Strongly Amenable Groups, Choquet-Deny Groups, and the Infinite Conjugacy Class Property

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ABSTRACT

This thesis consists of two main parts. In the first part, we study a space of symbolic dynamical systems for countable discrete ICC groups and show that minimal proximal actions in that space are generic. This study leads to a characterization of countable discrete strongly amenable groups; a countable discrete group is strongly amenable if and only if it has no ICC quotients.

In the second part, we show that a countable discrete group is Choquet-Deny if and only if it has no ICC quotients, where a group is called Choquet-Deny if the Poisson boundary of every non-degenerate measure on the group is trivial. Combining the aforementioned results, we get that a countable discrete group is Choquet-Deny if and only if it is strongly amenable.

In the case of finitely generated groups, by an old result due to McLain [McL56] and Duguid and McLain [DM56] and our classifications, we see that strongly amenable groups and Choquet-Deny groups are the same as virtually nilpotent groups.

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Chapter 1

INTRODUCTION

In this thesis, we study two group properties for countable discrete groups and show that they are both equivalent to the group not having ICC quotients.

The first property is strong amenability. Let G be a topological group and let $G \curvearrowright X$ be a continuous action of G on a compact Hausdorff space X. This action is said to be *proximal* if for any $x, y \in X$ there exists a net $\{g_i\}$ in G such that $\lim_i g_i x = \lim_i g_i y$. G is said to be *strongly amenable* if every such proximal action of G has a fixed point.

The second class of groups we study are Choquet-Deny groups. Let G be a countable discrete group. A probability measure μ on G is *non-degenerate* if its support generates G as a semigroup. A function $f : G \to \mathbb{R}$ is *μ-harmonic* if $f(k) = \sum_{g \in G} \mu(g) f(kg)$ for all $k \in G$. We say that the *measured group* (G, μ) is *Liouville* if all the bounded μ-harmonic functions are constant; this is equivalent to the triviality of the Poisson boundary $\Pi(G, \mu)$. G is called Choquet-Deny if (G, μ) is Liouville for *every* non-degenerate μ.

1.1 ICC Groups

Let G be a group. Recall that G has the infinite conjugacy class property (ICC) if each of its non-trivial elements has an infinite conjugacy class. For example, the group S_∞ of finite permutations of $\mathbb{N}$ is ICC.

One can define the *upper FC-series* of a group G as

$$1 \leqslant F_1 \leqslant F_2 \leqslant \cdots \leqslant F_\alpha \leqslant \cdots \leqslant G,$$

where $F_{\alpha+1}/F_\alpha$ is the normal subgroup of G/F_α consisting of the elements of the finite conjugacy classes, and $F_\beta = \cup_{\alpha<\beta} F_\alpha$ for β a limit ordinal [Hai53; McL56]. The group $\cup_\alpha F_\alpha$ is called the *hyper-FC center* of G.

Groups that have no ICC quotients are known as *hyper-FC-central* [McL56] or *hyper-FC* [Dug60]. Note that hyper-FC groups are precisely the groups whose hyper-FC center is the whole group. The class of hyper-FC groups is closed under forming subgroups, quotients, and finite index extensions.

It has been long known that a finitely generated group is hyper-FC if and only if it is virtually nilpotent. This result, due to McLain [McL56] and Duguid and McLain [DM56], simplifies our characterizations for strongly amenable groups and Choquet-Deny groups in the case of finitely generated groups. In §A, we bring a self-contained presentation of the original proof of this fact, which is divided between [McL56, Theorem 2] and [DM56, Theorem 2].

A key property of ICC groups is that for them *switching elements* exist; given an ICC group G and a finite subset $X \subset G$, there are infinitely many $g \in G$ such that $X \cap g^{-1}Xg \subseteq \{e\}$. We call any such $g \in G$ a switching element for X. Indeed, we show in §3.1 that if G is in addition amenable, then for any finite subset $X \subset G$, the set of switching elements for X has full measure for any invariant finitely additive probability measure on G. Switching elements play an important role in the proof of Lemma 2.4.2 and the proof of Proposition 3.0.2, which are the main ingredients for the proofs of our main results.

1.2 Strongly Amenable Groups

Let G be a topological group and let $G \curvearrowright X$ be a continuous action of G on a compact Hausdorff space X. This action is said to be *proximal* if for any $x, y \in X$ there exists a net $\{g_i\}$ in G such that $\lim_i g_i x = \lim_i g_i y$. G is said to be *strongly amenable* if every such proximal action of G has a fixed point. Similarly, the continuous action $G \curvearrowright X$ is said to be *strongly proximal* if for each regular Borel probability measure μ on X there exists a net $\{g_i\}$ in G such that $\lim_i g_i\mu$ is a point mass. Glasner in [Gla76b] introduced these notions and showed that a group is amenable if and only if all its strongly proximal actions have a fixed point, strongly amenable groups are amenable, and that every virtually nilpotent group is strongly amenable.

Our main result in §2 is a characterization of countable discrete strongly amenable groups; a countable discrete group is strongly amenable if and only if it has no ICC quotients, which is equivalent to hyper-FC. In particular, in the case of finitely generated groups, a countable discrete group is strongly amenable if and only if it is virtually nilpotent.

1.3 Choquet-Deny Groups

Our main result in §3 is a characterization of countable discrete Choquet-Deny groups; a countable discrete group is Choquet-Deny if and only if it has no ICC quotients. Using our other result, this is equivalent to strong amenability. This implies that a countable discrete group G is strongly amenable if and only if (G, μ) is Liouville for every non-degenerate μ, which is parallel to the following result: a countable discrete group G is amenable if and only if (G, μ) is Liouville for some non-degenerate μ [Fur73; KV83; Ros81].

Chapter 2

PROXIMAL FLOWS AND STRONGLY AMENABLE GROUPS

Let $G \curvearrowright X$ be a continuous action of a countable discrete group on a compact Hausdorff space. This action is said to be *proximal* if for any $x, y \in X$ there exists a net $\{g_i\}$ in G such that $\lim_i g_i x = \lim_i g_i y$. G is said to be *strongly amenable* if every such proximal action of G has a fixed point. Glasner introduced these notions in [Gla76b] and proved a number of results: he showed that every virtually nilpotent group is strongly amenable, and that non-amenable groups are not strongly amenable. He also gave some examples of amenable groups that are not strongly amenable.[1] Since then, a number of papers have studied strong amenability [DG17; GW02; Gla83; MVT15], but none have made significant progress on relating it to other group properties.

Let $G \curvearrowright X$ be a continuous action of a countable discrete group on a compact Hausdorff space. This action is said to be *strongly proximal* if for each regular Borel probability measure μ on X there exists a net $\{g_i\}$ in G such that $\lim_i g_i\mu$ is a point mass. This notion, as well as that of the related Furstenberg boundary [Fur03; Fur63a; Fur73], have been the object of a much larger research effort, in particular because a group is amenable if and only if all of its strongly proximal actions on compact spaces have fixed points.

Our main result in this chapter is a characterization of strongly amenable groups.

Theorem 1. *A countable discrete group is strongly amenable if and only if it has no ICC quotients, i.e. it is hyper-FC. In particular, a finitely generated group is strongly amenable if and only if it is virtually nilpotent.*

For example, this implies that the group S_∞ of finite permutations of $\mathbb{N}$ is not strongly amenable. Likewise, the alternating subgroup of S_∞ is not strongly amenable, as is every infinite simple group.

Recall from §1.1 that a finitely generated group is hyper-FC if and only if it is virtually nilpotent [DM56; McL56]. The second part of the theorem follows from this.

[1] Glasner attributes one of these examples to Furstenberg.

The case of groups with no ICC quotients is a straightforward consequence of Glasner's work. To prove that groups with ICC quotients are not strongly amenable, we consider an ICC group G and a certain class of symbolic dynamical systems for G. Using a topological genericity argument, we show that in this class there is a proximal action without a fixed point.

In §2.5, we look at Thompson's group F. Since it has an ICC quotient, we know from Theorem 1 that it is not strongly proximal. Our proof for Theorem 1 is an existence proof. In §2.5, we directly construct a proximal action of F that has no fixed points, and thus show directly that F is not strongly amenable. This action does admit an invariant measure, and thus does not provide any information about the amenability of F.

The Universal Minimal Proximal Action

In [Gla76b, Section II.4], Glasner defines the universal minimal proximal action of a group G; this is the unique minimal proximal action of G which has every minimal proximal action as a factor. We denote this action by $G \curvearrowright \partial_p G$. In Proposition 2.3.6, we show that every ICC group has a minimal proximal faithful action. On the other hand, the proof of Proposition 2.1.1 shows that the hyper-FC center of G acts trivially on $\partial_p G$. Combining these gives us:

Corollary 2.0.1. *For a countable discrete group G,* $\ker(G \curvearrowright \partial_p G)$ *is equal to the hyper-FC center of G.*

Glasner also defines the universal minimal *strongly proximal* action of a group G, which is the unique minimal strongly proximal action of G which has every minimal strongly proximal action as a factor. We denote this action by $G \curvearrowright \partial_s G$. Furman [Fur03, Proposition 7] shows that the kernel of $G \curvearrowright \partial_s G$ is the amenable radical of G.

The Group von Neumann Algebra

It is known that the group von Neumann algebra of a group G has a unique tracial state iff G is ICC, and we show that ICC groups are precisely the groups with faithful universal minimal proximal actions. We thus have the following dynamical characterization of the unique trace property of the group von Neumann algebra:

Corollary 2.0.2. *For a countable discrete group G, the following are equivalent:*

1. *The group von Neumann algebra of G has a unique tracial state.*

2. *$G \curvearrowright \partial_p G$ is faithful.*

Analogously, it has been recently shown by Breuillard, Kalantar, Kennedy, and Ozawa [Bre+17, Corollary 4.3] that the reduced C*-algebra of G has a unique tracial state if and only if $G \curvearrowright \partial_s G$ is faithful.

Following this analogy raises an interesting question. We know from [KK17, Theorem 1.5] that simplicity of the reduced C*-algebra of a group G is equivalent to the freeness of $G \curvearrowright \partial_s G$. We also know from [Bre+17, Corollary 4.3] that unique trace property of the reduced C*-algebra of a group G is equivalent to faithfulness of $G \curvearrowright \partial_s G$. On the other hand, for group von Neumann algebras of discrete groups, simplicity and the unique trace property are equivalent. So it is natural to ask whether freeness of $G \curvearrowright \partial_p G$ is equivalent to its faithfulness.

2.1 Overview of the Proof of Theorem 1

That a group with no ICC quotients is strongly amenable follows immediately from the following proposition.

Proposition 2.1.1. *Let G be a countable discrete group that acts faithfully, minimally, and proximally on a compact Hausdorff space X. Then each non-trivial element of G has an infinite conjugacy class.*

Proof. Let g be a non-trivial element of G. Assume by contradiction that g has a finite conjugacy class. Let H be the centralizer of g, so that H has finite index in G. By [Gla76b, Lemma 3.2], H acts proximally and minimally on X. Since g is in the center of H, it acts trivially on X, by [Gla76b, Lemma 3.3]. This contradicts the assumption that the action is faithful. □

Thus, to prove Theorem 1, we consider any G that is ICC, and prove that it has a proximal action that does not have a fixed point. This is without loss of generality, since if G has a proximal action without a fixed point, then so does any group that has G as a quotient.

2.2 Existence by Genericity

Our general strategy for the proof of Theorem 1 is to consider a certain space $\mathcal{S}$ of non-trivial actions of G. We show that this space includes a proximal action without a fixed point by showing that, in fact, a *generic* action in this space is minimal and proximal. Genericity here is in the Baire category sense.

To define the space $\mathcal{S}$, let A be a finite alphabet of size at least 2. The *full shift* A^G, equipped with the product topology, is a space on which G acts continuously by left translations. Enumerate elements of $G = \{g_1, g_2, \ldots\}$ and endow A^G with the metric $d(\cdot, \cdot)$ given by $d(s,t) = 1/k$ where $k = \inf\{n : s(g_n) \neq t(g_n)\}$. An element of A^G is called a *configuration*.

The closed, G-invariant non-empty subsets of A^G are called *shifts*. The space of shifts is endowed with the subspace topology of the Hausdorff topology (or Fell topology) on the closed subsets of A^G. This topology is also metrizable: take, for example, the metric that assigns to a pair of shifts $S, T \subseteq A^G$ the distance $1/(n+1)$, where n is the largest index such that S and T agree on $\{g_1, \ldots, g_n\}$; by agreement on a finite $X \subseteq G$ we mean that the restriction of the configurations in S to X is equal to the restrictions of the configurations in T to X. Note that for any shift $S \subseteq A^G$, the sets of the form $\{T \subseteq A^G \mid T \text{ agrees with } S \text{ on } X\}$ for different finite subsets $X \subseteq G$ form a basis of the neighborhoods for S.

We define the space $\mathcal{S}$ to be the closure, in the space of shifts, of the strongly irreducible shifts, with the $|A|$-many trivial (i.e., singleton) shifts removed. Strongly irreducible shifts are defined as follows:

Definition 2.2.1. *A shift $S \subseteq A^G$ is said to be strongly irreducible if there exists a finite $X \subseteq G$ including the identity such that for any two subsets $E_1, E_2 \subseteq G$ with $E_1X \cap E_2X = \emptyset$ and any two configurations $s_1, s_2 \in S$, there is a configuration $s \in S$ such that s restricted to E_1 equals s_1 restricted to E_1, and s restricted to E_2 equals s_2 restricted to E_2.*

To show that the proximal actions are generic in $\mathcal{S}$, we define ε-proximal actions; proximal actions will be the actions which are ε-proximal for each $\varepsilon > 0$.

Definition 2.2.2. *An action $G \curvearrowright X$ on a compact metric space with metric $d(\cdot, \cdot)$ is ε-proximal if for all $x, y \in X$ there exists a $g \in G$ such that $d(gx, gy) < \varepsilon$.*

To show that minimal actions are generic in $\mathcal{S}$, we similarly define the notion of ε-minimality.

Definition 2.2.3. *An action $G \curvearrowright X$ on a compact metric space with metric $d(\cdot, \cdot)$ is ε-minimal if for all $x, y \in X$ there exists a $g \in G$ such that $d(gx, y) < \varepsilon$.*

A subset of a topological space is generic (in the Baire category sense) if it contains a dense G_δ. To prove our main result, we show that the proximal actions are a dense

G_δ in $\mathcal{S}$. The proof of density is the main challenge of this chapter, while it is straightforward to prove that this subset is a G_δ.

Claim 2.2.4. *The set of ε-proximal shifts is an open set in $\mathcal{S}$. Thus the set of proximal shifts is a G_δ set in $\mathcal{S}$.*

Similarly, the set of ε-minimal shifts is an open set in $\mathcal{S}$. Thus the set of minimal shifts is a G_δ set in $\mathcal{S}$.

The Baire Category Theorem guarantees that for well behaved spaces (such as our locally compact space $\mathcal{S}$), a countable intersection of dense open sets is dense. Thus, to prove that the proximal shifts are dense in the closure of the strongly irreducible shifts, it suffices to show that the ε-proximal shifts are dense in $\mathcal{S}$ for each ε. That is, fixing ε, we must show that for each strongly irreducible shift $S \subseteq A^G$ and each finite subset $X \subseteq G$ there exists a strongly irreducible shift S' that agrees with S on X, and is ε-proximal.

To this end, we construct a class of shifts of $\{0,1\}^G$ (which we denote by 2^G) which are ε-proximal. Furthermore, for these shifts ε-proximality is witnessed by a particular configuration around the origin: one having a 1 at the origin and zeros close to it. For a finite symmetric subset $X \subset G$ and $g, h \in G$, we say that g and h are *X-apart* if $g^{-1}h \notin X$.

Definition 2.2.5. *Let X be a finite symmetric subset of G. A non-trivial shift $S \subset 2^G$ is an X-*witness *shift if*

1. *For each $s \in S$, $s(a) = 1$ and $s(b) = 1$ implies that a and b are X-apart.*
2. *For each $s, t \in S$ there exists an $a \in G$ such that $s(a) = t(a) = 1$.*

The construction of X-witness shifts in Propositions 2.3.1 and 2.3.2 contains the main technical effort of this chapter.

A Toy Example

To give the reader some intuition and explain the role of ICC in the construction of X-witness shifts, we now explain how to construct a single configuration in 2^G with an X-witness *orbit*, and show that such configurations do not exist for groups that are not ICC. Note that the closure of this orbit is not necessarily an X-witness shift; the construction of X-witness shifts requires more work and a somewhat different approach, which we pursue later, in the formal proofs.

Given a configuration $u \in 2^G$, we denote by $Gu = \{gu \,:\, g \in G\}$ the G-orbit of u. Given a finite symmetric $X \subset G$, we say that a configuration $u \in 2^G$ is an *X-witness configuration* if

1. For each $s \in Gu$, $s(a) = 1$ and $s(b) = 1$ implies that a and b are X-apart.

2. For each $s, t \in Gu$ there exists an $a \in G$ such that $s(a) = t(a) = 1$.

We now informally explain that when G is ICC, then for every such X there exist X-witness configurations, and that when G is not ICC, then there is a finite symmetric $X \subset G$ for which there are no such configurations.

Suppose first that G is not ICC. Then there cannot exist an X-witness shift for every X. To see this, suppose that $g \in G$ is an element with finitely many conjugates, and let X be a finite symmetric subset of G that contains all the conjugates of g. Assume towards a contradiction that there exists an X-witness configuration u. So, by the second property of X-witness configurations, there exists an $a \in G$, such that $[gu](a) = u(a) = 1$, which means $u(g^{-1}a) = u(a) = 1$. Now, by the first property of u, we need to have that $g^{-1}a$ and a are X-apart, which means $(g^{-1}a)^{-1}a = a^{-1}ga \notin X$. This is a contradiction, since we let X contain all the conjugates of g.

Consider now the case that G is ICC. Given a finite symmetric X, we choose a random configuration $u \in 2^G$ as follows. Assign to each element of G an independent uniform random variable in $[0,1]$. Let V_a be the random variable corresponding to $a \in G$. For each $a \in G$, let $u(a) = 1$ iff $V_a > V_{ax}$ for all $x \in X \setminus \{e\}$; i.e., $u(a) = 1$ if V_a is maximal in its X-neighborhood. Note that if g and h are X^2-apart, then the event $u(g) = 1$ and the event $u(h) = 1$ are independent.

We claim that u is, with probability one, an X-witness shift. By construction, u almost surely satisfies the first property: if $s = g^{-1}u$ and $s(a) = s(b) = 1$, then $u(ga) = u(gb) = 1$, hence ga and gb are X-apart, and so a and b are X-apart. To satisfy the second property, it must hold that for every $g \neq h \in G$ there is some $a \in G$ such that $u(ga) = u(ha) = 1$. By the ICC property, we can choose an $a \in G$ to make ga and ha arbitrarily far apart, as this corresponds to finding an a such that $a^{-1}(h^{-1}g)a$ is large. For such a choice of a, the events $u(ga) = 1$ and $u(ha) = 1$ are independent, and, since we have infinitely many such a's that we can use, with probability one at least one of them will give us the desired result.

2.3 Constructing X-witness Shifts

We now return to the construction of X-witness shifts, which are the main tool in our proof of Theorem 1. The first step is to construct a single configuration which is an X-witness in a large finite set.

Proposition 2.3.1. *Let G be an ICC group. For each finite symmetric $X \subset G$, there exists an $s \in 2^G$ and a finite symmetric $Y \supset X$ such that*

1. *For every $a, b \in G$, if $s(a) = s(b) = 1$, then a and b are X-apart.*
2. *For every $g, h \in Y^{100}$ there exists some $a \in Y$ such that $s(ga) = s(ha) = 1$.*

The proof of this proposition—along with Proposition 2.3.2 below—contains the main technical effort of this chapter. The proof elaborates on the ideas of the informal construction of §2.2: we choose the configuration s at random, and then show that it has the desired properties with positive probability. This stage crucially uses the assumption that the group is ICC, which translates to independence of some events that arise in the analysis of this random choice. This is the only step in the proof of Theorem 1 in which we use the ICC property of G.

We use the configuration constructed in Proposition 2.3.1 to construct X-witness shifts. These shifts will additionally (and importantly) be strongly irreducible.

Proposition 2.3.2. *Let G be a group for which, for each finite symmetric $X \subset G$, there exists a configuration that satisfies the conditions of Proposition 2.3.1. Then for each such X there also exists a strongly irreducible X-witness shift.*

The combination of Propositions 2.3.1 and 2.3.2 immediately yields the following.

Proposition 2.3.3. *Let G be an ICC group. Then for each finite symmetric $X \subset G$ there exists a strongly irreducible X-witness shift $S \subset 2^G$.*

ε-proximal Shifts

Finally, we use these strongly irreducible X-witness shifts to construct approximations to a given strongly irreducible shift S that are both ε-proximal and ε-minimal.

Proposition 2.3.4. *Let G be a group for which there exists, for each finite symmetric $X \subset G$, a strongly irreducible X-witness shift. Let $T \subseteq A^G$ be a strongly irreducible shift. Then for each ε and finite $X \subset G$ there exists a strongly irreducible shift $T' \subseteq 2^G$ that is ε-proximal, ε-minimal, and agrees with T on X.*

An immediate consequence of Proposition 2.3.3, Proposition 2.3.4, and Claim 2.2.4 is the following.

Proposition 2.3.5. *Let G be an ICC group. Then there is a dense G_δ set in $\mathcal{S}$ for which the action $G \curvearrowright S$ is minimal and proximal.*

In the next proposition, we show that this result can be strengthened to show that a generic shift is additionally faithful.

Proposition 2.3.6. *Let G be an ICC group. Then there is a dense G_δ set in $\mathcal{S}$ for which the action $G \curvearrowright S$ is* faithful*, minimal, and proximal.*

Given all this, the proof of our main theorem follows easily.

Proof of Theorem 1. That groups with no ICC quotients are strongly amenable follows immediately from Proposition 2.1.1. Let G be ICC. By Proposition 2.3.5, the proximal minimal shifts are a dense G_δ in $\mathcal{S}$, and in particular exist, since $\mathcal{S}$ is non-empty (e.g., the full shift A^G is strongly irreducible and non-constant). Since there are no trivial shifts in $\mathcal{S}$, and since non-trivial minimal shifts have no fixed points, we have proved that G is not strongly amenable. □

2.4 Proofs

Proof of Proposition 2.3.1

Let G be an ICC group, and let X be a finite, symmetric subset of G. We choose a random configuration $u \in 2^G$ as follows. Assign to each element of G an independent uniform random variable in $[0, 1]$. Let V_a be the random variable corresponding to $a \in G$. For each $a \in G$, let $u(a) = 1$ iff $V_a > V_{ax}$ for all $x \in X \setminus \{e\}$. That is, let $u(a) = 1$ if $V_a > V_b$ whenever $a^{-1}b \in X$ and $b \neq a$. The following claim is an immediate consequence of the definition of u.

Claim 2.4.1. *If $a_1, \ldots, a_n$ are X^2-apart*[2] *for $a_i \in G$, then $\{u(a_i) = 1\}$ are independent events.*

Clearly, for all values of the random configuration, $u(a) = u(b) = 1$ implies that $a^{-1}b \notin X$ for all $a, b \in G$, which means that a and b are X-apart. So the random configuration u almost surely satisfies the first part of the proposition. It thus remains

[2] Recall that given a finite symmetric subset $X \subset G$ and $g, h \in G$, we say that g and h are X-apart if $g^{-1}h \notin X$.

to find a finite symmetric subset $Y \supset X$ such that, with positive probability for the random configuration u, for each $g, h \in Y^{100}$ there exists some $a \in Y$ such that $u(ga) = u(ha) = 1$.

The next lemma claims that there exists a subset Y with certain useful properties. We use this lemma to prove our proposition, and then prove the lemma.

Lemma 2.4.2. *There exists a $Y \supset X$ with the following properties.*

1.
$$|Y|^{200}(1 - |X|^{-2})^{|Y|/(20|X^2|+5)} < 1.$$

2. *For each $g, h \in G$ there exists a subset $Y_{g,h} \subseteq Y$ with the following properties.*
   a) $|Y_{g,h}| \geq |Y|/(20|X^2| + 5)$.
   b) *For $y \in Y_{g,h}$, gy and hy are X^2-apart.*
   c) *For $y_1 \neq y_2 \in Y_{g,h}$, w_1 and w_2 are X^2-apart for any $w_1 \in \{gy_1, hy_1\}$ and $w_2 \in \{gy_2, hy_2\}$.*

For $c, d, y \in G$, let E_c be the event that $u(c) = 1$, and let $E^y_{c,d} = E_{cy} \cap E_{dy}$. Now fix $g, h \in G$.

1. By the second property of $Y_{g,h}$, gy and hy are X^2-apart for any $y \in Y_{g,h}$. Hence E_{gy} and E_{hy} are independent, by Claim 2.4.1.

2. $\mathbb{P}[E_c] = 1/|X|$ for all $c \in G$.

3. Combining the previous two results: $\mathbb{P}\left[E^y_{g,h}\right] = |X|^{-2}$ for all $y \in Y_{g,h}$. So $\mathbb{P}\left[\neg E^y_{g,h}\right] = 1 - |X|^{-2}$.

4. $E^y_{g,h}$ are independent events for different values of $y \in Y_{g,h}$. This is because (I) gy and hy are X^2-apart for any $y \in Y_{g,h}$, and (II) w_1 and w_2 are X^2-apart for any $y_1 \neq y_2 \in Y_{g,h}$, $w_1 \in \{gy_1, hy_1\}$, and $w_2 \in \{gy_2, hy_2\}$, which means $\{E_{gy}, E_{hy} \mid y \in Y_{g,h}\}$ are independent events. And finally, since $E^y_{g,h} = E_{gy} \cap E_{hy}$, we get that $E^y_{g,h}$ are independent events for $y \in Y_{g,h}$.

5. We say that the pair (g,h) *fails* if $E^y_{g,h}$ does not happen for any $y \in Y_{g,h}$. So, by the previous two results,

$$\begin{aligned}\mathbb{P}\left[(g,h) \text{ fails}\right] &= \mathbb{P}\left[E^y_{g,h} \text{ for no } y \in Y_{g,h}\right] \\ &= (1-|X|^{-2})^{|Y_{g,h}|} \\ &\leq (1-|X|^{-2})^{|Y|/(20|X^2|+5)},\end{aligned}$$

where the last inequality follows from the first property of $Y_{g,h}$.

By the last inequality, union bound, and the first property of Y:

$$\begin{aligned}\mathbb{P}\left[(g,h) \text{ fails for some } g,h \in Y^{100}\right] &\leq |Y^{100}|^2(1-|X|^{-2})^{|Y|/(20|X^2|+5)} \\ &\leq |Y|^{200}(1-|X|^{-2})^{|Y|/(20|X^2|+5)} < 1.\end{aligned}$$

So, there is at least one configuration, say s, for which no (g,h) fails for $g,h \in Y^{100}$. Therefore, for all $g,h \in Y^{100}$, there is an $a \in Y$ such that $s(ga) = s(ha) = 1$. So this s satisfies the second part of the proposition, which concludes the proof of Proposition 2.3.1, except the proof of Lemma 2.4.2, to which we turn now.

Proof of Lemma 2.4.2. We call an element $g \in G$ *switching* if for all non-identity $x \in X^2$ we have $g^{-1}xg \notin X^2$.

Claim 2.4.3. *There exists at least one switching element $g_s \in G$.*

Proof. Let C_x be the centralizer of x for each $x \in X^2$. Then there are finitely many cosets of C_x, say $g^x_1 C_x, \ldots, g^x_{n_x} C_x$, such that $g^{-1}xg \in X^2$ only if $g \in g^x_i C_x$ for some $i \in \{1,\ldots,n_x\}$. So, non-switching elements are in the union of finitely many cosets of subgroups with infinite index, i.e. g is non-switching only if $g \in g^x_i C_x$ for some $x \in X^2$ and some $i \in \{1,\ldots,n_x\}$. Since G is ICC, each C_x has infinite index in G. By [Neu54, Lemma 4.1], a finite collection of cosets of infinite index does not cover the whole group G, so there is at least one switching element in G. □

Let g_s be a switching element. We can choose an arbitrarily large finite subset $Y_1 \subseteq G$ which includes the identity and such that $Y_1 \cap Y_1 g_s = \emptyset$. Choose such a Y_1 that is large enough so that

$$(5|Y_1|)^{200}(1-|X|^{-2})^{2|Y_1|/(20|X^2|+5)} < 1 \quad \text{and} \quad |Y_1| \geq |X|$$

and let $Y = (Y_1 \cup Y_1 g_s) \cup (Y_1 \cup Y_1 g_s)^{-1} \cup X$. Note that Y is symmetric and $5|Y_1| \geq |Y| \geq 2|Y_1|$, which implies that

$$|Y|^{200}(1 - |X|^{-2})^{|Y|/(20|X^2|+5)} < 1.$$

This establishes the first property of Y.

Fix $g, h \in G$ with $g \neq h$. We say $y \in G$ is *distancing* for the pair (g, h) if gy and hy are X^2-apart.

Claim 2.4.4. *If $y \in G$ is not distancing for (g, h), then yg_s is distancing for (g, h).*

Proof. Since y is not distancing for (g, h), $(gy)^{-1}(hy) = y^{-1}g^{-1}hy \in X^2$. By the definition of a switching element $g_s^{-1}[(gy)^{-1}(hy)]g_s = (gyg_s)^{-1}(hyg_s) \notin X^2$, which means that yg_s is distancing for (g, h). □

By this observation, if $y_1 \in Y_1$ is not distancing for (g, h), then $y_1 g_s \in Y_1 g_s$ is distancing for (g, h). So at least half of the elements in $Y_1 \cup Y_1 g_s$ are distancing for (g, h) and thus at least one fifth of the elements in Y are distancing for (g, h). Let $Y'_{g,h}$ be the collection of elements in Y that are distancing for (g, h). We just saw that $|Y'_{g,h}| \geq |Y|/5$.

Now define a graph on $Y'_{g,h}$ by connecting $y_1 \neq y_2 \in Y'_{g,h}$ if w_1 and w_2 are not X^2-apart for some $w_1 \in \{gy_1, hy_1\}, w_2 \in \{gy_2, hy_2\}$. Call this graph $G'_{g,h}$. Note that the degree of each $y \in Y'_{g,h}$ in $G'_{g,h}$ is at most $4|X^2|$. So, we can find an independent set of size at least $|Y'_{g,h}|/(4|X^2| + 1) \geq |Y|/(20|X^2| + 5)$ in $G'_{g,h}$. Call this independent set $Y_{g,h}$.

Claim 2.4.5. $|Y_{g,h}| \geq |Y|/(20|X^2| + 5)$*, for $y \in Y_{g,h}$, gy and hy are X^2-apart, and for $y_1 \neq y_2 \in Y_{g,h}$, we have that w_1 and w_2 are X^2-apart for* $w_1 \in \{gy_1, hy_1\}, w_2 \in \{gy_2, hy_2\}$.

Proof. The bound on the size of $Y_{g,h}$ is established in the previous paragraph. Since $Y_{g,h} \subseteq Y'_{g,h}$ and all elements of $Y'_{g,h}$ are distancing for (g, h), the second property holds. The third property follows from independence of $Y_{g,h}$ in $G'_{g,h}$. □

This establishes the three properties of $Y_{g,h}$, and thus concludes the proof of the lemma. □

Saturated Packings

In this section, we prove some general claims regarding *saturated packings* (see, e.g., [TK93]).

Definition 2.4.6. *Let $Z_1, \ldots, Z_n$ be distinct non-empty finite subsets of G. A* $\{Z_1, \ldots, Z_n\}$-packing *is a $p \in \{Z_1, \ldots, Z_n, \emptyset\}^G$ with $h\,p(h) \cap g\,p(g) = \emptyset$ for all $g \neq h \in G$; note that $h\,p(h)$ and $g\,p(g)$ are each a translate, by h and g, respectively, of some element of $\{Z_1, \ldots, Z_n, \emptyset\}$. When $p(g) \neq \emptyset$, we call the translate $g\,p(g)$ a* block.

By an abuse of notation, we use the term Z-packing instead of $\{Z\}$-packing when we have only one subset.

Definition 2.4.7. *A $\{Z_1, \ldots, Z_n\}$-packing p is* saturated *if there is no $\{Z_1, \ldots, Z_n\}$-packing $p' \neq p$ such that $p(g) \neq \emptyset$ implies that $p'(g) = p(g)$.*

We say that p is a saturation of q if p is saturated and $q(g) \neq \emptyset$ implies that $p(g) = q(g)$.

Saturated packings are packings to which one cannot add any blocks. Note, however, that it may be possible to add more blocks by first removing some. Note also that by Zorn's Lemma, there exists for each $\{Z_1, \ldots, Z_n\}$-packing q a $\{Z_1, \ldots, Z_n\}$-packing p that saturates it.

The following claim shows the existence of strongly irreducible saturated packings, which will be useful in the construction of strongly irreducible X-witness shifts. A similar claim with a similar proof appears in [FT17, Lemma 2.2].

Given two distinct non-empty finite subsets Z_1 and Z_2 of G, we denote by $\pi\colon \{Z_1, Z_2, \emptyset\}^G \to \{Z_1, \emptyset\}^G$ the map

$$(\pi(p))(g) = \begin{cases} Z_1 & \text{if } p(g) = Z_1 \\ \emptyset & \text{otherwise.} \end{cases}$$

That is, π transforms a $\{Z_1, Z_2\}$-packing into a Z_1-packing by removing all the Z_2-blocks.

Claim 2.4.8. *Let Z_1 and Z_2 be two distinct non-empty finite subsets of G. Let P be the collection of all saturated $\{Z_1, Z_2\}$-packings p such that $\pi(p)$ is a saturated Z_1-packing. Then P is a non-empty strongly irreducible shift.*

Proof. The proof of the fact that P is a non-empty shift is standard. It thus remains to be shown that it is strongly irreducible.

Let $X = (Z_1 \cup Z_2) \cup (Z_1 \cup Z_2)^{-1}$. Let $F_1, F_2 \subset G$ be any two subsets of G that are X^{14}-apart. To prove the claim, it suffices to show that for any $p_1, p_2 \in P$, there is a $p \in P$ that agrees with p_1 on F_1 and with p_2 on F_2. We know that F_1X^6 and F_2X^6 are disjoint, and furthermore, if $a_1 \in F_1X^6$ and $a_2 \in F_2X^6$, then the blocks $a_1p_1(a_1)$ and $a_2p_2(a_2)$ are disjoint.

Let $q_1 = \pi(p_1)$ and $q_2 = \pi(p_2)$. We know that q_1 and q_2 are saturated Z_1-packings. We also know that if $a_1 \in F_1X^6$ and $a_2 \in F_2X^6$, then $a_1q_1(a_1)$ and $a_2q_2(a_2)$ are disjoint. Thus there is a Z_1-packing, say q', that is equal to q_1 on F_1X^6 and to q_2 on F_2X^6. Let q be a Z_1-packing that is a saturation of q'. Fix $i \in \{1, 2\}$ and $g \in F_iX^4$. We will show that $q_i(g) = q(g)$.

- If $q_i(g) = Z_1$, we know that $q'(g) = Z_1$, and hence $q(g) = Z_1$.
- If $q_i(g) = \emptyset$, since q_i is a saturated Z_1-packing, there exists $a \in gZ_1Z_1^{-1} \subseteq F_iX^6$ with $q_i(a) = Z_1$. So, $q'(a) = Z_1$, and hence $q(a) = Z_1$. Since $gZ_1 \cap aZ_1 \neq \emptyset$, this implies that $q(g) = \emptyset$.

So, q is a saturated Z_1-packing that agrees with q_1 on F_1X^4 and with q_2 on F_2X^4.

Since q agrees with $q_1 = \pi(p_1)$ on F_1X^4 and with $q_2 = \pi(p_2)$ on F_2X^4, it is easy to see that p', which is defined as follows, is a well-defined $\{Z_1, Z_2\}$-packing:

$$p'(g) = \begin{cases} p_1(g) & \text{if } g \in F_1X^2 \\ p_2(g) & \text{if } g \in F_2X^2 \\ q(g) & \text{otherwise.} \end{cases}$$

So, by definition, p' agrees with p_1 on F_1X^2 and with p_2 on F_2X^2. Furthermore, $\pi(p') = q$, since $\pi(p_i)$ agrees with q on F_iX^2.

Let p be a $\{Z_1, Z_2\}$-packing that is a saturation of p'. Since $\pi(p') = q$ is a saturated Z_1-packing, we have $\pi(p) = \pi(p') = q$. So, p is a saturated $\{Z_1, Z_2\}$-packing where $\pi(p)$ is a saturated Z_1-packing, which means $p \in P$. To complete the proof, we just need to show that p agrees with p_1 on F_1 and with p_2 on F_2. Fix $i \in \{1, 2\}$ and $g \in F_i$. We will show that $p_i(g) = p(g)$.

- If $p_i(g) \in \{Z_1, Z_2\}$, we know that $p'(g) = p_i(g) \in \{Z_1, Z_2\}$, and hence $p(g) = p_i(g)$.

- If $p_i(g) = \emptyset$, since p_i is a saturated $\{Z_1, Z_2\}$-packing, for any $j \in \{1,2\}$ there exist $\ell \in \{1,2\}$ and $a \in gZ_jZ_\ell^{-1} \subseteq F_iX^2$ with $p_i(a) = Z_\ell$. So, $p'(a) = Z_\ell$, and hence $p(a) = Z_\ell$. Since $gZ_j \cap aZ_\ell \neq \emptyset$, this implies that $p(g) = \emptyset$.

□

Proof of Proposition 2.3.2

We can now start the proof of proposition 2.3.2. Assume that X, a finite symmetric subset of G, is given. We now seek to construct a strongly irreducible X-witness shift T. Since G satisfies proposition 2.3.1, we can let Y and s be a finite symmetric subset of G and a configuration on G that satisfy the statement of proposition 2.3.1 for $X \subseteq G$.

Let P be the strongly irreducible shift given by Claim 2.4.8 for $Z_1 = Y^{100}X$ and $Z_2 = YX$.

Define $\psi : P \to 2^G$ by

$$[\psi(p)](g) = \begin{cases} s(h^{-1}g) & \text{if } g \in h\,Y^{100} \text{ for some } h \in G \\ & \quad \text{with } p(h) = Y^{100}X \\ s(h^{-1}g) & \text{if } g \in h\,Y \text{ for some } h \in G \\ & \quad \text{with } p(h) = YX \\ 0 & \text{otherwise.} \end{cases}$$

What ψ does is produce a configuration which is 0 outside of the X-interior[3] of blocks, and is equal to translates of $s|_{Y^{100}}$ and $s|_Y$ inside the interior of blocks.

It is again easy to see that ψ is continuous and equivariant, so $T = \psi(P)$ is a strongly irreducible shift. The following claim completes the proof of proposition 2.3.2.

Claim 2.4.9. *T is an X-witness shift.*

The claim follows immediately from the following two lemmas. The first of the lemmas is straightforward from our construction, while the second is less immediate.

Lemma 2.4.10. *For all $t \in T$, the 1's in t are X-apart.*

Proof. Let $t \in T$ and $a, b \in G$ with $t(a) = t(b) = 1$. Since $t \in T$, there is a $p \in P$ with $\psi(p) = t$. By the definition of ψ, since $[\psi(p)](a) = [\psi(p)](b) = 1$, we get

[3]The X-interiors of $Y^{100}X$ and YX are Y^{100} and Y.

that $a \in h\,p(h)$ and $b \in g\,p(g)$ for some $h, g \in G$. If $g = h$, i.e. a and b are in the same block of p, then $s(h^{-1}a) = t(a) = 1$ and $s(h^{-1}b) = t(b) = 1$. But, since s satisfies proposition 2.3.1 (in particular, the 1's in s are X-apart), $h^{-1}a$ and $h^{-1}b$ are X-apart, which implies that a and b are X-apart. If $g \neq h$, then $h\,p(h)$ and $g\,p(g)$ are disjoint, so the X-interior of $h\,p(h)$ and the X-interior of $g\,p(g)$ are X-apart. We also know that a is in the X-interior of $h\,p(h)$ and b is in the X-interior of $g\,p(g)$. Therefore, a and b are X-apart. □

Lemma 2.4.11. *For any $t_1, t_2 \in T$, there is an $a \in G$ with $t_1(a) = t_2(a) = 1$.*

Proof. We essentially prove this lemma by a series of reductions.

Let $t_1, t_2 \in T$. So there are $p_1, p_2 \in P$ with $\psi(p_1) = t_1$ and $\psi(p_2) = t_2$. Pick an $a_1 \in G$ with $p_1(a_1) = Y^{100}X$. This means that $a_1^{-1}p_1$ has a block of shape $Y^{100}X$ centered at the identity. Let $p_1' = a_1^{-1}p_1$, $p_2' = a_1^{-1}p_2$, and let $t_1' = \psi(p_1')$, $t_2' = \psi(p_2')$. So, p_1' has a block of shape $Y^{100}X$ centered at the identity.

Since p_2' is saturated, we know there is an $a_2 \in Y^4$ such that either (I) a_2 is in the YX-interior of a block of shape $Y^{100}X$ in p_2', or (II) a_2 is the center of a block of shape YX in p_2'. Let $p_1'' = a_2^{-1}p_1'$, $p_2'' = a_2^{-1}p_2'$, and let $t_1'' = \psi(p_1'')$, $t_2'' = \psi(p_2'')$. Observe that in p_1'' the identity is in the YX-interior of a block of shape $Y^{100}X$, say $e \in k_1Y^{99}$ for some $k_1 \in G$ with $p_1''(k_1) = Y^{100}X$. Moreover, in p_2'' the identity is either (I) in the YX-interior of a block of shape $Y^{100}X$ or (II) in the center of a block of shape YX.

In case (I), since in p_2'' the identity is in the YX-interior of a block of shape $Y^{100}X$, $e \in k_2\,Y^{99}$ for some $k_2 \in G$ with $p_2''(k_2) = Y^{100}X$. So $k_2^{-1} \in Y^{99}$. By the second part of proposition 2.3.1 applied to $g = k_1^{-1}$ and $h = k_2^{-1}$, we know that there is an $a_3 \in Y$ such that $s(k_1^{-1}a_3) = s(k_2^{-1}a_3) = 1$. So, by the definition of ψ, the fact that $k_1^{-1}a_3 \in Y^{100}$, and the fact that $p_1''(k_1) = Y^{100}X$, we get $t_1''(a_3) = s(k_1^{-1}a_3) = 1$, and similarly, we get $t_2''(a_3) = s(k_2^{-1}a_3) = 1$. Therefore, $t_1(a_1a_2a_3) = t_1''(a_3) = 1$ and $t_2(a_1a_2a_3) = t_2''(a_3) = 1$. Case (I) is schematically depicted in Figure 2.1.

In case (II), $p_2''(e) = YX$. Again, if we apply the second part of proposition 2.3.1 to $g = k_1^{-1}$ and $h = e$, we get that there is an $a_3 \in Y$ such that $s(k_1^{-1}a_3) = s(a_3) = 1$. So, by the definition of ψ, the fact that $k_1^{-1}a_3 \in Y^{100}$, and the fact that $p_1''(k_1) = Y^{100}X$, we get $t_1''(a_3) = s(k_1^{-1}a_3) = 1$. Also, by the definition of ψ, the fact that $a_3 \in Y$, and the fact that $p_2''(e) = YX$, we get $t_2''(a_3) = s(a_3) = 1$. Therefore, $t_1(a_1a_2a_3) = t_1''(a_3) = 1$ and $t_2(a_1a_2a_3) = t_2''(a_3) = 1$. Case (II) is schematically depicted in Figure 2.2.

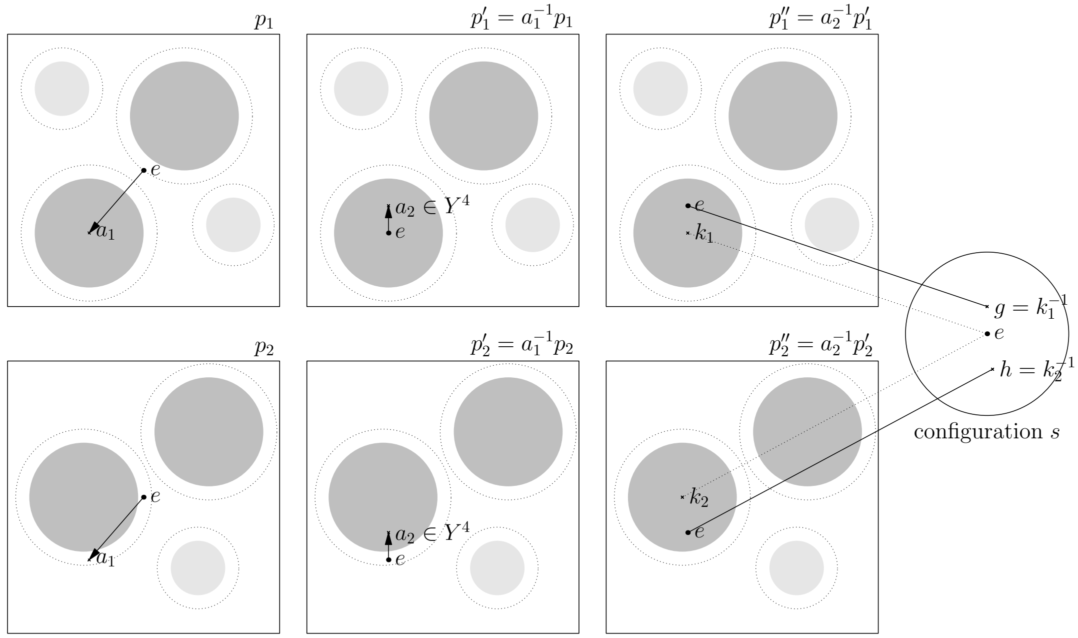


Figure 2.1: Case (I).

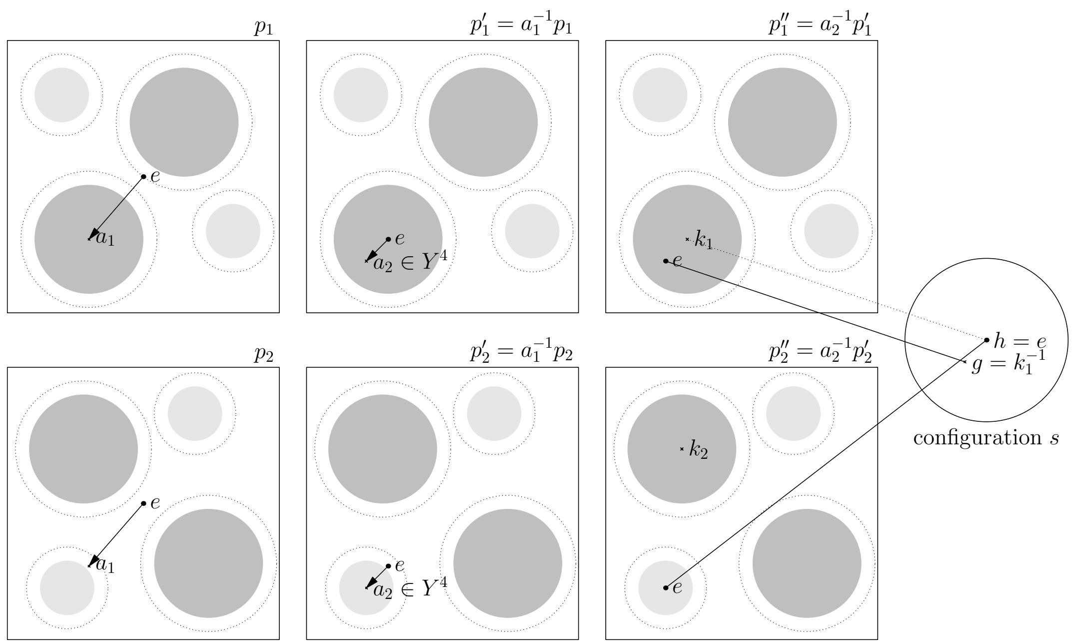


Figure 2.2: Case (II).

In both cases, we showed that there is an $a \in G$ with $t_1(a) = t_2(a) = 1$. This completes the proof. □

Proof of Proposition 2.3.4

Fix $\epsilon > 0$ and X a finite symmetric subset of G. Without loss of generality, we may assume that X includes the identity, and is large enough so that any two configurations that agree on X have distance less than ε.

Since $T \subseteq A^G$ is a strongly irreducible shift, there is a finite symmetric $U \subseteq G$ including the identity such that for any two subsets $E_1, E_2 \subseteq G$ with $E_1 U \cap E_2 U = \emptyset$ and any two configurations $t_1, t_2 \in T$, there is a configuration $t \in T$ such that t restricted to E_1 equals t_1 restricted to E_1, and t restricted to E_2 equals t_2 restricted to E_2

Given a shift $S \subseteq A^G$ and a finite $Y \subset G$, we call a map $p\colon Y \to A$ a *Y-pattern of S* if it is equal to $s|_Y$, the restriction of some $s \in S$ to Y. In this case, we say that s contains the Y-pattern p.

By strong irreducibility of T, we can find a $u \in T$ whose orbit $\{gu : g \in G\}$ contains all the X-patterns of T. Furthermore, since there are only finitely many X-patterns in T, there must be a finite $V \subset G$ (which we assume w.l.o.g. to be symmetric and contain the identity) such that $\{gu : g \in \text{ the } X\text{-interior of } V\}$ contains all the X-patterns of T. By making V even larger, we can assume that $d(t, t') < \varepsilon$ for any two configurations $t, t' \in T$ that agree on V, where $d(\cdot, \cdot)$ is the metric on T.

Let $Z = (VU^2X)(VU^2X)^{-1}$. By the assumption in the statement, there is a strongly irreducible Z-witness shift for G. Call this shift S.

Now, define a continuous equivariant function $\phi : S \times T \to A^G$. Let $s \in S, t \in T$. Let $t' = \phi(s, t)$ be defined as follows, in the following cases:

1. $g = kh$ for some $k \in G$ with $s(k) = 1$ and some $h \in V$:

   In this case, let $t'(g) = u(h)$.

2. $g = kh$ for some $k \in G$ with $s(k) = 1$ and some $h \in VU^2 \setminus V$:

   In this case let $E_1 = kV$ and $E_2 = kVU^2X^2 \setminus kVU^2$. Since $E_1 U \cap E_2 U = \emptyset$, there is a $v \in T$ with $v|_{E_1} = (ku)|_{E_1}$ and $v|_{E_2} = t|_{E_2}$. If there are multiple choices for v, choose the v such that the restriction of $k^{-1}v$ to $F = VU^2X^2 \setminus VU^2$ is lexicographically least for a fixed ordering of F and a fixed ordering of A. Let $t'(g) = v(kh)$.

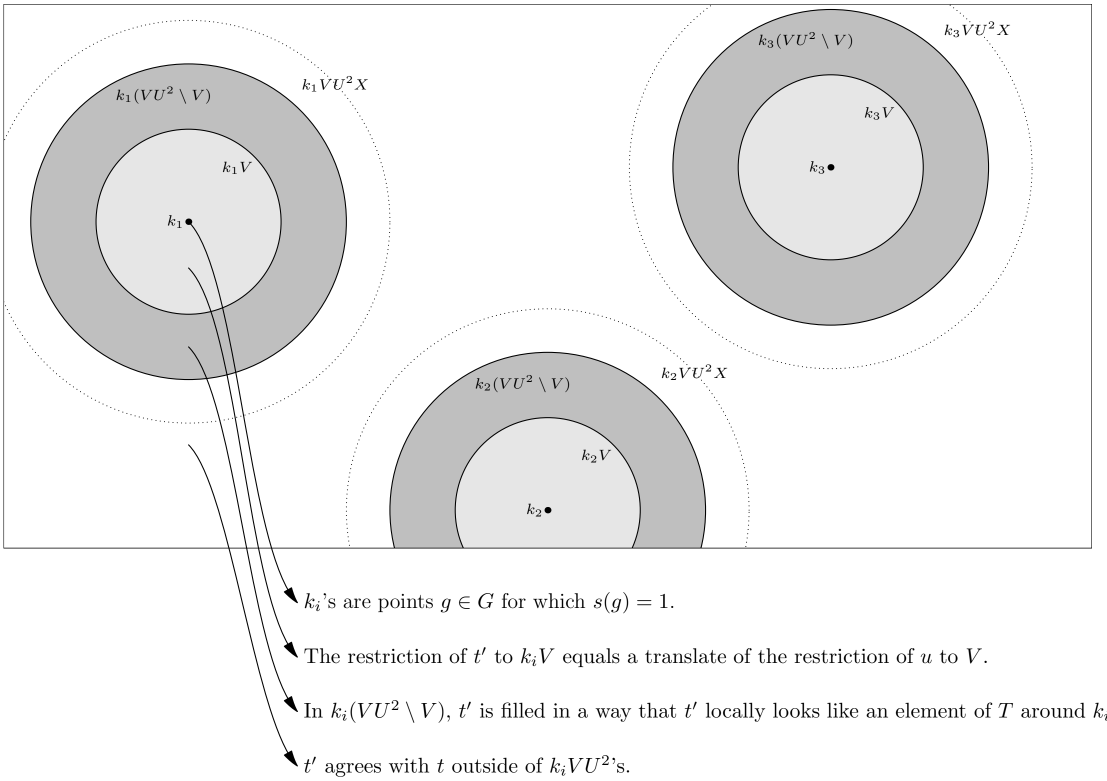


Figure 2.3: $t' = \phi(s, t)$ for $s \in S$ and $t \in T$.

3. $g \neq kh$ for $s(k) = 1$ and $h \in VU^2$:

   In this case, let $t'(g) = t(g)$.

Since the 1's in S are Z-apart, this leads to a well-defined definition for t'. Informally, t' is constructed from t as follows: the configuration t' mostly agrees with t. The first exceptions are the V-neighborhoods of any $k \in G$ such that $s(k) = 1$, where we set t' to equal the pattern that appears around the origin in u. The second exceptions are the borders of these V-neighborhoods, where some adjustments need to be made so that—as we explain below—t' and t agree on any translate of X. This construction is schematically depicted in Figure 2.3.

The following hold:

- ϕ is continuous and equivariant. So $T' = \phi(S \times T)$ is a shift.
- Since strong irreducibility is closed under taking products and factors, we see that T' is strongly irreducible.

- Let $t'_1 = \phi(s_1, t_1), t'_2 = \phi(s_2, t_2) \in T'$. Since S is a $(VU^2X)(VU^2X)^{-1}$-witness shift, there is a $g \in G$ with $s_1(g) = s_2(g) = 1$. So, $t'_1|_{gV}$ and $t'_2|_{gV}$ are both translates of $u|_V$, which means $(g^{-1}t'_1)|_V = (g^{-1}t'_2)|_V$. So from the definition of V, we get $d(g^{-1}t'_1, g^{-1}t'_2) < \epsilon$. Hence T' is ϵ-proximal.

- Now we claim that the set of X-patterns of T' and T are equal.

  First note that since $u|_V$ has all the X-patterns in T, and $u|_V$ appears in T', we get that all the X-patterns of T appear in T'.

  Now let $t' = \phi(s, t) \in T'$ and fix an X-pattern in t', located at gX. If gX does not meet any $k(VU^2)$ for $s(k) = 1$, then $t'|_{gX} = t|_{gX}$ and so the pattern appears in T. If, on the other hand, gX intersects $k(VU^2)$ for some k with $s(k) = 1$ (note that there is at most one such k), we have $gX \subseteq k(VU^2X^2)$, and by the definition of t' around k, we again see that the pattern in gX appears in T.

- To see ε-minimality of T', let $t'_1, t'_2 \in T'$. We know that $t'_2|_X$ is one of the X-patterns in T, so it appears somewhere in t'_1, i.e. there exists a $g \in G$ such that $(gt'_1)|_X$ agrees with $t'_2|_X$. We assumed that any two configurations that agree on X have distance less than ε. So, $d(gt'_1, t'_2) < \varepsilon$.

This concludes the proof of Proposition 2.3.4.

Proof of Claim 2.2.4

First we prove that proximal shifts are a G_δ. Given a shift S, an $\varepsilon > 0$ and a $g \in G$, let $P_g \subset S \times S$ be the set of pairs of configurations s_1, s_2 such that $d(gs_1, gs_2) < \varepsilon$. Since P_g is the preimage of an open set under a continuous map, we have that P_g is open. Thus, whenever S is ε-proximal, the collection $\{P_g : g \in G\}$ forms an open cover of $S \times S$ and thus, by compactness, whenever a shift is ε-proximal, there is a finite subset $X \subset G$ which suffice to demonstrate this. For each $X \subset G$, whether X demonstrates ε-proximality is determined by the restriction of S to a finite set of elements of G. But this is exactly the definition of a clopen set in the topology on the space of shifts. Thus the set of ε-proximal shifts is the union of a collection of clopen sets and is therefore open.

Now we prove that minimal shifts are a G_δ. To do this, since minimal shifts are exactly the shifts that are ε-minimal for all $\varepsilon > 0$, it is enough to show that the set of ε-minimal shifts is open. Note that by compactness of X, a shift is ε-minimal iff its ε-minimality is demonstrated by a finite set. Thus ε-minimality is determined by the set of Z-patterns for a finite large enough $Z \subseteq G$. So, as above, the set of ε-minimal shifts is a union of clopen sets, so it is open.

Proof of Proposition 2.3.6

By Proposition 2.3.5, the minimal proximal shifts are a dense G_δ in $\mathcal{S}$. It thus remains to be shown that faithfulness is also generic. Given an element $g \in G$, call a shift g-*faithful* if g acts non-trivially on the shift. It is easy to see that g-faithfulness is an open condition, and so the intersection over all non-trivial $g \in G$, which is faithfulness of the action of G, is a G_δ set. It remains to show that it is dense. To do this, we show that each non-trivial $g \in G$ acts non-trivially on every non-trivial strongly irreducible shift. Suppose g is not the identity and acts trivially on a shift S. Then all conjugates of g also act trivially on S, so that $hs = s$ for every h a conjugate of g and $s \in S$. In particular, $s(h^{-1})$ must be the same for every h a conjugate of g and every $s \in S$. Since g has an infinite conjugacy class, this holds for infinitely many such h. But if S is strongly irreducible and non-trivial, then there is some finite $X \subset G$ such that, if $h \notin X$, then there is an $s \in S$ such that $s(h) \neq s(e)$. Thus g must act non-trivially on every non-trivial strongly irreducible shift, and so we have proved the claim.

2.5 Thompson's Group F

Let F denote Thompson's group F. In the representation of F as a group of piecewise linear transformations of $\mathbb{R}$ (see, e.g., [Kai17, Section 2.C]), it is generated by a and b which are given by

$$a(x) = x - 1$$

$$b(x) = \begin{cases} x & x \leq 0 \\ x/2 & 0 \leq x \leq 2 \\ x - 1 & 2 \leq x. \end{cases}$$

The set of dyadic rationals $\Gamma = \mathbb{Z}[\frac{1}{2}]$ is the orbit of 0. The Schreier graph of the action $G \curvearrowright \Gamma$ with respect to the generating set $\{a, b\}$ is shown in Figure 2.4 (see [Kai17, Section 5.A, Figure 6]). The solid lines denote the a action and the dotted lines denote the b action; self-loops (i.e., points stabilized by a generator) are omitted. This graph consists of a tree-like structure (the blue and white nodes) with infinite chains attached to each node (the red nodes).

Equipped with the product topology, $\{-1, 1\}^\Gamma$ is a compact space on which F acts

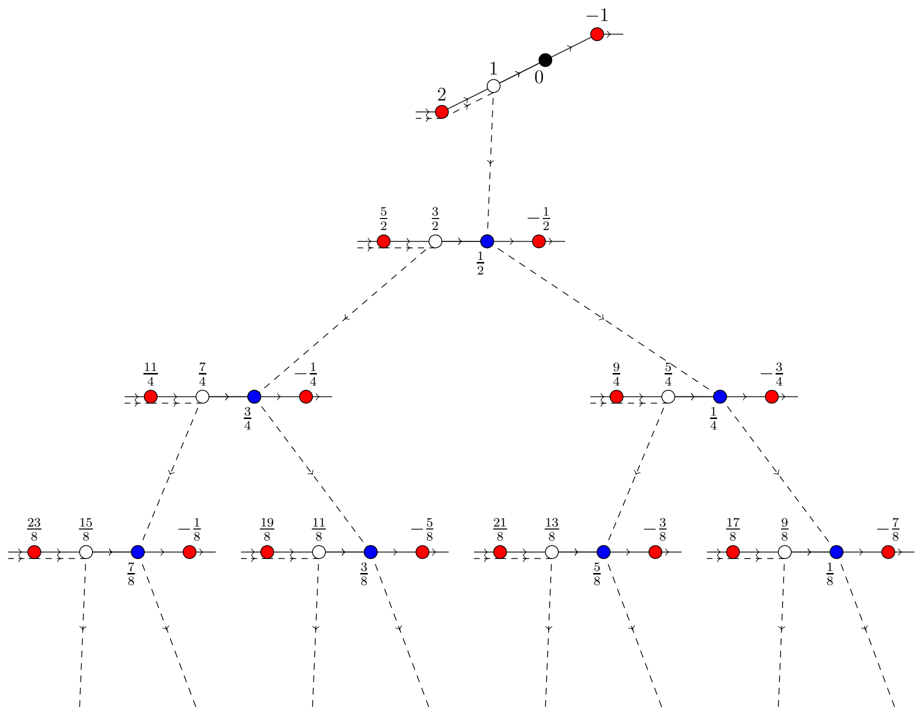


Figure 2.4: The action of F on Γ.

continuously by shifts:

$$[fx](\gamma) = x(f^{-1}\gamma). \tag{2.5.1}$$

Proposition 2.5.1. *Let $c_{-1}, c_{+1} \in \{-1,1\}^\Gamma$ be the constant functions. Then for any $x \in \{-1,1\}^\Gamma$, it holds that at least one of c_{-1}, c_{+1} is in the orbit closure $\overline{Fx}$.*

Proof. It is known that the action $F \curvearrowright \Gamma$ is highly-transitive (Lemma 4.2 in [CFP96]), i.e. for every finite $V, W \subset \Gamma$ of the same size, there exists a $f \in F$ such that $f(V) = W$. Let $x \in \{-1,1\}^\Gamma$. There is at least one of -1 and 1, say α, for which we have infinitely many $\gamma \in \Gamma$ with $x(\gamma) = \alpha$. Given a finite $W \subset \Gamma$, choose a $V \subset \Gamma$ of the same size and such that $x(\gamma) = \alpha$ for all $\gamma \in V$. Then there is some $f \in F$ with $f(V) = W$, and so fx takes the value α on W. Since W is arbitrary, we have that c_α is in the orbit closure of x. □

Given $x_1, x_2 \in \{-1,1\}^\Gamma$, let d be their pointwise product, given by $d(\gamma) = x_1(\gamma) \cdot x_2(\gamma)$. By Proposition 2.5.1, there exists a sequence $\{f_n\}$ of elements in F such that either $\lim_n f_n d = c_{+1}$ or $\lim_n f_n d = c_{-1}$. In the first case, $\lim_n f_n x_1 = \lim_n f_n x_2$, while in the second case $\lim_n f_n x_1 = -\lim_n f_n x_2$, and so this action resembles a proximal action. In fact, by identifying each $x \in \{-1,1\}^\Gamma$ with $-x$, one attains a proximal action, and indeed we do this below. However, this action has a fixed point — the constant functions — and therefore does not suffice to prove our result. We spend the remainder of this section in deriving a new action from this one. The new action retains proximality, but does not have fixed points.

Consider the path $({}^1/_2, {}^1/_4, {}^1/_8, \ldots, {}^1/_{2^n}, \ldots)$ in the Schreier graph of Γ (Figure 2.4); it starts in the top blue node and follows the dotted edges through the blue nodes on the rightmost branch of the tree. The pointed Gromov-Hausdorff limit of this sequence of rooted graphs[4] is given in Figure 2.5, and hence is also a Schreier graph of some transitive F-action $F \curvearrowright F/K$. In terms of the topology on the space $\mathrm{Sub}_F \subset \{0,1\}^F$ of the subgroups of F, the subgroup K is the limit of the subgroups K_n, where K_n is the stabilizer of ${}^1/_{2^n}$. It is easy to verify that K is the subgroup of F consisting of the transformations that stabilize 0 and have right derivative 1 at 0 (although this fact will not be important). Let $\Lambda = F/K$.

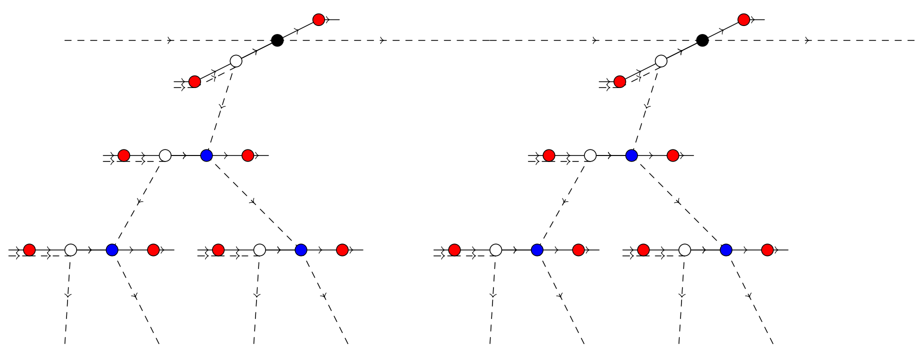


Figure 2.5: The action of F on Λ.

We can naturally identify with $\mathbb{Z}$ the chain black nodes at the top of Λ (see Figure 2.5). Let Λ' be the subgraph of Λ in which the dotted edges connecting the black nodes have been removed. Given a black node $n \in \mathbb{Z}$, denote by T_n the connected component of n in Λ'; this includes the black node n, the chain that can be reached from it using solid edges, and the entire tree that hangs from it. Each graph T_n is isomorphic to the Schreier graph of Γ, and so the graph Λ is a covering graph of Γ (in the category of Schreier graphs). Let

$$\Psi \colon \Lambda \to \Gamma$$

be the covering map. That is, Ψ is a graph isomorphism when restricted to each T_n, with the black nodes in Λ mapped to the black node $0 \in \Gamma$.

[4]The limit of a sequence of rooted graphs (G_n, v_n) is a rooted graph (G, v) if each ball of radius r around v_n in G_n is, for n large enough, isomorphic to the ball of radius r around v in G (see, e.g., [AL07, p. 1460]).

Using the map Ψ, we give names to the nodes in Λ. Denote the nodes in T_0 as $\{(0,\gamma)\ :\ \gamma\in\Gamma\}$ so that $\Psi(0,\gamma)=\gamma$. Likewise, in each T_n, denote by (n,γ) the unique node in T_n that Ψ maps to γ. Hence we identify Λ with

$$\mathbb{Z}\times\Gamma=\{(n,\gamma)\ :\ n\in\mathbb{Z},\gamma\in\Gamma\}$$

and the F-action is given by

$$a(n,\gamma)=(n,a\gamma) \tag{2.5.2}$$

$$b(n,\gamma)=\begin{cases}(n,b\gamma) & \text{if } \gamma\neq 0\\ (n+1,0) & \text{if } \gamma=0.\end{cases} \tag{2.5.3}$$

Equip $\{-1,1\}^{\Lambda}$ with the product topology to get a compact space. As usual, the F-action on Λ (given explicitly in 2.5.2 and 2.5.3) defines a continuous action on $\{-1,1\}^{\Lambda}$.

Consider $\pi\ :\ \{-1,1\}^{\Gamma}\to\{-1,1\}^{\Lambda}$, given by $\pi(x)(n,\gamma)=(-1)^n x(\gamma)$. Let $Y=\pi(\{-1,1\}^{\Gamma})\subseteq\{-1,1\}^{\Lambda}$.

Claim 2.5.2. *Y is compact and F-invariant.*

Proof. π is injective and continuous, so $Y=\pi(\{-1,1\}^{\Gamma})\subseteq\{-1,1\}^{\Lambda}$ is compact and isomorphic to $\{-1,1\}^{\Gamma}$. Moreover, Y is invariant to the action of F, because $a^{\pm1}\pi(x)=\pi(a^{\pm1}x)$ and $b^{\pm1}\pi(x)=\pi(b^{\pm}\bar{x})$ where $\bar{x}(\gamma)=\begin{cases}x(\gamma) & \text{if } \gamma\neq 0\\ -x(\gamma) & \text{if } \gamma=0\end{cases}$. □

The last F-space we define is Z, the set of pairs of mirror image configurations in Y:

$$Z=\{\{y,-y\}\ :\ y\in Y\}\,. \tag{2.5.4}$$

Now it is clear that, equipped with the quotient topology, Z is a compact and Hausdorff F-space. Furthermore, we now observe that Z admits an invariant measure. Consider the i.i.d. Bernoulli $1/2$ measure on $\{-1,1\}^{\Gamma}$, i.e. the unique Borel measure on $\{-1,1\}^{\Gamma}$, for which

$$X_\gamma\colon\{-1,1\}^{\Gamma}\to\{0,1\},\quad x\mapsto\frac{x(\gamma)+1}{2}$$

are independent Bernoulli $1/2$ random variables for all $\gamma\in\Gamma$. Clearly, it is an invariant measure and hence it is pushed forward to an invariant measure on Y, and then on Z. In particular, this shows that Z is not strongly proximal.

Claim 2.5.3. *The action $F \curvearrowright Z$ does not have any fixed points.*

Proof. Pick $\hat{y} = \{y, -y\} \in Z$. We have $[by](0,-1) = y(0,-1) \neq -y(0,-1)$, so $by \neq -y$. Similarly, $[by](0,0) = y(-1,0) = -y(0,0) \neq y(0,0)$, and so $by \neq y$. Hence $b\hat{y} \neq \hat{y}$. □

Proposition 2.5.4. *The action $F \curvearrowright Z$ is proximal.*

Proof. Let $\hat{y_1} = \{y_1, -y_1\}$ and $\hat{y_2} = \{y_2, -y_2\}$ be two points in Z, and let $y_i = \pi(x_i)$.

Let $x_1 \cdot x_2$ denote the pointwise product of x_1 and x_2. Now by Proposition 2.5.1, there is a sequence of elements $\{f_n\}_n$ in F such that $\{f_n(x_1 \cdot x_2)\}_n$ tends to either c_{-1} or c_{+1} in $\{-1,1\}^\Gamma$. Since Y is compact, we may assume that $\{f_n y_1\}_n$ and $\{f_n y_2\}_n$ have limits, by descending to a subsequence if necessary.

It is straightforward to check that $f_n y_1 \cdot f_n y_2 = f_n \pi(x_1) \cdot f_n \pi(x_2) = \pi(f_n x_1) \cdot \pi(f_n x_2)$. So:

$$\begin{aligned}
[f_n y_1 \cdot f_n y_2](n, \gamma) &= [\pi(f_n x_1) \cdot \pi(f_n x_2)](n, \gamma) \\
&= (-1)^{2n} \, [f_n x_1](\gamma) \, [f_n x_2](\gamma) \\
&= [f_n x_1 \cdot f_n x_2](\gamma) = [f_n(x_1 \cdot x_2)](\gamma).
\end{aligned}$$

So $\lim_n f_n y_1 = \pm \lim_n f_n y_2$, which implies that $\lim_n f_n \hat{y_1} = \lim_n f_n \hat{y_2}$. □

Theorem 2.5.5. *Thompson's group F is not strongly amenable.*

Proof. Since the space Z we constructed above is proximal (Proposition 2.5.4) and has no fixed points (Claim 2.5.3), we conclude that F has a proximal action with no fixed points, so F is not strongly amenable. □

Chapter 3

RANDOM WALKS AND CHOQUET-DENY GROUPS

Let G be a countable discrete group. A probability measure μ on G is *non-degenerate* if its support generates G as a semigroup.[1] A function $f\colon G \to \mathbb{R}$ is *μ-harmonic* if $f(k) = \sum_{g \in G} \mu(g) f(kg)$ for all $k \in G$. We say that the *measured group* (G, μ) is *Liouville* if all the bounded μ-harmonic functions are constant; this is equivalent to the triviality of the Poisson boundary $\Pi(G, \mu)$ [Fur63b; Fur71; Fur73] (also called the Furstenberg-Poisson boundary; for formal definitions see also, e.g., Furstenberg and Glasner [FG10], Bader and Shalom [BS06], or a survey by Furman [Fur02]).

When G is non-amenable, (G, μ) is not Liouville for every non-degenerate μ [Fur73]. Conversely, when G is amenable, then there exists some non-degenerate μ such that (G, μ) is Liouville, as shown by Kaimanovich and Vershik [KV83] and Rosenblatt [Ros81]. It is natural to ask for which groups G it holds that (G, μ) is Liouville for *every* non-degenerate μ. We call such groups *Choquet-Deny* groups; as we discuss later, there are a few variants of this definition (see, e.g., [Gla76a; Gla76b; Gui73], or [JR07]), which, however, we show to be equivalent.

The classical Choquet-Deny Theorem (which was first proved for $\mathbb{Z}^d$ by Blackwell [Bla55]) states that abelian groups are Choquet-Deny [CD60]; the same holds for virtually nilpotent groups [DM61]. There are many examples of amenable groups that are not Choquet-Deny: first examples of such groups[2] are due to Kaimanovich [Kai83] and Kaimanovich and Vershik [KV83], and include locally finite groups; Erschler shows that finitely generated solvable groups that are not virtually nilpotent are not Choquet-Deny [Ers04b], and that even some groups of intermediate growth are not Choquet-Deny [Ers04a]. Kaimanovich and Vershik [KV83, p. 466] conjecture that: "Given an exponential group G, there exists a symmetric (nonfinitary, in general) measure with non-trivial boundary." See Bartholdi and Erschler [BE17] for additional related results and further references and discussion.

Our main result in this chapter is a characterization of Choquet-Deny groups. We say that μ is *fully supported* if $\operatorname{supp}\mu = G$; obviously this implies that μ is non-

[1] In the context of Markov chains, such measures are called *irreducible*.

[2] In the Lie group setting, an example of an amenable group that is not Choquet-Deny was already known to Furstenberg [Fur63b].

degenerate.

Theorem 2. *A countable discrete group G is Choquet-Deny if and only if it has no ICC quotients, i.e. it is hyper-FC. Moreover, when G does have an ICC quotient, then there exists a fully supported, symmetric, finite entropy probability measure μ on G such that (G, μ) is not Liouville. In particular, if G is finitely generated, then it is Choquet-Deny if and only if it is virtually nilpotent.*

That a group with no ICC quotients is Choquet-Deny was shown by Jaworski [Jaw04, Theorem 4.8]. Our contribution is therefore in the proof of the converse.

Recall from §1.1 that a finitely generated group is hyper-FC if and only if it is virtually nilpotent [DM56; McL56]; this implies the result in Theorem 2 for finitely generated groups. Since finitely generated groups of exponential growth are not virtually nilpotent, Theorem 2 implies that the above mentioned conjecture of Kaimanovich and Vershik [KV83] is correct.

Different Possible Definitions of Choquet-Deny Groups

Our definition of Choquet-Deny groups is not the usual one, which states that a group is Choquet-Deny if (G, μ) is Liouville for every *adapted* measure μ, where μ is called adapted if its support generates G as a *group* (rather than as a semigroup, as in the non-degenerate case) [Gla76a; Gla76b; Gui73]. Yet another definition used in the literature requires that for *every* μ, every bounded μ-harmonic function is constant on the left cosets of G_μ, where G_μ is the subgroup of G generated by the support of μ [JR07].

While a priori these are different definitions, they are equivalent, as demonstrated by our result and by Jaworski's Theorem 4.8 in [Jaw04]. Jaworski's result shows that groups with no ICC quotients are Choquet-Deny according to any of these definitions. Since our construction of μ with a non-trivial boundary yields measures that are supported on all of G (hence non-degenerate, hence adapted), it shows that groups with ICC quotients are not Choquet-Deny according to any of these definitions. Moreover, our result shows that the class of Choquet-Deny groups (whether defined with adapted or with non-degenerate measures) is closed under taking subgroups, which, to the best of our knowledge, was also not previously known.

Proof of Theorem 2

In the remaining of this chapter, unless stated otherwise, we will assume that all groups are countable and discrete. Recall that a probability measure μ on G is symmetric if $\mu(g) = \mu(g^{-1})$ for all $g \in G$. Its Shannon entropy (or just entropy) is $H(\mu) = -\sum_{g\in G} \mu(g) \log \mu(g)$.

Our Theorem 2 is a direct consequence of [Jaw04, Theorem 4.8], which proves it for the case of groups with no ICC quotients, and of the following proposition, which handles the case of groups with ICC quotients.

Proposition 3.0.1. *Let G be a group with an ICC quotient. Then there exists a fully-supported, symmetric, finite entropy probability measure μ on G such that* $\Pi(G, \mu)$ *is non-trivial.*

The main technical effort in the proof of Proposition 3.0.1 is in the proof of the following proposition.

Proposition 3.0.2. *Let G be an amenable ICC group. For every* $h \in G \setminus \{e\}$*, there exists a fully supported, symmetric, finite entropy probability measure μ such that*

$$\lim_{m\to\infty} \|h\mu^{*m} - \mu^{*m}\| > 0. \tag{3.0.1}$$

Here μ^{*m} is the m-fold convolution $\mu * \cdots * \mu$. We will prove this Proposition later, and now turn to the proof of Proposition 3.0.1.

Proof of Proposition 3.0.1. The case of non-amenable G is known, so assume that G is amenable and has an ICC quotient Q. Let h be a non-identity element of Q. Applying Proposition 3.0.2 to Q and h yields a finite entropy, symmetric measure $\bar{\mu}$ on Q that is fully supported, and satisfies (3.0.1).

Since $\bar{\mu}$ has full support and satisfies (3.0.1), it follows from [Gla76a, Theorem 2] that $(Q, \bar{\mu})$ has a non-trivial Poisson boundary. Let μ be any symmetric, finite entropy non-degenerate probability measure on G that is projected to $\bar{\mu}$; the existence of such a μ is straightforward. Then (G, μ) has a non-trivial Poisson boundary. □

3.1 Switching Elements

Here we introduce two notions: switching elements and super-switching elements. We will use these notions in the proof of Proposition 3.0.2.

Definition 3.1.1. *Let X be a finite symmetric subset of a group G.*

- *We call $g \in G$ a* switching element for *X if*

$$X \cap gXg^{-1} \subseteq \{e\}.$$

- *We call $g \in G$ a* super-switching element for *X if*

$$X \cap \left(gXg \cup gXg^{-1} \cup g^{-1}Xg \cup g^{-1}Xg^{-1}\right) \subseteq \{e\}.$$

Note that since X is symmetric, $g \in G$ is a switching element for X if and only if g^{-1} is a switching element for X.

Claim 3.1.2. *Let X be a finite symmetric subset of a group G and let $g \in G$ be a super-switching element for X. If $g^{w_1}xg^{w_2} = y$ for $x, y \in X$ and $w_1, w_2 \in \{-1, +1\}$, then $x = y = e$.*

Proof. Let $g^{w_1}xg^{w_2} = y$ for $x, y \in X$ and $w_1, w_2 \in \{-1, +1\}$. Since

$$y = g^{w_1}xg^{w_2} \in \left(gXg \cup gXg^{-1} \cup g^{-1}Xg \cup g^{-1}Xg^{-1}\right)$$

and $y \in X$, it follows from the definition of a super-switching element for X that $y = e$.

From $g^{w_1}xg^{w_2} = y$, we get $g^{-w_1}yg^{-w_2} = x$. So, by symmetry, the same argument shows $x = e$. □

Proposition 3.1.3. *Let G be a discrete (not necessarily countable) amenable ICC group, and let X be a finite symmetric subset of G. The set of super-switching elements for X is infinite.*

Proof of Proposition 3.1.3. Fix an invariant finitely additive probability measure d on G. For $A \subseteq G$, we call $d(A)$ the density of A. We will need the fact that infinite index subgroups have zero density, and that $d(A) = 0$ for every finite subset $A \subset G$.

Let $C_G(x)$ be the centralizer of a non-identity $x \in X$. Then, since X is finite, there is a finite set of cosets of $C_G(x)$ that includes all $g \in G$ such that $g^{-1}xg \in X$. So, non-switching elements for X are in the union of finitely many cosets of subgroups with infinite index, since G is ICC. This means that the set of non-switching elements for X has zero density, and so the set S of switching elements for X has density one.

Let T be the set of all super-switching elements for X. Let $A \subseteq G$ be the set of involutions $\{g \in G \mid g^2 = e\}$.

If $d(A) > 0$, then $d(A \cap S) > 0$. On the other hand, for any $g \in A \cap S$, since g is switching for X and $g^{-1} = g$, g is super-switching for X. Hence $A \cap S \subseteq T$. This shows that if $d(A) > 0$, then $d(T) \geq d(A \cap S) > 0$, and so we are done.

So, we can assume that $d(A) = 0$. For any $x, y \in X$, let $S_{x,y} = \{g \in S \mid gxg = y\}$. Note that

$$T = S \setminus \bigcup_{\substack{x,y \in X \\ (x,y) \neq (e,e)}} S_{x,y}.$$

It is thus enough to be shown that each $S_{x,y}$ has zero density when $(x, y) \neq (e, e)$. So assume for the sake of contradiction that $d(S_{x,y}) > 0$. Fix $g \in S_{x,y}$. We have the following for all $h \in g^{-1}S_{x,y}$.

$$\begin{aligned} gxg = y = ghxgh &\implies (xg) = h(xg)h \\ &\implies (xg)^{-1}h^{-1}(xg) = h \\ &\implies h = (xg)^{-1}h^{-1}(xg) \\ &\qquad = (xg)^{-1}[(xg)^{-1}h^{-1}(xg)]^{-1}(xg) \\ &\qquad = (xg)^{-2}h(xg)^2 \\ &\implies h \text{ is in the centralizer of } (xg)^2. \end{aligned}$$

So, the centralizer of $(xg)^2$ includes $g^{-1}S_{x,y}$, which has a positive density. So, the centralizer of $(xg)^2$ has finite index. This implies that $(xg)^2 = e$, because in an ICC group only the identity can have a finite index centralizer. Hence $xg \in A$ for all $g \in S_{x,y}$. So $xS_{x,y} \subseteq A$. Hence $S_{x,y}$ also has zero density, which is a contradiction. □

3.2 A Heavy-Tailed Probability Distribution on $\mathbb{N}$

Here we state and prove a lemma about the existence of a probability distribution on $\mathbb{N} = \{1, 2, \ldots\}$ such that infinite i.i.d. samples from this measure have certain properties. We will use this distribution in the proof of Proposition 3.0.2.

Lemma 3.2.1. *Let p be the following probability measure on $\mathbb{N}$: $p(n) = cn^{-5/4}$, where $1/c = \sum_{n=1}^{\infty} n^{-5/4}$. Then p has finite entropy and the following property: for any $\varepsilon > 0$, there exist constants $K_\varepsilon, N_\varepsilon \in \mathbb{N}$ such that for any natural number $m \geq K_\varepsilon$, there exists an $E_{\varepsilon,m} \subseteq \mathbb{N}^m$ such that:*

1. *$p^{\times m}(E_{\varepsilon,m}) \geq 1 - \varepsilon$, where $p^{\times m}$ is the m-fold product measure $p \times \cdots \times p$.*

2. *For any $s = (s_1, \ldots, s_m) \in E_{\varepsilon,m}$, the maximum of $\{s_1, \ldots, s_{K_\varepsilon}\}$ is at most N_ε.*

3. *For any* $s = (s_1, \ldots, s_m) \in E_{\varepsilon,m}$ *and for any* $K_\varepsilon \leq k \leq m$, *the maximum of* $\{s_1, \ldots, s_k\}$ *is at least* k^2.

4. *For any* $s = (s_1, \ldots, s_m) \in E_{\varepsilon,m}$ *and for any* $K_\varepsilon \leq k \leq m$, *the maximum of* $\{s_1, \ldots, s_k\}$ *appears in* $(s_1, \ldots, s_k)$ *only once.*

Proof. It is straightforward to see that p has finite entropy.

Let $s = (s_1, s_2, \ldots) \in \mathbb{N}^\infty$ have distribution $p^{\times\infty}$; i.e., s is a sequence of i.d.d. random variables with distribution p. Since each s_i has distribution p, for each $n \in \mathbb{N}$, we have:

$$\mathbb{P}\left[s_i \geq n\right] = \sum_{m=n}^{\infty} p(m) = c \sum_{m=n}^{\infty} m^{-5/4} \geq c \int_n^\infty x^{-5/4} \mathrm{d}x = 4cn^{-1/4}. \tag{3.2.1}$$

For $k \geq 1$, let

$$M_k := \max\{s_1, \ldots, s_k\},$$

and let

$$\mathrm{next}(k) := \min\{i > k \mid s_i \geq M_k\}.$$

In words, $\mathrm{next}(k)$ is the first index $i > k$ for which s_i matches or exceeds M_k.

We first show that with probability one, $M_k \geq k^2$ for all k large enough. To this end, let A_k be the event that $M_k < k^2$. We have:

$$\begin{aligned} \mathbb{P}\left[A_k\right] &= \mathbb{P}\left[s_i < k^2 \ \forall i \in \{1, \ldots, k\}\right] \\ &= (1 - \mathbb{P}\left[s_1 < k^2\right])^k \\ &\leq (1 - 4c(k^2)^{-1/4})^k \\ &\leq e^{-4ck^{1/2}}. \end{aligned}$$

Since the sum of these probabilities is finite, by Borel-Cantelli we get that

$$\mathbb{P}\left[A_k \text{ infinitely often}\right] = 0.$$

Hence $M_k \geq k^2$ for all k large enough, almost surely. Furthermore, the expectation of $1/M_k$ is small:

$$\mathbb{E}\left[\frac{1}{M_k}\right] = \mathbb{E}\left[\frac{1}{M_k}\middle| A_k\right]\mathbb{P}\left[A_k\right] + \mathbb{E}\left[\frac{1}{M_k}\middle| \neg A_k\right]\mathbb{P}\left[\neg A_k\right] \leq e^{-4ck^{1/2}} + \frac{1}{k^2}. \tag{3.2.2}$$

Next, we show that, with probability one, $s_{\text{next}(k)} > M_k$ for all k large enough. That is, for large enough k, the first time that M_k is matched or exceeded after index k, it is in fact exceeded.

Let B_k be the event that $s_{\text{next}(k)} = M_k$. We would like to show that this occurs only finitely often. Note that

$$\begin{aligned}\mathbb{P}[B_k|M_k] &= \mathbb{P}\left[s_{\text{next}(k)} = M_k \middle| M_k\right] \\ &= \sum_{i=k+1}^{\infty} \mathbb{P}[s_i = M_k, \text{next}(k) = i|M_k].\end{aligned}$$

Applying the definition of $\text{next}(k)$ yields

$$\mathbb{P}[B_k|M_k] = \sum_{i=k+1}^{\infty} \mathbb{P}[s_i = M_k, s_{k+1}, \ldots, s_{i-1} < M_k|M_k].$$

By the independence of the s_i's, we can write this as

$$\begin{aligned}\mathbb{P}[B_k|M_k] &= \sum_{i=k+1}^{\infty} \mathbb{P}[s_i = M_k|M_k] \prod_{n=1}^{i-(k+1)} \mathbb{P}[s_{k+n} < M_k|M_k] \\ &= \sum_{i=k+1}^{\infty} \frac{c}{M_k^{5/4}} \mathbb{P}[s_{k+1} < M_k|M_k]^{i-(k+1)}.\end{aligned}$$

By (3.2.1), $\mathbb{P}[s_{k+1} < M_k|M_k] \leq 1 - 4cM_k^{-1/4}$. Hence

$$\mathbb{P}[B_k|M_k] \leq \frac{c}{M_k^{5/4}} \cdot \frac{1}{4cM_k^{-1/4}} = \frac{1}{4M_k}.$$

Using (3.2.2), it follows that

$$\mathbb{P}[B_k] = \mathbb{E}[\mathbb{P}[B_k|M_k]] \leq \mathbb{E}\left[\frac{1}{4M_k}\right] \leq \frac{1}{4}e^{-4ck^{1/2}} + \frac{1}{4k^2}.$$

Hence $\sum_k \mathbb{P}[B_k] < \infty$, and so by Borel-Cantelli B_k occurs only finitely often.

Since A_k and B_k both occur for only finitely many k, the (random) index ind' at which they stop occurring is almost surely finite, and is given by

$$\text{ind}' = \min\{\ell \in \mathbb{N} : s \notin A_k \cup B_k \text{ for all } k \geq \ell\}.$$

Let

$$\text{ind} = \text{next}(\text{ind}').$$

Hence for $k \geq \text{ind}$, $M_k \geq k^2$ and M_k appears in $(s_1, \ldots, s_k)$ only once.

Fix $\varepsilon > 0$. Since ind is almost surely finite, then for large enough constants $K_\varepsilon \in \mathbb{N}$ and $N_\varepsilon \in \mathbb{N}$, the event

$$E_\varepsilon = \{\text{ind} \leq K_\varepsilon \text{ and } M_{K_\varepsilon} \leq N_\varepsilon\}$$

has probability at least $1 - \varepsilon$, and additionally, conditioned on E_ε, it holds that $k \geq \text{ind}$ for all $k \geq K_\varepsilon$, and hence $M_k \geq k^2$ and M_k appears in $(s_1, \ldots, s_k)$ only once. Therefore, if for $m \geq K_\varepsilon$ we let $E_{\varepsilon,m}$ be the projection of E_ε to the first m coordinates, then $E_{\varepsilon,m}$ satisfies the desired properties. □

3.3 Proof of Proposition 3.0.2

Let $\frac{1}{8} > \varepsilon > 0$. Let $p, K_\varepsilon \in \mathbb{N}, N_\varepsilon \in \mathbb{N}$, and $E_{\varepsilon,m} \subseteq \mathbb{N}^m$ be the probability measure, the constants, and the events from Lemma 3.2.1. To simplify notation, let $N = N_\varepsilon$ and $K = K_\varepsilon$.

Let $G = \{a_1, a_2, \ldots\}$, where $a_1 = a_2 = \cdots = a_N = e$. We define $(g_n)_n$, $(A_n)_n$, $(B_n)_n$ and $(C_n)_n$ recursively. Given $g_1, \ldots, g_n$, let $A_n = \{g_n, g_n^{-1}, a_n, a_n^{-1}\}$ and $B_n = \cup_{i \leq n} A_i$. Denote $C_n = B_n \cup \{h^{-1}, h\}$. Note that A_n, B_n, and C_n are finite and symmetric for any $n \in \mathbb{N}$. Let $g_1 = g_2 = \ldots = g_N = e$. For $n + 1 > N$, given C_n, let $g_{n+1} \in G$ be a super-switching element for $(C_n)^{2n+1}$ which is not in $(C_n)^{8n+1}$. The existence of such a super-switching element is guaranteed by Proposition 3.1.3 and the facts that $(C_n)^{2n+1}$ is a finite symmetric subset of G and that $(C_n)^{8n+1}$ is finite.

For $n \in \mathbb{N}$, define a symmetric probability measure μ_n on A_n by

$$\mu_n = \varepsilon 2^{-n}(\frac{1}{2}\delta_{a_n} + \frac{1}{2}\delta_{a_n^{-1}}) + (1 - \varepsilon 2^{-n})(\frac{1}{2}\delta_{g_n} + \frac{1}{2}\delta_{g_n^{-1}}).$$

Here δ_g is the point mass on $g \in G$. Finally, let

$$\mu = \sum_{n=1}^{\infty} p(n)\mu_n.$$

Obviously μ is symmetric and $\text{supp}\,\mu = G$. Since p has finite entropy and each μ_n has support of size at most 4, it follows easily that μ has finite entropy.

We want to show that

$$\lim_{m\to\infty} \|h\mu^{*m} - \mu^{*m}\| > 0.$$

Fix $m \in \mathbb{N}$ larger than K and N. For each $n \in \mathbb{N}$, define $f_n : \{1, 2, 3, 4\} \to A_n$ by

$$f_n(1) = a_n,\ f_n(2) = a_n^{-1},\ f_n(3) = g_n,\ f_n(4) = g_n^{-1},$$

and define $\nu_n : \{1,2,3,4\} \longrightarrow [0,1]$ by

$$\nu_n(1) = \nu_n(2) = \frac{1}{2}\varepsilon 2^{-n},\ \nu_n(3) = \nu_n(4) = \frac{1}{2}(1-\varepsilon 2^{-n}).$$

Let

$$\Omega = \{(s,w) \mid s \in \mathbb{N}^m,\ w \in \{1,2,3,4\}^m\}.$$

We define the measure η on the countable set Ω by specifying its values on the singletons:

$$\eta(\{(s,w)\}) = p^{\times m}(s)\ \nu_{s_1}(w_1)\ \nu_{s_2}(w_2)\ \dots\ \nu_{s_m}(w_m).$$

It follows immediately from this definition that η is a probability measure.

Define $r : \Omega \to G$ by

$$r(s,w) = f_{s_1}(w_1) f_{s_2}(w_2) \dots f_{s_m}(w_m).$$

It is not difficult to see that $r_*\eta = \mu^{*m}$, and so we need to show that $\|hr_*\eta - r_*\eta\|$ is uniformly bounded away from zero for m larger than K and N.

Recall that $E_{\varepsilon,m} \subseteq \mathbb{N}^m$ is the event given by Lemma 3.2.1. Fix $s \in E_{\varepsilon,m}$. Define

$$\begin{aligned} i_{s,1} &= \min\{j \in \{1\dots,m\} \mid s_j > N\}, \\ i_{s,2} &= \min\{j > i_{s,1} \mid s_j \geq s_{i_{s,1}}\}, \\ &\vdots \\ i_{s,l(s)} &= \min\{j > i_{s,l(s)-1} \mid s_j \geq s_{i_{s,l(s)-1}}\}. \end{aligned}$$

Note that by the second property of $E_{\varepsilon,m}$ in Lemma 3.2.1, we know that

$$K < i_{s,1} < i_{s,2} < \dots < i_{s,l(s)},$$

and by the fourth property,

$$N < s_{i_{s,1}} < s_{i_{s,2}} < \dots < s_{i_{s,l(s)}} = \max\{s_1,\dots,s_m\}.$$

Let

$$W_\varepsilon^s = \{w \in \{1,2,3,4\}^m \mid \forall k \leq l(s)\ w_{i_{s,k}} = 3,4\}.$$

For $s \in \mathbb{N}^m$, let η_s be the measure η, conditioned on the first coordinate equalling s. I.e., let

$$\eta_s(A) = \frac{\eta(A \cap \Omega^s)}{\eta(\Omega^s)},$$

where $\Omega^s = \{s\} \times \{1,2,3,4\}^m \subseteq \Omega$.

Then

$$\begin{aligned}
\eta_s(\{s\} \times W^s_\varepsilon) &= 1 - \eta_s(\{w_{i_{s,1}} = 1,2;\ \text{or}\ w_{i_{s,2}} = 1,2;\ \ldots;\ \text{or}\ w_{i_{s,l(s)}} = 1,2\ \}) \\
&\geq 1 - \sum_{k=1}^{l(s)} \eta_s(\{w_{i_{s,k}} = 1,2\}) \\
&= 1 - \sum_{k=1}^{l(s)} \varepsilon 2^{-s_{i_{s,k}}} \\
&\geq 1 - \sum_{j=1}^{\infty} \varepsilon 2^{-j} \\
&= 1 - \varepsilon,
\end{aligned}$$

where the first inequality follows from the union bound, and the last inequality holds since $s_{i_{s,1}} < s_{i_{s,2}} < \cdots < s_{i_{s,l(s)}}$.

Finally, let

$$\Omega_\varepsilon = \{(s,w) \in \Omega \mid s \in E_{\varepsilon,m},\ w \in W^s_\varepsilon\}.$$

By the above, and since $\eta(E_{\varepsilon,m} \times \{1,2,3,4\}^m) \geq 1 - \varepsilon$ by Lemma 3.2.1, we have shown that

$$\eta(\Omega_\varepsilon) \geq\ (1-\varepsilon)(1-\varepsilon) >\ 1 - 2\varepsilon.$$

Claim 3.3.1. *For any $\alpha, \beta \in \Omega_\varepsilon$, we have $hr(\alpha) \neq r(\beta)$.*

We prove this claim after we finish the proof of the Proposition.

Let η_1 be equal to η conditioned on Ω_ε, and η_2 be equal to η conditioned on the complement of Ω_ε. We have $\eta = \eta(\Omega_\varepsilon)\eta_1 + (1 - \eta(\Omega_\varepsilon))\eta_2$, and by the above claim, we know $\|hr_*\eta_1 - r_*\eta_1\| = 2$. So for m larger than K and N

$$\begin{aligned}
\|h\mu^{*m} - \mu^{*m}\| &=\ \|hr_*\eta - r_*\eta\| \\
&=\ \|\eta(\Omega_\varepsilon)(hr_*\eta_1 - r_*\eta_1) + (1 - \eta(\Omega_\varepsilon))(hr_*\eta_2 - r_*\eta_2)\| \\
&\geq \eta(\Omega_\varepsilon)\, \|hr_*\eta_1 - r_*\eta_1\| - 2(1 - \eta(\Omega_\varepsilon)) \\
&\geq 2(1-2\varepsilon) - 2(2\varepsilon) = 2 - 8\varepsilon,
\end{aligned}$$

which is uniformly bounded away from zero since $\varepsilon < \frac{1}{8}$. Since $\|h\mu^{*m} - \mu^{*m}\|$ is a decreasing sequence, this completes the proof of Proposition 3.0.2.

Proof of Claim 3.3.1. Let $\alpha = (s, w)$, $\beta = (t, v) \in \Omega_\varepsilon$. Hence $\max\{K, N\} < m$, $s \in E_{\varepsilon,m}$, $t \in E_{\varepsilon,m}$, $w \in W_\varepsilon^s$, and $v \in W_\varepsilon^t$. Assume that $hr(\alpha) = r(\beta)$. So, we have

$$hf_{s_1}(w_1)\cdots f_{s_m}(w_m) = f_{t_1}(v_1)\cdots f_{t_m}(v_m).$$

Let $K < i_1 < i_2 < \cdots < i_{l(s)}$ and $K < j_1 < j_2 < \cdots < j_{l(t)}$ be the indices we defined for s and t in the proof of Proposition 3.0.2. We remind the reader that the unique maximum of $(s_1, \ldots, s_m)$ is attained at $i_{l(s)}$, with a corresponding statement for $(t_1, \ldots, t_m)$ and $j_{l(t)}$. So we have

$$\begin{aligned} &h\,\overbrace{f_{s_1}(w_1)\cdots f_{s_{i_{l(s)}-1}}(w_{i_{l(s)}-1})}^{b_1}\, f_{s_{i_{l(s)}}}(w_{i_{l(s)}})\, \overbrace{f_{s_{i_{l(s)}+1}}(w_{i_{l(s)}+1})\cdots f_{s_m}(w_m)}^{b_2} \\ &= \underbrace{f_{t_1}(v_1)\cdots f_{t_{j_{l(t)}-1}}(v_{j_{l(t)}-1})}_{c_1}\, f_{t_{j_{l(t)}}}(v_{j_{l(t)}})\, \underbrace{f_{t_{j_{l(t)}+1}}(v_{j_{l(t)}+1})\cdots f_{t_m}(v_m)}_{c_2}. \end{aligned}$$

Let $p = s_{i_{l(s)}} = \max\{s_1, \ldots, s_m\}$ and $q = t_{j_{l(t)}} = \max\{t_1, \ldots, t_m\}$. Since $w \in W_\varepsilon^s$ and $v \in W_\varepsilon^t$, we know $f_{s_{i_{l(s)}}}(w_{i_{l(s)}}) = g_p^{\pm 1}$ and $f_{t_{j_{l(t)}}}(v_{j_{l(t)}}) = g_q^{\pm 1}$, so

$$hb_1 g_p^{\pm 1} b_2 = c_1 g_q^{\pm 1} c_2. \tag{3.3.1}$$

Since $p = \max\{s_1, \ldots, s_m\}$, and since $m \geq K$, we know that $m \leq m^2 \leq p$. So $b_1, b_2 \in (B_{p-1})^{p-1} \subseteq (C_{p-1})^{p-1}$. Similarly, $c_1, c_2 \in (C_{q-1})^{q-1}$.

Consider the case that $p > q$. Then $c_1, c_2, g_q^{\pm 1} \in (C_q)^q \subseteq (C_{p-1})^{p-1}$. Hence $g_p^{\pm 1} = [b_1^{-1}]h^{-1}[c_1 g_q^{\pm 1} c_2 b_2^{-1}]$ by (3.3.1), and so

$$g_p \in (C_{p-1})^{4(p-1)}\{h, h^{-1}\}(C_{p-1})^{4(p-1)} \subseteq (C_{p-1})^{8(p-1)+1},$$

which is a contradiction with our choice of g_p, since $p > N$. Similarly, if $p < q$, we get a contradiction. So we can assume that $p = q$.

If $p = q$, then by (3.3.1) we have

$$hb_1 g_p^{\pm 1} b_2 = c_1 g_p^{\pm 1} c_2,$$

and $c_1, c_2, b_1, b_2 \in (C_{p-1})^{p-1}$. So, for $x = c_1^{-1} h b_1 \in (C_{p-1})^{2(p-1)+1}$, we have $g_p^{\pm 1} x g_p^{\pm 1} = c_2 b_2^{-1} \in (C_{p-1})^{2(p-1)} \subseteq (C_{p-1})^{2(p-1)+1}$. By the fact that g_p is a super-switching element for $(C_{p-1})^{2(p-1)+1}$ and from Claim 3.1.2, we get that x is the identity.

So $hb_1 = c_1$, i.e.

$$hf_{s_1}(w_1)\cdots f_{s_{i_{l(s)}-1}}(w_{i_{l(s)}-1}) = f_{t_1}(v_1)\cdots f_{t_{j_{l(t)}-1}}(v_{j_{l(t)}-1}).$$

By the exact same argument, we can see that this leads to a contradiction unless

$$hf_{s_1}(w_1)\cdots f_{s_{i_{l(s)-1}-1}}(w_{i_{l(s)-1}-1}) = f_{t_1}(v_1)\cdots f_{t_{j_{l(t)-1}-1}}(v_{j_{l(t)-1}-1}).$$

And again, this leads to a contradiction unless

$$hf_{s_1}(w_1)\cdots f_{s_{i_{l(s)-2}-1}}(w_{i_{l(s)-2}-1}) = f_{t_1}(v_1)\cdots f_{t_{j_{l(t)-2}-1}}(v_{j_{l(t)-2}-1}).$$

Note that if $l(s) \neq l(t)$, at some point in this process we get that either all the s_i's or all the t_i's are at most N while the other string has characters strictly greater than N. This leads to a contradiction similar to the case $p \neq q$, which we explained before. So, by continuing this process, we get a contradiction unless

$$hf_{s_1}(w_1)\cdots f_{s_{i_1-1}}(w_{i_1-1}) = f_{t_1}(v_1)\cdots f_{t_{j_1-1}}(v_{j_1-1}). \tag{3.3.2}$$

Note that $s_1, \ldots, s_{i_1-1} \leq N$, which implies that

$$f_{s_1}(w_1) = \cdots = f_{s_{i_1-1}}(w_{i_1-1}) = e.$$

Similarly, $t_1, \ldots, t_{j_1-1} \leq N$ implies that

$$f_{t_1}(v_1) = \cdots = f_{t_{j_1-1}}(v_{j_1-1}) = e.$$

So, from (3.3.2), we get $h = e$, which is a contradiction.

□

Appendix A

FINITELY GENERATED HYPER-FC GROUPS

Here we bring a self-contained presentation of the original proof that a finitely generated group is hyper-FC if and only if it is virtually nilpotent, which is divided between [McL56, Theorem 2] and [DM56, Theorem 2].

Let G be a group. An element $g \in G$ is said to be a *finite conjugacy (or FC) element* if it has only finitely many conjugates in G. The *FC-center* of G is the set of all FC-elements in G. The *upper FC-series* of G is defined as follows

$$\{e\} = F_0 \leqslant F_1 \leqslant \cdots \leqslant F_\alpha \leqslant \cdots,$$

where $F_{\alpha+1}/F_\alpha$ is the set of all FC-elements of G/F_α, and $F_\beta = \cup_{\alpha<\beta} F_\alpha$ for a limit ordinal β. This series will stabilize at some ordinal γ. F_γ is called the *hyper-FC center* of G and the least such γ is called the *FC-rank* of G. If G is equal to its hyper-FC center, G is called *hyper-FC*.

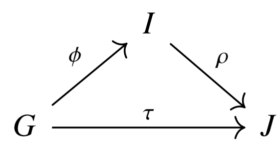
Theorem 3. *For a finitely generated group G, the following are equivalent.*

1. *G is virtually nilpotent.*
2. *G is hyper-FC.*
3. *G has no non-trivial ICC quotients.*

As a corollary, a finitely generated group is either virtually nilpotent or has an ICC quotient.

The following easy, but important, proposition shows that the obstruction to a group being ICC is the hyper-FC center of the group. Before we state the proposition, we define *a universal ICC quotient* of a group. This notion is useful to see the relation between the hyper-FC center of a group and its ICC quotients.

Definition A.0.1. *Let G be a group. A universal ICC quotient of G, which we denote by $\phi : G \to I$, is a quotient of G onto an ICC group I such that any quotient $\tau : G \to J$ of G onto an ICC group J lifts to a homomorphism $\rho : I \to J$ such that* *the following diagram commutes.*



Now we can state the following proposition.

Proposition A.0.2. *Let G be a group and let $H \trianglelefteq G$ be the hyper-FC center of G. The quotient map $\phi : G \to G/H$ is the unique, up to isomorphism, universal ICC quotient of G.*

Proof. First, we show that H is in the kernel of any ICC quotient $\tau : G \to J$. Let

$$\{e\} = F_0 \leqslant F_1 \leqslant \cdots \leqslant F_\alpha \leqslant \cdots$$

be the upper FC-series of G. If H is not in the kernel of τ, then there exists a minimum ordinal α such that $F_\alpha \nsubseteq \ker\tau$. Obviously α is not a limit ordinal. Let $h \in F_\alpha \setminus \ker\tau$. Since $hF_{\alpha-1}$ is FC in $G/F_{\alpha-1}$ and $F_{\alpha-1} \subseteq \ker\tau$ and τ is a surjective homomorphism, we get that $\tau(h)$ is FC in J. So, $\tau(h)$ is the identity in J, which is a contradiction. So, H is in the kernel of any ICC quotient of G.

Now, we show that the quotient map $\phi : G \to G/H$ is an ICC quotient. Let γ be the FC-rank of G. So $H = F_\gamma$. Note that since $F_\gamma = F_{\gamma+1}$, we know that any non-identity element of $G/H = G/F_\gamma$ has infinitely many conjugates, which shows that G/H is ICC.

Thus $\phi : G \to G/H$ is a universal ICC quotient. Uniqueness follows from a standard fact about universal properties. □

An immediate corollary of the above result is that hyper-FC groups are exactly those with no non-trivial ICC quotients. Theorem 3, which we prove next, gives a third equivalent condition when the group is finitely generated.

Proof of Theorem 3. The equivalence of (2) and (3) follows from the above corollary. We will show that (1) and (2) are equivalent. For that, we first show that the upper FC-series of a finitely generated hyper-FC group stabilizes at some finite ordinal, i.e. the FC-rank is finite.

Claim A.0.3. *Let G be a finitely generated hyper-FC group. The upper FC-series of G stabilizes at some $n \in \mathbb{N}$, i.e. its FC-rank is finite.*

Proof. Let S be a finite symmetric generating set for G. Let

$$\{e\} = F_0 \leqslant F_1 \leqslant \cdots \leqslant F_\alpha \leqslant \cdots \leqslant F_\gamma = G$$

be the upper FC-series of G. We need to show that for some $n \in \mathbb{N}$, we have $S \subseteq F_n$. For that, we will define a sequence $X_0, X_1, \ldots$ of finite subsets of G with the following properties:

1. $X_0 = S$.
2. If α_i is the least ordinal with $X_i \subseteq F_{\alpha_i}$, then either $\alpha_i = \alpha_{i-1} = 0$ or $\alpha_i = \alpha_{i-1} - 1$.

Given such a sequence, if none of the α_i's are 0, then $\alpha_0, \alpha_1, \ldots$ is an infinite strictly decreasing sequence of ordinals, which is a contradiction. So, some α_i is 0. Let n be the least index with $\alpha_n = 0$. Then $\alpha_0 = n$. By the definition of α_0, we get that $S = X_0 \subseteq F_n$. But since S generates G, we get that $G \subseteq F_n$. So the upper FC-series stabilizes at n.

Now we define the sequence $X_0, X_1, \ldots$ and prove that it has the properties we claimed. Let $X_0 = S$. Assume that $X_0, \ldots, X_i$ are defined. We want to define X_{i+1}. If $\alpha_i = 0$, then simply let $X_{i+1} = X_i$. And if $\alpha_i \neq 0$, we define X_{i+1} below. First, we make a few observations.

- Note that since α_i is the least ordinal such that F_{α_i} contains the finite set X_i, we get that α_i is not a limit ordinal.
- Since $F_{\alpha_i}/F_{\alpha_i-1}$ is the FC-center of G/F_{α_i-1} and $X_i \subseteq F_{\alpha_i}$, we have that xF_{α_i-1} is FC in G/F_{α_i-1} for each $x \in X_i$.
- For each $x \in X_i$ and each conjugate of xF_{α_i-1}, pick an element of G in that conjugate, and let Y_i be the union of X_i and the collection of all the elements we chose. So, $X_i \subseteq Y_i \subseteq F_{\alpha_i}$ and Y_i is finite.

Note that if $y \in Y_i$ and $g \in G$, then $A = (g^{-1}yg)F_{\alpha_i-1}$ is a conjugate of xF_{α_i-1} for some $x \in X_i$. Thus $A = zF_{\alpha_i-1}$ for some $z \in Y_i$, and so $z^{-1}(g^{-1}yg) \in F_{\alpha_i-1}$ for some $z \in Y_i$. Let

$$X_{i+1} = \{z^{-1}(g^{-1}yg) \mid z^{-1}(g^{-1}yg) \in F_{\alpha_i-1},\ g \in S,\ y, z \in Y_i\}.$$

Note that X_{i+1} is finite and $X_{i+1} \subseteq F_{\alpha_i-1}$. So, $\alpha_{i+1} \leq \alpha_i - 1$. To show that $\alpha_{i+1} = \alpha_i - 1$, we just need to show that $\alpha_i \leq \alpha_{i+1} + 1$, i.e. $X_i \subseteq F_{\alpha_{i+1}+1}$, which is the same as

showing that $xF_{\alpha_{i+1}}$ is FC in $G/F_{\alpha_{i+1}}$ for all $x \in X_i$. Since $X_i \subseteq Y_i$, it suffices to show that $yF_{\alpha_{i+1}}$ is FC in $G/F_{\alpha_{i+1}}$ for all $y \in Y_i$.
Since $X_{i+1} \subseteq F_{\alpha_{i+1}}$, for any $y \in Y_i$ and $s \in S$, there exists a $z \in Y_i$ with $z^{-1}(s^{-1}ys) \in F_{\alpha_{i+1}}$. Since S generates G and $F_{\alpha_{i+1}}$ is normal in G, the same holds for any $g \in G$ replacing $s \in S$. Thus $Y_iF_{\alpha_{i+1}} \subset G/F_{\alpha_{i+1}}$ is closed under taking conjugates. Since Y_i is finite, $yF_{\alpha_{i+1}}$ is thus FC in $G/F_{\alpha_{i+1}}$ for any $y \in Y_i$. This completes the proof. □

Now, we show that a finitely generated group has finite FC-rank if and only if it is virtually nilpotent. For $n \in \mathbb{N}$, denote the class of finitely generated hyper-FC groups of FC-rank less than or equal to n by $\mathcal{FC}_n$, and the class of finitely generated virtually nilpotent groups of rank less than or equal to n by $\mathcal{VN}_n$.

First we prove a useful lemma.

Lemma A.0.4. *Let G be a group, and H be a finitely generated subgroup of the FC-center of G. The centralizer of H in G, denoted by $C_G(H)$, has finite index in G.*

Proof. Let $\{h_1, ...h_n\}$ be a set of generators for H. Note that for each h_i, since it is FC in G, its centralizer $C_G(h_i)$ has finite index. Thus, the intersection of the centralizers $\cap_{i=1}^n C_G(h_i)$, which is the same as $C_G(H)$, has finite index in G. □

Claim A.0.5. *We have*

$$\mathcal{VN}_0 \subseteq \mathcal{FC}_1 \subseteq \mathcal{VN}_1 \subseteq \mathcal{FC}_2 \subseteq \cdots \subseteq \mathcal{FC}_n \subseteq \mathcal{VN}_n \subseteq \mathcal{FC}_{n+1} \subseteq \cdots .$$

Proof. First, we show that $\mathcal{VN}_{n-1} \subseteq \mathcal{FC}_n$ for any $n \in \mathbb{N}$. Let G be a group in $\mathcal{VN}_{n-1}$ for $n \in \mathbb{N}$. Let $N \trianglelefteq G$ be a finite index normal subgroup with the upper central series

$$\{e\} = Z_0 \leqslant Z_1 \leqslant \cdots \leqslant Z_m = N,$$

where $m \leq n-1$. Since Z_1 is the center of a normal subgroup of G, we get that Z_1 is normal in G. Similarly, we can show that each Z_k is normal in G. Since Z_k/Z_{k-1} is in the center of N/Z_{k-1}, we get that $N/Z_{k-1} \leqslant C_{G/Z_{k-1}}(zZ_{k-1})$ for any $z \in Z_k$, which means that $C_{G/Z_{k-1}}(zZ_{k-1})$ is of finite index in G/Z_{k-1} for any $z \in Z_k$. So, Z_k/Z_{k-1} is in the FC-center of G/Z_{k-1}. Obviously, since G/N is finite, we have that G/N is FC. So, we have that

$$\{e\} = Z_0 \leqslant Z_1 \leqslant \cdots \leqslant Z_m = N \leqslant G$$

is an FC-series for G with length $m + 1 \leq n$. So, G belongs to $\mathcal{FC}_n$.

Now, by induction on $n \in \mathbb{N}$ we show that $\mathcal{FC}_n \subseteq \mathcal{VN}_n$. Let G be a group that belongs to $\mathcal{FC}_1$. By Lemma A.0.4, the center of G has finite index in G. So, G is virtually abelian, which means that G belongs to $\mathcal{VN}_1$. Thus $\mathcal{FC}_1 \subseteq \mathcal{VN}_1$.

Let G be a group that belongs to $\mathcal{FC}_n$ for $n \geq 2$. Let

$$\{e\} = F_0 \leqslant F_1 \leqslant \cdots \leqslant F_m = G$$

be the upper FC-series of G, where $m \leq n$. Since G/F_1 is in $\mathcal{FC}_{m-1}$, by the induction hypothesis we know that G/F_1 is virtually nilpotent of rank at most $m-1$. So, there is a normal subgroup $N \trianglelefteq G$ with finite index such that $F_1 \leqslant N$ and N/F_1 is nilpotent of rank at most $m - 1$. We can make the following observations:

- Since N has finite index in a finitely generated group, N is finitely generated. Let S be a finite symmetric set of generators for N.
- Let $N = \Gamma_0 \trianglerighteq \Gamma_1 \trianglerighteq \cdots \trianglerighteq \Gamma_{m-1}$ be the first m subgroups in the lower central series of N. Since N/F_1 is nilpotent of rank at most $m - 1$, we know that $\Gamma_{m-1} \leqslant F_1$. So, Γ_{m-1} is FC.
- It is easy to see that Γ_{m-1} is the least normal subgroup of N that contains all the $(m-1)$-fold commutators $[s_1, s_2, \ldots, s_{m-1}]$, where s_i's are elements of S. Note that 1) since Γ_{m-1} is FC, we know that each of $[s_1, s_2, \ldots, s_{m-1}]$ has finitely many conjugates in N, and 2) since S is finite, we have finitely many elements of the form $[s_1, s_2, \ldots, s_{m-1}]$. So, Γ_{m-1} is finitely generated.

From the last two observations, we know that Γ_{m-1} is a finitely generated FC subgroup of N. By Lemma A.0.4, we know that $C_N(\Gamma_{m-1})$ has finite index in N. Obviously, $C_N(\Gamma_{m-1})$ has a normal subgroup M with finite index in N. It is clear that $Z = \Gamma_{m-1} \cap M$ is in the center of M. Since N has finite index in G, we get that M also has finite index in G.

By the second isomorphism theorem for groups, we have that

$$M/Z = M/(\Gamma_{m-1} \cap M) \cong (M\Gamma_{m-1})/\Gamma_{m-1} \leqslant N/\Gamma_{m-1}.$$

But we know that N/Γ_{m-1} is nilpotent with rank at most $m-1$. So, M/Z is nilpotent with rank at most $m - 1$. Also, Z is in the center of M. Hence, M is nilpotent with rank at most $m \leq n$. So, G is virtually nilpotent with rank at most n. □

Now, we can show that (1) and (2) are equivalent. Let G be a finitely generated group.

If G is a virtually nilpotent group of rank n, then by Claim A.0.5 we know that it is FC with FC-rank at most $n + 1$.

If, on the other hand, G is hyper-FC, then by Claim A.0.3 we know that its FC-rank is finite, say $n \in \mathbb{N}$. So, by Claim A.0.5 we know that it is virtually nilpotent of rank at most n. □

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