Immersed surfaces, Dehn surgery and essential laminations

Thesis by

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Abstract

Let M be an orientable and irreducible 3-manifold whose boundary is an incompressible torus. We are interested in immersed essential surfaces in closed 3-manifolds obtained from Dehn fillings on M. We show the following two things.

In Chapter 2, we suppose that M does not contain closed non-peripheral incompressible surfaces. We show that the immersed surfaces in M with the 4-plane property can realize only finitely many slopes. Moreover, we show that only finitely many Dehn fillings on M can yield 3-manifolds that admit non-positive cubing. This gives the first examples of hyperbolic 3-manifolds that cannot admit non-positive cubing.

In Chapter 3, we suppose M is hyperbolic. We show that all but finitely many Dehn fillings on M yield 3-manifolds that contain closed essential surfaces. Moreover, we give a bound on the number of exceptional Dehn fillings.

Contents

Chapter 1 Summary

A closed irreducible 3-manifold is called Haken if it contains a two-sided incompressible surface. Waldhausen has proved topological rigidity for Haken 3-manifolds [35], i.e., if two Haken 3-manifolds are homotopy equivalent, then they are homeomorphic. However, a theorem of Hatcher [19] implies that, in some sense, most 3-manifolds are not Haken. Immersed π_1-injective surfaces are a natural generalization of incompressible surfaces, and, conjecturally, 3-manifolds that contain π_1-injective surfaces have the same topological and geometric properties as Haken 3-manifolds. Another related major conjecture in 3-manifold topology is that any 3-manifold with infinite fundamental group contains a π_1-injective surface. In this thesis, we investigate immersed essential surfaces in closed 3-manifolds. In particular, we are interested in closed 3-manifolds obtained from Dehn surgery.

Dehn surgery on a 3-manifold is the operation that takes out a solid torus in the 3-manifold and glues it back using a different homeomorphism of the boundary torus. If we have a 3-manifold whose boundary is a torus, we can glue a solid torus to this 3-manifold along its boundary and get a closed 3-manifold. This operation is called Dehn filling. Dehn surgery is a useful way of constructing closed 3-manifolds. It has been known for a long time that any closed 3-manifold can be obtained from Dehn surgery on a link in S^3.

Let M be an irreducible 3-manifold whose boundary is an incompressible torus. Dehn filling on M has been used extensively to construct examples and counter-examples of closed 3-manifolds with certain properties. Thurston found first examples of non-Haken 3-manifolds that are not small Seifert fiber spaces by doing Dehn surgery on the figure eight knot. Later, Hatcher showed that, if M does not contain closed non-peripheral incompressible surfaces, only finitely many Dehn fillings on M yield Haken 3-manifolds. Along this line, Thurston has shown that, if M is hyperbolic, all but finitely many Dehn fillings on M yield closed hyperbolic 3-manifolds. This gives positive evidence for the hyperbolization conjecture.

We denote the closed manifold after Dehn filling (along slope s) by $M(s)$. Suppose $M(s)$ contains an essential surface, then there are two cases. Either there is an injective surface in M whose boundary is a union of closed curves of slope s, or M contains a closed essential surface that remains essential after the surgery.

In Chapter 2, we consider the boundary slopes of immersed surfaces with small complexity in M, namely surfaces with the 4-plane property (see Chapter 2 for the history of surfaces with the 4-plane property). The following theorem is a generalization of a theorem of Hatcher [19].

Theorem 1. *Let M be an orientable and irreducible 3-manifold whose boundary is an incompressible torus, and let $\mathcal{H}$ be the set of injective surfaces that are embedded along their boundaries and satisfy the 4-plane property. Suppose that M does not contain non-peripheral closed incompressible surfaces. Then the surfaces in $\mathcal{H}$ can realize only finitely many slopes.*

As a corollary of the theorem above, we give the first examples of hyperbolic 3-manifolds without non-positive cubing (see Chapter 2 for the history of non-positive cubing).

Theorem 2. *Let M be an orientable and irreducible 3-manifold whose boundary is an incompressible torus. Suppose that M does not contain closed non-peripheral incompressible surfaces. Then only finitely many Dehn fillings on M can yield 3-manifolds that admit non-positive cubing.*

In the proof of Theorem 1, we apply Hatcher's observation to immersed branched surfaces. The key part of the proof of Hatcher's theorem is a result of Floyd and Oertel [12]. We also generalize this result to immersed surfaces with the 4-plane property.

Theorem 3. *Let M be a closed, irreducible and non-Haken 3-manifold. Then surfaces with the 4-plane property in M are carried by finitely many immersed branched surfaces.*

A similar result of Theorem 3 for surfaces with 3-plane and 1-line properties has been shown by Choi [5] using similar approaches. The proofs of the theorems above are much more technical than the case of embedded surfaces. In particular, we used techniques of essential laminations that are not commonly used in the field of immersed surfaces.

In Chapter 3, we assume that M is hyperbolic and we construct closed immersed surfaces by closing up embedded surfaces with boundary using long annuli that wind around ∂M many times. We show that these immersed surfaces remain π_1-injective after most Dehn fillings.

Theorem 4. *Suppose M is a hyperbolic 3-manifold whose boundary is a single torus. Then all but finitely many Dehn fillings on M yield 3-manifolds that contain π_1-injective surfaces.*

Moreover, an explicit bound (depending on M) on the number of exceptional surgeries is also given in Chapter 3. Cooper and Long [8] have also proved Theorem 4 (earlier) using different methods, but no bound was given in [8].

Chapter 2 Surfaces with the 4-plane property

2.1 Introduction

Hass and Scott [18] have generalized Waldhausen's theorem by proving topological rigidity for 3-manifolds that contain π_1-injective surfaces with the 4-plane and 1-line properties. A surface in a 3-manifold is said to have n-plane property if its pre-image in the universal cover of the 3-manifold is a union of planes, and among any n planes, there is a disjoint pair. The n-plane property is a good way to measure combinatorially how complicated an immersed surface is. Incompressible surfaces are surfaces with the 2-plane property. It has been shown [33] that any immersed π_1-injective surface in a hyperbolic 3-manifold satisfies n-plane property for some n.

In this paper, we will use immersed branched surfaces to study surfaces with the 4-plane property. Branched surfaces have been used effectively in the study of incompressible surfaces and laminations [12, 15]. Many results in 3-manifold topology (e.g. Hatcher's theorem [19]) are based on the theory of branched surfaces. We define an immersed branched surface in a 3-manifold M to be a local embedding to M from a branched surface that can be embedded in some 3-manifold (see definition 2.2.1). Using lamination techniques, we will show:

Theorem 2.1. *Let M be a closed, irreducible and non-Haken 3-manifold. Then surfaces with the 4-plane property in M are carried by finitely many immersed branched surfaces.*

This theorem generalizes a fundamental result of Floyd and Oertel [12] in the theory of embedded branched surfaces. One important application of the theorem of Floyd and Oertel is the proof of a theorem of Hatcher [19], which says that incompressible surfaces in an orientable and irreducible 3-manifold can realize only finitely many slopes. However, Hatcher's theorem is not true for immersed π_1-injective surfaces in general, since there are many 3-manifolds [2, 31, 3, 27] that injective surfaces can realize infinitely many slopes, and in some cases, can realize every slope. Using Theorem 2.1, we will show that surfaces with the 4-plane property are, in a sense, like incompressible surfaces.

Theorem 2.2. *Let M be an orientable and irreducible 3-manifold whose boundary is an incompressible torus, and let $\mathcal{H}$ be the set of injective surfaces that are embedded along their boundaries and satisfy the 4-plane property. Suppose that M does not contain non-peripheral closed incompressible surfaces. Then the surfaces in $\mathcal{H}$ can realize only finitely many slopes.*

Aitchison and Rubinstein have shown that if a 3-manifold has a non-positive cubing, then it contains a surface with the 4-plane and 1-line properties, and hence topological rigidity holds for such 3-manifolds. Non-positive cubing was introduced by Gromov [16] as an example of non-positive polyhedral metric. A 3-manifold is said to have a non-positive cubing if it is obtained by gluing cubes together along their square faces under the following conditions: (1) For each edge, there are at least four cubes sharing this edge; (2) for each vertex, in its link sphere, any simple 1-cycle consisting of no more than three edges must consist of exactly three edges, and must bound a triangle. Mosher [29] has shown that if a 3-manifold has a non-positive cubing, then it satisfies the weak hyperbolization conjecture, i.e., either it is negatively curved in the sense of Gromov or its fundamental group has a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup.

So, non-positively cubulated 3-manifolds have very nice topological and geometric properties. A natural question, then, is how large the class of such 3-manifolds is. Aitchison and Rubinstein have constructed many examples of such 3-manifolds, and only trivial examples, such as manifolds with finite fundamental groups, were known without such cubing. In this paper, we will give the first non-trivial examples of 3-manifolds, in particular, first examples of hyperbolic 3-manifolds, that cannot have non-positive cubing. In fact, Theorem 2.3 says that, in some sense, most 3-manifolds do not have such cubing.

Theorem 2.3. *Let M be an orientable and irreducible 3-manifold whose boundary is an incompressible torus. Suppose that M does not contain closed non-peripheral incompressible surfaces. Then only finitely many Dehn fillings on M can yield 3-manifolds that admit non-positive cubing.*

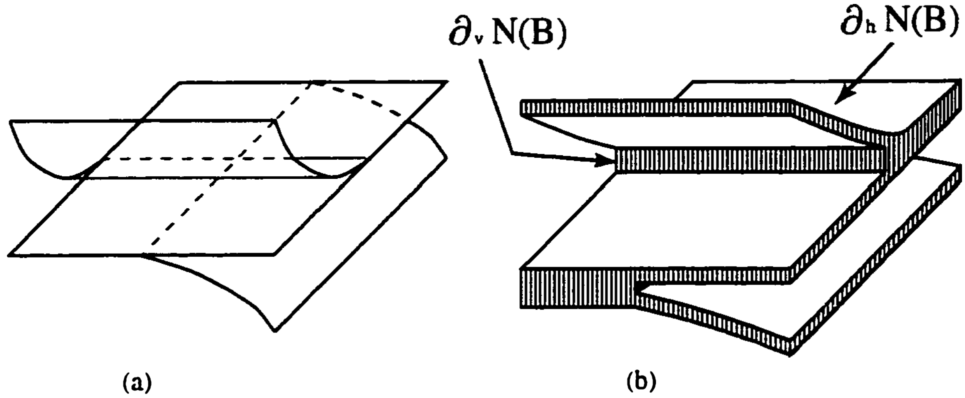


Figure 2.1:

2.2 Hatcher's trick

A branched surface in a 3-manifold is a closed subset locally diffeomorphic to the model in Figure 2.1 (a). A branched surface is said to carry a surface (or lamination) S if, after homotopies, S lies in a fibered regular neighborhood of B (as shown in Figure 2.1 (b)), which we denote by $N(B)$, and is transverse to the interval fibers of $N(B)$. We say that S is fully carried by a branched surface B if it meets every interval fiber of $N(B)$. A branched surface B is said to be incompressible if it satisfies the following conditions: (1) The horizontal boundary of $N(B)$, which we denote by $\partial_h N(B)$, is incompressible in the complement of $N(B)$, and $\partial_h N(B)$ has no sphere component; (2) B does not contain a disk of contact; (3) there is no monogon (see [12] for details).

Theorem 2.2.1 (Floyd-Oertel). *Let M be a compact and irreducible 3-manifold with incompressible boundary. Then there are finitely many incompressible branched surfaces such that every incompressible and ∂-incompressible surface is fully carried by one of these branched surfaces. Moreover, any surface fully carried by an incompressible surface is incompressible and ∂-incompressible.*

Using this theorem and a simple trick, Hatcher has shown [19] that given a compact, irreducible and orientable 3-manifold M whose boundary is an incompressible torus, the boundary curves of incompressible and ∂-incompressible surfaces in M can realize only finitely many slopes. An immediate consequence of Hatcher's theorem is that if M contains no closed non-peripheral incompressible surfaces, then all but finitely many Dehn fillings on M yield irreducible and non-Haken 3-manifolds. To prove Hatcher's theorem, we need the following lemma [19].

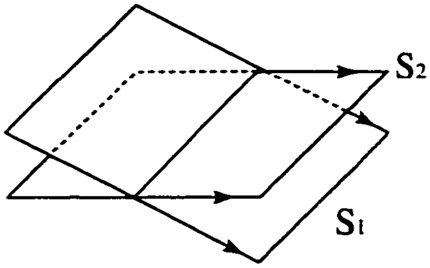
Lemma 2.2.1 (Hatcher). *Let T be a torus and τ be a train track in T that fully carries a union of disjoint and non-trivial simple closed curves. Suppose that τ does not bound a monogon. Then τ is transversely orientable.*

In Theorem 2.2.1, if ∂M is a torus, then by the definition of incompressible branched surfaces, the boundaries of those branched surfaces are train tracks that satisfy the hypotheses in Lemma 2.2.1. This lemma together with a trick of Hatcher prove the following.

Theorem 2.2.2 (Hatcher). *Let M be a compact, orientable and irreducible 3-manifold whose boundary is an incompressible torus. Suppose that $(B, \partial B) \subset (M, \partial M)$ is an incompressible branched surface. If S_1 and S_2 are two embedded surfaces carried by B, then ∂S_1 and ∂S_2 have the same slope. Moreover, the incompressible and ∂-incompressible surfaces can realize only finitely many slopes.*

Proof. Since M is orientable, the normal direction of ∂M and the transverse orientation of ∂B uniquely determine an orientation for every curve carried by ∂B. Now every component of ∂S_i ($i = 1$ *or* 2) with this induced orientation represents the same element in $H_1(\partial M)$. If ∂S_1 and ∂S_2 have different slopes, they must have a non-zero intersection number. There are two possible configurations for the induced orientations of ∂S_1 and ∂S_2 at endpoints of an arc α of $S_1 \cap S_2$, as shown in Figure 2.2. In either case, the two ends of α give points of $\partial S_1 \cap \partial S_2$ with opposite intersection numbers. Thus, the intersection number $\partial S_1 \cdot \partial S_2 = 0$. So, they must have the same slope. The last assertion of the theorem follows from the theorem of Floyd and Oertel. □

In order to apply the trick about intersection numbers, we do not need the surfaces S_1 and S_2 to be embedded. In fact, if S_1 and S_2 are immersed π_1-injective surfaces that are embedded along their boundaries and transversely intersect the interval fibers of $N(B)$, then ∂S_1 and ∂S_2 must have the same slope by the same argument. This is the starting point of this paper. Actually, even the branched surface B can be immersed. An obstruction to applying Hatcher's trick is the



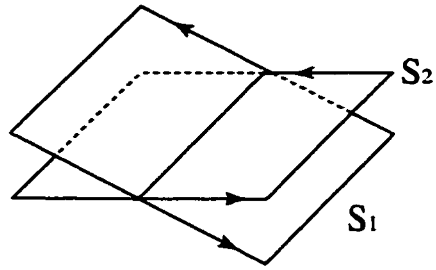


Figure 2.2:

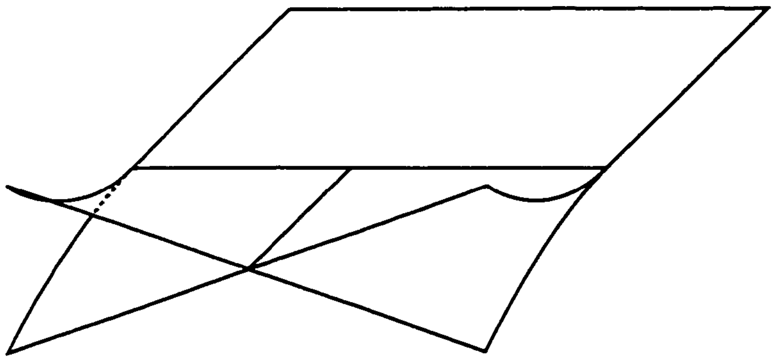


Figure 2.3:

existence of a local picture as in Figure 2.3 in B. Note that a branched surface with a local picture as in Figure 2.3 can never be embedded in any 3-manifold. Next, we will give our definition of immersed branched surfaces so that we can apply Hatcher's trick to immersed surfaces.

Definition 2.2.1. Let B be a branched surface properly embedded in some compact 3-manifold; i.e. the local picture of B in this manifold is as in Figure 2.1 (a). Let $i : B \to M$ (resp. $i : N(B) \to M$) be a map from B (resp. $N(B)$) to a 3-manifold M. We call $i(B)$ an *immersed branched surface* in M if the map i is a local embedding. An immersed surface $j : S \to M$ is said to be *carried* by $i(B)$ if, after some homotopy in M, $j = i \circ h$, where $h : S \to N(B)$ is an embedded surface that transversely intersects the interval fibers of $N(B)$.

The following proposition is an extension of Hatcher's theorem, and its proof is simply an application of Hatcher's trick to immersed branched surfaces.

Proposition 2.2.2. *Let M be a compact, orientable and irreducible 3-manifold whose boundary is an incompressible torus. Let S_1 and S_2 be immersed π_1-injective* *surfaces carried by an immersed branched surface B. Suppose that $i|_{\partial B}$ is an embedding and $i(\partial B)$ does not bound a monogon, where $i : B \to M$ is the immersion. Then ∂S_1 and ∂S_2 have the same slope.*

□

2.3 Cross disks

We have seen in section 2.2 that Hatcher's trick can be applied to immersed branched surfaces. However, we also need finiteness on the number of branched surfaces, as in the theorem of Floyd and Oertel, to get interesting results. This is impossible in general because there are many examples of 3-manifolds that immersed π_1-injective surfaces can realize infinitely many slopes. In this section, we will show that this can be done under certain assumptions.

By the normal surface theory, it is very easy to get finiteness (of the number of branched surfaces) in the case of embedded branched surfaces. For any triangulation of a 3-manifold, an incompressible surface can be put in Kneser-Haken normal form [26, 17]. The intersection of the surface with each tetrahedron is a union of normal disks in at most 7 normal disk types. By identifying all the normal disks of the same type to a branch sector, we can naturally construct a branched surface carrying this surface, and the finiteness follows from the compactness of the 3-manifold (see [12] for details). However, in the case of immersed surfaces, we cannot do this, although immersed π_1-injective surfaces can also be put in normal form. Given two normal disks (of the same normal disk type) that intersect each other, if we simply put them in the same branch sector, we may get a picture like that in Figure 2.3 in some tetrahedron along the double curves of the immersed surface, which makes Hatcher's argument fail.

Suppose that S is a π_1-injective surface in a 3-manifold M with a triangulation $\mathcal{T}$. By the normal surface theory, we can assume that S is in normal form. Let $\tilde{M}$ be the universal cover of M, $\pi : \tilde{M} \to M$ be the covering map, $\tilde{S} = \pi^{-1}(S)$, and $\tilde{\mathcal{T}}$ be the induced triangulation of $\tilde{M}$. For any arc α in M (or $\tilde{M}$) whose interior does not intersect the one skeleton $\mathcal{T}^{(1)}$, we define the *length* of α to be $|int(\alpha) \cap \mathcal{T}^{(2)}|$, where $int(E)$ denotes the interior of E and $|E|$ denotes the number of connected components of E. Moreover, we define the *distance* between points x and y, $d(x,y)$, to be the minimal length of all such arcs connecting x to y. In this paper, we will always use the distance defined above unless specified. A normal (immersed) surface $f : F \to M$ is said to have *least weight* if $|f^{-1}(\mathcal{T}^{(1)})|$ is minimal in the homotopy class of f. Let $g : F \to M$ be a π_1-injective map and F' be the universal cover of F. Suppose that $\tilde{g} : F' \to \tilde{M}$ is a lift of $g \circ p : F' \to M$ to $\tilde{M}$, where $p : F' \to F$ is the covering map. Then we call $\tilde{g}(F')$ a *component* of $\tilde{F}$ in $\tilde{M}$, where $\tilde{F}$ is the pre-image of $g(F)$ in $\tilde{M}$. We say that a component of $\tilde{F}$ has *least weight* if any disk in this component has least weight among all the disks in $\tilde{M}$ with the same boundary. A normal homotopy is defined to be a smooth map $H : F \times [0,1] \to M$ so that for each $t \in [0,1]$, the surface F_t given by $H|_{F\times\{t\}}$ is a normal surface. Note that the weight of F_t is fixed in a normal homotopy. A π_1-injective immersed surface $f : (F, \partial F) \to (M, \partial M)$ ($F \neq S^2$ or P^2) is said to have n-plane property if every component of the pre-image of $f(F)$ in $\tilde{M}$ is embedded and in any collection of n different components, there is a disjoint pair. From the PL-minimal surface theory [22], we know that for any π_1-injective surface $f : F \to M$, there is a normal surface $f_1 : F \to M$ of least weight in the homotopy class of f such that any component of the pre-image of $f_1(F)$ in $\tilde{M}$ is embedded. Moreover, it follows from Theorem 5 of [22] or Theorem 3.4 of [14] that f_1 can be chosen so that any component of the pre-image of $f_1(F)$ in $\tilde{M}$ has least weight. By Theorem 8 of [22] (or Theorem 6.3 of [14]), if f has n-plane property, so does f_1.

In this paper, we will assume that our 3-manifolds are compact and irreducible, and our immersed surfaces, when restricted to the boundary, are embedded. We will also assume that our injective surfaces are normal and have least weight, and any component of their pre-images in the universal cover of the 3-manifold has least weight. To simplify notation, we will not distinguish $f : F \to M$, F and $f(F)$ unless necessary.

Definition 2.3.1. Let $f : F \to M$ be an π_1-injective normal surface. Let F_1 and F_2 be two components of the pre-image of $f(F)$ in $\tilde{M}$. Suppose that D_1 and D_2 are two embedded sub-surfaces in F_1 and F_2 respectively. We say that D_1 and D_2 are *parallel* if there is a normal homotopy $H : D \times I \to \tilde{M}$ such that $H(D,0) = D_1$,

$H(D,1) = D_2$ and H fixes the 2-skeleton, i.e., if $H(x,y) \in \tilde{T}^{(i)}$ then $H(x,I) \subset \tilde{T}^{(i)}$ $(i = 1,2)$. We call $D_1 \cup D_2$ a *cross disk* if D_1 and D_2 are parallel disks, $F_1 \neq F_2$, and $F_1 \cap F_2 \neq \emptyset$. We call D_i $(i = 1,2)$ a component of the cross disk $D_1 \cup D_2$. Let H be the normal homotopy as above. We call $H(p,0) \cup H(p,1)$ a *pair of points* (resp. *arcs, disks*) in the cross disk, for any point (resp. arc, disk) p in D. A cross disk $D_1 \cup D_2$ (or the disk D_1) is said to have *size* at least R if there exists a point $x \in D_1$ such that $length(\alpha) \geq R$ for any normal arc $\alpha \subset D_1$ connecting x and $\partial D_1 - \partial\tilde{M}$, and we call the normal disk of $T \cap D_1$ that contains x a *center* of the cross disk, where T is a tetrahedron in the triangulation. To simplify notation, we will also call $\pi(D_1 \cup D_2)$ a cross disk and call the image (under the map π) of a pair of points (resp. arcs, disks) in $D_1 \cup D_2$ a pair of points (resp. arcs, disks) in the cross disk, where $\pi : \tilde{M} \to M$ is the covering map.

We denote by $\mathcal{F}$ the set of π_1-injective and ∂-injective surfaces in M whose boundaries are embedded in ∂M. Let $\mathcal{F}_R = \{F \in \mathcal{F} :$ there are no cross disks of size R in $\tilde{F}\}$, where $\tilde{F}$ is the pre-image of F in $\tilde{M}$. The following lemma is due to Choi [5].

Lemma 2.3.1. *The surfaces in $\mathcal{F}_R$ are carried by finitely many immersed branched surfaces.*

Proof. Let T be a tetrahedron in the triangulation $\mathcal{T}$ of M and $d_i \subset F \cap T$ be a normal disk $(i = 1,2,3)$, where $F \in \mathcal{F}_R$. Suppose that $\tilde{T}$ is a lift of T in $\tilde{M}$, $\tilde{d_i}$ is a lift of d_i in $\tilde{T}$, and F_i is a component of $\tilde{F}$ in $\tilde{M}$ that contains $\tilde{d_i}$ $(i = 1,2,3)$, where $\tilde{F}$ is the pre-image of F in $\tilde{M}$. We call $D_N(d_i) = \{x \in F_i | d(x,p) \leq N,$ where $p \in \tilde{d_i}\}$ a surface of radius N with center $\tilde{d_i}$. Note that, topologically, $D_N(d_i)$ may not be a disk under this discrete metric.

Next, we will define an equivalence relation. We say that d_1 is equivalent to d_2 if $D_{kR}(d_1)$ is parallel to $D_{kR}(d_2)$ and $F_1 \cap F_2 = \emptyset$ (or $F_1 = F_2$), where k is fixed. We assume that k is so large that $D_{kR}(d_i)$ contains a sub-disk of size R whose center is $\tilde{d_i}$ $(i = 1,2)$. Note that, since M is compact and every component of $\tilde{F}$ has least weight, k can be chosen to be independent from the choices of $F \in \mathcal{F}_R$ and the normal disk $d_i \subset F$, i.e., k depends only on R and the triangulation of M. Suppose that there are three normal disks d_1, d_2 and d_3 in $F \cap T$ so that d_1 is equivalent to d_2 and d_2 is equivalent to d_3. Then $D_{kR}(d_1)$ is parallel to $D_{kR}(d_3)$ by definition. If $F_1 \neq F_3$ and $F_1 \cap F_3 \neq \emptyset$, by the assumption on k, there is a cross disk of size R that consists of two disks from F_1 and F_3 respectively. This contradicts the hypothesis that $F \in \mathcal{F}_R$. Thus d_1 is equivalent to d_3, and the equivalence relation is well-defined.

Since M is compact, for any normal disk d in $\tilde{M}$, the number of non-parallel (embedded) normal surface of radius kR (with center d) is bounded by a constant C that depends only on R and the triangulation of M. As there are no cross disks of size R, there are at most C equivalence classes in the normal disks of $F \cap T$ of the same type, and hence at most $7C$ equivalence classes in $F \cap T$ (since there are 7 different types of normal disks). For any tetrahedron T, suppose there are C_T ($C_T \leq 7C$) equivalence classes in $F \cap T$. We put C_T products $D_i \times I$ ($i = 1, \dots, C_T$) in T such that $D_i \times \{t\}$ is a normal disk and the normal disks of $F \cap T$ in the same equivalence class lie in the same product $D_i \times I$. Along $T^{(2)}$, we can glue these products $D_i \times I$'s together according to the equivalence classes, as in the construction of embedded branched surfaces in [12]. In fact, we can abstractly construct a branched surface B and a map $f : N(B) \to M$ such that, for any tetrahedron T, $f(\partial_v N(B)) \subset T^{(2)}$ and $f(N(B) - p^{-1}(L)) \cap T$ is exactly the union of the products $int(D_i) \times I$'s in T, where L is the branch locus of B, $p : N(B) \to B$ is the map that collapses every interval fiber of $N(B)$ to a point, and $int(D_i)$ denotes the interior of D_i. By our construction, B does not contain a local picture like that in Figure 2.3, and hence it can be embedded in some 3-manifold [6]. Since the number of equivalence classes is bounded by a constant, there are only finitely many such immersed branched surfaces that carry surfaces in $\mathcal{F}_R$. □

Corollary 2.3.2. *Suppose M is a compact, orientable and irreducible 3-manifold whose boundary is an incompressible torus. Then the surfaces in $\mathcal{F}_R$ can realize only finitely many slopes.*

Proof. Suppose that $F_1, F_2 \in \mathcal{F}_R$ are carried by the same immersed branched surface $f : B \to M$. To simplify notation, we will also denote by f the correspondent map from $N(B)$ to M. Since the surfaces in $\mathcal{F}_R$ are embedded along their boundaries, after some normal homotopy if necessary, we can assume that $f|_{\partial B}$ is an embedding.

Since the surfaces in $\mathcal{F}_R$ are π_1-injective, the horizontal boundary of $f(\partial B)$ does not contain any component that is a trivial circle. Because of Lemma 2.3.1 and Proposition 2.2.2, we only need to show that $f(\partial B)$ does not bound a monogon. Since $f|_{\partial B}$ is an embedding, to simplify notation, we do not distinguish ∂B and $f(\partial B)$, and denote $f(N(\partial B))$ by $N(\partial B)$, where $N(\partial B)$ is a fibered neighborhood of the train track ∂B. By our definition of immersed branched surface, we can assume that $F_1 \subset f(N(B))$ and $f^{-1}(F_1)$ is an embedded surface carried by $N(B)$.

Suppose that $D \subset \partial M$ is a monogon, i.e., $\partial D = \alpha \cup \beta$, where α is a vertical arc of $\partial_v N(\partial B)$ and $\beta \subset \partial_h N(\partial B)$. The vertical boundary component of $f(\partial_v N(B))$ that contains α is a rectangle E whose boundary consists of two vertical arcs α, α' on ∂M and two arcs γ, γ' in $f(\partial_v N(B) \cap \partial_h N(B))$. By our definition, $f^{-1}(F_1)$ is embedded in $N(B)$. So, after some normal homotopy, we may assume that E is embedded, $\partial_h N(\partial B) \subset \partial F_1$, and $\gamma \cup \gamma' \subset F_1$. Then $\delta = \beta \cup \gamma \cup \gamma'$ is an embedded arc in F_1 with $\partial\delta \subset \partial F_1 \subset \partial M$, and δ can be homotoped rel $\partial\delta$ into ∂M. Since F_1 is ∂-injective, δ must be ∂-parallel in F_1, i.e., there is an arc $\delta' \subset \partial F_1$ such that $\delta \cup \delta'$ bounds a disk Δ in F_1. Moreover, $\alpha' \cup \delta'$ also bounds a disk D' in ∂M, since $\alpha' \cup \delta'$ forms a homotopically trivial curve in M. Note that Δ may not be embedded in M. So, $D \cup E \cup \Delta \cup D'$ forms an immersed sphere in M. Since M is irreducible, i.e., $\pi_2(M)$ is trivial, we can homotope the sphere $D \cup E \cup \Delta \cup D'$ (fixing E) into E. After this homotopy, we get an immersed surface in the same homotopy class as F_1 with less weight. This contradicts our least weight assumption on the surface F_1.

So, ∂B does not bound any monogon. By Proposition 2.2.2, ∂F_1 and ∂F_2 must have the same boundary slope, and the corollary follows from Lemma 2.3.1.

□

2.4 Limits of cross disks

Let $\mathcal{H}$ be the set of injective surfaces with the 4-plane property in M. If there is a $K \in \mathbb{R}$ such that $\mathcal{H} \in \mathcal{F}_K$, by Corollary 2.3.2, the surfaces in $\mathcal{H}$ can realize only finitely many slopes. Suppose no such number K exists. Then there must be a sequence of surfaces $F_1, F_2, \ldots, F_n, \cdots \in \mathcal{H}$ such that, in the pre-image of F_i in $\tilde{M}$ (denoted by $\tilde{F}_i$), there is a cross disk $D_i = D_i' \cup D_i''$ of size at least i, where $i \in \mathbb{N}$.

Since M is compact, after passing to a sub-sequence if necessary, we can assume that D_i' is parallel to a sub-disk Δ_i of D_{i+1}' such that $d(\partial\Delta_i - \partial\tilde{M}, \partial D_{i+1}' - \partial\tilde{M}) \geq 1$. We also assume that $\partial D_i'$ lies in the 2-skeleton.

Now we consider the image of D_i in M, i.e., $\pi(D_i)$, where $\pi : \tilde{M} \to M$ is the covering map.

Proposition 2.4.1. *The intersection of $\pi(D_i)$ with any tetrahedron does not contain two quadrilateral normal disks of different types.*

Proof. We know that any two quadrilateral normal disks of different types must intersect each other. Suppose that the intersection of $\pi(D_i)$ with a tetrahedron contains two different types of quadrilateral normal disks. Let T be a lift of this tetrahedron in $\tilde{M}$. Then, in each of the two quadrilateral disk types, there is a pair of normal disks in $\tilde{F}_i \cap T$ that belong to different components of a cross disk. So, by the definition of cross disk, the two components of $\tilde{F}_i$ that contain the two normal disks must intersect each other. The two different quadrilateral disk types give rise to 4 components of $\tilde{F}_i$ intersecting each other. Note that, since each component of $\tilde{F}_i$ is embedded, the 4 components above are different components of $\tilde{F}_i$. This contradicts the 4-plane property. □

Thus, as in [12], we can construct an embedded branched surface B_i that carries $\pi(D_i)$, i.e., $\pi(D_i)$ lies in $N(B_i)$ transversely intersecting every interval fiber of $N(B_i)$. In fact, for each normal disk type of $\pi(D_i) \cap T$, we construct a product $\delta \times I$, where T is a tetrahedron and $\delta \times \{t\}$ is a normal disk of this disk type ($t \in I$). Then, we can glue these products along $\mathcal{T}^{(2)}$ naturally to get a fibered neighborhood of a branched surface B_i, and $\pi(D_i)$ can be isotoped into $N(B_i)$ transversely intersecting every interval fiber of $N(B_i)$. Note that B_i may have non-trivial boundary. After some isotopy, we can assume that $\partial_v N(B_i) \cap \mathcal{T}^{(1)} = \emptyset$ and $N(B_i) \cap \mathcal{T}^{(2)}$ is a union of interval fibers of $N(B_i)$. Note that by the definition of cross disk, we can assume that the image of every pair of points in the cross disk lies in the same I-fiber of $N(B_i)$.

Proposition 2.4.2. *$N(B_i)$ can be split into an I-bundle over a compact surface such that, after isotopies, every pair of points in the cross disk $\pi(D_i)$ lies in the same I-fiber of this I-bundle.*

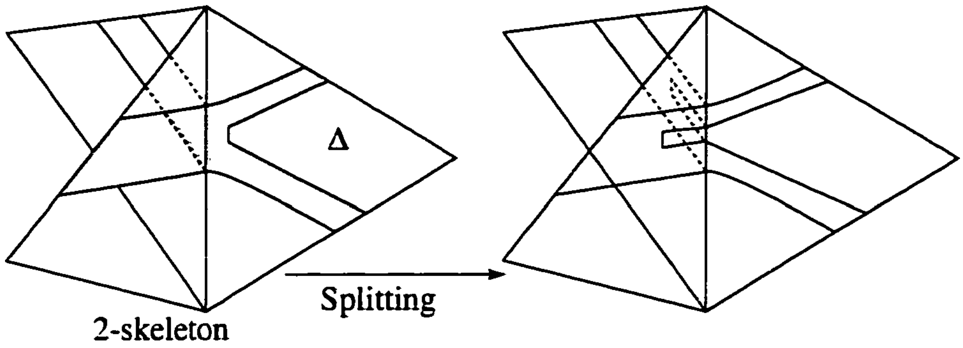


Figure 2.4:

Proof. By our construction above, $N(B_i) \cap \mathcal{T}^{(2)}$, when restricted to a 2-simplex in $\mathcal{T}^{(2)}$, is a fibered neighborhood of a union of train tracks. Suppose that $\partial_v N(B_i)$ transversely intersects $\mathcal{T}^{(2)}$. First, we split $N(B_i)$ near $N(B_i) \cap \mathcal{T}^{(2)}$ to eliminate $\partial_v N(B_i) \cap \mathcal{T}^{(2)}$.

Let Δ be a 2-simplex in $\mathcal{T}^{(2)}$, δ be a component of $\partial_v N(B_i) \cap \Delta$ and τ be a component of $N(B_i) \cap \Delta$ that contains δ. We associate every component δ' of $\partial_v N(B_i) \cap \Delta$ with a direction (in Δ) that is orthogonal to δ' and points into the interior of $N(B_i) \cap \Delta$. Let V be the union of interval fibers of τ that contain some components of $\partial_v N(B_i) \cap \Delta$. By some isotopies, we can assume that every interval fiber in V contains only one component of $\partial_v N(B_i) \cap \Delta$. We give every interval fiber in V a direction induced from the direction of $\partial_v N(B_i) \cap \Delta$. Now $\tau - V$ is a union of rectangles with two horizontal edges from $\partial_h N(B_i)$ and two vertical edges from V. Every vertical edge of a rectangle has an induced direction.

Case 1. For any rectangle of $\tau - V$, the direction of at most one vertical edge points inwards.

In this case, there is no ambiguity about the splitting near the rectangle. We split τ as shown in Figure 2.4, pushing a component of $\partial_v N(B)$ across an edge of Δ. During the splitting we also push some double curves of F_i across this edge. The effect of the splitting to $\pi(D_i)$ is just an isotopy. Thus, we can assume that any pair of points in the cross disk lies in the same interval fiber of the fibered neighborhood of the branched surface after this splitting.

Case 2. There is a rectangle in $\tau - V$ such that the directions of both vertical edges

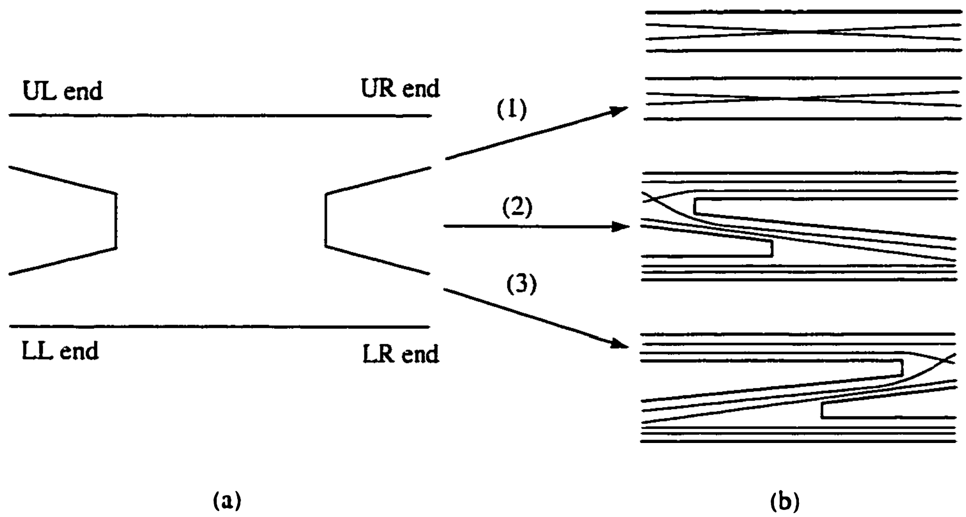


Figure 2.5:

point inwards.

The local picture of such a rectangle must be as in Figure 2.5 (a), and there are (locally) three different splittings as shown in Figure 2.5 (b). We denote the rectangle by R and the part of τ as in Figure 2.5 (a) by τ_R. Then $\tau_R - R$ consists of 4 components, and we call them UL (upper left) end, LL (lower left) end, UR (upper right) end and LR (lower right) end, as shown in Figure 2.5 (a). The intersection of the image of the cross disk and τ_R, i.e., $\pi(D_i) \cap \tau_R$, consists of arcs connecting the ends on the left side to the ends on the right side. An arc in $\pi(D_i) \cap \tau_R$ is called a diagonal arc if it connects an upper end to a lower end.

Claim. $\pi(D_i) \cap \tau_R$ does not contain two diagonal arcs, say α and β, such that α connects the UL end to the LR end, and β connects the LL end to the UR end.

Proof of the claim. Suppose that it contains such arcs α and β. Then there is another arc α' (resp. β') such that $\alpha \cup \alpha'$ (resp. $\beta \cup \beta'$) is a pair of arcs in the cross disk. So, α' (resp. β') also connects the UL end to the LR end (resp. the LL end to the UR end). Note that α (or α') and β (or β') must have non-trivial intersection in τ_R. Next we consider a lift of τ_R in $\tilde{M}$ and still use the same notation. By the definition of cross disk, the 4 components of $\tilde{F}_i$ that contain α, α', β and β' respectively must intersect each other in $\tilde{M}$. Since every component of $\tilde{F}_i$ is embedded in $\tilde{M}$, each is a different component of $\tilde{F}_i$. This contradicts the assumption that F_i has the 4-plane property. □

Now we split $N(B_i)$ near τ_R as follows. If there are no diagonal arcs in $\pi(D_i)\cap\tau_R$, we split $N(B_i)$ in a small neighborhood of τ_R as the splitting (1) in Figure 2.5. If there are diagonal arcs, we split it as the splitting (2) or (3) in Figure 2.5 according to the type of the diagonal arcs. Note that by the claim, diagonal arcs of different types cannot appear in τ_R at the same time. As in case 1, we can assume that any pair of points of the cross disk lies in the same I-fiber after the splitting. To simplify the notation, we will also denote the branched surface after the splitting by B_i. Since D_i is bounded, after finitely many such splittings, $\partial_v N(B_i) \cap \mathcal{T}^{(2)} = \emptyset$. Now $\partial_v N(B_i)$ is contained in the interior of the 3-simplices, i.e., in a collection of disjoint open 3-balls. So, every component of $\partial_v N(B_i)$ bounds a disk of contact (or a half disk of contact near the boundary). After we cut $N(B_i)$ along these (half) disks of contact, as in [12], $\partial_v N(B_i) = \emptyset$ and $N(B_i)$ becomes an I-bundle over a compact surface. As before, we can assume that, after isotopies if necessary, every pair of points in the cross disk lies in the same I-fiber. □

In the splittings above, we can preserve the intersection pattern of $\tilde{F}_i$. For any arc $\gamma \subset F_i \cap \Delta$, since every arc in $F_i \cap \Delta$ is a normal arc in the triangle Δ, we can assume that if an arc (in $F_i \cap \Delta$) does not intersect γ before the splitting, it does not intersect γ after the splitting. Moreover, since the intersection of F_i with any tetrahedron is a union of normal disks, we can assume that cutting the (half) disks of contact as above does not destroy the 4-plane property. The effect of the splitting on F_i is just a homotopy pushing some double curves out of the cross disk. So, after the splitting, F_i still satisfies the 4-plane property and has least weight. Therefore, we can assume for each i, $\pi(D_i)$ is carried by an I-bundle over a compact surface. We will still denote this I-bundle by $N(B_i)$.

After collapsing every I-fiber of $N(B_i)$ to a point, we get a piece of embedded normal surface, which we denote by S_i, in M. Furthermore, D_i' is parallel to a sub-surface of a component of $\tilde{S}_i$, where $\tilde{S}_i$ is the pre-image of S_i in $\tilde{M}$.

There are only finitely many embedded normal surfaces (up to normal isotopy) in M that are images (under the covering map π) of normal surfaces that are parallel to D_i'. So, after passing to a sub-sequence and doing some isotopies if necessary, we can assume that S_i is a sub-surface of S_{i+1}. By our assumption $d(\partial D_i - \partial\tilde{M}, \partial D_{i+1} -$

$\partial\tilde{M}) \geq 1$, it is easy to see that the direct limit of the sequence $\{S_i\}$ is a surface in M whose boundary lies in ∂M, and its closure is a lamination in M. We denote this lamination by λ. Next we will show that λ is an essential lamination. Before we proceed, we will prove a useful lemma.

Lemma 2.4.3. *Let F_0 be an injective normal surface in a 3-manifold M and F be a component of the pre-image of F_0 in $\tilde{M}$. Suppose that F has least weight and there are two disks D_1 and D_2 embedded in F and parallel to each other. Suppose that there is another embedded disk D with $\partial D = \alpha \cup \beta$, where $\beta = D \cap (F - D_1 \cup D_2)$, $\beta \cap D_1$ and $\beta \cap D_2$ are two endpoints of α, and α is an arc lying in a 2-simplex. Then $weight(D_1) \leq weight(D)$.*

Proof. Since F has least weight, we can assume that F is embedded in $\tilde{M}$. As D_1 and D_2 are parallel, there is an embedded region $D^2 \times [1,2]$ in $\tilde{M}$, where $D^2 \times \{t\}$ is parallel to D_1 for any $t \in [1,2]$ and $D^2 \times \{i\} = D_i$ for $i = 1, 2$. Moreover, by our hypothesis on α, we can assume that $\alpha = \{p\} \times [1,2]$, where $p \in \partial D^2$.

We take a parallel copy of D, say D', which is close to D. Let $D' = \alpha' \cap \beta'$ and $\alpha' = \{p'\} \times [1,2]$, where $p' \in \partial D^2$. Then $\partial D^2 - p \cup p'$ consists of two arcs γ and η. By choosing D' to be close to D, we can assume that η is the shorter one. The four arcs β, β' and $\eta \times \{1,2\}$ form a circle that bounds a disk δ in F. We can assume that D' is so close to D that the weight of δ is zero. $D_1 \cup D_2 \cup \delta$ is a disk in F whose boundary is $\beta \cup \beta' \cup (\gamma \times \{1,2\})$. The circle $\beta \cup \beta' \cup (\gamma \times \{1,2\})$ also bounds another disk $D \cup D' \cup (\gamma \times [1,2])$ in $\tilde{M}$. Since F has least weight, $weight(D_1 \cup D_2 \cup \delta) = 2weight(D_1) \leq weight(D \cup D' \cup \gamma \times [1,2]) = 2weight(D) + weight(\gamma \times [1,2])$. By our assumption, $weight(\gamma \times [1,2]) = 0$. Thus, $weight(D_1) \leq weight(D)$.

□

We call the disk D (in the lemma above) a monogon.

Lemma 2.4.4. *The lamination λ is an essential lamination.*

Proof. First we will show that every leaf of λ is π_1-injective. Otherwise, there is a compressing disk D embedded in $\tilde{M} - \tilde{\lambda}$ and ∂D lies in a leaf l, where $\tilde{\lambda}$ is the pre-image of λ in the universal cover $\tilde{M}$. By our construction of λ, there is a cross disk $D_K = D'_K \cup D''_K$ of size at least K that is parallel to a sub-surface of l. Since F_K is

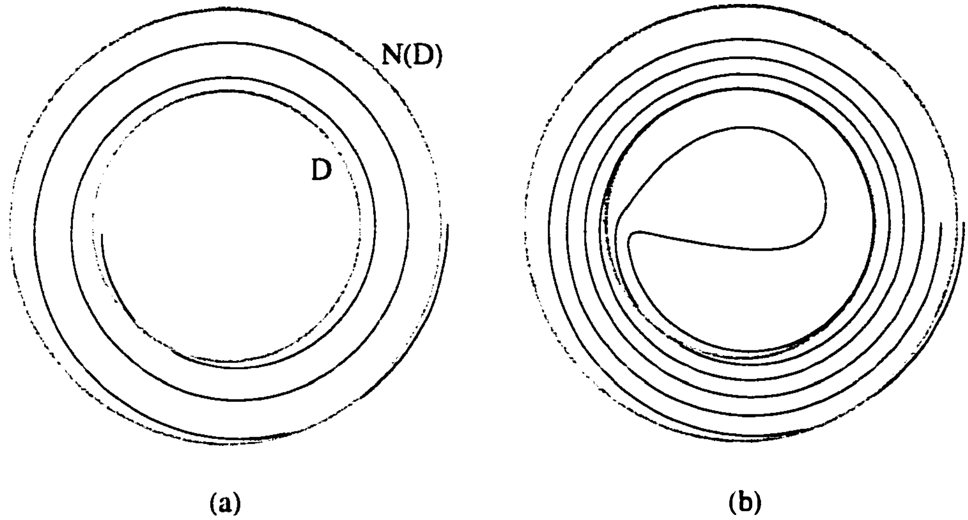


Figure 2.6:

π_1-injective and has least weight, and since ∂D is an essential curve in l, if K is large, D'_K does not contain a closed curve that is parallel to ∂D. By choosing K sufficiently large, we may assume that D'_K winds around ∂D (in a small neighborhood of D) many times, as shown in Figure 2.6 (a). Let $N(D)$ be an enlargement of D (i.e. an embedded disk in $\tilde{M}$ that contains D in its interior), and F be the component of $\tilde{F}_K$ that contains D'_K. Since F is embedded in $\tilde{M}$, the component of $F \cap N(D)$ that contains the spiral arc in Figure 2.6 (a) must form a monogon with a long 'tail' that consists of two parallel spiral arcs, winding around ∂D many times, as shown in Figure 2.6 (b). The weight of the monogon is at most $weight(D)$. If K is large enough, the length of each arc in the 'tail' of the monogon is very large and, in a neighborhood of the 'tail', we can choose two pieces of normal surfaces that are parallel to each other and have weight greater than $weight(D)$. This contradicts Lemma 2.4.3.

Next, we will show that every leaf of λ is ∂-injective. Otherwise, there is a ∂-compressing disk D' whose boundary consists of two arcs α and β, where $\alpha \subset \partial M$ and β is an essential arc in l. By our construction of λ, there is a cross disk $D_n = D'_n \cup D''_n$ of size at least n such that there are arcs $\alpha_n \subset \partial M$ and $\beta_n \subset \pi(D'_n)$ $(\partial\alpha_n = \partial\beta_n)$ that are parallel and close to α and β respectively. The two arcs α_n and β_n bound a disk d_n that is parallel and close to D'. Since the surface F_n is ∂-injective, there is an arc $\gamma_n \subset \partial F_n$ such that $\gamma_n \cup \beta_n$ bounds an immersed disk

Δ_n in F_n. Since β is an essential arc in l, by choosing n sufficiently large, we can assume $weight(\Delta_n) > weight(D') = weight(d_n)$. Note that $\gamma_n \cup \alpha_n$ bounds an immersed disk δ_n in ∂M and that $d_n \cup \Delta_n \cup \delta_n$ is an immersed 2-sphere in M. Since $\pi_2(M)$ is trivial, we can homotope $\Delta_n \cup \delta_n$ to d_n fixing d_n and get another immersed surface F_n' that is homotopic to F_n. Moreover, $weight(F_n') - weight(F_n) = weight(d_n) - weight(\Delta_n) < 0$, which contradicts the assumption that F_n has least weight.

It is easy to see from our construction that no leaf is a sphere. Also, if λ is not end-incompressible, there must be a monogon with a long 'tail', which contradicts Lemma 2.4.3 by the same argument as above. Therefore, λ is an essential lamination.

□

2.5 Measured sub-laminations

In this section, we will show that any minimal sub-lamination of λ has a transverse measure. A minimal lamination is a lamination that does not contain any proper sub-lamination. Using this result, we will prove Theorem 2.1 that can be viewed as a generalization of a theorem of Floyd and Oertel [12].

Let μ be a lamination in M and $i : I \times I \to M$ be an immersion that is transverse to μ, where $I = [0, 1]$. We will call $\{p\} \times I$ a *vertical arc*, for any $p \in I$, and call $i(I \times I)$ a *transverse rectangle* if $i(I \times \{0, 1\}) \subset \mu$ and the singular set of i is a collection of sub-arcs of the vertical arcs. To simplify notation, we will not distinguish $I \times I$ and its image in M.

Lemma 2.5.1. *Let μ be a minimal lamination. If μ has non-trivial holonomy, then there is a transverse rectangle $R : I \times I \to M$ such that $R(\{1\} \times I) \subset R(\{0\} \times \mathring{I})$, where $\mathring{I} = (0, 1)$.*

Proof. Since μ has non-trivial holonomy, there must be a map $g : S^1 \times I \to M$, which is transverse to μ, such that $g(S^1 \times \{0\}) \subset L \subset \mu$ (L is a leaf) and $g^{-1}(\mu)$ consists of a collection of spirals and one circle $S^1 \times \{0\}$ that is the limiting circle of these spirals. Moreover, for any spiral leaf l of $g^{-1}(\mu)$, there is an embedding $i : [0, \infty) \times I \to S^1 \times I$ such that $i^{-1}(l) = [0, \infty) \times \{1/2\}$ (see the shaded region in

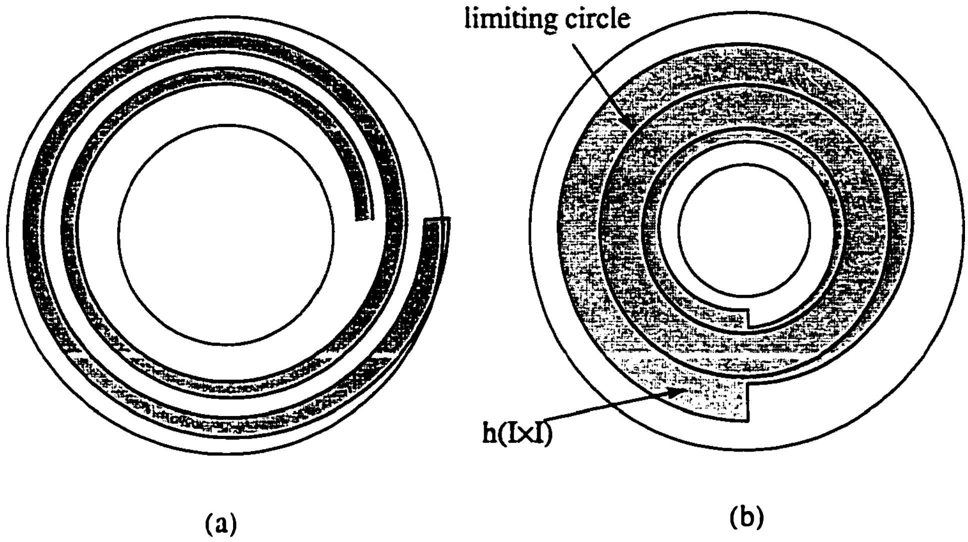


Figure 2.7:

Figure 2.7 (a)). Since $S^1 \times \{0\}$ is the limit circle of l, for any arc $\{p\} \times [0, \epsilon] \subset S^1 \times I$, there exists a number N, such that $i(\{N\} \times I) \subset \{p\} \times (0, \epsilon)$.

Since μ is a minimal lamination, every leaf is dense in μ. Thus, there is a path $\rho : I \rightarrow L$ such that $\rho(0) = g(p, 0)$, where $p \in S^1$, and $\rho(1) \in g \circ i(\{0\} \times \mathring{I})$. Moreover, if ϵ is small enough, there is a transverse rectangle $r : I \times I \rightarrow M$ such that $r|_{I \times \{0\}} = \rho$, $r(\{0\} \times I) = g(\{p\} \times [0, \epsilon])$, and $r(\{1\} \times I) = g \circ i(\{0\} \times [\delta_1, \delta_2])$, where $[\delta_1, \delta_2] \subset I$. The concatenation of the transverse rectangle r and $g \circ i([0, N] \times [\delta_1, \delta_2])$, i.e., $R : I \times I \rightarrow M$ where $R([0, 1/2] \times I) = r(I \times I)$ and $R([1/2, 1] \times I) = g \circ i([0, N] \times [\delta_1, \delta_2])$, is a transverse rectangle that we want. □

Remarks. 1. The kind of construction in Lemma 2.5.1 was also used in [20].

2. After connecting two copies of such transverse rectangles if necessary, we can assume that $R(\{1\} \times I) \subset R(\{0\} \times \mathring{I})$ in Lemma 2.5.1 preserves the orientation of the I-fibers. In other words, we may assume that there is a map $f : A \rightarrow M$ transverse to μ, where $A = S^1 \times I$, and an embedding (except for the boundary) $h : I \times I \rightarrow A$, as shown in Figure 2.7 (b), such that $R = f \circ h$ and $f(A)$ lies in a small neighborhood of $R(I \times I)$.

3. Let f, h, and R be the maps above. Suppose that L_0 and L_1 are leaves in μ containing $R(I \times \{0\})$ and $R(I \times \{1\})$ respectively. Then $f^{-1}(L_0 \cup L_1)$ contains two spirals of different directions whose limiting circles are meridian circles of A (see

Figure 2.7 (b)). Note that L_0 and L_1 may be the same leaf and the two spirals may have the same limiting circle.

4. If μ is carried by a branched surface B, we can assume that $h(\{p\} \times I)$ is a sub-arc of an interval fiber of $N(B)$.

Lemma 2.5.2. *Let λ be the lamination constructed in section 2.4 and μ be any minimal sub-lamination of λ. Then μ has trivial holonomy.*

Proof. Suppose that μ has non-trivial holonomy. Since μ is a minimal lamination, by the remarks above, there is an annulus $g : A = S^1 \times I \to M$ such that $g^{-1}(\mu)$ contains two spiral leaves, one clockwise and one counterclockwise, as shown in Figure 2.7 (b). From our construction of λ, there is a cross disk $D_N = D'_N \cup D''_N$ such that $g^{-1}(\pi(D'_N))$ (resp. $g^{-1}(\pi(D''_N)))$ contains two arcs parallel and close to the two spirals respectively. We denote these two arcs by α'_0 and α'_1 (resp. α''_0 and α''_1), as shown in Figure 2.8 (a). Now we consider $g^{-1}(F_N)$. Since F_N is compact, $g^{-1}(F_N)$ is compact. Denote the component of $g^{-1}(F_N)$ that contains α'_i (resp. α''_i) by c'_i (resp. c''_i), where $i = 0, 1$. Since F_N is a normal surface, by Remark 4 above, we can assume that $g^{-1}(F_N)$ is transverse to each vertical arc $\{p\} \times I$ in A.

If $c'_1 \cap S^1 \times \{0\} = \emptyset$, then c'_1 is either a closed curve, as shown in Figure 2.8 (c), or an arc with both endpoints on $S^1 \times \{1\}$, as shown in Figure 2.8 (b). Since λ is an essential lamination, $g(S^1 \times \{0\})$ must be an essential curve in M, and we have the following commutative diagram, where q is a covering map.

$$\begin{array}{ccc} \mathbb{R} \times I & \xrightarrow{\tilde{g}} & \tilde{M} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \pi} \\ A = S^1 \times I & \xrightarrow{g} & M \end{array}$$

The pictures of $q^{-1}(c'_1) \subset \tilde{g}^{-1}(\tilde{F_N})$ are shown in Figure 2.9 (a) or (b) depending on whether c'_1 is an arc with both endpoints on $S^1 \times \{1\}$ or a closed curve. If N is so large that α'_1 winds around A more than four times, there are four components of $q^{-1}(c'_1)$ intersecting each other, as shown in Figure 2.9 (a) and (b), which contradicts the assumption that F_N has the 4-plane property.

Thus, by the argument above, c'_1 , c''_1, c'_0 and c''_0 must be arcs with endpoints in different components of ∂A, as shown in Figure 2.8 (d). In this case, $q^{-1}(c'_0 \cup c'_1 \cup$

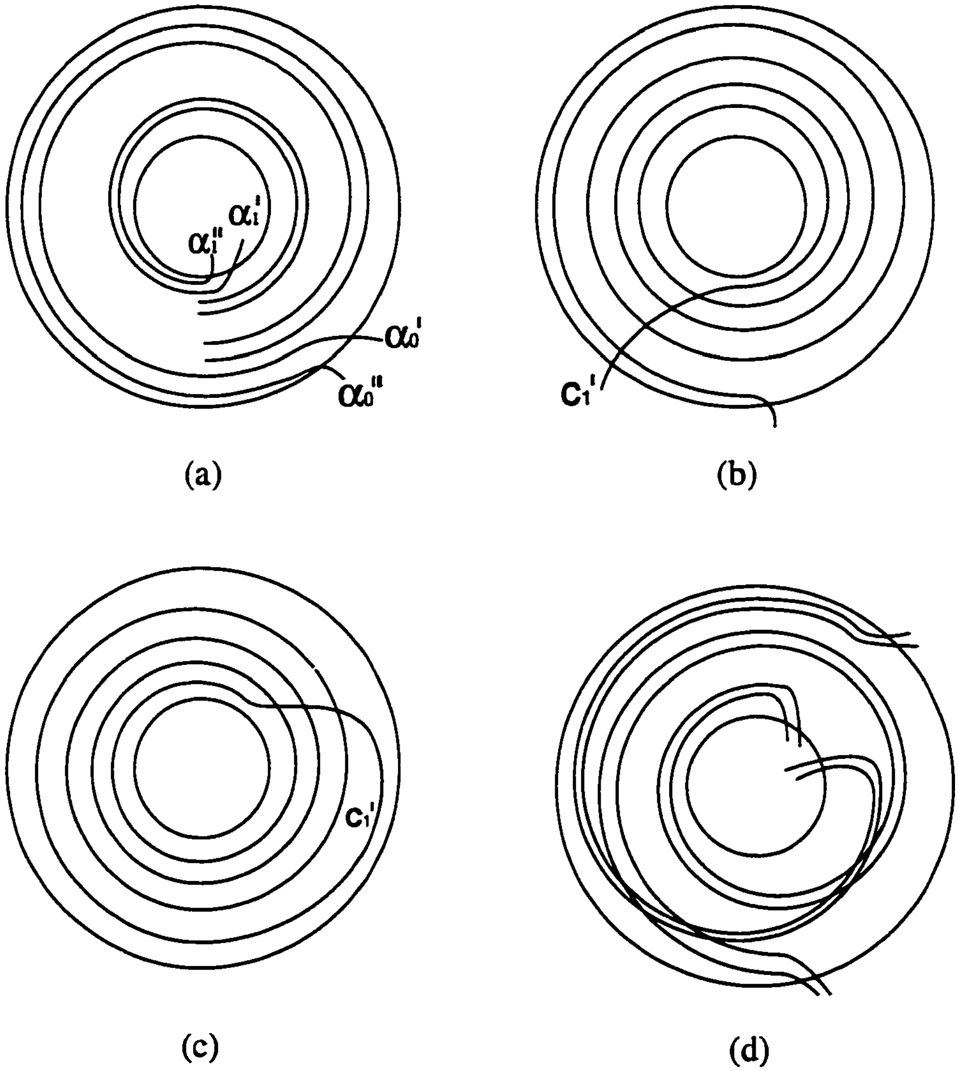


Figure 2.8:

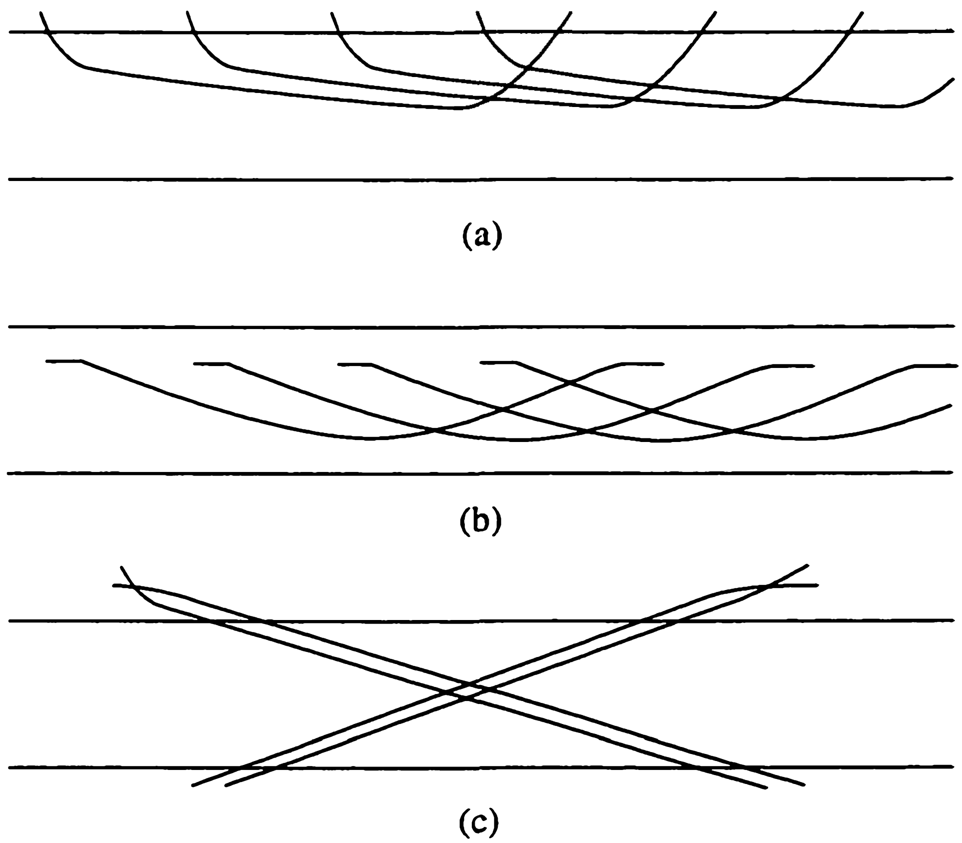


Figure 2.9:

$c_0'' \cup c_1''$) must contain 4 arcs d_0', d_0'', d_1', d_1'' as shown in Figure 2.9 (c), where $\tilde{g}(d_i' \cup d_i'')$ is the union of two arcs in different components of a cross disk ($i = 0, 1$). By the definition of cross disk, the 4 components of $\tilde{F}_N$ that contain $\tilde{g}(d_0')$, $\tilde{g}(d_0'')$, $\tilde{g}(d_1')$ and $\tilde{g}(d_1'')$ respectively must intersect each other, which contradicts the assumption that F_N has the 4-plane property.

□

The next theorem is a generalization of a theorem of Floyd and Oertel [12].

Theorem 2.1. *Let M be a closed, irreducible and non-Haken 3-manifold. Then the surfaces with the 4-plane property in M are carried by finitely many immersed branched surfaces*

Proof. If the set of immersed surfaces with the 4-plane property is a subset of $\mathcal{F}_R$ for some number R (see section 2.3 for the definition of $\mathcal{F}_R$), then by Lemma 2.3.1 the surfaces are carried by finitely many immersed branched surfaces.

If there is no such number R, by section 2.4, there are a sequence of cross disks that give rise to an essential lamination λ. Let μ be a minimal sub-lamination of λ. Since μ is also an essential lamination, by [15], we can assume that μ is carried by an embedded incompressible branched surface B. By Lemma 2.5.2, μ has no holonomy. A theorem of Candel [4] says that if a lamination has no holonomy then it has a transverse measure. So, μ has a transverse measure, and hence the system of branch equations of B (see [32]) has positive solutions. Since each branch equation is a linear homogeneous equation with integer coefficients, the system of branch equations of B must have positive integer solutions. Every positive integer solution corresponds to an embedded surface fully carried by B. But, by a theorem of Floyd and Oertel [12], any surface fully carried by an incompressible branched surface must be incompressible. This contradicts the hypothesis that M is non-Haken. □

2.6 Boundary curves

Let M be an irreducible 3-manifold whose boundary is an incompressible torus, λ be the lamination constructed in section 2.4 and μ be a minimal sub-lamination of λ. Let $\{D_i = D_i' \cup D_i''\}$ be the sequence of cross disks used in the construction of the lamination λ in section 2.4 and let F_i be the immersed surface that contains $\pi(D_i)$. We denote the pre-image of F_i in $\tilde{M}$ by $\tilde{F}_i$. Suppose that M does not contain non-peripheral closed incompressible surfaces.

Lemma 2.6.1. $\mu \cap \partial M \neq \emptyset$

Proof. Suppose that $\mu \cap \partial M = \emptyset$. Then μ is fully carried by an incompressible branched surface B and $B \cap \partial M = \emptyset$. As in the proof of Theorem 2.1 (see section 2.5), the linear system of branch equations must have integer solutions that correspond to incompressible surfaces. Since $B \cap \partial M = \emptyset$ and M does not contain non-peripheral closed incompressible surfaces, those incompressible surfaces correspondent to the integer solutions must be ∂-parallel tori.

Let $N(B)$ be a fibered neighborhood of B, C be the component of $M - N(B)$ that contains ∂M, and $T_1, T_2, \dots, T_n$ be a collection of ∂-parallel tori that correspond to a positive integer solution of the system of branch equations. After isotopies, we can assume that every T_i is transverse to the interval fibers of $N(B)$ and $\partial_h N(B) \subset \cup_{i=1}^n T_i$. Let A be a component of $\partial_h N(B)$ that lies in the closure of C.

Claim. The surface A must be a torus.

Proof of the claim. We first show that A is not a disk. Suppose A is a disk. Let ν be the component of $\partial_v N(B)$ that contains ∂A. Then $\partial \nu - \partial A$ is a circle in the boundary of a component D of $\partial_h N(B)$. Since $\partial_h N(B)$ is incompressible and A is a disk, D must be a disk. So $A \cup \nu \cup D$ is a 2-sphere. Since M is irreducible, $A \cup \nu \cup D$ must bound a 3-ball that contains $\cup_{i=1}^n T_i$, which contradicts the assumption that T_i is incompressible.

If $\partial A = \emptyset$, since $\partial_h N(B) \subset \cup_{i=1}^n T_i$, A must be a torus.

Suppose $\partial A \neq \emptyset$ and $A \subset T_1$. If there is a component of ∂A that is a trivial circle in T_1 then , since A is not a disk, there must be a trivial circle in ∂A that bounds a disk in $T_1 - A$. We can isotope this disk by fixing its boundary and pushing its interior into the interior of $N(B)$ so that it is still transverse to the I-fibers of $N(B)$. This gives us a disk of contact [12], which contradicts the assumption that B is an incompressible branched surface. So, every circle of ∂A must be an essential curve in T_1, and hence A must be an annulus.

Let c be a component of ∂A, ν' be a component of $\partial_v N(B)$ that contains c, and $c' = \partial\nu' - c$ be the other boundary component of ν'. Denote the component of $\partial_h N(B)$ containing c' by A'. By the argument above, A' must also be an annulus. If A and A' belong to different tori, then ν' is a vertical annulus in the product region $T^2 \times I$ bounded by the two tori. This contradicts the assumptions that those tori are ∂-parallel and $\partial M \subset C$. Thus, A and A' must belong to the same torus T_1. Then, ν' must be an annulus in the $T^2 \times I$ region bounded by T_1 and ∂M, and $\partial\nu' \subset T_1$. So, the vertical arcs of ν' can be homotoped rel $\partial\nu'$ into T_1. This contradicts the assumption that B is an incompressible branched surface [12]. Therefore, $\partial A = \emptyset$ and A must be a torus. □

By the claim and our assumptions, C must be a product region $T^2 \times I$ where $T^2 \times \{1\} = \partial M$ and $T^2 \times \{0\} = A \subset \partial_h N(B)$. Since μ is fully carried by B, we can assume that $A \subset \mu$ is a leaf. After choosing a sub cross disk if necessary, we can assume that there is a cross disk $D_K = D'_K \cup D''_K$ of size at least K such that $\pi(D'_K)$ lies in a small neighborhood of A that we denote by $T^2 \times J$, where $J = [-\epsilon, \epsilon]$ and $A = T^2 \times \{0\}$. By choosing ϵ small enough, we can assume $T^2 \times \{t\}$ is normal for any $t \in J$. Let E be the component of $F_K \cap (T^2 \times J)$ that contains $\pi(D'_K)$ and E' be a component of the pre-image of E in $\tilde{M}$. Let F' be the component of $\tilde{F_K}$ that contains E'. So E' is embedded in a region $\mathbb{R}^2 \times J$ in $\tilde{M}$, $\partial E' \subset \mathbb{R}^2 \times \{\pm\epsilon\}$. By choosing ϵ small enough, we can assume that E' is transverse to the J-fibers of $\mathbb{R}^2 \times J$.

If E' is a compact disk, then $\partial E'$ must be a circle in $\mathbb{R}^2 \times \{\pm\epsilon\}$ and D_K must be in the region bounded by $\partial E' \times J$. So, if K is large, the disk in $\mathbb{R}^2 \times \{\pm\epsilon\}$ bounded by $\partial E'$ is large. However, if the disk bounded by $\partial E'$ is large enough, $g^k(\partial E')$ ($k = 0, 1, 2, 3$) must intersect each other, where g is an element in $\pi_1(\partial M)$ that acts on $\tilde{M}$ and fixes $\mathbb{R}^2 \times J$. This violates the 4-plane property, and hence E' cannot be a compact disk.

Suppose that $\tilde{F_K} \cap (\mathbb{R}^2 \times \{\pm\epsilon\})$ contains circular components. Let e be an innermost such circle and F_e be the component of $\tilde{F_K}$ that contains e. Then e bounds a disk D in $\mathbb{R}^2 \times \{\pm\epsilon\}$ and bounds another disk D' in F_e. We can assume that $D' \cap \pi^{-1}(T^2 \times \{\pm\epsilon\}) = \partial D'$; otherwise, we can choose e to be a circle in

$D' \cap \pi^{-1}(T^2 \times \{\pm\epsilon\})$ that is innermost in D'. So, $D \cup D'$ bounds a 3-ball in $\tilde{M}$ and $\pi(D' - \partial D') \cap (T^2 \times J) = \emptyset$. Then, we can homotope $\pi(D')$ to $\pi(D)$ fixing $\pi(e)$. We denote the surface after this homotopy by F'_K and denote by F'_e the component of $\tilde{F'_K}$ (the pre-image of F'_K in $\tilde{M}$) that contains e. Let e' be a component of $\pi^{-1}(\pi(e))$ and $F_{e'}$ (resp. $F'_{e'}$) be the component of $\tilde{F_K}$ (resp. $\tilde{F'_K}$) that contains e'. Since D is innermost, if $F_e \cap F_{e'} = \emptyset$, then $F'_e \cap F'_{e'} = \emptyset$. Hence, F'_K is a surface homotopic to F_K and F'_K also has the 4-plane property. Note that since F_K has least weight and μ is the 'limit' of least weight cross disks, both D and D' have least weight and $weight(D) = weight(D')$. Thus, F'_K also has least weight and $F'_K \cap T^2 \times \{\pm\epsilon\}$ has fewer trivial circles after a small homotopy. So, we can assume that $\tilde{F_K} \cap \mathbb{R}^2 \times \{\pm\epsilon\}$ contains no trivial circles. Note that since E' can never be a compact disk as above, the homotopy above will not push the entire E' out of $\mathbb{R}^2 \times J$. Therefore, we can assume that E' is a non-compact and simply connected surface.

If $\partial E' \cap \mathbb{R}^2 \times \{\epsilon\}$ has more than one component, then since we have assumed that E' is transverse to the J-fibers of $\mathbb{R}^2 \times J$, $\partial E' \cap \mathbb{R}^2 \times \{\epsilon\}$ bounds a (non-compact) region Q in $\mathbb{R}^2 \times \{\epsilon\}$ and $D'_K \subset Q \times J$. Moreover, it is easy to see that, for any element $g \in \pi_1(\partial M)$ that acts on $\tilde{M}$ fixing $\mathbb{R}^2 \times J$, if $Q \neq g(Q)$ and $Q \cap g(Q) \neq \emptyset$ in $\mathbb{R}^2 \times \{\epsilon\}$, then $E' \cap g(E') \neq \emptyset$. If K is large, the distance between any two lines in ∂Q must be large. Thus, by assuming D'_K to be large, we can find a non-trivial element g in $\pi_1(\partial M)$ such that $g^k(Q)$, and hence the 4 components $g^k(E')$ $(k = 0, 1, 2, 3)$ intersect each other, which contradicts the 4-plane property.

Therefore, $\partial E' \cap \mathbb{R}^2 \times \{\epsilon\}$ must have only one component that is a line, and hence E must be an immersed annulus in $T^2 \times J$ with one boundary component in $T^2 \times \{\epsilon\}$ and the other boundary component in $T^2 \times \{-\epsilon\}$. By our construction, $weight(E)$ is large if K is large. We can always find an immersed annulus $A_E \subset T^2 \times J$ with $\partial A_E = \partial E$ and $weight(A_E)$ relatively small. So, the surface $(F_K - E) \cup A_E$ is homotopic to F_K and has less weight. This contradicts the assumption that F_K has least weight. So, $\mu \cap \partial M$ cannot be empty. □

Lemma 2.6.2. *$\mu|_{\partial M}$ is a lamination by circles.*

Proof. Suppose μ is fully carried by an incompressible branched surface B. Since μ is a measured lamination and ∂M is a torus, $\mu|_{\partial M}$ is either a lamination by circles or a lamination by lines with irrational slope. Let $\mathcal{S}$ be the solution space of the system of branch equations. Since the coefficients of branched equations are integers, there are finitely many integer solutions that generate $\mathcal{S}$, i.e., any point in $\mathcal{S}$ can be written as a linear combination of these integer solutions. Every integer solution gives rise to an incompressible surface carried by B. By Hatcher's theorem, these surfaces have the same slope. The boundary slope of any measured lamination μ carried by B is equal to the measure of a longitude of ∂M divided by the measure of a meridian. Hence, it can be expressed as a fraction with both nominator and denominator linear functions of the weights of the branch sectors. Since the solution in $\mathcal{S}$ that corresponds to μ is a linear combination of those integer solutions, and since the slopes of those integer solutions (plugging into the fraction described above) are the same, $\partial\mu$ must have the same slope as the slope of the compact surfaces carried by B. Therefore, any measured lamination carried by B, restricted to ∂M, is a lamination by circles with the same slope. □

Lemma 2.6.3. *Let $\{D_i = D_i' \cup D_i''\}$ be the sequence of cross disks used in the construction of λ, F_i be the immersed surface with the 4-plane property that contains $\pi(D_i)$, and μ be a minimal sub-lamination of λ. Then there exists a number N such that ∂F_i and $\partial\mu$ have the same slope if $i > N$.*

Proof. Let B be an incompressible branched surface that fully carries μ. Since $\partial\mu$ is a union of parallel circles, we can assume that ∂B is a union of circles. Let $N(B)$ be a fibered neighborhood of B, $\tilde{B} = \pi^{-1}(B)$ and $N(\tilde{B}) = \pi^{-1}(N(B))$. We denote $D_i' \cap N(\tilde{B})$ by E_i. Note that, since μ is a sub-lamination of λ, E_i is only a sub-disk of D_i'. Nevertheless, by our construction of λ, we can assume that the size of E_i is large if i is large. By the modification of the construction of λ above, we can assume that, after isotopies, $\pi(E_i) \cap \partial M \subset \pi(E_{i+1}) \cap \partial M$.

Suppose that ∂F_k has a different slope from $\partial\mu$. Then $\pi(E_k)$ is a piece of immersed surface in $N(B)$ transverse to every I-fiber, and $\pi(E_k) \cap \partial M$ is a union of spirals in $N(B) \cap \partial M$. We give each component of ∂B an orientation so that they represent the same element in $H_1(\partial M)$. This orientation of ∂B determines an orientation for each I-fiber of $N(B) \cap \partial M$. As in the proof of Hatcher's theorem, the orientation of the I-fibers and a normal direction of ∂M uniquely determine an

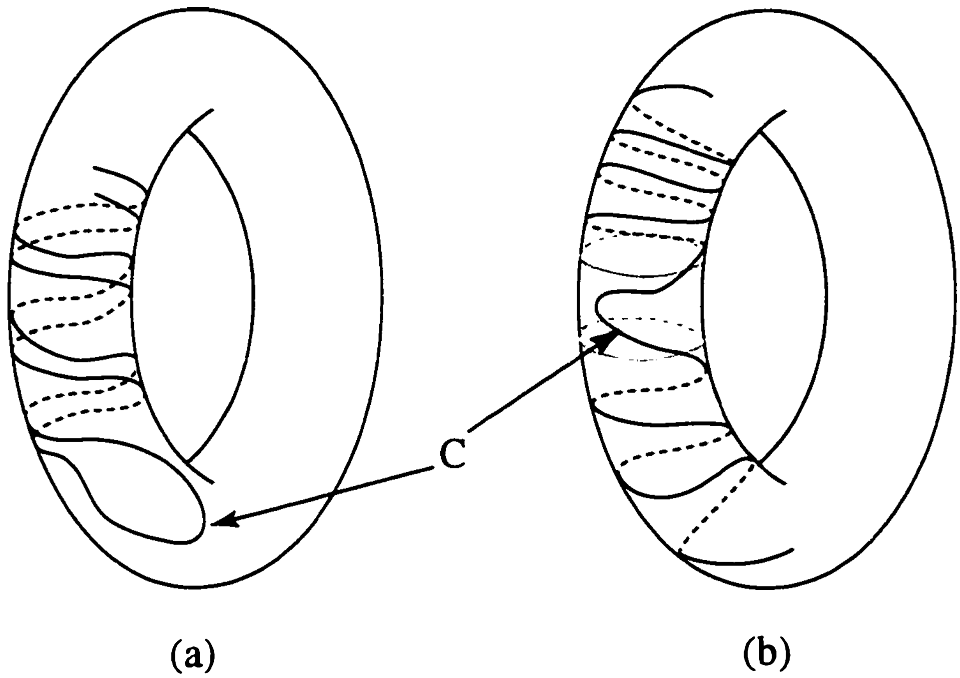


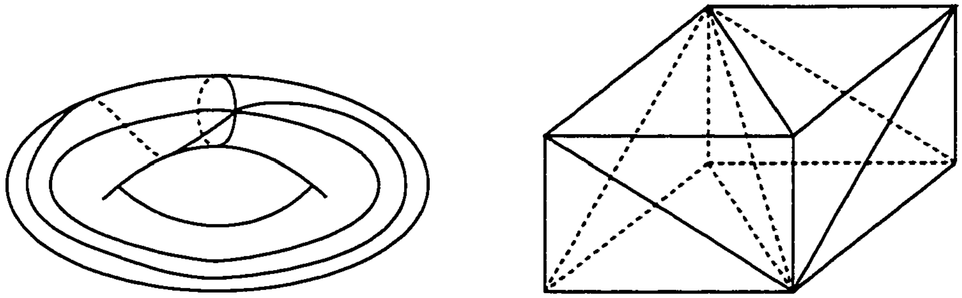
Figure 2.10:

orientation for every curve in $N(B) \cap \partial M$ that is transverse to the interval fibers.

Claim 1. Each circle in ∂F_k admits an orientation that agrees with the induced orientation of $\partial F_k \cap N(B)$.

Proof of claim 1. Suppose there is a circle in ∂F_k that does not admit such an orientation. Then there must be a sub-arc C (of the circle) outside $N(B) \cap \partial M$ connecting two spirals that are either in the same component of $N(B) \cap \partial M$, as shown in Figure 2.10 (a), or in different components of $N(B) \cap \partial M$ with incompatible induced orientations, as shown in Figure 2.10 (b). After assuming the size of the cross disk to be large, we can rule out the first possibility, i.e., Figure 2.10 (a), by Lemma 2.4.3. To eliminate the second possibility, i.e., Figure 2.10 (b), we use a certain triangulation of M as follows.

By [23], there is a one-vertex triangulation of M and this vertex must be on ∂M. Since $\partial M = T^2$, the induced triangulation of ∂M must consist of two triangles as shown in Figure 2.11 (a). Now we glue a product region $T^2 \times I$ ($I = [0,1]$) to M such that $T^2 \times \{0\} = \partial M$. Then $(\mathcal{T}^{(1)} \cap \partial M) \times I$ gives a cellulation of $T^2 \times I$ that consists of a pair of triangular prisms. Then we add a diagonal to each rectangular face of the prisms, which gives a triangulation of $T^2 \times I$. Figure 2.11 (b)



(a) (b)

Figure 2.11:

is a picture of the induced triangulation of a fundamental region in the universal cover of $T^2 \times I$. In this triangulation, there is only one vertex on $T^2 \times \{1\}$, and its link hemisphere H intersects the 1-skeleton in 10 points of which 6 points lie on ∂H. Since $M \cup (T^2 \times I)$ is homeomorphic to M, we can assume that M has a triangulation as that of $M \cup (T^2 \times I)$ above. We denote this triangulation also by $\mathcal{T}$. Note that $\mathcal{T}^{(0)} \cap \partial M$ is a single vertex v and that the intersection of its link hemisphere H and $\mathcal{T}^{(1)}$ consists of 10 points of which 6 points lie on $\partial H \subset \partial M$.

We assume that our immersed surfaces are normal and have least weight with respect to the triangulation above. Suppose the second case, i.e., Figure 2.10 (b), occurs. Let A be the annulus of $\partial M - N(B)$ that contains the arc C. Then we isotope F_k by pushing C along A to 'unwrap' the spirals in a small neighborhood of ∂M, as shown in Figure 2.12. If the vertex v is not in A, then after this isotopy, $|\partial F_k \cap \mathcal{T}^{(1)}|$ decreases and $|(F_k - \partial F_k) \cap \mathcal{T}^{(1)}|$ does not change. This contradicts the assumption that F_k has least weight. So $v \in A$. If every edge of $\mathcal{T}^{(1)} \cap \partial M$ intersects ∂A non-trivially, then after C passes through the vertex v during the isotopy, $|\partial F_k \cap \mathcal{T}^{(1)}|$ decreases by 6 and $|(F_k - \partial F_k) \cap \mathcal{T}^{(1)}|$ increases by 4. Hence, the total weight of F_k decreases, which also gives a contradiction. Therefore, there is an edge e of $\mathcal{T}^{(1)} \cap \partial M$ lying inside A, as shown in Figure 2.12 (a). Then by our construction of the triangulation, e forms a meridian circle of the annulus A and there is at most one such edge. After C passes through v in the isotopy above, $|\partial F_k \cap \mathcal{T}^{(1)}|$ decreases by 4, $|(F_k - \partial F_k) \cap \mathcal{T}^{(1)}|$ increases by 4, and the total weight does not change.

Now, we will see exactly what happens in a tetrahedron. Let T be a tetrahedron

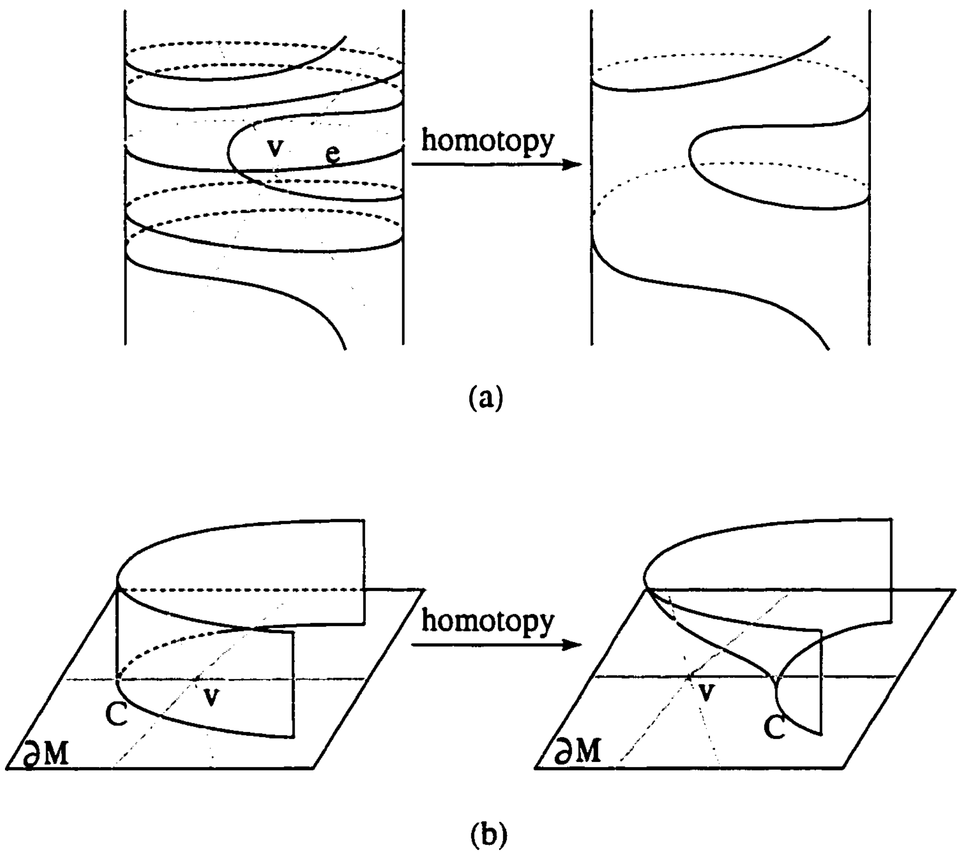


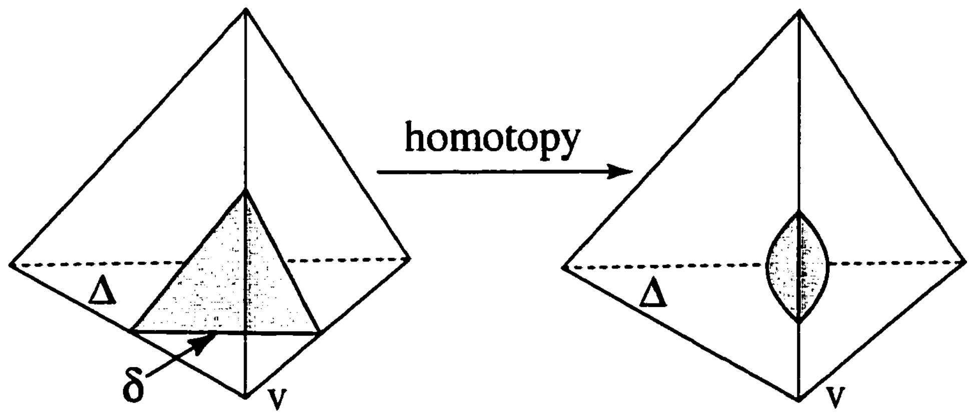
Figure 2.12:

with a face Δ on ∂M. There is a normal arc δ in $C \cap \Delta$ that cuts off a sub-triangle (in $\Delta \cap A$) that contains the vertex v. The normal disk of $F_k \cap T$ containing δ is either a triangle or a quadrilateral. If we do the isotopy above by pushing C across v, then the effect of this isotopy on the normal disk that contains δ is either as in Figure 2.13 (a), in which case the normal disk is a triangle, or as in Figure 2.13 (b), in which case the normal disk is a quadrilateral. In the first case, as shown in Figure 2.13 (a), the disk is no longer a normal disk after the isotopy. So, we can perform another homotopy to make F_k (after the first isotopy) a normal surface. This homotopy reduces $|F_k \cap \mathcal{T}^{(1)}|$ by at least 2 as we push the disk in Figure 2.13 (a) across the edge, which contradicts the assumption that F_k has least weight. Thus, every normal disk that contains such an arc (as δ) is a quadrilateral. Since there are only two triangles on ∂M, and since the edge e lies inside A, there must be two arcs δ_1 and δ_2 in C that cut off two corners of the same triangle (in the induced triangulation of ∂M). By the argument above, the two normal disks that contain δ_1 and δ_2 (respectively) must be two quadrilaterals of different normal disk types in the same tetrahedron. Note that, during the isotopy, we push parts of ∂F_k from $N(B) \cap \partial M$ to the annulus A. Suppose that the weight of F_k is fixed during these isotopies. After some isotopies as above, we can assume that there is a pair of normal disks of the cross disk in each of the two quadrilateral normal disk types. Since any two quadrilateral normal disks of different types must intersect each other, those 4 quadrilaterals give rise to 4 components of $\tilde{F}_k$ intersecting each other, which contradicts the hypothesis that F_k has the 4-plane property. So, if k is large enough, we can reduce the weight of F_k at a certain stage of the isotopy above. Therefore, Figure 2.10 (b) cannot occur and claim 1 holds. □

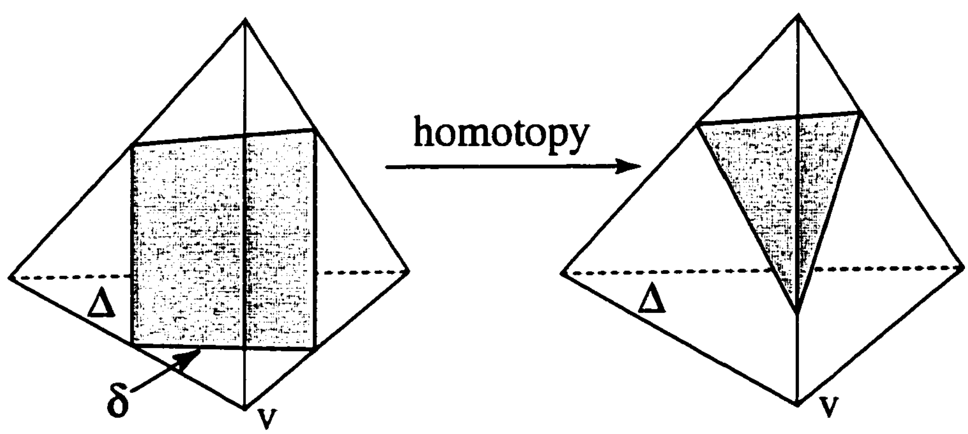
Case 1. μ is a compact orientable surface.

Let $\Gamma = \mu \times I \subset M$ ($I = [0, 1]$), and Γ' be a component of the pre-image of Γ in $\tilde{M}$. Suppose k is large; by our construction of the lamination, there is always a large sub cross disk of $D_k = D_k' \cup D_k''$ lying in Γ'. To simplify notation, we assume that $D_k \subset \Gamma'$; otherwise we use a large sub cross disk of D_k and the proof is the same.

Let F_k' be the component of $\tilde{F}_k$ that contains D_k', $H' = F_k' \cap \Gamma'$, $H = \pi(H')$, and



(a)



(b)

Figure 2.13:

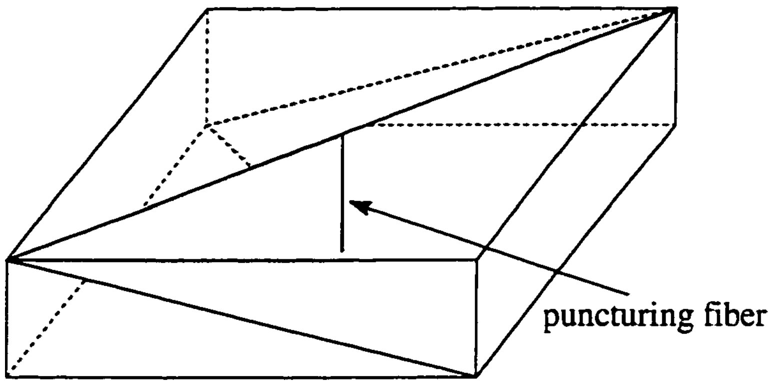


Figure 2.14:

H_0 be the smallest normal sub-surface of F_k that contains H. Since we can give every component of ∂F_k an orientation that agrees with the induced orientation of $\partial F_k \cap \Gamma$, the sign of every intersection point of $\partial F_k \cap \partial S$ is always the same, where $S = \mu \times \{t\}$ $(t \in I)$. Then, H cannot be transverse to every I-fiber of Γ, because otherwise, by the same argument as in the proof of Hatcher's theorem, ∂F_k and ∂S would have the same slope, which contradicts our assumptions. Figure 2.14 gives a local picture of H where it is not transverse to an I-fiber of Γ. In fact, it is not hard to see that, in some tetrahedron T, there must be two quadrilateral normal disks of different types in $T \cap S$ and $T \cap H_0$ respectively. Otherwise, by an argument in [12], H_0 and S lie in $N(B_T)$ and are transverse to the I-fibers of $N(B_T)$, where $N(B_T)$ is a fibered neighborhood of an embedded normal branched surface B_T. Hence, by the argument in the proof of Hatcher's theorem, F_k and S have the same boundary slope (although F_k is not embedded), which contradicts our assumption. Moreover, since all these surfaces are normal, after a small homotopy, we can assume that each I-fiber of Γ either transversely intersects H or lies in H, in which case the local picture of this fiber is as shown in Figure 2.14. We call such fibers *puncturing fibers.* Any arc of $F_k \cap S$ must pass through a puncturing fiber.

Let g be the genus of μ and b be the number of boundary components. Then there are g disjoint annuli $A_1, A_2, \ldots, A_g$ such that $\mu - \cup_{i=1}^{g} A_i$ is a planar surface. Let $A_{g+1}, \ldots, A_{g+b}$ be the collection of annuli that are regular neighborhoods of the boundary components of μ. Suppose $A_i \cap A_j = \emptyset$ if $i \neq j$. Let $c_1, \ldots, c_\kappa$ be a maximal collection of disjoint non-parallel and non ∂-parallel arcs in $\mu - \cup_{i=1}^{g+b} A_i$, and let $\Sigma = (\cup_{i=1}^{g+b} A_i) \cup (\cup_{j=1}^{\kappa} N(c_j))$, where $N(c_j)$ is a small neighborhood of c_j.

For each component a_i of ∂A_j, a component of $F_k \cap (a_i \times I)$ is either a closed curve, or an arc with endpoints on different boundary components of $a_i \times I$, or a ∂-parallel arc, i.e., an arc with both endpoints on the same boundary component of $a_i \times I$. By pushing the ∂-parallel arcs out of $a_i \times I$, we can assume that $F_k \cap (a_i \times I)$ does not contain ∂-parallel arcs. Similarly, after some homotopy, we can assume that $F_k \cap (A_i \times \partial I)$ has no ∂-parallel arc for any i. Note that this homotopy can be done without destroying the 4-plane property. Moreover, since such a ∂-parallel arc cannot wind around the annulus more than 4 times as in Figure 2.8 (b), the length of every ∂-parallel arc is short compared with k, and hence the parts of F_k that we push out of $\Sigma \times I$ cannot be parts of $\pi(D'_{[k/2]})$ (if k is large).

Using the homotopy in the proof of Lemma 2.6.1, we can assume that $F_k \cap (A_i \times \partial I)$ contains no trivial closed curves. Since both μ and F_k are normal surfaces and every component of $\tilde{F_k}$ is embedded in $\tilde{M}$, after some homotopies as above, we can assume that each component of $F_k \cap (A_i \times I)$ that intersects both components of $(\partial A_i) \times I$ is either an annulus or a quadrilateral with two edges on $(\partial A_i) \times I$ and two edges on $A_i \times \partial I$. Moreover, we can assume that every component of $F_k \cap (A_i \times I)$ that intersects only one component of $(\partial A_i) \times I$ is an annulus, since we have assumed that there are no ∂-parallel arcs in $(\partial A_i) \times I$.

If k is large, since the slopes of ∂F_k and $\partial \mu$ are different, $\pi(D'_{[k/2]}) \cap \partial M$ contains a long spiral, where $D'_{[k/2]}$ is a sub-disk of D'_k of size $[k/2]$. Suppose $(\pi(D'_{[k/2]}) \cap \partial M) \bigcap (\mu \times \{t\}) \neq \emptyset$ for some $t \in (0,1)$. By claim 1 and the proof of Hatcher's theorem, any double arc of $F_k \cap (\mu \times \{t\})$ must pass through a puncturing fiber. Let γ be such a double arc with at least one endpoint in $(\pi(D'_{[k/2]}) \cap \partial M) \bigcap (\mu \times \{t\})$. By enlarging the annuli A_i's and isotoping F_k (or $\mu \times \{t\}$) if necessary, we can assume that $F_k \cap (\mu \times \{t\}) \subset \Sigma \times I$ and each component of $F_k \cap (N(c_j) \times I)$ is a small neighborhood of $F_k \cap (c_j \times I)$ (though two components of $F_k \cap (N(c_j) \times I)$ may intersect each other). Let γ' be an arc in $F_k \cap (\mu \times \partial I)$. Then every component of $F_k \cap (A_i \times I)$ that contains a sub-arc of γ' cannot be an annulus. Therefore, after some isotopy, we can assume our γ as above also has this property, i.e., every component of $F_k \cap (A_i \times I)$ that contains a sub-arc of γ is not an annulus, for any i.

Let ω be a simple closed curve in μ and α be an arc in $\omega \times I$ with its endpoints on different boundary components of $\omega \times I$. We call α a *puncturing arc* if there is a point $p \in \omega$ such that α (up to homotopy fixing $\partial\alpha$) does not intersect $\{p\} \times I$. Let D be a component of $F_k \cap (A_i \times I)$ for some i. We say that D is a *level 0* surface in $A_i \times I$ if there exists an essential simple closed curve $\omega \subset A_i$ such that $D \cap (\omega \times I)$ contains a puncturing arc. Note that the curve ω can be chosen to have length no larger than the diameter of a fundamental region in the universal cover of A_i.

Notation. For any A_i, we denote by $\bar{A}_i$ a component of $\pi^{-1}(A_i)$ and by g_i a generator of $\pi_1(A_i)$. Suppose g_i acts on $\tilde{M}$ fixing $\bar{A}_i$, where $i = 1, \ldots, g+b$. For any component D of $F_k \cap (A_i \times I)$, we denote by $\bar{D}$ a component of $\pi^{-1}(D) \cap (\bar{A}_i \times I)$ and by F_D the component of $\tilde{F}_k$ that contains $\bar{D}$.

Claim 2. If D is a level 0 surface in $A_i \times I$, then $F_D \cap g_i(F_D) = \emptyset$.

Proof of claim 2. Suppose $F_D \cap g_i(F_D) \neq \emptyset$. Since D has level 0, there exists a simple closed curve ω in A_i such that $D \cap (\omega \times I)$ contains a puncturing arc α. Moreover, $\pi(D_k) \cap (\omega \times I)$ contains a pair of curves that are either closed essential curves in $\omega \times I$ or spirals winding around $\omega \times I$ many times (if k is large). In both cases, α intersects these two curves non-trivially. Therefore, there is a cross disk $D_k = D'_k \cup D''_k$ such that $D'_k \cap \bar{D}$ and $D''_k \cap \bar{D}$ are not empty. Furthermore, since $\bar{D}$ and $g_i(\bar{D})$ are not far away, if k is large, $D'_k \cap g_i(\bar{D})$ and $D''_k \cap g_i(\bar{D})$ are not empty. Thus, if $F_D \cap g_i(F_D) \neq \emptyset$, we have 4 components of $\tilde{F}_k$ intersecting each other, which contradicts the assumption that F_k has the 4-plane property. □

Claim 3. Let D be a component of $F_k \cap (A_i \times I)$ whose intersection with $(\partial A_i) \times I$ is non-empty. Suppose $F_D \cap g_i(F_D) = \emptyset$ or $F_D = g_i(F_D)$. Then D must be embedded in $A_i \times I$.

Proof of claim 3. We first consider the case that D is a disk. Since every component of $\tilde{F}_k$ is embedded in $\tilde{M}$, $\bar{D}$ is an embedded disk in $\bar{A}_i \times I$. Topologically, $\bar{A}_i \times I$ is an infinite solid cylinder and by our assumptions on $F_k \cap (A_i \times I)$, $\bar{D}$ must be a meridian disk. If $\bar{D} \cap g_i(\bar{D}) = \emptyset$ then $g_i(\bar{D}) \cap g_i^2(\bar{D}) = \emptyset$. Moreover, $\bar{D}$ and $g_i^2(\bar{D})$ lie in different components of $\bar{A}_i \times I - g_i(\bar{D})$, thus $\bar{D} \cap g_i^2(\bar{D}) = \emptyset$. Inductively, $\bar{D} \cap g_i^n(\bar{D}) = \emptyset$ for any n, and hence D is embedded in $A_i \times I$. Now we suppose that D is an annulus. Since F_D is embedded in $\tilde{M}$, it is clear that D must be embedded in $A_i \times I$. □

Let D be a component of $F_k \cap (A_i \times I)$, let c_s be a simple arc (defined before) connecting A_i to A_j, and let $c_s \cap A_i = x, c_s \cap A_j = y$, where $1 \leq i, j \leq g + b$ and $1 \leq s \leq \kappa$. Suppose $D \cap (\{x\} \times I) = \cup_{h=1}^{p} X_h$. We denote the components of $F_k \cap (c_s \times I)$ that contain $X_1, \ldots, X_p$ by $\alpha_1, \ldots, \alpha_p$ respectively ($X_h \in \alpha_h$). Let Y_h be the other end point of α_h ($h = 1, \ldots, p$). Now, we inductively define *level* and *extended level* of a component in $A_i \times I$ as follows. If D is a disk and $(\cup_{h=1}^{p} Y_h) \bigcap (\{y\} \times I) = \emptyset$ (i.e. $\cup_{h=1}^{p} Y_h \subset c_s \times \partial I$), we say that D has *extended level* (or simply *e-level*) 0 in $A_i \times I$. Moreover, if $A_i \times I$ contains a disk D_0 of level (resp. e-level) 0 and if $D \cap D_0 = \emptyset$, then we also say that D has level (resp. e-level) 0. We say that D is a surface with *level* (resp. *e-level*) at most n ($n \geq 1$) in $A_i \times I$ if $(\cup_{h=1}^{p} Y_h) \bigcap (\{y\} \times I) \neq \emptyset$ and at least one component of $F_k \cap (A_j \times I)$ that has non-empty intersection with $\cup_{h=1}^{p} Y_h$ is a surface with level (resp. e-level) at most $n - 1$ in $A_j \times I$. We define the *level* (resp. *e-level*) of D to be the minimum of such n with respect to all the $A_j \times I$'s and all c_s's. Note that A_i and A_j may be the same annulus.

Let D be a surface with level (resp. e-level) n in $A_i \times I$ as above and D_1 be a surface with level (resp. e-level) $n - 1$ (that contains Y_h for some h) in $A_j \times I$ as above. We say that D_1 is a level (resp. e-level) $n - 1$ surface attached to D. If D_2 is a level (resp. e-level) $n - 2$ surface attached to D_1, we say that D_2 is a level (resp. e-level) $n - 2$ surface attached to D. Repeatedly, for any $0 \leq t < n$, there is at least one surface with level (resp. e-level) t attached to D.

Suppose that $D \subset A_i \times I$ has level (or e-level) n. We say that D is *regular* if there is a sequence $\{K_t\}$ ($0 \leq t \leq n$) such that: (1) K_t is a component of $F_k \cap A_s \times I$ (for some s) and K_t has level (or e-level) t, (2) K_{t-1} is a surface attached to K_t ($K_n = D$), and (3) every K_t is a disk.

Remarks. Suppose that $D \subset A_i \times I$ has e-level 0. We can enlarge A_i along c_s (using the notation above). Namely, let A_i^c be a small neighborhood of $A_i \cup c'_s$ where c'_s is a sub-arc of c_s such that c'_s contains x and $c'_s \times I$ contains $\cup_{h=1}^{p} Y_h$. Then, since D is a disk by definition, the component of $F_k \cap (A_i^c \times I)$ that contains D must have a puncturing arc in $A_i^c \times I$, and hence D has level 0 in $A_i^c \times I$. In particular, $\bar{D}$ must puncture through a cross disk. Level and e-level can be considered as measures of the distance between $\bar{D}$ and an arc in F_D that puncture through a cross disk.

By the remarks above, for any disk D of level (or e-level) 0 in $A_i \times I$, D must contain a short arc that connects the two components of $A_i \times \partial I$. We also call such short arcs *puncturing arcs*. By our assumptions on the cross disk $D_k = D'_k \cup D''_k$, any puncturing arc cannot lie in $\pi(D_k)$.

Claim 4. Let D and D' be two components of $F_k \cap (A_i \times I)$ with level (or e-level) n and n' respectively. Suppose k is very large and n, n' are relatively small. Then, D and D' are embedded. Moreover, if D is an annulus, D' must also be an annulus and $D \cap D' = \emptyset$. If D has level (or e-level) 0, then $D' \cap D = \emptyset$, and hence D' also has level (or e-level) 0.

Proof of claim 4. We first show that D is embedded. By claim 3, it suffices to show that $F_D \cap g_i(F_D) = \emptyset$ (or $F_D = g_i(F_D)$). Suppose that $F_D \cap g_i(F_D) \neq \emptyset$ and $F_D \neq g_i(F_D)$. Then, since k is large and n, n' are relatively small, there is a cross disk $D_k = D'_k \cup D''_k$ such that $D'_k \cap F_D$, $D'_k \cap g_i(F_D)$, $D''_k \cap F_D$ and $D''_k \cap g_i(F_D)$ are not empty, as in the proofs of claim 2. This gives 4 components of $\tilde{F}_k$ intersecting each other, which contradicts the assumption that F_k has the 4-plane property.

So, D and D' are embedded. Suppose D is an annulus, then $\pi^{-1}(D) \cap (\tilde{A}_i \times I)$ has only one component, say $\tilde{D}$. Let $\tilde{D}'$ be a component of $\pi^{-1}(D') \cap (\tilde{A}_i \times I)$ (using the notation before claim refclaim2) and $F_{D'}$ be the component of $\tilde{F}_k$ that contains $\tilde{D}'$. If $D \cap D' \neq \emptyset$, then $\tilde{D} \cap g_i^m(\tilde{D}') \neq \emptyset$, and hence $F_D \cap g_i^m(F_{D'}) \neq \emptyset$ for any m. Since n and n' are relatively small, we can find a points $x \in \tilde{D}$ and $x' \in g_i^m(\tilde{D}')$ such that x and x' are closed to some puncturing arcs in F_D and $g_i^m(F_{D'})$ respectively. By choosing an appropriate m, we can assume that the distance between x and x' is short. Thus, if k is large, the two puncturing arcs (in F_D and $g_i^m(F_{D'})$) puncture through the same cross disk. Since $F_D \cap g_i^m(F_{D'}) \neq \emptyset$ for any m, we have 4 planes intersecting each other, which gives a contradiction. By our assumptions on $F_k \cap (A_i \times I)$, $D \cap D' = \emptyset$ implies that D' must also be an annulus.

If D has level (or e-level) 0, $\tilde{D}$ contains a puncturing arc. If $D \cap D' \neq \emptyset$, similarly to the argument above, we can find a cross disk and an m such that the cross disk intersects both $\tilde{D}$ and $g_i^m(F_{D'})$ and $\tilde{D} \cap g_i^m(F_{D'}) \neq \emptyset$. This also contradicts the assumption that F_k has the 4-plane property.

□

Claim 5. Let D and D' be two components of $F_k \cap (A_i \times I)$ with level (or e-level) n and n' respectively. Suppose that D is regular. If k is large and n, n' are relatively small, then $n' \leq n$ and D' is regular.

Proof of claim 5. By claim 4, D and D' are embedded. We prove claim 5 by induction on n. If $n = 0$, by claim 4, $D \cap D' = \emptyset$. Hence, $n' = 0$ by our definition. Now, we assume claim 5 holds for $n-1$. Since D is regular, D is a disk and there is another disk component K_{n-1} of $F_k \cap (A_j \times I)$ with level (or e-level) $n-1$ such that D and K_{n-1} are connected by an arc in $F_k \cap (c_s \times I)$. Since D is a disk, by claim 4, D' cannot be an annulus. So, D' must be a disk. Then either D' has e-level 0, or there is an arc in $F_k \cap (c_s \times I)$ connecting D' to a component K'_{n-1} in $F_k \cap (A_j \times I)$. By our induction, K'_{n-1} has level (or e-level) at most $n-1$ and K'_{n-1} is regular. Therefore, Claim 5 holds for D'.

□

We have assumed (before claim 2) that there is a double arc γ of $F_k \cap \mu \times \{t\}$ such that $\gamma \subset \Sigma \times I$ and γ has an end point lying in $\pi(D'_{[k/2]}) \cap \partial M$. Moreover, we have assumed that any component of $F_k \cap (A_i \times I)$ that contains a sub-arc of γ is not an annulus.

Claim 6. Let T be the union of the components of $F_k \cap (A_i \times I)$'s and $F_k \cap (N(c_s) \times I)$'s that contain sub-arcs of γ. Then every component of $T \cap (A_i \times I)$ has level (or e-level) at most $g+b-1$, for any i.

Proof of claim 6. The proof of claim 6 is an easy application of the claims 4 and 5. Since γ passes through a puncturing fiber, there is a component of $T \cap (A_i \times I)$, say T_0, contains a puncturing fiber, then it must have level 0. We can inductive define the level and e-level of each component of $T \cap (A_i \times I)$ for every i. By our assumptions on γ, every component of $T \cap (A_i \times I)$ is regular. Thus, claim 6 follows from repeated application of Claim 4 and 6. □

Claim 7. If k is large (compared with g and b), then $z \notin \pi(D'_{[k/2]})$ for any $z \in T$, where T is as in claim 6.

Proof of claim 7. By claim 6, the level (or e-level) of the surface that contains z is less than $g+b$. Thus, there is a short arc that contains z and connects both components of $\mu \times \partial I$. However, if k is large, any short arcs must lie in $\pi(D_k)$, which contradicts our assumption on D_k.

□

Therefore, Claim 7 and our assumption that γ has an end point lying in $\pi(D'_{[k/2]}) \cap \partial M$ give a contradiction. If μ is a non-orientable compact surface, we can take a double cover of μ and the proof is the same. So, we have proved Lemma 2.6.3 in the case that μ is a compact surface.

Case 2. μ is not a compact surface.

If μ is not a compact surface, then μ is carried by a normal incompressible branched surface B whose boundary is a union of circles. By blowing air into each leaf (i.e. replacing each leaf of μ by an I-bundle over this leaf and then deleting the interior of the I-bundle), we can assume that μ is nowhere dense. Since B carries a compact surface, $F_k \cap N(B)$ cannot be transverse to every I-fiber of $N(B)$. Same as the case that μ is a compact surface, we can assume that any I-fiber of $N(B)$ either transversely intersects F_k, or lies in F_k (we also call it a puncturing fiber). By assuming that the branch locus L of B lies in the 2-skeleton, we can view B as a union of normal disks. For each normal disk $D \subset B - L$, we denote $p^{-1}(D)$ by $N(D)$, where $p : N(B) \to B$ is the map that collapses every I-fiber to a point. Suppose that $N(D) = D^2 \times I$, $N(D) \cap \mu = D^2 \times C$, where C is a nowhere dense closed set in I, and suppose $N(D)$ contains some puncturing fiber K. For any $x \in K$, there is a non-trivial simple closed curve l_x on a leaf of μ such that $x \in l_x$ and $l_x \cap N(D)$ has only one component. Since μ has no holonomy, there is an embedding $b_x : S^1 \times [-1, 1] \to N(B)$ such that $b_x(S^1 \times \{0\}) = l_x$, $b_x(\{q\} \times [-1, 1])$ is a sub-arc of an I-fiber of $N(B)$ for any $q \in S^1$, and $b_x^{-1}(\mu)$ is a union of parallel circles. Suppose that $b_x(S^1 \times [-1, 1]) \cap N(D) = \alpha \times J_x$, where α is an arc of D^2 (in $N(D) = D^2 \times I$) and J_x is a sub-arc of an I-fiber of $N(D) = D^2 \times I$. Let b'_x be a small fibered neighborhood of the union of $b_x(S^1 \times [-1, 1])$ and $D^2 \times J_x$. Note that b'_x is a product of an annulus and an interval I. We can assume that each I-fiber of b'_x is a sub-arc of an interval fiber of $N(B)$ and $b'_x \cap \mu$ is a union of parallel annuli. We call b'_x a thick band. By compactness, there are finitely many disjoint thick bands in $N(B)$ such that the union of these bands contains $N(D) \cap \mu$. Moreover, there are finitely many disjoint thick bands $B_1, B_2, \ldots B_n$ in $N(B)$ such that, for every puncturing fiber K, $K \cap \mu$ belongs to $\cup_{i=1}^{n} B_i$. Let $B_i = A_i \times I$, where A_i is an annulus and $\{q\} \times I$ is a sub-arc of an interval fiber of $N(B)$ for any $q \in A_i$.

Let $B_{n+1}, \ldots, B_{n+b}$ be a small neighborhood of $p^{-1}(\partial B)$. Then $B_j = A_j \times I$ $(n+1 \leq j \leq n+b)$ where A_j is a small neighborhood of a boundary component of ∂B and $\{q\} \times I$ is an interval fiber of $N(B)$ for any $q \in A_j$. By our construction, we can assume that $\mu \cap B_i = A_i \times C_i$, where C_i is a closed infinite set in I for each i.

Similarly, since every leaf is dense in μ, there are finitely many disjoint embeddings $\alpha_i : E \times I \to N(B)$ $(1 \leq i \leq m$ and $E = l \times [0, \epsilon])$, where l is an arc, such that:

1. $\alpha_i^{-1}(\mu) = E \times C_i'$, where $C_i' \subset I$ is a closed and infinite set,
2. $\alpha_i(\{q\} \times I)$ is a sub-arc of an interval fiber of $N(B)$ for any $q \in E$,
3. $\alpha_i(((\partial l) \times [0, \epsilon]) \times I) = \alpha_i(E \times I) \cap (\cup_{j=1}^{n+b} A_j \times I) \subset \cup_{j=1}^{n+b} (\partial A_j) \times I$,
4. $\mu - \cup_{j=1}^{n+b} A_j \times I - \cup_{i=1}^{m} \alpha_i(E \times I)$ is a union of disks,
5. The intersection of each I-fiber of $\partial A_i \times I$ with $\cup_{i=1}^{m} \alpha_i(E \times I)$ has at most one component.

Let $\Omega = (\bigcup_{i=1}^{n+b} A_i \times I) \cup (\bigcup_{j=1}^{m} \alpha_i(E \times I))$ and $\Omega_v = (\bigcup_{i=1}^{n+b} (\partial A_i) \times I) \cup (\bigcup_{j=1}^{m} \alpha_i((\partial E) \times I))$. Since $\mu - \Omega$ is a union of disks, there are finitely many disjoint embeddings $\beta_i : D \times I \to N(B)$ (D is a disk and $1 \leq i \leq q$), such that:

1. $\beta_i^{-1}(\mu) = D \times C_i''$, where C_i'' is a closed and infinite set in I for each i,
2. $\beta_i(\{p\} \times I)$ is a sub-arc of an interval fiber of $N(B)$ for any $p \in D$,
3. $\beta_i(D \times I) \cap \Omega = \beta_i((\partial D) \times I) \subset \Omega_v$,
4. $\mu - \Omega \subset \bigcup_{i=1}^{q} \beta_i(D \times I)$.

Furthermore, we can assume that $(\bigcup_{i=1}^{q} \beta_i(D \times I)) \cup \Omega$ matches perfectly to form an I-bundle over another branched surface B' and $\partial_v N(B') \subset \Omega_v$. Otherwise, we can replace $A_i \times I$ by $A_i \times I'$ where I' is a closed sub-interval of I, or modify α_i and β_i. So, μ is fully carried by $N(B')$. Since μ is a measured lamination, by out previous discussion, $N(B')$ must fully carry a compact orientable surface, which we denote by S. It follows from our construction that $S - \Omega$ is a union of disks. We denote by $A_1', \ldots, A_{n'}'$ the collection of annuli in $S \cap (\bigcup_{i=1}^{n'} A_i' \times I)$, and denote by

$L_1, \ldots, L_{m'}$ the collection of disks in $S \cap (\bigcup_{i=1}^{m} \alpha_i(E \times I))$. Let $S \times J$ be a small neighborhood of S in $N(B')$, where J is a small interval. Since μ intersects each $\alpha_i(E \times I)$ infinitely many times, we can assume that $\mu \cap (A'_i \times J) = A'_i \times C$ (for each i), where C is a closed and infinite set in J. Moreover, we can assume that $\mu \cap (L_j \times J)$ (for each j) contains $L_j \times C$ for some closed infinite set C in J. Note that $\mu \cap (L_j \times J)$ may contain components that are not parallel to $L_j \times \{t\}$ $(t \in J)$. In other words, some components of $\mu \cap (L_j \times J)$ may intersect $L_i \times \partial J$.

Now, we apply the same argument in the case that μ is a compact surface. We only need to replace μ in that case by S, replace the $N(c_i) \times I$'s in that argument by $\alpha_i(E \times I)$'s in this case, and replace the $A_i \times I$'s in that argument by $A'_j \times I$'s here. We can assume that the image of a cross disk, i.e., $\pi(D_k)$, lies in a tiny neighborhood of a leaf of μ. Since the intersection of every leaf of μ with each $L_i \times J$ contains infinitely many components of the form $L_i \times \{t\}$ $(t \in J)$, after choosing a sub cross disk of D_k (with large size) if necessary, we can assume that $\pi(D_k)$ does not contain a puncturing arc (defined as before) in any $A'_i \times J$. Thus, $\pi(D_{[k/2]})$ must be 'far away' from any puncturing arc, and hence the argument in the case that μ is a compact surface also works here.

□

Theorem 2.2, which is a generalization of Hatcher's theorem, now follows easily from Corollary 2.3.2 and Lemmas 2.6.1 and 2.6.3.

Theorem 2.2. *Let M be an orientable and irreducible 3-manifold whose boundary is an incompressible torus, and let $\mathcal{H}$ be the set of injective surfaces that are embedded along their boundaries and satisfy the 4-plane property. Suppose that M does not contain non-peripheral closed incompressible surfaces. Then the surfaces in $\mathcal{H}$ can realize only finitely many slopes.*

Proof. Suppose that the surfaces can realize infinitely many slopes. Let $\{F_n\}$ be a sequence of surfaces in $\mathcal{H}$ with different slopes. Since they have different slopes, by Corollary 2.3.2, the surfaces in $\{F_n\}$ cannot be carried by finitely many immersed branched surfaces. Then, by the argument is section 2.4, there exists a sequence of cross disks from $\{F_n\}$ whose 'limit' is an essential lamination. However, Lemma 2.6.1

and 2.6.3 imply that the surfaces in $\{F_n\}$ cannot realize infinitely many slopes, which gives a contradiction. □

As an application of Theorem 2.2, we will show that, in some sense, most 3-manifolds do not have non-positive cubing. Theorem 2.3 gives the first non-trivial examples of 3-manifolds that do not have non-positive cubing. Before we proceed, we would like to prove a lemma.

Lemma 2.6.4. *Let M be a closed and irreducible 3-manifold, S be an immersed surface in M with the 4-plane property, and C be a homotopically non-trivial simple closed curve that intersects S non-trivially. Then $S-C$ is a surface with the 4-plane property in $M-C$.*

Proof. Let $\tilde{M}$ be the universal cover of M and $\tilde{C}$ be the pre-image of C in $\tilde{M}$. Then $\tilde{M}-\tilde{C}$ is a cover of $M-C$, and among any 4 components of $\tilde{S}-\tilde{C}$, there is a disjoint pair, where $\tilde{S}$ is the pre-image of S in $\tilde{M}$. Since each component of $\tilde{S}-\tilde{C}$ is embedded, among any 4 components of the pre-image of $\tilde{S}-\tilde{C}$ in the universal cover of $\tilde{M}-\tilde{C}$ (i.e. the universal cover of $M-C$), there is a disjoint pair. Therefore, $S-C$ satisfies the 4-plane property in $M-C$. □

Theorem 2.3. *Let M be an orientable and irreducible 3-manifold whose boundary is an incompressible torus. Suppose that M does not contain closed non-peripheral incompressible surfaces. Then only finitely many Dehn fillings on M can yield 3-manifolds that admit non-positive cubing.*

Proof. Let $M(s)$ be the closed 3-manifold after doing Dehn filling along slope s, and C_s be the core of the solid torus glued to M. Then, except for finitely many slopes, C_s is a homotopically non-trivial curve in $M(s)$. Suppose that $M(s)$ admits a non-positive cubing. For each cube in the cubing, there are 3 disks parallel to the square faces and that intersect the edges of the cube in their mid-points. These mid-disks from all the cubes in the cubing match up and yield a union of immersed surfaces. Moreover, the complement of these immersed surfaces in $M(s)$ is a union of 3-balls. Aitchison and Rubinstein have shown that these surfaces (and their double cover in $M(s)$ if they are one-sided) satisfy the 4-plane property. Since C_s is non-trivial and the complement of these surfaces is a union of 3-balls, C_s must non-trivially intersect at least one of these surfaces. Hence, by Lemma 2.6.4, there is an injective surface in M that satisfies the 4-plane property and has boundary slope s. By Theorem 2.2, there are only finitely many such slopes. Therefore, the theorem holds. □

Chapter 3 Immersed surfaces in hyperbolic 3-manifolds

3.1 Introduction

An important question in 3-manifold topology is whether a closed 3-manifold contains π_1-injective surfaces. Embedded π_1-injective surfaces give a lot of information about 3-manifolds. e.g. [35]. But unfortunately, in some sense, most 3-manifolds do not contain embedded π_1-injective surfaces [19]. The main goal of this paper is to prove the contrary to [19] for immersed surfaces. i.e. (in some sense) most 3-manifolds do contain a surface subgroup.

Theorem 3.1. *Suppose X is a hyperbolic 3-manifold whose boundary is a single torus. Then all but finitely many Dehn fillings on X produce 3-manifolds containing π_1-injective surfaces.*

This theorem was also proved by Cooper and Long [8] earlier using different methods. The proof that we give here is topological, and an advantage of this approach is that it gives an explicit bound on the number of exceptional surgeries. Theorem 3.1 follows directly from Theorem 3.2 by the deep results in [11, 10]. See below for definitions of $X(\mu)$ and $\Delta(\mu, s)$.

Theorem 3.2. *Suppose X is a hyperbolic 3-manifold whose boundary is a single torus, and S is a two-sided, embedded, incompressible and ∂-incompressible surface with boundary slope s, and S is not a virtual fiber of X. Then there exists a number Γ such that $X(\mu)$ contains π_1-injective surfaces for any boundary slope μ with $\Delta(\mu, s) \geq \Gamma$.*

Proof of Theorem 3.1 from Theorem 3.2. It follows from [11, 10] that X contains such incompressible surfaces with at least two distinct boundary slopes arising from the splitting of $\pi_1(X)$ associated with the ideal points of certain algebraic curves.

Then by Proposition 1.2.7 of [10], the fundamental groups of the splitting surfaces cannot be normal subgroups of $\pi_1(X)$, hence they are not virtual fibers and Theorem 3.2 implies Theorem 3.1. □

In this paper, we will mainly prove Theorem 3.2. Unlike [8], we do not actually use the hyperbolic structure. The only thing that we need is that $\pi_1(X)$ has no non-peripheral $\mathbb{Z} \oplus \mathbb{Z}$ subgroups, which is equivalent to saying that $\mathring{X}$ has a complete hyperbolic structure by Thurston [34]. Moreover, we will give an explicit bound to the number of exceptional surgeries.

Theorem 3.3. *In theorem 3.2, Γ can be chosen to be an explicit linear function of the genus and number of boundary components of S.*

The idea of the proof is to construct a closed surface from S by connecting pairs of the boundary components of S using long annuli that wind around ∂X. By some combinatorial arguments, we show that if both the number of times that the annuli wind around ∂X and the distance between the surgery slope and the slope of ∂S are large, then this closed surface is π_1-injective. Notice that the immersed surface constructed has no triple points.

The techniques in this paper have been used on embedded incompressible surfaces in various papers (e.g. [28, 10]). The simplicity of the immersion in our construction allows us to apply them to this case. The idea of closing up boundaries of surfaces using long annuli was introduced by B. Freedman and M. H. Freedman in [13], and extensively used in [9, 7, 8].

Notation. Let α, β be two slopes on the boundary torus of X. $X(\alpha)$ denotes the closed manifold by Dehn filling along α, i.e., by adding a two-handle to X along a simple closed curve with slope α and then capping off the resulting 2-sphere boundary component with a 3-cell. $\Delta(\alpha, \beta)$ denotes the minimal geometric intersection number between two closed curves representing α and β. $N(E)$ denotes a small regular neighborhood of E, and $|E|$ denotes the number of components of E. We use both $\mathring{E}$ and $int(E)$ to denote the interior of E.

3.2 I-bundle regions

Definition 3.2.1. Suppose that M is an irreducible 3-manifold with boundary, and $A_1, \dots, A_s$ are disjoint annuli in ∂M such that $\partial M - \bigcup_{i=1}^{s} A_i$ is incompressible in M and the vertical arcs of each annulus cannot be homotoped rel boundary into $\partial M - \bigcup_{i=1}^{s} A_i$. Let $i : D = I \times I \to M$ be a proper map. We call i a **product disk** of (M, A), if $i(\partial I \times I)$ is a pair of vertical arcs of $A = \bigcup_{i=1}^{s} A_i$ and $i(I \times \partial I)$ is a pair of immersed arcs in $\partial M - A$ which cannot be homotoped rel boundary into A. We call $\{p\} \times I$ a **vertical arc** of the product disk for any $p \in I$.

By our definition, any vertical arc of a product disk cannot be homotoped rel boundary into $M - A$.

The following lemma is a simple case of the characteristic pairs in [24, 21]; see also [25]. For completeness, we give a proof here.

Lemma 3.2.1. *Let (M, A) be as above. Then there is a maximal I-bundle region J in M such that any product disk can be homotoped into J.*

Proof. First, we will show that there exists such a region J for embedded product disks. Given any two embedded product disks, by the standard cutting and pasting argument, we can assume after isotopy that their intersection is a union of vertical arcs. So, in our proof, we always assume that the intersection of any two embedded product disks is a union of vertical arcs.

We start with A and an embedded product disk D_1. We thicken them a little to get a small neighborhood of the union of A and D_1, which is clearly an I-bundle. We call it J_1. Assume that we have constructed J_k, which is a neighborhood of the union of $D_1, D_2, \dots, D_k$ and A. If there is still an embedded product disk D_{k+1} that cannot be isotoped into J_k, then we let J_{k+1} be a neighborhood of the union of $D_1, \dots, D_{k+1}$ and A. Since D_{k+1} cannot be isotoped into J_k such operations increase the Euler characteristic of the non-disk components of $\partial M - J_k$. Thus the operations must stop at a certain stage, and we get an I-bundle J' such that any embedded product disk can be isotoped into J'.

Furthermore, suppose some component of $\partial M - J'$, say D, is a disk. Then by the definition of a product disk, each fiber of J' cannot be homotoped rel boundary into

$\partial M - A$, and hence D together with the fibers of J' incident to ∂D is a disk, denoted by D'. Since $\partial M - A$ is incompressible, $\partial D'$ bounds a disk D'' in $\partial M - A$, and $D' \cup D''$ bounds a 3-ball because M is irreducible. Therefore, by adding such 3-balls to J', we can enlarge J' to another I-bundle J with canonical fibration such that no component of $\partial M - J$ is a disk. We call $(\partial M - A) \cap J$ the horizontal boundary, which we denote by $\partial_h J$, and $\partial J - \partial_h J - A$ the vertical boundary of J, which we denote by $\partial_v J$. Notice that $M - J$ does not contain any embedded product disks with respect to $(M - J, \partial_v J)$.

Now we show that any product disk can be homotoped into J. Let $i : P = I \times I \to M$ be a product disk. Then $i^{-1}(\partial_v J)$ is a union of disjoint simple arcs and simple closed curves in P since ∂J is embedded in M.

If there are simple closed curves in $i^{-1}(\partial_v J)$, we choose an innermost one which bounds a disk Δ in P. Then $i(\partial \Delta)$ is a homotopically trivial curve in the vertical boundary of J, and $i(\Delta)$ lies in J (or $M - J$). Since $\pi_2(M)$ is trivial, we can homotope $i(\Delta)$ to a point on $\partial_v J$ and move it out of J (or $M - J$) reducing the number of components of $i^{-1}(\partial_v J)$. Hence, we can assume that $i^{-1}(\partial_v J)$ does not contain any simple closed curves.

Since $i(\partial I \times I) \subset A \subset J$, each component of $i^{-1}(\partial_v J)$ is either a vertical arc of P or an arc with both endpoints on the same component of $I \times \partial I$. In the latter case, we choose an outermost such arc, say α. α together with a subarc of $I \times \partial I$, say β, bounds a disk δ in P. Now $i(\alpha)$ is a ∂-parallel arc in $\partial_v J$ and, since $\partial M - A$ is incompressible and M is irreducible, $i(\beta)$ is a ∂-parallel arc in $\partial_h J$ (or $\partial M - J$). Since $\pi_2(M)$ is trivial, we can homotope $i(\delta)$ out of J (or $M - J$) reducing the number of components of $i^{-1}(\partial_v J)$. So we can assume that $i^{-1}(\partial_v J)$ consists of only disjoint vertical arcs in P.

If P cannot be homotoped into J, then $i^{-1}(M - J)$ is a collection of rectangles of the form $[a, b] \times I$ in P, where $[a, b]$ is a subinterval of I. i restricted to each of these rectangles is a product disk of $(M - J, \partial_v J)$. By doing some cutting and pasting to P and $\partial_v J$, we get an embedded product disk in $M - J$, which contradicts the assumption that J is maximal. Thus any product disk can be homotoped into J. □

Notation. Let S be an orientable surface and $R \subset S$ be a subsurface of S with $\partial S \subset R$. Let $R' = R \cup$(disk components of $S - R$), and $\hat{R} = R' -$(disk components of R').

We define an equivalence relation: $R_1 \sim R_2$, if $\hat{R}_1$ and $\hat{R}_2$ are isotopic in S. Denote the set of surfaces equivalent to R by $[R]$.

Proposition 3.2.2. *Suppose R_1, R_2 are subsurfaces of S, and $\partial S \subset R_1 \cap R_2$. Then there exist $R'_1 \in [R_1]$ and $R'_2 \in [R_2]$ such that if a non-trivial curve can be homotoped into each of R_1 and R_2, it can be homotoped into $R'_1 \cap R'_2$.*

Proof. If S is a disk or an annulus, then the proof is trivial. So, we can assume that S is a hyperbolic surface with geodesic boundary. For simplicity, we only consider the case that $\hat{R}_1$ and $\hat{R}_2$ are connected. By our definition, there are no disk components in $S - \hat{R}_i$. We isotope $\hat{R}_1$ and $\hat{R}_2$ to be subsurfaces of S with quasi-geodesic boundaries as follows. If $S - \hat{R}_i$ ($i = 1$ *or* 2) contains annular components, we isotope $\hat{R}_i$ such that the annular components are ϵ-neighborhood of geodesics for some small ϵ. For other boundary components of $\hat{R}_i$, we first isotope $\hat{R}_i$ so that these boundary components are geodesics, then enlarge $\hat{R}_i$ by adding a 2ϵ-neighborhood of the geodesics to it. By choosing ϵ small enough, we can assume that there is no overlapping of $\hat{R}_i$ with itself, $i = 1, 2$.

For any nontrivial curve of S which can be homotoped into both R_1 and R_2, we first homotope it to be a geodesic. It then lies either in both surfaces constructed above or in an annular component of $S - \hat{R}_i$, for some i. In the later case we homotope the curve out of the ϵ-neighborhood so that it still remains in the 2ϵ-neighborhood of the geodesics. By our construction, it lies in the intersection of the two surfaces. □

Let R_1, R_2, R'_1, R'_2 be the surfaces of Proposition 2.2. We denote $[R'_1 \cap R'_2]$ by $[R_1] \cap [R_2]$. To simplify our notation, we do not distinguish between $[R]$ and a properly chosen element in $[R]$.

Definition 3.2.2. Let X be an irreducible 3-manifold whose boundary components are tori, S be a two-sided, incompressible, ∂-incompressible, embedded surface in X, and M be the 3-manifold obtained by cutting X along S, i.e., $M \cong X - \mathring{N}(S)$ (M may not be connected). Let $A = \partial X - \mathring{N}(\partial S)$ be a union of annuli in ∂M. We call a map $i : I \times [0, n] \to X$ an **essential rectangle of length n**, if i intersects S transversely and $i|_{I\times[k,k+1]}$ is a product disk of (M, A) for each $k \in \{0, 1, \dots, n-1\}$.

The following lemma is important to our proof. The same result was proved in [7] for the non-separating case.

Lemma 3.2.3. *Let X be a hyperbolic 3-manifold whose boundary is a single torus and S be a two-sided, incompressible, ∂-incompressible surface in X. Suppose S is not a virtual fiber. Then there exists a number $P(S) \in \mathbb{N}$ such that the length of any essential rectangle is less than $P(S)$, where $P(S) = 6g + 4b - 6$ and g, b are the genus and the number of boundary components of S.*

Proof. We assume that S is separating. If not, we can take S together with a parallel copy of S (disconnected) to be our surface.

Let M_1 and M_2 be the closures of the two components of $X - S$, and let M be the disjoint union of M_1 and M_2. Let $A_i = M_i \cap \partial X$ for $i = 1, 2$. Then $\partial M_1 - A_1 \cong \partial M_2 - A_2$.

Let J_i be the maximal I-bundle region of (M_i, A_i) constructed in Lemma 3.2.1. Let $S_i = \overline{J_i \cap (\partial M_i - A_i)}$ be the horizontal boundary of J_i for $i = 1, 2$. Note that S_i and $\partial M_i - A_i - S_i$ have no disk components. We can also assume that $A_i \subset J_i$. Define τ_i to be an involution of S_i such that $\tau_i : p_0 \mapsto p_1$, where p_0 and p_1 are the endpoints of a fiber of J_i, $i = 1$ or 2.

If $J_i = M_i$ for both i, then both M_1 and M_2 are I-bundles. Hence $\pi_1(S)$ is a normal subgroup of $\pi_1(X)$ and S is a virtual fiber. So we can assume that $S_1 \neq \partial M_1 - A_1$. Let $\varphi : \partial M_1 - A_1 \to \partial M_2 - A_2$ be the gluing map, then $X \cong M_1 \cup M_2/x \sim \varphi(x)$. For any $S' \in [S_i]$, we can isotope J_i so that $J_i \cap (\partial M_i - A_i) = \hat{S}'$ and define τ_i coherently. So we do not distinguish between $[S_i]$ and an element in the equivalent class and always use J_i for the coherent I-bundle.

Let $T_1 = [S_1]$, $T'_k = \tau_1([S_1] \cap [\varphi^{-1} \circ \tau_2([S_2] \cap [\varphi(T_k)])])$, and $T_{k+1} = [T_k] \cap [T'_k]$.

Claim. $[T_k] \supsetneqq [T_{k+1}]$ for any k unless $[T_k] = [\partial A_1]$, where $[\partial A_1]$ is a small neighborhood of ∂A_1 in $\partial M_1 - A_1$.

Proof of the claim. We have $[T_k] \cong \varphi[T_k] \supseteq [S_2] \cap [\varphi(T_k)] \cong \varphi^{-1} \circ \tau_2([S_2] \cap [\varphi(T_k)]) \supseteq [S_1] \cap [\varphi^{-1} \circ \tau_2([S_2] \cap [\varphi(T_k)])] = [T'_k] \supseteq [T_k] \cap [T'_k] = [T_{k+1}]$. Equalities hold if and only if $[\varphi(T_k)] = [S_2] \cap [\varphi(T_k)]$, $[\varphi^{-1} \circ \tau_2([S_2] \cap [\varphi(T_k)])] = [S_1] \cap [\varphi^{-1} \circ \tau_2([S_2] \cap [\varphi(T_k)])]$, and $[T'_k] = [T_k]$.

If $[T_{k+1}] = [T_k]$ and $[T_k] \neq [\partial A_1]$ for some k, then there exists a boundary component γ of $[T_k]$ such that γ is not parallel to ∂A_1. Note that γ is a non-trivial curve by our construction. Hence $\varphi(\gamma)$ is a boundary component of $[\varphi(T_k)] = [S_2] \cap [\varphi(T_k)]$, and $\gamma_1 = \tau_1 \circ \varphi^{-1} \circ \tau_2 \circ \varphi(\gamma)$ is a boundary component of $[T'_k] = [T_k]$.

By our construction, $\delta \cup \tau_i(\delta)$ bound an annulus or Möbius band in J_i for any simple closed curve δ in S_i. Hence, if γ is isotopic to γ_1, we can close up the two annuli or Möbius bands bounded by $\varphi(\gamma) \cup \tau_2 \circ \varphi(\gamma)$ and $\varphi^{-1} \circ \tau_2 \circ \varphi(\gamma) \cup \gamma_1$ to get a torus or Klein bottle in M. If γ is not isotopic to γ_1 then $\gamma_2 = \tau_1 \circ \varphi^{-1} \circ \tau_2 \circ \varphi(\gamma_1)$ is also a boundary component of $[T_k]$. In this way, we can define γ_i for any i. Since $[T_k]$ has only finitely many boundary components, there exist $i \neq j \in \mathbb{N}$ such that γ_i is isotopic to γ_j. So we can always close up some annuli or Möbius bands to get an immersed torus or Klein bottle. We shall show that the immersed torus (or Klein bottle) is π_1-injective.

To simplify notation, we do not distinguish between the torus (or Klein bottle) and its image in X under the immersion and denote both by T. The γ_i's are parallel non-trivial curves in T and their images are non-trivial curves in S. Since S is two-sided and incompressible, the γ_i's are non-trivial in M. If the immersion is not π_1-injective, then there is a non-trivial curve, say γ', in T which intersects each γ_i non-trivially and is mapped to a trivial curve in M. By our construction of T, we can homotope γ' so that it consists of vertical arcs of J_1 and J_2. Now that γ' bounds a disk D' in M. The pull-back of the intersection of D' with S is a collection of simple curves in D'. By using homotopies as before we can assume that there are no simple closed curves in D'. We then choose an outermost arc. This arc together with a subarc of γ' bounds a subdisk in D'. This subdisk is mapped to either M_1 or M_2, which means that the subarc of γ' (i.e. a vertical arc of J_i, by assumption) can be homotoped rel boundary into $\partial M_i - A_i$. This contradicts our assumption on J_i. Hence the immersion is π_1-injective.

Now we show that γ_i is not homotopic into ∂X. Note that, since S is π_1-injective, any non-trivial and non ∂-parallel curve in $\partial M_h - A_h$ ($h = 1$ or 2) is not homotopic into A_h in M_h. Suppose that γ_i is homotopic into ∂X in X. Then there is an immersed annulus $f : E = S^1 \times [0,1] \to X$ such that f is transverse to S, $f^{-1}(\partial X) = S^1 \times \{0\}$, and $f(S^1 \times \{1\}) \subset int(M_1)$ is a curve parallel and close to γ_i. Thus $f^{-1}(S) \cap S^1 \times \{1\} = \emptyset$. Since S is incompressible and ∂-incompressible, by some homotopies, we can get rid of trivial circles in $f^{-1}(S)$ and those arcs in $f^{-1}(S)$ with both endpoints on $S^1 \times \{0\}$. Hence, we can assume that $f^{-1}(S)$ is a union of disjoint meridian circles in E. Since γ_i is non-trivial and non ∂-parallel in S, the image of each component of $f^{-1}(S)$ is non-trivial and non ∂-parallel in S. Let E_0 be the component of $E - f^{-1}(S)$ that contains $S^1 \times \{0\}$. Then $f|_{E_0}$ is an annulus connecting A_h ($h = 1$ or 2) to a non-trivial and non ∂-parallel curve in $\partial M_h - A_h$, which gives a contradiction. Therefore, we get a π_1-injective and non-peripheral torus in X, which contradicts the hypothesis that X is hyperbolic.

□

It is easy to see that if $[T_{k+1}] \subsetneqq [T_k]$ then there exists a non-trivial simple closed curve α_k in $[T_{k+1}] - [T_k]$. Moreover, we can choose the simple closed curves such that α_i is not parallel to α_j if $i \neq j$. By an Euler characteristic argument, there are at most $3g + 2b - 3$ disjoint, essential and non-parallel simple closed curves in $\partial M_1 - A_1$. By our assumption that $S_1 \neq \partial M_1 - A_1$, there is at least one non-trivial simple closed curve in $\partial M_1 - A_1 - S_1$. Hence if $k \geq 3g + 2b - 3$, $[T_k] = [\partial A_1]$.

Let $i : I \times [0, n] \to M$ be an essential rectangle of length n. Suppose

$$i|_{I\times[0,1]} \text{ is a product disk of } (M_1, A_1), \tag{3.1}$$

$$i|_{I\times[1,2]} \text{ is a product disk of } (M_2, A_2), \tag{3.2}$$

$$i|_{I\times[2,3]} \text{ is a product disk of } (M_1, A_1), \tag{3.3}$$

$$\cdots\cdots$$

$$\begin{aligned}
\text{By (1),}\quad & i(I\times\{1\})\in[T_1],\\
\text{by (2),}\quad & i(I\times\{1\})\in[S_2]\cap[\varphi(T_1)],\\
& i(I\times\{2\})\in\tau_2([S_2]\cap[\varphi(T_1)]),\\
\text{by (3),}\quad & i(I\times\{2\})\in[S_1]\cap\varphi^{-1}\circ\tau_2([S_2]\cap[\varphi(T_1)]),\\
& i(I\times\{3\})\in[T_1']\cap[T_1]=[T_2],\\
& \qquad\qquad \cdots\cdots
\end{aligned}$$

Thus, if $n \geq 2(3g+2b-4)+1$, $i(I\times\{2(3g+2b-4)+1\}) \in [T_{3g+2b-3}] = [\partial A_1]$, which contradicts our definition of a product disk.

$$\text{Hence we have}\quad n \leq 2(3g+2b-4).$$

Similarly, if $i(I\times[0,1])$ is a product disk of (M_2, A_2) and $J_2 = M_2$ then we get $n \leq 2(3g+2b-4)+1$. So in any case, we have

$$n \leq 6g+4b-7.$$

□

Corollary 3.2.4. *Let X be a hyperbolic 3-manifold whose boundary is a single torus and $i:(S,\partial S)\looparrowright(X,\partial X)$ a π_1-injective surface. Suppose there is a constant $C\in\mathbb{N}$ such that the genus of S is less than C. Then there are only finitely many possible slopes for the boundary circles of $i(\partial S)$.*

Proof. Let S' be an embedded, two-sided, incompressible, ∂-incompressible surface in X and suppose S' is not a virtual fiber. Let the boundary slope of $\partial S'$ be s and the boundary slope of $i(\partial S)$ be μ. As in the proof of Theorem 3.1 it suffices to show that $\Delta(\mu, s)$ is bounded.

The components of $i^{-1}(S')$ are disjoint simple arcs or simple closed curves in S because S' is embedded. Let g be the genus of S and b be the number of boundary components of S. Then $i^{-1}(S')$ consists of at least $\frac{1}{2}b\Delta(\mu,s)$ simple arcs in S. By an Euler characteristic argument, there are at most $6g + 3b - 6$ disjoint nonparallel nontrivial simple arcs in S. Since $g \leq C$, if $\Delta(\mu, s)$ is large, there are many parallel arcs in S and we get an essential rectangle with large length violating Lemma 3.2.3. So $\Delta(\mu, s)$ cannot be too large. □

Remarks. 1. In [2] and [31], it was shown that for many 3-manifolds with boundary a torus there are infinitely many boundary slopes realizing π_1-injective surfaces. Corollary 2.4 says that as the boundary slope increases, the genus of the surface increases.

2. Corollary 2.4 is not a deep result. The following elegant argument is due to Dave Gabai. Let X be a hyperbolic 3-manifold with a single cusp and let S be a π_1-injective surface mapping cusps to cusp. Suppose S has the least area in its homotopy class. Then, by Gauss-Bonnet, $Area(S) \leq -\chi(S) = 2g - 2 + b$. On the other hand, we let T be a horospherical torus in X. Then $S \cap T$ is a union of b closed curves (in T) of length at least l, where l depends on the slope of the closed curves. By hyperbolic geometry, the area of the cusps of S is at least kbl, where k is a constant and b is the number of cusps of S. Hence we have $kbl \leq Area(S) \leq 2g-2+b$. Since g is bounded, l cannot be too large and S can realize only finitely many slopes.

3.3 Construction of the injective surfaces

Let X be a hyperbolic 3-manifold whose boundary is a torus and S be a two-sided, incompressible, ∂-incompressible, embedded surface which is not a virtual fiber. As before, we assume that S is separating; otherwise we take S together with a parallel copy of S (disconnected) to be our surface. For simplicity we only consider the case that S has two boundary components. The proof is similar for the case that S has more than two boundary components.

Let $T^2 \times I$ be a product neighborhood of ∂X and S' be a parallel copy of S. We construct our immersed surface T by connecting the two circles of $\partial(S' - T^2 \times I)$ using an annulus that winds (in $T^2 \times I$) around ∂X K times as shown in Figure 3.1 (a). Thus $T \cap S$ is a collection of $2K$ ∂-parallel disjoint simple closed curves. We call this annulus the *long annulus*.

We define a retraction map $\pi : X \to \overline{X - T^2 \times I}$ by fixing points in $X - T^2 \times I$ and mapping every interval $\{p\} \times I$ of $T^2 \times I$ to the point $(p, 1)$, where $p \in T^2$.

Lemma 3.3.1. *If $K \geq P(S) + 1$ then T is π_1-injective in X.*

Proof. Suppose not, then there exists an immersed closed curve l in T such that l is contractible in X but not contractible in T.

Let p be the number of arcs in the intersection of l with the long annulus. Notice that $p \geq 1$ since S is incompressible and two-sided. We homotope l to minimize p and the number of points in its intersection with S.

Since l is contractible in X, there is a map $j : D \to X$ such that $j(\partial D) = l$, where D is a disk. We see that $|l \cap S| = 2Kp$ and $j^{-1}(S)$ is a collection of disjoint simple arcs in D since S is embedded.

The two circle components of $\partial(S' - T^2 \times I)$ divides l into $2p$ subarcs, namely $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_p, \beta_p$, where $\bigcup_{i=1}^{p}(\alpha_i \cup \beta_i) = \partial D$, $j(\beta_i)$ is a subarc of l lying entirely in the long annulus and $j(\alpha_i)$ is a subarc of l lying entirely in S'. Thus $j^{-1}(S) \cap \partial D \subset \bigcup_{i=1}^{p} \beta_i$ and $|j^{-1}(S) \cap \beta_i| = 2K$ for each i. We call the α_i's α-arcs, and the β_i's β-arcs. These α-arcs and β-arcs appear on ∂D alternately.

Claim 1. There are no arcs in $j^{-1}(S)$ whose endpoints are both in the same β_i.

Proof of claim 1. Suppose there are such arcs. We choose an outermost one, say γ; then γ together with a subarc $\hat{\beta}$ of β_i bounds a bigon in D. Hence $\pi \circ j(\gamma)$ must be a ∂-parallel arc in $\pi(S)$. Since γ is outermost and S is ∂-incompressible, both endpoints of $j(\hat{\beta})$ must lie in the same component of $T \cap S$ and $j(int(\hat{\beta})) \cap S = \emptyset$. So we can homotope l to have fewer points of intersection with S, which contradicts our assumption. This proves claim 1.

We call an arc in $j^{-1}(S)$ a *long arc* if it cuts D into two components such that each of them contains at least two α-arcs.

Claim 2. There exists a $k \in \mathbb{N}$ such that the endpoints of no long arc lie in β_k.

Proof of claim 2. Consider all the long arcs of $j^{-1}(S)$ in D and choose an outermost one. This together with a subarc of ∂D bounds a bigon that does not contain long arcs. Suppose the bigon contains arcs α_k and α_{k+1}, then β_k is as needed because $j^{-1}(S)$ consists of disjoint simple arcs. This proves claim 2.

Consider all the arcs with an endpoint in β_k (β_k as in claim 2). By claim 1, the

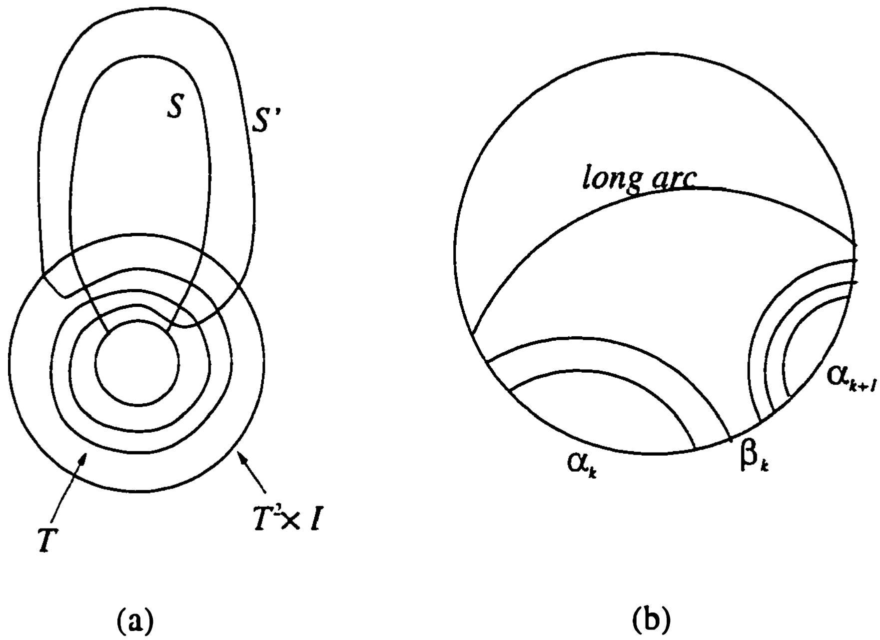


Figure 3.1:

other endpoint of such an arc must lie in either β_{k-1} or β_{k+1}. Since $|\beta_k \cap j^{-1}(S)| = 2K$, we have at least K parallel arcs which are all parallel to α_k (or α_{k+1}), as shown in Figure 3.1 (b).

Notice that $\pi \circ j(\alpha_k)$ is not a ∂-parallel arc in $\pi(S')$; otherwise we can homotope l to have fewer points of intersection with S. Thus the images of the K arcs which are parallel to α_k (or α_{k+1}) are essential arcs in $\pi(S)$. Hence we get an essential rectangle of length $K - 1 \geq P(S)$ with respect to the 3-manifold $\pi(X)$ and surface $\pi(S)$, which contradicts Lemma 3.2.3. □

Proof of Theorem 3.2 and Theorem 3.3. We will prove that the surface T constructed at the beginning of this section is π_1-injective in $X(\mu)$ if both $\Delta(\mu, s)$ and K are large.

Suppose not, then there exists a closed essential curve l in T contractible in $X(\mu)$. Hence for any $i \geq 1$ there is an immersion $j : P \looparrowright X$, where P is a planar surface with $k+1$ boundary components, $\partial P = \bigcup_{i=0}^{k} p_i$, $j(p_0) = l$ and $j(p_i)$ is an immersed curve of slope μ in ∂X. We assume that l has been homotoped to have the fewest points of intersection with S and k (≥ 1) is the minimal number for all such planar surfaces. The case where $k = 0$ follows from Lemma 3.3.1. Now $j^{-1}(S)$ is a union of disjoint simple arcs or simple closed curves in P because S is embedded. By the same argument as before we can assume that there are no trivial circles in P.

Claim. There are no ∂-parallel arcs in P with both endpoints on the same p_i for any $i \geq 1$.

Proof of the claim. Suppose there are such arcs. We choose an outermost one, say γ, which together with a subarc γ' of p_i bounds a bigon in P. Hence $j(\gamma)$ can be homotoped rel boundary into ∂X. Since S is ∂-incompressible, both endpoints of $j(\gamma')$ must lie in the same component of ∂S and $j(int(\gamma')) \cap \partial S = \emptyset$ (because γ is outermost). So we can homotope p_i to get fewer points of intersection with ∂S, which contradicts our assumption. □

Let $\mathcal{B}$ be the subset of $j^{-1}(S)$ consisting of arcs with at least one endpoint on p_i for some $i \geq 1$. Since $j(p_i)$ is a curve of slope μ in ∂X for $i \geq 1$, $j(p_i)$ intersects S in at least $2\Delta(\mu, s)$ points (we have assumed that S has two boundary components). Hence $|\mathcal{B}| \geq k\Delta(\mu, s)$. By an Euler characteristic argument, the maximal number of non-parallel arcs in P is $3k-3$ if $k > 1$, and 1 if $k = 1$. So, if $|\mathcal{B}| \geq k\Delta(\mu, s) \geq 3kN$, there are at least $N+1$ arcs in $\mathcal{B}$ which are parallel to each other. Let $\delta_0, \delta_1, \ldots, \delta_N$ be the $N+1$ parallel arcs.

Case 1. The $N+1$ parallel arcs have endpoints on p_i and p_j with both $i, j \geq 1$.

Recall that by our construction $j(p_i) \subset \partial X$ if $i \geq 1$.

Suppose $j(\delta_i)$ is a ∂-parallel arc in S for some i. Then we can homotope $j(\delta_i)$ to ∂X, then cut along δ_i to get a map of a planar surface with fewer boundary components, which contradicts our assumption.

Therefore $j(\delta_i)$ is an essential arc for every i and $\delta_0, \delta_1, \ldots, \delta_N$ form an essential rectangle of length N. By Lemma 3.2.3, if $N \geq P(S)$, no such essential rectangle exists.

Case 2. Each of the $N+1$ parallel arcs has one endpoint on p_0 and the other endpoint on p_i for some $i \geq 1$.

As in the proof of Lemma 3.3.1, we divide p_0 into segments $\alpha_1, \beta_1, \ldots, \alpha_q, \beta_q$, where $p_0 = \bigcup_{i=1}^{q}(\alpha_i \cup \beta_i)$ and each $j(\beta_i)$ is a subarc of l lying entirely in the long annulus, and each $j(\alpha_i)$ is a subarc of l lying entirely in the surface S'. We call the α_i's α-arcs and the β_i's β-arcs.

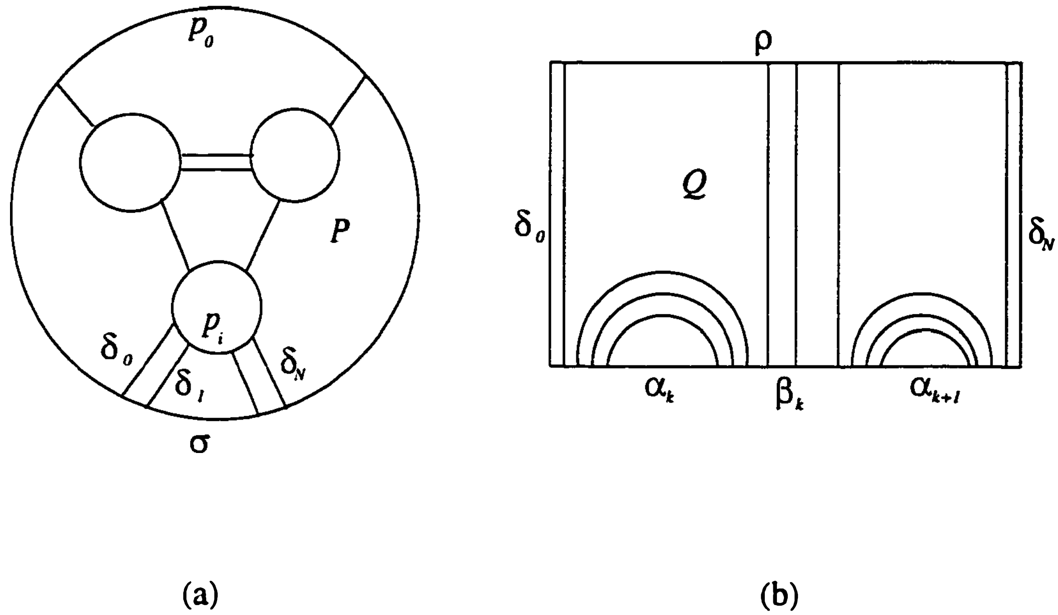


Figure 3.2:

$\delta_0 \cup \delta_N$ together with a subarc σ of p_0 and a subarc ρ of p_i form a quadrilateral Q in P (see Figure 3.2 (a)). By the claim. there are no arcs in Q with both endpoints on ρ. So the arcs of $j^{-1}(S) - \mathcal{B}$ in Q are arcs with both endpoints on σ. By the claim 1 in Lemma 3.3.1, there are no arcs in Q with both endpoints on the same β-arc. As in the proof of Lemma 3.3.1, we can assume that there is no long arc in Q, i.e., no arc cutting off a bigon in Q which contains at least two α-arcs.

Next we choose K and N such that $2K \geq 3P(S)+1$ and $N+1 \geq 2K+2P(S)+1$. Since each β-arc contains exactly $2K$ endpoints of arcs in $j^{-1}(S)$, there is at least one α-arc in Q.

Suppose there are at least two α-arcs, say α_k and α_{k+1}, in Q (by choosing N larger, we can always ensure that.). Then all the arcs with one endpoint on β_k are contained in Q. As in the proof of Lemma 3.3.1, there are at most $P(S)$ arcs parallel to α_k or α_{k+1}; otherwise we have an essential rectangle of length $P(S)$. So there are at least $2K - 2P(S) \geq P(S) + 1$ arcs with one endpoint on β_k and the other endpoint on ρ, as shown in Figure 3.2 (b). If the images of these $P(S)+1$ arcs under the map $\pi \circ j$ are not trivial in $\pi(S)$, we get an essential rectangle of length $P(S)$, which contradicts Lemma 3.2.3. Therefore we assume that they are trivial arcs in $\pi(S)$.

Since $N+1 \geq 2K+2P(S)+1$, we have at least $P(S)+1$ parallel arcs in $\mathcal{B}$ with one endpoint on β_{k+1} (or β_{k-1}). Again we assume the image of these parallel arcs under $\pi \circ j$ are trivial in $\pi(S)$; otherwise it contradicts Lemma 3.2.3.

Let $\bigcup_{i=1}^{2K} l_i = T \cap S$, where each l_i is a simple closed curve. Then by our construction of T and assumption on l, the intersection of each β_i with $T \cap S$ appears either in the order (with respect of a certain orientation of l) $l_1, l_2, \ldots, l_{2K}$ or in the order $l_{2K}, l_{2K-1}, \ldots, l_1$. An arc that has one endpoint in the central portion of β_k or β_{k+1} (or β_{k-1}) must have the other endpoint on ρ, and hence is one of the $N+1$ parallel arcs that we considered above (see Figure 3.2 (b)). The reason is that we cannot have too many arcs parallel to the two α-arcs adjacent to this β-arc; otherwise we will get a long essential rectangle. So it is easy to see that there must be an arc δ_i with $\delta_i \cap \beta_k = a$ and an arc δ_j with $\delta_j \cap \beta_{k\pm 1} = b$ such that $j(a)$ and $j(b)$ lie on the same simple closed curve component of $T \cap S$. Since by our assumption $\pi \circ j(\delta_i)$ and $\pi \circ j(\delta_j)$ are trivial arcs in $\pi(S)$, we can homotope $j(\delta_i)$, moving $j(a)$ along the simple closed curve component of $T \cap S$ to $j(b)$ and closing up $j(\delta_i)$ and $j(\delta_j)$, as shown in Figure 3.3, to get an immersed annulus in X.

One boundary component, say σ', of this immersed annulus is mapped into T and the other boundary component is mapped into ∂X with slope different from that of ∂S. Notice that there is exactly one α-arc between δ_i and δ_j in Q. If σ' is mapped to a trivial curve in T, then the α-arc between δ_i and δ_j must be a ∂-parallel arc in S', and we can homotope l to get fewer points of intersection with S, which contradicts our assumptions. So σ' is a non-trivial curve in T. Now the simple closed curve component of $T \cap S$ containing $j(a)$ and $j(b)$ together with a boundary component of ∂S bounds another annulus in S, and the intersections of the two annuli are vertical arcs in both of them. Therefore we get two elements in $\pi_1(T)$ simultaneously homotopic to two curves in ∂X of different slopes. Since T is π_1-injective in X and clearly T is not peripheral, $\mathbb{Z} \oplus \mathbb{Z}$ is a non-peripheral subgroup of $\pi_1(X)$. This contradicts the hypothesis that X is hyperbolic.

So, as long as $2K \geq 3P(S)+1$ and $N+1 \geq 2K+2P(S)+1$ (i.e. $N \geq 5P(S)+1$), T is π_1-injective in $X(\mu)$. Recall that we have chosen $k\Delta(\mu, s) \geq 3kN$ to get $N+1$ parallel arcs. Hence it suffices that $\Delta(\mu, s) \geq 3N \geq 15P(S)+3$.

If Q contains exactly one α-arc, then as in the proof of Lemma 3.3.1, there are

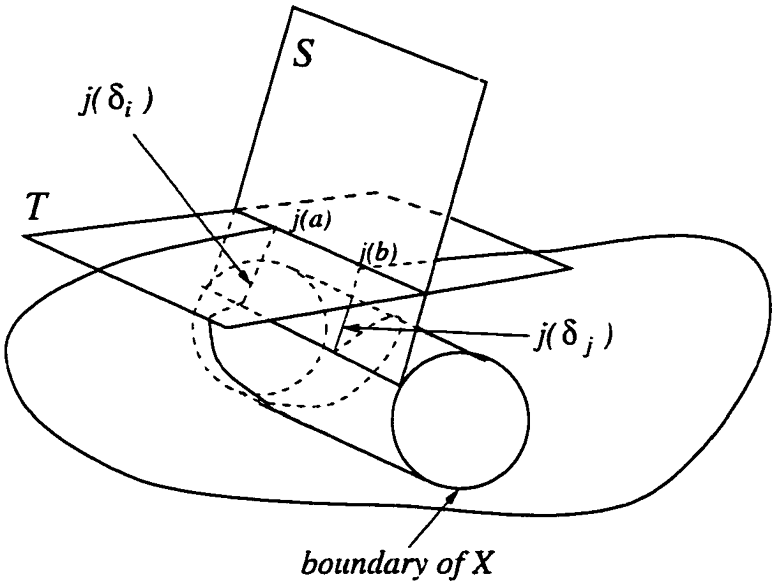


Figure 3.3:

at most $P(S)$ arcs parallel to this α-arc. Since $N \geq 2K + 2P(S)$, incident to each β-arc that is adjacent to this α-arc, there are at least $2P(S)$ arcs belonging to the set of the $N + 1$ parallel arcs that we considered above. Now the proof is as earlier.

□

Remarks. If S has more than two boundary components, then $|\mathcal{B}| \geq \frac{b}{2}k\Delta(\mu, s)$. In this case, there are $\frac{b}{2}$ long annuli and we need $j(a)$ and $j(b)$ to be on the same long annulus. Hence the factor b will be canceled and we get a bound for $\Delta(\mu, s)$ that is a linear function of g and b.

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