Some estimates of Fourier transforms

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Abstract

This work consists of two independent parts. In the first part we prove several results related to the theorem of Logvinenko and Sereda on determining sets for functions with Fourier transforms supported in a parallelepiped. We obtain a polynomial instead of exponential bound in this theorem, and we extend it to the case of functions with Fourier transforms supported in the union of a bounded number of parallelepipeds. When dimension $d = 1$ we also consider the case of infinitely many lacunary intervals. We generalize the Zygmund theorem for lacunary series whose Fourier coefficients are replaced with polynomials of uniformly bounded degree. We give also a necessary condition for the support of Fourier transforms for which the Logvinenko-Sereda theorem still holds.

In the second part we prove that the $L^2([0,1]^d \times SO(d))$ norm of periodizations of a function from $L^1(\mathbb{R}^d)$ is equivalent to the $L^2(\mathbb{R}^d)$ norm of the function itself in higher dimensions. We generalize the statement for functions from $L^p(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$ in the spirit of the Stein-Tomas theorem. We also show that the following theorem due to M. Kolountzakis and T. Wolff does not hold if dimension $d = 2$: if periodizations of a function from $L^1(\mathbb{R}^d)$ are constants, then the function is continuous and bounded provided that the dimension d is at least three.

Contents

0.1 Introduction

This work consists of two independent parts. We use various estimations of Fourier transforms to obtain results.

The first part considers the Logvinenko-Sereda theorem which is one of the basic examples of the so-called Uncertainty Principle in Fourier Analysis. This Principle states that a function and its Fourier transform can not be simultaneously supported on "small" sets. Although this formulation is rather vague, we obtain rigorous results. Intervals and compliments of "relatively dense" subsets play the role of small sets in our situation. The Logvinenko-Sereda theorem is based on properties of entire functions of exponential type. We will improve on this theorem by getting an optimal estimate and generalize it for the case of finitely many intervals. We will also investigate the case of infinitely many lacunary intervals. We will generalize a result of F. Nazarov for the Zygmund theorem on lacunary series. We will also give a necessary condition for the support of Fourier transforms for which "relatively dense" subsets are still determining sets.

The second part is devoted to periodizations of functions from $L^1(\mathbb{R}^d)$ in higher dimensions. Some results on the Steinhaus tiling problem due to M. Kolountzakis and T. Wolff are related to mine since periodizations naturally appear in the problem of Steinhaus. The main idea can be formulated in this way. If we are given some information on periodizations, what can we say about the function itself and vice versa? It is rather natural to formulate this problem in terms of various norms. Using some facts from Number Theory, we prove that the $L^2([0,1]^d \times SO(d))$ norm of periodizations of a function from $L^1(\mathbb{R}^d)$ is equivalent to the $L^2(\mathbb{R}^d)$ norm of the function itself in higher dimensions. We generalize the statement for functions from $L^p(\mathbb{R}^d)$ where $1 \le p < \frac{2d}{d+2}$ in the spirit of the Stein-Tomas theorem. We will also show that the result due to M. Kolountzakis and T. Wolff, which holds when dimension $d \ge 3$, does not hold when $d = 2$.

Chapter 1 Some results related to the Logvinenko-Sereda theorem

1.1 Overview

The purpose of this work is to study the behavior of functions whose Fourier transforms are supported in a parallelepiped or in a union of finitely many parallelepipeds on "thick" subsets of $\mathbb{R}^d$. A main result of this type was proven by Logvinenko and Sereda.

By a "thick" or "relatively dense" subset of $\mathbb{R}^d$ we mean a measurable set E for which there exist a parallelepiped Π with sides of length $a_1, a_2, ..., a_d$ parallel to coordinate axes and $\gamma > 0$ such that

$$|E \cap (\Pi + x)| \geq \gamma |\Pi| \tag{1.1}$$

for every $x \in \mathbb{R}^d$.

The Logvinenko-Sereda Theorem, $d = 1$: *let J be an interval with $|J| = b$. If $f \in L^p(\mathbb{R})$, $p \in [1, +\infty]$, and supp $\hat{f} \subset J$ and if a measurable set E satisfies (1.1), then*

$$\|f\|_{L^p(E)} \geq \exp(-C \cdot \frac{(ab+1)}{\gamma}) \cdot \|f\|_p. \tag{1.2}$$

It is a well-known fact that the condition (1.1) is also necessary for an inequality of the form

$$\|f\|_{L^p(E)} \geq C \cdot \|f\|_p$$

to hold. See for example ([7], p.113).

We will improve the estimate (1.2) by getting a polynomial dependence on γ and show that our estimate is optimal except for the constant C:

Theorem 1: *let J be a parallelepiped with sides of length $b_1, b_2, ..., b_d$ parallel to coordinate axes. If $f \in L^p(\mathbb{R}^d)$, $p \in [1, +\infty]$, and supp $\hat{f} \subset J$ and if a measurable set E satisfies (1.1), then*

$$\|f\|_{L^p(E)} \geq \left(\frac{\gamma}{C^d}\right)^{C(d+\sum_{j=1}^{d} a_j b_j)} \cdot \|f\|_p.$$

We will also generalize the Logvinenko-Sereda theorem to functions whose Fourier transforms are supported in a union of finitely many parallelepipeds:

Theorem 2: *let J_k be identical parallelepipeds with sides of length $b_1, b_2, ..., b_d$ parallel to coordinate axes. If $f \in L^p$, $p \in [1, +\infty]$, and supp $\hat{f} \subset \bigcup_{1}^{n} J_k$ and if a measurable set E satisfies (1.1), then*

$$\|f\|_{L^p(E)} \geq c(\gamma, n, \mathbf{a} \cdot \mathbf{b}, d, p) \cdot \|f\|_p$$

where $c(\gamma, n, \mathbf{a} \cdot \mathbf{b}, d, p) = \left(\frac{C^d}{\gamma}\right)^{-\mathbf{a}\cdot\mathbf{b}\left(\frac{C^d}{\gamma}\right)^n - n + \frac{p-1}{p}}$ depends only on the number of parallelepipeds but not how they are placed.

The next natural step is to consider the case of infinitely many parallelepipeds. We will conjecture the following theorem for infinitely many lacunary intervals when dimension $d = 1$:

Theorem 3: *let* $\Lambda = \{\lambda_k\}_{k=-\infty}^{\infty}$ *be lacunary and let* E *be "relatively dense." If* $f \in L^2$ *and supp* $\hat{f} \subset \bigcup_{k=-\infty}^{\infty} [\lambda_k - \frac{b}{2}, \lambda_k + \frac{b}{2}]$*, then*

$$\|f\|_{L^2(E)} \geq C(\gamma, N, a, b) \cdot \|f\|_2.$$

See section 1.4.1 for the definition of lacunarity. We will prove some partial results: the above **Theorem** holds when γ is large or when b is small. It holds too when E is periodic. We will also generalize Nazarov's result for the Zygmund theorem on lacunary series ([13], Theorem 3.6):

Nazarov's Theorem: *let* $\Lambda = \{\lambda_k\}_{k=-\infty}^{\infty}$ *be a lacunary sequence of integer numbers with the maximal number of representations* N*. If* $f \in L^2([0, 2\pi])$ *with spec*$f \in \Lambda$ *then*

$$\int_E |f|^2 \geq C(|E|, N)\|f\|_2^2$$

with $C(|E|, N) = e^{-\frac{C(N,\epsilon)}{|E|^{2+\epsilon}}}$ *for every* E *with positive measure.*

We extend the above theorem to generalized lacunary series with Fourier coefficients being replaced by polynomials of uniformly bounded degree:

Theorem 4: *let* I *be an interval of length* 1 *and* E *be a subset of* I *of positive measure. If* $\Lambda = \{\lambda_k\}_{k=-\infty}^{\infty}$ *is lacunary in the sense of (1.29) and* $f(x) = \sum_{k=-\infty}^{\infty} p_k(x) e^{i\lambda_k x}$ *where* $p_k(x)$ *are polynomials of degree at most* m*, then*

$$\int_E |f|^2 \geq C(|E|, N, m) \int_I |f|^2$$

where $C(|E|, N, m) = e^{-(\ln CN)N^2\left(\frac{C}{|E|}\right)^{C(m+1)}}$.

Note that the constant C below is not fixed and varies appropriately from one equality or inequality to another without being mentioned.

1.2 Case of one parallelepiped

Proof of Theorem 1:

First we treat the case when $p \in [1, +\infty)$. Without loss of generality we can always assume that J is centered at the origin:

$$J = [-\frac{b_1}{2}, \frac{b_1}{2}] \times ... \times [-\frac{b_d}{2}, \frac{b_d}{2}].$$

By considering $f(\frac{x_1}{a_1}, ..., \frac{x_d}{a_d})$ instead of f, we can also assume that $|E \cap \Pi| \geq \gamma$ for all cubes Π with sides of length 1 and parallel to coordinate axes and

$$supp\hat{f} \subset [-\frac{a_1 b_1}{2}, \frac{a_1 b_1}{2}] \times ... \times [-\frac{a_d b_d}{2}, \frac{a_d b_d}{2}],$$

just say

$$supp\hat{f} \subset [-\frac{b_1}{2}, \frac{b_1}{2}] \times ... \times [-\frac{b_d}{2}, \frac{b_d}{2}].$$

Define $b = (b_1, ..., b_d)$. Bernstein's inequality ([2], Theorem 11.3.3) applied to each variable separately gives that

$$\int |f^{(\alpha)}|^p \leq (C \cdot b)^{\alpha p} \cdot \int |f|^p$$

with $C = \frac{1}{2}$. Here $\alpha = (\alpha_1, ..., \alpha_d)$ is a multiindex in $\mathbb{R}^d$ with nonnegative integer coordinates. $|\alpha| = \sum_{j=1}^{d} \alpha_j$. $f^{(\alpha)} = \frac{\partial^{\alpha_1}}{\partial^{x_1}} ... \frac{\partial^{\alpha_d}}{\partial^{x_d}} f$. If $x \in \mathbb{R}^d$ then $x^\alpha = x_1^{\alpha_1} \cdot ... \cdot x_d^{\alpha_d}$. $\alpha! = \alpha_1! \cdot ... \cdot \alpha_d!$.

Divide the whole $\mathbb{R}^d$ into cubes Π with sides of length 1 and parallel to coordinate axes. Choose $A > 1$. Call a cube Π bad if $\exists \alpha \neq 0$ such that

$$\int_{\Pi} |f^{(\alpha)}|^p \geq 2^d A^{|\alpha|p} (C \cdot b)^{\alpha p} \cdot \int_{\Pi} |f|^p.$$

Then

$$\begin{aligned}
\int_{\bigcup_{\Pi \text{ is bad}} \Pi} |f|^p &\leq \int_{\bigcup_{\Pi \text{ is bad}} \Pi} \sum_{\alpha \neq 0} \frac{1}{2^d A^{|\alpha|p} (C \cdot b)^{\alpha p}} |f^{(\alpha)}|^p \\
&= \sum_{\alpha \neq 0} \frac{1}{2^d A^{|\alpha|p} (C \cdot b)^{\alpha p}} \int_{\bigcup_{\Pi \text{ is bad}} \Pi} |f^{(\alpha)}|^p \\
&\leq \sum_{\alpha \neq 0} \frac{1}{2^d A^{|\alpha|p} (C \cdot b)^{\alpha p}} \int |f^{(\alpha)}|^p \\
&\leq \sum_{\alpha \neq 0} \frac{1}{2^d A^{|\alpha|p}} \int |f|^p \\
&= \left(\frac{1}{(1 - \frac{1}{A^p})^d} - 1\right) / 2^d \int |f|^p. \qquad (1.3)
\end{aligned}$$

Choosing $A = 3$ and applying (1.3), we obtain

$$\int_{\bigcup_{\Pi \text{ is bad}} \Pi} |f|^p \leq \frac{1}{2} \int |f|^p.$$

Therefore,

$$\int_{\bigcup_{\Pi \text{ is good}} \Pi} |f|^p \geq \frac{1}{2} \int |f|^p. \qquad (1.4)$$

Replace $3C$ with C.

We claim that $\exists B > 1$ such that if Π is a good cube, then $\exists x \in \Pi$ with the property that

$$|f^{(\alpha)}(x)|^p \leq 4^d \cdot B^{|\alpha|p} (C \cdot b)^{\alpha p} \cdot \int_{\Pi} |f|^p \quad \forall \alpha.$$

Suppose towards a contradiction that this is not true. Then

$$2^d \cdot \int_\Pi |f|^p \le \sum_\alpha \frac{1}{B^{|\alpha|p} 2^d (C \cdot b)^{\alpha p}} |f^{(\alpha)}(x)|^p \quad \forall x \in \Pi. \tag{1.5}$$

Integrate both sides of (1.5) over Π:

$$\begin{aligned} 2^d \cdot \int_\Pi |f|^p &\le \sum_\alpha \frac{1}{B^{|\alpha|p} 2^d (C \cdot b)^{\alpha p}} \int_\Pi |f^{(\alpha)}(x)|^p \\ &\le \sum_\alpha \frac{1}{B^{|\alpha|p}} \int_\Pi |f|^p \\ &= \frac{1}{(1 - (\frac{1}{B})^p)^d} \int_\Pi |f|^p. \end{aligned} \tag{1.6}$$

Choose $B = 3$ and apply (1.6). So

$$2^d \cdot \int_\Pi |f|^p \le (3/2)^d \int_\Pi |f|^p.$$

This contradiction proves our claim. Replace $3C$ with C.

We will need to prove the following local estimate:

$$\int_{E \cap \Pi} |f|^p \ge \left(\frac{\gamma}{C^d}\right)^{3d + Cp \sum_{j=1}^{d} b_j} \int_\Pi |f|^p$$

for every good cube Π. Without loss of generality we can assume that

$$\Pi = [-\frac{1}{2}, \frac{1}{2}]^d$$

by considering a shift $f(x-n)$ which has

$$supp \widehat{f(x-n)} \subset [-\frac{b_1}{2}, \frac{b_1}{2}] \times \dots \times [-\frac{b_d}{2}, \frac{b_d}{2}].$$

Let $z = (z_1, ..., z_d) \in \mathbb{C}^d$ be a complex vector. Let $D_j(0, R) = \{z_j \in \mathbb{C} : |z_j| \leq R\}$ be a disk in the complex plane $\mathbb{C}$. If

$$z \in D_1(0, R) \times ... \times D_d(0, R) \subset D_1(x_1, R + \frac{1}{2}) \times ... \times D_d(x_d, R + \frac{1}{2})$$

then

$$\begin{aligned} |f(z)| &\leq \sum_{\alpha} \frac{|f^{(\alpha)}(x)|}{\alpha!} \cdot |z - x|^{\alpha} \\ &\leq \sum_{\alpha} 4^{\frac{d}{p}} \frac{(R + \frac{1}{2})^{|\alpha|} \cdot (Cb)^{\alpha}}{\alpha!} \|f\|_{L^p(\Pi)} \\ &= 4^{\frac{d}{p}} \exp(C(R + \frac{1}{2}) \sum_{j=1}^{d} b_j) \cdot \|f\|_{L^p(\Pi)}. \end{aligned} \tag{1.7}$$

We can assume that $\int_{\Pi} |f|^p = 1$. Then $\exists y \in \Pi$ such that $|f(y)| \geq 1$. Following an idea of F. Nazarov we can choose spherical coordinates centered at y and find a segment $I \in \Pi$, $y \in I$ and $\frac{|E \cap I|}{|I|} \geq C(d)\gamma$:

$$\begin{aligned} \gamma &\leq |E \cap \Pi| \\ &= \int_{\Pi} \chi_{E \cap \Pi}(x) dx \\ &= \int_{|\xi|=1} \int_{r=0}^{r(\xi)} \chi_{E \cap \Pi}(y + r\xi) r^{d-1} dr d\sigma(\xi). \end{aligned} \tag{1.8}$$

It follows from (1.8) that $\exists \eta \in S^{d-1}$ such that

$$\gamma \leq \sigma_{d-1} \int_{r=0}^{r(\eta)} \chi_{E \cap \Pi}(y + r\eta) r^{d-1} dr$$

where $\sigma_{d-1} = |S^{d-1}|$. Let I be the longest interval in Π centered at y in the direction of η:

$$I = y + t|I|\eta, \quad 0 \leq t \leq 1. \tag{1.9}$$

It is clear that $|I| \le d^{1/2}$. Therefore,

$$\begin{aligned}\frac{|E\cap I|}{|I|} &\ge \frac{\gamma}{\sigma_{d-1}\cdot d^{d/2}}\\ &\ge \frac{\gamma}{C^d}.\end{aligned} \tag{1.10}$$

It follows from (1.10) that

$$\begin{aligned}|\{t\in[0,1]: y+t|I|\eta \in E\cap I\}| &= \frac{|E\cap I|}{|I|}\\ &\ge \frac{\gamma}{C^d}.\end{aligned} \tag{1.11}$$

Define an analytic function of one complex variable $w \in \mathbb{C}$

$$F(w) = f(y + w|I|\eta). \tag{1.12}$$

Then $|F(0)| = |f(y)| \ge 1$. If $w \in D(0, R)$ then

$$z = y + w|I|\eta \in D_1(0, R + \frac{1}{2}) \times \ldots \times D_d(0, R + \frac{1}{2})$$

since $|y_j| \le \frac{1}{2}$ and $||I|\eta_j| \le 1$ for $j = 1, \ldots, d$. Then it follows from (1.7) that

$$|F(w)| \le 4^{\frac{d}{p}} \exp(C(R+1)\sum_{j=1}^{d} b_j). \tag{1.13}$$

Now we will give a local estimate for analytic functions of one complex variable.

Lemma 1: *Let $\phi(z)$ be analytic in $D(0,5)$ and let I be an interval of length 1 such that $0 \in I$ and let $E \subset I$ be a measurable set of positive measure. If $|\phi(0)| \ge 1$ and $M = \max_{|z|\le 4} |\phi(z)|$, then*

$$\sup_{x\in I} |\phi(x)| \le \left(\frac{C}{|E|}\right)^{\frac{\ln M}{\ln 2}} \sup_{x\in E} |\phi(x)|. \tag{1.14}$$

Proof of Lemma 1:

Let w_1, w_2,...w_n be the zeros of ϕ in $D(0,2)$. If

$$g(z) = \phi(z) \cdot \prod_{k=1}^{n} \frac{4 - \bar{w}_k z}{2(w_k - z)} = \phi(z) \cdot \frac{Q(z)}{P(z)}$$

then $|g(0)| \geq 1$ and $\max_{|z|\leq 2} |g(z)| \leq M$ by the property of Blaschke products. Applying Harnack's inequality to the positive harmonic function $\ln M - \ln|g(z)|$ in $D(0,2)$ we have:

$$\max_{|z|\leq 1}(\ln M - \ln|g(z)|) \leq 3\ln M.$$

Therefore,

$$\min_{|z|\leq 1} |g(z)| \geq M^{-2}.$$

This gives us

$$\frac{\max_{x\in I} |g(x)|}{\min_{x\in I} |g(x)|} \leq M^3.$$

We can give a similar estimate for Q:

$$\begin{aligned} \frac{\max_{x\in I} |Q(x)|}{\min_{x\in I} |Q(x)|} &\leq \frac{\max_{|z|\leq 1} \prod_{k=1}^{n} |4 - \bar{w}_k z|}{\min_{|z|\leq 1} \prod_{k=1}^{n} |4 - \bar{w}_k z|} \\ &\leq 3^n. \end{aligned}$$

From the Remez inequality for polynomials ([3], Theorem 5.1.1) it follows that:

$$\sup_{x\in I} |P(x)| \leq \left(\frac{4}{|E|}\right)^n \cdot \sup_{x\in E} |P(x)|.$$

Therefore,

$$\begin{aligned}
\sup_{x\in I}|\phi(x)| &\le \max_{x\in I}|g(x)|\cdot\frac{\max_{x\in I}|P(x)|}{\min_{x\in I}|Q(x)|}\\
&\le M^3\cdot 3^n\cdot\left(\frac{C}{|E|}\right)^n\cdot\min_{x\in I}|g(x)|\cdot\frac{\sup_{x\in E}|P(x)|}{\max_{x\in I}|Q(x)|}\\
&\le M^3\cdot\left(\frac{C}{|E|}\right)^n\cdot\sup_{x\in E}|\phi(x)|.
\end{aligned}$$

From Jensen's formula it follows that $n\le\frac{\ln M}{\ln 2}$. Therefore,

$$\sup_{x\in I}|\phi(x)|\le\left(\frac{C}{|E|}\right)^{\frac{\ln M}{\ln 2}}\sup_{x\in E}|\phi(x)|.$$

□

Now we are in a position to proceed with the proof of our theorem. Let $M=\max_{|w|\le 4}|F(w)|$. Applying **Lemma 1** to $F(w)$, interval $[0,1]$ and set $\{t\in[0,1]:y+t|I|\eta\in E\cap I\}$ and using (1.11), we have that

$$\begin{aligned}
\sup_{E\cap\Pi}|f(x)| &\ge \sup_{E\cap I}|f(x)|\\
&\ge \left(\frac{|E\cap\Pi|}{C^d}\right)^{\frac{\ln M}{\ln 2}}\|f\|_{L^p(\Pi)}.
\end{aligned}$$

Therefore,

$$|\{x\in\Pi:|f(x)|<\left(\frac{\epsilon}{C^d}\right)^{\frac{\ln M}{\ln 2}}\|f\|_{L^p(\Pi)}\}|\le\epsilon\quad\forall\epsilon>0.$$

If we put $\epsilon=\frac{|E\cap\Pi|}{2}$ then

$$|\{x\in\Pi:|f(x)|<\left(\frac{|E\cap\Pi|}{2C^d}\right)^{\frac{\ln M}{\ln 2}}\|f\|_{L^p(\Pi)}\}|\le\frac{|E\cap\Pi|}{2}.$$

Therefore,

$$\begin{aligned}
\int_{E\cap\Pi} |f|^p &\geq \int_{E\cap\Pi} \chi_{|f|\geq\left(\frac{|E\cap\Pi|}{2C^d}\right)^{\frac{\ln M}{\ln 2}}\|f\|_{L^p(\Pi)}} \cdot |f|^p \\
&\geq \frac{|E\cap\Pi|}{2}\cdot\left(\frac{|E\cap\Pi|}{2C^d}\right)^{p\frac{\ln M}{\ln 2}} \|f\|^p_{L^p(\Pi)} \\
&\geq \left(\frac{|E\cap\Pi|}{2C^d}\right)^{p\frac{\ln M}{\ln 2}+1}\cdot\int_{\Pi} |f|^p.
\end{aligned}$$

Using (1.13) we get

$$M \leq 4^{\frac{d}{p}} \exp(5C\sum_{j=1}^{d} b_j).$$

Hence we obtain the desired local estimate

$$\int_{E\cap\Pi} |f|^p \geq \left(\frac{\gamma}{C^d}\right)^{3d+Cp\sum\limits_{j=1}^{d} b_j} \int_{\Pi} |f|^p \tag{1.15}$$

for every good cube Π. Summing (1.15) over all good cubes and applying (1.4), we have

$$\begin{aligned}
\int_{E} |f|^p &\geq \int_{E\cap \bigcup\limits_{\Pi \text{ is good}} \Pi} |f|^p \\
&\geq \left(\frac{\gamma}{C^d}\right)^{3d+Cp\sum\limits_{j=1}^{d} b_j} \cdot \int_{\bigcup\limits_{\Pi \text{ is good}} \Pi} |f|^p \\
&\geq \frac{1}{2}\left(\frac{\gamma}{C^d}\right)^{3d+Cp\sum\limits_{j=1}^{d} b_j} \cdot \int |f|^p.
\end{aligned}$$

Replacing $\sum_{j=1}^{d} b_j$ with $\sum_{j=1}^{d} a_j b_j$ and choosing a new C, we have:

$$\int_E |f|^p \geq \left(\frac{\gamma}{C^d}\right)^{3d+Cp\sum_{j=1}^{d} a_j b_j} \cdot \int |f|^p.$$

If $p = \infty$ then the proof is almost the same: $\|f\|_{L^\infty(\bigcup_{\Pi \text{ is good}} \Pi)} = \|f\|_\infty$. If Π is good then $\|f\|_{L^\infty(E\cap\Pi)} \geq \left(\frac{\gamma}{C^d}\right)^{2d+C\sum_{j=1}^{d} a_j b_j} \cdot \|f\|_{L^\infty(\Pi)}$. Hence

$$\|f\|_{L^\infty(E)} \geq \left(\frac{\gamma}{C}\right)^{2d+C\sum_{j=1}^{d} a_j b_j} \cdot \|f\|_\infty.$$

□

If we keep track of all the constants and do the calculations more accurately, then we can get that if $d = 1$ and $p \in [1, \infty)$:

$$\|f\|_{L^p(E)} \geq \left(\frac{\gamma}{300}\right)^{33ab+\frac{2}{p}} \cdot \|f\|_p,$$

if $p = \infty$:

$$\|f\|_{L^\infty(E)} \geq \left(\frac{\gamma}{100}\right)^{33ab+1} \cdot \|f\|_\infty.$$

However, if we try to minimize the factor in front of ab, then we can get the following estimate:

$$\|f\|_{L^p(E)} \geq \left(\frac{\gamma}{C}\right)^{(\frac{(1+\epsilon)}{2}+\epsilon)\cdot ab+A(\epsilon)} \cdot \|f\|_p \quad \forall \epsilon > 0.$$

The following example suggests that the right behavior of the estimate in the Logvinenko-Sereda Theorem is γ to the power of a linear function of $\sum_{j=1}^{d} a_j b_j$:
Let J be a parallelepiped with sides of length $b_1, ..., b_d$ parallel to coordinate axes and let Π be a cube with sides of length 2 parallel to coordinate axes. Denote $\mathbf{b} = (b_1, ..., b_d)$. Let $T = Proj_{\mathbf{b}}(1, 1, ..., 1) = \sum_{j=1}^{d} b_j / |\mathbf{b}|$. Choose another system of

coordinates $y_1, ..., y_d$ with axis y_1 parallel to $\mathbf{b}$. Define

$$\hat{f}(y) = \hat{f}_1(y_1)\hat{\psi}(y_2, ..., y_d)$$

where

$$\hat{f}_1(y_1) = (\chi_{[-\frac{2\pi}{T}, \frac{2\pi}{T}]} * ... * \chi_{[-\frac{2\pi}{T}, \frac{2\pi}{T}]})(y_1)$$

with the number of convolutions equal to $[|\mathbf{b}|T/8\pi] = [\sum_{j=1}^{d} b_j/8\pi]$ and $\hat{\psi}$ is supported in a small enough ball in $\mathbb{R}^{d-1}$. Then $\hat{f}$ lives in a cylinder with axis along the main diagonal of J parallel to $\mathbf{b}$ with small enough radius so that it is inside of J. We have

$$f(y) = \left(\frac{\sin(2\pi y_1/T)}{y_1}\right)^{[\sum_{j=1}^{d} b_j/8\pi]} \cdot \psi(y_2, ..., y_d).$$

Let $E = E_1 \times \mathbb{R}^{d-1}$ where E_1 is a periodic subset of y_1 axis with period T:

$$E_1 \cap [-\frac{T}{2}, \frac{T}{2}] = [-\frac{T}{2}, -\frac{T}{2} + \frac{\gamma}{2}] \cap [\frac{T}{2} - \frac{\gamma}{2}, \frac{T}{2}].$$

Then

$$|E \cap (\Pi + x)| \geq C(d)\gamma|\Pi| \quad \forall x \in \mathbb{R}^d.$$

If $\sum_{j=1}^{d} b_j$ is large enough we have:

$$\|f_1\|_{L^p(E_1)} \leq \left(\frac{\gamma}{C}\right)^{-1+\sum_{j=1}^{d} b_j/8\pi} \|f_1\|_p$$

and therefore

$$\begin{aligned} \|f\|_{L^p(E)} &= \|f_1\|_{L^p(E_1)}\|\psi\|_p \\ &\leq \left(\frac{\gamma}{C}\right)^{-1+\sum_{j=1}^{d} b_j/8\pi} \|f_1\|_p\|\psi\|_p \\ &= \left(\frac{\gamma}{C}\right)^{-1+\sum_{j=1}^{d} b_j/8\pi} \|f\|_p. \end{aligned}$$

Remark 1: when $\sum_{j=1}^{d} a_j b_j$ is sufficiently small the proof of the theorem is much simpler: if $\sum_{j=1}^{d} a_j b_j \le 1/p$ then $\|f\|_{L^p(E)} \ge (\frac{\gamma}{C})^{\frac{1}{p}} \|f\|_p$ for certain p. This can be proven very easily. If $p = 1$ we have

$$|\Pi| \cdot |f(x)| \ \ge \ \int_{\Pi} |f| - \sum_{\alpha \ne 0 : \alpha_j \le 1} a^{\alpha} \int_{\Pi} |f^{(\alpha)}| \tag{1.16}$$

and

$$|\Pi| \cdot |f(x)| \ \le \ \int_{\Pi} |f| + \sum_{\alpha \ne 0 : \alpha_j \le 1} a^{\alpha} \int_{\Pi} |f^{(\alpha)}| \tag{1.17}$$

where $x \in \Pi$, by induction on the dimension d. Hence, using (1.16), we have

$$|\Pi| \cdot \int_{E \cap \Pi} |f(x)| dx \ \ge \ \gamma |\Pi| \left(\int_{\Pi} |f| - \sum_{\alpha \ne 0 : \alpha_j \le 1} a^{\alpha} \int_{\Pi} |f^{(\alpha)}| \right).$$

Therefore,

$$\frac{1}{\gamma} \cdot \int_{E \cap \Pi} |f(x)| dx \ \ge \ \int_{\Pi} |f| - \sum_{\alpha \ne 0 : \alpha_j \le 1} a^{\alpha} \int_{\Pi} |f^{(\alpha)}|.$$

Summing over all parallelepipeds Π we have

$$\begin{aligned} \frac{1}{\gamma} \cdot \int_{E} |f| \ &\ge \ \int |f| - \sum_{\alpha \ne 0 : \alpha_j \le 1} a^{\alpha} \int |f^{(\alpha)}| \\ &\ge \ \int |f| - \sum_{\alpha \ne 0 : \alpha_j \le 1} (\frac{b}{2})^{\alpha} a^{\alpha} \int |f| \\ &= \ (2 - \prod_{j=1}^{d} (1 + a_j b_j / 2)) \int |f| \\ &\ge \ (2 - e^{\sum_{j=1}^{d} a_j b_j / 2}) \int |f| \\ &\ge \ C \int |f|. \end{aligned}$$

If p is an integer then repeat the previous argument with f^p instead of f and take into account that $\operatorname{supp}\widehat{f^p} \in p \cdot J$. We can also repeat the above argument when $p \geq d$ where p is not necessarily an integer since if $|\alpha| \leq p$ then $\int |f^{p(\alpha)}| \leq (pb/2)^\alpha \int |f^p|$ (combine Holder's and Bernstein's inequalities to get it). It is even easier to prove that if $p = \infty$ and $\sum_{j=1}^{d} a_j b_j \leq 1$, then $\|f\|_{L^\infty(E)} \geq \frac{1}{2}\|f\|_\infty$.

Alternatively, using Holder's inequality, we can obtain

$$\frac{1}{\gamma} \cdot \int_{E\cap\Pi} |f|^p \geq \frac{1}{2^{p-1}} \int_{\Pi} |f|^p - \left(\sum_{\alpha\neq 0:\alpha_j\leq 1} a^{p\alpha} c_\alpha^{-p} \int_{\Pi} |f^{(\alpha)}|^p \right) \cdot \left(\sum_{\alpha\neq 0:\alpha_j\leq 1} c_\alpha^{p'} \right)^{\frac{p}{p'}}.$$

Put $c_\alpha = a^{\alpha/2}(b/2)^{\alpha/2}$. Then we are done provided $(e^{\sum_{j=1}^{d} a_j b_j p/4} - 1)^2 \leq \frac{1}{2^p}$. It shows that if $\sum_{j=1}^{d} a_j b_j \leq C$ then $\|f\|_{L^p(E)} \geq \frac{\gamma^{\frac{1}{p}}}{2}\|f\|_p$ for $p \in [1, \infty]$.

In a similar way using inequalities analogous to (1.17) we can treat the case when $1-\gamma$ is sufficiently small depending on $\sum_{j=1}^{d} a_j b_j$: if $p \in [1, \infty)$ and $1 - \gamma \leq e^{-p(C + \sum_{j=1}^{d} a_j b_j/2)}$ then $\|f\|^p_{L^p(E)} \geq \frac{1}{2}\|f\|_p^p$.

When $d = 1$ we get better results.

1.3 Case of finitely many parallelepipeds

Proof of Theorem 2:

Let

$$J = [-\frac{b_1}{2}, \frac{b_1}{2}] \times \ldots \times [-\frac{b_d}{2}, \frac{b_d}{2}].$$

Then $J_k = J + \lambda_k$ where $\lambda_k \in \mathbb{R}^d$, $k = 1, 2, \ldots, n$. Denote by

$$\tilde{J}_k = 2J + \lambda_k, \quad k = 1, 2, \ldots, n.$$

First we will prove a special case of **Theorem 2:**

Theorem 2′: *if* $\tilde{J}_k$, $k = 1, 2, ..., n$, *are disjoint then*

$$\|f\|_{L^p(E)} \geq c(\gamma, n, \mathbf{a}\cdot\mathbf{b}, d, p) \cdot \|f\|_p$$

where $c(\gamma, n, \mathbf{a}\cdot\mathbf{b}, d, p) = \left(\frac{\gamma}{C^d}\right)^{\mathbf{a}\cdot\mathbf{b}\left(\frac{C^d}{\gamma}\right)^n + n - \frac{p-1}{p}}$.

Proof of Theorem 2′:

Let $\hat{f}(x) = \sum_{k=1}^{n} \hat{f}_k(x - \lambda_k)$ where $\operatorname{supp}\hat{f}_k \subset J$ and $f(x) = \sum_{k=1}^{n} f_k(x) e^{i\lambda_k\cdot \mathbf{x}}$. The following lemma gives an estimate of $\|f_k\|_p$ from above.

Lemma 2:

$$\|f_k\|_p \leq C^d \|f\|_p \quad (k = 1, 2, ..., n). \tag{1.18}$$

Proof of Lemma 2:

Let ϕ be a Schwartz function such that $\operatorname{supp}\hat{\phi} \subset [-1, 1]^d$ and $\hat{\phi}(x) = 1$ for $x \in [-\frac{1}{2}, \frac{1}{2}]^d$. Then $\hat{f}_k(x) = \hat{f}\cdot\hat{\phi}(\frac{(x-\lambda_k)_1}{b_1}, ..., \frac{(x-\lambda_k)_d}{b_d})$. Therefore, $f_k = f * (|J|\phi(b_1x_1, ..., b_dx_d)e^{i\lambda_k\cdot\mathbf{x}})$. Applying Young's inequality we have $\|f_k\|_p \leq \|f\|_p \cdot \|\phi\|_1$. □

We will also need the following auxiliary lemma on local estimates of generalized trigonometric polynomials of one real variable:

Lemma 3: *if* $r(t) = \sum_{k=1}^{n} p_k(t) e^{i\mu_k t}$ *where* $p_k(t)$ *is a polynomial of degree* $\leq m - 1$ *and* $E \subset I$ *is measurable subset of an interval* I *with* $|E| > 0$, *then*

$$\|r\|_{L^\infty(I)} \leq \left(\frac{C|I|}{|E|}\right)^{nm-1} \cdot \|r\|_{L^\infty(E)}. \tag{1.19}$$

Proof of Lemma 3:

If g is a pure trigonometric polynomial of order n, i.e.,

$$g(t) = \sum_{k=1}^{n} c_k e^{i\mu_k t},$$

then it follows from a theorem on trigonometric polynomials by F. Nazarov ([13], Theorem 1.5) that

$$\|g\|_{L^\infty(I)} \leq \left(\frac{C|I|}{|E|}\right)^{n-1} \cdot \|g\|_{L^\infty(E)}. \tag{1.20}$$

If $p(t) = \sum_{l=0}^{m-1} a_l t^l$ is a polynomial of degree $m-1$, then it can be approximated uniformly on an interval with a trigonometric polynomial of order $\leq m$

$$\tilde{p}(t) = \sum_{l=0}^{m-1} a_l \left(\frac{e^{i\mu t} - 1}{i\mu}\right)^l = \sum_{l=0}^{m-1} \tilde{a}_l e^{il\mu t}$$

because $t = \lim_{\mu \to 0} \frac{e^{i\mu t}-1}{i\mu}$ uniformly on an interval. Applying (1.20) to the trigonometric polynomial of order mn

$$\tilde{r}(t) = \sum_{k=1}^{n} \tilde{p}_k(t) e^{i\mu_k t}$$

and taking the limit we have the desired result:

$$\|r\|_{L^\infty(I)} \leq \left(\frac{C|I|}{|E|}\right)^{nm-1} \cdot \|r\|_{L^\infty(E)}.$$

□

Now we are in a position to proceed with the proof of **Theorem 2′**.

First we assume that $p \in [1, \infty)$. Divide the whole $\mathbb{R}^d$ into parallelepipeds Π.

Consider one of them. Suppose $|f|$ attains its maximum in Π at point $y \in \Pi$. We can find an interval $I \in \Pi$, $y \in \Pi$ and $\frac{|E \cap I|}{|I|} \geq C^d \gamma$ (see argument before **Lemma 1**):

$$I = y + t|I|\eta, \quad 0 \leq t \leq 1. \tag{1.21}$$

$$\begin{aligned} |\{t \in [0,1] : y + t|I|\eta \in E \cap I\}| &= \frac{|E \cap I|}{|I|} \\ &\geq \frac{\gamma}{C^d}. \end{aligned} \tag{1.22}$$

Define

$$\begin{aligned} F(t) &= f(y + t|I|\eta) \\ &= \sum_{k=1}^{n} f_k(y + t|I|\eta) e^{i\lambda_k \cdot (y + t|I|\eta)}. \end{aligned} \tag{1.23}$$

Using the Taylor formula

$$\begin{aligned} g(t) &= \sum_{l=0}^{m-1} \frac{g^l(0)}{l!} t^l + \frac{1}{(m-1)!} \int_0^t g^{(m)}(s)(t-s)^{m-1} ds \\ &= p(t) + \frac{1}{(m-1)!} \int_0^t g^{(m)}(s)(t-s)^{m-1} ds \end{aligned}$$

where $p(t)$ is a polynomial of degree $m-1$, we obtain from (1.23)

$$\begin{aligned} F(t) &= \sum_{k=1}^{n} p_k(t) e^{i\mu_k t} + \frac{1}{(m-1)!} \sum_{k=1}^{n} e^{i\lambda_k \cdot (y + t|I|\eta)} \int_0^t g_k^{(m)}(s)(t-s)^{m-1} ds \\ &= r(t) + T(t) \end{aligned}$$

where $g_k(s) = f_k(y + s|I|\eta)$. Applying (1.17) to $f_k^{(\alpha)}$ we have that

$$\begin{aligned}\max_{t\in[0,1]}|T(t)| &\leq \frac{1}{(m)!}\sum_{k=1}^{n}\sum_{|\alpha|=m}\frac{m!}{\alpha!}\max_{x\in\Pi}|f_k^{(\alpha)}(x)|a^{\alpha} \\ &\leq \frac{1}{(m)!}\sum_{k=1}^{n}\sum_{|\alpha|=m}\frac{m!}{\alpha!}\sum_{\beta,\beta_i\leq 1}\|f_k^{(\alpha+\beta)}\|_{L^1(\Pi)}a^{\alpha+\beta}/|\Pi|.\end{aligned}$$

Denote the last quantity by M. Applying Holder's inequality we have

$$M^p|\Pi| \leq \frac{(2^d m^d n)^{p-1}}{[m!]^p}\sum_{k=1}^{n}\sum_{|\alpha|=m}\sum_{\beta,\beta_i\leq 1}\left(\frac{m!a^{\alpha+\beta}}{\alpha!}\right)^p\|f_k^{(\alpha+\beta)}\|^p_{L^p(\Pi)}. \tag{1.24}$$

Summing (1.24) over all parallelepipeds Π

$$\begin{aligned}\sum_{\Pi}M^p|\Pi| &\leq \frac{(2^d m^d n)^{p-1}}{[m!]^p}\sum_{k=1}^{n}\sum_{|\alpha|=m}\sum_{\beta,\beta_i\leq 1}\left(\frac{m!a^{\alpha+\beta}}{\alpha!}\right)^p\|f_k^{(\alpha+\beta)}\|^p_p \\ &\leq \frac{(2^d m^d n)^{p-1}}{[m!]^p}\sum_{k=1}^{n}\sum_{|\alpha|=m}\sum_{\beta,\beta_i\leq 1}\left(\frac{m!a^{\alpha+\beta}}{\alpha!}\right)^p(Cb)^{(\alpha+\beta)p}\|f_k\|^p_p \\ &\leq \frac{(2^d m^d n)^{p-1}(C\mathbf{a}\cdot\mathbf{b})^{mp}e^{Cp\mathbf{a}\cdot\mathbf{b}}}{[m!]^p}\sum_{k=1}^{n}\|f_k\|^p_p \\ &\leq \frac{(Cmn)^{dp}(C\mathbf{a}\cdot\mathbf{b})^{mp}e^{Cp\mathbf{a}\cdot\mathbf{b}}}{[m!]^p}\|f\|^p_p\end{aligned} \tag{1.25}$$

where the last inequality follows from (1.18).

Lemma 3 applied to r, interval [0,1] and subset $E_1 = \{t \in [0,1] : y + t|I|\eta \in E \cap I\}$ gives the following local estimate:

$$\begin{aligned}\|f\|_{L^\infty(\Pi)} &= \|F\|_{L^\infty([0,1])} \\ &\leq \|r\|_{L^\infty([0,1])} + M \\ &\leq \left(\frac{C^d}{\gamma}\right)^{nm-1}\cdot\|r\|_{L^\infty(E_1)} + M \\ &\leq \left(\frac{C^d}{\gamma}\right)^{nm-1}\cdot\|F\|_{L^\infty(E_1)} + \left(\frac{C^d}{\gamma}\right)^{nm-1}\cdot M \\ &\leq \left(\frac{C^d}{\gamma}\right)^{nm-1}\cdot\|f\|_{L^\infty(E)} + \left(\frac{C^d}{\gamma}\right)^{nm-1}\cdot M.\end{aligned} \tag{1.26}$$

An argument similar to the one after **Lemma 1** shows that the following can be obtained from (1.26)

$$\|f\|^p_{L^p(\Pi)} \leq 2^{p-1}\left(\frac{C^d}{\gamma}\right)^{pnm-(p-1)} \cdot \|f\|^p_{L^p(E\cap\Pi)} + 2^{p-1}\left(\frac{C^d}{\gamma}\right)^{pnm-(p-1)} \cdot M^p|\Pi|.$$

Summing over all parallelepipeds Π we have:

$$\begin{aligned}\int |f|^p &\leq \left(\frac{C^d}{\gamma}\right)^{pnm-(p-1)} \cdot \|f\|^p_{L^p(E)} + \left(\frac{C^d}{\gamma}\right)^{pnm-(p-1)} \cdot \sum_{\Pi} M^p|\Pi| \\ &\leq \left(\frac{C^d}{\gamma}\right)^{pnm-(p-1)} \cdot \|f\|^p_{L^p(E)} + \left(\frac{C^d}{\gamma}\right)^{pnm-(p-1)} \cdot \frac{(Cmn)^{dp}(C\mathbf{a}\cdot\mathbf{b})^{mp}e^{Cp\mathbf{a}\cdot\mathbf{b}}}{[m!]^p}\|f\|^p_p \\ &\leq \left(\frac{C^d}{\gamma}\right)^{pnm-(p-1)} \cdot \|f\|^p_{L^p(E)} + \left(\frac{C^d}{\gamma}\right)^{pnm} \frac{(\mathbf{a}\cdot\mathbf{b})^{mp}e^{Cp\mathbf{a}\cdot\mathbf{b}}}{m^{pm}}\|f\|^p_p. \qquad (1.27)\end{aligned}$$

The second inequality follows from (1.25). The last inequality is due to Stirling's formula for $m!$ and the fact that $t \leq 2^t$.

Choose m such that it is a positive integer and $\left(\frac{C^d}{\gamma}\right)^n \frac{\mathbf{a}\cdot\mathbf{b}}{m} \leq \frac{1}{C}$, e.g., $m = 1 + [\mathbf{a}\cdot\mathbf{b}\left(\frac{C^d}{\gamma}\right)^n]$ with so large $C > 0$ that the factor in front of $\|f\|^p_p$ in the last inequality in (1.27) is less than $\frac{1}{2}$. Therefore,

$$\begin{aligned}\int |f|^p &\leq \left(\frac{C^d}{\gamma}\right)^{pn(1+\mathbf{a}\cdot\mathbf{b}\left(\frac{C^d}{\gamma}\right)^n)-(p-1)} \cdot \int_E |f|^p \\ &\leq \left(\frac{C^d}{\gamma}\right)^{p\mathbf{a}\cdot\mathbf{b}\left(\frac{C^d}{\gamma}\right)^n+pn-(p-1)} \cdot \int_E |f|^p.\end{aligned}$$

The proof for $p = \infty$ is similar and even simpler. □

Now we can proceed with the proof of **Theorem 2**. We will apply induction on n. For $n = 1$ the theorem follows from **Theorem 2′** or the usual Logvinenko-Sereda Theorem. Suppose the statement is true for $n \leq m$. Let $n = m + 1$.

If $\tilde{J}_k$, $k = 1, 2, ..., n$, are disjoint then the result follows from **Theorem 2′**.

If $\tilde{J}_k$ and $\tilde{J}_l$ intersect each other for some k and l, then we can replace J with $3J$ reducing the number of frequencies λ_k and replacing $\mathbf{b}$ with $3\mathbf{b}$. Therefore, by induction:

$$\begin{aligned} \|f\|_{L^p(E)} &\geq \left(\frac{C^d}{\gamma}\right)^{-3\mathbf{a}\cdot\mathbf{b}\left(\frac{C^d}{\gamma}\right)^m - m + \frac{p-1}{p}} \cdot \|f\|_p \\ &\geq \left(\frac{C^d}{\gamma}\right)^{-\mathbf{a}\cdot\mathbf{b}\left(\frac{C^d}{\gamma}\right)^{(m+1)} - (m+1) + \frac{p-1}{p}} \cdot \|f\|_p. \end{aligned}$$

□

The purpose of this theorem is to prove the existence of a constant $c(\gamma, n, \mathbf{ab}, d, p) > 0$ depending only on the number of parallelepipeds and not how they are placed rather than to get the best possible estimate. We can conjecture that the right behavior of the constant $c(\gamma, n, \mathbf{ab}, d, p)$ is the following:

$$c(\gamma, n, \mathbf{a}\cdot\mathbf{b}, d, p) = \left(\frac{\gamma}{C^d}\right)^{Cn\mathbf{a}\cdot\mathbf{b} + n - \frac{p-1}{p}}. \tag{1.28}$$

The estimate (1.28) is suggested by an example similar to the one after **Theorem 1**: choose $J_k = J + k \cdot (1 - \epsilon)\mathbf{b}$, $k = 1, ..., n$, so that neighborhoods of two corners of J_k and J_{k+1}, $k = 1, ..., n-1$, intersect. Then we can choose $\hat{f}$ supported in a cylinder with axis along $\mathbf{b}$ of length $n(1-\epsilon)|\mathbf{b}|$. The rest is the same as in the former example.

1.4 Case of infinitely many lacunary intervals

1.4.1 Conjectured theorem

The goal is to generalize a result of F. Nazarov for the Zygmund theorem on lacunary trigonometric series ([13], Theorem 3.6). Instead of a trigonometric series we will consider the following sum $f(x) = \sum_{k=-\infty}^{\infty} f_k(x)e^{i\lambda_k x}$ where $f_k \in L^2(\mathbb{R})$ with supp $\hat{f}_k \subset [-\frac{b}{2}, \frac{b}{2}]$ and $\{\lambda_k\}_{k=-\infty}^{\infty}$ is a lacunary sequence. A sequence of real numbers $\Lambda = \{\lambda_k\}_{k=-\infty}^{\infty}$ is lacunary if there exists $N < \infty$ such that

$$N = \max_{k \neq l} Card\{(k', l') : |\lambda_k - \lambda_l - (\lambda_{k'} - \lambda_{l'})| \leq 1\}. \quad (1.29)$$

Let $I_{k,l} = [\lambda_k - \lambda_l - \frac{1}{2}, \lambda_k - \lambda_l + \frac{1}{2}]$. Then

$$N = \max_{k \neq l} Card\{(k', l') : I_{k,l} \cap I_{k',l'} \neq \emptyset\}. \quad (1.30)$$

Another equivalent definition:

$$\begin{aligned} N &= \max_x Card\{(k,l), k \neq l : x \in I_{k,l}\} \\ &= \max_x Card\{(k,l), k \neq l : \lambda_k - \lambda_l \in [x - \frac{1}{2}, x + \frac{1}{2}]\}. \end{aligned} \quad (1.31)$$

The conjectured theorem is the following:

Theorem 3: *let* $\Lambda = \{\lambda_k\}_{k=-\infty}^{\infty}$ *be lacunary and let* E *be "relatively dense." If* $f \in L^2$ *and supp* $\hat{f} \subset \bigcup_{k=-\infty}^{\infty} [\lambda_k - \frac{b}{2}, \lambda_k + \frac{b}{2}]$ *then*

$$\|f\|_{L^2(E)} \geq C(\gamma, N, a, b) \cdot \|f\|_2.$$

We will prove some partial results.

1.4.2 Large E, small support

Proposition 1. Theorem 3 *is true if* $1 - \gamma$ *is small enough depending on* N, a, b:

if $1-\gamma \leq \frac{1}{16(1+ab)^2(1+1/a)N}$ *then*

$$\int_E |f|^2 \;\geq\; \frac{1}{2}\int |f|^2.$$

Proof of Proposition 1: We can choose f_k such that $\mathrm{supp}\hat{f}_k \in [-\frac{b}{2}, \frac{b}{2}]$, supp $\hat{f}_k(x-\lambda_k)$ are disjoint and $\hat{f}(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k(x-\lambda_k)$. Then

$$f(x) = \sum_{k=-\infty}^{\infty} f_k(x)e^{i\lambda_k x}$$

and

$$\|f\|_2^2 = \sum_{k=-\infty}^{\infty} \|f_k\|_2^2.$$

Divide the whole real line into intervals I of length a each. Consider one of them. We will need to prove the following local estimate:

$$\begin{aligned}\int_{I\cap E^c} |f|^2 \;\leq\; & (1-\gamma)\sum_{k=-\infty}^{\infty}\int_I (|f_k|^2 + a|(f_k^2)'|) \\ & + \frac{1}{a}\sum_{k=-\infty}^{\infty}\sqrt{(1-\gamma)a(1+a)N}\int_I |f_k|^2 \\ & + \sum_{k=-\infty}^{\infty}\sqrt{(1-\gamma)a(1+a)N}\int_I (|f_k'|^2/(b/2) + |f_k|^2 b/2).\end{aligned}$$

We can assume that $I = [-\frac{a}{2}, \frac{a}{2}]$.

$$\int_{I\cap E^c} |f|^2 = \sum_{k=-\infty}^{\infty}\int_{I\cap E^c} |f_k|^2 + \sum_{k\neq l}\int_{I\cap E^c} f_k(x)\bar{f}_l(x)e^{i(\lambda_k-\lambda_l)x}dx. \tag{1.32}$$

The first term is bounded by

$$(1-\gamma)\sum_{k=-\infty}^{\infty}\int_I (|f_k|^2 + a|(f_k^2)'|). \tag{1.33}$$

To estimate the second term, we will rewrite it in the following way:

$$\begin{aligned}
\sum_{k\neq l}\int_{I\cap E^c}\left(f_k(y)\bar{f}_l(y)+\int_y^x (f_k(t)\bar{f}_l(t))'dt\right)e^{i(\lambda_k-\lambda_l)x}dx &= \\
\sum_{k\neq l} f_k(y)\bar{f}_l(y)\hat{\chi}_{I\cap E^c}(\lambda_k-\lambda_l) &+ \\
\sum_{k\neq l}\int_y^{\frac{a}{2}} (f_k\bar{f}_l)'(t)\hat{\chi}_{[t,a/2]\cap E^c}(\lambda_k-\lambda_l)dt &- \\
\sum_{k\neq l}\int_{-\frac{a}{2}}^{y} (f_k\bar{f}_l)'(t)\hat{\chi}_{[-a/2,t]\cap E^c}(\lambda_k-\lambda_l)dt. & \qquad (1.34)
\end{aligned}$$

The first term in this expression is bounded by

$$\sum_{k=-\infty}^{\infty}\sqrt{(1-\gamma)a(1+a)N}|f_k(y)|^2.$$

To show this we will apply Holder's inequality to the first term in (1.34):

$$\begin{aligned}
\sum_{k\neq l} f_k(y)\bar{f}_l(y)\hat{\chi}_{I\cap E^c}(\lambda_k-\lambda_l) &\leq \\
\sqrt{\sum_{k\neq l}|f_k(y)f_l(y)|^2}\cdot\sqrt{\sum_{k\neq l}|\hat{\chi}_{I\cap E^c}(\lambda_k-\lambda_l)|^2} &\leq \\
\sum_{k=-\infty}^{\infty}|f_k(y)|^2\sqrt{\sum_{k\neq l}\int_{I_{k,l}}(|\hat{\chi}_{I\cap E^c}|^2+|((\hat{\chi}_{I\cap E^c})^2)'|)} &\leq \\
\sum_{k=-\infty}^{\infty}|f_k(y)|^2\sqrt{N\int_{\bigcup_{k\neq l} I_{k,l}}(|\hat{\chi}_{I\cap E^c}|^2+|((\hat{\chi}_{I\cap E^c})^2)'|)} &\leq \\
\sum_{k=-\infty}^{\infty}|f_k(y)|^2\sqrt{N(1+a)\int|\hat{\chi}_{I\cap E^c}|^2} &= \\
\sum_{k=-\infty}^{\infty}|f_k(y)|^2\sqrt{N(1+a)(1-\gamma)a}. & \qquad (1.35)
\end{aligned}$$

We used here Bernstein's inequality: $\|g'\|_1 \leq L\|g\|_1$ if supp$\hat{g} \subset [-L, L]$ and that

$\text{supp}(\widehat{\chi_{I\cap E^c}})^2 \in [-a,a]$. In a similar way we can bound the second term in (1.34) by

$$\int_y^{\frac{a}{2}} \sqrt{4(\sum_{k=-\infty}^{\infty} |f_k'|^2)(\sum_{k=-\infty}^{\infty} |f_k|^2) \cdot \sqrt{\sum_{k\neq l} |\hat{\chi}_{[t,a/2]\cap E^c}(\lambda_k - \lambda_l)|^2 dt}} \leq$$

$$\sum_{k=-\infty}^{\infty} \int_y^{\frac{a}{2}} (|f_k'|^2/(b/2) + |f_k|^2 b/2)\sqrt{N(1+a)(1-\gamma)a}. \tag{1.36}$$

The third term in (1.34) is treated analogously. Integrating (1.32) over $y \in I$ and applying (1.33), (1.35) and (1.36) we obtain

$$\begin{aligned} \int_{I\cap E^c} |f|^2 \;&\leq\; (1-\gamma)\sum_{k=-\infty}^{\infty} \int_I (|f_k|^2 + a|(f_k^2)'|) \\ &+ \frac{1}{a}\sum_{k=-\infty}^{\infty} \sqrt{(1-\gamma)a(1+a)N} \int_I |f_k|^2 \\ &+ \sum_{k=-\infty}^{\infty} \sqrt{(1-\gamma)a(1+a)N} \int_I (|f_k'|^2/(b/2) + |f_k|^2 b/2). \end{aligned} \tag{1.37}$$

Summing (1.37) over all intervals I and applying Bernstein's inequality, we get

$$\int_{E^c} |f|^2 \;\leq\; (1+ab)(1-\gamma+\sqrt{(1-\gamma)(1+1/a)N}) \int |f|^2. \tag{1.38}$$

If $1-\gamma \leq \frac{1}{16(1+ab)^2(1+1/a)N}$ then the factor in the right side of (1.38) is less than $\frac{1}{2}$. Therefore,

$$\int_E |f|^2 \;\geq\; \frac{1}{2}\int |f|^2.$$

□

Remark:

1. We can always take a larger a and try to minimize $(1+ab)^2(1+1/a)$. Then we will get $C(1+ab)^2(1+\min(b,1/a))$, so $1-\gamma \leq \frac{1}{C(1+ab)^2(1+\min(b,1/a))}$.

2. Using **Theorem 4** with $m = 0$ we can prove in a similar way that **Theorem 3** holds if b is sufficiently small depending on γ, a and N:
if $b(1+a) \le e^{-(\ln CN)N^2(\frac{C}{\gamma})^C}$ *then*

$$\int_E |f|^2 \ \ge \ e^{-(\ln CN)N^2(\frac{C}{\gamma})^C} \int |f|^2.$$

Sketch of the proof: We can assume that $a \ge C$ (otherwise if $a < C$ we can put $a = C$ and replace γ with $\gamma/2$). Rescaling we can assume that $|E \cap I| \ge \gamma$ for every interval I of length 1 and replace Λ with $a\Lambda$ and b with ab.

$$\begin{aligned}
\int_{E\cap I} |f(x)|^2 dx &= \int_{E\cap I} |\sum_{k=-\infty}^{\infty} f_k(y)e^{i\lambda_k x} + \int_y^x f_k'(t)dte^{i\lambda_k x}|^2 dx \\
&\ge \int_{E\cap I} |\sum_{k=-\infty}^{\infty} f_k(y)e^{i\lambda_k x}|^2 dx - \int_I |\sum_{k=-\infty}^{\infty} \int_y^x f_k'(t)dte^{i\lambda_k x}|^2 dx \\
&\ge e^{-(\ln CN)N^2(\frac{C}{\gamma})^C} \sum_{k=-\infty}^{\infty} |f_k(y)|^2 - C\sqrt{N} \sum_{k=-\infty}^{\infty} \int_I |f_k'|^2.
\end{aligned}$$

Integrating over $y \in I$ and summing over all intervals I we get

$$\begin{aligned}
\int_E |f|^2 &\ge e^{-(\ln CN)N^2(\frac{C}{\gamma})^C} \sum_{k=-\infty}^{\infty} \int |f_k|^2 - C\sqrt{N} \sum_{k=-\infty}^{\infty} \int |f_k'|^2 \\
&\ge e^{-(\ln CN)N^2(\frac{C}{\gamma})^C} \int |f|^2 - C\sqrt{N}(ab)^2 \int |f|^2.
\end{aligned}$$

Thus we obtain the desired estimate if $ab \le e^{-(\ln CN)N^2(\frac{C}{\gamma})^C}$. □

1.4.3 Periodic E

The next case we consider is when E is a periodic set with period a. Let

$$E \cap [0, a] = \gamma \cdot a.$$

Proposition 2: Theorem 3 *holds in this periodic case*

$$\int_E |f|^2 \geq C(\gamma, a, b, N) \int |f|^2.$$

Proof of Proposition 2: We will start with some results on periodizations. Define a family of periodizations of a function $f \in L^1$:

$$g_t(x) = \sum_{k=-\infty}^{\infty} f(x+ka)e^{-i2\pi t(x+ka)} \tag{1.39}$$

where $t \in [-\frac{1}{2a}, \frac{1}{2a}]$. Then $g_t(x)$ is periodic with period a and its Fourier coefficients are:

$$\hat{g}_t(l) = \frac{1}{a}\hat{f}(\frac{l}{a}+t). \tag{1.40}$$

Now we assume that $f \in L^1 \cap L^2$. The next argument shows an important relation between the average of L^2 norm of periodizations and the L^2 norm of f:

$$\begin{aligned} a\int_{-\frac{1}{2a}}^{\frac{1}{2a}} \int_{E\cap[0,a]} |g_t(x)|^2 dxdt &= \sum_{k,l} \int_{E\cap[0,a]} a \int_{-\frac{1}{2a}}^{\frac{1}{2a}} f(x+ka)\bar{f}(x+l\cdot a)e^{-i2\pi t(k-l)a} dtdx \\ &= \sum_{k,l} \delta_{k\,l} \int_{E\cap[0,a]} f(x+ka)\bar{f}(x+la)dx \\ &= \sum_{k=-\infty}^{\infty} \int_{(E-ka)\cap[0,a]} |f(x+ka)|^2 dx \\ &= \int_E |f|^2. \end{aligned} \tag{1.41}$$

In particular, it follows that

$$a\int_{-\frac{1}{2a}}^{\frac{1}{2a}} \int_{[0,a]} |g_t(x)|^2 dxdt = \int |f|^2. \tag{1.42}$$

In the next lemma we extend these results to functions from L^2. For simplicity we assume that $a = 1$.

Lemma 4: *if $f \in L^2$ then there exists a family of periodic functions $g_t(x)$: $x \in [0,1]$, $t \in [-\frac{1}{2}, \frac{1}{2}]$ with period 1 such that $g_t(x) \in L^2([0,1] \times [-\frac{1}{2}, \frac{1}{2}])$,*

$$\int_{\frac{1}{2}}^{\frac{1}{2}} \int_{E \cap [0,1]} |g_t(x)|^2 dx dt = \int_E |f|^2$$

and

$$\hat{g}_t(l) = \hat{f}(l + t)$$

for almost all t.

Proof of Lemma 4: Consider a cut of f:

$$f^n(x) = \chi_{[-n,n]} f(x).$$

Since $f^n \in L^1 \cap L^2$ and converge to f in L^2 we can define corresponding families of periodizations $g_t^n(x)$ which form a Cauchy sequence in $L^2([0,1] \times [-\frac{1}{2}, \frac{1}{2}])$:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{[0,1]} |g_t^n(x) - g_t^m(x)|^2 dx dt = \int |f^n - f^m|^2.$$

Let $g_t(x)$ be the limit of $g_t^n(x)$. Then we obtain the first statement of **Lemma 4**

$$\begin{aligned}\int_{\frac{1}{2}}^{\frac{1}{2}} \int_{E\cap[0,1]} |g_t(x)|^2 dx dt &= \lim_{n\to\infty} \int_{\frac{1}{2}}^{\frac{1}{2}} \int_{E\cap[0,1]} |g_t^n(x)|^2 dx dt \\ &= \lim_{n\to\infty} \int_E |f^n|^2 \\ &= \int_E |f|^2.\end{aligned}$$

To obtain the second statement of **Lemma 4** we consider the following sum:

$$\begin{aligned}\sum_{l=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{g}_t(l) - \hat{f}(l+t)|^2 dt &= \sum_{l=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{g}_t(l) - \hat{g}_t^n(l) + \hat{f}^n(l+t) - \hat{f}(l+t)|^2 dt \\ &\leq 2 \sum_{l=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{g}_t(l) - \hat{g}_t^n(l)|^2 + |\hat{f}^n(l+t) - \hat{f}(l+t)|^2 dt \\ &= 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{[0,1]} |g_t(x) - g_t^n(x)|^2 dx dt + 2 \int |f^n - f|^2 \\ &\leq \epsilon\end{aligned}$$

where ϵ can be arbitrarily small if n is large enough. □

Rescaling we can assume that the set E has period 1, $|E \cap [0,1]| = \gamma$, $\text{supp}\hat{f}_k \in [-\frac{ab}{2}, \frac{ab}{2}]$ and Λ is lacunary with 1 being replaced by a in definitions (1.29), (1.30) and (1.31). Let g_t be a family of periodizations of f as defined in (1.39). It follows from **Lemma 4** that $\hat{g}_t(l) = \hat{f}(l+t)$ for almost all $t \in [-\frac{1}{2}, \frac{1}{2}]$. Assume for simplicity that $t = 0$ is among them. Let n_k be the smallest integer in $[\lambda_k - \frac{ab}{2}, \lambda_k + \frac{ab}{2}]$ if such exists. Denote $\tilde{\Lambda} = \{n_k\}_{k=-\infty}^{\infty}$. Then

$$Spec\ g_0 \subset \bigcup_{m=0}^{m=[ab]} (\tilde{\Lambda} + m).$$

Next lemma generalizes a result of F. Nazarov for the Zygmund theorem on lacunary Fourier series.

Lemma 5:

$$\int_{E\cap[0,1]} |g_0|^2 \geq C(\gamma, a, b, N) \int_0^1 |g_0|^2.$$

Proof of Lemma 5: Denote $M = [ab]$. Let

$$\begin{aligned} R &= \sup_{n\in\mathbf{Z}} Card\{(k,k',m,m'), k \neq k', 0 \leq m, m' \leq M : n = n_k + m - (n_{k'} + m')\} \\ &\leq (M+1) \sup_{n\in\mathbf{Z}} Card\{(k,k'), k \neq k' : n \in [\lambda_k - \lambda_{k'} - ab, \lambda_k - \lambda_{k'} + ab]\} \\ &\leq (M+1) \cdot N \cdot (1 + [2b]). \end{aligned}$$

To obtain the last inequality we used (1.31) with 1 being replaced by a. We can write g_0 in the following way:

$$g_0(x) = \sum_{k=-\infty}^{\infty} p_k(x) e^{i2\pi n_k x}$$

where p_k are trigonometric polynomials of degree M:

$$p_k(x) = \sum_{m=0}^{M} c_m^{(k)} e^{i2\pi m x}.$$

Therefore,

$$\begin{aligned} \int_{E\cap[0,1]} |g_0|^2 &= \sum_{k=-\infty}^{\infty} \int_{E\cap[0,1]} |p_k|^2 + \sum_{k\neq l} \int_{E\cap[0,1]} p_k \bar{p}_l \\ &= \sum_{k=-\infty}^{\infty} \int_{E\cap[0,1]} |p_k|^2 \\ &+ \sum_{k\neq k'} \sum_{0\leq m,m'\leq M} c_m^{(k)} \bar{c}_{m'}^{(k')} \hat{\chi}_{E\cap[0,1]}(n_k + m - (n_{k'} + m')). \qquad (1.43) \end{aligned}$$

The first term in (1.43) is bounded from below by

$$\sum_{k=-\infty}^{\infty}\left(\frac{\gamma}{C}\right)^{2M+1}\sum_{m=0}^{M}|c_m^{(k)}|^2=\left(\frac{\gamma}{C}\right)^{2M+1}\int_0^1|g_0|^2. \tag{1.44}$$

The second term in (1.43) can be written in the form $\langle Tg_0, g_0\rangle$ where T is a Hilbert-Schmidt operator on $L^2([0,1])$:

$$\widehat{Tg_0}(n_{k'}+m')=\sum_{k\neq k'}{}^{'}\sum_{0\leq m\leq M}\hat{\chi}_{E\cap[0,1]}(n_k+m-(n_{k'}+m'))\hat{g}_0(n_k+m)$$

and $\widehat{Tg_0}(n)=0$ for the rest of n. Its Hilbert-Schmidt norm is

$$\sqrt{\sum_{k\neq k'}\sum_{0\leq m,m'\leq M}|\hat{\chi}_{E\cap[0,1]}(n_k+m-(n_{k'}+m'))|^2}\leq\sqrt{R\gamma(1-\gamma)}. \tag{1.45}$$

We won't proceed further since the rest of the argument is the same as in Nazarov's proof of the Zygmund theorem ([13], Theorem 3.6). See details in the proof of the next **Theorem**. This leads us to the desired inequality

$$\int_{E\cap[0,1]}|g_0|^2\geq C(\gamma,a,b,N)\int_0^1|g_0|^2$$

where

$$C(\gamma,a,b,N)=e^{-(\ln R)R^2\left(\frac{C}{\gamma}\right)^{C(M+1)}}\geq e^{-\ln((1+ab)(1+2b)N)((1+ab)(1+2b)N)^2\left(\frac{C}{\gamma}\right)^{C(1+ab)}}.$$

□

In a similar way we have that

$$\int_{E\cap[0,1]}|g_t|^2\geq C(\gamma,a,b,N)\int_0^1|g_t|^2 \tag{1.46}$$

for almost all $t \in [-\frac{1}{2}, \frac{1}{2}]$. Applying **Lemma 4** and (1.46) we obtain

$$\int_E |f|^2 \geq C(\gamma, a, b, N) \int |f|^2. \tag{1.47}$$

□

1.4.4 Generalized lacunary series

The next theorem generalizes Nazarov's result for the Zygmund theorem for series whose coefficients are replaced with polynomials of uniformly bounded degree.

Theorem 4: *let I be an interval of length 1 and E be a subset of I of positive measure. If $\Lambda = \{\lambda_k\}_{k=-\infty}^{\infty}$ is lacunary in the sense of (1.29) and $f(x) = \sum_{k=-\infty}^{\infty} p_k(x)e^{i\lambda_k x}$ where $p_k(x)$ are polynomials of degree at most m, then*

$$\int_E |f|^2 \geq C(|E|, N, m) \int_I |f|^2$$

$$C(|E|, N, m) = e^{-(\ln CN)N^2\left(\frac{C}{|E|}\right)^{C(m+1)}}.$$

Proof of Theorem 4: Without loss of generality we can assume that $I = [-\frac{1}{2}, \frac{1}{2}]$. Let $p(x) = \sum_{j=0}^{m} b_j x^j$ be a polynomial of degree m. Denote $\mathbf{b} = (b_0, ..., b_m)$. Then $\|p\|_{L^2(I)} \sim \|\mathbf{b}\|_2$. In fact,

$$\frac{\|\mathbf{b}\|_2^2}{C^m} \leq \int_I |p|^2 \leq C\|\mathbf{b}\|_2^2.$$

First we will treat the case when gaps $|\lambda_k - \lambda_l|$ are large: $|\lambda_k - \lambda_l| \geq C(1 + m^3)$ for $k \neq l$. We need this condition to prove that $\int_I |f|^2 \sim \sum_{k=-\infty}^{\infty} \int_I |p_k|^2$. Note that we don't need the lacunarity of Λ for the lemma below.

Lemma 6: *If* $|\lambda_k - \lambda_l| \geq C(1+m^3)$ *for* $k \neq l$ *then*

$$\frac{C}{1+m^2} \sum_{k=-\infty}^{\infty} \int_I |p_k|^2 \leq \int_I |f|^2 \leq C(1+m^2) \sum_{k=-\infty}^{\infty} \int_I |p_k|^2.$$

Proof of Lemma 6: We will prove only the left inequality. The right one can be proved similarly. Let p be a polynomial of degree m. Hence, using the following inequality

$$\int_I |p'| \leq Cm^2 \int_I |p|, \tag{1.48}$$

we obtain that

$$\int_A |p|^2 \geq C \int_I |p|^2 \tag{1.49}$$

for any set $A \subset I$ provided $|I \cap A^c| \leq \frac{C}{1+m^2}$.
Let $\phi(x) = (\frac{1}{4} - x^2)\chi_I(x)$. Then

$$\begin{aligned} \int_I |f|^2 &\geq \int_I \phi |f|^2 \\ &= \sum_{k=-\infty}^{\infty} \int_I \phi |p_k|^2 + \sum_{k \neq l} \int_I \phi p_k \bar{p}_l e^{i(\lambda_k - \lambda_l)x} dx. \end{aligned} \tag{1.50}$$

Using (1.49) we can bound the first term in (1.50) from below by

$$\frac{C}{1+m^2} \sum_{k=-\infty}^{\infty} \int_I |p_k|^2. \tag{1.51}$$

We claim that each summand in the second term in (1.50) is bounded from above by

$$\frac{C(1+m^4)}{(\lambda_k-\lambda_l)^2}\int\limits_I (|p_k|^2+|p_l|^2). \tag{1.52}$$

To show this we consider $\int\limits_I p(x)\phi(x)e^{i\lambda x}dx$ for $\lambda \neq 0$ where $p = p_k\bar{p}_l$ is a polynomial of degree $2m$. If we write

$$p(x) = p(y) + p'(y)(x-y) + \int\limits_y^x p''(t)(x-t)dt,$$

then $\int\limits_I p(x)\phi(x)e^{i\lambda x}dx$ can be written in the following way:

$$p(y)\hat{\phi}(\lambda) + p'(y)\widehat{(x-y)\phi(x)}(\lambda)$$
$$-\int\limits_{-\frac{1}{2}}^{y}\int\limits_{x}^{y}\phi(x)e^{i\lambda x}p''(t)(x-t)dtdx + \int\limits_{y}^{\frac{1}{2}}\int\limits_{y}^{x}\phi(x)e^{i\lambda x}p''(t)(x-t)dtdx. \tag{1.53}$$

The first and second terms in (1.53) are bounded from above by

$$\frac{C}{1+\lambda^2}(|p(y)|+|p'(y)|).$$

Meanwhile the third term in (1.53)

$$\int\limits_{-\frac{1}{2}}^{y}\int\limits_{-\frac{1}{2}}^{t}\phi(x)e^{i\lambda x}p''(t)(x-t)dxdt$$

is bounded from above by

$$\frac{C}{1+\lambda^2}\int\limits_{-\frac{1}{2}}^{y}|p''(t)|dt.$$

The fourth term in (1.53) can be treated analogously.

Integrating (1.53) over $y \in I$ and applying (1.48) twice, we obtain (1.52). If $|\lambda_k - \lambda_l| \geq$

$C(1+m^3)$ with large enough C then

$$\begin{aligned}\int_I |f|^2 &\geq \frac{C}{1+m^2}\sum_{k=-\infty}^{\infty}\int_I |p_k|^2 - \sum_{k\neq l}\frac{C(1+m^4)}{(\lambda_k-\lambda_l)^2}\int_I(|p_k|^2+|p_l|^2)\\ &\geq \frac{C}{1+m^2}\sum_{k=-\infty}^{\infty}\int_I |p_k|^2(1-\sum_{l\neq k}{}' \frac{1}{100(l-k)^2})\\ &\geq \frac{C}{1+m^2}\sum_{k=-\infty}^{\infty}\int_I |p_k|^2.\end{aligned}$$

The right inequality can be proven similarly with $\phi(\frac{x}{1+\frac{C}{1+m^2}})$ instead of ϕ. □

Repeating an argument similar to the one used in **Proposition 1** to prove a local estimate, we have that **Theorem 4** holds if E is large enough:

If $|I \cap E^c| \leq \frac{C}{(1+m^8)N}$ *then*

$$\int_E |f|^2 \geq \frac{1}{2}\int_I |f|^2. \tag{1.54}$$

Now we consider the case when $|I \cap E^c| \geq \frac{C}{(1+m^8)N}$. The same argument shows (see also **Lemma 5**) that

$$\int_E |f|^2 = \sum_{k=-\infty}^{\infty}\int_E |p_k|^2 + \langle T_E f, f\rangle$$

where

$$
\begin{aligned}
\langle T_E f, f\rangle &= \sum_{k\neq l}\int_E p_k\bar{p}_l e^{i(\lambda_k-\lambda_l)x}dx \\
&= \sum_{k\neq l}\int_I p_k(y)\bar{p}_l(y)\hat{\chi}_E(\lambda_k-\lambda_l)dy \\
&- \int_I\sum_{k\neq l}\int_{-\frac{1}{2}}^{y}(p_k\bar{p}_l)'(t)\hat{\chi}_{[-1/2,t]\cap E}(\lambda_k-\lambda_l)dtdy \\
&+ \int_I\sum_{k\neq l}\int_{y}^{\frac{1}{2}}(p_k\bar{p}_l)'(t)\hat{\chi}_{[t,1/2]\cap E}(\lambda_k-\lambda_l)dtdy \\
&= \sum_{k\neq l}\int_I p_k(t)\bar{p}_l(t)\hat{\chi}_E(\lambda_k-\lambda_l)dt \\
&- \sum_{k\neq l}\int_I (p_k\bar{p}_l)'(t)(1/2-t)\hat{\chi}_{[-1/2,t]\cap E}(\lambda_k-\lambda_l)dt \\
&+ \sum_{k\neq l}\int_I (p_k\bar{p}_l)'(t)(1/2+t)\hat{\chi}_{[t,1/2]\cap E}(\lambda_k-\lambda_l)dt.
\end{aligned}
$$

We can view T_E as an operator acting on the space of sequences of pairs of functions from $L^2(I)$:

$$\mathbf{G} = (G_1, G_2, G_3, \ldots),\quad G_k = (g_k^1, g_k^2),\ g_k^i \in L^2(I), i = 1, 2,\quad k = 1, 2, 3, \ldots$$

with a scalar product defined in the following way:

$$\langle F, G\rangle = \sum_{k=1}^{\infty}\sum_{i=1}^{2}\int_I f_k^i \bar{g}_k^i.$$

Then

$$(T_E F)_l = \sum_{k\neq l} T_{lk} F_k$$

where each T_{lk} is a 2×2 matrix whose $(1,1)$ entry is

$$\hat{\chi}_E(\lambda_k-\lambda_l),$$

$(1,2)$ and $(2,1)$ entries are the same and equal to

$$-(1/2-t)\hat{\chi}_{[-1/2,t]\cap E}(\lambda_k-\lambda_l)+(1/2+t)\hat{\chi}_{[t,1/2]\cap E}(\lambda_k-\lambda_l)$$

and $(2,2)$ entry is 0. In our case $F_k=(p_k,p_k')$. We claim that T_E is a Hilbert-Schmidt operator since

$$\sup_{t\in I}\sum_{k,l}\sum_{i,j=1}^{2}|T_{lk}^{ij}|^2 \ \leq\ CN|E|$$

by an argument similar to the one used to prove a local estimate in **Proposition 1.**

Our proof will follow the one of F. Nazarov for the Zygmund theorem. Let $\sigma_1,\sigma_2,..$ be the eigenvalues of T_E enumerated in the descending order of their absolute values: $|\sigma_1|\geq|\sigma_2|\geq ....$ Since

$$\begin{aligned}\sum_{s=1}^{\infty}|\sigma_s|^2 \ &\leq\ \sup_{t\in I}\sum_{k,l}\sum_{i,j=1}^{2}|T_{lk}^{ij}|^2\\ &\leq\ CN|E|\end{aligned}$$

we have that $|\sigma_{n+1}|^2\leq\frac{CN|E|}{n+1}$. If V_n is the space spanned by the first n eigenvectors of T_E, then the norm of T_E on $V_n^{\perp}$ is equal to $|\sigma_{n+1}|$. Recall that

$$\int_E |f|^2=\sum_{k=-\infty}^{\infty}\int_E |p_k|^2+\langle T_E f,f\rangle. \tag{1.55}$$

The first term in (1.55) is bounded from below by

$$\left(\frac{|E|}{C}\right)^{2m+1}\sum_{k=-\infty}^{\infty}\int_I |p_k|^2. \tag{1.56}$$

If $f \perp V_{n(E)}$ with

$$n = \left[N \left(\frac{C}{|E|} \right)^{4m+1} \right] \tag{1.57}$$

with large enough C then the second term in (1.55) is bounded from above by $\frac{1}{2}$ of the expression in (1.56). Hence, if $f \perp V_n$, then

$$\begin{aligned} \int_E |f|^2 &\geq \left(\frac{|E|}{C} \right)^{2m+1} \sum_{k=-\infty}^{\infty} \int_I |p_k|^2 \\ &\geq \left(\frac{|E|}{C} \right)^{2m+1} \int_I |f|^2. \end{aligned} \tag{1.58}$$

The last inequality follows from **Lemma 6**. Therefore, **Theorem 4** holds if f is orthogonal to certain subspaces. Formula (1.57) gives that $n \geq 1$. We can think that $n = 0$ if E is large enough (see (1.54)).

Now we will do the general case. The main idea is to construct a set $\delta E \in I \backslash E$ such that $|\delta E| \geq \delta(|E|) > 0$ if $|E| < 1$ and

$$\int_E |f|^2 \geq C(m, N, |E|) \int_{E \cup \delta E} |f|^2.$$

Then we will iterate until we get a large enough set so that we can apply (1.54). In fact, we will reach I automatically at some iteration (see the end of the proof).

Choose n as in (1.57) but with twice larger constant C so that it will work for sets of measure at least $|E|/2$. Define

$$E_t = \bigcap_{j=0}^{n} (E - jt).$$

We can choose $\tau > 0$ so small that $|E_t| \geq |E|/2$ for $t \in [0, \tau]$. It is clear that $\tau < 1$. We will pick τ in a special manner. Consider the continuous function $\phi(x) = |E^c \cap (E + x)|$.

$\phi(0) = 0$, $\phi(1) = |E|$. Let $n\tau = \inf\{x, 0 \le x \le 1 : \phi(x) = \frac{|E|}{2n}\}$ then

$$|E^c \cap (E + n\tau)| = \frac{|E|}{2n}$$

and

$$|E^c \cap (E + kt)| \le \frac{|E|}{2n} \quad \textit{for } k = 1, ..., n \textit{ and } t \in [0, \tau].$$

Hence

$$\begin{aligned}
|E_t| &= |\bigcap_{k=0}^{n}(E - kt)| \\
&= |E \backslash \bigcup_{k=1}^{n}(E - kt)^c \cap E| \\
&\ge |E| - \sum_{k=1}^{n} |(E - kt)^c \cap E| \\
&= |E| - \sum_{k=1}^{n} |E^c \cap (E + kt)| \\
&\ge \frac{|E|}{2}.
\end{aligned}$$

Let g be the following linear combination:

$$g(x) = \sum_{j=0}^{n} a_j(t) f(x + jt)$$

with $\sum_{j=0}^{n} |a_j(t)|^2 = 1$ then

$$\begin{aligned}\int_{E_t} |g|^2 &\le (\sum_{j=0}^{n} |a_j(t)|^2)(\sum_{j=0}^{n} \int_{E_t} |f(x+jt)|^2 dx)\\ &\le \sum_{j=0}^{n} \int_{E-jt} |f(x+jt)|^2 dx\\ &= \sum_{j=0}^{n} \int_{E} |f(x)|^2 dx\\ &= (n+1) \int_{E} |f|^2.\end{aligned}$$

Since $|E_t| \ge |E|/2$ we can choose such $a_j(t)$ that $g \perp V_{E_t}$. Then

$$\begin{aligned}\int_{I} |g|^2 &\le \left(\frac{C}{|E_t|}\right)^{2m+1} \int_{E_t} |g|^2\\ &\le (n+1)\left(\frac{C}{|E|}\right)^{2m+1} \int_{E} |f|^2 = \epsilon^2.\end{aligned}$$

Let $p(x) = \sum_{l=0}^{m} b_l x^l$ be a polynomial of degree m. Define

$$\tilde{p}(x) = \sum_{j=0}^{n} a_j(t) e^{i\lambda jt} p(x+jt) = \sum_{l=0}^{m} \tilde{b}_l x^l.$$

Then $\tilde{\mathbf{b}} = A\mathbf{b}$ where A is a certain $(m+1) \times (m+1)$ upper triangular matrix with $\sum_{j=0}^{n} a_j(t) e^{i\lambda jt}$ on the diagonal. If this sum is not zero, then A is invertible. A^{-1} is also upper triangular and $A_{ij}^{-1} = \frac{(-1)^{i+j} M_{ji}}{detA}$. In fact,

$$A_{kl} = \binom{l}{k} \sum_{j=0}^{n} a_j(t) e^{i\lambda jt} (jt)^{l-k}, \quad 0 \le k \le l \le m.$$

A trivial estimate shows that $|M_{ji}| \le m!\,(2^m \sqrt{n} n^m)^m \le (Cn^{3/2})^{m^2}$. However, we can give a better estimate by noticing that M_{ji} is the sum of at most 2^m terms (open determinants via rows or columns containing at most two nonzero entries) whose absolute values do not exceed

$$\frac{\prod_{i=1}^{m} l_i}{\prod_{i=1}^{m} k_i!(l_i - k_i)!} n^{\sum_{i=1}^{m} l_i - \sum_{i=1}^{m} k_i} n^{m/2}, \quad 0 \le k_i \le l_i \le m.$$

We used the fact that $\tau < 1$. Therefore,

$$|M_{ji}| \le 2^m m! n^{3m/2} \le (C(m+1)n^{3/2})^m.$$

Hence

$$\|A^{-1}\|_2 \le \frac{(C(m+1)n^{3/2})^m}{|\sum_{j=0}^{n} a_j(t) e^{i\lambda j t}|^{m+1}}.$$

Note that

$$g(x) = \sum_{j=0}^{n} a_j(t) f(x+jt) = \sum_{k=-\infty}^{\infty} \tilde{p}_k(x) e^{i\lambda_k x}.$$

Therefore,

$$\begin{aligned}
\int_I |g|^2 &\ge \frac{C}{1+m^2} \sum_{k=-\infty}^{\infty} \int_I |\tilde{p}_k|^2 \\
&\ge \frac{1}{C^m} \sum_{k=-\infty}^{\infty} \|\tilde{\mathbf{b}}_{\mathbf{k}}\|_2^2 \\
&\ge \frac{1}{C^m \|A^{-1}\|_2^2} \sum_{k=-\infty}^{\infty} \|\mathbf{b}_{\mathbf{k}}\|_2^2.
\end{aligned}$$

Integrating over $t \in [0,\tau]$ and applying Holder's inequality, we have

$$\begin{aligned}(Cmn^{3/2})^m\epsilon^2 &\geq \sum_{k=-\infty}^{\infty} \|\mathbf{b_k}\|_2^2 \frac{1}{\tau}\int_0^\tau |\sum_{j=0}^n a_j(t)e^{i\lambda_k jt}|^{2(m+1)}dt \\ &\geq \sum_{k=-\infty}^{\infty} \|\mathbf{b_k}\|_2^2 \left(\frac{1}{\tau}\int_0^\tau |\sum_{j=0}^n a_j(t)e^{i\lambda_k jt}|^2 dt\right)^{m+1} \\ &= \sum_{k=-\infty}^{\infty} \|\mathbf{b_k}\|_2^2 \rho^{2(m+1)}(\lambda_k).\end{aligned}$$

It is known that $S = \{\lambda : \rho^2(\lambda) < (\frac{C}{n^2})^{2n}\} \subset \bigcup_{j=1}^n I_j$ where I_j are some intervals of length $\frac{1}{4n^2\tau}$ (see the proof of Lemma 3.1 in ([13])). Thus we obtain a lemma analogous to ([13], Lemma 3.1).

Lemma 7: *There are n intervals* $I_1, ..., I_n$ *of length* $\frac{1}{4n^2\tau}$ *such that*

$$\sum_{\lambda_k \in \Lambda \setminus \bigcup_{j=1}^n I_j} \|\mathbf{b_k}\|_2^2 \leq \left((m+1)(Cn)^{4n+3/2}\right)^{m+1} \epsilon^2.$$

Remark: if $\tau \geq C(m, N, |E|)$ then we are done since the cardinality of $\Lambda \cap \bigcup_{j=1}^n I_j$ is bounded by $n(1+\frac{1}{4n^2 C(m,N,|E|)})$ and f can be approximated by $\sum_{\lambda_k \in \Lambda \cap \bigcup_{j=1}^n I_j} p_k(x)e^{i\lambda_k x}$.

It is also known that there are real numbers $\mu_1, ..., \mu_n$ and $t \in [0,\tau]$ such that

$$|\sum_{j=0}^n a_j(t)e^{i\lambda_k jt}| \geq \left(\frac{1}{8n^2}\right)^{2n} \prod_{j=1}^n \theta_\tau(\lambda - \mu_j)$$

where $\theta_\tau(x) = \min(1, \tau|x|)$ and $\lambda \in \bigcup_{j=1}^n I_j$ (see arguments after Lemma 3.1 in ([13])).

Thus we obtain the following result:

$$\sum_{\lambda_k \in \Lambda \cap \bigcup_{j=1}^{n} I_j} \|\mathbf{b_k}\|_2^2 \left(\prod_{j=1}^{n} \theta_\tau(\lambda_k - \mu_j)\right)^{2(m+1)} \leq \left((m+1)(Cn)^{4n+3/2}\right)^{m+1} \epsilon^2.$$

Combining this result with **Lemma 4** and the fact that $0 \leq \theta_\tau \leq 1$, we obtain

Lemma 8: *There are n real numbers $\mu_1, ..., \mu_n$ such that*

$$\sum_{k=-\infty}^{\infty} \|\mathbf{b_k}\|_2^2 \left(\prod_{j=1}^{n} \theta_\tau(\lambda_k - \mu_j)\right)^{2(m+1)} \leq \left(C(m+1)n^{4n+3/2}\right)^{m+1} \epsilon^2 = \epsilon'^2.$$

Let $D = \left(\prod_{j=1}^{n} (e^{i\mu_j x} \frac{d}{dx} e^{-i\mu_j x})\right)^{m+1} = \prod_{j=1}^{n} (e^{i\mu_j x} \frac{d^{m+1}}{dx^{m+1}} e^{-i\mu_j x})$ be a differential operator. Any solution of the homogenous equation $Df = 0$ is of the form $\sum_{j=1}^{n} p_j(x) e^{i\mu_j x}$ with p_j being polynomials of degree at most m. A partial solution of $Df = g$ is of the form $\phi = \left(\prod_{j=1}^{n} (e^{i\mu_j x} \int e^{-i\mu_j x})\right)^{m+1} g$. Note that $\max_{x \in J} |\phi(x)| \leq \frac{|J|^{n(m+1)}}{[n(m+1)]!} \frac{1}{|J|} \int_J |g|$ for any interval J. Hence there exist n polynomials p_j of degree at most m such that

$$\max_{x \in J} |f(x) - \sum_{j=1}^{n} p_j(x) e^{i\mu_j x}| \leq \frac{|J|^{n(m+1)}}{[n(m+1)]!} \frac{1}{|J|} \int_J |g|. \tag{1.59}$$

Define a set of $2n$ intervals $I_k = (\mu_k - \frac{1}{\tau}, \mu_k + \frac{1}{\tau})$ and $\tilde{I}_k = (\mu_k - \frac{2}{\tau}, \mu_k + \frac{2}{\tau})$. For some fixed constant C which will be chosen later, define $\tilde{\Lambda} = \bigcup_{k=1}^{n} \{\lambda \in \Lambda : |\lambda - \mu_k| \leq Cm^2\}$. The cardinality of $\tilde{\Lambda}$ is bounded from above by Cn because of our gap assumption (see **Lemma 6**). Note that we don't need the lacunarity of Λ for this. Put $\Lambda' = \Lambda \setminus \tilde{\Lambda}$. Split Λ' into $n+1$ disjoint subsets in the following way. $\Lambda_0 = \Lambda' \setminus \bigcup_{j=1}^{n} I_j$, $\Lambda_k =$

$I_k \setminus \bigcup_{j=k+1}^{n} I_j$, $k = 1, ..., n$. Then we can decompose

$$f = \tilde{f} + \sum_{j=0}^{n} f_j$$

where $\tilde{f} = \sum_{\lambda_k \in \tilde{\Lambda}} p_k(x) e^{i\lambda_k x}$ and $f_j = \sum_{\lambda_k \in \Lambda_j} p_k(x) e^{i\lambda_k x}$. It follows from **Lemma 8** that

$$\sum_{\lambda_k \in \Lambda_l} \|\mathbf{b_k}\|_2^2 \left(\prod_{j=1}^{n} \theta_\tau(\lambda_k - \mu_j) \right)^{2(m+1)} \leq \epsilon'^2, \quad l = 0, ..., n.$$

In particular, it means that

$$\frac{C}{1+m^2} \int_I |f_0|^2 \leq \sum_{\lambda_k \in \Lambda_0} \|\mathbf{b_k}\|_2^2 \leq \epsilon'^2 \tag{1.60}$$

since $\prod_{j=1}^{n} \theta_\tau(\lambda - \mu_j) = 1$ if $\lambda \in \Lambda_0$.

If $1 \leq k \leq n$ then f_k has only finitely many terms and therefore it is infinitely many times differentiable. If $\lambda \in \Lambda_k$ then

$$\prod_{j=1}^{n} \theta_\tau(\lambda - \mu_j) = \prod_{\mu_j \in \tilde{I}_k} \theta_\tau(\lambda - \mu_j).$$

Correspondingly, instead of D we will define another differential operator

$$D_k = \prod_{\mu_j \in \tilde{I}_k} \left(e^{i\mu_j x} \frac{d^{m+1}}{dx^{m+1}} e^{-i\mu_j x}\right).$$

Let $D_k f_k = g_k$ and let n_k denote the number of $\mu_j \in \tilde{I}_k$. We have that $1 \leq n_k \leq n$. Let p be a polynomial of degree at most m, then

$$\left(e^{i\mu x} \frac{d^{m+1}}{dx^{m+1}} e^{-i\mu x}\right) p(x) e^{i\lambda x} = \sum_{k=0}^{m+1} \left(\binom{m+1}{k} p^{(k)}(x) (i(\lambda - \mu))^{m+1-k} \right) e^{i\lambda x} = \tilde{p}(x) e^{i\lambda x}.$$

Since $\|p^{(k)}\|_{L^2(I)} \le (Cm^2)^k \|p\|_{L^2(I)}$ we have

$$\|\tilde{p}\|_{L^2(I)} \le (|\lambda - \mu| + Cm^2)^{m+1} \|p\|_{L^2(I)}.$$

Actually we can give a better estimate because

$$\|p^{(k)}\|_{L^2(I)} \le C^m m^k \|p\|_{L^2(I)}$$

which follows from well-known Markov's inequality ([3], p.256):

$$\|p^{(k)}\|_{L^\infty([-1,1])} \le \frac{m^2 \cdot (m^2-1) \cdot (m^2-2^2) \cdot \ldots \cdot (m^2-(k-1)^2)}{1 \cdot 3 \cdot \ldots \cdot (2k-1)} \|p\|_{L^\infty([-1,1])}$$

but what we already have is enough. Therefore,

$$\begin{aligned} \frac{C}{1+m^2} \int_I |g_k|^2 &\le \sum_{\lambda_l \in \Lambda_k} \prod_{\mu_j \in \tilde{I}_k} (|\lambda_l - \mu_j| + Cm^2)^{2(m+1)} \int_I |p_l|^2 \\ &\le \sum_{\lambda_l \in \Lambda_k} \prod_{\mu_j \in \tilde{I}_k} (C|\lambda_l - \mu_j|)^{2(m+1)} \int_I |p_l|^2 \end{aligned}$$

since $|\lambda - \mu_j| \ge Cm^2$ if $\lambda \in \Lambda_k \subset \Lambda \backslash \tilde{\Lambda}$. It is left to notice that $|\lambda - \mu_j| \le \frac{3}{\tau}\theta_\tau(\lambda - \mu_j)$ if $\lambda \in \Lambda_k \subset I_k$ and $\mu_j \in \tilde{I}_k$ to conclude that

$$\begin{aligned} \frac{C}{1+m^2} \int_I |g_k|^2 &\le \sum_{\lambda_l \in \Lambda_k} \prod_{\mu_j \in \tilde{I}_k} (C\frac{3}{\tau}\theta_\tau(\lambda_l - \mu_j))^{2(m+1)} \|\mathbf{b_k}\|_2^2 \\ &\le \left(\frac{C}{\tau}\right)^{2n_k(m+1)} \epsilon'^2. \end{aligned}$$

(1.59) shows that for each $1 \le k \le n$ and each interval J there exists $\phi_k^J = \sum_{\mu_j \in \tilde{I}_k} p_j(x) e^{i\mu_j x}$ such that

$$\begin{aligned} |f_k(x) - \phi_k(x)| &\le \frac{|J|^{n_k(m+1)}}{[n_k(m+1)]!} \frac{1}{|J|} \int_J |g_k| \\ &\le \frac{|J|^{n_k(m+1)}}{[n_k(m+1)]!} Mg_k(x) \end{aligned}$$

for every $x \in J$. We will consider only $J \subset I$. Mg_k is the maximal function of g_k on I. Then

$$\|Mg_k\|_{L^2(I)} \leq C\|g_k\|_{L^2(I)} \leq \left(\frac{C}{\tau}\right)^{n_k(m+1)} \epsilon'. \tag{1.61}$$

Let $|J| = A\tau$ with $A \geq 1$ and $\phi^J(x) = \tilde{f}(x) + \sum_{k=1}^{n} \phi_k^J(x)$ then

$$|f(x) - \phi^J(x)| \leq |f_0(x)| + \sum_{k=1}^{n} \frac{(A\tau)^{n_k(m+1)}}{[n_k(m+1)]!} Mg_k(x) = R(x)$$

for every $x \in J$. Note that ϕ^J can be written in the following way:

$$\phi^J(x) = \sum_{j=1}^{Cn} p_j^J(x) e^{i\beta_j x}$$

where p_j^J are polynomials of degree at most m. Applying (1.60) and (1.61) we can estimate the $L^2(I)$ norm of the remainder R by

$$\|R\|_{L^2(I)} \leq (C \cdot A)^{n(m+1)} \epsilon'.$$

Now we will construct $\delta E \in I \cap E^c$. Divide I into intervals of length $|J| = A\tau$. We will choose A later so that so that $\frac{1}{A\tau}$ is an integer. Fix some $0 < \gamma \leq |E| < 1$. Call an interval J good if $|E \cap J| \geq \gamma |J|$ and correspondingly bad if $|E \cap J| < \gamma |J|$.

Denote $\delta E = E^c \cap \bigcup_{J \text{ is good}} J$. Then

$$\begin{aligned}
|E^c \cap (E+n\tau)| &\leq n\tau + |\delta E| + |E^c \cap \bigcup_{J \text{ is bad}} J \cap (E+n\tau)| \\
&\leq n\tau + |\delta E| + \sum_{J \text{ is bad}} |J \cap (E+n\tau)| \\
&\leq n\tau + |\delta E| + \sum_{J \text{ is bad}} (|J \backslash (J+n\tau)| + |(J+n\tau) \cap (E+n\tau)|) \\
&\leq |\delta E| + \frac{1}{|J|}(n\tau + \gamma|J|) \\
&\leq |\delta E| + n\tau\frac{1}{|J|} + \gamma
\end{aligned}$$

since the number of bad intervals is bounded by $\frac{1}{|J|} - 1$. Recall that $|E^c \cap (E+n\tau)| = \frac{|E|}{2n}$. If we put $\gamma = \frac{|E|}{8n} < |E|$ then

$$\frac{3|E|}{8n} \leq |\delta E| + n\tau\frac{1}{|J|}.$$

There are two possible situations:

1. $n\tau \leq \frac{|E|}{8n}$. Then we can put $\frac{1}{|J|} = [\frac{|E|}{8n^2\tau}]$ and obtain that

$$|\delta E| \geq \frac{|E|}{4n}.$$

We also have that

$$A = \frac{|J|}{\tau} = \frac{1}{\tau[\frac{|E|}{8n^2\tau}]} \leq \frac{16n^2}{|E|}.$$

It is interesting to note that the condition $n\tau \leq \frac{|E|}{8n}$ implies that $|E| \leq 1 - \frac{3}{11n}$ since $\frac{|E|}{2n} = |E^c \cap (E+n\tau)| \leq n\tau + 1 - |E| \leq \frac{|E|}{8n} + 1 - |E|$.

2. $n\tau > \frac{|E|}{8n}$. Then we put $|J| = 1$, $\gamma = |E|$ and therefore I is a good interval itself. In this case

$$A = \frac{1}{\tau} < \frac{8n^2}{|E|}.$$

Alternatively, we can use **Lemma 7** directly in this case since $\frac{1}{\tau} < \frac{8n^2}{|E|}$ and the cardinality of $\Lambda \cap \bigcup_{j=1}^{n} I_j$ is bounded by $n + \frac{n}{4n^2\tau} < \frac{3n}{|E|}$ and we can approximate f with $\sum_{\lambda_k \in \bigcup_{j=1}^{n} I_j} p_k(x)e^{i\lambda_k x}$.

Now we will explain how to proceed in each case.

1.

$$\begin{aligned}
\int_{\delta E} |f|^2 &\leq \int_{\bigcup\limits_{J \text{ is good}}} |f|^2 \\
&\leq 2 \sum_{J \text{ is good}} \int_J |\phi^J|^2 + \int_J |R|^2 \\
&\leq \sum_{J \text{ is good}} \left(\frac{C|J|}{|E \cap J|}\right)^{2Cn(m+1)-1} \int_{E\cap J} |\phi^J|^2 + \int_J |R|^2 \\
&\leq \sum_{J \text{ is good}} \left(\frac{C}{\gamma}\right)^{2Cn(m+1)-1} \int_{E\cap J} |f|^2 + \left(\frac{C}{\gamma}\right)^{2Cn(m+1)-1} \int_J |R|^2 \\
&\leq \left(\frac{Cn}{|E|}\right)^{2Cn(m+1)-1} \int_E |f|^2 + \left(\frac{Cn}{|E|}\right)^{2Cn(m+1)-1} \int_I |R|^2.
\end{aligned}$$

Recall that

$$\begin{aligned}
\int_I |R|^2 &\leq (C \cdot A)^{2n(m+1)} \epsilon'^2 \\
&\leq (C \cdot A)^{2n(m+1)} \left(C(m+1)n^{4n+3/2}\right)^{m+1} \epsilon^2 \\
&\leq (C \cdot A)^{2n(m+1)} \left(C(m+1)n^{4n+3/2}\right)^{m+1} (n+1) \left(\frac{C}{|E|}\right)^{2m+1} \int_E |f|^2.
\end{aligned}$$

Hence

$$\begin{aligned}\int_{\delta E\cap E} |f|^2 &\le (m+1)^{m+1}\left(\frac{C(n+1)}{|E|}\right)^{C(n+1)(m+1)}\int_E |f|^2\\ &\le \left(\frac{CN}{|E|}\right)^{N\left(\frac{C}{|E|}\right)^{C(m+1)}}\int_E |f|^2\\ &\le e^{(\ln CN)N\left(\frac{C}{|E|}\right)^{C(m+1)}}\int_E |f|^2\end{aligned}$$

with

$$\begin{aligned}|\delta E| &\ge \frac{|E|}{4n}\\ &\ge \frac{1}{N}\left(\frac{|E|}{C}\right)^{C(m+1)}.\end{aligned}$$

2. In a similar way we can get

$$\int_I |f|^2 \le e^{(\ln CN)N\left(\frac{C}{|E|}\right)^{C(m+1)}}\int_E |f|^2$$

and we are done in this case.

If we iterate $N\left(\frac{C}{|E|}\right)^{C(m+1)}$ times in case 1 we will reach I and therefore

$$\int_I |f|^2 \le e^{(\ln CN)N^2\left(\frac{C}{|E|}\right)^{C(m+1)}}\int_E |f|^2. \tag{1.62}$$

Now we will drop the assumption that gaps $|\lambda_k-\lambda_l|$ are large, i.e., $|\lambda_k-\lambda_l|\ge C(1+m^3)$ for $k\neq l$. Since Λ is lacunary it has no more than $CN(1+m^3)$ pairs of (λ_k,λ_l) with $k\neq l$ such that $|\lambda_k-\lambda_l|\le C(1+m^3)$. To show this we will split these pairs in $[C(1+m^3)]$ groups such that $d<|\lambda_k-\lambda_l|\le d+1$ with $d=0,1,2,...,[C(1+m^3)]$ and apply definition (1.29). Denote by

$$\Lambda'=\{\lambda_k\in\Lambda : \exists\lambda_l\in\Lambda, l\neq k, |\lambda_k-\lambda_l|\le C(1+m^3)\}.$$

Then the cardinality of Λ' is bounded by $CN(1+m^3)$. Let

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} p_k(x)e^{i\lambda_k x} \\ &= \sum_{\lambda_k \in \Lambda \setminus \Lambda'} p_k(x)e^{i\lambda_k x} + \sum_{\lambda_k \in \Lambda'} p_k(x)e^{i\lambda_k x} \\ &= h(x) + r(x). \end{aligned}$$

Assume for a while that $r(x)$ is a trigonometric polynomial of order $n' = (1+m)|\Lambda'| \leq (1+m)CN(1+m^3)$. The proof goes in the same way with few changes. We will only discuss the changes. Let n_1 be the same as in (1.57):

$$n_1 = \left[N\left(\frac{C}{|E|}\right)^{4m+1} \right].$$

Put

$$\begin{aligned} n &= n_1 + n' \\ &\leq \left[N\left(\frac{C}{|E|}\right)^{4m+1} \right] + [CN(1+m^3)] \\ &\leq N\left(\frac{C}{|E|}\right)^{4m+1}. \end{aligned}$$

Let

$$\begin{aligned} g(x) &= \sum_{j=0}^{n} a_j(t)f(x+jt) \\ &= \sum_{j=0}^{n} a_j(t)h(x+jt) + \sum_{j=0}^{n} a_j(t)r(x+jt) \end{aligned}$$

with $\sum_{j=0}^{n} |a_j(t)|^2 = 1$ such that $\sum_{j=0}^{n} a_j(t)r(x+jt) = 0$ and $\sum_{j=0}^{n} a_j(t)h(x+jt) = g(x) \perp V_{n_1(E_t)}$ which is exactly $n_1 + n' = n$ linear homogenous equations with $n+1$

variables a_j. Then

$$\begin{aligned}\int_I |\sum_{j=0}^{n} a_j(t)h(x+jt)|^2 dx &= \int_I |g|^2 \\ &\leq \left(\frac{C}{|E_t|}\right)^{2m+1} \int_{E_t} |g|^2 \\ &\leq (n+1)\left(\frac{C}{|E|}\right)^{2m+1} \int_E |f|^2 = \epsilon^2.\end{aligned}$$

Therefore, there are ϕ^J

$$|h(x) - \phi^J(x)| \leq R(x)$$

for every $x \in J$. Note that ϕ^J can be written in the following way:

$$\phi^J(x) = \sum_{j=1}^{Cn} p_j^J(x)e^{i\beta_j x}$$

where p_j^J are polynomials of degree at most m and

$$\|R\|_{L^2(I)} \leq (C \cdot A)^{n(m+1)}\epsilon'.$$

Then

$$|f(x) - \phi^J(x) - r(x)| \leq R(x).$$

It is left to note that

$$\begin{aligned}\int_J |\phi^J + r|^2 &\leq \left(\frac{C|J|}{|E\cap J|}\right)^{2Cn(m+1)+2n'-1} \int_{E\cap J} |\phi^J + r|^2 \\ &\leq \left(\frac{C|J|}{|E\cap J|}\right)^{2Cn(m+1)-1} \int_{E\cap J} |\phi^J + r|^2.\end{aligned}$$

The rest of the proof is the same. Thus we obtain an estimate similar to (1.62)

$$\int_I |f|^2 \le e^{(\ln CN)N^2\left(\frac{C}{|E|}\right)^{C(m+1)}} \int_E |f|^2. \tag{1.63}$$

Since any polynomial of degree m can be uniformly approximated on an interval by a trigonometric polynomial of order $m+1$ we can drop our assumption that $r(x)$ is a trigonometric polynomial and obtain the same estimate as (1.63). □

Remark: Unfortunately, the factor in (1.63) grows much faster than $m!$ and we can not use this result to prove the conjectured **Theorem 3** by approximating f_k on each interval I with corresponding Taylor polynomials as we did in the case when $\hat{f}$ is supported on a union of finitely many intervals in **Theorem 2.**

1.5 Necessary condition for support

Recall that a set $E \subset \mathbb{R}$ is called "relatively dense" if there exist $a > 0$ and $\gamma > 0$ such that

$$|E \cap I| \ge \gamma \cdot a \tag{1.64}$$

for every interval I of length a.

It is a well-known fact that "relative density" is also necessary for an inequality of the form

$$\|f\|_{L^p(E)} \ge C(E, \Sigma, p) \cdot \|f\|_p \tag{1.65}$$

to hold for every $f \in L^p$ with $\operatorname{supp}\hat{f} \subset \Sigma$. Now we will consider an inverse problem. We will give a necessary condition for $\Sigma \supset \hat{f}$ so that (1.65) holds.

Theorem 5: *Suppose that for a given open set $\Sigma \in \mathbb{R}$ there exist $0 < \gamma < 1$, $a > 0$ and $p \in [1, \infty]$ such that for every "relatively dense" set E satisfying (1.64) we* *have*

$$\|f\|_{L^p(E)} \geq C(E,p) \cdot \|f\|_p \tag{1.66}$$

for every $f \in L^p$ with $\mathrm{supp}\hat{f} \in \Sigma$

then for every $b_0 > 0$ Σ can not contain arbitrarily long arithmetic progressions with steps at least b_0.

Proof of Theorem 5: Note that the parameter a is fixed during the whole proof. First we will prove that there is a uniform constant C such that (1.66) holds for large enough sets E, i.e., there exists $\gamma_0 \in [\gamma, 1)$ and $C(p) > 0$ such that

$$\|f\|_{L^p(E)} \geq C(p) \cdot \|f\|_p \tag{1.67}$$

holds for every "relatively dense" set E with density γ_0 and every $f \in L^p$ with $\mathrm{supp}\hat{f} \in \Sigma$. Suppose towards a contradiction that this is not true. Then there exists a sequence of f_n and corresponding "relatively dense" sets E_n with density $\gamma_n > 1 - \frac{1-\gamma}{2^n}$ such that

$$\|f_n\|_{L^p(E_n)} \leq \frac{1}{n}\|f_n\|_p.$$

Let $E = \bigcap_{n=1}^{\infty} E_n$. Then E is "relatively dense" with density γ:

$$\begin{aligned} |E \cap I| &= a - |\bigcup_{n=1}^{\infty} E_n^c| \\ &\geq a - \sum_{n=1}^{\infty} a(1-\gamma_n) \\ &\geq a - a\sum_{n=1}^{\infty} \frac{1-\gamma}{2^n} = \gamma a \end{aligned}$$

for every interval I of length a. On the other hand we have

$$\|f_n\|_{L^p(E)} \leq \|f_n\|_{L^p(E_n)} \leq \frac{1}{n}\|f_n\|_p$$

which contradicts to (1.66). The next lemma plays a crucial role.

Lemma 9: *Let $\Sigma \in \mathbb{R}$ be an open set for which there exists $b_0 > 0$ such that Σ contains arbitrarily long arithmetic progressions with steps at least b_0 then for every $0 < \gamma < 1$ (meaning arbitrarily small $1-\gamma$) and every $a > 0$ (meaning arbitrarily small a) there exist a sequence of "relatively dense" sets E_n satisfying (1.64) and a corresponding sequence of functions $f_n \in L^p$ with supp$\hat{f} \subset \Sigma$ such that*

$$\lim_{n\to\infty} \frac{\int\limits_{E_n} |f_n|^p}{\int |f_n|^p} = 0$$

for every $1 \leq p \leq \infty$.

Remark: If Σ contains an infinite arithmetic progression then we can take only one set E instead of the sequence of E_n. Here is an example of such Σ which has a finite measure: $\Sigma = \bigcup\limits_{k=1}^{\infty} (k - \frac{1}{2^k}, k + \frac{1}{2^k})$. Compare with the Amrein-Berthier theorem (see for example [7], pp. 97, 455). Note also that small $1-\gamma$ and small a are typical cases of "easy" proofs of the Logvinenko-Sereda theorem.

Proof of Lemma 9: Define $L = \left[\frac{2\pi}{ab_0}\right] + 1$, a positive integer such that $\frac{2\pi}{Lb} \leq a$ for every $b \geq b_0$. Suppose we have an arithmetic progression with step $b \geq b_0$ of length $Ln+1$: $x_0, x_0+b, x_0+2b, ..., x_0+Lnb \in \Sigma$ hence there exists $\epsilon > 0$ such that $\bigcup\limits_{k=0}^{n} [x_0 + kLb - \epsilon, x_0 + kLb + \epsilon] \subset \Sigma$ because Σ is open. We can assume that $\epsilon < \frac{Lb}{2}$ so that these intervals are disjoint. Let ϕ be a Schwartz function with supp$\hat{\phi} \subset [-1,1]$ such that $\phi(0) = 1$, e.g., if $\hat{\phi} \geq 0$. Define

$$\hat{f}_n(x) = \sum_{k=0}^{n} \hat{\phi}(\frac{x - x_0 - kLb}{\epsilon})$$

so that $\text{supp}\hat{f}_n \subset \Sigma$. Then

$$|f_n(y)| = |\epsilon \sum_{k=0}^{n} e^{i(x_0+kLb)y}\phi(\epsilon y)| \tag{1.68}$$
$$= \epsilon|D_n(Lby)\phi(\epsilon y)| \tag{1.69}$$

where the 2π-periodic function $D_n(t) = \frac{\sin(n+1)t/2}{\sin t/2}$ is the Dirichlet kernel with the following easily verified property:

$$\lim_{n\to\infty} \frac{\|D_n\|_{L^p([-\delta,\delta]^c)}}{\|D_n\|_{L^p([-\pi,\pi])}} = 0 \quad \forall\, 0 < \delta < \pi, \quad 1 \le p \le \infty.$$

This is true since $\|D_n\|_{L^p([-\pi,\pi])} \sim n^{\frac{1}{p'}}$ for $1 < p \le \infty$ and $\|D_n\|_{L^1([-\pi,\pi])} \sim \ln n$ and $|D_n(x)| \le \frac{1}{\sin\frac{\delta}{2}}$ if $\delta \le |x| \le \pi$. Now we will construct a $\frac{2\pi}{Lb}$-periodic set E:

$$E \cap [-\frac{\pi}{Lb}, \frac{\pi}{Lb}] = [-\frac{\pi}{Lb}, \frac{\pi}{Lb}] \setminus [-\frac{(1-\gamma)\pi}{2Lb}, \frac{(1-\gamma)\pi}{2Lb}].$$

Then

$$\begin{aligned} |E^c \cap I| &\le (\left\lceil \frac{aLb}{2\pi} \right\rceil + 1) \cdot \frac{(1-\gamma)\pi}{Lb} \\ &\le \frac{2aLb}{2\pi} \cdot \frac{(1-\gamma)\pi}{Lb} \\ &= (1-\gamma)a \end{aligned}$$

for every interval I of length a. We used here that $\frac{2\pi}{Lb} \le a$ to get the second inequality.

Let $1 \le p < \infty$. $|\phi(x)| \le \frac{C}{1+x^2}$ since ϕ is a Schwartz function. Applying (1.69) we have

$$\begin{aligned}
\int_E |f_n|^p &= \sum_{k=-\infty}^{\infty} \int_{E\cap[-\frac{(2k-1)\pi}{Lb},\frac{(2k+1)\pi}{Lb}]} |\epsilon D_n(Lby)\phi(\epsilon y)|^p dy \\
&\leq \sum_{k=-\infty}^{\infty} \epsilon^p \frac{C^p}{(1+(\epsilon k/Lb)^2)^p} \frac{1}{Lb} \int_{[-\pi,\pi]\setminus[-\frac{(1-\gamma)\pi}{2},\frac{(1-\gamma)\pi}{2}]} |D_n(y)|^p dy \\
&\leq C\epsilon^p(1+\frac{Lb}{\epsilon})\frac{1}{Lb}\|D_n\|^p_{L^p([-\delta,\delta]^c)} \\
&\leq \frac{C\epsilon^p}{\epsilon}\|D_n\|^p_{L^p([-\delta,\delta]^c)} \qquad (1.70)
\end{aligned}$$

where $\delta = \frac{(1-\gamma)\pi}{2}$. We used the fact that $\epsilon < \frac{Lb}{2}$ to get the last inequality.

Now we need to estimate $\|f_n\|_p^p$. Recall that $\phi(0) = 1$. Since ϕ is a continuous function there is $d > 0$ such that $\phi(x) \geq 1/2$ if $|x| \leq d$. We can assume that $\epsilon < \frac{Lbd}{\pi}$. Then applying (1.69) we have

$$\begin{aligned}
\int |f_n|^p &\geq \epsilon^p \frac{1}{2^p} \int_{-d/\epsilon}^{d/\epsilon} |D_n(Lby)|^p dy \\
&= \frac{\epsilon^p}{2^p}\frac{1}{Lb} \int_{-dLb/\epsilon}^{dLb/\epsilon} |D_n(y)|^p dy \\
&\geq \frac{\epsilon^p}{2^p}\frac{1}{Lb}\left[\frac{Lbd}{\epsilon\pi}\right] \|D_n\|^p_{L^p([-\pi,\pi])} \\
&\geq \frac{\epsilon^p}{2^p}\frac{1}{Lb}\frac{Lbd}{2\epsilon\pi}\|D_n\|^p_{L^p([-\pi,\pi])} \\
&\geq \frac{d\epsilon^p}{\epsilon 2^{p+1}\pi}\|D_n\|^p_{L^p([-\pi,\pi])}. \qquad (1.71)
\end{aligned}$$

We used here that $\epsilon < \frac{Lbd}{\pi}$ to get the last but one inequality.

Hence dividing (1.70) by (1.71) we obtain the desired result

$$\frac{\int_{E_n} |f_n|^p}{\int |f_n|^p} \leq C\frac{\|D_n\|_{L^p([-\delta,\delta]^c)}}{\|D_n\|_{L^p([-\pi,\pi])}} \to 0$$

as $n \to \infty$.

The proof for $p = \infty$ is similar and even easier. □

Remark: If $p = 2$ we can use periodizations to prove **Lemma 9** (see the proof of **Proposition 2**).

Theorem 5 follows from **Lemma 9** since otherwise we would get a contradiction to (1.67). □

Chapter 2 Periodizations of functions

2.1 Overview

Let f be a function from $L^1(\mathbb{R}^d)$. Define a family of its periodizations with respect to a rotated integer lattice:

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x-\nu)) \tag{2.1}$$

for all rotations $\rho \in SO(d)$. The main object of our study is G, the $L^2([0,1]^d \times SO(d))$ norm of the family of periodizations,

$$\begin{aligned} G^2 &= \int_{\rho \in SO(d)} \int_{[0,1]^d} |g_\rho(x)|^2 dx d\rho \\ &= \int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho. \end{aligned} \tag{2.2}$$

The purpose of this work is to show how G can give an estimate of the $L^2(\mathbb{R}^d)$ norm of a function from $L^1(\mathbb{R}^d)$ in higher dimensions. Some results on the Steinhaus tiling problem are related to **Theorem 1** since periodizations naturally appear in the problem of Steinhaus. M. Kolountzakis ([9]) proves that if a function $f \in L^1(\mathbb{R}^2)$ and $|x|^\alpha f \in L^1(\mathbb{R}^2)$ where $\alpha > \frac{10}{3}$ and its periodizations are constants, then the function is continuous. Another result is obtained by M. Kolountzakis and T. Wolff ([10], Theorem 1). It says that if periodizations of a function from $L^1(\mathbb{R}^d)$ are constants, then the function is continuous provided that the dimension d is at least three. We will show that this result is false when dimension $d = 2$.

The main theorems are the following:

Theorem 1: *let $d \geq 4$ and let $f \in L^1(\mathbb{R}^d)$. If periodizations of f*

$$g_\rho(x) = \sum_{\nu \in \mathbf{Z}^d} f(\rho(x-\nu))$$

are in $L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$G^2 = \int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho < \infty,$$

then $f \in L^2(\mathbb{R}^d)$:

$$\|f\|_2 \leq C(G + \|f\|_1) \tag{2.3}$$

where C depends only on d.

We will also obtain an inverse theorem.

Theorem 1′: *let $d \geq 5$ and let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and let g_ρ be periodizations of f*

$$g_\rho(x) = \sum_{\nu \in \mathbf{Z}^d} f(\rho(x-\nu)),$$

then $g_\rho \in L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$\int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho \leq C(\|f\|_2 + \|f\|_1)^2 \tag{2.4}$$

where C depends only on d.

We will generalize **Theorems 1** and **1′** in the spirit of the Stein-Tomas Theorem ([4], Chapter 6.5).

Theorem 2: *let $d \geq 4$ and let $f \in L^p(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$. If periodizations of f*

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x-\nu))$$

are in $L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$G^2 = \int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho < \infty,$$

then $f \in L^2(\mathbb{R}^d)$:

$$\|f\|_2 \leq C(G + \|f\|_p) \tag{2.5}$$

where C depends only on d and p.

We will also obtain an inverse theorem.

Theorem 2′: *let $d \geq 5$ and let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $1 \leq p < \frac{2d}{d+2}$ and let g_ρ be periodizations of f*

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x-\nu)),$$

then $g_\rho \in L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$\int_{\rho \in SO(d)} \|g_\rho - \hat{g}_\rho(0)\|_2^2 d\rho \leq C(\|f\|_2 + \|f\|_p)^2 \tag{2.6}$$

where C depends only on d and p.

Note that the constant C below is not fixed and varies appropriately from one equality or inequality to another without being mentioned.

2.2 Case $p = 1$

Proof of Theorem 1:

We will denote $\tilde{f}(x) = \bar{f}(-x)$ and $F(x) = f * \tilde{f}(x)$. Then $F \in L^1(\mathbb{R}^d)$ and

$$\|F\|_1 \le \|f\|_1^2. \tag{2.7}$$

We will define the following functions $h, h_1, h_2 : \mathbb{R}^+ \to \mathbb{C}$

$$\begin{aligned} h(t) &= \int |\hat{f}(\xi)|^2 d\sigma_t(\xi) && (2.8)\\ &= \int_{\mathbb{R}^d} f * \tilde{f}(x) \widehat{d\sigma_t}(x) dx \\ &= \int_{\mathbb{R}^d} F(x) \widehat{d\sigma_t}(x) dx, && (2.9) \end{aligned}$$

$$h_1(t) = \int_{|x| \le 1} F(x) \widehat{d\sigma_t}(x) dx, \tag{2.10}$$

$$h_2(t) = \int_{|x| > 1} F(x) \widehat{d\sigma_t}(x) dx. \tag{2.11}$$

Clearly $h = h_1 + h_2$.

Lemma 1: *Let $q : \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[\frac{1}{2}, 2]$, and let $b \in [0, 1)$. Define $H_1 : \mathbb{R} \to \mathbb{C}$*

$$H_1(t) = \frac{1}{\sqrt{t+b}} h_1(\sqrt{t+b}) q(\frac{\sqrt{t+b}}{N}).$$

Then for large enough N we have

$$\sum_{\nu\neq 0} |\hat{H}_1(\nu)| \leq \frac{C\|F\|_1}{N} \tag{2.12}$$

where C depends only on q and d.

Proof of Lemma 1:

First we will estimate derivatives of $h_1(t)$ when $t \geq 1$

$$|h_1^{(k)}(t)| \leq Ct^{d-1}\|F\|_1 \tag{2.13}$$

where C depends only on k and d. This follows from (2.11)

$$\begin{aligned} h_1(t) &= \int_{|x|\leq 1} F(x)\widehat{d\sigma_t}(x)dx \\ &= t^{d-1}\int_{|x|\leq 1} F(x)\int_{|\xi|=1} e^{-i2\pi t x\cdot\xi}d\sigma(\xi)dx \end{aligned}$$

by differentiating the last equality k times.

We can easily prove by induction that

$$\frac{d^k}{dt^k}\left(\frac{h_1(\sqrt{t+b})}{\sqrt{t+b}}\right) = \sum_{i=0}^{k} C_{i,k}\frac{h_1^{(i)}(\sqrt{t+b})}{(\sqrt{t+b})^{2k+1-i}}. \tag{2.14}$$

It follows from (2.14) and (2.13) that when $t \sim N^2$ we have

$$\left|\frac{d^k}{dt^k}\left(\frac{h_1(\sqrt{t+b})}{\sqrt{t+b}}\right)\right| \leq CN^{d-k-2}\|F\|_1 \tag{2.15}$$

with C depending only on k and d.

Since $q(\frac{(\sqrt{t+b})}{N}) = q(\sqrt{t'+b'}) = \tilde{q}(t')$ with $t' = \frac{t}{N^2}$ and $b' = \frac{b}{N^2}$ and $\tilde{q}(t')$ is a

Schwartz function supported in $t' \sim 1$, then we have

$$\begin{aligned} \left|\frac{d^k}{dt^k} q(\frac{(\sqrt{t+b})}{N})\right| &= N^{-2k} |\frac{d^k}{dt'^k}\tilde{q}(t')| \\ &\leq CN^{-2k} \end{aligned} \tag{2.16}$$

with C depending only on k and q.

Since $q(\frac{(\sqrt{t+b})}{N})$ is supported in $t \sim N^2$, it follows from (2.15) and (2.16) that

$$\begin{aligned} \left|\frac{d^k}{dt^k} H_1(t)\right| &= \left|\frac{d^k}{dt^k}\left(\frac{h_1(\sqrt{t+b})}{\sqrt{t+b}} q(\frac{\sqrt{t+b}}{N})\right)\right| \\ &\leq CN^{d-2-k}\|F\|_1 \end{aligned} \tag{2.17}$$

with C depending only on k, d and q. Since $H_1(t)$ is also supported in $t \sim N^2$ we have

$$\|H_1^{(k)}\|_1 \leq CN^{d-k}\|F\|_1.$$

Therefore,

$$\begin{aligned} |\hat{H}_1(\nu)| &\leq \frac{C}{|\nu|^k}\|H_1^{(k)}\|_1 \\ &\leq \frac{CN^{d-k}\|F\|_1}{|\nu|^k} \end{aligned} \tag{2.18}$$

for every $\nu \neq 0$.

Summing (2.18) over all $\nu \neq 0$ and putting $k = d+1$, we get our desired result

$$\sum_{\nu \neq 0} |\hat{H}_1(\nu)| \leq \frac{C\|F\|_1}{N} \tag{2.19}$$

where C depends only on q and d. □

In the next lemma we will use an approach related to ([10], Lemma 1.1).

Lemma 2: *Let $q : \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[\frac{1}{2}, 2]$, and let $b \in [0, 1)$. Define $H_2 : \mathbb{R} \to \mathbb{C}$*

$$H_2(t) = \frac{1}{\sqrt{t+b}} h_2(\sqrt{t+b}) q(\frac{\sqrt{t+b}}{N}).$$

Then for large enough N we have

$$\sum_{\nu \neq 0} |\hat{H}_2(\nu)| \leq \int_{|x| \geq 1} |F(x)| \cdot |D_N(x)| \tag{2.20}$$

where $D_N : \mathbb{R}^d \to \mathbb{C}$

$$|D_N(x)| \leq C \begin{cases} (\frac{N}{|x|})^{\frac{d-2}{2}} & \text{if } |x| \geq \frac{N}{2} \\ \frac{1}{N} & \text{if } 1 \leq |x| \leq \frac{N}{2} \end{cases} \tag{2.21}$$

with C depending only on q and d.

Proof of Lemma 2:

We have

$$\begin{aligned} & \hat{H}_2(\nu) \\ = & \int H_2(t) e^{-i2\pi\nu t} dt \\ = & \; 2e^{i2\pi\nu b} \int N q(t) h_2(tN) e^{-i2\pi\nu(Nt)^2} dt \\ = & \; 2e^{i2\pi\nu b} \int N q(t) e^{-i2\pi\nu(Nt)^2} \int_{|x|>1} F(x) \widehat{d\sigma_{Nt}}(x) dx dt \\ = & \; 2e^{i2\pi\nu b} \int_{|x|>1} F(x) \int N q(t) e^{-i2\pi\nu(Nt)^2} (Nt)^{d-1} \widehat{d\sigma}(Ntx) dt dx. \end{aligned} \tag{2.22}$$

We will use a well-known fact that $\widehat{d\sigma}(x) = Re(B(|x|))$ with $B(r) = a(r)e^{i2\pi r}$ and

$a(r)$ satisfying estimates

$$|a^k(r)| \leq \frac{C}{r^{\frac{d-1}{2}+k}} \tag{2.23}$$

with C depending only on k and d. Now we will need to estimate the inner integral in (2.22) with $B(|x|)$ instead of $\widehat{d\sigma}(x)$

$$\begin{aligned}
&\int Nq(t)e^{-i2\pi\nu(Nt)^2}(Nt)^{d-1}a(N|x|t)e^{i2\pi N|x|t}dt \\
&= \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}}\int q(t)e^{-i2\pi\nu(Nt)^2}t^{d-1}a(N|x|t)(N|x|)^{\frac{d-1}{2}}e^{i2\pi N|x|t}dt \\
&= \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}}e^{i2\pi\frac{|x|^2}{4\nu}}\int q(t)a(N|x|t)(N|x|)^{\frac{d-1}{2}}t^{d-1}e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu N})^2}dt \\
&= \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}}e^{i2\pi\frac{|x|^2}{4\nu}}\int \phi(t)e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu N})^2}dt
\end{aligned} \tag{2.24}$$

where $\phi(t) = q(t)a(N|x|t)(N|x|)^{\frac{d-1}{2}}t^{d-1}$ is a Schwartz function supported in $[\frac{1}{2},2]$ whose derivatives and the function itself are bounded uniformly in t, x and N because of (2.23). Note that we used here the fact that $N|x| \geq 1$. We can say even more. Note that in fact $\phi(t) = \phi(t,|x|)$. Let $|x| = c \cdot r$ where $c \geq 2$ and $r \geq \frac{1}{2}$. Then all partial derivatives of $\phi(t, c \cdot r)$ with respect to t and r are also bounded uniformly in t, r, c and N. However, we will use that $\phi(t)$ depends also on x only in formula (2.66) from the proof of **Lemma 4** and therefore we will keep writing just $\phi(t)$.

From the method of stationary phase ([8], Theorem 7.7.3) it follows that if $k \geq 1$ then

$$|\int \phi(t)e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu N})^2}dt - \sum_{j=0}^{k-1} c_j(\nu N^2)^{-j-\frac{1}{2}}\phi^{(2j)}(\frac{|x|}{2\nu N})| \leq c_k(|\nu|N^2)^{-k-\frac{1}{2}} \tag{2.25}$$

where c_j are some constants.

Since ϕ is supported in $[\frac{1}{2}, 2]$ we conclude from (2.25) that

$$|\int \phi(t)e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu N})^2}dt| \leq \begin{cases} C(|\nu|N^2)^{-\frac{1}{2}} & \text{if } \nu \in [\frac{|x|}{4N}, \frac{|x|}{N}] \\ C_k(|\nu|N^2)^{-k-\frac{1}{2}} & \text{if } \nu \notin [\frac{|x|}{4N}, \frac{|x|}{N}] \end{cases}. \tag{2.26}$$

If $\frac{|x|}{N} \leq \frac{1}{2}$, then there are no ν in $[\frac{|x|}{4N}, \frac{|x|}{N}]$ and therefore if we sum (2.26) over all $\nu \neq 0$ we will get

$$|\int \phi(t)e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu N})^2}dt| \leq C_k N^{-2k-1}. \tag{2.27}$$

If $\frac{|x|}{N} \geq \frac{1}{2}$ then the number of ν in $[\frac{|x|}{4N}, \frac{|x|}{N}]$ is bounded by $\frac{|x|}{N}$ and therefore if we sum (2.26) over all $\nu \neq 0$ we will get

$$\begin{aligned} \sum_{\nu\neq 0} |\int \phi(t)e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu N})^2}dt| &\leq C\frac{|x|}{N}(|x|N)^{-\frac{1}{2}} + C_k N^{-2k-1} \\ &\leq C_k \frac{|x|^{\frac{1}{2}}}{N^{\frac{3}{2}}}. \end{aligned} \tag{2.28}$$

Summing (2.24) over all $\nu \neq 0$ and applying (2.27) or (2.28) we conclude

$$\sum_{\nu\neq 0} |\int Nq(t)e^{-i2\pi\nu(Nt)^2}(Nt)^{d-1}B(N|x|t)dt| \leq \begin{cases} C_k(\frac{N}{|x|})^{\frac{d-2}{2}} & \text{if } \frac{|x|}{N} \geq \frac{1}{2} \\ C_k \frac{N^{\frac{d+1}{2}-2k-1}}{|x|^{\frac{d-1}{2}}} & \text{if } \frac{|x|}{N} \leq \frac{1}{2} \end{cases}. \tag{2.29}$$

Replacing in (2.22) $\widehat{d\sigma}(x)$ with $\frac{B(|x|)+\bar{B}(|x|)}{2}$, summing over all $\nu \neq 0$ and applying (2.29) with $k \geq \frac{d+1}{4}$ we get the desired result

$$\sum_{\nu\neq 0} |\hat{H}_2(\nu)| \leq \int_{|x|\geq 1} |F(x)| \cdot |D_N(x)| \tag{2.30}$$

where $D_N : \mathbb{R}^d \to \mathbb{C}$

$$|D_N(x)| \le C \begin{cases} (\frac{N}{|x|})^{\frac{d-2}{2}} & \text{if } |x| \ge \frac{N}{2} \\ \frac{1}{N} & \text{if } 1 \le |x| \le \frac{N}{2} \end{cases} \tag{2.31}$$

with C depending only on q and d. □

Now we are in a position to proceed with the proof of **Theorem 1**. From (2.1) it follows that

$$\hat{g}_\rho(m) = \hat{f}(\rho m) \tag{2.32}$$

for every $m \in \mathbb{Z}^d$. Scaling we can assume that

$$\hat{g}_\rho(m) = \hat{f}(\frac{\rho m}{\sqrt{2}}) \tag{2.33}$$

for every $m \in \mathbb{Z}^d$. It follows that

$$\|g_\rho\|_2^2 = \sum_{m \in \mathbb{Z}^d} |\hat{g}_\rho(m)|^2 = \sum_{m \in \mathbb{Z}^d} |\hat{f}(\frac{\rho m}{\sqrt{2}})|^2. \tag{2.34}$$

Let $r_d(n)$ denote the number of representations of an integer n as sums of d squares. It is a well-known fact from Number Theory that if $d \ge 5$ then

$$r_d(n) \ge C n^{\frac{d-2}{2}} \tag{2.35}$$

and if $d = 4$ and n is odd then

$$r_4(n) \ge Cn \tag{2.36}$$

where $C > 0$ depends only on d. See for example ([5], p.30, p.155, p.160).

Integrating (2.34) with respect to the Haar measure $d\rho$ and applying (2.2) we have

$$\begin{aligned}
G^2 &= \int_{\rho\in SO(d)} \sum_{m\in\mathbf{Z}^d} |\hat{f}(\frac{\rho m}{\sqrt{2}})|^2 d\rho \\
&= \int_{|\xi|=1} \sum_{m\in\mathbf{Z}^d} |\hat{f}(\frac{|m|}{\sqrt{2}}\xi)|^2 d\sigma(\xi) \\
&= \sum_{n\geq 0} \sum_{|m|^2=n} \int_{|\xi|=1} |\hat{f}(\frac{|m|}{\sqrt{2}}\xi)|^2 d\sigma(\xi) \\
&\geq \sum_{n\geq 0} \sum_{|m|^2=2n+1} \int_{|\xi|=1} |\hat{f}(\frac{|m|}{\sqrt{2}}\xi)|^2 d\sigma(\xi) \\
&= \sum_{n\geq 0} r_d(2n+1) \int_{|\xi|=1} |\hat{f}(\sqrt{n+\frac{1}{2}}\xi)|^2 d\sigma(\xi) \\
&= \sum_{n\geq 0} \frac{r_d(2n+1)}{(n+\frac{1}{2})^{\frac{d-1}{2}}} \int |\hat{f}(\xi)|^2 d\sigma_{\sqrt{n+\frac{1}{2}}}(\xi). \qquad (2.37)
\end{aligned}$$

Using (2.8) and (2.35) or (2.36) we conclude from (2.37) that

$$\sum_{n\geq 0} \frac{1}{\sqrt{n+\frac{1}{2}}} h(\sqrt{n+\frac{1}{2}}) \leq CG^2. \qquad (2.38)$$

Let $q : \mathbb{R} \to \mathbb{R}$ be a fixed non-negative Schwartz function supported in $[\frac{1}{2}, 2]$ such that

$$q(x) + q(x/2) = 1$$

when $x \in [1, 2]$. It follows that

$$\sum_{j\geq 0} q(\frac{x}{2^j}) = 1 \qquad (2.39)$$

when $x \geq 1$.

Applying the Poisson summation formula to

$$\begin{aligned} H(t) &= \frac{1}{\sqrt{t+\frac{1}{2}}}h(\sqrt{t+\frac{1}{2}})q(\frac{\sqrt{t+\frac{1}{2}}}{N}) \\ &= \frac{1}{\sqrt{t+\frac{1}{2}}}h_1(\sqrt{t+\frac{1}{2}})q(\frac{\sqrt{t+\frac{1}{2}}}{N}) + \frac{1}{\sqrt{t+\frac{1}{2}}}h_2(\sqrt{t+\frac{1}{2}})q(\frac{\sqrt{t+\frac{1}{2}}}{N}) \\ &= H_1(t) + H_2(t) \end{aligned}$$

we have

$$\begin{aligned} \sum_n H(n) &= \sum_\nu \hat{H}(\nu) \\ &= \hat{H}(0) + \sum_{\nu\neq 0} \hat{H}_1(\nu) + \sum_{\nu\neq 0} \hat{H}_2(\nu). \end{aligned} \tag{2.40}$$

Note that

$$\begin{aligned} \hat{H}(0) &= \int \frac{1}{\sqrt{t+\frac{1}{2}}}h(\sqrt{t+\frac{1}{2}})q(\frac{\sqrt{t+\frac{1}{2}}}{N})dt \\ &= 2\int h(t)q(\frac{t}{N})dt. \end{aligned} \tag{2.41}$$

Substituting (2.41) into (2.40) we get that

$$\begin{aligned} 2\int h(t)q(\frac{t}{N})dt &\leq \\ \sum_n H(n) + \sum_{\nu\neq 0}|\hat{H}_1(\nu)| + \sum_{\nu\neq 0}|\hat{H}_2(\nu)| &\leq \\ \sum_{n\geq 0}\frac{1}{\sqrt{n+\frac{1}{2}}}h(\sqrt{n+\frac{1}{2}})q(\frac{\sqrt{n+\frac{1}{2}}}{N}) + \frac{C\|F\|_1}{N} + \int_{|x|\geq 1} &|F(x)D_N(x)|dx \end{aligned} \tag{2.42}$$

where the last inequality follows from **Lemma 1** and **Lemma 2.**

From the definition of $D_N(x)$ (2.21) it follows that

$$\begin{aligned}\sum_{j\geq 0}|D_{2^j}(x)| &= \sum_{2^j\leq 2|x|}|D_{2^j}(x)| + \sum_{2^j>2|x|}|D_{2^j}(x)| \\ &\leq \sum_{2^j\leq 2|x|} C(\frac{2^j}{|x|})^{\frac{d-2}{2}} + \sum_{2^j>2|x|}\frac{C}{2^j} \leq C \qquad (2.43)\end{aligned}$$

for every $|x|\geq 1$.

Putting $N = 2^j$ in (2.42), summing over all $j \geq 0$ and applying (2.39) we get by Lebesgue Monotone Convergence Theorem

$$\begin{aligned}2\int_1^\infty h(t)dt &\leq \sum_{n\geq 0}\frac{1}{\sqrt{n+\frac{1}{2}}}h(\sqrt{n+\frac{1}{2}}) + C\|F\|_1 + C\int_{|x|\geq 1}|F(x)|dx \\ &\leq C(G^2 + \|F\|_1) \qquad (2.44)\end{aligned}$$

where the last inequality follows from (2.38). From the definition of $h(t)$ (2.8) it follows that

$$h(t) \leq C\|f\|_1^2 \qquad (2.45)$$

for $t \leq 1$. Therefore, we have

$$\begin{aligned}\int|\hat{f}(x)|^2dx &= \int_0^\infty |\hat{f}(\xi)|^2 d\sigma_t(\xi)dt \\ &= \int_0^\infty h(t)dt \\ &\leq C(G^2 + \|f\|_1^2) \qquad (2.46)\end{aligned}$$

where the last inequality is obtained from (2.44), (2.45) and (2.7). From (2.46) it follows that $f \in L^2$ and

$$\|f\|_2 \leq C(G + \|f\|_1)$$

with C depending only on d. □

If $d \geq 5$ then

$$r_d(n) \leq Cn^{\frac{d-2}{2}} \tag{2.47}$$

where $C > 0$ depends only on d. See for example ([5], p.155, p.160). An argument similar to the one used to get (2.37) but without scaling shows that

$$\begin{aligned} G^2 &= \int_{\rho \in SO(d)} \sum_{m \in \mathbb{Z}^d} |\hat{f}(\rho m)|^2 d\rho \\ &= |\hat{f}(0)|^2 + \sum_{n \geq 1} r_d(n) \int_{|\xi|=1} |\hat{f}(\sqrt{n}\xi)|^2 d\sigma(\xi) \\ &= |\hat{f}(0)|^2 + \sum_{n \geq 1} \frac{r_d(n)}{n^{\frac{d-1}{2}}} \int |\hat{f}(\xi)|^2 d\sigma_{\sqrt{n}}(\xi). \end{aligned} \tag{2.48}$$

Using (2.8) and (2.47) we conclude from (2.48) that

$$G^2 \leq \|f\|_1^2 + C \sum_{n \geq 1} \frac{1}{\sqrt{n}} h(\sqrt{n}). \tag{2.49}$$

Repeating arguments which we used to obtain (2.44) we get

$$\begin{aligned} \sum_{n \geq 1} \frac{1}{\sqrt{n}} h(\sqrt{n}) &\leq 2 \int_0^\infty h(t) dt + C\|F\|_1 \\ &\leq C(\|\hat{f}\|_2^2 + \|f\|_1^2). \end{aligned} \tag{2.50}$$

Hence we can formulate an inverse theorem to **Theorem 1**:

Theorem 1′: *let $d \geq 5$ and let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and let g_ρ be periodizations of f*

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu)) \tag{2.51}$$

then $g_\rho \in L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$\int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho \le C(\|f\|_2 + \|f\|_1)^2$$

where C depends only on d.

Corollary: interpolating between the trivial $p = 1$ and $p = 2$, we obtain the following generalization of the previous theorem for $1 \le p \le 2$: let $d \ge 5$ and let $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and let g_ρ be periodizations of f

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x-\nu))$$

then $g_\rho \in L^p([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$\int_{\rho \in SO(d)} \|g_\rho\|_p^{p'} d\rho \le C(\|f\|_p + \|f\|_1)^{p'}$$

where C depends only on d.

2.3 Case $1 \le p < \frac{2d}{d+2}$

We will generalize **Theorems 1 and 1′** in the spirit of the Stein-Tomas Theorem ([4], Chapter 6.5).

Theorem 2: *let $d \ge 4$ and let $f \in L^p(\mathbb{R}^d)$ where $1 \le p < \frac{2d}{d+2}$. If periodizations of f*

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x-\nu))$$

are in $L^2([0,1]^d)$ *for almost all rotations* $\rho \in SO(d)$ *and*

$$G^2 = \int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho < \infty$$

then $f \in L^2(\mathbb{R}^d)$*:*

$$\|f\|_2 \leq C(G + \|f\|_p) \tag{2.52}$$

where C *depends only on* d *and* p*.*

It will follow from the proof (see (2.37)) that we can replace $\int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho$ with $\int_{\rho \in SO(d)} \|g_\rho - \hat{g}(0)\|_2^2 d\rho$ in **Theorem 2**. We will also obtain an inverse theorem.

Theorem 2′: *let* $d \geq 5$ *and let* $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ *and* $1 \leq p < \frac{2d}{d+2}$ *and let* g_ρ *be periodizations of* f

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))$$

then $g_\rho \in L^2([0,1]^d)$ *for almost all rotations* $\rho \in SO(d)$ *and*

$$\int_{\rho \in SO(d)} \|g_\rho - \hat{g}_\rho(0)\|_2^2 d\rho \leq C(\|f\|_2 + \|f\|_p)^2 \tag{2.53}$$

where C *depends only on* d *and* p*.*

Since Schwartz functions are dense in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ it follows from **Theorem 2′** that we can define periodizations g_ρ of $f \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$ for a.e. $\rho \in SO(d)$ as elements of the quotient space of $L^2([0,1]^d)$ modulo constants.

Remarks: 1. As the following example shows, we can not replace $\int_{\rho \in SO(d)} \|g_\rho -$

$\hat{g}(0)\|_2^2 d\rho$ with $\int_{\rho\in SO(d)} \|g_\rho\|_2^2 d\rho$ in **Theorem 2′** when $p > 1$. Let $\phi : \mathbb{R}^d \to \mathbb{C}$ be a Schwartz function supported in $B(0,1)$ such that $\phi(0) = 1$. Put $\hat{f}(x) = \phi(\frac{x}{\epsilon})$. Then

$$g_\rho \equiv \hat{f}(0) = 1$$

but

$$\|f\|_p = \epsilon^{\frac{d}{p'}} \|\check{\phi}\|_p.$$

2. The next example from ([4], Chapter 6.3) shows that p can not be greater than $\frac{2d+2}{d+3}$ in **Theorem 2′**. Put

$$\hat{f}(x_1, ..., x_d) = \phi(\frac{x_1 - 1}{\epsilon^2}, \frac{x_2}{\epsilon}, ..., \frac{x_d}{\epsilon}) \tag{2.54}$$

where $\phi : \mathbb{R}^d \to \mathbb{C}$ is a Schwartz function supported in $B(0,2)$ such that $\phi = 1$ in $B(0,1)$. Then

$$\begin{aligned} \int_{\rho\in SO(d)} \|g_\rho - \hat{g}(0)\|_2^2 d\rho &= 2d \int_{|\xi|=1} |\hat{f}(\xi)|^2 d\sigma(\xi) \\ &\geq C\epsilon^{d-1} \end{aligned}$$

but

$$\|f\|_p^2 = \epsilon^{\frac{2d+2}{p'}} \|\check{\phi}\|_p.$$

It is an open question whether **Theorems 2** and **2′** are valid when $\frac{2d}{d+2} \leq p < \frac{2d+2}{d+3}$. We discuss this further in **Remark 2** at the end of the section.

Proof of Theorem 2:

The proof goes quite similarly to the one of **Theorem 1**. We will replace **Lemma 1** with

Lemma 3: *Let $q : \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[\frac{1}{2}, 2]$, let $f \in$*

$L^p(\mathbb{R}^d)$ where $1 \leq p \leq 2$ and let $b \in [0,1)$. Define $H_1 : \mathbb{R} \to \mathbb{C}$

$$H_1(t) = \frac{1}{\sqrt{t+b}} h_1(\sqrt{t+b}) q(\frac{\sqrt{t+b}}{N}).$$

Then for large enough N we have

$$\sum_{\nu \neq 0} |\hat{H}_1(\nu)| \leq \frac{C\|f\|_p^2}{N} \tag{2.55}$$

where C depends only on q and d.

Proof of Lemma 3:

The only difference in the proof is to obtain an inequality analogous to (2.13). Using Young's inequality we have $\|f * \tilde{f}\|_q \leq \|f\|_p^2$ where $1 + \frac{1}{q} = \frac{2}{p}$. Therefore, $|\int (f * \tilde{f})(x) w(x) dx| \leq \|f\|_p^2 \|w\|_{q'}$. Substituting derivatives of $\widehat{d\sigma_t}(x)\chi_{\{|x|\leq 1\}}$ with respect to t instead of w, we get the desired inequality

$$|h_1^{(k)}(t)| \leq Ct^{d-1}\|f\|_p^2 \tag{2.56}$$

where $t \geq 1$ and C depends only on k and d. □

The main difficulty is to prove a lemma analogous to **Lemma 2**:

Lemma 4: *Let $q : \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[\frac{1}{2}, 2]$, let $f \in L^p(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$ and let $b \in [0,1)$. Define $H_{2,N} : \mathbb{R} \to \mathbb{C}$*

$$H_{2,N}(t) = \frac{1}{\sqrt{t+b}} h_2(\sqrt{t+b}) q(\frac{\sqrt{t+b}}{N}).$$

Then we have

$$\sum_{\nu\neq 0} |\sum_{j\geq 0} \hat{H}_{2,2^j}(\nu)| \leq C\|f\|_p^2 \tag{2.57}$$

with C depending only on p, q and d.

Proof of Lemma 4:

Recall from (2.22) that

$$\hat{H}_{2,N}(\nu)| = 2\int (f * \tilde{f})(x) D_{N,\nu}(x)dx$$

where

$$D_{N,\nu}(x) = \chi_{\{|x|>1\}} e^{i2\pi\nu b} \int Nq(t) e^{-i2\pi\nu(Nt)^2} (Nt)^{d-1} \widehat{d\sigma}(Ntx)dt. \tag{2.58}$$

Denote by

$$K_\nu(x) = \sum_{l\geq 0} D_{2^l,\nu}(x). \tag{2.59}$$

Then

$$\begin{aligned} |\sum_{l\geq 0} \hat{H}_{2,2^l}(\nu)| &= 2|\int (f * \tilde{f})(x) \sum_{l\geq 0} D_{2^l,\nu}(x)dx| \\ &= 2|\int \tilde{f}(x)(K_\nu * f)(x)dx| \\ &\leq 2\|f\|_p \|K_\nu * f\|_{p'}. \end{aligned} \tag{2.60}$$

If $p' = \infty$ or $p' = 2$ we have

$$\|K_\nu * f\|_\infty \le \|K_\nu\|_\infty \|f\|_1$$
$$\|K_\nu * f\|_2 \le \|\hat{K}_\nu\|_\infty \|f\|_2.$$

First we will show that

$$\begin{aligned} \|K_\nu\|_\infty &\le \|\sum_{l\ge 0} |D_{2^l,\nu}|(x)\|_\infty \\ &\le C|\nu|^{-\frac{d}{2}}. \end{aligned} \tag{2.61}$$

It follows from (2.26) that

$$|D_{N,\nu}(x)| \le \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} \begin{cases} C(|\nu|N^2)^{-\frac{1}{2}} & \text{if } N \in [\frac{|x|}{4\nu}, \frac{|x|}{\nu}] \\ C_k(|\nu|N^2)^{-k-\frac{1}{2}} & \text{if } N \notin [\frac{|x|}{4\nu}, \frac{|x|}{\nu}] \end{cases}. \tag{2.62}$$

If $\nu > 0$ then the number of diadic $N \in [\frac{|x|}{4\nu}, \frac{|x|}{\nu}]$ is at most 3. If $\nu < 0$ then there are no N in $[\frac{|x|}{4\nu}, \frac{|x|}{\nu}]$. Therefore, choosing $k \ge \frac{d-1}{2}$ and summing (2.62) over all diadic N, we have

$$\sum_{l\ge 0} |D_{2^l,\nu}(x)| \le C|\nu|^{-\frac{d}{2}}$$

with C depending only on d and q.

Now we will show that

$$\|\hat{K}_\nu\|_\infty \le \|\sum_{l\ge 0} |\hat{D}_{2^l,\nu}|(y)\|_\infty \le C. \tag{2.63}$$

Since supp $\phi \in [\frac{1}{2}, 2]$ we can re-write (2.25) for a stronger version of the method of stationary phase ([8], Theorems 7.6.4, 7.6.5, 7.7.3)

$$|\int \phi(t)e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu N})^2}dt - \sum_{j=0}^{k-1} c_j(\nu N^2)^{-j-\frac{1}{2}}\phi^{(2j)}(\frac{|x|}{2\nu N})| \le \frac{c_k(|\nu|N^2)^{-k-\frac{1}{2}}}{\max(1, \frac{|x|}{8N|\nu|})^k}$$

where c_j are some constants. Therefore,

$$D_{N,\nu}(x) = \chi_{\{|x|>1\}}\frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}}e^{i2\pi\frac{|x|^2}{4\nu}}\sum_{j=0}^{k-1} c_j(\nu N^2)^{-j-\frac{1}{2}}\phi^{(2j)}(\frac{|x|}{2\nu N}) + \phi_k(x) \tag{2.64}$$

where $|\phi_k(x)| \le \chi_{\{|x|>1\}}\frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}}\frac{c_k(|\nu|N^2)^{-k-\frac{1}{2}}}{\max(1,\frac{|x|}{8N|\nu|})^k}$. Choosing $k \ge \frac{d+2}{2}$ we have

$$\begin{aligned} \|\hat{\phi}_k\|_\infty &\le \|\phi_k\|_1 \\ &= \int\limits_{|x|\le 8|\nu|N} |\phi_k|dx + \int\limits_{|x|>8|\nu|N} |\phi_k|dx \\ &\le \frac{C}{N} \end{aligned} \tag{2.65}$$

where C depends only on d and q. We can assume that $\nu > 0$ since $D_{N,\nu}(x) = \phi_k(x)$ for $\nu < 0$. We can also ignore $\chi_{\{|x|>1\}}$ in front of the sum in (2.64) because if $\frac{|x|}{2\nu N} \in [\frac{1}{2}, 2]$, then $|x| \ge \nu N \ge 1$. We will consider only the zero term in the sum. The other terms can be treated similarly. The Fourier transform of

$$\frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}}e^{i2\pi\frac{|x|^2}{4\nu}}(\nu N^2)^{-\frac{1}{2}}\phi(\frac{|x|}{2\nu N})$$

at point y is equal to

$$\begin{aligned} N^{\frac{d+1}{2}}(2\nu N)^{\frac{d+1}{2}}(\nu N^2)^{-\frac{1}{2}} \int_{\mathbb{R}^d} \psi(|x|)e^{i2\pi\nu N^2|x|^2}e^{-i2\pi 2\nu N x\cdot y}dx = \\ C(\nu N^2)^{\frac{d}{2}}e^{-i2\pi\nu|y|^2}\int_{\mathbb{R}^d}\psi(|x|)e^{i2\pi\nu N^2|x-\frac{y}{N}|^2}dx \end{aligned} \tag{2.66}$$

where $\psi(t) = \phi(t, 2\nu Nt)t^{-\frac{d-1}{2}}$ is a Schwartz function supported in $[\frac{1}{2}, 2]$ whose derivatives and the function itself are bounded uniformly in t, ν and N (see remark after

(2.24)). The same is true about partial derivatives of $\psi(|x|)$. Applying the stationary phase method for $\mathbb{R}^d$ ([8], Theorem 7.7.3) we get

$$|\int_{\mathbb{R}^d} \psi(|x|)e^{i2\pi\nu N^2|x-\frac{y}{N}|^2}dx| \leq \begin{cases} C(\nu N^2)^{-\frac{d}{2}} & \text{if } N \in [\frac{|y|}{2}, 2|y|] \\ C_k(\nu N^2)^{-k-\frac{d}{2}} & \text{if } N \notin [\frac{|y|}{2}, 2|y|] \end{cases}. \tag{2.67}$$

Therefore, the absolute value of (2.66) can be bounded from above by:

$$\leq \begin{cases} C & \text{if } N \in [\frac{|y|}{2}, 2|y|] \\ C_k(\nu N^2)^{-k} & \text{if } N \notin [\frac{|y|}{2}, 2|y|] \end{cases}. \tag{2.68}$$

Similar inequalities hold for Fourier transforms for the rest of the terms in the sum in (2.64). The number of diadic $N \in [\frac{|y|}{2}, 2|y|]$ is bounded by 3. Using (2.65), choosing $k \geq 1$ in (2.68) and summing over all diadic N, we get

$$\sum_{l \geq 0} |\hat{D}_{2^l,\nu}(y)| \leq C \tag{2.69}$$

with C depending only on d and q. Using (2.61) and (2.63) and interpolating between $p = 1$ and $p = 2$, we obtain

$$\|K_\nu * f\|_{p'} \leq C|\nu|^{-\alpha_p}\|f\|_p$$

where $\alpha_p = \frac{d}{2}\frac{2-p}{p}$. $\alpha_p > 1$ if $p < \frac{2d}{d+2}$. Summing (2.60) over all $\nu \neq 0$, we get the desired inequality

$$\sum_{\nu \neq 0} |\sum_{j \geq 0} \hat{H}_{2,2^j}(\nu)| \leq C\|f\|_p^2.$$

□

Now we are in a position to proceed with the proof of **Theorem 2**. The proof is almost the same as the one of **Theorem 1**. We need also to replace inequality (2.45) with the following one:

$$\begin{aligned}\int_0^1 h(t)dt &= \int_{|y|\le 1} |\hat{f}(y)|^2 dy \\ &\le C\|\hat{f}\|_{p'}^2 \\ &\le C\|f\|_p^2\end{aligned}$$

where $p \le 2$ and C depends only on d. An argument similar to the one used to get (2.42), (2.44) and (2.46) (note that the interchange of summation by ν and N is not a problem) yields the desired inequality

$$\begin{aligned}\int |\hat{f}(x)|^2 dx &= \int_0^\infty |\hat{f}(\xi)|^2 d\sigma_t(\xi) dt \\ &= \int_0^\infty h(t)dt \\ &\le C(G^2 + \|f\|_p^2)\end{aligned}$$

with C depending on d and p. □

The proof of **Theorem 2′** is the same (see the argument after the proof of **Theorem 1**). The important thing is that we exclude $\hat{g}_\rho(0) = \hat{f}(0)$ in (2.48) now.

Final remarks: 1. We can further generalize **Theorem 2′**. Fix some $q \in [1, \frac{2d}{d+2})$. Interpolating between the trivial $p = 1$ and $p = 2$, we obtain the following generalization of the **Theorem 2′** for $1 \le p \le 2$: let $d \ge 5$ and let $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and let g_ρ be periodizations of f

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu)) \tag{2.70}$$

then $g_\rho \in L^p([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$\int\limits_{\rho\in SO(d)} \|g_\rho - \hat{g}_\rho(0)\|_p^{p'} d\rho \le C(\|f\|_p + \|f\|_q)^{p'}$$

where C depends only on d and q.

We can restate this result in the following way: if $1 \le p < \frac{2d}{d+2}$ then

$$\int\limits_{\rho\in SO(d)} \|g_\rho - \hat{g}_\rho(0)\|_p^{p'} d\rho \le C\|f\|_p^{p'},$$

if $1 \le q < \frac{2d}{d+2}$ and $\frac{2d}{d+2} \le p \le 2$ then

$$\int\limits_{\rho\in SO(d)} \|g_\rho - \hat{g}_\rho(0)\|_p^{p'} d\rho \le C(\|f\|_p + \|f\|_q)^{p'}.$$

2. Conditionally on the exponent pair conjecture ([12], Chapter 4, Conjecture 2) we can clarify what happens when $\frac{2d}{d+2} \le p < \frac{2d+2}{d+3}$. In our case the conjecture says that

$$|\sum_{n\le\nu\le m} e^{i\frac{|x|^2}{\nu}}| \le C_\epsilon |x|^\epsilon n^{\frac{1}{2}} \tag{2.71}$$

where $m \le 2n$ and $|x| \ge n$. Let $\beta(x) = \max(1, |x|)$.

Proposition. Theorems 2 and 2′ hold if we replace $\|f\|_p$ with $\|\beta^\epsilon f\|_p$ and if $p < \frac{2d+2}{d+3}$, provided the conjecture is valid.

Using the example (2.54) we can show that the **Proposition** is sharp up to ϵ in the range of p for the estimate (2.53).

Proof of Proposition:

The main issue is to improve the result of **Lemma 4.** Denote by

$$L_j(x) = \sum_{2^j \leq \nu \leq 2^{j+1}} K_\nu(x). \tag{2.72}$$

Using summation by parts we obtain from (2.64), (2.59) and (2.71) that

$$\begin{aligned} |L_j(x)| &= |\sum_{2^j \leq \nu \leq 2^{j+1}} K_\nu(x)| \\ &\leq C_\epsilon |x|^\epsilon 2^{-\frac{d-1}{2}j}. \end{aligned}$$

We will deal with the following expression instead of (2.60)

$$\begin{aligned} |\int \tilde{f}(x)(L_j * f)(x)dx| &= |\int \tilde{f}(x)\beta^\epsilon \frac{(L_j * f)(x)}{\beta^\epsilon}dx| \\ &\leq \|\beta^\epsilon f(x)\|_p \|\frac{(L_j * f)}{\beta^\epsilon}\|_{p'}. \end{aligned}$$

If $p' = \infty$ or $p' = 2$ we have

$$\begin{aligned} \|\frac{(L_j * f)}{\beta^\epsilon}\|_\infty &\leq \|\frac{\int_{|y|\leq|x|} |L_j(x-y)| \cdot |f(y)|dy + \int_{|y|\geq|x|} |L_j(x-y)| \cdot |f(y)|dy}{\beta^\epsilon(x)}\|_\infty \\ &\leq \|\frac{L_j}{\beta^\epsilon}\|_\infty \|\beta^\epsilon f\|_1 \\ &\leq C_\epsilon 2^{-\frac{d-1}{2}j} \|\beta^\epsilon f\|_1, \\ \|\frac{(L_j * f)}{\beta^\epsilon}\|_2 &\leq \|L_j * f\|_2 \\ &\leq \|\hat{L}_j\|_\infty \|f\|_2 \\ &\leq C 2^j \|f\|_2. \end{aligned}$$

Interpolating between $p = 1$ and $p = 2$, we obtain

$$\|\frac{(L_j * f)}{\beta^\epsilon}\|_{p'} \leq C_\epsilon 2^{-j\alpha_p} \|\beta^\epsilon f\|_p$$

where $\alpha_p = \frac{d+1}{p} - \frac{d+3}{2}$. $\alpha_p > 0$ if $p < \frac{2d+2}{d+3}$. □

3. Concerning the lower dimensional cases we can use the following results from the Number Theory:

$$r_3(n) \le Cn^{\frac{1}{2}} \ln n \ln \ln n,$$

$$r_4(n) \le Cn \ln \ln n.$$

See for example ([2]). There is an infinite arithmetic progression, e.g., $n = 8k+1$, such that

$$r_3(n) \ge C_\epsilon n^{\frac{1}{2}-\epsilon}.$$

See for example ([6]). Then **Theorem 1** holds when $d = 3$ and **Theorem 1′** holds when $d = 3$ or $d = 4$ if we replace

$$\|g_\rho\|_2^2 = \sum_{m \in \mathbf{Z}^d} |\hat{g}_\rho(m)|_2^2$$

with

$$\sum_{m \in \mathbf{Z}^d} |m|^\epsilon |\hat{g}_\rho(m)|_2^2,$$

$$\sum_{m \in \mathbf{Z}^d, |m|>3} \frac{|\hat{g}_\rho(m)|_2^2}{\ln |m| \ln \ln |m|}$$

or

$$\sum_{m \in \mathbf{Z}^d, |m|>3} \frac{|\hat{g}_\rho(m)|_2^2}{\ln \ln |m|}$$

correspondingly.

2.4 Case $d = 2$ and $p = \infty$

Some results on the Steinhaus tiling problem are related to **Theorem 1** since periodizations naturally appear in the problem of Steinhaus. In particular, showing that there are no measurable Steinhaus sets in dimensions greater than two, M. Kolountzakis and T. Wolff proved the following theorem on periodizations in higher dimensions

([10], Theorem 1) which can be viewed as some version of **Theorem 1** for $p = \infty$:

Kolountzakis and Wolff's Theorem *If* $f \in L^1(\mathbb{R}^d)$ *and its periodizations* g_ρ *are constants for almost all rotations* $\rho \in SO(d)$, *then* f *is continuous and, in fact,*

$$\|f\|_\infty \leq C_d \|f\|_1$$

provided that the dimension d is at least 3.

Obviously, this statement is false when $d = 1$. We will show that it doesn't hold either if $d = 2$. The fact that g_ρ are constant means that $\hat{f}(\rho(k,l)) = \hat{g}_\rho(k,l) = 0$ for all $(k,l) \in \mathbb{Z}^2 \backslash (0,0)$ and almost all $\rho \in SO(d)$ which means that $\hat{f}$ vanishes on all circles of radii $\sqrt{l^2+k^2} > 0$. Denote by X the Banach space of functions from $L^1(\mathbb{R}^2)$ whose Fourier transforms vanish on all circles of radii $\sqrt{l^2+k^2} > 0$

$$X = \{f \in L^1(\mathbb{R}^2) : \hat{f}(\mathrm{r}) = 0 \text{ if } |\mathrm{r}| = \sqrt{l^2+k^2}, (k,l) \in \mathbb{Z}^2 \backslash (0,0)\}.$$

We will use the notation $x \lesssim y$ meaning $x \leq Cy$, and $x \sim y$ meaning that $x \lesssim y$ and $y \lesssim x$ for some constant $C > 0$ independent from x and y.

The next lemma crucially depends on the following fact from the Number Theory ([5], p.22):
The number of integers in $[n, 2n]$ *which can be represented as sums of two squares is* $n\epsilon_n$ *where* $\epsilon_n \lesssim \frac{1}{\ln^{1/2} n} \to 0$ *as* $n \to \infty$.

Lemma 5: *There exists a sequence of Schwartz functions* $f_n \in X$ *such that*

$$\lim_{n\to\infty} \frac{\|f_n\|_1}{|f_n(0)|} = 0.$$

Proof of Lemma 5: Let $a_1 < a_2 < a_3 < \ldots$ be the enumeration of numbers

$a_m = \sqrt{l^2+k^2}$ in ascending order. Denote $\delta_m = a_{m+1} - a_m$. As we already said the number of a_m in $[\sqrt{n}, 2\sqrt{n}]$ is $n\epsilon_n$. Let a_{m_0} and a_{m_1} be correspondingly the smallest and the largest such a_m. Then

$$\sum_{m=m_0}^{m_1-1} \delta_m = a_{m_1} - a_m \sim \sqrt{n}.$$

Let $\delta = \frac{C}{\sqrt{n\epsilon_n}}$ with small enough constant $C > 0$ so that if

$$M = \{m, m_0 \le m < m_1 : \delta_m \ge \delta\}$$

then

$$\sqrt{n} \lesssim \sum_{m\in M} \delta_m$$

since $m_1 - m_0 \sim n\epsilon_n$. Choose coordinate axes x and y. We will construct $\hat{f}_n$ supported in $\bigcup_{m\in M} R_m$ where R_m is a largest possible rectangle inscribed between circles of radius a_m and a_{m+1} with sides parallel to the coordinate axes. Then R_m is of size $\sim \delta_m \times \sqrt{\delta_m a_m} \gtrsim \delta_m \times \sqrt{\delta\sqrt{n}}$. We will split each rectangle R_m further into smaller $[\frac{\delta_m}{\delta}]$ rectangles r of the same size $\sim \delta \times \sqrt{\delta\sqrt{n}}$. The number of these rectangles r is

$$\begin{aligned} N &= \sum_{m\in M} \left[\frac{\delta_m}{\delta}\right] \\ &\sim \sum_{m\in M} \frac{\delta_m}{\delta} \\ &\sim \frac{\sqrt{n}}{\frac{1}{\sqrt{n\epsilon_n}}} = n\epsilon_n \end{aligned}$$

since $\delta_m \ge \delta$ for $m \in M$. Enumerate these rectangles r_k, $k = 1, ..., N$. Let r_k be centered at $(\lambda_k, 0)$ It is clear that $|\lambda_k - \lambda_l| \ge \delta$ for $k \ne l$. Let ϕ be a nonnegative Schwartz function on $\mathbb{R}$ supported in $[-\frac{1}{2}, \frac{1}{2}]$. Define $\hat{f}_n$ as the following sum:

$$\hat{f}_n(x,y) = \sum_{k=1}^{N} \phi(\frac{x-\lambda_k}{\delta})\phi(\frac{y}{\sqrt{\delta\sqrt{n}}}). \tag{2.73}$$

The k-th term in (2.73) is supported in r_k. Therefore, $\hat{f}_n$ is a Schwartz function supported in $\bigcup_{m\in M} R_m$. Hence $\hat{f}_n$ vanishes on all circles of radii a_l. Taking the inverse Fourier transform of (2.73), we get

$$f_n(\xi,\eta) = \delta\hat{\phi}(\xi\delta)\sqrt{\delta\sqrt{n}}\hat{\phi}(\eta\sqrt{\delta\sqrt{n}})\sum_{k=1}^{N} e^{i\lambda_k\xi}. \tag{2.74}$$

Then

$$\begin{aligned} f_n(0) &= \delta\hat{\phi}(0)\sqrt{\delta\sqrt{n}}\hat{\phi}(0)N \\ &\sim \frac{1}{\sqrt{n}\epsilon_n}\sqrt{\frac{1}{\sqrt{n}\epsilon_n}\sqrt{n}n\epsilon_n} \\ &= \frac{\sqrt{n}}{\sqrt{\epsilon_n}}. \end{aligned} \tag{2.75}$$

We used here that $\hat{\phi}(0) = \int \phi > 0$ is some nonzero constant. Denote

$$g(x) = \sum_{k=1}^{N} e^{i\frac{\lambda_k}{\delta}\xi}.$$

Since $|\frac{\lambda_k - \lambda_l}{\delta}| \geq \frac{\delta}{\delta} = 1$ for $k \neq l$ we have

$$\int_I |g|^2 \sim N$$

for any interval I of length 4π (see ([14], Theorem 9.1)). Therefore,

$$\begin{aligned} \int_I |g| &\leq \sqrt{|I|}\sqrt{\int_I |g|^2} \\ &\lesssim \sqrt{N} \end{aligned} \tag{2.76}$$

for any interval I of length 4π. Since $\hat{\phi}$ is a Schwartz function, we have that

$$|\hat{\phi}(x)| \lesssim \frac{1}{1+x^2}.$$

The L^1 norm of (2.74) is

$$
\begin{aligned}
\int |f_n(\xi,\eta)|d\xi d\eta &= \|\hat{\phi}\|_1 \int |\hat{\phi}(\xi)| \cdot |\sum_{k=1}^{N} e^{i\frac{\lambda_k}{\delta}\xi}|d\xi \\
&= C \sum_{l=-\infty}^{\infty} \int_{l4\pi}^{(l+1)4\pi} |\hat{\phi}(\xi)| \cdot |g(\xi)|d\xi \\
&\lesssim \sum_{l=-\infty}^{\infty} \frac{1}{1+l^2}\sqrt{N} \\
&\lesssim \sqrt{n\epsilon_n}. \qquad (2.77)
\end{aligned}
$$

Dividing (2.77) by (2.75) we obtain the desired result

$$
\begin{aligned}
\frac{\|f_n\|_1}{|f_n(0)|} &\leq \frac{\sqrt{n\epsilon_n}}{\frac{\sqrt{n}}{\sqrt{\epsilon_n}}} \\
&= \epsilon_n \to 0
\end{aligned}
$$

as $n \to \infty$.

□

Corollary: It follows immediately from the lemma that

$$\sup_{f \in X} \frac{\|f\|_{L^\infty(D(0,1))}}{\|f\|_1} = \infty.$$

We claim that there exists a function $f \in X$ such that $\|f\|_{L^\infty(D(0,1))} = \infty$. Suppose towards a contradiction that this is not true. Then the restriction operator

$$T : f \to f|_{D(0,1)}$$

maps X to $L^\infty(D(0,1))$. Note that if $f_n \to f$ in L^1 and $f_n \to g$ in $L^\infty(D(0,1))$, then $f = g$ a.e. on $D(0,1)$. An application of the Closed Graph theorem shows that T is a bounded operator acting from X to $L^\infty(D(0,1))$. This contradicts to the **Corollary**. Thus we proved our claim. Obviously, this function f is not continuous. Therefore, it can serve as a counterexample to the **Theorem** when $d = 2$.

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