Dade's Ordinary Conjecture for the Finite Unitary Groups in the Defining Characteristic

Thesis by

Chao Ku

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To My Parents

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Abstract

There has been rising interest in the study of Dade's conjectures, which not only generalize Alperin's weight conjecture, but unify some other major conjectures in (modular) representation theory, such as Brauer's height conjecture in abelian blocks and McKay's conjecture. In this thesis we verify Dade's ordinary conjecture for the finite unitary groups in the defining characteristic. Dade's conjectures involve proving the vanishing of the alternating sum of certain G-stable function over the p-group complex of a finite group G. We develop some machinery to treat alternating sums which we hope will serve as part of a general approach to such problems. This includes extending some of the existing techniques in a functorial way. We also show how to make use of the topological properties of p-group complexes to reduce the alternating sums. While this work is mainly intended for the unitary groups, it should also apply to other groups of Lie type, and part of the work can be generalized to treat a much wider class of groups. Among other things, we also obtain a formula which expresses the McKay's numbers of the finite unitary groups in terms of partitions of integers.

Contents

Chapter 1 Introduction

Background.

The description of character values, in terms of structure given by a prime p, has been a basic topic of the representation theory of finite groups. In particular, since Brauer's pioneering work, the question of the number of irreducible characters in a p-block and the divisibility properties of their degrees is of fundamental concern.

In the 1986 Arcata conference ([A]), J.L. Alperin proposed his famous weight conjecture which expresses the number of irreducible modular characters of a finite group in terms of p-local information. Since then it has become one of the central problems in the modular representation theory of finite groups, as the conjecture implies many new results and old theorems as well as outstanding conjectures. Alperin also pointed out that the weight conjecture, which has been verified in many cases but hasn't yet been proved, could be a special case of even more sweeping conjectures.

Later an intriguing reformulation of Alperin's weight conjecture was obtained by G. Robinson and R. Knorr ([KR]), who introduced the chains of so-called radical subgroups and stated a form of the conjecture in terms of alternating sums. This inspired E.C. Dade ([D1], [D2]) to propose his spectacular series of conjectures. Dade's conjectures not only generalize Alperin's weight conjecture, but unify some other major conjectures in (modular) representation theory, such as Brauer's height conjecture in abelian blocks and McKay's conjecture. The weakest of the conjectures, Dade's Ordinary Conjecture, already implies Alperin's weight conjecture. The strongest of Dade's conjectures, Dade's Final Conjecture, reportedly has the property that if it is true for the finite simple groups, then it is true for all the finite groups, i.e., its verification can be reduced to the case of the finite simple groups. In certain cases Dade's Final Conjecture is equivalent to the Ordinary Conjecture.

There has been rising interest in the study of Dade's conjectures ever since they appeared. In general, however, Dade's conjectures are very difficult to verify. For instance, while there is an very elegant bijective proof of Alperin's conjecture for the finite groups of Lie type in the defining characteristic (see [A]), an analogous proof of Dade's conjectures in those cases has yet to be found. Indeed, only Dade's Ordinary Conjecture has been verified for the general linear group, in work of J.B. Olsson and K. Uno ([OU]).

In this thesis we only study Dade's Ordinary Conjecture. We include an incomplete list due to J. An ([An]) of the cases for which Dade's Ordinary Conjecture has been verified.

(a) $GL_n(q)$ $(p|q)$, ${}^2F_4(2^{2n+1})$ $(p \neq 2)$, $G_2(q)$ $(p \nmid q)$ (by Olsson, Uno, An).

(b) S_n (by Olsson and Uno when p odd, An when $p = 2$).

(c) Ru (by Dade).

(d) Unipotent abelian defect blocks (by Broue, Malle and Michel).

(e) Abelian defect principal 2-blocks (by Fong and Harris).

(f) All abelian defect blocks with small inertia index (by Usami).

Results and Goals.

Dade's work has already shown the importance of verifying his conjectures for the finite simple groups and groups close to being simple. Among other things, we prove that Dade's Ordinary Conjecture holds for the finite unitary groups in the defining characteristic. However, our goal is beyond that.

It is believed that the study of conjectures like those of Alperin and Dade which express the number of irreducible complex/modular characters of a finite group G in terms of the p-local structure of G fits into a larger picture where one can consider how to evaluate the alternating sum of certain G-stable function f over the p-local geometry of G. More explicitly, it involves proving the vanishing of the alternating sum of certain G-stable function over the p-group complex of G. Alternatively, J. Thevenaz (see [Th2]) showed that (given a prime divisor p of $|G|$) any G-stable function f can be decomposed as the sum of the "p-part" f_p and the "p'-part" $f_{p'}$, where

f_p is a p-locally determined G-stable function in the sense that the alternating sum of f_p over the p-group complex of G is 0, and $f_{p'}$ is a G-stable function which vanishes on p-local subgroups. Thus the question turns to how to determine the "p-part," as well as the "p'-part" of a given G-stable function f.

We feel that the study of these questions may proceed in two directions. First, in order to show a function f is p-locally determined, one may consider which properties f should have. These properties should be independent of the explicit group structure if f is defined for any finite group. Subsequently techniques for manipulating the G-stable functions may be developed and applied to show the corresponding alternating sum is 0. Second, one may proceed by studying the topological properties of the p-group complexes for finite groups, as questions of this kind can be interpreted as showing the "weighted" Euler characteristic of the p-group complex for any finite group G by f is 0. The study of such properties by analyzing the p-local structure of G has been active. See for instance D. Quillen's work on the homotopy properties of p-group complexes (see [Q]), M. Aschbacher and S.D. Smith's work on Quillen's conjecture (see [AS]) and Aschbacher's work on the simple connectivity of p-group complexes (see [As2]). In our case, we need to utilize certain properties of the p-group complex or of various subcomplexes to reduce the alternating sums to a stage where we can either proceed by an inductive argument or make use of the explicit properties of f.

In this thesis, we make efforts in both directions. Namely we develop some machinery to treat the alternating sums which we hope will serve as part of a general approach to such problems. This includes extending some of the existing techniques in a functorial way; and in particular we are interested in the study of the case when the function f is decomposable in a certain sense. Moreover, in proving Main Theorem 1 which is stated below, we display how to make use of the topological properties of the p-group complexes to reduce the alternating sums. While this work is mainly intended for the unitary groups, we feel optimistic that it applies to other groups of Lie type and have good reason to believe that part of the work can be generalized to treat a much wider class of groups. As an application, we prove the following results.

Throughout this thesis, Dade's Ordinary Conjecture is abbreviated as **DOC**. Let $G = GU_n(q)$ be the full isometry group of an n-dimensional unitary space over a field $\mathbb{F}_{q^2}$ of order q^2 where $q = p^e$ is an integral power of a prime p.

Main Theorem 1. *DOC holds for G at the prime p.*

Main Theorem 1 follows from Lemma 3.1.6 and Theorem 3.3.5. We also obtain the formula for the McKay numbers for the general unitary group:

Main Theorem 2. *The number of irreducible complex characters φ of G such that the p-part of $\varphi(1)$ is p^h is $\sum' q^{l(\mu)-\delta(\mu)}$.*

Here the sum $\sum'$ is taken over all the partitions μ of n with $n(\mu) = h/e$. The parameters $l(\mu)$, $\delta(\mu)$ and $n(\mu)$ are defined in section 2.3.

Main Theorem 2 is restated as Theorem 3.3.4, which follows from Theorem 3.3.3. Two immediate consequences of Main Theorem 1 are the following:

Corollary 1.0.1. *DOC holds for the finite projective unitary group $PGU_n(q)$ at p.*

Corollary 1.0.2. *If $(n, q+1) = 1$, then* ***DOC*** *holds for the finite simple unitary group $U_n(q)$ at p.*

The corollaries are proved in section 3.3.

The Organization of the Thesis.

The thesis is organized as follows. In Chapter 2, after we state **DOC**, we set up some notation which will be used throughout the thesis and prove some preliminary lemmas. Then we prove some results on partitions of integers which will be used in section 4.3 as well as in 9.3.

In Chapter 3 we give a reformulation of the conjecture for the finite groups of Lie type. The reformulation is an extension of Olsson and Uno's reformulation for the general linear groups. We then present a strategy for verifying the conjecture for the finite groups of Lie type. In particular, we show that Main Theorem 1 follows from Proposition 3.3.6 and give a proof of Corollary 1.0.1 and 1.0.2 assuming Main Theorem 1.

In Chapter 4, we start by introducing some key facts from the Deligne-Lusztig theory on the representation theory of the finite groups of Lie type, and then prove Theorem 3.3.3, and hence also Main Theorem 2. In the end, we prove part (1) of Proposition 3.3.6.

In Chapter 5 we discuss some general techniques for treating alternating sums. In particular we extend some combinatorial ideas in a functorial way to obtain some results on cancellation in alternating sums. We also set up some machinery to deal with the alternating sum of certain decomposable functions. Some interesting examples are discussed to illustrate how the general results can be applied.

In Chapter 6 we study the action of the direct product of two groups on the tensor product of two modules, one for each factor, and obtain a parameterization of the orbits as well as information on the stabilizers. This serves as the basis for the analysis of the action of the parabolic subgroups of a finite group of Lie type on some of their internal modules, namely the so-called linear modules and unitary modules.

Chapter 7 is devoted to the study of the action on a twisted version of the tensor modules, namely the action of a sub-parabolic subgroup of a maximal parabolic P of the general unitary group on the center of the unipotent radical of P, which we call the central module of P.

We begin our reduction toward a proof of part (2) of Proposition 3.3.6 in Chapter 8. We introduce a system to parameterize the stabilizers in the parabolics of certain characters of the internal modules, and show that the count of the alternating sum of the number of irreducible characters of the parabolics of the unitary group lying over the set of non-trivial characters of a fixed internal module can be reduced to the count of characters over some special subset of characters in that module whose stabilizers are well understood and labeled by the system we introduced.

We continue the discussion in Chapter 9. By a recursive analysis we show that only certain characters of the parabolics lying over characters in the unitary and central modules need to be counted. Then we are able to implement an inductive argument in section 9.3 to the proof of part(2) of Proposition 3.3.6.

Chapter 2 The Conjecture and Preliminary Lemmas

2.1 The Conjecture

Let G be a finite group, p a prime. A chain

$$c: U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_r$$

of p-subgroups of G is called *radical* if $U_0 = O_p(G)$ and

$$U_i = O_p(\bigcap_{j=0}^{i} N_G(U_j))$$

for all i with $1 \leqslant i \leqslant r$. For such a chain c we denote $\bigcap_{j=0}^{m} N_G(U_j)$ by G_c and the length r of c by $|c|$. For a p-block S of G and a non-negative integer i, let $k(G_c, S, i)$ be the number of irreducible characters φ of G_c such that φ lies in a p-block of G_c inducing up to S and such that p^i is the highest power of p dividing $|G_c|/\varphi(1)$. Note that if two radical chains c and c' are G-conjugate, then $k(G_c, S, i) = k(G_{c'}, S, i)$ for all S and i. Dade proposed the following

Dade's Ordinary Conjecture. Let G be a finite group with $O_p(G) = 1$ and S a p-block of G of positive defect. Then

$$\sum (-1)^{|c|} k(G_c, S, i) = 0$$

for all $i \geqslant 0$, where $\sum$ denotes the sum over a set of representatives of the G-conjugacy classes of p-radical chains of G.

Remark 2.1.1. (1) A p-subgroup $P \leqslant G$ is *radical* if $P = O_p(N_G(P))$. By definition, the second term of a radical p-chain must be a radical p-subgroup. In general, however, the radical p-chains are not the chains of radical p-subgroups and do not constitute a simplicial complex;

(2) It can be shown that (see Proposition 3.3 in [KR] and Proposition 3.7 in [D1]) the alternating sum in the conjecture remains the same if the set of radical chains is replaced by any of the following: The Brown complex, Quillen complex, Bouc complex or Robinson complex.

2.2 Notation and Preliminary Lemmas

We fix some notation used throughout the thesis.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{C}$ the set of complex numbers. Let $\mathbb{C}^*$ be the multiplicative group of the nonzero complex numbers and $\mathbb{C}_n$ the unique cyclic subgroup of $\mathbb{C}$ of order n. Similarly we write $\mathbb{F}_q$ for the field of q elements and $\mathbb{F}_q^*$ for the multiplicative group of $\mathbb{F}_q$. We write $M_{m,n}(\mathbb{F})$ for the set of $m \times n$ matrices over a field $\mathbb{F}$.

For $x \geqslant 1$, let $[x] = \{1, 2, \cdots, n\}$ where n is the largest integer less than or equal to x. If $x < 1$, set $[x] = \emptyset$. For a subset $J \subseteq [n]$, denote the minimal (resp. maximal) member of J by $\min(J)$ (resp. $\max(J)$). By convention we set $\min(\emptyset) = \max(\emptyset) = 0$.

A *simplicial complex* is a set X whose elements are called vertices, together with a collection $\mathcal{X}$ of subsets of X called simplices, such that if $B \in \mathcal{X}$ and $A \subseteq B$, then $A \in \mathcal{X}$. A G-complex is a simplicial complex with a group G acting on the vertex set and mapping simplices to simplices.

If X is a topological space together with a continuous action of G on X, then X is called a G-space. Recall that a *G-homotopy* between two G-spaces X and X' is a continuous map $F : [0,1] \times X \to X'$ such that $F_t : X \to X'$ is a G-map for every $t \in [0,1]$ (where $F_t(x) = F(t,x)$). We may set $F_0 = f$, $F_1 = g$ and write $f \sim_G g$. Two G-spaces X and X' are *G-homotopy equivalent* if there are continuous G-maps $f : X \to X'$ and $g : X' \to X$ such that $f \circ g \sim_G 1_{X'}$ and $g \circ f \sim_G 1_X$. In particular a space is G-contractible if it is G-homotopy equivalent to a point. These concepts apply in an obvious way to G-complexes since the geometric realization $|\Delta|$ of a G-complex Δ is a G-space.

In this thesis we only discuss finite posets. We require that all the posets have a unique minimal element 0 and that all the chains involved start with 0, unless otherwise mentioned. The set of chains in $\mathcal{P}$ starting with 0, including the chain $\{0\}$, is denoted by $\Delta(\mathcal{P})$. For a chain

$$c \in \Delta(\mathcal{P}) : 0 = x_0 < x_1 < x_2 < \cdots < x_k,\ k \geqslant 0,$$

$|c| = k$ is called the *length* of c. By our convention $c = \{0\}$ is the unique chain in $\Delta(\mathcal{P})$ with $|c| = 0$. The *order complex* $\mathcal{O}(\mathcal{P})$ of $\mathcal{P}$ is the simplicial complex whose k-simplices are the chains in $\Delta(\mathcal{P})$ of length $k+1$. By convention $\{0\}$ is the unique (-1)-simplex of $\mathcal{O}(\mathcal{P})$. Recall $\mathcal{P}$ is a G-poset if $\mathcal{P}$ affords a G-action which preserves ordering. Two G-posets are *G-homotopy equivalent* if their order complexes are G-homotopy equivalent.

If G acts on $\mathcal{P}$, the stabilizer in G of $c \in \Delta(\mathcal{P})$ is $G_c = \cap_{1\leq i\leq k} G_{x_i}$. The orbit space of $\Delta(\mathcal{P})$ under G is denoted by $\Delta(\mathcal{P})/G$. Throughout this thesis, $c \in \Delta(\mathcal{P})/G$ always means that c is a representative of the G-orbit c^G. So $\sum_{c\in\Delta(\mathcal{P})/G}$ means the sum is taken over a set of representatives of $\Delta(\mathcal{P})/G$. Similarly if $H_i \leqslant G$, $i = 1, 2$, $\sum_{h\in H_1\backslash G/H_2}$ means the sum is taken over a set of representatives of $H_1\backslash G/H_2$. By abuse of notation, we denote by $\Delta([n])$ the set of subsets of $[n]$.

If V is a vector space, we set $\mathcal{P}(V)$ to be the poset of proper subspaces of V ordered by inclusion. The chains in $\mathcal{P}(V)$ are also called flags. For $c \in \Delta(\mathcal{P}(V))$, $\{\dim(U); 0 \neq U \in c\}$ is called the *type* of c. Clearly if G acts on V then G preserves type.

If G is a group, we write $\mathrm{Irr}(G)$ for the set of irreducible complex characters of G.

Let N be a normal subgroup of G and $\rho \in \mathrm{Irr}(N)$. We say $\varphi \in \mathrm{Irr}(G)$ *lies over* ρ if ρ is a direct summand of the restriction of φ to N, or equivalently $\langle \rho, \varphi|_N \rangle \neq 0$. G acts on $\mathrm{Irr}(N)$ in a natural way. The stabilizer in G of ρ is denoted by G_ρ or $N_G(\rho)$. We write $\mathrm{Irr}(G, \rho)$ for the set of $\varphi \in \mathrm{Irr}(G)$ lying over ρ. Let $\mathrm{Irr}^1(G)$ be the set of non-trivial irreducible complex characters of G.

For $X \subseteq \mathrm{Irr}(N)$, $\mathrm{Irr}(G, X)$ denotes the set of φ lying over some $\tau \in X$. In particular we denote by $\mathrm{Irr}^0(G, N) = \mathrm{Irr}(G, 1_N)$ the set of irreducible characters of G whose restriction to N is trivial, and denote by $\mathrm{Irr}^1(G, N) = \mathrm{Irr}(G, \mathrm{Irr}^1(N))$ the set of irreducible characters of G lying over a non-trivial irreducible character of N. Here 1_N is the trivial character of N. Clearly $\mathrm{Irr}(G)$ is the disjoint union of $\mathrm{Irr}^0(G, N)$ with $\mathrm{Irr}^1(G, N)$.

Similarly for $N_i \trianglelefteq G$ and $\rho_i \in \mathrm{Irr}(N_i)$, $i = 1, 2$, $\mathrm{Irr}(G, \rho_1, \rho_2)$ denotes the set of irreducible characters of G lying over ρ_1 and ρ_2. If $N_1 \leqslant N_2$ and $V = N_2/N_1$, we denote by $\mathrm{Irr}^0(G, V)$ the set of irreducible characters of G whose restriction to N_2 (and hence to V) is trivial, and denote by $\mathrm{Irr}^1(G, V)$ the set of irreducible characters of G whose restriction to N_1 is trivial but whose restriction to N_2 is non-trivial. In other words, $\varphi \in \mathrm{Irr}^1(G, V)$ if and only if $N_1 \leqslant \ker(\varphi) \not\geqslant N_2$.

In the above notation, we replace Irr by k to denote the size of the corresponding set of characters. For instance $k(G) = |\mathrm{Irr}(G)|$, $k^1(G, N) = |\mathrm{Irr}^1(G, N)|$, etc.

Let p be a prime. The *p-height* of $\varphi \in \mathrm{Irr}(G)$ is the exponent of p in the prime factorization of $\varphi(1)$. Similarly if q is a power of p, the q-height of φ is the exponent of q in the prime factorization of $\varphi(1)$. In this thesis, we always use $\mathrm{Irr}_d(G)$ to denote the set of irreducible characters of G whose q-height is d for some given q, and set $k_d(G) = |\mathrm{Irr}_d(G)|$.

If N is normal in G, the inflation operator

$$\mathrm{Inf}^G : \mathrm{Irr}(G/N) \to \mathrm{Irr}(G),\ \lambda \mapsto \mathrm{Inf}^G(\lambda)$$

is the map defined by letting $\mathrm{Inf}^G(\lambda)$ be the character such that $\mathrm{Inf}^G(\lambda)(g) = \lambda(gN)$ for $g \in G$.

Lemma 2.2.1. *Let G be a finite group and N a normal subgroup of G. Let $\tau \in$ $Irr(N)$, $H = G_\tau$ and $\varphi \in Irr(G, \tau)$.*

(1)

$$\varphi|_N = m \sum_{g \in G/H} \tau^g \tag{2.1}$$

for some $m \in \mathbb{N}$.

(2) *If $\tau \in X \subseteq Irr(N)$ is a subset of G-conjugates of τ, then*

$$Irr(G, X) = Irr(G, \tau).$$

Proof. Part (1) is Clifford Theorem. See for instance Theorem 3 in [BZ], Chapter 7.1. By definition, $\mathrm{Irr}(G, X)$ consists of the set of irreducible characters of G lying over some member of X. So by part (1), $\varphi \in \mathrm{Irr}(G)$ lies over some $\tau^g \in X$ if and only if it lies over τ. Therefore, part (2) holds.

□

Lemma 2.2.2. *Let G be a finite group and N a normal subgroup of G. Let $\tau \in Irr(N)$, $H = G_\tau$ and $H \leqslant P \leqslant G$:*

(1) *There is a 1-1 correspondence between $Irr(P, \tau)$ and $Irr(G, \tau)$ given by inducing characters. In particular, $k(G, \tau) = k(P, \tau)$. Moreover, if $q = p^e$ is a prime power, then for each $d \geqslant 0$, $k_d(G, \tau) = k_{d-d'}(P, \tau)$ where d' is the exponent of q in the prime factorization of $|G|/|P|$.*

(2) *Let Z be a normal subgroup of G contained in H and $\rho \in Irr(Z)$. Then the bijection in* (1) *restricts to a bijection between $Irr(P, \tau, \rho)$ and $Irr(G, \tau, \rho)$.*

Proof. Fix a P with $H \leqslant P \leqslant G$. Then $H = P_\tau$. Let $\varphi \in \mathrm{Irr}(P, \tau)$.

By Theorem (6c) in [BZ], Chapter 7,

$$\theta_P : \varphi \mapsto \varphi^P$$

defines a 1-1 correspondence from $\mathrm{Irr}(H, \tau)$ to $\mathrm{Irr}(P, \tau)$. As this holds for any $P \leqslant G$ with $H \leqslant P$, $\theta_G \circ \theta_P^{-1}$ is the desired 1-1 correspondence between $\mathrm{Irr}(P, \tau)$ and $\mathrm{Irr}(G, \tau)$. Therefore, $k(P, \tau) = k(G, \tau)$. Under this map, $\varphi \in \mathrm{Irr}(P, \tau)$ has q-height $d - d'$ where d' is as in the hypothesis if and only if φ^G has q-height d as $\varphi^G(1) = \varphi(1)|G|/|P|$. So part (1) holds. Also φ lies over $\rho \in \mathrm{Irr}(Z)$ if and only if φ^G does. So Part (2) holds.

□

Lemma 2.2.3. *Let G be a finite group and N a normal subgroup of G. Let $\tau \in Irr(N)$ be G-stable, that is $G = G_\tau$. Assume τ is extendable to G, i.e., there is $\psi \in Irr(G)$ with $\psi|_N = \tau$. Then there is a 1-1 correspondence between $Irr(G/N)$ and $Irr(G,\tau)$ given by $\lambda \mapsto Inf^G(\lambda)\psi$. In particular if q is a prime power, then for each $d \geqslant 0$, $k_{d-d'}(G/N) = k_d(G,\tau)$ where d' is the q-height of ψ.*

Proof. By definition, $\varphi \in \mathrm{Irr}(G,\tau)$ if and only if $\langle \varphi|_N, \tau\rangle > 0$, and hence if and only if $\langle \varphi, \tau^G\rangle > 0$ by Frobenieus reciprocity. But by (11.5) in [CR], $Inf^G(\lambda)\psi \in \mathrm{Irr}(G,\tau)$ for each $\lambda \in \mathrm{Irr}(G/N)$ and

$$\tau^G = \sum_{\lambda \in \mathrm{Irr}(G/N)} \lambda(1)(\mathrm{Inf}^G(\lambda)\psi).$$

In other words,

$$\mathrm{Irr}(G,\tau) = \{\mathrm{Inf}^G(\lambda)\psi \mid \lambda \in \mathrm{Irr}(G/N)\}.$$

Moreover, $(\mathrm{Inf}^G(\lambda)\psi)(1) = \lambda(1)\psi(1)$, so λ has q-height $d - d'$ with d' as in the hypothesis if and only if $\mathrm{Inf}^G(\lambda)\psi$ has q-height d. Therefore, the lemma holds.

□

Lemma 2.2.4. *Let $G = AB$ be a finite group with A a normal abelian subgroup and B a complement to A. Let $\tau \in Irr(A)$, $H = G_\tau$, and $K = B_\tau$.*

(1) $H = AK$;

(2) τ can be extended to H, i.e., $\tau = \psi|_A$ for some $\psi \in Irr(H)$;

(3) There is a 1-1 correspondence between $Irr(K)$ and $Irr(H,\tau)$ given by Lemma 2.2.3. In particular if q is a prime power, then for each $d \geqslant 0$, $k_d(K) = k_d(H,\tau)$.

Proof. Part(1) is obvious. Part (2) follows from Proposition (11.8) in [CR]. Part (3) follows from part (2) and Lemma 2.2.3 and the fact that ψ is of degree 1, which is true because A is abelian and hence τ is of degree 1. □

Lemma 2.2.5. *Let G_i, $i = 1, 2$, be a finite group, $Z_i \leqslant Z(G_i)$ and $Z \leqslant Z_1 \times Z_2$. There is a natural 1-1 correspondence between $Irr(G_1) \times Irr(G_2)$ and $Irr(G_1 \times G_2)$ given by*

$$(\varphi_1, \varphi_2) \mapsto \varphi = \varphi_1 \varphi_2,$$

where φ is the character afforded by the tensor product of the modules afforded by φ_1 and φ_2 with $\varphi(g) = \varphi_1(g_1)\varphi_2(g_2)$ for $g = (g_1, g_2) \in G_1 \times G_2$. In particular $\varphi(1) = \varphi_1(1)\varphi_2(1)$. Moveover, if φ_i lies over $\rho_i \in Irr(Z_i)$, $i = 1, 2$, then φ lies over $\rho = (\rho_1\rho_2)|_Z$. If q is a prime power and $d \geqslant 0$, then

$$k_d(G_1 \times G_2, \rho) = \sum_{\substack{\rho_1, \rho_2 \\ \rho = (\rho_1\rho_2)|_Z}} \sum_{\substack{d_1, d_2 \\ d = d_1 + d_2}} k_{d_1}(G_1, \rho_1) k_{d_2}(G_2, \rho_2).$$

The proof is trivial. As both ρ_1 and ρ_2 are linear, so is ρ. In this case write $\rho = \rho_1\rho_2$ by abuse of notation.

The following lemma exhibits an elementary technique for simplifying the computation in the later chapters of the thesis.

For $i = 1, 2$, let G^i be a group with $Z^i \leqslant Z(G^i)$. Let $I_i = [n_i]$ and $\{G^i_J; J \subseteq I_i\}$ a collection of subgroups of G^i containing Z^i. Let $G = G^1 \times G^2$, $Z \leqslant Z^1 \times Z^2$, and $I = [n_1 + n_2]$. For $J \subseteq I$, let $G_J = G^1_{J(1)} \times G^2_{J(2)}$ where

$$J(1) = J(\leqslant n_1), \quad J(2) = \{\, j - n_1 \mid n_1 < j \in J\}. \tag{2.2}$$

Finally let N be a normal subgroup of G^1, and $X_1, X_2 \subseteq \mathrm{Irr}(N)$. Let q be a prime power and $d \geqslant 0$. Recall $k_d(G^i, X_i, \rho_i)$ counts the number of irreducible characters of G^i lying over some character in X_i as well as over ρ_i whose q-height is d.

Lemma 2.2.6. *Assume for all $d_1 \geqslant 0$ and all $\rho_1 \in Irr(Z_1)$,*

$$\sum_{J_1 \subseteq I_1} (-1)^{|J_1|} k_{d_1}(G^1_{J_1}, X_1, \rho_1) = \sum_{J_1 \subseteq I_1} (-1)^{|J_1|} k_{d_1}(G^1_{J_1}, X_2, \rho_1).$$

Then for all $d \geqslant 0$ and all $\rho \in Irr(Z)$,

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(G_J, X_1, \rho) = \sum_{J \subseteq I} (-1)^{|J|} k_d(G_J, X_2, \rho).$$

Proof. By Lemma 2.2.5, there is a natural 1-1 correspondence

$$\lambda : \mathrm{Irr}(G) \to \mathrm{Irr}(G^1) \times \mathrm{Irr}(G^2).$$

By hypothesis, G^2 centralizes N, so G^2 acts trivially on X_1 and X_2. Thus λ restricts to a 1-1 correspondence between $\mathrm{Irr}(G, X_i)$ and $\mathrm{Irr}(G^1, X_i) \times \mathrm{Irr}(G^2)$, $i = 1, 2$. By construction of λ, we have for each $d \geqslant 0$, $\rho \in \mathrm{Irr}(Z)$ and for $i = 1, 2$,

$$k_d(G, X_i, \rho) = \sum_{\substack{d_1, d_2 \\ d_1 + d_2 = d}} \sum_{\substack{\rho_1, \rho_2 \\ \rho_1 \rho_2 = \rho}} k_{d_1}(G^1, X_i, \rho_1) k_{d_2}(G^2, \rho_2).$$

Next $J \mapsto (J(1), J(2))$ where $(J(1), J(2))$ is defined as in equation (2.2), defines a natural bijection of $\Delta(I)$ with $\Delta(I_1) \times \Delta(I_2)$ such that $|J| = |J(1)| + |J(2)|$. Similarly there is also a bijection of $\mathrm{Irr}(G_J, X_i)$ with $\mathrm{Irr}(G^1_{J(1)}, X_i) \times \mathrm{Irr}(G^2_{J(2)})$, $i = 1, 2$. Therefore,

$$\begin{aligned}
\sigma &= \sum_{J \subseteq I} (-1)^{|J|} k_d(G_J, X_1, \rho) \\
&= \sum_{J \subseteq I} \sum_{\substack{d_1, d_2 \\ d_1 + d_2 = d}} \sum_{\substack{\rho_1, \rho_2 \\ \rho_1 \rho_2 = \rho}} (-1)^{|J(1)| + |J(2)|} k_{d_1}(G^1_{J(1)}, X_1, \rho_1) k_{d_2}(G^2_{J(2)}, \rho_2) \\
&= \sum_{\substack{d_1, d_2 \\ d_1 + d_2 = d}} \sum_{\substack{\rho_1, \rho_2 \\ \rho_1 \rho_2 = \rho}} \Big(\sum_{J(1) \subseteq I_1} (-1)^{|J(1)|} k_{d_1}(G^1_{J(1)}, X_1, \rho_1) \Big) \Big(\sum_{J(2) \subseteq I_2} (-1)^{|J(2)|} k_{d_2}(G^2_{J(2)}, \rho_2) \Big).
\end{aligned}$$

But by hypothesis

$$\sum_{J(1) \subseteq I_1} (-1)^{|J(1)|} k_{d_1}(G^1_{J(1)}, X_1, \rho_1) = \sum_{J(1) \subseteq I_1} (-1)^{|J(1)|} k_{d_1}(G^1_{J(1)}, X_2, \rho_1),$$

hence

$$\begin{aligned}\sigma &= \sum_{\substack{d_1,d_2\\ d_1+d_2=d}} \sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}} \Big(\sum_{J(1)\subseteq I_1} (-1)^{|J(1)|} k_{d_1}(G^1_{J(1)}, X_2, \rho_1)\Big)\Big(\sum_{J(2)\subseteq I_2} (-1)^{|J(2)|} k_{d_2}(G^2_{J(2)}, \rho_2)\Big) \\ &= \sum_{J\subseteq I} \sum_{\substack{d_1,d_2\\ d_1+d_2=d}} \sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}} (-1)^{|J(1)|+|J(2)|} k_{d_1}(G^1_{J(1)}, X_2, \rho_1) k_{d_2}(G^2_{J(2)}, \rho_2) \\ &= \sum_{J\subseteq I} (-1)^{|J|} k_d(G_J, X_2, \rho).\end{aligned}$$

The lemma is proved. □

2.3 Partition of Integers

We need to establish some notation involving the partitions of integers. Let $n \in \mathbb{N}$. We say $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ with $\lambda_i \geqslant \lambda_{i+1} > 0$ for all i is a *partition* of n if $\sum_{i=1}^{l} \lambda_i = n$. In this situation we write $\mu \vdash n$ and $|\mu| = n$. λ_i are called the *parts* of μ. By convention we set (0) to be the unique partition of 0. The *diagram* $[\lambda]$ of λ is defined to be the set of ordered pairs (i, j) with $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant \lambda_i$, and we always regard $[\lambda]$ as an array of "nodes" as in the following example:

$$\lambda = (5,3,2) \qquad [\lambda] = \begin{matrix} X & X & X & X & X \\ X & X & X & & \\ X & X & & & \end{matrix}$$

Let λ'_j be the number of nodes in column j of $[\lambda]$. Then $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$ is a partition and called the *conjugate partition* of λ. For instance, if $\lambda = (5,3,2)$, then $\lambda' = (3,3,2,1,1)$. Clearly the diagram of λ' is obtained by interchanging the rows and columns of the diagram of λ. Therefore, $(\lambda')' = \lambda$. Notice $|\lambda| = |\lambda'|$.

Let $\mu = (a_1^{m_1}, a_2^{m_2}, \cdots, a_r^{m_r})$ be a partition of n. Here $a_1 > a_2 > \cdots > a_r > 0$ and $m_r > 0$ is the multiplicity of a_i as a part of μ. So $\sum_i a_i m_i = n$. We introduce some parameters associated to μ. Let $l(\mu) = \sum_{1\leqslant i\leqslant r} m_i$ be the number of parts of μ, and let $\delta(\mu) = r$ be the number of *distinct* parts of μ. Also define

$$n(\mu) = \sum_{i=1}^{r} m_i \binom{a_i}{2}.$$

Here $\binom{x}{2} = 0$ if $x < 2$.

Lemma 2.3.1. *(1) For $a, b \geqslant 0$,*

$$\binom{a+b}{2} = \binom{a}{2} + \binom{b}{2} + ab.$$

So in particular

$$\binom{a+b}{2} \geqslant \binom{a}{2} + \binom{b}{2},$$

and the equality holds if and only if $ab = 0$.

(2) If $\mu \vdash n$, then $n(\mu) \leqslant \binom{n}{2}$, and the equality holds if and only if $\mu = (n)$.

Proof.

$$\begin{aligned} 2\binom{a+b}{2} &= (a+b)(a+b-1) = (a+b)^2 - (a+b) \\ &= (a^2 - a) + (b^2 - b) + 2ab = 2(\binom{a}{2} + \binom{b}{2} + ab). \end{aligned}$$

So part (1) holds. If $\mu = (\lambda_1, \lambda_2, \ldots) \vdash n$, then by part (1),

$$n(\mu) = \sum_i \binom{\lambda_i}{2} \leqslant \binom{\sum_i \lambda_i}{2} = \binom{n}{2},$$

and the equality holds if and only if all but one of the parts of μ are 0, that is, if and only if $\mu = (n)$. So part (2) is true.

□

We denote the multiplicative subgroup $\mathbb{F}_q^*$ by $\mathbb{H}_{q-1}$. Similarly we denote $\mathbb{F}_{q^2}^*$ by $\mathbb{H}_{q^2-1}$, and the cyclic subgroup of order $q+1$ in $\mathbb{H}_{q^2-1}$ by $\mathbb{H}_{q+1}$.

We define a function $\alpha(\mu, a)$, where μ is a partition and a is an element of $\mathbb{H}_{q-1}$. If μ is written as above, then

$$\alpha(\mu, a) = \#\{ (x_1, x_2, \cdots, x_r) \in \mathbb{H}_{q-1}^r \mid (-1)^n \prod_{i=1}^{r} x_i^{a_i} = a \}$$

where -1 is the additive inverse of 1 in $\mathbb{F}_q$. Notice $-1 \in \mathbb{H}_{q-1}$.

Lemma 2.3.2. *(1)*

$$\sum_{a \in \mathbb{H}_{q-1}} \alpha(\mu, a) = (q-1)^r;$$

(2) In the above notation, let $m = \gcd(a_1, a_2, \cdots, a_r)$. *Then*

$$\alpha(\mu, a) = (q-1)^{r-1}\xi,$$

where ξ *is the number of solutions in* $\mathbb{H}_{q-1}$ *to the equation* $x^m = a$.

This is Lemma 2.5 in Olsson and Uno's paper [OU].

Similarly we define a function $\beta(\mu, a)$ where μ is a partition and $a \in \mathbb{H}_{q+1}$. If μ is as above, then

$$\beta(\mu, a) = \#\{ (x_1, x_2, \cdots, x_r) \in \mathbb{H}_{q+1}^r \mid (-1)^n \prod_{i=1}^{r} x_i^{a_i} = a \}$$

where -1 denotes the additive inverse of 1 in $\mathbb{F}_{q^2}$. Observe $-1 \in \mathbb{H}_{q+1}$.

Lemma 2.3.3. *(1)*

$$\sum_{a \in \mathbb{H}_{q+1}} \beta(\mu, a) = (q+1)^r;$$

(2) In the above notation, let $m = \gcd(a_1, a_2, \cdots, a_r)$. *Then*

$$\beta(\mu, a) = (q+1)^{r-1}\xi,$$

where ξ is the number of solutions in $\mathbb{H}_{q+1}$ to the equation $x^m = a$.

The proof is identical to the proof of Lemma 2.3.2.

Lemma 2.3.4. *Let $\mu = (a_i^{m_i})$ as above.*

(1) Let $a \in \mathbb{H}_{q-1}$.

$$\alpha(\mu, a) = \#\{ (x_1, x_2, \cdots, x_r) \in \mathbb{H}_{q-1}^r \mid \prod_{i=1}^{r} x_i^{a_i} = a \}.$$

(2) Let $a \in \mathbb{H}_{q+1}$.

$$\beta(\mu, a) = \#\{ (x_1, x_2, \cdots, x_r) \in \mathbb{H}_{q+1}^r \mid \prod_{i=1}^{r} x_i^{a_i} = a \}.$$

Proof. The proof of part (1) and (2) are identical. Here we prove part (2). Define

$$f : \mathbb{H}_{q+1}^r \to \mathbb{H}_{q+1}^r, \quad (x_1, \cdots, x_r) \mapsto (y_1, \cdots, y_r)$$

where $y_i = (-1)^{m_i} x_i$, $1 \leqslant i \leqslant r$. Recall that $-1 \in \mathbb{H}_{q+1}$. It is easy to check that this is a well defined bijection. Observe

$$(-1)^n \prod_{i=1}^{r} x_i^{a_i} = \prod_{i=1}^{r} y_i^{a_i}.$$

So $(-1)^n \prod_{i=1}^{r} x_i^{a_i} = a$ if and only if $\prod_{i=1}^{r} y_i^{a_i} = a$. Therefore, by definition of β,

$$\beta(\mu, a) = \#\{ (y_1, y_2, \cdots, y_r) \in \mathbb{H}_{q+1}^r \mid \prod_{i=1}^{r} y_i^{a_i} = a \}.$$

Done.

□

Example 2.3.5. Assume $\mu = (a_i^{m_i}) \vdash n$ with $\delta(\mu) = r$. Set $l(\mu) = l$. Let $s > l$ and $k \in \mathbb{N}$. So (k^s) is a partition of sk. Define $\lambda = \mu + (k^s) = (b_j^{m_j})$ such that

$$b_j = a_j + k \text{ for } 1 \leqslant j \leqslant r;\ b_{r+1} = k, \text{ and } m_{r+1} = s - l.$$

Then $\lambda \vdash (n + sk)$ with $\delta(\lambda) = r + 1$, $l(\lambda) = s$, and by Lemma 2.3.1,

$$n(\lambda) - n(\mu) = \sum_{j=1}^{r} m_j(\binom{b_j}{2} - \binom{a_j}{2}) + (s - l)\binom{k}{2} = s\binom{k}{2} + kn.$$

Observe $\mu \mapsto \mu + (k^s)$ defines a bijection from the set of partitions μ of n with $l(\mu) < s$ to the set of partitions λ of $n + sk$ with $l(\lambda) = s$ and $\min(\lambda) = k$, where $\min(\mu)$ denotes the minimal part of λ.

Lemma 2.3.6. *Let $a \in \mathbb{H}_{q+1}$. Then for (k^s), μ, and $\lambda = \mu + (k^s)$ as above,*

$$\sum_{\substack{a_1, a_2 \in \mathbb{H}_{q+1} \\ a_1 a_2 = a}} \beta((k), a_1)\beta(\mu, a_2) = \beta(\lambda, a).$$

Proof. Define

$$f : \mathbb{H}_{q+1}^{r+1} \to \mathbb{H}_{q+1}^{r+1}, \quad (x_1, \cdots, x_r, x_{r+1}) \mapsto (x_1, \cdots, x_r, \prod_{j=1}^{r+1} x_j).$$

This is a well defined bijection. Set $y = \prod_{j=1}^{r+1} x_j$. As $b_j = a_j + k$ for $1 \leqslant j \leqslant r$,

$$\prod_{j=1}^{r+1} x_j^{b_j} = (\prod_{j=1}^{r} x_j^{a_j}) y^k.$$

So $\prod_{j=1}^{r+1} x_j^{b_j} = a$ if and only if $y^k = a_1$ and $\prod_{j=1}^{r} x_j^{a_j} = a_2$ for some $(a_1, a_2) \in \mathbb{H}_{q+1}^2$ with $a_1 a_2 = a$. The lemma then follows from Lemma 2.3.4.

□

Finally we define another function so that we can deal with α and β together. If a is an element of $\mathbb{H}_{q+1} \leqslant \mathbb{H}_{q^2-1}$, define

$$\bar{\beta}(\mu, a) = \sum_{\substack{b \in \mathbb{H}_{q^2-1} \\ b^{q-1} = a}} \alpha(\mu, b).$$

For the rest of this section, we fix $n \in \mathbb{N}$. Then for $0 \leqslant n' \leqslant n$, a partition μ of n' may be denoted by $\mu = (t^{m_t})$ with $1 \leqslant t \leqslant n$ and $m_t \geqslant 0$ such that $n' = \sum_{t=1}^{n} tm_t$.

If $r_i \in \mathbb{N}$, $0 \leqslant n_i \leqslant n$, $i = 1, 2$, such that $r_1n_1 + r_2n_2 \leqslant n$, and $\mu_i = (t^{m_{i,t}})$ is a partition of n_i, then we define $\mu = r_1\mu_1 \cup r_2\mu_2$ to be the partition $\mu = (t^{r_1m_{1,t}+r_2m_{2,t}})$ of $r_1n_1 + r_2n_2$. Check $n(\mu) = r_1n(\mu_1) + r_2n(\mu_2)$. The decomposition $\mu = r_1\mu_1 \cup r_2\mu_2$ is parameterized by the $(2 \times n)$-matrix

$$A = A(\mu_1, \mu_2) = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \end{bmatrix}, \qquad \text{where } m_{ij} \geqslant 0,$$

such that $r_1m_{1t} + r_2m_{2t} = m_t$ for all t. Notice $\mu_i = 0$ if and only if $m_{it} = 0$ for all t.

Given such a matrix A define the shadow $(2 \times n)$-matrix $C = C(\mu_1, \mu_2) = (c_{it})$ of A by

$$c_{it} = \begin{cases} 1 & \text{if } m_{it} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

That is $c_{it} = 0$ if and only if $m_{it} = 0$. Observe that $\sum_{t=1}^{n} c_{it} = \delta(\mu_i)$, $i = 1, 2$. Set $c = c(\mu_1, \mu_2) = \sum_{t=1}^{n} c_{1t}c_{2t}$.

Lemma 2.3.7. *Assume $r_i \in \mathbb{N}$, $0 \leqslant n_i \leqslant n$, $i = 1, 2$, such that $r_1n_1 + r_2n_2 \leqslant n$, $\mu_i = (t^{m_{i,t}})$ is a partition of n_i, and $\mu = r_1\mu_1 \cup r_2\mu_2$. Let $a \in \mathbb{H}_{q+1}$. Then*

$$\sum_{\substack{a_1, a_2 \in \mathbb{H}_{q+1} \\ a_1a_2 = a}} \bar{\beta}(\mu_1, a_1)\beta(\mu_2, a_2) = (q-1)^{\delta(\mu_1)}(q+1)^c\beta(\mu, a).$$

Proof. As $\mu_i = (t^{m_{it}})$, for $b \in \mathbb{H}_{q^2-1}$, we may deduce from Lemma 2.3.4 that

$$\alpha(\mu_1, b) = \#\{\, (x_1, x_2, \cdots, x_n) \in \mathbb{H}_{q^2-1}^n \mid \prod_{t=1}^{n} (x_t)^t = b, \text{ and } x_t = 1 \text{ if } m_{1t} = 0 \,\}.$$

By definition of $\bar{\beta}$, for $a_1 \in \mathbb{H}_{q+1}$, we have

$$\bar{\beta}(\mu_1, a_1) = \sum_{\substack{b \in \mathbb{H}_{q^2-1} \\ b^{q-1}=a_1}} \#\{\ (x_1, x_2, \cdots, x_n) \in \mathbb{H}^n_{q^2-1} \mid \prod_{t=1}^{n} x_t^t = b, \text{ and } x_t = 1 \text{ if } m_{1t} = 0\ \}$$

$$= \#\{\ (x_1, x_2, \cdots, x_n) \in \mathbb{H}^n_{q^2-1} \mid \prod_{t=1}^{n} x_t^{(q-1)t} = a_1, \text{ and } x_t = 1 \text{ if } m_{1t} = 0\ \}.$$

Similarly for $a_2 \in \mathbb{H}_{q+1}$, we deduce from Lemma 2.3.4 that

$$\beta(\mu_1, a_2) = \#\{\ (y_1, y_2, \cdots, y_n) \in \mathbb{H}^n_{q+1} \mid \prod_{t=1}^{n} y_t^t = a_2, \text{ and } y_t = 1 \text{ if } m_{2t} = 0\ \}.$$

For $x_t \in \mathbb{H}_{q^2-1}$ and $y_t \in \mathbb{H}_{q+1}$, set $z_t = x_t^{q-1} y_t$. So $z_t \in \mathbb{H}_{q+1}$. We then conclude

$$\sum_{\substack{a_1, a_2 \\ a_1 a_2 = a}} \bar{\beta}(\mu_1, a_1)\beta(\mu_2, a_2) = \#\{\ (x_1, x_2 \cdots, x_n) \times (y_1, y_2, \cdots, y_n) \in \mathbb{H}^n_{q^2-1} \times \mathbb{H}^n_{q+1} \mid$$
$$\prod_{t=1}^{n} z_t^t = a; \text{ and } z_t = x_t^{q-1} y_t, x_t = 1 \text{ if } m_{1t} = 0,\ y_t = 1 \text{ if } m_{2t} = 0\ \}. \tag{2.3}$$

For a fixed choice of $z_t \in \mathbb{H}_{q+1}$, the number of choices of $(x_t, y_t) \in \mathbb{H}_{q^2-1} \times \mathbb{H}_{q+1}$ such that $z_t = x_t^{q-1} y_t$ is

$$\begin{cases} (q-1)(q+1), & \text{if } c_{1t} = c_{2t} = 1; \\ q-1, & \text{if } c_{1t} = 1 \text{ and } c_{2t} = 0; \\ 1, & \text{if } c_{1t} = 0 \text{ and } c_{2t} = 1; \\ 1, & \text{if } c_{1t} = c_{2t} = 0 \text{ and } z_t = 1; \\ 0, & \text{if } c_{1t} = c_{2t} = 0 \text{ and } z_t \neq 1. \end{cases}$$

In other words, the number is $(q-1)^{c_{1t}}(q+1)^{c_{1t}c_{2t}}$ unless $c_{1t} = c_{2t} = 0$ and $z_t \neq 1$. So for a fixed choice of $(z_1, \cdots, z_n)$, the number of choices of $(x_1, \cdots, x_n) \times (y_1, \cdots, y_n)$

is

$$\prod_{t=1}^{n}(q-1)^{c_{1t}}(q+1)^{c_{1t}c_{2t}} = (q-1)^{\delta(\mu_1)}(q+1)^{c}$$

unless $c_{1t} = c_{2t} = 0$ and $z_t \neq 1$ for some t, where the number of choices is 0. Moreover, $z_t = 1$ if $m_{1t} = m_{2t} = 0$, or equivalently $m_t = r_1 m_{1t} + r_2 m_{2t} = 0$. Therefore, the right-hand side of equation (2.3) is

$$(q-1)^{\delta(\mu_1)}(q+1)^{c} \cdot \#\{\ (z_1, z_2, \cdots, z_n) \in \mathbb{H}_{q+1}^{n} \mid \prod_{t=1}^{n} z_t^t = a, \text{ and } z_t = 1 \text{ if } m_t = 0\ \}$$

$$= (q-1)^{\delta(\mu_1)}(q+1)^{c}\beta(\mu, a).$$

This completes the proof.

□

We close the section by proving the following technical lemma.

Lemma 2.3.8. *Let μ be a partition of n. Then*

$$\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2}} q^{2(l(\mu_1)-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)} = q^{l(\mu)-\delta(\mu)}. \tag{2.4}$$

Proof. Recall pairs (μ_1, μ_2) of partitions $\mu_i = (t^{m_{it}})$ of n_i with $2n_1 + n_2 = n$ are parameterized by the $(2 \times n)$-matrices $A = A(\mu_1, \mu_2)$. Given such an A, define

$$P_A(q) = \prod_{t=1}^{n} q^{2(m_{1t}-c_{1t})}(q-1)^{c_{1t}}(q+1)^{c_{1t}c_{2t}}.$$

Then as $\sum_{t=1}^{n}(m_{1t} - c_{1t}) = l(\mu_1) - \delta(\mu_1)$, $\sum_{t=1}^{n} c_{1t} = \delta(\mu_1)$ and $c = \sum_t c_{1t}c_{2t}$, we have

$$P_A(q) = q^{2(l(\mu_1)-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)}.$$

Consequently equality (2.4) is equivalent to

$$\sum_A P_A(q) = q^{l(\mu)-\delta(\mu)} \tag{2.5}$$

where the sum is taken over all the possible matrices parameterizing pairs (μ_1, μ_2) with $\mu = 2\mu_1 \cup \mu_2$. Define

$$h_0(q) = h_1(q) = 1;$$
$$h_{2r}(q) = 1 + \sum_{j=1}^{r-1} q^{2(j-1)}(q^2-1) + q^{2(r-1)}(q-1);$$
$$h_{2r+1}(q) = 1 + \sum_{j=1}^{r} q^{2(j-1)}(q^2-1)$$

for $r \in \mathbb{N}$. We claim that

(i) For each t,

$$h_{m_t}(q) = \sum_{0 \leqslant m_{1t} \leqslant m_t/2} q^{2(m_{1t}-c_{1t})}(q-1)^{c_{1t}}(q+1)^{c_{1t}c_{2t}}.$$

(ii)

$$\sum_A P_A(q) = \prod_{t=1}^{n} h_{m_t}(q)$$

where the sum is taken over all the possible matrices parameterizing pairs (μ_1, μ_2) with $\mu = 2\mu_1 \cup \mu_2$.

(iii) $h_m(q) = q^{m-1}$ for all $m \geqslant 1$.

Then

$$\sum_A P_A(q) = \prod_{t=1}^{n} h_{m_t}(q) = \prod_{t=1}^{n} q^{m_t-1} = q^{l(\mu)-\delta(\mu)}.$$

Therefore, equality (2.5) and hence the lemma is proved.

So it remains to prove that the claims are true. Fix $1 \leqslant t \leqslant n$. Assume $m_t \leqslant 1$. Then $m_{1t} = 0$ as $2m_{1t} + m_{2t} = m_t$ and $m_{2t} \geqslant 0$. Consequently $c_{1t} = 0$. So by definition of h,

$$h_{m_t}(q) = 1 = \sum_{0 \leqslant m_{1t} \leqslant m_t/2} q^{2(m_{1t}-c_{1t})}(q-1)^{c_{1t}}(q+1)^{c_{1t}c_{2t}}$$

for $m_t \leqslant 1$.

Assume $m_t = 2r$ for some $r \in \mathbb{N}$. Then $0 \leqslant m_{1t} \leqslant r$. If $m_{1t} = 0$, then $c_{1t} = 0$. So the contribution to the sum in (i) from the term $m_{1,t} = 0$ is 1. If $1 \leqslant m_{1t} < r$, then $c_{1t} = 1$ and $m_{2t} > 0$ and hence $c_{2t} = 1$. So the contribution from each such term is $q^{2(m_{1,t}-1)}(q^2-1)$. Finally if $m_{1t} = r$, then $m_{2t} = c_{2t} = 0$. So the contribution from this term is $q^{2(r-1)}(q-1)$. Therefore,

$$\begin{aligned}\sum_{0 \leqslant m_{1t} \leqslant m_t/2} & q^{2(m_{1t}-c_{1t})}(q-1)^{c_{1t}}(q+1)^{c_{1t}c_{2t}} \\ &= 1 + \sum_{m_{1t}=1}^{r-1} q^{2(m_{1t}-1)}(q-1)(q+1) + q^{2(r-1)}(q-1) \\ &= h_{2r}(q).\end{aligned}$$

Finally assume $m_t = 2r+1$ for some $r \in \mathbb{N}$. Then $0 \leqslant m_{1t} \leqslant r$. If $m_{1t} = 0$, then $c_{1t} = 0$. So the contribution to the sum in (i) from the term $m_{1,t} = 0$ is 1. If $1 \leqslant m_{1t} \leqslant r$, then $c_{1t} = 1$ and $m_{2t} > 0$ and hence $c_{2t} = 1$. So the contribution to the sum from each such term is $q^{2(m_{1,t}-1)}(q-1)(q+1)$. Therefore,

$$\begin{aligned}\sum_{0 \leqslant m_{1t} \leqslant m_t/2} q^{2(m_{1t}-c_{1t})}(q-1)^{c_{1t}}(q+1)^{c_{1t}c_{2t}} &= 1 + \sum_{m_{1t}=1}^{r} q^{2(m_{1t}-1)}(q-1)(q+1) \\ &= h_{2r+1}(q).\end{aligned}$$

Therefore, claim (i) holds.

As each m_t is fixed, the choice of a matrix A parameterizing (μ_1, μ_2) with $\mu = 2\mu_1 \cup \mu_2$ is completely determined by the choice of $(m_{11}, \cdots, m_{1n})$ with $0 \leqslant m_{1t} \leqslant m_t/2$, and the choice of m_{1i} is independent of the choice of m_{1j} for $i \neq j$. Therefore, by definition of $P_A(q)$,

$$\begin{aligned}\sum_A P_A(q) &= \sum_{(m_{11},\cdots,m_{1n})} \prod_{t=1}^{n} q^{2(m_{1t}-c_{1t})}(q-1)^{c_{1t}}(q+1)^{c_{1t}c_{2t}} \\ &= \prod_{t=1}^{n} \sum_{0\leqslant m_{1t}\leqslant m_t/2} q^{2(m_{1t}-c_{1t})}(q-1)^{c_{1t}}(q+1)^{c_{1t}c_{2t}} \\ &= \prod_{t=1}^{n} h_{m_t}(q);\end{aligned}$$

That is, claim (ii) holds.

Finally $h_1(q) = 1 = q^0$,

$$\begin{aligned}h_{2r}(q) &= 1 + (q^2-1)\frac{q^{2(r-1)}-1}{q^2-1} + q^{2(r-1)}(q-1) \\ &= q^{2(r-1)}(1+q-1) = q^{2r-1},\end{aligned}$$

and

$$h_{2r+1}(q) = 1 + (q^2-1)\frac{q^{2r}-1}{q^2-1} = q^{2r}.$$

So claim (iii) is true. Therefore, the proof is complete.

□

Chapter 3 Refinement and General Discussion

3.1 A Refinement for The Finite Groups of Lie Type

Let $G = G(q)$ be a finite group of Lie type of rank n over a finite field of order $q = p^e$. That is, G is either an ordinary Chevalley group defined over $\mathbb{F}_q$ or the subgroup of fixed points of an ordinary Chevalley group defined over $\mathbb{F}_{q^{|\sigma|}}$ or $\mathbb{F}_q$ by some automorphism σ. Let (G, B, N, S) be a Tits system for some root system Σ, $H = B \cap N$, and $B = UH$ where $U = O_p(B)$. Set $I = [n]$. Let $\{P_i; i \in I\}$ be the set of maximal parabolic subgroups of G over B and $P_J = \bigcap_{j \in J} P_j$ for $J \subseteq I$. In particular $P_I = B$. By convention $P_\emptyset = G$. For $J \subseteq I$, set $U_J = O_p(P_J)$. Let P_J^- be the parabolic subgroup opposite to P_J and $U_J^- = O_p(P_J^-)$ the opposite unipotent radical. Clearly $P_J = U_J L_J$ where $L_J = P_J \cap P_J^-$ is a Levi factor of P_J. The following is an example (when the base field $\mathbb{F}$ is finite), which we will refer to from time to time.

Example 3.1.1. Let V be an n-dimensional vector space over a field $\mathbb{F}$ and $G = GL(V)$. Then G is a group of rank $n-1$ and in this case $I = [n-1]$. We fix a basis $\{e_j; j \in [n]\}$ for V. Set $V_0 = 0$, $V_j = \langle e_i; 1 \leqslant i \leqslant j\rangle$ for $1 \leqslant j \leqslant n-1$, and $V_n = V$. For

$$\emptyset \neq J \subseteq I: \quad j_1 < j_2 < \cdots < j_s,$$

let c_J be the flag

$$0 < V_{j_1} < V_{j_2} < \cdots < V_{j_s}.$$

We may choose $B = N_G(c_I)$ and $P_J = N_G(c_J)$. Consequently for $1 \leqslant r \leqslant n-1$, $P_r^- = N_G(V_r^-)$ where $V_r^- = \langle e_i; r+1 \leqslant i \leqslant n \rangle$.

Let Δ be the set of chains of the poset on $\{U_J; J \subseteq I\}$ ordered by inclusion.

Lemma 3.1.2. *Δ is a set of representatives of the G-orbits on radical p-chains of G. The stabilizer in G of $c \in \Delta$ is the stabilizer of the final term of c.*

Proof. If $0 < U_{J_1} < \cdots < U_{J_m}$ is a chain in $\mathcal{P}$, then $N_G(U_{J_i}) = P_{J_i} \leqslant P_{J_j}$ for $i \geqslant j$, so the second statement holds. Let $c: U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_r$ be a radical p-chain of G. So $U_0 = O_p(G) = 1$. By definition, U_1 is a radical p-subgroup of G. However, it is well known that the radical p-subgroups are precisely the unipotent radicals of the parabolics and each parabolic subgroup is conjugate to a unique member of $\{P_J; J \subset I\}$. Therefore, replacing c by a G-conjugate if necessary, we may assume $U_1 = U_{J_1}$ for some J_1, $J_1 \subseteq I$. Set $P = P_{J_1} = N_G(U_1)$.

Next by definition U_2 is a radical p-subgroup of P and therefore, U_2/U_1 is a radical p-subgroup of P/U_1. But $\bar{P} = P/U_1$ is a finite group of Lie type, possibly with a disconnected diagram, and the radical p-subgroups of $\bar{P}$ are the images in $\bar{P}$ of the radical p-subgroups of G contained in P, which are conjugates of U_J for $J_1 \subseteq J \subseteq I$. Therefore, replacing c by a P-conjugate if necessary, we may assume $U_2 = O_p(P_{J_2})$ for some J_2, $J_1 \subseteq J_2 \subseteq I$.

By definition U_i is a radical p-subgroup of $\cap_{1 \leqslant j \leqslant i-1} N_G(U_i)$ for all i. We may proceed by induction to conclude that c can be conjugated to a chain of $\mathcal{P}$. As distinct parabolics over B are not conjugate, no two chains in $\mathcal{P}$ are conjugate. The proof is complete. □

Lemma 3.1.3. *Set $Z = Z(G)$.*

(1) G has $|G/G^{(1)}|$ p-blocks of defect 0 and $|Z|$ p-blocks of full defect and no other blocks. There is a 1-1 correspondence between $Irr(Z)$ and the set of p-blocks of G of full defect given by $\rho \mapsto S_\rho$, such that $\varphi \in Irr(G)$ lies in S_ρ if and only if φ lies over ρ and $|\varphi(1)|_p < |G|_p$.

(2) Let $\emptyset \neq J \subseteq I$. P_J has only p-blocks of full defect. Moreover, the Brauer map gives a 1-1 correspondence between the set of p-blocks of P_J and the set of p-blocks of G of full defect. If S is a p-block of P_J and the Brauer correspondent S^G corresponds to $\rho \in Irr(Z)$ in (1), then $\varphi \in Irr(P_J)$ lies in S if and only if φ lies over ρ.

These are either well known results (see [H]) on the block theory of the finite groups of Lie type or direct consequence of Lemma 2.1 in [OU].

For $d \geqslant 0$, recall $\mathrm{Irr}_d(G)$ is the set of characters $\varphi \in \mathrm{Irr}(G)$ whose q-height is d and $k_d(G) = |\mathrm{Irr}_d(G)|$.

Proposition 3.1.4. *Let $\rho \in Irr(Z)$, $|G|_p = q^{d_0}$. The following are equivalent:*

*(1) **DOC** holds for (G, p, S_ρ) and $i > 0$. Here S_ρ is the p-block of full defect of G corresponding to ρ.*

(2) $\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, \rho) = 0$, where $d = d_0 - i/e$.

Proof. By part (1) of Lemma 3.1.3, the p-blocks of G of positive defect are of full defect and parameterized by the central characters of G. Fix our choice of p-block $S = S_\rho$ corresponding to $\rho \in \mathrm{Irr}(Z)$. Let Δ be as in Lemma 3.1.2. So Δ is a set of representatives of the G-orbits on the set of p-radical chains. For $J \subseteq I$, let $\Delta(J)$ be the set of chains in Δ whose final term is U_J. By Lemma 3.1.2, $G_c = P_J$ for all $c \in \Delta(J)$. As $|P_J|_p = |G|_p$ and $i > 0$, by Lemma 3.1.3 we have $k(G_c, S, i) = k_d(P_J, \rho)$. Finally the Euler characteristic $\chi(K)$ of the complex K of all chains of proper nonempty subsets of J is $1 + (-1)^{|J|}$. Then as

$$\Delta(J) = \{\, U_\emptyset < \cdots < U_{J_{|c|-1}} < U_J \mid J_1 < \cdots < J_{|c|-1} \in K \cup \{\emptyset\} \,\},$$

it follows that

$$\sum_{c \in \Delta(J)} (-1)^{|c|} = \sum_{b \in K} (-1)^{|b|-1} - 1 = \chi(K) - 1 = (-1)^{|J|}.$$

Therefore, (1) is true if and only if

$$0=\sum_{c\in\Delta}(-1)^{|c|}k(G_c,S,i)=\sum_{J\subseteq I}\sum_{c\in\Delta(J)}(-1)^{|c|}k_d(P_J,\rho)=\sum_{J\subseteq I}(-1)^{|J|}k_d(P_J,\rho),$$

i.e., if and only if (2) is true. Done.

□

Remark 3.1.5. (1) The reason that we don't have a reformulation when $i=0$ is because $k(G,S_\rho,0)\neq k_{d_0}(G,\rho)$. The p-defect 0 case will be handled below.

(2) We consider the q-height of characters of G rather than p-height to simplify notation. In the case of $G=GU_n(q)$, it follows from Proposition 4.2.2 that the p-part of the degree of $\varphi(1)$ for $\varphi\in\mathrm{Irr}(G)$ is an integral power of q, so that the q-heights involved are all integers. In general, according to Deligne-Lusztig theory, the p-part of $\varphi(1)$ is always an integral power of q (including the case $G=GL_n(q)$) except in some cases when p is small.

Lemma 3.1.6. *(1)* ***DOC*** *holds for G at p when $i=0$.*

(2) ***DOC*** *holds for G at p if and only if*

$$\sum_{J\subseteq I}(-1)^{|J|}k_d(P_J,\rho)=0,\qquad \forall\rho\in Irr(Z),\forall d<d_0. \tag{3.1}$$

Proof. An irreducible character φ of G lies in a p-block of defect 0 if and only $\varphi(1)$ is divisible by the p-part of $|G|$, or equivalently if and only φ has p-defect 0. So as S_ρ is of positive defect, $k(G,S_\rho,0)=0$. On the other hand, for all $\emptyset\neq J\subseteq I$, $O_p(P_J)\neq 1$. So it is easy to show that $k(P_J,S_\rho,0)=0$, i.e., $|\varphi(1)|_p<q^{d_0}$ for $\varphi\in\mathrm{Irr}(P_J)$. Therefore, **DOC** holds for G at p when $i=0$. Part (2) then follows immediately from part (1) and Proposition 3.1.4.

□

3.2 Strategy

We continue with the notation in section 3.1. By Lemma 3.1.6, to prove **DOC** for G, it suffices to prove equation (3.1). The general strategy of proving equation (3.1) presented here is essentially a generalization of Olsson and Uno's approach for dealing with the general linear group. But technically there are more difficulties in the unitary case than in the linear case.

For $J \subseteq I$, $\mathrm{Irr}(P_J, \rho)$ is naturally partitioned into two subsets $\mathrm{Irr}^0(P_J, U_J, \rho)$ and $\mathrm{Irr}^1(P_J, U_J, \rho)$ using the notation established in section 2.2. We have

$$k_d(P_J, \rho) = k_d^0(P_J, U_J, \rho) + k_d^1(P_J, U_J, \rho).$$

where as usual $k_d^i(P_J, U_J, \rho) = |\mathrm{Irr}_d^i(P_J, U_J, \rho)|$.

But $k_d^0(P_J, U_J, \rho) = k_d(L_J, \rho)$. As L_J is a group of Lie type, we may apply the Deligne-Lusztig theory to evaluate $\sum_{J \subseteq I} (-1)^{|J|} k_d(L_J, \rho)$ and describe it in terms of information controlled by the Weyl group of G. Whatever the outcome is, say X, we then need to show that $\sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_J, \rho) = -X$.

Let $0 = W_0 < W_1 < \cdots < W_r = U_J$ be the lower central series of U_J. The factor groups W_i/W_{i-1} are the so-called *internal modules* of P_J. We further partition $\mathrm{Irr}^1(P_J, U_J, \rho)$ into the subsets $\mathrm{Irr}^1(P_J, (W_i/W_{i-1}), \rho)$ for $1 \leqslant i \leqslant r$ again using the notation established in section 2.2. Cancellation occurs when we count characters in $\mathrm{Irr}^1(P_J, W, \rho)$ for those P_J which have a common internal module W appeared as a factor of its lower central series, so that we only need to count the number of irreducible characters of P_J lying over a subset $S(W) \subseteq \mathrm{Irr}(W)$ of very special characters of W. Then by a recursive analysis we show that only the characters in a subset $\mathrm{Irr}(P_J, S'(W), \rho)$ of $\mathrm{Irr}(P_J, S(W), \rho)$ need to be counted. This leads to a surprising cancellation among the number of characters of P_J over $\mathrm{Irr}(S'(W))$ for different internal modules. Most characters in the count are eliminated during the process, and essentially only the characters of P_J over certain characters in the so-called unitary and central modules, which are parameterized by certain chains of a unitary space, need to be counted. The stabilizers in P_J of such characters of W are well understood. And by implementing a dimension argument, we show that the outcome is indeed $-X$.

The key to the success of this approach is to develop some general tools for dealing with alternating sums and to find a smart way to partition the characters of the parabolic subgroups so that we can achieve substantial cancellation.

3.3 The General Linear and Unitary Groups

Let $G = GL_n(q)$. Then $Z = Z(G)$ is a cyclic group of order $q-1$. We fix a generator z of Z and an isomorphism θ from $\mathbb{C}_{q-1}$ to the multiplicative group $\mathbb{H}_{q-1}$ of $\mathbb{F}_q$. Consequently we obtain an isomorphism between $\mathrm{Irr}(Z)$ and $\mathbb{H}_{q-1}$ given by $\rho \mapsto a_\rho = \theta(\rho(z))$.

Olsson and Uno proved the following results which imply equation (3.1) in section 3.1 and hence verified **DOC** for the general linear groups in the defining characteristic.

Lemma 3.3.1. *Let $G = GL_n(q)$ and $Z = Z(G)$. Assume the notation in section 3.1. Let $\rho \in Irr(Z)$ and $d \geqslant 0$. Then*

(1) $k_d(G,\rho) = \sum' q^{l(\mu)-\delta(\mu)}\alpha(\mu, a_\rho)$. *In particular,*

$$k_d(G) = \sum{}' q^{l(\mu)-\delta(\mu)}(q-1)^{\delta(\mu)}.$$

Here $\sum'$ is taken over all the partitions μ of n such that $n(\mu) = d$.

(2)

$$\sum_{J\subseteq I}(-1)^{|J|}k_d(P_J,\rho) = \begin{cases} \alpha((n), a_\rho), & \text{if } d = \binom{n}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Part (1) is Proposition 3.1 in [OU] and Part 2 is a corollary to the following result:

Lemma 3.3.2. *Let $G = GL_n(q)$ and $Z = Z(G)$. Assume the notation in section 3.1. Let $\rho \in Irr(Z)$ and $d \geqslant 0$. Then*

(1) $\sum_{J\subseteq I}(-1)^{|J|}k_d^0(P_J,U_J,\rho) = -\sum'(-1)^{\delta(\mu)}\alpha(\mu,a_\rho)$.

(2)

$$\sum_{J\subseteq I}(-1)^{|J|}k_d^1(P_J,U_J,\rho) = \begin{cases}\sum'(-1)^{\delta(\mu)}\alpha(\mu,a_\rho), & \text{if } d < \binom{n}{2};\\ 0, & \text{otherwise.}\end{cases}$$

Here $\sum'$ is the same as in Lemma 3.3.1.

This is Theorem 2.6 in [OU].

In this thesis we will prove the following results which complete the verification of **DOC** for the general unitary groups in the defining characteristic.

Let $G = GU_n(q)$, $n = 2m$ or $2m+1$. So as a linear group, G is a subgroup of $GL_n(q^2)$. $Z = Z(G) \cong \mathbb{H}_{q+1}$ is the unique subgroup of $Z(GL_n(q^2))$ of order $q+1$. We fix a generator z of $Z(GL_n(q^2))$, and hence also a generator z^{q-1} of Z, and an isomorphism θ from $\mathbb{C}_{q^2-1}$ to $\mathbb{H}_{q^2-1}$. Consequently we obtain an isomorphism between $\mathrm{Irr}(Z)$ and $\mathbb{H}_{q+1}$ given by $\rho \mapsto a_\rho = \theta(\rho(z^{q-1}))$.

Assume the notation in section 3.1. In particular $I = [m]$. We refer to the set-up in the beginning of section 7.1. So P_J is a parabolic subgroup of G stabilizing a chain of totally isotropic subspaces of the natural module V of G of type J.

Theorem 3.3.3. *Let $G = GU_n(q)$ and $Z = Z(G)$. Assume the notation in section 3.1. Let $\rho \in Irr(Z)$ and $d \geqslant 0$. Then*

$$k_d(G,\rho) = \sum{}' q^{l(\mu)-\delta(\mu)}\beta(\mu,a_\rho).$$

Here $\sum'$ is the same as in Lemma 3.3.1.

The *McKay numbers* for a finite group and some prime p are the number of irreducible complex characters of a given p-height. By Theorem 3.3.3 and part (1) of Lemma 2.3.3 we immediately obtain the following formula on the McKay numbers for G at p:

Theorem 3.3.4. (McKay numbers) *Let $G = GU_n(q)$. Then*

$$k_d(G) = \sum{}' q^{l(\mu)-\delta(\mu)}(q+1)^{\delta(\mu)}.$$

Here $\sum'$ *is the same as in Lemma 3.3.1.*

Theorem 3.3.3 will be proved in section 4.3.

Theorem 3.3.5. *Let* $G = GU_n(q)$ *and* $Z = Z(G)$. *Assume the notation in section 3.1. Let* $\rho \in Irr(Z)$ *and* $d \geqslant 0$. *Then*

$$\sum_{J\subseteq I}(-1)^{|J|}k_d(P_J,\rho) = \begin{cases} \beta((n),a_\rho), & \text{if } d = \binom{n}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly Theorem 3.3.5 implies equation (3.1) in section 3.2. Therefore, Main Theorem 1 follows. We need the formula for the case when $d = \binom{n}{2}$ for the purpose of induction. Theorem 3.3.5 is a consequence of the following proposition.

Proposition 3.3.6. *Let* $G = GU_n(q)$ *and* $Z = Z(G)$. *Assume the notation in section 3.1. Let* $\rho \in Irr(Z)$ *and* $d \geqslant 0$. *Then*

(1) $\sum_{J\subseteq I}(-1)^{|J|}k_d^0(P_J,U_J,\rho) = \sum' \beta(\mu,a_\rho)$.

(2)

$$\sum_{J\subseteq I}(-1)^{|J|}k_d^1(P_J,U_J,\rho) = \begin{cases} -\sum' \beta(\mu,a_\rho), & \text{if } d < \binom{n}{2}; \\ 0. & \text{otherwise.} \end{cases}$$

Here $\sum'$ *is the same as in Lemma 3.3.1.*

Part(1) is Proposition 4.3.7 and Part (2) will be proved in Chapter 9.

Proof of Corollary 1.0.1. Assume Theorem 3.3.5. Let $G = PGU_n(q)$. By definition $Z(G) = 1$. So by Lemma 3.1.3, G and the parabolics of G have a unique p-block of positive defect. So equation (3.1) becomes

$$\sum_{J\subseteq I}(-1)^{|J|}k_d(P_J) = 0, \qquad \forall d < d_0. \tag{3.2}$$

On the other hand, there is a natural bijection preserving degrees between $\mathrm{Irr}(G)$ and $\mathrm{Irr}(GU_n(q), 1)$ (where 1 denotes the trivial central character) which restricts to a bijection between the set of characters of corresponding parabolics. So in Theorem 3.3.5 if we let ρ be the trivial character, we then obtain equation (3.2).

□

Proof of Corollary 1.0.2. This follows from Corollary 1.0.1 as $PGU_n(q) \cong U_n(q)$ when $(n, q+1) = 1$.

□

Chapter 4 On the McKay Numbers

In this chapter, we apply the Deligne-Lusztig theory on the representation of finite groups of Lie type to the finite unitary groups and prove Theorem 3.3.3 as well as part (1) of Proposition 3.3.6. We record most of the well known results without proof. The proofs are similar to Olsson and Uno's proofs of the corresponding results for the general linear groups. The material in this section is independent of the rest of the thesis. For this reason, the reader may temporarily choose to skip this section and return to it later.

4.1 General Facts

We recall some well known facts from the Deligne-Lusztig theory. See for instance [Ca] for relevant terminology or [DL] for more details.

Let $\bar{G}$ be a connected reductive group over the closure $\bar{\mathbb{F}}_q$ of a finite field $\mathbb{F}_q$ of order $q = p^e$ associated with a Frobenius endomorphism σ in the general sense. The group $G = \bar{G}_\sigma$ of fixed points by σ is then a finite reductive group. The irreducible representations of G are studied by Deligne and Lusztig in terms of the so-called Deligne-Lusztig generalized characters $R_{T,\theta}$ of G where $T = \bar{T}_\sigma$, $\bar{T}$ is a σ-stable maximal torus of $\bar{G}$, and $\theta \in \mathrm{Irr}(T)$. It turns out that each irreducible character of G occurs as a constituent of $R_{T,\theta}$ for some pair (T, θ).

The set of pairs (T, θ) can be partitioned into equivalence classes called the *geometric conjugacy classes* with the following properties:

(1) The geometric conjugacy classes are in 1-1 correspondence with the semisimple conjugacy classes in G^*, where G^* is the dual group of G. We write Λ_s for the geometric conjugacy class corresponding to the semisimple class s^{G^*};

(2) For each $\varphi \in \mathrm{Irr}(G)$, there is a unique geometric conjugacy class Λ_s such that

$\langle\varphi, R_{T,\theta}\rangle = 0$ if $(T,\theta) \notin \Lambda_s$. In other words, if φ is a constituent of $R_{T,\theta}$, then $(T,\theta) \in \Lambda_s$. This gives a partition of $\mathrm{Irr}(G)$ into subsets labeled by the semisimple classes of G^*. We write Γ_s for the set of $\varphi \in \mathrm{Irr}(G)$ corresponding to Λ_s.

The characters in Γ_1 corresponding to the identity element of G^* are the so-called *unipotent characters* of G; Λ_1 consists of all $(T, 1_T)$ where 1_T is the trivial character.

Further it turns out that Γ_s is in 1-1 correspondence with the set of unipotent characters of $H(s) = (C_{G^*}(s))^*$, the dual group of the centralizer in G^* of s. If $\varphi \in \Gamma_s$ corresponds to a unipotent character $\lambda \in \mathrm{Irr}(H(s))$, then $\varphi(1) = \lambda(1)|G/H(s)|_{p'}$.

In summary, the set of irreducible characters of G are in 1-1 correspondence with the set of pairs (s,λ) where s is a representative of a semisimple class of G^* and λ is a unipotent character of $(C_{G^*}(s))^*$; And the p-height of the character corresponding to (s,λ) is equal to the p-height of λ .

The dual group of a finite reductive group G and the semisimple classes of G, as well as their centralizers, are all well known; See for instance [Ca]. The unipotent representations are studied by Lusztig and others in terms of the representations of the Weyl group. This make it clear how to calculate the number $k_d(G,\rho)$ and hence $\sum_{J\subseteq I}(-1)^{|J|}k_d(L_J,\rho)$.

4.2 Unitary groups

We discuss in more details the irreducible representation of finite unitary groups. It is well known that both finite general linear and unitary groups are self-dual. The semisimple classes of a finite unitary group can be described in terms of the rational canonical form, and the centralizer of a semisimple element is a direct product of general linear and unitary groups. See for instance [FS]. For completeness, we include the necessary information as follows.

For a polynomial $f(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0 \in \mathbb{F}_{q^2}[x]$ with $a_0 \neq 0$, let $\bar{f}(x) = x^m + a_{m-1}^q x^{m-1} + \cdots + a_1^q x + a_0^q$. Define

$$\tilde{f}(x) = a_0^{-q} x^m \bar{f}(\frac{1}{x}) = x^m + a_0^{-q} a_1^q x^{m-1} + a_0^{-q} a_2^q x^{m-2} + \cdots + a_0^{-q}.$$

Notice $\tilde{\tilde{f}} = f$ for all f. Set

$$\mathbf{F}' = \{f \mid x \neq f(x) \in \mathbb{F}_{q^2}[x] \text{ is monic, irreducible and } f = \tilde{f}\},$$
$$\mathbf{F}'' = \{f\tilde{f} \mid x \neq f(x) \in \mathbb{F}_{q^2}[x] \text{ is monic, irreducible and } f \neq \tilde{f}\},$$

and $\mathbf{F} = \mathbf{F}' \cup \mathbf{F}''$. Set $d_f = \deg(f)$.

The conjugacy classes of elements in $GL_n(q)$ are described by *elementary divisors* which are powers of monic irreducible polynomials in $\mathbb{F}_q[x]\backslash\{x\}$. Similarly if we regard $GU_n(q)$ as a subgroup of $GL_n(q^2)$, the conjugacy classes of elements in $GU_n(q)$ can also be described by elementary divisors which turn out to be the powers of members in $\mathbf{F}$.

For $r \in \mathbb{N}$ and $f \in \mathbf{F}$, let (f) be the companion matrix of f and $r(f)$ be the matrix direct sum of r copies of (f). So if $d_f = d$, then (f) is a $d \times d$ matrix and $r(f)$ is a $(rd) \times (rd)$ matrix consisting of r blocks with each block equal to (f).

For the rest of this section assume $G = GU_n(q)$. It turns out that the rational canonical form of a conjugacy class g^G in G is the matrix direct sum

$$\bigoplus_{f \in \mathbf{F}} \bigoplus_i m_{f^i}(g)(f^i)$$

such that $\sum_{f\in\mathbf{F}} \deg(f^i) m_{f^i}(g) = n$, where $m_{f^i}(g)$ is the multiplicity of f^i appearing as elementary divisors of g. g is semisimple if and only if $m_{f^i}(g) = 0$ for $i > 1$, in which case g corresponds to $\prod_{f\in\mathbf{F}} m_f(g)(f)$.

Let V be the unitary space on which G acts, and $g \in G$ a semisimple element. The primary decomposition will be denoted by $g = \prod_{f\in\mathbf{F}} g_f$, where g_f is the element in G whose rational canonical form is $m_f(g)(f)$. Correspondingly V decomposes as $V = \bigoplus_{f\in\mathbf{F}} V_f$ where V_f are non-singular subspaces of V; g_f acts on V_f and centralizes the other components. $C_G(g)$ is the direct product of $C(g)_f$ for all elementary divisors of g where

Lemma 4.2.1. *(1) If $f \in \mathbf{F}'$, then $C(g)_f = GU_{m_f(g)}(\mathbb{F}_f)$, where $\mathbb{F}_f$ is an extension field of $\mathbb{F}_q$ with $|\mathbb{F}_f : \mathbb{F}_q| = d_f$;*

(2) If $f = f_1\tilde{f}_1 \in \mathbf{F}''$, then $C(g)_f = GL_{m_f(g)}(\mathbb{F}_f)$, where $\mathbb{F}_f$ is an extension field of $\mathbb{F}_q$ with $|\mathbb{F}_f : \mathbb{F}_q| = d_f$.

This is Proposition (1A) in [FS].

On the other hand, the unipotent characters of $GL_n(q)$ or $GU_n(q)$ are in 1-1 correspondence with the conjugacy classes of the Weyl group $W = S_n$ of $GL_n(\bar{\mathbb{F}}_q)$, and hence with the partitions of n. Moreover, if a unipotent character $\boldsymbol{\lambda}$ of $GL_n(q)$ or $GU_n(q)$ corresponds to $\mu \vdash n$, then $\boldsymbol{\lambda}(1)|_p = q^{n(\mu')}$ where μ' is the conjugate partition of μ defined in section 2.3. Finally if we denote the set of unipotent characters of G by $\mathrm{Irr}^u(G)$, then for finite reductive groups H_1 and H_2, there is a natural 1-1 correspondence

$$\psi : \mathrm{Irr}^u(H_1) \times \mathrm{Irr}^u(H_2) \to \mathrm{Irr}^u(H_1 \times H_2), \quad (\varphi_1, \varphi_2) \mapsto \varphi = \varphi_1\varphi_2$$

where $\varphi(h_1, h_2) = \varphi_1(h_1)\varphi_2(h_2)$.

Let $\mathbf{P}$ be the union of the set of all partitions of n for all $n \geqslant 0$.

Proposition 4.2.2. *There is a 1-1 correspondence $\boldsymbol{\lambda} \mapsto \varphi_{\boldsymbol{\lambda}}$ between the set of maps $\boldsymbol{\lambda} : \mathbf{F} \to \mathbf{P}$ with*

$$\sum_{f \in \mathbf{F}} |\boldsymbol{\lambda}(f)|\, d_f = n \tag{4.1}$$

and $Irr(G)$. Moreover, if $\varphi_{\boldsymbol{\lambda}} \in Irr(G)$ corresponds to $\boldsymbol{\lambda}$, then the q-height of $\varphi_{\boldsymbol{\lambda}}$ is $\sum_{f \in \mathbf{F}} d_f n(\boldsymbol{\lambda}(f)')$.

Proof. Let $\varphi \in \mathrm{Irr}(G)$. Then φ is uniquely determined by (g, φ_g), where g is a representative of a semisimple class of G and $\varphi_g \in \mathrm{Irr}^u(C_G(g))$. (g, φ_g) is in turn determined by $\{(g_f, \varphi_f) \mid f \in \mathbf{F}\}$ where $g = \prod_{f \in \mathbf{F}} g_f$, $\varphi_f \in \mathrm{Irr}^u(C(g)_f)$, and $\varphi_g =$

$\prod_{f\in\mathbf{F}}\varphi_f$. However, by Lemma 4.2.1 $C(g)_f$ is either an $m_f(g)$-dimensional general linear group or an $m_f(g)$-dimensional general unitary group, hence φ_f is uniquely determined by a partition μ_f of $m_f(g)$. We now define the map $\boldsymbol{\lambda}:\mathbf{F}\to\mathbf{P}$ by letting $f\mapsto\mu_f$. As $\sum_{f\in\mathbf{F}}d_fm_f(g)=n$, we have

$$\sum_{f\in\mathbf{F}}|\boldsymbol{\lambda}(f)|\,d_f=n.$$

We let φ correspond to $\boldsymbol{\lambda}$. It is easy to check that each step of our construction of $\boldsymbol{\lambda}$ is bijective, and therefore we obtain a 1-1 correspondence between $\mathrm{Irr}(G)$ and the set of maps from $\mathbf{F}$ to $\mathbf{P}$ with desired properties.

Let's turn to the degrees. We have seen that the p-height of φ is the same as that of $\varphi_g=\prod\varphi_f$. By Lemma 4.2.1, if $f\in\mathbf{F}'$, then $C(g)_f=GU_{m_f(g)}(q^{d_f})$, so the q-height of φ_f is $d_fn(\mu'_f)$. If $f\in\mathbf{F}''$, then $C(g)_f=GL_{m_f(g)}(q^{d_f})$, so the q-height of φ_f is also $d_fn(\mu'_f)$. Therefore, the q-height of φ is $\sum_{f\in\mathbf{F}}d_fn(\boldsymbol{\lambda}(f)')$.

□

Indeed, the irreducible characters of $GU_n(q)$ are constructed by Lusztig and Srinivasan ([LS], also see [FS]). By directly checking with their construction, we can deduce the following:

Lemma 4.2.3. *If $\varphi_{\boldsymbol{\lambda}}\in Irr(G)$ corresponds to the map $\boldsymbol{\lambda}$ as in Proposition 4.2.2, then $\varphi_{\boldsymbol{\lambda}}$ lies over $\rho\in Irr(Z(G))$ where a_ρ is equal to the product of the roots of $\prod_{f\in\mathbf{F}}f^{|\boldsymbol{\lambda}(f)|}$.*

4.3 The McKay Numbers

Let $\bar{\mathbf{F}}$ be the set of polynomials which are the product of members of $\mathbf{F}$. Then $\bar{\mathbf{F}}$ consists of monic polynomials $x\neq f(x)\in\mathbb{F}_{q^2}[X]$ with $f=\tilde{f}$.

Lemma 4.3.1. *Let $f(x)=x^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0\in\mathbb{F}_{q^2}[X]$. Then $f\in\bar{\mathbf{F}}$ if and only if $a_0\in\mathbb{H}_{q+1}$ and $a_{m-i}=a_0a_i^q$, $1\leqslant i\leqslant m-1$.*

The proof is trivial. Notice that $a_{m-i}=a_0a_i^q$ if and only if $a_i=a_0a_{m-i}^q$.

Proof of Theorem 3.3.3. Fix $r \geqslant 0$ and $\rho \in \mathrm{Irr}(Z(G))$. We need to count $k_r(G,\rho)$.

Let n_d be the number of elements in $\mathbf{F}$ of degree d and let $f_{d,1}, f_{d,2}, \ldots, f_{d,n_d}$ be the polynomials in $\mathbf{F}$ of degree d. Let Λ be the set of maps $\boldsymbol{\lambda} : \mathbf{F} \to \mathbf{P}$ satisfying equation (4.1) of Proposition 4.2.2. Given $\boldsymbol{\lambda} \in \Lambda$, we put

$$\boldsymbol{\lambda}(f_{d,i}) = (j^{m^j_{d,i}}). \tag{4.2}$$

Also we denote by $\boldsymbol{\lambda}'$ the map from $\mathbf{F}$ to $\mathbf{P}$ such that $\boldsymbol{\lambda}'(f)$ is the conjugate partition to $\boldsymbol{\lambda}(f)$ for all $f \in \mathbf{F}$. By Proposition 4.2.2, $\boldsymbol{\lambda}$ corresponds to $\varphi_{\boldsymbol{\lambda}} \in \mathrm{Irr}(G)$. Set

$$\Lambda(\rho) = \{\, \boldsymbol{\lambda} \in \Lambda \mid \varphi_{\boldsymbol{\lambda}} \text{ lies over } \rho \,\}.$$

Lemma 4.3.2. $\boldsymbol{\lambda} \in \Lambda(\rho)$ *if and only if* $\boldsymbol{\lambda}' \in \Lambda(\rho)$.

Proof. Recall from section 2.3 that for each partition μ, $|\mu| = |\mu'|$. Then the lemma follows from Proposition 4.2.2 and Lemma 4.2.3.

□

Define $h_q(\boldsymbol{\lambda}) = h_q(\varphi_{\boldsymbol{\lambda}})$ to be the q-height of $\varphi_{\boldsymbol{\lambda}}$.

Lemma 4.3.3.

$$k_r(G,\rho) = |\, \{\, \boldsymbol{\lambda} \in \Lambda(\rho) \mid h_q(\boldsymbol{\lambda}) = r \,\} \,| = |\, \{\, \boldsymbol{\lambda} \in \Lambda(\rho) \mid h_q(\boldsymbol{\lambda}') = r \,\} \,|.$$

Proof. The first equality follows from the definition of $k_r(G,\rho)$ and the second follows from Lemma 4.3.2.

□

Now for $\boldsymbol{\lambda} \in \Lambda$, we construct a partition $\mu = \mu(\boldsymbol{\lambda})$ of n by defining $\mu = (j^{x_j})$, where

$$x_j = \sum_{d,i} d m^j_{d,i} = \sum_d d(\sum_i m^j_{d,i}).$$

Observe μ is a partition of n because by Proposition 4.2.2,

$$n = \sum_{f \in \mathbf{F}} |\boldsymbol{\lambda}(f)| d_f = \sum_d d \sum_{i=1}^{n_d} |(j^{m^j_{d,i}})| = \sum_{d,i,j} dj m^j_{d,i} = \sum_j j \sum_{d,i} dm^j_{d,i} = \sum_j j x_j.$$

Lemma 4.3.4. $h_q(\boldsymbol{\lambda}') = n(\mu(\boldsymbol{\lambda}))$.

Proof. By Proposition 4.2.2,

$$h_q(\boldsymbol{\lambda}') = \sum_{f \in \mathbf{F}} d_f n(\boldsymbol{\lambda}(f)) = \sum_d d(\sum_{i=1}^{n_d} n(\boldsymbol{\lambda}(f_{d,i})) = \sum_{j,d,i} \binom{j}{2} dm^j_{d,i} = n(\mu(\boldsymbol{\lambda})). \quad (4.3)$$

□

Let

$$\mathcal{U}(r) = \{ \mu \vdash n \mid n(\mu) = r \}.$$

For $\mu \vdash n$, let

$$\Lambda(\mu) = \{ \boldsymbol{\lambda} \in \Lambda \mid \mu(\boldsymbol{\lambda}) = \mu \}$$

and $\Lambda(\mu, \rho) = \Lambda(\mu) \cap \Lambda(\rho)$. By Lemma 4.3.3 and 4.3.4:

Lemma 4.3.5.

$$k_r(G, \rho) = | \{ \boldsymbol{\lambda} \in \Lambda(\rho) \mid n(\mu(\boldsymbol{\lambda})) = r \} |.$$

Let $\mu = (j^{x_j})$ be a partition of n, we next obtain a parameterization of $\Lambda(\mu)$ and $\Lambda(\mu, \rho)$. Let $S(\mu)$ be the set of sequences $(f_1, f_2, \ldots, f_n)$ such that $f_j \in \bar{\mathbf{F}}$ and $\deg(f_j) = x_j$.

Lemma 4.3.6. *There is a bijection*

$$s : \Lambda(\mu) \to S(\mu), \quad s(\boldsymbol{\lambda}) = (f_1, f_2, \ldots, f_n)$$

such that $\boldsymbol{\lambda} \in \Lambda(\mu, \rho)$ if and only if

$$(-1)^n a_1 a_2^2 a_3^3 \ldots a_n^n = a_\rho. \tag{$*$}$$

where $a_\rho \in \mathbb{H}_{q+1}$ is defined in section 3.3 and for $1 \leqslant j \leqslant n$, a_j is the constant term of f_j.

Proof. For $\boldsymbol{\lambda} \in \Lambda(\mu)$ define $s(\boldsymbol{\lambda}) = (f_1, \ldots, f_n)$ by $f_j = \prod_{d,i} f_{d,i}^{m^j_{d,i}}$. As $f_{d,i} \in \mathbf{F}$, $f_j \in \bar{\mathbf{F}}$. Further

$$\deg(f_j) = \sum_{d,i} d m^j_{d,i} = x_j,$$

so indeed $s(\boldsymbol{\lambda}) \in S(\mu)$. To see that $s : \Lambda(\mu) \to S(\mu)$ is a bijection, we define an inverse $t : S(\mu) \to \Lambda(\mu)$ for s. Namely given $\underline{f} = (f_1, f_2, \ldots, f_n) \in S(\mu)$, f_j has a unique factorization

$$f_j = \prod_{d,i} f_{d,i}^{m^j_{d,i}}$$

and we define $t(\underline{f}) = \boldsymbol{\lambda} \in \Lambda$ using equation (4.2). Now

$$\sum_f |\boldsymbol{\lambda}(f)| d_f = \sum_{d,i,j} j d m^j_{d,i} = \sum_j j x_j = n,$$

so indeed $\boldsymbol{\lambda} \in \Lambda$. Further for each j,

$$\sum_{d,i} d m^j_{d,i} = \deg(f_j) = x_j$$

as $\underline{f} \in S(\mu)$, so $\boldsymbol{\lambda} \in \Lambda(\mu)$. By construction, s and t are inverses.

Next by Lemma 4.2.3, $\boldsymbol{\lambda} \in \Lambda(\rho)$ if and only if a_ρ is the product of the roots of $\prod_{f \in \mathbf{F}} f^{|\boldsymbol{\lambda}(f)|}$. But $|\boldsymbol{\lambda}(f_{d,i})| = \sum_j j m^j_{d,i}$, so

$$\prod_{f \in \mathbf{F}} f^{|\boldsymbol{\lambda}(f)|} = \prod_{d,i} f_{d,i}^{|\boldsymbol{\lambda}(f_{d,i})|} = \prod_{d,i,j} f_{d,i}^{j m^j_{d,i}} = \sum_j f_j^j.$$

Further the product of the roots of f_j is $(-1)^{\deg(f_j)}a_j$, so $\boldsymbol{\lambda} \in \Lambda(\rho)$ if and only if

$$a_\rho = \prod_j (-1)^{\deg(f_j)j} a_j^j = (-1)^n \prod_j a_j^j$$

as claimed.

□

We can now complete the proof of Theorem 3.3.3. We must show

$$k_r(G,\rho) = \sum_{\mu \in \mathcal{U}(r)} q^{l(\mu)-\delta(\mu)} \beta(\mu, a_\rho).$$

So by Lemma 4.3.5, it suffices to show

$$|\Lambda(\mu,\rho)| = q^{l(\mu)-\delta(\mu)} \beta(\mu, a_\rho). \tag{4.4}$$

To establish equation (4.4) we use the parameterization of Lemma 4.3.6, and count the number of $\underline{f} = (f_1, \ldots, f_n) \in S(\mu)$ satisfying $(*)$. Let

$$f(x) = x^m + b_{m-1}x^{m-1} + \ldots b_1 x + b_0 \in \mathbb{F}_{q^2}[x].$$

For $1 \leqslant i \leqslant (m-1)/2$, there are q^2 choices for b_i, and then by Lemma 4.3.1, $f \in \bar{\mathbf{F}}$ if and only if $b_0 \in \mathbb{H}_{q+1}$, $b_{m-i} = b_0 b_i^q$, and if m is even, $b_{m/2}^{1-q} = b_0$. Thus there are $q^{2(\deg(f_j)-1)/2} = q^{x_j - 1}$ choices for the coefficients $b_{j,1}, \ldots, b_{j,x_j-1}$, and the coefficients $a_j = b_{j,0}$ must satisfy $(*)$. By Lemma 2.3.3, there are $\beta(\mu, a_\rho)$ tuples $(a_1, \ldots, a_n)$ satisfying $(*)$, so

$$|\Lambda(\mu,\rho)| = \prod_j q^{x_j-1} \beta(\mu, a_\rho) = q^{l(\mu)-\delta(\mu)} \beta(\mu, a_\rho).$$

as desired. This completes the proof of Theorem 3.3.3.

□

We now prove the following proposition which is equivalent to Proposition 3.3.6.1.

Recall $n = 2m$ or $2m + 1$ and $I = [m]$.

Proposition 4.3.7. *Let $G = GU_n(\mathbb{F}_q)$. Then*

$$\sum_{J\subseteq I}(-1)^{|J|}k_d(L_J, \rho) = \sum{}' \beta(\mu, a_\rho)$$

where $\sum'$ is the same as in Lemma 3.3.1.

Proof. Use induction on m. If $m = 0$, then $n = 1$, so $G = GU_1(q) = Z(G) \cong \mathbb{H}_{q+1}$. As $I = [0] = \emptyset$ and $\mathbb{H}_{q+1}$ has $q + 1$ irreducible representations, all of degree 1, if follows that

$$\sum_{J\subseteq I}(-1)^{|J|}k_d(L_J, \rho) = k_d(G, \rho) = \begin{cases} 1, & \text{if } d = 0; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, 1 has a unique partition $\mu = (1)$, with $n(\mu) = 0$ and $\beta(\mu, a_\rho) = 1$. So the right-hand side of the proposition is 1 if $d = 0$ and 0 otherwise. Therefore, the proposition holds for $m = 0$.

Assume $m \geqslant 1$. Let $\Delta = \Delta(I)$. For $0 \leqslant l \leqslant m$, let Δ_l be the set of $J \subseteq I$ whose minimal member $\min(J)$ is equal to l. Here we set $\min(\emptyset) = 0$. Then Δ is the disjoint union of Δ_l, $0 \leqslant l \leqslant m$.

If $J \in \Delta_l$ and $l \geqslant 1$, then $L_J = G^{+l} \times L_{J'}^{n-2l}$ where $G^{+l} \cong GL_l(q^2)$ and $L_{J'}^{n-2l}$ is the Levi subgroup of $GU_{n-2l}(q)$ corresponding to the subset $J' = \{j - l \mid l < j \in J\}$.

Let Z_1 be the subgroup of $Z(G^{+l})$ of order $q + 1$ and $Z_2 = Z(GU_{n-2l}(q))$. Then $Z(G) \leqslant Z_1 \times Z_2$. So by Lemma 2.2.5 we have

$$k_d(L_J, \rho) = \sum_{j=0}^{d} \sum_{\substack{\rho_i \in Z_i \\ \rho_1\rho_2=\rho}} k_j(G^{+l}, \rho_1)k_{d-j}(L_{J'}^{n-2l}, \rho_2).$$

Let $Z = Z(G^{+l})$. Fix $\rho \in \mathrm{Irr}(Z_1)$. $\varphi \in \mathrm{Irr}(G^{+l})$ lies over ρ if and only if φ lies over some $\tau \in \mathrm{Irr}(Z)$ with τ lying over ρ. So $\mathrm{Irr}(G^{+l}, \rho)$ is the disjoint union of $\mathrm{Irr}(G^{+l}, \tau)$ for $\tau \in \mathrm{Irr}(Z)$ lying over ρ. As $Z \cong \mathbb{H}_{q^2-1}$, there are $(q - 1)$ choices of $\tau \in \mathrm{Irr}(Z)$

lying over ρ. In this case τ lies over ρ if and only if the restriction of τ to Z_1 is ρ. So

$$a_\rho = \rho(z^{q-1}) = \tau(z^{q-1}) = \tau(z)^{q-1} = a_\tau^{q-1}.$$

Therefore, by Lemma 3.3.1,

$$\begin{aligned} k_d(G^{+l}, \rho) &= \sum_{\substack{\tau \in \mathrm{Irr}(Z) \\ \tau|_{Z_1} = \rho}} k_d(G^{+l}, \tau) = \sum_{\substack{b \in \mathbb{H}_{q^2-1} \\ b^{q-1} = a_\rho}} \sum_{\substack{\mu_1 \vdash l \\ n(\mu_1) = d/2}} q^{2(l(\mu_1) - \delta(\mu_1))} \alpha(\mu_1, b) \\ &= \sum_{\substack{\mu_1 \vdash l \\ n(\mu_1) = d/2}} q^{2(l(\mu_1) - \delta(\mu_1))} \bar{\beta}(\mu_1, a_\rho). \end{aligned}$$

Therefore, by induction

$$\begin{aligned} \sum_{J \in \Delta_l} (-1)^{|J|} k_d(L_J, \rho) &= -\sum_{j=0}^{d} \sum_{\substack{\rho_1, \rho_2 \\ \rho_1 \rho_2 = \rho}} k_j(G^{+l}, \rho_1) \sum_{J' \subseteq I'} (-1)^{|J'|} k_{d-j}(L_{J'}, \rho_2) \\ &= -\sum_{j=0}^{d} \sum_{\substack{\rho_1, \rho_2 \\ \rho_1 \rho_2 = \rho}} \Gamma(\sum_{\substack{\mu_2 \vdash n - 2l \\ n(\mu_2) = d - j}} \beta(\mu_2, a_{\rho_2})) \end{aligned}$$

for all $1 \leqslant l \leqslant m$, where

$$\Gamma = \sum_{\substack{\mu_1 \vdash l \\ n(\mu_1) = j/2}} q^{2(l(\mu_1) - \delta(\mu_1))} \bar{\beta}(\mu_1, a_{\rho_1}).$$

Recall from section 2.3 that if $\mu_1 \vdash l$ with $n(\mu_1) = j/2$ and $\mu_2 \vdash n - 2l$ with $n(\mu_2) = d - j$, then $\mu = 2\mu_1 \cup \mu_2 \vdash n$ with $n(\mu) = 2n(\mu_1) + n(\mu_2) = d$. So when j runs over all possibilities, μ and (μ_1, μ_2) run over all $\mu \vdash n$ with $n(\mu) = d$, and such that $\mu = 2\mu_1 \cup \mu_2$ for some $\mu_1 \vdash l$ and $\mu_2 \vdash n - 2l$. Then let l run over all possibilities from 1 to m, μ and (μ_1, μ_2) run over all $\mu \vdash n$ with $n(\mu) = d$ such that μ can be written as $2\mu_1 \cup \mu_2$, except that $\mu_1 \neq (0)$, as $1 \leqslant |\mu_1| \leqslant m$. Therefore, by exchanging the order of summation,

$$\sum_{\emptyset\neq J\subseteq I}(-1)^{|J|}k_d(L_J,\rho)=\sum_{l=1}^{m}\sum_{J\in\Delta_l}(-1)^{|J|}k_d(L_J,\rho)$$

$$=-\sum_{\substack{\mu\vdash n\\ n(\mu)=d}}\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2\\ |\mu_1|\neq 0}}\sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}}q^{2(l(\mu_1)-\delta(\mu_1))}\bar{\beta}(\mu_1,a_{\rho_1})\beta(\mu_2,a_{\rho_2}).$$

On the other hand, we need to show

$$\sum_{J\subseteq I}(-1)^{|J|}k_d(L_J,\rho)=\sum{}'\beta(\mu,a_\rho)=\sum_{\substack{\mu\vdash n\\ n(\mu)=d}}\beta(\mu,a_\rho)$$

and $\sum'\beta(\mu,a_\rho)$ can be regarded as the term corresponding to $|\mu_1|=0$ in the sum σ, where

$$\sigma=\sum_{\substack{\mu\vdash n\\ n(\mu)=d}}\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2}}\sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}}q^{2(l(\mu_1)-\delta(\mu_1))}\bar{\beta}(\mu_1,a_{\rho_1})\beta(\mu_2,a_{\rho_2}).$$

So as $G=L_\emptyset$, it suffices to show that

$$\sigma=k_d(G,\rho)=\sum_{\substack{\mu\vdash n\\ n(\mu)=d}}q^{l(\mu)-\delta(\mu)}\beta(\mu,a_\rho)$$

where μ_1 is allowed to be 0. However, for each $\mu\vdash n$, by Lemma 2.3.7 and Lemma 2.3.8,

$$\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2}}\sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}}q^{2(l(\mu_1)-\delta(\mu_1))}\bar{\beta}(\mu_1,a_{\rho_1})\beta(\mu_2,a_{\rho_2})=$$

$$\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2}}q^{2(l(\mu_1)-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)}\beta(\mu,a_\rho)=q^{l(\mu)-\delta(\mu)}\beta(\mu,a_\rho).$$

Therefore, the proposition is proved.

□

Chapter 5 Evaluating Alternating Sums

5.1 Overview

Let G be a finite group, E an abelian group and $\mathcal{S}(G)$ the set of subgroups of G.

Definition 5.1.1. A function

$$f : \mathcal{S}(G) \to E$$

is called a *G-stable function* if it is constant on the conjugacy classes of subgroups of G.

Let $\mathcal{P}$ be a G-poset. The alternating sum of f over $\mathcal{P}/G$ is defined as

$$A(f, \mathcal{P}/G) = \sum_{c \in \Delta(\mathcal{P})/G} (-1)^{|c|} f(G_c).$$

This may also be written as $A(f, \Delta(\mathcal{P})/G)$. The alternating sum of f over a G-set of chains in $\mathcal{P}$ can be similarly defined. If $H \leqslant G$, then clearly $f|_{\mathcal{S}(H)}$ is an H-stable function. So we may define

$$A(f, \mathcal{P}/H) = \sum_{c \in \Delta(\mathcal{P})/H} (-1)^{|c|} f(H_c).$$

Seeking techniques for evaluating the alternating sum of a G-stable function f over $\mathcal{P}/G$ where $\mathcal{P}$ is a G-poset, has become more and more important in the recent study of the representation theory of finite groups, as many important conjectures in this area can be stated in a form which asserts the vanishing of certain alternating sums.

There are two well known approaches for dealing with alternating sums by studying the structure of the poset: the topological approach and the combinatorial approach. The idea of the first approach is discussed in the work of P. Webb ([W]), Thevenaz ([Th2]), and others. Recall that the *Burnside Ring* $B(G)$ of G is the Grothendieck group of the category of finite G-sets with respect to relations given by disjoint union decompositions. That is $B(G)$ is the free abelian group with the set of equivalence classes of transitive G-sets as a basis, two transitive G-sets G/H and G/K being equivalent if and only if H and K are conjugate in G.

A G-stable function $f : \mathcal{S}(G) \to E$ can naturally be viewed as a $\mathbb{Z}$-linear map from $B(G)$ to E by defining $f(G/H) = f(H)$ for any subgroup $H \leqslant G$, and extending $\mathbb{Z}$-linearly to $B(G)$. Since f is constant on conjugacy classes of subgroups, this function is well defined. Let $\mathcal{P}$ be a G-poset. For $k \geqslant 0$, let Δ_k be the set of chains of length k in $\Delta(\mathcal{P})$. Clearly Δ_k is a G-set. The alternating sum

$$\Lambda_G(\mathcal{P}) = \sum_k (-1)^k \Delta_k = \sum_{c \in \Delta(\mathcal{P})/G} (-1)^{|c|} G/G_c$$

is an element of $B(G)$ called the *reduced Lefschetz G-set* of $\mathcal{P}$, which is G-homotopy invariant (see [Th]). That is, if $\mathcal{P}$ is G-homotopy equivalent to $\mathcal{Q}$ for some G-poset $\mathcal{Q}$, then $\Lambda_G(\mathcal{P}) = \Lambda_G(\mathcal{Q})$ in $B(G)$.

As f is $\mathbb{Z}$-linear, applying f to $\Lambda_G(\mathcal{P})$ we have

$$f(\Lambda_G(\mathcal{P})) = \sum_{c \in \Delta(\mathcal{P})/G} (-1)^{|c|} f(G_c) = A(f, \mathcal{P}/G).$$

Therefore, the following is true:

Lemma 5.1.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be G-posets. If $\mathcal{P}$ and $\mathcal{Q}$ are G-homotopy equivalent, then $A(f, \mathcal{P}/G) = A(f, \mathcal{Q}/G)$.*

An immediate consequence to Lemma 5.1.2 is that the alternating sum of a G-stable function over the Brown complex for G and some p is the equal to the alternating sum over the Quillen complex or Bouc complex or Robinson complex, as these complexes are all G-homotopy equivalent.

A special case to Lemma 5.1.2 is the following:

Corollary 5.1.3. *If $\mathcal{P}$ is G-contractible, then $A(f, \mathcal{P}/G) = 0$.*

Proof. Recall from section 2.2 that we demand each poset $\mathcal{P}$ has a unique minimal element 0 and the k-simplices of the order complex $\mathcal{O}(\mathcal{P})$ are the chains in $\Delta(\mathcal{P})$ of length $k+1$ starting from 0. In particular the vertex set of $\mathcal{O}(\mathcal{P})$ is $\mathcal{P}\backslash\{0\}$ with $0 < x \in \mathcal{P}$ identified with $\{0 < x\}$. Assume $\mathcal{P}$ is contractible. By Lemma 5.1.2, we may assume the order complex of $\mathcal{P}$ is a point, so $\mathcal{P} = \{0, 1\}$. Then $\Delta(\mathcal{P})$ consists of two chains, $\{0\}$ and $\{0, 1\}$, both stabilized by G. Therefore,

$$A(f, \mathcal{P}/G) = f(G) - f(G) = 0.$$

□

Thus in order to prove the vanishing of an alternating sum, it suffices to show that $\mathcal{P}$ is G-contractible. On the other hand, when the poset is not G-contractible, one still can reduce the alternating sum to a smaller poset which is G-homotopy equivalent to $\mathcal{P}$, or eliminate most chains in $\mathcal{P}$ by applying Lemma 5.1.2 to certain subposets of $\mathcal{P}$.

The combinatorial approach is based on a pretty simple-minded idea. Namely one can achieve cancelation by pairing certain chains which have the same normalizers but different length parity. Interesting examples exhibiting this idea can be found in [KR]. In this section we extend some of the existing techniques in a functorial way and later study some useful examples.

5.2 The Combinatorial Approach

Let G be a finite group, $\mathcal{P}$ a G-poset, and $f : \mathcal{S}(G) \to E$ a G-stable function. Let $\tau : \mathcal{P} \to \mathcal{P}$ be a G-equivariant map of posets. So τ preserves ordering and for all $g \in G$ and $x \in \mathcal{P}$, $\tau(gx) = g(\tau(x))$. Consequently, $G_x \leqslant G_{\tau(x)}$ for all $x \in \mathcal{P}$. We say τ is *admissible* if one of the following holds:

(1) $\tau(x) \geqslant x$ for all $x \in \mathcal{P}$;

(2) $\tau(x) \leqslant x$ for all $x \in \mathcal{P}$ and $\tau(x) > 0$ if $x > 0$.

In the first case, it is possible that $\tau(0) > 0$. In the second case, $\tau(0) \leqslant 0$ so $\tau(0) = 0$.

Recall τ is *idempotent* if $\tau(\tau(x)) = \tau(x)$ for $x \in \mathcal{P}$. The identity map is a trivial example of G-equivariant, admissible and idempotent map.

A subset X of elements in $\mathcal{P}$ is *non-degenerate* with respect to τ if the restriction of τ to X is not the identity map, in which case we say τ is non-degenerate on X. In particular a chain $c \in \Delta(\mathcal{P})$ is non-degenerate with respect to τ if it is non-degenerate as a subset of elements of $\mathcal{P}$.

Example 5.2.1. Let $\mathcal{S} = \mathcal{S}_p(G)$ be the Brown complex of G at p, that is $\mathcal{S}$ is the poset of p-subgroups of G ordered by inclusion. Fix a non-trivial p-subgroup P of G and set $H = N_G(P)$. Then $\mathcal{S}(\leqslant P)$ is an H-poset. Define

$$\tau : \mathcal{S}(\leqslant P) \to \mathcal{S}(\leqslant P), \quad Q \mapsto Q\Phi(P)$$

where $\Phi(P)$ is the Frattini subgroup of P. As $\Phi(P)$ is a characteristic subgroup of P, $Q\Phi(P) \in \mathcal{S}(\leqslant P)$ and $\tau(Q^h) = Q^h\Phi(P) = (Q\Phi(P))^h$ for $h \in H$, so τ is an H-equivariant map. τ is admissible because $\tau(Q) \geqslant Q$. Apparently $\tau(\tau(Q)) = \tau(Q)$ so τ is idempotent.

Similarly we can define

$$\lambda : \mathcal{S}(\leqslant P) \to \mathcal{S}(\leqslant P), \quad Q \mapsto Q\Omega_1(Z(P)).$$

Again as $\Omega_1(Z(P))$ is a characteristic subgroup of P, one can check that λ is a well defined H-equivariant, admissible and idempotent map on $\mathcal{S}(\leqslant P)$.

Example 5.2.2. Let $\mathcal{P} = \mathcal{P}(V)$ where V is a unitary space and $G \leqslant GU(V)$. Fix a subspace U of V and set $H = N_G(U)$. Then $\mathcal{P}(\leqslant U)$ is an H-poset. Define

$$\tau : \mathcal{P}(\leqslant U) \to \mathcal{P}(\leqslant U), \quad W \mapsto W + \mathrm{Rad}(U).$$

τ preserves inclusion and hence is a map of posets. Moreover it is H-equivariant as H stabilizes $\mathrm{Rad}(U)$. τ is admissible because $\tau(W) \geqslant W$. Apparently $\tau(\tau(W)) = \tau(W)$

so τ is idempotent. Observe that τ is the identity map if U is non-degenerate, and τ maps all the elements to U if U is totally isotropic.

Lemma 5.2.3. (Pairing Lemma). *Let $\tau : \mathcal{P} \to \mathcal{P}$ be a G-equivariant, idempotent, admissible map. Let $\Delta \subseteq \Delta(\mathcal{P})$ be the set of non-degenerate chains with respect to τ in $\mathcal{P}$.*

(1) τ induces a G-equivariant permutation $\zeta = \zeta_\tau$ on Δ of order 2 such that for any $C \in \Delta$, $|\zeta(C)| = |C| \pm 1$ and $G_C = G_{\zeta(C)}$.

(2) $A(f, \Gamma/G) = 0$ for any ζ-stable G-subset $\Gamma \subseteq \Delta$. In particular $A(f, \Delta/G) = 0$.

(3) Let $H \leqslant G$ and Γ a ζ-stable H-subset of Δ. Then $A(f, \Gamma/H) = 0$.

(4) $A(f, \mathcal{P}/G) = A(f, \mathcal{P}_\tau/G)$. Here $\mathcal{P}_\tau$ is the subposet consisting of elements in $\mathcal{P}$ fixed by τ.

Proof. For $x \in \mathcal{P}$, set $\tau(x) = x'$. Let $C : 0 = c_0 < c_1 < \cdots < c_s$ be a chain in Δ.

Assume $\tau(x) \geqslant x$ for all $x \in \mathcal{P}$. By the choice of C, τ is non-degenerate on C. So there is some $c_r \in C$, $0 \leqslant r \leqslant s$, with $c_r' > c_r$. Choose r to be maximal with this property. If $r < s$, by the maximality of r, $c_{r+1}' = c_{r+1}$; hence $c_r' \leqslant c_{r+1}' = c_{r+1}$. Now if $r < s$ and $c_r' = c_{r+1}$, set $C' = C \backslash \{c_r'\}$; Otherwise (i.e., if $r = s$, or if $r < s$ and $c_r' \neq c_{r+1}$), set $C' = C \cup \{c_r'\}$. Then we define $\zeta : \Delta \to \Delta$ by $\zeta(C) = C'$.

Now assume $\tau(x) \leqslant x$ for all $x \in \mathcal{P}$. As τ is admissible, by definition $\tau(x) > 0$ for $x > 0$. By the choice of C there is some c_r, such that $c_r > c_r' > 0$. Choose r to be minimal with property. As $\tau(0) = 0$, it follows that $1 \leqslant r \leqslant s$. Then by the minimality of r, $c_{r-1} = c_{r-1}' \leqslant c_r'$. If $c_r' > c_{r-1}$, set $C' = C \cup \{c_r'\}$. If $c_r' = c_{r-1}$, set $C' = C \backslash \{c_r'\}$. Notice that in the second case C' still begins with 0 as $c_r' > 0$. Again we define $\zeta : \Delta \to \Delta$ by $\zeta(C) = C'$.

By hypothesis τ is G-equivariant. Hence if C is non-degenerate with respect to τ, so is C^g for any $g \in G$. Therefore, Δ is a G-set. In both cases, $c_r \in C'$ with $\tau(c_r) \neq c_r$. So C' is non-degenerate, hence $C' \in \Delta$. Therefore, ζ acts on Δ. As τ is G-equivariant, so is ζ. By the choices of c_r and the fact that τ is idempotent, it is easy to check that $\zeta(C') = C$. So ζ is a permutation of order 2. Obviously $|\zeta(C)| = |C| \pm 1$. Finally, as τ is admissible and in both cases the inserted or deleted element is $\tau(c_\tau)$, we have $G_C = G_{\zeta(C)}$. Therefore, ζ is a well defined permutation on Δ which satisfies all the desired properties. Thus part (1) is proved.

Let Γ be a G-subset of Δ with $\zeta(\Gamma) = \Gamma$. Then by part (1), for any $C \in \Gamma$, $f(G_C) = f(G_{\zeta(C)})$; So the contribution in $A(f, \Gamma/G)$ of C cancels with the contribution of $\zeta(C)$. As ζ is G-equivariant and involutary, it follows that $A(f, \Gamma/G) = 0$. Part (2) is proved.

For $H \leqslant G$, regard $\mathcal{P}$ as an H-poset. Certainly τ is H-equivariant and admissible, so by applying part (1) and part (2), we have $A(f, \Gamma/H) = 0$. Part (3) is proved.

Observe that $\Delta(\mathcal{P})$ is the disjoint union of Δ with $\Delta(\mathcal{P}_\tau)$. So Part (4) follows from part (2).

□

Remark 5.2.4. (1) We call (τ, Γ) as in part (2) of Lemma 5.2.3 a G-equivariant *canceling pair.* Lemma 5.2.3 says that one can reduce an alternating sum by finding a canceling pair.

(2) Under some extra hypothesis, one can apply a stronger version of Quillen's Fibre Theorem (see Proposition 1.6 in [Q] and Theorem 1 in [ThW]) to show that τ is indeed a G-homotopy equivalence (which happens in many cases). So Lemma 5.2.3.4 follows from Lemma 5.1.2.

Corollary 5.2.5. *Let $\mathcal{P}$ be a G-poset and Δ a G-set of chains in $\mathcal{P}$. If (τ_i, Δ_i), $1 \leqslant i \leqslant r$, are G-equivariant canceling pairs such that Δ is the disjoint union of Δ_i, then $A(f, \Delta/G) = 0$.*

Proof. This follows from the proceeding lemma as $A(f, \Delta/G) = \sum_i A(f, \Delta_i/G)$.

□

Corollary 5.2.6. *Let $\mathcal{P}$ be a G-poset and f a G-stable function.*

(1) Let $\mathcal{P}_0 = \mathcal{P}$. For $1 \leqslant i \leqslant n$, let τ_i be a G-equivariant admissible idempotent map on $\mathcal{P}$ and $\mathcal{P}_i = (\mathcal{P}_{i-1})_{\tau_i}$. Then

$$A(f, \mathcal{P}/G) = A(f, \mathcal{P}_n/G).$$

(2) Let τ_i, $1 \leqslant i \leqslant n$, be G-equivariant admissible idempotent maps on $\mathcal{P}$ which commute with each other. That is, $\tau_i(\tau_j(x)) = \tau_j(\tau_i(x))$ for all $x \in \mathcal{P}$ and all i, j. Then

$$A(f, \mathcal{P}/G) = A(f, (\bigcap_{i=1}^{n} \mathcal{P}_{\tau_i})/G).$$

Proof. In part (1), observe that for each $i \geqslant 1$, by Lemma 5.2.3.4,

$$A(f, \mathcal{P}_{i-1}/G) = A(f, \mathcal{P}_i/G).$$

So part (1) holds. As for part (2), set $\mathcal{P}_0 = \mathcal{P}$ and

$$\mathcal{P}_i = \bigcap_{j=1}^{i} \mathcal{P}_{\tau_j}$$

for $1 \leqslant i \leqslant n$. As τ_i commutes with τ_j for all i, j, it follows that τ_i acts on $\mathcal{P}_{i-1}$ and indeed a G-equivariant admmisible idempotent map on $\mathcal{P}_{i-1}$. Moreover, $\mathcal{P}_i = (\mathcal{P}_{i-1})_{\tau_i}$. So Part (2) follows from part (1).

□

For $x \in \mathcal{P}$, recall the *star* of x is

$$\mathrm{St}(x) = \mathcal{P}(\leqslant x) \cup \mathcal{P}(\geqslant x).$$

If Δ is a set of chains in $\mathcal{P}$, then a map $\theta : \Delta \to \mathcal{P}$ is called a *signalizer* for Δ if $\theta(C) \in C$ for all $C \in \Delta$.

Lemma 5.2.7. *Let $\mathcal{P}$ be a G-poset, Δ a G-set of chains in $\mathcal{P}$, and $\theta : \Delta \to \mathcal{P}$ a G-equivariant signalizer. Assume for each $x \in \theta(\Delta)$, there is a G_x-equivariant, admissible idempotent map $\tau_x : St(x) \to St(x)$, such that $(\tau_x, \theta^{-1}(x))$ is a G_x-equivariant canceling pair. Then $A(f, \Delta/G)$=0.*

Here $\theta^{-1}(x)$ is the set of pre-images of x.

Proof. As θ is G-equivariant and $\Delta = \coprod_{x\in\theta(\Delta)} \theta^{-1}(x)$, it follows that

$$\Delta/G = \coprod_{x\in\mathcal{P}/G} \theta^{-1}(x)/G_x.$$

Therefore, as $G_C = N_{G_{\theta(C)}}(C)$ for all $C \in \Delta$,

$$A(f,\Delta/G) = \sum_{x\in\mathcal{P}/G} \sum_{C\in\theta^{-1}(x)/G_x} f(G_C) = \sum_{x\in\mathcal{P}/G} \sum_{C\in\theta^{-1}(x)/G_x} f(N_{G_{\theta(C)}}(C))$$

$$= \sum_{x\in\mathcal{P}/G} A(f,\theta^{-1}(x)/G_x).$$

By hypothesis $(\tau_x, \theta^{-1}(x))$ is a G_x-equivariant canceling pair, so by Lemma 5.2.3,

$$A(f,\theta^{-1}(x)/G_x) = 0$$

for all $x \in \mathcal{P}$. Therefore, $A(f,\Delta/G) = 0$.

□

Remark 5.2.8. (1) Lemma 5.2.7 can be combined with Corollary 5.2.5. That is, one may define more than one admissible idempotent map on $\mathrm{St}(x)$ so that $\theta^{-1}(x)$ satisfies the hypothesis of Corollary 5.2.5 to make the cancelation.

(2) It turns out that in most existing examples we have worked on, τ_x is obtained by extending a G_x-equivariant admissible idempotent map λ_x on $\mathcal{P}(\geqslant x)$ (resp. $\mathcal{P}(\leqslant x)$) to $\mathrm{St}(x)$ by letting $\lambda_x(y) = y$ for all $y < x$ (resp. $y > x$).

Example 5.2.9. Assume a G-poset $\mathcal{P}$ contains a maximal element $1 \neq 0$. It is easy to show that $\mathcal{P}$ is G-contractible and hence $A(f,\mathcal{P}/G) = 0$. Alternatively define $\tau : \mathcal{P} \to \mathcal{P}$ by mapping all elements to 1. As G fixes 1, τ is a well defined G-equivariant map on $\mathcal{P}$ and admissible. $\tau(\tau(x)) = 1 = \tau(x)$ for all $x \in \mathcal{P}$. Each chain $C \in \Delta(\mathcal{P})$ is non-degenerate as C contains 0 and $\tau(0) = 1 > 0$. Therefore, by Lemma 5.2.3, $A(f,\mathcal{P}/G) = 0$.

If $\mathcal{P}$ is a G-poset and $x, y \in \mathcal{P}$, the *join* $x \vee y$ is the minimal upper bound for x and y, if it exists. It may or may not exist in $\mathcal{P}$. Example 5.2.9 is a special case of the following lemma.

Lemma 5.2.10. *Let $\mathcal{P}$ be a G-poset. Assume $0 < z \in \mathcal{P}$ is fixed by G and $x \vee z$ exists in $\mathcal{P}$ for all $x \in \mathcal{P}$. Then $A(f, \mathcal{P}/G)=0$.*

Proof. $\mathcal{P}$ is conically contractible in Quillen's sense (see [Q]) and hence contractible. It can be proved that $\mathcal{P}$ is G-contractible. So the lemma follows from Corollary 5.1.3. Alternatively we can prove the lemma combinatorially.

Define $\tau : \mathcal{P} \to \mathcal{P}$ by $\tau(x) = x \vee z$. By hypothesis τ is a well defined map of posets. For $g \in G$ and $x \in \mathcal{P}$, claim $g(x \vee z) = (gx) \vee z$.

Indeed, as $x \vee z \geqslant x, z$, $g(x \vee z) \geqslant gx$ and $g(x \vee z) \geqslant gz = z$, so $g(x \vee z)$ is an upper bound for gx and z. For any $y \in \mathcal{P}$ with $y \geqslant gx, z$, we have $g^{-1}y \geqslant x, z$, so $g^{-1}y \geqslant x \vee z$. It follows that $y \geqslant g(x \vee z)$. Therefore, by definition of the join, $g(x \vee z) = (gx) \vee z$. So the claim is true.

Therefore, τ is G-equivariant. By construction τ is admissible. τ is idempotent as $(x \vee z) \vee z = x \vee z$. Each chain $c \in \Delta(\mathcal{P})$ is non-degenerate as $0 \in c$ and $\tau(0) = z > 0$. Therefore, by Lemma 5.2.3, $A(f, \mathcal{P}/G) = 0$. Done.

□

Example 5.2.11. Let p be a prime, $\mathcal{S} = \mathcal{S}_p(G)$ as in Example 5.2.1 and $\mathcal{A} = \mathcal{A}_p(G)$ the Quillen complex of G. That is $\mathcal{A}$ is the poset of the elementary abelian p-subgroups of G ordered by inclusion. Then $A(f, \mathcal{S}/G) = A(f, \mathcal{A}/G)$. We have seen that this can be proved by showing that $\mathcal{S}$ and $\mathcal{A}$ are G-homotopy equivariant. It can also be proved combinatorially. For instance, Knorr and Robinson's proof in Proposition 3.3 of [KR] can be interpretated as follows. Let $\Delta = \Delta(\mathcal{S}) \backslash \Delta(\mathcal{A})$. It suffices to show $A(f, \Delta/G) = 0$.

Notice that $C \in \Delta$ if and only if the final term of C is not elementary abelian. Define $\theta : \Delta \to \mathcal{S}$ by letting $\theta(C)$ be the final term of C. This is a G-equivariant signalizer. Fix $C \in \Delta$ and set $\theta(C) = P$. Extend $\tau = \tau_P$ in Example 5.2.1 to $\mathrm{St}(P)$ as in part (2) of Remark 5.2.8, that is define $\tau(Q) = Q = Q\Phi(P)$ for $P \leqslant Q$.

Then τ is a well defined $N_G(P)$-equivariant, admissible idempotent map on $\mathrm{St}(P)$. $\theta^{-1}(P)$ consists of all the chains in $\mathcal{S}$ whose final term is P. τ is non-degenerate on $D \in \theta^{-1}(P)$ because $1 \in D$ and $\tau(1) = \Phi(P) > 1$. As required in Lemma 5.2.3, ζ_τ acts on $\theta^{-1}(P)$ since $\zeta_\tau(D)$ is obtained by inserting or deleting a proper subgroup of P into or from D and consequently P remains the final term of $\zeta_\tau(D)$. So $(\tau_P, \theta^{-1}(P))$ is an $N_G(P)$-equivariant canceling pair. By Lemma 5.2.7, $A(f, \Delta/G) = 0$.

Alternatively we can define $\theta : \Delta \to \mathcal{S}$ by letting $\theta(C)$ be the first nonzero term of C which is not elementary abelian. Fix $C \in \Delta$ and set $\theta(C) = P$. Extend $\lambda = \lambda_P$ in Example 5.2.1 to $\mathrm{St}(P)$ as in part (3) of Remark 5.2.8. It is an easy exercise to check that the hypothesis in Lemma 5.2.7 is satisfied, and hence $A(f, \Delta/G) = 0$.

Example 5.2.12. Let $\mathcal{P}$ be the poset of proper subspaces of a unitary space V ordered by inclusion and $G \leqslant GU(V)$. Let $\mathcal{P}_1$ be the subposet of $\mathcal{P}$ on the set of subspaces U such that $\mathrm{Rad}(U) > 0$ together with 0, and $\mathcal{P}_2$ the subposet of $\mathcal{P}_1$ on the set of totally isotropic subspaces including 0. Then $A(f, \mathcal{P}_1/G) = A(f, \mathcal{P}_2/G)$. As an application of Lemma 5.1.2, this can be proved by showing the embedding $i : \mathcal{P}_2 \to \mathcal{P}_1$ is a G-homotopy equivalence. We omit the proof here. It can also be proved combinatorially.

Set $\Delta = \Delta(\mathcal{P}_1)\backslash\Delta(\mathcal{P}_2)$. So $C \in \Delta$ if and only if C contains a member U with $0 < \mathrm{Rad}(U) < U$. Define $\theta : \Delta \to \mathcal{P}_1$ by letting $\theta(C)$ be the first (minimal) member which is not totally isotropic. As G is an isometry group, θ is a G-equivariant signalizer. Fix $C \in \Delta$ and set $\theta(C) = U$. Extend $\tau = \tau_U$ in Example 5.2.2 to $\mathrm{St}(U)$ as in part (3) of Remark 5.2.8. It is a well defined $N_G(U)$-equivariant, admissible idempotent map on $\mathrm{St}(U)$. Obviously $\theta^{-1}(U)$ consists of all the chains D in $\mathcal{P}_1$ containing U such that $D(< U)$ consists of totally isotropic subspaces. τ is non-degenerate on $D \in \theta^{-1}(U)$ because $0 \in D$ and $\tau(0) = \mathrm{Rad}(U) > 0$. As required in Lemma 5.2.3, ζ_τ acts on $\theta^{-1}(U)$ since $\zeta_\tau(D)$ is obtained by inserting or deleting a totally isotropic subspace into or from D and therefore $\zeta_\tau(D) \in \theta^{-1}(U)$. So $(\tau_U, \theta^{-1}(U))$ is a $N_G(U)$-equivariant canceling pair. By Lemma 5.2.7, $A(f, \Delta/G) = 0$.

The following two results, Lemma 5.2.13 and Proposition 5.2.16 are of key importance in our reduction theorems proved in the later chapters.

Lemma 5.2.13. *Let V be a finite dimensional vector space over a field $\mathbb{F}$, $\mathcal{P} = \mathcal{P}(V)$, $G = GL(V)$, and $H \leqslant N_G(W)$ for some subspace $0 < W < V$. Let f be an H-stable function. Then*

$$A(f, \mathcal{P}/H) = A(f, \Gamma/H)$$

where Γ consists of all the chains in $\mathcal{P}$ containing a complement to W.

Proof. Let $\mathcal{Q}$ be the subposet of $\mathcal{P}$ on the set of subspaces which are not the complements of W. Observe $\mathcal{Q}$ is an H-poset and H acts on Γ. Moreover $\Delta(\mathcal{P}) = \Delta(\mathcal{Q}) \coprod \Gamma$. So the lemma is equivalent to $A(f, \mathcal{Q}/H) = 0$. But this follows from Lemma 5.2.10 as $0 < W \in \mathcal{Q}$ is fixed by H and for each $U \in \mathcal{Q}$, $U \vee W = U + W \in \mathcal{Q}$. The lemma is proved.

□

For the rest of this section, let V be an n-dimensional unitary space over a field $\mathbb{F}$ and $G = GU(V)$. Let $\mathcal{P} = \mathcal{P}(V)$ and $\Delta = \Delta(\mathcal{P})$.

Definition 5.2.14. A chain

$$c \in \Delta : 0 < V_1 < V_2 < \cdots < V_s \tag{5.1}$$

is called a *normal* chain if there exists $0 = i_0 < i_1 < \cdots < i_k \leqslant s$, $k \geqslant 0$, such that

(1) V_{i_j} is non-degenerate or 0 for all $0 \leqslant j \leqslant k$;

(2) For each $0 \leqslant j \leqslant k$ and any $i_j < i < i_{j+1}$ (assume $i_{k+1} = s + 1$ and $V_{s+1} = V$), $V_i = V_{i_j} \bigoplus \mathrm{Rad}(V_i)$.

Clearly chains of totally isotropic subspaces are normal. If c is a normal chain, it is *singular* if c consists of totally isotropic subspaces. Otherwise we say c is *non-singular*. If c is non-singular, the *non-singular rank* of c is the dimension of the minimal non-degenerate member of c.

Let Γ be the set of normal chains in Δ. Let Δ_n be the set of chains in $\mathcal{P}$ which don't contain a non-degenerate subspace. Set $\Gamma_n = \Delta_n \cap \Gamma$. Observe Γ_n consists of the singular normal chains in Δ. For $1 \leqslant r \leqslant n-1$, let Δ_r be the set of chains in $\mathcal{P}$ whose minimal non-degenerate member has dimension r. Set $\Gamma_r = \Delta_r \cap \Gamma$. Observe Γ_r consist of the non-singular normal chains of non-singular rank r in Δ. Also observe that Δ (resp. Γ) is the disjoint union of Δ_r (resp. Γ_r) for $1 \leqslant r \leqslant n$. As G preserves isometry type, Γ, Δ_r and Γ_r are G-sets.

Lemma 5.2.15. *Fix $r < n$ and an r-dimensional non-degenerate subspace U of V. We write $\mathcal{P}^r$, Δ^r, Γ^r, G^r, etc. for the corresponding sets or isometry group defined for U. For instance $G^r = GU(U)$. Set $H = N_G(U)$. Then $H = G^r \times G^{n-r}$. Denote the set of chains in Δ_r containing U by $\Delta_r(U)$. and set $\Gamma_r(U) = \Delta_r(U) \cap \Gamma_r$.*

(1) Δ_r/G is in 1-1 correspondence with $\Delta_r(U)/N_G(U)$.

(2) Define

$$\theta : \Delta_r(U) \to \Delta_r^r \times \Delta^{n-r}$$

as follows. If $c \in \Delta_r(U)$ is represented as in (5.1) with $U = V_j \in c$ for some j, then $\theta(c) = (c_1, c_2)$ where

$$c_1 \in \Delta_r^r : 0 < V_1 < \cdots < V_{j-1} \text{ and}$$
$$c_2 \in \Delta^{n-r} : 0 < V_{j+1} \cap U^\perp < \cdots < V_{|c|} \cap U^\perp.$$

Then θ is an H-equivariant 1-1 correspondence with $|c| = |c_1| + |c_2| - 1$. In particular $G_c = G^r_{c_1} \times G^{n-r}_{c_2}$.

(3) The statements in part (1) and (2) hold for the corresponding set of normal chains in Δ_r. That is, Γ_r/G is in 1-1 correspondence with $\Gamma_r(U)/N_G(U)$, and θ restricts to a H-equivariant 1-1 correspondence between $\Gamma_r(U)$ and $\Gamma_r^r \times \Gamma^{n-r}$.

The proof is straightforward and omitted.

Proposition 5.2.16. $A(f, \mathcal{P}/G) = A(f, \Gamma/G)$.

Proof. We induct on n. If $n \leqslant 2$, the statement is trivial as $\Delta = \Gamma$. Assume the lemma for spaces of dimension less than n.

We show

$$A(f, \Delta_r/G) = A(f, \Gamma_r/G) \tag{5.2}$$

for each $1 \leqslant r \leqslant n$. Then as

$$A(f, \mathcal{H}/G) = \sum_{r=1}^{n} A(f, \mathcal{H}_r/G)$$

for $\mathcal{H} = \Delta, \Gamma$, the statement follows.

Observe Δ_n is the set of chains in $\mathcal{P}$ which consists of subspaces U with $\mathrm{Rad}(U) > 0$ and Γ_n is the set of chains of totally isotropic subspaces of V. So by Example 5.2.12, equality (5.2) holds for $r = n$.

Fix $r < n$ and an r-dimensional non-degenerate subspace U of V. Set $H = N_G(U)$. Then by Lemma 5.2.15, Δ_r/G is in 1-1 correspondence with $(\Delta_r^r \times \Delta^{n-r})/H$ which is in 1-1 correspondence with $\Delta_r^r/G^r \times \Delta^{n-r}/G^{n-r}$. And a similar statement holds for Γ_r. Therefore,

$$\begin{aligned}\sigma = A(f, \Delta_r/G) &= \sum_{c \in \Delta_r/G} (-1)^{|c|} f(G_c) \\ &= - \sum_{c_1 \in \Delta_r^r/G^r} (-1)^{|c_1|} \sum_{c_2 \in \Delta^{n-r}/G^{n-r}} (-1)^{|c_2|} f(G^r_{c_1} \times G^{n-r}_{c_2}).\end{aligned}$$

But for a fixed $c_1 \in \Delta_r^r$, $f_{c_1}(K) = f(G^r_{c_1} \times K)$ for $K \leqslant G^{n-r}$ defines a G^{n-r}-stable function. Therefore, by induction,

$$\sigma = - \sum_{c_1 \in \Delta_r^r/G^r} (-1)^{|c_1|} A(f_{c_1}, \Delta^{n-r}/G^{n-r}) = - \sum_{c_1 \in \Delta_r^r/G^r} (-1)^{|c_1|} A(f_{c_1}, \Gamma^{n-r}/G^{n-r}).$$

By exchanging the order of summation, we have

$$\sigma = -\sum_{d_2 \in \Gamma^{n-r}/G^{n-r}} (-1)^{|d_2|} \sum_{c_1 \in \Delta_r^r/G^r} (-1)^{|c_1|} f(G^r_{c_1} \times G^{n-r}_{d_2}).$$

Similarly for each fixed $d_2 \in \Delta^{n-r}/G^{n-r}$, $f_{d_2}(K) = f(K, G^{n-r}_{d_2})$ for $K \leqslant G^r$ defines a G^r-stable function. By Example 5.2.12,

$$\sigma = -\sum_{d_2 \in \Gamma^{n-r}/G^{n-r}} (-1)^{|d_1|} A(f_{d_2}, \Delta^r_r/G^r) = -\sum_{d_2 \in \Gamma^{n-r}/G^{n-r}} (-1)^{|d_1|} A(f_{d_2}, \Gamma^r_r/G^r).$$

Finally

$$\sigma = \sum_{(d_1 \times d_2) \in \Gamma^r_r/G^r \times \Gamma^{n-r}/G^{n-r}} (-1)^{|d_1|+|d_2|-1} f(G^r_{d_1} \times G^{n-r}_{d_2}) = A(f, \Gamma_r/G).$$

Therefore, equality (5.2) is proved. This completes the proof of the lemma.

□

Next we consider the truncation of Δ. For $0 \leqslant m < n$, let $\Delta(m)$ be the set of chains of subspaces of V whose final term has dimension m. Set $\Delta(n) = \Delta$. For $0 \leqslant m \leqslant n$, let $\Gamma(m) = \Delta(m) \cap \Gamma$.

Proposition 5.2.17. *For each $m \leqslant n$, $A(f, \Delta(m)/G) = A(f, \Gamma(m)/G)$.*

The proof is identical to the proof of Proposition 5.2.16. The only reason we chose to prove Proposition 5.2.16 instead of proving Proposition 5.2.17 directly was to minimize notation.

5.3 Decomposable functions

We now discuss some techniques for evaluating alternating sums through the study of the properties of f. We develop some machinery to deal with the alternating sum of a function which can be decomposed. All sets here are finite.

Let G be a group. We have seen that a G-stable function $f : \mathcal{S}(G) \to E$ can be viewed as a $\mathbb{Z}$-linear map from $B(G)$ to E. In practice, however, we are only interested in the values of f on certain collections $\mathcal{G}$ of conjugacy classes of subgroups of G (instead of on all subgroups). Often $\mathcal{G}$ is the collection of stabilizers in G of the elements in some G-set. Therefore, we may define and work with G-stable functions on a G-set X, which is defined in a weaker sense comparing with Definition 5.1.1.

Definition 5.3.1. Let X be a G-set and E an abelian group. A function $f : X \to E$ is *G-stable on X* if it is contstant on G-orbits on X.

Remark 5.3.2. Clearly a G-stable function $f : X \to E$ is a G-stable function on X, but not so conversely. Definition 5.3.1 is weaker than Definition 5.1.1 in the sense that the values of f on the subgroups of G which are not the stabilizers of elements of X are not defined, and that it does not require $f(x) = f(y)$ if $G_x = G_y$.

We write E^X for the set of G-stable functions on X. X is *graded* if there is a G-stable function $r \in \mathbb{Z}^X$. $r(x)$ is usually denoted as $|x|$ and called the rank of x.

If X is a G-set and $f \in E^X$, the sum of f over X/G is:

$$S(f, X/G) = \sum_{x \in X/G} f(x).$$

If X is a graded G-set, the alternating sum of f over X/G is:

$$A(f, X/G) = \sum_{x \in X/G} (-1)^{|x|} f(x).$$

We may abbreviate $S(f, X/G)$ and $A(f, X/G)$ as $S(f)$ and $A(f)$, respectively. Notice if $X = \Delta(\mathcal{P})$ for some G-poset $\mathcal{P}$, then X is a graded G-set with the rank function being the length function. In this case our definition of the alternating sum of f over X/G coincides with the definition in the beginning of section 5.1.

Remark 5.3.3. Let X be a G-set and $f \in E^X$. We say f can be extended to a G-stable function $h : \mathcal{S}(G) \to E$ if $f(x) = h(G_x)$ for any $x \in X$. Observe this is possible if and only if $f(x) = f(y)$ whenever $G_x = G_y$ for $x, y \in X$. Clearly in this case

$$A(f, X/G) = A(h, X/G).$$

Let X be a graded G-set and Y a G-set. Let $f \in E^{X \times Y}$. Then for $x \in X$, $y \in Y$, f induces the following functions:

$$f_x : Y \to E, \quad y \mapsto f(x,y); \quad f_x \text{ is a } G_x\text{-stable function on } Y;$$
$$f_y : X \to E, \quad x \mapsto f(x,y); \quad f_y \text{ is a } G_y\text{-stable function on } X.$$

We then define $B : E^{X \times Y} \to E^Y$ by

$$B(f) : Y \to E, \quad y \mapsto A(f_y, X/G_y)$$

and $T : E^{X \times Y} \to E^X$ by

$$T(f) : X \to E, \quad x \mapsto S(f_x, Y/G_x).$$

Clearly $B(f)$ is a G-stable function on Y and $T(f)$ is a G-stable function on X.

Lemma 5.3.4. *Let X be a graded G-set and Y a G-set, $f, g \in E^{X \times Y}$.*

(1) $A \circ T = S \circ B$;

(2) If $B(f) = B(g)$, then $A(T(f)) = S(B(f))$.

Proof.

$$\begin{aligned} A(T(f)) &= \sum_{x\in X/G} (-1)^{|x|} T(f)(x) = \sum_{x\in X/G} (-1)^{|x|} S(f_x, Y/G_x) \\ &= \sum_{x\in X/G} (-1)^{|x|} \sum_{y\in Y/G_x} f(x,y) = \sum_{(x,y)\in (X\times Y)/G} (-1)^{|x|} f(x,y) \\ &= \sum_{y\in Y/G} \sum_{x\in X/G_y} (-1)^{|x|} f(x,y) = \sum_{y\in Y/G} A(f_y, X/G_y) \\ &= \sum_{y\in Y/G} B(f)(y) = S(B(f)). \end{aligned}$$

So part (1) is proved. Part (2) then follows immediately. □

Lemma 5.3.5. *Let Y, Z be two G-sets, $\phi : Y \to Z$ a G-equivariant map. f is a G-stable function on Y.*

(1) ϕ induces a map $\bar{\phi} : E^Y \to E^Z$ by

$$\bar{\phi}(f): \ Z \to E, \quad z \to S(f|_{\phi^{-1}(z)}, \phi^{-1}(z)/G_z), \ for\, f \in E^Y.$$

(2) $S(f, Y/G) = S(\bar{\phi}(f), Z/G)$. In particular if $S(\bar{\phi}(f)) = S(\bar{\phi}(g))$ for some $g \in E^Y$, then $S(f) = S(g)$.

Proof. Part (1) is easy. As for part (2),

$$\begin{aligned} S(f, Y/G) &= \sum_{y\in Y/G} f(y) = \sum_{y\in Y} \frac{|G_y|}{|G|} f(y) \\ &= \sum_{z\in Z} \sum_{y\in \phi^{-1}(z)} \frac{|G_y|}{|G|} f(y) = \sum_{z\in Z} \sum_{y\in \phi^{-1}(z)/G_z} \frac{|G_y|}{|G|} \frac{|G_z|}{|G_y|} f(y) \\ &= \sum_{z\in Z} \frac{|G_z|}{|G|} \sum_{y\in\phi^{-1}(z)/G_z} f(y) = \sum_{z\in Z} \frac{|G_z|}{|G|} \bar{\phi}(f)(z) \\ &= \sum_{z\in Z/G} \bar{\phi}(f)(z) = S(\bar{\phi}(f), Z/G). \end{aligned}$$

□

Proposition 5.3.6. *Let X be a graded G-set. Let Y, Z be G-sets and $\phi : Y \to Z$ a G-equivariant map. Assume f, g are two G-stable functions on $X \times Y$.*

(1) ϕ induces a G-equivariant map

$$\Phi = 1 \otimes \phi : X \times Y \to X \times Z$$

by $\Phi(x, y) = (x, \phi(y))$, and hence induces

$$\bar{\Phi} : E^{X \times Y} \to E^{X \times Z}$$

with

$$\bar{\Phi}(f)(x, z) = S(f|_{\Phi^{-1}(x,z)}, \Phi^{-1}(x, z)/G_z).$$

(2) $T(\bar{\Phi}(f)) = T(f)$.

(3) If $B(\bar{\Phi}(f)) = B(\bar{\Phi}(g))$, then $A(T(f)) = A(T(g))$.

Proof. By Lemma 5.3.5, part (1) is clear. By definition of $\bar{\Phi}(f)$, we have

$$\begin{aligned} T(\bar{\Phi}(f))(x) &= \sum_{z \in Z/G_x} \bar{\Phi}(f)(x, z) = \sum_{z \in Z/G_x} \sum_{y \in \phi^{-1}(z)/G_{x,z}} f(x, y) \\ &= \sum_{y \in Y/G_x} f(x, y) = T(f)(x) \end{aligned}$$

for all $x \in X$, so $T(\bar{\Phi}(f)) = T(f)$. Consequently

$$A(T(f), X/G) = A(T(\bar{\Phi}(f)), X/G).$$

Assume $B(\bar{\Phi}(f)) = B(\bar{\Phi}(g))$, then by Lemma 5.3.4, $A(T(\bar{\Phi}(f)) = A(T(\bar{\Phi}(g))$. Hence $A(T(f)) = A(T(g))$.

□

Lemma 5.3.7. *Let G be a finite group, $\mathcal{P}$ a G-poset and Y a transitive G-set. Assume g is a G-stable function on $\Delta(\mathcal{P}) \times Y$ and $f = T(g)$. Then*

$$A(f, \Delta(\mathcal{P})/G) = A(g_z, \Delta(\mathcal{P})/H)$$

where $H = G_z$ for some $z \in Y$.

Proof. By definition

$$A(f, \Delta(\mathcal{P})/G) = A(T(g), \Delta(\mathcal{P})/G) = A \circ T(g, (\Delta(\mathcal{P}) \times Y)/G).$$

By Lemma 5.3.4,

$$\begin{aligned} A \circ T(g, (\Delta(\mathcal{P}) \times Y)/G) &= S \circ B(g, (\Delta(\mathcal{P}) \times Y)/G) \\ &= S(B(g), Y/G) = \sum_{y \in Y/G} B(g)(y). \end{aligned}$$

But G is transitive on Y, so z is a representative of the unique orbit in Y/G. Therefore, we have

$$A(f, \Delta(\mathcal{P})/G) = B(g)(z) = A(g_z, \Delta(\mathcal{P})/H).$$

□

Chapter 6 On Parabolic Actions, I

In this chapter, we study the action of a parabolic subgroup P of a finite unitary group on the successive quotients of the lower central series of its unipotent radical, which are called the internal modues for P. As we are mainly interested in the information on the orbits and stabilizers, often the statements are true up to conjugation.

Starting from section 6.2 we need to treat certain subgroups of general linear groups and unitary groups on vector spaces of different dimensions over $\mathbb{F}_{q^2}$ where q is prime power. We use superscripts to denote the dimension of the space on which the ambient group acts. To distinguish a subgroup of a linear group from a subgroup of a unitary group, we use a + sign in the superscript to indicate that the ambient group is a linear group. For instance, V^l is an l-dimensional vector space over $\mathbb{F}_{q^2}$, $G^{+n} = GL_n(q^2)$, $G^r = GU_r(q)$, P_J^r is a parabolic subgroup of G^r which is the stabilizer of a chain of totally isotropic subspaces of the natural module for G^r of type J, and L_J^{+r} is a Levi factor of the parabolic subgroup P_J^{+r} of $GL_r(q^2)$. Set $G^{\pm 0} = 1$.

6.1 Tensor Modules

In this section, $\mathbb{F}$ is an abitrary field, Ω_i is a group and V_i is an n_i-dimensional $\mathbb{F}\Omega_i$-module, $i = 1, 2$. Let $G = \Omega_1 \times \Omega_2$ and $V = V_1 \otimes_{\mathbb{F}} V_2$. For $g = (g_1, g_2) \in G$ and $v = v_1 \otimes v_2 \in V$, define $gv = g_1 v_1 \otimes g_2 v_2$ and extend this definition linearly to V. Under this construction V becomes a module for G. We are interested in parameterizing the orbits of G on $\mathrm{Irr}(V)$ and describing the stabilizer of orbits up to conjugation.

Example 6.1.1. Let V_1 be the set of n_1-dimensional column vectors and Ω_1 a subgroup of $GL(V_1)$ acting on V_1 by multiplication from the left; Let V_2 be the set of n_2-dimensional row vectors and Ω_2 a subgroup of $GL(V_2)$ acting from the right. So V_1 is the restriction to Ω_1 of the natural module for $GL_{n_1}(\mathbb{F})$ and V_2 is the restriction to Ω_2 of the dual module of the natural module for $GL_{n_2}(\mathbb{F})$. Fix the natural basis

$X^t = \{e^t_j; 1 \leqslant j \leqslant n_t\}$ for V_t, $t = 1, 2$. That is, e^t_j is the column/row vector all of whose entries are 0 except the j-th entry, which is 1. So $e^1_i \times e^2_j = e_{ij}$ is the ij-th matrix unit in $M_{n_1,n_2}(\mathbb{F})$, the set of $n_1 \times n_2$-matrices over $\mathbb{F}$. $\sum a_{ij}e^1_i \otimes e^2_j \mapsto \sum a_{ij}e_{ij} = (a_{ij})$ defines an $\mathbb{F}$-space isomorphism between V and $M_{n_1,n_2}(\mathbb{F})$. For $g = (g_1, g_2) \in G$ and $v \in V$, $gv = g_1 v g_2{}^{-1}$, where on the right-hand side the operation is matrix multiplication.

Lemma 6.1.2. *The following are G-isomorphic as abelian groups:*

$$Irr(V) = Hom(V, \mathbb{C}^*) \cong Hom(V, \mathbb{C}_p) \cong Hom_{\mathbb{F}_p}(V, \mathbb{F}_p) \cong Hom_{\mathbb{F}}(V, \mathbb{F}) \cong L(V_1, V_2; \mathbb{F}).$$

Here $L(V_1, V_2, \mathbb{F})$ is the space of $\mathbb{F}$-bilinear maps from $V_1 \times V_2$ to $\mathbb{F}$.

Proof. $\mathrm{Hom}(V, \mathbb{C}^*)$ is isomorphic to $\mathrm{Hom}(V, \mathbb{C}_p)$ because the values of any character of V are in $\mathbb{C}_p$. As $\mathbb{C}_p$ can be identified with the additive group of $\mathbb{F}_p$, $\mathrm{Hom}(V, \mathbb{C}_p)$ is G-isomorphic to $\mathrm{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p)$.

Regard $\mathbb{F}$ as a vector space over $\mathbb{F}_p$. Let $X = \{x_i; i \in A\}$ be a $\mathbb{F}_p$-basis of $\mathbb{F}$ where A is an index set. Assume $0 \in A$ and $x_0 = 1$. Define $\theta : \mathbb{F} \to \mathbb{F}_p$ by $\theta(\sum a_i x_i) = a_0$. Then $\theta \in \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}, \mathbb{F}_p)$ as θ is a projection. Fix an $\mathbb{F}$-basis $Y = \{u_1, \dots, u_k\}$ of V and let U be the $\mathbb{F}_p$-space spanned by Y. So $V = \mathbb{F} \otimes_{\mathbb{F}_p} U$ and $\{x_i \otimes u_j\}$ is an $\mathbb{F}_p$-basis of V. Now for $\varphi \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$, define $\bar{\varphi} \in \mathrm{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p)$ by $\bar{\varphi} = \theta \circ \varphi$. Check $\bar{\varphi}$ is well defined and $\varphi \mapsto \bar{\varphi}$ defines a G-isomorphism between the two abelian groups.

The fact that $\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ is G-isomorphic to $L(V_1, V_2, \mathbb{F})$ follows from the universal property of tensor product. Namely there is a bilinear map $\pi \in L(V_1, V_2, V)$ such that for any $\alpha \in L(V_1, V_2, \mathbb{F})$, there is a unique $\beta \in \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ with $\alpha = \beta\pi$. So $\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ is in 1-1 correspondence with $L(V_1, V_2, \mathbb{F})$. By checking the G-action directly, they are G-isomorphic.

□

Remark 6.1.3. Observe that the proof of the first three G-isomorphisms does not use the fact that G is a product of two groups or V is a tensor module for G. It applies to any group G and any $\mathbb{F}G$-module V.

Lemma 6.1.2 suggests that we can use the G-equivariant identification of each of these sets with $\mathrm{Irr}(V)$ and study the G-action on whichever set is most convenient to obtain information on the action of G on $\mathrm{Irr}(V)$. For this reason, we denote $L(V_1, V_2; \mathbb{F})$ by V^* which normally would denote the dual space $\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$, although we occasionally still use V^* to denote the dual space, in which case we will say so explicitly. Also from time to time, when we say $\tau \in \mathrm{Irr}(V)$ has a certain property, we may actually mean that the element corresponding to τ in V^* has that property.

Recall the action of G on V^* is defined as follows: For $f \in V^*$, $(v_1, v_2) \in V_1 \times V_2$ and $g = (g_1, g_2) \in G$ as above, $(gf)(v_1, v_2) = f(g_1^{-1}v_1, g_2^{-1}v_2)$. Set

$$R_1(f) = R_{\Omega_1}(f) = \{u \in V_1 \mid f(u, v) = 0, \ \forall v \in V_2\}.$$

$R_2(f)$ is defined similarly. It is easy to check that $R_i(f) = V_i$ if $f = 0$ and $R_i(f)$ is a proper subspace of V_i if $f \neq 0$, $i = 1, 2$.

For $W \leqslant V$, set $C_G(V/W) = C_P(V/W)$ and $\mathrm{Aut}_G(V/W) = P/C_P(V/W)$ where $P = N_G(W)$.

In the following proposition V_2^* denotes the dual space of V_2 (while V^* always denotes $L(V_1, V_2, \mathbb{F})$).

Proposition 6.1.4. *(1) $\mathit{codim}(R_1(f)) = \mathit{codim}(R_2(f))$, $\forall f \in V^*$.*

(2) $R_i(gf) = g_iR_i(f)$, $\forall f \in V^*, \forall g = (g_1, g_2) \in G$, $i = 1, 2$.

(3) Let $f \in V^*$, $R_i = R_i(f)$ and $\bar{V}_i = V_i/R_i$, $i = 1, 2$. Then

$$C_{\Omega_1}(\bar{V}_1) \times C_{\Omega_2}(\bar{V}_2) \leqslant N_G(f) \leqslant N_{\Omega_1}(R_1) \times N_{\Omega_2}(R_2).$$

(4) Let $1 \leqslant r \leqslant \min(n_1, n_2)$, and R_i be a co-dimension r subspace of V_i, $i = 1, 2$. Set $\bar{V}_i = V_i/R_i$, $P_i = N_{\Omega_i}(R_i)$, and $P = P_1 \times P_2$. Then there is a P-isomorphism between the set X of members f of V^* with $R_i(f) = R_i$, and the set Y of isomorphisms from $\bar{V}_1$ to $\bar{V}_2^*$.

(5) Let (r, R_1, R_2) and f be as in (4). Fix some $\beta \in Y$. Set $H_1 = Aut_{\Omega_1}(\bar{V}_1)$, $H_2 = \beta^{-1} Aut_{\Omega_2}(\bar{V}_2^*)\beta$. Then there is a 1-1 correspondence between the P-orbits on X and the double cosets of $GL(\bar{V}_1)$ on (H_2, H_1), such that if f corresponds to $H_2 g H_1$, then

$$N_G(f)/(C_{P_1}(\bar{V}_1) \times C_{P_2}(\bar{V}_2)) \cong H_1 \cap H_2^g.$$

Proof. Pick $0 \neq f \in V^*$ and let $\bar{V}_i = V_i/R_i(f)$. We define $\alpha = \alpha_f : \bar{V}_1 \to \bar{V}_2^*$ as follows:

$$\alpha(\bar{v}_1)(\bar{v}_2) = f(v_1, v_2) \quad \text{for } v_i \in V_i, i = 1, 2.$$

It is easy to check that α is a well defined linear map between the two vector spaces. We claim α is injective. In fact, if $\alpha(\bar{v}_1) = 0$ for some $\bar{v}_1 \in \bar{V}_1$, then for all $\bar{v}_2 \in \bar{V}_2$, by definition we have $\alpha(\bar{v}_1)(\bar{v}_2) = f(v_1, v_2) = 0$. Therefore, $v_1 \in R_1(f)$, i.e., $\bar{v}_1 = 0$.

Consequently $\dim(\bar{V}_1) \leq \dim(\bar{V}_2^*) = \dim(\bar{V}_2)$. By symmetry, $\dim(\bar{V}_2) \leq \dim(\bar{V}_1)$. Hence $\dim(\bar{V}_1) = \dim(\bar{V}_2)$ and α is an isomorphism. So (1) holds.

By definition

$$\begin{aligned} R_1(gf) &= \{u \in V_1 \mid f({g_1}^{-1}u, {g_2}^{-1}v) = 0, \forall v \in V_2\} \\ &= \{g_1 u \in V_1 \mid f(u, v) = 0, \forall v \in V_2\} = g_1 R_1(f). \end{aligned}$$

Similarly $R_2(gf) = g_2 R_2(f)$. Part (2) holds.

By part (2), if $gf = f$, then $R_i(f) = R_i(gf) = g_i R_i(f)$, so $g_i \in N_{g_i}(R_i)$ and hence $N_G(f) \leqslant N_{\Omega_1}(R_1) \times N_{\Omega_2}(R_2)$. If $g_i \in C_G(\bar{V}_i)$, then for all $v_i \in V_i$,

$$gf(v_1, v_2) = gf(\bar{v}_1, \bar{v}_2) = f({g_1}^{-1}\bar{v}_1, {g_2}^{-1}\bar{v}_2) = f(\bar{v}_1, \bar{v}_2).$$

So $g \in N_G(f)$. Thus part (3) is proved.

As for part (4), we claim that $\phi : f \mapsto \alpha_f$, where α_f is defined in the proof of part (1), defines a 1-1 correspondence between X and Y.

Certainly ϕ is well defined. Assume $\alpha_f = \alpha_g$ for some f, g with $R_i(f) = R_i(g) = R_i$. Then by definition $\alpha_f(\bar{v}_1)(\bar{v}_2) = \alpha_g(\bar{v}_1)(\bar{v}_2)$ for all $v_i \in V_i$. Consequently $f(v_1, v_2) = g(v_1, v_2)$, i.e., $f = g$. Hence ϕ is injective. On the other hand, given $\alpha \in Y$, we can construct $f \in V^*$ by letting $f(v_1, v_2) = \alpha(\bar{v}_1)(\bar{v}_2)$ for all $v_i \in V_i$. Check $\alpha = \alpha_f$. So ϕ is surjective. Therefore, the claim is true.

By part (2), $P = N_{\Omega_1}(R_1) \times N_{\Omega_2}(R_2)$ acts on X. Also P acts on Y by $g\alpha = g_2(\alpha \circ g_1^{-1})$ for $g = (g_1, g_2) \in P$. To show the map is actually a P-isomorphism, we need to verify that for $g = (g_1, g_2) \in P$ and $f \in X$, $\alpha_{gf} = g(\alpha_f)$. But

$$\alpha_{gf}(\bar{v}_1)(\bar{v}_2) = f(g_1^{-1}v_1, g_2^{-1}v_2) = \alpha_f(g_1^{-1}\bar{v}_1)(g_2^{-1}\bar{v}_2),$$

so $\alpha_{gf} = g_2(\alpha_f g_1^{-1}) = g\alpha_f$. This proves (4).

Set $H = GL(\bar{V}_1)$. It is clear that $H_i \leqslant H, i = 1, 2$, and $\lambda : \alpha \mapsto \beta^{-1}\alpha$ defines a 1-1 correspondence between Y and H. Denote the projection of $g \in P$ in $\bar{P}$ by $\bar{g}$ where $\bar{P} = P/(C_{P_1}(\bar{V}_1) \times C_{P_2}(\bar{V}_2))$. $\bar{P}$ acts on Y by $\bar{g}\alpha = g\alpha$. If $\gamma = \bar{g}\alpha$ for $\alpha, \gamma \in Y$ and $g = (g_1, g_2) \in P$, then by the way $\bar{g}$ acts on Y, we have

$$\beta^{-1}\gamma = (\beta^{-1}\bar{g}_2\beta)(\beta^{-1}\alpha)\bar{g}_1^{-1}.$$

This shows that two elements α, γ in Y are conjugate by P if and only if their images $\lambda(\alpha)$ and $\lambda(\gamma)$ in H are in the same double coset of H on (H_2, H_1). Moreover, if $\alpha = g\alpha$, then

$$\bar{g}_1 = (\beta^{-1}\alpha)^{-1}(\beta^{-1}\bar{g}_2\beta)(\beta^{-1}\alpha) \in H_1 \cap H_2^{(\beta^{-1}\alpha)}.$$

It then follows from part (4) that $\lambda \circ \phi$ is the 1-1 correspondence from X to the double coset of H on (H_2, H_1) with the desired property. This completes the proof.

□

$R_i(f)$ is called the *Ω_i-radical* of f or radical for short. $\operatorname{codim}(R_1(f))$ is called the *rank* of f. According to Lemma 6.1.2, if $\tau \in \operatorname{Irr}(V)$ is identified with f, we say $R_i(\tau) = R_i(f)$ is the radical of τ and τ has rank r if f does. The set of rank r characters in $\operatorname{Irr}(V)$ is denoted by $\operatorname{Irr}(V, r)$.

Remark 6.1.5. (1) Fix $r \leqslant \min(n_1, n_2)$. Let $X = \operatorname{Irr}(V, r)$. Let $\mathbf{R}_i$ be the set of co-dimension r subspaces of V_i, $i = 1, 2$. Proposition 6.1.4.2 says

$$\theta : \tau \mapsto (R_1(\tau), R_2(\tau))$$

defines a G-equivariant map from X to $\mathbf{R}_1 \times \mathbf{R}_2$. By Proposition 6.1.4.4, it is surjective. Equivalently, if we regard X as a Ω_i-set, then

$$\theta_i : \tau \mapsto R_i(\tau)$$

defines a surjective Ω_i-equivariant map from X to $\mathbf{R}_i$.

(2) Fix $R_1 \in \mathbf{R}_1$ and let $X(R_1)$ be the set of characters $\tau \in X$ with $R_1(\tau) = R_1$. By Proposition 6.1.4.3, $C_{\Omega_1}(V_1/R_1)$ acts trivially on $X(R_1)$. The projection $pr : V_1 \to V_1/R_1$ induces an $\operatorname{Aut}_{\Omega_1}(V_1/R_1) \times \Omega_2$-equivariant isomorphism from $X(R_1)$ to the set of $\bar{f} \in \bar{V}^* = ((V_1/R_1) \otimes V_2)^*$ with $R_1(\bar{f}) = 0$ via $f \mapsto \bar{f}$ where $\bar{f}(pr(v_1), v_2) = f(v_1, v_2)$.

Lemma 6.1.6. *Let $\Omega_2 = GL_{n_2}(\mathbb{F})$ and $1 \leqslant r \leqslant \min(n_1, n_2)$. There is a 1-1 correspondence between the G-orbits on $\operatorname{Irr}(V, r)$ and the Ω_1-orbits on the set of codimension r subspaces of V_1 given by $\tau^G \mapsto R_1(\tau)^{\Omega_1}$.*

Proof. Let X, $\mathbf{R}_1$, $\mathbf{R}_2$, and θ be as in Remark 6.1.5. By Remark 6.1.5, θ is surjective and G-equivariant. So X/G is in 1-1 correspondence with

$$\coprod_{(R_1,R_2)\in(\mathbf{R}_1\times\mathbf{R}_2)/G} X(R_1, R_2)/(N_{\Omega_1}(R_1) \times N_{\Omega_2}(R_2))$$

via $\tau^G \mapsto \tau^P$ where $X(R_1, R_2) = \{\tau \in \operatorname{Irr}(V) \mid R_i(\tau) = R_i\}$ and $P = N_{\Omega_1}(R_1(\tau)) \times N_{\Omega_2}(R_2(\tau))$.

Fix $R_i \in \mathbf{R}_i$, $i = 1, 2$, and let $P = N_{\Omega_1}(R_1) \times N_{\Omega_2}(R_2)$. As $\Omega_2 = GL_{n_2}(\mathbb{F})$, it follows that $H_2 = \mathrm{Aut}_{\Omega_2}(V_2/R_2)$ is the full general linear group $GL_r(\mathbb{F})$ on V_2/R_2. Consequently, as in the proof of Proposition 6.1.4, $H_1\backslash H/H_2$ contains a unique element. Therefore, by Proposition 6.1.4.5, $X(R_1, R_2)/P$ contains a unique member. Thus the G-orbits on $\mathrm{Irr}(V, r)$ are in 1-1 correspondence with $(\mathbf{R}_1 \times \mathbf{R}_2)/G$ via $\tau^G \mapsto (R_1(\tau)^{\Omega_1}, R_2(\tau)^{\Omega_2})$, which is in 1-1 correspondence with $\mathbf{R}_1/\Omega_1$ by the natural projection, as $\Omega_2 = GL(V_2)$ is transitive on $\mathbf{R}_2$. Therefore, the proof is complete.

□

Lemma 6.1.7. *Let $\Omega_i = GL(V_i)$, $i = 1, 2$.*

(1) G is transitive on $Irr(V, r)$ and consequently G has $1 + \min(n_1, n_2)$ orbits on $Irr(V)$.

(2) Let (r, R_1, R_2) be as in Proposition 6.1.4.4 and $\tau \in Irr(V, r)$ with $R_i(\tau) = R_i$, $i = 1, 2$. Then $C_{\Omega_i}(V_i/R_i)$ is the semi-direct product of $C_{\Omega_i}(R_i) \cap C_{\Omega_i}(V_i/R_i)$, which is isomorphic to $M_{r,n_i-r}(\mathbb{F})$ as an abelian group, by $N_{\Omega_i}(R_i) \cap C_{\Omega_i}(R_i')$ with R_i' being a complement to R_i in V_i, which is isomorphic to $GL_{n_i-r}(\mathbb{F})$. So $Aut_{\Omega_i}(V_i/R_i)$ is isomorphic to $GL_r(\mathbb{F})$. For $\tau \in Irr(V, r)$, $N_G(\tau)$ is the semi-direct product of $C_{\Omega_1}(V_1/R_1) \times C_{\Omega_2}(V_2/R_2)$ by a subgroup of $Aut_{\Omega_1}(V_1/R_1) \times Aut_{\Omega_2}(V_2/R_2)$ isomorphic to $GL_r(\mathbb{F})$.

Proof. Part (1) follows from Lemma 6.1.6 and and the fact that $\Omega_1 = GL(V_1)$ is transitive on the co-dimension r subspaces of V_1. Part (2) follows from Proposition 6.1.4.5 and well known facts on the structure of the general linear groups.

□

6.2 Action on the Linear Modules

Throughout this section, $\mathbb{F} = \mathbb{F}_{q^2}$ for some prime power q, and $\Omega_1 = GL(V_1)$. Fix $1 \leqslant r \leqslant \min(n_1, n_2)$ and set $X = \mathrm{Irr}(V, r)$.

Refer to the set-up as in Example 3.1.1 in discussing the action of Ω_1 on V_1. Let $\mathcal{P} = \mathcal{P}(V_1)$. So $\{P_J^{+n_1}; J \subseteq [n_1 - 1]\}$ is a set of standard parabolics of Ω_1 which are the normalizers of a set $\{c_J\}$ of representatives of $\Delta(\mathcal{P})/\Omega_1$. Let Y be the set of co-dimension r subspaces of V_1, and for $y \in Y$, let $X(y)$ be the set of $\tau \in X$ whose Ω_1-radical is y.

Form the semi-direct product H of V by G. Extend the action of Ω_1 on V_1 to H by letting $V \rtimes \Omega_2$ act trivially. So H acts on $\mathcal{P}$, X and Y, as well as on $\mathcal{P} \times X$. Also for each $J \subseteq [n_1 - 1]$, let $G_J = P_J^{+n_1} \times \Omega_2$, and $H_J = V \rtimes G_J$.

Proposition 6.2.1. *Let $d \geqslant 0$, $Z \leqslant Z(G)$ centralizing V and $\rho \in Irr(Z)$.*

(1) If $r = n_1$, then as the only co-dimension r subspace in V_1 is 0, $Y = \{0\}$ and $X = X(0)$.

(2) If $r < n_1$, then

$$\sum_{J \subseteq [n_1-1]} (-1)^{|J|} k_d(H_J, X, \rho) = \sum_{\tau \in J \subseteq [n_1-1]} (-1)^{|J|} k_d(H_J, X(w), \rho)$$

where $w \in Y$ is a complement in V_1 to the r-dimensional subspace stabilized by $P_r^{+n_1}$.

Proof. Part (1) is trivial. So Assume $r < n_1$. Recall from section 2.2 that $k_d(H_J, X, \rho)$ counts the number of irreducible characters of H_J of q-height d lying over ρ and some $\tau \in X$. Define $f : \Delta(\mathcal{P}) \times X \to \mathbb{Z}$ such that for $J \subseteq [n_1 - 1]$ and $\tau \in X$, $f(c_J, \tau) = k_d(H_J, \tau, \rho)$. We show f is a well defined H-stable function on $\Delta(\mathcal{P}) \times X$. Observe $\{c_J; J \subseteq [n_1 - 1]\}$ is a set of representatives of $\Delta(\mathcal{P})/H$, and

$$k_d(H_J, \tau, \rho) = k_d(H_J, \tau^g, \rho) \text{ for } g \in H_J.$$

So f is a well defined H-stable function on $\Delta(\mathcal{P}) \times X$.

Recall the definition of $T(f)$ from section 5.3. We have

$$T(f)(c_J) = \sum_{\tau \in X/H_J} f(c_J, \tau) = \sum_{\tau \in X/H_J} k_d(H_J, \tau, \rho) = k_d(H_J, X, \rho).$$

The last equality holds by Lemma 2.2.1.2. Consequently,

$$\sum_{J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, X, \rho) = A(T(f), \Delta(\mathcal{P})/H). \tag{6.1}$$

By Remark 6.1.5.2 and by our hypothesis preceeding this proposition,

$$\phi : \tau \mapsto R_1(\tau)$$

defines a surjective H-equivariant map from X to Y. So by Proposition 5.3.6.1, ϕ induces an H-equivariant map Φ from $\mathcal{P} \times X$ to $\mathcal{P} \times Y$, and consequently f induces an H-stable function $f' = \bar{\Phi}(f)$ on $\mathcal{P} \times Y$ defined by

$$f'(c_J, y) = S(f|_{\Phi^{-1}(c_J,y)}, \Phi^{-1}(c_J, y)/N_H(y)) = k_d(H_J, X(y), \rho).$$

By Proposition 5.3.6.2, $T(f) = T(f')$ as H-stable functions on $\mathcal{P}$. So

$$A(T(f), \Delta(\mathcal{P})/H) = A(T(f'), \Delta(\mathcal{P})/H). \tag{6.2}$$

But Ω_1 and hence H is transitive on Y, so applying Lemma 5.3.7, we have

$$A(T(f'), \Delta(\mathcal{P})/H) = A(f'_w, \Delta(\mathcal{P})/N_H(w)) \tag{6.3}$$

where w is given by the hypothesis. But recall from Proposition 6.1.4.3 that for each $V \leqslant K \leqslant H$ and $\tau \in X(w)$, $N_K(\tau) \leqslant N_K(w) \leqslant N_H(\tau)$. So by Lemma 2.2.1.2 and 2.2.2, for each J,

$$\begin{aligned} f'_w(c_J) &= k_d(H_J, X(w), \rho) \\ &= \sum_{\tau\in X(w)/N_{H_J}(w)} k_d(H_J, \tau, \rho) \\ &= \sum_{\tau\in X(w)/N_{H_J}(w)} k_{d-d'}(N_{H_J}(w), \tau, \rho) \\ &= k_{d-d'}(N_{H_J}(w), X(w), \rho) \end{aligned}$$

where d' is exponent of q in the p-part of $|H_J|/|N_{H_J}(w)|$. Observe H_J has the same q-height as H.

Define $g : \mathcal{S}(N_H(w)) \to \mathbb{Z}$ by

$$g(K) = \begin{cases} k_{d-d_0+d(K)}(K, X(w), \rho), & \text{if } K \geqslant V; \\ 0, & \text{otherwise.} \end{cases}$$

Here d_0 is the q-height of H and $d(K)$ is the q-height of K. It is easy to see that g is a $N_H(w)$-stable function. Moreover, $f'_w(c_J) = g(N_{H_J}(w))$ for each J and consequently

$$A(f'_w, \Delta(\mathcal{P})/N_H(w)) = A(g, \Delta(\mathcal{P})/N_H(w)). \tag{6.4}$$

Finally as $N_H(w)$ acts as $N_{GL(V_1)}(w)$ on V_1, so by Lemma 5.2.13,

$$A(g, \Delta(\mathcal{P})/N_H(w)) = A(g, \Delta(\mathcal{P}, w)/N_H(w)) \tag{6.5}$$

where $\Delta(\mathcal{P}, w)$ consists of the chains in $\mathcal{P}$ containing a complement to w. But $N_{\Omega_1}(w)$ is a maximal parabolic of Ω_1 and acts transitively on the complements to w. It follows that $\{c_J : r \in J \subseteq [n_1 - 1]\}$ is a set of representatives of $\Delta(\mathcal{P}, w)/N_H(w)$. So

$$A(g, \Delta(\mathcal{P}, w)/N_H(w)) = \sum_{r \in J \subseteq [l-1]} (-1)^{|J|} f'(c_J, w) = \sum_{r \in J \subseteq [l-1]} (-1)^{|J|} k_d(H_J, X(w), \rho).$$

Therefore, the proposition follows from the above equation and equations (6.1)-(6.5).

□

For the rest of this section, assume $\Omega_2 = GL(V_2)$. In this case, we say V is a linear module.

Lemma 6.2.2. *Let $\Omega_i = GL(V_i)$, $i = 1, 2$. Let d, Z, and ρ be as in Proposition 6.2.1. Let $w = 0$ be the 0 subspace if $r = n_1$, or as in Proposition 6.2.1 a complement in V_1 to the r-dimensional subspace R stabilized by $P_r^{+n_1}$ if $r < n_1$.*

(1) $N_{P_r^{+n_1}}(w) = N_{G^{+n_1}}(R) \cap N_{G^{+n_1}}(w) = G^{+(n_1-r)} \times G^{+r}$.

Moreover, $C_{P_r^{+n_1}}(V_1/w) = G^{+(n_1-r)}$ and $Aut_{P_r^{+n_1}}(V_1/w) = G^{+r}$.

(2) $N_{G_r}(w)$ *is transitive on* $X(w)$. *For* $\tau \in X(w)$,

$$N_{G_r}(\tau) = \begin{cases} G^{+(n_1-r)} \times G^{+n_2}, & \text{if } r = n_2, \\ G^{+(n_1-r)} \times P_r^{+n_2}, & \text{if } r < n_2. \end{cases}$$

(3) Let $J \subseteq [n_1 - 1]$ *with* $r \in J \cup \{n_1\}$, *that is* $r = n_1$ *or* $r \in J$. *Then*

$$N_{P_J^{+n_1}}(w) = P_J^{+n_1} \cap N_{P_r^{+n_1}}(w) = P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+r}$$

with $J_1 = \{j - r \mid r < j \in J\}$ *and* $J_2 = J(< r)$. *Moreover,* $C_{P_J^{+n_1}}(V_1/w) = P_{J_1}^{+(n_1-r)}$ *and* $Aut_{P_J^{+n_1}}(V_1/w) = P_{J_2}^{+r}$.

(4) Let J *be as in part (3). Then* $N_{G_J}(w)$ *is transitive on* $X(w)$. *For* $\tau \in X(w)$,

$$N_{G_J}(\tau) = \begin{cases} P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+n_2}, & \text{if } r = n_2, \\ P_{J_1}^{+(n_1-r)} \times P_{J_2 \cup \{r\}}^{+n_2}, & \text{if } r < n_2. \end{cases}$$

Proof. Part (1) follows from our choice of w. As $G_r = P_r^{+n_1} \times GL(V_2)$, by Lemma 6.1.6, the G_r-orbits on X are in 1-1 correspondence with the $P_r^{+n_1}$-orbits on the codimension r subspaces of V_1 via $\tau^{G_r} \mapsto R_1(\tau)^{P_r^{+n_1}}$. In particular as w is fixed, any two members in $X(w)$ are conjugate under G_r, and hence under $N_{G_r}(w)$ by Proposition 6.1.4.2. So $N_{G_r}(w)$ is transitive on $X(w)$. Also

$$N_{G_r}(\tau) = G_r \cap N_G(\tau)$$

and $N_G(\tau)$ is given by Lemma 6.1.7.2. Assume $R_2(\tau) = w'$. By part (1), and as $\Omega_2 = G^{+n_2}$,

$$\mathrm{Aut}_{P_r^{+n_1}}(V_1/w) \cong G^{+r} \cong \mathrm{Aut}_{\Omega_2}(V_2/w').$$

So by Proposition 6.1.4.5, $N_{G_r}(\tau)$ is the semi-direct product of $C_{P_r^{+n_1}}(V_1/w) \times C_{\Omega_2}(V_2/w')$ by the diagonal subgroup D in $\mathrm{Aut}_{P_r^{+n_1}}(V_1/w) \times \mathrm{Aut}_{\Omega_2}(V_2/w')$ isomorphic to G^{+r}. But $C_{P_r^{+n_1}}(V_1/w)$ commutes with D and Ω_2, and the extension of $C_{\Omega_2}(V_2/w')$

by D is isomorphic to G^{+n_2} if $r = n_2$ and isomorphic to $P_r^{+n_2}$ if $r < n_2$, therefore, part (2) holds.

Part (3) is easy. As for part (4), we may argue as in part (2) to show that $N_{G_J}(w)$ is transitive on $X(w)$. $N_{G_J}(\tau)$ can be worked out by the same argument in the proof of part (2).

□

Proposition 6.2.3. *Assume the hypothesis in Proposition 6.2.1.*

(1) If $r = n_1$, then

$$\sum_{J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, X, \rho) =$$
$$\begin{cases} \sum_{J\subseteq[r-1]} (-1)^{|J|} k_{d-d'}(P_J^{+n_2}, \rho), & \text{if } r = n_2, \\ \sum_{J\subseteq[r-1]} (-1)^{|J|} k_{d-d'}(P_{J\cup\{r\}}^{+n_2}, \rho), & \text{if } r < n_2. \end{cases}$$

(2) If $r < n_1$, then

$$\sum_{J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, X, \rho) =$$
$$\begin{cases} \sum_{J_1\subseteq[n_1-r-1]} \sum_{J_2\subseteq[r-1]} (-1)^{|J_1|+|J_2|+1} & k_{d-d'}(P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+n_2}, \rho), \\ & \text{if } r = n_2, \\ \sum_{J_1\subseteq[n_1-r-1]} \sum_{J_2\subseteq[r-1]} (-1)^{|J_1|+|J_2|+1} & k_{d-d'}(P_{J_1}^{+(n_1-r)} \times P_{J_2\cup\{r\}}^{+n_2}, \rho), \\ & \text{if } r < n_2. \end{cases}$$

In any case $d' = 2(\binom{n_1}{2} - \binom{n_1-r}{2})$.

Proof. Let w be as in Lemma 6.2.2. Let $J \subseteq [n_1]$ be as in part (3) of the same lemma. By Lemma 6.2.2.4, $N_{H_J}(w) = V \rtimes N_{G_J}(w)$ is transitive on $X(w)$. So by Lemma 2.2.1.2,

$$k_d(H_J, X(w), \rho) = k_d(H_J, \tau, \rho)$$

for some $\tau \in X(w)$. By Lemma 2.2.2,

$$k_d(H_J, \tau, \rho) = k_{d-d'}(N_{H_J}(\tau), \tau, \rho)$$

where d' is the exponent of q in the p-part of $|H_J|/|N_{H_J}(\tau)|$. But $H_J = V \rtimes G_J$ with V being abelian. So by Lemma 2.2.4,

$$k_{d-d'}(N_{H_J}(\tau), \tau, \rho) = k_{d-d'}(N_{G_J}(\tau), \rho).$$

Therefore,

$$k_d(H_J, X(w), \rho) = k_{d-d'}(N_{G_J}(\tau), \rho). \tag{6.6}$$

Recall $N_{G_J}(\tau)$ is given by Lemma 6.2.2.4. Also recall a parabolic subgroup of a general linear group contains a Sylow p-subgroup of the general linear group. So in any case the exponent of q in the p-part of $|N_{G_J}(\tau)|$ is

$$2(\binom{n_1-r}{2} + \binom{n_2}{2}).$$

On the other hand, the q-height of $|G_J| = |P_J^{+n_1} \times G^{+n_2}|$ is

$$2(\binom{n_1}{2} + \binom{n_2}{2}).$$

As $N_{H_J}(\tau) = V \rtimes N_{G_J}(\tau)$,

$$|H_J|/|N_{H_J}(\tau)| = |G_J|/|N_{G_J}(\tau)|.$$

So $d' = 2(\binom{n_1}{2} - \binom{n_1-r}{2})$. Observe d' does not depend on the choice of J.

Assume $r = n_1$. Then by Proposition 6.2.1, $X = X(0)$. So by equation (6.6),

$$\sum_{J \subseteq [n_1-1]} (-1)^{|J|} k_d(H_J, X, \rho) = \sum_{J \subseteq [n_1-1]} (-1)^{|J|} k_{d-d'}(N_{G_J}(\tau), \rho).$$

Now part (1) follows from Lemma 6.2.2.4, where we observe that $N_{G_J}(\tau) = P_J^{+n_2}$ if $r = n_2$ or $P_{J\cup\{r\}}^{+n_2}$ if $r < n_2$..

Similarly if $r < n_1$, then by Proposition 6.2.1.2 and equation (6.6),

$$\sum_{J \subseteq [n_1-1]} (-1)^{|J|} k_d(H_J, X, \rho) = \sum_{r \in J \subseteq [n_1-1]} (-1)^{|J|} k_{d-d'}(N_{G_J}(\tau), \rho).$$

Then again the proposition follows from Lemma 6.2.2.4, as we observe $J \mapsto (J_1, J_2)$ there defines a 1-1 correspondence from $\Delta([n_1-1])$ to $\Delta([n_1-r-1])\times\Delta([r-1])$ with $|J| = |J_1| + |J_2| + 1$.

□

6.3 Action on the Unitary Modules

Continue with the notation in the beginning of section 6.2. So in particular $\Omega_1 = G^{+n_1}$. In addition we assume $\Omega_2 = G^{n_2}$ throughout this section. We call V a unitary module in this case. Recall as in Example 6.1.1 that the dual space V_2^* is the natural module for Ω_2 and becomes a unitary space. For $R \leqslant V_2$, we set R^* to be the subspace of V_2^* consisting of functions which vanish on R. Then $\dim(R^*) = \mathrm{codim}(R)$, $N_{\Omega_2}(R) = N_{\Omega_2}(R^*)$, $C_G(V_2/R) = C_G(R^*)$ and $\mathrm{Aut}_{\Omega_2}(V_2/R) = \mathrm{Aut}_{\Omega_2}(R^*)$. We first study the special case when $n_1 = r$.

Lemma 6.3.1. *Let $n_1 = r$ and $J \subseteq [r-1]$. Then there is a 1-1 correspondence between the G_J-orbits on X with the Ω_2-orbits on flags of type $J\cup\{r\}\backslash\{n_2\}$ in $\mathcal{P}(V_2^*)$, such that if $\tau \in X/G_J$ corresponds to $c \in \Delta(\mathcal{P}(V_2^*))/\Omega_2$, then $N_{G_J}(\tau) = N_{G^{n_2}}(c)$ up to conjugation.*

Proof. By hypothesis $r \leqslant n_2$. First assume $n_2 = r$. So $J \cup \{r\}\backslash\{n_2\} = J$. As the only co-dimension r subspace in V_i is the 0 space, which is of course stabilized by Ω_i, any member in X must have $(0,0)$ as radicals, and $\Omega_i = \mathrm{Aut}_{\Omega_i}(V_i/0)$. By Proposition 6.1.4.5, the G_J-orbits on X are in 1-1 correspondence with the double cosets in $G^r\backslash G^{+r}/P_J^{+r}$, such that if τ corresponds to $G^r g P_J^{+r}$, then $N_{G_J}(\tau) = P_J^{+r} \cap (G^r)^g$. On the other hand, recall that V_2^* is the natural module for G^r and P_J^{+r} is the stabilizer in G^{+r} of a flag of type J. So as G^{+r} is transitive on the flags of type J, it follows that the G^r-orbits are in 1-1 correspondence with the double cosets $P_J^{+r}\backslash G^{+r}/G^r$, such that if c corresponds to $P_J^{+r}g^{-1}G^r$, then $N_{G^r}(c) = G^r \cap (P_J^{+r})^{g^{-1}}$. Therefore, as $G^r g P_J^{+r} \mapsto P_J^{+r} g^{-1} G^r$ is a bijection of $G^r\backslash G^{+r}/P_J^{+r}$ with $P_J^{+r}\backslash G^{+r}/G^r$ the proposition holds in this case.

Next assume $n_2 > r$. So $J \cup \{r\} \backslash \{n_2\} = J \cup \{r\}$. Let $\mathbf{S}$ be the set of subspaces of dimension r in V_2^*. For $R \in \mathbf{S}$, let $X(R) = \{\tau \in X \mid (R_2(\tau))^* = R\}$. Recall $R_1(\tau) = 0$ for each $\tau \in X$. By Remark 6.1.5.1, $\tau \mapsto (R_1(\tau), R_2(\tau))$ defines a G_J-equivariant surjective map from X to $\{0\} \times \{R^* \mid R \in \mathbf{S}\}$. Therefore, as $N_{G^r}(R) = N_{G^r}(R^*)$,

$$X/G_J = \coprod_{R \in \mathbf{S}/G^r} X(R)/(P_J^{+r} \times N_{G^r}(R)).$$

Let $\Lambda \subseteq \Delta(\mathcal{P})$ be the set of flags in $\mathcal{P}(V_2^*)$ of type $J \cup \{r\}$. For $R \in \mathbf{S}$, let $\Lambda(R)$ be the set of flags in Λ whose final term is R. Then

$$\Lambda/G^r = \coprod_{R \in \mathbf{S}/G^r} \Lambda(R)/N_{G^r}(R).$$

Fix $R \in \mathbf{S}$. Set $P = N_{G^r}(R)$ and $\bar{P} = \mathrm{Aut}_{G^r}(R)$. As $R_1(\tau) = 0$ for all $\tau \in X$, $C_{\Omega_1}(V_1/R_1(\tau)) = 0$ and $\mathrm{Aut}_{P_J^{+r}}(V_1/R_1(\tau)) = P_J^{+r}$. By Proposition 6.1.4.5, $X(R)/(P_J^{+r} \times P)$ is in 1-1 correspondence with $\bar{P} \backslash G^{+r}/P_J^{+r}$ and the stabilizer in G_J of the orbit corresponding to $\bar{P} g P_J^{+r}$ is the extension of $C_P(R)$ by $\bar{P}^g \cap P_J^{+r}$.

On the other hand, there is an P-isomorphism between $\Lambda(R)$ and the set of flags of type J in $\mathcal{P}(R)$ given by $c \mapsto c \backslash \{R\}$, where $\mathcal{P}(R)$ is the poset of proper subspaces of R. As G^{+r} is transitive on the chains of type J in $\mathcal{P}(R)$ and P_J^{+r} is the stabilizer in G^{+r} of a chain of type J, it follows that $\Lambda(R)/P = \Lambda(R)/\bar{P}$ is in 1-1 correspondence with the $\bar{P}$-orbits on G^{+r}/P_J^{+r}, or equivalently $P_J^{+r} \backslash G^{+r}/\bar{P}$, such that the stabilizer in G^r of the orbit corresponding to $P_J^{+r} g^{-1} \bar{P}$ is the extension of $C_P(R)$ by $\bar{P} \cap (P_J^{+r})^{g^{-1}}$. We have established a 1-1 correspondence between $X(R)/(P_J^{+r} \times N_{G^r}(R))$ and $\Lambda(R)/N_{G^r}(R)$ for each $R \in \mathbf{S}/G^r$ with the desired property. So the lemma follows.

□

Definition 6.3.2. Assume $n_1 = r$ and $J \subseteq [n_1 - 1]$. If $\tau^{G_J} \in \mathrm{Irr}(V, r)/G_J$ corresponds to $c^{G^r} \in \Delta(\mathcal{P}(V_2^*))$ of type $J \cup \{r\} \backslash \{n_2\}$ as in Lemma 6.3.1, we say τ^{G_J} is *labeled* by c^{G^r}. By abuse of notation, we may also say τ is labeled by c and write $\tau = \tau_c$.

Let $\Delta = \Delta(\mathcal{P}(V_2^*))$, and Γ the set of normal chains in $\mathcal{P}(V_2^*)$ as defined in section 5.2. Recall from the paragraph preceeding Proposition 5.2.17 that $\Delta(r)$, $\Gamma(r)$ are the r-th truncations of Δ, Γ, respectively, and in particular $\Delta(n_2) = \Delta$, $\Gamma(n_2) = \Gamma$.

Lemma 6.3.3. *Assume $n_1 = r$. Let E be an abelian group and $f : \mathcal{P} \times X \to E$ a G-stable function on $\mathcal{P} \times X$.*

(1) *There is a 1-1 correspondence $\phi : (\mathcal{P} \times X)/G \to \Delta(r)/\Omega_2$, such that if*

$$(c_J, x)^G \mapsto c^{\Omega_2},$$

then d is of type $J \cup \{r\} \backslash \{n_2\}$ and $G_{c_J,x} = N_{\Omega_2}(c)$.

(2) *Let g be the Ω_2-stable function on Δ defined by $g = f \circ \phi^{-1}$. Assume g can be extended to an Ω_2-stable function in the sense of Remark 5.3.3. If $r = n_2$, then*

$$A(T(f), \mathcal{P}/G) = A(g, \Delta/\Omega_2) = A(g, \Gamma/\Omega_2).$$

If $r < n_2$, then

$$A(T(f), \mathcal{P}/G) = -A(g, \Delta(r)/\Omega_2) = -A(g, \Gamma(r)/\Omega_2).$$

Proof. Recall from Example 3.1.1 that c_J is the flag stabilized by P_J^{+r}, and

$$\{c_J; J \subseteq [n_1 - 1]\}$$

is a set of representatives of $\mathcal{P}/\Omega_1$. So

$$(\mathcal{P} \times X)/G = \coprod_{J \subseteq [r-1]} (\{c_J\} \times X)/(P_J^{+r} \times \Omega_2) \cong X/G_J.$$

Here $\cong$ means 1-1 correspondence. Without loss we may identify these sets.

Set $\tilde{J} = J \cup \{r\} \backslash \{n_2\}$. By Lemma 6.3.1, there is an 1-1 correspondence

$$\phi_J : X/G_J \to \Delta_{\tilde{J}}(r)/\Omega_2$$

$$x^{G_J} \mapsto \phi_J(x)^{\Omega_2}$$

where $\Delta_{\tilde{J}}(r)$ denotes the set of chains of type $\tilde{J}$ in $\Delta(r)$, such that $G_{c_J,x} = N_{\Omega_2}(\phi_J(x))$. Therefore, as $\Delta(r)$ is the disjoint union of $\Delta_{\tilde{J}}$, $\phi = \cup_J \phi_J$ with $\phi((c_J, x)^{G_J}) = \phi_J(x)^{\Omega_2}$ is the desired 1-1 correspondence. Part(1) holds. As the value of each Ω_2-orbit on

$\Delta(r)$ under g is determined by f and ϕ, g is a well defined Ω_2-stable function. We have

$$A(T(f), \mathcal{P}/G) = \sum_{(c,x)\in(\Delta(\mathcal{P})\times X)/G} (-1)^{|c|} f(c,x) = \pm \sum_{d\in\Delta/\Omega_2} (-1)^{|d|} f(\phi^{-1}(d))$$
$$= \pm A(g, \Delta/\Omega_2)$$

where the $+$ is taken if $r = n_2$, and $-$ is taken if $r < n_2$. This is because when (c, x) corresponds to d under ϕ with c of type J, the d is of type $\tilde{J}$. But $\tilde{J} = J$ if $r = n_2$ and $\tilde{J} = J \cup \{r\}$ if $r < n_2$.

This proves the first equality of part (2). The second equality follows from Remark 5.3.3 and Proposition 5.2.16. So the proof is complete.

□

For each $J \subseteq [n_1 - 1]$, we denote by $S^u(V, J, r) \subseteq X/G_J$ the set of $\tau \in X/G_J$ labeled by normal chains of type $J \cup \{r\} \backslash \{n_2\}$ in $\mathcal{P}(V_2^*)$.

Lemma 6.3.4. *Assume $n_1 = r$. Let $d \geqslant 0$, $Z \leqslant Z(G)$ centralizing V and $\rho \in Irr(Z)$. Then*

$$\sum_{J\subseteq[r-1]} (-1)^{|J|} k_d(H_J, X, \rho) = \sum_{J\subseteq[r-1]} (-1)^{|J|} k_d(H_J, S^u(V, J, r), \rho).$$

Proof. We define a G-stable function f on $\mathcal{P} \times X$ as follows. For $J \subseteq [r-1]$ and $\tau \in X/H_J$, let $f(c_J, \tau) = k_d(H_J, \tau, \rho)$. As in the proof of Proposition 6.2.1, this is a well defined G-stable function on $\mathcal{P} \times X$ with $T(f)(c_J) = k_d(H_J, X, \rho)$, so that

$$A(T(f), \mathcal{P}/G) = \sum_{J\subseteq[r-1]} (-1)^{|J|} k_d(H_J, X, \rho).$$

Let ϕ and $g = f \circ \phi^{-1}$ be as in Lemma 6.3.3.

If $c \in \Gamma(r)$ is of type $\tilde{J}$ as in the preceeding lemma, then $g(c) = k_d(H_J, \tau, \rho)$ where $\phi(c_J, \tau)^G = c^{\Omega_2}$. Consequently as ϕ is a 1-1 correspondence, and $S^u(V, J, r)$ is in 1-1 correspondence with the $\Gamma(r)/\Omega_2$,

$$A(g,\Gamma/\Omega_2) = \pm \sum_{J\subseteq[r-1]} (-1)^{|J|} k_d(H_J, S^u(V,J,r),\rho).$$

On the other hand, as H_J splits over V with V abelian, so by Lemma 2.2.3 and 2.2.4, $g(c) = k_{d-d_1}(N_{\Omega_2}(c),\rho)$ where d_1 is the exponent of q in

$$|G_J/N_G(c_J,\tau)| = |G/N_{\Omega_2}(c)|$$

by Lemma 6.3.3.1. Therefore, if $G_c = G_{c'}$, then $g(c) = g(c')$. So by Remark 5.3.3, g can be extended to a Ω_2-stable function. Therefore the lemma follows from Lemma 6.3.3.

□

We now discuss the case when $n_1 > r$. Pick $w \in Y$ as in Proposition 6.2.1. Let $\bar{V} = (V_1/w)\otimes V_2$. Recall $\dim(V_1/w) = r$ and $\mathrm{Aut}_{\Omega_1}(V_1/w) = G^{+r}$. Then $\bar{V}$ is a tensor module for $G^{+r}\times\Omega_2$. So the above discussion applies to $\bar{X} = \mathrm{Irr}(\bar{V},r)$.

Form the semi-direct product $\bar{H} = \bar{V} \rtimes (G^{+r}\times\Omega_2)$, and let $\bar{H}_{J_2} = \bar{V}\rtimes\bar{G}_{J_2}$, $G_{J_2} = P^{+r}_{J_2}\times\Omega_2$ for $J_2 \subseteq [r-1]$. By Lemma 6.3.1, $\bar{X}/\bar{H}_{J_2}$ is labeled by G^{n_2}-orbits of chains of type $J_2\cup\{r\}\backslash\{n_2\}$ in $\mathcal{P}(V_2^*)$.

Lemma 6.3.5. *Assume $r < n_1$. There is a $Aut_{\Omega_1}(V_1/w)\times\Omega_2$-equivariant isomorphism θ between $X(w)$ and $\bar{X}$ given by $\theta(\tau) = \bar{\tau}$, such that for $r\in J\subseteq[n_1-1]$ and $\tau\in X(w)$,*

$$N_{G_J}(\tau) = P^{+(n_1-r)}_{J_1}\times N_{P^{+r}_{J_2}\times\Omega_2}(\tau) \text{ with } N_{P^{+r}_{J_2}\times\Omega_2}(\tau)\cong N_{\bar{G}_{J_2}}(\bar{\tau})$$

where $J_1 = \{j-r \mid j\in J(>r)\}$ and $J_2 = J(<r)$.

Proof. By Lemma 6.1.7, $\mathrm{Aut}_{\Omega_1}(V_1/w) = G^{+r}$. The existence of θ follows from Remark 6.1.5.2. Let J and τ be as in the hypothesis. By Lemma 6.2.2.3,

$$N_{P^{+n_1}_J}(w) = P^{+(n_1-r)}_{J_1}\times P^{+r}_{J_2}$$

with

$$C_{P_J^{+n_1}}(V_1/w) = P_{J_1}^{+(n_1-r)} \text{ and } \mathrm{Aut}_{P_J^{+n_1}}(V_1/w) = P_{J_2}^{+r}.$$

By Proposition 6.1.4.5, $C_{P_J^{+n_1}}(V_1/w) \leqslant N_{G_J}(\tau)$. As $C_{P_J^{+n_1}}(V_1/w)$ commutes with the diagonal subgroup $D \cong \mathrm{Aut}_{P_J^{+n_1}}(V_1/w)$ (described in the proof of Lemma 6.2.2) and Ω_2, and as $N_{G_J}(\tau) \leqslant N_{P_J^{+n_1}}(w) \times \Omega_2$, we have

$$N_{G_J}(\tau) = P_{J_1}^{+(n_1-r)} \times N_{P_{J_2}^{+r}\times\Omega_2}(\tau).$$

Now as $\bar{G}_{J_2} = P_{J_2}^{+r} \times \Omega_2$ and θ is $\mathrm{Aut}_{\Omega_1}(V_1/w) \times \Omega_2$-equivariant, it follows that

$$N_{P_{J_2}^{+r}\times\Omega_2}(\tau) \cong N_{\bar{G}_{J_2}}(\bar{\tau}).$$

Therefore, the lemma is proved.

□

Definition 6.3.6. Let $r < n_1$ and θ be as in Lemma 6.3.5. For $r \in J \subseteq [n_1 - 1]$, we say $\tau^{N_{G_J}(w)} \in X(w)/N_{G_J}(w)$ is *labeled* by $c^{\Omega_2} \in \mathcal{P}(V_2^*)/G^r$ of type $J_2 \cup \{r\}\backslash\{n_2\}$ if $\theta(\tau)^{\bar{G}_{J_2}}$ is. By abuse of notation, we also say τ is labeled by c and write $\tau = \tau_c$.

Let $r < n_1$. We denote by $S^u(V, J, r) \subseteq X(w)/N_{G_J}(w)$ the set of orbits labeled by a normal chain of type $J_2 \cup \{r\}\backslash\{n_2\}$ in $\mathcal{P}(V_2^*)$.

Remark 6.3.7. In general, each $x \in S^u(V, J, r)$ is not a member of X/G_J, but $x \subseteq y$ for some $y \in X/G_J$. Hence for $\tau \in x$, by Lemma 2.2.1.2,

$$k_d(H_J, x, \rho) = k_d(H_J, \tau, \rho) = k_d(H_J, y, \rho).$$

So for this purpose, we may regard $S^u(V, J, r)$ as a subset of X/G_J.

Lemma 6.3.8. *Assume $r < n_1$. Let d, Z and ρ be as in Lemma 6.3.1. Then*

$$\sum_{J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, X, \rho) = \sum_{r\in J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, S^u(V, J, r), \rho). \tag{6.7}$$

Proof. By Proposition 6.2.1, we have

$$\sum_{J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, X, \rho) = \sum_{r\in J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, X(w), \rho) \tag{6.8}$$

where w is defined in that proposition. Pick $\tau \in X(w)$. By Lemma 2.2.2,

$$k_d(H_J, \tau, \rho) = k_{d-d'}(N_{H_J}(\tau), \tau, \rho)$$

where d' is the exponent of q in the p-part of $|H_J/N_{H_J}(\tau)|$. But $H_J = V \rtimes G_J$ with V abelian, so by Lemma 2.2.4,

$$k_{d-d'}(N_{H_J}(\tau), \tau, \rho) = k_{d-d'}(N_{G_J}(\tau), \tau, \rho).$$

$N_{G_J}(\tau)$ is given by Lemma 6.3.5. So

$$k_d(H_J, \tau, \rho) = k_{d-d'}(P_{J_1}^{+(n_1-r)} \times N_{\bar{G}_{J_2}}(\bar{\tau}), \rho)$$

where J_1, J_2 are as in that lemma. Similarly as $P_{J_1}^{+(n_1-r)}$ acts trivially on $\bar{X}$ and $\bar{H}_{J_2} = \bar{V} \rtimes \bar{G}_{J_2}$ with $\bar{V}$ abelian, by Lemma 2.2.2 and 2.2.4,

$$k_{d-d'+d''}(P_{J_1}^{+(n_1-r)} \times \bar{H}_{J_2}, \bar{\tau}, \rho) = k_{d-d'}(P_{J_1}^{+(n_1-r)} \times N_{\bar{G}_{J_2}}(\bar{\tau}), \rho)$$

where d'' is the exponent of q in the p-part of $|\bar{H}_{J_2}/N_{\bar{H}_{J_2}}(\bar{\tau})|$. Therefore,

$$k_d(H_J, \tau, \rho) = k_{d-d'+d''}(P_{J_1}^{+(n_1-r)} \times \bar{H}_{J_2}, \bar{\tau}, \rho). \tag{6.9}$$

Recall $H_J = V \rtimes G_J$ with $V \leqslant N_{H_J}(\tau)$ and $G_J = P_J^{+n_1} \times \Omega_2$. So

$$|H_J|/|N_{H_J}(\tau)| = |G_J|/|N_{G_J}(\tau)|.$$

Similarly $\bar{H}_{J_2} = \bar{V} \rtimes \bar{G}_{J_2}$ with $\bar{V} \leqslant N_{\bar{H}_{J_2}}(\bar{\tau})$ and $\bar{G}_{J_2} = P_{J_2}^{+(n_1-r)} \times \Omega_2$. So

$$|\bar{H}_J|/|N_{\bar{H}_J}(\bar{\tau})| = |\bar{G}_J|/|N_{\bar{G}_J}(\bar{\tau})|.$$

But by Lemma 6.3.5,

$$N_{G_J}(\tau) \cong P_{J_1}^{+(n_1-r)} \times N_{\bar{G}_{J_2}}(\bar{\tau}).$$

Therefore, $d' - d''$ is the exponent of q in the p-part of $|P_J^{+n_1}/(P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+r})|$, which is $2(\binom{n_1}{2} - \binom{n_1-r}{2} - \binom{r}{2})$, and does not depend on the choice of J.

Equality (6.9) holds for any $\tau \in X(w)$, so as $S^u(V, J, r)$ (resp. X) is in 1-1 correspondence with $S^u(\bar{V}, J_2, r)$ (resp. $\bar{X}$),

$$k_d(H_J, S^u(V, J, r), \rho) = k_{d-d'+d''}(P_{J_1}^{+(n_1-r)} \times \bar{H}_{J_2}, S^u(\bar{V}, J_2, r), \rho)$$

and

$$k_d(H_J, X(w), \rho) = k_{d-d'+d''}(P_{J_1}^{+(n_1-r)} \times \bar{H}_{J_2}, \bar{X}, \rho).$$

But by Lemma 6.3.4, we have

$$\sum_{J_2\subseteq[r-1]} (-1)^{|J_2|} k_d(\bar{H}_{J_2}, \bar{X}, \rho) = \sum_{J_2\subseteq[r-1]} (-1)^{|J_2|} k_d(\bar{H}_{J_2}, S^u(\bar{V}, J_2, r), \rho).$$

Applying Lemma 2.2.6, we obtain that for any $d \geqslant 0$,

$$\begin{aligned} &\sum_{J_1\subseteq[n_1-r-1]} \sum_{J_2\subseteq[r-1]} (-1)^{|J_1|+|J_2|} k_d(P_{J_1}^{+(n_1-r)} \times \bar{H}_{J_2}, \bar{X}, \rho) = \\ &\sum_{J_1\subseteq[n_1-r-1]} \sum_{J_2\subseteq[r-1]} (-1)^{|J_1|+|J_2|} k_d(P_{J_1}^{+(n_1-r)} \times \bar{H}_{J_2}, S^u(\bar{V}, J_2, r), \rho). \end{aligned} \tag{6.10}$$

Recall that $J \mapsto (J_1, J_2)$ defines a 1-1 correspondence between the subsets of $[n_1 - 1]$ containing r and $\Delta([n_1 - r - 1]) \times \Delta([r - 1])$ with $|J| = |J_1 + |J_2| + 1$. So

$$\begin{aligned} &\sum_{r\in J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, X(w), \rho) = \\ &-\sum_{J_1\subseteq[n_1-r-1]} \sum_{J_2\subseteq[r-1]} (-1)^{|J_1|+|J_2|} k_{d-d'+d''}(P_{J_1}^{+(n_1-r)} \times \bar{H}_{J_2}, \bar{X}, \rho). \end{aligned}$$

Similarly

$$\begin{aligned} &\sum_{r\in J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, S^u(V, J, r), \rho) = \\ &-\sum_{J_1\subseteq[n_1-r-1]} \sum_{J_2\subseteq[r-1]} (-1)^{|J_1|+|J_2|} k_{d-d'+d''}(P_{J_1}^{+(n_1-r)} \times \bar{H}_{J_2}, S^u(\bar{V}, J_2, r), \rho). \end{aligned}$$

By (6.10), the last two double sums are equal, so

$$\sum_{r\in J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, X(w), \rho) = \sum_{r\in J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, S^u(V, J, r), \rho).$$

Hence the lemma follows from (6.8).

□

Fix $J \subseteq [n_1 - 1]$ and let $w = 0$ (if $r = n_1$) or w be as in Proposition 6.2.1. We have labeled $X(w)/N_{G_J}(w)$ by G^r-orbits of chains of type $J \cup \{r\} \backslash \{n_2\}$ in $\mathcal{P}(V_2^*)$ when $r = n_1$ or $r \in J$, or equivalently when $r \in J \cup \{n_1\}$. In particular $S^u(V, J, r)$ is defined when $r \in J \cup \{n_1\}$ and in 1-1 correspondence with the G^{n_2}-orbits on the normal chains in $\mathcal{P}(V_2^*)$ of type $J \cup \{r\} \backslash \{n_2\}$, in which case $S^u(V, J, r)$ is non-empty as normal chains of any type exists in $\mathcal{P}(V_2^*)$. Set $S^u(V, J, r) = \emptyset$ if $r \notin J \cup \{n_1\}$, and set

$$S^u(V, J) = \bigcup_{r=1}^{\min(n_1, n_2)} S^u(V, J, r).$$

Furthermore, let $S^u(V, J, 0)$ consist of the trivial character of V.

Recall that $\mathrm{Irr}^1(V)$ is the disjoint union of $\mathrm{Irr}(V, r)$ for $1 \leqslant r \leqslant \min(n_1, n_2)$ and each $\mathrm{Irr}(V, r)$ is a G-set. So it follows from Lemma 6.3.4 and 6.3.8 by summing over all r with $1 \leqslant r \leqslant \min(n_1, n_2)$ that

Proposition 6.3.9.

$$\sum_{J \subseteq [n_1 - 1]} (-1)^{|J|} k_d^1(H_J, V, \rho) = \sum_{J \subseteq [n_1 - 1]} (-1)^{|J|} k_d(H_J, S^u(V, J), \rho). \tag{6.11}$$

Recall the definition of singular chains as well as non-singular chains from Definition 5.2.14.

Definition 6.3.10. Let $J \subseteq [n_1 - 1]$, $1 \leqslant r \leqslant \min(n_1, n_2)$ and $r \in \tilde{J}$ with

$$\tilde{J} = J(< r) \cup \{r\} \backslash \{n_2\}.$$

(1) Assume $r < n_2$; so $\tilde{J} = J(\leqslant r)$. Let $S^{su}(V, J, r)$ be the set of $\tau \in S^u(V, J, r)$ labeled by a singular normal chain of type $J(\leqslant r)$ in $\mathcal{P}(V_2^*)$. Observe $S^{su}(V, J, r)$ is non-empty if and only if $r \leqslant n_2/2$, in which case it consists of a unique member. Let $S^{nu}(V, J, r)$ be the set of $\tau \in S^u(V, J, r)$ labeled by a non-singular normal chain in $\mathcal{P}(V_2^*)$. Clearly

$$S^u(V, J, r) = S^{su}(V, J, r) \coprod S^{nu}(V, J, r).$$

For $1 \leqslant r' \leqslant r$, let $S^{nu}_{r'}(V, J, r)$ be the set of members in $S^{nu}(V, J, r)$ labeled by non-singular normal chains of non-singular rank r'. Observe $S^{nu}_{r'}(V, J, r)$ is non-empty if and only if $r' \in \tilde{J} = J(\leqslant r)$ and $J(< r') \subseteq [r'/2]$. Clearly

$$S^{nu}(V, J, r) = \coprod_{r'=1}^{r} S^{nu}_{r'}(V, J, r).$$

(2) Assume $r = n_2$; so $\tilde{J} = J(< r)$. For $1 \leqslant r' < r$, let $S^{nu}_{r'}(V, J, r)$ be the set of members in $S^{u}(V, J, r)$ labeled by non-singular normal chains of type $J(< r)$ in $\mathcal{P}(V_2^*)$ non-singular rank r'. Observe $S^{nu}_{r'}(V, J, r)$ is non-empty if and only if $r' \in \tilde{J} = J(< r)$. Let $S^{nu}_{r}(V, J, r)$ be the set of members in $S^{u}(V, J, r)$ labeled by singular normal chains of type $\tilde{J} = J(< r)$ in $\mathcal{P}(V_2^*)$. Observe $S^{u}_{r}(V, J, r)$ is non-empty if and only if $J(< r) \subseteq [r/2]$. Set

$$S^{nu}(V, J, r) = \coprod_{r'=1}^{r} S^{nu}_{r'}(V, J, r).$$

So in this case $S^{u}(V, J, r) = S^{nu}(V, J, r)$, and we may set $S^{su}(V, J, r) = \emptyset$.

(3) Set

$$S^{su}(V, J) = \bigcup_{r=1}^{\min(n_1, n_2)} S^{su}(V, J, r),$$

$$S^{nu}_{r'}(V, J) = \bigcup_{r=r'}^{\min(n_1, n_2)} S^{nu}_{r'}(V, J, r),$$

and

$$S^{nu}(V, J) = \bigcup_{r=1}^{\min(n_1, n_2)} S^{nu}(V, J, r).$$

Remark 6.3.11. Observe in either case of Definition 6.3.10, $S^{su}(V, J, r)$ is non-empty if and only if $1 \leqslant r \leqslant n_2/2$, in which case it consists of a unique member.

$S^{nu}_{r'}(V, J, r)$ is non-empty if and only if $r' \in J(\leqslant r)$ and $J(< r') \subseteq [r'/2]$. Moreover,

$$S^u(V, J, r) = S^{su}(V, J, r) \coprod S^{nu}(V, J, r)$$

and

$$S^u(V, J) = S^{su}(V, J) \coprod S^{nu}(V, J).$$

We make the following observations.

Remark 6.3.12. Let $J \subseteq [n_1 - 1]$ with $r \in J \cup \{n_1\}$.

(1) Assume $\tau \in S^u(V, J, r)$. If $r < n_1$, by Lemma 6.3.5,

$$N_{G_J}(\tau) = P^{+(n_1-r)}_{J_1} \times N_{\bar{G}_{J_2}}(\bar{\tau}).$$

Here $J_1 = \{j - r \mid j \in J(> r)\}$ and $J_2 = J(< r)$. If $r = n_1$, w as in Lemma 6.3.5 may be chosen to be 0, so that $\bar{H}$, $\bar{V}$, $\bar{H}_{J_2}$, $\bar{\tau}$ are identified with H, V, H_J, τ, respectively. So the above equation holds trivially in this case as $P^{+(n_1-r)}_{J_1} = 1$ and $N_{G_J}(\tau) = N_{\bar{G}_{J_2}}(\bar{\tau})$. In either case, $\bar{\tau}$ is labeled by a normal chain c in $\mathcal{P}(V_2^*)$ of type $\tilde{J} = J_2 \cup \{r\} \backslash \{n_2\}$. Recall from the discussion preceeding Lemma 6.3.5 that $\bar{\tau} \in \mathrm{Irr}(\bar{V}, r)$ with $\bar{V} \cong M_{r,n_2}(\mathbb{F}_{q^2})$. So we may apply Lemma 6.3.1 to get

$$N_{\bar{G}_{J_2}}(\bar{\tau}) = N_{G^{n_2}}(c).$$

Consequently

$$N_{G_J}(\tau) = P^{+(n_1-r)}_{J_1} \times N_{G^{n_2}}(c).$$

(2) Assume $\tau = \tau_c \in S^{su}(V, J, r)$ is labeled by a singular normal chain c in $\mathcal{P}(V_2^*)$, then $N_{\Omega_2}(c) = P^{n_2}_{\tilde{J}}$. Therefore,

$$N_{G_J}(\tau) = P^{+(n_1-r)}_{J_1} \times P^{n_2}_{\tilde{J}}. \tag{6.12}$$

(3) Assume $\tau = \tau_c \in S^{nu}_{r'}(V, J, r)$. Then c is a non-singular normal chain of type $\tilde{J}$ in $\mathcal{P}(V_2^*)$ of non-singular rank r'. So $1 \leqslant r' \leqslant r$ and $\tilde{J}(< r') \subseteq [r'/2]$. Let c correspond to (c_1, c_2) as in Lemma 5.2.15.3, where c_1 is a singular normal chain in $\mathcal{P}(V^{r'})$ and c_2 is a normal chain in $\mathcal{P}(V^{n_2-r'})$. Here $V^{r'}$ is the natural module for $G^{r'}$ and $V^{n_2-r'}$ is similarly defined. c_1 is of type $\tilde{J}' = \tilde{J}(< r') = J(< r')$ and c_2 is of type $\tilde{J}'' = \{j - r' \mid j \in \tilde{J}_2(> r')\}$ with $|\tilde{J}| = |\tilde{J}'| + |\tilde{J}''| + 1$. So by Lemma 5.2.15.3 and by part (1),

$$N_{G_J}(\tau) = P_{J_1}^{+(n_1-r)} \times N_{G^{r'}}(c_1) \times N_{G^{n_2-r'}}(c_2)$$

with $N_{G^{r'}}(c_1) = P^{r'}_{J(<r')}$ being a parabolic subgroup of $G^{r'}$.

Proposition 6.3.13. *(1) If $r = n_1$, then*

$$\sum_{J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, S^{su}(V, J, r), \rho) =$$
$$\begin{cases} 0, & \text{if } r = n_2; \\ \sum_{J\subseteq[n_2/2]} (-1)^{|J|} k_{d-d'}(P^{n_2}_{J\cup\{r\}}, \rho), & \text{if } r < n_2. \end{cases}$$

(2) If $r < n_1$, then

$$\sum_{J\subseteq[n_1-1]} (-1)^{|J|} k_d(H_J, S^{su}(V, J, r), \rho) =$$
$$\begin{cases} 0, & \text{if } r = n_2; \\ \sum_{J_1\subseteq[n_1-r-1]} \sum_{J_2\subseteq[n_2/2]} (-1)^{|J_1|+|J_2|+1} & k_{d-d'}(P_{J_1}^{+(n_1-r)} \times P^{n_2}_{J_2\cup\{r\}}, \rho), \\ & \text{if } r < n_2. \end{cases}$$

In any case $d' = 2(\binom{n_1}{2} - \binom{n_1-r}{2})$.

Proof. If $r = n_2$, $S^{su}(V, J, r) = \emptyset$ by definition. So the statement holds in this case. Without loss assume $r < n_2$.

First assume $r = n_1$. From Remark 6.3.11, $S^{su}(V, J, r)$ consists of a unique member if $J \subseteq [r/2]$, and is empty otherwise, in which case $k_d(H_J, S^{su}(V, J, r), \rho) = 0$.

For $\tau \in S^{su}(V, J, r)$ (and hence $J \subseteq [r/2]$), from Remark 6.3.12.2, $N_{G_J}(\tau) = P_J^{n_2}$. So as H_J is the semi-direct product of an abelian normal subgroup V by G_J, by Lemma 2.2.1.2, 2.2.3 and 2.2.4, we have

$$k_d(H_J, S^{su}(V, J, r), \rho) = k_d(H_J, \tau, \rho) = k_{d-d'}(N_{G_J}(\tau), \rho) = k_{d-d'}(P_J^{n_2}, \rho)$$

where d' is the exponent of q in $|G_J|/|N_{G_J}(\tau)|$. As $G_J = P_J^{+n_1} \times G^{n_2}$ and $N_{G_J}(\tau) = P_J^{n_2}$ and the $P_J^{n_2}$ has the same q-height as G^{n_2}, it follows that

$$d' = 2\binom{n_1}{2}.$$

Therefore, the part (1) holds. The case $r < n_1$ can be proved similarly.

□

To end this section, we prove the following technical lemma which will be used in section 9.2.

Let $J \subseteq [n_1 - 1]$, $1 \leqslant r \in J \cup \{n_1\}$ and $r \leqslant n_2$. Let $\tilde{J} = J(< r) \cup \{r\} \backslash \{n_2\}$ and $r' \in \tilde{J}$. Let $\bar{V} = M_{n_1 - r', n_2 - r'}(\mathbb{F})$ be a tensor module for $\bar{G} = G^{+(n_1 - r')} \times G^{n_2 - r'}$. Form the semi-direct product $\bar{H}$ of $\bar{V}$ by $\bar{G}$, and for $J' \subseteq [n_1 - r' - 1]$, set $\bar{H}_{J'} = \bar{V}\bar{G}_{J'}$ where $\bar{G}_{J'} = P_{J'}^{+(n_1 - r')} \times G^{n_2 - r'}$.

Lemma 6.3.14. *There is a 1-1 correspondence γ from $S_{r'}^{nu}(V, J, r)$ to $S^{u}(\bar{V}, J', r - r')$ where $J' = \{j - r' \mid r' < j \in J\}$, such that for $\tau \in S_{r'}^{nu}(V, J, r)$,*

$$N_{G_J}(\tau) = P_{J(<r')}^{r'} \times N_{\bar{G}_{J'}}(\gamma(\tau)).$$

Proof. By definition, there is a 1-1 correspondence $a_{V,J}$ from $S_{r'}^{nu}(V, J, r)$ to the set A of G^r-orbits on the nonsingular normal chains of nonsingular rank r' in $\mathcal{P}(V^{n_2})$ of type $\tilde{J}$. Moreover, by Remark 6.3.12.1, if $\tau = \tau_c$, then

$$N_{G_J}(\tau) = P_{J_1}^{+(n_1 - r)} \times N_{G^{n_2}}(c) \tag{6.13}$$

where $J_1 = \{j - r \mid r < j \in J\}$. By Lemma 5.2.15.3, and as $G^{r'}$ is transitive on the set of singular normal chains of a given type in $\mathcal{P}(V^{r'})$, there is a natural 1-1 correspondence θ from A to the set B of $G^{n_2 - r'}$-orbits on the set of normal chains in $\mathcal{P}(V^{n_2 - r})$ of type

$$\bar{J}' = \{j - r' \mid r' < j \in \bar{J}\} = J' \cup \{r - r'\} \cup \{n_2 - r'\},$$

such that if $c \in A$ with $c' = \theta(c)$, then

$$N_{G^{n_2}}(c) = P^{r'}_{J(<r')} \times N_{G^{n_2 - r'}}(c'). \tag{6.14}$$

Finally $b_{\bar{V},J'}$ is a 1-1 correspondence from $S^u(\bar{V}, J', r - r')$ to B such that if $\tau' = \tau'_{c'} \in S^u(\bar{V}, J', r - r')$, then

$$N_{\bar{G}_{J'}}(\tau') = P^{+((n_1 - r') - (r - r'))}_{J'_1} \times N_{G^{n_2 - r'}}(c').$$

with

$$J'_1 = \{j' - (r - r') \mid j' \in J'(> r - r')\} = \{(j - r') - (r - r') \mid j \in J\} = J_1.$$

That is,

$$N_{\bar{G}_{J'}}(\tau') = P^{+(n_1 - r)}_{J_1} \times N_{G^{n_2 - r'}}(c') \tag{6.15}$$

Now it is easy to check that $\gamma = a_{V,J} \circ \theta \circ b^{-1}_{\bar{V},J'}$ is a 1-1 correspondence from $S^{nu}_{r'}(V, J, r)$ to $S^u(\bar{V}, J', r - r')$. If $\tau \mapsto \bar{\tau}$ under this map, then by equations (6.13)-(6.15),

$$N_{G_J}(\tau) = P^{+(n_1 - r)}_{J_1} \times P^{r'}_{J(<r')} \times N_{G^{n_2 - r'}}(c') = P^{r'}_{J(<r')} \times N_{\bar{G}_{J'}}(\tau').$$

Therefore, the proof is complete. □

Chapter 7 On Parabolic Actions, II

In this chapter we first state a few facts about the unitary group, then study the action of a maximal parabolic subgroup on the center of its unipotent radical.

7.1 The Structure of the Unipotent Radicals

The structure of the parabolic subgroups as well as the unipotent radicals are discussed in [ABG]. We state the most well known facts without proof. Recall that $M_{r,s}(\mathbb{F})$ is the set of r by s matrices whose entries are elements in $\mathbb{F}$.

In this section, let V be an n-dimensional unitary space over $\mathbb{F}_{q^2}$ with $n = 2m$ or $2m+1$. Let $G = GU(V)$. We fix a basis $X = \{e_1, \ldots, e_n\}$ for V such that

$$(e_i, e_j) = \begin{cases} 1, & \text{if } i + j = n + 1 \\ 0, & \text{otherwise.} \end{cases}$$

Here $(\ ,\ )$ is the unitary form on V. Define $M = M(X)$ to be the n by n matrix $M = M(a_{i,j})$ with $a_{i,j} = (e_i, e_j)$. Clearly for $g \in M_{n,n}(\mathbb{F}_{q^2})$, $g \in G$ if and only if $M = gMg^{T\theta}$, where g^T denotes the transpose of g, and g^θ is the matrix obtained by raising every entry of g to its q-th power.

Let $I = [m]$. For $j \in I$, let V_j be the subspace of V spanned by $\{e_i; 1 \leqslant i \leqslant j\}$. By our choice of X, each V_j is a totally isotropic subspace. In particular V_m is a maximal totally isotropic subspace of V.

Let $\emptyset \neq J \subseteq I$. Assume

$$J = \{j_1 < j_2 < \cdots < j_s\}. \tag{7.1}$$

Let c_J be the flag

$$0 < V_{j_1} < V_{j_2} < \cdots < V_{j_s}.$$

Set $B = N_G(c_I)$, $P_J = N_G(c_J)$, and $P_\emptyset = G$. Then B is a Borel group of G and $\{P_J; J \subseteq I\}$ is a set of parabolics of G over B. Let $P_J = U_J L_J$ be the Levi decomposition of P_J, with U_J and L_J being the unipotent radical and Levi factor, respectively, as in section 3.1. As usual, we write P_j for $P_{\{j\}}$. It is well known that

$$U_j = C_G(V_j) \cap C_G(V_j^\perp / V_j) \cap C_G(V/V_j^\perp).$$

Lemma 7.1.1. *Fix $j \in I$.*

(1) $Z(U_j) = C_{U_j}(V_j^\perp) = C_{U_j}(V_j')$ where $V_j' = \langle e_{j+1}, \ldots, e_{n-j} \rangle$. In particular, as an additive group, $Z(U_j)$ is isomorphic to $M_{j,j}(\mathbb{F}_q)$ and $U_j/Z(U_j)$ is isomorphic to $M_{j,n-2j}(\mathbb{F}_{q^2})$. ($U_j = Z_j$ when $n = 2j$) $Z(U_j)$ is an $\mathbb{F}_q L_j$-module while $U_j/Z(U_j)$ is an $\mathbb{F}_{q^2} L_j$-module, both induced from the conjugation of L_j on U_j.

(2) $L_j = N_G(V_j) \cap N_G(V_j') \cap N_G(V_j'')$ where $V_j'' = \langle e_{n-j+1}, \ldots, e_n \rangle$. In particular $L_j = L_j^+ \times L_j^-$ where $L_j^+ = C_{L_j}(V_j') \cong G^{+j}$ and $L_j^- = C_{L_j}(V_j) \cong G^{n-2j}$. Moreover, L_j^- acts trivially on $Z(U_j)$; $U_j/Z(U_j)$ is the tensor module for L_j described in the beginning of section 6.1.

Proof. Pick $u \in U_j$. As a matrix, u can be written as:

$$u = \begin{pmatrix} I_{j,j} & 0 & 0 \\ A & I_{n-2j,n-2j} & 0 \\ C & B & I_{j,j} \end{pmatrix}$$

where $I_{j,j}$ is the identity $j \times j$ matrix, A is a $(n-2j) \times j$ matrix, etc. Certainly we must have $M = uMu^{T\theta}$. Indeed all the assertions in this lemma can easily be deduced from direct calculation, and we omit the proof.

□

Similarly for $J \subseteq I$ represented as in equation (7.1), it is well known that U_J is the subgroup of G centralizing the successive quotients of the following series:

$$0 < V_{j_1} < V_{j_2} < \ldots V_{j_s} \leqslant V_{j_s}^{\perp} < V_{j_2}^{\perp} < V_{j_1}^{\perp} < V.$$

Lemma 7.1.2. *(1) For $1 \leqslant i \leqslant s$, $U_{J(\geqslant j_i)}$ is a normal subgroup of P_J as it is the unipotent radical of a subparabolic of P_J. Consequently the following is a chain of normal p-subgroups of P_J:*

$$U_J = U_{J(\geqslant j_1)} > U_{J(\geqslant j_2)} > \cdots > U_{J(\geqslant j_s)} = U_{j_s} \geqslant Z_{j_s} > 1. \tag{7.2}$$

(2) For $1 \leqslant i < s$,

$$V(j_i, j_{i+1}) = U_{J(\geqslant j_i)}/U_{J(\geqslant j_{i+1})} \cong M_{j_i, j_{i+1}-j_i}(\mathbb{F}_{q^2})$$

as an abelian group. $V(j_i, j_{i+1})$ is an $\mathbb{F}_{q^2}P_J/U_{J(\geqslant j_i)}$-module induced by conjugation. $P_J/U_{J(\geqslant j_i)}$ is isomorphic to

$$L_{J_0}^{(n-2j_{i+1})} \times P_{J_1}^{+j_i} \times G^{+(j_{i+1}-j_i)} \tag{7.3}$$

where $J_0 = \{j - j_{i+1} \mid j_{i+1} < j \in J\}$ and $J_1 = J(< j_i)$. $L_{J_0}^{(n-2j_{i+1})}$ acts trivially on $V(j_i, j_{i+1})$ by conjugation. When regarded as a module for $P_{J_1}^{+j_i} \times G^{+(j_{i+1}-j_i)}$, $V(j_i, j_{i+1})$ is a tensor module as in Example 6.1.1. Consequently P_J is the semi-direct product of $U_{J(\geqslant j_{i+1})}$ by

$$L_{J_0}^{(n-2j_{i+1})} \times P_{J_1 \cup \{j_i\}}^{+j_{i+1}}.$$

(3) As an abelian group, $Z(U_{j_s}) \cong M_{j_s, j_s}(\mathbb{F}_q)$ while $U_{j_s}/Z(U_{j_s}) \cong M_{j_s, n-2j_s}(\mathbb{F}_{q^2})$. $Z(U_{j_s})$ is an $\mathbb{F}_q P_J/U_{j_s}$-module while $U_{j_s}/Z(U_{j_s})$ is an $\mathbb{F}_{q^2}P_J/U_{j_s}$-module, both induced by conjugation. Moreover, P_J/U_{j_s} is isomorphic to

$$P_{J_1}^{+j_s} \times G^{n-2j_s} \tag{7.4}$$

where $J_1 = J(< j_s)$. G^{n-2j_s} *acts trivially on* $Z(U_{j_s})$. $U_{j_s}/Z(U_{j_s})$ *is a tensor module for* P_J/U_{j_s} *as in Example 6.1.1.*

Again these assertions are either well known or can easily be deduced from direct calculation.

We study a "twisted" variation of a tensor module.

Let $G = GU_{2m}(q)$ for some $m \in \mathbb{N}$. Let $P = P_m$. So $L = L_m \cong GL_m(q^2)$. Let $V = U_m = O_p(P)$. From section 7.1, we know V is an $\mathbb{F}_qL$-module via conjugation. So by Lemma 6.1.2 and Remark 6.1.3, $\mathrm{Irr}(V)$ is L-isomorphic to the dual module V^*. On the other hand, the unipotent radical V^- of the opposite parabolic P^- is known to be isomorphic to the dual module for V. Therefore, in this situation we may as well identify V^- with $\mathrm{Irr}(V)$.

Let $\bar{G} = GL_{2m}(\bar{\mathbb{F}})$, where $\bar{\mathbb{F}}$ is the algebraic closure of $\mathbb{F}_q$. Fixing a basis for the natural module on which $\bar{G}$ acts, we may view $\bar{G}$ as a matrix group. Define $\sigma : \bar{G} \to \bar{G}$ by $(a_{ij}) \mapsto M^{-1}(a_{ij}^q)^{-T}M$ where X^T is the transpose of X and $M = (m_{ij})$ with

$$m_{ij} = \begin{cases} 1, & \text{if } i+j = 2m+1; \\ 0, & \text{otherwise.} \end{cases}$$

So σ is an extended Frobenius endomorphism of $\bar{G}$ and we may assume $\bar{G}_\sigma = G$. We can choose a σ-stable maximal parabolic subgroup $\bar{P}$ with $\bar{P}_\sigma = P$. Then $\bar{V} = O_p(\bar{P}) \cong M_{m,m}(\bar{\mathbb{F}}_q)$ is σ-stable and $\bar{V}_\sigma = V$. Also $\bar{V}$ has a σ-stable Levi complement $\bar{L} \cong GL_m(\bar{\mathbb{F}}_q) \times GL_m(\bar{\mathbb{F}}_q)$ with $\bar{L}_\sigma = L$. As $\bar{V}^-$ is characteristic in $\bar{P}^-$ and $\bar{P}^-$ is σ-stable, $\bar{V}^-$ is σ-stable and $(\bar{V}^-)_\sigma = V^-$.

By the set-up in Example 3.1.1 and the choice of σ, we have $\bar{L} = \bar{L}_1 \times \bar{L}_2$ with

$$\bar{L}_1 \cong \bar{L}_2 \cong GL_m(\bar{\mathbb{F}}_{q^2})$$

with $\sigma(\bar{L}_i) = \bar{L}_{3-i}$, $i = 1, 2$. Written as a $(2m) \times (2m)$-matrix, a typical element

$g_1 \in \bar{L}_1$ has the form

$$\begin{pmatrix} A & 0 \\ 0 & I_{m,m} \end{pmatrix}$$

where A is a non-singular $m \times m$-matrix. A typical element $g_2 \in \bar{L}_2$ has the form

$$\begin{pmatrix} I_{m,m} & 0 \\ 0 & B \end{pmatrix}$$

where B is a non-singular $m \times m$-matrix. Consequently a typical element $g \in \bar{L}$ has the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

$\bar{V}^-$ can be either upper triangular or lower triangular. We assume the former. Then a typical element $x \in \bar{V}^-$ has the form

$$\begin{pmatrix} I_{m,m} & C \\ 0 & I_{m,m} \end{pmatrix}$$

with $C \in M_{m,m}(\bar{\mathbb{F}}_q)$. So each x is uniquely determined by an $m \times m$-matrix. Clearly gxg^{-1} has the form

$$\begin{pmatrix} I_{m,m} & ACB^{-1} \\ 0 & I_{m,m} \end{pmatrix}.$$

Moreover, if $x \in \bar{V}^-$ corresponds to $C = (c_{ij})$ as above, then $\sigma(x)$ has the form

$$\begin{pmatrix} I_{m,m} & (d_{i,j}) \\ 0 & I_{m,m} \end{pmatrix}$$

with $d_{ij} = -c^q_{(m+1-j),(m+1-i)}$.

As both $\bar{V}^-$ and $\mathrm{Irr}(\bar{V})$ are dual to $\bar{V}$ as $\bar{\mathbb{F}}\bar{L}$-modules, we may identify them with each other. So the results apply when we study the action of $\bar{L}$ on $\bar{V}^-$, as $\bar{L} = \bar{L}_1 \times \bar{L}_2$ and $\bar{V}$ is a tensor module for $\bar{L}$. In particular, Lemma 6.1.7 says $\bar{L}$ has $1+m$ orbits on $\bar{V}^-$ and describes the orbit stabilizers.

On the other hand, we may identify $\bar{V}^-$ as $M_{m,m}(\bar{\mathbb{F}})$, as we do in Example 6.1.1. As the action of $\bar{L}$ on $\bar{V}^-$ preserves the rank of matrices, we conclude that the $\bar{L}$-orbits on $\bar{V}^-$ are determined by the ranks.

Next we define a set of representatives of $\bar{V}^-/\bar{L}$ as follows. Fix $0 \neq \epsilon \in \mathbb{F}_q$ with $\epsilon^q + \epsilon = 0$. For $1 \leqslant r \leqslant m$, let $x_r = (a_{ij}) \in \bar{V}^-$ be defined as:

$$a_{ij} = \begin{cases} \epsilon, & \text{if } j - i = m - r; \\ 0, & \text{otherwise.} \end{cases}$$

By definition x_r has rank r as a matrix. So $\{0, x_r; 1 \leqslant r \leqslant m\}$ is indeed a set of representatives of $\bar{V}^-/\bar{L}$. Moreover, it is easy to check that x_r is σ-stable for all r.

Assume $\bar{V}^- = V_1 \otimes V_2$ as in Example 6.1.1. So V_1 is the natural module for $\bar{L}_1$ while V_2 is the dual of the natural module for $\bar{L}_2$. Fix $1 \leqslant r \leqslant m$ and let $\bar{H} = N_{\bar{L}}(x_r)$. So by Lemma 6.1.7.2,

$$\bar{C} = \bar{C}_1 \times \bar{C}_2 \leqslant \bar{H} \leqslant \bar{H}_1 \times \bar{H}_2$$

such that $\bar{H}_i = N_{\bar{L}_i}(R_i)$ for some co-dimension r subspace R_i of V_i, and $\bar{C}_i = C_{\bar{L}_i}(V_i/R_i)$. Moreover,

$$\bar{H}_i = \bar{U}_i \rtimes (\bar{K}_i \times \bar{K}_i')$$

where

$$\bar{U}_i = C_{\bar{C}_i}(R_i) \cong M_{r,m-r}(\bar{\mathbb{F}}),$$

$\bar{K}_i \cong GL_{m-r}(\bar{\mathbb{F}})$ is the stabilizer in $\bar{C}_i$ of a complement R_i' to R_i in V_i,

$$\bar{K}_i' = C_{\bar{H}_i}(R_i') \cong GL_r(\bar{\mathbb{F}}),$$

$\bar{C}_i = \bar{U}_i\bar{K}_i$, and

$$\bar{H} = (\bar{C}_1 \times \bar{C}_2) \rtimes \bar{D},$$

where $\bar{D} \cong GL_r(\bar{\mathbb{F}})$ is a full diagonal subgroup of $\bar{K}_1' \times \bar{K}_2'$.

Now as x_r is σ-stable, and $\sigma(\bar{L}_i) = \bar{L}_{3-i}$ for $i = 1, 2$, it follows that $\sigma(\bar{H}_i) = \bar{H}_{3-i}$, $\sigma(\bar{K}_i) = \bar{K}_{3-i}$, and $\sigma(\bar{K}_i') = \bar{K}_{3-i}'$, so $\sigma(\bar{C}_i) = \bar{C}_{3-i}$, and as x_r is σ-stable, $\sigma(\bar{D}) = \bar{D}$. We deduce that

$$C = \bar{C}_\sigma = U \rtimes K$$

where

$$U = (\bar{U}_1\bar{U}_2)_\sigma \cong M_{r,m-r}(\mathbb{F}_{q^2}),$$

$$K = (\bar{K}_1\bar{K}_2)_\sigma \cong GL_{m-r}(\mathbb{F}_{q^2}),$$

and

$$H = N_L(x_r) = \bar{H}_\sigma = CD,$$

where

$$D = \bar{D}_\sigma \cong GU_r(\mathbb{F}_{q^2}).$$

By construction,

$$C = C_L(V_0/R_0) \leqslant H \leqslant N_L(R_0)$$

where R_0 is a co-dimension r subspace of the natural module V_0 for L. We have shown

Lemma 7.1.3. *Let* $1 \leqslant r \leqslant m$.

(1) There is a unique $\bar{L}$*-orbit on the elements of* $\bar{V}^-$ *of rank* r*, and* $\{0, x_r; 1 \leqslant r \leqslant m\}$ *is a set of representatives of* $\bar{L}$*-orbits on* $\bar{V}^-$*.*

(2) $N_{\bar{L}}(x_r)$ *is isomorphic to*

$$(\,(M_{r,r}(\bar{\mathbb{F}}) \rtimes GL_{m-r}(\bar{\mathbb{F}})) \times (M_{r,r}(\bar{\mathbb{F}}) \rtimes GL_{m-r}(\bar{\mathbb{F}}))\,) \rtimes GL_r(\bar{\mathbb{F}}).$$

(3) $\sigma(x_r) = x_r$, *and there is a co-dimension* r *subspace* R *of the natural module* V^m *for* L *such that*

$$N_L(x_r) = U \rtimes (K \times D)$$

stabilizes R,

$$U = C_L(R) \cap C_L(V^m/R) \cong M_{r,m-r}(\mathbb{F}_{q^2}),$$
$$K = N_L(R) \cap C_L(R') \cong GL_{m-r}(\mathbb{F}_{q^2}),$$

where $V^m = R \oplus R'$, *and*

$$D = N_L(x_r) \cap N_L(R') \cap C_L(R) \cong GU_r(\mathbb{F}_q).$$

By Lemma 7.1.3, σ acts on each $\bar{L}$-orbit on $\bar{V}^-$. Therefore, studying the L-orbits on V^- is equivalent to studying how the fixed points by σ in each orbits breaks into orbits of V^- under L-action.

Lemma 7.1.4. *For* $1 \leqslant r \leqslant m$, x_r^L *is the unique* L*-orbit on the set of elements of rank* r *of* V^-. *Consequently* L *has* $1+m$ *orbits on* V^- *with* $\{0, x_r; 1 \leqslant r \leqslant m\}$ *being a set of representatives of* V^-/L.

The proof is an easy application of the following well known lemmas.

Lemma 7.1.5. *Let* G *be a group acting transitively on a set* X, $x \in X$, $H = G_x$ *and* $K \leqslant H$. *Then* $N_G(K)$ *is transitive on* $Fix(K)$ *if and only if* $K^G \cap H = K^H$.

Proof. This is (5.21) on page 19 in [As].

□

Lemma 7.1.6. (Lang-Steinberg Theorem.)

If H *is a connected algebraic group over* $\bar{\mathbb{F}}_q$ *and* σ *is an endomorphism of* H, *then the map* $g \mapsto \sigma(g)g^{-1}$ *from* H *to itself is surjective.*

Proof. This is Steinberg's generalization of Lang's Theorem. See the discussion on page 32 in [Ca]. For the proof, see [S].

□

Proof of Lemma 7.1.4. Let $X = x^{\bar{L}}$ where $x = x_r$. It suffices to show that L is transitive on X_σ.

Set $\tilde{G} = \langle \bar{L}, \sigma \rangle$ such that $g^\sigma = \sigma(g)$ for $g \in \bar{L}$. Then $\tilde{G}$ acts transitively on X with $\tilde{G}_x = \bar{L}_x T$ where $T = \langle \sigma \rangle$. Then $N_{\tilde{G}}(T) = LT$, $Fix(T) = X_\sigma$. By Lemma 7.1.5, LT is transitive on X_σ if and only if $T^{\tilde{G}} \cap \tilde{G}_x = T^{\tilde{G}_x}$ if and only if $T^{\bar{L}} \cap \bar{L}_x T = T^{\bar{L}_x}$ if and only if $\sigma^{\bar{L}} \sigma^{-1} \cap \bar{L}_x = \sigma^{\bar{L}_x} \sigma^{-1}$. Here $\sigma^{\bar{L}} \sigma^{-1} = \{g^{-1} \sigma g \sigma^{-1} \mid g \in \bar{L}\}$.

Certainly $\bar{L}$ is a connected algebraic group. By Lemma 7.1.3, $\bar{L}_x$ is the semi-direct product of connected algebraic groups and hence connected. So by Lemma 7.1.6, $\sigma^{\bar{L}} \sigma^{-1} = \bar{L}$ and $\sigma^{\bar{L}_x} \sigma^{-1} = \bar{L}_x$. Therefore, the final equality indeed holds. Hence the lemma is proved.

□

Recall the rank of $x \in V^-$ as a matrix is L-invariant. For each $1 \leqslant r \leqslant m$, we set $\mathrm{Irr}(V, r)$ to be the set of $\tau \in \mathrm{Irr}(V)$ identified with the rank r elements in V^-. we let $\tau_r \in \mathrm{Irr}(V, r)$ be identified with x_r.

7.2 Representations of U_l

In this section, $G = GU_n(q)$ with $n = 2m$ or $2m + 1$. Fix $1 \leqslant l \leqslant m$ and let $P = P_l$ be the maximal parabolic subgroup of G stabilizing an l-dimensional totally isotropic subspace in the natural module for G, $U = U_l$, and $P = LU$ where L is a Levi complement of U. Set $Z = Z(U)$. We study the irreducible representations of U. If $n = 2m = 2l$, then $U = Z$ and $\mathrm{Irr}(U)$ is studied in section 7.1. So without loss we assume $U \neq Z$. In this case, U is a special p-group, Z is an $\mathbb{F}_q L$-module of dimension l^2 and $U/Z \cong M_{l,n-2l}(q^2)$ is a tensor module for $L = G^{+l} \times G^{n-2l}$. The linear representations of U are precisely those which contain Z in the kernel, and hence can be identified with the representations of U/Z, which are studied in section 6.2. So it remains to study the non-linear representations of U, i.e. those lying over some non-trivial character of Z.

Recall that there is an L-equivariant 1-1 correspondence between $\mathrm{Irr}(Z)$ and $Z(O_p(P_l^-))$. So as the G^{n-2l} factor of L acts trivially on Z, by Lemma 7.1.4, L

has l orbits on the non-trivial characters in $\mathrm{Irr}(Z)$ with $\{1, \tau_r; 1 \leqslant r \leqslant l\}$ being a set of representatives of L-orbits on $\mathrm{Irr}(Z)$, where τ_r is identified with x_r in Lemma 7.1.3.

Let $\varphi \in \mathrm{Irr}(U)$ lying over τ_r, for some $1 \leqslant r \leqslant l$. Set $N = \ker(\varphi)$ and $K = \ker(\tau_r)$. As $Z = Z(U)$, $\varphi|_Z$ is a multiple of τ_r. So $K \leqslant N$. As Z is abelian, Z/K is cyclic of order p. Hence as $Z \nleqslant N$, it follows that $N \cap Z = K$. Set $\bar{U} = U/K$ and $\tilde{U} = U/Z$. As P is irreducible on $\tilde{U}$ and Z and U is non-abelian, $Z = U^{(1)} = \Phi(U)$. Thus $\bar{Z} = \bar{U}^{(1)} = \Phi(\bar{U})$ and as $1 \neq \bar{Z} = \bar{U}^{(1)}$, $\bar{U}$ is non-abelian.

Claim $\Phi(Z(\bar{U})) = 1$. If not, as $\Phi(\bar{U}) = \bar{Z}$, $|Z(\bar{U}) : \Omega_1(Z(\bar{U}))| = p$ and hence $Z(\bar{U})/\Omega_1(Z(\bar{U}))$ is centralized by some $e \in E^{\#}$, $E = Z(G^{n-2l})$, contradicting $\tilde{U} = [\tilde{U}, e]$. Let $\hat{U} = U/N$. As $\hat{U}$ has a faithful irreducible representation, $Z(\hat{U})$ is cyclic, so as $Z(\hat{U})$ is an homomorphic image of $Z(\bar{U})$ and $\Phi(Z(\bar{U})) = 1$, $Z(\hat{U}) = \hat{Z}$ and hence $\hat{U}$ is extraspecial.

Now observe φ and τ_r can be regarded as a character of $\bar{U}$ and $\bar{Z}$, respectively. But the irreducible representations of extraspecial p-groups are well known. See for instance (34.9) in [As]. Explicitly, regarded as a character of $\bar{U}$, φ is faithful and the unique irreducible character of $\bar{U}$ lying over τ_r. Equivalently φ is the unique member of $\mathrm{Irr}(U, \tau_r)$ with $\ker(\varphi) = N$. Clearly $N_L(\varphi) = N_L(\tau_r) \cap N_L(N)$. We have proved that

Lemma 7.2.1. *Let $1 \leqslant r \leqslant l$ and $\varphi \in Irr(U, \tau_r)$ with $N = \ker(\varphi)$. Then U/N is an extraspecial p-subgroup, φ is uniquely determined by N, and $N_L(\varphi) = N_L(\tau_r) \cap N_L(N)$.*

We now divide the discussion into two cases, namely $r = l$ and $r < l$.

Assume $r = l$. We claim that $N = K$ so $\bar{U}$ is extraspecial. To prove the claim, we need to show $\bar{Z} = Z(\bar{U})$. Clearly $\bar{Z} \leqslant Z(\bar{U})$. If $\bar{Z} \lneqq Z(\bar{U})$, then there exists $u \in U - Z$ with $\bar{u} \in Z(\bar{U})$, so $[u, U] \leqslant K$. However, $N_L(\tau_l) = G^l \times G^{n-2l}$ stabilizes K and acts irreducibly on $\tilde{U}$. Therefore, $U = \langle u^{N_L(\tau_l)}, Z \rangle$. It follows that $[U, U] \leqslant K$, a contradiction. Hence the claim follows.

As $N_L(K) = N_L(\tau_l)$, by Lemma 7.2.1, $N_L(\varphi) = N_L(\tau_l)$. Moreover, as $|\bar{U}| = pq^{2l(n-2l)}$, $\varphi(1) = q^{l(n-2l)}$. We have shown that

Lemma 7.2.2. *Irr(U,τ_l) consists of a unique member φ. Moreover,* $\ker(\varphi)=\ker(\tau_l)$, $\varphi(1)=q^{l(n-2l)}$, *and*

$$N_L(\varphi)=N_L(\tau_l)=G^l\times G^{n-2l}.$$

Now assume $1\leqslant r<l$. Recall that $\tilde{U}=V^l\rtimes(V^{n-2l})^*$ where V^k is the natural module for G^{+k}, $k=l,n-2l$. By Lemma 7.1.3.3, $N_L(\tau_r)=L_1\times G^{n-2l}$ where G^{n-2l} centralizes Z, and L_1 stabilizes a co-dimension r subspace R of V^l with

$$L_1=C\rtimes(L_1'\times L_1'')$$

where

$$C=C_{G^l}(R)\cap C_{G^l}(V^l/R)\cong M_{r,l-r}(\mathbb{F}_{q^2}),$$
$$L_1'=C_{L_1}(R)\cap N_{G^l}(R')\cong G^r$$

and

$$L_1''=C_{G^l}(R')\cap N_{G^l}(R)\cong G^{+(l-r)}.$$

Here $V^l=R\oplus R'$. Therefore, R is the unique proper nontrivial L_1-submodule, and consequently $\tilde{W}=R\otimes V^{n-2l}\leqslant\tilde{U}$ is the unique proper nontrivial $N_L(\tau_r)$-submodule where W is the preimage of $\tilde{W}$ in U. Moreover, as $Z(\bar{U})\neq\bar{Z}$, it follows that $Z(\bar{U})=\bar{W}$. Let $E=Z(G^{n-2l})$. Recall that for each $e\in E^\#$, $C_U(e)=Z$. So $\bar{Z}=C_{\bar{U}}(e)=C_{\bar{U}}(E)$, $[e,W]=[E,W]$ and $\bar{W}=\bar{Z}\times[E,\bar{W}]$. Set $\bar{N}_0=[E,\bar{W}]$ and let N_0 be the full preimage of $\bar{N}_0$ in W. So $W=N_0Z$ with $N_0\cap Z=K$ and $\bar{W}=\bar{N}_0\times\bar{Z}$.

As U/N is extraspecial, $W=NZ$. As $Z\cap N=K$, it follows that $\bar{W}=\bar{N}\times\bar{Z}$. So as φ is uniquely determined by N, Irr(U,τ_r) is in 1-1 correspondence with the set X of complements of $\bar{Z}$ in $\bar{W}$ as $\Phi(\bar{W})=1$. Moreover, X is in 1-1 correspondence with

$$\mathrm{Hom}(\bar{N}_0,\bar{Z})\cong\mathrm{Hom}(\bar{N}_0,\mathbb{F}_p)\cong\mathrm{Irr}(\bar{N}_0)$$

via a map $\bar{Y}\mapsto\phi_{\bar{Y}}$ with $\ker(\phi_{\bar{Y}})=\bar{Y}\cap\bar{N}_0$. Now $N_L(\tau_r)$ centralizes τ_r and hence also $\bar{Z}$, so these bijections are all $N_P(\tau_r)$-equivariant. Let φ correspond to $\phi\in\mathrm{Irr}(\bar{N}_0)$. Observe

(1) $C_{L_1}(R) = C \rtimes L_1'$ acts trivially on $\bar{N}$, and $C_{L_1}(V^l/R) = C \rtimes L_1''$ acts trivially on $\hat{U} = U/N$. So $C_{L_1}(\hat{U}) = Z(G)C_{L_1}(V^l/R)$. Moreover, $|\hat{U}| = pq^{2r(n-2l)}$. So $\varphi(1) = q^{r(n-2l)}$.

(2) $\bar{N}_0 \cong V^{l-r} \rtimes V^{n-2l}$ is a tensor module for $L_1'' \times G^{n-2l}$.

(3) If ϕ is the trivial character, then $N = N_0\ N_{L_1'' \times G^{n-2l}}(\phi) = L_1'' \times G^{n-2l}$. Consequently $N_L(\varphi) = N_L(\tau_r)$ and

$$N_L(\varphi)/C_{L_1}(V^l/R) \cong L_1' \times G^{n-2l}. \tag{7.5}$$

(4) If ϕ is non-trivial, then $N_{L_1'' \times G^{n-2l}}(\phi)$ is described in Proposition 6.1.4.5. Assume the radicals of ϕ are (R^1, R^2) as in the proposition. Then in this case,

$$N_L(\varphi)/C_{L_1}(V^l/R) \cap N_L(\varphi) \cong L_1' \times L_2 \tag{7.6}$$

with $L_2 \cong N_{G^{n-2l}}(R^2)$.

In summary,

Lemma 7.2.3. *Let $r < l$ and $P' = N_P(\tau_r)$. There is a P'-equivariant bijection from $Irr(U, \tau_r)$ to $Irr(\bar{N}_0)$, such that if φ corresponds to ϕ, then $N_P(\varphi) = N_{P'}(\phi)$. Moreover, $\varphi(1) = q^{r(n-2l)}$ for all $\varphi \in Irr(U, \tau_r)$.*

Observe that in the above lemma, if $r = l$, then $\bar{N} = 1$. And hence Lemma 7.2.3 coincides with Lemma 7.2.2.

Lemma 7.2.4. *Let $1 \leqslant r \leqslant l$, and $\varphi \in Irr(U, \tau_r)$. Then φ is extendable to $N_P(\varphi)$. That is, there exists $\tilde{\varphi} \in Irr(N_P(\varphi))$ such that $\tilde{\varphi}|_U = \varphi$.*

The proof is an application of the following result due to Dade (see [D3]).

Lemma 7.2.5. *Let E be an extraspecial p-group and $G = E \rtimes H$ with $Z(E) \leqslant Z(G)$. Assume that for each normal p'-subgroup K of H, $[E, K] = E$. Let $\varphi \in Irr(E)$ with $\varphi(1) > 1$. Then φ is extendable to G.*

Proof of Lemma 7.2.4. Let $\ker(\varphi) = N$ and $\hat{U} = U/N$. We may regard φ as a character of $\hat{U}$. To prove the lemma, it suffices to show that φ can be extended to $H\hat{U}$ where $H = N_L(\varphi)$. Recall from Lemma 7.2.1 that $\hat{U}$ is extraspecial.

Assume $r = l$. Then by Lemma 7.2.2, $H = G^l \times G^{n-2l}$. As H is irreducible on $\hat{U}/\hat{Z} = U/Z$, $[X, \hat{U}] = \hat{U}$ for any normal p'-subgrup of H with $X \nleq Z(G)$. Therefore, $[K, \hat{U}] = \hat{U}$ for any normal p'-subgroup of $H/Z(G)$. Hence by Lemma 7.2.5, φ is extendable to $(H/Z(G))\hat{U}$. But $[Z(G), \hat{U}] = 1$, so φ can be extended to $N_P(\varphi)$.

Assume $r < l$. Adopt the notation of the proof of Lemma 7.2.3. Let φ correspond to $\phi \in \mathrm{Irr}(\bar{N}_0)$. If $\phi = 1$, then from equation 7.5,

$$H/H \cap C_{L_1}(V^l/R) = G^r \times G^{n-2l}.$$

But $\hat{U}$ is an extraspecial p-group. So as in the preceeding case, φ is extendable to $\hat{U}(H/H \cap C_{L_1}(V^l/R))$. But $[H \cap C_{L_1}(V^l/R), \hat{U}] = 1$, so φ can be extended to $H\hat{U}$, or equivalently to $N_P(\varphi)$ in this case.

Assume $\phi \neq 1$ and that the radicals of ϕ are (R^1, R^2). Then $H/H \cap C_{L_1}(V^l/R)$ is described in equation (7.6). If R^2 is nondegenerate, then L_2 is the product of two general linear groups, and $\bar{W} = [\bar{W}, L_2] \oplus C_{\bar{W}}(L_2)$, so there is a $N_L(\varphi)$-invariant automorphism of $\bar{U}$ mappping $\bar{N}_0$ to $\bar{N}$. Thus U/N_0 is $N_L(\varphi)$-isomorphic to U/N, so we can take $N = N_0$, a case already handled (i.e., $\phi = 1$).

If R^2 is degenerate, then

$$L_2 = C_{G^{n-2l}}(\mathrm{Rad}(R^2)) \cap C_{G^{n-2l}}(T)$$

where T is a complement to $\mathrm{Rad}(R^2)$ to R^2, so $F^*(L_2) = O_p(L_2)$ and the only normal p'-subgroups of $\mathrm{Aut}_H(\hat{U})$ are the normal p'-subgroup X of L_1'. But as $L_1' \times G^{n-2l}$ is irreducible on $U/W \cong \hat{U}/Z(\hat{U})$, $\hat{U} = [\hat{U}, X]$, so again by Lemma 7.2.5, φ can be extended to $N_P(\varphi)$. Done.

□

7.3 Action On the Central Modules

In this section $G = GU_n(q)$ with $n = 2m$ or $2m+1$ and $l \in I = [m]$. Let $V = Z(U_l)$ where U_l is the unipotent radical of P_l. Let V^l be the natural module for G^{+l}. Denote the set of $J \subseteq [m]$ with $l = \max(J)$ by $\mathbf{J}(l)$. By Lemma 7.1.2.3, if $J \in \mathbf{J}(l)$, then P_J is the semi-direct product of U_l by $P^{+l}_{J(<l)} \times G^{n-2l}$ with G^{n-2l} acting trivially on V. We call V a central module for P_l.

Fix $1 \leqslant r \leqslant l$ and set $X = \mathrm{Irr}(V, r)$. Let $\tau_r \in X$ be identified as x_r in Lemma 7.1.3 and $K = K_r = N_{G^{+l}}(\tau_r)$. So K is described in Lemma 7.1.3.3. That is

$$K \cong M_{r,l-r}(\mathbb{F}_{q^2}) \rtimes (G^{+(l-r)} \times G^r).$$

Lemma 7.3.1. *Let $J \in \mathbf{J}(l)$. There is a 1-1 correspondence between the P_J-orbits on X and the K-orbits on the set of chains of type $J(<l)$ in $\mathcal{P}(V^l)$, such that if $\tau^{P_J} \mapsto c^K$, then $N_{P^{+l}_{J(<l)}}(\tau) = N_K(c)$ up to conjugation.*

Proof. Recall P_J acts as $P^{+l}_{J(<l)}$ on X as U_l and G^{n-2l} act trivially on V. By Lemma 7.1.4, G^{+l} is transitive on X. So as $\tau_r \in X$ and $K = N_{G^{+l}}(\tau_r)$, the $P^{+l}_{J(<l)}$-orbits are in 1-1 correspondence with $K\backslash G^{+l}/P^{+l}_{J(<l)}$ which is in turn in 1-1 correspondence with $P^{+l}_{J(<l)}\backslash G^{+l}/K$, which is in 1-1 correspondence with the K-orbits on chains of type $J(<l)$ in $\mathcal{P}(V^l)$, and if

$$\tau^{P^{+l}_{J(<l)}} \mapsto KgP^{+l}_{J(<l)} \mapsto P^{+l}_{J(<l)}g^{-1}K \mapsto c^K,$$

then $N_{P^{+l}_{J(<l)}}(\tau) = P^{+l}_{J(<l)} \cap K^g$ and $N_K(c) = (P^{+l}_{J(<l)})^{g^{-1}} \cap K$. Therefore, the lemma holds.

□

Definition 7.3.2. Let $r = l$ and $J \in \mathbf{J}(l)$. So by Lemma 7.1.3.3, $K = G^l$. For $\tau \in X$, we say τ^{P_J} is *labeled* by a chain $c^{G^l} \subseteq \Delta(\mathcal{P}(V^l))$ if $\tau^{P^{+l}_{J(<l)}}$ corresponds to c^{G^l} of type $J(<l)$ as in Lemma 7.3.1.2. By abuse of notation, we also say τ is labeled by c in this case and write $\tau = \tau_c$. Moreover, let $S^{z}_{l}(V, J)$ be the set of $\tau^{P_J} \in X/P_J$ labeled by a normal chain in $\mathcal{P}(V^l)$ of type $J(<l)$.

Lemma 7.3.3. *Assume $r = l$. Let $d \geqslant 0$ and $\rho \in Irr(Z(G))$. Then*

$$\sum_{J \in \mathbf{J}(l)} (-1)^{|J|} k_d(P_J, X, \rho) = \sum_{J \in \mathbf{J}(l)} (-1)^{|J|} k_d(P_J, S^z(V, J, l), \rho)$$

Proof. Certainly $P_l = U_l \rtimes (G^{+l} \times G^{n-2l})$ acts on $X = \mathrm{Irr}(V, l)$ with U_l and G^{n-2l} acting trivially. Extend the action of G^{+l} on $\mathcal{P} = \mathcal{P}(V^l)$ to P_l by letting U_l and G^{n-2l} act trivially. Set $H = P_l$. So H acts on both $\mathcal{P}$ and X. For $J \subseteq [l-1]$, set $H_J = P_{J \cup \{l\}}$.

Define an H-stable function on $\mathcal{P} \times X$ as follows. For $J \subseteq [l-1]$ and $\tau \in X$,

$$f(c_J, \tau) = k_d(P_{J \cup \{l\}}, \tau, \rho).$$

Argued as in the proof of Proposition 6.2.1, we see that f is well defined, and

$$T(f)(c_J) = k_d(P_{J \cup \{l\}}, X, \rho).$$

So as $\Delta[l-1]$ is in 1-1 correspondence with $\mathbf{J}(l)$, we have

$$A(T(f), \mathcal{P}/H) = -\sum_{J \in \mathbf{J}(l)} (-1)^{|J|} k_d(P_J, X, \rho).$$

But by Lemma 7.1.4, G^{+l} and hence H is transitive on X. By Lemma 7.1.3.3, $N_{G^{+l}}(\tau_l) = G^l$ and hence $N_H(\tau_l) = U_l \rtimes (G^l \times G^{n-2l})$. So by Lemma 5.3.7,

$$A(T(f), \mathcal{P}/H) = A(f_{\tau_l}, \mathcal{P}/N_H(\tau_l)).$$

But U_l and G^{n-2l} act trivially on $\mathcal{P}$, so $N_H(\tau_l)$ acts as G^l on $\mathcal{P}$.

Define $g : \mathcal{S}(G^l) \to \mathbb{Z}$ by

$$g(K) = k_d(U \rtimes (K \times G^{n-2l}), \tau_l, \rho) \text{ for } K \leqslant G^l.$$

It is straightforward to check that g is an G^l-stable function, and

$$A(f_{\tau_l}, \mathcal{P}/N_H(\tau_l)) = A(g, \mathcal{P}/G^l).$$

Therefore, by applying Proposition 5.2.16 to g, we obtain

$$A(f_{\tau_l}, \mathcal{P}/N_H(\tau_l)) = A(f_{\tau_l}, \Gamma/N_H(\tau_l))$$

where Γ consists of the normal chains in $\mathcal{P}$. For $J \subseteq [l-1]$, let Γ_J be the set of chains of type J in Γ. Then

$$A(f_{\tau_l}, \Gamma/N_H(\tau_l)) = \sum_{J \subseteq [l-1]} (-1)^{|J|} \sum_{c \in \Gamma_J/N_H(\tau_l)} f(c, \tau_l).$$

By Lemma 7.3.1 and Definition 7.3.2, $S^z(V, J \cup \{l\}, l)$ is in 1-1 correspondence with the G^l-orbits on the set Γ_J of normal chains in $\mathcal{P}$ of type J, that is with Γ_J/G^l. So as $\Gamma_J/G^l = \Gamma_J/N_H(\tau_l)$,

$$k_d(P_{J\cup\{l\}}, S^z(V, J \cup \{l\}, l), \rho) = \sum_{c \in \Gamma_J/N_H(\tau_l)} f(c, \tau_l).$$

Consequently

$$A(f_{\tau_l}, \Gamma/N_H(\tau_l)) = -\sum_{J \in \mathbf{J}(l)} (-1)^{|J|} k_d(P_J, S^z(V, J, l), \rho).$$

Therefore, the lemma holds.

□

Now we discuss the case $1 \leqslant r < l$. By Lemma 7.1.3.3, K is contained in a maximal parabolic subgroup of G^{+l} stabilizing a co-dimension r subspace of V^l. Without loss assume $K \leqslant N_{G^{+l}}(w)$ with $V^l = R \oplus w$ and R being the r-dimensional subspace stabilized by P_r^{+l}.

Lemma 7.3.4. *Set $x = \tau_r$. Let Γ be the set of chains in $\mathcal{P}$ containing a complement to w and $\Gamma(R)$ consists of the chains in $\mathcal{P}$ containing R. Let f be a G^{+l}-stable function on $\mathcal{P} \times X$ such that f_x can be extended to a K-stable function in the sense of Remark 5.3.3.*

(1) K is transitive on the complements to w in V^l. So in particular Γ/K can be identified with $\Gamma(R)/N_K(R)$. Moreover, $N_K(R) = G^r \times G^{+(l-r)}$ with $G^r = N_K(R) \cap C_K(w)$ and $G^{+(l-r)} = C_K(R) \cap N_K(w)$.

(2) $A(f, \mathcal{P} \times X/G^{+l}) = A(f_x, \Gamma(R)/N_K(R))$.

(3) Define $\theta : \Gamma(R) \to \Delta(\mathcal{P}(R)) \times \Delta(\mathcal{P}(w))$, $c \mapsto (c_1, c_2)$, *as follows. If*

$$c = 0 < V_1 < \cdots < V_i < \cdots < V_s$$

with $R = V_i$, *then*

$$c_1 = 0 < V_1 < \cdots < V_{i-1} \text{ and } c_2 = 0 < V_{i+1} \cap w < \cdots < V_s \cap w.$$

Then θ *is a* $N_K(R)$*-equivariant isomorphism. So in particular*

$$N_K(c) = N_{G^r}(c_1) \times N_{G^{+(l-r)}}(c_2).$$

Observe if c *is of type* $J \subseteq [l-1]$, *then* c_1 *is of type* $J(< r)$ *and* c_2 *is of type* $\{j - r \mid j \in J(> r)\}$.

(4) Define $\theta_1 : \Gamma(R) \to \Delta(\mathcal{P}(R))$ *as the projection of* θ *to* $\Delta(\mathcal{P}(R))$. *That is if* $\theta(c) = (c_1, c_2)$, *then* $\theta_1(c) = c_1$. *Let* Λ *be the set of* $c \in \Gamma(R)$ *such that* $\theta_1(c)$ *is a normal chain in* $\mathcal{P}(R)$ *regarded as a* G^l*-poset. Then*

$$A(f_x, \Gamma(R)/N_K(R)) = A(f_x, \Lambda/N_K(R)).$$

In summary, $A(f, \mathcal{P} \times X/G^{+l}) = A(f_x, \Lambda/N_K(R))$.

Proof. Part (1) follows from Lemma 7.1.3.3 and the choice of w. As G^{+l} is transitive on X, by Lemma 5.3.7,

$$A(f, \mathcal{P} \times X/G^{+l}) = A(f_x, \Delta(\mathcal{P})/K).$$

Then as $K \leqslant N_{G^{+l}}(w)$, by Lemma 5.2.13,

$$A(f_x, \Delta(\mathcal{P})/K) = A(f_x, \Gamma/K).$$

By part (1), K is transitive on the complements to w, and Γ/K can be identified with $\Gamma(R)/N_K(R)$. More explicitly, $c^{N_K(R)} \mapsto c^K$ defines a 1-1 correspondence from $\Gamma(R)/N_K(R)$ to Γ/K such that $N_{N_K(R)}(c) = N_K(c)$. Therefore,

$$A(f_x, \Gamma/K) = A(f_x, \Gamma(R)/N_K(R)).$$

So part (2) holds.

Part (3) is easy. Part (4) can be proved by the argument used in the proof of Proposition 5.2.16.

□

Definition 7.3.5. Let $r < l$ and $r \in J \in \mathbf{J}(l)$. By Lemma 7.3.1, X/P_J is in 1-1 correspondence with the G^{+l}-orbits on $\Delta = \Delta(\mathcal{P}(V^l))$ of type $J(< l)$. We also see in Lemma 7.3.4 that

$$\Lambda/N_K(R) \subseteq \Gamma(R)/N_K(R) = \Gamma/K \subseteq \Delta/G^{+l}$$

subject to suitable identification. Here Λ is as in Lemma 7.3.4.4. For $\tau \in X$, if τ^{P_J} corresponds to $c^{N_K(R)} \in \Gamma(R)/N_K(R)$ of type $J(< l)$ with $\theta(c) = (c_1, c_2)$, we say τ^{P_J} is *labeled* by a $c_1^{G^r} \in \Delta(\mathcal{P}(V^r))$. By Lemma 7.3.4, c_1 is of type $J(< r)$. By abuse of notation, we also say τ is labeled by c_1 in this case and write $\tau = \tau_{c_1}$. Moreover, let $S^z(V, J, r)$ be the set of $\tau^{P_J} \in X/P_J$ labeled by $c_1^{G^r} \in \mathcal{P}(V^r)/G^r$ of type $J(< r)$ with c_1 normal. By definition, $S^z(V, J, r)$ is in 1-1 correspondence with the G^r-orbits on the set of normal chains in $\mathcal{P}(V^r)$ of type $J(< r)$.

Lemma 7.3.6. *Let $r < l$. Let $d \geqslant 0$ and $\rho \in Irr(Z(G))$. Then*

$$\sum_{J \in \mathbf{J}(l)} (-1)^{|J|} k_d^1(P_J, X, \rho) = \sum_{r \in J \in \mathbf{J}(l)} (-1)^{|J|} k_d(P_J, S^z(V, J, r), \rho). \tag{7.7}$$

Proof. This is a direct application of Lemma 7.3.4. Namely as $\{c_J; J \subseteq [l-1]\}$ is set of representatives of $\Delta(\mathcal{P}(V^l))/G^{+l}$, we define a G^{+l}-stable function f on $\mathcal{P}(V^l) \times X$ by

$$f(c_J, \tau) = k_d^1(P_{J \cup \{l\}}, \tau, \rho).$$

It is easy to check that f_x can be extended to a K-stable function. Then we may proceed as in the proof of Lemma 7.3.3 and apply Lemma 7.3.4 to show the statement is true. □

Let $J \in \mathbf{J}(l)$. By Definition 7.3.2 and 7.3.5, we have labeled a subset of $\mathrm{Irr}(V,r)/P_J$ by chains of type $J(<r)$ in $\mathcal{P}(V^r)$ when $r \in J$. In particular we defined $S^z(V,J,r)$ for such (r,J) and $S^z(V,J,r)$ is in 1-1 correspondence with the G^r-orbits on the normal chains of type $J(<r)$ in $\mathcal{P}(V^r)$. Obviously $S^z(V,J,r)$ is non-empty as there are always normal chains of type $J(<r)$. We set $S^z(V,J,r)=\emptyset$ if $r \notin J$, and set

$$S^z(V,J)=\bigcup_{1\leqslant r\leqslant l} S^z(V,J,r).$$

Furthermore, let $S^z(V,J,0)$ consist of the trivial character of V.

It follows from Lemma 7.3.3 and 7.3.6 by summing over all r with $1 \leqslant r \leqslant l$ that

Proposition 7.3.7.

$$\sum_{J\in\mathbf{J}(l)}(-1)^{|J|}k_d^1(P_J,V,\rho)=\sum_{J\in\mathbf{J}(l)}(-1)^{|J|}k_d(P_J,S^z(V,J),\rho). \tag{7.8}$$

Remark 7.3.8. Let $J \in \mathbf{J}(l)$ and $r \in J$.

(1) Assume $\tau \in S^z(V,J,r)$. If $r=l$, by Definition 7.3.2, $\tau=\tau_c$ with $c \in \Delta(\mathcal{P}(V^l))$ of type $J(<l)$. By Lemma 7.3.1, and as $K=N_{G^{+l}}(\tau_l)=G^l$ in that lemma,

$$N_{P^{+l}_{J(<l)}}(\tau)=N_{G^l}(c).$$

If $r<l$, then by Definition 7.3.5, $\tau=\tau_c$ where $d \in \Delta(\mathcal{P}(V^l))$ is of type $J(<l)$ with $\theta(d)=(c,c')$ with $c \in \Delta(\mathcal{P}(V^r))$ being of type $J(<r)$ as in Lemma 7.3.4.3, and $c' \in \Delta(\mathcal{P}(V^{l-r}))$ of type $J'=\{j-r \mid j \in J(<l)\}$. By Lemma 7.3.1, $N_{P^{+l}_{J(<l)}}(\tau)=N_K(d)$. So by 7.3.4.3,

$$N_{P^{+l}_{J(<l)}}(\tau)=N_{G^r}(c)\times N_{G^{+(l-r)}}(c').$$

As c' is a chain of subspaces of type J' in $\Delta(\mathcal{P}(V^{l-r}))$, $N_{G^{+(l-r)}}(c')\cong P^{+(l-r)}_{J'}$.

In either case $N_{P_J}(\tau)$ is the semi-direct product of U_l by

$$N_{P^{+l}_{J(<l)}}(\tau) \times G^{n-2l} = N_{G^r}(c) \times N_{G^{+(l-r)}}(c') \times G^{n-2l}.$$

Observe this is true even when $r = l$, as we may set c' to be the 0 chain when $r = l$.

As G^l is transitive on $\mathrm{Irr}(V, r)$, we may assume $\tau_c = \tau_r$ as in Lemma 7.2.3. Let N_0 be defined as in that lemma and let $\varphi \in \mathrm{Irr}(U_l, \tau_c)$ correspond to $\phi \in \mathrm{Irr}(\tilde{N}_0)$. Observe that $N_{G^r}(c) \leqslant N_{P_J}(\varphi)$, D acts on N_0 where $D = N_{G^{+(l-r)}}(c') \times G^{n-2l}$, and $N_{P_J}(\varphi)$ is the semi-direct product of U_l by

$$N_{G^r}(c) \times N_D(\phi).$$

(2) For $r' < r$, let $S^z_{r'}(V, J, r)$ consist of the set of members in $S^z(V, J, r)$ labeled by a normal chain c of type $J(< r)$ in $\mathcal{P}(V^r)$ of non-singular rank r'; That is, the first non-degenerate member of c has dimension r'. Observe $S^z_{r'}(V, J, r)$ is non-empty if and only if $r' \in J(< r)$ and $J(< r') \subseteq [r'/2]$. As in Lemma 5.2.15.3, c corresponds to (c_1, c_2), where c_1 is a chain of totally isotropic subspaces in $\mathcal{P}(V^{r'})$ of type $J(< r')$, and c_2 is a normal chain in $\mathcal{P}(V^{r-r'})$ of type $\{j - r' \mid j \in J \text{ and } r' < j < r\}$. Also from Lemma 5.2.15.3,

$$N_{G^r}(c) = N_{G^{r'}}(c_1) \times N_{G^{(r-r')}}(c_2) \text{ with } N_{G^{r'}}(c_1) = P^{r'}_{J(<r')}.$$

Similarly let $S^z_r(V, J, r)$ be the set of members of $S^z(V, J, r)$ labeled by a singular normal chain c of type $J(< r)$, that is c consists of totally isotropic subspaces. Observe $S^z_r(V, J, r)$ is non-empty if and only if $J(< r) \subseteq [r/2]$, in which case $N_{G^r}(c) = P^r_{J(<r)}$.

To end this section, we prove the following technical lemma which is analogus to Lemma 7.3.9 and will be used in section 9.2. Let $J \in \mathbf{J}(l)$, $r \in J$ and $r' \in J(\leqslant r)$. Let $V' = Z(U^{n-2r'}_{l-r'})$ where $U^{n-2r'}_{l-r'} = O_p(P^{n-2r'}_{l-r'})$.

Lemma 7.3.9. *(1) There is a 1-1 correspondence*

$$\gamma : S^z_{r'}(V, J, r) \to S^z(V', J', r - r')$$

where $J' = \{j - r' \mid r' < j \in J\}$, *such that for* $\tau \in S^z_{r'}(V, J, r)$,

$$N_{P_J/U_l}(\tau) = P^{r'}_{J(<r')} \times N_{P^{n-2r'}_{J'}/U^{n-2r'}_{l-r'}}(\gamma(\tau)).$$

(2) Let $\tau \in S^z_{r'}(V, J, r)$ *and* $\gamma(\tau) = \tau'$. *Then there is a 1-1 correspondence between* $Irr(U_l, \tau)$ *and* $Irr(U^{n-2r'}_{l-r'}, \tau')$, *such that if* $\varphi \in Irr(U_l, \tau)$ *corresponds to* $\varphi' \in Irr(U^{n-2r'}_{l-r'}, \tau')$, *then*

$$N_{P_J/U_l}(\varphi) = P^{r'}_{J(<r')} \times N_{P^{n-2r'}_{J'}/U^{n-2r'}_{l-r'}}(\varphi').$$

(3) Let $d \geqslant 0$ *and* $\rho \in Irr(Z(G))$. *Let* τ *and* τ' *be as in part (2). Then*

$$k_d(P_J, \tau, \rho) = \sum_{\substack{\rho_1, \rho_2 \\ \rho_1\rho_2 = \rho}} \sum_{\substack{d_1, d_2 \\ d_1 + d_2 = d'}} k_{d_1}(P^{r'}_{J(<r')}, \rho_1) k_{d_2}(P^{n-2r'}_{J'}, \tau', \rho_2)$$

where $d' = d - \binom{r'}{2} - r'(n - 2r')$.

Proof. Part (1) can be proved by the same argument as in the proof of Lemma 6.3.14 by applying Remark 7.3.8. Part (2) follows easily from Remrk 7.3.8, part (1), Lemma 7.2.2 and 7.2.3. As for part (3), observe that $\lambda \in \mathrm{Irr}(P_J, \tau, \rho)$ if and only if $\lambda \in \mathrm{Irr}(P_J, \varphi, \rho)$ for some $\varphi \in \mathrm{Irr}(U_l, \tau)$. So by Lemma 2.2.1,

$$k_d(P_J, \tau, \rho) = k_d(P_J, \mathrm{Irr}(U_l, \tau), \rho) = \sum_{\varphi \in \mathrm{Irr}(U_l, \tau)/N_{P_J}(\tau)} k_d(P_J, \varphi, \rho).$$

However, by Lemma 7.2.4, each $\varphi \in \mathrm{Irr}(U_l, \tau)$ is extendable to $N_{P_J}(\varphi)$. Therefore, by Lemma 2.2.3 and the fact that $\varphi(1) = q^{r(n-2l)}$,

$$k_d(P_J, \varphi, \rho) = k_{d-d_1}(N_{P_J/U_l}(\varphi), \rho)$$

where $d_1 = r(n - 2l) + x$ and x is the exponent of q in the p-part of $|P_J/N_{P_J}(\varphi)|$. But $N_{P_J/U_l}(\varphi) = N_K(\phi)$ where $K = N_{P_J/U_l}(\tau)$ and φ corresponds to ϕ in the sense of Lemma 7.2.3. Similar statement holds for $k_{d_2}(P^{n-2r'}_{J'}, \varphi', \rho_2)$. The rest of the proof is then an easy application of part (1) and (2) as well as Lemma 2.2.5. We leave it as an exercise.

□

Chapter 8 The Reduction Theorems

In this chapter, $G = GU_n(q)$, $n = 2m$ or $2m+1$, $I = [m]$. Denote the power set $\Delta(I)$ of I by $\mathbf{J}$.

8.1 General Discussion

Let A, B be sets. Recall a relation between A and B is just a subset $R = R_{A,B}$ of $A \times B$. For $a \in A$, set

$$B(a) = \{b \in B \mid (a,b) \in R\}.$$

Similarly for $b \in B$, set $A(b) = \{a \in A \mid (a,b) \in R\}$.

Let A_i, $i = 1,2,3$, be sets. Again a relation among the A_i is a subset $R = R_{A_1,A_2,A_3}$ of $A_1 \times A_2 \times A_3$. For $i = 1,2,3$, let R_i be a relation between A_j and A_k, where $1 \leqslant j < k \leqslant 3$ and $i \neq j,k$. Then R_i, $i = 1,2,3$, naturally define a relation R among the A_i, $i = 1,2,3$, as follows:

$$R = \{(a_1,a_2,a_3) \mid \text{for all}\, i,\ (a_j,a_k) \in R_i\} \subseteq A_1 \times A_2 \times A_3$$

where i,j,k run over all the possibilities described in the preceeding paragraph. In other words, a triple (a_1,a_2,a_3) is in R if and only if any pair of coordinates have the corresponding binary relation. For $a_j \in A_j$ and $a_k \in A_k$, we set

$$A_i(a_j,a_k) = \{a_i \in A_i \mid (a_i,a_j,a_k) \in R\}.$$

Observe from the definition that $A_i(a_j,a_k) = \emptyset$ if $(a_j,a_k) \notin R_i$, and $A_i(a_j,a_k) = A_i(a_j) \cap A_i(a_k)$ if $(a_j,a_k) \in R_i$.

Let $\mathbf{C}$ be the set of subsets $C \subseteq I$ where either $C = \emptyset$ or

$$C : l_1 < l_2 < \cdots < l_s,\ s \geqslant 1 \tag{8.1}$$

such that

$$l_1 \geqslant l_2 - l_1 \geqslant l_3 - l_2 \geqslant \ldots \geqslant l_s - l_{s-1}. \tag{8.2}$$

Denote the sequence in (8.2) by ∂C. We call $\mathbf{C}$ the set of *convex sequences*. Set $\partial l_1 = l_1$ and $\partial l_i = l_i - l_{i-1}$ for $2 \leqslant i \leqslant s$. Set $\partial^2 l_i = \partial l_i - \partial l_{i+1}$ for $1 \leqslant i \leqslant s-1$. Notice $\partial^2 l_i \geqslant 0$ for all i. Observe that if $\emptyset \neq C \in \mathbf{C}$, then ∂C can be regarded as a partition of l_s with ∂l_i, $1 \leqslant i \leqslant s$, being the parts of ∂C. Notice that as a partition, ∂C has s parts, i.e., $l(\partial C) = |C|$.

Definition 8.1.1. Let $C, C' \in \mathbf{C}$. We say C' *covers* C, written as $C \prec C'$, if C' is the disjoint union of C with $\{\max(C')\}$.

Remark 8.1.2. (1) Each non-empty element $C \in \mathbf{C}$ is uniquely determined by its final member l_s and the partition $\partial C \vdash l_s$, because for each i, $1 \leqslant i \leqslant s$,

$$l_i = \textstyle\sum_{j=1}^{i} \partial l_j.$$

Conversely given a positive integer k and a partition $\mu : m_1 \geqslant m_2 \geqslant \ldots \geqslant m_s$ of k, define $C = (l_1 < l_2 < \cdots < l_s)$ by $l_i = \sum_{j=1}^{i} m_j$. Then $C \in \mathbf{C}$ and $\partial C = \mu$. In particular, the set of $C \in \mathbf{C}$ whose final term is a given integer k is in 1-1 correspondence with the set of partitions of k.

(2) Each $\emptyset \neq C' \in \mathbf{C}$ covers a unique element in C, namely $C' \backslash \{\max(C')\}$. If $\emptyset \neq C \in \mathbf{C}$ is represented as in (8.1), then by definition, $C \prec C' = C \cup \{l_{s+1}\} \in \mathbf{C}$ if and only if $l_{s+1} \leqslant m$ and $0 < l_{s+1} - l_s \leqslant \partial l_s$, that is if and only if $0 < l_{s+1} - l_s \leqslant \min(m - l_s, \partial l_s)$. Therefore, for each $\emptyset \neq C \in \mathbf{C}$,

$$\{C' \in \mathbf{C} \mid C \prec C'\} = \{C \cup \{l_s + r\} \mid 0 < r \leqslant \min(m - l_s, \partial l_s)\}.$$

8.2 First Reduction

Let $\mathbf{T}$ be the set of ordered pairs $t = (l, l')$ with $1 \leqslant l < l' \leqslant m$.

We now apply the above discussion on relations among sets to the triple $(\mathbf{J}, \mathbf{C}, \mathbf{T})$.

Definition 8.2.1. (1) Let $R_{\mathbf{J},\mathbf{T}}$ be the set of $(J, t) \in \mathbf{J} \times \mathbf{T}$ such that

(1a) If $t = (l, l')$, then $l, l' \in J$;

(1b) there is no $j \in J$ with $l < j < l'$.

(2) Let $R_{\mathbf{J},\mathbf{C}}$ be the set of $(J, C) \in \mathbf{J} \times \mathbf{C}$ such that either $(J, C) = (\emptyset, \emptyset)$, or $C \neq \emptyset$ and representing C as in (8.1)

(2a) $l_1 \in J$, but there is no $j \in J$ with $l_1 < j < l_s$; And

(2b) $\partial C \subseteq J$. That is, $\partial l_i \in J$ for all $1 \leqslant i \leqslant s$.

(3) Let $R_{\mathbf{C},\mathbf{T}}$ be the set of $(C, t) \in \mathbf{C} \times \mathbf{T}$ such that $C \neq \emptyset$, and representing C as in (8.1) and setting $t = (l, l')$

(3a) $l = l_1$; and

(3b) $l_s \leqslant l'$.

(4) Set $\mathbf{E} = R_{\mathbf{J},\mathbf{C},\mathbf{T}}$ to be the relation on $\mathbf{J} \times \mathbf{C} \times \mathbf{T}$ induced from the three binary relations we just defined, and make $\mathbf{E}$ a graded set by letting $|e| = |J|$ for $e = (J, C, t) \in \mathbf{E}$. We also define a length function on $\mathbf{E}$ by $l(e) = |C|$.

We make a few observations.

Remark 8.2.2. (1) By definition, $t \in \mathbf{T}(J)$ if and only if t consists of two consecutive members of J. In particular $\mathbf{T}(J)$ is non-empty if and only if J contains at least two elements. Moreover $\mathbf{T}(J)$ has $|J| - 1$ members. On the other hand, $J \in \mathbf{J}(t)$ if and only if J contains both l and l', and no elements in between. In particular $\mathbf{J}(t)$ is in 1-1 correspondence with $\Delta([m - l']) \times \Delta([l - 1])$ via the following map:

$$J \mapsto (J_0, J_1)$$

where $J_0 = \{j - l' \mid l' < j \in J\}$ and $J_1 = J(< l)$.

(2) If $|C| = 1$ and $C \in \mathbf{C}(t)$, then $C = \{l\}$. In this case, it can be deduced from the definition of $R_{\mathbf{J},\mathbf{C}}$ that $J \in \mathbf{J}(C)$ if and only if $l \in J$. Consequently, from part (1) we have

$$\mathbf{J}(C,t) = \mathbf{J}(C) \cap \mathbf{J}(t) = \mathbf{J}(t).$$

(3) For $t = (l, l') \in \mathbf{T}$, if $C' \in \mathbf{C}(t)$ with $|C'| \geqslant 2$ and $C \prec C'$, then by checking the conditions directly, $C \in \mathbf{C}(t)$. Conversely, assume $C \in \mathbf{C}(t)$ is represented as in (8.1) and $C' = C \cup \{l_s + r\}$. Then by Remark 8.1.2 and Definition 8.2.1, $C' \in \mathbf{C}(t)$ if and only if $0 < r \leqslant \min(m - l_s, \partial l_s)$ and $l_s + r \leqslant l'$, and hence if and only if $0 < r \leqslant \min(l' - l_s, \partial l_s)$.

(4) Assume $C, C' \in \mathbf{C}$ with $|C| \geqslant 1$, $C \prec C'$ and $e' = (J, C', t) \in \mathbf{E}$. Claim $e = (J, C, t) \in \mathbf{E}$. Indeed, as $e' \in \mathbf{E}$, $(J, t) \in R_{\mathbf{J},\mathbf{T}}$. In order to show $e \in \mathbf{E}$, we must check that $C \in \mathbf{C}(J, t)$. By part (3), $C \in \mathbf{C}(t)$. Also $C' \in \mathbf{C}(J)$ implies $\partial C' \subseteq J$. By hypothesis, C is obtained by deleting the maximal element of C'. So $\partial C \subset \partial C' \subseteq J$. Therefore, (J, C) satisfies the conditions in the definition of $R_{\mathbf{J},\mathbf{C}}$. That is $C \in \mathbf{C}(J)$. Hence $C \in \mathbf{C}(J) \cap \mathbf{C}(t) = \mathbf{C}(J, t)$. So the claim holds.

Recall from section 2.2 that $[n] = \emptyset$ when $n \leqslant 0$. Recall from section 2.3 that for $C \in \mathbf{C}$, $\delta(\partial C)$ counts the number of distinct elements in ∂C regarded as a partition.

Lemma 8.2.3. *For $t = (l, l') \in \mathbf{T}$ and $C \in \mathbf{C}(t)$ represented as in (8.1), there is a 1-1 correspondence*

$$\phi = \phi_{C,t} : \mathbf{J}(C,t) \to \Delta([m - l']) \times \Delta([\partial^2 l_1 - 1]) \times \cdots \times \Delta([\partial^2 l_{s-1} - 1]) \times \Delta([\partial l_s - 1])$$

$$J \mapsto (J_0, J_1, \ldots, J_s)$$

where

$$J_0 = \{j - l' \mid j \in J(> l')\},$$

$$J_i = \{j - \partial l_{i+1} \mid j \in J \text{ and } \partial l_{i+1} < j < \partial l_i\}, \quad 1 \leqslant i \leqslant s - 1,$$

and

$$J_s = J(< \partial l_s).$$

Moreover

$$|J| = \sum_{i=0}^{s} |J_i| + \delta(\partial C) + 1. \tag{8.3}$$

Proof. Observe that if $|C| = 1$, then $C = \{l\}$ and $\mathbf{J}(C,t) = \mathbf{J}(t)$ by part (2) of Remark 8.2.2. So the lemma follows from part (1) of the same remark, except we have to check (8.3). Notice in this case $\delta(\partial C) = 1$. Therefore, $|J| = |J_0| + |J_1| + 2$ as J is the disjoint union of J_0, $\{l, l'\}$, and $\{j + l' \mid j \in J_1\}$.

In general, as $C \in \mathbf{C}(t)$, $l_1 = l$ and $l_s \leqslant l'$. By definition of $R_{\mathbf{J},\mathbf{C}}$, $J \in \mathbf{J}(C)$ if and only if $\partial C \subseteq J$ and there is no $j \in J$ between l_1 and l_s. By definition of $R_{\mathbf{J},\mathbf{T}}$, $J \in \mathbf{J}(t)$ if and only if $l, l' \in J$ and there is no $j \in J$ between l and l'. As $\mathbf{J}(C,t) = \mathbf{J}(C) \cap \mathbf{J}(t)$, we conclude that $J \in \mathbf{J}(C,t)$ if and only if

(1) $\partial C \cup \{l'\} \subseteq J$; and

(2) there is no $j \in J$ with $l < j < l'$.

That is the sequence

$$S : \partial l_s \leqslant \partial l_{s-1} \leqslant \ldots \partial l_1 = l_1 = l \leqslant l'$$

is a subset of any $J \in \mathbf{J}(C,t)$ and breaks $J \backslash S$ into segments $\hat{J}_s = J(< \partial l_s)$, $\hat{J}_i = \{j \in J \mid \partial l_{i+1} < j < \partial l_i\}$ for $1 \leqslant i \leqslant s-1$, and $\hat{J}_0 = J(> l')$, with $J \cap \{l, l+1, \ldots, l'\} = \emptyset$. Each segment $\hat{J}_i$, $0 \leqslant i \leqslant s$, can be any subset of $[\partial l_s - 1]$ or $\{\partial l_{i+1} + 1, \partial l_{i+1} + 2, \ldots, \partial l_i - 1\}$ or $\{l' + 1, \ldots, m\}$. Therefore,

$$\phi : \quad J \mapsto (\hat{J}_0, \ldots, \hat{J}_s) \mapsto (J_0, \ldots, J_s)$$

where J_i is defined as in the hypothesis is the desired 1-1 correspondence.

Finally to count the number of elements of J, we observe that as $\partial l_1 = l < l'$, J is the disjoint union of ∂C regarded as a set, $\{l'\}$, and $\hat{J}_i$, $0 \leqslant i \leqslant s$. Consequently, as ∂C contains $\delta(\partial C)$ distinguished elements and $|\hat{J}_i| = |J_i|$, we have $|J| = \sum_i |J_i| + \delta(\partial C) + 1$.

□

Remark 8.2.4. Let $C \in \mathbf{C}$ with $|C| \geqslant 1$ and $t \in \mathbf{T}$ be as in Lemma 8.2.3. Let $C \prec C' = C \cup \{l_{s+1}\} \in \mathbf{C}$. Assume $(J, C', t) \in \mathbf{E}$ for some J. Then by part (4) of Remark 8.2.2, $(J, C, t) \in \mathbf{E}$. Assume $\phi_{C,t}(J) = (J_0, J_1, \ldots, J_s)$ and $\phi_{C',t}(J) = (J'_0, J'_1, \ldots, J'_s, J'_{s+1})$. Then $J_i = J'_i$ for $0 \leqslant i \leqslant s-1$. This can be done by checking directly with the definition in Lemma 8.2.3. Also by definition,

$$J_s = J(< \partial l_s) \subseteq [\partial l_s - 1],$$

while

$$J'_s = \{j - \partial l_{s+1} \mid j \in J \text{ and } \partial l_{s+1} < j < \partial l_s\} \subseteq [\partial l_s - \partial l_{s+1} - 1]$$

and

$$J'_{s+1} = J(< \partial l_{s+1}) \subseteq [\partial l_{s+1} - 1].$$

So J_s, J'_s, and J'_{s+1} are related by

$$J'_s = \{j - \partial l_{s+1} \mid j \in J_s(> \partial l_{s+1})\}$$

and

$$J'_{s+1} = J_s(< \partial l_{s+1}).$$

To each $e = (J, C, t) \in \mathbf{E}$ with $t = (l, l')$ and C represented as in (8.1), we associate a group $P(e)$, an elementary abelian normal subgroup $V(e)$ of $P(e)$, and an integer $d(e)$. Let $\phi = \phi_{C,t}$ be the map defined in Lemma 8.2.3 with $\phi(J) = (J_0, \ldots, J_s)$. Recall from part (3b) of Definition 8.2.1 that $l_s \leqslant l'$. So $\partial l_s = l_s - l_{s-1} \leqslant l' - l_s$ and consequently, $J_s = J(< \partial l_s) \subseteq [l' - l_s - 1]$.

Definition 8.2.5. (1) Define $P(e)$ to be a group isomorphic to

$$L_{J_0(e)}^{n_0(e)} \times P_{J_1(e)}^{+n_1(e)} \times P_{J_2(e)}^{+n_2(e)} \times \cdots \times P_{J_{s-1}(e)}^{+n_{s-1}(e)} \times P_{J_s(e)}^{+n_s(e)} \tag{8.4}$$

where

$$n_0(e) = n - 2l', \quad J_0(e) = J_0,$$

$$n_i(e) = \partial^2 l_i, \quad J_i(e) = J_i \quad \text{for } 1 \leqslant i \leqslant s-1,$$

$$n_s(e) = l' - l_{s-1},$$

and:

(1a) If $l_s < l'$, then $J_s(e) = J_s \cup \{\partial l_s\}$;

(1b) If $l_s = l'$, then $J_s(e) = J_s$.

(2) (2a) If $l_s < l'$, then by (1a), $\partial l_s \in J_s(e)$, so $P_{J_s(e)}^{+n_s(e)} \leqslant P_{\partial l_s}^{+n_s(e)} \leqslant G^{+n_s(e)}$. Set $V(e) = U_{\partial l_s}^{+n_s(e)}$. So as an abelian group, $V(e) \cong M_{\partial l_s, l'-l_s}(\mathbb{F}_{q^2})$. Also $V(e)$ is a normal subgroup of $P_{J_s(e)}^{+n_s(e)}$, and hence of $P(e)$. So $V(e)$ becomes an $\mathbb{F}_{q^2}P(e)$-module induced by conjugation, with all but one factors of $P(e)$ acting trivially on $V(e)$. Observe that $P_{J_s(e)}^{+n_s(e)}$ is the semi-direct product of $V(e)$ by

$$K = P_{J_s}^{+\partial l_s} \times G^{+(l'-l_s)},$$

For $g = (g_1, g_2) \in K$ and $v \in V(e)$, ${}^g v = g_1 v g_2^{-1}$. So $V(e)$ is a tensor module for K as in Example 6.1.1.

(2b) If $l_s = l'$, let $V(e) = 1$, the trivial group, which of course affords the trivial action by $P(e)$. Observe in this case

$$P_{J_s(e)}^{+n_s(e)} = P_{J_s}^{+\partial l_s}.$$

$$P_{J_s(e)}^{+n_s(e)} = P_{J_s}^{+\partial l_s}.$$

(3) Define

$$d(e) = 2\sum_{i=1}^{s-1}(\binom{\partial l_i}{2} - \binom{\partial^2 l_i}{2}).$$

Remark 8.2.6. (1) As an abelian group, $V(e)$ depends only on (C,t). That is, if $e_i = (J_i, C, t) \in \mathbf{E}$, $i = 1,2$, then $V(e_1) \cong V(e_2)$. Also $d(e)$ depends only on C.

(2) Assume both $e = (J, C, t) \in \mathbf{E}$ and $e' = (J, C', t) \in \mathbf{E}$, with $C \prec C' = C \cup \{l_{s+1}\}$. By definition and by Remark 8.2.4, $n_i(e) = n_i(e')$ and $J_i(e) = J_i(e')$ for $0 \leqslant i \leqslant s-1$. That is the i-th factor of $P(e)$ is isomorphic to the i-th factor of $P(e')$ for $0 \leqslant i \leqslant s-1$.

(3) Continue the assumption in part (2). As $l_s < l_{s+1} \leqslant l'$, by part (1a) and (2a) of Definition 8.2.1, $J_s(e) = J(\leqslant \partial l_s)$ and $V(e)$ is non-trivial. Moreover $P^{+n_s(e)}_{J_s(e)}$ is the semi-direct product of $V(e)$ by

$$K = P^{+\partial l_s}_{J(<\partial l_s)} \times G^{+(l'-l_s)}.$$

So we are now in the situation of Lemma 6.2.2.4 with $P^{+n_s(e)}_{J_s(e)}$, $P^{+(l_s-l_{s-1})}_{J(<(l_s-l_{s-1}))}$, $G^{+(l'-l_s)}$, $V(e)$, $l_s - l_{s-1}$, and $l' - l_s$ playing the roles of H, $P^{+n_1}_J$, G_2, V, n_1, and n_2, respectively. Set $r = l_{s+1} - l_s$. As $C' \in \mathbf{C}(J,t)$, it follows from Definition 8.2.1 and Remark 8.2.2.3 that $0 < r \leqslant \min(l' - l_s, l_s - l_{s-1})$ and $r \in J$, and hence $r \in J_s(e)$. Pick $\tau \in \mathrm{Irr}(V(e), r)$ as in Lemma 6.2.2.4. Then $N_K(\tau)$ is given there, with two possibilities depending on whether $r = l' - l_s$ or $r < l' - l_s$, or equivalently depending on $l_{s+1} = l'$ or $l_{s+1} < l'$. If $l_{s+1} = l'$, then $P(e')$ is defined by case (1b) of Definition 8.2.5. If $l_{s+1} < l'$, then $P(e')$ is defined by case (1a) of Definition 8.2.5. In both cases

$$N_K(\tau) \cong P^{+n_s(e')}_{J_s(e')} \times P^{+n_{s+1}(e')}_{J_{s+1}(e')}.$$

By part (2), the first s factors of $P(e)$ are isomorphic to the first s factors of $P(e')$. We conclude that $N_{P(e)}(\tau)$ is the semi-direct product of $V(e)$ by

$$L^{n_0(e)}_{J_0(e)} \times P^{+n_1(e)}_{J_1(e)} \times P^{+n_2(e)}_{J_2(e)} \times \cdots \times P^{+n_{s-1}(e)}_{J_{s-1}(e)} \times N_K(\tau) = P(e').$$

In other words, $P(e')$ can be embedded into $P(e)$.

(4) In part (2), the q-height of $|P(e)|/|P(e')||V(e)|$ is $d(e') - d(e)$. Indeed, as the first s factors of $P(e)$ are isomorphic to the first s factors of $P(e')$,

$$q^x = |P(e)|/|P(e')||V(e)| = |P^{+n_s(e)}_{J_s(e)}|/|P^{+n_s(e')}_{J_s(e')} \times P^{+n_{s+1}(e')}_{J_{s+1}(e')}||V(e)|.$$

But a parabolic subgroup of a general linear group contains a Sylow p-subgroup of the general linear group, and the q-height of $G^{+n} = GL_n(\mathbb{F}_{q^2})$ is $2\binom{n}{2}$. So

$$x = 2(\binom{n_s(e)}{2} - \binom{n_s(e')}{2} - \binom{n_{s+1}(e')}{2} - (l' - l_s)(l_s - l_{s-1})).$$

By Definition 8.2.5, $n_s(e) = l' - l_{s-1}$. We need to be a little careful when plugging in the formula for $n_s(e')$ and $n_{s+1}(e')$. Recall that $e' = (J, C', t)$ and $C' = \{l_1, \ldots, l_s, l_{s+1}\}$. That is C' consists of $s+1$ elements. So as the $n_{s+1}(e')$ appears in the last factor of $P(e')$, $n_{s+1}(e') = l' - l_s$. As $n_s(e')$ appears in the second to last factor of $P(e')$, $n_s(e') = (l_s - l_{s-1}) - (l_{s+1} - l_s)$. Applying Lemma 2.3.1, we obtain

$$x = 2(\binom{l_s - l_{s-1}}{2} - \binom{(l_s - l_{s-1}) - (l_{s+1} - l_s)}{2}).$$

Then it can be checked directly from the definition of $d(e)$ that $d(e') - d(e) = x$.

Example 8.2.7. Let $t = (l, l') \in \mathbf{T}$ and $C \in \mathbf{C}(t)$ with $|C| = 1$. Then $C = \{l\}$, and by Remark 8.2.2.2, $e = (J, C, t) \in \mathbf{E}$ if and only if $J \in \mathbf{J}(t)$. So the set of $e \in \mathbf{E}$ with $l(e) = 1$ is in natural 1-1 correspondence with $R_{\mathbf{J},\mathbf{T}}$. Pick such an $e \in \mathbf{E}$. By definition $P(e) = L^{n-2l'}_{J_0(e)} \times P^{+l'}_{J_1(e)}$. On the other hand, as (l, l') are consecutive members of J, we may let $(l, l') = (j_i, j_{i+1})$ as in Lemma 7.1.2.2. Then the lemma says a complement to $U_{J(\geqslant l')}$ in P_J is isomorphic to $L^{n-2l'}_{J_0(e)} \times P^{+l'}_{J_1(e)}$. Therefore, $P(e)$ can be identified as this subgroup of P_J containing $Z(G)$, in which case $V(e)$ is identified with $V(l, l')$. Recall from section 2.2 that $\mathrm{Irr}^1(P_J, V(l, l'))$ denotes the set of $\tau \in$

$\mathrm{Irr}(P_J)$ with $U_{J(\geqslant l')} \leqslant \ker(\tau) \not\geqslant U_{J(\geqslant l)}$. With this identification, $\mathrm{Irr}^1(P(e), V(e)) = \mathrm{Irr}^1(P_J, V(l, l'))$. As $|C| = 1$, it is easy to see that $d(e) = 0$. So in particular $k_d^1(P_J, V(l, l'), \rho) = k_{d-d(e)}^1(P(e), V(e), \rho)$ for $d \geqslant 0$ and $\rho \in \mathrm{Irr}(Z(G))$.

Lemma 8.2.8. *For any $e \in \mathbf{E}$, $P(e)$ can be embedded into G such that $Z(G) \leqslant P(e)$.*

Proof. Fix $t = (l, l') \in \mathbf{T}$. We prove the lemma by induction on $|C|$ where $e = (J, C, t) \in \mathbf{E}$. If $|C| = 1$, the lemma follows from the preceeding example. Assume the lemma for all $e = (J, C, t)$ with $|C| \leqslant s$. If $C' \in \mathbf{C}(t)$ with $|C'| = s + 1$ and $e' = (J, C', t) \in \mathbf{E}$. Let $C \prec C'$. By Remark 8.2.2.4, $e = (J, C, t) \in \mathbf{E}$. By induction we may assume $Z(G) \leqslant P(e) \leqslant G$. Then by Remark 8.2.6.3, $P(e')$ can be embedded as a complement to $V(e)$ in the stabilizer in $P(e)$ of a linear character τ of $V(e)$. As $Z(G)$ is normal in G and stabilizes τ, $Z(G) \leqslant P(e')$. That is, $Z(G) \leqslant P(e') < P(e) \leqslant G$. The lemma is proved.

□

Proposition 8.2.9. *Let $d \geqslant 0$ and $\rho \in \mathrm{Irr}(Z(G))$. For each $(C, t) \in R_{\mathbf{C},\mathbf{T}}$ with C represented as in* (8.1) *and $t = (l, l')$,*

$$\sum_{e=(J,C,t)\in\mathbf{E}} (-1)^{|e|} k_{d-d(e)}^1(P(e), V(e), \rho) = \sum_{\substack{e'=(J,C',t)\in\mathbf{E}\\ C\prec C'}} (-1)^{|e'|} k_{d-d(e')}(P(e'), \rho). \quad (8.5)$$

Proof. Denote the left-hand (right-hand) side of (8.5) by Γ_L (Γ_R). Recall from section 2.2 that $k_{d-d(e)}^1(P(e), V(e), \rho)$ counts the number of irreducible characters of $P(e)$ of q-height $d - d(e)$ lying over ρ and a non-trivial character in $\mathrm{Irr}(V(e))$. If $l_s = l'$, then by part (2a) of Definition 8.2.1, $V(e) = 1$ and $\mathrm{Irr}(V(e))$ consists of the trivial character. Therefore, $k_{d-d(e)}^1(P(e), V(e), \rho) = 0$ for each $e = (J, C, t) \in \mathbf{E}$. We conclude $\Gamma_L = 0$. Also if $l_s = l'$, then by Remark 8.2.2.3, there is no $C' \in \mathbf{C}(t)$ with $C \prec C'$. Therefore, $\Gamma_R = 0$, too. So the proposition holds in this case. We now assume $l_s < l'$.

By Remark 8.2.2.3, the set of $C' \in \mathbf{C}(t)$ with $C \prec C'$ is indexed by the set of τ

with $1 \leqslant r \leqslant \min(\partial l_s, l' - l_s)$. So

$$\Gamma_R = \sum_{r=1}^{\min(\partial l_s, l'-l_s)} \sum_{\substack{e'=(J,C',t)\in \mathbf{E} \\ C'=C\cup\{l_s+r\}}} (-1)^{|e'|} k_{d-d(e')}(P(e'), \rho).$$

By Remark 8.2.6.1, all the groups $V(e)$ involved on the left-hand side of (8.5) are isomorphic as abelian groups, and we denote $V(e)$ by V. $V = M_{\partial l_s, l'-l_s}(\mathbb{F}_{q^2})$ as an abelian group. Recall from section 6.1 that each $\tau \in \mathrm{Irr}(V)$ is assigned a rank which is invariant under the action of $P(e)$, and $\mathrm{Irr}(V, r)$ consists of the set of rank r characters in $\mathrm{Irr}(V)$. Therefore, $\mathrm{Irr}^1(V)$ is the disjoint union of $\mathrm{Irr}(V, r)$, $1 \leqslant r \leqslant \min(\partial l_s, l' - l_s)$, and each $\mathrm{Irr}(V, r)$ is a $P(e)$-set. Consequently,

$$\Gamma_L = \sum_{r=1}^{\min(\partial l_s, l'-l_s)} \sum_{e=(J,C,t)\in \mathbf{E}} (-1)^{|e|} k^1_{d-d(e)}(P(e), \mathrm{Irr}(V, r), \rho).$$

We claim that for each $1 \leqslant r \leqslant \min(\partial l_s, l' - l_s)$,

$$\sum_{e=(J,C,t)\in \mathbf{E}} (-1)^{|e|} k^1_{d-d(e)}(P(e), \mathrm{Irr}(V, r), \rho) = \sum_{\substack{e'=(J,C',t)\in \mathbf{E} \\ C'=C\cup\{l_s+r\}}} (-1)^{|e'|} k_{d-d(e')}(P(e'), \rho). \quad (8.6)$$

Then the proposition follows by summing over all r. So it remains to prove (8.6). This is a direct application of Proposition 6.2.3, Lemma 2.2.5, 8.2.3 and Remark 8.2.4. To simplify notation and to make it easier for the readers to understand the proof, we only prove (8.6) for $s = 1$. The general case can be proved by using the same argument.

As $s = 1$, $C = \{l_1\}$ with $l_1 = l$ and $C' = \{l_1, l_2\}$ with $r = l_2 - l_1$. Recall $l < l'$ by assumption.

Let $\phi_{C,t}$ be as in Lemma 8.2.3. For $e = (J, C, t) \in \mathbf{E}$, assume $\phi_{C,t}(J) = (J_0, J_1)$. So $J_0 = \{j - l' \mid j \in J(> l')\}$ and $J_1 = J(< l)$. By definition, $n_0(e) = n - 2l'$, $J_0(e) = J_0$, $n_1(e) = l'$ and $J_1(e) = J_1 \cup \{l\} = J(\leqslant l)$ as $l < l'$. Observe $n_i(e)$ does not depend on J. We have

$$P(e) = L_{J_0}^{n-2l'} \times P_{J_1\cup\{l\}}^{+l'}.$$

As V is normal in $P_{J_1\cup\{l\}}^{+l'}$, by Lemma 2.2.5 we have

$$k_{d-d(e)}(P(e), X, \rho) = \sum_{\substack{d_1,d_2\\ d_1+d_2=d}} \sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}} k_{d_1}(L_{J_0}^{n-2l'}, \rho_1) k_{d_2-d(e)}(P_{J_1\cup\{l\}}^{+l'}, X, \rho_2) \rho_2).$$

Recall from Lemma 8.2.3 that $|e| = |J| = |J_0| + |J_1| + 1 + \delta(\partial C)$, and $\phi_{C,t}$ is a 1-1 correspondence. So as the set of $(J, C, t) \in \mathbf{E}$ (with (C, t) fixed) is in 1-1 correspondence with $\mathbf{J}(C, t)$, when e runs over all the possibilities, J_0 and J_1 run over all the subsets of $[m - l']$ and $[l - 1]$. Therefore, the left-hand side of (8.6) can be written as

$$(-1)^{\delta(\partial C)+1} \sum_{\substack{d_1,d_2\\ d_1+d_2=d}} \sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}} \sigma(d_1, \rho_1)\eta(d_2 - d(e), \rho_2) \tag{8.7}$$

where

$$\sigma(d_1, \rho_1) = \sum_{J_0\subseteq[m-l']} (-1)^{|J_0|} k_{d_1}(L_{J_0}^{n-2l'}, \rho_1)$$

and

$$\eta(d_2 - d(e), \rho_2) = \sum_{J_1\subseteq[l-1]} (-1)^{|J_1|} k_{d_2-d(e)}(P_{J_1\cup\{l\}}^{+l'}, X, \rho_2).$$

Recall that $d(e)$ depends only on C, so it is a constant as C is fixed.

Now pick $e' = (J, C', t) \in \mathbf{E}$. Assume $\phi_{C',t}(J) = (J_0', J_1', J_2')$. Then by Lemma 8.2.3, $J_0' = \{j - l' \mid j \in J(> l')\}$, $J_1' = \{j - r \mid j \in J \text{ and } r < j < l\}$, and $J_2' = J(< r)$. By definition, $n_0(e') = n - 2l'$, $J_0(e') = J_0'$, $n_1(e') = l - r$, $J_1(e') = J_1'$, $n_2(e') = l' - l$, $J_2(e') = J_2'$ if $l_2 = l'$ and $J_2(e') = J_2' \cup \{r\}$ if $l_2 < l'$. Observe $n_i(e')$ does not depend on J. Observe (as we did in general in Remark 8.2.6.2) that $n_0(e) = n_0(e')$ and $J_0 = J_0'$ for $e = (J, C, t)$. Also J_1, J_1' and J_2' are related as in Remark 8.2.4. We have

$$P(e') = L_{J_0'}^{n-2l'} \times P_{J_1'}^{+(l-r)} \times P_{J_2'}^{+(l'-l)}$$

where $\tilde{J}_2' = J_2(e')$. Again by Lemma 2.2.5 we have

$$k_{d-d(e)}(P(e'), X, \rho) = \sum_{\substack{d_1,d_2\\ d_1+d_2=d}} \sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}} k_{d_1}(L_{J_0}^{n-2l'}, \rho_1) k_{d_2-d(e')}(P_{J_1}^{+(l-r)} \times P_{\tilde{J}_2'}^{+(l'-l)}, \rho_2).$$

As above, the right-hand side of (8.6) can be written as

$$(-1)^{\delta(\partial C')+1} \sum_{\substack{d_1,d_2\\ d_1+d_2=d}} \sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}} \sigma'(d_1,\rho_1)\eta'(d_2 - d(e'), \rho_2) \tag{8.8}$$

where

$$\sigma'(d_1,\rho_1) = \sum_{J_0'\subseteq[m-l']} (-1)^{|J_0'|} k_{d_1}(L_{J_0}^{n-2l'}, \rho_1)$$

and

$$\eta'(d_2 - d(e'), \rho_2) = \sum_{J_1'\subseteq[l-r]} \sum_{J_2'\subseteq[l'-l]} (-1)^{|J_1'|+|J_2'|} k_{d_2-d(e')}(P_{J_1}^{+(l-r)} \times P_{\tilde{J}_2'}^{+(l'-l)}, \rho_2).$$

Recall $d(e')$ depends only on C', so it is a constant as C' is fixed.

Clearly for any d_1 and ρ_1,

$$\sigma(d_1,\rho_1) = \sigma'(d_1,\rho_1).$$

Now if we can prove that for any d and ρ,

$$(-1)^{\delta(\partial C)}\eta(d,\rho) = (-1)^{\delta(\partial C')}\eta'(d-d',\rho) \tag{8.9}$$

where $d' = d(e') - d(e)$, then (8.7) is equal to (8.8), thus the claim is proved. So it remains proved (8.9).

Recall $V \cong M_{l,l'-l}(\mathbb{F}_{q^2})$ as an abelian group. We identify V with the module $V = V_1 \otimes V_2$ in Proposition 6.2.3. So $n_1 = l$, $n_2 = l' - l$, and $r = l_2 - l$. Also $H_{J_1} = P_{J_1\cup\{l\}}^{+l'}$ for each $J_1 \subseteq [l-1]$. Recall $\partial C = \{l\}$ and $\partial C' = \{l, r\}$, so $\delta(\partial C) = \delta(\partial C')$ if $l = r$

and $\delta(\partial C) = \delta(\partial C') - 1$ if $r < l$. Also observe that by Lemma 2.3.1 we always have

$$2(\binom{n_1}{2} - \binom{n_1 - r}{2}) = 2(\binom{l}{2} - \binom{l-r}{2}) = d(e') - d(e),$$

that is d' in Proposition 6.2.3 coincides with d' in (8.9).

Assume $r = l$. So $\delta(\partial C) = \delta(\partial C')$. If $r = l' - l$, that is if $l_2 = l'$, then $\hat{J}_2' = J_2'$. In this case (8.9) follows from the first equation of Proposition 6.2.3, as $J_1' = \emptyset$ and $P_{J_1'}^{+(l-r)} = 1$. If $r < l' - l$, that is if $l_2 < l'$, then $\hat{J}_2' = J_2 \cup \{r\}$. In this case (8.9) follows from the second equation of the proposition.

Assume $r < l$. So $\delta(\partial C) = \delta(\partial C') - 1$. If $r = l' - l$, (8.9) follows from the third equation of the proposition. If $r < l' - l$, (8.9) follows from the fourth equation of the proposition. The proof is complete.

□

Corollary 8.2.10.

$$\sum_{\substack{e \in \mathbf{E} \\ l(e)=1}} (-1)^{|e|} k^1_{d-d(e)}(P(e), V(e), \rho) = \sum_{\substack{e \in \mathbf{E} \\ l(e)\geqslant 2}} (-1)^{|e|} k^0_{d-d(e)}(P(e), V(e), \rho).$$

Proof. Fix $s \geqslant 1$. As (8.5) holds for each $|C|$ with $|C| = s$ and $t \in \mathbf{C}(t)$, and as each $C' \in \mathbf{C}$ with $|C'| = s + 1$ covers a unique $C \in \mathbf{C}$ by Remark 8.1.2.2, we obtain

$$\sum_{\substack{e \in \mathbf{E} \\ l(e)=s}} (-1)^{|e|} k^1_{d-d(e)}(P(e), V(e), \rho) = \sum_{\substack{e' \in \mathbf{E} \\ l(e')=s+1}} (-1)^{|e'|} k_{d-d(e')}(P(e'), \rho).$$

Then taking the sum over all possible $s \geqslant 1$, we obtain

$$\sum_{\substack{e \in \mathbf{E} \\ l(e)\geqslant 1}} (-1)^{|e|} k^1_{d-d(e)}(P(e), V(e), \rho) = \sum_{\substack{e' \in \mathbf{E} \\ l(e')\geqslant 2}} (-1)^{|e'|} k_{d-d(e')}(P(e'), \rho).$$

But for each $e' \in \mathbf{E}$ with $l(e') \geqslant 2$,

$$k_{d-d(e')}(P(e'), \rho) = k^0_{d-d(e')}(P(e'), V(e'), \rho) + k^1_{d-d(e')}(P(e'), V(e'), \rho),$$

It follows that

$$\sum_{\substack{e\in\mathbf{E}\\ l(e)=1}} (-1)^{|e|} k^1_{d-d(e)}(P(e),V(e),\rho) = \sum_{\substack{e\in\mathbf{E}\\ l(e)\geqslant 2}} (-1)^{|e|} k^0_{d-d(e)}(P(e),V(e),\rho).$$

□

8.3 Second Reduction

We now move to the discussion of a relation on $\mathbf{J}\times\mathbf{C}\times\mathbf{I}$ where $\mathbf{I}=I\cup\{0\}$.

Definition 8.3.1. (1) $R_{\mathbf{J},\mathbf{C}}$ remains as in Definition 8.2.1.

(2) Set $R_{\mathbf{J},\mathbf{I}}$ to be the set of $(J,l)\in\mathbf{J}\times\mathbf{I}$ with $l=\max(J)$.

(3) Set $R_{\mathbf{C},\mathbf{I}}$ to be the set of $(C,l)\in\mathbf{C}\times\mathbf{I}$ with $l=\min(C)$.

(4) Set $\mathbf{F}=R_{\mathbf{J},\mathbf{C},\mathbf{I}}$ and make $\mathbf{F}$ a graded set by letting $|f|=|J|$ for $f=(J,C,l)$. We also define a length function on $\mathbf{F}$ by setting $l(f)=|C|$.

The following observations and lemmas are similar in nature to Remark 8.2.2 and Lemma 8.3.3.

Remark 8.3.2. (1) Recall $\min(\emptyset)=\max(\emptyset)=0$. By definition, if $f=(J,C,l)\in\mathbf{F}$, then either $f=(\emptyset,\emptyset,0)$ or both J and C are non-empty. If $f\neq(\emptyset,\emptyset,0)$, then $f\in\mathbf{F}$ if and only if $\max(J)=\min(C)=l$ and $\partial C\subseteq J$, as condition (2a) in Definition 8.2.1 is automatically satisfied when $\max(J)=l$.

(2) Assume $C\in\mathbf{C}(l)$ with $|C|=1$. Then $C=\{l\}$. In this case,

$$\mathbf{J}(C,l)=\mathbf{J}(C)\cap\mathbf{J}(l)=\mathbf{J}(l)=\{J\in\mathbf{J}\mid\max(J)=l\}.$$

(3) If $C,C'\in\mathbf{C}$ with $C\prec C'$, $|C|\geqslant 1$, and $f'=(J,C',l)\in\mathbf{F}$, then $f=(J,C,l)\in\mathbf{F}$. To show $f\in\mathbf{F}$, we have to check $C\in\mathbf{C}(J,l)$. As in part (4) of Remark 8.2.2, $C\in\mathbf{C}(J)$. $l=\min(C')$ since $C'\in\mathbf{C}(l)$. But $\min(C)=\min(C')$, given

that C is obtained by deleting $\max(C')$ and the fact that $|C'| = |C| + 1 \geqslant 2$. So $l = \min(C)$. That is, $C \in \mathbf{C}(l)$. It follows that $C \in \mathbf{C}(J, l)$.

(4) $\mathbf{C}(J)$ is the disjoint union of $\mathbf{C}(J, \max(J))$ and $\mathbf{C}(J, t)$, $t \in \mathbf{T}(J)$. Indeed, $\mathbf{C}(J, \max(J))$ and $\mathbf{C}(J, t)$ are disjoint with each other as the minimal members of the convex sequences from different sets are different. On the other hand, if $C \in \mathbf{C}(J)$ with $\min(C) = l$, then either $l = \max(J)$ so that $C \in \mathbf{C}(J, l)$, or there is a unique $t = (l, l') \in \mathbf{T}(J)$ whose minimal member is l, in which case it can be checked from the definition that $C \in \mathbf{C}(t)$ and hence $C \in \mathbf{C}(J, t) = \mathbf{C}(J) \cap \mathbf{C}(t)$.

Lemma 8.3.3. *For $l \in I$ and $C \in \mathbf{C}(l)$ represented as in* (8.1), *there is a 1-1 correspondence*

$$\phi = \phi_{C,l} : \mathbf{J}(C, l) \to \Delta([\partial^2 l_1 - 1]) \times \cdots \times \Delta([\partial^2 l_{s-1} - 1]) \times \Delta([\partial l_s - 1])$$

$$J \mapsto (J_1, \ldots, J_s)$$

where

$$J_i = \{j - \partial l_{i+1} \mid j \in J \text{ and } \partial l_{i+1} < j < \partial l_i\},\ 1 \leqslant i \leqslant s - 1,$$

and

$$J_s = J(< \partial l_s).$$

Moreover,

$$|J| = \textstyle\sum_{i=1}^{s} |J_i| + \delta(\partial C).$$

Notice if $|C| = 1$, then the lemma is an easy consequence of part (2) of the Remark 8.3.2.

Proof. The proof is similar to that of Lemma 8.2.3. From part (1) of Remark 8.3.2 we know that $J \in \mathbf{J}(C, l)$ if and only if $\max(J) = l$ and $\partial C \subseteq J$. Thus ∂C breaks $J \backslash \partial C$ into several segments $\hat{J}_i = \{j \in J \mid \partial l_{i+1} < j < \partial l_i\}$ for $1 \leqslant i \leqslant s - 1$, and

$\hat{J}_s = J(< \partial l_s)$, with $J(> l) = \emptyset$. Each segment $\hat{J}_i$, $1 \leqslant i \leqslant s$, can be any subset of $\Delta([\partial l_s - 1])$ or $\{\partial l_{i+1} + 1, \ldots, \partial l_i - 1\}$. Therefore,

$$\phi : \quad J \mapsto (\hat{J}_1, \ldots, \hat{J}_s) \mapsto (J_0, \ldots, J_s)$$

where J_i is defined as in the hypothesis is the desired 1-1 correspondence.

To count the number of elements of J, we observe that J is the disjoint union of ∂C regarded as a set, and $\hat{J}_i$, $1 \leqslant i \leqslant s$. Consequently, as ∂C contains $\delta(\partial C)$ distinguished elements and $|\hat{J}_i = |J_i|$, we have $|J| = \sum_i |J_i| + \delta(\partial C)$.

□

Remark 8.3.4. Let $C \in \mathbf{C}$ and $l \in I$ be as in Lemma 8.3.3. Let $C \prec C' = C \cup \{l_{s+1}\} \in \mathbf{C}$. Assume $(J, C', l) \in \mathbf{F}$ for some J. Then by Remark 8.3.2.3, $(J, C, l) \in \mathbf{F}$. Assume $\phi_{C,l}(J) = (J_1, \ldots, J_s)$ and $\phi_{C',l}(J) = (J'_1, \ldots, J'_s, J'_{s+1})$. Then $J_i = J'_i$ for $1 \leqslant i \leqslant s - 1$. This can be seen by checking directly with the definition in Lemma 8.3.3. Also by definition,

$$J_s = J(< \partial l_s)) \subseteq [\partial l_s - 1],$$

while

$$J'_s = \{j - \partial l_{s+1} \mid j \in J \,\text{and}\, \partial l_{s+1} < j < \partial l_s\} \subseteq [\partial_s^2 C' - 1]$$

and

$$J'_{s+1} = J(< \partial l_{s+1}) \subseteq [\partial l_{s+1} - 1].$$

So J_s, J'_s, and J'_{s+1} are related by

$$J'_s = \{j - \partial l_{s+1} \mid j \in J_s(> \partial l_{s+1})\}$$

and

$$J'_{s+1} = J_s(< \partial l_{s+1}).$$

To each $f \in \mathbf{F}$, we associate a group $P(f)$, a normal p-subgroup $U(f)$ of $P(f)$, and an integer $d(f)$. Notice $f = (\emptyset, \emptyset, 0)$ is the only member in $\mathbf{F}$ with $l(f) = 0$. Set $P(f) = G$, $U(f) = 1$, and $d(f) = 0$. Next let $f = (J, C, l) \in \mathbf{F}$ with $l \in I$ and C as in equation (8.1). Assume $\phi(J) = (J_1, \ldots, J_s)$ as in Lemma 8.3.3.

Definition 8.3.5. (1) Define $P(f)$ to be a group isomorphic to

$$P_{J_1(f)}^{+n_1(f)} \times P_{J_2(f)}^{+n_2(f)} \times \cdots \times P_{J_{s-1}(f)}^{+n_{s-1}(f)} \times P_{J_s(f)}^{n_s(f)} \tag{8.10}$$

where

$$n_i(f) = \partial^2 l_i,\ J_i(f) = J_i \text{ for } 1 \leqslant i \leqslant s-1,$$

and

$$n_s(f) = n - 2l_{s-1},\ J_s(f) = J_s \cup \{\partial l_s\} = J(\leqslant \partial l_s).$$

(2) As $\max(J_s(f)) = \partial l_s$, $P_{J_s(f)}^{n_s(f)} \leqslant P_{\partial l_s}^{n_s(f)} \leqslant G^{n-2l_{s-1}}$. Set $U(f) = U_{\partial l_s}^{n-2l_{s-1}}$. So $U(f)$ is a normal p-subgroup of $P_{J_s(f)}^{n_s(f)}$ and hence of $P(f)$. Also set $Z(f) = Z(U(f))$ and $V(f) = U(f)/Z(f)$.

(3) Define

$$d(f) = 2\sum_{i=1}^{s-1}\left(\binom{\partial l_i}{2} - \binom{\partial^2 l_i}{2}\right).$$

The following remarks are similar to Remark 8.2.6.

Remark 8.3.6. (1) $U(f)$ depends only on (C, l). That is, if $f_i = (J_i, C, l) \in \mathbf{F}$, $i = 1, 2$, then $U(f_1) \cong U(f_2)$. $d(f)$ depends only on C.

(2) Assume both $f = (J, C, l) \in \mathbf{F}$ and $f' = (J, C, l) \in \mathbf{F}$, with $C \prec C' = C \cup \{l_{s+1}\}$. By definition and by Remark 8.3.4, $n_i(f) = n_i(f')$ and $J_i(f) = J_i(f')$ for $1 \leqslant i \leqslant s-1$. That is the i-th factor of $P(f)$ is isomorphic to the i-th factor of $P(f')$.

(3) Continue the discussion in part (2). Set $\bar{P}(f) = P(f)/Z(f)$. Then by Lemma 7.1.2.3 (where we replace n, j_s, by $n_s(f)$, ∂l_s, respectively, and observe $n_s(f) - 2\partial l_s = n - 2l_s$), $\bar{P}^{n_s(f)}_{J_s(f)}$ is a semi-direct product of $V(f)$ by $K = P^{+\partial l_s}_{J(<\partial l_s)} \times G^{n-2l_s}$. The conjugation action makes $V(f)$ a tensor module for K. So we may apply the discussion in section 6.2. Set $r = l_{s+1-l_s}$. Then as C' is convex, $1 \leqslant r \leqslant \partial l_s$. As $l_s < l_{s+1} \leqslant m$, $2r \leqslant n - 2l_s$. Moreover $r \in \partial C' \subseteq J$, so $r \in J_s(f) = J(\leqslant \partial l_s)$. So $S^u(V(f), J_s(f), r) \subseteq \mathrm{Irr}(V(f), r)/P^{n_s(f)}_{J_s(f)}$ is defined by Definition 6.3.2 (in case $r = \partial l_s$) or Definition 6.3.6 (in case $r < \partial l_s$), and it is non-empty. As $2r \leqslant n - 2l_s$, $J(\leqslant r) \subseteq [m - l_s]$. So by Remark 6.3.12.2, $S^{su}(V(f), J_s(f), r)$ consists of a single orbit. Let $\tau \in S^{su}(V(f), J_s(f), r)$. By Remark 6.3.12.2, it can be computed that

$$N_K(\tau) = P^{+n_s(f')}_{J_s(f')} \times P^{n_{s+1}(f')}_{J_{s+1}(f')}.$$

By part (2), the first $s-1$ factors of $P(f)$ are isomorphic to the first $s-1$ factors of $P(e')$. Therefore, viewing τ as a linear character of $U(f)$, $N_{P(f)}(\tau)$ is the semi-direct product of $U(f)$ by $P(f')$. That is $P(f')$ can be embedded into $P(f)$.

(4) In part (2), the q-height of $|P(f)|/|P(f')|$ is $q^{d(f')-d(f)}$. Indeed, by part (2),

$$q^x = |P(f)|/|P(f')||U(f)| = |P^{+n_s(f)}_{J_s(f)}|/|P^{+n_s(f')}_{J_s(f')} \times P^{+n_{s+1}(f')}_{J_{s+1}(f')}||U(f)|.$$

But a parabolic subgroup of a general unitary contains a Sylow p-subgroup of the unitary group, and the q-height of $G^n = GU_n(\mathbb{F}_q)$ is $\binom{n}{2}$. So

$$x = (\binom{n_s(f)}{2} - 2\binom{n_s(f')}{2} - \binom{n_{s+1}(f')}{2} - (l_s - l_{s-1})^2 - 2\binom{(l_s - l_{s-1}) - (l_{s+1} - l_s)}{2}).$$

By definition, $n_s(f) = n - 2l_{s-1}$. Again we need to be careful when plugging in the formula for $n_s(f')$ and $n_{s+1}(f')$. Recall $f' = (J, C', l)$ and $C' = \{l_1, \ldots, l_s, l_{s+1}\}$ consisting of $s+1$ elements. So as $n_{s+1}(f')$ appears in the last factor of $P(f')$, $n_{s+1}(f') = n - 2l_s$. As $n_s(f')$ appears in the second to last factor of $P(f')$, $n_s(f') = (l_s - l_{s-1}) - (l_{s+1} - l_s)$. Applying Lemma 2.3.1, we obtain

$$x = 2\left(\binom{l_s - l_{s-1}}{2} - \binom{(l_s - l_{s-1}) - (l_{s+1} - l_s)}{2}\right).$$

Then it can be checked directly from the definition of $d(f)$ that $d(f') - d(f) = x$.

Example 8.3.7. Let $l \in I$ and $C \in \mathbf{C}(l)$ with $|C| = 1$. Then $C = \{l\}$, and by part (2) of Remark 8.3.2, $f = (J, C, l) \in \mathbf{F}$ if and only if $J \in \mathbf{J}(l)$. By definition of $P(f)$, $P(f) = P_J$ and $U(f) = U_l$. In particular $Z(G) \leqslant P(f) \leqslant G$. Also $d(f) = 0$. So in this case $\mathrm{Irr}(P_J, U_l, \rho) = \mathrm{Irr}(P(f), U(f), \rho)$. Consequently,

$$k_d^1(P_J, Z(U_l), \rho) = k_d^1(P(f), Z(f), \rho) \text{ and}$$
$$k_d^1(P_J, U_l/Z(U_l), \rho) = k_{d-d(f)}^1(P(f), V(f), \rho).$$

Observe there is a 1-1 correspondence between the set of $\emptyset \neq J \subseteq I$ and the set of $f \in \mathbf{F}$ with $l(f) = 1$, given by $J \mapsto (J, \{\max(J)\}, \max(J))$.

Lemma 8.3.8. *For any $f \in \mathbf{F}$, $P(f)$ can be embedded into G such that $Z(G) \leqslant P(f)$.*

Proof. We prove the lemma by induction on $|C|$ where $f = (J, C, l) \in \mathbf{F}$. If $f = (\emptyset, \emptyset, 0)$, by definition $P(f) = G$ so the statement is true. If $f = (J, C, l) \in \mathbf{F}$ with $|C| = 1$, then the lemma follows from the preceeding example. Assume the lemma for all $f = (J, C, l)$ with $|C| \leqslant s$. Pick $C' \in \mathbf{C}(t)$ with $|C'| = s + 1$ and $f' = (J, C', l) \in \mathbf{F}$. Let $C \prec C'$. By part (3) of Remark 8.3.2, $f = (J, C, l) \in \mathbf{F}$. By induction we may assume $Z(G) \leqslant P(f) \leqslant G$. Then by Remark 8.3.6.3, $P(f')$ can be embedded as a complement to $U(f)$ in the stabilizer in $P(f)$ of a linear character τ of $U(f)/Z(f)$ labeled by a singular normal chain in $\Delta(\mathcal{P}(V^{n-2l_s})$, where V^{n-2l_s} is the natural module for G^{n-2l_s} (assuming $\max(C) = l_s$). As $Z(G)$ is normal and stabilizes τ, $Z(G) \leqslant P(f')$. That is $Z(G) \leqslant P(f') \leqslant G$. The lemma is proved.

□

Definition 8.3.9. Let $f = (J, C, l) \in \mathbf{F}$ be as in Definition 8.3.5. By definition of $P(f)$, all but one factors act trivially on $U(f)$ and hence on $V(f)$ and $Z(f)$. Moreover $V(f)$ is a unitary module for $P_{J_s(f)}^{n_s(f)}/U(f)$ as we studied in section 6.3. We define $S^u(f)$ (resp. $S^{nu}(f)$, $S^{su}(f)$, $S^{nu}(f, r)$, $S_\tau^{nu}(f)$, etc.) to be $S^u(V(f), J_s(f))$ (resp. $S^{nu}(V(f), J_s(f))$, $S^{su}(V(f), J_s(f))$, $S^{nu}(V(f), J_s(f), r)$, $S_\tau^{nu}(V(f), J_s(f))$, etc.).

Similarly as in Remark 8.3.6.3,

$$P^{n_s(f)}_{J_s(f)}/U(f) \cong \bar{P}^{n_s(f)}_{J_s(f)}/V(f) \cong P^{+\partial l_s}_{J(<\partial l_s)} \times G^{n-2l_s},$$

G^{n-2l_s} acts trivially on $Z(f)$, and $Z(f)$ is a central module for $P^{+(\partial l_s)}_{J(<\partial l_s)}$ as we studied in section 7.3. So we define $S^z(f)$ (resp. $S^z(f,r)$, $S^z_r(f)$, etc.) to be $S^z(Z(f), J(\leqslant \partial l_s))$ (resp. $S^z(Z(f), J(\leqslant \partial l_s), r)$, $S^z_r(Z(f), J(\leqslant \partial l_s))$, etc.)

Proposition 8.3.10. *Let $d \geqslant 0$ and $\rho \in Irr(Z(G))$. Fix $\emptyset \neq C \in \mathbf{C}$ represented as in* (8.1) *with $l = \min(C)$, and let $\mathbf{F}_0$ be the set of $f = (J, C, l) \in \mathbf{F}$.*

(1)

$$\sum_{f \in \mathbf{F}_0} (-1)^{|f|} k^1_{d-d(f)}(P(f), Z(f), \rho) = \sum_{f \in \mathbf{F}_0} (-1)^{|f|} k^1_{d-d(f)}(P(f), S^z(f), \rho).$$

(2)

$$\sum_{f \in \mathbf{F}_0} (-1)^{|f|} k^1_{d-d(f)}(P(f), V(f), \rho) = \sum_{f \in \mathbf{F}_0} (-1)^{|f|} k^1_{d-d(f)}(P(f), S^u(f), \rho).$$

(3)

$$\sum_{f \in \mathbf{F}_0} (-1)^{|f|} k^1_{d-d(f)}(P(f), S^{su}(f), \rho) = \sum_{\substack{f'=(J,C',l) \in \mathbf{F} \\ C \prec C'}} k_{d-d(f')}(P(f'), \rho).$$

(4)

$$\sum_{f=(J,C,l) \in \mathbf{F}} (-1)^{|f|} k^1_{d-d(f)}(P(f), U(f), \rho) = \sum_{\substack{f'=(J,C',l) \in \mathbf{F} \\ C \prec C'}} k_{d-d(f')}(P(f'), \rho) +$$
$$\sum_{f=(J,C,l) \in \mathbf{F}} (-1)^{|f|} (k_{d-d(f)}(P(f), S^{nu}(f), \rho) + k_{d-d(f)}(P(f), S^z(f), \rho)).$$

Proof. We prove part (1). First assume $s = |C| = 1$. As (C, l) is given, by Example

8.3.7,

$$\sum_{f=(J,C,l)\in\mathbf{F}} (-1)^{|f|} k^1_{d-d(f)}(P(f), Z(f), \rho) = \sum_{J\in\mathbf{J}(l)} (-1)^{|J|} k^1_d(P_J, Z(U_l), \rho).$$

Similarly

$$\sum_{f=(J,C,l)\in\mathbf{F}} (-1)^{|f|} k^1_{d-d(f)}(P(f), S^z(f), \rho) = \sum_{J\in\mathbf{J}(l)} (-1)^{|J|} k_d(P_J, S^z(Z(U_l), J), \rho).$$

Therefore, part (1) follows directly from Proposition 7.3.7.

For $s \geqslant 2$, each $P(f)$ is a product of $s-1$ parabolics of general linear groups with a parabolic subgroup of a general unitary group. Ther proof of part (1) is an application of Proposition 7.3.7 and Lemma 2.2.6. To simplify notation, we only present a proof for the case $s = 2$. The same argument applies for the general case.

Assume $C = \{l, l'\}$. Let $\phi_{C,l}$ be as in Lemma 8.3.3. Let $f = (J, C, l) \in \mathbf{F}$ with $\phi_{C,l}(J) = (J_1, J_2)$. So $|J| = |J_1| + |J_2| + \delta(\partial C)$. Observe that $n_1(f) = l - (l' - l)$ and $n_2(f) = n - 2l$ do not depend on J. By definition

$$P(f) = P^{+(l-(l'-l))}_{J_1} \times P^{n-2l}_{J_2\cup\{l'-l\}}.$$

As (C, l) is fixed, the set of $(J, C, l) \in \mathbf{F}$ is in 1-1 correspondence with $\mathbf{J}(C, l)$, which is in 1-1 correspondence with $\Delta([l - (l' - l) - 1]) \times \Delta([l - 1])$ via $\phi_{C,l}$.

Recall that the $Z(f)$ are isomorphic as abelian groups for all $f = (J, C, l) \in \mathbf{F}$, which we may record as Z, and the first factor $P^{+n_1(f)}_{J_1}$ acts trivially on Z. Finally by Proposition 7.3.7 (where the sum is taken over all $J \in \mathbf{J}(l' - l) = \{J \mid \max(J) = l' - l\}$), for any d, ρ,

$$\sum_{J_2\subseteq[x]} (-1)^{|J_2|} k^1_d(P^{n-2l}_{J_2\cup\{l'-l\}}, Z, \rho) = \sum_{J_2\subseteq[x]} (-1)^{|J_2|} k_d(P^{n-2l}_{J_2\cup\{l'-l\}}, S^z(Z, J_2 \cup \{l' - l\}), \rho)$$

where $x = l' - l - 1$. Therefore, as $\phi_{J,l}$ is a bijection, and by definition

$$k^1_d(P^{n-2l}_{J_2\cup\{l\}}, Z, \rho) = k_d(P^{n-2l}_{J_2\cup\{l\}}, \mathrm{Irr}^1(Z), \rho),$$

part (1) follows from Lemma 2.2.6.

The same proof applies to part (2), as it is a direct application of Proposition 6.3.9 and Lemma 2.2.6. The argument in the proof of Proposition 8.2.9 can be used to prove part (3), which is an application of Proposition 6.3.13, Lemma 2.2.5, 8.3.3 and Remark 8.3.4. Finally it remains to prove part (4).

For each $f = (J, C, l) \in \mathbf{F}$, as $V(f) = U(f)/Z(f)$, we have

$$k_d^1(P(f), U(f), \rho) = k_d^1(P(f), Z(f), \rho) + k_d^1(P(f), V(f), \rho).$$

Summing over all $f = (J, C, l) \in \mathbf{F}$ and applying part (1) and (2), we have

$$\sum_{f=(J,C,l)\in\mathbf{F}} (-1)^{|f|} k_{d-d(f)}^1(P(f), U(f), \rho) = \sum_{f=(J,C,l)\in\mathbf{F}} (-1)^{|f|} (k_{d-d(f)}(P(f), S^z(f), \rho) + k_{d-d(f)}(P(f), S^u(f), \rho)). \tag{8.11}$$

On the other hand, by Definition 8.3.9 and 6.3.10,

$$k_d(P(f), S^u(f), \rho) = k_d(P(f), S^{su}(f), \rho) + k_d(P(f), S^{nu}(f), \rho).$$

Consequently,

$$\sum_{f=(J,C,l)\in\mathbf{F}} (-1)^{|f|} k_{d-d(f)}(P(f), S^u(f), \rho) = \sum_{f=(J,C,l)\in\mathbf{F}} (-1)^{|f|} k_{d-d(f)}(P(f), S^{su}(f), \rho) + \sum_{f=(J,C,l)\in\mathbf{F}} (-1)^{|f|} k_{d-d(f)}(P(f), S^{nu}(f), \rho). \tag{8.12}$$

Therefore, part (4) follows by substituting (8.11) in (8.12) and applying part (3).

□

Corollary 8.3.11. *For each $d \geqslant 0$ and $\rho \in Irr(Z(G))$,*

$$\sum_{\substack{f\in\mathbf{F}\\ l(f)=1}} (-1)^{|f|} k^1_{d-d(f)}(P(f),U(f),\rho) = \sum_{\substack{f\in\mathbf{F}\\ l(f)\geqslant 2}} k^0_{d-d(f)}(P(f),U(f),\rho) + \sum_{\substack{f\in\mathbf{F}\\ l(f)\geqslant 1}} (-1)^{|f|}(k_{d-d(f)}(P(f),S^{nu}(f),\rho) + k_{d-d(f)}(P(f),S^{z}(f),\rho)).$$

Proof. This can be proved by the same argument as in the proof of Corollary 8.2.10. For each $f \in \mathbf{F}$, set

$$\sigma(f) = k_{d-d(f)}(P(f),S^{nu}(f),\rho) + k_{d-d(f)}(P(f),S^{z}(f),\rho).$$

Fix $s \geqslant 1$. As Proposition 8.3.10.4 holds for each $|C|$ with $|C| = s$, and as each $C' \in \mathbf{C}$ with $|C'| = s+1$ covers a unique $C \in \mathbf{C}$ by Remark 8.1.2.2, we obtain

$$\sum_{\substack{f\in\mathbf{F}\\ l(f)=s}} (-1)^{|f|} k^1_{d-d(f)}(P(f),U(f),\rho) = \sum_{\substack{f'\in\mathbf{F}\\ l(f')=s+1}} (-1)^{|f'|} k_{d-d(f')}(P(f'),\rho) + \sum_{\substack{f\in\mathbf{F}\\ l(f)=s}} (-1)^{|f|}\sigma(f).$$

Then taking the sum over all possible $s \geqslant 1$, we obtain

$$\sum_{\substack{f\in\mathbf{E}\\ l(f)\geqslant 1}} (-1)^{|f|} k^1_{d-d(f)}(P(f),U(f),\rho) = \sum_{\substack{f'\in\mathbf{F}\\ l(f')\geqslant 2}} (-1)^{|f'|} k_{d-d(f')}(P(f'),\rho) + \sum_{\substack{f\in\mathbf{F}\\ l(f)\geqslant 1}} (-1)^{|f|}\sigma(f).$$

But for each $f' \in \mathbf{F}$ with $l(f') \geqslant 2$,

$$k_{d-d(f')}(P(f'),\rho) = k^0_{d-d(f')}(P(f'),U(f'),\rho) + k^1_{d-d(f')}(P(f'),U(f'),\rho),$$

It follows that

$$\sum_{\substack{f\in\mathbf{F}\\ l(f)=1}} (-1)^{|f|} k^1_{d-d(f)}(P(f),U(f),\rho) = \sum_{\substack{f\in\mathbf{F}\\ l(f)\geqslant 2}} (-1)^{|f|} k^0_{d-d(f)}(P(f),U(f),\rho) + \sum_{\substack{f\in\mathbf{F}\\ l(f)\geqslant 1}} (-1)^{|f|}\sigma(f).$$

The proof is complete.

□

Chapter 9 Completion of the Verification

9.1 Further Reduction

Let $\mathbf{E}'$ be the set of $e \in \mathbf{E}$ with $l(e) \geqslant 2$. Similarly let $\mathbf{F}'$ be the set of $f \in \mathbf{F}$ with $l(f) \geqslant 2$. Set $\mathbf{H} = \mathbf{E}' \cup \mathbf{F}'$. We define $\xi : \mathbf{H} \to \mathbf{H}$ as follows and then show it is well defined.

Definition 9.1.1. Let $h \in \mathbf{H}$. So either $h = (J, C, t) \in \mathbf{E}'$, or $h = (J, C, l) \in \mathbf{F}'$. In either case, we assume C is represented as in equation (8.1). By hypothesis, $l(h) = |C| \geqslant 2$.

(1) If $h = f = (J, C, l) \in \mathbf{F}'$, then define

$$\xi(h) = (J', C, t') \in \mathbf{E}', \text{ where } J' = J \cup \{l_s\} \text{ and } t' = (l, l_s).$$

If $h = e = (J, C, t) \in \mathbf{E}'$ with $t = (l, l')$, then by definition of $\mathbf{E}'$, $l < l_s \leqslant l'$. Also as $l' \in J$, $l' \leqslant \max(J)$.

(2) If $l_s < l'$, then define

$$\xi(h) = (J', C, t'), \text{ where } J' = J \cup \{l_s\} \text{ and } t' = (l, l_s).$$

(3) If $l_s = l' < \max(J)$, then as $l' \in J$, there is a unique $l'' \in J$ with $(l', l'') \in \mathbf{T}(J)$. Define

$$\xi(h) = (J', C, t'), \text{ where } J' = J \backslash \{l'\} \text{ and } t' = (l, l'').$$

(4) If $l_s = l' = \max(J)$, then define

$$\xi(h) = (J', C, l) \in \mathbf{F}', \text{ where } J' = J \backslash \{l'\}.$$

We now verify that ξ is a well defined.

In case (1), as $l = \min(C) = \max(J)$, $l = l_1$, and there is no $j \in J$ between $l_1 = l$ and l_s. Hence $(C, t') \in R_{\mathbf{C},\mathbf{T}}$ and $(J', C) \in R_{\mathbf{J},\mathbf{C}}$ since $J' = J \cup \{l_s\}$. Also there is no $j \in J'$ between l and l_s. Therefore, as $l, l_s \in \mathbf{J}'$, $t' \in \mathbf{T}(J')$. Hence $\xi(h) \in \mathbf{E}$.

In case (2), as $(l, l') \in \mathbf{T}(J)$, there is no $j \in J$ between l and l'. but $l < l_s < l'$ and $l, l_s \in J' = J \cup \{l_s\}$, so there is no $j \in J'$ between l and l_s. Consequently, $(J', t') \in R_{\mathbf{J},\mathbf{T}}$, $(J', C) \in R_{\mathbf{J},\mathbf{C}}$, and $(C, t') \in R_{\mathbf{C},\mathbf{T}}$. Therefore, $\xi(h) \in \mathbf{E}$.

In case (3), as $(l, l'), (l', l'') \in \mathbf{T}(J)$, there is no $j \in J$ between l and l', or between l' and l''. As $J' = J \backslash \{l'\}$, it follows that there is no $j \in J'$ between l and l''. So $(J, t') \in R_{\mathbf{J},\mathbf{T}}$ as $l, l'' \in J$. As $|C| > 1$, $l' = l_s > l_1 = \max(\partial C)$. So $l' \notin \partial C$. Consequently, $\partial C \subseteq J' = J \backslash \{l'\}$ since $\partial C \subseteq J$. As there is no $j \in J$, and hence no $j \in J' \subset J$, between $l_1 = l$ and $l_s = l'$, $(J', C) \in R_{\mathbf{J},\mathbf{C}}$. Finally as $t = (l, l'')$ with $l_s = l' < l''$, it follows that $(C, t') \in R_{\mathbf{C},\mathbf{T}}$. So $\xi(h) \in \mathbf{E}$.

In case (4), as there is no $j \in J$ between l and $l' = \max(J)$, and as $J' = J \backslash \{l'\}$, it follows that $l = \max(J')$. $t \in \mathbf{T}(C)$ implies $l = \min(C)$. Finally $\partial C \subseteq J$ with $\max(\partial C) = l < l'$, so $\partial C \subseteq J'$. Therefore, by part (1) of Remark 8.3.2, $\xi(h) \in \mathbf{F}$.

Observe in all cases, $l(\xi(h)) = |C| \geqslant 2$, so $\xi(h) \in \mathbf{H}$. That is, ξ is well defined.

Proposition 9.1.2. *(1) ξ is a permutation on $\mathbf{H}$ of order 2 with $|h| = |\xi(h)| \pm 1$ for $h \in \mathbf{H}$.*

(2) If E is an abelian group and $\varphi : \mathbf{H} \to E$ is a function with $\varphi(h) = \varphi(\xi(h))$ for all $h \in \mathbf{H}$, then

$$\sum_{h \in \mathbf{H}} (-1)^{|h|} \varphi(h) = 0.$$

Proof. Part (2) is a direct consequence of part (1) as the contribution of h and $\xi(h)$ in the alternating sum cancel with each other. As for part (1), it follows from the definition of ξ that $|h| = |\xi(h)| \pm 1$. So $\xi(h) \neq h$. Therefore, it suffices to check that $\xi(\xi(h)) = h$ for all $h \in \mathbf{H}$.

Let $h = (J, C, l)$ be as in case (1) of Definition 9.1.1 with $\xi(h) = (J', C, t')$. Then by construction, $t' = (l, l_s)$ and $l_s = \max(J')$. So $\xi(h)$ satisfies the conditions in case (4). Consequently,

$$\xi(\xi(h)) = (J'\backslash\{l_s\}, C, l) = (J, C, l) = h.$$

If $h = (J, C, t)$ satisfies the conditions in case (4), we can check $\xi(\xi(h)) = h$ similarly.

Let $h = (J, C, t)$ be as in case (2) of Definition 9.1.1 with $\xi(h) = (J', C, t')$. Then by construction, $t' = (l, l_s)$ and $\max(C) = l_s < l' \leqslant \max(J')$. So $\xi(h)$ satisfies the conditions in case (3). Now $(l, l_s), (l_s, l') \in \mathbf{T}(J')$. So by definition

$$\xi(\xi(h)) = (J'\backslash\{l_s\}, C, (l, l')) = (J, C, t) = h.$$

If $h = (J, C, t)$ satisfies the conditions in case (3), we can check $\xi(\xi(h)) = h$ similarly. The proof is complete.

□

Corollary 9.1.3. *In Proposition 9.1.2, define $\varphi : \mathbf{H} \to \mathbb{Z}$ as follows. Let $d \geqslant 0$ and $\rho \in Irr(Z(G))$. For $e \in \mathbf{E}'$, let*

$$\varphi(e) = k^0_{d-d(e)}(P(e), V(e), \rho).$$

For $f \in \mathbf{F}'$, let

$$\varphi(f) = k^0_{d-d(f)}(P(f), U(f), \rho).$$

Then $\varphi(\xi(h)) = \varphi(h)$ for all $h \in \mathbf{H}$. In particular,

$$\sum_{e\in\mathbf{E}'}(-1)^{|e|}k^0_{d-d(e)}(P(e), V(e), \rho) + \sum_{f\in\mathbf{F}'}(-1)^{|f|}k^0_{d-d(f)}(P(f), U(f), \rho) = 0.$$

Proof. For $e \in \mathbf{E}$, let $L(e)$ be a complement to $V(e)$ in $P(e)$ containing $Z(G)$. For $f \in \mathbf{F}$, let $L(f)$ be a complement to $U(f)$ in $P(f)$ containing $Z(G)$. So by definition of φ, $\varphi(h) = k_{d-d(h)}(L(h), \rho)$ for all $h \in H$. To show $\varphi(\xi(h)) = \varphi(h)$, it suffices to show $L(\xi(h)) \cong L(h)$ with $d(\xi(h)) = d(h)$. Then the final remark in Corollary 9.1.3 follows from Proposition 9.1.2.2.

Let $h = (J, C, l) \in \mathbf{F}'$ be as in case (1) of Definition 9.1.1; then as in the proof of Proposition 9.1.2, $h' = \xi(h) = (J', C, t) \in \mathbf{E}'$ satisfies the conditions in case (4).

Recall the definition of $d(h)$ from Definition 8.3.5. Observe $d(h)$ depends only on C as $h = (J, C, l)$. Recall the definition of $d(h')$ from Definition 8.2.5. Similarly $d(h')$ depends only on C where $h' = (J', C, t')$. As the defining formulas for $d(h)$ and $d(h')$ are identical, $d(h) = d(h')$.

Represent C as in equation (8.1). Recall the definition of $P(h)$ from Definition 8.3.5. Recall from Remark 8.3.6.3 that a complement to $U(h)$ in $P^{n_s(h)}_{J_s(h)}$ is isomorphic to $P^{+\partial l_s}_{J(<\partial l_s)} \times G^{n-2l_s}$. So

$$L(h) = P^{+n_1(h)}_{J_1(h)} \times \cdots \times P^{+n_{s-1}(h)}_{J_{s-1}(h)} \times P^{+\partial l_s}_{J(<\partial l_s)} \times G^{n-2l_s}.$$

Similarly as $h' = (J', C, t')$ with $J' = J \cup \{l_s\}$ and $t' = (l, l_s)$, $P(h')$ is defined by case (1b) in part (1) of Definition 8.2.5, while $V(h') = 1$ by part (2b) of the definition. So $L(h') = P(h')$ and

$$L(h') = L^{n_0(h')}_{J'_0(h')} \times P^{+n_1(h')}_{J'_1(h')} \times \cdots \times P^{+n_s(h')}_{J'_s(h')}.$$

Observe for $1 \leqslant i \leqslant s-1$, $n_i(h) = \partial^2 l_i = n_i(h')$ and $J_i(h) = J_i = J'_i = J'_i(h')$ where J_i is defined in Lemma 8.3.3 while J'_i is defined in Lemma 8.2.3. Also $n_0(h') = n - 2l_s$ as $t' = (l, l_s)$ and $J'_0(h') = J'(> l_s) = \emptyset$; Finally $n_s(h') = l_s - l_{s-1} = \partial l_s$ and $J'_s(h') = J'(< \partial l_s) = J(< \partial l_s)$. Therefore,

$$L^{n_0(h')}_{J'_0(h')} \times P^{+n_s(h')}_{J'_s(h')} = P^{+\partial l_s}_{J(<\partial l_s)} \times G^{n-2l_s}.$$

Consequently, $L(h) \cong L(h')$ when $h \in \mathbf{H}$ satisfying the conditions in part (1) of Definition 9.1.1.

The case when $h \in \mathbf{H}$ satisfies the conditions in part (2) of Definition 9.1.1 can be similarly handled. This completes the proof. □

Theorem 9.1.4. *Let $d \geqslant 0$ and $\rho \in Irr(Z(G))$. Then*

$$\sum_{J \subseteq I} (-1)^{|J|} k^1_d(P_J, U_J, \rho) = \Gamma_1 + \Gamma_2$$

where

$$\Gamma_1 = \sum_{\substack{f\in\mathbf{F}\\ l(f)\geqslant 1}} (-1)^{|f|} k_{d-d(f)}(P(f), S^{nu}(f), \rho)$$

and

$$\Gamma_2 = \sum_{\substack{f\in\mathbf{F}\\ l(f)\geqslant 1}} (-1)^{|f|} k_{d-d(f)}(P(f), S^{z}(f), \rho).$$

Proof. If $J = \emptyset$, $P_J = G$ and $U_J = 1$, so $k_d^1(G, 1, \rho) = 0$. Hence the sum on the left hand side can be taken over all $\emptyset \neq J \subseteq I$.

For $\emptyset \neq J = \{j_1 < \cdots < j_s\} \subseteq I$, by Lemma 7.1.2.1, equation (7.2) is a chain of normal p-subgroups of P_J. So $\mathrm{Irr}^1(U_J)$ is the disjoint union of $\mathrm{Irr}^1(V(j_i, j_{i+1}))$, $i \geqslant 1$, and $\mathrm{Irr}^1(U_{j_s})$. Here $V(J_i, j_{i+1})$ is defined in Lemma 7.1.2.2. Certainly P_J acts on each of these subsets of $\mathrm{Irr}^1(U_J)$. So

$$k_d^1(P_J, U_J, \rho) = \sum_{i=1}^{s-1} k_d^1(P_J, V(j_i, j_{i+1}), \rho) + k_d^1(P_J, U_{j_s}, \rho).$$

But from Example 8.2.7,

$$k_d^1(P_J, V(j_i, j_{i+1}), \rho) = k_{d-d(e)}^1(P(e), V(e), \rho)$$

with $e = (J, \{j_i\}, (j_i, j_{i+1})) \in \mathbf{E}$. From Example 8.3.7,

$$k_d^1(P_J, U_{j_s}, \rho) = k_{d-d(f)}^1(P(f), U(f), \rho)$$

with $f = (J, \{j_s\}, j_s) \in \mathbf{F}$.

Recall $\mathbf{T}(J)$ consists of the pairs which are consecutive members of J. Therefore, the set of modules $V(j_i, j_{i+1})$ is in 1-1 correspondence with $\mathbf{T}(J)$, which is in 1-1 correspondence with the set of $f = (J, C, t) \in \mathbf{E}$ with J given and $|C| = 1$. So

$$k_d^1(P_J, U_J, \rho) = \sum_{\substack{e=(J,C,t)\in\mathbf{E}\\ l(e)=1}} k_{d-d(e)}^1(P(e), V(e), \rho) + k_{d-d(f)}^1(P(f), U(f), \rho)$$

where $f = (J, \{j_s\}, j_s)$. We have observed in Example 8.2.7 and 8.3.7 that the set of $e \in \mathbf{E}$ with $l(e) = 1$ is in 1-1 correspondence with $R_{\mathbf{J},\mathbf{T}}$, and the set of $f \in \mathbf{F}$ with $l(f) = 1$ is in 1-1 correspondence with the set of $\emptyset \neq J \subseteq I$. So summing over all possible J, we obtain

$$\sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_J, \rho) = \Lambda_1 + \Lambda_2$$

where

$$\Lambda_1 = \sum_{\substack{e \in \mathbf{E} \\ l(e)=1}} (-1)^{|e|} k_{d-d(e)}^1(P(e), V(e), \rho)$$

and

$$\Lambda_2 = \sum_{\substack{f \in \mathbf{F} \\ l(f)=1}} (-1)^{|f|} k_{d-d(f)}^1(P(f), U(f), \rho).$$

By Corollary 8.2.10,

$$\Lambda_1 = \sum_{\substack{e \in \mathbf{E} \\ l(e)\geqslant 2}} (-1)^{|e|} k_{d-d(e)}^0(P(e), V(e), \rho).$$

By Corollary 8.3.11,

$$\Lambda_2 = \sum_{\substack{f \in \mathbf{F} \\ l(f)\geqslant 2}} (-1)^{|f|} k_{d-d(f)}^0(P(f), U(f), \rho) + \Gamma_1 + \Gamma_2.$$

The theorem then follows from Corollary 9.1.3.

□

9.2 A Dimension Argument

In order to complete the verification of **DOC** for G^n, we will proceed by induction on $n \in \mathbb{N}$. In this section, we prove some technical results which relate the representations of certain subgroups of G^n to the representations of certain subgroups of a unitary group of lower dimension.

Recall from Chapter 8 that the definitions of **J**, **C**, **I**, and **F** depend on n (and hence on m as $n = 2m$ or $2m+1$). From now on we denote them by $\mathbf{J}^n$, $\mathbf{C}^n$, $\mathbf{I}^n$ $\mathbf{F}^n$, respectively, to distinguish corresponding sets defined for different n. Recall that $\mathbf{F}^n = R_{\mathbf{J}^n,\mathbf{C}^n,\mathbf{I}^n}$ is a subset of $\mathbf{J}^n \times \mathbf{C}^n \times \mathbf{I}^n$. Observe that for $n < n'$, $\mathbf{C}^n \subset \mathbf{C}^{n'}$ and $\mathbf{F}^n \subset \mathbf{F}^{n'}$. On the other hand, observe from Definition 8.3.5 that $P(f)$ for $f \in \mathbf{F}^n$ also depends on n. So in order to avoid confusion,we write $P^n(f)$, $U^n(f)$, $V^n(f)$, and $Z^n(f)$ for $P(f)$, $U(f)$, $V(f)$, and $Z(f)$, respectively, if necessary. So in particular for $f \in \mathbf{F}^n \subset \mathbf{F}^{n'}$ with $n < n'$, $P^n(f)$ and $P^{n'}(f)$ are different groups. Recall from Remark 8.3.6.1 that $d(f)$ depends only on C for $f = (J, C, l)$.

For $r, s \in [m]$, let

$$\mathbf{C}^n(s,r) = \{C \in \mathbf{C}^n \mid |C| = s \text{ and } \min(\partial C) \geqslant r\}$$

and

$$\mathbf{C}^n(\leqslant s) = \{C \in \mathbf{C}^n \mid |C| \leqslant s\}.$$

Lemma 9.2.1. *Define*

$$\theta = \theta(s,r) : \mathbf{C}^n(s,r) \to \mathbf{C}^{n-2sr}(\leqslant s)$$

$$C \mapsto \{l_i - ir; 1 \leqslant i \leqslant s\} \backslash \{0\}$$

where C is represented as in equation (8.1). Then θ is a 1-1 correspondence. Specifically,

$$\min(\partial C) > r \textit{ if and only if } |\theta(C)| = s;$$
$$\partial C = \{r\} \textit{ if and only if } |\theta(C)| = 0;$$
$$\{r\} \subsetneq \partial C \textit{ if and only if } 0 < |\theta(C)| < s.$$

Proof. We must show first that θ is well defined. Let C be as in the hypothesis. For $1 \leqslant i \leqslant s$, set $l'_i = l_i - ir$. Recall that as C is convex,

$$l_1 = \partial l_1 \geqslant \partial l_2 \ldots \geqslant \partial l_s = l_s - l_{s-1}.$$

Observe $l'_{i+1} > l'_i$ if and only if $\partial l_{i+1} > r$. We need to discuss three cases.

Assume $\min(\partial C) > r$, i.e., $\partial l_s = l_s - l_{s-1} > r$. Then $\partial l_i > r$ for all i, so as we just observed, it follows that $l'_{i+1} > l'_i$ for $1 \leqslant i \leqslant s-1$ and $l'_1 = l_1 - r > 0$. Therefore, $\theta(C)$ is a chain. Clearly $\partial l'_i = \partial l_i - r$ for all i. So $\theta(C)$ is convex, and $2l'_s = 2(l_s - sr) \leqslant n - 2sr$, so $\theta(C) \in \mathbf{C}^{n-2sr}$. As $|\theta(C)| = s$, $\theta(C) \in \mathbf{C}^{n-2sr}(\leqslant s)$. So in this case $\theta(C) \in \mathbf{C}^{n-2sr}(\leqslant s)$ with $|\theta(C)| = s$.

Assume $\min(\partial C) = r$. Let $j = \min\{i \mid \partial l_i = r\}$. Then $1 \leqslant j \leqslant s$. If $j = 1$, then $\partial C = \{r\}$ and $l_i = ir$ for all i. By definition $\theta(C) = \emptyset \in \mathbf{C}^{n-2sr}(\leqslant s)$. So in this case $\theta(C) \in \mathbf{C}^{n-2sr}(\leqslant s)$ with $|\theta(C)| = 0$.

If $j > 1$, then $\{r\} \subsetneq \partial C$. By our choice of j, and an earlier observation $l'_i > l'_{i-1}$ for $i < j$, so

$$0 < l'_1 < \cdots < l'_{j-1} = l'_j = \cdots = l'_s.$$

By definition $\theta(C) = \{l'_1, \cdots, l'_{j-1}\}$. Again as $\partial l'_i = \partial l_i - r$, $\partial l'_1 \geqslant \cdots \geqslant \partial l'_{j-1}$. So $\theta(C)$ is convex. Finally as $2l'_{j-1} = 2l'_s \leqslant n - 2sr$ and $|\theta(C)| = j - 1 < s$, we conclude $\theta(C) \in \mathbf{C}^{n-2sr}(\leqslant s)$. So in this case $\theta(C) \in \mathbf{C}^{n-2sr}(\leqslant s)$ with $0 < |\theta(C)| < s$.

In summary, θ is well defined. Conversely, we define $\zeta : \mathbf{C}^{n-2sr}(\leqslant s) \to \mathbf{C}^n(s, r)$ as follows. Let $D \in \mathbf{C}^{n-2sr}(\leqslant s)$.

If $|D| = 0$, then $D = \emptyset$. Define $\zeta(D) = \{ir; 1 \leqslant i \leqslant s\}$. By definition $\zeta(D)$ is a chain. As the difference of the successive members of $\zeta(D)$ is a constant, ζ is convex. As $2sr \leqslant n$, $\zeta(D) \in \mathbf{C}^n$. As $\min(\zeta(D)) = r$ and $|\zeta(D)| = s$, $\zeta(D) \in \mathbf{C}^n(s, r)$.

If $0 < |D| < s$, then $D = \{d_1 < \cdots < d_j\}$ for some $j < s$. Define

$$\zeta(D) = \{d_1 + r < d_2 + 2r < \cdots < d_j + jr < d_j + (j+1)r < \cdots < d_j + sr\}.$$

By definition $\zeta(D)$ is a convex sequence. Observe $|\zeta(D)| = s$. As $D \in \mathbf{C}^{n-2sr}(\leqslant s)$, $0 < d_1 \leqslant d_s \leqslant m - sr$. So

$$r < \min(\zeta(D)) = d_1 + r \leqslant d_j + sr \leqslant m.$$

It follows that $\zeta \in \mathbf{C}^n(s,r)$.

Finally if $|D| = s$, then $D = \{d_1 < \cdots < d_s\}$, and we define $\zeta(D) = \{d_i + ir; 1 \leqslant i \leqslant s\}$. As in the second case, it is easy to see that $\zeta(D) \in \mathbf{C}^n(s,r)$.

Therefore, ζ is a well defined map from $\mathbf{C}^{n-2sr}(\leqslant s)$ to $\mathbf{C}^n(s,r)$. It remains to show ζ is the inverse of θ.

Observe $C \in \mathbf{C}^n(s,r)$ with $\min(\partial l_s)$ if and only if $|\theta(C)| = s$, so checking with the definitions directly shows that θ and ζ are inverses of each other in this case.

Observe $C \in \mathbf{C}^n(s,r)$ with $\partial C = \{r\}$ if and only if $\theta(C) = \emptyset$. Also observe that $\{r\} \subsetneq \partial C$ if and only if $0 < |\theta(C)| < s$. Again checking directly with these two cases shows that θ and ζ are inverses to each other. So θ is a 1-1 correspondence.

□

Remark 9.2.2. Observe if $\mathbf{C}^n(s,r)$ is non-empty, say $C \in \mathbf{C}^n(s,r)$ is represented as in (8.1), then as $\partial l_i \geqslant r$,

$$n \geqslant 2m \geqslant 2l_s = 2\sum_{i=1}^{s} \partial l_i \geqslant 2sr.$$

Conversely, if $n \geqslant 2sr$, then $\{ir \mid 1 \leqslant i \leqslant s\} \in \mathbf{C}^n(s,r)$. So $\mathbf{C}^n(s,r)$ is non-empty if and only if $n \geqslant 2sr$.

Let

$$\mathbf{F}^n(s,r) = \{f = (J,C,l) \in \mathbf{F}^n \mid \ r \in J \text{ and } C \in \mathbf{C}^n(s,r)\ \}$$

and

$$\mathbf{F}^n(\leqslant s) = \{f = (J,C,l) \in \mathbf{F}^n \mid C \in \mathbf{C}^{n-2sr}(\leqslant s)\}.$$

Lemma 9.2.3. *Define*

$$\gamma = \gamma(s,r) : \mathbf{F}^n(s,r) \to \Delta([r-1]) \times \mathbf{F}^{n-2sr}(\leqslant s),$$

$$f = (J,C,l) \mapsto (J'', f')$$

where $J'' = J(< r)$, *and* $f' = (J', \theta(C), l - r)$ *with* $J' = \{j - r \mid r < j \in J\}$. *Then* γ *is a 1-1 correspondence. Moreover,*

$$d(f) = d(f') + 2(s-1)\binom{r}{2} + 2r(l_{s-1} - (s-1)r).$$

Here C *is represented as in equation (8.1).*

Proof. Fix $f = (J, C, l) \in \mathbf{F}^n(s, r)$ and set $\gamma(f) = (J'', f')$ with $f' = (J', \theta(C), l - r)$. By definition $J'' \subseteq [r-1]$. We need to verify $f' \in \mathbf{F}^{n-2sr}(\leqslant s)$ in order to show γ is well defined. But $\theta(C) \in \mathbf{C}^{n-2sr}(\leqslant s)$, so it remains to show f' satisfies the conditions in Definition 8.3.1. As $f \in \mathbf{F}^n$, $\max(J) = l = \min(C)$. By definition of $\mathbf{C}^n(s, r)$, $r \leqslant l$.

If $l = r$, then as $l = \max(J)$, $J' = \emptyset$. As $C \in \mathbf{C}^n(s, r)$ with $\min(C) = l = r$, it follows from the proof of Lemma 9.2.1 that $C = \{ir; 1 \leqslant i \leqslant s\}$ and $\theta(C) = \emptyset$. So $f' = (\emptyset, \emptyset, 0) \in \mathbf{F}^{n-2sr}(\leqslant s)$.

Assume $l > r$. As $l = \max(J)$, by definition of J', $l - r = \max(J')$. Also $l - r = \min(\theta(C))$. So it remains to show $\partial(\theta(C)) \subseteq J'$. But $\partial(\theta(C)) = \{\partial l_i - r; 1 \leqslant i \leqslant s\} \backslash \{0\}$. As $f \in \mathbf{F}^n$, $\partial C \subseteq J$. That is, each $\partial l_i \in J$. Therefore, each non-zero $\partial l_i - r \in J'$, i.e., $\partial(\theta(C)) \subseteq J'$. So γ is well defined.

Assume $\gamma(f) = \gamma(\hat{f})$ for some $\hat{f} = (\hat{J}, \hat{C}, \hat{l}) \in \mathbf{F}^n(s, r)$. So $J' = \hat{J}'$, $J'' = \hat{J}''$, $\theta(C) = \theta(\hat{C})$, and $l - r = \hat{l} - r$. So $l = \hat{l}$. Also

$$J = J'' \cup \{r\} \cup \{j + r \mid j \in J'\} \tag{9.1}$$

and $\hat{J}$ can be similarly expressed. So $J = \hat{J}$. Finally as θ is a bijection, $C = \hat{C}$. So γ is injective.

Given $f' = (J', C', l') \in \mathbf{F}^{n-2sr}(\leqslant s)$ and $J'' \subseteq [r-1]$, we set $f = (J, C, l)$ where J is defined by (9.1), $C = \zeta(C')$, ζ being the inverse of θ, and $l = l' + r$. We show $f \in \mathbf{F}^n(s, r)$. As $\max(J') = l' \leqslant m - sr$, by (9.1) J is a subset of $[m]$ and $\max(J) = l$. As $C' \in \mathbf{C}^{n-2sr}(\leqslant s)$, by Lemma 9.2.1, $C = \zeta(C') \in \mathbf{F}^n(s, r)$ with $\min(C) = l' + r = l$. As $\partial C' \in J'$, $\partial C = \{x + r \mid x \in \partial C'\} \subseteq J$. So $f \in \mathbf{F}$. By definition of J, $r \in J$. So $f \in \mathbf{F}^n(s,r)$. By construction, $\gamma(f) = (J'', f')$. So γ is surjective. Hence γ is a bijection.

To calculate $d(f) - d(f')$, notice by definition

$$d(f) = 2\sum_{i=1}^{s-1}(\binom{\partial l_i}{2} - \binom{\partial^2 l_i}{2})$$

and

$$d(f') = 2\sum_{i=1}^{s-1}(\binom{\partial l'_i}{2} - \binom{\partial^2 l'_i}{2}).$$

But $l_i = l'_i + ir$ for all i (l'_i, $\partial l'_i$ are allowed to be 0 for this purpose). So $\partial l_i = \partial l'_i + r$ and $\partial^2 l_i = \partial^2 l'_i$. Finally by Lemma 2.3.1,

$$\binom{a+b}{2} = \binom{a}{2} + \binom{b}{2} + ab.$$

It follows that

$$d(f) - d(f') = 2(s-1)\binom{r}{2} + 2r\sum_{i=1}^{s-1}\partial l'_i = 2(s-1)\binom{r}{2} + 2r(l_{s-1} - (s-1)r).$$

□

Remark 9.2.4. Assume $f = (J, C, l) \mapsto (J'', f')$ under γ as in Lemma 9.2.3. Then as $J'' = J(< r)$, the following statements are equivalent:

(i) There is no $j \in J$ between $r/2$ and r;

(ii) There is no $j \in J''$ between $r/2$ and r.

Moreover, replacing n by $n-r$, we see that γ restricts to a 1-1 correspondence from $\mathbf{F}^{n-r}(s,r)$ to $\Delta([r-1]) \times \mathbf{F}^{n-(2s+1)r}(\leqslant s)$. That is, $f \in \mathbf{F}^{n-r}(s,r) \subset \mathbf{F}^n(s,r)$ if and only if $f' \in \mathbf{F}^{n-(2s+1)r}(\leqslant s)$.

Proposition 9.2.5. *Fix $n \in \mathbb{N}$, and $1 \leqslant r \leqslant m$. Let $d \geqslant 0$ and $\rho \in Irr(Z(G))$. Define*

$$\iota_{d,\rho} = \iota_{n,d,\rho,s,r} : \mathbf{F}^n(s,r) \to \mathbb{Z}$$

$$f \mapsto \iota_{d,\rho}(f) = k_{d-d(f)}(P^n(f), S^{nu}_r(f), \rho)$$

and

$$h_{d,\rho} = h_{n,d,\rho,s,r} : \mathbf{F}^{n-(2s+1)r}(\leqslant s) \to \mathbb{Z}$$

$$f' \mapsto h_{d,\rho}(f') = \begin{cases} k^0_{d-d(f')}(P^{n-(2s+1)r}(f'), U^{n-(2s+1)r}(f'), \rho), & \text{if } 0 \leqslant l(f') < s, \\ k^0_{d-d(f')}(P^{n-(2s+1)r}(f'), Z^{n-(2s+1)r}(f'), \rho), & \text{if } l(f') = s. \end{cases}$$

Then

$$A(\iota_{d,\rho}, \mathbf{F}^n(s,r)) = - \sum_{\substack{\rho_1,\rho_2 \\ \rho_1\rho_2=\rho}} \sum_{\substack{d_1,d_2 \\ d_1+d_2=d'}} \Big(\sum_{J\subseteq[r/2]} (-1)^{|J|} k_{d_1}(P^r_J, \rho_1)\Big) A(h_{d_2,\rho_2}, \mathbf{F}^{n-(2s+1)r}(\leqslant s)),$$

and

$$d' = d - 2s\tbinom{r}{2} - r(n-(2s+1)r).$$

Here we regard $\mathbf{F}^n$ as a graded set for all $n \in \mathbb{N}$ where the rank of f is $|f| = |J|$ for $f = (J, C, l)$. Recall from section 5.3 that $A(\iota_{d,\rho}, \mathbf{F}^n(s,r))$ is the alternating sum of $\iota_{d,\rho}$ over $\mathbf{F}^n(s,r)$.

Proof. Let $f = (J, C, l) \in \mathbf{F}^n(s,r)$ with $C \in \mathbf{C}^n(s,r)$ represented as in (8.1). Let γ be as in Lemma 9.2.3, and $\gamma(f) = (J'', f')$ with $f' = (J', C', l-r)$. By definition, $J'' = J(< r)$. Let M^k be the natural module for G^{+k}.

The proof consists of several steps.

(I) Set $P^n(f) = Q(f) \times H(f)$, where

$$Q(f) = \prod_{i=1}^{s-1} P^{+n_i(f)}_{J_i(f)} \quad \text{and} \quad H(f) = P^{n_s(f)}_{J_s(f)}.$$

Let $L(f)$ be a complement in $H(f)$ to $U^n(f)$. Recall from Definition 8.3.5 and Remark 8.3.6.3 that $n_s(f) = n - 2l_{s-1}$, $J_s(f) = J(\leqslant \partial l_s)$, $V^n(f) \cong M_{\partial l_s, n-2l_s}(\mathbb{F}_{q^2})$, and $L(f) = P^{+\partial l_s}_{J(<\partial l_s)} \times G^{n-2l_s}$.

(II) By Definition 8.3.9 (which depends on Definition 6.3.10),

$$S_r^{nu}(f) = S_r^{nu}(V^n(f), J(< \partial l_s)) = \bigcup_{r'=r}^{\min(\partial l_s, n-2l_s)} S_r^{nu}(V^n(f), J(< \partial l_s), r'),$$

and $S_r^{nu}(V^n(f), J(< \partial l_s), r')$ consists of a subset of $\mathrm{Irr}(V^n(f))/H(f)$ which are labeled bijectively by the G^{n-2l_s}-orbits on the set of non-singular normal chains of type $J(< \partial l_s) \cup \{r'\} \backslash \{n-2l_s\}$ in $\mathcal{P}(M^{n-2l_s})$ of non-singular rank r (In particular these chains contain subchains of totally isotropic subspaces of type $J(< r)$). If $r > n - 2l_s$, then there is no chain in $\mathcal{P}(M^{n-2l_s})$ containing an r-dimensional subspace, hence $S_r^{nu}(f) = \emptyset$, and the contribution of f in $A(\iota_{d,\rho}, \mathbf{F}^n(s,r))$ is 0. From now on we assume $r \leqslant n - 2l_s$, or equivalently $2l_s \leqslant n - r$. So $C \in \mathbf{C}^{n-r}(s,r)$, or equivalently $f \in \mathbf{F}^{n-r}(s,r)$. By Remark 9.2.4, $f \in \mathbf{F}^{n-r}(s,r)$ if and only if $f' \in \mathbf{F}^{n-(2s+1)r}(\leqslant s)$. So $P^{n-(2s+1)r}(f')$ is well defined. For the rest of the proof, set $P(f) = P^n(f)$ and $P(f') = P^{n-(2s+1)r}(f')$.

(III) In (II), notice that there is a singular normal chain of type $J(< r)$ in $\mathcal{P}(M^{n-2l_s})$ only if $J(< r) \subseteq [r/2]$. Otherwise $S_r^{nu}(f) = \emptyset$, and the contribution of f is 0. So from now on assume $J'' = J(< r) \subseteq [r/2]$.

(IV) Assume $r = \partial l_s$. Then by (II), $S_r^{nu}(f) = S_r^{nu}(f,r)$ and it consists of a unique member. In this case $V(f) \cong M_{r,n-2l_s}(\mathbb{F}_{q^2})$ by (I). Let $\tau = \tau_c \in S_r^{nu}(f,r)$. Then by Definition 6.3.10, either $r = n - 2l_s$ and c is a singular normal chain of type $J(< r)$ in $\mathcal{P}(V^r)$, or $r < n - 2l_s$ and c is a non-singular normal chain of type $J(\leqslant r)$ such that all but the final member of c are totally isotropic, while the final member is non-degenerate of dimension r. In either case, by Remark 6.3.12.3 (where c_2 is forced to be $\emptyset$),

$$N_{L(f)}(\tau) = P^r_{J(<r)} \times G^{n-2l_s-r}.$$

(V) Assume $r < \partial l_s$. By Lemma 6.3.14, $S_r^{nu}(f,r')$ is in 1-1 correspondence with $S^{nu}(V', J', r'-r)$, where $V' \cong M_{\partial l_s - r, n-2l_s-r}(\mathbb{F}_{q^2})$ is a tensor module for $G^{+(\partial l_s - r)} \times G^{n-2l_s-r}$ and J' is as in the above hypothesis. On the other hand, as $r < \partial l_s = \min(\partial C)$, $l(f') = |C'| = s$ by Lemma 9.2.1. Recall from Lemma 9.2.1, $C' =$

$\{l'_1, \ldots, l'_s\}$ with $l'_i = l_i - ir$. So by the definition of $P(f') = P^{n-(2s+1)r}(f')$ (as we assumed in (II)), $V(f')$ is the restriction to

$$L(f') = P^{+\partial l'_s}_{J'(<\partial l'_s)} \times G^{n-(2s+1)r-2l'_s}$$

of a tensor module for

$$G^{+\partial l'_s} \times G^{n-(2s+1)r-2l'_s}.$$

But $\partial l'_s = \partial l_s - r$ and $l'_s = l_s - sr$. So

$$S^u(V', J', r' - r) = S^u(f', r' - r). \tag{9.2}$$

Recall from the paragraph preceeding Proposition 6.3.9 that $S^u(f', 0)$ consists of the trivial character of $V(f')$. So taking the union of equatility (9.2) for all r' with $r \leqslant r' \leqslant \min(\partial l_s, n - 2l_s)$, we deduce that $S^{nu}_r(f)$ is in 1-1 correspondence with $\{1\} \cup S^u(f')$. Also by Lemma 6.3.14, if $\tau \in S^{nu}_r(f, r') \mapsto \tau' \in S^u(f', r' - r)$, then

$$N_{L(f)}(\tau) = P^r_{J''} \times N_{L(f')}(\tau').$$

(VI) Define

$$h'_{d,\rho} : \mathbf{F}^{n-(2s+1)r}(\leqslant s) \to \mathbb{Z}$$

$$f' \mapsto h'_{d,\rho}(f') = \begin{cases} h_{d,\rho}(f') & \text{if } 0 \leqslant l(f') < s, \\ k_{d-d(f')}(P(f'), \{1\} \cup S^u(f'), \rho), & \text{if } l(f') = s. \end{cases}$$

We show

$$A(h_{d,\rho}, \mathbf{F}^{n-(2s+1)r}(\leqslant s)) = A(h'_{d,\rho}, \mathbf{F}^{n-(2s+1)r}(\leqslant s)).$$

To see this, it suffices to prove

$$\sum_{\substack{f' \in \mathbf{F}^{n-(2s+1)r} \\ l(f')=s}} (-1)^{|f'|} h_{d,\rho}(f') = \sum_{\substack{f' \in \mathbf{F}^{n-(2s+1)r} \\ l(f')=s}} (-1)^{|f'|} h'_{d,\rho}(f'). \tag{9.3}$$

However, if $l(f') = s$, then by definition of h,

$$h_{d,\rho}(f') = k^0_{d-d(f')}(P(f'), V(f'), \rho) + k^1_{d-d(f')}(P(f'), V(f'), \rho).$$

Recall from section 2.2 that $k^0_d(P(f'), V(f'), \rho) = k_d(P(f'), 1, \rho)$. So for each $C' \in \mathbf{C}^{n-(2s+1)r}$ with $|C'| = s$, by applying Proposition 8.3.10.2, we obtain

$$\sum_{f'=(J',C',l')\in\mathbf{F}^{n-(2s+1)r}} (-1)^{|f'|} h_{d,\rho}(f') = \sum_{f'=(J',C',l')\in\mathbf{F}^{n-(2s+1)r}} (-1)^{|f'|} h'_{d,\rho}(f').$$

Equation (9.3) then follows by summing over all $C' \in \mathbf{C}^{n-(2s+1)r}$ with $|C'| = s$.

(VII) We claim that

$$\eta_{d,\rho}(f) = \sum_{\substack{\rho_1,\rho_2 \\ \rho_1\rho_2=\rho}} \sum_{\substack{d_1,d_2 \\ d_1+d_2=d'}} k_{d_1}(P^r_{J''}, \rho_1) h'_{d_2,\rho_2}(f') \tag{9.4}$$

where d' is defined in the statement of the proposition. Then by (II) and (III), when we sum over all $f \in \mathbf{F}^n(s,r)$, only those $f \in \mathbf{F}^{n-r}(s,r) \subset \mathbf{F}^n(s,r)$ with $J(< r) \subset [r/2]$ contribute to $A(\eta_{d,\rho}, \mathbf{F}^n(s,r))$, and consequently (J'', f') runs over all subsets in $\Delta([r/2]) \times \mathbf{F}^{n-(2s+1)r}(\leqslant s)$. Also notice that $|f| = |J''| + |f'| + 1$. So the proposition follows by summing over all f and by (VI). Therefore, it remains to establish the claim.

(VIII) Assume $\partial C = \{r\}$. So by (I), $P(f) = H(f)$ and $L(f) = P^{+r}_{J(<r)} \times G^{n-(2s+1)r}$. By (IV), $S^{nu}_r(f)$ consists of one member, say τ. Then $N_{L(f)}(\tau) = P^r_{J(<r)} \times G^{n-(2s+1)r}$.

As $H(f)$ splits over $V(f)$ with $V(f)$ abelian, by Lemma 2.2.3 and 2.2.4,

$$k_{d-d(f)}(P(f), \tau, \rho) = k_{d'}(Q(f) \times N_{L(f)}(\tau), \rho)$$

such that $d - d(f) - d'$ is the exponent of q in the p-part of $|L(f)|/|N_{L(f)}(\tau)|$. But by Lemma 9.2.1, $C = \{ir; 1 \leqslant i \leqslant s\}$. It is easy to see that $Q(f) = 1$. So by Lemma 2.2.5,

$$\eta_{d,\rho}(f) = \sum_{\substack{\rho_1,\rho_2 \\ \rho_1\rho_2=\rho}} \sum_{\substack{d_1,d_2 \\ d_1+d_2=d'}} k_{d_1}(P^r_{J''}, \rho_1) k_{d_2}(G^{n-(2s+1)r}, \rho_2)$$

But $C = \{ir; 1 \leqslant i \leqslant s\}$, $C' = \emptyset$, so $f' = (\emptyset, \emptyset, 0)$ and $P(f') = G^{n-(2s+1)r}$ with

$d(f') = 0$. Also $U(f') = 1$ in this case. So

$$h'_{d_2,\rho_2}(f') = k_{d_2}(G^{n-(2s+1)r}, \rho_2).$$

Finally, $d(f) = 2(s-1)\binom{r}{2}$ by Definition 8.3.5. So

$$d' = d - 2s\binom{r}{2} - 2\binom{r}{2} - \binom{n-2sr}{2} + \binom{r}{2} + \binom{n-(2s+1)r}{2} = d - 2s\binom{r}{2} - r(n-(2s+1)r).$$

Therefore, the claim holds for this case.

(IX) Assume $\{r\} \subsetneq \partial C$. Again by (II), $S^{nu}_{r}(f)$ consists of a unique member τ, and

$$N_{L(f)}(\tau) = P^{r}_{J(<r)} \times G^{n-2l_s-r}.$$

We claim

$$Q(f) \times G^{n-2l_s-r} \cong Q(f') \times L(f'). \tag{9.5}$$

So

$$Q(f) \times N_{L(f)}(\tau) \cong P^{r}_{J(<r)} \times (Q(f') \times L(f')).$$

Again as $H(f)$ splits over $V(f)$ with $V(f)$ abelian, so as $Q(f') \times L(f')$ is a complement in $P(f')$ to $U(f')$, by Lemma 2.2.3, 2.2.4 and 2.2.5,

$$\begin{aligned} k_{d-d(f)}(P(f), \tau, \rho) &= k_{d'-d(f')}(Q(f) \times N_{L(f)}(\tau), \rho) \\ &= k_{d'-d(f')}(P^{r}_{J(<r)} \times (Q(f') \times L(f')), \rho) \\ &= \sum_{\substack{\rho_1,\rho_2 \\ \rho_1\rho_2=\rho}} \sum_{\substack{d_1,d_2 \\ d_1+d_2=d'}} k_{d_1}(P^{r}_{J''}, \rho_1) k_{d_2-d(f')}(Q(f') \times L(f'), \rho_2) \end{aligned}$$

such that $d - d(f) - (d' - d(f'))$ is the exponent of q in the p-part of $|L(f)|/|N_{L(f)}(\tau)|$. But $Q(f') \times L(f')$ is a complement to $U(f')$ in $P(f')$, Therefore,

$$\begin{aligned} k_{d-d(f)}(P(f), \tau, \rho) &= \sum_{\substack{\rho_1,\rho_2 \\ \rho_1\rho_2=\rho}} \sum_{\substack{d_1,d_2 \\ d_1+d_2=d'}} k_{d_1}(P^{r}_{J''}, \rho_1) k^{0}_{d_2-d(f')}(P(f'), U(f'), \rho_2) \\ &= \sum_{\substack{\rho_1,\rho_2 \\ \rho_1\rho_2=\rho}} \sum_{\substack{d_1,d_2 \\ d_1+d_2=d'}} k_{d_1}(P^{r}_{J''}, \rho_1) h'_{d_2,\rho_2}(f'). \end{aligned}$$

As $\partial l_s = r$, the exponent of q in the p-part of $|L(f)|/|N_{L(f)}(\tau)|$ is

$$x = (2\binom{\partial l_s}{2} + \binom{n-2l_s}{2}) - (\binom{r}{2} + \binom{n-2l_s-r}{2}) = 2\binom{r}{2} + r(n-2l_s-r).$$

By Lemma 9.2.3,

$$d(f) - d(f') = 2(s-1)\binom{r}{2} + 2r(l_{s-1} - (s-1)r).$$

So direct calculation shows that

$$d' = d - x - (d(f) - d(f')) = d - 2s\binom{r}{2} - r(n-(2s+1)r)$$

as required in the proposition. Therefore, the claim in (VII) holds if we can establish (9.5). And we do so now.

Recall $\{r\} \subsetneq \partial C$. So by Lemma 9.2.1, there is $1 \leqslant i < s$, such that $\partial l_i > r$, and $\partial l_j = r$ or equivalently $l_j = l_i + (j-i)r$ for $j > i$. Consequently, $C' = \{l'_1, l'_2, \ldots, l'_i\}$. where $l'_j = l_j - jr$. Recall $\partial l'_j = \partial l_j - r$ and $\partial^2 l'_j = \partial^2 l_j$ for $1 \leqslant j \leqslant i-1$. Moreover, $\partial^2 l_j = 0$ for $i < j < s-1$. Also recall $J' = \{j - r \mid r < j \in J\}$.

Now by hypothesis, as $n_j(f) = \partial^2 l_j = 0$ for $i < j \leqslant s-1$, we have

$$Q(f) = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_i(f)}_{J_i(f)}.$$

On the other hand,

$$Q(f') = P^{+n_i(f')}_{J_1(f')} \times \cdots \times P^{+n_{i-1}(f')}_{J_{i-1}(f')}.$$

But for $1 \leqslant j \leqslant i-1$, $n_j(f) = \partial^2 l_j = \partial^2 l'_j = n_j(f')$, and

$$\begin{aligned} J_j(f) &= \{x - \partial l_{j+1} \mid x \in J \text{ and } \partial l_{j+1} < x < \partial l_j\} \\ &= \{x' - \partial l'_{j+1} \mid x' \in J' \text{ and } \partial l_{j+1} < x' < \partial l'_j\} \\ &= J'_i(f'). \end{aligned}$$

That is, $Q(f) \cong Q(f') \times P^{+n_i(f)}_{J_i(f)}$. Therefore, to establish (9.5), it suffices to show

$$P^{+n_i(f)}_{J_i(f)} \times G^{n-2l_s-r} \cong L(f'). \tag{9.6}$$

By definition,

$$L(f') = P^{+\partial l'_i}_{J'(<\partial l'_i)} \times G^{(n-(2s+1)r)-2l'_i}.$$

But recall $\partial l_j = r$ for $j > i$. So

$$n_i(f) = \partial^2 l_i = \partial l_i - \partial l_{i+1} = \partial l_i - r = \partial l'_i$$

and

$$\begin{aligned} J_i(f) &= \{x - \partial l_{i+1} \mid x \in J \text{ and } \partial l_{i+1} < x < \partial l_i\} \\ &= \{x' \in J' \mid x' < \partial l'_j\} \\ &= J'(< \partial l'_i). \end{aligned}$$

Moreover, as $l_s = l_i + (s-i)r$ and $l'_i = l_i - ir$, $(n - 2l_s - r) = (n - (2s+1)r - 2l'_i)$. Therefore, (9.6) indeed holds, thus the claim in (VII) holds when $\{r\} \subsetneq \partial C$.

(X) Assume $\min(\partial C) > r$. so $l(f') = s$ by Lemma 9.2.1. By part (V), $S^{nu}_r(f)$ is in 1-1 correspondence with $\{1\} \cup S^u(f')$. So to prove the claim in (VII), it suffices to show that for each $\tau \mapsto \tau'$,

$$k_{d-d(f)}(P(f), \tau, \rho) = \sum_{\substack{\rho_1, \rho_2 \\ \rho_1\rho_2 = \rho}} \sum_{\substack{d_1, d_2 \\ d_1 + d_2 = d'}} k_{d_1}(P^{\tau}_{J''}, \rho_1) k_{d_2 - d(f')}(P(f'), \tau', \rho_2) \tag{9.7}$$

where d' is as required. But argued as before,

$$k_d(P(f), \tau, \rho) = k_{d-a(f)}(Q(f) \times N_{L(f)}(\tau), \rho)$$

where $a(f)$ is the exponent of q in $|L(f)|/N_{L(f)}(\tau)|$. Similar statement holds for f'. But by (V),

$$N_{L(f)}(\tau) = P^{\tau}_{J''} \times N_{L(f')}(\tau'). \tag{9.8}$$

Also by checking with the definitions as we did in (IX), we deduce $Q(f) \cong Q(f')$. So

$$Q(f) \times N_{L(f)}(\tau) \cong P^{\tau}_{J''} \times (Q(f') \times N_{L(f')}(\tau')).$$

Consequently,

$$\begin{aligned} k_{d-d(f)}(P(f),\tau,\rho) &= k_{d-d(f)-a(f)}(Q(f)\times N_{L(f)}(\tau),\rho) \\ &= k_{d-d(f)-a(f)}(P^{\tau}_{J''}\times (Q(f')\times N_{L(f')}(\tau'),\rho) \\ &= k_{d-d(f)-a(f)+a(f')}(P^{\tau}_{J''}\times P(f'),\tau',\rho) \\ &= \sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}} \sum_{\substack{d_1,d_2\\ d_1+d_2=d'}} k_{d_1}(P^{\tau}_{J''},\rho_1)k_{d_2-d(f')}(P(f'),\tau',\rho_2) \end{aligned}$$

where $d' = d - a(f) - d(f) + a(f') + d(f')$. Therefore, (9.7) will hold if we can show d' is as required by the proposition. By (9.8), $a(f) - a(f')$ is the exponent of q in the p-part of $|L(f)|/(|L(f')||P^{\tau}_{J''}|)$. So as $l_s = l'_s + sr$ and $\partial l_s = \partial l'_s + r$,

$$\begin{aligned} a(f)-a(f') &= 2\binom{\partial l_s}{2} + \binom{n-2l_s}{2} - 2\binom{\partial l'_s}{2} - \binom{n-(2s+1)r-2l'_s}{2} - \binom{r}{2} \\ &= 2\binom{r}{2} + 2r(\partial l_s - r) + r(n-2l_s-r) \\ &= 2\binom{r}{2} + r(n-2l_{s-1}-3r). \end{aligned}$$

Now as $d(f) - d(f')$ is given in Lemma 9.2.3, it is easy to check d' is indeed as required. Therefore, the claim in (VII) holds in this case.

This completes the proof of the proposition. □

Proposition 9.2.6. *Fix $n \in \mathbb{N}$, and $1 \leqslant s, r \leqslant m$. Let $d \geqslant 0$ and $\rho \in Irr(Z(G))$. Define*

$$\eta_{d,\rho} = \eta_{n,d,\rho,s,r} : \mathbf{F}^n(s,r) \to \mathbb{Z}$$

$$f \mapsto \eta_{d,\rho}(f) = k_{d-d(f)}(P^n(f), S^z_r(f), \rho)$$

and

$$g_{d,\rho} = g_{n,d,\rho,s,r} : \mathbf{F}^{n-2sr}(\leqslant s) \to \mathbb{Z}$$

$$f' \mapsto g_{d,\rho}(f') = \begin{cases} k^0_{d-d(f')}(P^{n-2sr}(f'), U^{n-2sr}(f'),\rho), & \text{if } 0 \leqslant l(f') < s, \\ k_{d-d(f')}(P^{n-2sr}(f'),\rho), & \text{if } l(f') = s. \end{cases}$$

Then

$$A(\eta_{d,\rho},\mathbf{F}^n(s,r)) = -\sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}}\ \sum_{\substack{d_1,d_2\\ d_1+d_2=d'}} \Big(\sum_{J\subseteq[r/2]} (-1)^{|J|}k_{d_1}(P_J^r,\rho_1)\Big)A(g_{d_2,\rho_2},\mathbf{F}^{n-2sr}(\leqslant s)),$$

and

$$d' = d-(2s-1)\binom{r}{2} - r(n-2sr).$$

Proof. The proof is analogous to the proof of Proposition 9.2.5 so the proof is omitted. We do point out that the only difference is that each $\tau \in \mathrm{Irr}(V(f))$ is extendable to $N_{P(f)}(\tau)$, while there is no such property for $\tau\in\mathrm{Irr}(Z(f))$. To resolve this problem, in part (VIII)-(X) of the proof of Proposition 9.2.5, we may apply Lemma 7.3.9.3 instead of applying Lemma 2.2.3 and 2.2.4.

□

Lemma 9.2.7. *For a fixed n and s with $1\leqslant s\leqslant m$,*

(1)

$$\sum_{\substack{f\in\mathbf{F}^n\\ l(f)=s}} (-1)^{|f|}k_{d-d(f)}(P^n(f),S^z(f),\rho) = \sum_{r\geqslant 1} A(\eta_{n,d,\rho,s,r},\mathbf{F}^n(s,r));$$

(2)

$$\sum_{\substack{f\in\mathbf{F}^n\\ l(f)=s}} (-1)^{|f|}k_{d-d(f)}(P^n(f),S^{nu}(f),\rho) = \sum_{r\geqslant 1} A(\iota_{n,d,\rho,s,r},\mathbf{F}^n(s,r)).$$

Proof. This is straightforward as for each $f\in\mathbf{F}^n$, $S^z(f)$ (resp. $S^{nu}(f)$) is the disjoint union of $S^z_r(f)$ (resp. $S^{nu}_r(f)$) for $r\geqslant 1$, so that for a fixed $f\in\mathbf{F}^n$ with $l(f)=s$, we have

$$k_{d-d(f)}(P^n(f),S^z(f),\rho) = \textstyle\sum_{r\geqslant 1} k_{d-d(f)}(P^n(f),S^z_r(f),\rho)$$

and

$$k_{d-d(f)}(P^n(f), S^{nu}(f), \rho) = \sum_{r\geqslant 1} k_{d-d(f)}(P^n(f), S^{nu}_r(f), \rho).$$

Therefore, the lemma follows from the definition of $\eta_{n,d,\rho,s,r}$ and $\iota_{n,d,\rho,s,r}$.

□

Remark 9.2.8. Observe that $g_{n,d,\rho,s,r}$ and $A(g_{n,d,\rho,s,r}, \mathbf{F}^{n-2sr}(\leqslant s))$ are well defined for $r = 0$. Moreover, observe that for each $f \in \mathbf{F}^{n-2sr}(\leqslant s)$,

$$g_{n,d,\rho,s,r}(f) = g_{n-2sr,d,\rho,s,0}(f).$$

So

$$A(g_{n,d,\rho,s,r}, \mathbf{F}^{n-2sr}(\leqslant s)) = A(g_{n-2sr,d,\rho,s,0}, \mathbf{F}^{n-2sr}(\leqslant s)).$$

Similarly observe that $h_{n,d,\rho,s,r}$ and $A(h_{n,d,\rho,s,r}, \mathbf{F}^{n-2sr}(\leqslant s))$ are well defined for $r = 0$, and for each $f \in \mathbf{F}^{n-2sr}(\leqslant s)$,

$$h_{n,d,\rho,s,r}(f) = h_{n-2sr,d,\rho,s,0}(f).$$

Hence

$$A(h_{n,d,\rho,s,r}, \mathbf{F}^{n-(2s+1)r}(\leqslant s)) = A(h_{n-(2s+1)r,d,\rho,s,0}, \mathbf{F}^{n-(2s+1)r}(\leqslant s)).$$

9.3 Completion of the Verification

In this section, $G = GU_n(q)$, $n \in \mathbb{N}$ with $n = 2m$ or $2m+1$. So $I = [m]$. Recall from section 3.3 that in order to show **DOC** holds for G, we need to prove Proposition 3.3.6.2, that is,

$$\sum_{J\subseteq I}(-1)^{|J|} k_d^1(P_J, U_J, \rho) = \begin{cases} -\sum' \beta(\mu, a_\rho), & \text{if } d < \binom{n}{2}; \\ 0. & \text{otherwise.} \end{cases} \tag{9.9}$$

Here the sum $\sum'$ is taken over all partitions $\mu \vdash n$ with $n(\mu) = d$. Recall the definition of $n(\mu)$ from section 2.3. By Lemma 2.3.1.2, $n(\mu) \leqslant \binom{n}{2}$, with equality holding if and only if $\mu = (n)$, that is, if and only if $l(\mu) = 1$. So in either case, the right-hand side of (9.9) can be written as

$$-\sum_{\substack{\mu \vdash n \\ n(\mu)=d \\ l(\mu)\geqslant 2}} \beta(\mu, a_\rho).$$

So by Theorem 9.1.4, Proposition 3.3.6.2 is equivalent to the following:

Proposition 9.3.1. *Let $n \in \mathbb{N}$.*

$$\sum_{\substack{f\in \mathbf{F} \\ l(f)\geqslant 1}} (-1)^{|f|}(k_{d-d(f)}(P^n(f), S^{nu}(f), \rho) + k_{d-d(f)}(P^n(f), S^z(f), \rho)) = -\sum_{\substack{\mu \vdash n \\ n(\mu)=d \\ l(\mu)\geqslant 2}} \beta(\mu, a_\rho).$$

Observe Proposition 9.3.1 follows directly from the next proposition by summing over all $s \geqslant 1$:

Proposition 9.3.2. *For a fixed n and s with $1 \leqslant s \leqslant m$,*

(1)

$$\sum_{\substack{f\in \mathbf{F} \\ l(f)=s}} (-1)^{|f|}k_{d-d(f)}(P^n(f), S^z(f), \rho) = -\sum_{\substack{\mu \vdash n \\ n(\mu)=d \\ l(\mu)=2s}} \beta(\mu, a_\rho);$$

(2)

$$\sum_{\substack{f\in \mathbf{F} \\ l(f)=s}} (-1)^{|f|}k_{d-d(f)}(P^n(f), S^{nu}(f), \rho) = -\sum_{\substack{\mu \vdash n \\ n(\mu)=d \\ l(\mu)=2s+1}} \beta(\mu, a_\rho).$$

Then by Lemma 9.2.7, Proposition 9.3.2 follows from the following proposition by summing over all $r \geqslant 1$.

Proposition 9.3.3. *Fix n and $1 \leqslant s, r \leqslant m$.*

(1) Let $\eta_{n,d,\rho,s,r}$ *be as in Proposition 9.2.6. Then*

$$A(\eta_{n,d,\rho,s,r}, \mathbf{F}^n(s,r)) = -\sum_{\substack{\mu\vdash n \\ n(\mu)=d \\ l(\mu)=2s \\ \min(\mu)=r}} \beta(\mu, a_\rho);$$

(2) Let $\iota_{n,d,\rho,s,r}$ *be as in Proposition 9.2.5. Then*

$$A(\iota_{n,d,\rho,s,r}, \mathbf{F}^n(s,r)) = -\sum_{\substack{\mu\vdash n \\ n(\mu)=d \\ l(\mu)=2s+1 \\ \min(\mu)=r}} \beta(\mu, a_\rho).$$

Before we prove Proposition 9.3.3, we prove the following corollary to Proposition 9.3.3. Actually Corollary 9.3.4 is proved only under the assumption that Proposition 9.3.3 holds at n. Then later as we prove Proposition 9.3.3 by induction on n, we can assume the corollary is valid for all $n' < n$.

Corollary 9.3.4. *Fix* n *and assume Proposition 9.3.3 for* n*. Then the following is true for each* $1 \leqslant s \leqslant m$*.*

(1) Let $g_{d,\rho,s} = g_{n,d,\rho,s,0}$ *be as in Proposition 9.2.6 with* $r = 0$*. Then*

$$A(g_{d,\rho,s}, \mathbf{F}^n(\leqslant s)) = \sum_{\substack{\mu\vdash n \\ n(\mu)=d \\ l(\mu)\leqslant 2s-1}} \beta(\mu, a_\rho). \tag{9.10}$$

(2) Let $h_{d,\rho,s} = h_{n,d,\rho,s,0}$ *be as in Proposition 9.2.5 with* $r = 0$*. Then*

$$A(h_{d,\rho,s}, \mathbf{F}^n(\leqslant s)) = \sum_{\substack{\mu\vdash n \\ n(\mu)=d \\ l(\mu)\leqslant 2s}} \beta(\mu, a_\rho). \tag{9.11}$$

Proof. Throughout the proof, set $P(f) = P^n(f)$ for $f \in \mathbf{F}^n$.

As Proposition 9.3.3 holds for n, so do Proposition 9.3.1, 9.3.2, and 3.3.6.2. Hence Main Theorem 1 and Theorem 3.3.5 hold for n. We prove the corollary by induction on s. Recall from Definition 8.3.1 that if $f = (J, C, l) \in \mathbf{F}^n$, then $l(f) = |C|$ and $|f| = |J|$.

Assume $s = 1$. Let $f = (J, C, l) \in \mathbf{F}^n(\leqslant 1)$. If $l(f) = 0$, then $f = (\emptyset, \emptyset, 0)$ by definition of $\mathbf{F}^n$. In this case $P(f) = G$, $U(f) = 1$, and $d(f) = 0$ by Definition 8.3.5. So the contribution to $A = A(g_{d,\rho,s}, \mathbf{F}^n(\leqslant s))$ from f is $k_d(G, \rho)$. If $l(f) = 1$, then by Remark 8.3.2.2, $f = (J, \{l\}, l)$ with $\emptyset \neq J \subseteq I$ and $l = \max(J)$. In this case it follows from Example 8.3.7 that $P(f) = P_J$, $U(f) = U_l$, and $d(f) = 0$. So the contribution to A from such an f is $(-1)^{|J|} k_d(P_J, \rho)$. Thus as the set of $f \in \mathbf{F}^n$ with $l(f) = 1$ consists of all such $(J, \{l\}, l)$ with J running over all non-empty subsets of I, the left-hand side of equation (9.10) is equal to

$$k_d(G, \rho) + (-1)^{|J|} \sum_{\emptyset \subsetneq J \subseteq I} k_d(P_J, \rho).$$

By Theorem 3.3.5, this is equal to 0 if $d < \binom{n}{2}$, and $\beta((n), a_\rho)$ if $d = \binom{n}{2}$. We show this is equal to the right-hand side of (9.10). As $s = 1$, $l(\mu) \leqslant 2s - 1 = 1$, so $\mu = (n)$ and $n(\mu) = \binom{n}{2}$. So the right-hand side of (9.10) is nonzero if and only if $d = n(\mu) = \binom{n}{2}$, in which case it is $\beta((n), a_\rho)$. So part (1) holds when $s = 1$.

As for part (2), notice that for $f \in \mathbf{F}^n$, $h_{d,\rho,s}(f) = g_{d,\rho,s}(f)$ if $l(f) < 1$, while if $l(f) = 1$,

$$\begin{aligned} h_{d,\rho,s}(f) &= k^0_{d-d(f)}(P(f), Z(f), \rho) \\ &= k_{d-d(f)}(P(f), \rho) - k^1_{d-d(f)}(P(f), Z(f), \rho) \\ &= g_{d,\rho,s}(f) - k^1_{d-d(f)}(P(f), Z(f), \rho). \end{aligned} \tag{9.12}$$

But by Proposition 9.3.2.1 and Proposition 8.3.10.1,

$$x = \sum_{\substack{f \in \mathbf{F}^n \\ l(f)=1}} (-1)^{|f|} k^1_{d-d(f)}(P(f), Z(f), \rho) = - \sum_{\substack{\mu \vdash n \\ n(\mu)=d \\ l(\mu)=2}} \beta(\mu, a_\rho).$$

Therefore, as part (1) holds for $s = 1$, by (9.12),

$$A(h_{d,\rho,1}, \mathbf{F}^n(\leqslant 1)) = A(g_{d,\rho,1}, \mathbf{F}^n(\leqslant 1)) - x = \sum_{\substack{\mu \vdash n \\ n(\mu)=d \\ l(\mu)\leqslant 2}} \beta(\mu, a_\rho).$$

So part (2) and hence the corollary is true when $s = 1$.

Assume the corollary for $s' < s$. Observe for each $f \in \mathbf{F}^n$ with $l(f) \leqslant s-2$,

$$g_{d,\rho,s}(f) = h_{d,\rho,s-1}(f).$$

Also for each $f \in \mathbf{F}^n$,

$$k^0_{d-d(f)}(P(f), Z(f), \rho) = k^0_{d-d(f)}(P(f), U(f), \rho) + k^1_{d-d(f)}(P(f), V(f), \rho).$$

We have

$$\begin{aligned} y &= A(g_{d,\rho,s}, \mathbf{F}^n(\leqslant s)) - A(h_{d,\rho,s-1}, \mathbf{F}^n(\leqslant (s-1))) \\ &= \sum_{\substack{f \in \mathbf{F}^n \\ l(f)=s}} (-1)^{|f|} k_{d-d(f)}(P(f), \rho) + \sum_{\substack{f \in \mathbf{F}^n \\ l(f)=s-1}} (-1)^{|f|} k^0_{d-d(f)}(P(f), U(f), \rho) \\ &- \sum_{\substack{f \in \mathbf{F}^n \\ l(f)=s-1}} (-1)^{|f|} k^0_{d-d(f)}(P(f), Z(f), \rho) \\ &= \sum_{\substack{f \in \mathbf{F}^n \\ l(f)=s}} (-1)^{|f|} k_{d-d(f)}(P(f), \rho) - \sum_{\substack{f \in \mathbf{F}^n \\ l(f)=s-1}} (-1)^{|f|} k^1_{d-d(f)}(P(f), V(f), \rho). \end{aligned}$$

But for each $f \in \mathbf{F}^n$, by Definition 6.3.10,

$$k_{d-d(f)}(P(f), S^u(f), \rho) = k_{d-d(f)}(P(f), S^{su}(f), \rho) + k_{d-d(f)}(P(f), S^{nu}(f), \rho),$$

so by part (2) and (3) of Proposition 8.3.10, we obtain

$$\sum_{\substack{f\in\mathbf{F}^n\\ l(f)=s-1}} (-1)^{|f|}k^1_{d-d(f)}(P(f),V(f),\rho) = \sum_{\substack{f\in\mathbf{F}^n\\ l(f)=s}} (-1)^{|f|}k_{d-d(f)}(P(f),\rho) + \sum_{\substack{f\in\mathbf{F}^n\\ l(f)=s-1}} (-1)^{|f|}k^1_{d-d(f)}(P(f),S^{nu}(f),\rho).$$

So by Proposition 9.3.2.2,

$$y = -\sum_{\substack{f\in\mathbf{F}^n\\ l(f)=s-1}} (-1)^{|f|}k_{d-d(f)}(P(f),S^{nu}(f),\rho) = \sum_{\substack{\mu\vdash n\\ n(\mu)=d\\ l(\mu)=2s-1}} \beta(\mu,a_\rho).$$

By induction, $A(h_{d,\rho,s-1},\mathbf{F}^n(\leqslant (s-1)))$ is given by (9.11). Therefore, part (1) holds for s.

Observe that for $f\in\mathbf{F}^n$, $h_{d,\rho,s}(f) = g_{d,\rho,s}(f)$ if $l(f) < s$, while if $l(f) = s$, (9.12) still holds. Consequently,

$$w = A(h_{d,\rho,s},\mathbf{F}^n(\leqslant s)) - A(g_{d,\rho,s},\mathbf{F}^n(\leqslant s)) = -\sum_{\substack{f\in\mathbf{F}^n\\ l(f)=s}} (-1)^{|f|}k^1_{d-d(f)}(P(f),Z(f),\rho).$$

But by Proposition 9.3.2.1 and Proposition 8.3.10.1,

$$w = -\sum_{\substack{f\in\mathbf{F}^n\\ l(f)=s}} (-1)^{|f|}k^1_{d-d(f)}(P(f),Z(f),\rho) = \sum_{\substack{\mu\vdash n\\ n(\mu)=d\\ l(\mu)=2s}} \beta(\mu,a_\rho);$$

Therefore, as part (1) holds for s,

$$A(h_{d,\rho,s},\mathbf{F}^n(\leqslant s)) = A(g_{d,\rho,s},\mathbf{F}^n(\leqslant s)) + w = \sum_{\substack{\mu\vdash n\\ n(\mu)=d\\ l(\mu)\leqslant 2s}} \beta(\mu,a_\rho).$$

So part (2) and hence the corollary is true for s. The proof is complete.

□

Proof of Proposition 9.3.3. Recall $n = 2m$ or $2m+1$. We prove the proposition by induction on m, or equivalently by n. This breaks the proof into two steps, namely the small case and the induction process.

Step 1: Small case.

If $m = 0$, then $[m]$ is empty, so the proposition is vacuously true.

Let $m = 1$. Then $n = 2$ or 3, and $[m] = \{1\}$. So $s = r = 1$. In this case if $f = (J, C, l) \in \mathbf{F}$ with $l(f) = 1$, then $l = 1$. Also $C = \{1\}$, as $l = \min(C)$ and $|C| = 1$. Then $J = \{1\}$ as $l = \max(J)$ and $\partial C \subseteq J$. That is $f = (\{1\}, \{1\}, 1)$. By Example 8.3.7, $P = P(f)$ is the Borel subgroup of G with $U(f)$ being the unipotent radical of P. Also $d(f) = 0$ by definition. We suppress the notation (f) and write U, Z for $U(f)$ and $Z(f)$.

If n=2, then $U = Z$ is elementary abelian of order q. It follows that $V = U/Z = 1$ and $S_1^{nu}(f) = \emptyset$. On the other hand, the only partitions of 2 are (2) and (1^2). So there is no $\mu \vdash 2$ with $l(\mu) = 3$. Hence part (2) holds vacuously. As for part (1), observe in this case $S_1^z(f) = \mathrm{Irr}^1(Z)$ and P is transitive on $\mathrm{Irr}^1(Z)$. Let $\tau \in \mathrm{Irr}^1(Z)$. Then $N_P(\tau) = Z \times Z(G)$. As p does not divide $|P|/|N_P(\tau)|$, by Lemma 2.2.2, the left-hand side of part (1) becomes $-k_d(P, \tau, \rho) = -k_d(N_P(\tau), \tau, \rho)$. By Lemma 2.2.5, $\varphi = \tau\rho$ is the only character in $\mathrm{Irr}(N_P(\tau))$ lying over $\tau \in \mathrm{Irr}(Z)$ and $\rho \in \mathrm{Irr}(Z(G))$. Moreover, as both Z and $Z(G)$ are abelian, $\varphi(1) = 1$. Therefore, $-k_d(N_P(\tau), \tau, \rho) = -1$ if $d = 0$ and 0 otherwise. On the other hand, the only $\mu \vdash 2$ with two parts is $\mu = (1^2)$, in which case $n(\mu) = 0$, $\min(\mu) = 1$, and $\beta(\mu, a_\rho) = 1$. So the right-hand side is -1 if $d = 0$ and 0 otherwise. Thus the proposition is true if $n = 2$.

If $n = 3$, then U is a special p-group with $Z \cong \mathbb{F}_q$ and $V = U/Z \cong \mathbb{F}_{q^2}$. Moreover $P/U \cong \mathbb{C}_{q^2-1} \times \mathbb{C}_{q+1}$. Observe in this case $S_1^{nu}(f) = \mathrm{Irr}^1(V)$ and P is transitive on $\mathrm{Irr}^1(V)$. Let $\tau \in \mathrm{Irr}^1(V)$; then $N_P(\tau) = U \times Z(G)$. Let $\bar{P} = P/Z$; then $N_{\bar{P}}(\tau) = V \times Z(G)$. As p does not divide $|\bar{P}|/|N_{\bar{P}}(\tau)|$, by Lemma 2.2.2, the left-hand side of part (2) becomes

$$-k_d(P, \tau, \rho) = -k_d(\bar{P}, \tau, \rho) = -k_d(N_{\bar{P}}(\tau), \tau, \rho).$$

By Lemma 2.2.5, $\varphi = \tau\rho$ is the only character in $\mathrm{Irr}(N_P(\tau)$ lying over $\tau \in \mathrm{Irr}(V)$

and $\rho \in \mathrm{Irr}(Z(G))$. Moreover, as both V and $Z(G)$ are abelian, $\varphi(1) = 1$. Therefore, $-k_d(N_P(\tau), \tau, \rho) = -1$ if $d = 0$ and 0 otherwise. On the other hand, the only $\mu \vdash 3$ with $l(\mu) = 3$ is $\mu = (1^3)$, in which case $n(\mu) = 0$, $\min(\mu) = 1$ and $\beta(\mu, a_\rho) = 1$. So the right-hand side is -1 if $d = 0$ and 0 otherwise. So part (2) holds if $n = 3$.

Next observe $S_1^z(f) = \mathrm{Irr}^1(Z)$ and P is transitive on $\mathrm{Irr}^1(Z)$. Pick $\tau \in \mathrm{Irr}^1(Z)$. By Lemma 7.2.2 there is a unique $\phi \in \mathrm{Irr}(U)$ lying over τ with $\phi(1) = q$, and $\varphi \in \mathrm{Irr}(P)$ lies over τ if and only if it lies over ϕ. Notice $N_P(\phi) = N_P(\tau) = UH$ with $Z(G) \leqslant H \cong \mathbb{C}_{q+1} \times \mathbb{C}_{q+1}$. Therefore,, as p does not divide $|P|/|N_P(\phi)|$, the left-hand side of part (1) becomes

$$-k_d(P, \tau, \rho) = -k_d(P, \phi, \rho) = -k_d(N_P(\phi), \phi, \rho).$$

But ϕ is extendable to $N_P(\phi)$ by Lemma 7.2.4. So by Lemma 2.2.3,

$$-k_d(N_P(\phi), \phi, \rho) = -k_{d-1}(H, \rho),$$

which is $-(q+1)$ if $d = 1$ and 0 otherwise. Finally the only $\mu \vdash 3$ with $l(\mu) = 2$ is $\mu = (21)$, in which case $n(\mu) = 1$, $\min(\mu) = 1$ and $\beta(\mu, a_\rho) = q+1$. Consequently, the right-hand side of part (1) is equal to $-(q+1)$ if $d = 1$ and 0 otherwise. Therefore, part (1) and hence the proposition holds when $n = 3$.

Step 2: The induction process.

Assume the proposition for dimension less than $m \in \mathbb{N}$, $m \geqslant 2$, with $n = 2m$ or $2m+1$. Hence Proposition 9.3.1, 9.3.2 and Main Theorem 1 hold for dimension less than n. Also Corollary 9.3.4 holds for dimension less than n.

Denote the left-hand side of part (1) by σ. Then by Proposition 9.2.6,

$$\sigma = -\sum_{\substack{\rho_1,\rho_2 \\ \rho_1\rho_2=\rho}} \sum_{\substack{d_1,d_2 \\ d_1+d_2=d'}} \sigma'\sigma''$$

where $d' = d - (2s-1)\binom{r}{2} - r(n-2sr)$ is as in Proposition 9.2.6,

$$\sigma' = \sum_{J''\subseteq[r/2]} (-1)^{|J''|} k_{d_1}(P^r_{J''}, \rho_1)$$

and

$$\sigma'' = A(g_{n,d_2,\rho_2,s,r}, \mathbf{F}^{n-2sr}(\leqslant s)).$$

By induction, Theorem 3.3.5 holds for r. So

$$\sigma' = \begin{cases} \beta((r), a_{\rho_1}), & \text{if } d_1 = \binom{r}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

By induction, Corollary 9.3.4.1 holds for $n - 2sr$. So by Remark 9.2.8,

$$\sigma'' = A(g_{n-2sr,d_2,\rho_2,s,0}, \mathbf{F}^{n-2sr}(\leqslant s)) = \sum_{\substack{\mu \vdash (n-2sr) \\ n(\mu)=d_2 \\ l(\mu)\leqslant 2s-1}} \beta(\mu, a_{\rho_2}).$$

Therefore,

$$\sigma = -\sum_{\substack{\rho_1,\rho_2 \\ \rho_1\rho_2=\rho}} \beta((r), a_{\rho_1}) \sum_{\substack{\mu \vdash (n-2sr) \\ n(\mu)=d'' \\ l(\mu)\leqslant 2s-1}} \beta(\mu, a_{\rho_2})$$

where $d'' = d' - \binom{r}{2} = d - 2s\binom{r}{2} - r(n - 2sr)$.

Recall from Example 2.3.5 that the set of $\mu = (a_i^{m_i}) \vdash n - 2sr$ with $l(\mu) \leqslant 2s - 1$ is in 1-1 correspondence with the set of $\mu' \vdash n$ with $l(\mu') = 2s$ and $\min(\mu') = r$ such tht if $n(\mu) = d''$, then $n(\mu') = d'' + 2s\binom{r}{2} + r(n - 2sr) = d$ via $\mu \mapsto \mu' = (r^{2s}) + \mu$. So by Lemma 2.3.6,

$$\sigma = -\sum_{\substack{\mu \vdash n \\ n(\mu)=d \\ \min(\mu)=r \\ l(\mu)=2s}} \beta(\mu, a_\rho).$$

Part (1) is proved.

Part (2) can be proved similarly. Denote the left-hand side of part (2) by σ. Then by Proposition 9.2.5,

$$\sigma = - \sum_{\substack{\rho_1,\rho_2 \\ \rho_1\rho_2=\rho}} \sum_{\substack{d_1,d_2 \\ d_1+d_2=d'}} \sigma'\sigma''$$

where $d' = d - 2s\binom{r}{2} - r(n-(2s+1)r)$ is as in Proposition 9.2.5, σ' is as in the preceeding case, and

$$\sigma'' = A(h_{n,d_2,\rho_2,s,r}, \mathbf{F}^{n-(2s+1)r}(\leqslant s)).$$

Again by induction, Corollary 9.3.4.2 holds for $n-(2s+1)r$. So by Remark 9.2.8,

$$\sigma'' = A(h_{n-(2s+1)r,d_2,\rho_2,s,0}, \mathbf{F}^{n-(2s+1)r}(\leqslant s)) = \sum_{\substack{\mu\vdash(n-(2s+1)r) \\ n(\mu)=d_2 \\ l(\mu)\leqslant 2s}} \beta(\mu, a_{\rho_2}).$$

Therefore,

$$\sigma = - \sum_{\substack{\rho_1,\rho_2 \\ \rho_1\rho_2=\rho}} \beta((r), a_{\rho_1}) \sum_{\substack{\mu\vdash(n-(2s+1)r) \\ n(\mu)=d'' \\ l(\mu)\leqslant 2s}} \beta(\mu, a_{\rho_2})$$

where $d'' = d' - \binom{r}{2} = d - (2s+1)\binom{r}{2} - r(n-(2s+1)r)$.

Applying Lemma 2.3.6 and Example 2.3.5 once more, we have

$$\sigma = - \sum_{\substack{\mu\vdash n \\ n(\mu)=d \\ \min(\mu)=r \\ l(\mu)=2s+1}} \beta(\mu, a_\rho).$$

So part (2) is proved. This completes the proof.

□

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