Some estimates of Fourier transforms

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Abstract

This work consists of two independent parts. In the first part we prove several results related to the theorem of Logvinenko and Sereda on determining sets for functions with Fourier transforms supported in a parallelepiped. We obtain a polynomial instead of exponential bound in this theorem, and we extend it to the case of functions with Fourier transforms supported in the union of a bounded number of parallelepipeds. When dimension $d = 1$ we also consider the case of infinitely many lacunary intervals. We generalize the Zygmund theorem for lacunary series whose Fourier coefficients are replaced with polynomials of uniformly bounded degree. We give also a necessary condition for the support of Fourier transforms for which the Logvinenko-Sereda theorem still holds.

In the second part we prove that the $L^2([0,1]^d \times SO(d))$ norm of periodizations of a function from $L^1(\mathbb{R}^d)$ is equivalent to the $L^2(\mathbb{R}^d)$ norm of the function itself in higher dimensions. We generalize the statement for functions from $L^p(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$ in the spirit of the Stein-Tomas theorem. We also show that the following theorem due to M. Kolountzakis and T. Wolff does not hold if dimension $d = 2$: if periodizations of a function from $L^1(\mathbb{R}^d)$ are constants, then the function is continuous and bounded provided that the dimension $d$ is at least three.
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0.1 Introduction

This work consists of two independent parts. We use various estimations of Fourier transforms to obtain results.

The first part considers the Logvinenko-Sereda theorem which is one of the basic examples of the so-called Uncertainty Principle in Fourier Analysis. This Principle states that a function and its Fourier transform can not be simultaneously supported on "small" sets. Although this formulation is rather vague, we obtain rigorous results. Intervals and compliments of "relatively dense" subsets play the role of small sets in our situation. The Logvinenko-Sereda theorem is based on properties of entire functions of exponential type. We will improve on this theorem by getting an optimal estimate and generalize it for the case of finitely many intervals. We will also investigate the case of infinitely many lacunary intervals. We will generalize a result of F. Nazarov for the Zygmund theorem on lacunary series. We will also give a necessary condition for the support of Fourier transforms for which "relatively dense" subsets are still determining sets.

The second part is devoted to periodizations of functions from $L^1(\mathbb{R}^d)$ in higher dimensions. Some results on the Steinhaus tiling problem due to M. Kolountzakis and T. Wolff are related to mine since periodizations naturally appear in the problem of Steinhaus. The main idea can be formulated in this way. If we are given some information on periodizations, what can we say about the function itself and vice versa? It is rather natural to formulate this problem in terms of various norms. Using some facts from Number Theory, we prove that the $L^2([0,1]^d \times SO(d))$ norm of periodizations of a function from $L^1(\mathbb{R}^d)$ is equivalent to the $L^2(\mathbb{R}^d)$ norm of the function itself in higher dimensions. We generalize the statement for functions from $L^p(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$ in the spirit of the Stein-Tomas theorem. We will also show that the result due to M. Kolountzakis and T. Wolff, which holds when dimension $d \geq 3$, does not hold when $d = 2$. 

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Chapter 1  Some results related to the
Logvinenko-Sereda theorem

1.1  Overview

The purpose of this work is to study the behavior of functions whose Fourier trans­
forms are supported in a parallelepiped or in a union of finitely many parallelepipeds
on "thick" subsets of $\mathbb{R}^d$. A main result of this type was proven by Logvinenko and
Sereda.

By a "thick" or "relatively dense" subset of $\mathbb{R}^d$ we mean a measurable set $E$
for which there exist a parallelepiped $\Pi$ with sides of length $a_1, a_2, \ldots, a_d$ parallel to
coordinate axes and $\gamma > 0$ such that

$$|E \cap (\Pi + x)| \geq \gamma |\Pi| \quad (1.1)$$

for every $x \in \mathbb{R}^d$.

The Logvinenko-Sereda Theorem, $d = 1$: let $J$ be an interval with $|J| = b$. If $f \in L^p(\mathbb{R})$, $p \in [1, +\infty]$, and $\text{supp} \, \hat{f} \subset J$ and if a measurable set $E$ satisfies (1.1),
then

$$\|f\|_{L^p(E)} \geq \exp(-C \cdot \frac{(ab + 1)}{\gamma}) \cdot \|f\|_p \quad (1.2)$$

It is a well-known fact that the condition (1.1) is also necessary for an inequality
of the form
\[ \|f\|_{L^p(E)} \geq C \cdot \|f\|_p \]
to hold. See for example ([7], p.113).

We will improve the estimate (1.2) by getting a polynomial dependence on \(\gamma\) and show that our estimate is optimal except for the constant \(C\):

**Theorem 1:** let \(J\) be a parallelepiped with sides of length \(b_1, b_2, \ldots, b_d\) parallel to coordinate axes. If \(f \in L^p(\mathbb{R}^d), p \in [1, +\infty], \) and \(\text{supp} \ f \subseteq J\) and if a measurable set \(E\) satisfies (1.1), then

\[ \|f\|_{L^p(E)} \geq \left( \frac{\gamma}{C^d} \right)^{C(d + \sum_{j=1}^d a_j b_j)} \cdot \|f\|_p. \]

We will also generalize the Logvinenko-Sereda theorem to functions whose Fourier transforms are supported in a union of finitely many parallelepipeds:

**Theorem 2:** let \(J_k\) be identical parallelepipeds with sides of length \(b_1, b_2, \ldots, b_d\) parallel to coordinate axes. If \(f \in L^p, p \in [1, +\infty], \) and \(\text{supp} \ f \subset \bigcup_{k=1}^n J_k\) and if a measurable set \(E\) satisfies (1.1), then

\[ \|f\|_{L^p(E)} \geq c(\gamma, n, a \cdot b, d, p) \cdot \|f\|_p \]

where \(c(\gamma, n, a \cdot b, d, p) = (\frac{C^d}{\gamma})^{-a \cdot b (\frac{C^d}{\gamma})^{n - n + \frac{p-1}{p}}} \) depends only on the number of parallelepipeds but not how they are placed.

The next natural step is to consider the case of infinitely many parallelepipeds. We will conjecture the following theorem for infinitely many lacunary intervals when dimension \(d = 1\):
Theorem 3: let \( \Lambda = \{\lambda_k\}_{k=-\infty}^{\infty} \) be lacunary and let \( E \) be "relatively dense." If \( f \in L^2 \) and \( \text{supp} \hat{f} \subset \bigcup_{k=-\infty}^{\infty} [\lambda_k - \frac{b}{2}, \lambda_k + \frac{b}{2}] \), then

\[
\|f\|_{L^2(E)} \geq C(\gamma, N, a, b) \cdot \|f\|_2.
\]

See section 1.4.1 for the definition of lacunarity. We will prove some partial results: the above Theorem holds when \( \gamma \) is large or when \( b \) is small. It holds too when \( E \) is periodic. We will also generalize Nazarov's result for the Zygmund theorem on lacunary series ([13], Theorem 3.6):

Nazarov's Theorem: let \( \Lambda = \{\lambda_k\}_{k=-\infty}^{\infty} \) be a lacunary sequence of integer numbers with the maximal number of representations \( N \). If \( f \in L^2([0,2\pi]) \) with \( \text{spec} f \in \Lambda \) then

\[
\int_E |f|^2 \geq C(|E|, N)\|f\|_2^2
\]

with \( C(|E|, N) = e^{-C(N, x)|E|^{2+\varepsilon}} \) for every \( E \) with positive measure.

We extend the above theorem to generalized lacunary series with Fourier coefficients being replaced by polynomials of uniformly bounded degree:

Theorem 4: let \( I \) be an interval of length 1 and \( E \) be a subset of \( I \) of positive measure. If \( \Lambda = \{\lambda_k\}_{k=-\infty}^{\infty} \) is lacunary in the sense of (1.29) and \( f(x) = \sum_{k=-\infty}^{\infty} p_k(x) e^{i\lambda_k x} \) where \( p_k(x) \) are polynomials of degree at most \( m \), then

\[
\int_E |f|^2 \geq C(|E|, N, m) \int_I |f|^2
\]

where \( C(|E|, N, m) = e^{-\left(\ln CN\right)N^2 \left(\frac{E}{m}\right)^{C(m+1)}} \).
Note that the constant $C$ below is not fixed and varies appropriately from one equality or inequality to another without being mentioned.

1.2 Case of one parallelepiped

Proof of Theorem 1:
First we treat the case when $p \in [1, +\infty)$. Without loss of generality we can always assume that $J$ is centered at the origin:

$$J = \left[\frac{-b_1}{2}, \frac{b_1}{2}\right] \times \cdots \times \left[\frac{-b_d}{2}, \frac{b_d}{2}\right].$$

By considering $f(\frac{x_1}{a_1}, \ldots, \frac{x_d}{a_d})$ instead of $f$, we can also assume that $|E \cap \Pi| \geq \gamma$ for all cubes $\Pi$ with sides of length 1 and parallel to coordinate axes and

$$\text{supp} \hat{f} \subset \left[\frac{-a_1b_1}{2}, \frac{a_1b_1}{2}\right] \times \cdots \times \left[\frac{-a_d b_d}{2}, \frac{a_d b_d}{2}\right],$$

just say

$$\text{supp} \hat{f} \subset \left[\frac{-b_1}{2}, \frac{b_1}{2}\right] \times \cdots \times \left[\frac{-b_d}{2}, \frac{b_d}{2}\right].$$

Define $b = (b_1, \ldots, b_d)$. Bernstein's inequality ([2], Theorem 11.3.3) applied to each variable separately gives that

$$\int |f^{(\alpha)}|^p \leq (C \cdot b)^{\alpha}. \int |f|^p$$

with $C = \frac{1}{2}$. Here $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multiindex in $\mathbb{R}^d$ with nonnegative integer coordinates. $|\alpha| = \sum_{j=1}^d \alpha_j$. $f^{(\alpha)} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f$. If $x \in \mathbb{R}^d$ then $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. $\alpha! = \alpha_1! \cdots \alpha_d!$.

Divide the whole $\mathbb{R}^d$ into cubes $\Pi$ with sides of length 1 and parallel to coordinate
axes. Choose $A > 1$. Call a cube $\Pi$ bad if $\exists \alpha \neq 0$ such that

$$\int_\Pi |f^{(\alpha)}|^p \geq 2^d A^{|\alpha|p} (C \cdot b)^{\alpha p} \cdot \int_\Pi |f|^p.$$  

Then

$$\int_{\bigcup_{\Pi \text{ is bad}}} |f|^p \leq \int_{\bigcup_{\Pi \text{ is bad}}} \sum_{\alpha \neq 0} \frac{1}{2^d A^{|\alpha|p} (C \cdot b)^{\alpha p}} |f^{(\alpha)}|^p$$

$$= \sum_{\alpha \neq 0} \frac{1}{2^d A^{|\alpha|p} (C \cdot b)^{\alpha p}} \int_{\bigcup_{\Pi \text{ is bad}}} |f^{(\alpha)}|^p$$

$$\leq \sum_{\alpha \neq 0} \frac{1}{2^d A^{|\alpha|p}} \int_{\bigcup_{\Pi \text{ is bad}}} |f|^p$$

$$= \left( \frac{1}{(1 - \frac{1}{A^p})^d} - 1 \right)/2^d \int_{\bigcup_{\Pi \text{ is bad}}} |f|^p. \quad (1.3)$$

Choosing $A = 3$ and applying (1.3), we obtain

$$\int_{\bigcup_{\Pi \text{ is bad}}} |f|^p \leq \frac{1}{2} \int_{\bigcup_{\Pi \text{ is bad}}} |f|^p.$$  

Therefore,

$$\int_{\bigcup_{\Pi \text{ is good}}} |f|^p \geq \frac{1}{2} \int_{\bigcup_{\Pi \text{ is good}}} |f|^p. \quad (1.4)$$

Replace $3C$ with $C$.

We claim that $\exists B > 1$ such that if $\Pi$ is a good cube, then $\exists x \in \Pi$ with the property that

$$|f^{(\alpha)}(x)|^p \leq 4^d \cdot B^{|\alpha|p} (C \cdot b)^{\alpha p} \cdot \int_\Pi |f|^p \ \forall \alpha.$$
Suppose towards a contradiction that this is not true. Then

\[
2^d \cdot \int_{\Pi} |f|^p \leq \sum_{\alpha} \frac{1}{B^{\dim p} (C \cdot b)^{\alpha p}} |f^{(\alpha)}(x)|^p \quad \forall x \in \Pi. \tag{1.5}
\]

Integrate both sides of (1.5) over \(\Pi\):

\[
2^d \cdot \int_{\Pi} |f|^p \leq \sum_{\alpha} \frac{1}{B^{\dim p} (C \cdot b)^{\alpha p}} \int_{\Pi} |f^{(\alpha)}(x)|^p \\
\leq \sum_{\alpha} \frac{1}{B^{\dim p}} \int_{\Pi} |f|^p \\
= \frac{1}{(1 - (\frac{1}{B})^p)^d} \int_{\Pi} |f|^p. \tag{1.6}
\]

Choose \(B = 3\) and apply (1.6). So

\[
2^d \cdot \int_{\Pi} |f|^p \leq (3/2)^d \int_{\Pi} |f|^p.
\]

This contradiction proves our claim. Replace \(3C\) with \(C\).

We will need to prove the following local estimate:

\[
\int_{E \cap \Pi} |f|^p \geq \left( \frac{\gamma}{C^d} \right)^{3d + C_p} \sum_{j=1}^{d} b_j \int_{\Pi} |f|^p
\]

for every good cube \(\Pi\). Without loss of generality we can assume that

\[
\Pi = [-\frac{1}{2}, \frac{1}{2}]^d
\]

by considering a shift \(f(x - n)\) which has

\[
\text{supp} f(x - n) \subset [-\frac{b_1}{2}, \frac{b_1}{2}] \times ... \times [-\frac{b_d}{2}, \frac{b_d}{2}].
\]
Let \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \) be a complex vector. Let \( D_j(0, R) = \{ z_j \in \mathbb{C} : |z_j| \leq R \} \) be a disk in the complex plane \( \mathbb{C} \). If

\[
z \in D_1(0, R) \times \cdots \times D_d(0, R) \subset D_1(x_1, R + \frac{1}{2}) \times \cdots \times D_d(x_d, R + \frac{1}{2})
\]

then

\[
|f(z)| \leq \sum_{\alpha} \frac{|f^{(\alpha)}(x)|}{\alpha!} \cdot |z - x|^\alpha \\
\leq \sum_{\alpha} 4^d \frac{(R + \frac{1}{2})^{\alpha}}{\alpha!} (Cb)^\alpha \|f\|_{L^p(\Pi)} \\
= 4^d \exp(C(R + \frac{1}{2}) \sum_{j=1}^{d} b_j) \cdot \|f\|_{L^p(\Pi)}. \tag{1.7}
\]

We can assume that \( \int_{\Pi} |f|^p = 1 \). Then \( \exists y \in \Pi \) such that \( |f(y)| \geq 1 \). Following an idea of F. Nazarov we can choose spherical coordinates centered at \( y \) and find a segment \( I \in \Pi, y \in I \) and \( \frac{|E \cap \Pi|}{|\Pi|} \geq C(d) \gamma \):

\[
\gamma \leq |E \cap \Pi| \\
= \int_{\Pi} \chi_{E \cap \Pi}(x) dx \\
= \int_{\xi} \int_{r=0}^{r(\xi)} \chi_{E \cap \Pi}(y + r\xi) r^{d-1} dr d\sigma(\xi). \tag{1.8}
\]

It follows from (1.8) that \( \exists \eta \in S^{d-1} \) such that

\[
\gamma \leq \sigma_{d-1} \int_{r=0}^{r(\eta)} \chi_{E \cap \Pi}(y + r\eta) r^{d-1} dr
\]

where \( \sigma_{d-1} = |S^{d-1}| \). Let \( I \) be the longest interval in \( \Pi \) centered at \( y \) in the direction of \( \eta \):

\[
I = y + t|I|\eta, \quad 0 \leq t \leq 1. \tag{1.9}
\]
It is clear that $|I| \leq d^{1/2}$. Therefore,

$$\frac{|E \cap I|}{|I|} \geq \frac{\gamma}{\sigma_{d-1} \cdot d^{d/2}} \geq \frac{\gamma}{C_d}. \tag{1.10}$$

It follows from (1.10) that

$$|\{t \in [0, 1] : y + t|\eta \in E \cap I\}| = \frac{|E \cap I|}{|I|} \geq \frac{\gamma}{C_d}. \tag{1.11}$$

Define an analytic function of one complex variable $w \in \mathbb{C}$

$$F(w) = f(y + w|\eta). \tag{1.12}$$

Then $|F(0)| = |f(y)| \geq 1$. If $w \in D(0, R)$ then

$$z = y + w|\eta \in D_1(0, R + \frac{1}{2}) \times \ldots \times D_d(0, R + \frac{1}{2})$$

since $|y_j| \leq \frac{1}{2}$ and $|\eta_j| \leq 1$ for $j = 1, ..., d$. Then it follows from (1.7) that

$$|F(w)| \leq 4^d \exp(C(R + 1) \sum_{j=1}^{d} b_j). \tag{1.13}$$

Now we will give a local estimate for analytic functions of one complex variable.

**Lemma 1:** Let $\phi(z)$ be analytic in $D(0, 5)$ and let $I$ be an interval of length 1 such that $0 \in I$ and let $E \subset I$ be a measurable set of positive measure. If $|\phi(0)| \geq 1$ and $M = \max_{|x| \leq 4} |\phi(x)|$, then

$$\sup_{x \in I} |\phi(x)| \leq \left(\frac{C}{|E|}\right)^{\frac{b}{2a}} \sup_{x \in E} |\phi(x)|. \tag{1.14}$$
Proof of Lemma 1:

Let \( w_1, w_2, \ldots, w_n \) be the zeros of \( \phi \) in \( D(0, 2) \). If

\[
g(z) = \phi(z) \cdot \prod_{k=1}^{n} \frac{4 - \bar{w}_k z}{2(w_k - z)} = \phi(z) \cdot \frac{Q(z)}{P(z)}
\]

then \( |g(0)| \geq 1 \) and \( \max_{|z| \leq 2} |g(z)| \leq M \) by the property of Blaschke products. Applying Harnack's inequality to the positive harmonic function \( \ln M - \ln |g(z)| \) in \( D(0, 2) \) we have:

\[
\max_{|z| \leq 1} (\ln M - \ln |g(z)|) \leq 3 \ln M.
\]

Therefore,

\[
\min_{|z| \leq 1} |g(z)| \geq M^{-2}.
\]

This gives us

\[
\frac{\max_{x \in I} |g(x)|}{\min_{x \in I} |g(x)|} \leq M^3.
\]

We can give a similar estimate for \( Q \):

\[
\frac{\max_{x \in I} |Q(x)|}{\min_{x \in I} |Q(x)|} \leq \frac{\max_{|z| \leq 1} \prod_{k=1}^{n} |4 - \bar{w}_k z|}{\min_{|z| \leq 1} \prod_{k=1}^{n} |4 - \bar{w}_k z|} \leq 3^n.
\]

From the Remez inequality for polynomials ([3], Theorem 5.1.1) it follows that:

\[
\sup_{x \in I} |P(x)| \leq \left( \frac{4}{|E|} \right)^n \sup_{x \in E} |P(x)|.
\]
Therefore,
\[
\sup_{x \in \mathcal{I}} |\phi(x)| \leq \max_{x \in \mathcal{I}} |g(x)| \cdot \frac{\max_{x \in \mathcal{I}} |P(x)|}{\min_{x \in \mathcal{I}} |Q(x)|}
\leq M^3 \cdot 3^n \cdot \left( \frac{C}{|E|} \right)^n \cdot \min_{x \in \mathcal{I}} |g(x)| \cdot \frac{\max_{x \in \mathcal{I}} |P(x)|}{\max_{x \in \mathcal{I}} |Q(x)|}
\leq M^3 \cdot \left( \frac{C}{|E|} \right)^n \cdot \sup_{x \in \mathcal{E}} |\phi(x)|.
\]

From Jensen's formula it follows that \( n \leq \frac{\ln M}{\ln 2} \). Therefore,
\[
\sup_{x \in \mathcal{I}} |\phi(x)| \leq \left( \frac{C}{|E|} \right)^{\frac{\ln M}{\ln 2}} \sup_{x \in \mathcal{E}} |\phi(x)|.
\]

\[\square\]

Now we are in a position to proceed with the proof of our theorem. Let \( M = \max_{|w| \leq 4} |F(w)| \). Applying Lemma 1 to \( F(w) \), interval \([0, 1]\) and set \( \{ \xi \in E \cap I \} \) and using (1.11), we have that
\[
\sup_{E \cap \Pi} |f(x)| \geq \sup_{E \cap \Pi} |f(x)| \geq \left( \frac{|E \cap \Pi|}{C^d} \right)^{\frac{\ln M}{\ln 2}} \|f\|_{L^p(\Pi)}.
\]

Therefore,
\[
|\{ x \in \Pi : |f(x)| < \left( \frac{\varepsilon}{C^d} \right)^{\frac{\ln M}{\ln 2}} \|f\|_{L^p(\Pi)} \}| \leq \varepsilon \quad \forall \varepsilon > 0.
\]

If we put \( \varepsilon = \frac{|E \cap \Pi|}{2} \) then
\[
|\{ x \in \Pi : |f(x)| < \left( \frac{|E \cap \Pi|}{2C^d} \right)^{\frac{\ln M}{\ln 2}} \|f\|_{L^p(\Pi)} \}| \leq \frac{|E \cap \Pi|}{2}.
\]
Therefore,

\[
\int_{E \cap \Pi} |f|^p \geq \int_{E \cap \Pi} \chi_{|f| \geq \left( \frac{\text{len}(E)}{2C^d} \right)^{\frac{1}{4} + \frac{1}{2}}} \cdot |f|^p \\
\geq \frac{|E \cap \Pi|}{2} \cdot \left( \frac{|E \cap \Pi|}{2C^d} \right)^{\frac{1}{4} + \frac{1}{2}} \left\| f \right\|_{L^p(\Pi)}^p \\
\geq \left( \frac{|E \cap \Pi|}{2C^d} \right)^{\frac{1}{4} + \frac{1}{2} + 1} \cdot \int_{\Pi} |f|^p.
\]

Using (1.13) we get

\[
M \leq 4^d \exp(5C \sum_{j=1}^{d} b_j).
\]

Hence we obtain the desired local estimate

\[
\int_{E \cap \Pi} |f|^p \geq \left( \frac{\gamma}{C^d} \right)^{3d+Cp} \sum_{j=1}^{d} b_j \int_{\Pi} |f|^p
\]

(1.15)

for every good cube \(\Pi\). Summing (1.15) over all good cubes and applying (1.4), we have

\[
\int_{E} |f|^p \geq \int_{E} \bigcup_{\Pi \text{ is good}} \int_{\Pi} |f|^p
\]

\[
\geq \left( \frac{\gamma}{C^d} \right)^{3d+Cp} \sum_{j=1}^{d} b_j \cdot \int_{\bigcup_{\Pi \text{ is good}}} |f|^p
\]

\[
\geq \frac{1}{2} \left( \frac{\gamma}{C^d} \right)^{3d+Cp} \sum_{j=1}^{d} b_j \cdot \int |f|^p.
\]
Replacing $\sum_{j=1}^{d} b_j$ with $\sum_{j=1}^{d} a_j b_j$ and choosing a new $C$, we have:

$$\int_{E} |f|^p \geq \left( \frac{\gamma}{C^d} \right)^{3d+Cp} \sum_{j=1}^{d} a_j b_j \cdot \int_{E} |f|^p.$$

If $p = \infty$ then the proof is almost the same: $\|f\|_{L^\infty(\bigcup_{\Pi \text{ is good}} \Pi)} = \|f\|_{\infty}$. If $\Pi$ is good then $\|f\|_{L^\infty(E \cap \Pi)} \geq \left( \frac{\gamma}{C^d} \right)^{2d+C} \sum_{j=1}^{d} a_j b_j \cdot \|f\|_{L^\infty(\Pi)}$. Hence

$$\|f\|_{L^\infty(E)} \geq \left( \frac{\gamma}{C} \right)^{2d+C} \sum_{j=1}^{d} a_j b_j \cdot \|f\|_{\infty}.$$

If we keep track of all the constants and do the calculations more accurately, then we can get that if $d = 1$ and $p \in [1, \infty)$:

$$\|f\|_{L^p(E)} \geq \left( \frac{\gamma}{300} \right)^{3ab+\frac{2}{p}} \cdot \|f\|_{p},$$

if $p = \infty$:

$$\|f\|_{L^\infty(E)} \geq \left( \frac{\gamma}{100} \right)^{3ab+1} \cdot \|f\|_{\infty}.$$

However, if we try to minimize the factor in front of $ab$, then we can get the following estimate:

$$\|f\|_{L^p(E)} \geq \left( \frac{\gamma}{C} \right)^{(\frac{1}{2}+\epsilon)+ab+A(\epsilon)} \cdot \|f\|_{p} \ \forall \epsilon > 0.$$
coordinates $y_1, ..., y_d$ with axis $y_1$ parallel to $b$. Define

$$f(y) = \hat{f}_1(y_1) \hat{\psi}(y_2, ..., y_d)$$

where

$$\hat{f}_1(y_1) = (\chi_{[-\pi, \pi]} * ... * \chi_{[-\pi, \pi]})(y_1)$$

with the number of convolutions equal to $|[b]T/8\pi| = \left[ \sum_{j=1}^{d} b_j/8\pi \right]$ and $\hat{\psi}$ is supported in a small enough ball in $\mathbb{R}^{d-1}$. Then $\hat{f}$ lives in a cylinder with axis along the main diagonal of $J$ parallel to $b$ with small enough radius so that it is inside of $J$. We have

$$f(y) = \left( \frac{\sin(2\pi y_1/T)}{y_1} \right)^{\left[ \sum_{j=1}^{d} b_j/8\pi \right]} \hat{\psi}(y_2, ..., y_d).$$

Let $E = E_1 \times \mathbb{R}^{d-1}$ where $E_1$ is a periodic subset of $y_1$ axis with period $T$:

$$E_1 \cap [-\frac{T}{2}, \frac{T}{2}] = [-\frac{T}{2}, \frac{T}{2} + \frac{\gamma}{2}] \cap [-\frac{T}{2}, \frac{T}{2}].$$

Then

$$|E \cap (\Pi + x)| \geq C(d) \gamma |\Pi| \quad \forall x \in \mathbb{R}^d.$$ 

If $\sum_{j=1}^{d} b_j$ is large enough we have:

$$\|f_1\|_{L^p(E_1)} \leq \left( \frac{\gamma}{C} \right)^{-1+\sum_{j=1}^{d} b_j/8\pi} \|f_1\|_p$$

and therefore

$$\|f\|_{L^p(E)} = \|f_1\|_{L^p(E_1)} \|f_2\|_p \\
\leq \left( \frac{\gamma}{C} \right)^{-1+\sum_{j=1}^{d} b_j/8\pi} \|f_1\|_p \|f_2\|_p \\
= \left( \frac{\gamma}{C} \right)^{-1+\sum_{j=1}^{d} b_j/8\pi} \|f\|_p.$$
Remark 1: when \( \sum_{j=1}^{d} a_j b_j \) is sufficiently small the proof of the theorem is much simpler: if \( \sum_{j=1}^{d} a_j b_j \leq 1/p \) then \( \|f\|_{L^p(E)} \geq (\frac{1}{C})^{1/p} \|f\|_{p} \) for certain \( p \). This can be proven very easily. If \( p = 1 \) we have

\[
|\Pi| \cdot |f(x)| \geq \int_{\Pi} |f(x)| - \sum_{\alpha \neq 0: \alpha_j \leq 1} a^\alpha \int_{\Pi} |f^{(\alpha)}| \tag{1.16}
\]

and

\[
|\Pi| \cdot |f(x)| \leq \int_{\Pi} |f(x)| + \sum_{\alpha \neq 0: \alpha_j \leq 1} a^\alpha \int_{\Pi} |f^{(\alpha)}| \tag{1.17}
\]

where \( x \in \Pi \), by induction on the dimension \( d \). Hence, using (1.16), we have

\[
\frac{1}{\gamma} \cdot \int_{E \cap \Pi} |f(x)| dx \geq \int_{\Pi} |f| - \sum_{\alpha \neq 0: \alpha_j \leq 1} a^\alpha \int_{\Pi} |f^{(\alpha)}|.
\]

Therefore,

\[
\frac{1}{\gamma} \cdot \int_{E \cap \Pi} |f(x)| dx \geq \int_{\Pi} |f| - \sum_{\alpha \neq 0: \alpha_j \leq 1} a^\alpha \int_{\Pi} |f^{(\alpha)}|.
\]

Summing over all parallelepipeds \( \Pi \) we have

\[
\frac{1}{\gamma} \cdot \int_{E} |f| \geq \int_{\Pi} |f| - \sum_{\alpha \neq 0: \alpha_j \leq 1} a^\alpha \int_{\Pi} |f^{(\alpha)}| \\
\geq \int_{\Pi} |f| - \sum_{\alpha \neq 0: \alpha_j \leq 1} (\frac{b}{2})^\alpha a^\alpha \int_{\Pi} |f| \\
= (2 - \prod_{j=1}^{d}(1 + a_j b_j / 2)) \int_{\Pi} |f| \\
\geq (2 - \sum_{j=1}^{d} a_j b_j / 2) \int_{\Pi} |f| \\
\geq C \int_{\Pi} |f|.
\]
If $p$ is an integer then repeat the previous argument with $f^p$ instead of $f$ and take into account that $\text{supp} f^p \subseteq p \cdot J$. We can also repeat the above argument when $p \geq d$ where $p$ is not necessarily an integer since if $|\alpha| \leq p$ then $\int |f^{p(\alpha)}| \leq (pb/2)^{p} \int |f^p|$ (combine Holder’s and Bernstein’s inequalities to get it). It is even easier to prove that if $p = \infty$ and $\sum_{j=1}^{d} a_j b_j \leq 1$, then $\|f\|_{L^\infty(E)} \geq \frac{1}{2} \|f\|_{\infty}$.

Alternatively, using Holder’s inequality, we can obtain

$$\frac{1}{\gamma} \cdot \int_{\mathbb{R}^d} |f|^p \geq \frac{1}{2^{p-1}} \prod_{\alpha \neq 0: \alpha_j \leq 1} a_\alpha c_\alpha^{-p} \int_{\mathbb{R}^d} \frac{|f(\alpha)|^p}{\prod_{\alpha \neq 0: \alpha_j \leq 1} c_\alpha^p}.$$ 

Put $c_\alpha = a^{\alpha/2} (b/2)^{\alpha/2}$. Then we are done provided $(e^{\sum_{j=1}^{d} a_j b_j p/4} - 1)^2 \leq \frac{1}{2}$. It shows that if $\sum_{j=1}^{d} a_j b_j \leq C$ then $\|f\|_{L^p(E)} \geq \frac{1}{2} \|f\|_{p}$ for $p \in [1, \infty]$.

In a similar way using inequalities analogous to (1.17) we can treat the case when $1 - \gamma$ is sufficiently small depending on $\sum_{j=1}^{d} a_j b_j$: if $p \in [1, \infty)$ and $1 - \gamma \leq e^{-p(C + \sum_{j=1}^{d} a_j b_j/2)}$ then $\|f\|_{L^p(E)} \geq \frac{1}{2} \|f\|_{p}$.

When $d = 1$ we get better results.

1.3 Case of finitely many parallelepipeds

Proof of Theorem 2:

Let

$$J = [-b_1/2, b_1/2] \times \ldots \times [-b_d/2, b_d/2].$$

Then $J_k = J + \lambda_k$ where $\lambda_k \in \mathbb{R}^d$, $k = 1, 2, \ldots, n$. Denote by

$$\tilde{J}_k = 2J + \lambda_k, \quad k = 1, 2, \ldots, n.$$ 

First we will prove a special case of Theorem 2:
Theorem 2': if \( \tilde{f}_k, k = 1, 2, \ldots, n, \) are disjoint then

\[
\|f\|_{L^p(E)} \geq c(\gamma, n, a \cdot b, d, p) \cdot \|f\|_p
\]

where \( c(\gamma, n, a \cdot b, d, p) = \left( \frac{\gamma}{E^d} \right)^n \cdot b \left( \frac{c^d}{r} \right)^{n - \frac{p - 1}{p}}. \)

Proof of Theorem 2':

Let \( \hat{f}(x) = \sum_{k=1}^{n} \hat{f}_k(x - \lambda_k) \) where \( \text{supp} \hat{f}_k \subset J \) and \( f(x) = \sum_{k=1}^{n} f_k(x) e^{i\lambda_k \cdot x}. \) The following lemma gives an estimate of \( \|f_k\|_p \) from above.

Lemma 2:

\[
\|f_k\|_p \leq C^d \|f\|_p \quad (k = 1, 2, \ldots, n).
\]

Proof of Lemma 2:

Let \( \phi \) be a Schwartz function such that \( \text{supp} \hat{\phi} \subset [-1, 1]^d \) and \( \hat{\phi}(x) = 1 \) for \( x \in [-\frac{1}{2}, \frac{1}{2}]^d. \) Then \( \hat{f}_k(x) = \hat{f} \cdot \hat{\phi}(\frac{x - \lambda_k}{b_1}, \ldots, \frac{x - \lambda_k}{b_d}). \) Therefore, \( f_k = f \ast (|J| \phi(b_1 x_1, \ldots, b_d x_d) e^{i\lambda_k \cdot x}). \) Applying Young's inequality we have \( \|f_k\|_p \leq \|f\|_p \cdot \|\phi\|_1. \)

We will also need the following auxiliary lemma on local estimates of generalized trigonometric polynomials of one real variable:

Lemma 3: if \( r(t) = \sum_{k=1}^{n} p_k(t) e^{i\mu_k t} \)
where \( p_k(t) \) is a polynomial of degree \( \leq m - 1 \) and \( E \subset I \) is measurable subset of an interval \( I \) with \( |E| > 0, \) then

\[
\|r\|_{L^\infty(t)} \leq \left( \frac{C|I|}{|E|} \right)^{nm-1} \cdot \|r\|_{L^\infty(E)}.\]

(1.19)
Proof of Lemma 3:

If \( g \) is a pure trigonometric polynomial of order \( n \), i.e.,

\[
g(t) = \sum_{k=1}^{n} c_k e^{i\mu_k t},
\]

then it follows from a theorem on trigonometric polynomials by F. Nazarov ([13], Theorem 1.5) that

\[
\|g\|_{L^\infty(E)} \leq \left( \frac{|I|}{|E|} \right)^{n-1} \cdot \|g\|_{L^\infty(E)}. \tag{1.20}
\]

If \( p(t) = \sum_{l=0}^{m-1} a_l t^l \) is a polynomial of degree \( m - 1 \), then it can be approximated uniformly on an interval with a trigonometric polynomial of order \( \leq m \)

\[
\tilde{p}(t) = \sum_{l=0}^{m-1} a_l \left( e^{i\mu t} - \frac{1}{i\mu} \right)^l = \sum_{l=0}^{m-1} \tilde{a}_l e^{i\mu t}
\]

because \( t = \lim_{\nu \to 0} \frac{e^{i\mu t} - 1}{i\nu} \) uniformly on an interval. Applying (1.20) to the trigonometric polynomial of order \( mn \)

\[
\tilde{r}(t) = \sum_{k=1}^{n} \tilde{p}_k(t)e^{i\mu_k t}
\]

and taking the limit we have the desired result:

\[
\|r\|_{L^\infty(E)} \leq \left( \frac{|I|}{|E|} \right)^{nm-1} \cdot \|r\|_{L^\infty(E)}. \]

\[\square\]

Now we are in a position to proceed with the proof of Theorem 2'.

First we assume that \( p \in [1, \infty) \). Divide the whole \( \mathbb{R}^d \) into parallelepipeds \( \Pi \).
Consider one of them. Suppose $|f|$ attains its maximum in $\Pi$ at point $y \in \Pi$. We can find an interval $I \in \Pi$, $y \in I$ and $\frac{|E \cap I|}{|I|} \geq C^d \gamma$ (see argument before Lemma 1):

$$I = y + t|I|\eta, \quad 0 \leq t \leq 1.$$  \tag{1.21}

$$|\{t \in [0, 1] : y + t|I|\eta \in E \cap I\}| = \frac{|E \cap I|}{|I|} \geq \frac{\gamma}{C^d}. \tag{1.22}$$

Define

$$F(t) = f(y + t|I|\eta) = \sum_{k=1}^{n} f_k(y + t|I|\eta)e^{i\lambda_k(y + t|I|\eta)}. \tag{1.23}$$

Using the Taylor formula

$$g(t) = \sum_{l=0}^{m-1} \frac{g^{(l)}(0)}{l!} t^l + \frac{1}{(m-1)!} \int_{0}^{t} g^{(m)}(s)(t-s)^{m-1} ds$$

$$= p(t) + \frac{1}{(m-1)!} \int_{0}^{t} g^{(m)}(s)(t-s)^{m-1} ds$$

where $p(t)$ is a polynomial of degree $m - 1$, we obtain from (1.23)

$$F(t) = \sum_{k=1}^{n} p_k(t)e^{i\mu_k t} + \frac{1}{(m-1)!} \sum_{k=1}^{n} e^{i\lambda_k(y + t|I|\eta)} \int_{0}^{t} g_k^{(m)}(s)(t-s)^{m-1} ds$$

$$= r(t) + T(t).$$
where \( g_k(s) = f_k(y + s|I|\eta) \). Applying (1.17) to \( f^{(\alpha)}_k \) we have that

\[
\max_{t \in [0,1]} |T(t)| \leq \frac{1}{(m)!} \sum_{k=1}^{n} \sum_{|a|=m} \frac{m!}{\alpha!} \max_{x \in \Omega} |f^{(\alpha)}_k(x)| a^\alpha \leq \frac{1}{(m)!} \sum_{k=1}^{n} \sum_{|a|=m} \frac{m!}{\alpha!} \sum_{\beta, \delta, \leq 1} \|f^{(\alpha + \beta)}_k\|_{L^1(\Omega)} a^\alpha b^\beta \leq |\Pi|.
\]

Denote the last quantity by \( M \). Applying Holder’s inequality we have

\[
M^p|\Pi| \leq \frac{(2d m^d n)^{p-1}}{[m!]^p} \sum_{k=1}^{n} \sum_{|a|=m} \sum_{\beta, \delta, \leq 1} \left( \frac{m! a^\alpha b^\beta}{\alpha!} \right)^p \|f^{(\alpha + \beta)}_k\|_{L^p(\Pi)}.
\] (1.24)

Summing (1.24) over all parallelepipeds \( \Pi \)

\[
\sum_{\Pi} M^p|\Pi| \leq \frac{(2d m^d n)^{p-1}}{[m!]^p} \sum_{k=1}^{n} \sum_{|a|=m} \sum_{\beta, \delta, \leq 1} \left( \frac{m! a^\alpha b^\beta}{\alpha!} \right)^p \|f^{(\alpha + \beta)}_k\|_{L^p(\Pi)} \leq \frac{(2d m^d n)^{p-1}}{[m!]^p} \sum_{k=1}^{n} \sum_{|a|=m} \sum_{\beta, \delta, \leq 1} \left( \frac{m! a^\alpha b^\beta}{\alpha!} \right)^p (C\beta)^{(\alpha + \beta)p} \|f_k\|_{L^p} \leq \frac{(2d m^d n)^{p-1} (Ca \cdot b)^{mp} e^{cpa \cdot b}}{[m!]^p} \sum_{k=1}^{n} \|f_k\|_{L^p} \leq \frac{(Cnm)^d (Ca \cdot b)^{mp} e^{cpa \cdot b}}{[m!]^p} \|f\|_{L^p}.
\] (1.25)

where the last inequality follows from (1.18).

Lemma 3 applied to \( r \), interval \([0,1]\) and subset \( E_1 = \{ t \in [0,1] : y + t|I|\eta \in E \cap I \} \) gives the following local estimate:

\[
\|f\|_{L^\infty(\Pi)} = \|F\|_{L^\infty([0,1])} \leq \|r\|_{L^\infty([0,1])} + M \leq \left( \frac{C^d}{\gamma} \right)^{nm-1} \|r\|_{L^\infty(E_1)} + M \leq \left( \frac{C^d}{\gamma} \right)^{nm-1} \|F\|_{L^\infty(E_1)} + \left( \frac{C^d}{\gamma} \right)^{nm-1} - M \leq \left( \frac{C^d}{\gamma} \right)^{nm-1} \|f\|_{L^\infty(E)} + \left( \frac{C^d}{\gamma} \right)^{nm-1} - M.
\] (1.26)
An argument similar to the one after Lemma 1 shows that the following can be obtained from (1.26)

\[ \|f\|_{L^p(\Pi)}^p \leq 2^{p-1} \left( \frac{C_d}{\gamma} \right)^{pnm-(p-1)} \cdot \|f\|_{L^p(E\cap \Pi)}^p + 2^{p-1} \left( \frac{C_d}{\gamma} \right)^{pnm-(p-1)} \cdot M^p|\Pi|. \]

Summing over all parallelepipeds \( \Pi \) we have:

\[
\int |f|^p \leq \left( \frac{C_d}{\gamma} \right)^{pnm-(p-1)} \cdot \|f\|_{L^p(E)}^p + \left( \frac{C_d}{\gamma} \right)^{pnm-(p-1)} \cdot \sum_{\Pi} M^p|\Pi|
\leq \left( \frac{C_d}{\gamma} \right)^{pnm-(p-1)} \cdot \|f\|_{L^p(E)}^p + \left( \frac{C_d}{\gamma} \right)^{pnm-(p-1)} \cdot \frac{(Cmn)^{dp(Ca \cdot b)^{mp}e^{Cpa\cdot b}}}{m^m} \cdot \|f\|_p^p
\leq \left( \frac{C_d}{\gamma} \right)^{pnm-(p-1)} \cdot \|f\|_{L^p(E)}^p + \left( \frac{C_d}{\gamma} \right)^{pnm} \cdot (a \cdot b)^{mp}e^{Cpa\cdot b} \cdot \|f\|_p^p.
\]

(1.27)

The second inequality follows from (1.25). The last inequality is due to Stirling's formula for \( m! \) and the fact that \( t \leq 2^t \).

Choose \( m \) such that it is a positive integer and \( \left( \frac{C_d}{\gamma} \right)^n \frac{a+b}{m} \leq \frac{1}{C} \), e.g.,

\( m = 1 + [a \cdot b \left( \frac{C_d}{\gamma} \right)^n] \) with so large \( C > 0 \) that the factor in front of \( \|f\|_p^p \) in the last inequality in (1.27) is less than \( \frac{1}{2} \). Therefore,

\[
\int |f|^p \leq \left( \frac{C_d}{\gamma} \right)^{n+1+a \cdot b \left( \frac{C_d}{\gamma} \right)^n-(p-1)} \cdot \int_{E} |f|^p
\leq \left( \frac{C_d}{\gamma} \right)^{n+1+a \cdot b \left( \frac{C_d}{\gamma} \right)^n} \cdot \int_{E} |f|^p.
\]

The proof for \( p = \infty \) is similar and even simpler.

Now we can proceed with the proof of Theorem 2. We will apply induction on \( n \). For \( n = 1 \) the theorem follows from Theorem 2' or the usual Logvinenko-Sereda Theorem. Suppose the statement is true for \( n \leq m \). Let \( n = m + 1 \).
If $\tilde{J}_k, k = 1, 2, \ldots, n,$ are disjoint then the result follows from Theorem 2'.

If $\tilde{J}_k$ and $\tilde{J}_l$ intersect each other for some $k$ and $l$, then we can replace $J$ with $3J$ reducing the number of frequencies $\lambda_k$ and replacing $b$ with $3b$. Therefore, by induction:

$$\|f\|_{L^p(E)} \geq \left( \frac{C^d}{\gamma} \right)^{-3a \cdot b \left( \frac{c^d}{\gamma} \right)^{(m+1)p^-1} \|f\|_p} \cdot \|f\|_p$$

$$\geq \left( \frac{C^d}{\gamma} \right)^{-a \cdot b \left( \frac{c^d}{\gamma} \right)^{(m+1)p^-1} \|f\|_p} \cdot \|f\|_p.$$

The purpose of this theorem is to prove the existence of a constant $c(\gamma, n, ab, d, p) > 0$ depending only on the number of parallelepipeds and not how they are placed rather than to get the best possible estimate. We can conjecture that the right behavior of the constant $c(\gamma, n, ab, d, p)$ is the following:

$$c(\gamma, n, a \cdot b, d, p) = \left( \frac{\gamma}{C^d} \right)^{Cn \cdot b + n - \frac{p^-1}{p}}.$$

The estimate (1.28) is suggested by an example similar to the one after Theorem 1: choose $J_k = J + k \cdot (1 - \epsilon)b, k = 1, \ldots, n$, so that neighborhoods of two corners of $J_k$ and $J_{k+1}, k = 1, \ldots, n - 1$, intersect. Then we can choose $f$ supported in a cylinder with axis along $b$ of length $n(1 - \epsilon)|b|$. The rest is the same as in the former example.

1.4 Case of infinitely many lacunary intervals

1.4.1 Conjectured theorem

The goal is to generalize a result of F. Nazarov for the Zygmund theorem on lacunary trigonometric series ([13], Theorem 3.6). Instead of a trigonometric series we will
consider the following sum \( f(x) = \sum_{k=-\infty}^{\infty} f_k(x)e^{i\lambda_k x} \) where \( f_k \in L^2(\mathbb{R}) \) with supp \( \hat{f}_k \subset [-\frac{\alpha}{2}, \frac{\alpha}{2}] \) and \( \{\lambda_k\}_{k=-\infty}^{\infty} \) is a lacunary sequence. A sequence of real numbers \( \Lambda = \{\lambda_k\}_{k=-\infty}^{\infty} \) is lacunary if there exists \( N < \infty \) such that

\[
N = \max_{k \neq l} \text{Card}\{(k', l') : |\lambda_k - \lambda_l - (\lambda_{k'} - \lambda_{l'})| \leq 1\}. \tag{1.29}
\]

Let \( I_{k,l} = [\lambda_k - \lambda_l - \frac{1}{2}, \lambda_k - \lambda_l + \frac{1}{2}] \). Then

\[
N = \max_{k \neq l} \text{Card}\{(k', l') : I_{k,l} \cap I_{k', l'} \neq \emptyset\}. \tag{1.30}
\]

Another equivalent definition:

\[
N = \max_x \text{Card}\{(k, l), k \neq l : x \in I_{k,l}\}
= \max_x \text{Card}\{(k, l), k \neq l : \lambda_k - \lambda_l \in [x - \frac{1}{2}, x + \frac{1}{2}]\}. \tag{1.31}
\]

The conjectured theorem is the following:

**Theorem 3:** let \( \Lambda = \{\lambda_k\}_{k=-\infty}^{\infty} \) be lacunary and let \( E \) be "relatively dense." If \( f \in L^2 \) and supp \( \hat{f} \subset \bigcup_{k=-\infty}^{\infty} [\lambda_k - \frac{\alpha}{2}, \lambda_k + \frac{\alpha}{2}] \) then

\[
\|f\|_{L^2(E)} \geq C(\gamma, N, a, b) \cdot \|f\|_2.
\]

We will prove some partial results.

### 1.4.2 Large \( E \), small support

**Proposition 1.** Theorem 3 is true if \( 1 - \gamma \) is small enough depending on \( N, a, b \):
if \( 1 - \gamma \leq \frac{1}{16(1+ab)^2(1+a)N} \) then

\[
\int_{E} |f|^2 \geq \frac{1}{2} \int |f|^2.
\]

**Proof of Proposition 1:** We can choose \( f_k \) such that \( \text{supp} \, \hat{f}_k \subset [-\frac{a}{2}, \frac{a}{2}] \), \( \text{supp} \, \hat{f}_k(x - \lambda_k) \) are disjoint and \( \hat{f}(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k(x - \lambda_k). \) Then

\[
f(x) = \sum_{k=-\infty}^{\infty} f_k(x)e^{i\lambda_k x}
\]

and

\[
\|f\|_2^2 = \sum_{k=-\infty}^{\infty} \|f_k\|_2^2.
\]

Divide the whole real line into intervals \( I \) of length \( a \) each. Consider one of them. We will need to prove the following local estimate:

\[
\int_{I \cap E^c} |f|^2 \leq (1 - \gamma) \sum_{k=-\infty}^{\infty} \int_{I} (|f_k|^2 + |a|(|f_k'|^2))
\]

\[
+ \frac{1}{a} \sum_{k=-\infty}^{\infty} \sqrt{(1 - \gamma)a(1+a)N} \int_{I} |f_k|^2
\]

\[
+ \sum_{k=-\infty}^{\infty} \sqrt{(1 - \gamma)a(1+a)N} \int_{I} (|f_k'|^2/(b/2) + |f_k|^2 b/2).
\]

We can assume that \( I = [-\frac{a}{2}, \frac{a}{2}] \).

\[
\int_{I \cap E^c} |f|^2 = \sum_{k=-\infty}^{\infty} \int_{I \cap E^c} |f_k|^2 + \sum_{k \neq \mu} \int_{I \cap E^c} f_k(x) \tilde{f}_\mu(x) e^{i(\lambda_k - \lambda_\mu) x} dx. \quad (1.32)
\]

The first term is bounded by

\[
(1 - \gamma) \sum_{k=-\infty}^{\infty} \int_{I} (|f_k|^2 + |a|(|f_k'|^2)). \quad (1.33)
\]
To estimate the second term, we will rewrite it in the following way:

\[
\sum_{k \neq l} \int_{t \in E^c} \left( f_k(y) \tilde{f}_l(y) + \int_y^x (f_k(t) \tilde{f}_l(t))' dt \right) e^{i(\lambda_k - \lambda_l)z} dx = \sum_{k \neq l} f_k(y) \tilde{f}_l(y) \tilde{x}_{l \cap E^c}(\lambda_k - \lambda_l) + \sum_{k \neq l} \int_y^y (f_k \tilde{f}_l)'(t) \tilde{x}_{l, a/2 \cap E^c}(\lambda_k - \lambda_l) dt - \sum_{k \neq l} \int_{-y}^y (f_k \tilde{f}_l)'(t) \tilde{x}_{l-a/2, l \cap E^c}(\lambda_k - \lambda_l) dt. \tag{1.34}
\]

The first term in this expression is bounded by

\[
\sum_{k=-\infty}^{\infty} \sqrt{(1 - \gamma)a(1 + a)N|f_k(y)|^2}.
\]

To show this we will apply Holder's inequality to the first term in (1.34):

\[
\sum_{k \neq l} f_k(y) \tilde{f}_l(y) \tilde{x}_{l \cap E^c}(\lambda_k - \lambda_l) \leq \sqrt{\sum_{k \neq l} |f_k(y)\tilde{f}_l(y)|^2} \sqrt{\sum_{k \neq l} |\tilde{x}_{l \cap E^c}(\lambda_k - \lambda_l)|^2} \leq \sum_{k=-\infty}^{\infty} |f_k(y)|^2 \sqrt{\sum_{k \neq l, l \neq k} \left( |\tilde{x}_{l \cap E^c}|^2 + |(\tilde{x}_{l \cap E^c})'|^2 \right)} \leq \sum_{k=-\infty}^{\infty} |f_k(y)|^2 \sqrt{N \int_{l, l \neq k} \left( |\tilde{x}_{l \cap E^c}|^2 + |(\tilde{x}_{l \cap E^c})'|^2 \right)} \leq \sum_{k=-\infty}^{\infty} |f_k(y)|^2 \sqrt{N(1 + a) \int_{l, l \neq k} |\tilde{x}_{l \cap E^c}|^2} = \sum_{k=-\infty}^{\infty} |f_k(y)|^2 \sqrt{N(1 + a)(1 - \gamma)a}. \tag{1.35}
\]

We used here Bernstein's inequality: \( \|g'\|_1 \leq L\|g\|_1 \) if \( \text{supp} \tilde{g} \subset [-L, L] \) and that

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supp(\hat{\chi}_{\mathbb{N} \mathbb{C}})^2 \in [-a, a]. In a similar way we can bound the second term in (1.34) by

\[ \int_{\mathbb{T}} \sqrt{4\left( \sum_{k=-\infty}^{\infty} |f_k'|^2 \right) \left( \sum_{k=-\infty}^{\infty} |f_k|^2 \right) \cdot \sum_{k \neq l} |\hat{\chi}_{[t, a/2]}| \mathbb{N} \mathbb{C}^2 |(\lambda_k - \lambda_l)|^2 dt} \leq \]

\[ \sum_{k=-\infty}^{\infty} \int_{\mathbb{T}} (|f'_k|^2 (b/2) + |f_k|^2 b/2) \sqrt{N(1+a)(1-\gamma)a}. \] (1.36)

The third term in (1.34) is treated analogously. Integrating (1.32) over \( y \in I \) and applying (1.33), (1.35) and (1.36) we obtain

\[ \int_{\mathbb{N} \mathbb{C}^2} |f|^2 \leq (1-\gamma) \sum_{k=-\infty}^{\infty} \int_I (|f_k|^2 + a |f_k^2'|) + \frac{1}{a} \sum_{k=-\infty}^{\infty} \sqrt{(1-\gamma)a(1+a)N} \int_I |f_k|^2 + \sum_{k=-\infty}^{\infty} \sqrt{(1-\gamma)a(1+a)N} \int_I (|f'_k|^2 (b/2) + |f_k|^2 b/2). \] (1.37)

Summing (1.37) over all intervals \( I \) and applying Bernstein's inequality, we get

\[ \int_{\mathbb{N} \mathbb{C}^2} |f|^2 \leq (1+ab)(1-\gamma + \sqrt{(1-\gamma)(1+1/a)N}) \int |f|^2. \] (1.38)

If \( 1-\gamma \leq \frac{1}{16a+b^2(1+1/a)N} \) then the factor in the right side of (1.38) is less than \( \frac{1}{2} \). Therefore,

\[ \int_{\mathbb{N} \mathbb{C}^2} |f|^2 \geq \frac{1}{2} \int |f|^2. \]

Remark:

1. We can always take a larger \( a \) and try to minimize \( (1+ab)^2(1+1/a) \). Then we will get \( C(1+ab)^2(1+\min(b, 1/a)) \), so \( 1-\gamma \leq \frac{1}{C(1+ab)^2(1+\min(b, 1/a))} \).
2. Using Theorem 4 with \( m = 0 \) we can prove in a similar way that Theorem 3 holds if \( b \) is sufficiently small depending on \( \gamma, a \) and \( N \):

if \( b(1 + a) \leq e^{-(\ln CN)N^2(\frac{\xi}{\gamma})^C} \) then

\[
\int_{\mathcal{E}} |f|^2 \geq e^{-(\ln CN)N^2(\frac{\xi}{\gamma})^C} \int |f|^2.
\]

Sketch of the proof: We can assume that \( a \geq C \) (otherwise if \( a < C \) we can put \( a = C \) and replace \( \xi \) with \( \xi/2 \)). Rescaling we can assume that \( |E \cap I| \geq \gamma \) for every interval \( I \) of length 1 and replace \( \Lambda \) with \( a\Lambda \) and \( b \) with \( ab \).

\[
\int_{E \cap I} |f(x)|^2 dx = \int_{E \cap I} \sum_{k=-\infty}^{\infty} f_k(y)e^{i\lambda_k x} + \int_{y}^{x} f_k'(t)dt e^{i\lambda_k x} |^2 dx
\]

\[
\geq \int_{E \cap I} \sum_{k=-\infty}^{\infty} |f_k(y)|^2 dx - \int_{E \cap I} \sum_{k=-\infty}^{\infty} \int_{y}^{x} |f_k'(t)|^2 dt e^{i\lambda_k x} |^2 dx
\]

\[
\geq e^{-(\ln CN)N^2(\frac{\xi}{\gamma})^C} \sum_{k=-\infty}^{\infty} |f_k(y)|^2 - C\sqrt{N} \sum_{k=-\infty}^{\infty} \int |f_k'|^2.
\]

Integrating over \( y \in I \) and summing over all intervals \( I \) we get

\[
\int_{\mathcal{E}} |f|^2 \geq e^{-(\ln CN)N^2(\frac{\xi}{\gamma})^C} \sum_{k=-\infty}^{\infty} \int |f_k|^2 - C\sqrt{N} \sum_{k=-\infty}^{\infty} \int |f_k'|^2
\]

\[
\geq e^{-(\ln CN)N^2(\frac{\xi}{\gamma})^C} \int |f|^2 - C\sqrt{N}(ab)^2 \int |f|^2.
\]

Thus we obtain the desired estimate if \( ab \leq e^{-(\ln CN)N^2(\frac{\xi}{\gamma})^C} \). \( \square \)

### 1.4.3 Periodic \( E \)

The next case we consider is when \( E \) is a periodic set with period \( a \). Let

\[
E \cap [0, a] = \gamma \cdot a.
\]
Proposition 2: Theorem 3 holds in this periodic case

\[ \int_E |f|^2 \geq C(\gamma, a, b, N) \int |f|^2. \]

Proof of Proposition 2: We will start with some results on periodizations. Define a family of periodizations of a function \( f \in L^1 \):

\[ g_t(x) = \sum_{k=-\infty}^{\infty} f(x + ka)e^{-i2\pi(x+ka)} \]  (1.39)

where \( t \in [-\frac{1}{2a}, \frac{1}{2a}] \). Then \( g_t(x) \) is periodic with period \( a \) and its Fourier coefficients are:

\[ \hat{g}_t(l) = \frac{1}{a} \hat{f}(\frac{l}{a} + t). \]  (1.40)

Now we assume that \( f \in L^1 \cap L^2 \). The next argument shows an important relation between the average of \( L^2 \) norm of periodizations and the \( L^2 \) norm of \( f \):

\[ a \int_{-\frac{1}{2a}}^{\frac{1}{2a}} \int_{E \cap [0,a]} |g_t(x)|^2 dx dt = \sum_{k,l} a \int_{E \cap [0,a]}^{\frac{1}{2a}} f(x + ka)\tilde{f}(x + l \cdot a)e^{-i2\pi(k-l)a} dt dx \]

\[ = \sum_{k,l} \delta_{k,l} \int_{E \cap [0,a]} f(x + ka)\tilde{f}(x + la) dx \]

\[ = \sum_{k=-\infty}^{\infty} \int_{E \cap [0,a]} |f(x + ka)|^2 dx \]

\[ = \int_E |f|^2. \]  (1.41)

In particular, it follows that

\[ a \int_{-\frac{1}{2a}}^{\frac{1}{2a}} \int_{[0,a]} |g_t(x)|^2 dx dt = \int |f|^2. \]  (1.42)
In the next lemma we extend these results to functions from $L^2$. For simplicity we assume that $a = 1$.

**Lemma 4:** If $f \in L^2$ then there exists a family of periodic functions $g_t(x): x \in [0, 1], \ t \in [-\frac{1}{2}, \frac{1}{2}]$ with period 1 such that $g_t(x) \in L^2([0, 1] \times [-\frac{1}{2}, \frac{1}{2}]),$

$$\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{E \cap [0,1]} |g_t(x)|^2 dx dt = \int \frac{|f|^2}{E}$$

and

$$\hat{g}_t(l) = \hat{f}(l + t)$$

for almost all $t$.

**Proof of Lemma 4:** Consider a cut of $f$:

$$f^n(x) = \chi_{[-n, n]} f(x).$$

Since $f^n \in L^1 \cap L^2$ and converge to $f$ in $L^2$ we can define corresponding families of periodizations $g_t^n(x)$ which form a Cauchy sequence in $L^2([0, 1] \times [-\frac{1}{2}, \frac{1}{2}]):$

$$\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{[0,1]} |g_t^n(x) - g_t^m(x)|^2 dx dt = \int |f^n - f^m|^2.$$
Let \( g_t(x) \) be the limit of \( g_t^n(x) \). Then we obtain the first statement of Lemma 4

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \int E \, |g_t(x)|^2 \, dx \, dt = \lim_{n \to \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int E \, |g_t^n(x)|^2 \, dx \, dt
\]
\[
= \lim_{n \to \infty} \int E \, |f^n|^2
\]
\[
= \int E \, |f|^2.
\]

To obtain the second statement of Lemma 4 we consider the following sum:

\[
\sum_{l=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{g}_t(l) - \hat{f}(l + t)|^2 \, dt = \sum_{l=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{g}_t(l) - \hat{g}_t^n(l) + \hat{f}(l + t) - \hat{f}(l + t)|^2 \, dt
\]
\[
\leq 2 \sum_{l=-\infty}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{g}_t(l) - \hat{g}_t^n(l)|^2 + |\hat{f}(l + t) - \hat{f}(l + t)|^2 \, dt
\]
\[
= 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} |g_t(x) - g_t^n(x)|^2 \, dx \, dt + 2 \int E \, |f^n - f|^2
\]
\[
\leq \epsilon
\]

where \( \epsilon \) can be arbitrarily small if \( n \) is large enough. \( \square \)

Rescaling we can assume that the set \( E \) has period 1, \( |E \cap [0, 1]| = \gamma \), \( \text{supp} \hat{f}_k \in [-\frac{ab}{2}, \frac{ab}{2}] \) and \( \Lambda \) is lacunary with 1 being replaced by \( a \) in definitions (1.29), (1.30) and (1.31). Let \( g_t \) be a family of periodizations of \( f \) as defined in (1.39). It follows from Lemma 4 that \( \hat{g}_t(l) = \hat{f}(l + t) \) for almost all \( t \in [-\frac{1}{2}, \frac{1}{2}] \). Assume for simplicity that \( t = 0 \) is among them. Let \( n_k \) be the smallest integer in \( [\lambda_k - \frac{ab}{2}, \lambda_k + \frac{ab}{2}] \) if such exists. Denote \( \tilde{\Lambda} = \{n_k\}_{k=-\infty}^{\infty} \). Then

\[
\text{Spec } g_0 \subset \bigcup_{m=0}^{m=[ab]} (\tilde{\Lambda} + m).
\]
Next lemma generalizes a result of F. Nazarov for the Zygmund theorem on lacunary Fourier series.

**Lemma 5:**
\[ \int_{E \cap [0,1]} |g_0|^2 \geq C(\gamma, a, b, N) \int_0^1 |g_0|^2. \]

**Proof of Lemma 5:** Denote \( M = [ab] \). Let

\[
R = \sup_{n \in \mathbb{Z}} \text{Card}\{(k, k', m, m'), k \neq k', 0 \leq m, m' \leq M : n = n_k + m - (n_{k'} + m')\}
\]

\[ \leq (M + 1) \sup_{n \in \mathbb{Z}} \text{Card}\{(k, k') : n \in [\lambda_k - \lambda_{k'} - ab, \lambda_k - \lambda_{k'} + ab]\}
\]

\[ \leq (M + 1) \cdot N \cdot (1 + [2b]). \]

To obtain the last inequality we used (1.31) with \( 1 \) being replaced by \( a \). We can write \( g_0 \) in the following way:

\[ g_0(x) = \sum_{k=-\infty}^{\infty} p_k(x)e^{i2\pi n_kx} \]

where \( p_k \) are trigonometric polynomials of degree \( M \):

\[ p_k(x) = \sum_{m=0}^{M} c_{m}^{(k)} e^{i2\pi m x}. \]

Therefore,

\[
\int_{E \cap [0,1]} |g_0|^2 = \sum_{k=-\infty}^{\infty} \int_{E \cap [0,1]} |p_k|^2 + \sum_{k \neq l} \int_{E \cap [0,1]} p_k \bar{p}_l
\]

\[ = \sum_{k=-\infty}^{\infty} \int_{E \cap [0,1]} |p_k|^2 + \sum_{k \neq l} \sum_{0 \leq m, m' \leq M} c_{m}^{(k)} \bar{c}_{m}^{(l)} \chi_{E \cap [0,1]}(n_k + m - (n_{k'} + m')). \quad (1.43) \]
The first term in (1.43) is bounded from below by
\[
\sum_{k=-\infty}^{\infty} \left( \frac{\gamma}{C} \right)^{2M+1} \sum_{m=0}^{M} |c_{m}^{(k)}|^2 = \left( \frac{\gamma}{C} \right)^{2M+1} \int_0^1 |g_0|^2.
\] (1.44)

The second term in (1.43) can be written in the form \( \langle Tg_0, g_0 \rangle \) where \( T \) is a Hilbert-Schmidt operator on \( L^2([0,1]) \):
\[
\widetilde{Tg_0}(n_k + m') = \sum_{k \neq k'} \sum_{0 \leq m \leq M} \tilde{x}_{E\cap[0,1]}(n_k + m - (n_{k'} + m')) \tilde{g}_0(n_k + m)
\]
and \( \widetilde{Tg_0}(n) = 0 \) for the rest of \( n \). Its Hilbert-Schmidt norm is
\[
\sqrt{\sum_{k \neq k'} \sum_{0 \leq m, m' \leq M} \left| \tilde{x}_{E\cap[0,1]}(n_k + m - (n_{k'} + m')) \right|^2} \leq \sqrt{R\gamma(1 - \gamma)}.
\] (1.45)

We won't proceed further since the rest of the argument is the same as in Nazarov's proof of the Zygmund theorem ([13], Theorem 3.6). See details in the proof of the next Theorem. This leads us to the desired inequality
\[
\int_{E \cap [0,1]} |g_0|^2 \geq C(\gamma, a, b, N) \int_0^1 |g_0|^2
\]
where
\[
C(\gamma, a, b, N) = e^{-(\ln R)R^2(\frac{C}{\gamma})^{C(M+1)}} \geq e^{-\ln((1+ab)(1+2b)N)((1+ab)(1+2b)N)^2(\frac{C}{\gamma})^{C(1+ab)}}.
\]

In a similar way we have that
\[
\int_{E \cap [0,1]} |g_t|^2 \geq C(\gamma, a, b, N) \int_0^1 |g_t|^2
\] (1.46)
for almost all $t \in [-\frac{1}{2}, \frac{1}{2}]$. Applying Lemma 4 and (1.46) we obtain
\[
\int_{E} |f|^2 \geq C(\gamma, a, b, N) \int_{I} |f|^2.
\] (1.47)

\[ \square \]

### 1.4.4 Generalized lacunary series

The next theorem generalizes Nazarov's result for the Zygmund theorem for series whose coefficients are replaced with polynomials of uniformly bounded degree.

**Theorem 4:** let $I$ be an interval of length 1 and $E$ be a subset of $I$ of positive measure. If $\Lambda = \{\lambda_k\}_{k=-\infty}^{\infty}$ is lacunary in the sense of (1.29) and $f(x) = \sum_{k=-\infty}^{\infty} p_k(x)e^{\lambda_k x}$ where $p_k(x)$ are polynomials of degree at most $m$, then
\[
\int_{E} |f|^2 \geq C(|E|, N, m) \int_{I} |f|^2
\]

\[ C(|E|, N, m) = e^{-(\ln CN)^2(\frac{C}{|E|})^{C(m+1)}}. \]

**Proof of Theorem 4:** Without loss of generality we can assume that $I = [-\frac{1}{2}, \frac{1}{2}]$. Let $p(x) = \sum_{j=0}^{m} b_j x^j$ be a polynomial of degree $m$. Denote $b = (b_0, ..., b_m)$. Then $\|p\|_{L^2(I)} \sim \|b\|_2$. In fact,
\[
\frac{\|b\|_2^2}{C^m} \leq \int_{I} |p|^2 \leq C\|b\|_2^2.
\]

First we will treat the case when gaps $|\lambda_k - \lambda_l|$ are large: $|\lambda_k - \lambda_l| \geq C(1 + m^3)$ for $k \neq l$. We need this condition to prove that $\int_{I} |f|^2 \sim \sum_{k=-\infty}^{\infty} \int_{I} |p_k|^2$. Note that we don't need the lacunarity of $\Lambda$ for the lemma below.
Lemma 6: If $|\lambda_k - \lambda_l| \geq C(1 + m^3)$ for $k \neq l$ then

$$\frac{C}{1 + m^2} \sum_{k=-\infty}^{\infty} \int_{I} |p_k|^2 \leq \int_{I} |f|^2 \leq C(1 + m^2) \sum_{k=-\infty}^{\infty} \int_{I} |p_k|^2.$$  

Proof of Lemma 6: We will prove only the left inequality. The right one can be proved similarly. Let $p$ be a polynomial of degree $m$. Hence, using the following inequality

$$\int_{I} |p'| \leq Cm^2 \int_{I} |p|,$$  

we obtain that

$$\int_{A} |p|^2 \geq C \int_{I} |p|^2$$  

(1.49)  

for any set $A \subset I$ provided $|I \cap A^c| \leq \frac{C}{1 + m^2}$.  

Let $\phi(x) = (\frac{1}{2} - x^2)\chi_{I}(x)$. Then

$$\int_{I} |f|^2 \geq \int_{I} \phi |f|^2$$

$$= \sum_{k=-\infty}^{\infty} \int_{I} \phi |p_k|^2 + \sum_{k \neq l} \int_{I} \phi p_k \overline{p_l} e^{i(\lambda_k - \lambda_l)x} dx.$$  

(1.50)

Using (1.49) we can bound the first term in (1.50) from below by

$$\frac{C}{1 + m^2} \sum_{k=-\infty}^{\infty} \int_{I} |p_k|^2.$$  

(1.51)
We claim that each summand in the second term in (1.50) is bounded from above by

$$\frac{C(1 + m^4)}{(\lambda_k - \lambda_l)^2} \int_I (|p_k|^2 + |p_l|^2). \tag{1.52}$$

To show this we consider \( \int_I p(x)\phi(x)e^{i\lambda x}dx \) for \( \lambda \neq 0 \) where \( p = p_k\bar{p}_l \) is a polynomial of degree \( 2m \). If we write

\[
p(x) = p(y) + p'(y)(x - y) + \int_y^x p''(t)(x - t)dt,
\]

then \( \int_I p(x)\phi(x)e^{i\lambda x}dx \) can be written in the following way:

\[
p(y)\phi(\lambda) + p'(y)(x - y)\phi(x)(\lambda)
- \int_y^x \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x)e^{i\lambda x}p''(t)(x - t)dt \ dx \ dx + \int_y^x \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x)e^{i\lambda x}p''(t)(x - t)dt \ dx. \tag{1.53}
\]

The first and second terms in (1.53) are bounded from above by

$$\frac{C}{1 + \lambda^2}(|p(y)| + |p'(y)|).$$

Meanwhile the third term in (1.53)

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x)e^{i\lambda x}p''(t)(x - t)dxdt
\]

is bounded from above by

$$\frac{C}{1 + \lambda^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |p''(t)|dt.$$

The fourth term in (1.53) can be treated analogously.

Integrating (1.53) over \( y \in I \) and applying (1.48) twice, we obtain (1.52). If \( |\lambda_k - \lambda_l| \geq \)
$C(1 + m^3)$ with large enough $C$ then

$$\int |f|^2 \geq \frac{C}{1 + m^2} \sum_{k=-\infty}^{\infty} \int |p_k|^2 - \sum_{k \neq l} \frac{C(1 + m^4)}{(\lambda_k - \lambda_l)^2} \int (|p_k|^2 + |p_l|^2)$$

$$\geq \frac{C}{1 + m^2} \sum_{k=-\infty}^{\infty} \int |p_k|^2 (1 - \sum_{l \neq k} \frac{1}{100(l - k)^2})$$

$$\geq \frac{C}{1 + m^2} \sum_{k=-\infty}^{\infty} \int |p_k|^2.$$  

The right inequality can be proven similarly with $\phi(\frac{-x}{1 + m^2})$ instead of $\phi$. □

Repeating an argument similar to the one used in Proposition 1 to prove a local estimate, we have that Theorem 4 holds if $E$ is large enough:

If $|I \cap E^c| \leq \frac{C}{(1 + m^3)N}$ then

$$\int_E |f|^2 \geq \frac{1}{2} \int_I |f|^2.$$  \hspace{1cm} (1.54)

Now we consider the case when $|I \cap E^c| \geq \frac{C}{(1 + m^3)N}$. The same argument shows (see also Lemma 5) that

$$\int_E |f|^2 = \sum_{k=-\infty}^{\infty} \int_E |p_k|^2 + \langle T_E f, f \rangle$$
where

\[
\langle T_E f, f \rangle = \sum_{k \neq l} \int_E \overline{p_k \bar{p}_l} e^{i(\lambda_k - \lambda_l)^2} dx
\]

\[
= \sum_{k \neq l} \int_I p_k(y) \bar{p}_l(y) \hat{\chi}_E(\lambda_k - \lambda_l) dy
\]

\[
- \int \sum_{k \neq l} \int_{-\frac{1}{2}}^{\frac{1}{2}} (p_k \bar{p}_l)'(t) \hat{\chi}_{[-1/2, 1]} \cap E(\lambda_k - \lambda_l) dt dy
\]

\[
+ \int \sum_{k \neq l} \int_I (p_k \bar{p}_l)'(t) \hat{\chi}_{[1/2, 1]} \cap E(\lambda_k - \lambda_l) dt dy
\]

\[
= \sum_{k \neq l} \int_I p_k(t) \bar{p}_l(t) \hat{\chi}_E(\lambda_k - \lambda_l) dt
\]

\[
- \sum_{k \neq l} \int_I (p_k \bar{p}_l)'(t)(1/2 - t) \hat{\chi}_{[-1/2, 1]} \cap E(\lambda_k - \lambda_l) dt
\]

\[
+ \sum_{k \neq l} \int_I (p_k \bar{p}_l)'(t)(1/2 + t) \hat{\chi}_{[1/2, 1]} \cap E(\lambda_k - \lambda_l) dt.
\]

We can view $T_E$ as an operator acting on the space of sequences of pairs of functions from $L^2(I)$:

\[ G = (G_1, G_2, G_3, ...), \quad G_k = (g_k^1, g_k^2), \quad g_k^i \in L^2(I), i = 1, 2, k = 1, 2, 3, ...
\]

with a scalar product defined in the following way:

\[
\langle F, G \rangle = \sum_{k=1}^{\infty} \sum_{i=1}^{2} \int_I f_k^i \overline{g_k^i}.
\]

Then

\[
(T_E F)_l = \sum_{k \neq l} T_{lk} F_k
\]
where each $T_{ik}$ is a $2 \times 2$ matrix whose $(1, 1)$ entry is

$$\hat{\chi}_{E}(\lambda_k - \lambda_l),$$

$(1, 2)$ and $(2, 1)$ entries are the same and equal to

$$-(1/2 - t)\hat{\chi}_{[-1/2,1]}E(\lambda_k - \lambda_l) + (1/2 + t)\hat{\chi}_{[t,1/2]}E(\lambda_k - \lambda_l)$$

and $(2, 2)$ entry is $0$. In our case $F_k = (p_k, p_k')$. We claim that $T_E$ is a Hilbert-Schmidt operator since

$$\sup_{t \in I} \sum_{k,l} \sum_{i,j=1}^{2} |T_{ik}^{ij}|^2 \leq CN|E|$$

by an argument similar to the one used to prove a local estimate in Proposition 1.

Our proof will follow the one of F. Nazarov for the Zygmund theorem. Let $\sigma_1, \sigma_2, \ldots$ be the eigenvalues of $T_E$ enumerated in the descending order of their absolute values: $|\sigma_1| \geq |\sigma_2| \geq \ldots$. Since

$$\sum_{s=1}^{\infty} |\sigma_s|^2 \leq \sup_{t \in I} \sum_{k,l} \sum_{i,j=1}^{2} |T_{ik}^{ij}|^2$$

$$\leq CN|E|$$

we have that $|\sigma_{n+1}|^2 \leq \frac{CN|E|}{n+1}$. If $V_n$ is the space spanned by the first $n$ eigenvectors of $T_E$, then the norm of $T_E$ on $V_n^\perp$ is equal to $|\sigma_{n+1}|$. Recall that

$$\int_{E} |f|^2 = \sum_{k=-\infty}^{\infty} \int_{E} |p_k|^2 + \langle T_{E}f, f \rangle. \quad (1.55)$$

The first term in $(1.55)$ is bounded from below by

$$\left( \frac{|E|}{C} \right)^{2m+1} \sum_{k=-\infty}^{\infty} \int_{E} |p_k|^2. \quad (1.56)$$
If $f \perp V_n(E)$ with

$$n = \left\lfloor N \left( \frac{C}{|E|} \right)^{4m+1} \right\rfloor$$

(1.57)

with large enough $C$ then the second term in (1.55) is bounded from above by $\frac{1}{2}$ of the expression in (1.56). Hence, if $f \perp V_n$, then

$$\int_E |f|^2 \geq \left( \frac{|E|}{C} \right)^{2m+1} \sum_{k=-\infty}^\infty \int_I |p_k|^2$$

$$\geq \left( \frac{|E|}{C} \right)^{2m+1} \int_I |f|^2.$$  

(1.58)

The last inequality follows from Lemma 6. Therefore, Theorem 4 holds if $f$ is orthogonal to certain subspaces. Formula (1.57) gives that $n \geq 1$. We can think that $n = 0$ if $E$ is large enough (see (1.54)).

Now we will do the general case. The main idea is to construct a set $\delta E \in I \setminus E$ such that $|\delta E| \geq \delta(|E|) > 0$ if $|E| < 1$ and

$$\int_E |f|^2 \geq C(m, N, |E|) \int_{E \cup \delta E} |f|^2.$$  

Then we will iterate until we get a large enough set so that we can apply (1.54). In fact, we will reach $I$ automatically at some iteration (see the end of the proof).

Choose $n$ as in (1.57) but with twice larger constant $C$ so that it will work for sets of measure at least $|E|/2$. Define

$$E_t = \bigcap_{j=0}^n (E - jt).$$

We can choose $\tau > 0$ so small that $|E_\tau| \geq |E|/2$ for $t \in [0, \tau]$. It is clear that $\tau < 1$. We will pick $\tau$ in a special manner. Consider the continuous function $\phi(x) = |E^c \cap (E + x)|$. 

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\[ \phi(0) = 0, \phi(1) = |E|. \] Let \( n\tau = \inf\{x, 0 \leq x \leq 1 : \phi(x) = \frac{|E|}{2n}\} \) then

\[ |E^c \cap (E + n\tau)| = \frac{|E|}{2n} \]

and

\[ |E^c \cap (E + kt)| \leq \frac{|E|}{2n} \text{ for } k = 1, \ldots, n \text{ and } t \in [0, \tau]. \]

Hence

\[
|E_t| = \bigcap_{k=0}^n (E - kt) \\
= |E \setminus \bigcup_{k=1}^n (E - kt)^c \cap E| \\
\geq |E| - \sum_{k=1}^n |(E - kt)^c \cap E| \\
= |E| - \sum_{k=1}^n |E^c \cap (E + kt)| \\
\geq \frac{|E|}{2}.
\]

Let \( g \) be the following linear combination:

\[ g(x) = \sum_{j=0}^n a_j(t) f(x + j\tau) \]
with \( \sum_{j=0}^{n} |a_j(t)|^2 = 1 \) then

\[
\int_{E_t} |g|^2 \leq \left( \sum_{j=0}^{n} |a_j(t)|^2 \right) \left( \sum_{j=0}^{n} \int_{E_t} |f(x + jt)|^2 \, dx \right)
\]

\[
\leq \sum_{j=0}^{n} \int_{E - jE} |f(x + jt)|^2 \, dx
\]

\[
= \sum_{j=0}^{n} \int_{E} |f(x)|^2 \, dx
\]

\[
= (n + 1) \int_{E} |f|^2.
\]

Since \( |E_t| \geq |E|/2 \) we can choose such \( a_j(t) \) that \( g \perp V_{E_t} \). Then

\[
\int_{E} |g|^2 \leq \left( \frac{C}{|E_t|} \right)^{2m+1} \int_{E_t} |g|^2 \leq (n + 1) \left( \frac{C}{|E|} \right)^{2m+1} \int_{E} |f|^2 = \varepsilon^2.
\]

Let \( p(x) = \sum_{l=0}^{m} b_l x^l \) be a polynomial of degree \( m \). Define

\[
\tilde{p}(x) = \sum_{j=0}^{n} a_j(t) e^{i\lambda_j t} p(x + jt) = \sum_{l=0}^{m} \tilde{b}_l x^l.
\]

Then \( \tilde{b} = Ab \) where \( A \) is a certain \( (m + 1) \times (m + 1) \) upper triangular matrix with \( \sum_{j=0}^{n} a_j(t) e^{i\lambda_j t} \) on the diagonal. If this sum is not zero, then \( A \) is invertible. \( A^{-1} \) is also upper triangular and \( A_{ij}^{-1} = \frac{(-1)^{i+j} M_{ji}}{\det A} \). In fact,

\[
A_{kl} = \binom{l}{k} \sum_{j=0}^{n} a_j(t) e^{i\lambda_j t} (jt)^{l-k}, \quad 0 \leq k \leq l \leq m.
\]

A trivial estimate shows that \( |M_{ji}| \leq m! (2^m \sqrt{m^n})^m \leq (Cn^{3/2})^m \). However, we can give a better estimate by noticing that \( M_{ji} \) is the sum of at most \( 2^m \) terms (open...
determinants via rows or columns containing at most two nonzero entries) whose absolute values do not exceed

\[
\frac{\prod_{i=1}^{m} l_i}{\prod_{i=1}^{m} k_i! (l_i - k_i)!} \sum_{n=1}^{m} k_i - \sum_{i=1}^{m} k_i n^{m/2}, \quad 0 \leq k_i \leq l_i \leq m.
\]

We used the fact that \( \tau < 1 \). Therefore,

\[
|M_{ji}| \leq 2^m m! n^{3m/2} \leq (C(m + 1)n^{3/2})^m.
\]

Hence

\[
\|A^{-1}\|_2 \leq \frac{(C(m + 1)n^{3/2})^m}{|\sum_{j=0}^{n} a_j(t)e^{i\lambda_j t}|^{m+1}}.
\]

Note that

\[
g(x) = \sum_{j=0}^{n} a_j(t)f(x + jt) = \sum_{k=-\infty}^{\infty} \tilde{p}_k(x)e^{i\lambda_k x}.
\]

Therefore,

\[
\int |g|^2 \geq \frac{C}{1 + m^2} \sum_{k=-\infty}^{\infty} \int |\tilde{p}_k|^2
\]

\[
\geq \frac{1}{C^m} \sum_{k=-\infty}^{\infty} \|b_k\|_2^2
\]

\[
\geq \frac{1}{C^m \|A^{-1}\|_2^2} \sum_{k=-\infty}^{\infty} \|b_k\|_2^2.
\]
Integrating over \( t \in [0, \tau] \) and applying Holder's inequality, we have

\[
(Cmn^{3/2})^m \epsilon^2 \geq \sum_{k=-\infty}^{\infty} \|b_k\|_2^2 \left( \frac{1}{\tau} \int_0^\tau \left| \sum_{j=0}^n a_j(t)e^{i\lambda_kjt} \right|^2 dt \right)^{m+1} \geq \sum_{k=-\infty}^{\infty} \|b_k\|_2^2 \rho^{2(m+1)}(\lambda_k). \]

It is known that \( S = \{ \lambda : \rho^2(\lambda) < (\frac{C}{N^2})^{2n} \} \subset \bigcup_{j=1}^n I_j \) where \( I_j \) are some intervals of length \( \frac{1}{4n\tau} \) (see the proof of Lemma 3.1 in ([13])). Thus we obtain a lemma analogous to ([13], Lemma 3.1).

**Lemma 7:** There are \( n \) intervals \( I_1, \ldots, I_n \) of length \( \frac{1}{4n^3\tau} \) such that

\[
\sum_{\lambda_k \in \Lambda \setminus \bigcup_{j=1}^n I_j} \|b_k\|_2^2 \leq \left( (m + 1)(Cn)^{4n+3/2} \right)^{m+1} \epsilon^2.
\]

**Remark:** if \( \tau \geq C(m, N, |E|) \) then we are done since the cardinality of \( \Lambda \cap \bigcup_{j=1}^n I_j \) is bounded by \( n(1 + \frac{1}{4n^2C(m, N, |E|)}) \) and \( f \) can be approximated by \( \sum_{\lambda_k \in \Lambda \cap \bigcup_{j=1}^n I_j} p_k(x)e^{i\lambda_kx} \).

It is also known that there are real numbers \( \mu_1, \ldots, \mu_n \) and \( t \in [0, \tau] \) such that

\[
\left| \sum_{j=0}^n a_j(t)e^{i\lambda_kjt} \right| \geq \left( \frac{1}{8n^2} \right)^{2n} \prod_{j=1}^n \theta_r(\lambda - \mu_j)
\]

where \( \theta_r(x) = \min(1, |x|) \) and \( \lambda \in \bigcup_{j=1}^n I_j \) (see arguments after Lemma 3.1 in ([13])).
Thus we obtain the following result:

$$\sum_{\lambda_k \in A \cap \bigcup_{j=1}^{n} I_j} \| b_k \|_2^2 \left( \prod_{j=1}^{n} \theta_r(\lambda_k - \mu_j) \right)^{2(m+1)} \leq ((m + 1)(Cn)^{4n+3/2})^{m+1} \varepsilon^2.$$ 

Combining this result with Lemma 4 and the fact that $0 \leq \theta_r \leq 1$, we obtain

**Lemma 8:** There are $n$ real numbers $\mu_1, \ldots, \mu_n$ such that

$$\sum_{k=-\infty}^{\infty} \| b_k \|_2^2 \left( \prod_{j=1}^{n} \theta_r(\lambda_k - \mu_j) \right)^{2(m+1)} \leq (C(m + 1)n^{4n+3/2})^{m+1} \varepsilon^2 = \varepsilon^2.$$ 

Let $D = \left( \prod_{j=1}^{n} \left( e^{i\mu_j x} \frac{d}{dx} e^{-i\mu_j x} \right) \right)^{m+1} = \prod_{j=1}^{n} \left( e^{i\mu_j x} \frac{d^{m+1}}{dx^{m+1}} e^{-i\mu_j x} \right)$ be a differential operator. Any solution of the homogenous equation $Df = 0$ is of the form $\sum p_j(x)e^{i\mu_j x}$ with $p_j$ being polynomials of degree at most $m$. A partial solution of $Df = g$ is of the form $\phi = \left( \prod_{j=1}^{n} \left( e^{i\mu_j x} \int e^{-i\mu_j x} \right) \right)^{m+1} g$. Note that $\max_{x \in J} |\phi(x)| \leq \frac{J^{n(m+1)}}{[n(m+1)]!} \int_{J} |g|$, for any interval $J$. Hence there exist $n$ polynomials $p_j$ of degree at most $m$ such that

$$\max_{x \in J} |f(x) - \sum_{j=1}^{n} p_j(x)e^{i\mu_j x}| \leq \frac{J^{n(m+1)}}{[n(m+1)]!} \frac{1}{|J|} \int_{J} |g|. \quad (1.59)$$

Define a set of $2n$ intervals $I_k = (\mu_k - \frac{1}{r}, \mu_k + \frac{1}{r})$ and $\tilde{I}_k = (\mu_k - \frac{2}{r}, \mu_k + \frac{2}{r})$. For some fixed constant $C$ which will be chosen later, define $\Lambda = \bigcup_{k=1}^{n} \{ \lambda \in \Lambda : |\lambda - \mu_k| \leq Cm^2 \}$. The cardinality of $\Lambda$ is bounded from above by $Cn$ because of our gap assumption (see Lemma 6). Note that we don’t need the lacunarity of $\Lambda$ for this. Put $\Lambda' = \Lambda \setminus \tilde{\Lambda}$. Split $\Lambda'$ into $n + 1$ disjoint subsets in the following way. $\Lambda_0 = \Lambda \setminus \bigcup_{j=1}^{n} I_j$, $\Lambda_k =$
\[ I_k \setminus \bigcup_{j=k+1}^{n} I_j, \quad k = 1, \ldots, n. \] Then we can decompose

\[ f = \tilde{f} + \sum_{j=0}^{n} f_j \]

where \( \tilde{f} = \sum_{\lambda_k \in \tilde{\Lambda}} p_k(x) e^{i\lambda_k x} \) and \( f_j = \sum_{\lambda_k \in \Lambda_j} p_k(x) e^{i\lambda_k x} \). It follows from Lemma 8 that

\[ \sum_{\lambda_k \in \Lambda_{l+1}} \left\| b_k \right\|^2_2 \left( \prod_{j=1}^{n} \theta_r(\lambda_k - \mu_j) \right)^{2(m+1)} \leq \epsilon^2, \quad l = 0, \ldots, n. \]

In particular, it means that

\[ \frac{C}{1 + m^2} \int |f_0|^2 \leq \sum_{\lambda_k \in \Lambda_0} \|b_k\|^2_2 \leq \epsilon^2 \quad (1.60) \]

since \( \prod_{j=1}^{n} \theta_r(\lambda - \mu_j) = 1 \) if \( \lambda \in \Lambda_0 \).

If \( 1 \leq k \leq n \) then \( f_k \) has only finitely many terms and therefore it is infinitely many times differentiable. If \( \lambda \in \Lambda_k \) then

\[ \prod_{j=1}^{n} \theta_r(\lambda - \mu_j) = \prod_{\mu_j \in \tilde{\Lambda}_k} \theta_r(\lambda - \mu_j). \]

Correspondingly, instead of \( D \) we will define another differential operator

\[ D_k = \prod_{\mu_j \in \tilde{\Lambda}_k} (e^{i\mu_j x} (x^m + 1) \frac{d^m}{dx^m} e^{-i\mu_j x}). \]

Let \( D_k f_k = g_k \) and let \( n_k \) denote the number of \( \mu_j \in \tilde{\Lambda}_k \). We have that \( 1 \leq n_k \leq n \). Let \( p \) be a polynomial of degree at most \( m \), then

\[ e^{i\mu_j x} (x^m + 1) \frac{d^m}{dx^m} e^{-i\mu_j x} p(x) e^{i\lambda x} = \sum_{k=0}^{m+1} \binom{m+1}{k} p^{(k)}(x) (i(\lambda - \mu_j)^{m+1-k}) e^{i\lambda x} = \bar{p}(x) e^{i\lambda x}. \]
Since \( \|p^{(k)}\|_{L^2(I)} \leq (Cm^2)^k \|p\|_{L^2(I)} \) we have
\[
\|\tilde{p}\|_{L^2(I)} \leq (|\lambda - \mu| + Cm^2)^{m+1}\|p\|_{L^2(I)}.
\]

Actually we can give a better estimate because
\[
\|p^{(k)}\|_{L^2(I)} \leq Cm^k\|p\|_{L^2(I)}
\]
which follows from well-known Markov's inequality ([3], p.256):
\[
\|p^{(k)}\|_{L^\infty([-1,1])} \leq \frac{m^2 \cdot (m^2 - 1) \cdot (m^2 - 2^2) \cdot \ldots \cdot (m^2 - (k - 1)^2)}{1 \cdot 3 \cdot \ldots \cdot (2k - 1)}\|p\|_{L^\infty([-1,1])}
\]
but what we already have is enough. Therefore,
\[
\frac{C}{1 + m^2} \int_I |g_k|^2 \leq \sum_{\lambda_1 \in \Lambda_k} \prod_{\mu_j \in \bar{I}_k} (|\lambda_1 - \mu_j| + Cm^2)^{2(m+1)} \int_I |p_l|^2
\]
\[
\leq \sum_{\lambda_1 \in \Lambda_k} \prod_{\mu_j \in \bar{I}_k} (C|\lambda_1 - \mu_j|)^{2(m+1)} \int_I |p_l|^2
\]
since \(|\lambda - \mu_j| \geq Cm^2\) if \(\lambda \in \Lambda_k \subset \Lambda \setminus \tilde{\Lambda}\). It is left to notice that \(|\lambda - \mu_j| \leq \frac{3\theta_\tau(\lambda - \mu_j)}{\tau}\) if \(\lambda \in \Lambda_k \subset I_k\) and \(\mu_j \in \bar{I}_k\) to conclude that
\[
\frac{C}{1 + m^2} \int_I |g_k|^2 \leq \sum_{\lambda_1 \in \Lambda_k} \prod_{\mu_j \in \bar{I}_k} (C\frac{3\theta_\tau(\lambda_1 - \mu_j)}{\tau})^{2(m+1)}\|b_k\|_2^2
\]
\[
\leq \left(\frac{C}{\tau}\right)^{2n_k(m+1)} \varepsilon^2.
\]
(1.59) shows that for each \(1 \leq k \leq n\) and each interval \(J\) there exists \(\phi_k' = \sum_{\mu_j \in \bar{I}_k} p_j(x)e^{i\mu_j x}\) such that
\[
|f_k(x) - \phi_k(x)| \leq \frac{|J|^{n_k(m+1)}}{[n_k(m+1)]!} \frac{1}{|J|} \int_J |g_k|
\]
\[
\leq \frac{|J|^{n_k(m+1)}}{[n_k(m+1)]!} Mg_k(x)
\]
for every $x \in J$. We will consider only $J \subset I$. $Mg_k$ is the maximal function of $g_k$ on $I$. Then

$$\|Mg_k\|_{L^2(I)} \leq C\|g_k\|_{L^2(I)} \leq \left(\frac{C}{\tau}\right)^{n_k(m+1)} \epsilon'.$$

(1.61)

Let $|J| = A\tau$ with $A \geq 1$ and $\phi^J(x) = \tilde{f}(x) + \sum_{k=1}^n \phi_k^J(x)$ then

$$|f(x) - \phi^J(x)| \leq |f_0(x)| + \sum_{k=1}^n \frac{(A\tau)^{n_k(m+1)}}{[n_k(m+1)]!} Mg_k(x) = R(x)$$

for every $x \in J$. Note that $\phi^J$ can be written in the following way:

$$\phi^J(x) = \sum_{j=1}^{Cn} p_j^J(x)e^{i\varphi_j x}$$

where $p_j^J$ are polynomials of degree at most $m$. Applying (1.60) and (1.61) we can estimate the $L^2(I)$ norm of the remainder $R$ by

$$\|R\|_{L^2(I)} \leq (C \cdot A)^{n(m+1)} \epsilon'.$$

Now we will construct $\delta E \in I \cap E^c$. Divide $I$ into intervals of length $|J| = A\tau$. We will choose $A$ later so that so that $\frac{1}{\delta\tau}$ is an integer. Fix some $0 < \gamma \leq |E| < 1$. Call an interval $J$ good if $|E \cap J| \geq \gamma |J|$ and correspondingly bad if $|E \cap J| < \gamma |J|$.
Denote $\delta E = E^c \cap \bigcup_{J \text{ is good}} J$. Then

$$|E^c \cap (E + n\tau)| \leq n\tau + |\delta E| + |E^c \cap \bigcup_{J \text{ is bad}} J \cap (E + n\tau)|$$

$$\leq n\tau + |\delta E| + \sum_{J \text{ is bad}} |J \cap (E + n\tau)|$$

$$\leq n\tau + |\delta E| + \sum_{J \text{ is bad}} (|J \setminus (J + n\tau)| + |(J + n\tau) \cap (E + n\tau)|)$$

$$\leq |\delta E| + \frac{1}{|J|} (n\tau + \gamma |J|)$$

$$\leq |\delta E| + n\tau \frac{1}{|J|} + \gamma$$

since the number of bad intervals is bounded by $\frac{1}{|J|} - 1$. Recall that $|E^c \cap (E + n\tau)| = \frac{|E|}{2n}$. If we put $\gamma = \frac{|E|}{8n} < |E|$ then

$$\frac{3|E|}{8n} \leq |\delta E| + n\tau \frac{1}{|J|}.$$

There are two possible situations:

1. $n\tau \leq \frac{|E|}{8n}$. Then we can put $\frac{1}{|J|} = \frac{|E|}{8n^2 \tau}$ and obtain that

$$|\delta E| \geq \frac{|E|}{4n}.$$  

We also have that

$$A = \frac{|J|}{\tau} = \frac{1}{\tau \frac{|E|}{8n^2 \tau}} \leq \frac{16n^2}{|E|}.$$

It is interesting to note that the condition $n\tau \leq \frac{|E|}{8n}$ implies that $|E| \leq 1 - \frac{3}{11n}$ since $\frac{|E|}{2n} = |E^c \cap (E + n\tau)| \leq n\tau + 1 - |E| \leq \frac{|E|}{8n} + 1 - |E|.$

2. $n\tau > \frac{|E|}{8n}$. Then we put $|J| = 1, \gamma = |E|$ and therefore $I$ is a good interval itself. In this case

$$A = \frac{1}{\tau} < \frac{8n^2}{|E|}.$$
Alternatively, we can use Lemma 7 directly in this case since \( \frac{1}{\tau} < \frac{8n^2}{|E|} \) and the cardinality of \( \Lambda \cap \bigcup_{j=1}^{n} I_j \) is bounded by \( n + \frac{n}{4n^2 \tau} < \frac{3n}{|E|} \) and we can approximate \( f \) with
\[
\sum_{\lambda_k \in \bigcup_{j=1}^{n} I_j} p_k(x)e^{i\lambda_k x}.
\]
Now we will explain how to proceed in each case.

1.
\[
\int_{\delta E} |f|^2 \leq \int_{\bigcup_{J \text{ is good}}} |f|^2
\]
\[
\leq 2 \sum_{J \text{ is good}} \int_{J} \left( |\phi|^2 + \int |R|^2 \right)
\]
\[
\leq \sum_{J \text{ is good}} \left( \frac{C|J|}{|E \cap J|} \right)^{2Cn(m+1) - 1} \int_{E \cap J} |\phi|^2 + \left( \frac{C}{\gamma} \right)^{2Cn(m+1) - 1} \int_{J} |R|^2
\]
\[
\leq \left( \frac{Cn}{|E|} \right)^{2Cn(m+1) - 1} \int_{E} |f|^2 + \left( \frac{Cn}{|E|} \right)^{2Cn(m+1) - 1} \int_{E} |R|^2.
\]

Recall that
\[
\int_{I} |R|^2 \leq (C \cdot A)^{2n(m+1)} \epsilon^2
\]
\[
\leq (C \cdot A)^{2n(m+1)} (C(m+1)n^{4n+3/2})^{m+1} \epsilon^2
\]
\[
\leq (C \cdot A)^{2n(m+1)} (C(m+1)n^{4n+3/2})^{m+1} (n + 1) \left( \frac{C}{|E|} \right)^{2m+1} \int_{E} |f|^2.
\]
Hence

\[
\int_{\delta E \cap E} |f|^2 \leq (m + 1)^{m+1} \left( \frac{C(n + 1)}{|E|} \right)^{C(n+1)(m+1)} \int_E |f|^2
\]

\[
\leq \left( \frac{CN}{|E|} \right)^{N(\frac{E}{|E|})^{C(m+1)}} \int_E |f|^2
\]

\[
\leq e^{(\ln CN)N(\frac{E}{|E|})^{C(m+1)}} \int_E |f|^2
\]

with

\[
|\delta E| \geq \frac{|E|}{4n}
\]

\[
\geq \frac{1}{N} \left( \frac{|E|}{C} \right)^{C(m+1)}
\]

2. In a similar way we can get

\[
\int_{I} |f|^2 \leq e^{(\ln CN)N(\frac{E}{|E|})^{C(m+1)}} \int_{E} |f|^2
\]

and we are done in this case.

If we iterate \(N(\frac{E}{|E|})^{C(m+1)}\) times in case 1 we will reach \(I\) and therefore

\[
\int_{I} |f|^2 \leq e^{(\ln CN)N^2(\frac{E}{|E|})^{C(m+1)}} \int_{E} |f|^2. \tag{1.62}
\]

Now we will drop the assumption that gaps \(|\lambda_k - \lambda_l|\) are large, i.e., \(|\lambda_k - \lambda_l| \geq C(1 + m^3)\) for \(k \neq l\). Since \(\Lambda\) is lacunary it has no more than \(CN(1 + m^3)\) pairs of \((\lambda_k, \lambda_l)\) with \(k \neq l\) such that \(|\lambda_k - \lambda_l| \leq C(1 + m^3)\). To show this we will split these pairs in \([C(1 + m^3)]\) groups such that \(d < |\lambda_k - \lambda_l| \leq d + 1\) with \(d = 0, 1, 2, ..., [C(1 + m^3)]\) and apply definition (1.29). Denote by

\[
\Lambda' = \{\lambda_k \in \Lambda : \exists \lambda_l \in \Lambda, l \neq k; |\lambda_k - \lambda_l| \leq C(1 + m^3)\}.
\]
Then the cardinality of $\Lambda'$ is bounded by $CN(1 + m^3)$. Let

\[
f(x) = \sum_{k=-\infty}^{\infty} p_k(x)e^{i\lambda_k x}
\]

\[
= \sum_{\lambda_k \in \Lambda \setminus \Lambda'} p_k(x)e^{i\lambda_k x} + \sum_{\lambda_k \in \Lambda'} p_k(x)e^{i\lambda_k x}
\]

\[
= h(x) + r(x).
\]

Assume for a while that $r(x)$ is a trigonometric polynomial of order $n' = (1 + m)|\Lambda'| \leq (1 + m)CN(1 + m^3)$. The proof goes in the same way with few changes. We will only discuss the changes. Let $n_1$ be the same as in (1.57):

\[
n_1 = \left[N \left(\frac{C}{|E|}\right)^{4m+1}\right].
\]

Put

\[
n = n_1 + n'
\]

\[
\leq \left[N \left(\frac{C}{|E|}\right)^{4m+1}\right] + [CN(1 + m^3)]
\]

\[
\leq N \left(\frac{C}{|E|}\right)^{4m+1}.
\]

Let

\[
g(x) = \sum_{j=0}^{n} a_j(t)f(x + jt)
\]

\[
= \sum_{j=0}^{n} a_j(t)h(x + jt) + \sum_{j=0}^{n} a_j(t)r(x + jt)
\]

with $\sum_{j=0}^{n} |a_j(t)|^2 = 1$ such that $\sum_{j=0}^{n} a_j(t)r(x + jt) = 0$ and $\sum_{j=0}^{n} a_j(t)h(x + jt) = g(x)\perp V_{n_1(E)}$ which is exactly $n_1 + n' = n$ linear homogenous equations with $n + 1$
variables $a_j$. Then

$$\int \left| \sum_{j=0}^{n} a_j(t)h(x + jt) \right|^2 dx = \int |g|^2$$

$$\leq \left( \frac{C}{|E|} \right)^{2m+1} \int _E |g|^2$$

$$\leq (n + 1) \left( \frac{C}{|E|} \right)^{2m+1} \int _E |f|^2 = \varepsilon^2.$$

Therefore, there are $\phi^J$

$$|h(x) - \phi^J(x)| \leq R(x)$$

for every $x \in J$. Note that $\phi^J$ can be written in the following way:

$$\phi^J(x) = \sum_{j=1}^{C_n} p^J_j(x) e^{i\alpha_j x}$$

where $p^J_j$ are polynomials of degree at most $m$ and

$$\|R\|_{L^2(E)} \leq (C \cdot A)^{n(m+1)} \varepsilon^t.$$

Then

$$|f(x) - \phi^J(x) - r(x)| \leq R(x).$$

It is left to note that

$$\int _J |\phi^J + r|^2 \leq \left( \frac{C|J|}{|E \cap J|} \right)^{2Cn(m+1)+2n' - 1} \int _{E \cap J} |\phi^J + r|^2$$

$$\leq \left( \frac{C|J|}{|E \cap J|} \right)^{2Cn(m+1)-1} \int _{E \cap J} |\phi^J + r|^2.$$
The rest of the proof is the same. Thus we obtain an estimate similar to (1.62)

$$
\int \int |f|^2 \leq e^{(\ln C_N)N^2(\frac{C}{|E|})(m+1)} \int_{E} |f|^2.
$$

(1.63)

Since any polynomial of degree $m$ can be uniformly approximated on an interval by a trigonometric polynomial of order $m + 1$ we can drop our assumption that $\tau(x)$ is a trigonometric polynomial and obtain the same estimate as (1.63). □

Remark: Unfortunately, the factor in (1.63) grows much faster than $m!$ and we can not use this result to prove the conjectured Theorem 3 by approximating $f_k$ on each interval $I$ with corresponding Taylor polynomials as we did in the case when $\hat{f}$ is supported on a union of finitely many intervals in Theorem 2.

1.5 Necessary condition for support

Recall that a set $E \subset \mathbb{R}$ is called "relatively dense" if there exist $a > 0$ and $\gamma > 0$ such that

$$
|E \cap I| \geq \gamma \cdot a
$$

(1.64)

for every interval $I$ of length $a$.

It is a well-known fact that "relative density" is also necessary for an inequality of the form

$$
\|f\|_{L^p(E)} \geq C(E, \Sigma, p) \cdot \|f\|_p
$$

(1.65)

to hold for every $f \in L^p$ with $\text{supp} \hat{f} \subset \Sigma$. Now we will consider an inverse problem. We will give a necessary condition for $\Sigma \supset \hat{f}$ so that (1.65) holds.

**Theorem 5:** Suppose that for a given open set $\Sigma \in \mathbb{R}$ there exist $0 < \gamma < 1$, $a > 0$ and $p \in [1, \infty]$ such that for every "relatively dense" set $E$ satisfying (1.64) we
have

\[ \|f\|_{L^p(E)} \geq C(E, p) \cdot \|f\|_p \]  \hspace{1cm} (1.66)

for every \( f \in L^p \) with \( \text{supp} \hat{f} \subseteq \Sigma \)

then for every \( b_0 > 0 \) \( \Sigma \) cannot contain arbitrarily long arithmetic progressions with steps at least \( b_0 \).

**Proof of Theorem 5:** Note that the parameter \( a \) is fixed during the whole proof. First we will prove that there is a uniform constant \( C \) such that (1.66) holds for large enough sets \( E \), i.e., there exists \( \gamma_0 \in [\gamma, 1) \) and \( C(p) > 0 \) such that

\[ \|f\|_{L^p(E)} \geq C(p) \cdot \|f\|_p \]  \hspace{1cm} (1.67)

holds for every "relatively dense" set \( E \) with density \( \gamma_0 \) and every \( f \in L^p \) with \( \text{supp} \hat{f} \subseteq \Sigma \). Suppose towards a contradiction that this is not true. Then there exists a sequence of \( f_n \) and corresponding "relatively dense" sets \( E_n \) with density \( \gamma_n > 1 - \frac{1}{2^n} \) such that

\[ \|f_n\|_{L^p(E_n)} \leq \frac{1}{n} \|f_n\|_p. \]

Let \( E = \bigcap_{n=1}^{\infty} E_n \). Then \( E \) is "relatively dense" with density \( \gamma \):

\[ |E \cap I| = a - \left| \bigcup_{n=1}^{\infty} E_n^c \right| \]

\[ \geq a - \sum_{n=1}^{\infty} a(1 - \gamma_n) \]

\[ \geq a - a \sum_{n=1}^{\infty} \frac{1 - \gamma}{2^n} = \gamma a \]
for every interval $I$ of length $a$. On the other hand we have

$$
\|f_n\|_{L^p(E)} \leq \|f_n\|_{L^p(E_n)} \leq \frac{1}{n} \|f_n\|_p
$$

which contradicts to (1.66). The next lemma plays a crucial role.

**Lemma 9:** Let $\Sigma \in \mathbb{R}$ be an open set for which there exists $b_0 > 0$ such that $\Sigma$ contains arbitrarily long arithmetic progressions with steps at least $b_0$ then for every $0 < \gamma < 1$ (meaning arbitrarily small $1 - \gamma$) and every $a > 0$ (meaning arbitrarily small $a$) there exist a sequence of “relatively dense” sets $E_n$ satisfying (1.64) and a corresponding sequence of functions $f_n \in L^p$ with $\text{supp} \tilde{f} \subset \Sigma$ such that

$$
\lim_{n \to \infty} \frac{\int_{E_n} |f_n|^p}{\int |f_n|^p} = 0
$$

for every $1 \leq p < \infty$.

**Remark:** If $\Sigma$ contains an infinite arithmetic progression then we can take only one set $E$ instead of the sequence of $E_n$. Here is an example of such $\Sigma$ which has a finite measure: $\Sigma = \bigcup_{k=1}^{\infty} (k - \frac{1}{2^k}, k + \frac{1}{2^k})$. Compare with the Amrein-Berthier theorem (see for example [7], pp. 97, 455). Note also that small $1 - \gamma$ and small $a$ are typical cases of “easy” proofs of the Logvinenko-Sereda theorem.

**Proof of Lemma 9:** Define $L = \left\lfloor \frac{2\pi}{ab_0} \right\rfloor + 1$, a positive integer such that $\frac{2\pi}{Lb} \leq a$ for every $b \geq b_0$. Suppose we have an arithmetic progression with step $b \geq b_0$ of length $Ln + 1$: $x_0, x_0 + b, x_0 + 2b, ..., x_0 + Lnb \in \Sigma$ hence there exists $\epsilon > 0$ such that $\bigcup_{k=0}^{n} [x_0 + kLb - \epsilon, x_0 + kLb + \epsilon] \subset \Sigma$ because $\Sigma$ is open. We can assume that $\epsilon < \frac{Lb}{2}$ so that these intervals are disjoint. Let $\phi$ be a Schwartz function with $\text{supp} \hat{\phi} \subset [-1, 1]$ such that $\phi(0) = 1$, e.g., if $\hat{\phi} \geq 0$. Define

$$
\hat{f_n}(x) = \sum_{k=0}^{n} \hat{\phi}(\frac{x - x_0 - kLb}{\epsilon})
$$
so that supp $f_n \subset \Sigma$. Then

$$|f_n(y)| = |e^{i(x_0 + kLb)y} \phi(ey)|$$

(1.68)

$$= e|D_n(Lby)\phi(ey)|$$

(1.69)

where the $2\pi$-periodic function $D_n(t) = \frac{\sin((n+1)t/2)}{\sin t/2}$ is the Dirichlet kernel with the following easily verified property:

$$\lim_{n \to \infty} \frac{\|D_n\|_{L^p([0,\pi])}}{\|D_n\|_{L^p([-\pi,\pi])}} = 0 \quad 0 < \delta < \pi, \quad 1 \leq p \leq \infty.$$

This is true since $\|D_n\|_{L^p([-\pi,\pi])} \sim n^{1/p}$ for $1 < p \leq \infty$ and $\|D_n\|_{L^1([-\pi,\pi])} \sim \ln n$ and $|D_n(x)| \leq \frac{1}{\sin \frac{x}{2}}$ if $\delta \leq |x| \leq \pi$. Now we will construct a $\frac{2\pi}{Lb}$-periodic set $E$:

$$E \cap \left[ -\pi \frac{Lb}{Lb}, \pi \frac{Lb}{Lb} \right] = \left[ -\pi \frac{Lb}{Lb}, \pi \frac{Lb}{Lb} \right] \setminus \left[ -\frac{(1-\gamma)\pi}{2Lb}, \frac{(1-\gamma)\pi}{2Lb} \right].$$

Then

$$|E^c \cap I| \leq \left( \left\lfloor \frac{aLb}{2\pi} \right\rfloor + 1 \right) \frac{(1-\gamma)\pi}{Lb}$$

$$\leq \frac{2aLb}{2\pi} \frac{(1-\gamma)\pi}{Lb}$$

$$= (1-\gamma)a$$

for every interval $I$ of length $a$. We used here that $\frac{2\pi}{Lb} \leq a$ to get the second inequality.

Let $1 \leq p < \infty$. $|\phi(x)| \leq \frac{C}{1+x^2}$ since $\phi$ is a Schwartz function. Applying (1.69) we
have

\[
\int_E |f_n|^p = \sum_{k=-\infty}^{\infty} \int_{E \cap [-\frac{(2k-1)\pi}{Lb}, \frac{(2k+1)\pi}{Lb}]} |\varepsilon D_n(Lby)\phi(ey)|^p dy
\]

\[
\leq \sum_{k=-\infty}^{\infty} e^p \left( \frac{\varepsilon^p}{1 + (ek/Lb)^2} \right)^p \frac{1}{Lb} \int_{[-\pi, \pi] \setminus [-\frac{(1-\varepsilon)\pi}{2}, \frac{(1-\varepsilon)\pi}{2}]} |D_n(y)|^p dy
\]

\[
\leq C e^p \left(1 + \frac{Lb}{\varepsilon}\right)^p \frac{1}{Lb} \|D_n\|^p_{\mathcal{L}^p([-\delta, \delta])}
\]

\[
\leq \frac{C e^p}{\varepsilon} \|D_n\|^p_{\mathcal{L}^p([-\delta, \delta])}
\]

(1.70)

where \( \delta = \frac{(1-\varepsilon)\pi}{2} \). We used the fact that \( \varepsilon < \frac{Lb}{2} \) to get the last inequality.

Now we need to estimate \( \|f_n\|^p \). Recall that \( \phi(0) = 1 \). Since \( \phi \) is a continuous function there is \( d > 0 \) such that \( \phi(x) \geq 1/2 \) if \( |x| \leq d \). We can assume that \( \varepsilon < \frac{Lbd}{\pi} \).

Then applying (1.69) we have

\[
\int |f_n|^p \geq e^p \frac{1}{2p} \int_{-d/e}^{d/e} |D_n(Lby)|^p dy
\]

\[
= e^p \frac{1}{2p} \frac{Lbd}{\varepsilon \pi} \|D_n\|^p_{\mathcal{L}^p([-\pi, \pi])}
\]

\[
\geq e^p \frac{1}{2p} \frac{Lbd}{\varepsilon \pi} 2\pi \|D_n\|^p_{\mathcal{L}^p([-\pi, \pi])}
\]

\[
\geq \frac{C e^p}{\varepsilon^{2p+1}} \|D_n\|^p_{\mathcal{L}^p([-\pi, \pi])}.
\]

(1.71)

We used here that \( \varepsilon < \frac{Lbd}{\pi} \) to get the last but one inequality.

Hence dividing (1.70) by (1.71) we obtain the desired result

\[
\frac{\int_{E_n} |f_n|^p}{\int |f_n|^p} \leq C \frac{\|D_n\|^p_{\mathcal{L}^p([-\delta, \delta])}}{\|D_n\|^p_{\mathcal{L}^p([-\pi, \pi])}} \to 0
\]
as \( n \to \infty \).

The proof for \( p = \infty \) is similar and even easier. \( \square \)

Remark: If \( p = 2 \) we can use periodizations to prove Lemma 9 (see the proof of Proposition 2).

Theorem 5 follows from Lemma 9 since otherwise we would get a contradiction to (1.67). \( \square \)
Chapter 2  Periodizations of functions

2.1  Overview

Let $f$ be a function from $L^1(\mathbb{R}^d)$. Define a family of its periodizations with respect to a rotated integer lattice:

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))$$

for all rotations $\rho \in SO(d)$. The main object of our study is $G$, the $L^2([0, 1]^d \times SO(d))$ norm of the family of periodizations,

$$G^2 = \int_{\rho \in SO(d)} \int_{[0,1]^d} |g_\rho(x)|^2 dxd\rho$$

$$= \int_{\rho \in SO(d)} \|g_\rho\|^2 L d\rho. \quad (2.2)$$

The purpose of this work is to show how $G$ can give an estimate of the $L^2(\mathbb{R}^d)$ norm of a function from $L^1(\mathbb{R}^d)$ in higher dimensions. Some results on the Steinhaus tiling problem are related to Theorem 1 since periodizations naturally appear in the problem of Steinhaus. M. Kolountzakis ([9]) proves that if a function $f \in L^1(\mathbb{R}^2)$ and $|x|^\alpha f \in L^1(\mathbb{R}^2)$ where $\alpha > \frac{10}{3}$ and its periodizations are constants, then the function is continuous. Another result is obtained by M. Kolountzakis and T. Wolff ([10], Theorem 1). It says that if periodizations of a function from $L^1(\mathbb{R}^d)$ are constants, then the function is continuous provided that the dimension $d$ is at least three. We will show that this result is false when dimension $d = 2$.

The main theorems are the following:
Theorem 1: let $d \geq 4$ and let $f \in L^1(\mathbb{R}^d)$. If periodizations of $f$

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))$$

are in $L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$G^2 = \int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho < \infty,$$

then $f \in L^2(\mathbb{R}^d)$:

$$\|f\|_2 \leq C(G + \|f\|_1) \quad (2.3)$$

where $C$ depends only on $d$.

We will also obtain an inverse theorem.

Theorem 1': let $d \geq 5$ and let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and let $g_\rho$ be periodizations of $f$

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu)),$$

then $g_\rho \in L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$\int_{\rho \in SO(d)} \|g_\rho\|_2^2 d\rho \leq C(\|f\|_2 + \|f\|_1)^2 \quad (2.4)$$

where $C$ depends only on $d$.

We will generalize Theorems 1 and 1' in the spirit of the Stein-Tomas Theorem ([4], Chapter 6.5).
Theorem 2: let $d \geq 4$ and let $f \in L^p(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$. If periodizations of $f$

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))$$

are in $L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$G^2 = \int_{\rho \in SO(d)} \|g_\rho\|_{L^2}^2 d\rho < \infty,$$

then $f \in L^2(\mathbb{R}^d)$:

$$\|f\|_2 \leq C(G + \|f\|_p)$$  \hspace{1cm} (2.5)

where $C$ depends only on $d$ and $p$.

We will also obtain an inverse theorem.

Theorem 2': let $d \geq 5$ and let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $1 \leq p < \frac{2d}{d+2}$ and let $g_\rho$ be periodizations of $f$

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu)),$$

then $g_\rho \in L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$\int_{\rho \in SO(d)} \|g_\rho - \tilde{g}_\rho(0)\|_{L^2}^2 d\rho \leq C(\|f\|_2 + \|f\|_p)^2$$  \hspace{1cm} (2.6)

where $C$ depends only on $d$ and $p$.

Note that the constant $C$ below is not fixed and varies appropriately from one equality or inequality to another without being mentioned.
2.2 Case $p = 1$

Proof of Theorem 1:

We will denote $\tilde{f}(x) = \tilde{f}(-x)$ and $F(x) = f \ast \tilde{f}(x)$. Then $F \in L^1(\mathbb{R}^d)$ and

$$\|F\|_1 \leq \|f\|^2_1. \quad (2.7)$$

We will define the following functions $h, h_1, h_2 : \mathbb{R}^+ \to \mathbb{C}$

$$h(t) = \int |\tilde{f}(\xi)|^2 d\sigma_t(\xi) \quad (2.8)$$

$$= \int_{\mathbb{R}^d} f \ast \tilde{f}(x) d\sigma_t(x) dx$$

$$= \int_{\mathbb{R}^d} F(x) d\sigma_t(x) dx, \quad (2.9)$$

$$h_1(t) = \int_{|x| \leq 1} F(x) d\sigma_t(x) dx, \quad (2.10)$$

$$h_2(t) = \int_{|x| > 1} F(x) d\sigma_t(x) dx. \quad (2.11)$$

Clearly $h = h_1 + h_2$.

Lemma 1: Let $q : \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[\frac{1}{2}, 2]$, and let $b \in [0, 1)$. Define $H_1 : \mathbb{R} \to \mathbb{C}$

$$H_1(t) = \frac{1}{\sqrt{t + b}} h_1(\sqrt{t + b}) q(\sqrt{t + b} \frac{N}{N}).$$

Then for large enough $N$ we have...
\[ \sum_{\nu \neq 0} |\hat{H}_1(\nu)| \leq \frac{C\|F\|_1}{N} \]  

(2.12)

where \( C \) depends only on \( q \) and \( d \).

**Proof of Lemma 1:**

First we will estimate derivatives of \( h_1(t) \) when \( t \geq 1 \)

\[ |h_1^{(k)}(t)| \leq C t^{d-1} \|F\|_1 \]  

(2.13)

where \( C \) depends only on \( k \) and \( d \). This follows from (2.11)

\[
h_1(t) = \int_{|x| \leq 1} F(x) \widehat{\sigma_i}(x) \, dx = t^{d-1} \int_{|x| \leq 1} F(x) \int_{|\xi| = 1} e^{-i2\pi x_\xi d\sigma(\xi)} \, dx
\]

by differentiating the last equality \( k \) times.

We can easily prove by induction that

\[
\frac{d^k}{dt^k} \left( \frac{h_1(\sqrt{t} + b)}{\sqrt{t} + b} \right) = \sum_{i=0}^{k} C_{i,k} \frac{h_1^{(i)}(\sqrt{t} + b)}{(\sqrt{t} + b)^{2k+1-i}}.
\]  

(2.14)

It follows from (2.14) and (2.13) that when \( t \sim N^2 \) we have

\[
\left| \frac{d^k}{dt^k} \left( \frac{h_1(\sqrt{t} + b)}{\sqrt{t} + b} \right) \right| \leq C N^{d-k-2} \|F\|_1
\]  

(2.15)

with \( C \) depending only on \( k \) and \( d \).

Since \( q(\frac{\sqrt{k+b}}{N}) = q(\sqrt{t' + b'}) = \bar{q}(t') \) with \( t' = \frac{k}{N^2} \) and \( b' = \frac{b}{N^2} \) and \( \bar{q}(t') \) is a
Schwartz function supported in $t' \sim 1$, then we have
\[
\left| \frac{d^k}{dt^k} q\left( \frac{\sqrt{t+b}}{N} \right) \right| = N^{-2k} \left| \frac{d^k}{dt^k} \bar{q}(t') \right| 
\leq CN^{-2k} \tag{2.16}
\]
with $C$ depending only on $k$ and $q$.

Since $q\left( \frac{\sqrt{t+b}}{N} \right)$ is supported in $t \sim N^2$, it follows from (2.15) and (2.16) that
\[
\left| \frac{d^k}{dt^k} H_1(t) \right| = \left| \frac{d^k}{dt^k} \left( \frac{h_1(\sqrt{t+b})}{\sqrt{t+b}} q\left( \frac{\sqrt{t+b}}{N} \right) \right) \right| 
\leq CN^{d-2-k} \| F \|_1 \tag{2.17}
\]
with $C$ depending only on $k$, $d$ and $q$. Since $H_1(t)$ is also supported in $t \sim N^2$ we have
\[
\| H_1^{(k)} \|_1 \leq CN^{d-k} \| F \|_1.
\]

Therefore,
\[
| \hat{H}_1(\nu) | \leq \frac{C}{|\nu|^k} \| H_1^{(k)} \|_1 
\leq \frac{CN^{d-k} \| F \|_1}{|\nu|^k} \tag{2.18}
\]
for every $\nu \neq 0$.

Summing (2.18) over all $\nu \neq 0$ and putting $k = d + 1$, we get our desired result
\[
\sum_{\nu \neq 0} | \hat{H}_1(\nu) | \leq \frac{C \| F \|_1}{N} \tag{2.19}
\]
where $C$ depends only on $q$ and $d$. \hfill \Box

In the next lemma we will use an approach related to ([10], Lemma 1.1).
Lemma 2: Let $q : \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[\frac{1}{2}, 2]$, and let $b \in [0, 1)$. Define $H_2 : \mathbb{R} \to \mathbb{C}$

$$H_2(t) = \frac{1}{\sqrt{t+b}}h_2(\sqrt{t+b})q(\frac{\sqrt{t+b}}{N}).$$

Then for large enough $N$ we have

$$\sum_{\nu \neq 0} |\hat{H}_2(\nu)| \leq \int |F(x)| \cdot |D_N(x)|$$

(2.20)

where $D_N : \mathbb{R}^d \to \mathbb{C}$

$$|D_N(x)| \leq C \begin{cases} \left( \frac{N}{|x|} \right)^{\frac{d-2}{2}} & \text{if } |x| \geq \frac{N}{2} \\ \frac{1}{N} & \text{if } 1 \leq |x| \leq \frac{N}{2} \end{cases}$$

(2.21)

with $C$ depending only on $q$ and $d$.

Proof of Lemma 2:

We have

$$\hat{H}_2(\nu) = \int H_2(t)e^{-2\pi i \nu t} dt$$

$$= 2e^{i2\pi \nu b} \int Nq(t)h_2(tN)e^{-i2\pi \nu (Nt)^2} dt$$

$$= 2e^{i2\pi \nu b} \int Nq(t)e^{-i2\pi \nu (Nt)^2} \int_{|x| > 1} F(x) \overline{\sigma_{Nt}(x)} dx dt$$

$$= 2e^{i2\pi \nu b} \int_{|x| > 1} F(x) \int Nq(t)e^{-i2\pi \nu (Nt)^2} (Nt)^{d-1} \overline{\sigma(Ntx)} dtdx.$$  

(2.22)

We will use a well-known fact that $\overline{\sigma(x)} = Re(B(|x|))$ with $B(r) = \alpha(r)e^{2\pi r}$ and
a(r) satisfying estimates
\[ |a^k(r)| \leq \frac{C}{r^{\frac{d-1}{2}+k}} \] (2.23)

with \( C \) depending only on \( k \) and \( d \). Now we will need to estimate the inner integral in (2.22) with \( B(|x|) \) instead of \( \widehat{d\sigma(x)} \)

\[
\int N q(t) e^{-i2\pi \nu (Nt)^2} (Nt)^{d-1} a(N|x|t)e^{i2\pi N|x|t} dt
\]
\[
= \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} \int \frac{q(t)}{|x|^{\frac{d-1}{2}}} e^{i2\pi \nu (Nt)^2 t^{d-1}} a(N|x|t) (N|x|)^{\frac{d-1}{2}} e^{i2\pi N|x|t} dt
\]
\[
= \frac{N^{\frac{d+1}{2}}}{|x|^{\frac{d-1}{2}}} e^{i2\pi \nu \frac{|x|^2}{4v}} \int \frac{q(t)}{|x|^{\frac{d-1}{2}}} a(N|x|t) (N|x|)^{\frac{d-1}{2}} t^{d-1} e^{-i2\pi \nu N^2 (t - \frac{|x|^2}{2\nu})^2} dt
\]
\[
= \frac{N^{d+1}}{|x|^{\frac{d-1}{2}}} e^{i2\pi \nu \frac{|x|^2}{4v}} \int \phi(t) e^{-i2\pi \nu N^2 (t - \frac{|x|^2}{2\nu})^2} dt \] (2.24)

where \( \phi(t) = q(t) a(N|x|t) (N|x|)^{\frac{d-1}{2}} t^{d-1} \) is a Schwartz function supported in \([\frac{1}{2}, 2]\) whose derivatives and the function itself are bounded uniformly in \( t, x \) and \( N \) because of (2.23). Note that we used here the fact that \( N|x| \geq 1 \). We can say even more.

Note that in fact \( \phi(t) = \phi(t, |x|) \). Let \( |x| = c \cdot r \) where \( c \geq 2 \) and \( r \geq \frac{1}{2} \). Then all partial derivatives of \( \phi(t, c \cdot r) \) with respect to \( t \) and \( r \) are also bounded uniformly in \( t, r, c \) and \( N \). However, we will use that \( \phi(t) \) depends also on \( x \) only in formula (2.66) from the proof of Lemma 4 and therefore we will keep writing just \( \phi(t) \).

From the method of stationary phase ([8], Theorem 7.7.3) it follows that if \( k \geq 1 \) then

\[
|\int \phi(t) e^{-i2\pi \nu N^2 (t - \frac{|x|^2}{2\nu})^2} dt - \sum_{j=0}^{k-1} c_j (\nu N^2)^{-j-\frac{1}{2}} \phi^{(2j)}(\frac{|x|}{2\nu N})| \leq c_k |\nu| N^{2-k-\frac{1}{2}} \] (2.25)

where \( c_j \) are some constants.
Since \( \phi \) is supported in \([\frac{1}{2}, 2]\) we conclude from (2.25) that

\[
| \int \phi(t)e^{-i2\pi \nu N^2(t-\frac{|x|}{2N})^2} dt | \leq \begin{cases} 
C(|\nu|N^2)^{-\frac{1}{2}} & \text{if } \nu \in \left[ \frac{|x|}{4N}, \frac{|x|}{N} \right] \\
C_k(|\nu|N^2)^{-k-\frac{1}{2}} & \text{if } \nu \notin \left[ \frac{|x|}{4N}, \frac{|x|}{N} \right]
\end{cases}.
\tag{2.26}
\]

If \( \frac{|x|}{N} \leq \frac{1}{2} \), then there are no \( \nu \) in \( \left[ \frac{|x|}{4N}, \frac{|x|}{N} \right] \) and therefore if we sum (2.26) over all \( \nu \neq 0 \) we will get

\[
| \int \phi(t)e^{-i2\pi \nu N^2(t-\frac{|x|}{2N})^2} dt | \leq C_k N^{-2k-1}.
\tag{2.27}
\]

If \( \frac{|x|}{N} \geq \frac{1}{2} \) then the number of \( \nu \) in \( \left[ \frac{|x|}{4N}, \frac{|x|}{N} \right] \) is bounded by \( \frac{|x|}{N} \) and therefore if we sum (2.26) over all \( \nu \neq 0 \) we will get

\[
\sum_{\nu \neq 0} | \int \phi(t)e^{-i2\pi \nu N^2(t-\frac{|x|}{2N})^2} dt | \leq C \frac{|x|}{N} (|x|N)^{-\frac{1}{2}} + C_k N^{-2k-1} \\
\leq C_k \frac{|x|^\frac{1}{2}}{N^{\frac{3}{2}}}.
\tag{2.28}
\]

Summing (2.24) over all \( \nu \neq 0 \) and applying (2.27) or (2.28) we conclude

\[
\sum_{\nu \neq 0} \left| \int Nq(t)e^{-i2\pi \nu(Nt)^2} (Nt)^{d-1} B(N|x|t) dt \right| \leq \begin{cases} 
C_k \left( \frac{N}{|x|} \right)^{\frac{d-2}{4}} & \text{if } \frac{|x|}{N} \geq \frac{1}{2} \\
C_k \left( \frac{N}{|x|} \right)^{\frac{d+1-2k}{4}} & \text{if } \frac{|x|}{N} \leq \frac{1}{2}
\end{cases}.
\tag{2.29}
\]

Replacing in (2.22) \( \tilde{\sigma}(x) \) with \( \frac{B(|x|)+\tilde{B}(|x|)}{2} \), summing over all \( \nu \neq 0 \) and applying (2.29) with \( k \geq \frac{d+1}{4} \) we get the desired result

\[
\sum_{\nu \neq 0} |\tilde{H}_2(\nu)| \leq \int \int_{|x| \geq 1} |F(x)| \cdot |D_N(x)|
\tag{2.30}
\]
where $D_N : \mathbb{R}^d \to \mathbb{C}$

\[
|D_N(x)| \leq C \begin{cases} 
\left( \frac{N}{|x|} \right)^{\frac{d+2}{2}} & \text{if } |x| \geq \frac{N}{2} \\
\frac{1}{N} & \text{if } 1 \leq |x| \leq \frac{N}{2}
\end{cases}
\]  \hspace{1cm} (2.31)

with $C$ depending only on $q$ and $d$.  \hfill \Box

Now we are in a position to proceed with the proof of Theorem 1. From (2.1) it follows that

\[
\hat{g}_\rho(m) = \hat{f}(\rho m)
\]  \hspace{1cm} (2.32)

for every $m \in \mathbb{Z}^d$. Scaling we can assume that

\[
\hat{g}_\rho(m) = \hat{f}\left(\frac{pm}{\sqrt{2}}\right)
\]  \hspace{1cm} (2.33)

for every $m \in \mathbb{Z}^d$. It follows that

\[
\|g_\rho\|_2^2 = \sum_{m \in \mathbb{Z}^d} |\hat{g}_\rho(m)|^2 = \sum_{m \in \mathbb{Z}^d} |\hat{f}\left(\frac{pm}{\sqrt{2}}\right)|^2.
\]  \hspace{1cm} (2.34)

Let $r_d(n)$ denote the number of representations of an integer $n$ as sums of $d$ squares. It is a well-known fact from Number Theory that if $d \geq 5$ then

\[
r_d(n) \geq C n^{\frac{d+2}{2}}
\]  \hspace{1cm} (2.35)

and if $d = 4$ and $n$ is odd then

\[
r_4(n) \geq C n
\]  \hspace{1cm} (2.36)

where $C > 0$ depends only on $d$. See for example ([5], p.30, p.155, p.160).
Integrating (2.34) with respect to the Haar measure $d\rho$ and applying (2.2) we have

\[
C^2 = \int_{\xi \in \mathbb{Z}^d} \sum_{\xi, m} |\hat{f}(\xi/m)|^2 d\rho
= \int_{|\xi|=1} \sum_{m \in \mathbb{Z}^d} |\hat{f}(\xi/m)|^2 d\sigma(\xi)
\geq \sum_{n \geq 0} \sum_{|m|^2=2n+1} \int_{|\xi|=1} |\hat{f}(\xi/m)|^2 d\sigma(\xi)
\geq \sum_{n \geq 0} r_d(2n+1) \int_{|\xi|=1} |\hat{f}(\xi/n + \frac{1}{2})|^2 d\sigma(\xi)
\geq \sum_{n \geq 0} r_d(2n+1) \int_{n+\frac{1}{2}} |\hat{f}(\xi)|^2 d\sigma(\sqrt{n+\frac{1}{2}}(\xi)). \tag{2.37}
\]

Using (2.8) and (2.35) or (2.36) we conclude from (2.37) that

\[
\sum_{n \geq 0} \frac{1}{\sqrt{n + \frac{1}{2}}} h(\sqrt{n + \frac{1}{2}}) \leq CG^2. \tag{2.38}
\]

Let $q : \mathbb{R} \to \mathbb{R}$ be a fixed non-negative Schwartz function supported in $[\frac{1}{2}, 2]$ such that

\[
q(x) + q(x/2) = 1
\]

when $x \in [1, 2]$. It follows that

\[
\sum_{j \geq 0} q(\frac{x}{2^j}) = 1 \tag{2.39}
\]

when $x \geq 1$. 

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Applying the Poisson summation formula to

\[ H(t) = \frac{1}{\sqrt{t + \frac{1}{2}}} h(\sqrt{t + \frac{1}{2}}) q(\frac{\sqrt{t + \frac{1}{2}}}{N}) \]

we have

\[ \sum_n H(n) = \sum_\nu \hat{H}(\nu) = \hat{H}(0) + \sum_{\nu \neq 0} \hat{H}_1(\nu) + \sum_{\nu \neq 0} \hat{H}_2(\nu). \]  

(2.40)

Note that

\[ \hat{H}(0) = \int \frac{1}{\sqrt{t + \frac{1}{2}}} h(\sqrt{t + \frac{1}{2}}) q(\frac{\sqrt{t + \frac{1}{2}}}{N}) dt \]

\[ = 2 \int h(t) q(\frac{t}{N}) dt. \]  

(2.41)

Substituting (2.41) into (2.40) we get that

\[ 2 \int h(t) q(\frac{t}{N}) dt \leq \sum_n H(n) + \sum_{\nu \neq 0} |\hat{H}_1(\nu)| + \sum_{\nu \neq 0} |\hat{H}_2(\nu)| \leq \sum_{n \geq 0} \frac{1}{\sqrt{n + \frac{1}{2}}} h(\sqrt{n + \frac{1}{2}}) q(\frac{\sqrt{n + \frac{1}{2}}}{N}) + \frac{C\|F\|_1}{N} + \int_{|x| \geq 1} |F(x) D_N(x)| dx \]

(2.42)

where the last inequality follows from Lemma 1 and Lemma 2.
From the definition of $D_N(x)$ (2.21) it follows that

$$\sum_{j \geq 0} |D_{2^j}(x)| = \sum_{2^j \leq |x|} |D_{2^j}(x)| + \sum_{2^j > |x|} |D_{2^j}(x)|$$

$$\leq \sum_{2^j \leq |x|} C(\frac{2^j}{|x|})^{\frac{d-2}{2}} + \sum_{2^j > |x|} \frac{C}{2^j} \leq C$$

(2.43)

for every $|x| \geq 1$.

Putting $N = 2^j$ in (2.42), summing over all $j \geq 0$ and applying (2.39) we get by Lebesgue Monotone Convergence Theorem

$$2 \int_{1}^{\infty} h(t) dt \leq \sum_{n \geq 0} \frac{1}{\sqrt{n + \frac{1}{2}}} h(\sqrt{n + \frac{1}{2}}) + C\|F\|_1 + C \int_{|x| \geq 1} |F(x)| dx$$

$$\leq C(G^2 + \|F\|_1)$$

(2.44)

where the last inequality follows from (2.38). From the definition of $h(t)$ (2.8) it follows that

$$h(t) \leq C\|f\|_1^2$$

(2.45)

for $t \leq 1$. Therefore, we have

$$\int |\hat{f}(x)|^2 dx = \int_{0}^{\infty} |\hat{f}(\xi)|^2 d\sigma(\xi) dt$$

$$= \int_{0}^{\infty} h(t) dt$$

$$\leq C(G^2 + \|f\|_1^2)$$

(2.46)

where the last inequality is obtained from (2.44), (2.45) and (2.7). From (2.46) it follows that $f \in L^2$ and

$$\|f\|_2 \leq C(G + \|f\|_1)$$
with $C$ depending only on $d$. □

If $d \geq 5$ then

$$r_d(n) \leq C n^{\frac{d-3}{2}}$$

(2.47)

where $C > 0$ depends only on $d$. See for example ([5], p.155, p.160). An argument similar to the one used to get (2.37) but without scaling shows that

$$G^2 = \int \sum_{\rho \in \text{SO}(d)} |\hat{f}(\rho m)|^2 d\rho$$

$$= |\hat{f}(0)|^2 + \sum_{n \geq 1} r_d(n) \int |\hat{f}(\sqrt{n}\xi)|^2 d\sigma(\xi)$$

$$= |\hat{f}(0)|^2 + \sum_{n \geq 1} \frac{r_d(n)}{n^{\frac{d-1}{2}}} \int |\hat{f}(\xi)|^2 d\sigma(\sqrt{n}\xi).$$

(2.48)

Using (2.8) and (2.47) we conclude from (2.48) that

$$G^2 \leq \|f\|_2^2 + C \sum_{n \geq 1} \frac{1}{\sqrt{n}} h(\sqrt{n}).$$

(2.49)

Repeating arguments which we used to obtain (2.44) we get

$$\sum_{n \geq 1} \frac{1}{\sqrt{n}} h(\sqrt{n}) \leq 2 \int_0^\infty h(t) dt + C \|F\|_1$$

$$\leq C(\|\hat{f}\|_2^2 + \|f\|_1^2).$$

(2.50)

Hence we can formulate an inverse theorem to Theorem 1:

**Theorem 1':** let $d \geq 5$ and let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and let $g_\rho$ be periodizations of $f$

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))$$

(2.51)
then \( g_\rho \in L^2([0,1]^d) \) for almost all rotations \( \rho \in SO(d) \) and

\[
\int_{\rho \in SO(d)} \|g_\rho\|_2^2 \, d\rho \leq C(\|f\|_2 + \|f\|_1)^2
\]

where \( C \) depends only on \( d \).

Corollary: interpolating between the trivial \( p = 1 \) and \( p = 2 \), we obtain the following generalization of the previous theorem for \( 1 \leq p \leq 2 \): let \( d \geq 5 \) and let \( f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \) and let \( g_\rho \) be periodizations of \( f \)

\[
g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))
\]

then \( g_\rho \in L^p([0,1]^d) \) for almost all rotations \( \rho \in SO(d) \) and

\[
\int_{\rho \in SO(d)} \|g_\rho\|_p^p \, d\rho \leq C(\|f\|_p + \|f\|_1)'^p
\]

where \( C \) depends only on \( d \).

2.3 Case \( 1 \leq p < \frac{2d}{d+2} \)

We will generalize Theorems 1 and 1' in the spirit of the Stein-Tomas Theorem ([4], Chapter 6.5).

Theorem 2: let \( d \geq 4 \) and let \( f \in L^p(\mathbb{R}^d) \) where \( 1 \leq p < \frac{2d}{d+2} \). If periodizations of \( f \)

\[
g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))
\]
are in $L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$G^2 = \int_{\rho \in SO(d)} \|g_\rho\|^2 \, d\rho < \infty$$

then $f \in L^2(\mathbb{R}^d)$:

$$\|f\|_2 \leq C(G + \|f\|_p) \quad (2.52)$$

where $C$ depends only on $d$ and $p$.

It will follow from the proof (see (2.37)) that we can replace $\int_{\rho \in SO(d)} \|g_\rho\|^2 \, d\rho$ with $\int_{\rho \in SO(d)} \|g_\rho - \tilde{g}(0)\|^2 \, d\rho$ in Theorem 2. We will also obtain an inverse theorem.

**Theorem 2':** let $d \geq 5$ and let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $1 \leq p < \frac{2d}{d+2}$ and let $g_\rho$ be periodizations of $f$

$$g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))$$

then $g_\rho \in L^2([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$\int_{\rho \in SO(d)} \|g_\rho - \tilde{g}_\rho(0)\|^2 \, d\rho \leq C(\|f\|_2 + \|f\|_p)^2 \quad (2.53)$$

where $C$ depends only on $d$ and $p$.

Since Schwartz functions are dense in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ it follows from Theorem 2' that we can define periodizations $g_\rho$ of $f \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$ for a.e. $\rho \in SO(d)$ as elements of the quotient space of $L^2([0,1]^d)$ modulo constants.

**Remarks:** 1. As the following example shows, we cannot replace $\int_{\rho \in SO(d)} \|g_\rho - \tilde{g}_\rho(0)\|^2 \, d\rho$ with $\|f\|^2$.
\[ g(0) \| g_0 \|_{2d}^2 \rho \text{ with } \int_{\rho \in \text{SO}(d)} \| g_\rho \|_{2d}^2 \rho \text{ in Theorem 2' when } p > 1. \]

Let \( \phi : \mathbb{R}^d \rightarrow \mathbb{C} \) be a Schwartz function supported in \( B(0, 1) \) such that \( \phi(0) = 1 \). Put \( \hat{f}(x) = \phi(i) \). Then
\[
g_\rho \equiv \hat{f}(0) = 1
\]

but
\[
\|f\|_p = \epsilon^{\frac{d}{2}} \|\phi\|_p.
\]

2. The next example from ([4], Chapter 6.3) shows that \( p \) can not be greater than \( \frac{2d+2}{d+3} \) in Theorem 2'. Put
\[
\hat{f}(x_1, \ldots, x_d) = \phi\left(\frac{x_1 - 1}{\epsilon^2}, \frac{x_2}{\epsilon}, \ldots, \frac{x_d}{\epsilon}\right) \quad (2.54)
\]

where \( \phi : \mathbb{R}^d \rightarrow \mathbb{C} \) is a Schwartz function supported in \( B(0, 2) \) such that \( \phi = 1 \) in \( B(0, 1) \). Then
\[
\int_{\rho \in \text{SO}(d)} \| g_\rho - \hat{g}(0) \|_{2d}^2 \rho = 2d \int_{|\xi| = 1} |\hat{f}(\xi)|^2 d\sigma(\xi)
\]
\[
\geq C\epsilon^{d-1}
\]

but
\[
\|f\|^2_p = \epsilon^{\frac{2d+2}{d+3}} \|\phi\|_p.
\]

It is an open question whether Theorems 2 and 2' are valid when \( \frac{2d}{d+2} \leq p < \frac{2d+2}{d+3} \). We discuss this further in Remark 2 at the end of the section.

**Proof of Theorem 2:**

The proof goes quite similarly to the one of Theorem 1. We will replace Lemma 1 with

**Lemma 3:** Let \( q : \mathbb{R} \rightarrow \mathbb{R} \) be a Schwartz function supported in \( [\frac{1}{2}, 2] \), let \( f \in \)
$L^p(\mathbb{R}^d)$ where $1 \leq p \leq 2$ and let $b \in [0, 1)$. Define $H_1 : \mathbb{R} \rightarrow \mathbb{C}$

$$H_1(t) = \frac{1}{\sqrt{t+b}} h_1(\sqrt{t+b}) q(\frac{\sqrt{t+b}}{N}).$$

Then for large enough $N$ we have

$$\sum_{\nu \neq 0} |\hat{H}_1(\nu)| \leq \frac{C \|f\|_p^2}{N}$$

where $C$ depends only on $q$ and $d$.

**Proof of Lemma 3:**

The only difference in the proof is to obtain an inequality analogous to (2.13). Using Young's inequality we have $\|f \ast \hat{f}\|_q \leq \|f\|_p^2$ where $1 + \frac{1}{q} = \frac{2}{p}$. Therefore,

$$|\int (f \ast \hat{f})(x)w(x)dx| \leq \|f\|_p^2 \|w\|_q.$$  Substituting derivatives of $d\sigma_t(x)\chi_{\{|x| \leq 1\}}$ with respect to $t$ instead of $w$, we get the desired inequality

$$|h_1^{(k)}(t)| \leq C t^{d-1} \|f\|_p^2$$

where $t \geq 1$ and $C$ depends only on $k$ and $d$.  

The main difficulty is to prove a lemma analogous to Lemma 2:

**Lemma 4:** Let $q : \mathbb{R} \rightarrow \mathbb{R}$ be a Schwartz function supported in $[\frac{1}{2}, 2]$, let $f \in L^p(\mathbb{R}^d)$ where $1 \leq p < \frac{2d}{d+2}$ and let $b \in [0, 1)$. Define $H_{2,N} : \mathbb{R} \rightarrow \mathbb{C}$

$$H_{2,N}(t) = \frac{1}{\sqrt{t+b}} h_2(\sqrt{t+b}) q(\frac{\sqrt{t+b}}{N}).$$

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Then we have

$$\sum_{\nu \neq 0} | \sum_{j \geq 0} \hat{H}_{2,2j}(\nu)| \leq C \| f \|_p^2$$

(2.57)

with $C$ depending only on $p$, $q$ and $d$.

**Proof of Lemma 4:**

Recall from (2.22) that

$$\hat{H}_{2,N}(\nu) = 2 \int (f * \tilde{f})(x) D_{N,\nu}(x) dx$$

where

$$D_{N,\nu}(x) = \chi_{\{|x|>1\}} e^{i2\pi \nu b} \int Nq(t) e^{-i2\pi(Nt)^2} (Nt)^{d-1} \tilde{\sigma}(Ntx) dt.$$  

(2.58)

Denote by

$$K_{\nu}(x) = \sum_{l \geq 0} D_{2l,\nu}(x).$$  

(2.59)

Then

$$\left| \sum_{l \geq 0} \hat{H}_{2,l}(\nu) \right| = 2 \left| \int (f * \tilde{f})(x) \sum_{l \geq 0} D_{2l,\nu}(x) dx \right|$$

$$= 2 \left| \int \tilde{f}(x) (K_{\nu} * f)(x) dx \right|$$

$$\leq 2 \| f \|_p \| K_{\nu} * f \|_{\mathcal{F}}.$$  

(2.60)
If $p' = \infty$ or $p' = 2$ we have

$$\|K_\nu \ast f\|_\infty \leq \|K_\nu\|_\infty \|f\|_1$$
$$\|K_\nu \ast f\|_2 \leq \|\hat{K}_\nu\|_\infty \|f\|_2.$$

First we will show that

$$\|K_\nu\|_\infty \leq \|\sum_{l \geq 0} |D_{2^l, \nu}|(x)\|_\infty \leq C|\nu|^{-\frac{d}{2}}.$$ (2.61)

It follows from (2.26) that

$$|D_{N, \nu}(x)| \leq \frac{N^{\frac{d+1}{2}}}{|x|^{d+1}} \begin{cases} C(|\nu|N^2)^{-\frac{1}{2}} \quad \text{if } N \in \left[\frac{|\nu|}{4}, \frac{|\nu|}{\nu}\right] \\ C_k(|\nu|N^2)^{-k-\frac{1}{2}} \quad \text{if } N \notin \left[\frac{|\nu|}{4}, \frac{|\nu|}{\nu}\right]. \end{cases}$$ (2.62)

If $\nu > 0$ then the number of diadic $N \in \left[\frac{|\nu|}{4}, \frac{|\nu|}{\nu}\right]$ is at most 3. If $\nu < 0$ then there are no $N$ in $\left[\frac{|\nu|}{4}, \frac{|\nu|}{\nu}\right]$. Therefore, choosing $k \geq \frac{d-1}{2}$ and summing (2.62) over all diadic $N$, we have

$$\sum_{l \geq 0} |D_{2^l, \nu}(x)| \leq C|\nu|^{-\frac{d}{2}}$$

with $C$ depending only on $d$ and $q$.

Now we will show that

$$\|\hat{K}_\nu\|_\infty \leq \|\sum_{l \geq 0} |\hat{D}_{2^l, \nu}|(y)\|_\infty \leq C.$$ (2.63)

Since supp $\phi \in \left[\frac{1}{2}, 2\right]$ we can re-write (2.25) for a stronger version of the method of
stationary phase ([8], Theorems 7.6.4, 7.6.5, 7.7.3)

\[ \left| \int \phi(t)e^{-i2\pi \nu N^2(t-\frac{f}{2\nu N})^2} dt - \sum_{j=0}^{k-1} c_j(\nu N^2)^{-j-\frac{1}{2}} \phi^{(2j)}(\frac{|x|}{2\nu N}) \right| \leq \frac{c_k(|\nu|N^2)^{-k-\frac{1}{2}}}{\max(1, \frac{|x|}{8N|\nu|})} \]

where \( c_j \) are some constants. Therefore,

\[ D_{N,\nu}(x) = \chi_{\{|x|>1\}} \frac{N^{d+1}}{|x|^{d+1}} e^{i2\pi \frac{|x|^2}{4\nu}} \sum_{j=0}^{k-1} c_j(\nu N^2)^{-j-\frac{1}{2}} \phi^{(2j)}(\frac{|x|}{2\nu N}) + \phi_k(x) \]

(2.64)

where \( |\phi_k(x)| \leq \chi_{\{|x|>1\}} \frac{N^{d+1}}{|x|^{d+1}} \frac{c_k(|\nu|N^2)^{-k-\frac{1}{2}}}{\max(1, \frac{|x|}{8N|\nu|})} \). Choosing \( k \geq \frac{d+2}{2} \) we have

\[ \|\phi_k\|_\infty \leq \|\phi_k\|_1 \]

\[ = \int_{|x| \leq 2N|\nu|} |\phi_k| dx + \int_{|x| > 2N|\nu|} |\phi_k| dx \]

\[ \leq \frac{C}{N} \]

(2.65)

where \( C \) depends only on \( d \) and \( q \). We can assume that \( \nu > 0 \) since \( D_{N,\nu}(x) = \phi_k(x) \) for \( \nu < 0 \). We can also ignore \( \chi_{\{|x|>1\}} \) in front of the sum in (2.64) because if \( \frac{|x|}{2\nu N} \in [\frac{1}{2}, 2] \), then \( |x| > \nu N \geq 1 \). We will consider only the zero term in the sum. The other terms can be treated similarly. The Fourier transform of

\[ \frac{N^{d+1}}{|x|^{d+1}} e^{i2\pi \frac{|x|^2}{4\nu}} (\nu N^2)^{-\frac{1}{2}} \phi(\frac{|x|}{2\nu N}) \]

at point \( y \) is equal to

\[ N^{d+1}(2\nu N)^{d+1}(\nu N^2)^{-\frac{1}{2}} \int_{\mathbb{R}^d} \psi(|x|) e^{i2\pi \nu N^2|x|^2} e^{-i2\pi 2\nu N x \cdot y} dx = \]

\[ C(\nu N^2)^{\frac{d}{2}} e^{-i2\pi \nu |y|^2} \int_{\mathbb{R}^d} \psi(|x|) e^{i2\pi \nu N^2|x|^2-x^2} dx \]

(2.66)

where \( \psi(t) = \phi(t, 2\nu N t) t^{-\frac{d-1}{2}} \) is a Schwartz function supported in \([\frac{1}{2}, 2]\) whose derivatives and the function itself are bounded uniformly in \( t, \nu \) and \( N \) (see remark after
The same is true about partial derivatives of $\psi(|x|)$. Applying the stationary phase method for $\mathbb{R}^d$ ([8], Theorem 7.7.3) we get

$$\left| \int_{\mathbb{R}^d} \psi(|x|) e^{i2\nu N^2|x| - \frac{\nu}{N} |x|^2} \, dx \right| \leq \begin{cases} C(\nu N^2)^{-\frac{3}{4}} & \text{if } N \in \left[ \frac{1}{2}, 2|y| \right] \\ C_k(\nu N^2)^{-k \frac{3}{4}} & \text{if } N \notin \left[ \frac{1}{2}, 2|y| \right] \end{cases}. \quad (2.67)$$

Therefore, the absolute value of (2.66) can be bounded from above by:

$$\leq \begin{cases} C & \text{if } N \in \left[ \frac{1}{2}, 2|y| \right] \\ C_k(\nu N^2)^{-k} & \text{if } N \notin \left[ \frac{1}{2}, 2|y| \right] \end{cases}. \quad (2.68)$$

Similar inequalities hold for Fourier transforms for the rest of the terms in the sum in (2.64). The number of diadic $N \in \left[ \frac{1}{2}, 2|y| \right]$ is bounded by 3. Using (2.65), choosing $k \geq 1$ in (2.68) and summing over all diadic $N$, we get

$$\sum_{N \geq 0} |\hat{a}_{2^j,\nu}(y)| \leq C \quad (2.69)$$

with $C$ depending only on $d$ and $q$. Using (2.61) and (2.63) and interpolating between $p = 1$ and $p = 2$, we obtain

$$\|K_\nu * f\|_{p'} \leq C|\nu|^{-\alpha_p}\|f\|_p$$

where $\alpha_p = \frac{d+2-p}{2}$. $\alpha_p > 1$ if $p < \frac{2d}{d+2}$. Summing (2.60) over all $\nu \neq 0$, we get the desired inequality

$$\sum_{\nu \neq 0} \sum_{j \geq 0} |\hat{a}_{2^j,\nu}(y)| \leq C\|f\|_p^2.$$

Now we are in a position to proceed with the proof of Theorem 2. The proof is almost the same as the one of Theorem 1. We need also to replace inequality (2.45)
with the following one:

\[
\int_0^1 h(t) dt = \int_{|y| \leq 1} |\hat{f}(y)|^2 dy \\
\leq C\|\hat{f}\|_p^2 \\
\leq C\|f\|_p^2
\]

where \( p \leq 2 \) and \( C \) depends only on \( d \). An argument similar to the one used to get (2.42), (2.44) and (2.46) (note that the interchange of summation by \( \nu \) and \( N \) is not a problem) yields the desired inequality

\[
\int |\hat{f}(x)|^2 dx = \int_0^\infty |\hat{f}(\xi)|^2 d\sigma_1(\xi) dt \\
= \int_0^\infty h(t) dt \\
\leq C(C^2 + \|f\|_p^2)
\]

with \( C \) depending on \( d \) and \( p \). □

The proof of Theorem 2' is the same (see the argument after the proof of Theorem 1). The important thing is that we exclude \( \hat{g}_p(0) = \hat{f}(0) \) in (2.48) now.

**Final remarks:** 1. We can further generalize Theorem 2'. Fix some \( q \in [1, \frac{2d}{d+2}] \). Interpolating between the trivial \( p = 1 \) and \( p = 2 \), we obtain the following generalization of the Theorem 2' for \( 1 \leq p \leq 2 \): let \( d \geq 5 \) and let \( f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \) and let \( g_p \) be periodizations of \( f \)

\[
g_p(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu))
\]
then $g_{\rho} \in L^p([0,1]^d)$ for almost all rotations $\rho \in SO(d)$ and

$$\int_{\rho \in SO(d)} \|g_{\rho} - \hat{g}_{\rho}(0)\|_p^p \, d\rho \leq C(\|f\|_p + \|f\|_q)^p$$

where $C$ depends only on $d$ and $q$.

We can restate this result in the following way: if $1 \leq p < \frac{2d}{d+2}$ then

$$\int_{\rho \in SO(d)} \|g_{\rho} - \hat{g}_{\rho}(0)\|_p^p \, d\rho \leq C\|f\|_p^p,$$

if $1 \leq q < \frac{2d}{d+2}$ and $\frac{2d}{d+2} \leq p \leq 2$ then

$$\int_{\rho \in SO(d)} \|g_{\rho} - \hat{g}_{\rho}(0)\|_p^p \, d\rho \leq C(\|f\|_p + \|f\|_q)^p.$$

2. Conditionally on the exponent pair conjecture ([12], Chapter 4, Conjecture 2) we can clarify what happens when $\frac{2d}{d+2} \leq p < \frac{2d+2}{d+3}$. In our case the conjecture says that

$$| \sum_{n \leq \nu \leq m} e^{i\lambda x_n^2} \nu | \leq C_4 |x| \nu n^\frac{1}{2}$$

(2.71)

where $m \leq 2n$ and $|x| \geq n$. Let $\beta(x) = \max(1, |x|)$. 

**Proposition.** Theorems 2 and 2' hold if we replace $\|f\|_p$ with $\|\beta f\|_p$ and if $p < \frac{2d+2}{d+3}$, provided the conjecture is valid.

Using the example (2.54) we can show that the Proposition is sharp up to $\epsilon$ in the range of $p$ for the estimate (2.53).

**Proof of Proposition:**
The main issue is to improve the result of Lemma 4. Denote by

\[ L_j(x) = \sum_{2^j \leq \nu \leq 2^{j+1}} K_{\nu}(x). \]  

(2.72)

Using summation by parts we obtain from (2.64), (2.59) and (2.71) that

\[ |L_j(x)| = \left| \sum_{2^j \leq \nu \leq 2^{j+1}} K_{\nu}(x) \right| \leq C \epsilon |x|^{\epsilon - \frac{d-1}{2} j}. \]

We will deal with the following expression instead of (2.60)

\[ \left| \int f(x)(L_j * f)(x) \, dx \right| = \left| \int f(x) \beta^* \frac{(L_j * f)(x)}{\beta^*} \, dx \right| \leq \|\beta^* f(x)\|_p \left\| \frac{(L_j * f)}{\beta^*} \right\|_{p'}. \]

If \( p' = \infty \) or \( p' = 2 \) we have

\[ \left\| \frac{(L_j * f)}{\beta^*} \right\|_{\infty} \leq \left\| \int_{|y| \leq |z|} |L_j(x - y)| \cdot |f(y)| \, dy + \int_{|y| \geq |z|} |L_j(x - y)| \cdot |f(y)| \, dy \right\|_{\infty} \]

\[ \leq \left\| L_j \right\|_{\infty} \|\beta^* f\|_1 \leq C \epsilon 2^{-\frac{d-1}{2} j} \|\beta^* f\|_1, \]

\[ \left\| \frac{(L_j * f)}{\beta^*} \right\|_2 \leq \|L_j * f\|_2 \leq \|\tilde{L}_j\|_{\infty} \|f\|_2 \leq C \epsilon 2^{j} \|f\|_2. \]

Interpolating between \( p = 1 \) and \( p = 2 \), we obtain

\[ \left\| \frac{(L_j * f)}{\beta^*} \right\|_{p'} \leq C \epsilon 2^{-j\alpha_p} \|\beta^* f\|_p \]

where \( \alpha_p = \frac{d+1}{p} - \frac{d+3}{2} \). \( \alpha_p > 0 \) if \( p < \frac{2d+2}{d+3} \). \( \square \)
3. Concerning the lower dimensional cases we can use the following results from the Number Theory:

\[ r_3(n) \leq C n^{3 \frac{1}{2}} \ln n \ln \ln n, \]

\[ r_4(n) \leq C n \ln \ln n. \]

See for example ([2]). There is an infinite arithmetic progression, e.g., \( n = 8k + 1 \), such that

\[ r_3(n) \geq C_e n^{\frac{1}{2}-\varepsilon}. \]

See for example ([6]). Then Theorem 1 holds when \( d = 3 \) and Theorem 1' holds when \( d = 3 \) or \( d = 4 \) if we replace

\[ \|g_p\|_2^2 = \sum_{m \in \mathbb{Z}^d} |\hat{g}_p(m)|_2^2 \]

with

\[ \sum_{m \in \mathbb{Z}^d} |m| |\hat{g}_p(m)|_2^2, \]

\[ \sum_{m \in \mathbb{Z}^d, |m| > 1} \frac{|\hat{g}_p(m)|_2^2}{\ln |m| \ln \ln |m|} \]

or

\[ \sum_{m \in \mathbb{Z}^d, |m| > 3} \frac{|\hat{g}_p(m)|_2^2}{\ln \ln |m|} \]

correspondingly.

2.4 Case \( d = 2 \) and \( p = \infty \)

Some results on the Steinhaus tiling problem are related to Theorem 1 since periodizations naturally appear in the problem of Steinhaus. In particular, showing that there are no measurable Steinhaus sets in dimensions greater than two, M. Kolountzakis and T. Wolff proved the following theorem on periodizations in higher dimensions.
([10], Theorem 1) which can be viewed as some version of Theorem 1 for \( p = \infty \):

**Kolountzakis and Wolff’s Theorem** If \( f \in L^1(\mathbb{R}^d) \) and its periodizations \( g_\rho \) are constants for almost all rotations \( \rho \in SO(d) \), then \( f \) is continuous and, in fact,

\[
\|f\|_\infty \leq C_d \|f\|_1
\]

provided that the dimension \( d \) is at least 3.

Obviously, this statement is false when \( d = 1 \). We will show that it doesn’t hold either if \( d = 2 \). The fact that \( g_\rho \) are constant means that \( \hat{f}(\rho(k,l)) = \hat{g}_\rho(k,l) = 0 \) for all \( (k,l) \in \mathbb{Z}^2 \setminus (0,0) \) and almost all \( \rho \in SO(d) \) which means that \( \hat{f} \) vanishes on all circles of radii \( \sqrt{l^2 + k^2} > 0 \). Denote by \( X \) the Banach space of functions from \( L^1(\mathbb{R}^2) \) whose Fourier transforms vanish on all circles of radii \( \sqrt{l^2 + k^2} > 0 \)

\[
X = \{ f \in L^1(\mathbb{R}^2) : \hat{f}(r) = 0 \text{ if } |r| = \sqrt{l^2 + k^2}, (k,l) \in \mathbb{Z}^2 \setminus (0,0) \}.
\]

We will use the notation \( x \preceq y \) meaning \( x \leq Cy \), and \( x \asymp y \) meaning that \( x \preceq y \) and \( y \preceq x \) for some constant \( C > 0 \) independent from \( x \) and \( y \).

The next lemma crucially depends on the following fact from the Number Theory ([5], p.22):

The number of integers in \([n, 2n]\) which can be represented as sums of two squares is \( n \epsilon_n \) where \( \epsilon_n \leq \frac{1}{\ln \sqrt{n}} \to 0 \) as \( n \to \infty \).

**Lemma 5:** There exists a sequence of Schwartz functions \( f_n \in X \) such that

\[
\lim_{n \to \infty} \frac{\|f_n\|_1}{|f_n(0)|} = 0.
\]

**Proof of Lemma 5:** Let \( a_1 < a_2 < a_3 < ... \) be the enumeration of numbers
\[ a_m = \sqrt{m^2 + k^2} \] in ascending order. Denote \( \delta_m = a_{m+1} - a_m \). As we already said the number of \( a_m \) in \([\sqrt{n}, 2\sqrt{n}]\) is \( n\varepsilon_n \). Let \( a_{m_0} \) and \( a_{m_1} \) be correspondingly the smallest and the largest such \( a_m \). Then

\[
\sum_{m=m_0}^{m_{1}-1} \delta_m = a_{m_1} - a_{m} \sim \sqrt{n}.
\]

Let \( \delta = \frac{C}{\sqrt{n}\varepsilon_n} \) with small enough constant \( C > 0 \) so that if

\[ M = \{m, m_0 \leq m < m_1 : \delta_m \geq \delta\} \]

then

\[
\sqrt{n} \lesssim \sum_{m \in M} \delta_m
\]

since \( m_1 - m_0 \sim n\varepsilon_n \). Choose coordinate axes \( x \) and \( y \). We will construct \( \hat{\Phi}_n \) supported in \( \bigcup R_m \) where \( R_m \) is a largest possible rectangle inscribed between circles of radius \( a_m \) and \( a_{m+1} \) with sides parallel to the coordinate axes. Then \( R_m \) is of size \( \sim \delta_m \times \sqrt{\delta_m a_m} \gtrsim \delta_m \times \sqrt{\delta \sqrt{n}} \). We will split each rectangle \( R_m \) further into smaller \( \left[ \frac{\delta_m}{\delta} \right] \) rectangles \( r \) of the same size \( \sim \delta \times \sqrt{\delta \sqrt{n}} \). The number of these rectangles \( r \) is

\[
N = \sum_{m \in M} \left[ \frac{\delta_m}{\delta} \right] \sim \sum_{m \in M} \frac{\delta_m}{\delta} \sim \frac{\sqrt{n}}{1/n\varepsilon_n} = n\varepsilon_n
\]

since \( \delta_m \geq \delta \) for \( m \in M \). Enumerate these rectangles \( r_k, k = 1, \ldots, N \). Let \( r_k \) be centered at \((\lambda_k, 0)\) It is clear that \( |\lambda_k - \lambda_l| \geq \delta \) for \( k \neq l \). Let \( \phi \) be a nonnegative Schwartz function on \( \mathbb{R} \) supported in \([-\frac{1}{2}, \frac{1}{2}]\). Define \( \hat{\Phi}_n \) as the following sum:

\[
\hat{\Phi}_n(x, y) = \sum_{k=1}^{N} \phi\left(\frac{x - \lambda_k}{\delta}\right)\phi\left(\frac{y}{\sqrt{\delta \sqrt{n}}}ight).
\] (2.73)
The $k$-th term in (2.73) is supported in $r_k$. Therefore, $\hat{f}_n$ is a Schwartz function supported in $\bigcup_{m \in M} R_m$. Hence $\hat{f}_n$ vanishes on all circles of radii $a_i$. Taking the inverse Fourier transform of (2.73), we get

$$f_n(\xi, \eta) = \delta \hat{\phi}(\xi \delta) \sqrt{\delta \sqrt{n} \hat{\phi}(\eta \sqrt{\delta \sqrt{n}}} \sum_{k=1}^{N} e^{i \lambda_k \xi}.$$  \hspace{1cm} (2.74)

Then

$$f_n(0) = \delta \hat{\phi}(0) \sqrt{\delta \sqrt{n} \hat{\phi}(0) n} \sum_{k=1}^{N} e^{i \lambda_k} \xi.$$  \hspace{1cm} (2.75)

We used here that $\hat{\phi}(0) = \int \phi > 0$ is some nonzero constant. Denote

$$g(x) = \sum_{k=1}^{N} e^{i \lambda_k \xi}.$$  \hspace{1cm} (2.76)

Since $|\lambda_k - \lambda_l| \geq \frac{\delta}{2} = 1$ for $k \neq l$ we have

$$\int_I |g|^2 \sim N$$

for any interval $I$ of length $4\pi$ (see ([14], Theorem 9.1)). Therefore,

$$\int_I |g| \leq \sqrt{|I|} \sqrt{\int_I |g|^2} \leq \sqrt{N}$$  \hspace{1cm} (2.76)

for any interval $I$ of length $4\pi$. Since $\hat{\phi}$ is a Schwartz function, we have that

$$|\hat{\phi}(x)| \lesssim \frac{1}{1 + x^2}.$$
The $L^1$ norm of (2.74) is

$$\int |f_n(\xi, \eta)| d\xi d\eta = \|\hat{f}_n\|_1 \int |\hat{f}(\xi)| \cdot | \sum_{k=1}^{N} e^{ik\xi} | d\xi$$

$$= C \sum_{l=\infty}^{1} \int_{4\pi}^{(l+1)4\pi} |\hat{f}(\xi)| \cdot |g(\xi)| d\xi$$

$$\leq \sum_{l=\infty}^{\infty} \frac{1}{1 + l^2} \sqrt{N}$$

$$\leq \sqrt{\frac{\epsilon_n}{N}}. \quad (2.77)$$

Dividing (2.77) by (2.75) we obtain the desired result

$$\frac{\|f_n\|_1}{|f_n(0)|} \leq \frac{\sqrt{\frac{\epsilon_n}{N}}}{\sqrt{\frac{\epsilon_n}{\epsilon_n}}}$$

$$= \epsilon_n \to 0$$

as $n \to \infty$.

$\square$

**Corollary:** It follows immediately from the lemma that

$$\sup_{f \in X} \frac{\|f\|_{L^\infty(D(0,1))}}{\|f\|_1} = \infty.$$ 

We claim that there exists a function $f \in X$ such that $\|f\|_{L^\infty(D(0,1))} = \infty$. Suppose towards a contradiction that this is not true. Then the restriction operator

$$T : f \to f|_{D(0,1)}$$

maps $X$ to $L^\infty(D(0,1))$. Note that if $f_n \to f$ in $L^1$ and $f_n \to g$ in $L^\infty(D(0,1))$, then $f = g$ a.e. on $D(0,1)$. An application of the Closed Graph theorem shows that $T$ is a bounded operator acting from $X$ to $L^\infty(D(0,1))$. This contradicts to the Corollary. Thus we proved our claim. Obviously, this function $f$ is not continuous. Therefore, it can serve as a counterexample to the Theorem when $d = 2$. 

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Bibliography


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