(3, 1)-Surfaces via branched Surfaces

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Abstract

Loosely speaking, a $(n, 1)$-surface is a very nicely immersed π_1-injective surface in a 3-manifold. Its concept was born around 1981 by Peter Scott in his work on Seifet fibered spaces. It has been shown that if a 3-manifold M contains a $(4, 1)$-surface, then its universal cover is $\mathbf{R}^3$ and $\pi_1(M)$ determines M up to homeomorphism. Homotopic homeomorphisms are isotopic on a 3-manifold containing a $(3, 1)$-surface. On the other hand, some class of 3-manifolds, such as manifolds with nonpositive cubing, by Aitchison and Rubinstein, are known to contain $(4, 1)$-surfaces. One natural question, then, is how 'big' is the set of 3-manifolds with $(4, 1)$-surfaces in the set of all 3-manifolds. Similar question for embedded π_1-injective surfaces, called *incompressible* surfaces, has been answered in a work of Floyd and Oertel around 1980. They showed that the set of incompressible surfaces in a 3-manifold is carried by a finite number of branched surfaces. Combining this with a theorem of Hatcher, one can reasonably argue that 3-manifolds containing incompressible surfaces, called Haken manifolds, are scarce. In this paper we prove a similar result in the context of $(3, 1)$-surfaces and non Haken 3-manifolds.

Theorem 1 *If M is a non Haken 3-manifold, then the set of $(3, 1)$-surfaces in M are embeddedly carried by a finite number of branched surfaces.*

'Embeddedly carried' is a precise generalization of 'carried' in the context of immersed surfaces. Careful examination of when the theorem is not true will lead one to obtain a sequence of least area embedded disks in M that limits to an essential measured lamination of M. Such lamination always approximates an incompressible surface in M. In some cases euler characteristic of the lamination is zero, hence M has an essential torus. We strongly suspect this is actually true in all cases. We hope that this method generalizes to the context of $(4, 1)$-surfaces in any 3-manifold. This would establish some kind of finiteness property for $(4, 1)$-surfaces in a 3-manifold, as in the case of incompressible surfaces.

Contents

1 Introduction

Concept of $(n\,,1)$-surfaces in the theory of 3-manifolds was implicitly developed by Peter Scott in [Sc] while studying immersed least area torus in Seifert fibered spaces. Since then it has been shown that if M is a 3-manifold containing $(4\,,1)$-surfaces, then its universal cover is $\mathbf{R}^3$ [HRS] and $\pi_1(M)$ determines M up to homeomorphism [HS1]. Also homotopic homeomorphism is isotopic on a 3-manifold containing $(3\,,1)$-surfaces [HS2]. There have been a few approaches showing that certain class of 3-manifolds contain $(4\,,1)$-surfaces. Most notably, Aitchison and Rubinstein showed that if M has a cubing of nonpositive curvature, then M has a $(4\,,1)$-surface [AR1]. This was used in [AR2] to show that compliment of certain alternating links carries $(4\,,1)$-surface. Although many 3-manifolds constructed there does not contain embedded π_1-injective surfaces, called *incompressible* surfaces, a reasonable description on the 'size' of the set of 3-manifolds containing $(4\,,1)$-surfaces among all 3-manifolds has not been given. A good approach for this problem is to follow the work of Floyd and Oertel [FO]. They showed that embedded π_1-injective surfaces in a 3-manifold is carried by a finite number of, what is called, incompressible branched surfaces. Using this result, Hatcher showed in [H] that boundary of incompressible surfaces in a knot compliment M realize only a finite number of slopes on ∂M. This makes it reasonable to say that 3-manifolds with incompressible surfaces, called *Haken* manifolds, are, in a sense, scarce in the set of all 3-manifolds.

On the other hand, there is a 3-manifold M with a torus boundary such that M has an infinite number of π_1-injective immersed surfaces whose boundary curves realize infinitely many slopes on ∂M [O]. Therefore, the theorem of Floyd and Oertel may not generalize to the set of arbitrary π_1-injective surfaces. However, purpose of this paper is to show that a theorem similar to that of Floyd and Oertel is true if we restrict our attention to the set of $(3,1)$-surfaces in a non Haken manifold. Specifically, our main—and the only—theorem is following.

Theorem 1 *If M is a non Haken 3-manifold, then the set of $(3,1)$-surfaces in M are embeddedly carried by a finite number of branched surfaces.*

The notion of being *embeddedly carried* is an exact and natural generalization of being *carried* as to the context of immersed surfaces. Incompressibility of a branched surface when it's immersed doesn't seem to be clear at this point. But certain properties such as 'no disk of contact after finite splitting' are easily seen to be true for an immersed branched surface that fully carries a $(3,1)$-surface. Our theorem is proved by showing that if the theorem is not true, there is a sequence of least area embedded disks in M that limits to an essential measured lamination of M. Such lamination always approximates an incompressible surface in M. So M has to be Haken. A careful reader will see that 'non Haken' assumption looks very likely to be replaced by 'atoroidal'. This part is being worked on now but not proved yet.

Here is a brief outline of this paper. In section 2 we define what is called a crossed disk and prove a splitting lemma which will be a building block of this paper. In section 3 we study possible wrappings of a crossed disk in M and define a contraction. Using this, another splitting lemma, this time specifically for $(3\,,1)$-surfaces, is proved in section 4. Then a couple of technical lemmas are stated and proved. In section 5 combining previous results we obtain a sequence of embedded disks in M limiting to a measured essential lamination of M. This completes our argument. One hope that this method generalizes to the set of $(4\,,1)$-surfaces for any 3-manifold, hence establishing some sort of finiteness result for $(4\,,1)$-surfaces in a 3-manifold, similar to the theorem of Hatcher mentioned above. We will define basic notations and terminologies before we start.

For an n-manifold X with a triangulation Y and a subspace E, ∂X will denote boundary of X, $N(E)$ denotes a regular neighborhood of E in X and $|E|$ denotes the number of components of E. $\mathring{E}$ denotes the interior of E, and $\overline{X \setminus E}$ will denote the closure of $X \setminus E$ in X under the path metric. $Y^{(i)}$ denotes the i skeleton of Y. X will be equipped with the simplicial metric induced by Y, i.e., length of a simplicial curve γ, denoted by *length* (γ), in X will be computed to be $|\, \gamma \cap Y^{(n-1)} \,|$. If X is an oriented manifold of the form $X' \times I$ for a closed $n-1$ manifold X', then $\partial_+ X$ ($\partial_- X$) will denote the component of ∂X on which the normal vector points outward (inward).

M is a closed and irreducible 3-manifold with a fixed triangulation $\mathcal{T}$. For simplicity, we'll assume that every 3-simplex of $\mathcal{T}$ is embedded in M. It is always possible to get such a triangulation—for example—by the barycentric subdivision. $\tilde{M}$ denotes the universal covering of M with $\pi: \tilde{M} \rightarrow M$ denoting the covering projection. The triangulation on $\tilde{M}$ will refer to the one induced by $\mathcal{T}$.

A *surface* F in M means an immersion $f: F \rightarrow M$ of a compact surface F transverse to $\mathcal{T}$. $\tilde{F}$ or $\tilde{F}_i$ will denote a lift of $f(F)$ in $\tilde{M}$. F is *normal* if $|f(F) \cap \Delta^{(1)}|$ is minimized for each 3-simplex Δ of M. F is *least weight* if $|f(F) \cap \mathcal{T}^{(1)}|$ is minimal in its homotopy class. We shall not distinguish F and $f(F)$ unless necessary. A normal surface in M (or $\tilde{M}$) will be equipped with the induced cellular structure where each 2-cell is a normal disk of M. Two normal surfaces in M (or $\tilde{M}$) are *parallel* if there is a normal homotopy taking one to the other.

A set of embedded planes in $\tilde{M}$ has *n-plane property* if there is a mutually disjoint pair for any set of n planes. It has *1-line property* if intersection of any two planes is empty or an embedded line. A closed surface F in M is a *$(n, 1)$-surface* if lifts of F are embedded planes in $\tilde{M}$ satisfying n-plane and 1-line property.

A *branched surface* is a compact space locally modeled by charts of the form described in Figure 1.1 a). A *branched surface in M* is an immersion of branched surface B into M. It naturally induces a regular neighborhood

$N(B)$ of B with an I–bundle structure (Figure 1.1 b)).

By a *lamination* in M we mean an embedded lamination, i.e., a foliation of a closed subset of M. For definition and properties of an *essential* lamination in M, please refer to [GO].

2 A Splitting Lemma

Definition 2.1 A surface F in M ($f\colon F \longrightarrow M$) is called *embeddedly carried* by a branched surface B in M ($\phi\colon N(B) \longrightarrow M$) if there is an embedding $\psi\colon F \longrightarrow N(B)$ such that $\psi(F)$ is transverse to the I–fibers of $N(B)$ and $f = \phi \circ \psi$ (Figure 2.1).

Definition 2.2 Let $\mathcal{A}$ be a subset of the set of all surfaces in M. $\mathcal{A}$ is *embeddedly carried by a finite number of branched surfaces* if there is a finite number of branched surfaces $B_1, \ldots, B_n$ in M such that any element of $\mathcal{A}$ can be isotoped to be embeddedly carried by one of B_i's.

Definition 2.3 Suppose F is a normal surface in M whose lifts in $\tilde{M}$ are embedded planes. Let $\tilde{F}$ be a lift of F. A *simplicial (metric) disk of radius* R in $\tilde{F}$ centered at a 2-cell x of $\tilde{F}$ is the union of 2-cells of $\tilde{F}$ distance $\leq R$ from x. Note that this is in general a planar surface. A *crossed disk* of F is a pair of parallel simplicial disks (of some radius) D^1 and D^2 in $\tilde{M}$ such that $D^1 \subset \tilde{F}_1$, $D^2 \subset \tilde{F}_2$ for distinct lifts $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_1 \cap \tilde{F}_2 \neq \emptyset$ (Figure 2.3). For a 1-simplex δ of $\mathcal{T}$, let v_δ be the number of 2-simplices adjacent to δ in M

and $v(M)$ be the $max\,\{\, v_\delta \mid \delta$ is a 1-simplex of $\mathcal{T}\,\}$. Let R be any positive integer $> v(M)$. A surface F in M is said to satisfy *R-disk condition* if F is closed and lifts to embedded planes in $\tilde{M}$ containing no crossed disk of radius $\geq R$.

Lemma 2.4 *The set of surfaces in M satisfying R-disk condition is embeddedly carried by a finite number of branched surfaces.*

Proof Suppose F satisfies R-disk condition. Let d_1 and d_2 be normal disks of F in a 3-simplex Δ. Fix a lift $\tilde{\Delta}$ of Δ in $\tilde{M}$ and let $\tilde{d_i}$ be the lifts of d_i in $\tilde{\Delta}$ $(i = 1, 2)$. $\tilde{d_i}$ is contained in some lift $\tilde{F_i}$ of F. Let's denote the simplicial disk of radius X in $\tilde{F_i}$ centered at $\tilde{d_i}$ by $D_X(d_i)$ $(i = 1, 2)$. Now define

$$d_1 \text{ and } d_2 \text{ are } \textit{same R type} \text{ if}$$
$$D_{3R}(d_1) \text{ and } D_{3R}(d_2) \text{ are parallel and either } \tilde{F_1} = \tilde{F_2} \text{ or } \tilde{F_1} \cap \tilde{F_2} = \emptyset.$$

R-disk condition of F implies that this is an equivalence relation on normal disks of F. Each equivalence class will be called a *normal sector* of F. Number of nonparallel embedded disks of a fixed radius in $\tilde{M}$ centered at a same 3-simplex is bounded by a constant which is a function of $\mathcal{T}$. Hence the number of normal sectors of any surface satisfying R-disk condition is uniformly bounded. We will define some words before we proceed further. Two normal sectors s_1 and s_2 lying in a same 3-simplex Δ will be called *separate* if for any $d_1 \in s_1$ and e_1, $e_2 \in s_2$, d_1 doesn't intersect the 3-ball region of Δ bounded by $\partial\Delta$ and $e_1 \cup e_2$ (Figure 2.4 a)). Let σ denote a 2-simplex of M. Two normal sectors s_1 and s_2 will be called *glued along* σ if there are $d_1 \in s_1$, $d_2 \in s_2$ such that $d_1 \cap \sigma$ and $d_2 \cap \sigma$ are identified in F (Figure 2.4 b)). Let δ denote a 1-simplex of M and let σ_i, Δ_i ($i = 1, \ldots, m = v_\delta$) be the 2, 3-simplices of M adjacent to δ. Two normal sectors s and t will be called *glued along* δ if there are normal sectors s_i for $i = 1, \ldots, n$ such that $s = s_1$, $s_n = t$ and s_{i-1} and s_i is glued along σ_{j_i} for some j_i ($1 \leq j_i \leq m$) ($i = 2, \ldots, n$).

Claim If distinct normal sectors s and t in Δ_k ($1 \leq k \leq m$) are glued along δ, then $D_R(d)$ and $D_R(e)$ are parallel and disjoint for any $d \in s$ and $e \in t$. Moreover, s and t are separate.

Proof of Claim By definition there is a sequence of normal sectors $s = t_1$, $t_2, \ldots, t_n = t$ such that t_{i-1} and t_i are glued along some σ_{j_i} ($1 \leq j_i \leq m$) for $i = 2, \ldots, n$. Thus we can find normal disks $d = d_1$, d'_1, d_2, $d'_2, \ldots, d_n$, $d'_n = e$ such that $d_i, d'_i \in t_i$ ($1 \leq i \leq n$) and $d'_{i-1} \cap \sigma_{j_i}$ is identified with $d_i \cap \sigma_{j_i}$ in F for $i = 2, \ldots, n$. Let $\tilde{F}_i$ ($i = 1, \ldots, n$) be the lifts of F containing d_i. Note that $\tilde{F}_i$ also contains d'_{i-1} if $i > 1$ and $F_i \cap F_j = \emptyset$ or $F_i = F_j$ for any $1 \leq i, j \leq n$. This implies that $D_R(d)$ and $D_R(e)$ are disjoint. Also $D_{3R}(d_i)$ is parallel to $D_{3R}(d'_i)$ for all i ($1 \leq i \leq n$) and the distance between d_i and d_j are uniformly bounded by $v(M)$ for any i, j ($1 \leq i, j \leq n$). But $v(M) < R$ by assumption. Therefore, $D_R(d)$ and $D_R(e)$ are parallel. If s and t are not separate, there are normal disks $d_1 \in s$ and d_1, $d_2 \in t$ such that $D_R(d_1)$ lies in between $D_R(d_1)$ and $D_R(d_2)$. But s and t are distinct, so $D_{3R}(d_1)$ is not parallel to $D_{3R}(d_1) \cup D_{3R}(d_2)$, which is impossible. $\square$

Note that same statement is true for normal sectors glued along σ for a 2-simplex σ. For a 1 or 2 simplex x of M and a normal sector s intersecting x, let F_s^x be the set of components of $F \cap N(x)$ that intersects a normal disk d contained in a normal sector t glued to s along x. By the claim, F_s^x is a set of mutually disjoint embedded disks and there are distinct normal sectors s_i ($i = 1, \ldots, k$) intersecting x so that $F \cap N(x) = \cup_{i=1}^k F_{s_i}^x$. Moreover, normal sectors in $F_{s_i}^x$ and $F_{s_j}^x$ are never glued along x if $i \neq j$. Therefore, a common method of constructing a fibered neighborhood, for example in [FO], will work in our setting with R types of normal disks. To be precise, let $G = \partial N(F)$ and we shall construct a branched surface embeddedly carrying G. Same R type normal disks in a 3-simplex of M are disjoint and parallel by definition. Pick two adjacent such disks, say d_0 and d_1, in Δ and provide the 3-ball region between them in Δ with a product I–bundle structure $d \times [0,1]$ so that $d \times 0 = d_0$, $d \times 1 = d_1$ and $\partial d \times [0,1] \subset \partial\Delta$. Do this for all normal disks of G in every 3-simplex of M so that two product structures in adjacent 3-simplices intersect in the I–fibers of each on their boundaries. The singular object obtained this way is immersed. Let B_1 be obtained by crushing each I–fiber to a point. Then we have an immersion $\phi_1 \colon N(B_1) \to M$ that embeddedly carries G. In terms of normal sectors, B_1 is obtained by gluing normal sectors along their boundary to agree with the gluing of corresponding normal disks. By slightly smoothing the singular branched locus of B_1 (Figure 2.4 c)), we get a genuine branched surface B and an immersion ϕ: $N(B) \rightarrow M$. G is embeddedly carried by B. If we had a fixed set of normal sectors to start with, this procedure guarantees a finite number of distinct immersed branched surfaces. Since the number of normal sectors was uniformly bounded, there is a finite set of branched surfaces in M arising from this construction. □

3 Wrapping of Crossed Disks

From now on, F will always denote a normal $(3,1)$-surface. A crossed disk is assumed to be a pair of topological disks rather than planar surfaces. In the beginning of the next section, it will be explained why this causes no problem in our context.

Notation Warning: An object in a crossed disk $D = D^1 \cup D^2$ always consists of a pair $X^1 \cup X^2$ with the convention $X^1 \subset D^1$, $X^2 \subset D^2$. But $X^1 \cup X^2$ may be omitted to just X often for simplicity. Object X in $\tilde{M}$ and its image $\pi(X)$ in M shall not be distinguished unless absolutely necessary. For example, '*D is a subset of F in M*' means '*$\pi(D)$ is a subset of F in M*'.

Definition 3.1 A *wrapping path* of a crossed disk D ($D^1 \cup D^2$) is an oriented path γ ($\gamma^1 \cap \gamma^2$) in D connecting p ($p^1 \cup p^2$) and q ($q^1 \cup q^2$) (p, $q \in D$) with the following property.

1) If d_p $(d_p^1 \cup d_p^2)$ and d_q $(d_q^1 \cup d_q^2)$ are the 2-cells of D containing p and q, their images in M lie in a same 3-simplex Δ and parallel.

2) The 3-ball region of Δ bounded by $d_p^1 \cup d_p^2$ and $\partial\Delta$ has nonempty intersection with $d_q^1 \cup d_q^2$ (Figure 3.1).

Definition 3.2 A *product band* is a compact annulus or mobius band A with an immersion $i\colon A \longrightarrow M$ such that the pull back cellular structure is a product, i.e., $i(A) \cap \mathcal{T}^{(i)} \neq \emptyset$ only for $i = 2, 3$ and the preimages of $i(A) \cap \mathcal{T}^{(2)}$ in A consists of essential embedded arcs. A has an I–bundle structure with components of $i(A) \cap \mathcal{T}^{(2)}$ as its fibers (Figure 3.2).

Definition 3.3 A *wrapping band* A_γ of a wrapping path γ in D is a product band transverse to F containing $\pi(\gamma)$. An I–fiber of A_γ denoted by v_γ containing $\pi(\partial\gamma)$ shall be chosen. A_γ^o will denote the orientable double cover of A_γ when A_γ is nonorientable. We'll assume that v_γ (or a lift in A_γ^o) is oriented so that $\partial_- v \in \partial_- A_\gamma$, $\partial_+ v \in \partial_+ A_\gamma$ (replace A_γ with A_γ^o if nonorientable). The components of $A_\gamma \cap F$ containing $\pi(\gamma)$ will be denoted by α^1 and α^2, meaning $\alpha^1 = \alpha^2$ if there's only one such component (Figure 3.3 a)). Let $\alpha = \alpha^1 \cup \alpha^2$ with a convention $\pi(\gamma^i) \subset \alpha^i$ $(i = 1, 2)$. For two points p, q in v_γ, we say $p > q$ or *p is higher than q* if the subarc v' of v_γ with end points p and q are such that $\partial_- v' = q$, $\partial_+ v' = p$ (Figure 3.3 b)). We say γ is *trivial* if the core of A_γ is null homotopic M.

Lemma 3.4 *For any $r > 0$ and any orientable least weight surface F in M, there is $t(r) \geq 0$ such that if D is a crossed disk of F of radius r, then a subdisk of D of radius $t(r)$ without trivial wrapping path exist in D. Moreover, $t(r) \to \infty$ as $r \to \infty$.*

Proof Suppose otherwise. Then there is a fixed $l > 0$ and very large $N > 0$, such that a crossed disk D of radius $N \cdot l$ exist in F but D has a trivial wrapping path γ of length l lying near the center of D. Since γ is trivial, the 3-plane property forces F to wrap A_γ at least—approximately—N times. This provides an immersed monogon whose area is very small compared to length of its tail which is about $N \cdot l$. The tail touches at least about N distinct 2-cells of F. Since F is orientable the loop theorem will provide an embedded monogon with tail of length at least about N. This is a contradiction to F being least weight. □

We will consider only nontrivial wrapping path for the rest of this section. Also, by enlarging I–fibers and perturbing its position a little bit, A_γ will always be assumed to satisfy the following property.

Suppose Δ is 3-simplex of M intersecting A_γ. A normal disk of F containing components of $(\cup_{i=1}^{2} \alpha^i) \cap \Delta$ will be called a *boundary disk* if $\partial\alpha^i$ is in it ($i = 1$ or 2) and an *interior disk* if otherwise. Then *interior disks are mutually parallel and no boundary disk is parallel to interior disks.*

Lemma 3.5 *Let $x = \alpha^i$ for $i = 1$ or 2 and $|x \cap v_\gamma| = w$ ($w \geq 1$).*

1) When $\partial x = \emptyset$. If A_γ is orientable then $w = 1$. If not, $w = 1$ or 2.

2) When $\partial x \neq \emptyset$. If A_γ is orientable then x is an essential arc unless $w \leq 2$. If not, x lifts to an essential arc in A_γ^o unless $w \leq 4$.

3) Suppose A_γ is orientable. If there is a subarc x' of x such that $\partial x' = \{p, q\} \subset v$, $|v \cap x'| = 3$ and $p > q$, then components of $\overline{x \setminus x'}$ connects p to $\partial_+ A_\gamma$ and q to $\partial_- A_\gamma$.

Proof Suppose A_γ is an annulus. If $\partial x = \emptyset$ or x is an inessential arc, then lifts of x in $\tilde{M}$ show a set of w planes with no disjoint pair. For $w \geq 3$, it violates 3-plane property. When $w = 2$ and $\partial x = \emptyset$, 1-line property provides a double curve ϵ of F such that $\epsilon \cap A_\gamma$ contains double points of x and core of A_γ is homotopic to some power of ϵ. Let M_γ be the covering space of M corresponding to the core of A_γ. A lift of F in M_γ is an immersed noncompact annulus with a compact, hence closed, double curve. But since x has odd number of double points, this is impossible. When A_γ is a mobius band, a lift of x and v_γ in A_γ^o intersect w times if x is essential. If $\partial x = \emptyset$ then w is odd if and only if x is essential, hence $w \leq 2$. If $\partial x \neq \emptyset$ and $w \geq 4$ then x lifts to an essential arc in A_γ^o because of 3-plane property. To prove *3)*, first note that x is essential in A_γ by *2)*. There are three possible configuration of x' depending on the position of the point of x' hitting v_γ between p and q. But each results in a violation of 3-plane property if *3)* is not true. □

Definition 3.6 (Contraction) Suppose γ is a nontrivial wrapping path. Note that γ was oriented, so α has a naturally inherited orientation. With this in mind, a nontrivial wrapping path γ (equivalently, A_γ) is called a *contraction* if either of the following is true.

1) A_γ is orientable and $\alpha^1 \neq \alpha^2$. If $\partial\alpha^i \neq \emptyset$ for $i = 1, 2$ then each α^i is an essential arc in A_γ such that $\partial_-\alpha \cap \partial_- A_\gamma \neq \emptyset$ and $\partial_-\alpha \cap \partial_+ A_\gamma \neq \emptyset$ (Figure 3.6 a)).

2) A_γ is nonorientable. If $\alpha^1 \neq \alpha^2$ and $\partial\alpha^i \neq \emptyset$ for $i = 1, 2$ then α^i is essential in A_γ for $i = 1, 2$ (Figure 3.6 b)).

A_γ or γ is an *open* (*closed*) contraction if it is a contraction and $\partial\alpha^i \neq \emptyset$ for $i = 1$ or 2 ($\partial\alpha^i = \emptyset$ for $i = 1$ and 2).

N will be a large integer and D is a crossed disk large enough so that a subdisk of radius $(1/N) \cdot radius(D)$ concentric in D is defined. $(1/N)\,D$ will denote such a subdisk. Suppose γ is a wrapping path in $(1/N)\,D$. If $\partial D \cap \alpha \neq \emptyset$ then α will be called *long*. Otherwise, α is called *short*. Then the following can be easily verified.

1) If $\partial\alpha^i = \emptyset$ for some $i = 1$ or 2, then α^i is a simple closed curve.

2) If α has one component and $\partial\alpha \neq \emptyset$, then α is long and essential in A_γ.

3) If $\alpha^1 \neq \alpha^2$, $\partial\alpha^1 = \emptyset$ and $\partial\alpha^2 \neq \emptyset$, then α is long.

4) Suppose α is a pair of arcs in A_γ. If γ is a contraction, then α is long and each α^i $(i = 1, 2)$ is embedded in A_γ. If γ is a not a contraction and A_γ is nonorientable, then α is short and not essential in A_γ.

Lemma 3.7 (Homotopy) *For $j = 0, 1$ let $\gamma_j \left(\gamma_j^1 \cup \gamma_j^2\right)$ be wrapping paths in $(1/N)\,D$ connecting $p\ (p^1 \cup p^2)$ and $q\ (q^1 \cup q^2)$. A_j, α_j, v_j denote the corresponding objects for γ_j defined as before. Assume $v_1 = v_2 = v$. Then the following is true:*

1) γ_0 is a contraction if and only if γ_1 is.

2) Suppose A_0 is orientable and not a contraction. α_0 connects $\partial_- A_0$ to $\partial_+ A_0$ if and only if α_1 does (Figure 3.7).

Proof *1)* A_0 is orientable if and only if A_1 is since they are homotopic in $(1/N)\,D$. If A_0 is a contraction then α_0 is long, i.e., D wraps around A_0 at least N times. By 3-plane property, then, the whole subdisk $(1/N)\,D$ of D also wraps itself at least N times, thus α_1 is long, too. Therefore, we can assume that A_0 is orientable. For $i = 1$ or 2, if α_0^i is a closed curve in A_0, so is α_1^i in A_1. If α_0^i is an essential arc in A_0, then, since γ_0 and γ_1 are homotopic, α_0^i and α_1^i intersects v at the same points x_k $(k = 0, \ldots, N)$. Suppose x_k is linearly ordered on α_0^i and α_1^i so that $p^i = x_0$, $q^i = x_1$ and so on. By Lemma 3.5 3) x_0, x_2 and x_4 are linearly ordered on v_0 and that determines how α_0^i and α_1^i connects components of ∂A_0 and ∂A_1 respectively.

2) If both α_0 and α_1 are long, the latter half argument in the proof of

1) works similarly. If not, simply tracking down $\alpha_0 \cup \alpha_1$ in $A_0 \cup A_1$ shows the same conclusion. □

4 A Splitting Lemma for (3 , 1)-Surfaces

If D is a crossed disk in $\tilde{F}_1 \cup \tilde{F}_2$, let $\widehat{D}$ be the object obtained by capping off D with compact components of $\left(\tilde{F}_1 \cup \tilde{F}_2\right) \setminus D$. $\widehat{D}$ is a pair of topological disks. Let n be any positive integer. If D is a crossed disk of radius n in a least weight $(3\,,1)$-surface in M, then there is $k\,(n)\,(\leq n)$ such that for the concentric subdisk E of radius $k\,(n)$, $\widehat{E}$ is also a crossed disk. Moreover, $k\,(n) \to \infty$ as $n \to \infty$. This is due to the fact that $\tilde{M}$ has a bounded geometry as it covers a compact manifold M. Therefore, in any argument used in this paper, without loss of generality, a crossed disk can be assumed to be a pair of topological disks.

Definition 4.1 An *open (closed) contraction radius* of a crossed disk D, denoted by $ocr\,(D)$ ($ccr\,(D)$), is the maximum radius of a sub crossed disk of D without open (closed) contraction path.

Let R be a positive integer $\geq v(M)$ as in section 2. F satisfies *R-contraction condition* if F has no crossed disk D with $ocr\,(D) \geq R$.

Lemma 4.2 *The set of surfaces in M satisfying R-contraction condition is embeddedly carried by a finite number of branched surfaces.*

Proof We will proceed in a similar manner as in the proof of Lemma 2.4. Let F be a least weight $(3,1)$-surface in M satisfying R-contraction condition. Let d_1 and d_2 be normal disks of F in a 3-simplex Δ. Let K be an integer $> 5R$. Notion of d_1 and d_2 being same R type is defined exactly the same as in Lemma 2.4.

$$d_1 \text{ and } d_2 \text{ are same } R \text{ type if}$$
$$D_K(d_1) \text{ and } D_K(d_2) \text{ are parallel and either } \tilde{F}_1 = \tilde{F}_2 \text{ or } \tilde{F}_1 \cap \tilde{F}_2 = \emptyset.$$

We need to show that this is an equivalence relation. The reflexivity and symmetry are clear. To show it's transitive, suppose d_1, d_2 and d_3 are normal disks in Δ such that d_1, d_2 are same R type and so are d_2, d_3. By definition $D_K(d_1)$, $D_K(d_2)$ and $D_K(d_3)$ are mutually parallel. Therefore, if d_1 and d_3 are not same R type, then $\tilde{F}_1 \cap \tilde{F}_3 \neq \emptyset$ which means that $D = D_R(d_1) \cup D_R(d_3)$ and $E = D_K(d_1) \cup D_K(d_3)$ are crossed disks. R-contraction condition says D has an open contraction path. By homotoping the path a bit assume that it goes through $d_1 \cup d_3$ and let A denote the corresponding contraction band. A can be chosen so that components of $(\cup_{i=1}^{3} D_K(d_i)) \cap A$ containing $d_i \cap A$ $(i = 1, 2, 3)$ are long parallel arcs in A. Then the long components of $F \cap A$ fall into two classes where two components are in the same class if and only if either both are closed curves or both are arcs connecting ∂A in the same way. Furthermore, two components are in the same class if and only if they are disjoint in A (3-plane property). But $\tilde{F}_1 \cap \tilde{F}_3 \neq \emptyset$ so either $\tilde{F}_1 \cap \tilde{F}_2 \neq \emptyset$ or $\tilde{F}_2 \cap \tilde{F}_3 \neq \emptyset$.

In the above it is showed that if there is a set of large parallel disks in $\tilde{M}$, then the lifts of F containing them fall into at most two classes where any plane in one class intersects every plane in the other. The same claim as in the proof of Lemma 2.4 can be proved using this property and 3-plane property implies uniform boundedness of the number of normal sectors. $\square$

The rest of this section is devoted to a couple of technical lemmas that will be used in the next section.

Definition 4.3 Let $\tilde{F}_1$ and $\tilde{F}_2$ be distinct lifts of F intersecting in an embedded line $\tilde{l}$. A *flat* arc λ ($\lambda^1 \cup \lambda^2$) of F is a pair of parallel embedded arcs in $\tilde{F}_1 \cup \tilde{F}_2$ such that (Figure 4.3)

1) $\lambda^i \subset \tilde{F}_i$ for $i = 1, 2$. $\partial\lambda = \partial\lambda \cap \tilde{l} = \lambda \cap \tilde{l}$.

2) Let $\tilde{l}_\lambda$ be the compact component of $\tilde{l} \setminus \partial\lambda$ and X be the closed 3-ball region in $\tilde{M}$ bounded by $\tilde{F}_1 \cup \tilde{F}_2$ and $\tilde{l}_\lambda \cup \lambda$. Then either $\tilde{F}_1$ is parallel to $\tilde{F}_2$ in X or no lift of F intersects $\mathring{X}$.

$\tilde{l}_\lambda$ will be called the *companion* of λ. X will be called a *flat region.*

Lemma 4.4 (Flat Arc Shortening) *Suppose F and M are orientable and F minimizes the length of double curves in its homotopy class. If a flat arc λ and its companion $\tilde{l}_\lambda$ lie in $(1/N)D$ for a large crossed disk D of F and large N, then length $(\lambda) \geq$ length $(\tilde{l}_\lambda)$.*

Proof Let $\tilde{F}_i$ $(i = 1, 2)$ and X be as in the definition. Let $l = \pi(\tilde{l})$ and $D^i \subset \tilde{F}_i$ $(i = 1, 2)$. Without loss of generality, we assume that length of λ is minimized in its relative homotopy class in D.

Case 1) No lift of F intersects $\overset{\circ}{X}$.

Let $a = \partial_- \tilde{l}_\lambda$ and a' be the translate of a in $\tilde{l}$ nearest to $\partial_+ \tilde{l}_\lambda$ outside $\tilde{l}_\lambda$. Let $\tilde{l}'_\lambda$ be the subarc of $\tilde{l}$ bounded by $\partial_+\lambda$ and a'. Let $\tilde{m}$ be the arc $\lambda \cup \tilde{l}'_\lambda$ slightly pushed off of $\tilde{l}$ to meet $\tilde{l}$ only in a and a'. $\tilde{m}$ is a pair of arcs and will be denoted by $\tilde{m}_i$ $(i = 1, 2)$. By construction $\pi(\tilde{m}_i) = l^k$ $(i = 1, 2)$ in $\pi_1(M)$ for some integer $k\,(> 0)$. If $\pi(\tilde{m}_1)$ is not embedded in F, then by [HS3, Theorem 2], there is a singular 2-gon E_1 in F bounded by arcs of $\pi(\tilde{m}_1)$ which lifts to an innermost embedded 2-gon in $\tilde{F}_1$. Since F is orientable and no lift of F intersect $\overset{\circ}{X}$, $\pi(\tilde{m}_2)$ bounds a parallel singular 2-gon E_2 in F also innermost and embedded in $\tilde{F}_2$. We claim that $E_1 \cap E_2 = \emptyset$ in M. If it were not, in $\tilde{M}$, a covering translate $E' = E'_1 \cup E'_2$ of E intersect E in such a way that $E'_1 \subset \tilde{F}_2$, $E'_2 \not\subset \tilde{F}_1$ and $E'_1 \cap E_2 \neq \emptyset$ (Figure 4.4 a)). Let t be the element of $\pi_1(M)$ taking E to E'. Since E was in $(1/N)D$ with large N, 3-plane property implies that t and l has to commute. But then action of l on $\tilde{F}_1$ would not be discrete. Therefore, a cut and paste can be performed on ∂E to reduce double points of $\pi(\tilde{m}_i)$ $(i = 1, 2)$ without changing its homotopy class. After removing excessive self intersections of each $\pi(\tilde{m}_i)$ this way, suppose $\pi(\tilde{m}_1) \cap \pi(\tilde{m}_2) \neq \emptyset$. An argument as above says no translate of $\overset{\circ}{X}$ intersects $\overset{\circ}{X}$, in particular, $n = 1$ so we can assume that each $\pi(\tilde{m}_i)$ is embedded and there is an innermost embedded 2-gon E_1 in F bounded by arcs b_1 of $\pi(\tilde{m}_1)$ and b_2 of $\pi(\tilde{m}_2)$. Note that λ was length minimizing, so $length\,(b_1) = length\,(b_2)$. Let $\{\, p_1, p_2 \,\} = b_1 \cap b_2$ with $\partial_+ b_1 = p_1$, $\partial_- b_1 = p_2$. In $\tilde{M}$, assume $E_1 \subset \tilde{F}_1$. Let E_2 be the piece on $\tilde{F}_2$ parallel to E_1, $\partial E_2 = c_1 \cup c_2$ where c_i is parallel to b_i and $\{\, q_1, q_2 \,\} = c_1 \cap c_2$ with p_i parallel to q_i $(i = 1, 2)$. As before the only possible intersection of $E = E_1 \cup E_2$ with its covering translate E' occurs by $t \in \pi_1(M)$ taking $\tilde{F}_2$ to $\tilde{F}_1$ so that $E'_2 \cap E_1 \neq \emptyset$. If $\overset{\circ}{E'_2} \cap \overset{\circ}{E_1} \neq \emptyset$ then t fixes a 3-simplex containing a 2-cell of E_1 so $\overset{\circ}{E'_2} \cap \overset{\circ}{E_1} = \emptyset$. If b'_i, c'_i, p'_1, q'_i denote translates of b_i, c_i, p_i, q_i by t then $b_2 \cap c'_1 = \emptyset$, $b_2 \cap c'_2 = p_1 \in c'_2$, $b_1 \cap c'_1 = q'_2 \in b_1$ and $b_1 \cap c'_2$ is a connected arc with end points p_1 and q'_2 (Figure 4.4 b)). Image of this in M looks as in Figure 4.4 c). Now isotope $b_1 \cup c_1$ across E to be slightly off of $b_2 \cup c_2$ to remove p_1 and p_2 (Figure 4.4 d)). This isotopy preserves length and number of double points of $\pi(\tilde{m})$ and reveals a new pair of innermost 2-gons G sitting on top of E such that $length\,(\partial G) \leq length\,(\partial E)$. If G is not embedded, then repeat the same isotopy on G. In a finite step an embedded pair of 2-gons, also denoted by G, has to appear. Therefore, a cut and paste on ∂G reduce $|\,\pi(\tilde{m}_1) \cap \pi(\tilde{m}_2)\,|$. Let $m = m_1 \cup m_2$ be the pair of closed curves in F homotopic to $\pi(\tilde{m})$ with the minimum number of double points obtained by this process.

When $k = 1$, m is a pair of simple closed curves in F. Hence l and m meet only in $\pi(a)$ and $l \cup m$ bound a pair of embedded disks in F. l can be isotoped to m in F but $length\ (m_1) \geq length\ (l)$ by assumption. Hence $length\ (\lambda) \geq length\ (m_1) - length\,(\tilde{l}'_\lambda) \geq length\ (l) - length\,(\tilde{l}'_\lambda) = length\,(\tilde{l}_\lambda)$. When $k > 1$, m_i is homotopic to $k{\cdot}l$ with k–1 crossings ($i = 1, 2$). A proper cut and paste on double points of m produce k simple closed curves $s_1, \ldots, s_k$, each homotopic to l. Again $\sum_{i=1}^{k} length\ (s_i) \geq n \cdot length\ (l)$ and $\sum_{i=1}^{k} length\ (s_i) = length\,(m_1) \leq length\ (\lambda) + length\,(\tilde{l}'_\lambda)$.

Case 2) Some lift of F intersects $\overset{\circ}{X}$.

$\tilde{F}_1$ and $\tilde{F}_2$ are parallel in X this case. Choose an innermost—sort of—flat region X_1 in X which could have a multi component flat arc (Figure 4.4 e)). Let λ_i and l_i ($i = 1, \ldots ,n$) be the flat arcs and arcs of double curves in ∂X_1. 1-line property says that an embedded line contains $\cup_{i=1}^{n} l_i$. Hence there is some j ($1 \leq j \leq n$), say $j = 1$, and a flat region X_2 containing X_1 with λ_1 as its flat arc. If l'_1 denotes the companion of λ_1 then $\cup_{i=1}^{n} l_i \subset l'_1$. X_1 was innermost so $\overset{\circ}{X}_2$ is clean therefore $length\,(l'_1) \leq length\,(\lambda_1)$. If l'_1 contains l_λ then we are done since $length\,(\lambda_1) < length\,(\lambda)$. Otherwise choose l_p ($1 \leq p \leq n$) such that the region between l_p and l_λ contains no other l_i's ($1 \leq i \leq n$) and replace λ by an arc starting at $\partial_-\lambda$ and following λ till hit l_k then follow l_k till hit λ again then follow λ to end at $\partial_+\lambda$ (Figure 4.4 f)). Length of this new flat arc is $\leq length\,(\lambda)$ with the same companion l_λ. So an induction on the number of planes hitting X completes proof. $\square$

Definition 4.5 An orientable wrapping path γ *non height preserving* if it is a non contraction, $|gg^1 \cap gg^2|$ is even and the relative ordering of $\partial_+\gamma^i$ and $\partial_-\gamma^i$ in v_γ is not the same as that of $\partial_+\alpha^i$ and $\partial_-\alpha^i$ for $i = 1$ or 2. Otherwise it is *height preserving.* (Figure 4.5)

Lemma 4.6 *Let D be a crossed disk of large radius. Suppose there is an orientable wrapping path μ in $(1/N)D$ for large N such that $\mu^1 \cap \mu^2 \neq \emptyset$ in $\tilde{M}$ and $\partial_-\mu^2 = \partial_+\mu^1$ in M. Then every orientable wrapping path in $(1/N)D$ is height preserving.*

Proof Let $D^i \subset \tilde{F}_i$ $(i = 1, 2)$. Suppose γ is a wrapping path in $(1/N)D$ not preserving height. By inspection $\alpha^1 \neq \alpha^2$ and α is long. Pick a point p in $\mu^1 \cap \mu^2$ and choose a path θ in $(1/N)D$ joining p to $\partial_-\gamma$. Then $\theta^{-1} \circ \gamma \circ \theta$ slightly perturbed to be transverse to F will be a wrapping path denoted by η (Figure 4.6 a)). Note that θ^{-1} was meant to denote the path starting at $\partial_+\gamma$ and staying parallel to θ. By construction $|\eta^1 \cap \eta^2|$ is even. Let $l = \tilde{F}_1 \cap \tilde{F}_2$. $p \in l$. Thinking of η as an element of $\pi_1(M)$, let l', $\tilde{F}_i'$ and l'', $\tilde{F}_i''$ $(i = 1, 2)$ be the translate of l by η and η^2 respectively. Let B be the thin transverse neighborhood of η in $\tilde{M}$ since N was large, 3-plane property says that $\tilde{F}_1'' \cup \tilde{F}_2''$ intersects B in a pair of parallel arcs which has to coincide with η since otherwise the action of t would not be discrete. See (Figure 4.6 b)). Therefore, $\tilde{F}_i'' = \tilde{F}_i$ $(i = 1, 2)$ and $l'' = l$. $l' \neq l$ since l is not a closed curve, so η and $\pi(l)$ are non homotopic commuting elements of $\pi_1(M)$. But that is not compatible with the orientation on l and l' by the action of η. □

5 Construction of a Lamination

In this section we complete our main theorem.

Theorem 1 *If M is non Haken, then the set of $(3,1)$-surfaces in M are embeddedly carried by a finite number of branched surfaces.*

Proof We will assume that M is orientable. Suppose the theorem is not true. Then, by Lemma 2.4 and Lemma 4.2, there is an infinite sequence of $(3,1)$-surfaces $\{F_k\}_{k=1}^{\infty}$ in M such that F_k has a crossed disk D_k without open contraction for all $k \in Z_+$ and *radius* $(D_k) \to \infty$ as $k \to \infty$. Since any $(3,1)$-surface in M is isotopic to a least weight representative, F_k can be assumed to be least weight for all $k \in Z_+$. We divide this into two cases depending whether the set $\{ccr\,(D_k) \mid k \in Z_+\}$ is bounded or not. Then Proposition 5.1, Proposition 5.2 and Lemma 5.5 that follows below will prove that M has to be Haken in both case. □

Proposition 5.1 *If $ccr\,(D_k)$ is uniformly bounded for all $k \in Z_+$, then M contains an incompressible torus.*

Proof Let C be an upper bound of $\{ccr\,(D_k) \mid k \in Z_+\}$. Let N be a fixed large integer and k be large enough so that *radius* $(1/N)D_k > C$. Let

$F = F_k$, $D = D_k$, $D^i = D^i_k$, $G = (1/N)\, D_k$ and $G^i = (1/N)\, D^i_k$ $(i = 1, 2)$. Note that 1-line property says that any two closed contraction paths in D are homotopic. A closed contraction γ exists in G by hypothesis and A_γ is orientable by 1-line property. Let M_γ be the covering space of M generated by γ. Image of D in M_γ is a pair of euclidean annulus $T = T_1 \cup T_2$. In other words there is a constant C', a function of N, such that T_i $(i = 1, 2)$ is a fiber bundle over a connected arc whose fibers are closed curves of length $< C'$. Let S_i $(i = 1, 2)$ be the image of G^i in M_γ. $S_i \subset T_i$ $(i = 1, 2)$. If a covering translate S'_i of S_i intersects S_i, then the double curve of T_i containing $S_i \cap S'_i$ is a closed curve homotopic to the core of T_i (use Lemma 4.4). Now having no open contraction allows two possible configuration of $S \cup S'$ and each were shown to be impossible in the proof of Lemma 4.4 and Lemma 4.6 respectively. Therefore, by taking subdisks and passing to a subsequence if necessary, the sequence $\{D^1_k\}_{k=1}^{\infty}$ can be assumed to be such that D^1_k is embedded in M and D^1_k is normally homotopic to a concentric subdisk of D^1_{k+1}, for all $k \in Z_+$. This limits to a measured essential lamination of M with euler characteristic zero by [MO]. It is well known that such measure is always approximated by an integral measure. Although it's not clear who should be credited for it, it was apparently implicit as early as in a work of Imanishi [Im]. At any rate, an incompressible torus exist in M. □

Proposition 5.2 *If $ccr(D_k) \to \infty$ as $k \to \infty$, then there is an essential lamination λ in M with a plane leaf.*

Proof The sequence $\{D_k\}_{k=1}^{\infty}$ can be chosen so that every wrapping path in D_k is height preserving for all $k \in Z_+$ as follows. When D_k has a wrapping path γ near its center not preserving height but radius of D_k is huge compared to the length of γ, then there is i (1 or 2) such that D_k^i and its translate $(D_k^i)'$ by γ constitute a crossed disk intersecting in the interior of each. Replace D_k with $D_k^i \cup (D_k^i)'$. Any wrapping path on this new crossed disk is height preserving by Lemma 4.6.

Let N be a fixed large number. We will construct an embedded disk in M parallel to $(1/N)\, D_k^i$ $(i = 1, 2)$ for any large k. Let $F = F_k$, $D = D_k$, $D_k^i = D^i$ and $G = (1/N)\, D_k$ and $G^i = (1/N)\, D_k^i$ as before. Let Δ be a 3-simplex of M such that $G \cap \Delta \neq \emptyset$. For each 2-cell $d = d^1 \cup d^2$ of G in Δ we shall assign a normal disk e, a *model disk* of d, parallel to d in Δ. Suppose $d_0\,(d_0^1 \cup d_0^2)$ and $d_1\,(d_1^1 \cup d_1^2)$ are distinct 2-cells of G in Δ that are parallel. Position of model disk e_0 of d_0 with respect to e_1 of d_1 in Δ is defined as follows.

Ordering of model disks.

Pick an oriented path γ in G connecting d_0 to d_1 with $\partial_-\gamma = d_0$. The wrapping band A, vertical fiber v and α is defined as before. Convention was $\partial_- v \in \partial_- A$, $\partial_+ v \in \partial_+ A$ when A is orientable. Since $d_0 \cup d_1$ intersects v transversely so does e_0 and e_1. We shall write $e_0 < e_1$ if $(e_0 \cap v) < (e_1 \cap v)$.

1) e_0 and e_1 are disjoint in Δ.

2) When A is orientable, $e_1 > e_0$ if $\partial_-\alpha \in \partial_- A$, i.e., α connects $\partial_- A$ to $\partial_+ A$. Otherwise it is the opposite.

3) When A is nonorientable, either α^1 or α^2 is embedded in A. Suppose α^1 is. Then $e_1 > e_0$ if and only if $(d_1^1 \cap v) > (d_0^1 \cap v)$.

Since G^i $(i = 1, 2)$ was an immersed disk in M, Lemma 3.7 implies this ordering does not depend on the choice of path γ or its orientation. Since any wrapping path is height preserving, this becomes a well defined local ordering relation among parallel normal disks in Δ. Moreover if Δ and Δ' are 3-simplices of M glued along a 2-simplex σ, then gluing model disks in Δ and Δ' along σ same way as their 2-cells were glued will introduce no self intersection. Therefore, an embedded disk E in M will be obtained by gluing the boundaries of model disks as just described. E is normally homotopic to G^i $(i = 1, 2)$ by construction.

So by taking subdisks and a subsequence if necessary, $\{D_k\}_{k=1}^{\infty}$ can be assumed so that a model disk E_k of D_k exist and E_k is normally isotopic to a concentric subdisk of E_{k+1} for all $k \in Z_+$. A lamination can be constructed as a proper limit of $\{E_k\}_{k=1}^{\infty}$. It goes as follows. Since E_k is embedded for all $k \in Z_+$, $\{E_k\}_{k=1}^{\infty}$ is carried by an embedded branched surface B in M. Fix a fibered neighborhood $N(B)$ of B once and for all. Let $t_k = max\,\{\, |E_k \cap \delta| \mid \delta$ is a 1-simplex of $M\,\}$. For each $k \in Z_+$, remove a thin neighborhood of E_k from $N(B)$ (Figure 5.2) to create a fibered neighborhood $N(B_k)$ so that $\partial_h N(B_k)$ contains 2 copies of E_k. A chain of inclusions $N(B_{k+1}) \hookrightarrow N(B_k)$

can be chosen so that the copy of E_k in $N(B_k)$ is isotopic to a subdisk of E_{k+1} in $N(B_{k+1})$. If this were defined at the k–th level and ϵ is the minimum length of I–fibers in $N(B_k)$, then the inclusion in the $(k+1)$–th level is defined by taking lengths of I–fibers in $N(B_{k+1})$ to be smaller than $\epsilon/2t_{k+1}$. The fact that E_k was nested and $radius(E_k) \to \infty$ assures that $\lambda = \cap_{k=1}^{\infty} N(B_k)$ is a lamination of M.

Since E_k was a disk for all $k \in Z_+$, E_k itself limits to a plane leaf l_p of λ. By construction l_p is dense in λ and every leaf of λ is two sided in M due to 3-plane property of F_k. λ is essential because it is a limit of least weight disks. If d is a compressing disk of a leaf l of λ, by the loop theorem d can be assumed to be embedded. Then, since l_p is dense, E_k approximates ∂d many times for large k. This produces an embedded monogon with a long tail which contradicts F_k being least weight. Note that we didn't need F_k to be orientable here. Similarly, neither end compressing disk nor sphere leaf exist in λ. $M - \lambda$ is irreducible since M is. □

Definition 5.3 A leaf l of a lamination is *resilient* if there is closed curve γ in l, a short arc Q transversely intersecting l in x_0 and a path $h\colon [0, 1] \to l$ such that (Figure 5.3)

1) $h(0) = x_0$, $h(1) \neq x_0$ and $h(1) \in Q$

2) If g is the holonomy of λ based at x_0, then $\lim_{n\to\infty} g^n(h(1)) = x_0$.

Definition 5.4 A lamination has a *contraction* if there is a transverse product band $i\colon A \to M$ such that the induced lamination on A has a pair of noncompact leaves taking an arc of I–fiber (of A) to a proper subarc of itself (Figure 5.4).

Lemma 5.5 *Under the same hypothesis of Proposition 5.2, M is haken.*

Proof Suppose λ has no compact leaf and μ is a minimal sublamination of λ. Suppose μ has a contraction with a product band A as in the definition and a_1, a_2 be the pair of noncompact leaves in A inducing the contraction. l_p is dense in λ so a_i $(i = 1, 2)$ is approximated by components of $E_k \cap A$ for large k. But since $a_1 \cup a_2$ formed a contraction, some components of $F_k \cap A$ form two distinct non contraction wrapping paths of D_k connecting components of ∂A in the opposite way (Figure 5.5). That is a violation of 3-plane property. Therefore, μ has no contraction. Since existence of a resilient leaf implies a contraction in μ, μ does not have a leaf with nontrivial holonomy. If l were such a leaf, then let γ be a closed curve in l and Q, $x_0 = Q \cap \gamma$ and g be the same as in the definition of resilient leaf. If x_0 is an isolated fixed point of g, then l is resilient. Otherwise, there is fixed point x_1 of g in Q isolated at least on one side because μ is nowhere dense in M. Then the leaf of μ passing through x_1 is resilient. An extension of Sacksteder's theorem [Sa] by Candel in [Ca, Theorem 3] says that μ is measured hence M contains an incompressible surface. □

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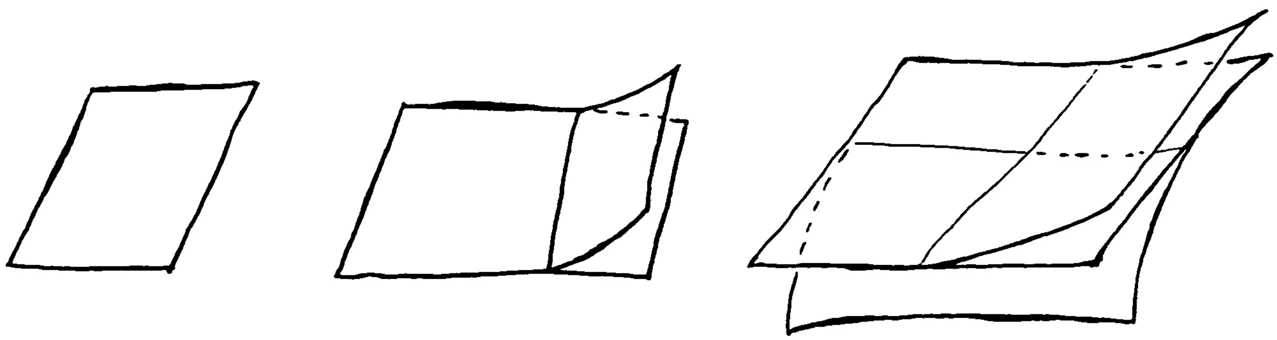


Figure 1.1 a)

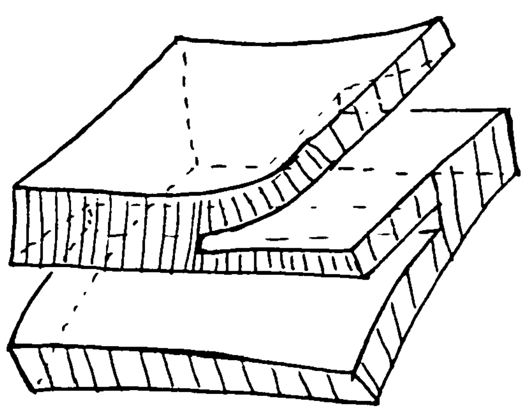


Figure 1.1 b)

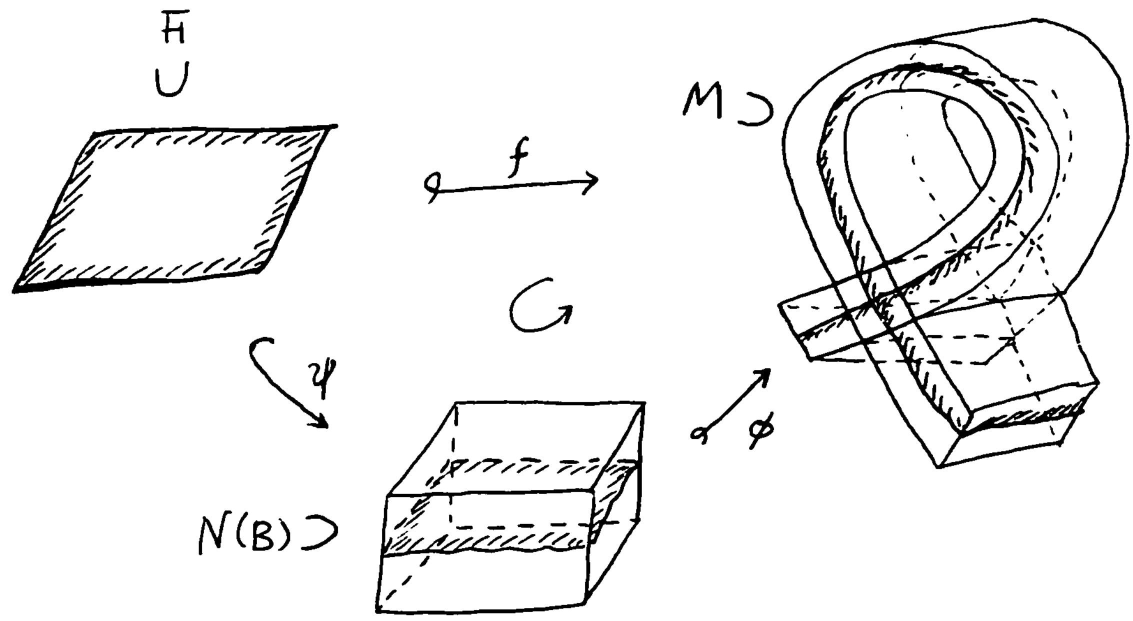


Figure 2.1

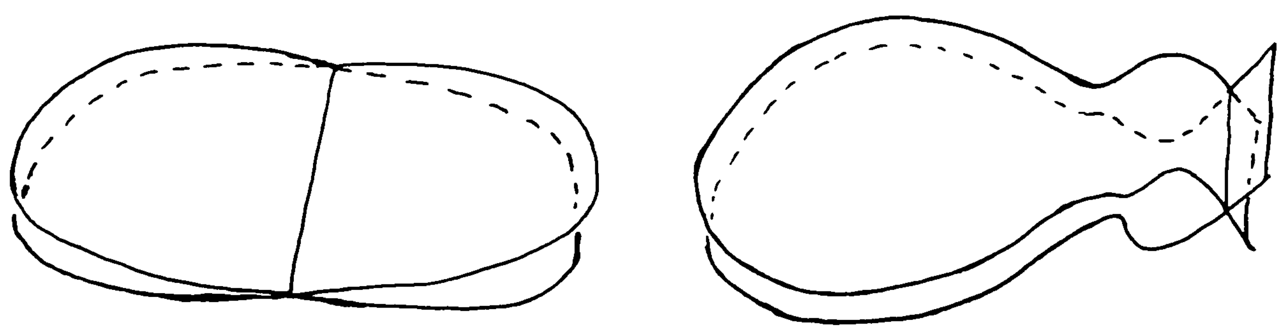


Figure 2.3

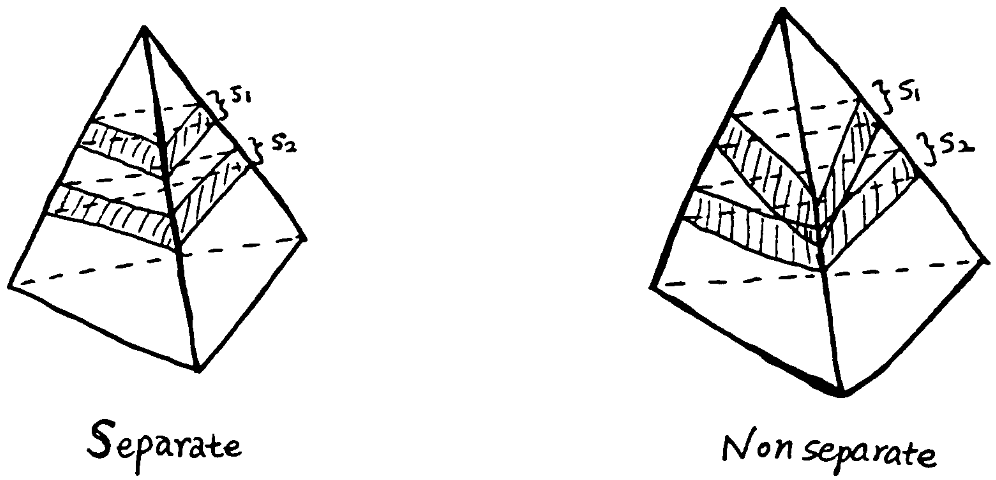


Figure 2.4 a)

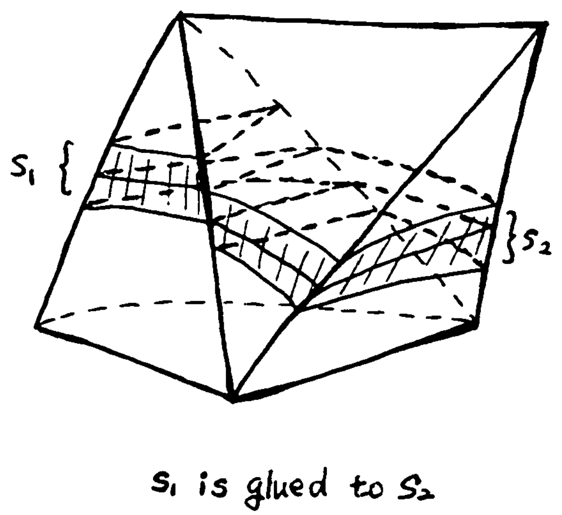
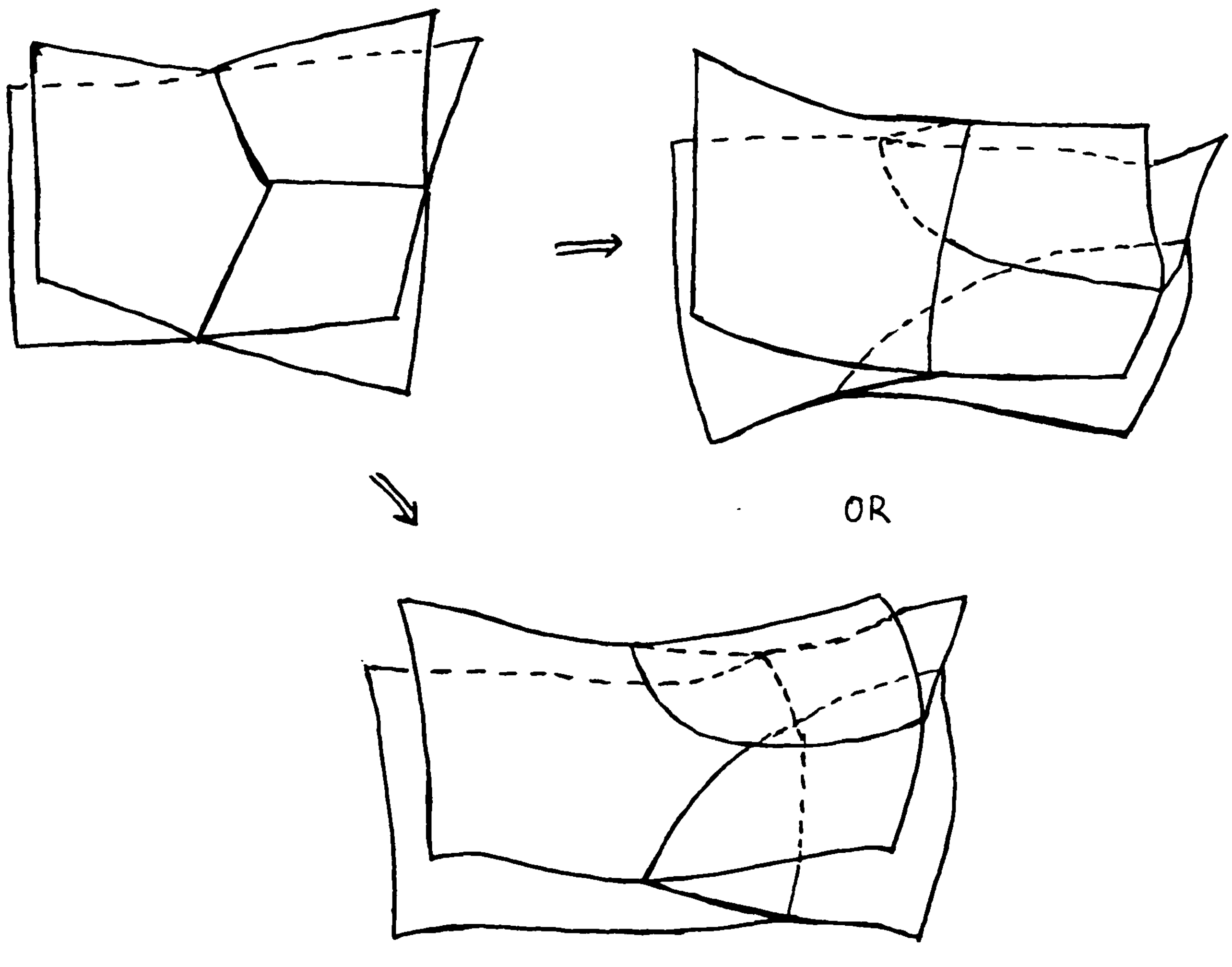


Figure 2.4 b)



Splitting of B_1

Figure 2.4 c)

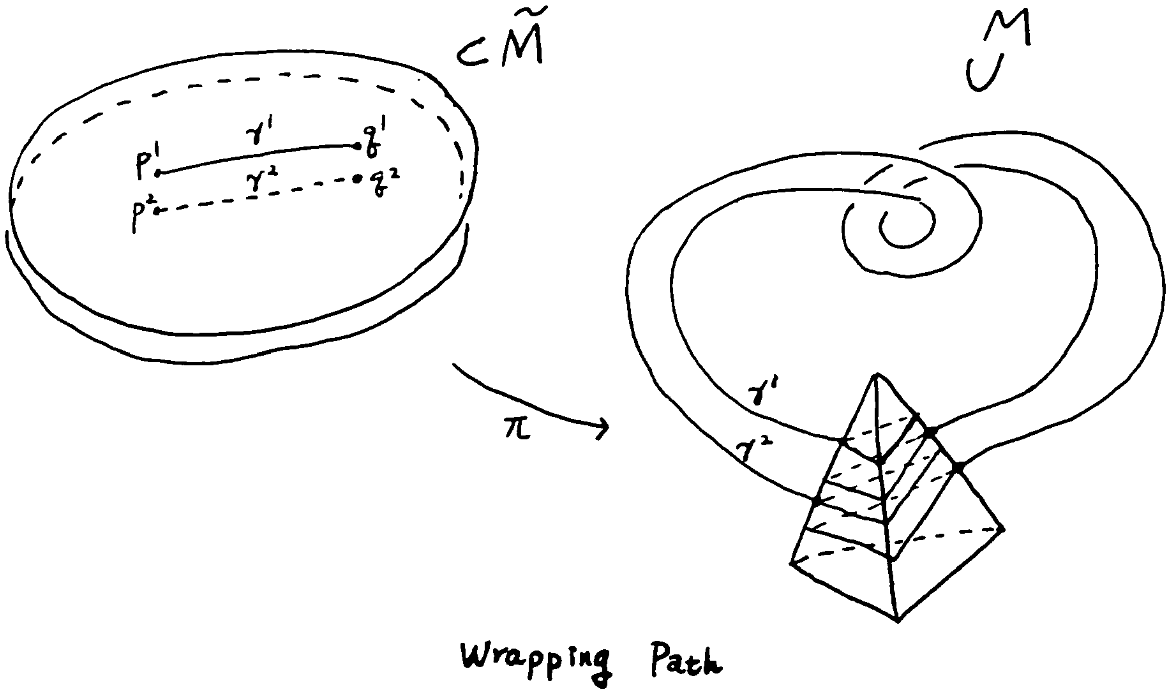


Figure 3.1

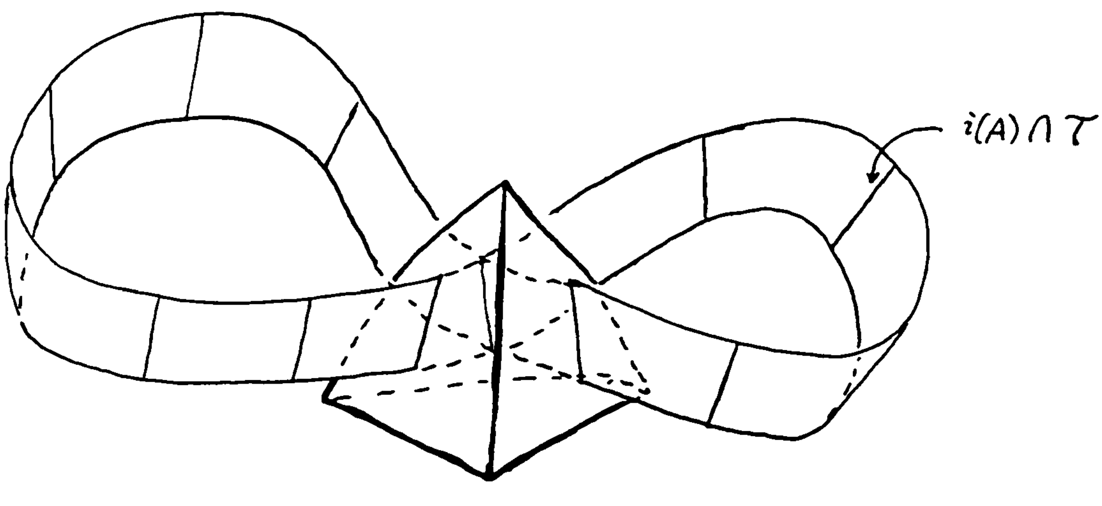


Figure 3.2

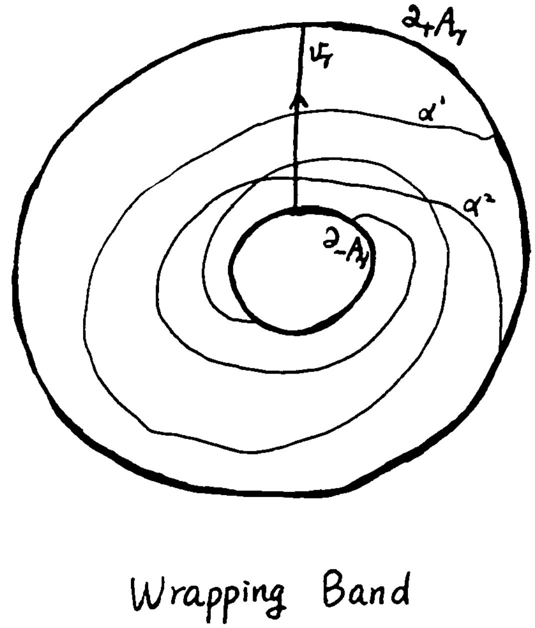


Figure 3.3 a)

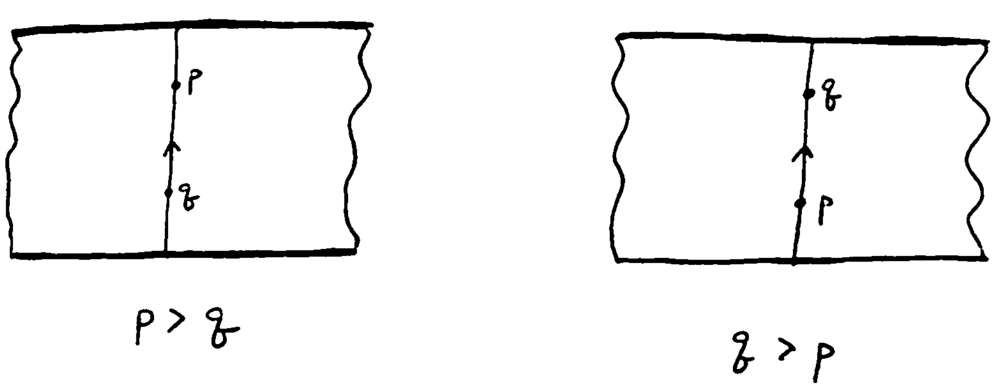
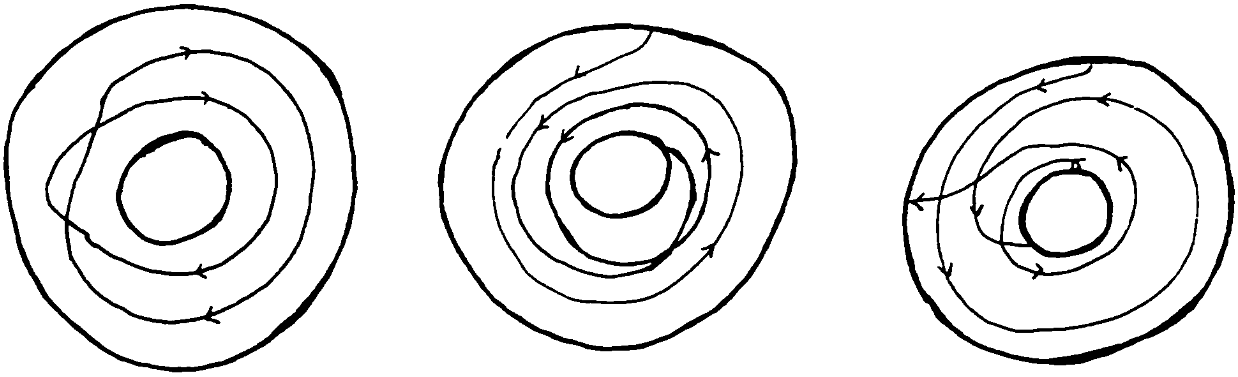
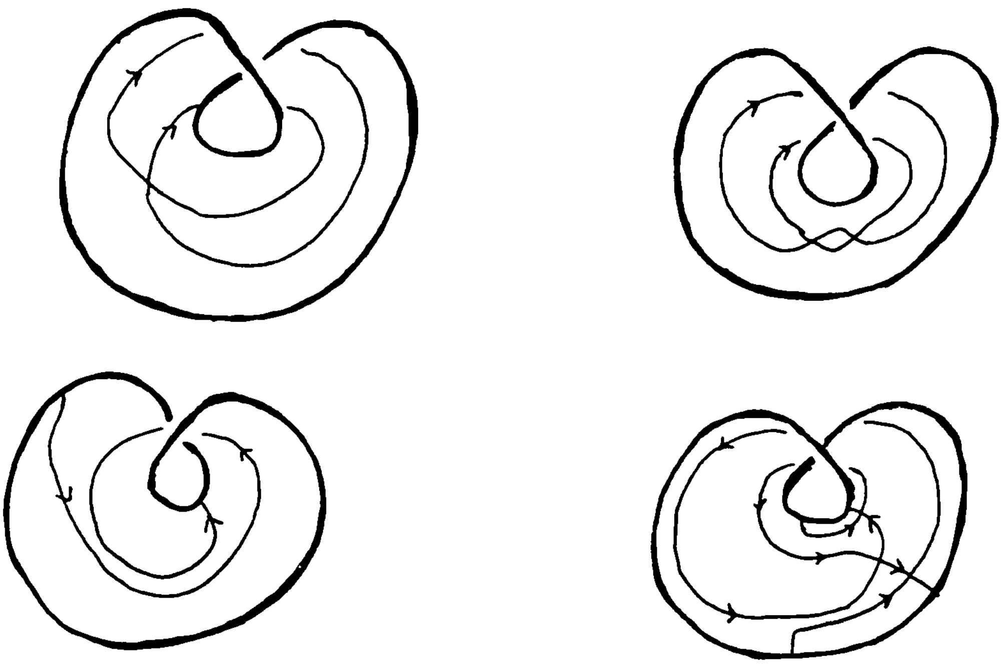


Figure 3.3 b)



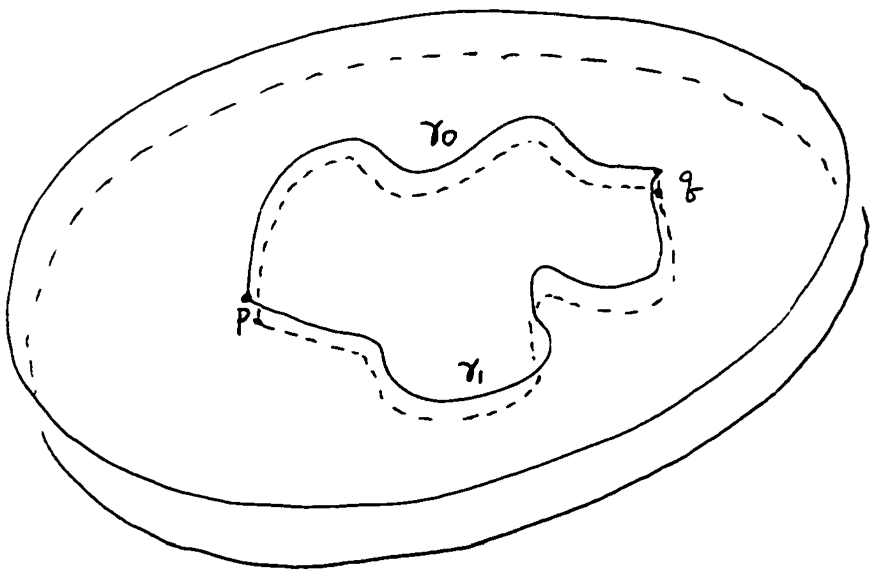
Contraction, when A_γ is orientable

Figure 3.6 a)



Contraction, when A_γ is nonorientable

Figure 3.6 b)



Homotopic Wrapping Paths

Figure 3.7

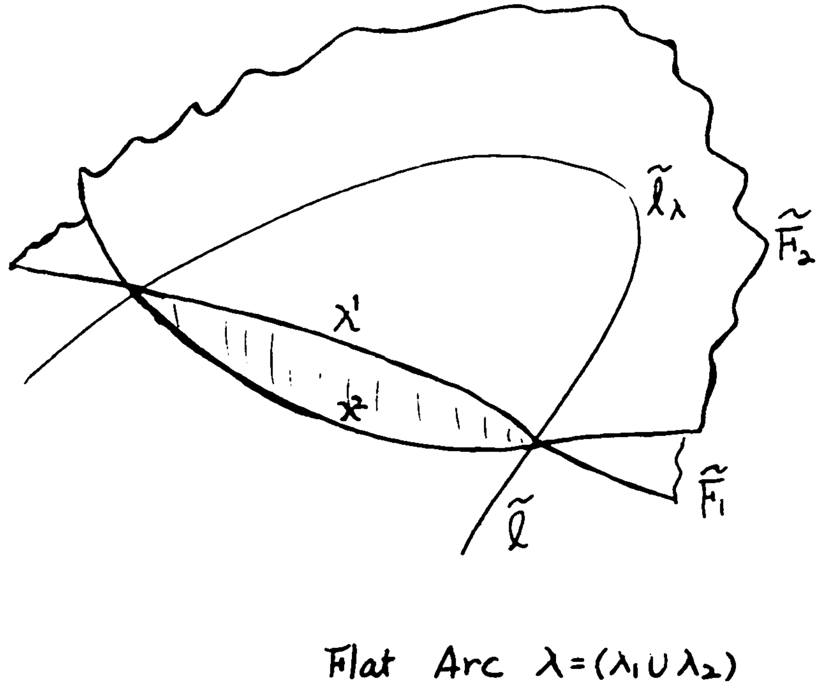


Figure 4.3

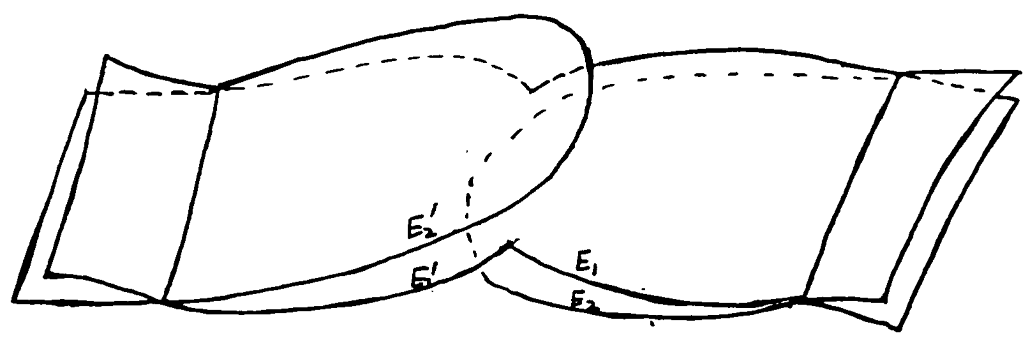


Figure 4.4 a)

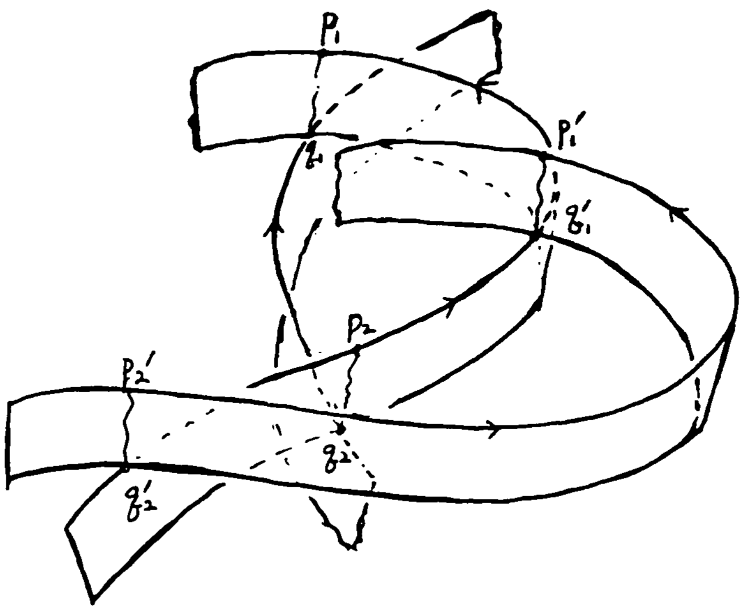


Figure 4.4 b)

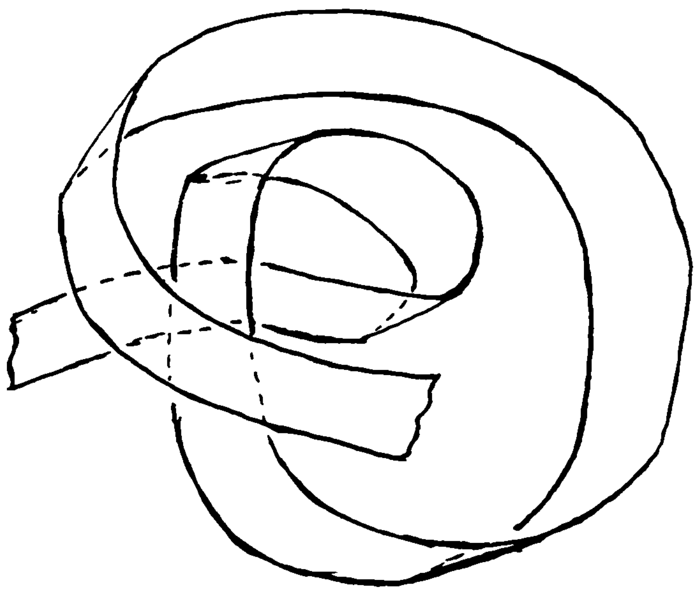


Figure 4.4 c)

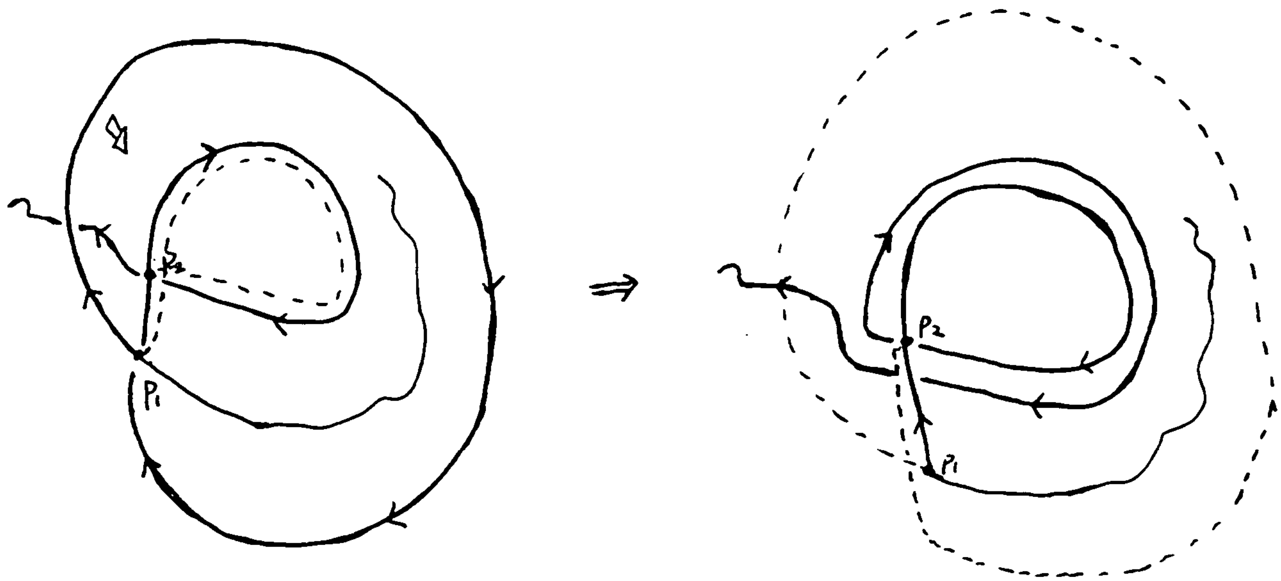


Figure 4.4 d)

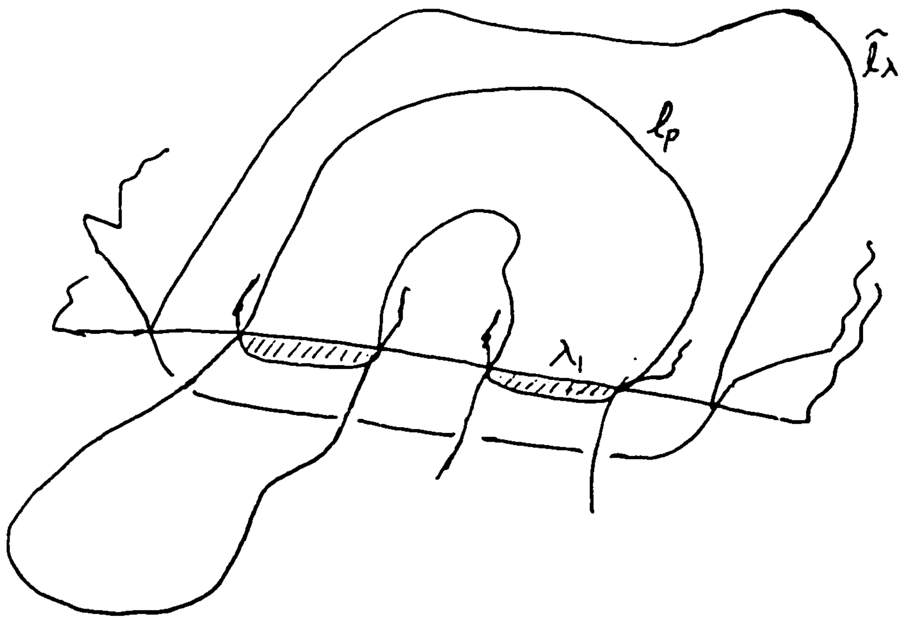


Figure 4.4 e)

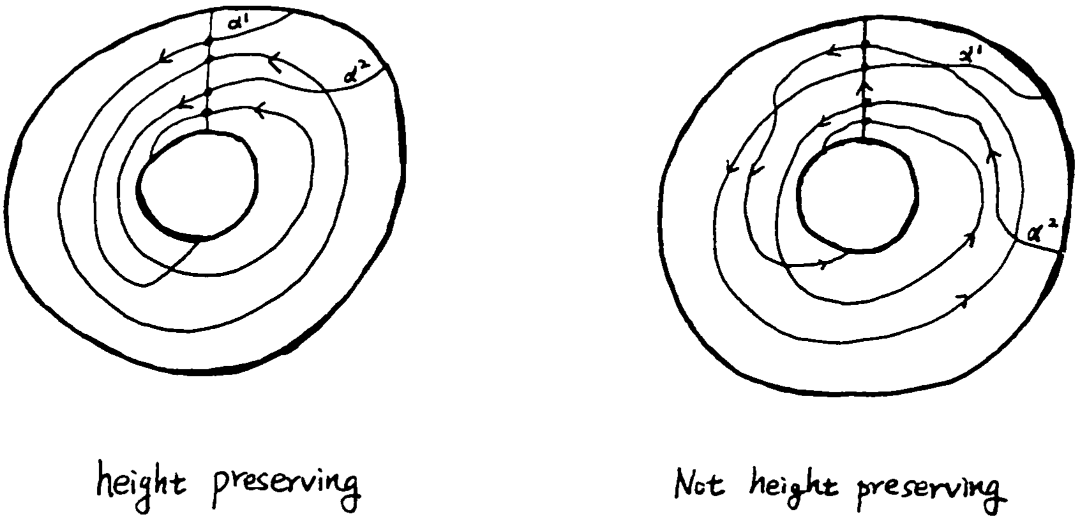


Figure 4.5

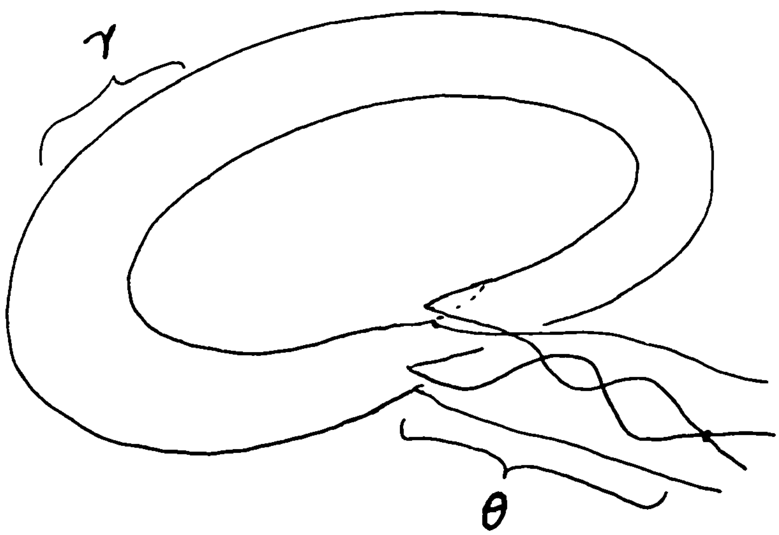


Figure 4.6 a)

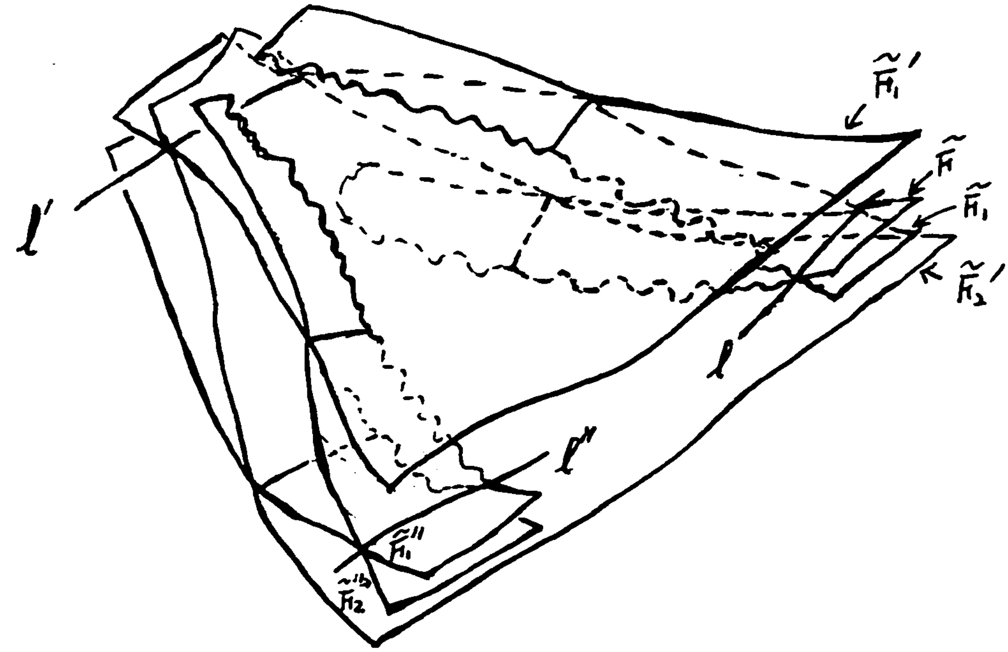


Figure 4.6 b)

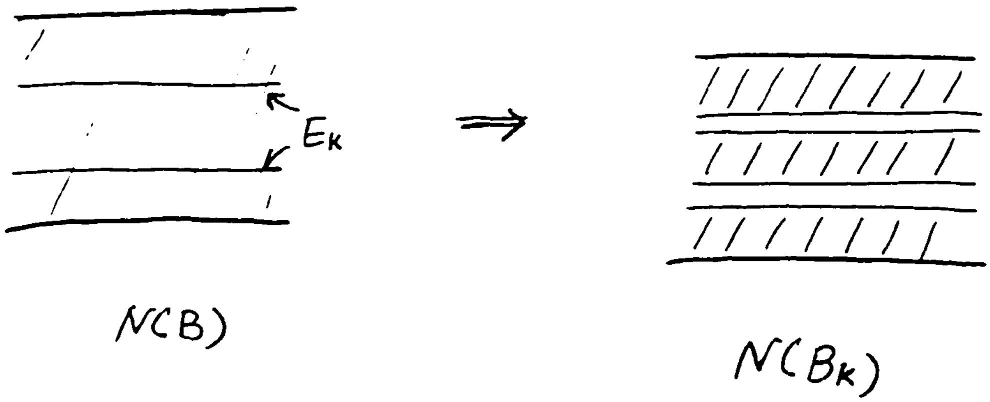


Figure 5.2

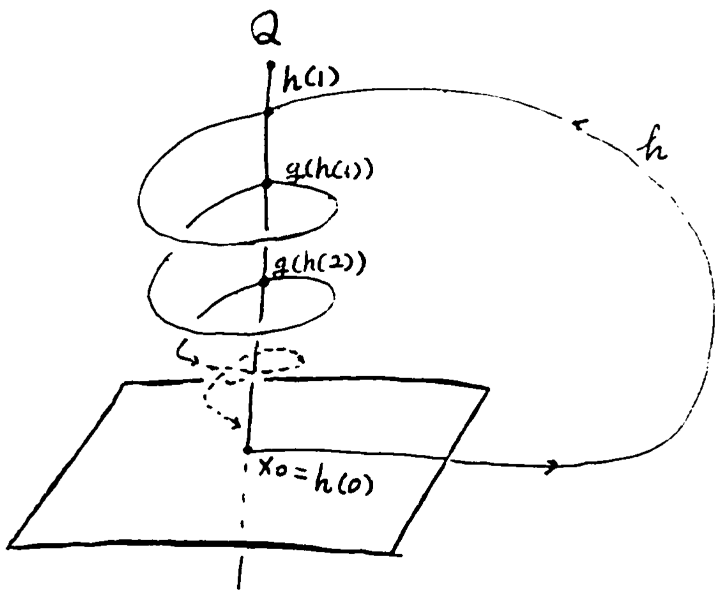


Figure 5.3

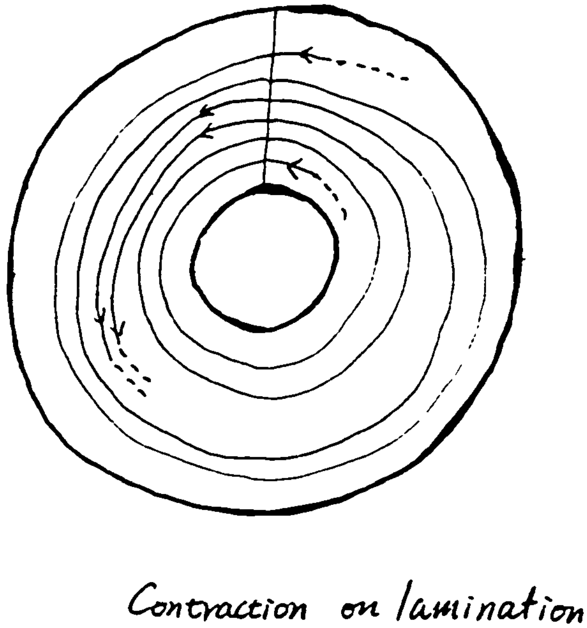


Figure 5.4

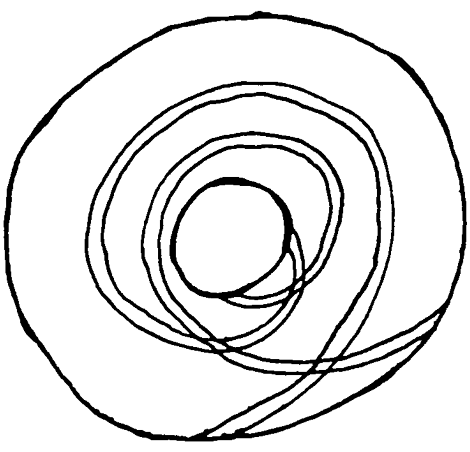


Figure 5.5