On the Geometric Simple-Connectivity of 4-manifolds

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Abstract

This work concerns the question of when contractible 4-manifolds need 1-handles. In high-dimensions, eliminating handles whenever permitted by homotopy type has been a very fruitful approach. The question addressed concerns the applicability of these methods in low dimensions.

Specifically, we study when the interiors of compact, contractible 4-manifolds have a handle-decomposition without 1-handles. An argument of Casson shows that the compact manifolds themselves 'usually' need 1-handles. This argument depends essentially on finiteness of the handle-decomposition.

We show that any handle-decomposition without 1-handles must be of a particularly nice form, which involves surgery along surfaces representing homology. We show that we have 'Casson finiteness', i.e., Casson's argument can be used to show that such a handle-decomposition cannot exist, whenever there are embedded, disjoint surfaces satisfying a certain property. We then show that there are immersed surfaces satisfying this property. Finally, we show that the obstruction to cutting and pasting the surfaces to get embedded ones is non-trivial.

As a corollary to the methods, we give an example of an open manifold with an infinite handle-decomposition without 1-handles that is not the interior of a compact manifold, and thus has no finite handle-decomposition.

A relative version of this question is also considered. In this case, Donaldson's theorem leads to obstructions to the existence of finite handle decomposition without 1-handles.

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Chapter 1 Introduction

A fruitful approach in high-dimensional topology has been to simplify the handle decomposition of a given manifold as much as possible. For instance, Smale's proof of the h-cobordism theorem and the Poincaré conjecture in high dimensions is along these lines. These methods don't work so well in low dimensions. In the case of 3-manifolds, one has all the intricacies of Heegard theory.

On the other hand, for simply-connected 4-manifolds, one may ask whether one can at least eliminate 1-handles. The simple-connectivity means that there is no obstruction to this from homotopy theory. In the closed case, there are no known obstructions. In several cases one does have a handle decomposition without 1-handles, for instance for non-singular complex hypersurfaces in $\mathbb{CP}^3$ [10] [4]. We consider here the case of contractible 4-manifolds.

A motivation for the results here is Poènaru's program to prove the Poincaré conjecture, which involves first eliminating the 1-handles of $\Delta \times I$, where Δ is a homotopy 4-ball. As part of this program, Poénaru conjectured that if the interior of a contractible 4-manifold $(N, \partial N)$ has a handle decomposition without 1-handles, then so does $(N, \partial N)$ itself. This is to be applied to $\Delta \times I$, after showing that its interior does have a handle decomposition without 1-handles.

An argument of Casson [6, page 253] shows that for a contractible manifold $(N, \partial N)$ with boundary ∂N that has, for instance, a residually finite fundamental group, we cannot have a handle decomposition with no 1-handles. This argument does not rule out the interior having a handle decomposition without 1-handles. We study the applicability of this result to the interior of the manifold.

Casson's result was based on partial positive solutions to the following algebraic conjecture [5, page 117] [7, page 403].

Conjecture (Kervaire Conjecture). *Suppose one adds an equal number of generators* $\alpha_1, \ldots, \alpha_n$ *and relations* $r_1, \ldots, r_n$ *to a non-trivial group* G, *then the group* $\frac{G * <\alpha_1, \ldots, \alpha_n>}{<<r_1, \ldots, r_n>>}$ *that one obtains is also non-trivial.*

Casson showed that certain 4-manifolds $(N, \partial N)$ have no handle decompositions without 1-handles by showing that if they did, then $\pi_1(\partial N)$ violates the Kervaire conjecture.

Theorem (Casson). *If the contractible* 4*-manifold* $(N, \partial N)$ *with boundary not a homotopy sphere has a handle decomposition without* 1*-handles, then* $\pi_1(\partial N)$ *violates the Kervaire conjecture.*

Proof. Since the manifold $(N, \partial N)$ is contractible, the number of 2-handles is equal to the number of 3-handles in a handle decomposition without 1-handles. If we invert the handle decomposition, then 2-handles and 3-handles become 2-handles and 1-handles respectively. It follows that $\pi_1(N)$ is obtained from $\pi_1(\partial N)$ by adding an equal number of generators and relations, as these correspond respectively to the 1-handles and 2-handles of the inverted handle decomposition. Since N is contractible, $\pi_1(N)$ is trivial, and hence $\pi_1(\partial N)$ is either trivial or violates the Kervaire conjecture. □

Casson's argument works to the extent that the Kervaire conjecture is known to be true. Casson originally applied it using a theorem of Gerstenhaber and Rothaus [3], which said that the Kervaire conjecture holds for subgroups of a compact Lie group. Subsequently, Rothaus [9] showed that the conjecture in fact holds for residually finite groups. Since residual finiteness for all 3-manifold groups is implied by the geometrisation conjecture, Casson's argument works in particular for all manifolds satisfying the geometrisation conjecture. A simple argument (Proposition 1.1 below) extends the class of groups for which the Kervaire conjecture is known further.

We shall study to what extent Casson's theorem extends to a statement about the interior of the manifold. We start with the following simple proposition extends the class of groups for which the Kervaire conjecture is known to be true. The methods of this proposition will also give us criteria under which the argument can be applied to a handle-decomposition of the interior.

Proposition 1.1. *If any non-trivial quotient Q of a group G satisfies the Kervaire conjecture, then so does G.*

Proof. Let $\phi : G \twoheadrightarrow Q$ be the quotient map. Assume that Q satisfies the Kervaire conjecture. Suppose that G violates the Kervaire conjecture. Then we have generators $\alpha_1, \ldots, \alpha_n$ and relations such that $\frac{G * <\alpha_1, \ldots, \alpha_n>}{<<r_1, \ldots, r_n>>}$ is the trivial group. Let $\tilde{\phi} : G* < \alpha_1, \ldots, \alpha_n > \to Q* < \bar{\alpha}_1, \ldots, \bar{\alpha}_n >$ be the map extending ϕ by mapping α_i to $\bar{\alpha}_i$. This is clearly a surjection, and induces a surjective map $\bar{\phi} : \frac{G * <\alpha_1, \ldots, \alpha_n>}{<<r_1, \ldots, r_n>>} \twoheadrightarrow \frac{Q * <\bar{\alpha}_1, \ldots, \bar{\alpha}_n>}{<<\tilde{\phi}(r_1), \ldots, \tilde{\phi}(r_n)>>}$. But since the domain of the surjection $\bar{\phi}$ is trivial, so is the codomain. But this means that $\frac{Q * <\bar{\alpha}_1, \ldots, \bar{\alpha}_n>}{<<\tilde{\phi}(r_1), \ldots, \tilde{\phi}(r_n)>>}$ is trivial, and so Q violates the Kervaire conjecture, a contradiction. □

Corollary 1.2. *If a finitely generated group G has a proper finite-index subgroup, then G satisfies the Kervaire conjecture.*

Proof. This follows from the above proposition since finite groups satisfy the Kervaire conjecture by the theorem of Gerstenhaber and Rothaus. □

Remark 1.3. The above corollary shows that the fundamental group of a manifold satisfies the Kervaire conjecture as long the manifold has some non-trivial finite cover.

Suppose we do have a contractible 4-manifold $(N, \partial N)$ with a handle decomposition without 1-handles of its interior. Since there may be infinitely many handles, we cannot use Casson's argument. However, we note that we can use Casson's argument if we can show that

- $(N, \partial N)$ has a (finite) handle decomposition without 1-handles.
- Some $(N', \partial N')$ has a handle decomposition without 1-handles, where N' is compact, contractible with $\pi_1(\partial N') = \pi_1(\partial N)$.

- Some $(N', \partial N')$ has a handle decomposition without 1-handles, where N' is compact, contractible and there is a surjection $\pi_1(\partial N') \twoheadrightarrow \pi_1(\partial N)$ (by the above proposition).

Thus, we can apply Casson's argument if we show finiteness, or some weak form of finiteness such as the latter statements above.

We consider the level sets of the handle decomposition, which are 3-manifolds. We see that in a certain 'uniform' sense, the limit of the fundamental groups of the level sets tends to $\pi_1(\partial N)$.

We then show that the handle decomposition can be assumed to be of a canonical form. This canonical form involves surfaces representing the homology of level sets. We show that if the surfaces are embedded, disjoint and of a particular form, then we have 'Casson finiteness', i.e., we can conclude that the Kervaire conjecture is violated. We then show that we have surfaces that are in general immersed and intersect that do satisfy the required conditions.

Thus the obstruction is to find embedded surfaces of this form where we have immersed surfaces. In the case of a finite handle-decomposition, such surfaces exist. We shall show that this obstruction, namely being able to obtain embedded, disjoint surfaces from the immersed ones, is non-trivial. More precisely, we do not have such embedded, disjoint surfaces in a neighbourhood of the attaching regions of the handles in our handle-decomposition.

We shall also construct an example of an open manifold with an infinite handle decomposition without 1-handles that is not the interior of a compact manifold. This implies that the manifold has no finite handle decomposition (in particular no finite handle-decompostion without 1-handles).

We also consider a relative version of the same question. Namely, we take a partial handle-decomposition without 1-handles and ask when it can be extended to a handle-decomposition without 1-handles for the whole manifold. Donaldson's theorem leads to subtle obstructions in the finite case.

Chapter 2 Properties of the handle decomposition

We assume henceforth that we have a handle decomposition of the interior $\mathring{N}$ of $(N, \partial N)$, a contractible four manifold with boundary a homology 3-sphere. This handle decomposition shall be regarded as coming from a Morse function, which we shall refer to as 'time', with words like 'past' and future' having obvious meanings. The inverse images of regular points are 3-manifolds, which we shall refer to as the 'manifold at that time'.

Now let $(K_i, \partial K_i), i \in \mathbb{N}$ denote the 4-manifolds obtained by successively attaching handles to the zero handle (B^4, S^3), that is if $t : (N, \partial N) \to \mathbb{R}$ is the Morse function time, then $(K_i, \partial K_i) = t^{-1}((\infty, a_i])$, with a_i being points lying between pairs of critical values of the Morse function.

Lemma 2.1. *∂K_{i+1} is obtained from ∂K_i by one of the following:*

- *A 0-frame surgery about a homologically trivial knot in ∂K_i.*
- *Cutting along a non-separating 2-sphere in ∂K_i and capping off the result by attaching a 3-ball.*

These correspond respectively to attaching 2-handles and 3-handles to $(K_i, \partial K_i)$.

Proof. Since attaching 2-handles and 3-handles always correspond to surgery and cutting along 2-spheres respectively, we merely have to show that the surgery is 0-frame about a homologically trivial curve and the spheres are non-separating.

First note that since there are no 1-handles, $H_1(K_i) = 0 = \pi_1(K_i) \forall i$. Further, each ∂K_i is connected, for, suppose it were not, since there are no 1-handles, $\pi_1(\mathring{N} \setminus K_i, \partial K_i) = 0$. Hence $\mathring{N} \setminus K_i$ is not connected, so $\mathring{N}$ has more than one end, a contradiction. Thus the

2-spheres along which any ∂K_i is split have to be non-separating. This in particular means that attaching a 3-handle decreases the rank of $H_2(M_i)$.

Now, since $\mathring{N}$ is contractible, by a Mayer-Vietoris argument the inclusion maps induce surjections $H_2(\partial K_i) \twoheadrightarrow H_2(K_i)$, hence $rk(H_2(\partial K_i)) \geq rk(H_2(K_i)$. Since there are no 1-handles, each 2-handle increases the rank of $H_2(K_i)$ by one, and that of $H_2(\partial K_i)$ by at most one. Also a 3-handle decreases the rank of $H_2(\partial K_i)$ by one and that of $H_2(K_i)$ by at most one. Since for (B^4, S^3), the ranks are equal, it follows that the ranks are always equal, and that every surgery decreases the rank of $H_2(\partial K_i)$. But this means that the surgery must be a zero-frame surgery about a homologically trivial curve. □

Since ∂N has a neighbourhood of the form $\partial N \times I$, $\mathring{N}$ has an end of the form $\partial N \times [0, \infty)$. For i large enough, ∂K_i lies in this set, hence we have a map $f_i : \partial K_i \to \partial N$ which is the composition of the inclusion with the projection.

Lemma 2.2. *The map f_i is a degree-one map and hence induces a surjection ϕ_i between the fundamental groups. Further, the image under f_i of a curve along which surgery is performed is homotopically trivial in ∂N.*

Proof. We have an inclusion map $i : (K_i, \partial K_i) \to (\mathring{N}, \partial N \times [0, \infty))$. This induces maps between the long-exact sequences of the pairs $(K_i, \partial K_i)$ and $(\mathring{N}, \partial N \times [0, \infty))$. From this we see that f_i is degree-one iff $i_* : H_4(K_i, \mathring{K}_i) \to H_4(\mathring{N}, \partial N \times [0, \infty))$ is an isomorphism. This in turn follows as i is an inclusion and degree maybe computed locally, or more formally by looking at exact sequences of triples $(M, M \setminus \{x\}, \partial M)$ for manifolds M with boundary, where $x \in \mathring{M}$.

Further, if a surgery is performed along a curve γ, this means that a two handle is attached along the curve in the 4-manifold. Hence γ bounds a disk in $\partial N \times [0, \infty)$, which projects to a disk bounded by $f_i(\gamma)$ in ∂N. □

Remark 2.3. The maps ϕ_i and ϕ_{i+1} are related in a natural way. To define the map ϕ_{i+1}, take a generic curve γ representing any given element of $\pi_1(\partial K_{i+1})$. If ∂K_{i+1} is obtained from ∂K_i by splitting along a sphere, then γ is a curve in ∂K_i, and so we can simply take its image. On the other hand, if a surgery was performed, then we may assume that γ lies off the solid torus that has been attached, and hence lies in ∂K_i, so we can take its image as before. This map is well-defined by lemma 2.2.

We now make some key definitions. Suppose henceforth that we have a sequence of 3-manifolds M_i with degree-one maps onto ∂N that satisfies the properties of ∂K_i stated above, i.e., in lemmata 2.1, 2.2 and remark 2.3. We note that if we take a curve γ in M_i and perform surgery on M_i, then γ, regarded as an element of the fundamental group of ∂N, has several 'descendants', i.e., curves homotopic to γ in M_i (though not in general homotopic to γ after the surgery). On the other hand, if we split M_i along a sphere, we may or may not be able to homotope γ to be disjoint from the sphere. In the latter case, we say that γ does not persist.

Definition 2.1. A curve γ in M_i is said to persist till M_{i+n} if some descendant of γ persists, i.e., we can homotope γ in M_i so that it is disjoint from all the future 2-spheres that are attached while passing from M_i to M_{i+1}.

Definition 2.2. A curve γ in M_i is said to die by M_{i+n} if it is homotopically trivial in the 4-manifold obtained by attaching 2-handles to $M_i \times [0,1]$ along the curves in $M_i \times \{1\} = M_i$ where surgeries are performed in the process of passing to M_{i+n}, or equivalently, γ is trivial in the group obtained by adding relations to $\pi_1(M_i)$ corresponding to curves along which the surgery is performed.

It must be emphasised that the above does not mean that the curve is homotopically trivial in M_{i+n}, or in any $M_{i+j}, 0 \leq j \leq n$.

We now prove a key property of the sequence ∂K_i.

Lemma 2.4. *Given i, there is a uniform n such that any curve γ in ∂K_i that is in the kernel of ϕ_i that persists till ∂K_{i+n} dies by ∂K_{i+n}.*

Proof. The set $\partial N \times [0, \infty)$ has a proper Morse function, time, on it, as well as a foliation by compact leaves $\partial N \times \{x\}$, and the ∂K_j's are level sets of time. We can find $x \in [0, \infty)$ so that $\partial N \times \{x\}$ is entirely after ∂K_i, and n_1 so that ∂K_{i+n_1} is in turn entirely after $\partial N \times \{x\}$. We then define n by repeating this process once.

Now suppose the curve γ in the kernel of ϕ_i persists till ∂K_{i+n_1}. This means that in $\partial N \times [0, \infty)$, there is an annulus bounding γ and a curve $\tilde{\gamma}$ in ∂K_{i+n} so that the annulus is entirely in the present and future with respect to ∂K_i. Since $\tilde{\gamma}$ in ∂K_{i+n}, it is entirely in $\partial N \times [x, \infty)$. Since it is in the kernel of ϕ_i, it follows that it bounds a disc in $\partial N \times [x, \infty)$. This disc together with the above annulus ensure that γ dies by ∂K_{i+n}, as they bound together a disc entirely in the present and future w.r.t. ∂K_i, and 3-handles do not affect the fundamental group. $\square$

Henceforth we also assume that the sequence M_i satisfies the conclusion of lemma 2.4. Indeed in our applications, we shall take M_i to be ∂K_i, with perhaps a modified handle decomposition.

Chapter 3 Re-ordering of the handle decomposition

We show in this chapter that, after possibly changing the order of attaching handles, any handle decomposition without 1-handles is of a particular form.

We first describe a procedure for attempting to construct a handle decomposition for $\mathring{N}$ starting with a partial handle decomposition, with boundary M_i. In general, M_i has non-trivial homology. It follows readily from the proof of lemma 2.1 that $H_1(M_i)$ is a torsion free abelian group. The only way we can remove homology is by splitting along spheres. To this end, we take a collection of surfaces representing the homology, perform surgeries along curves in these surfaces so that they compress down to spheres, and then split along these spheres. By doing the surgeries, we have created new homology, and hence have to take new surfaces representing this homology and continue this procedure. In addition to this, we may need to perform other surgeries to get rid of the 'homologically trivial portion' of the kernel of $\phi_i : M_i \to \partial N$.

The above construction may meet obstructions, since the surgeries have to be performed about curves that are homologically trivial as well as lie in the kernel of ϕ_i, hence it may not be always possible to perform enough of them to compress the surfaces to spheres. The construction terminates at some finite stage if at that stage all the homology is represented by spheres and no surgery off these surfaces is necessary.

Remark 3.1. A surface in M_{i+n}, after isotopy, can be assumed to intersect all the 2-handles added between times i and $i+n$ in a union of horizontal discs $D^2 \times \{i\} \subset D^2 \times [0,1]$. On deleting these discs, one gets a surface in $M_{i+n} \cap M_i$ with boundary on the loci of the surgeries between times i and $i+n$. Thus, the surface pulls back to a surface with boundary in M_i.

Theorem 3.2. *After possibly changing the order of attaching handles, any handle decomposition without* 1*-handles may be described as follows. We have a collection of surfaces, with disjoint simple closed curves on them which shall be called seams, and embeddings from the surfaces to* M_i *so that:*

- *The surfaces represent the homology of* M_i.
- *The only intersections of the surfaces are along the seams.*
- *When compressed along the seams the surfaces become spheres.*
- *The seams are homologically trivial curves in* M_i *and lie in the kernel of* ϕ_i.
- *The pull backs of surfaces at any future time, which are in general surfaces with boundary in* M_i*, can only intersect the surfaces in* M_i *either transversely at the seams or by having some boundary components along the seams.*

We attach 2*-handles along all the seams of* M_i*, and possibly also along some curves that are completely off the surfaces in* M_i *and have no intersection with any future surface. We then attach* 3*-handles along the sphere obtained by compressing the surface. Iterating this procedure gives us the handle decomposition.*

We shall see that once we construct the surfaces, all of the properties follow automatically. First we need some preliminary results.

Lemma 3.3. *Let* $\alpha : \pi_1(M_i) \to H_1(M_i)$ *be the Hurewicz map. Then* $ker(\alpha)$ *surjects onto* $\pi_1(\partial N)$ *under* ϕ_i.

Proof. We first show that $ker(\phi_i)$ surjects onto $H_1(M_i)$. This follows as if $\pi_1(M_i)$ is expressed in terms of generators and relations, then we get its quotient $\pi_1(\partial N)$ by adding some further relations. Since $\pi_1(\partial N)$ when abelianised is trivial, it follows that the extra relations on abelianising must normally generate $H_1(M_i)$, the abelianisation of $\pi_1(M_i)$.

These relations are in $ker(\phi_i)$ by hypothesis, and as $H_1(M_i)$ is abelian they generate (as they normally generate) $H_1(M_i)$. The claim follows.

The lemma now follows from a simple diagram chase. Given any element in $x \in \pi_1(\partial N)$, we take some element $y \in \pi_1(M_i)$ that maps to it under ϕ_i, which must exist as ϕ_i surjects. Now by the above we can find $z \in ker(\phi)$ with $\alpha(z) = \alpha(y)$, and yz^{-1} is the required element in $ker(\alpha)$ that maps to x under ϕ_i. □

Next, observe that spheres in some M_{i+j} look like planar surfaces in M_i, with boundary components being the loci of future surgeries. Further, since we have a surjection ϕ_{i+j}, we have curves in M_{i+j} mapping to every element in $\pi_1(\partial N)$, and hence curves in the complement of the planar surfaces mapping to every element. Moreover, by the above lemma, we have such curves that are homologically trivial in M_{i+j}, and hence in M_i as all relations added are trivial on abelianising.

Now, let n be as in the conclusion of lemma 2.4, and pull back the spheres up to time i to get a collection of planar surfaces Σ. By the above, homologically-trivial curves disjoint from these surfaces map to every element of $\pi_1(\partial N)$.

Lemma 3.4. *$i_* : H_1(M_i \setminus \Sigma) \to H_1(M_i)$ is the zero map.*

Proof. Suppose not. Then we have an element $H_1(M_i \setminus \Sigma)$, which we regard as a curve in M_i, that represents a non-trivial element of $H_1(M_i)$. Using the above remarks and modifying by a homologically trivial element if necessary, we may assume that $\gamma \in ker(\phi_i)$. But, by hypothesis γ persists. Also γ does not die as it represents a non-trivial element on abelianisation (i.e., on projecting to $H_1(M_i)$), and all the relations are trivial on abelianisation (i.e., they project onto 0 in $H_(M_i)$). This gives the required contradiction. □

We are now in a position to prove the theorem. The surfaces will be obtained from a sub collection of the collection Σ of planar surfaces by 'stitching together' along the knots on which the boundaries of these lie. These knots shall be the seams of the surfaces. It is clear by construction that we have all the desired properties as soon as we show that there are enough planar surfaces to be stitched together to represent all the homology.

To see this, we consider the reduced homology exact-sequence of the pair $(M_i, M_i \setminus \Sigma)$, and use the fact that $M_i \setminus \Sigma$ is connected, since M_{i+n} is, as well as lemma 3.4. Thus, we have the exact sequence

$$\cdots \to H_1(M_i \setminus \Sigma) \to H_1(M_i) \to H_1(M_i, M_i \setminus \Sigma) \to \tilde{H}_0(M_i \setminus \Sigma)$$

which gives the exact sequence

$$0 \to H_1(M_i) \to H_1(M_i, M_i \setminus \Sigma) \to 0$$

which together with an application of Alexander duality gives $H_1(M_i) \cong H_1(M_i, M_i \setminus \Sigma) \cong H^2(\Sigma)$. Further, as the isomorphisms $H_1(M) \cong H^2(M)$ and $H_1(M_i, M_i \setminus \Sigma) \cong H^2(\Sigma)$, given respectively by Poincare and Alexander duality, are obtained by taking cup products with the fundamental class, the diagram

$$\begin{array}{ccc} H^2(M_i) & \longrightarrow & H^2(\Sigma) \\ \downarrow & & \downarrow \\ H_1(M_i) & \longrightarrow & H_1(M_i, M_i \setminus \Sigma) \end{array}$$

commutes.

Thus, $H^2(M_i) \overset{\cong}{\to} H^2(\Sigma)$ by the map induced by the inclusion of Σ in M_i. Since $H_2(M)$ and $H_2(\Sigma)$ have no torsion, the cap product induces perfect pairings $H_2(M_i) \times H^2(M_i) \to \mathbb{Z}$ and $H_2(\Sigma) \times H^2(\Sigma) \to \mathbb{Z}$. Thus the map $H_2(\Sigma) \to H_2(M_i)$ induced by inclusion, which is the dual of the above isomorphism $H^2(M_i) \overset{\cong}{\to} H^2(\Sigma)$, is also an isomorphism, i.e., the above sub collection of surfaces in Σ carry all the homology of M_i.

Now take a basis for $H^2(\Sigma)$. Each element of this basis can be looked at as an integral linear combination of the planar surfaces (as in cellular homology), with trivial boundary. We obtain a surface corresponding to each such homology class by taking copies of the planar surfaces, with the number and orientation determined by the coefficient. Since the homology classes are cycles, these planar surfaces can be glued together at the boundaries to form closed, oriented, immersed surfaces. Without loss of generality, we can assume these to be connected.

Remark 3.5. The surfaces representing the homology that is created by the surgeries come essentially from capping-off Seifert surfaces of the loci of the surgeries, except that they might also have other boundary components along the seams that need to be capped off.

Chapter 4 On Casson finiteness

We now assume that the handle decomposition is as in the conclusion of Theorem 3.2. We shall change our measures of time so that passing from M_i to M_{i+1} consists of performing all the surgeries required to compress the surfaces, splitting along the surfaces, and also performing the necessary surgeries off the surface.

In M_i, we have a collection of embedded surfaces representing all the homology of M_i. We see that we have Casson finiteness in a special case.

Theorem 4.1. *If for some M_i, the surfaces are embedded and their fundamental groups map to the trivial group under ϕ_i, then $\pi_1(\partial N)$ violates the Kervaire conjecture.*

Proof. Let k be the rank of $H_1(M_i)$ and $P_j, 1 \leq j \leq k$ be the fundamental groups of the surfaces. Since the surfaces are disjoint, $\pi_1(M_i)$ is obtained by HNN extesions from the fundamental group G of the complement of the surfaces. Thus, if ψ_j are the gluing maps, we have

$$\pi_1(M_i) = \langle G, t_1, \ldots, t_k; t_j x t_j^{-1} = \psi_j(x) \forall x \in P_j \rangle$$

Now, since P_j's map to the trivial group under ϕ_i, and G surjects onto $\pi_1(\partial N)$, $\pi_1(M_i)$ surjects onto $\langle \pi_1(\partial N), t_1, \ldots, t_n \rangle$, the group obtained by adding k generators to $\pi_1(\partial N)$. But, M_i is obtained by using n 2-handles and $n-k$ 3-handles. Thus, as in Casson's theorem, $\pi_1(M_i)$ is killed by adding $n-k$ generators and n relators. As in proposition 1.1, this implies that $\pi_1(\partial N)$ is killed by adding n generators and n relators. □

Theorem 4.2. *There is a collection of immersed, not necessarily disjoint surfaces, carried by Σ, whose fundamental groups map to the trivial group under ϕ_i.*

Proof. Observe that half the curves on the surface do map to homotopically trivial curves. Further, if the Siefert surfaces have groups that map to the trivial group, then compressing the surfaces along these, we obtain surfaces representing homology with trivial π_1 images. Thus, it suffices to show that we obtain this condition for all surfaces at some time in the future.

Thus, we look at the surgeries that need to be performed at some stage far out in the future. These are along curves that must have died between $M_{i+n'}$ and M_{i+n}, but may be non-trivial in the M_j in which they live. Here, we choose n' so that any 2-sphere after time $i+n'$ that bounds a 3-sphere does so after i, and then n by applying the lemma to time $i+n'$. We now use an analogue of lemma 2.4 for π_2 rather than π_1. Let γ be one of the curves in M_j where the surgery was performed. Then it bounds a disc in the 2-handle attached to it. Further, as it can be pulled back, say along an annulus, to time M_i, and then dies by M_{i+n}, it bounds another disc consisting of the annulus and the disc by which it dies. These discs together bound a 2-sphere, which must bound a 3-ball if ∂N is irreducible, or more generally after modifying by a sphere in a collar $\partial N \times I$ contained between $M_{i+n'}$ and M_{i+n}.

Thus, we have a 2-sphere which gives a trivial element of π_2 of the 4-manifold between times i and $i+m$,for some m. We shall call this manifold the collar. Thus, it must also be a trivial element in the homology of the universal cover of this manifold. Thus, we have a finite number of 3-handles (with multiplicity) whose boundary consists of this 2-handle together with a surface that intersects each 2-handle algebraically 0 times in the cover.

We observe some basic facts. The boundary of a 3-handle consists of a planar surface in M_i together with some discs attached along the seams, with the coefficient of a 2-handle in the boundary being that of the seam in the planar surface. Thus, we may identify 2-handles with loci of surgery and 3-handles with the planar surfaces. Further, the coefficient of a 2-handle vanishing in a 3-cycle is equivalent to the boundaries of the surfaces gluing together to close up at the corresponding surgery locus.

Thus, the above 3-handles give a surface in M_i with boundary the curve with which we started plus some curves along which surgery is performed by time $i+n$. Further, as this is in fact a cycle in the universal cover, the surface lifts to one, with a single boundary component being a curve not surgered by time $i+n$, in the universal cover of the collar.

As the fundamental group of the collar maps to $\pi_1(\partial N)$, the image of the surface group in $\pi_1(\partial N)$ is trivial.

Thus, after surgering along the curves up to the beginning of the collar, we do have the required Seifert surfaces to compress to get embedded surfaces with trivial $\pi_1(\partial N)$ image. □

Next, we shall attempt to eliminate intersections between the surfaces. To do this, we find appropriate Siefert surfaces along intersection loci and compress along these.

First, we make an observation that shall be used in the following. If two surfaces S_1 and S_2 intersect transversally in a family of curves, this may or may not bound a subsurface in S_2. If it does not bound a surface, in S_2, then we can connect the two sides of a curve of intersection by an arc in S_2 whose interior is disjoint from the curves of intersection, which gives a closed curve. This curve intersects S_1 exactly once, implying that both S_1 and this curve are non-trivial in homology, and in particular S_1 is in a homology class that gives 1 upon evaluation with respect to a particular cohomology class.

Since, we have immersed surfaces of the required form, the obstruction we encounter is in making these surfaces disjoint at some finite stage. Note that for a finite decomposition, we do indeed have disjoint surfaces representing the homology after finitely many surgeries, since we in fact have a family of such spheres.

Proposition 4.3. *There is a 2-complex Σ with intersections along double-curves, coming from a handle-decomposition as above, where all the seams are trivial in homology, but which does not carry disjoint, embedded surfaces representing all of the homology.*

Proof. For the first stage, take two surfaces of genus 2, and let them intersect trasversely along two curves (which we shall call seams) that are disjoint and homologically independent in each surface. Next, take as Seifert surfaces for these curves once punctured surfaces of genus 2 intersecting in a similar manner, and glue their boundary to the abovementioned curves of intersection. Repeat this process to obtain the complex.

At the first stage, we cannot have embedded, disjoint surfaces representing the homology as the cup product of the surfaces is non-trivial. As the surfaces are compact, we must terminate at some finite stage. We shall prove that if we can have disjoint surfaces at the stage $k+1$, then we do at stage k. This will suffice to give the contradiction.

Now, we know the complex cannot be embedded in the first stage. Suppose we did have disjoint embedded surface S_1 and S_2 at stage $k+1$. Since these form a basis for the homology, they contain curves on them that are the seams at the first stage with algerbraically non-zero multiplicity, i.e., the collection of curves representing the seam is not homologically trivial in the intersection of the first stage with the surface. Further, some copy of the first seam must bound a subsurface S_i' in each of the surfaces, for otherwise the surface contains a curve dual to the seam. For, the cup product of such a dual curve with the homology class of the other surface is non-trivial, hence it must intersect the other surface, contradicting the hypothesis that the surfaces are disjoint. Similarly, at the other seam we get surfaces S_i''.

By deleting the first stage surfaces and capping off the first stage seams by attaching discs, we get a complex exactly as before with the $(j+1)$th stage having become the jth stage. Further, the S_1' and S_2'' now give disjoint, embedded surfaces representing the homology that are supported by stages up to k. This suffices as above to complete the induction argument.

It is easy to see that this complex can in fact be embedded in $S^2 \times S^2$. Figure 4.1 shows a construction of tori with one curve of intersection. Here we have used the notation of Kirby calculus, with the thickened curves being an unlink along each component of which 0-frame surgery has been performed. It is easy to see that the same construction can give surfaces of genus 2 intersecting in 2 curves. Since the curves of intersection are unknots, after surgery they bound spheres. Further, it is easy to see by cutting along these that we get $S^2 \times S^2$ after surgery as well. Repeating this process, we obtain our embedding.

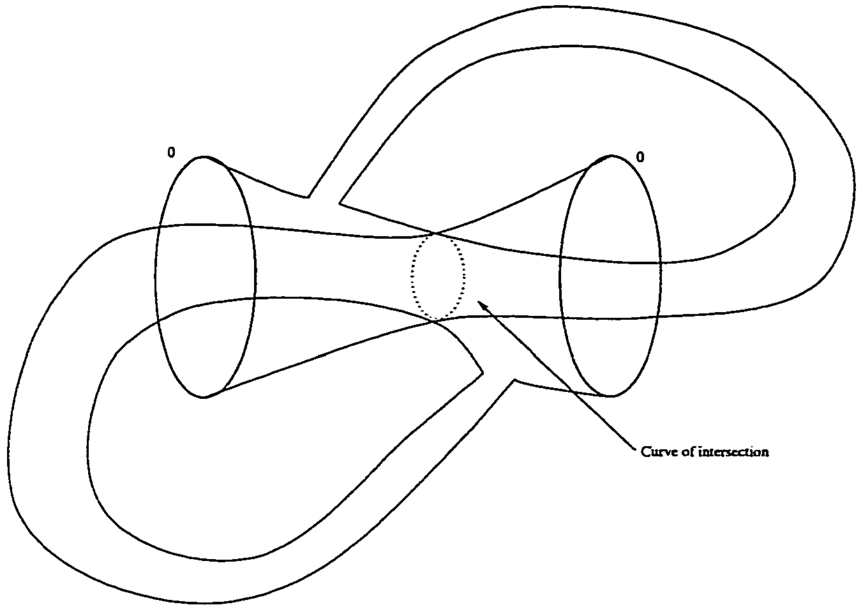

Figure 4.1:

Thus we have an infinite handle-decomposition satisfying our hypothesis for which this 2-complex is Σ. □

Chapter 5 A wild example.

We shall construct an example of an open, contractible 4-manifold that is not tame, and that has a handle-decomposition without 1-handles.

Theorem 5.1. *There is a proper handle-decomposition of an open, contractible 4-manifold N such that N is not the interior of a compact 4-manifold. In particular N does not have a finite handle-decomposition.*

Proof. We shall take a variant of the example in the last section. Namely, we take three surfaces of genus 2 at each stage, starting with $S^2 \times S^2 \times S^2$, and make each pair intersect in a single curve. Attach 2-handles along the seams and 3-handles along the surfaces as before.

Observe that the 3-manifold obtained at each stage is $S^2 \times S^2 \times S^2$. Thus, we may iterate this process to get N. Further, the fundamental group of the 3-manifold at any stage is the free group generated by the commutators of the generators of the previous stage, since the curves dual to the intersection curves are the generators of the previous stage. This in particuar implies that the fundamental group at each stage injects into that of the previous stage.

Suppose N is in fact tame. Then, we may use the results of the previous chapters. Now, by construction no curve dies as only trivial relations have been added. Thus every element in kernel(ϕ_i) must fail to persist by some uniform time. In particular, the image of the group after that time in the present (curves that persist beyond that time) must inject under ϕ_i. But we know that it also surjects. Thus, we must have an isomorphism.

Thus, there is a unique element mapping onto each element of $\pi_1(\partial N)$. Thus, this element must persist till infinity as we have a surjection at all times. On the other hand, since the limit of the lower central series of the free group is trivial, no non-trivial element persists. This gives a contradiction unless $\pi_1(\partial N)$ is trivial.

But there are non-trivial elements that do persist beyond any give time. As no element dies, we again get a contradiction.

□

Chapter 6 Further obstructions from Gauge theory

To explore some of the subtleties that one might encounter in trying to construct a handle decomposition without 1-handles for a contractible manifold, given one for its interior, we consider a more general situation. We shall consider sequences of 3-manifolds M_i that begin with S^3. As before, we require that each manifold comes from the previous one by 0-frame surgery about a homologically trivial curve or by splitting along a non-separating S^2 and capping off. Also, we require degree-one maps f_i to a common manifold N, related as before. We shall say that the sequence limits to N if 'any curve that persists dies' as in lemma 2.4.

In this situation, our main question generalises to a 'relative version,' namely, given any such sequence $\{M_i\}$, with M_k an element in the sequence, is there a finite sequence that agrees up to M_k with the old sequence and whose final term is N?

We shall show that there is an obstruction to completing certain sequences to finite sequences when $N = S^3$. I do not know whether there are infinite sequences limiting to N in this case.

Let $\mathfrak{P}$ denote the Poincaré homology sphere. Observe that we cannot pass from this to S^3 by 0-frame surgery about homologically trivial curves and capping-off non-separating spheres. For, if we could, $\mathfrak{P}$ would bound a manifold with $H^2 = \oplus_k \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, which is impossible as $\mathfrak{P}$ has Rochlin invariant 1. On the other hand, for the same reason, $\mathfrak{P}$ cannot be part of any sequence of the above form.

Using Donaldson's theorem [1], we have a similar result for the connected sum $\mathfrak{P}\#\mathfrak{P}$ of $\mathfrak{P}$ with itself. The main part of the proof of this lemma was communicated to me by R. Gompf.

Lemma 6.1. *One cannot pass from $\mathfrak{P}\#\mathfrak{P}$ to S^3 by 0-frame surgery along homologically trivial curves and capping off non-separating S^2's.*

Proof. If we did have such a sequence of surgeries, then $\mathfrak{P}\#\mathfrak{P}$ bounds a 4-manifold M_1 with $H^2 = \oplus_k \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, with a half-basis formed by embedded spheres. Now glue this to a manifold with form $E_8 \oplus E_8$ which is bounded by $\mathfrak{P}\#\mathfrak{P}$ to get M.

We can surger out the disjoint family of S^2's from M to get a 4-manifold with form $E_8 \oplus E_8$ and trivial H_1. This contradicts Donaldson's theorem. □

We still do not have a sequence as claimed. For, Cassson's argument shows that $\mathfrak{P}\#\mathfrak{P}$ cannot be part of a sequence. To obtain such a sequence, we shall construct a manifold M that can be obtained by 0-frame surgery on algebraically unlinked 2-handles from each of S^3 and $\mathfrak{P}\#\mathfrak{P}$. Thus, M is part of a sequence. On the other hand, if we had a sequence starting at M that terminated at S^3, then we would have one starting at $\mathfrak{P}\#\mathfrak{P}$, which contradicts the above lemma.

To construct M, take a contractible 4-manifold K that bounds $\mathfrak{P}\#\mathfrak{P}$. By Freedman's theorem [2], this exists, and can moreover be smoothed after taking connected sums with sufficiently many copies of $S^2 \times S^2$. Take a handle decomposition of M. This may include 1-handles, but these must be boundaries of 2-handles. Hence, by handle-slides, we can ensure that each 1-handle is, at the homological level, a boundary of a 2-handle and is not part of the boundary of any other 2-handle. Replacing the 1-handle by a 2-handle does not change the boundary, and changes $H^2(M)$ to $H^2(M) \oplus (\oplus_k \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$. We do this dually with 3-handles too. Sliding 2-handles over the new ones, we can ensure that the attaching maps of the 2-handles having the same algebraic linking (and framing) structure as a disjoint union of Hopf links.

Now take M obtained from S^3 by attaching half the links, so that these are pairwise algebraically unlinked. The manifold M has the required properties.

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