m,\ c))$ is decreasing in m if and only if $e_J^B(m,\ c) \leq 1/2$. We prove this statement by contradiction.

$$\begin{aligned} e_J^B(m,\ c) > \frac{1}{2} \quad &\Longrightarrow \quad \frac{1}{2^{m-1}} > (1-e_J^B(m,\ c))^{m-1} \\ &\Longrightarrow \quad \frac{e_J^B(m,\ c)}{(1-e_J^B(m,\ c))^{m-1}} > 2^{m-2} \\ &\Longrightarrow \quad \frac{c(1-c)}{2} > 2^{m-2} \quad \text{[follows from (3.3)]}, \end{aligned}$$

which is a contradiction since the maximum value of $c(1-c)/2$ is $1/8$ for $c \in [0, 1]$, and the minimum value of 2^{m-1} is $1/4$ for $m \geq 0$. □

C.3 Numerical Implementation

In order to show $G(c) < e_I(m) < F(c)$ for every $c \in (0, \hat{c})$, $n \geq 1$, $m \geq 1$ and the validity of Implication 2, in Section 3.5, we compute the model numerically. We define a grid over the parameter space $(c,\ n,\ m) \in \mathcal{C} \times \mathcal{N} \times \mathcal{M}$, where $\mathcal{C}$ is a grid of (0, 1), $\mathcal{N} = \{1,\ 2,\ \ldots,\ 50\}$ and $\mathcal{M} = \{0,\ 1, \ldots, 50\}$. For each $(n,\ m) \in \mathcal{N} \times \mathcal{M}$, we solve numerically for the upper bound of c, given by $\hat{c}(n,\ m)$. For each $(n,\ m) \in \mathcal{N} \times \mathcal{M}$, we then show the validity of the claims at every point $c \in \mathcal{C}(n, m) \equiv \{c = \hat{c}(n,\ m) * H/51 : H = 1, \ldots, 50\}$.

C.4 Social Welfare

Expected Consumer Surplus

As explained in the text, the expected consumer surplus is given by $\mathbb{E}CS = 0.5 * \mathbb{E}Q^2$. Straightforward computations show that

$$\mathbb{E}Q^2 = n\mathbb{E}q_i^2 + n(n-1)\left(\mathbb{E}q_i\right)^2 + m\mathbb{E}q_j^2 + m(m-1)\left(\mathbb{E}q_j\right)^2 + 2nm\mathbb{E}q_i\mathbb{E}q_j, \quad \text{(C.32)}$$

where $\mathbb{E}q_i$ and $\mathbb{E}q_j$ are the expected incumbent and entrant outputs in equilibrium, respectively.[1] Computing the expected output of an incumbent in equilibrium, $\mathbb{E}q_i$,

[1]To see that (C.32) obtains, it suffices to note:

$$\begin{aligned} \mathbb{E}Q^2 &= \mathbb{E}\left(\sum_{i\in I} q_i + \sum_{j\in J} q_j\right)^2 = \mathbb{E}\left(\sum_i q_i^2 + 2\sum_{i\neq i'} q_i q_{i'} + \sum_j q_j^2 + 2\sum_{j\neq j'} q_j q_{j'} + 2\sum_{i,j} q_i q_j\right) \\ &= n\mathbb{E}q_i^2 + 2\binom{n}{2}\left(\mathbb{E}q_i\right)^2 + m\mathbb{E}q_j^2 + 2\binom{m}{2}\left(\mathbb{E}q_j\right)^2 + 2nm\mathbb{E}q_i\mathbb{E}q_j. \end{aligned}$$

is straightforward using expressions derived in the text. Namely,

$$\mathbb{E}q_i = e_I(m)q_i(0, e_I(m), m) + (1 - e_I(m))q_i(c, e_I(m), m).$$

To compute the expected output of an entrant in equilibrium, one needs to compute the expectation over the incumbents aggregate output along the equilibrium path, Q_I. Along the equilibrium path, Q_I is a random variable determined by the realized number of incumbents that attain the cost reduction, which we denote by L. Let $Q_I(L)$ be the aggregate incumbent output conditional on L incumbents attaining the cost reduction, i.e.,

$$Q_I(L) = Lq_i(0, e_I(m), m) + (n - L)q_i(c, e_I(m), m).$$

Note that L is a random variable with support $\{0, 1, \ldots, n\}$ and probability distribution

$$p_L(l) \equiv \mathbb{P}[L = l] = \binom{n}{l} e_I(m)^l (1 - e_I(m))^{n-l}, \quad l \in \{0, 1, \ldots, n\}.$$

Let $q_j(c_j, e, L) = q_j(c_j, e, Q_I(L))$, and $e_J(L, m) = e_J(Q_I(L), m)$. Then, the expected output of an entrant in equilibrium is given by:

$$\begin{aligned}
\mathbb{E}q_j &= \mathbb{E}\Big[\mathbb{E}\left[q_j \mid L\right]\Big] \\
&= \mathbb{E}\Big[e_J(L, m)q_j(c, e_J(L, m), L) + (1 - e_J(L, m))q_j(0, e_J(L, m), L)\Big] \\
&= \sum_{l=0}^{n} p_L(l)\Big[e_J(l, m)q_j(c, e_J(l, m), l) + (1 - e_J(l, m))q_j(0, e_J(l, m), l)\Big].
\end{aligned}$$

The expectations $\mathbb{E}q_i^2$ and $\mathbb{E}q_j^2$ can be computed analogously.

Expected Producer Surplus of Incumbents

The expected producer surplus of an incumbent can be computed directly from its definition. Note that the expected market profits are given by $\Pi_i(m)$, which is provided in Section 3.4. Furthermore, at equilibrium, the effort cost of incumbents is given by $\psi(e_I(m))$.

Expected Producer Surplus of Entrants

To compute the expected producer surplus of an entrant, one needs to take the expectation over the incumbents aggregate output along the equilibrium path, Q_I. Following the same reasoning as in Section C.4, the expected market profits of entrant j can be expressed as:

$$\Pi_j(m) = \sum_{l=0}^{n} p_L(l)\Big[e_J(l, m)\pi_j(c, e_J(l, m), l) + (1 - e_J(l, m))\pi_j(0, e_J(l, m), l)\Big],$$

where $\pi_j(c_j, e, L)$ denotes the expected market profits of an entrant conditional on having realized marginal cost c_j, at a common effort level $e_j = e$ for all $j \in J$, when $Q_I = Q_I(L)$, see Section 3.4. Similarly, the effort cost of entrants along the equilibrium path depends on the incumbents aggregate output. Hence, the expected producer surplus of an entrant at equilibrium is given by:

$$PS_j(m) = \Pi_j(m) - \mathbb{E}\psi(e_J(L, m)),$$

where

$$\mathbb{E}\psi(e_J(L, m)) = \sum_{l=0}^{n} p_L(l) \left[\frac{e_J(L, m)^2}{2} \right].$$