Sun-dual characterizations of the Translation Group of $\mathbb{R}$

Thesis by

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Abstract

Let E be a Banach space. The mapping $t \to T(t)$ of $\mathbb{R}$ (real numbers) into $\mathcal{L}_b(E)$, the Banach algebra of all bounded linear operators on E, is called a *strongly continuous group* or a C_0-group, if $G = \{T(t) : t \in \mathbb{R}\}$ defines a group representation of $(\mathbb{R}, +)$ into the multiplicative group of $\mathcal{L}_b(E)$, and if $\forall f \in E$,

$$\lim_{t \to 0} \|T(t)f - f\| = 0.$$

For example, if $E = C_0(\mathbb{R})$, the function space which consists of all continuous, complex functions that vanish at infinity, then $(\forall t \in \mathbb{R})$ $(\forall f \in C_0(\mathbb{R}))$, the function $T(t)f(x) = f(x+t)$, $x \in \mathbb{R}$, defines a strongly continuous group, since each $f \in E$ is uniformly continuous; this group is called the *translation group*. If we now consider $E = B(\mathbb{R})$, the space of bounded, continuous complex functions on $\mathbb{R}$, then although the translation group on E is not strongly continuous, it is strongly continuous on the subspace $BUC(\mathbb{R})$ of E, which consists of bounded , uniformly continuous functions. $BUC(\mathbb{R})$ is the largest subspace of E on which the translation group is strongly continuous.

The *adjoint family* of a C_0-group defined on a Banach space E, need not be strongly continuous on the Banach dual E^* of E. Let $E^{\odot}$ (pronounced E-sun) be the largest linear subspace of E^* relative to which the adjoint family is a C_0-group:

$$E^{\odot} = \{\mu \in E^* : \lim_{t \to 0} \|T^*(t)\mu - \mu\| = 0\}.$$

$E^{\odot}$ is called the *sun-dual* or *sun-space* of E. If $E = C_0(\mathbb{R})$, then it follows from a well-known result of A. Plessner that $E^{\odot} = L^1(\mathbb{R})$ ([Ple]). This research paper contains a characterization of the sun-dual of $BUC(\mathbb{R})$ and of the subspace $WAP(\mathbb{R})$ of $BUC(\mathbb{R})$, which consists of weakly almost periodic functions on $\mathbb{R}$.

Contents

Chapter 1 Introduction

Let $\mathbb{R}$ be the field of real numbers. Suppose $G = \{T(t) : t \in \mathbb{R}\}$ is a family of maps such that for each $t \in \mathbb{R}$, $T(t)$ translates $\mathbb{R}$ by t, i.e., $T(t)x = x + t$, $x \in \mathbb{R}$. Then G is called a *regular representation* of the additive group of $\mathbb{R}$.

Suppose X is a real or complex linear function space which has the property that if $f(x) \in X$, then $f(x+t) \in X$. Then we can extend the family G to a group of operators defined on X in such a way that the action of G on X is given by translation:

$$\forall f \in X, \quad T(t)f(x) = f(x+t).$$

Note that G now has the following properties:

(1) $T(0) = I$, where I is the identity operator,

(2) $\forall s, t \in \mathbb{R}$, $T(s+t) = T(s)\,T(t) = T(t)\,T(s)$.

Let $\mathcal{L}(X)$ be the group of all linear operators that act on X. Since $\mathbb{R}$ is a group, then $\forall t \in \mathbb{R}$, the inverse $(T(t))^{-1}$ exists in $\mathcal{L}(X)$ and

(3) $(T(t))^{-1} = T(-t)$.

Observe that G is a group representation of $(\mathbb{R}, +)$ into the multiplicative group of $\mathcal{L}(X)$.

Suppose that $(X, \|\cdot\|)$ is now a Banach space of functions defined on $\mathbb{R}$, and that G is now a family of bounded, linear operators on X. If in addition to properties (1-3) above, G also satisfies,

(4) $\forall f \in X$, $\lim_{t\to 0} \|T(t)f - f\| = 0$,

then G is called a *strongly continuous translation group* or a *translation C_0-group*. For example, if $X = C_0(\mathbb{R})$, the space of continuous functions which vanish at infinity, and the norm on X is the supremum norm, then

$$\|T(t)f - f\| = \sup_{x \in \mathbb{R}} |f(x+t) - f(x)| \to 0 \text{ as } t \to 0, \tag{1.1}$$

since each $f \in C_0(\mathbb{R})$ is uniformly continuous.

Let $B(\mathbb{R})$ be the space of bounded, continuous, complex-valued functions on $\mathbb{R}$. The function $g(x) = \sin(e^x)$ is a bounded, continuous function on $\mathbb{R}$, but it is not uniformly continuous. For if $x_k = \log(\pi k)$ and $x'_k = \log(\pi k + \frac{\pi}{2})$, then $|x_k - x'_k| \to 0$, but $|g(x_k) - g(x'_k)| = 1$. It follows that the quantity $\|T(t)g - g\|$, does not tend to zero as $t \to 0$.

We can conclude from the above example that a function $f \in B(\mathbb{R})$ satisfies properties (1-4) if and only if f is uniformly continuous. Consider the subspace $BUC(\mathbb{R})$ of $B(\mathbb{R})$ consisting of all bounded, uniformly continuous, complex-valued functions on $\mathbb{R}$; then $B(\mathbb{R}) \supseteq BUC(\mathbb{R}) \supseteq C_0(\mathbb{R})$; moreover, $BUC(\mathbb{R})$ is the largest subspace of $B(\mathbb{R})$ for which the translation group is a C_0-group.

The adjoint family of a C_0-group defined on a Banach space X, need not be a C_0-group on the dual space X^* of X. To see this, consider again the space $C_0(\mathbb{R})$. The dual $C_0^*(\mathbb{R})$ of $C_0(\mathbb{R})$ is identified with the space $M(\mathbb{R})$, which consists of all bounded, regular, complex Borel measures on $\mathbb{R}$, with norm equal to total variation over $\mathbb{R}$ (Riesz representation theorem). If G is the translation group and δ_x is the point measure at $x \in \mathbb{R}$, then $\delta_x \in C_0^*(\mathbb{R})$. The action of the dual family $\{T^*(t) : t \in \mathbb{R}\}$ on δ_x is given by the dual relation:

$$\langle f, \delta_{x+t} \rangle = \langle T(t)f, \delta_x \rangle = \langle f, T^*(t)\delta_x \rangle, \quad f \in C_0(\mathbb{R}),$$

so $T^*(t)\delta_x = \delta_{x+t}$; from whence it follows,

$$\|T^*(t)\delta_x - \delta_x\| = 2 \quad \text{when } t \neq 0.$$

Let $X^\odot$ (pronounced X-sun) be the collection of all bounded, linear functionals in X^* with the following property:

$$X^{\odot} = \{\mu \in X^* : \lim_{t \to 0} \|T^*(t)\mu - \mu\| = 0\}.$$

Then $X^{\odot}$ is a closed, linear subspace of X^* ([Phi]). For example, consider the space $C_0(\mathbb{R})$; it is well-known that $C_0^{\odot}(\mathbb{R})$ is identified with the space $L^1(\mathbb{R})$. That is, $\mu \in C_0^*(\mathbb{R})$ exhibits 'the strong continuity property' if and only if μ is absolutely continuous with respect to Lebesgue measure ([Ple]). Note that the 'sufficiency portion' of this result follows from the fact that Lebesgue integrable functions are L^1-mean-continuous, i.e., if $f \in L^1(\mathbb{R})$, then

$$\lim_{t \to 0} \int_{-\infty}^{\infty} |f(x+t) - f(x)| \, dx = 0.$$

The primary focus of this work is to characterize $BUC^{\odot}(\mathbb{R})$; Chapters 2-4 and Chapters 6 and 8 are devoted to this objective. Chapter 2 is divided into two parts; the first part contains definitions and theorems in Banach lattice theory and the second part contains an application of this theory to the space $BUC(\mathbb{R})$. It is via lattice theory that we determine an important property of the norm on $BUC^*(\mathbb{R})$ that enables us to provide the first of three characterizations of $BUC^*(\mathbb{R})$:

$$BUC^*(\mathbb{R}) = BUC^{\odot}(\mathbb{R}) \oplus (BUC^{\odot}(\mathbb{R}))^d,$$

where $(BUC^{\odot}(\mathbb{R}))^d$ is the disjoint complement of $BUC^{\odot}(\mathbb{R})$ (Definition (v)b). We utilize the above representation in the proof of our main result, Theorem A. In Chapter 3 we employ the theory of commutative Banach algebras to determine a second characterization of $BUC^*(\mathbb{R})$: Let Ω be the maximal ideal space of $BUC(\mathbb{R})$ and let $C^*(\Omega)$ be the Banach dual of the space of continuous functions defined on Ω; then $C^*(\Omega)$ is identified with the space $M(\Omega)$, which consists of bounded, regular, Borel measures on Ω, and

$$BUC^*(\mathbb{R}) = C^*(\Omega) = M(\Omega).$$

Chapter 4 is pivotal; we not only give a third characterization of $BUC^*(\mathbb{R})$:

$$BUC^*(\mathbb{R}) = C_0^\perp(\mathbb{R}) \oplus C_0^*(\mathbb{R}),$$

but we use this characterization to show the existence of nonzero linear functionals $\phi \in BUC^*(\mathbb{R})$ that vanish on $C_0(\mathbb{R})$, and which have the property:

$$\forall t \in \mathbb{R}, \quad T^*(t)\phi = \phi. \tag{1.2}$$

We call any ϕ with property (1.2) *translation invariant* and we denote Fix as the collection of all such ϕ. Note that $Fix \subseteq BUC^\odot(\mathbb{R})$. The end of Chapter 4, specifically Lemma 4.7, contains a 'preliminary description' of $BUC^\odot(\mathbb{R})$ in terms of $\{Fix\}$, the band generated by Fix (Definition (v)b and Remark 4.8), and the sun- dual $C_0^\odot(\mathbb{R})$:

$$BUC^\odot(\mathbb{R}) = (C_0^\odot(\mathbb{R}) \oplus \{Fix\}) \oplus (C_0^\odot(\mathbb{R}) \oplus \{Fix\})^d \cap BUC^\odot(\mathbb{R}).$$

This Lemma is crucial to our work; thus a great deal of effort has gone into proving it. In Chapter 5 we digress a bit, by giving examples of elements in Fix. Chapter 7 may appear to the reader to be just as disconnected from our primary focus as Chapter 5. Afterall, we define a 'convolution operation' on $M(\Omega)$ which renders it a commutative Banach algebra with unit, and we show $M(\Omega)$, now regarded as a 'measure algebra,' contains $BUC^\odot(\mathbb{R})$ as a closed algebraic ideal, but $M(\Omega) \neq BUC^\odot(\mathbb{R})$. Although none of this information is used in proving Theorem A, all of the ideas in this chapter will be incorporated in the proof of Theorem B. In Chapters 6 and 8 we 'gather the tools' to prove $(C_0^\odot(\mathbb{R}) \oplus \{Fix\})^d \cap BUC^\odot(\mathbb{R}) = \{0\}$. Once we have shown this, we conclude $BUC^\odot(\mathbb{R}) = C_0^\odot(\mathbb{R}) \oplus \{Fix\}$ (Theorem A).

In the final chapter we prove Theorem B—our 'peripheral goal'. We obtain a representation of the sun-dual, relative to the translation group G, of the *space of weakly almost periodic functions on* $\mathbb{R}$ ($WAP(\mathbb{R})$). To do so, we rely on several theorems and definitions due to W.F. Eberlein ([Eb1] and [Eb2]); yet unlike Eberlein, we avoid any discussions of $WAP(\mathbb{R})$ in the context of abstract ergodic theory.

There are several points that we make before we prove Theorem B. The first is that

$$C_0(\mathbb{R}) \subseteq WAP(\mathbb{R}) \subseteq BUC(\mathbb{R}).$$

And the second is that there exists a *unique*, translation invariant, linear functional on $WAP(\mathbb{R})$; we designate this functional U_∞. If

$$F = \{m \in Fix : \|m\| = 1\},$$

then it turns out that the restriction of the elements in F to $WAP(\mathbb{R})$ is U_∞. From this and other observations (for complete details, see Chapter 9), we obtain a representation of $WAP^{\odot}(\mathbb{R})$, using the same ideas that we used to obtain a characterization of $BUC^{\odot}(\mathbb{R})$.

Chapter 2 Vector lattice terminology

2.1 Definitions

The following definitions were taken from [LuZ] and [Zan].

i) A vector lattice S is called *Archimedean* if $(\forall\ 0 \le u, v \in S)$ and $(\forall n,\ n = 1, 2, \dots)$, $nv \le u \Rightarrow v = 0$.

ii) a) The indexed subset $\{x_\alpha : \alpha \in \{\alpha\}\}$ in S is said to be *directed upwards*, if $(\forall \alpha, \beta \in \{\alpha\})$ $(\exists \alpha_0 \in \{\alpha\})$ such that $x_\alpha \le x_{\alpha_0}$ and $x_\beta \le x_{\alpha_0}$. This is denoted by $x_\alpha \uparrow$. If $x_\alpha \uparrow$ and $x = \sup_\alpha x_\alpha$, then we write $x_\alpha \uparrow x$.

b) S has *order continuous norm* ρ if $0 \le x_\alpha \uparrow x \Rightarrow \rho(x_\alpha - x) \to 0$.

c) S has *σ-order continuous norm* if the definition in (b) holds for increasing sequences.

iii) a) S is *Dedekind complete* if every nonempty subset which is bounded from above (below) has a supremum (infimum).

b) S is *σ-Dedekind complete* if every nonempty at most countable subset which is bounded from above has a supremum.

c) S is *order separable* if every nonempty subset D possessing a supremum d contains an at most countable subset which possesses d as a supremum.

d) S is *super Dedekind complete* if it is Dedekind complete and order separable.

iv) An *order ideal* U in S is a linear subspace with the property that if $x \in U$ and $y \in S$, then $|y| \le |x|$ implies that $y \in U$. Observe that every order ideal U in S has the property : $x \in U \Rightarrow |x| \in U$.

v) a) A *band is an order ideal* B such that if $u = \sup\{D : D \subseteq B\}$ exists, then $u \in B$. Equivalently, B is a band if and only if every positive, upward directed system $(u_\tau) \subseteq B$ has the property that if $u = \sup_\tau(u_\tau)$ exists in S, then $u \in B$.

b) $D^d = \{f \in S : \forall g \in D, \inf(|f|, |g|) = 0\}$. D^d is called the *disjoint complement* of the set D.

c) Any band B in S such that $S = B \oplus B^d$, is called a *projection band.*

d) The *band generated by a set* $D \subseteq S$, which we denote $\{D\}$, is the intersection of all bands that contain D.

2.2 Properties of vector lattices

Definition 2.1. *For the remainder of this chapter, (X, ρ) is a Banach lattice (i.e., X is a Banach space with the property that if $x, y \in X$ and $|x| \leq |y|$, then $\|x\| \leq \|y\|$). The indexed subset $\{u_\tau : \tau \in \{\tau\}\}$ in X is said to be ρ-Cauchy if $(\forall \epsilon > 0)$ $(\exists \tau_0 \in \{\tau\})$ such that $\rho(u_{\tau_1} - u_{\tau_2}) < \epsilon$ whenever $u_{\tau_1} \geq u_{\tau_0}$ and $u_{\tau_2} \geq u_{\tau_0}$ ([Zan]).*

Definition 2.2. *(X, ρ) satisfies the ρ-Cauchy condition if the following equivalent conditions hold:*

i) Every order bounded increasing sequence in X is ρ-Cauchy.

ii) Every order bounded upwards directed system in X is ρ-Cauchy.

Theorem 2.3. *The following conditions on (X, ρ) are equivalent:*

i) ρ is order continuous.

ii) ρ is σ-order continuous and X is Dedekind σ-complete.

In fact, X is super Dedekind complete if these equivalent conditions hold ([Zan]).

Theorem 2.4. *The norm ρ is order continuous if and only if ρ is σ-order continuous and X satisfies the ρ-Cauchy condition ([Zan]).*

Theorem 2.5. *If S is a Dedekind complete vector lattice, then the following hold:*

i) If A_1 and A_2 are bands and $A_1 \perp A_2$, then $A_1 \oplus A_2$ is a band in S.

ii) Every band B in S is a projection band, i.e., $S = B \oplus B^d$ ([LuZ]).

It follows from Theorem 2.3 and the definition of super Dedekind completeness, that if X has order continuous norm ρ , then X is Dedekind complete. Thus, by Theorem 2.5, if B is a *band* in X, then

$$X = B \oplus B^d. \tag{2.1}$$

Let $BUC^*(\mathbb{R})$ be the Banach dual of $BUC(\mathbb{R})$. We claim that the Banach lattice $(BUC^*(\mathbb{R}), \|\cdot\|_*)$ has *order continuous norm.* To see this, first observe that the space $(BUC(\mathbb{R}), \|\cdot\|)$ is an abstract **M**- space, i.e., $BUC(\mathbb{R})$ is a Banach lattice with the property that $\|f+g\| = \max\{\|f\|, \|g\|\}$ whenever $\inf(f,g) = 0$. That $BUC(\mathbb{R})$ is an **M**-space follows from the expressions $2\sup(f,g) = (f+g) + |f-g|$ and $2\inf(f,g) = (f+g) - |f-g|$, and from the fact that $BUC(\mathbb{R})$ is equipped with the supremum norm. Therefore, $BUC^*(\mathbb{R})$ is an abstract **L**-space ([Kal]), which implies that the norm on $BUC^*(\mathbb{R})$ has the following property: $(\forall\, 0 \le \mu, \nu \in BUC^*(\mathbb{R}))$,

$$\|\mu + \nu\|_* = \|\mu\|_* + \|\nu\|_*. \tag{2.2}$$

We shall show $BUC^*(\mathbb{R})$ has σ-order continuous norm and satisfies the ρ-Cauchy condition; we can then conclude that it has order continuous norm (Theorem 2.4).

To show $(BUC^*(\mathbb{R}), \|\cdot\|_*)$ satisfies the ρ-Cauchy condition (with $\rho = \|\cdot\|_*$), suppose $0 \le \phi_n, \phi \in BUC^*(\mathbb{R})$ such that $0 \le \phi_n \uparrow \phi$; or equivalently, given $0 \le f \in BUC(\mathbb{R})$, suppose $0 \le \phi_n(f) \uparrow \phi(f)$. Then ϕ_n is a norm-Cauchy sequence. Indeed, since $BUC^*(\mathbb{R})$ is a Banach lattice, then $0 \le \|\phi_n\|_* \uparrow \|\phi\|_*$. This implies that the sequence $(\|\phi_n\|_*)$ is an increasing, bounded sequence of real numbers; thus, $\lim_n(\|\phi_n\|_*)$ exists. If $n \ge m$, then

$$\|\phi_n\|_* = \|\phi_m + \phi_n - \phi_m\|_* = \|\phi_m\|_* + \|\phi_n - \phi_m\|_*,$$

by relation (2.2), so (ϕ_n) is Cauchy and the ρ-Cauchy condition is satisfied in $BUC^*(\mathbb{R})$.

To show that $BUC^*(\mathbb{R})$ is σ-order continuous, note that since (ϕ_n) is a Cauchy sequence and $BUC^*(\mathbb{R})$ is a Banach space, then for some $0 \le \phi_0 \in BUC^*(\mathbb{R})$,

$$\lim_{n\to\infty} \|\phi_n - \phi_0\|_* = 0.$$

Thus, for any $0 \le f \in L$, the following inequality holds:

$$|\phi_n(f) - \phi_0(f)| \le \|\phi_n - \phi_0\|_* \|f\|.$$

But this implies that $0 \le \phi_n \uparrow \phi_0$, and since we also have $0 \le \phi_n \uparrow \phi$, then $\phi = \phi_0$. Therefore, $\|\cdot\|_*$ is σ-order continuous.

Let $G = \{T(t) : t \in \mathbb{R}\}$ be the *translation* C_0*-group* defined on $BUC(\mathbb{R})$. We establish important properties of G (and its dual G^*) that we will find useful in later chapters.

Definition 2.6. *Let V and W be vector lattices.*

i) A linear map π of V onto W is called a lattice isomorphism, if π is a bijection and if $\forall f, g \in V$, $\inf(f,g) = 0 \Rightarrow \inf(\pi f, \pi g) = 0$. Equivalently, π is a lattice isomorphism if it is a bijection and if $\pi \sup(f,g) = \sup(\pi f, \pi g)$.

ii) π is said to be positive if $\pi f \ge 0$ whenever $f \ge 0$ ([LuZ]).

It follows from Definition 2.6, that every *lattice homomorphism* is positive. For if $f \ge 0$, then we can write $0 = \inf(f, 0)$, which implies that $0 = \inf(\pi f, 0)$ or that $\pi f \ge 0$.

Theorem 2.7. *The 1-1 positive linear mapping π of V onto W is a lattice isomorphism if and only if π^{-1} is positive ([LuZ]).*

Recall that for each $t \in \mathbb{R}$, the operator $T(t) \in G$ is defined on $BUC(\mathbb{R})$ by translation, i.e.,

$$T(t)f(x) = f(x+t), \quad x \in \mathbb{R}.$$

Then by Definition 2.6, G is a family of *lattice isomorphisms*. Therefore, so is the inverse family $\{T(t)^{-1}\}$. And since the dual of any positive operator is itself positive, then it follows by

$$(T(t)^{-1})^* = (T^*(t))^{-1},$$

and by Theorem 2.7, that the dual family G^* is also a family of *lattice isomorphisms.*

We end this chapter by stating the following result due to [Pag].

Theorem 2.8. *Suppose E is a Banach lattice with the property that its Banach dual, E^*, has order continuous norm ρ. If $H = \{U(t) : t \geq 0\}$ is a C_0-semigroup of positive linear operators on E, then the sun-dual $E^{\odot}$ of E is a band in E^*.*

Since $BUC^*(\mathbb{R})$ has *order continuous norm* $\|\cdot\|_*$, and since the group G is a family of lattice isomorphisms, then by Theorem 2.8, $BUC^{\odot}(\mathbb{R})$ is a *band* in $BUC^*(\mathbb{R})$. Thus, by equation (2.1):

$$BUC^*(\mathbb{R}) = BUC^{\odot}(\mathbb{R}) \oplus (BUC^{\odot}(\mathbb{R}))^d. \tag{2.3}$$

Chapter 3 A characterization of the Banach dual of BUC($\mathbb{R}$)

To obtain a characterization of $BUC^*(\mathbb{R})$ we shall employ the Gelfand theory of commutative Banach algebras.

Let A be a commutative Banach algebra with *unit* e (i.e., $e(a) = 1$, $a \in A$), and let $\mathcal{M}$ be the set of all maximal ideals in A. Recall that if $M \in \mathcal{M}$, then the Banach algebra A/M is a division algebra that equals $\mathbb{C}$, the set of all complex numbers ([Ru]). If $a \in A$, we denote by $\hat{a}$ the complex function defined on $\mathcal{M}$ by

$$\hat{a}(M) = a(M) = a + M,$$

and we put $\hat{A} = \{\hat{a} : a \in A\}$. Clearly, $\hat{A} \subseteq C(\mathcal{M})$, where $C(\mathcal{M})$ is the space of all continuous complex functions on $\mathcal{M}$. The mapping $a \mapsto \hat{a}$ is called the Gelfand mapping, and the weakest topology on $\mathcal{M}$ relative to which every function $\hat{a}$ is continuous is called the Gelfand topology ([Ru]). We call the topological space $\mathcal{M}$ the *maximal ideal space.*

Recall that a *multiplicative functional* h on A is a nonzero element in the dual A^* of A, which satisfies:

$$h(ab) = h(a)\ h(b) \text{ and } h(e) = 1.$$

It is well-known that there is a 1-1 correspondence between the elements of $\mathcal{M}$ and the set of all multiplicative functionals on A. This correspondence enables us to regard $\mathcal{M}$ as a subset of A^*. Moreover, since each multiplicative functional has norm 1, then $\mathcal{M}$ is also a subset of the unit ball B_1 in A^*. Recall that B_1 is a compact Hausdorff space with respect to the weak*-topology, which is the weak topology generated by all functions F_a defined on B_1 by $F_a(h) = h(a)$. Observe that F_a restricted to $\mathcal{M}$ is precisely $\hat{a}$; that is,

$$F_a(h) = h(a) = \hat{a}(h) \quad (h \in \mathcal{M}).$$

Then the topology which $\mathcal{M}$ has as a *subset* of B_1 is exactly its Gelfand topology, so we can regard $\mathcal{M}$ as a *subspace* of B_1. In fact, it is a weak*-closed subspace of B_1.

Theorem 3.1. *$\mathcal{M}$ is a compact Hausdorff space ([Ru]).*

Definition 3.2. *Let A be a Banach algebra with unit.*

1. *A is called a B^*- algebra if it has an involution , i.e., if there exists a mapping of A into itself with the following properties:*
   - $(a+b)^* = a^* + b^*$
   - $(\alpha b)^* = \bar{\alpha} b^* \quad (\alpha \in \mathbb{C})$
   - $(ab)^* = b^* a^*$
   - $a^{**} = a$
   - $\|a^* a\| = \|a\|^2$

   (note that the last property implies that $\|a^*\| = \|a\|$).

2. *if A' is another B^*-algebra and if p is an isomorphism of A onto A', then p is called a $*$ -isomorphism if*

$$p(a^*) = (p(a))^*.$$

Note, if X is any topological space, then $C(X)$, the space of continuous complex functions on X, is a commutative B^*-algebra with an involution defined by complex conjugation: $f^*(x) = \bar{f}(x)$, $f \in C(X)$.

Theorem 3.3. *(Gelfand-Naimark). If A is a commutative B^*-algebra, then the Gelfand mapping $a \mapsto \hat{a}$ is an isometric $*$-isomorphism g of A onto $C(\mathcal{M})$. Thus,*

$$\overline{(\hat{a})} = (\hat{a})^* = g(a^*) = \widehat{(a^*)}, \tag{3.1}$$

or equivalently,

$$\overline{(h(a))} = h(a^*), \quad a \in A, \ h \in \mathcal{M} \quad ([GeN]).$$

One can easily verify that under the pointwise operations of addition, scalar multiplication, and multiplication, $BUC(\mathbb{R})$ equipped with sup-norm, is *a commutative Banach algebra* with unit e. The mapping $f \mapsto \bar{f}$, where $f \in BUC(\mathbb{R})$ and $\bar{f}$ is the complex conjugate of f, is easily seen to be an involution that satisfies Definition 3.2; thus, $BUC(\mathbb{R})$ is a B^*-algebra (in fact, it is a C^*-algebra).

Let Ω be *the maximal ideal space* of $BUC(\mathbb{R})$. By Theorem 3.3, $BUC(\mathbb{R})$ is isometrically $*$-isomorphic to $C(\Omega)$. Consequently, if $f \in BUC(\mathbb{R})$ is real-valued, then by equation (3.1),

$$\overline{(\hat{f})} = \widehat{(f^*)} = \widehat{(\bar{f})} = \hat{f},$$

so that the conjugate of $\hat{f}$ is equal to $\hat{f}$, i.e., $\hat{f}$ is real-valued. Hence, we identify the real-valued elements in $BUC(\mathbb{R})$ with the real-valued elements in $C(\Omega)$.

We can now identify $BUC^*(\mathbb{R})$ with the space $C^*(\Omega)$; recall that $C^*(\Omega)$ is identified with the space $M(\Omega)$, which consists of all bounded, regular, complex Borel measures on Ω, with norm equal to total variation (by a complex Borel measure we mean a complex-valued, countably additive function defined on the σ -algebra of Borel subsets).

Remark 3.4. Note that the space Ω contains $\mathbb{R}$ as a weak*-dense subspace. The embedding of $\mathbb{R}$ into Ω is defined by the point-evaluation map $x \mapsto g_x$ of $\mathbb{R}$ into $BUC^*(\mathbb{R})$, where g_x denotes the linear function $g_x(f) = f(x)$, $(f \in BUC(\mathbb{R}))$. Ω can now be identified as the weak*-closure of $\mathbb{R}$ in the weak*-compact unit ball of $BUC^*(\mathbb{R})$ (see Chapter 6 for complete details).

Chapter 4 Translation invariant linear functionals on BUC($\mathbb{R}$)

Let Ω be the maximal ideal space of $BUC(\mathbb{R})$. If $f \in C_0(\mathbb{R})$ and $g \in BUC(\mathbb{R})$, then $fg \in C_0(\mathbb{R})$; that is, $C_0(\mathbb{R})$ is a (norm-closed) algebraic ideal in $BUC(\mathbb{R})$. Consequently, $\exists\ 0 \leq h \in \Omega$, with $\|h\| = 1$, such that $C_0(\mathbb{R}) \subseteq ker h$ ([Ru]). Stated precisely, if

$$C_0^{\perp}(\mathbb{R}) = \{g \in BUC^*(\mathbb{R}) : g(C_0(\mathbb{R})) = 0\},$$

then $h \in C_0^{\perp}(\mathbb{R})$.

Recall that the dual $C_0^*(\mathbb{R})$ of $C_0(\mathbb{R})$ is the space $M(\mathbb{R})$, which consists of all bounded, regular, complex Borel measures on $\mathbb{R}$, with norm equal to total variation over $\mathbb{R}$. We assert that h can not be represented by any nonzero $\lambda \in M(\mathbb{R})$. For if this representation were possible, then by the *Riesz representation theorem*,

$$h(f) = \int_{\mathbb{R}} f\ d\lambda, \qquad f \in C_0(\mathbb{R}),$$

and $|\lambda|(\mathbb{R}) = \|h\|$. But $C_0(\mathbb{R}) \subseteq ker h$, so the fact that

$$h(f) = \int_{\mathbb{R}} f d\lambda = 0, \qquad f \in C_0(\mathbb{R}),$$

and the fact that λ is a countably additive set function, imply

$$0 = |\lambda|(\mathbb{R}) = \|h\| = 1.$$

Evidently, associated with this $h \in BUC^*(\mathbb{R})$, there is a measure $\mu \in C^*(\Omega)$ with the property that $supp(\mu) \subseteq (\Omega \setminus \mathbb{R})$ ($supp(\mu)$ is the support of μ).

Lemma 4.1. *$C_0^{\perp}(\mathbb{R})$ is a band in $BUC^*(\mathbb{R})$, so $BUC^*(\mathbb{R}) = C_0^{\perp}(\mathbb{R}) \oplus (C_0^{\perp}(\mathbb{R}))^d$. In fact, $(C_0^{\perp}(\mathbb{R}))^d = C_0^*(\mathbb{R})$ (all equalities are isometric isomorphisms); thus,*

$$BUC^*(\mathbb{R}) = C_0^{\perp}(\mathbb{R}) \oplus C_0^*(\mathbb{R}). \tag{4.1}$$

Proof.

In Chapter 2 we determined that $BUC^*(\mathbb{R})$ has order continuous norm; thus, $BUC^*(\mathbb{R})$ is Dedekind complete. It follows that every norm-closed order ideal in $BUC^*(\mathbb{R})$ is a *band* ([And]); consequently,

$$BUC^*(\mathbb{R}) = C_0^{\perp}(\mathbb{R}) \oplus (C_0^{\perp}(\mathbb{R}))^d \quad \text{(Theorem 2.5)}. \tag{4.2}$$

If M is a closed subspace of a Banach space E, then the dual M^* of M is given by

$$M^* = E^*/(M)^{\perp} \quad \text{([Ru])}.$$

Thus, $C_0^*(\mathbb{R}) = BUC^*(\mathbb{R})/(C_0^{\perp}(\mathbb{R}))$, so by (4.2), $(C_0^{\perp}(\mathbb{R}))^d = C_0^*(\mathbb{R})$. □

Definition 4.2. *Let $0 \neq \phi \in BUC^*(\mathbb{R})$ and let $G^* = \{T^*(t) : t \in \mathbb{R}\}$ be the dual family of the translation group G defined on $BUC(\mathbb{R})$. Then ϕ is said to be translation invariant if*

$$\forall t \in \mathbb{R}, \quad T^*(t)\phi = \phi.$$

Remark 4.3. We determined at the end of Chapter 2, that G and G^* are families of lattice isomorphisms. If ϕ is translation invariant, then it follows from

$$T^*(t)(\phi^+) = T^*(t)(\sup(\phi, 0)) = \sup(T^*(t)\phi, 0) = \sup(\phi, 0) = \phi^+,$$

that the *positive part* of ϕ, the *negative part* of ϕ, and the *variation* of ϕ are also translation invariant.

Lemma 4.4. *If* $0 \neq \phi \in BUC^*(\mathbb{R})$ *is translation invariant, then* $\phi(C_0(\mathbb{R})) = 0$.

Proof. Since $C_c(\mathbb{R})$, the space of continuous, complex functions with compact support, is dense in $C_0(\mathbb{R})$, it suffices to show that ϕ vanishes on $V([a,b])$, the space of continuous, complex functions which vanish outside of a closed, bounded interval $[a,b]$. We first establish the result for positive, translation invariant linear functionals; the result for complex ϕ will then follow from Remark 4.3 and the inequality

$$|\phi(f)| \leq |\phi|(|f|), \quad f \in BUC(\mathbb{R}) \quad ([\text{Zan}]).$$

Let $0 \leq f \in V[a,b]$ and $\phi \geq 0$. Choose $c > (b-a)$ so that the translates $f(\cdot + kc)$, $k = 0, 1, 2, \ldots, n$, have disjoint support. Then

$$0 \leq \langle \sum_{k=0}^{n} f(\cdot + kc), \phi \rangle = \langle \sum_{k=0}^{n} T(kc)f, \phi \rangle = \langle f, \sum_{k=0}^{n} T^*(kc)\phi \rangle = (n+1)\langle f, \phi \rangle;$$

thus,

$$0 \leq (n+1)\phi(f) \leq \|\phi\| \sup_{x \in \mathbb{R}} f(x).$$

Dividing the above inequality by $(n+1)$ and letting $n \to \infty$, shows that $\phi(f) = 0$.

For real-valued $f \in V[a,b]$, write: $f = f^+ - f^-$. Observe that $0 \leq f^+, f^- \in V[a,b]$ (since $V[a,b]$ is a *vector lattice*). Now repeat the argument above, with f^+ and f^- taking the place of f. □

Theorem 4.5. *(Markov-Kakutani) Let K be a nonempty, compact convex subset of a topological vector space X, and let $\mathcal{F}$ be a commuting family of continuous linear mappings defined on X such that $\forall T \in \mathcal{F}$, $T(K) \subseteq K$. Then $\exists p \in K$ such that $T(p) = p$ for each $T \in \mathcal{F}$ ([Mar], [Ka2]).*

Lemma 4.6. *There exist nonzero, translation invariant $\phi \in BUC^*(\mathbb{R})$.*

Proof. Consider the set

$$F = \{\phi \in BUC^*(\mathbb{R}) : \phi(C_0(\mathbb{R})) = 0, \phi(e) = 1, \|\phi\| = 1\}.$$

Then F is a nonempty, convex, weak*-compact subset of the unit ball in $BUC^*(\mathbb{R})$. Furthermore, if $\phi \in F$ and $f \in BUC(\mathbb{R})$, then $\langle f, T^*(t)\phi\rangle = \langle T(t)f, \phi\rangle$ shows,

- $(\forall f \in C_0(\mathbb{R}))\ \ (\forall t \in \mathbb{R}),\ \ T^*(t)\phi(f) = 0,$
- $T^*(t)\phi(e) = 1$ and $\|T^*(t)\phi\| = 1.$

Thus, $\forall t \in \mathbb{R}$: $T^*(t)(F) \subseteq F$. And since the dual family $G^* = \{T^*(t) : t \in \mathbb{R}\}$ is a commuting family of weak*-continuous linear operators, then we can apply Theorem 4.5, with $\mathcal{F} = G^*$, to conclude that F has a fixed point. □

We shall denote Fix as the collection of all *translation invariant* $\phi \in BUC^*(\mathbb{R})$. Observe that Fix is a norm-closed, vector sublattice of $BUC^*(\mathbb{R})$, and

$$BUC^{\odot}(\mathbb{R}) \supseteq Fix.$$

Lemma 4.7. *Suppose $\{Fix\}$ is the band generated by the set Fix and $M = \{Fix\} \oplus C_0^{\odot}(\mathbb{R})$. Then*

1. *M is a direct sum,*
2. *M is a band in $BUC^*(\mathbb{R})$. Thus, $BUC^*(\mathbb{R}) = M \oplus M^d$ and $BUC^{\odot}(\mathbb{R}) = M \oplus\ (M^d \cap BUC^{\odot}(\mathbb{R}))$.*

Proof. By Lemma 4.4 , $C_0^{\perp}(\mathbb{R}) \supseteq (Fix)$; and since $C_0^{\perp}(\mathbb{R})$ is a band (Lemma 4.1), then $C_0^{\perp}(\mathbb{R}) \supseteq \{Fix\}$. But $(C_0^{\perp}(\mathbb{R}))^d = C_0^*(\mathbb{R}) \supseteq C_0^{\odot}(\mathbb{R})$, so M is a direct sum.

To prove part (2), we first define an isometric isomorphic mapping of $C_0^{\odot}(\mathbb{R})$ into $BUC^*(\mathbb{R})$. We then show that the image of $C_0^{\odot}(\mathbb{R})$ under this mapping is a band in $BUC^*(\mathbb{R})$ and $BUC^{\odot}(\mathbb{R})$; hence, $C_0^{\odot}(\mathbb{R})$ is a band in $BUC^*(\mathbb{R})$ and $BUC^{\odot}(\mathbb{R})$. Now we argue as follows: Since $\{Fix\}$ and $C_0^{\odot}(\mathbb{R})$ are disjoint bands in $BUC^*(\mathbb{R})$ by part (1), then $M = C_0^{\odot}(\mathbb{R}) \oplus \{Fix\}$ is a band in $BUC^*(\mathbb{R})$ (Theorem 2.5). Therefore,

$$BUC^*(\mathbb{R}) = M \oplus M^d \text{ and } BUC^{\odot}(\mathbb{R}) = M \oplus (M^d \cap BUC^{\odot}(\mathbb{R})).$$

We remind the reader that $C_0^{\odot}(\mathbb{R}) = L^1(\mathbb{R})$ ([Phi]). Hence, if $\mu \in C_0^{\odot}(\mathbb{R})$, then there exists a unique $f \in L^1(\mathbb{R})$ such that $d\mu = f\ dx$ (Radon-Nikodym theorem). Define

$$\mu \xrightarrow{\Theta} \tilde{\mu}, \text{ where } \tilde{\mu}(g) = \int_{\mathbb{R}} fg\ dx, \quad g \in BUC(\mathbb{R}).$$

Since f is unique, then the map Θ is well-defined. A trivial verification shows that $\tilde{\mu}$ and Θ are bounded and linear. Θ is also 1-1; to see this, suppose $\|\cdot\|_*$ is the norm on $BUC^*(\mathbb{R})$ and $\|\cdot\|_0$ is the norm on $C_0^*(\mathbb{R})$. If $\mu_1, \mu_2 \in C_0^{\odot}(\mathbb{R})$, then $(\mu_1 - \mu_2) \in C_0^{\odot}(\mathbb{R})$, so $\exists h \in L^1(\mathbb{R})$ such that $d(\mu_1 - \mu_2) = h\ dx$. We have

$$\|\mu_1 - \mu_2\|_0 = \sup_{\substack{p \in C_0(\mathbb{R}) \\ \|p\| \leq 1}} \left| \int_{\mathbb{R}} p\ d(\mu_1 - \mu_2) \right| \leq$$

$$\sup_{\substack{g \in BUC(\mathbb{R}) \\ \|g\| \leq 1}} \left| \int_{\mathbb{R}} g\ d(\mu_1 - \mu_2) \right| = \|\tilde{\mu}_1 - \tilde{\mu}_2\|_* = \sup_{\substack{g \in BUC(\mathbb{R}) \\ \|g\| \leq 1}} \left| \int_{\mathbb{R}} gh\ dx \right| \leq$$

$$\int_{\mathbb{R}} |h|\ dx = |\mu_1 - \mu_2|(\mathbb{R}) = \|\mu_1 - \mu_2\|_0.$$

This shows $\|\mu_1 - \mu_2\|_0 = \|\tilde{\mu}_1 - \tilde{\mu}_2\|_*$; it also shows Θ is 1-1.

We assert that $\Theta(C_0^{\odot}(\mathbb{R})) \subseteq BUC^{\odot}(\mathbb{R}) \subseteq BUC^*(\mathbb{R})$. Indeed,

$$\langle g, T^*(t)\tilde{\mu}\rangle = \langle T(t)g, \tilde{\mu}\rangle = \int_{\mathbb{R}} g(x+t)\ f(x)\ dx = \int_{\mathbb{R}} g(x)\ f(x-t)\ dx.$$

Hence,

$$d(T^*(t)\tilde{\mu}) = f(x-t)\ dx,$$

so

$$\|T^*(t)\tilde{\mu} - \tilde{\mu}\|_* = \sup_{\substack{g\in BUC(\mathbb{R})\\ \|g\|\leq 1}} \left| \int_{\mathbb{R}} (f(x) - f(x-t))\ g(x)\ dx \right|$$

$$\leq \int_{\mathbb{R}} |f(x) - f(x-t)|\ dx = \|T^*(t)\mu - \mu\|_0.$$

Letting $t \to 0$, we see that $\tilde{\mu} \in BUC^{\odot}(\mathbb{R})$ and that Θ is an isometric isomorphism.

Since $\Theta(C_0^{\odot}(\mathbb{R})$ is the isometric isomorphic image of the Banach space $C_0^{\odot}(\mathbb{R})$, it is a norm-closed linear subspace. We claim that it is also an order ideal; that is, $\Theta(C_0^{\odot}(\mathbb{R}))$ has the following property:

(a) $\lambda \in BUC^*(\mathbb{R}),\ \ \tilde{\nu} \in \Theta(C_0^{\odot}(\mathbb{R}))$, and $0 \leq |\lambda| \leq |\tilde{\nu}| \Rightarrow \lambda \in \Theta(C_0^{\odot}(\mathbb{R}))$.

The first step is to show (a) holds for positive elements in $BUC^*(\mathbb{R})$ and $\Theta(C_0^{\odot}(\mathbb{R}))$; we assume

$$0 \leq \lambda \leq \tilde{\nu},\ \text{ and show } \lambda \in \Theta(C_0^{\odot}(\mathbb{R})).$$

Note that

$$0 \leq \lambda \leq \tilde{\nu} \Rightarrow (\forall\ 0 \leq g \in BUC(\mathbb{R})),\ 0 \leq \lambda(g) \leq \tilde{\nu}(g).$$

Since $\tilde{\nu}$ is identified with an element $\nu \in C_0^{\odot}(\mathbb{R})$ which has the property that $d\nu = f\ dx$

for some unique $f \in L^1(\mathbb{R})$, then

$$0 \leq \lambda(g) \leq \tilde{\nu}(g) = \int_{\mathbb{R}} gf\ dx. \tag{4.3}$$

If we restrict λ to $C_0(\mathbb{R})$, then by the Riesz representation theorem, there is a positive, bounded, regular Borel measure τ whose representation as a functional on $C_0(\mathbb{R})$ is

$$\lambda(p) = \int_{\mathbb{R}} p\ d\tau, \quad p \in C_0(\mathbb{R}).$$

The inequality in (4.3), together with the above representation of λ, imply that

$$0 \leq \lambda(p) = \int_{\mathbb{R}} p\ d\tau \leq \int_{\mathbb{R}} pf\ dx, \quad p \in C_0(\mathbb{R}).$$

It follows that $\tau \ll \nu$. Thus, $d\tau = h\ d\nu$ for some $h \in L^1(\nu)$, so

$$d\tau = h\ d\nu = hf\ dx \leq d\nu.$$

This last inequality implies that $0 \leq h \leq 1$; thus, $hf \in L^1(\mathbb{R})$.

If we define $d\mu = hf\ dx$, then it is clear $\mu \in C_0^{\odot}(\mathbb{R})$, and

$$\Theta(\mu)(g) = \tilde{\mu}(g) = \int_{\mathbb{R}} g\ d\mu = \int_{\mathbb{R}} g\ hf\ dx = \lambda(g), \quad g \in BUC(\mathbb{R}).$$

Therefore, $\lambda \in \Theta(C_0^{\odot}(\mathbb{R}))$.

The second step is to show **(a)** holds for complex-valued elements in $BUC^*(\mathbb{R})$ and $\Theta(C_0^{\odot}(\mathbb{R}))$. Since $0 \leq \lambda^+ \leq |\tilde{\nu}|$ (similarly for λ^-), we can assume $0 \leq \lambda \leq |\tilde{\nu}|$. With this assumption in place, it suffices to show

(b) $\tilde{\nu} \in \Theta(C_0^{\odot}(\mathbb{R})) \Rightarrow |\tilde{\nu}| \in \Theta(C_0^{\odot}(\mathbb{R}))$;

we can then apply the above argument to conclude that $\lambda \in \Theta(C_0^{\odot}(\mathbb{R}))$.

Observe that $\nu \in C_0^{\odot}(\mathbb{R}) \Rightarrow |\nu| \in C_0^{\odot}(\mathbb{R})$. Indeed, $C_0^*(\mathbb{R})$ has order continuous norm, and $\forall t \in \mathbb{R}$, $T(t)$ is a positive operator (Chapter 2); it follows by Theorem 2.8 that $C_0^{\odot}(\mathbb{R})$ is a band in $C_0^*(\mathbb{R})$. Thus, $|\nu| \in C_0^{\odot}(\mathbb{R})$.

We claim that Θ is a lattice isomorphism, i.e., Θ has the following property:

$$\widetilde{|\nu|} = \Theta(|\nu|) = |\Theta(\nu)| = |\tilde{\nu}|,$$

which is precisely the property we need to show that (**b**) holds. To show the above equality, we use the following result of F. Riesz ([Zan]):

$$(\forall \tau \in BUC^*(\mathbb{R}))\ (\forall\ 0 \le g \in BUC(\mathbb{R})),\quad |\tau|(g) = \sup_{\substack{h \in BUC(\mathbb{R}) \\ |h| \le g}} \{|\tau(h)|\}. \tag{4.4}$$

We will use (4.4) to show

$$\sup_{\substack{h \in BUC(\mathbb{R}) \\ |h| \le g}} \left| \int_{\mathbb{R}} hf\ dx \right| = |\tilde{\nu}|(g) = \widetilde{|\nu|}(g) = \int_{\mathbb{R}} g|f|\ dx.$$

Suppose g_n is a sequence of functions with compact support, and $0 \le g_n \uparrow g$. Then

$$\sup_{\substack{h \in BUC(\mathbb{R}) \\ |h| \le g}} \left| \int_{\mathbb{R}} hf\ dx \right| \le \int_{\mathbb{R}} g|f|\ dx = \lim_{n \to \infty} \int_{\mathbb{R}} g_n|f|\ dx,$$

and

$$\lim_{n \to \infty} \left(\int_{\mathbb{R}} g_n\ |f|\ dx \right) = \lim_{n \to \infty} (|\tilde{\nu}|(g_n)).$$

However, by (4.4),

$$\lim_{n \to \infty} (|\tilde{\nu}|(g_n)) = \sup_n \Big(\sup_{\substack{h \in BUC(\mathbb{R}) \\ |h| \le g_n}} \left| \int_{\mathbb{R}} hf\ dx \right| \Big) \le \sup_{\substack{h \in BUC(\mathbb{R}) \\ |h| \le g}} \left| \int_{\mathbb{R}} hf\ dx \right|,$$

so $|\Theta(\nu)| = \Theta(|\nu|)$, i.e., Θ is a lattice isomorphism. Consequently, $|\tilde{\nu}| \in \Theta(C_0^{\odot}(\mathbb{R}))$, condition (**b**) is satisfied, and $\Theta(C_0^{\odot}(\mathbb{R}))$ is a norm-closed order ideal in $BUC^*(\mathbb{R})$ and $BUC^{\odot}(\mathbb{R})$. □

Remark 4.8. Lemma 4.7 plays a crucial role in our characterization of $BUC^{\odot}(\mathbb{R})$. While the representation of $BUC^{\odot}(\mathbb{R})$ as

$$BUC^{\odot}(\mathbb{R}) = M \oplus (M^d \cap BUC^{\odot}(\mathbb{R}))$$

is important, it is still a preliminary characterization. In Chapter 8 we show $M^d \cap BUC^{\odot}(\mathbb{R}) = \{0\}$.

In part (1) of Lemma 4.7, we use the fact that $C_0^*(\mathbb{R}) = (C_0^{\perp}(\mathbb{R}))^d$; this implies that $C_0^*(\mathbb{R})$ is a band in $BUC^*(\mathbb{R})$, since $(C_0^{\perp}(\mathbb{R}))^d$ is a band. Actually, to show that $C_0^*(\mathbb{R})$ is a band, we use an approach similar to the method used to embed $C_0^{\odot}(\mathbb{R})$ in $BUC^*(\mathbb{R})$, but with a slight variation. Instead of using the fact that $C_0^{\odot}(\mathbb{R}) = L^1(\mathbb{R})$, we make use of the characterization of $C_0^*(\mathbb{R})$ provided by the Riesz representation theorem .

We end this chapter by offering the reader a more precise description of the structure of $\{Fix\}$. Let A_{Fix} denote the order ideal generated by the set Fix. That is, $\nu \in A_{Fix}$ if for suitable $m_i \in Fix$ and $a_i \in \mathbb{R}$, we have

$$|\nu| \leq |a_1 m_1| + \ldots + |a_n m_n| \quad ([LuZ]).$$

Note that A_{Fix} is the smallest ideal which contains Fix. Clearly, $\nu \in A_{Fix}$ if for some $m \in Fix$, $\nu \ll m$; moreover, $\{A_{Fix}\} = \{Fix\}$. Also, if $0 \leq \mu \in \{A_{Fix}\}$, then $\mu = \sup_\alpha \nu_\alpha$ for some upward directed system $(\nu_\alpha : \alpha \in \{\alpha\})$ contained in A_{Fix} ([LuZ]). It follows that if $0 \leq \mu \in \{Fix\}$, then

$$\mu = \sup_\alpha \nu_\alpha = \sup_\alpha f_\alpha \, dm_\alpha,$$

where $m_\alpha \in Fix$ and $f_\alpha \in L^1(m_\alpha)$.

Chapter 5 Examples of elements in Fix

In Lemma 4.6 we proved that nonzero, translation invariant linear functionals exist in $BUC^*(\mathbb{R})$, and we designated the collection of all such functionals Fix. In this chapter we construct examples of elements in Fix. We proceed as follows: $(\forall x, y \in \mathbb{R})$ $(\forall f \in BUC(\mathbb{R}))$, consider

$$U_y(f(x)) = \frac{1}{y}\int_0^y f(x+s)\, ds. \tag{5.1}$$

It is obvious that U_y is linear on $BUC(\mathbb{R})$. Moreover,

$$|U_y(f(x))| \leq \frac{1}{|y|}\int_0^y |f(x+s)|\, ds \leq \|f\|, \tag{5.2}$$

shows that $\|U_y\| \leq 1$. If e is the unit in $BUC(\mathbb{R})$, then equation (5.1) shows that $|U_y e(x)| = 1$, so in fact $\|U_y\| = 1$. Hence, U_y is a bounded, positive linear functional on $BUC(\mathbb{R})$.

Let x be a fixed element in $\mathbb{R}$. We claim the following: If we regard $U_y(f(x))$ as a function of y, then

1. $U_y(f(x)) \in BUC(\mathbb{R})$ if $f \in BUC(\mathbb{R})$, and

2. $U_y(f(x)) \in C_0(\mathbb{R})$ if $f \in C_0(\mathbb{R})$.

To show (1) and (2), we first transform equation (5.1) by making the substitution $s = uy$, so that

$$U_y(f(x)) = \int_0^1 f(x+uy)\, du. \tag{5.3}$$

Let f be an element in $BUC(\mathbb{R})$. Then $(\forall\epsilon > 0)$ $(\exists\delta > 0)$ such that for every $y, y' \in \mathbb{R}$ with $|y - y'| < \delta$, we have $|f(y) - f(y')| < \epsilon$. If $0 \leq u \leq 1$, then

$$|(x + uy) - (x + uy')| = |u|\ |y - y'| \leq |y - y'| < \delta,$$

so by the uniform continuity of f, we have

$$|f(x + uy) - f(x + uy')| < \epsilon.$$

It follows that $U_y(f(x)) \in BUC(\mathbb{R})$ (note that $U_y(f(x))$ is jointly uniformly continuous in x and y).

If $f \in C_0(\mathbb{R})$, then we claim that

$$\lim_{y \to \infty} \int_0^1 f(x + uy)\ du = 0.$$

Suppose this were not the case; then $(\exists y_n \in \mathbb{R})$ $(\exists\epsilon > 0)$ such that

$$\lim_{n \to \infty} y_n = \infty \Rightarrow |\int_0^1 f(x + uy_n)\ du| \geq \epsilon. \tag{5.4}$$

However, since $f \in C_0(\mathbb{R})$, then $f(x + uy_n) \to 0$ as $y_n \to \infty$, so by *dominated convergence*,

$$\lim_{n \to \infty} \int_0^1 f(x + uy_n)\ du = 0,$$

which is a contradiction of (5.4). Therefore, $\lim_{y \to \infty} U_y(f(x)) = 0$, so $U_y(f(x))$, as a function of y, is an element in $C_0(\mathbb{R})$.

In Chapter 3 we showed that $BUC(\mathbb{R})$ is *- isometrically isomorphic to the function space $C(\Omega)$. Thus, if $f \in BUC(\mathbb{R})$, then we can continuously extend $U_y(f(x))$, as a function of y, to Ω. Let $\hat{U}_\alpha(f(x))$, $\alpha \in \Omega$, denote the extension. Then there exists a net $(y_\tau) \subseteq \mathbb{R}$ such that

$$\hat{U}_\alpha(f(x)) = \lim_\tau U_{y_\tau}(f(x)).$$

Clearly, $\hat{U}_\alpha$ is linear on $BUC(\mathbb{R})$. And since $U_{y_\tau} \geq 0$ and $U_{y_\tau}e(x) = 1$, $\forall\tau$, then $\hat{U}_\alpha \geq 0$ and $\hat{U}_\alpha(e(x)) = 1$. It follows that $\|\hat{U}_\alpha\| = 1$.

Lemma 5.1. *Let $f \in C_0(\mathbb{R})$ and let $\hat{f}$ be its corresponding function in $C(\Omega)$. Then $\forall\alpha \in (\Omega \setminus \mathbb{R})$, $\hat{f}(\alpha) = 0$.*

Proof. By hypothesis, there exists a net $(t_l) \subseteq \mathbb{R}$ such that $\hat{f}(\alpha) = \lim_l f(t_l)$. Hence, $(\forall\epsilon > 0)$ $(\exists l_0 \in \{l\})$, such that

$$|\hat{f}(\alpha) - f(t_l)| < \frac{\epsilon}{2}, \quad l \geq l_0.$$

Since $f \in C_0(\mathbb{R})$, then there exists a compact set K such that

$$|f(z)| < \frac{\epsilon}{2}, \quad \forall z \notin K.$$

Let $A = \{l : l \geq l_0\}$ and $B = \{l : t_l \notin K\}$. Observe that both B and $A \cap B$ are nonempty. Indeed, compact sets in $\mathbb{R}$ with respect to the usual topology, are also compact in $\mathbb{R}$ with respect to the weak*- topology (recall that $\Omega \supseteq \mathbb{R}$ as a weak*-dense subspace). It follows that K is compact in Ω, which means that the complement of K, $\Omega \setminus K$, is an open set which contains $\Omega \setminus \mathbb{R}$; that is, $\Omega \setminus K$ is an open neighborhood which contains α. And since $t_l \xrightarrow{w*} \alpha$, then there are plenty of $l \in \{l\}$ for which $t_l \notin K$. Ergo, B and $A \cap B$ are nonempty.

Let $d \in A \cap B$; then

$$|\hat{f}(\alpha)| \leq |\hat{f}(\alpha) - f(t_d)| + |f(t_d)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

We determined in (2) that $U_y(f(x)) \in C_0(\mathbb{R})$ if $f \in C_0(\mathbb{R})$. Consequently, we can apply Lemma 5.1 to the extension $\hat{U}_\alpha(f(x))$:

$$\hat{U}_\alpha(f(x)) = 0, \quad f \in C_0(\mathbb{R}), \quad \alpha \in \Omega \setminus \mathbb{R}.$$

We claim that $\hat{U}_\alpha$ is translation invariant, i.e.,

$$\forall t \in \mathbb{R}, \quad T^*(t)\hat{U}_\alpha = \hat{U}_\alpha.$$

Indeed, $(\forall t \in \mathbb{R})$ $(\forall f \in BUC(\mathbb{R}))$,

$$|U_y(T(t)f(x)) - U_y(f(x))| = |U_y[T(t)f(x) - f(x)]| = \frac{1}{|y|}\left|\int_t^{t+y} f(x+s)\,ds - \int_0^y f(x+s)\,ds\right| \le \frac{2|t|\,\|f\|}{|y|}.$$

Letting $|y| \longrightarrow \infty$ shows that $U_y[T(t)f(x) - f(x)] \in C_0(\mathbb{R})$; thus, $\hat{U}_\alpha[T(t)f(x) - f(x)] = 0$, by Lemma 5.1. We have

$$\langle f, \hat{U}_\alpha \rangle = \langle T(t)f, \hat{U}_\alpha \rangle = \langle f, T^*(t)\hat{U}_\alpha \rangle,$$

which shows $\hat{U}_\alpha \in Fix$.

In Chapter 9 we characterize the *sun-dual* of the space of *weakly almost periodic functions* on $\mathbb{R}$ $(WAP(\mathbb{R}))$. The ideas of this chapter are central to that characterization.

Chapter 6 Ω admits a semitopological semigroup structure

Let Ω be the *maximal ideal space* of $BUC(\mathbb{R})$. In this chapter we focus on the map

$$\varphi : (x,t) \longrightarrow (x+t) \text{ of } \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

It is clear that φ is a *jointly continuous, binary operation* on $\mathbb{R}$. It is also clear that this operation renders the additive group of $\mathbb{R}$ a topological group. Ideally, we would like to extend φ over $\Omega \times \Omega$ so that it is jointly continuous, since this would eliminate the necessity of this chapter, which would in turn simplify the proof of our main theorem (Theorem A). As we shall see presently, there exists a binary operation φ on Ω that is both *commutative* and *associative*, which renders Ω an 'additive' semigroup; the additive group of $\mathbb{R}$ is a subgroup of Ω with respect to this 'additive semigroup structure.' Unfortunately, Ω is neither a topological group nor a topological semigroup. However, it is a semitopological semigroup, i.e., Ω is a Hausdorff semigroup equipped with a binary operation that is only continuous in each variable separately.

Theorem A can still be proved, notwithstanding the lack of joint continuity of φ over $\Omega \times \Omega$, by employing the weaker condition that φ be separately continuous in each variable. How can we extend φ so that it is even separately continuous? We accomplish this in the following manner. First, we define a *uniformity* $\mathcal{U}_{\mathbb{R}}$ on $\mathbb{R}$ in which the *completion* of $(\mathbb{R}, \mathcal{U}_{\mathbb{R}})$ is $(\Omega, \mathcal{U}_{\mathbb{R}^*})$ ($\mathcal{U}_{\mathbb{R}^*}$ will be defined directly). Having done this, we provide a definition of uniform continuity with respect to uniform spaces, and point out that for fixed $t \in \mathbb{R}$, $\varphi(\cdot, t)$ is uniformly continuous on $\mathbb{R}$ according to this definition. We then use the fact that $\Omega \supseteq \mathbb{R}$ as a weak*-dense subspace, in order to extend $\varphi(\cdot, t)$ to a unique, uniformly continuous function defined on Ω. Finally, for fixed $\alpha \in \Omega$, we show that $\varphi(\alpha, \cdot)$ is uniformly continuous on $\mathbb{R}$; ergo, $\varphi(\alpha, \cdot)$ extends continuously to Ω.

Definition 6.1. *A uniformity for a set X is a nonempty family $\mathcal{S}$ of subsets of $X \times X$ such that*

- *if $A \in \mathcal{S}$, then $A \supseteq \{(x,x) : x \in X\}$*
- *if $A \in \mathcal{S}$, then $\{(x,y) : (y,x) \in X\} \in \mathcal{S}$*
- *if $A \in \mathcal{S}$, then $\{(x,z) : \exists y$ such that $(x,y),(y,z) \in A\} \in \mathcal{S}$*
- *if $A, B \in \mathcal{S}$, then $(A \cap B) \in \mathcal{S}$*
- *if $A \in \mathcal{S}$ and $A \subseteq B \subseteq (X \times X)$, then $B \in \mathcal{S}$.*

Let $\mathbf{B}$ denote the *unit ball* in $BUC^*(\mathbb{R})$. We define a uniformity on $\mathbf{B}$, which in turn induces a uniformity on $\mathbb{R}$: $(\forall f \in BUC(\mathbb{R}))$ $(\forall \epsilon > 0)$, define

$$U_{f,\epsilon} = \{(\psi, \chi) \in \mathbf{B} \times \mathbf{B} : |\langle f, \psi \rangle - \langle f, \chi \rangle| < \epsilon\}.$$

Then the sets

$$U_{f_1,\dots,f_n,\epsilon} = \{(\psi, \chi) : \max_{1 \le i \le n} |\langle f_i, \psi \rangle - \langle f_i, \chi \rangle| < \epsilon\}, \;\; f_i \in BUC(\mathbb{R}),$$

form a uniformity $\mathcal{U}$ on $\mathbf{B}$. We refer to $\mathcal{U}$ as the *weak*-uniformity* on $\mathbf{B}$. The topology $\tau_{\mathcal{U}}$ of the uniformity $\mathcal{U}$ or the *uniform topology*, is the *weak*-topology* on $\mathbf{B}$.

Note that via the embedding $x \mapsto \delta_x$ of $\mathbb{R}$ into $\mathbf{B}$ given by

$$\delta_x(f) = f(x), \quad f \in BUC(\mathbb{R}),$$

$\mathcal{U}$ induces a uniformity on $\mathbb{R}$; we shall designate this uniformity $\mathcal{U}_{\mathbb{R}}$. That is, $V \in \mathcal{U}_{\mathbb{R}}$ if $(\forall \epsilon > 0)$ $(\forall g_i \in BUC(\mathbb{R}))$,

$$V = \{(\delta_x, \delta_y) \in \mathbf{B} \times \mathbf{B} : \max_{1 \leq i \leq m} |\langle g_i, \delta_x \rangle - \langle g_i, \delta_y \rangle| < \epsilon\} =$$
$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : \max_{1 \leq i \leq m} |g_i(x) - g_i(y)| < \epsilon\}.$$

Recall that each Hausdorff uniform space is uniformly isomorphic to a dense subspace of a complete Hausdorff uniform space ([Kel]). Accordingly, define $(\mathbb{R}^*, \mathcal{U}_{\mathbb{R}^*})$ as the completion of $(\mathbb{R}, \mathcal{U}_{\mathbb{R}})$. Since $\mathbf{B}$ is compact with respect to the weak*-topology on $BUC^*(\mathbb{R})$, then the weak*-closure of $(\mathbb{R}, \mathcal{U}_{\mathbb{R}})$ in $\mathbf{B}$ is compact, i.e., $(\mathbb{R}^*, \mathcal{U}_{\mathbb{R}^*})$ is compact. It follows immediately that since $\Omega \supseteq \mathbb{R}$ as a weak*-dense subspace, then $\Omega = \mathbb{R}^*$; moreover, $\mathcal{U}_{\mathbb{R}^*}$ consists of finite intersections of sets of the form:

$$W_{\hat{f},\epsilon} = \{(\alpha, \beta) \in \Omega \times \Omega : |\hat{f}(\alpha) - \hat{f}(\beta)| < \epsilon\}.$$

Definition 6.2. *Let $(X_1, \mathcal{U}_1)$ and $(X_2, \mathcal{U}_2)$ be uniform spaces and let*

$$g : (X_1, \mathcal{U}_1) \to (X_2, \mathcal{U}_2).$$

Then g is uniformly continuous if $(\forall W \in \mathcal{U}_2)$ $(\exists V \in \mathcal{U}_1)$ such that

$$(x, y) \in V \Rightarrow (g(x), g(y)) \in W.$$

Theorem 6.3. *Let g be a function whose domain is a subset A of a uniform space $(X_1, \mathcal{U}_1)$ and whose values lie in a complete Hausdorff uniform space $(X_2, \mathcal{U}_2)$. If g is uniformly continuous on A, then there is a unique uniformly continuous extension $\bar{g}$ of g whose domain is the closure of A ([Kel]).*

One can easily verify the following:

1. For fixed $t \in \mathbb{R}$, the map $\varphi(\cdot, t) : (\mathbb{R}, \mathcal{U}_{\mathbb{R}}) \to (\mathbb{R}, \mathcal{U}_{\mathbb{R}})$ given by $\varphi(x, t) = (x + t)$, is uniformly continuous according to Definition 6.2.

2. Let $f \in BUC(\mathbb{R})$ and let $\mathcal{U}_M$ be the metric uniformity on the complex numbers $\mathbb{C}$, i.e.,

$$V \in \mathcal{U}_M \text{ if V has the form : } V = \{(x, y) \in \mathbb{C} \times \mathbb{C} : |x - y| < \epsilon\}.$$

   Then $f : (\mathbb{R}, \mathcal{U}_{\mathbb{R}}) \to (\mathbb{C}, \mathcal{U}_M)$ satisfies Definition 6.2.

According to properties (1) and (2) above, we may apply Theorem 6.3 to the map $\varphi(\cdot, t)$ and to each function $f \in BUC(\mathbb{R})$, to claim that each has a unique, continuous extension to all of Ω.

We also claim that if $\alpha \in (\Omega \setminus \mathbb{R})$ and $t \in \mathbb{R}$, then $\varphi(\alpha, t) \in (\Omega \setminus \mathbb{R})$. Suppose this is not the case, i.e., suppose $\varphi(\alpha, t) = b \in \mathbb{R}$. Then

$$b = \varphi(\alpha, t) = \lim_l (u_l + t),$$

where (u_l) is a net in $\mathbb{R}$ such that $u_l \xrightarrow{w*} \alpha$. It follows that $\alpha = \lim_l (u_l) = (b - t) \in \mathbb{R}$, which contradicts the fact that $\alpha \in (\Omega \setminus \mathbb{R})$.

Lemma 6.4. *The map φ has the following properties:*

i) $\forall \alpha \in \Omega$,

$$\varphi(\alpha, \cdot) : (\mathbb{R}, \mathcal{U}_{\mathbb{R}}) \to (\Omega, \mathcal{U}_{\mathbb{R}^*}) \textit{ satisfies Definition 6.2.}$$

ii) Let $\hat{f}$ be the continuous extension to Ω of any $f \in BUC(\mathbb{R})$. Then as a function of $t \in \mathbb{R}$,

$$\hat{f}(\varphi(\alpha, t)) \in BUC(\mathbb{R}).$$

Therefore, $\varphi(\alpha, t)$ and $\hat{f}(\varphi(\alpha, t))$ extend continuously to Ω.

iii) $\forall \alpha, \beta, \omega \in \Omega$,

(a) $\varphi(\alpha, 0) = \alpha$,

(b) $\varphi(\alpha, \beta) = \varphi(\beta, \alpha)$,

(c) $\varphi(\alpha, \varphi(\beta, \omega)) = \varphi(\varphi(\alpha, \beta), \omega)$;

thus, (Ω, φ) is an abelian semigroup.

Proof. We will prove (i) and (ii) simultaneously. We must show

$$(\forall W \in \mathcal{U}_{\mathbb{R}^*})\ (\exists V \in \mathcal{U}_{\mathbb{R}}) \text{ such that } (t, t') \in V \Rightarrow (\varphi(\alpha, t), \varphi(\alpha, t')) \in W.$$

Recall that each $W \in \mathcal{U}_{\mathbb{R}^*}$ is of the form:

$$W = \{(\alpha, \beta) \in \Omega \times \Omega : \max_{1 \leq i \leq p} |\hat{f}_i(\alpha) - \hat{f}_i(\beta)| < \epsilon\},$$

where each $\hat{f}_i$ is the extension of an $f_i \in BUC(\mathbb{R})$. Note that in each variable separately, the function $f_i(x + t)$ is a bounded, uniformly continuous function on $\mathbb{R}$. Consequently, for some real-valued net (u_l), we have

$$\forall \alpha \in \Omega, \quad \hat{f}_i(\varphi(\alpha, t)) = \lim_l f_i(u_l + t).$$

Let $(D, \succeq)$ be the directed set of the net (u_l). Then $f_i(u_l + t) \in BUC(\mathbb{R})$, for every $l \in D$. Thus, $(\forall \epsilon > 0)$ $(\exists d \in D)$ $(\exists \delta > 0)$ such that

$$|t - t'| < \delta \Rightarrow |f_i(u_d + t) - f_i(u_d + t')| < \frac{\epsilon}{3},\ 1 \leq i \leq p.$$

The set

$$V = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \max_{1 \leq i \leq p} |f_i(u_d + x) - f_i(u_d + y)| < \frac{\epsilon}{3}\},$$

is clearly a set in $\mathcal{U}_{\mathbb{R}}$. It follows that

$$(t,t') \in V \Rightarrow |\hat{f}_i(\varphi(\alpha,t)) - \hat{f}_i(\varphi(\alpha,t'))| < \epsilon, \quad 1 \le i \le p.$$

Thus, $(\varphi(\alpha,t),\varphi(\alpha,t')) \in W$ and $\hat{f}(\varphi(\alpha,t)) \in BUC(\mathbb{R})$. This proves (i) and (ii).

To show $(iii)a)$, let (t_l) be a real-valued net such that $t_l \xrightarrow{w*} \alpha$. Then

$$\varphi(\alpha,0) = \lim_l \varphi(t_l,0) = \lim_l (t_l+0) = \alpha;$$

this proves (a). To prove (b), we will show that if $f \in BUC(\mathbb{R})$ and $\hat{f}$ is the unique extension of f to all of Ω, then $\hat{f}(\varphi(\alpha,\beta)) = \hat{f}(\varphi(\beta,\alpha))$. Now we suppose $\varphi(\alpha,\beta) \neq \varphi(\beta,\alpha)$. Since Ω is a Hausdorff space, $\exists g \in C(\Omega)$ such that $g(\varphi(\alpha,\beta)) \neq g(\varphi(\beta,\alpha))$. Let h be the restriction of g to $\mathbb{R}$. Since the restriction of $\mathcal{U}_{\mathbb{R}^*}$ to $\mathbb{R}$ is $\mathcal{U}_{\mathbb{R}}$, and since $\mathcal{U}_{\mathbb{R}}$ is the weakest uniformity on $\mathbb{R}$ with respect to which all the $f \in BUC(\mathbb{R})$ are uniformly continuous in the sense of Definition 6.2, then $h \in BUC(\mathbb{R})$. But this implies $\hat{h}(\varphi(\alpha,\beta)) = \hat{h}(\varphi(\beta,\alpha))$; moreover, $\hat{h} = g$. Thus, we obtain a contradiction of the hypothesis on g.

Let $f \in BUC(\mathbb{R})$; then $\forall x,t \in \mathbb{R}$, $f(x+t) = f(t+x)$. In each variable separately, $f(x+t) \in BUC(\mathbb{R})$, so we can extend f to Ω:

$$\hat{f}(\varphi(x,\alpha)) = \lim_l f(x+t_l) = \lim_l f(t_l+x) = \hat{f}(\varphi(\alpha,x)).$$

By part (ii), $\forall x \in \mathbb{R}$, $\hat{f}(\varphi(\alpha,x)) \in BUC(\mathbb{R})$; hence, we can extend $\hat{f}(\varphi(\alpha,x))$ to Ω. It follows that

$$\hat{f}(\varphi(\alpha,\beta)) = \hat{f}(\varphi(\beta,\alpha)).$$

This shows (b); a similar argument shows (c). □

Let $E = C(\Omega)$. Consider the map $\alpha \mapsto T(\alpha)$ of $\Omega \to \mathcal{L}_b(E,E)$, where $\mathcal{L}_b(E,E)$ is the Banach algebra consisting of all bounded, linear operators from E into E. For each $\alpha \in \Omega$, let $T(\alpha) : E \to E$ be given by

$$T(\alpha)\hat{f}(\omega) = \hat{f}(\varphi(\alpha, \omega)), \quad \omega \in \Omega. \tag{6.1}$$

We end this chapter by determining certain properties of the family $\{T(\alpha) : \alpha \in \Omega\}$ and the dual family $\{T^*(\alpha) : \alpha \in \Omega\}$, that we will use in the proof of our main theorem (Theorem A).

Lemma 6.5. *Let H be the family of operators that satisfies equation (6.1). Then*

1. $\forall \alpha, \beta \in \Omega$,

$$T(\alpha)T(\beta) = T(\beta)T(\alpha) = T(\varphi(\alpha, \beta)).$$

   Therefore, H is a commuting family.

2. $\forall \alpha \in \Omega, \quad \|T(\alpha)\| = 1.$

Proof. By equation (6.1), $(\forall \alpha, \beta \in \Omega)$ $(\forall \hat{f} \in E)$,

$$T(\alpha)T(\beta)\hat{f}(\omega) = T(\alpha)(\hat{f}(\varphi(\beta, \omega))) = \hat{f}(\varphi(\alpha, \varphi(\beta, \omega))).$$

By part $(iii)c)$ of Lemma 6.4,

$$\hat{f}(\varphi(\alpha, \varphi(\beta, \omega))) = \hat{f}(\beta, \varphi(\alpha, \omega)) = T(\beta)T(\alpha)\hat{f}(\omega)$$

and

$$T(\varphi(\alpha, \beta)) = \hat{f}(\varphi(\varphi(\alpha, \beta), \omega)) = T(\alpha)\hat{f}(\varphi(\beta, \omega)) = T(\alpha)T(\beta)\hat{f}(\omega),$$

$\omega \in \Omega$. This proves (1).

To show (2), note that $(\forall \alpha \in \Omega)$ $(\forall \hat{f} \in E)$,

$$\|T(\alpha)\hat{f}\| = \sup_{\omega \in \Omega}\{|\hat{f}\varphi(\alpha, \omega)|\} \leq \sup_{\omega \in \Omega}\{|\hat{f}(\omega)|\} = \|\hat{f}\|.$$

Hence, $\|T(\alpha)\| \le 1$. If $\hat{e}$ is the *unit* in E, then $\|T(\alpha)\hat{e}\| = 1$; thus, $\|T(\alpha)\| = 1$. □

For each $\alpha \in \Omega$, there exists a unique $T^*(\alpha) \in \mathcal{L}_b(E^*, E^*)$ which satisfies $(\forall \hat{f} \in E)$ $(\forall \mu \in E^*)$, the dual relation $\langle T(\alpha)\hat{f}, \mu\rangle = \langle \hat{f}, T^*(\alpha)\mu\rangle$. The action of $T^*(\alpha)$ on an arbitrary $\mu \in E^*$ is given by

$$\langle \hat{f}, T^*(\alpha)\mu\rangle = \langle T(\alpha)\hat{f}, \mu\rangle = \int_\Omega \hat{f}(\varphi(\alpha,\omega))d\mu(\omega) = \int_\Omega \hat{f}(\omega)d\mu(\varphi^{-1}(\alpha,\omega)) = \langle \hat{f}, \mu(\varphi^{-1}(\alpha,\cdot))\rangle.$$

Thus,

$$T^*(\alpha)\mu(\cdot) = \mu(\varphi^{-1}(\alpha,\cdot)). \tag{6.2}$$

Remark 6.6. We summarize the ideas in this chapter.

- Note that we can regard the translation group on $BUC(\mathbb{R})$ as a subgroup of the family of operators outlined in Lemma 6.5.
- We consider the uniformity $\mathcal{U}_\mathbb{R}$ on $\mathbb{R}$ because it is the only uniformity on $\mathbb{R}$ in which Ω is the completion of $\mathbb{R}$; it is also the weakest uniformity on $\mathbb{R}$ in which each $f \in BUC(\mathbb{R})$ is uniformly continuous in the sense of Definition 6.2.
- In Chapter 2 we determined that $BUC(\mathbb{R})$ is *$*$-isometrically isomorphic* to the space $C(\Omega)$ of all continuous, complex functions on Ω. In this chapter we verify this isometry in another way; that is, the fact that Ω is the completion of $(\mathbb{R}, \mathcal{U}_\mathbb{R})$ also illustrates the isometry between $BUC(\mathbb{R})$ and $C(\Omega)$.
- Note that the uniformity $\mathcal{U}_\mathbb{R}$ is weaker than the natural uniformity on $\mathbb{R}$ given by the metric $d(x,y) = |x-y|$, $x, y \in \mathbb{R}$, in spite of the fact the uniform topology of $\mathbb{R}$, $\tau_{\mathcal{U}_\mathbb{R}}$, coincides with the usual topology of $\mathbb{R}$. However, $\mathbb{R}$ is not complete with respect to $\tau_{\mathcal{U}_\mathbb{R}}$.

- We asserted that φ is not a group operation on Ω; to show this, we need the following:

  1. Suppose G is a *compact Hausdorff group*. Then G is a topological group if and only if the group operation on G is separately continuous ([Ell]).

  2. Let G be a locally compact abelian group. Suppose α is a continuous homomorphism defined on G, with the property that $|\alpha(g)| = 1$, $g \in G$. If in addition α satisfies the equation

  $$\alpha(hg^{-1}) = \alpha(h)(\alpha(g))^{-1} \quad (h, g \in G),$$

  then α maps G onto the multiplicative group of complex numbers of absolute value 1. Define $\Gamma(G)$ as the *group* of all such homomorphisms on G; $\Gamma(G)$ is called the *group of characters* on G. If $G = (\mathbb{R}, +)$, then every character $\alpha(x)$ on $\mathbb{R}$ is of the form $\alpha(x) = e^{iyx}$, and $\Gamma(\mathbb{R})$ is homeomorphic and isomorphic to $\mathbb{R}$ under the correspondence $e^{iyx} \leftrightarrow y$ ([Loo]).

  3. G is a compact abelian topological group if and only if $\Gamma(G)$ is discrete ([Loo]).

  Now, we argue as follows: Suppose φ is a group operation on Ω. Then, by (1), Ω is a compact abelian topological group; by (3), this implies that $\Gamma(\Omega)$ is discrete. Since $(\mathbb{R}, +) \subseteq (\Omega, \varphi)$, then $\Gamma(\mathbb{R}) \subseteq \Gamma(\Omega)$. But $\Gamma(\mathbb{R}) = \mathbb{R}$, so $\Gamma(\Omega)$ induces the discrete topology on $\mathbb{R}$. However, this contradicts the fact that the topology on $\mathbb{R}$, as a subspace of Ω, is the usual topology.

The above argument not only shows that φ is not a group operation on Ω, it also shows that there is no extension g of the map

$$(x, t) \rightarrow (x + t) \text{ of } \mathbb{R} \times \mathbb{R} \rightarrow (\mathbb{R}, +),$$

which renders (Ω, g) a topological group.

Chapter 7 Convolutions

Let $E = C(\Omega)$, the space of continuous functions on Ω. We remind the reader that $E^* = M(\Omega)$, the space of all bounded, regular, complex Borel measures on Ω. Let $\mu_1, \mu_2 \in M(\Omega)$; in this chapter we define their convolution. But first,

Theorem 7.1. *([Bej]). Let X and Y be locally compact Hausdorff spaces; let f be a bounded, separately continuous function on $X \times Y$, and let $M(X \times Y)$ be the space of bounded Borel measures on $X \times Y$. If $\lambda \in M(X \times Y)$, then f is λ- measurable.*

Let $\hat{f} \in E$ and let φ be the semigroup operation defined on Ω (Chapter 6). By Lemma 6.4, the composition function $\hat{f}(\varphi(\alpha, \beta))$, $\alpha, \beta \in \Omega$, is a bounded, separately continuous function on $\Omega \times \Omega$. If $\mu_1, \mu_2 \in E^*$, then Theorem 7.1 asserts that $\hat{f} \circ \varphi$ is $\mu_1 \times \mu_2$-integrable. Thus, by Fubini:

$$\int_{\Omega\times\Omega} \hat{f}(\varphi(\alpha,\beta))\, d(\mu_1 \times \mu_2)(\alpha,\beta) = \int_\Omega \int_\Omega \hat{f}(\varphi(\alpha,\beta))\, d\mu_1(\alpha) d\mu_2(\beta) =$$
$$\int_\Omega \int_\Omega \hat{f}(\varphi(\alpha,\beta))\, d\mu_2(\beta)\, d\mu_1(\alpha). \quad (7.1)$$

Given $\mu_1, \mu_2 \in M(\Omega)$, consider the mapping

$$\hat{f} \to \int_\Omega \int_\Omega \hat{f}(\varphi(\alpha,\beta))\, d\mu_1(\alpha)\, d\mu_2(\beta),$$

of $E \to \mathbb{C}$. The mapping defines a bounded linear functional Φ on E; hence, by the Riesz representation theorem, $\exists \nu \in E^*$ such that

$$\Phi(\hat{f}) = \int_\Omega \hat{f}\, d\nu = \langle \hat{f}, \nu \rangle.$$

Definition 7.2. *We define the convolution of μ_1 and μ_2 as the linear functional*

$\nu = \mu_1 * \mu_2$ *given by*

$$\langle \hat{f}, \mu_1 * \mu_2 \rangle = \int_\Omega \int_\Omega \hat{f}(\varphi(\alpha, \beta))\, d\mu_1(\alpha)\, d\mu_2(\beta) = \int_\Omega \int_\Omega \hat{f}(\varphi(\alpha, \beta))\, d\mu_2(\beta)\, d\mu_1(\alpha) = \langle \hat{f}, \mu_2 * \mu_1 \rangle, \quad \hat{f} \in E. \quad (7.2)$$

Lemma 7.3. *Let $\mu_1, \mu_2 \in E^*$. Then*

1. $\|\mu_1 * \mu_2\| \leq \|\mu_1\|\ \|\mu_2\|$
2. $\mu_1 * \mu_2 = \mu_2 * \mu_1$ *and* $(\mu_1 * \mu_2) * \mu_3 = \mu_1 * (\mu_2 * \mu_3)$
3. $(\mu_1 + \mu_2) * \mu_3 = (\mu_1 * \mu_3) + (\mu_2 * \mu_3)$
4. $(E^*,\ *,\ +)$ *is a commutative Banach algebra with unit.*

Proof. The proof of $(1-3)$ follows from equation (7.2). That $(E^*,\ *,\ +)$ is a commutative Banach algebra follows from $(1-3)$. We show that $(E^*,\ *,\ +)$ has a unit.

Let δ_0 be the Dirac-δ measure concentrated at 0. Then $(\forall \hat{f} \in E)$ $(\forall \beta \in \Omega)$,

$$\delta_0(\hat{f}(\varphi(\cdot, \beta))) = \hat{f}(\varphi(0, \beta)) = \hat{f}(\beta).$$

If Σ_B is the Borel σ-algebra on Ω and $A \in \Sigma_B$, then

$$\delta_0(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A. \end{cases}$$

Clearly, $\delta_0 \in\ (E^*,\ *,\ +)$. Moreover, $\forall \mu \in E^*$,

$$\langle \hat{f}, \delta_0 * \mu \rangle = \int_\Omega \int_\Omega \hat{f}(\varphi(\alpha, \beta))\, d\delta_0(\alpha)\, d\mu(\beta) =$$
$$\int_\Omega \langle \hat{f}(\varphi(\cdot, \beta)), \delta_0 \rangle\, d\mu(\beta) = \int_\Omega \hat{f}(\varphi(0, \beta))\, d\mu(\beta) = \int_\Omega \hat{f}(\beta)\, d\mu(\beta) = \langle \hat{f}, \mu \rangle,$$

which proves that δ_0 is a unit. □

We can conclude from equation (7.2) that

$$\langle \hat{f}, \mu_1 * \mu_2 \rangle = \langle \hat{f}, \int_\Omega T^*(\beta)\mu_1 \, d\mu_2(\beta) \rangle.$$

Indeed, by equation (6.1), $T^*(\beta)\mu_1(\cdot) = \mu_1(\varphi^{-1}(\cdot, \beta))$. It follows that

$$\begin{aligned}\langle \hat{f}, \int_\Omega T^*(\beta)\mu_1 \, d\mu_2(\beta)\rangle &= \int_\Omega \hat{f} d(\int_\Omega T^*(\beta)\mu_1 \, d\mu_2(\beta)) = \\ &\int_\Omega \int_\Omega \hat{f}(u) \, d\mu_1(\varphi^{-1}(u,\beta)) \, d\mu_2(\beta) = \\ &\qquad \int_\Omega \int_\Omega \hat{f}(\varphi(\alpha,\beta)) \, d\mu_1(\alpha) \, d\mu_2(\beta) = \langle \hat{f}, \mu_1 * \mu_2 \rangle.\end{aligned}$$

Remark 7.4. Let $M(\mathbb{R})$ be the Banach space of all complex Borel measures λ on $\mathbb{R}$, with $\|\lambda\| = |\lambda|(\mathbb{R})$. Then the mapping $C_0(\mathbb{R}) \to \mathbb{C}$ given by

$$f \to \int_\mathbb{R} \int_\mathbb{R} f(x+y) \, d\mu(x) \, d\lambda(y), \quad \lambda, \mu \in M(\mathbb{R}),$$

is a bounded, linear functional on $C_0(\mathbb{R})$. Thus, by the Riesz representation theorem, there exists a unique element $\nu = \mu * \lambda$ which satisfies:

$$\int_\mathbb{R} f \, d(\mu * \lambda) = \int_\mathbb{R} \int_\mathbb{R} f(x+y) \, d\mu(x) \, d\lambda(y).$$

But $(\mathbb{R}, +)$ is abelian, so by Fubini's theorem, $\int_\mathbb{R} f \, d(\mu * \lambda) = \int_\mathbb{R} f \, d(\lambda * \mu)$. Ergo, properties $(1-3)$ of Lemma 7.3 are satisfied, and since $\delta_0 \in (M(\mathbb{R}), *, +)$, then it follows that $(M(\mathbb{R}), *, +)$ is a *closed subalgebra* of $(M(\Omega), *, +)$.

Theorem 7.5. *If $\nu \in BUC^{\odot}(\mathbb{R})$ and $\mu \in E^* = M(\Omega)$, then $(\nu * \mu) \in BUC^{\odot}(\mathbb{R})$, i.e., $BUC^{\odot}(\mathbb{R})$ is a closed algebraic ideal in $(E^*, *, +)$.*

Proof. We first show , $(\forall \hat{f} \in E)$ $(\forall \alpha \in \Omega)$,

$$\textbf{(a)} \quad \langle \hat{f}, T^*(\alpha)(\nu * \mu)\rangle = \langle \hat{f}, T^*(\alpha)\nu * \mu\rangle.$$

The above equality follows from

$$\langle \hat{f}, T^*(\alpha)\nu * \mu\rangle = \langle \hat{f}, \int_\Omega T^*(\beta)\, T^*(\alpha)\nu \, d\mu(\beta)\rangle = \langle \hat{f}, \int_\Omega T^*(\alpha)\, T^*(\beta)\nu \, d\mu(\beta)\rangle =$$
$$\langle T(\alpha)\hat{f}, \int_\Omega T^*(\beta)\nu \, d\mu(\beta)\rangle = \langle T(\alpha)\hat{f}, \nu * \mu\rangle = \langle \hat{f}, T^*(\alpha)(\nu * \mu)\rangle.$$

If $t \in \mathbb{R}$, then

$$T^*(t)(\nu * \mu) - \nu * \mu = T^*(t)\nu * \mu - \nu * \mu, \tag{7.3}$$

by (a). Applying Lemma 7.3 -3 to the right-hand side of equation (7.3) yields:

$$T^*(t)\nu * \mu - \nu * \mu = (T^*(t)\nu - \nu) * \mu.$$

Therefore,

$$\|T^*(t)(\nu * \mu) - \nu * \mu\| \le \|T^*(t)\nu - \nu\| \; \|\mu\|;$$

and since the right-hand side of the above inequality tends to zero as $t \longrightarrow 0$, the result follows. □

Remark 7.6. Note that $\delta_0 \notin BUC^{\odot}(\mathbb{R})$. This follows from the equations

$$\langle f, \delta_t\rangle = \langle T(t)f, \delta_0\rangle = \langle f, T^*(t)\delta_0\rangle \quad (f \in BUC(\mathbb{R})),$$

which show that $T^*(t)\delta_0 = \delta_t$, $t \in \mathbb{R}$. Hence, $\|T^*(t)\delta_0 - \delta_0\| = \|\delta_0 - \delta_t\| = 2$ if $t \neq 0$. It follows that $BUC^{\odot}(\mathbb{R}) \neq BUC^*(\mathbb{R})$.

Chapter 8 A characterization of $BUC^{\odot}(\mathbb{R})$

In this chapter we characterize $BUC^{\odot}(\mathbb{R})$ (Theorem A); we will need the following results.

Definition 8.1. *Let (X, Σ, μ) be a finite measure space and let (E, ρ) be a Banach space.*

a) *A function $f : X \to E$ is called μ- measurable if there exists a sequence of simple functions f_n such that*

$$\lim_{n \to \infty} \rho(f_n - f) = 0 \quad \mu.a.e.$$

b) *$f : X \to E$ is weakly μ-measurable if for each $x^* \in E^*$, the scalar function $x^* f$ is μ-measurable.*

c) *A μ-measurable function $f : X \to E$ is Bochner μ-integrable if there exists a sequence of simple functions f_n such that*

$$\lim_{n \to \infty} \left(\int_X \rho(f - f_n) d\mu \right) = 0 \quad ([DUl]).$$

We state the following well-known results.

Theorem 8.2. *(Pettis) A function $f : X \to E$ is μ-measurable if and only if*
i) $\exists W \in \sum$ with $\mu(W) = 0$ such that $f(X \setminus W)$ is norm-separable, and
ii) f is weakly μ-measurable ([Pet]).

Theorem 8.3. *A μ-measurable function $f : X \to E$ is Bochner integrable if and* *only if*

$$\int_X \|f\| d\mu < \infty \quad ([Boc]).$$

Let $0 \leq \mu \in BUC^{\odot}(\mathbb{R})$. Define $F : (\mathbb{R}, |\cdot|) \to (BUC^{\odot}(\mathbb{R}), \|\cdot\|)$ by $F(t;\mu) = T^*(t)\mu$. Then F is a bounded, uniformly continuous function on $(\mathbb{R}, |\cdot|)$, since $\mu \in BUC^{\odot}(\mathbb{R})$. Thus, $(\forall \epsilon > 0)$ $(\exists \delta > 0)$ such that $\|F(t;\mu) - F(t';\mu)\| < \epsilon$ whenever $|t - t'| < \delta$. We claim that F is a uniformly continuous mapping (in the sense of Definition 6.2) of $(\mathbb{R}, \mathcal{U}_{\mathbb{R}})$ into $(BUC^{\odot}(\mathbb{R}), \|\cdot\|)$. Indeed, if $g(t) = \|F(t;\mu)\|$, then $g(t) \in BUC(\mathbb{R})$, and $|g(t) - g(t')| \leq \|F(t;\mu) - F(t';\mu)\| < \epsilon$. It follows that if $V = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |g(x) - g(y)| < \epsilon\}$, then $V \in \mathcal{U}_{\mathbb{R}}$, and

$$|t - t'| < \delta \Rightarrow (t, t') \in V \Rightarrow \|F(t;\mu) - F(t';\mu)\| < \epsilon.$$

This proves our claim; we can therefore extend F continuously to Ω (Theorem 6.3).

Let Σ_B be the σ-algebra of Borel subsets of Ω and let $F(\alpha;\mu)$ be the extension of F. Note that for each $m \in Fix$, the space (Ω, Σ_B, m) is a finite measure space. We claim that F is Bochner m-integrable. To see this, observe that $\forall x^{**} \in BUC^{**}(\mathbb{R})$, the function g on Ω given by $g(\alpha) = x^{**}(F(\alpha;\mu))$, is a bounded, continuous, complex-valued function. Hence, g is m-measurable, so F is weakly m-measurable. Furthermore, the compactness of $F(\Omega;\mu)$, together with the fact that $F(\Omega;\mu) \subseteq (BUC^{\odot}(\mathbb{R}), \|\cdot\|)$, imply that $F(\Omega;\mu)$ is norm-separable. Thus, by Theorem 8.2, F is m-measurable. Consequently, $\|F(\alpha;\mu)\| \leq 1$; thus,

$$\int_\Omega \|F(\alpha;\mu)\| dm(\alpha) \leq (m(\Omega))(\|\mu\|) = \|\mu\| < \infty.$$

It follows by Theorem 8.3 that F is Bochner m-integrable (in the above inequality we assume $m(\Omega) = 1$ (Theorem 4.6)).

Let $0 \leq \mu \in BUC^{\odot}(\mathbb{R})$ and let $A = co\{F(\alpha;\mu) : \alpha \in \Omega\}$, where A is the convex hull of the set $\{F(\alpha;\mu) : \alpha \in \Omega\}$. Then $A \subseteq BUC^{\odot}(\mathbb{R})$. Indeed, if $y \in A$, then $y = \sum_{i=1}^{p} c_i F(\alpha_i;\mu)$ with $\sum_{i=1}^{p} c_i = 1$ and $c_i \geq 0$. But $F(\alpha_i;\mu) \in BUC^{\odot}(\mathbb{R})$,

$1 \leq i \leq p$, so $A \subseteq BUC^{\odot}(\mathbb{R})$. In fact, $BUC^{\odot}(\mathbb{R})$ contains the norm-closure $\bar{A}$ of A, since $BUC^{\odot}(\mathbb{R})$ is a norm-closed subspace of $BUC^{*}(\mathbb{R})$.

Lemma 8.4. $(\forall m \in Fix)$ $(\forall \mu \in BUC^{\odot}(\mathbb{R}))$,

$$\mu_m = \int_\Omega F(\alpha;\mu)\ dm(\alpha) \in \bar{A}.$$

Proof. We will first show, $(\forall \alpha \in \Omega)$ $(\forall y^{**} \in BUC^{**}(\mathbb{R}))$,

$$\left\langle \int_\Omega F(\alpha;\mu)\ dm(\alpha), y^{**} \right\rangle = \int_\Omega \langle F(\alpha;\mu), y^{**}\rangle\, dm(\alpha) \tag{8.1}$$

Recall that there exist simple functions

$s_n = \sum_{i=1}^{p_n} c_{i,n}\chi_{E_{i,n}}$ $n = 1, 2, \ldots,$ and a positive integer N, such that

$$\int_\Omega \|F(\alpha;\mu) - \sum_{i=1}^{p_n} \chi_{E_{i,n}}(\alpha)\| dm(\alpha) < \epsilon, \quad n \geq N.$$

Thus,

$$[\langle \int_\Omega F(\alpha;\mu)\ dm(\alpha), y^{**}\rangle] - [\int_\Omega \langle F(\alpha;\mu), y^{**}\rangle\ dm(\alpha)] =$$
$$\left[\left\langle \int_\Omega (F(\alpha;\mu) - \sum_{i=1}^{p_N} c_{i,N}\chi_{E_{i,N}}(\alpha))\ dm(\alpha), y^{**} \right\rangle\right] +$$
$$\left[\int_\Omega \left\langle \sum_{i=1}^{p_N} c_{i,N}\chi_{E_{i,N}}(\alpha) - F(\alpha;\mu), y^{**} \right\rangle\right].$$

It follows that

$$\|\left(\left\langle \int_\Omega F(\alpha;\mu)\ dm(\alpha), y^{**} \right\rangle\right) - \left(\int_\Omega \langle F(\alpha;\mu)\ , y^{**}\rangle\ dm(\alpha)\right)\| \leq 2\epsilon\|y^{**}\|.$$

Letting $\epsilon \longrightarrow 0$, we see that equation (8.1) holds.

Suppose $\mu_m \notin \bar{A}$. Then by the Hahn-Banach theorem, $\exists y^{**} \in BUC^{**}(\mathbb{R})$ such that the following dual relations hold:

$$\langle \mu_m, y^{**} \rangle = 1 \text{ and } (\langle y, y^{**} \rangle) < 1, \; y \in \bar{A}.$$

In particular, $(\langle F(\alpha;\mu), y^{**} \rangle) < 1, \; \alpha \in \Omega$. Thus,

$$1 = (\langle \mu_m, y^{**} \rangle) = \left(\int_\Omega \langle F(\alpha;\mu) \; dm(\alpha), y^{**} \rangle \right) = \left(\int_\Omega \langle F(\alpha;\mu), y^{**} \rangle \; dm(\alpha) \right) < \int_\Omega \; dm(\alpha) = m(\Omega) = 1.$$

Therefore, $\mu_m \in \bar{A}$. □

Let $M = C_0^{\odot}(\mathbb{R}) \oplus \{Fix\}$. In Lemma 4.7 we determined that

$$BUC^{\odot}(\mathbb{R}) = M \oplus (M^d \cap BUC^{\odot}(\mathbb{R})).$$

Lemma 8.5. *If* $0 \le \mu \in (M^d \cap BUC^{\odot}(\mathbb{R}))$, *then* $\forall m \in Fix$,

$$\inf\{\mu_m, m\} = 0, \; \textit{i.e.}, \; \mu_m \perp m.$$

Proof. We will apply the following (Birkhoff) identity:

Let V be a vector lattice and let $\lambda, \beta, \tau \in V$. Then

$$|\sup(\lambda,\tau) - sup(\beta,\tau)| + |\inf(\lambda,\tau) - \inf(\beta,\tau)| = |\lambda - \beta| \quad ([\text{LuZ}]).$$

$\mu \in (M^d \cap BUC^{\odot}(\mathbb{R})) \Rightarrow \mu \perp C_0^{\odot}(\mathbb{R})$ and $\mu \perp \{Fix\}$. In particular, $\mu \perp Fix$. But the operators $T^*(t)$, $t \in \mathbb{R}$, are lattice isomorphisms (Chapter 2), so $T^*(t)\mu \perp T^*(t)m, \; m \in Fix$. Since $T^*(t)m = m$, by the definition of Fix, then

$$\forall t \in \mathbb{R}, \quad (F(t;\mu) = T^*(t)\mu) \perp m. \tag{8.2}$$

As previously mentioned, the function $F(\alpha;\mu)$, $\alpha \in \Omega$, $\mu \in BUC^{\odot}(\mathbb{R})$, is continuous. Hence,

$$\|F(t_l;\mu) - F(\alpha;\mu)\| \to 0 \text{ for every net } (t_l) \subseteq \mathbb{R} \text{ with } t_l \overset{w*}{\to} \alpha. \tag{8.3}$$

Using the Birkhoff identity, with $\lambda = F(t_l;\mu)$, $\beta = F(\alpha;\mu)$, and $\tau = m$, yields:

$$|\inf(F(t_l;\mu), m) - \inf(F(\alpha;\mu), m)| \le |F(t_l;\mu) - F(\alpha;\mu)|.$$

Since $\inf\{F(t_l;\mu), m\} = 0$, by equation (8.2), then

$$|\inf(F(\alpha;\mu), m)| \le |F(t_l;\mu) - F(\alpha;\mu)|.$$

But, the right-hand side of the above inequality tends to 0 as $t_l \overset{w*}{\to} \alpha$; therefore, $(\forall m \in Fix)$ $(\forall \alpha \in \Omega)$, $F(\alpha;\mu) \perp m$. But,

$$\mu_m = \int_\Omega F(\alpha;\mu)\, dm(\alpha) \in \bar{A}, \quad \text{by Lemma 8.4,}$$

so $\mu_m \perp m$. □

Lemma 8.6. $(\forall m \in Fix)$ $(\forall\ 0 \le \mu \in BUC^{\odot}(\mathbb{R}))$, $\mu_m \ll m$, *i.e., μ_m is absolutely continuous with respect to m.*

Proof. Note that for each $\alpha \in \Omega$, $F(\alpha;\mu)$ is a bounded linear operator on $BUC^{\odot}(\mathbb{R})$; to reflect this property, we will write $F(\alpha)\mu$ in place of $F(\alpha;\mu)$.

For each $\hat{f} \in C(\Omega)$,

$$\langle \hat{f}, \mu_m \rangle = \langle \hat{f}, \int_\Omega F(\alpha)\mu\, dm(\alpha)\rangle = \int_\Omega \langle \hat{f}, F(\alpha)\mu\rangle\, dm(\alpha) = \int_\Omega \langle F^*(\alpha)\hat{f}, \mu\rangle\, dm(\alpha) = \int_\Omega (\int_\Omega F^*(\alpha)\hat{f}(\beta)\, d\mu(\beta))dm(\alpha).$$

Observe that $(\forall f \in BUC(\mathbb{R}))$ $(\forall x, t \in \mathbb{R})$, $f(x+t) = T(t)f(x) = T(x)f(t)$ implies

$F^*(\alpha)\hat{f}(\beta) = F^*(\beta)\hat{f}(\alpha)$; thus, it follows by Fubini's theorem,

$$\int_\Omega (\int_\Omega F^*(\alpha)\hat{f}(\beta)\; d\mu(\beta))dm(\alpha) = \int_\Omega (\int_\Omega F^*(\beta)\hat{f}(\alpha)\; d\mu(\beta))dm(\alpha) = \int_\Omega (\int_\Omega F^*(\beta)\hat{f}(\alpha)\; dm(\alpha))d\mu(\beta).$$

But, $\langle F^*(\beta)\hat{f}, m\rangle = \langle \hat{f}, F(\beta)m\rangle = \langle \hat{f}, m\rangle$; therefore,

$$\langle \hat{f}, \mu_m\rangle = \int_\Omega (\int_\Omega F^*(\beta)\hat{f}(\alpha)\; dm(\alpha))d\mu(\beta) = \mu(\Omega)\int_\Omega \hat{f}(\alpha)\; dm(\alpha) = \mu(\Omega)\langle \hat{f}, m\rangle.$$

This shows $\mu_m \ll m$. □

We are now in a position to complete the proof of our main theorem.

Theorem A. $BUC^{\odot}(\mathbb{R}) = M$.

Proof. Having determined that $BUC^{\odot}(\mathbb{R}) = M \oplus M^d \cap BUC^{\odot}(\mathbb{R})$ (Lemma 4.7), we need only show $M^d \cap BUC^{\odot}(\mathbb{R}) = \{0\}$. First, assume $0 \leq \mu \in M^d \cap BUC^{\odot}(\mathbb{R})$. On the one hand, $\mu_m \perp m$, by Lemma 8.5. But on the other hand, $\mu_m \ll m$, by Lemma 8.6. Therefore, $\mu_m = 0$. But if $\hat{e}$ is the *unit* in $C(\Omega)$, then

$$\langle \hat{e}, \mu_m\rangle = \int_\Omega \int_\Omega dm(z)\; d\mu(\alpha) = m(\Omega)\mu(\Omega) \neq 0.$$

Now consider complex $\mu \in M^d \cap BUC^{\odot}(\mathbb{R})$. Note that $|\mu|$ is contained in $M^d \cap BUC^{\odot}(\mathbb{R})$; this follows from the fact that $M^d \cap BUC^{\odot}(\mathbb{R})$ is a band in $BUC^*(\mathbb{R})$. Thus, we obtain a contradiction by applying the previous argument to $|\mu|$. □

Chapter 9 Weakly almost periodic functions on $\mathbb{R}$

Our objective in this chapter is to characterize the sun-dual of the Banach space of *weakly almost periodic functions* on $\mathbb{R}$. Several results due to [Eb1] (Definition 9.1 and Theorems 9.2-9.4) will help us accomplish this.

Definition 9.1. *Let $B(\mathbb{R})$ be the space of bounded, continuous, complex functions on $\mathbb{R}$ (with sup-norm). Then $f \in B(\mathbb{R})$ is a weakly almost periodic function on $\mathbb{R}$ ($WAP(\mathbb{R})$), if the set of translates $\{T(t)f(x) : t \in \mathbb{R}\}$ is a conditionally weakly compact subset of $B(\mathbb{R})$.*

Because there is no descriptive representation of the linear functionals on $B(\mathbb{R})$, it is not always simple to determine the weak almost periodicity of a given function. This is true, in spite of the fact that in the weak topology of any Banach space $\mathcal{E}$, you need only distinguish between compactness and conditional compactness ([Eb2] and [Smu]). Of course, if we consider $f \in B(\mathbb{R})$ which vanish at infinity, then not only does the Riesz representation theorem give a precise description of the linear functionals associated with these functions, but as a consequence of this theorem we obtain the result that in $C_0(\mathbb{R})$ weak convergence is equivalent to boundedness and pointwise convergence. If we apply this result to the set of $C_0(\mathbb{R})$-translates, $\{f(x+t) : t \in \mathbb{R}\}$, we obtain:

Theorem 9.2. $C_0(\mathbb{R}) \subseteq WAP(\mathbb{R})$.

The countable additivity and translation invariance of Lebesgue measure on $\mathbb{R}$ imply that every $f \in WAP(\mathbb{R})$ is uniformly continuous. Thus,

Theorem 9.3. *If $f \in WAP(\mathbb{R})$, then $f \in BUC(\mathbb{R})$.*

Theorem 9.4. *$WAP(\mathbb{R})$ is a closed, invariant C^*-subalgebra of $B(\mathbb{R})$ (and $BUC(\mathbb{R})$), with the involution operation given by complex conjugation (by invariant we mean $f(x) \in WAP(\mathbb{R}) \Rightarrow \forall t \in \mathbb{R},\ T(t)f(x) = f(x+t) \in WAP(\mathbb{R})$).*

Consider

$$U_y(f(x)) = \frac{1}{y}\int_0^y f(x+s)\,ds, \quad x, y \in \mathbb{R}, \quad f \in BUC(\mathbb{R}).$$

In Chapter 5 we determined that $\|U_y\| = 1$, and $\lim_{y\to\infty} U_y(f(x)) = 0$ if $f \in C_0(\mathbb{R})$. In fact, if $f \in WAP(\mathbb{R})$, then $\lim_{y\to\infty} U_y f(x)$ exists uniformly in x, and is constant ([Eb1]). We outline a sketch of this fact: By Definition 9.1, for each $f \in WAP(\mathbb{R})$, the set of translates $A = \{T(t)f(x) : t \in \mathbb{R}\}$, is conditionally weakly compact. Then by [Eb2] and [Smu], $B_f = \overline{coA}$ = closed convex hull of A in $B(\mathbb{R})$, is weakly compact. Note that B_f is a convex subset of $WAP(\mathbb{R})$, which is invariant under the translation group $G = \{T(t) : t \in \mathbb{R}\}$, i.e.,

$$\forall t \in \mathbb{R}, \quad T(t)(B_f) \subseteq B_f.$$

Thus, by a fixed point theorem of [Mar] and [Ka2], there exists a $g \in B_f \subseteq WAP(\mathbb{R})$ such that $T(t)g = g$. Note that g is a constant, since the only fixed points of G are the constant functions.

[Eb1] showed that $g = \lim_{y\to\infty} U_y(f(x))$, uniformly in x. To illustrate this we will need the following two facts:

1. $\forall f \in WAP(\mathbb{R}),\ U_y(f) \in B_f$,
2. $(\forall t \in \mathbb{R})\ (\forall f \in WAP(\mathbb{R}))$, $\lim_{y\to\infty} U_y(T(t)f) = \lim_{y\to\infty} U_y(f)$.

The proof of (1) requires uniform continuity to guarantee the uniform in x approximation of $U_y(f(x))$ by Riemann sums. That (2) is true follows by the inequality

$$\|U_y(T(t)f) - U_y(f)\| \leq 2\,\frac{\|f\|\ |t|}{|y|}.$$

Now we argue as follows: Since B_f is a weakly closed, convex subset of the locally convex space $WAP(\mathbb{R})$, then B_f is closed in the norm-topology of $WAP(\mathbb{R})$ ([Ru]). Hence, $g \in B_f$ implies,

$$(\forall \epsilon > 0) \quad (\exists \sum_{i=1}^{p} c_i\, T(t_i)f(\cdot) \in B_f) \text{ such that } \|g - \sum_{i=1}^{p} c_i T(t_i)f(\cdot)\| < \frac{\epsilon}{2}.$$

By fact number (2), $\exists y_0 \in \mathbb{R}$ such that

$$\|U_y(T(t_i)f) - U_y(f)\| < \frac{\epsilon}{2}, \quad y \geq y_0, \quad 1 \leq i \leq p.$$

Since $U_y(g) \in B_g$ by (1), then $U_y(g) = g$, so

$$\|g - U_y(f)\| \leq [\|U_y(g - \sum_{i=1}^{p} c_i\, T(t_i)f)\| + \sum_{i=1}^{p} c_i \|U_y(T(t_i)f) - U_y(f)\|] < \epsilon,$$

$y \geq y_0$, $1 \leq i \leq p$. Thus, $g = \lim_{y \to \infty} U_y f(x)$, uniformly in x.

Define

$$U_\infty(f) = \lim_{y \to \infty} [\frac{1}{y} \int_0^y f(x+u)\, du].$$

Then U_∞ is a linear transformation of $WAP(\mathbb{R})$ into itself; $\|U_\infty\| = 1$; $U_\infty(C_0(\mathbb{R})) = 0$, and by (2),

$$U_\infty(T(t)f) = \lim_{y \to \infty} U_y(T(t)f) = \lim_{y \to \infty} U_y(f) = U_\infty(f).$$

Therefore, U_∞ is a unique, translation invariant linear functional on $WAP(\mathbb{R})$.

Let $0 \leq m \in Fix$, with $\|m\| = 1$. Clearly, the restriction of all such m to $WAP(\mathbb{R})$ is the *unique* mean U_∞. We claim that

$$WAP^{\odot}(\mathbb{R}) = C_0^{\odot}(\mathbb{R}) \oplus \{U_\infty\}, \text{ where } \{U_\infty\} \text{ is the band generated by } U_\infty.$$

Before we prove this, we give a synopsis of the pertinent information regarding

$WAP(\mathbb{R})$ and $WAP^*(\mathbb{R})$, that we implement to support our claim.

Let B_0 denote the unit ball in $WAP^*(\mathbb{R})$; define a uniformity $\mathcal{U}_0$ on B_0 with respect to functions in $WAP(\mathbb{R})$, which in turn induces a uniformity $\mathcal{U}_{\mathbb{R}_0}$ on $\mathbb{R}$ (observe that if $\mathcal{U}$ is the uniformity on the unit ball B in $BUC^*(\mathbb{R})$, defined in terms of functions in $BUC(\mathbb{R})$, then $\mathcal{U}_0$ is weaker than $\mathcal{U}$, since $BUC(\mathbb{R}) \supseteq WAP(\mathbb{R})$ (Theorem 9.3)). Using the same approach as in Chapter 6, let $({\mathbb{R}_0}^*, \mathcal{U}_{{\mathbb{R}_0}^*})$ be the completion of $(\mathbb{R}, \mathcal{U}_{\mathbb{R}_0})$. Then ${\mathbb{R}_0}^*$ is a weak*-closed subspace of B_0; hence it is compact. If Ω_0 is the maximal ideal space of $WAP(\mathbb{R})$, and $\mathbb{R}$ is embedded in Ω_0 as a weak*-dense subspace, then $\Omega_0 = {\mathbb{R}_0}^*$ (note that Ω, the maximal ideal space of $BUC(\mathbb{R})$, is contained in Ω_0).

Consider the map

$$(x,t) \xrightarrow{\varphi} (x+t) \text{ of } \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

Then φ can be extended over $\Omega_0 \times \Omega_0$ so that it is separately continuous in each variable (Lemma 6.4). Let ζ be the extension. If $\hat{f} \in C(\Omega_0)$, then the composition of $\hat{f}$ and ζ, $\hat{f}(\zeta)$, is a separately continuous function on $\Omega_0 \times \Omega_0$ (Lemma 6.4).

Let $\tau_1, \tau_2 \in C^*(\Omega_0)$. Recall that $C^*(\Omega_0)$ is identified with $M(\Omega_0)$—the space of bounded, regular, Borel measures on Ω_0. The linear functional defined on $C(\Omega_0)$ by

$$\hat{f} \xrightarrow{\Psi} \int_{\Omega_0} \int_{\Omega_0} \hat{f}(\zeta(\alpha,\beta))\, d\tau_1(\alpha)\, d\tau_2(\beta),$$

is bounded; thus, $\exists \nu \in C^*(\Omega_0)$ such that $\langle \hat{f}, \nu \rangle = \Psi(\hat{f})$. Let $\nu = \tau_1 * \tau_2$. Since $\hat{f}(\zeta)$ is measurable with respect to the product measure $\tau_1 \times \tau_2$ (Theorem 7.1), then

$$\langle \hat{f}, \tau_1 * \tau_2 \rangle = \int_{\Omega_0 \times \Omega_0} \hat{f}(\zeta(\alpha,\beta))\, d(\tau_1 \times \tau_2) = \langle \hat{f}, \tau_2 * \tau_1 \rangle.$$

We define $\tau_1 * \tau_2$ as the convolution of τ_1 and τ_2 (Definition 7.2); and we conclude that $M(\Omega_0, *, +)$ is a commutative Banach algebra with unit, which contains $WAP^{\odot}(\mathbb{R})$ as a closed algebraic ideal.

We now have everything in place to prove

Theorem B. $WAP^{\odot}(\mathbb{R}) = C_0^{\odot}(\mathbb{R}) \oplus \{U_\infty\}$.

Proof. We first summarize essential properties of $WAP(\mathbb{R})$ and $WAP^*(\mathbb{R})$. The Banach lattice $WAP(\mathbb{R})$ is an abstract **M**-space, so $WAP^*(\mathbb{R})$ is an abstract **L**-space (Chapter 2) $\Rightarrow$ $WAP^*(\mathbb{R})$ has order continuous norm (Chapter 2) $\Rightarrow$ $WAP^*(\mathbb{R})$ is Dedekind complete (*Definition (iii)*) $\Rightarrow$ every band $S \subseteq WAP^*(\mathbb{R})$ is a projection band (Theorem 2.5(b)). By Theorem 2.8, $WAP^{\odot}(\mathbb{R})$ is a band in $WAP^*(\mathbb{R})$; thus,

$$U_\infty \in WAP^{\odot}(\mathbb{R}) \Rightarrow \{U_\infty\} \subseteq WAP^{\odot}(\mathbb{R}).$$

Using the fact that $C_0(\mathbb{R}) \subseteq WAP(\mathbb{R})$, we embed the *band* $C_0^{\odot}(\mathbb{R})$ in $WAP^*(\mathbb{R})$, and assert that it also a band in $WAP^{\odot}(\mathbb{R})$ (Lemma 4.7). Consequently, the direct sum $\{U_\infty\} \oplus C_0^{\odot}(\mathbb{R})$ is a band in $WAP^*(\mathbb{R})$ and $WAP^{\odot}(\mathbb{R})$ (Theorem 2.5); hence, à la Lemma 4.7, if $\widetilde{M} = C_0^{\odot}(\mathbb{R}) \oplus \{U_\infty\}$, then

$$WAP^*(\mathbb{R}) = \widetilde{M} \oplus (\widetilde{M})^d \text{ and}$$

$$WAP^{\odot}(\mathbb{R}) = \widetilde{M} \oplus ((\widetilde{M})^d \cap WAP^{\odot}(\mathbb{R})).$$

It remains to show $(\widetilde{M})^d \cap WAP^{\odot}(\mathbb{R}) = \{0\}$.

Suppose $0 \leq \tau \in (\widetilde{M})^d \cap WAP^{\odot}(\mathbb{R})$. Consider the map

$$G : (\mathbb{R}, |\cdot|) \to (WAP^{\odot}(\mathbb{R}), \|\cdot\|),$$

given by $G(t)\tau = T^*(t)\tau$. Then $\forall t \in \mathbb{R}$, $G(t)$ is a bounded linear operator on $WAP^{\odot}(\mathbb{R})$. Note that as a function of t, $G(t)\tau$ is a bounded, continuous function on $\mathbb{R}$; thus, we can extend G to Ω_0. We claim that as a function defined on Ω_0, G has the following form:

$$G(\alpha)\tau = \lim_{l} T^*(t_l)\tau = T^*(\alpha)\tau,$$

where the collection $\{T^*(\alpha) : \alpha \in \Omega_0\}$ is the dual of the family $H = \{T(\alpha) : \alpha \in \Omega_0\}$, and where each element in H is defined on $C(\Omega_0)$ by

$$T(\alpha)\hat{f}(\beta) = \hat{f}(\zeta(\alpha, \beta)).$$

To see this, recall that $(\forall x \in \mathbb{R})$ $(\forall f \in WAP(\mathbb{R}))$, $\hat{f}(\zeta(\alpha, x)) = \lim_l f(x + t_l)$, where (t_l) is a real-valued net such that $t_l \stackrel{w*}{\rightarrow} \alpha$, and $\hat{f}$ is the extension of f to Ω_0. It follows that since $f \in WAP(\mathbb{R})$, then the net $(f(x + t_l))$ has a weakly convergent subnet $(f(x + t_{l(\nu)}))$. Hence,

$$\langle f, G(t_{l(\nu)})\tau\rangle = \langle f, T^*(t_{l(\nu)})\tau\rangle = \langle T(t_{l(\nu)})f, \tau\rangle \rightarrow \langle \hat{f}(\zeta(\alpha, \cdot)), \tau\rangle = \langle T(\alpha)\hat{f}, \tau\rangle = \langle \hat{f}, T^*(\alpha)\tau\rangle.$$

From Chapter 7 ("Convolutions") and the above discussion, if m_∞ is the regular Borel measure on Ω_0 corresponding to U_∞, then

$$\int_{\Omega_0} T^*(\alpha)\tau(\cdot)\, dm_\infty(\alpha) = (\tau * m_\infty)(\cdot) = (m_\infty * \tau)(\cdot) = \int_{\Omega_0} T^*(\alpha)m_\infty(\cdot)\, d\tau(\alpha) = \int_{\Omega_0} m_\infty(\cdot)\, d\tau(\alpha).$$

Thus, $(\tau * m_\infty) \ll m_\infty$. Furthermore, a similar argument to the one used in Lemma 8.5 shows,

$$(\tau * m_\infty) \perp m_\infty.$$

Consequently, $\tau * m_\infty = 0$. But $\langle e, \tau * m_\infty\rangle \neq 0$, if e is the unit of $WAP(\mathbb{R})$; ergo,

$$(\widetilde{M})^d \cap WAP^{\odot}(\mathbb{R}) = \{0\}.$$

For complex $\tau \in (\widetilde{M})^d \cap WAP^{\odot}(\mathbb{R})$, observe that since $\widetilde{M})^d \cap WAP^{\odot}(\mathbb{R})$ is a band in $WAP^*(\mathbb{R})$, then $|\tau| \in \widetilde{M})^d \cap WAP^{\odot}(\mathbb{R})$. We therefore obtain a contradiction by applying the above argument to $|\tau|$. □

Remark 9.5. For each $\mu \in BUC^{\odot}(\mathbb{R})$, let us again consider the map

$$F : (\mathbb{R}, |\cdot|) \to (BUC^{\odot}(\mathbb{R}), \|\cdot\|),$$

defined by $F(t)\mu = F(t;\mu) = T^*(t)\mu$ (Chapter 8). Recall that we can extend $F(t)\mu$ continuously to Ω. If $F(\alpha)\mu$ is the extension, we claim

$$F(\alpha)\mu \neq T^*(\alpha)\mu,$$

where $\forall \alpha \in \Omega$, $T^*(\alpha)$ is the dual of the operator $T(\alpha)$, and

$$\forall \hat{f} \in C(\Omega), \quad T(\alpha)\hat{f}(\beta) = \hat{f}(\varphi(\alpha,\beta)).$$

Indeed, if $m, m_0 \in Fix$ with $m(\Omega) = m_0(\Omega) = 1$, and $F(\alpha)\mu = T^*(\alpha)\mu$, then $\forall \hat{f} \in C(\Omega)$,

$$\langle \hat{f}, m\rangle = \langle \hat{f}, \int_\Omega T^*(\alpha)m\, dm_0(\alpha)\rangle = \langle \hat{f}, m * m_0\rangle =$$
$$\langle \hat{f}, m_0 * m\rangle = \langle \hat{f}, \int_\Omega T^*(\alpha)m_0\, dm(\alpha)\rangle = \langle \hat{f}, m_0\rangle.$$

Therefore, $\langle \hat{f}, m\rangle = \langle \hat{f}, m_0\rangle$, so that $m = m_0$, which implies that the set Fix has one element. However, in Chapter 5 we showed that Fix has plenty of elements; therefore, $F(\alpha)\mu \neq T^*(\alpha)\mu$.

Let $AP(\mathbb{R})$ be the space of almost periodic functions on $\mathbb{R}$, i.e., $f \in AP(\mathbb{R})$ if the set of translates $\{f(x+t) : t \in \mathbb{R}\}$ is conditionally compact. Then

$$AP(\mathbb{R}) \subseteq WAP(\mathbb{R}) \subseteq BUC(\mathbb{R}) \subseteq B(\mathbb{R}).$$

Note that $AP(\mathbb{R}) \cap C_0(\mathbb{R}) = \emptyset$. If $b(\mathbb{R})$ is the Bohr compactification of $\mathbb{R}$, then

$AP^*(\mathbb{R}) = M(b(\mathbb{R}))$, the space of bounded, regular, complex Borel measures on $b(\mathbb{R})$.

It was shown by [Neu] that there exists a unique, translation invariant mean κ on $AP(\mathbb{R})$. It follows immediately that $\kappa = U_\infty$. Moreover, a proof analogous to the one that shows $C_0^{\odot}(\mathbb{R}) = L^1(\mathbb{R})$, yields:

$$AP^{\odot}(\mathbb{R}) = L^1(m_\infty).$$

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