Geometric Invariants in Contact Structures on 3-manifolds

Thesis by

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Abstract

> Manifolds that have serious reasons to be odd-dimensional usually carry natural contact structures. – V. I. Arnold

Contact structures are important geometric structures on smooth, odd-dimensional manifolds. This thesis studies contact structures on 3-manifolds, where the theory is enriched by the study of knots and connections to the geometry of 4-manifolds.

In this thesis, we construct a geometric invariant of (framed) knots transverse to contact structures, *size*, and investigate its connections with Dehn surgeries, symplectic cobordisms, and geometric intersection invariants under contactomorphisms.

In chapter 2, we characterize tight and overtwisted contact structures in terms of the sizes of their transverse knots. We define surgeries of contact 3-manifolds along transverse curves, and give some upperbounds for sizes of knots after such surgeries. We show that certain *reducing* surgeries are precisely the results of certain symplectic cobordisms. Any transverse knot in a coorientable contact structure has such a family of such surgeries. This suggests that certain contact surgeries preserve tightness, and in section 3.2 we present some partial results in this direction. We provide some methods for obtaining lower bounds for the sizes of knots.

In chapter 3, we study the intersections of knots with surfaces under contactomorphisms. We study the local action of contactomorphisms on arcs near surfaces in contact 3-manifolds and the intersections of overtwisted discs with knots and overtwisted unknots with surfaces.

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Chapter 1 Historical Sketch

This chapter reviews basic definitions and the history of contact structures on 3-manifolds. It includes some technical lemmas used in subsequent chapters.

1.1 Basic Definitions

Definition. A *contact structure* on a 3-manifold is a nowhere integrable tangent plane field. Locally, this can be described as the kernel of a 1-form α so that $\alpha \wedge d\alpha \neq 0$.

The obstruction to extending this 1-form globally measures whether a local coorientation is reversed along paths. This obstruction is an element of $H^1(M; \mathbf{Z}_{/2\mathbf{Z}})$ which vanishes if the contact structure is coorientable, and any non-coorientable contact structure has a coorientable double cover. Henceforth, except where explicitly noted, all contact structures are assumed to be coorientable, and thus definable globally as the kernel of a 1-form. The form is unique up to multiplication by a nowhere-0 function.

That the 3-form $\alpha \wedge d\alpha$ is never 0 means that it provides an orientation on the 3-manifold. This 3-form depends on the choice of α, but the orientation does not. The correct category seems to be that of *oriented* 3-manifolds, rather than just orientable 3-manifolds; see Theorem 1.6.14, for example. Henceforth, contact structures on oriented 3-manifolds are assumed to induce the orientation.

A smooth 2-dimensional foliation of a smooth 3-manifold is a tangent plane field which is integrable everywhere. The theory of foliations on 3-manifolds is rich and well-developed. Although the definitions of foliations and contact structures seem to be opposites, there are deep connections and analogies between them.

Example 1. The *standard contact structure on* $\mathbf{R}^3$ is the kernel of $ydx - dz$.

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$(ydx - dz) \wedge d(ydx - dz) = (ydx - dz) \wedge dy \wedge dx = dx \wedge dy \wedge dz.$

Example 2. The *standard contact structure on* S^3 is the field of complex lines tangent to the unit $S^3 \hookrightarrow \mathbf{C}^2$.

Example 3. Identify $SO(3) = \mathbf{RP}^3$ with the orientations of a round sphere on a plane. The tangent space at each orientation has a tangent plane of infinitesimal rolling motions. This tangent plane field is a contact structure. In fact, this contact structure is the quotient of the standard contact structure on $SU(2) = S^3$ by the antipodal map: both are orthogonal to the circles of a Seifert fibration and the circles of the Hopf fibration on S^3 double cover the circles of $SO(3)$ consisting of orientations with a particular point in contact with the plane. Any quotient by a finite subgroup of $SU(2)$ similarly inherits a contact structure from the standard contact structure on S^3.

1.2 Contactomorphisms

Definition. A *contactomorphism* is a diffeomorphism between contact 3-manifolds which sends contact planes to contact planes.

Contactomorphisms need not preserve the contact forms.

Example 4. Let $\phi : \mathbf{R}^2 \to \mathbf{R}^2$ be an area-preserving diffeomorphism sending (x, y) to $(\phi_x(x,y), \phi_y(x,y)) = (\phi_x, \phi_y)$. Let O be an arbitrarily chosen base point in $\mathbf{R}^3$. Then the map $\Phi : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$\Phi : (x, y, z) \mapsto (\phi_x, \phi_y, z + \int_\gamma (\phi_y d\phi_x - ydx))$$

is a contactomorphism from the standard contact structure on $\mathbf{R}^3$ to itself, where γ is any smooth path from O to (x, y). This is well-defined because $\int_\gamma (\phi_y d\phi_x - ydx)$ over any closed curve γ is 0; choosing a different base point just translates parallel to the z-axis.

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Example 5. Let $\phi : \mathbf{R}^2 \to \mathbf{R}^2$ be a diffeomorphism sending (x, z) to (ϕ_x, ϕ_z) preserving the partition of the plane into vertical lines, i.e., $\frac{\partial}{\partial z}\phi_x = 0$. Then the map $\Phi : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$(x, y, z) \mapsto (\phi_x, \frac{\frac{\partial}{\partial x}\phi_z + y\frac{\partial}{\partial z}\phi_z}{\frac{\partial}{\partial x}\phi_x}, \phi_z)$$

is an automorphism of the standard contact structure on $\mathbf{R}^3$.

Example 6. Let f be an area-preserving diffeomorphism of the xy-plane equal to the identity outside of a compact region R. Let g be a diffeomorphism of the xz-plane preserving vertical lines and equal to the identity outside a strip $z \in [z_0, z_1]$. Let F and G be the automorphisms of the standard contact structure on $\mathbf{R}^3$ constructed as in examples 4 and 5, with the base point for F chosen outside R. Then $FGF^{-1}G^{-1}$ is a contactomorphism which is compactly supported, that is, equal to the identity outside a compact region.

These examples of automorphisms of the standard contact structure on $\mathbf{R}^3$ are sufficiently general to show the following lemma:

Lemma 1.2.1. Compactly supported contactomorphisms act transitively on ordered n-tuples of points in the standard contact structure on $\mathbf{R}^3$.

Proof. By Darboux's theorem (1.2.2 below) which does not depend on this result, it suffices to prove that compactly supported contactomorphisms act transitively on points. It suffices to show that any point can be moved to $(0,0,0)$ by a compactly supported contactomorphism.

Let the point be (x_0, y_0, z_0). Let ϕ_0 be an area-preserving diffeomorphism of the xy-plane which takes (x_0, y_0) to $(0,0)$ supported in a compact region R_0, and let O_0 be a base point outside R_0. This induces a contactomorphism Φ_0 taking (x_0, y_0, z_0) to $(0,0,z_1)$, as in example 4. Let $f : \mathbf{R}^+ \to \mathbf{R}$ be a smooth function supported on $[\epsilon, 2\epsilon]$ satisfying $\int_\epsilon^{2\epsilon} rf(r)dr = z_1$. Let ϕ_1 be the area-preserving diffeomorphism which, in polar coordinates, sends (r, θ) to $(r, \theta + f(r))$. This is supported on the disc $R_1 = \{(r,\theta) | r \leq 2\epsilon\}$; let O_1 be a base point outside of the support. This induces the contactomorphism Φ_1 that takes $(0,0,z_1)$

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to $(0,0,0)$, since the signed area enclosed by the segment from $(\epsilon,0)$ to $(2\epsilon,0)$ together with its image under ϕ_1 is z_1. Then $\Phi = \Phi_1\Phi_0$ sends (x_0,y_0,z_0) to $(0,0,0)$ and is supported on a region whose projection to the xy-plane is contained in $R_0 \cup R_1$., In particular, Φ is supported on a region where $|y| < L$ for some L.

Let ψ be a smooth function on the xz-plane such that $\frac{\partial}{\partial x}\psi(0,0) > L$, $\frac{\partial}{\partial z}\psi(x,z) > -1$, and $\psi(x,z) = 0$ for $|z| > L'$. Let Ψ be the contactomorphism induced as in example 5 by the map sending (x,z) to $(x, z+\psi(x,z))$. Ψ sends $(0,0,0)$ to a point with y-coordinate greater than L, hence outside the support of Φ, and is supported on the region $|z| < L'$.

$\Psi^{-1}\Phi^{-1}\Psi\Phi(x_0,y_0,z_0) = (0,0,0)$ and is compactly supported, as required. □

Theorem 1.2.2. (Darboux) *Contact structures have no local structure: Every point has a neighborhood contactomorphic to the standard contact structure after possibly reversing orientation.*

It is interesting to study the standard contact structure on $\mathbf{R}^3$ both as a local structure induced upon submanifolds of a contact 3-manifold and as a global structure.

Theorem 1.2.3. (Gray) [21] *Smooth homotopies of tangent plane fields through contact structures equal smooth isotopies through contactomorphisms.*

Here, "homotopies" refers to continuous choices of contact structures and the "isotopies" are through families of diffeomorphisms, all of which preserve the contact plane field.

1.3 Links in Contact Structures

In the following, all knots and links are assumed to be smooth.

Definition. A knot in a contact 3-manifold is called *Legendrian* (resp. *transverse*) if it is tangent (resp. transverse) to the contact planes.

Example 7. Each Hopf fiber, $\ell \cap S^3$ with ℓ a complex line through $(0,0)$, is transverse to the standard contact structure on $S^3 \hookrightarrow \mathbf{C}^2$.

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Example 8. The real circle $\{(z, w) \in \mathrm{S}^3 \mid z, w \text{ real}\}$ is Legendrian in the standard contact structure on S^3, as are its images under the action of SU(2).

Example 9. Given an immersed curve in the xy-plane, up to translation parallel to the z-axis there is a unique Legendrian curve in the standard contact structure on $\mathbf{R}^3$ whose projection is the given curve. The z-coordinate is given by $\int y dx$. The projection of a generic Legendrian curve to the yz-plane is a piecewise smooth curve whose singularities are cusps and with no vertical tangencies. The Legendrian curve can be recovered from the projection: $y = \frac{dz}{dx}$.

Lemma 1.3.1. Every link is ambient isotopic to a Legendrian link. Similarly, every link is ambient isotopic to a transverse one, where the sign of each component can be chosen arbitrarily. In each case, the ambient isotopy can be chosen to be arbitrarily small in the C^0 topology.

Definition. Two links are *contact isotopic* if there is an isotopy through contactomorphisms taking one link to the other.

By Gray's Theorem, links are contact isotopic if there is a smooth (ambient) isotopy from one link to the other which preserves the Legendrian or transverse nature of each component: The family of contact structures pulled back along the isotopy is a homotopy of plane fields through contact structures, which Gray's theorem converts to contactomorphisms. For a transverse link, note that contact isotopies do not necessarily preserve the angles the link makes with the contact planes.

1.4 Characteristic Foliation

One may describe contact structures by their behavior on smooth surfaces.

Definition. Let S be a surface immersed in a contact 3-manifold. The intersection of the contact planes with the tangent space of the surface is a singular foliation on the surface, called the *characteristic foliation.*

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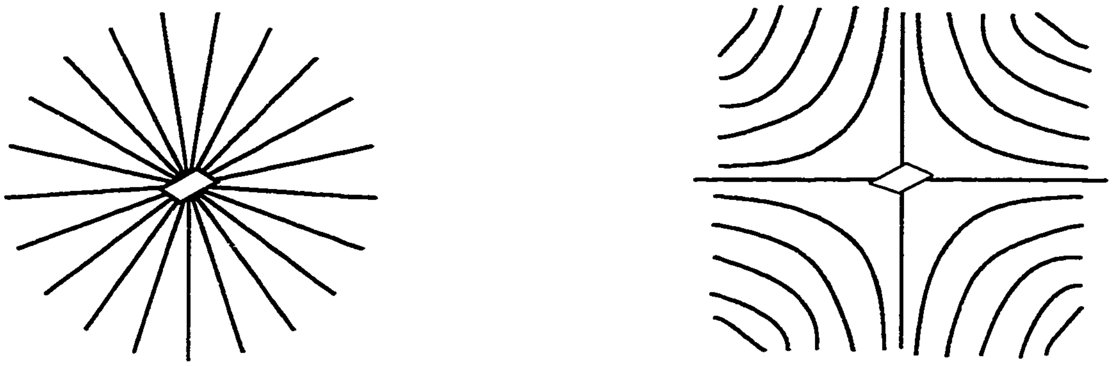

Figure 1.1: Elliptic and hyperbolic singularities

The leaves of the characteristic foliation are Legendrian curves. The singularities are where the contact planes are tangent to the surface. The nonintegrability of the contact planes implies that the set of singularities can have no interior, and generically the singularities are a discrete set of points.

Definition. An isolated singularity in a characteristic foliation is called *elliptic* if some neighborhood of the point is foliated radially. It is called *hyperbolic* if there are two pairs of leaves with ends at the singularity, and other leaves are topologically like the components of the curves $xy = c$ near the origin. See figure 1.1.

Example 10. Consider the standard contact structure on $\mathbf{R}^3$, the kernel of $ydx - dz$. The characteristic foliation on the plane $z = z_0$ is singular along the line $y = 0$, and away from this the leaves are lines of the form $x = x_0$. The characteristic foliation on the plane $y = y_0$ has no singularities, and the leaves are lines of the form $z = y_0 x + b$. The characteristic foliation on the plane $x = x_0$ has no singularities, and the leaves are lines of the form $z = z_0$.

Example 11. Consider the contact structure $\ker(dz + r^2 d\theta)$. The characteristic foliation on the cylinder $r = r_0$ has no singularities. The leaves are helices of slope $-r_0$ which can be parametrized by $(r_0 \cos \frac{t}{r_0^2}, -r_0 \sin \frac{t}{r_0^2}, t)$, as depicted in figure 1.2.

Example 12. The contact structure $\ker(dz + (r \tan r) d\theta)$ is cylindrically symmetric. The characteristic foliation on the plane $z = z_0$ is singular when r is a multiple of π and the leaves are the radial segments connecting these circles, as show in figure 1.3.

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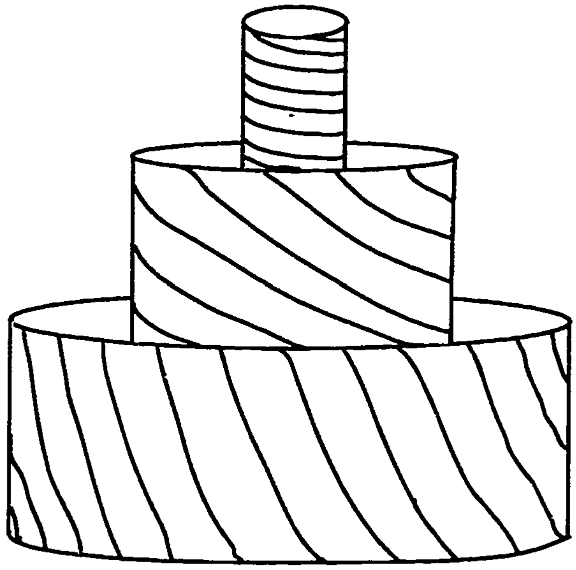

Figure 1.2: Characteristic foliation on cylinders from example 11

The map $(r, \theta, z) \mapsto (\tan r, \theta, z)$ is a contactomorphism from the cylinder $r < \frac{\pi}{2}$ in this example to the previous one. This can be checked by noting that the rays perpendicular to the z-axis are preserved by this map, and the characteristic foliations on cylinders are sent to each other.

More generally, the characteristic foliations on the leaves of a 2-dimensional foliation of a 3-manifold together with a Legendrian 1-dimensional foliation transverse to the leaves of the 2-dimensional foliation determine the contact structure. They determine two linearly independent tangent vectors on a dense set of points.

1.5 Fillability

Contact and symplectic geometry are closely related. There are several ways in which a contact structure on the boundary of a symplectic 4-manifold can be viewed as induced by the symplectic structure.

Definition. Let (W, ω) be a symplectic 4-manifold with a compatible almost complex structure J. Let M be a smooth 3-dimensional submanifold. By evaluating ω on a field of normal vectors to M satisfying $J(V) \subset TM$, one obtains a 1-form whose kernel may be locally a

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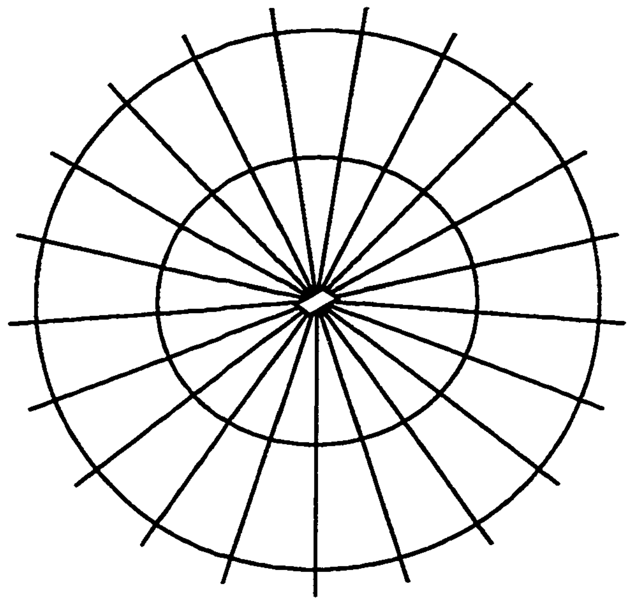

Figure 1.3: Characteristic foliation on planes from example 12

positive contact structure, a foliation, or a negative contact structure. Foliations occur when the 3-manifold is *Levi-flat*, and contact structures occur when the 3-manifold is *psuedoconvex* or *pseudoconcave* with respect to J.

If a contact form is the induced structure on the pseudoconvex boundary of a compact symplectic 4-manifold, then its contact structure is called *symplectically fillable.* A connected component of a fillable contact structure is called *symplectically semifillable.*

Note that one can always find a symplectic structure and almost complex structure on $M \times \mathbf{R}$, so the assumption that the 4-manifold is compact is essential.

Example 13. The unit S^3 in $\mathbf{C}^2$ with the standard symplectic structure is the pseudoconvex boundary of the unit ball. The standard contact structure on S^3 is the induced structure, hence is symplectically fillable.

Note that the level surfaces of the energy function on the phase space of a mechanical system are pseudoconvex.

Lemma 1.5.1. For $i = 1, 2$ let M_i be a connected component of the pseudoconvex boundary of a symplectic 4-manifold W_i, i.e., M_1 and M_2 are semifillable. Then there is a symplectic 1-

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handle which can be attached to W_1 and W_2 so that the resulting component, the connected sum $M_1 \# M_2$, is pseudoconvex hence also semifillable.

Theorem 1.5.2. (Weinstein) [28] *Let $M \subset \partial W$ be a semifillable contact 3-manifold, and let K be a Legendrian knot in M. Then there is a symplectic 2-handle which can be attached to M in a neighborhood of K so that the resulting boundary M' is pseudoconvex, and M' is an integral Dehn surgery on M along K with framing determined by (one less than) the Thurston–Bennequin invariant of K, defined in section 1.6.*

1.6 Algebraic Invariants of Links and Global Contact Structures

There are two main problems in the study of contact structures on 3-manifolds. First, classify the contact structures on particular 3-manifolds up to contactomorphisms isotopic to the identity. Second, classify the Legendrian and transverse links in particular contact structures up to isotopies preserving the contact structure. Perhaps surprisingly, these are closely related.

Definition. The *Thurston–Bennequin invariant* of a Legendrian knot K is the framing of the knot given by the contact plane field, i.e., a section of the unit normal bundle given by a family of vectors along the knot transverse to the contact planes.

A null-homologous knot has a canonical framing: the unique framing such that the knot has 0 linking number with its push-off in that direction. In this case one can identify framings by their difference with this one, hence denote the Thurston–Bennequin invariant by an integer.

The following two invariants of knots in a contact 3-manifold are functions of isotopy classes of trivializations of the unit tangent bundle of the contact structure (a circle bundle over M). For homology spheres, the trivialization is unique. The difference between two

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isotopy classes of trivializations is an element of $H^1(M;\mathbf{Z})$. If $H^2(M;\mathbf{Z})$ is trivial, then every coorientable contact structure is parallelizable. If there is no trivialization of the unit tangent bundle to the contact structure, then analogous invariants can be defined as functions of Seifert surfaces of nullhomologous knots by choosing a trivialization in a neighborhood of the Seifert surface.

Definition. The *rotation number* of a Legendrian knot counts the number of times the normalized tangent vector to the knot winds around the S^1 of unit tangents to the contact planes with respect to a trivialization of the unit tangent bundle of the contact structure.

Definition. The *Thurston–Bennequin* invariant of a transverse knot is the framing induced by a chosen trivialization of the unit tangent bundle to the contact structure.

The Thurston–Bennequin invariant for a transverse knot or the pair of Thurston-Bennequin invariant and rotation number for a Legendrian knot will be called the *algebraic information* of the knot, in contrast to the *topological information* which is the knot type.

Theorem 1.6.1. (Bennequin) [2] *In the standard contact structure on* $\mathbf{R}^3$*, there are no Legendrian unknots with Thurston–Bennequin invariant equal to the framing given by any disc bounded by the unknot.*

By the conventions in this thesis, Bennequin proved that the Thurston–Bennequin invariant of a Legendrian unknot in the standard contact structure on $\mathbf{R}^3$ is always negative or counterclockwise compared to the framing by a disc it bounds. Some people use an equivalent contact structure with the opposite orientation, in which case the Thurston–Bennequin invariant is always clockwise compared with a disc.

Corollary 1.6.2. [2] There exists a nonstandard contact structure on $\mathbf{R}^3$.

Proof. Example 12, $\ker(dz + r\tan r d\theta)$, contains the Legendrian unknots $\{(r,\theta,z) | r = n\pi, z = z_0\}$ which have horizontal framings by the contact planes and bound horizontal discs. □

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Note that one can decrease the Thurston–Bennequin invariant by adding a small kink to the knot, but one cannot, in general, increase the Thurston–Bennequin invariant of a topological knot.

Definition. A contact structure is *overtwisted* if there is a Legendrian unknot with Thurston–Bennequin invariant equal to the framing from a disc it bounds. If no such unknot exists, the contact structure is called *tight*.

Note that tightness passes to quotients, but does not necessarily lift to covers. Any contact structure covered by a tight contact structure is tight since embedded discs lift to embedded discs. Tight contact structures may be covered by overtwisted ones (see [20]).

Definition. Let D be a smooth disc whose boundary is a Legendrian unknot of Thurston–Bennequin invariant equal to the framing from D. If the characteristic foliation of D has a single elliptic singularity and leaves with an end at the elliptic singularity and at the boundary, then D is called an *overtwisted disc*.

Definition. An unknot with Thurston–Bennequin invariant equal to the framing by a disc it bounds will be called an *overtwisted unknot*.

An overtwisted unknot might not bound an overtwisted disc, but every disc it bounds contains an overtwisted disc.

One can assume that the tangent planes on the boundary of an overtwisted disc are all transverse or all equal to the contact planes. In the former case, the overtwisted disc's characteristic foliation looks like the characteristic foliation on the disc $r < \pi$ in example 12. In the latter case, the leaves spiral out to the boundary.

Definition. If the universal cover of a contact structure is tight, then the contact structure is called *universally tight*.

In section 1.7, we will discuss *Lutz twists* (or *Lutz surgeries*) on contact 3-manifolds. These are alterations of contact structures along transverse knots which result in overtwisted

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contact structures carried by the same topological manifold. These are common enough and sufficiently flexible to prove the following:

Theorem 1.6.3. (Martinet) [25] *There exist contact structures in every homotopy type of tangent plane field on every oriented compact 3-manifold.*

This may be thought of as analogous to the result that every homotopy class of tangent plane field on every oriented compact 3-manifold contains a foliation. In fact, there are many nonisotopic foliations in every homotopy class. Taut foliations are much more restricted; see Theorem 1.6.9.

Some details of the alteration's effects on the homotopy class of tangent plane field are given in section 1.7. Although there are many sequences of Lutz twists which leave the homotopy class of tangent plane field unchanged, the following fundamental theorem of Eliashberg showed that the apparent differences are illusory. The many contact structures in a given homotopy class of tangent plane field that one can obtain in this fashion are all contactomorphic.

Theorem 1.6.4. (Eliashberg) [8] Classification of overtwisted contact structures: *There exists a unique overtwisted contact structure in every homotopy class of tangent plane field on every oriented 3-manifold.*

Martinet's construction and this theorem of Eliashberg reduce the classification of contact structures on compact 3-manifolds to the classification of tight contact structures.

Theorem 1.6.5. (Eliashberg) [14] *The standard contact structures on $\mathbf{R}^3$ and S^3 are the unique positive tight contact structures on these 3-manifolds.*

Theorem 1.6.6. (Eliashberg and Gromov) [10] *Symplectically fillable contact structures are tight.*

Theorem 1.6.1 is a corollary of this, since the round surface in $\mathbf{C}^2$ with the standard contact structure is a pseudo-convex hypersurface.

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Theorem 1.6.7. (Eliashberg and Thurston) [6] *C^2 taut foliations can be perturbed to symplectically fillable contact structures.*

Corollary 1.6.8. By Gabai's construction of atoroidal C^2 taut foliations in these cases [18], there are tight contact structures on any compact, irreducible, oriented 3-manifolds with nontrivial H_2.

Theorem 1.6.9. (Kronheimer and Mrowka) [23] *There are only finitely many homotopy types of tangent plane fields which contain fillable contact structures.*

This result is stronger than the restrictions known for tight, but not necessarily fillable contact structures.

Theorem 1.6.10. (Bennequin and Eliashberg) [12] *There are only finitely many Euler classes of tangent plane fields which contain tight contact structures.*

The distinction between the Euler class and the homotopy type of tangent plane field is the 3 dimensional obstruction to constructing a homotopy between two tangent plane fields which agree on a 2-skeleton of M. This obstruction naturally lives in $H^3(M; \pi_3(S^2))$, or $\mathbf{Z}$, if M is connected.

Theorem 1.6.11. (Colin) [3] *Index one surgeries, e.g., the connected sum of two components, are well-defined and preserve tightness. One can delete small balls about the attaching points such that the characteristic foliations have two elliptic points and no other singularities. The sum along discs preserves tightness if the characteristic foliation of the disc has a single elliptic singularity in the middle, and no other singularities.*

Theorem 1.6.12. (Giroux) [19] *A tight contact structure on a reducible 3-manifold splits canonically as the connected sum of tight contact structures on its irreducible summands.*

Together, these results reduce the classification of contact structures on compact 3-manifolds to those on irreducible 3-manifolds.

The following are some relatively recent results which indicate the current state of knowledge:

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Theorem 1.6.13. (Gompf) [20] *Every oriented Seifert-fibered 3-manifold with base other than* S^2 *carries a fillable (hence tight) contact structure. Every orientable Seifert-fibered 3-manifold has a fillable contact structure in at least one orientation.*

Theorem 1.6.14. (Lisca) [24] *The Poincaré Dodecahedral Space has no symplectically fillable contact structure in the orientation opposite that inherited from the standard contact structure on* S^3.

The techniques of the proof seem very specific to the Poincaré Dodecahedral Space.

Theorem 1.6.15. (Eliashberg and Fraser) [5] *Let* M *be a connected tight contact 3-manifold. Two Legendrian unknots are classified up to contact isotopy by their algebraic information.*

Theorem 1.6.16. (Fuchs and Tabachnikov) [17] *The only additional finite-type invariants of Legendrian and transverse knots (beyond those for topological knots) are the Thurston–Bennequin invariant and rotation number.*

These results suggested that knots in a topological type may be classified by the algebraic invariants. However, the following results indicate otherwise:

Theorem 1.6.17. (Chekanov; Eliashberg and Hofer) [4] *There exist pairs of Legendrian knots in the standard contact structure on* $\mathbf{R}^3$ *with the same topological type and algebraic information, but which are not contact isotopic.*

Theorem 1.6.18. (Traynor) [27] *There exist pairs of Legendrian links in a tight contact structure on* $\mathbf{D}^2 \times S^1$ *which are topologically isotopic and whose components are respectively contact isotopic, but which are not contact isotopic as links.*

Theorem 1.6.19. (Fraser) [16] *There exist transverse knots in an overtwisted contact structure which have the same topological and algebraic information, but are not contact isotopic.*

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1.7 Lutz Surgeries

In this section we discuss Lutz surgeries on contact manifolds and the changes these make in the homotopy class of tangent plane field.

Definition. Let K be a transverse knot in a contact 3-manifold (M, ξ_1). Consider a (singular) map $f : M \mapsto M$ such that the preimage of each point p of K is a disc transverse to K at p, and which is a diffeomorphism from $M \setminus f^{-1}(K) \mapsto M \setminus K$. The pullback of $\xi_1|_{M\setminus K}$ is a contact structure on $M \setminus f^{-1}(K)$. Define ξ_2 to be equal to this contact structure outside the solid torus $f^{-1}(K)$ and equal to $\ker(dz + r \tan r d\theta)$ on $r < \pi$ inside the solid torus. It is always possible to choose f so that ξ_2 is a contact structure. If so, then we call (M, ξ_2) the *Lutz twist* of (M, ξ_1) about K. This is uniquely defined up to contactomorphism.

In the terminology of chapter 2, this is an enlarging surgery of size π. Note that this produces an overtwisted contact structure, since the boundary of the disc which is any preimge of a point of K is a Legendrian unknot and the framings given by the disc and contact structures agree.

The core of the solid torus is topologically ambient isotopic to K, but has the opposite sign of intersection with the contact planes. A subsequent Lutz twist about this core produces a knot in another contact structure with the same sign as the original.

A Lutz twist can change the homotopy class of tangent plane field of the contact structure.

Theorem 1.7.1. *There is a sequence of Lutz twists which changes any homotopy class of tangent plane field on a compact oriented 3-manifold M to any other.*

This is true for homotopy classes relative to the boundary if the plane fields are equal on the boundary.

Before proving this, note that all orientable 3-manifolds are parallelizable, that is, the tangent space of M can be identified continuously if not canonically with $M \times \mathbf{R}^3$. We shall fix a parallelization, so the bundle of oriented tangent planes, identified with the bundle of

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unit vectors, is given the coordinates of $M \times S^2$. Tangent plane fields correspond to sections of this bundle.

If all lower-dimensional obstructions vanish, the n-dimensional obstruction to homotoping one section to another is an element of $H^n(M; \pi_n(S^2))$. One may view this as the obstruction to extending over the n-skeleton of a cellular decomposition of M.

The obstruction in dimension 1 is trivial since $\pi_1(S^2)$ is trivial.

The 2-dimensional obstruction lives in $H^2(M; \pi_2(S^2)) = H^2(M; \mathbf{Z})$. When evaluated on a surface Σ, this obstruction is the index of the map from Σ to the S^2-coordinate of $M \times S^2$.

If the 2-dimensional invariant vanishes, the 3-dimensional obstruction corresponds to an element of $H^3(M; \pi_3(S^2)) = H^3(M; \mathbf{Z})$. When evaluated on M, this produces the linking number of preimages of two generic points in S^2 (the preimages are homologically trivial because the 2-dimensional invariant is 0).

These obstructions can be described without choosing a parallelization of the tangent space of M (see [20]).

Proof. (of Theorem 1.7.1)

Suppose one wishes to adjust the 2-dimensional invariant of the contact structure. Consider the effect of a Lutz twist about a transverse knot K. The 2-dimensional invariant assigns an integer (index) to each oriented surface; let Σ be an oriented surface.

We may assume that K is oriented to intersect the contact planes positively with respect to the coorientation of the contact planes, but the intersections with Σ may be either positive or negative. (Recall that K can have any topological ambient isotopy class.) Let P be an intersection.

We may assume that the parallelization of the tangent space sends the positive unit normal vectors to the contact planes near P to the north pole of S^2, and the unit vectors tangent to the contact planes to the equator. Perform the Lutz twist so that one meridianal disc is contained in Σ. The index of the map from Σ to S^2 can be determined by the signs of the points of preimages of the south pole, and there is only the center within the meridianal

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disc is sent to the south pole. The signs may be taken to be +1 for positive intersections of K with Σ, in which case the sign is -1 for negative intersections.

There are links Poincaré dual to the difference in 2 dimensional invariants one wishes to produce, and Lutz twists about the components of any of these links suffice.

Suppose one only needs to adjust the 3-dimensional invariant. One can do this by Lutz surgeries in a small ball B; this is equivalent to taking a connected sum with a contact structure on S^3.

We may assume that the parallelization sends the positive unit normal to the contact planes in B to the north pole. The Legendrian unit normals are sent to the equator. Within B, this provides a trivialization of the unit tangent bundle to the contact structure.

Let K be a transverse knot contained within B. Consider the effect of a Lutz twist about K. The contact planes in the surgered contact structure are sent to the north pole except in a neighborhood of K; there, the planes in a meridianal disc wrap about the sphere once. The preimage of two points P_1 and P_2 on the equator is unaffected by the Lutz twist outside of B, and within it for each P_i there is a new knot K_i parallel to K, pushed off in the direction of the trivialization of the unit tangent bundle of the contact planes on K. The linking number of the preimage of P_1 with the preimage of P_2 is changed only by the linking number of K_1 with K_2, since the K_i are linked with no components of the preimage except possibly each other. The linking number of K with K_i and hence K_1 with K_2 is the Thurston–Bennequin invariant of K with respect to this local trivialization.

Note that there are transverse knots in the standard contact structure on $\mathbf{R}^3$ with any specified Thurston–Bennequin invariant. Positive and negative values can be realized even by trefoil knots (though not arbitrary positive ones), so one can adjust the 3-dimensional invariant by Lutz twisting about a finite set of trefoil knots.

If one has two homotopy classes of tangent plane fields, one of which is represented by a contact structure, one can produce a contact structure in the second homotopy class by adjusting the 2-dimensional invariant first and then the 3-dimensional invariant. $\square$

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Note that there are many apparently distinct sequences of Lutz surgeries that allow one to move from a given homotopy class to another, but Eliashberg's classification of overtwisted contact structures (Theorem 1.6.4) shows that the resulting contact structures are contactomophic.

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Chapter 2 Size

In this chapter, we define an invariant of framed transverse links and study this invariant in both tight and overtwisted contact structures.

2.1 Definitions

Definition. The *cylindrically symmetric contact structure on* $\mathbf{R}^3$ is the kernel of $dz + \tan r d\theta$, or $d\theta$ when r is an odd multiple of $\frac{\pi}{2}$. This is not smooth at the z-axis, but by the discussion of example 12 there is a unique contact structure up to contactomorphism defined everywhere and contactomorphic to the cylindrically symmetric contact structure away from the z-axis by a map which preserves the rays perpendicular to the z-axis as sets. For example, the kernel of $dz + r \tan r d\theta$ is such a contact structure.

We use the singular structure because it simplifies notation and calculations. The characteristic foliation on the cylinder of radius R has no singularities and the leaves consist of helices which wind $-\cot R$ radians for every unit increase in z. In other words, if one takes the quotient by translating 2π parallel to the z-axis, the leaves of the characteristic foliation are intrinsically of angle R from the horizontal if one identifies the torus with the square torus. We'll abuse notation and say that contact structures contactomorphic to this singular contact structure away from the z-axis are actually contactomorphic.

The *cylindrically symmetric contact structure of radius* R is the restriction of the cylindrically symmetric contact structure on $\mathbf{R}^3$ to the cylinder $\{(r, \theta, z) \mid r \leq R\}$ or the solid torus quotient of this by the translation $z \mapsto z + 2\pi$. The longitude of the solid torus of size R is the quotient of a vertical line in the boundary.

If one ignores the longitude, then there are contactomorphisms between cylindrically

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symmetric contact structures on solid tori of different radii [11].

A consequence of Darboux's theorem is that every transverse knot has a neighborhood contactomorphic to the cylindrically symmetric contact structure of some radius. The characteristic foliation on the boundary of a cylindrically symmetric contact structure on a solid torus of radius θ is a linear foliation of angle θ, or slope $-\cot\theta$.

Definition. The *size* of a framed transverse knot K is the supremum of the set of θ so that there is contactomorphism from a neighborhood of K to the cylindrical contact structure on $\mathrm{S}^1 \times \mathbf{D}^2$ of taking longitude to longitude.

Changing the framing of the knot preserves the ordering of sizes and sizes which are multiples of π, so it makes sense to say whether an unframed knot has size at least π.

Remark. A contact 3-manifold with a transverse knot of size greater than π is overtwisted. The cylindrically symmetric contact structure of any radius greater than π contains an overtwisted disc. That this is an overtwisted disc is preserved under diffeomorphism.

Remark. Size is related to the *twisting* (*fr. torsion*) invariant of E. Giroux, also investigated by V. Colin. Twisting is a function of tori. Certain basic results are equivalent, but their emphasis seems to be on the twisting of incompressible tori.

Example 14. There is a unique tight (positive, coorientable) contact structure on $\mathrm{S}^1 \times \mathrm{S}^2$. This may be obtained from the cylindrically symmetric contact structure on $\mathrm{S}^1 \times \mathbf{D}^2$ of radius π by identifying the Legendrian circles on the boundary to points. The core, the image of the z-axis, has size π, with any framing: For any radius $\theta < \pi$, the inclusion of the concentric solid torus of radius θ to $\mathrm{S}^1 \times \mathrm{S}^2$ shows that the size is at least θ, hence the size is at least π, and the size is at most π because this contact structure is tight.

Definition. Let K be a framed transverse knot in a contact 3-manifold with size at least $\tan^{-1}\frac{-q}{p}$. A *reducing surgery of slope* $\frac{p}{q}$ is a Dehn surgery of slope $\frac{p}{q}$ which deletes an open solid torus contactomorphic to the cylindrical contact structure of radius $\tan^{-1}\frac{-q}{p}$, then identifies the leaves of the characteristic foliation of the boundary to points.

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Theorem 2.1.1. *Let K be a framed transverse knot in a contact 3-manifold M of size θ. Let M' be the result of performing a reducing surgery on K of size $\tan^{-1}\frac{-q}{p}$ on K so that the surgery locus is $K' \subset M'$. Choose a framing of K' so that the longitude is α meridians and β longitudes of K. Then with respect to this framing, the size of K' is at most $\nu = \tan^{-1}(\frac{q+p\tan\theta}{\beta+\alpha\tan\theta})$.*

Proof. Any solid torus about $K' \hookrightarrow M'$ can be extended to a solid torus about $K \hookrightarrow M$ by reattaching the solid torus that was removed. If the solid torus about K' had size ν with respect to the new framing, then the solid torus about K has size $\theta = \tan^{-1}(\frac{-q+\beta\tan\nu}{p-\alpha\tan\nu})$.

□

If one chooses a consistent basis, then reducing surgeries decrease the size of knots, and a reducing surgery on $K' \subset N$ is in fact also a reducing surgery on $K \subset M$, of larger size than the reducing surgery which produced N.

Corollary 2.1.2. Any reducing surgery on a knot of size less than or equal to $n\pi$ produces a knot of size strictly less than $n\pi$.

Corollary 2.1.3. Reducing surgeries on tight contact manifolds produce manifolds with at least one knot of size less than π.

Note that a reducing surgery of size π will reverse a Lutz twist. The opposite of a reducing surgery may be called an enlarging surgery.

Theorem 2.1.4. *Let $K \in M$ be a transverse knot. Let N branch-cover M with branch locus K. Let K' be a connected component of the preimage of K. Let the meridian of K' be sent to μ times the meridian of K under the covering map and let the longitude of K' be sent to α meridians plus β longitudes. If K has size θ, then K' has size at least $\tan^{-1}(\frac{k\tan\theta}{\beta+\alpha\tan\theta})$.*

Proof. A cylindrically symmetric solid torus about K lifts to a cylindrically symmetric solid torus about K'. □

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Example 15. The quotient of the standard contact structure on S^3 by a finite cyclic subgroup of SU(2) is a contact structure on a lens space. One can also obtain this contact structure by reducing surgery on the tight contact structure on $S^1 \times S^2$.

2.2 Symplectic Cobordisms

Theorem 2.2.1. *Let M be a compact contact 3-manifold. Suppose N is a contact 3-manifold obtained by a reducing surgery on $K \in M$ of size $\tan^{-1} n$, for $n \in \mathbf{Z}$. Then there is a symplectic cobordism between M and N so that M is pseudoconcave and N is pseudoconvex.*

Proof. Any contact structure is the pseudoconcave and pseudoconvex boundary of a symplectic structure on $M \times [-1, 1]$. What we need to attach to obtain a cobordism between M and N is a symplectic 2-handle. The boundary of a 2-handle is a non-smooth S^3. The corner is the boundary of the attaching region, a solid torus.

To produce integral reducing surgeries, it suffices to construct a reducing surgery of size $\frac{\pi}{2}$, since one can obtain the others by changing the framing of the knot.

Let $x_i = \mathrm{Re}(z_i)$, $y_i = \mathrm{Im}(z_i)$, and $R_i = |z_i|$ for $i = 1, 2$.

Consider the region $R_1^2 + c^2 R_2^2 \leq 1$. The boundary of this region is pseudoconvex in the standard complex structure with symplectic form ω.

Consider the function $f(x_1, y_1, x_2, y_2) = (R_1 - \gamma)^2 + R_2^2$. Evaluating ω on the gradient vector field produces a 1-form α. The 3-form $\alpha \wedge d\alpha$, restricted to the level set $(R_1 - \gamma)^2 + R_2^2 = \rho^2$, is the 3-form which may be expressed

$$\frac{-2}{y_2}(4(R_1 - \gamma)^2 + R_2^2(4 - \frac{2\gamma}{R_1}))dx_1 \wedge dy_1 \wedge dx_2$$

when $y_2 \neq 0$. We can choose the parameters so that the coefficient is never 0.

The intersection of $R_1^2 + c^2 R_2^2 \leq 1$ with $(R_1 - \gamma)^2 + R_2^2 \geq \rho^2$, for appropriate choices of c, γ, and ρ, will be a connected component of the difference between a filled ellipse and two

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internally tangent filled circles. For this tangency, $\rho^2 = \frac{1}{c^2} - \frac{\gamma^2}{c^2-1}$. We can choose γ almost arbitrarily within $[0, 1]$ for c large enough; let $\gamma = 1/2$ and consider the regions as $c \to \infty$.

Checking the characteristic foliations on the tori of fixed values of (R_1, R_2) on the boundary of this region confirms that the pseudoconcave solid torus has size approaching $\pi/2$ from above, and the pseudoconvex solid torus has size approaching 0 from above.

Finally, note that we can approximate this handle by one such that the result is not just C^1 but smooth.

□

Conjecture. Symplectic cobordisms from pseudoconcave to pseudoconvex contact structures preserve tightness.

Corollary 2.2.2. Integral reducing surgeries on fillable contact structures produce fillable contact structures.

For any transverse knot K, choosing the framing can give K a size greater than $\frac{\pi}{2}$.

The following corollary was also proved by Gompf in [20].

Corollary 2.2.3. Let M be a symplectically fillable contact 3-manifold. Let K be a topological knot in M. Then there is a non-empty interval in Dehn surgery space such that manifolds corresponding to these Dehn surgeries upon K carry fillable contact structures.

2.3 Sizes of Knots in Overtwisted Contact Structures

Theorem 2.3.1. *Let M be a connected contact 3-manifold. If the complement of a transverse knot K is overtwisted, then the knot has infinite size.*

Proof. For any even n, we can find a solid torus of size $n\pi$ about K: Perform n Lutz twists about K. This may change the homotopy type of the induced contact structure on $M \setminus K$. Change it back by Lutz twists in $M \setminus K$ which miss the added solid torus of size $n\pi$. By Eliashberg's classification, these contact structures on $M \setminus K$, which agree at the boundary, are contactomorphic. Hence we can find a solid torus of size $n\pi$ about K. □

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Corollary 2.3.2. Let M be a connected contact 3-manifold. In every topological type of knot, there is a transverse knot which has infinite size.

Proof. Topologically, there is no obstruction to sliding a knot off a particular overtwisted disc. One can then approximate the knot with an ambient isotopic transverse knot. □

Theorem 2.3.3. *Let K be a transverse knot in an overtwisted contact 3-manifold. If there is an overtwisted disc which intersects a transverse knot K only once, then K has size at least π.*

Proof. Let D be the overtwisted disc, given local coordinates to agree with a neighborhood of the disc $\{(r, \theta, z)|z = 0, r <= \pi\}$. Coorient D so that the coorientation of the contact plane field agrees with the coorientation of D at the center and disagrees at the boundary.

Either the intersection of K with the disc has the same sign as the intersection with the tangent planes, or the signs disagree. If they disagree, then by Theorem 3.1.2 one can isotope K off of D, and apply Theorem 2.3.1. Otherwise we can contact isotope K so that $D \cap K$ is the center of D and K is orthogonal to D there. (These can be checked explicitly or by using Theorem 3.1.2.)

Consider a small solid torus T about K whose induced contact structure is cylindrically symmetric and which consists of a small right cylinder near $D \cap K$.

Let D_r be the subdisc of D of radius r, for each $r < \pi$. An increasing family N_r of strictly convex rotationally symmetric neighborhoods of the D_r ($D_r \subset N_r$) can be chosen so that the characteristic foliation of the boundary of N_r has elliptic singularities on the z-axis and whose leaves are radial near the poles and spin $f(r)$ radians from the south pole to the north pole, where $\lim_{r \to \pi} f(r) = \infty$.

A smoothed out union of T with D_r may be chosen so that the characteristic foliation of the boundary has no singularities. In the boundary of $T \setminus D_r$, the leaves spin at least c radians when moving from the north pole to the south pole, for c some large negative constant. Altogether, each leaf covers at least $c + f(r)$ times the meridian for each longitude,

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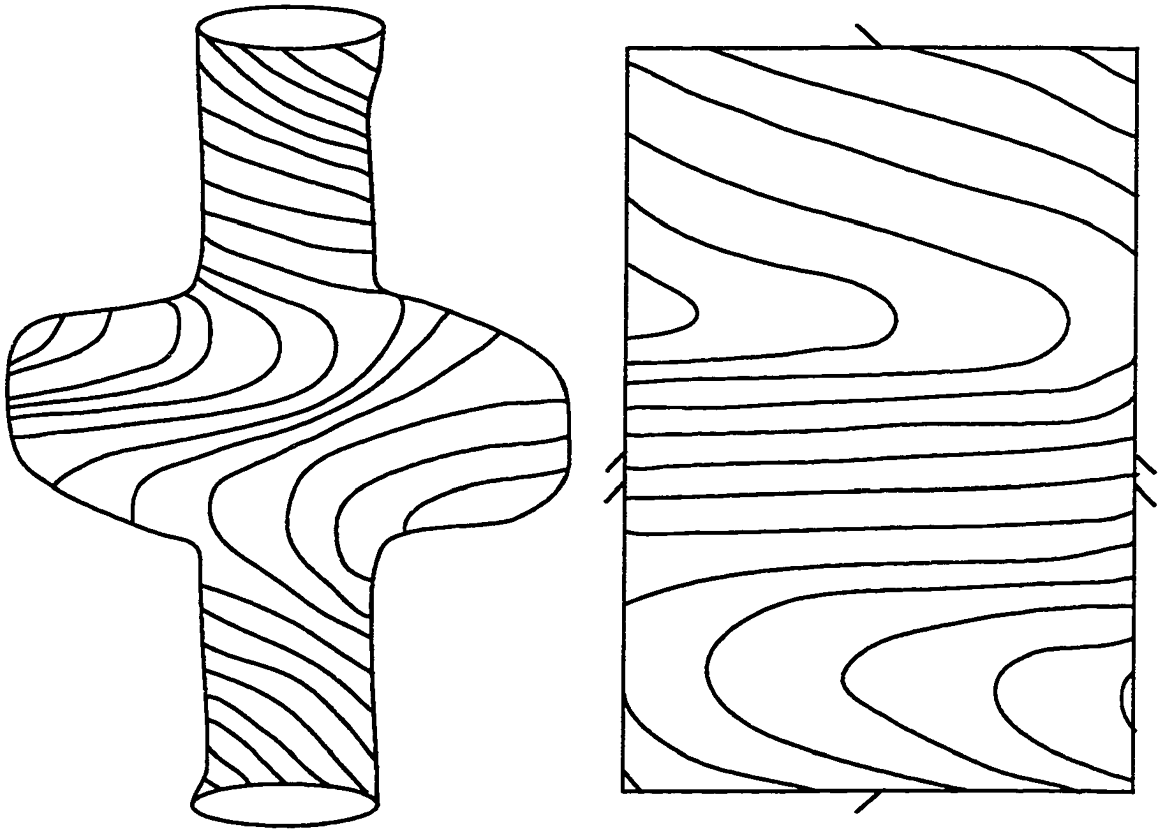

Figure 2.1: Characteristic foliation on $\partial(T \cup D_r)$ in the proof of 2.3.3

so the slope of the characteristic foliation is $c + f(r)$, which goes to ∞ as $r \to \pi$. See figure 2.1.

The condition of strict convexity allows us to construct a Legendrian flow transverse to the tori with these slopes, hence as in the discussion of example 12 contactomorphisms from the cylindrically symmetric contact structure of any radius less than π into $T \cup \bigcup_{r<\pi} D_r$.

$\square$

Theorem 2.3.4. *Let K be a transverse knot. Suppose there is a Legendrian $\frac{p}{q}$-fold cabling K' of K and a smooth helicoid H connecting K with K' so that the characteristic foliation of H has no singularities and whose leaves have ends at K and K'. Then the size of K is at least* $\tan^{-1}\frac{-q}{p}$.

Proof. Note that the characteristic foliation on the boundary of a solid torus of that size consists of such Legendrian cablings and the union of the radial segments joining the core

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with each leaf of the characteristic foliation produce such a helicoid.

One proof is exactly analogous to that of Theorem 2.3.3, which was the case of a $\frac{0}{1}$-cabling. Let T be a small cylindrically symmetric region about K. Let the family $\{H_r\}$ be subhelicoids which exhaust H. Let N_r be a family of neighborhoods of H_r so that the characteristic foliations of $\partial H_r \setminus T$ have no singularities and have leaves which wind increasingly about the slope $\frac{p}{q}$ curve. The torus $(\partial H_r \setminus T) \cup (\partial T \setminus H_r)$ has a characteristic foliation of slope which approaches $\frac{p}{q}$.

One could also perform an enlarging surgery so that K' is sent to a meridianal unknot and H is converted to an overtwisted disc, and then apply the proof of 2.3.3. $\square$

Corollary 2.3.5. Let K be a knot of size $\theta \geq \tan^{-1}\frac{-q}{p}$. Then for any slope $\frac{p'}{q'}$ with $\tan^{-1}\frac{-q}{p} \geq \tan^{-1} q'p'$, there is a cabling K' of K with that slope such that K' with the outward framing has size $\frac{\pi}{2}$.

Proof. Consider a helicoid H as in the statement of Theorem 2.3.4 of slope $\frac{p'}{q'}$. The intersection with the boundary of a small solid torus T about K is a transverse knot, K', which is a cabling as described. $H \setminus T$ is a helicoid of slope $\frac{1}{0}$ about K' given the outward framing. $\square$

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Chapter 3 Rigidity

In this chapter, we study geometric intersection properties invariant under contactomorphisms, primarily of knots with surfaces.

In section 3.2, we consider intersections of knots with overtwisted discs and solid tori from Lutz surgeries. These imply restrictions on the flexibility of contactomorphisms of tight submanifolds.

3.1 Germ Equivalence

In this section we consider the local flexibility near surfaces.

Every point has a standard neighborhood, which is tight. One can put together an overtwisted contact structure from these neighborhoods. As the following example demonstrates, it is possible to glue together two balls with tight contact structures along a disc so that the result is overtwisted.

Example 16. Consider the cylindrically symmetric contact structure with radius $\pi + \epsilon$ where $0 < \epsilon < \frac{\pi}{2}$. This is overtwisted. The xz-plane divides this contact structure into 2 regions, and the induced contact structures are tight. The characteristic foliation is singular on the lines $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$, and the leaves are the segments in the complement of the singularities described by $z = c$. See figure 3.1.

In contrast, we have Theorem 1.6.11 of Colin:

Theorem. *Let M_1 and M_2 be tight contact 3-manifolds. Let S_1 and S_2 be discs in the boundaries of M_1 and M_2, respectively, whose characteristic foliations consist of a single elliptic singularity and radial leaves, or 2-spheres with two elliptic singularities and leaves*

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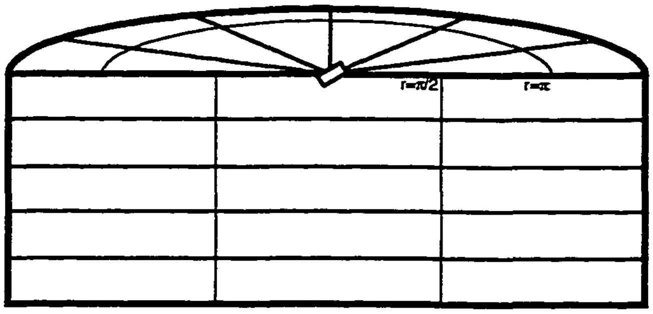

Figure 3.1: Half of an overtwisted disk's neighborhood, with a tight contact structure

running between the singularities. Then the contact structure obtained by gluing the M_i together along the S_i is tight.

In example 16, every overtwisted unknot has to intersect the xz-plane. One may prove Theorem 1.6.11 by showing that, if there were an overtwisted unknot, then not every overtwisted unknot would intersect the surface.

First, we study the local properties of surfaces in contact 3-manifolds.

Definition. Let S be a smooth surface in a contact 3-manifold together with its characteristic foliation. Consider the net of neighborhoods of S. The *germ* of a smooth curve A transverse to S is the direct limit of restrictions of A to neighborhoods of S.

Definition. Let S be as above. Two germs of arcs A_1, A_2 transverse to S are *germ equivalent* if in every neighborhood of S such that A_1 and A_2 are defined, there is a contact isotopy supported on that neighborhood taking A_1 to an arc B_1 so that $A_2 = B_1$ as germs, that is, B_1 restricted to some neighborhood of S equals A_2 restricted to that neighborhood with no added components.

Considering germs of arcs together with isotopy classes of paths is equivalent to considering germs of arcs transverse to the universal cover of S. One can also orient the arcs.

Lemma 3.1.1. Two germs of Legendrian arcs transverse to S in the same direction with the same points of intersection are germ equivalent. Similarly, two germs of arcs positively transverse to the contact planes and transverse to S at the same set of points are germ equivalent.

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This is an easy corollary of Darboux's theorem (applied to the points of intersection), and Gray's theorem.

In example 16, a Legendrian arc positively transverse to the xz-plane in the region $x < -\frac{\pi}{2}$ can be isotoped to a Legendrian arc which intersects the xz-plane several times including once, positively, in the region $x > \frac{\pi}{2}$. The germs of positively transverse Legendrian arcs in the negative region and the positive region are *not* germ equivalent.

Theorem 3.1.2. *Let S be a surface in a contact 3-manifold. Two germs of Legendrian arcs transverse to S are germ equivalent if (and not necessarily only if) they are connected by a leaf of the characteristic foliation of S.*

Proof. Let L be a leaf of the charcteristic foliation through connecting the two points of intersection. There is a neighborhood of L contactomorphic to the standard contact structure on $\mathbf{R}^3$ which takes a neighborhood of L in S to the plane $x = 0$ and L to part of the leaf $z = 0$ within that plane [26].

It suffices to provide contactomorphisms supported on compact regions in arbitrarily small neighborhoods of the plane $x = 0$ which send $(0, y_0, 0)$ to $(0,0,0)$. These may be constructed as in the proof of Lemma 1.2.1, by choosing R_0, R_1, and L appropriately. More explicitly, we have the following:

Let f be a smooth function supported on $(-\epsilon, \epsilon)$ satisfying $f(0) = 2y_0$. Let α be the area-preserving map taking (x, y) to $(x, y + f(x))$.

Let g be a smooth diffeomorphism from $\mathbf{R}$ to itself satisfying $g(-y_0) = -y_0$, $g(y_0) = 0$, $g'(y) \leq 1$, and $g' - 1$ is supported on a compact region. Let β be the area-preserving map taking (x, y) to $(\frac{x}{g'(y)}, g(y))$.

Then $\Phi = \beta\alpha\beta^{-1}\alpha^{-1}$ sends $(0, y_0)$ to $(0,0)$, preserves area, is supported on a compact region within $x \in (-\epsilon, \epsilon)$, and preserves the y-axis as a set. Choose a basepoint outside the support but on the y-axis. As in example 4, this induces a contactomorphism of the standard contact on $\mathbf{R}^3$ preserving the partition into lines parallel to the z-axis. The choice of basepoint means that $(0, y_0, 0)$ is sent to $(0,0,0)$. Let the support of Φ be contained in

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the region $y < L$.

Let ψ be a function on the xz-plane supported on the region $|z| < L'$ so that $\frac{\partial}{\partial x}\psi(0,0) > L$.

Then the contactomorphism Ψ induced as in example 5 is supported in the region $|z| < L'$, preserves the plane $x = 0$, and moves $(0,0,0)$ outside the support of Φ. Hence the commutator $\Psi^{-1}\Phi^{-1}\Psi\Phi$ is a compactly supported contactomorphism which preserves the plane $x = 0$ and sends $(0, y_0, 0)$ to $(0,0,0)$.

□

Corollary 3.1.3. Let S be a cooriented smooth surface in a contact 3-manifold. Let P be a piecewise Legendrian path in S with vertices at singularities of the characteristic foliation which all have the same coorientation. Let A_1 and A_2 be germs of arcs positively transverse to S through the endpoints of P and positively or negatively transverse to the contact planes as the singularities of P are (and each other). Then A_1 and A_2 are germ equivalent.

Let S be a surface containing an overtwisted disc D such that the contact planes are tangent to S at the boundary of D. Let S be cooriented with a coorientation which agrees with the coorientation of the contact plane at the elliptic singularity in D and disagrees with the coorientation of the contact planes at the boundary of D.

Corollary 3.1.4. Let A and B be the germs of arcs transverse to D and transverse to the contact planes so that the sign of the intersection of A with the contact planes agrees with both and B agrees with only one. Then B can be contact isotoped off of D, but A cannot. The complement of A in a neighborhood of D is a tight contact structure while the complement of B in a neighborhood of D is overtwisted.

Proof. There is a Legendrian path in S from the intersection point $B \cap S$ through the boundary of D. By Corollary 3.1.3, B is germ equivalent to the germ of an arc which passes through a point outside of D. That there are contactomorphisms moving B off D means that their inverses move D into the complement of B, hence the complement of B is

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overtwisted.

A is germ equivalent to the germ of an arc passing through the elliptic singularity of D orthogonally, and the following lemma shows that the complement of A in a neighborhood of D is tight.

Lemma 3.1.5. The complement of the z-axis in the contact structure $\ker(dz + \tan r d\theta)$ is universally tight.

Proof. The universal cover may be taken to have coordinates (r, θ, z) where $r > 0$ and θ and z are arbitrary. The map which rotates the plane $r = r_0$ by an angle of r_0, counterclockwise, is

$$(r, \theta, z) \mapsto (r, \theta \cos r - z \sin r, \theta \sin r - z \cos r)$$

This is a contactomorphism to the standard contact structure on $\mathbf{R}^3$ with image the half-space $x > 0$. $\square$

This completes the proof of Corollary 3.1.4. $\square$

Note that Lemma 3.1.5 implies that every overtwisted unknot in is geometrically linked with the z-axis, i.e., the unknot must intersect each half-plane $\theta = \theta_0$.

Definition. The *Thurston–Bennequin invariant* of a topological type of knot is the greatest value the framing of any Legendrian knot of that topological type.

This can be extended naturally to arcs with fixed endpoints on a smooth surface S which are transverse to the surface. Theorem 3.1.2 shows that this invariant does not depend upon the precise location of the endpoints, but depends upon the leafs containing the endpoints, and the order if the endpoints are in the same leaf.

Remark. Deleting a region R from M changes the Thurston–Bennequin invariant of a topological knot K or a relative Thurston–Bennequin invariant of a topological arc if and only if every knot or arc which had the extreme framing must intersect R.

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3.2 Rigidity in Overtwisted Contact Structures

One contrapositive of Theorem 2.3.3 is the following:

Theorem. *If K is a transverse knot of size less than π, then any overtwisted disc must intersect K at least twice.*

By applying Corollary 3.1.3, one can contact isotope intersections which have negative sign off of the disc.

Corollary 3.2.1. Let K be a positively transverse knot of size less than π. Let D be an overtwisted disc D cooriented so that the coorientation agrees with the coorientation of the elliptic singularity, and disagrees with the coorientation at the singularities at the boundary. Then K must intersect D at least twice with positive sign.

Note that reducing surgeries on knots of finite size produce contact 3-manifolds with knots which have size less than π.

This restriction on the behavior of overtwisted discs together with the analogue with Corollary 2.2.2 suggest the following conjecture:

Conjecture. Reducing surgeries on tight contact structures produce tight contact structures.

Theorem 3.2.2. *Let M be a tight contact structure, and let N be obtained from M by a Lutz twist about a knot $K \subset M$, inserting a solid torus T. Then every overtwisted unknot in N intersects T. If K is not an unknot, then every overtwisted unknot must intersect the boundary of T.*

Proof. There is a map $f : N \to M$ which induces a natural contactomorphism of $N \setminus T$ to $M \setminus K$. This map takes unknots in $N \setminus T$ to unknots in $M \setminus K$ and preserves both the framing by the contact planes and the sets of framings of discs bounded by these unknots. Hence any overtwisted unknot in $N \setminus T$ would result in an overtwisted unknot in M, but M is tight by assumption.

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The interior of T is abstractly tight, since it is a cylindrically symmetric contact structure of size π. If K is an unknot then some knots in T might bound discs in N though they do not bound discs contained in T. If K is not an unknot, then any disc bounded by a knot in T can be isotoped completely inside T, possibly after taking the connected sum with an essential sphere if N is reducible. □

The same can be said of an analogue of the Lutz twist about a properly embedded arc transverse to the contact structure, in which one inserts a cylindrically symmetric cylinder of radius π along the transverse arc.

Note that the cylinder of radius π in the cylindrically symmetric contact structure of radius 2π is not invariant under contactomorphisms. Any point on a meridianal unknot in the characteristic foliation can be moved into or out of the cylinder, but not all points simultaneously. This may be viewed as a contact analogue of some elementary symplectic non-squeezing results.

A trivial corollary of Theorem 3.2.2 is that there are no contactomorphism which pushes an overtwisted unknot completely out of the torus of a single Lutz twist, even if one restricts the support of the contactomorphism to a subset N of M. This is interesting, however, because it makes statements about relative isotopy classes of arcs in N up to contactomorphisms of N, even though N might be tight, as in example 16.

By the results in section 1.7, the same overtwisted contact structure can be produced from a tight contact structure by infinitely many Lutz surgeries about topologically distinct knots. Hence, if there is a tight contact structure on a 3-manifold, there are infinitely many solid tori whose boundaries have meridianal characteristic foliations and such that every leaf must intersect every other solid torus. The extent to which these tori can be described effectively with a finite amount of information, e.g., in the Legendrian loop space, warrants further attention.

Although Eliashberg's classification completely solves the question of what the overtwisted contact structures are, overtwisted contact structures nevertheless contain interest-

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ing structures which are as yet poorly understood.

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