Aspects of Non-Abelian Many Body Physics

Thesis by

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Abstract

The general formulation of quantum statistical mechanics hints at interesting generalizations of the usual Bose/Fermi framework in two spatial dimensions. Anyon statistics, which is essentially a continuous interpolation between Bose and Fermi statistics, is relevant to the Fractional Quantum Hall Effect in two-dimensional (i.e., thin layer) condensed matter systems. In addition, the possibility of *non-abelian* statistics, in which the multi-particle wavefunction transforms as a representation of a non-abelian group under the exchange of indistinguishable particles, has been explored. Spontaneously broken non-abelian gauge theories in $(2+1)$ dimensions often have stable topological defects, called non-abelian vortices, that experience non-abelian statistics. In addition, it has been suggested that degenerate quasihole multiplets in Quantum Hall systems also transform as non-abelian representations of the braid group under particle exchange. In this thesis, I explore the braiding properties of systems of two-cycle flux vortices in a residual S_3 discrete gauge group. The individual vortices are uncharged, but multi-vortex states can have Cheshire charge. The uncharged sectors all have non-vanishing bosonic subspaces, as do the non-abelian charged trivial flux sectors. A kinetic Hamiltonian for vortices on a periodic lattice is constructed. There is a modification to the translational symmetry in the periodically identified direction for non-trivial Z_2 charged sectors. The ground state energies for various three and four vortex sectors is numerically determined. Typically, the ground state is bosonic, with a gap separating it from a non-abelian subspace.

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Chapter 1 Introduction

Generalizations of Bose and Fermi statistics in $2+1$ dimensions have been studied for many years. Most familiar is anyon statistics, in which the wave function acquires an arbitrary phase upon the interchange of two indistinguishable anyons [24]. Anyon statistics are exhibited in nature by the quasiparticles of the Fractional Quantum Hall Effect (FQHE) [21]. The behavior of many-anyon systems may be probed through Mean Field Theory[3] or exact numerical analysis of finite-size systems[10, 11].

This thesis is concerned with a broader generalization of quantum statistics, namely non-abelian statistics [16, 32, 8]. Specifically, we consider collections of non-abelian vortices[4, 28, 25, 23], particles which are labeled with a quantum number called flux (the reason for this terminology will be made clear in Section 1.1) that takes its value in a non-abelian group. Non-abelian particles call for a generalized notion of indistinguishability. The flux associated with a particular particle in the many-body system depends on its history. There is at present no Mean Field Theory for many-body non-abelian systems, and even the problem of three bodies is not well understood[25].

In this chapter, we review the basic properties of non-abelian vortices and their interactions, most notably the holonomy interaction and Cheshire charge. We briefly discuss the general framework of quantum statistical mechanics, in particular braid statistics in $2+1$ dimensions.

In Chapter 2, we discuss the algebraic properties of many-vortex systems. The model that is the basis for the calculations in this thesis is introduced, followed by the definition of a sector group. We then discuss the three- and four-vortex sectors of the model in some detail, followed by a generalization to n-vortex sectors.

In Chapter 3 , we introduce a lattice-gas model suitable for numerical studies of non-abelian systems. The ground state structure of three and four vortex systems on the lattice is discussed.

Chapter 4 contains a summary and concluding remarks, as well as possible future directions for work in this field.

1.1 Non-abelian Vortices

Non-abelian vortices arise as topological defects in spontaneously broken non-abelian gauge theories in $(2+1)$ dimensions. Consider a model with a scalar Higgs field Φ coupled to a non-abelian gauge field with gauge group G[4, 5]. With the hermitian generators T_a of G in the representation of the Higgs field and gauge coupling e, the gauge-covariant derivative operator is $D_\mu = \partial_\mu + ieA^a_\mu T_a$. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + (D_\mu\Phi)^\dagger D^\mu\Phi + V(\Phi) \tag{1.1}$$

containing kinetic terms for the gauge and Higgs fields and the invariant potential for the Higgs. In the Higgs phase, $V(\Phi)$ is minimized for a non-vanishing Higgs field, hence Φ acquires a vacuum expectation value (vev) $\langle\Phi\rangle$ and the symmetry is broken to the symmetry group H of the Higgs vev. Due to the gauge symmetry, any element of G acting on a vev produces a vev, and any element of H acts trivially on a vev by definition, so the vacuum manifold is the coset space G/H.

We now look for time-independent finite-energy solutions to the field equations of Eqn. 1.1. The requirement that the energy be finite implies that the fields must approach the vacuum configuration at spatial infinity. Therefore, the gauge potentials A^a_μ must approach a pure gauge and the Higgs field at infinity must take its values in the vacuum manifold. The field equation for Φ is $D_\mu\Phi = 0$, which can be integrated to yield

$$\Phi(\mathbf{r}') = a(\mathbf{r}', \mathbf{r}, C)\Phi(\mathbf{r}) \tag{1.2}$$

where

$$a(\mathbf{r}', \mathbf{r}, C) = P\exp\left[ie\int_C \mathbf{A}\right] \tag{1.3}$$

Here P denotes path-ordering, and C is a path from $\mathbf{r}$ to $\mathbf{r}'$. Consider a circle of very large radius parameterized by an angle θ; since the Higgs field is covariantly constant, we have

$$\langle\Phi(\theta)\rangle = a(\theta)\langle\Phi(0)\rangle \tag{1.4}$$

Single-valuedness of the Higgs condensate requires that $a(2\pi) \in H$.

We can now determine the conditions under which stable topological defects exist. $a(\theta)$ is a map from S_1 to the vacuum manifold, so there are stable defects if the fundamental group $\pi_1(G/H)$ is non-trivial. Any path for which $a(2\pi) \in H$ is closed in the vacuum manifold, and hence is an element of $\pi_1(G/H)$. If G is not simply-connected, we consider the universal covering group $\bar{G}$ and the lift $\bar{H}$ of the unbroken gauge group into the covering group; then $\pi_1(G/H) \cong \pi_1(\bar{G}/\bar{H})$. Two closed paths for which $a(2\pi)$ are h_1 and h_2 are homotopic if and only if h_1 and h_2 are in the same connected component of $\bar{H}$; therefore, $\pi_1(\bar{G}/\bar{H}) \cong \pi_0(\bar{H})$.

If $\bar{H}$ is a discrete group, every connected component of $\bar{H}$ consists of a single group element, so we can endow π_0 with a group structure; $\pi_0(\bar{H}) \cong \bar{H}$. To summarize, if G is spontaneously broken to a discrete group H, there will be topological defects characterized by a magnetic flux taking its value in $\bar{H}$ [1]. A non-abelian vortex is a topological defect in (2 + 1) dimensions associated with a non-abelian discrete unbroken gauge group $\bar{H}$.

In the case where $\bar{H}$ is discrete, all of the gauge bosons pick up a mass of order ev, where v sets the scale of the Higgs vev. For processes that take place at energies well below v, we can take v to be infinite; the vortices are effectively point-like, and there are no long-range gauge fields. That does not mean that there are no long-range interactions between the vortices; we shall see in Sec. 1.1.2 that vortices interact topologically through a sort of generalized Aharonov-Bohm interaction.

[1] The identification of the group element $a(2\pi)$ with a magnetic flux follows from the abelian case $G = U(1)$, in which case $a(2\pi) = \exp(ie\Phi_B)$, where Φ_B is the magnetic flux linked by the loop.

1.1.1 Electric Charge

The defects discussed in the previous section can carry electric charge as well as magnetic flux. The electric charge of a dyon (a flux/charge composite) is specified by the transformation properties of the state under a global gauge transformation; a state with definite charge is associated with an irreducible representation of the global gauge group. In particular, a state with zero charge always transforms according to the trivial representation.

If the unbroken gauge group is non-abelian, in general the globally defined gauge group in the presence of a vortex is a subgroup of the local unbroken gauge group $\bar{H}$ [6, 1]. This can be seen as follows; consider a circle around the vortex, parameterized by an angle θ. At every θ there is a subgroup $\bar{H}(\theta) \subset \bar{G}$ that stabilizes the Higgs condensate at θ; all of the $\bar{H}(\theta)$ are isomorphic to the same abstract group $\bar{H}$, but the embedding of $\bar{H}$ in $\bar{G}$ varies smoothly with θ [2]. Let $S_a(0)$ be a basis for the generators of $\bar{H}(0)$. Then

$$S_a(\theta) = a(\theta) S_a(0) a^{-1}(\theta) \tag{1.5}$$

For a transformation to be globally well defined, it is necessary for $a(2\pi)$ to commute with $S_a(0)$; therefore, the global gauge group consists of all elements of $\bar{H}$ that commute with the vortex flux $a(2\pi)$, *i.e.*, the global gauge group is the normalizer [3] $N(a(2\pi))$ of the magnetic flux.

1.1.2 Vortex-Vortex Interactions

We now consider the problem of patching single vortices together into a multi-vortex configuration. This will involve establishing a number of conventions to uniquely specify the state of the multi-vortex system[4, 8].

Start with an isolated vortex. First, we select an arbitrary basepoint x_0 and a standard path C that encircles the vortex in a definite sense (*cf.* Fig. 1.1). The flux

[2]In other, words, $\bar{H}(\theta)$ is a section of a $\bar{G}$ principle fiber bundle over S_1.

[3]The normalizer of a group element is the set of all elements of the group that commute with the given element: $N(a) = \{h \in H : ha = ah\}$.

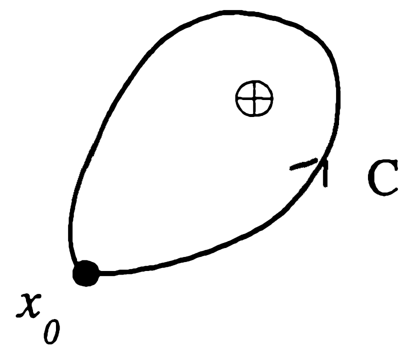

Figure 1.1: A standard path to define the flux of an isolated vortex.

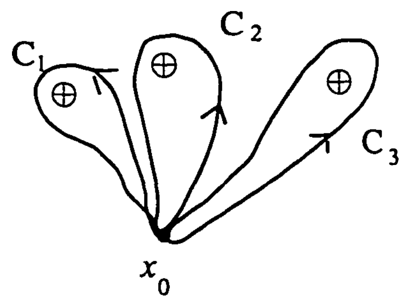

Figure 1.2: A set of standard paths.

of the enclosed vortex is

$$a(C, x_0) = P \exp \left(\int_{C, x_0} A \cdot dx \right) \in \bar{H} \tag{1.6}$$

Since the gauge connection is flat everywhere outside the vortex core, the flux linked by any loop that can be smoothly deformed to C without crossing the vortex is the same.

If there are multiple vortices, it is necessary to choose a set of standard paths that encircle the vortices in a definite order (Fig. 1.2). The *total* flux of the multi-vortex configuration is specified by the product (in the group $\bar{H}$) of the individual fluxes, in the order they are encircled. For example, the total flux in Fig. 1.2 is $a(C_1, x_0)a(C_2, x_0)a(C_3, x_))$. The importance of assigning a definite order to the single-particle fluxes in the multi-vortex configuration is readily apparent if the group $\bar{H}$ is non-abelian.

The fact that the gauge connection is flat outside of curvature singularities at

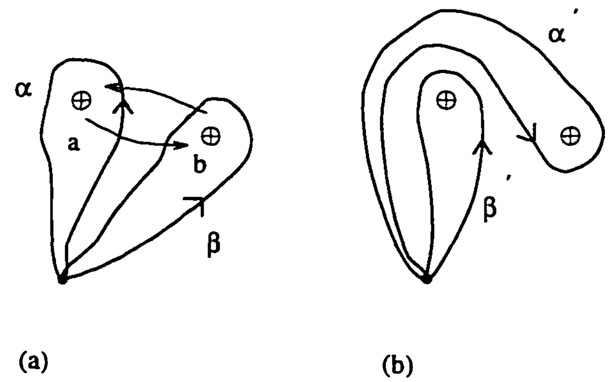

Figure 1.3: (a) The vortices a and b before the exchange. (b) The paths α' and β' deform to the correct paths after the exchange.

the vortex cores implies that the locations of the vortices must be excised from the space. Consider n vortices on a connected, oriented two-dimensional space Σ, and let $\Sigma(n)$ be the space Σ with holes at the vortex positions. It is clear that our choice of basepoint and standard paths on $\Sigma(n)$ amounts to a homomorphism ρ from the fundamental group of $\Sigma(n)$ to the group $\bar{H}$[23]:

$$\rho : \pi_1(\Sigma(n), x_0) \mapsto \bar{H} \tag{1.7}$$

In general, there will be a number of topologically inequivalent sets of standard paths to specify the individual vortex fluxes (and the total flux). This is the *patching ambiguity*. By prescribing a set of standard paths, we are fixing a convention to resolve this ambiguity; it is somewhat analogous to the procedure of fixing a gauge. Once we choose a set of standard paths, we must continue to measure fluxes relative to this choice. This has an important physical consequence if the vortices are free to move on $\Sigma(n)$, as we shall now see.

In Fig. 1.3, a vortex whose flux is $a \in \bar{H}$ relative to the standard path α is about to trade places in a counterclockwise sense with a vortex whose flux is b relative to the path β. In order to specify the particle fluxes *after* the exchange, it is necessary to construct paths that will deform to standard paths after the exchange, without crossing the core of any vortex. These paths are labeled α' and β'. Homotopically, we see that

$$\beta' \sim \alpha, \quad \alpha' \sim \alpha^{-1}\beta\alpha \tag{1.8}$$

Hence we come to the conclusion that, after the exchange, the particle whose flux originally was b now has flux aba^{-1}, which is distinct (unless a and b commute). This long range effect is termed the "holonomy interaction." It is essentially a non-abelian variant of the Aharonov-Bohm effect[4].

The holonomy interaction can be expressed in terms of the *braid operator* R; if we express the original state of Fig. 1.3(a) as $|a, b\rangle$, the braid operator performs a counterclockwise exchange:

$$R|a, b\rangle = |aba^{-1}, a\rangle \tag{1.9}$$

Clearly, a clockwise exchange is achieved with the operator R^{-1}. The total flux ab of the state is preserved by the braid operation.

A global gauge transformation h acts on a vortex flux by conjugation: $a \mapsto hah^{-1}$. It may seem, then, that a flux should merely be labeled by a conjugacy class. This is not true, however; $a \neq b \Rightarrow hah^{-1} \neq hbh^{-1}$, so vortices distinct in one gauge are distinct in every gauge. One may conclude that the conjugacy classes in $\bar{H}$ form degenerate multiplets.

1.1.3 Cheshire Charge

Non-abelian topological defects have a curious property; it is possible for them to carry electric charge that is *nonlocalizable*, *i.e.,* it is not associated with a physical, gauge invariant charge density localized on the object. This sort of charge, which is called *Cheshire charge*, is a global, topological effect.

A simple example will suffice to illustrate the concept of Cheshire charge. Consider an unbroken gauge group $\bar{H} = S_3$, and the two-vortex state $|(12), (23)\rangle$. The vortices are assumed to be pure fluxes, not dyons, so individually they carry no electric charge.

[4]It is distinguished from the original Aharonov-Bohm effect in that the holonomy interaction is classical, not intrinsically quantum mechanical.

The total flux of this state is (321). The globally defined gauge group is the centralizer of (321) in S_3 (*cf.* Sec. 1.1.1), which is just the Z_3 subgroup $\{e, (123), (321)\}$. Under a global gauge transformation by the element (123), the state transforms to $|(23), (31)\rangle$. The latter state is transformed to $|(31), (12)\rangle$ by a (123) gauge transformation. We may now combine these states into three states that are invariant under the global gauge group ($\phi \equiv \exp(2i\pi/3)$):

$$|a\rangle = \frac{1}{\sqrt{3}}(|(12),(23)\rangle + |(23),(31)\rangle + |(31),(12)\rangle) \tag{1.10}$$

$$|b\rangle = \frac{1}{\sqrt{3}}(|(12),(23)\rangle + \phi|(23),(31)\rangle + \phi^2|(31),(12)\rangle) \tag{1.11}$$

$$|c\rangle = \frac{1}{\sqrt{3}}(|(12),(23)\rangle + \phi^2|(23),(31)\rangle + \phi|(31),(12)\rangle) \tag{1.12}$$

Observe that state $|a\rangle$ transforms according to the trivial representation of Z_3, and hence is uncharged, whereas states $|b\rangle$ and $|c\rangle$ transform as the *nontrivial* representations of Z_3, and therefore carry electric charge. This charge is not localized on either or both of the particles, but is a global property of the two-vortex state; this is Cheshire charge.

1.2 Braid Statistics

Here, we briefly discuss the possible varieties of quantum statistics possible in two spatial dimensions[33].

The quantum statistics of indistinguishable particles can be expressed in a general way as follows; consider n indistinguishable particles, moving on a manifold M (typically R^d, where d is the spatial dimension). If the particles cannot coincide (like fermions or hard-core bosons), the classical configuration space of the system is $[M^n - D_n]$, where D_n is the subspace of M^n where two or more particles have the same coordinates. In quantum mechanics, it is necessary to identify all configurations that differ by a permutation of the coordinates of indistinguishable particles, so the configuration space is

$$C_n = [M^n - D_n]/S_n, \tag{1.13}$$

where S_n is the permutation group on n objects.

In the path integral formalism of quantum statistical mechanics, one sums over all histories connecting a given initial and final state (if we are calculating a thermal average, for example, the initial and final states are identified). If the configuration space C_n is not simply connected, the path integral decomposes into a sum of path integrals over disjoint sectors, corresponding to the homotopy classes of C_n. The terms in the sum corresponding to different homotopy classes need not be weighted equally, as long as overall unitarity is respected.

If one defines a set of "exchange operators" that carry the final configuration of the system around a closed path in C_n, clearly the different homotopy sectors are mixed by the operators, and if the sectors are weighted differently, the amplitude is transformed. By considering applying two such exchanges in turn, it is clear that the exchange operators generate a unitary representation of the fundamental group $\pi_1(C_n)$. For $M = R^d$, $d \geq 3$, $\pi_1(C_n) \cong S_n$, and exotic statistics are not allowed.

However, for $d = 2$, $\pi_1(C_n) \cong B_n$, the braid group on n strands. B_n is generated by $(n-1)$ braid generators σ_i, which obey the Yang-Baxter relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| \geq 2 \tag{1.14}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \leq i \leq n-2 \tag{1.15}$$

These are the *only* defining relations for B_n, which is an infinite, discrete, non-abelian group. Unitary, one-dimensional representations of B_n exist, in which each generator is represented by the same phase, $\sigma_i = e^{i\theta}$, $1 \leq i \leq n-1$ (these are the only abelian representations consistent with the Yang-Baxter relations). The phase θ is arbitrary, and hence interpolates between Bose and Fermi statistics. Systems of particles that transform as a one-dimensional representation of B_n with $\theta \neq 0, \pi$ are called *anyons*; anyon statistics is known to be relevant in certain two-dimensional Quantum Hall systems (the *Fractional Quantum Hall Effect*, FQHE).

Non-abelian vortices transform according to non-abelian representations of B_n. If one considers an arbitrary three-vortex state $|a, b, c\rangle$, along with two braid operators R_1, R_2 that braid the first and second pair, respectively, it is easy to demonstrate that $R_1R_2R_1 = R_2R_1R_2$. For n particles, there are $n-1$ braid operators R_i, and all of the Yang-Baxter relations are easily verified. In systems with non-commuting vortices, multi-dimensional irreducible representations of B_n arise. These systems are said to obey *non-abelian statistics.*

As emphasized by Lo and Preskill[25], non-abelian vortices may be treated in the above general framework, *if* one extends the usual concept of "indistinguishability." Two vortices are considered indistinguishable if their fluxes lie in the same conjugacy class of $\bar{H}$, and they are in the same representation of the normalizer of their respective fluxes[5] (*i.e.*, they have the same charge). In other words, vortices are indistinguishable if it is *possible* (through topological interactions with other vortices) for an exchange amplitude to interfere with a direct amplitude. Two vortices a and b, with the same charge but fluxes belonging to distinct elements of the same conjugacy class, are not identical, in that a b vortex will not be annihilated by an a anti-vortex, but they are treated as indistinguishable for the purposes of statistics.

As shown by Bais *et al.*[5], the classification of indistinguishable vortices is intimately related to the representation theory of the quasi-triangular Hopf algebra, or "quantum double" [7, 13]. The quantum double $D(\bar{H})$ associated with the group $\bar{H}$ is an algebra of order $|\bar{H}|^2$ generated by the operators (using the notation of Refs. [25, 23]) $P_h g$, $h, g \in \bar{H}$: here, g performs a global gauge transformation, and P_h projects out the total flux h. The projection operators obey

$$P_h P_g = \delta_{h,g} P_h, \quad a P_h a^{-1} = P_{aha^{-1}} \tag{1.16}$$

and hence

$$(P_h a)(P_g b) = \delta_{h,aga^{-1}} (P_h ab) \tag{1.17}$$

[5]It is easily shown that the normalizers of two group elements in the same class are isomorphic.

Following Ref. [5], let ${}^{A}C$ be a conjugacy class of $\bar{H}$, and ${}^{\alpha}\Gamma$ the α-th irreducible representation of the normalizer ${}^{A}N$ of ${}^{A}C$ in $\bar{H}$. A flux/charge sector is labeled $|{}^{A}C, {}^{\alpha}\Gamma\rangle$. The flux/charge sectors are irreducible representations of the quantum double $D(\bar{H})$[5, 23, 25]. In fact, the sectors form a complete set of irreducible representations of $D(\bar{H})$. The particles belonging to a sector should be considered a degenerate multiplet[5]. It should be clear from the proceeding discussion that two vortices are to be considered indistinguishable if and only if they belong to the same irreducible representation of $D(\bar{H})$[25].

1.3 Exotic Statistics in Condensed Matter Systems

Thin-layer condensed matter systems, such as two-dimensional inversion layers or MOSFETS, provide essentially two-dimensional systems in which braid statistics may be realized. Indeed, it is strongly believed that quasiparticles with fractional statistics arise in systems that exhibit the Fractional Quantum Hall Effect[21].

The incompressible Laughlin wavefunction[21] Ψ_m for N electrons is

$$\Psi_m(z_1, \ldots, z_N) = \prod_{1 \leq j < k \leq N} (z_j - z_k)^m \exp\{-\sum_{j=1}^{N} \frac{|z_j|^2}{4l_0^2}\} \tag{1.18}$$

Here z_j are the complex coordinates of the electrons, m is an odd integer, and l_0 is the magnetic length. This state is a good approximation to the ground state of a Quantum Hall system at filling factor $\nu = 1/m$. There are excitations around this state called *quasiholes*. The state with a quasihole at position z_0 is, in terms of the Laughlin state Ψ_m,

$$\Psi_m^{(+)}(z_0, z_1, \ldots, z_N) = \prod_{(j=1)}^{N} (z_j - z_0)\Psi_m(z_1, \ldots, z_N) \tag{1.19}$$

The quasihole has fractional charge e/m, where e is the electron charge[14, 22]. Considering multi-quasihole solutions, we may expect that the Aharonov-Bohm effect on a fractionally charged excitation slowly dragged around another quasihole will yield fractional statistics[20]. Arovas, Schrieffer and Wilczek demonstrated this using a Berry phase calculation[2].

For non-abelian statistics to be possible, there must be a set of degnerate excitations, that transform among each other by non-trivial matrices under adiabatic exchange. Moore and Read[26] have argued, using conformal field theory, that quasihole excitations of the paired Pfaffian Hall state obey non-abelian statistics. Recent advances in evaluating the degneracies of multi-quasihole paired Pfaffian states[27, 30] have strengthened this conclusion.

Chapter 2 Algebraic Aspects of the Many-Vortex Problem

In this chapter, we discuss the structure of the braid algebra acting on many-vortex systems. Because of the extreme mathematical complexity of the problem, we will for the most part restrict attention to a specific model, described below. This model is simple enough to be somewhat tractable, yet still exhibits most of the interesting features of the general non-abelian problem.

Specifically, we will take the unbroken gauge group to be S_3, the permutation group on three objects. The individual particles will be taken to be elements of the two-cycle class of S_3, with zero charge (however, configurations with more than one vortex may have a total charge, due to Cheshire charge). Unless otherwise stated, the results in this thesis apply to this S_3 model.

2.1 Preliminary Definitions

We may assume that a convention has been prescribed that resolves the patching ambiguity for multi-vortex states described in Sec. 1.1.2. Therefore, there exist states in which a definite flux may be assigned to each particle, so a typical n-vortex state is

$$|a, b, \ldots, k\rangle \quad a, b, \ldots, k \in \bar{H} \tag{2.1}$$

The ordering prescribed by the patching convention allows one to specify the total flux Φ_t associated with the state 2.1,

$$\Phi_t = ab \ldots k \in \bar{H} \tag{2.2}$$

In an n-vortex sector, we may define $n-1$ braid operators R_i,

$$R_i|a,\ldots,\overset{i}{f},\overset{i+1}{g},\ldots,k\rangle = |a,\ldots,\overset{i}{fgf^{-1}},\overset{i+1}{f},\ldots,k\rangle \tag{2.3}$$

It is easy to verify that the braid operators obey the Yang-Baxter relations:

$$R_iR_j = R_jR_i, \quad |i-j| \geq 2 \tag{2.4}$$

$$R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1} \tag{2.5}$$

It is also convenient to define the operator $\mathcal{C}(g)$, which conjugates the entire state by g:

$$\mathcal{C}(g)|a,b,\ldots,k\rangle = |gag^{-1},gbg^{-1},\ldots,gkg^{-1}\rangle \quad g\in\bar{H} \tag{2.6}$$

If g is in the centralizer of the total flux Φ_t, $\mathcal{C}(g)$ implements a global gauge transformation. The conjugation operator commutes with all of the braid operators:

$$[\mathcal{C}(g),R_i] = 0 \tag{2.7}$$

2.2 The Sector Group

Having specified the unbroken gauge group to be $\bar{H}\cong S_3$, and the fluxes of the individual vortices, a sector is determined by the number of vortices n, the total flux $\Phi_t\in S_3$, and the total charge, which is the representation of the centralizer of the total flux, $N(\Phi_t)$. If, following Ref. [5], we denote the α-th representation of $N(\Phi_t)$ by ${}^{\alpha}\Gamma(N(\Phi_t))$, a flux/charge sector can be specified as $(n,\Phi_t,{}^{\alpha}\Gamma(N(\Phi_t)))$.

On a given n-vortex sector, it is a straightforward exercise to calculate the braid matrices R_i $(1\leq i\leq n-1)$ associated with the action of the generators σ_i of B_n on the many-vortex states. These matrices themselves generate a group, which we will define to be the *sector group* $K(n,\Phi_t,{}^{\alpha}\Gamma)$. The sector $(n,\Phi_t,{}^{\alpha}\Gamma(N(\Phi_t)))$ is a (generically reducible) representation space of its corresponding sector group.

It is important to note the differences between the sector group K and the braid group B_n. The braid group for a sector is determined entirely by the number of particles (and the topology of the underlying configuration space), whereas the sector group also depends on the unbroken gauge group $\bar{H}$, and the total flux and total charge of the sector. While the braid group is an infinite, discrete group, we shall see that the sector group for a finite number of vortices is of finite order.

The generators R_i of the sector group obey the Yang-Baxter relations determined by their parent braid group, plus additional defining relations imposed by the structure of the underlying model (and required by the fact that K is finite). One defining relation is determined by considering the repeated application of a braid generator to a two-vortex state. Either the fluxes associated with the individual vortices commute, in which case R acts trivially on the state, or the fluxes do not commute, and the fluxes associated with the vortices cycle through the conjugacy class of $\bar{H}$ to which they belong. Since the conjugacy class has a finite number of elements, the original state is restored after some finite number k of braidings. Thus the defining relation may be written $R_i^k = e$ (for$(1 \leq i \leq n-1)$).

In the S_3 model underlying this thesis, the single particle fluxes take their values in the two-cycle class of S_3, which has three elements, hence $k = 3$ for this model:

$$R_i^3 = e \tag{2.8}$$

An additional defining relation for K that involves all of the generators for a given sector will be introduced in Sec. 2.5.

2.3 Sectors Of The Three Vortex Problem

The product of three two-cycles in S_3 is again a two-cycle, so there are three possible values of the total flux Φ_t. Since it is always possible to arrange for the total flux to be any given element in a conjugacy class of $\bar{H}$ by an appropriate gauge choice, the three possible total fluxes are equivalent, and we can arbitrarily take Φ_t to be (31).

The centralizer of (31) is $\{e,(31)\} \cong Z_2$, which has two irreducible representations. so there are two possible charge states, the uncharged state (corresponding to the trivial representation ${}^1\Gamma(Z_2)$), and the charged state, (corresponding to the non-trivial representation ${}^2\Gamma(Z_2)$).

There are 9 states of three two-cycle vortices that have $\Phi_t = (31)$. This 9-dimensional space decomposes under the action of the global gauge group into a 5-dimensional uncharged space and a 4-dimensional charged space.

The Uncharged Sector

One state in the uncharged sector may be immediately singled out:

$$|B_1\rangle = |(31),(31),(31)\rangle \tag{2.9}$$

This state is clearly trivial under any braiding, so it bosonic under vortex exchange. The other four states in the uncharged sector are, in arbitrary order,

$$|1\rangle = \frac{1}{\sqrt{2}}(|(12),(12),(31)\rangle + |(23),(23),(31)\rangle) \tag{2.10}$$

$$|2\rangle = \frac{1}{\sqrt{2}}(|(12),(31),(23)\rangle + |(23),(31),(12)\rangle) \tag{2.11}$$

$$|3\rangle = \frac{1}{\sqrt{2}}(|(23),(12),(23)\rangle + |(12),(23),(12)\rangle) \tag{2.12}$$

$$|4\rangle = \frac{1}{\sqrt{2}}(|(31),(12),(12)\rangle + |(31),(23),(23)\rangle) \tag{2.13}$$

Applying the braid operators R_1 and R_2 to the states above simply permute the states, so it is clear that the sector group for this sector $K(3,(31),{}^1\Gamma) \subseteq S_4$. With the states numbered as above, we can represent R_1 and R_2 as follows:

$$\begin{aligned} R_1 &= (1)(234) \\ R_2 &= (4)(132) \end{aligned} \tag{2.14}$$

Since R_1 and R_2 are both even permutations, we see that the sector group is a subgroup of A_4, the even permutations on four objects. In fact, it is quite simple to verify that the sector group *is* A_4:

$$K(3,(31),^1\Gamma) \cong A_4 \tag{2.15}$$

There is another bosonic state in this sector, namely

$$|B_2\rangle = \frac{1}{2}(|1\rangle + |2\rangle + |3\rangle + |4\rangle) \tag{2.16}$$

The remaining 3-dimensional subspace in the uncharged sector corresponds to the 3-dimensional irreducible representation of A_4, and hence is truly non-abelian in nature.

The Charged Sector

The charged sector is spanned by the states

$$|1\rangle = \frac{1}{\sqrt{2}}(|(12),(12),(31)\rangle - |(23),(23),(31)\rangle) \tag{2.17}$$

$$|2\rangle = \frac{1}{\sqrt{2}}(|(12),(31),(23)\rangle - |(23),(31),(12)\rangle) \tag{2.18}$$

$$|3\rangle = \frac{1}{\sqrt{2}}(|(23),(12),(23)\rangle - |(12),(23),(12)\rangle) \tag{2.19}$$

$$|4\rangle = \frac{1}{\sqrt{2}}(|(31),(23),(23)\rangle - |(31),(12),(12)\rangle) \tag{2.20}$$

Due to the minus signs in the states above, one cannot say that R_1 and R_2 are simply permutations in S_4, because a braid operator acting on a state may produce a sign change as well as a permutation. The signs of the states above were chosen so that R_1 does not produce any such sign flips, so that it is possible to write $R_1 = (1)(234)$ as before. However, it is no longer possible to write R_2 as a permutation. The action of R_2 on the charged subspace is

$$
\begin{aligned}
R_2|1\rangle &= -|3\rangle \\
R_2|2\rangle &= |1\rangle \\
R_2|3\rangle &= -|2\rangle \\
R_2|4\rangle &= |4\rangle
\end{aligned}
\tag{2.21}
$$

If we restrict ourselves to ordinary representations, we discover that the braid operators generate the group T^D, the double group of the point group of the tetrahedron (the point group of the tetrahedron $T \cong A_4$). But it is also possible to view the representation generated by the R's as a *projective* representation of A_4 (a projective representation differs from an ordinary representation in that the representatives corresponding to group elements g_1 and g_2 obey $\rho(g_1)\rho(g_2) = \omega(1,2)\rho(g_1 g_2)$, where $\omega(1,2)$ is a phase factor - for the case at hand, $\omega = \pm 1$). Since the latter possibility proves to be more easily generalizable, that is the point of view we will take:

$$
K(3,(31),^2\Gamma) = A_4 \quad \text{(projectively realized)} \tag{2.22}
$$

In the uncharged sector, we discovered that there were was a two-dimensional invariant subspace that obeys Bose statistics. That there are no bosonic states in the charged sector can be demonstrated by the following argument (which is generalized in Sec. 2.5):

Consider the effect of the operator $R_1 R_2$ on a three-vortex state:

$$
R_1 R_2|a,b,c\rangle = |\Phi_t c \Phi_t^{-1}, a, b\rangle \quad (\Phi_t = abc) \tag{2.23}
$$

Applying $R_1 R_2$ two more times conjugates the original state by the total flux:

$$
(R_1 R_2)^3|a,b,c\rangle = |\Phi_t a \Phi_t^{-1}, \Phi_t b \Phi_t^{-1}, \Phi_t c \Phi_t^{-1}\rangle \tag{2.24}
$$

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
$^5\Gamma$	2	ω	$-\omega^2$	0	-2	$-\omega$	ω^2
$^6\Gamma$	2	$-\omega^2$	ω	0	-2	ω^2	$-\omega$

Table 2.1: Characters of the two "extra" representations of T^D contained in the charged subspace: $\omega = exp(\frac{1}{3}\pi i)$

	C_1	C_2	C_3
$^1\Gamma$	1	1	1
$^2\Gamma$	1	ϕ	ϕ^2
$^3\Gamma$	1	ϕ^2	ϕ

Table 2.2: Character table of Z_3: $\phi = exp(\frac{2}{3}\pi i)$

Since $\Phi_t \in N(\Phi_t)$, this is a global gauge transformation by the total flux. If the global gauge group is a cyclic group generated by the total flux (as is the case for this sector), every non-trivial global gauge transformation can be effected by a braiding. Therefore, a bosonic state, which is trivial under all braidings, must also be trivial under all gauge transformations, hence it must be in the uncharged sector.

The charged subspace is reducible when considered as an ordinary representation of T^D. The space decomposes into $^5\Gamma \oplus ^6 \Gamma$, whose characters are listed in Table 2.1. It is clear from the characters that these two representations are related by an automorphism.

2.4 Sectors Of The Four Vortex Problem

The product of four two-cycles is either a three-cycle or the identity. Just as in the case of two-cycle total flux, the physics can't depend on which three-cycle is chosen, so we will arbitrarily take the total flux to be (123) in that sector.

2.4.1 $\Phi_t = (123)$

There are 27 states of four two-cycle vortices with $\Phi_t = (123)$. The centralizer of (123) is $\{e, (123), (321)\} \cong Z_3$. The 27 states are divided equally among the three inequivalent irreducible representations of Z_3 (table 2.2).

$\Phi_t = (123)$, Uncharged sector

A typical state in the uncharged sector is

$$|1\rangle = \frac{1}{\sqrt{3}}(|(12),(12),(12),(23)\rangle + |(23),(23),(23),(31)\rangle + |(31),(31),(31),(12)\rangle) \tag{2.25}$$

As in the uncharged three-vortex sector, the braid generators may be represented as permutations on the states. In some basis, the braid generators for this sector may be written as

$$\begin{aligned} R_1 &= (1)(2)(3)(465)(798) \\ R_2 &= (1)(6)(9)(275)(348) \\ R_3 &= (4)(5)(6)(132)(789) \end{aligned} \tag{2.26}$$

These are even permutations, so the sector group is a subgroup of A_9. Direct computation (using a computer, *cf.* App. B) shows that $K(4,(123),{}^1\Gamma)$ is a proper subgroup of A_9, of order 216.

There is a bosonic state in this sector, namely

$$|B\rangle = \frac{1}{3}(|1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle + |7\rangle + |8\rangle + |9\rangle) \tag{2.27}$$

Unlike the uncharged three-vortex sector, there is only one bosonic state in this sector.

$\Phi_t = (123)$, Charged sectors

Table 2.2 shows that the ${}^2\Gamma$ and ${}^3\Gamma$ sectors give essentially the same physics.

The braid operators in these sectors cannot be simply represented by permutations, since unremovable phases arise in the braiding. The phases are cube roots of unity, and the charged sectors can be considered to be Z_3 projective representations of the sector group of the uncharged sector. There are no bosonic states in this sector.

	C_1	C_2	C_3
$^1\Gamma$	1	1	1
$^2\Gamma$	1	-1	1
$^3\Gamma$	2	0	-1

Table 2.3: Character table of S_3.

2.4.2 $\Phi_t = e$

The centralizer of the trivial flux is all of S_3 (Table 2.3). There are 27 states of four two-cycle vortices with trivial total flux.

$\Phi_t = e$, Uncharged Sector

There are 5 states in this sector. One bosonic state may be written down immediately:

$$|B_1\rangle = \frac{1}{\sqrt{3}}(|(12),(12),(12),(12),\rangle + |(23),(23),(23),(23),\rangle + |(31),(31),(31),(31)\rangle) \tag{2.28}$$

It is possible to write the other four states in such a way that the braid generators are as follows:

$$\begin{aligned} R_1 &= (1)(234) \\ R_2 &= (4)(132) \\ R_3 &= R_1 \end{aligned} \tag{2.29}$$

Remarkably, there are only two distinct braid generators in this sector, and comparison with Eqn. 2.14 shows that the space of sector $(4, e, {}^1\Gamma)$ is isomorphic to the space of sector $(3, (31), {}^1\Gamma)$. In particular, we note that the sector group

$$K(4, e, {}^1\Gamma) \cong K(3, (31), {}^1\Gamma) \cong A_4 \tag{2.30}$$

and that the bosonic subspace is two dimensional.

$\Phi_t = e$, Charge $^2\Gamma$

There are 4 states in the non-trivial one dimensional irreducible representation of S_3. Just as in the uncharged sector, there are only two distinct braid generators, and the space is isomorphic to the charged three-vortex space:

$$R_1 = R_3$$
$$K(4, e, {}^2\Gamma(S_3)) \cong K(3, (31), {}^2\Gamma(Z_2)) \tag{2.31}$$

$\Phi_t = e$, Charge $^3\Gamma$

This subspace is 18 dimensional. Unlike the other charged sectors considered so far, it has a two dimensional bosonic subspace. The sector group is order 216, like the sector group for the uncharged, $\Phi_t = (123)$ sector.

2.5 Many-vortex Sectors

First we will present several formulae and lemmas that are valid for an arbitrary number of vortices. Proofs are given in App. A.

The dimensions of the various sectors of the S_3 vortex model are given in Lemma 1:

Lemma 1 *For an odd number n of vortices, the dimensions of the sectors are*

$$\dim(n, (31), {}^1\Gamma(Z_2)) = \frac{1}{2}(3^n + 1) \tag{2.32}$$
$$\dim(n, (31), {}^2\Gamma(Z_2)) = \frac{1}{2}(3^n - 1) \tag{2.33}$$

For an even vortex number n, the sectors with trivial total flux have dimensions

$$\dim(n, e, {}^1\Gamma(S_3)) = \frac{1}{2}(3^{n-2} + 1) \tag{2.34}$$
$$\dim(n, e, {}^2\Gamma(S_3)) = \frac{1}{2}(3^{n-2} - 1) \tag{2.35}$$
$$\dim(n, e, {}^3\Gamma(S_3)) = 2(3^{n-2}) \tag{2.36}$$

The sectors with even vortex number n, three-cycle flux have dimensions

$$\dim(n,(123),^1\Gamma(Z_3)) = \dim(n,(123),^2\Gamma(Z_3)) = \dim(n,(123),^3\Gamma(Z_3)) = 3^{n-2} \tag{2.37}$$

The following two lemmas are more general than Lemma 1, in that they are not specific to the S_3 model. Lemma 2 is the additional defining relation alluded to in Section 2.2, and Lemma 3 follows immediately from Lemma 2

Lemma 2 *In any n vortex sector with trivial total charge, and in sectors with trivial total flux and arbitrary charge, the generators of the sector group obey the relation*

$$(R_1 R_2 \ldots R_{n-1})^n = e \tag{2.38}$$

In any sector with n vortices, the generators of the sector group must obey

$$(R_1 R_2 \ldots R_{n-1})^{n \cdot n_G} = e \tag{2.39}$$

where n_G is the order of the global gauge group for that sector.

Lemma 3 *In any sector whose global gauge group is a cyclic group generated by the total flux, there are no bosonic states in sectors with nontrivial charge.*

The next two lemmas apply to the S_3 vortex model. They generalize the results of the preceeding sections.

Lemma 4 *In any d-dimensional uncharged sector of the S_3 vortex model, the sector group is a subgroup (not necessarily proper) of the alternating group on d objects, that is to say,*

$$K(n,\Phi_t,^1\Gamma) \subseteq A_d \quad d = \dim(n,\Phi_t,^1\Gamma) \tag{2.40}$$

Lemma 5 *In any sector whose global gauge group is a cyclic group Z_N generated by the total flux, the space of any non-trivially charged sector is a Z_N projective representation of the sector group of the corresponding uncharged sector.*

In Sec. 2.4.2, it was noted that the representation of the sector group on on the uncharged, trivial total flux four vortex sector is isomorphic to the representation of the sector group on the uncharged three vortex sector with $\Phi_t = (31)$, and likewise there is an isomorphism between the three vortex ${}^2\Gamma(Z_2)$ sector and the four vortex, trivial flux ${}^2\Gamma(S_3)$ sector. Using Lemma 1, one sees that the dimension of the uncharged, $\Phi_t = (31)$ n-vortex sector is always equal to the dimension of the uncharged, $\Phi_t = e$ $(n+1)$-vortex sector, and likewise for the corresponding charged sectors. It turns out that the isomorphism between sector group representations discovered in Sec. 2.4.2 generalizes for the S_3 model:

Theorem 1 *The representation space, with respect to the sector group, of the uncharged, $\Phi_t = (31)$, n-vortex sector (n odd) is isomorphic to the sector group representation space of the uncharged, $\Phi_t = e$, $(n+1)$- vortex sector: that is to say, the corresponding sector groups are isomorphic:*

$$K(n, (31), {}^1\Gamma(Z_2)) \cong K(n+1, e, {}^1\Gamma(S_3)) \quad n \text{ odd} \tag{2.41}$$

and the representations (typically reducible) of the sector group on these sections are equivalent. A corresponding isomorphism exists between the charged, $\Phi_t = (31)$, n-vortex representation space and the non-trivial one dimensional charged $({}^2\Gamma(S_3))$, $\Phi_t = e$ $(n+1)$-vortex space.

The final lemma concerns bosonic subspaces. The Bose states in non-abelian vortex sectors are particularly interesting from a physics standpoint, for one expects that the ground state of any sector that has a bosonic subspace will be a Bose state.

Lemma 6 *Let $bose(n, \Phi_t, {}^\alpha\Gamma)$ denote the bosonic subspace of the given sector. For any uncharged sector with an odd number of vortices, and for any uncharged sector* *with an even number of vortices and trivial total flux,*

$$\dim(\text{bose}(n,(31),^1\Gamma)) \geq 2 \quad n \text{ odd}, \tag{2.42}$$

$$\dim(\text{bose}(n,e,^1\Gamma)) \geq 2 \quad n \text{ even}, \tag{2.43}$$

$$\dim(\text{bose}(n,e,^2\Gamma)) \geq 2 \quad n \text{ even} \tag{2.44}$$

For an uncharged sector with an even number of vortices and three-cycle total flux,

$$\dim(\text{bose}(n,(123),^1\Gamma)) \geq 1 \quad n \text{ even}. \tag{2.45}$$

Theorem 1 and Lemma 3 combined show that there are no bosonic states in the non-trivial one-dimensional charged states ($^2\Gamma(S_3)$) with trivial total flux.

Lemma 6 only yields lower bounds on the bosonic dimensions of the uncharged sectors. The results of the numerical procedures described in App. B may help shed light on sectors with more vortices. One finds, for $n \leq 7$ vortices, that

- The bosonic dimension of trivial flux sectors with non-abelian charge $^3\Gamma(S_3)$ is two.
- The bosonic dimension of uncharged sectors with $\Phi_t = (31)$ and with $\Phi_t = e$ is exactly two.
- The bosonic dimension of uncharged sectors with $\Phi_t = (123)$ is exactly one.

It seems likely, therefore, that the bounds in Lemma 6 are saturated.

The sector groups grow in size quite rapidly with increasing numbers of vortices. The order of the sector group in the uncharged five vortex sector is at least 25,920, and the order in the seven vortex uncharged sector is greater than 100,000. This is to be expected, since Lemma 1 combined with Lemma 4 show that the groups which contain the sector groups for uncharged sector grow combinatorially in the dimension of the sector, which in turn grows exponentially. There can also be a wide gap in the size of sector groups for different sectors with the same number of vortices, as Theorem 1 shows.

The sector group for the trivial-flux, non-abelian charge sector with four particles is isomorphic to the sector group of the four vortex zero charge, three-cycle flux sector. This is not surprising, since they have the same number of generators and the same defining relations (including the defining relation implied by Lemma 2). We haven't proved that no new defining relations occur for larger vortex numbers, so we can't be certain that the corresponding sector groups will be isomorphic for any number of particles, but it seems likely. In addition, the characters of the representation of the sector group on the non-abelian charge are twice the characters of the three-cycle flux sector. Lemma 1 shows that the non-abelian charge sector is always twice as large as the uncharged three-cycle flux sector for any number of vortices; furthermore, the lower bound on bosonic dimension for the non-abelian charge sectors is again twice the uncharged three-cycle flux lower bound. It is likely, perhaps, that that the non-abelian charge sectors are always isomorphic to two copies of the corresponding uncharged, three-cycle flux sectors.

Chapter 3 A Non-abelian Vortex Lattice Gas

In this chapter, we investigate the ground state properties of a lattice gas of non-abelian vortices. In doing so, we are extending the work of Canright and Girvin[10, 11] on abelian anyons to non-abelian particles. They considered identical anyons on a two-dimensional lattice, with Hamiltonian

$$H = -t \sum_{\langle i,j \rangle} [c_i^\dagger c_j \exp(i\phi_{ij}) + h.c.] + u \sum_{\langle i,j \rangle} n_i n_j \tag{3.1}$$

Here, ϕ_{ij} is the anyon phase (with a possible contribution from an externally applied magnetic field), and u is a constant that sets the strength of a nearest-neighbor interaction. The lattice has periodic boundary conditions in one direction and hard walls in the other direction, so topologically it is equivalent to a cylinder. The ground state was iteratively determined using the modified Lanczos technique[15, 12].

3.1 The Patching Convention

In order to define many-vortex states, it is necessary to define a convention for patching the single vortex states together. Essentially, this defines an ordering on the single particle fluxes, to specify a unique total flux.

On the lattice, we define the following ordering: vortices live on the vertices of the lattice, which may be specified by a row number and a column number. Vortices are numbered starting at the bottom of the leftmost column, moving to the top of the column, then starting again at the bottom of the next column. In other words, a particle has a higher number than another if the number of its column is larger, or if they are in the same column and the number of its row is greater. This order

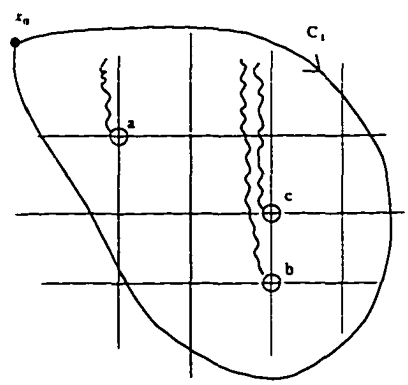

Figure 3.1: Ordering convention for vortices on the lattice. x_0 is the basepoint for flux measurements.

determines the order in which the single particle fluxes fill the multi-particle state, and also the value of the total flux.

The meaning of this convention is show in Fig.3.1 for three vortices with single particle fluxes a, b, and c. The basepoint x_0 anchors the path C that defines the total flux. This path can be deformed to the path $C_1C_2C_3$, travelling path C_3 first, then C_2, then C_1. This specifies the state and total flux

$$|a, b, c\rangle \quad \Phi_t = abc \tag{3.2}$$

The vector potential of a vortex is taken to vanish everywhere except for a delta-function singularity on a string that starts on the core of the vortex and ends at infinity (or a wall). The ordering convention for vortices in the same column determines how the strings must be deformed to maintain the proper order. The total flux associated with the loop C may be determined by the order in which the strings intersect the loop.

As the particles hop around the lattice, the order changes, but the total flux remains constant due to the action of the braid operators. If the particle with flux c in Fig.3.1 hops to the left across the string of the b vortex, it becomes conjugated by b's flux; this is a counterclockwise braiding on the last pair of particles:

$$|a, b, c\rangle \mapsto R_2|a, b, c\rangle = |a, bcb^{-1}, b\rangle \tag{3.3}$$

If the same particle hops to the left again, its string sweeps across the vortex with flux a, leading to a braiding of the first pair of particles in a *clockwise* sense:

$$|a, bcb^{-1}, b\rangle \mapsto R_1^{-1}|a, bcb^{-1}, b\rangle = |bcb^{-1}, (bcb^{-1})^{-1}a(bcb^{-1}), b\rangle \tag{3.4}$$

3.2 The Multi-vortex Hilbert Space

Having defined a patching convention, the Hilbert space factors into an internal space, specifying the vortex fluxes, and the position space. A typical three-vortex state on a 2×3 lattice may be written

$$|\psi\rangle = |internal\rangle \otimes |posn.\rangle = |a, b, c\rangle \otimes \left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array} \right\rangle \tag{3.5}$$

Here, $a, b, c \in \bar{H}$ are the single particle fluxes, a 1 denotes an occupied lattice site and a 0 denotes an unoccupied site. The patching convention unambiguously specifies which single particle flux is associated with which lattice site.

A basis for the Hilbert space of n vortices with a given total flux on a specified lattice may be constructed using states like Eqn. 3.5. This space can further be decomposed into sectors irreducible under the global guage group and any spatial symmetries in the problem (translation symmetry for periodic boundary conditions is discussed in Sec. 3.4).

3.2.1 Position Space

Here we will define several operators that act on the position part of the states. A lattice site is specified by a $2d$ vector $\mathbf{r} = (r, c)$, where r and c are the row and column of the site.

We define creation/annihilation operators $a^\dagger(\mathbf{r})$ and $a(\mathbf{r})$, respectively, at every site. The vortices have hard cores, so the operators obey

$$(a^\dagger(\mathbf{r}))^2 = (a(\mathbf{r}))^2 = 0 \tag{3.6}$$

As usual, one defines a number operator $n(\mathbf{r})$:

$$n(\mathbf{r}) = a^{\dagger}(\mathbf{r})a(\mathbf{r}) \tag{3.7}$$

The *ordering operator* $\Omega(\mathbf{r})$ is defined in terms of the patching convention of Sec. 3.1. Acting on a state like Eqn. 3.5, it yields the order a flux at site $\mathbf{r}$ would take in the internal space. It does this by counting the number of vortices that precede that site according to the patching convention.

$$\Omega(\mathbf{r}) = \Omega((r', c')) = \sum_{r_i} \sum_{c_i < c'} n((r_i, c_i)) + \sum_{r_i < r'} n(r_i, c') + 1 \tag{3.8}$$

3.2.2 The Internal Space

Several operators that act on the internal space have been introduced in previous chapters. These are the braid operators R_i, which implement a counterclockwise braid on the i-th pair of vortices, and the conjugation operator $C(h)$, which conjugates the entire state by $h \in \bar{H}$. In addition, it is useful to define the operators $z_f^{\dagger}(i)$ and $z_f(i)$, $f \in \bar{H}$, which create and destroy, respectively, a flux f at the i-th position in the vortex internal state:

$$z_f^{\dagger}(i)|a, \ldots, \overset{i}{h}, \ldots, \overset{n}{k}\rangle = |a, \ldots, \overset{i}{f}, \overset{i+1}{h}, \ldots, \overset{n+1}{k}\rangle \tag{3.9}$$

$$z_f(i)|a, \ldots, \overset{i}{g}, \overset{i+1}{h}, \ldots, \overset{n}{k}\rangle = \delta_{f,g}|a, \ldots, \overset{i}{h}, \ldots, \overset{n-1}{k}\rangle \tag{3.10}$$

The operator $\Lambda_h(i)$ conjugates only the flux in position i by the group element $h \in \bar{H}$:

$$\Lambda_h(i)|a, \ldots, \overset{i}{g}, \ldots, k\rangle = |a, \ldots, \overset{i}{hgh^{-1}}, \ldots, k\rangle \tag{3.11}$$

Using the ordering operator from Sec. 3.2.1, it is now possible to define operators $c_f^{\dagger}(\mathbf{r})$ and $c_f(\mathbf{r})$, acting on both the position and internal space, that create/destroy a particle of flux f at position $\mathbf{r}$ on the lattice:

$$c_f(\mathbf{r}) = z_f(\Omega(\mathbf{r}))a(\mathbf{r})$$
$$c_f^\dagger(\mathbf{r}) = z_f^\dagger(\Omega(\mathbf{r}))a^\dagger(\mathbf{r}) \tag{3.12}$$

3.3 The Lattice Hamiltonian

We may now construct a Hamiltonian analogous to Eqn. 3.1 for non-abelian vortices on a lattice. The vortices are allowed to hop to nearest-neighbor sites, leading to a kinetic term analogous to the first term of Eqn. 3.1. There is no explicit interaction term, but the kinetic term must take into account the statistical interactions of the vortices, *i.e.*, the braiding properties.

To construct the Hamiltonian, consider its effect on an n-particle flux/position eigenstate (like Eqn. 3.5):

$$|\psi\rangle = |g_1, g_2, \ldots, g_n\rangle \otimes |posn.\rangle \tag{3.13}$$

Say a vortex at site $\mathbf{r}_i = (r_i, c_i)$ and flux $g_{\Omega(\mathbf{r}_i)}$ hops to the unoccupied site one unit to the left $\mathbf{r}_j = (r_i, c_i - 1)$. Two things happen: first the moving particle is successively conjugated by the fluxes of all the vortices below it in column c_i, then all of the vortices above the vortex in column $(c_i - 1)$ are conjugated by the resultant flux. It is convenient to define operators that count particles above or below a lattice site $\mathbf{r} = (r, c)$:

$$n_a(\mathbf{r}) = \sum_{r_i > r} n(r_i, c)$$
$$n_b(\mathbf{r}) = \sum_{r_i < r} n(r_i, c) \tag{3.14}$$

The total flux that conjugates the moving vortex is then

$$k(\mathbf{r}_i) = g_{\Omega(\mathbf{r}_i) - n_b(\mathbf{r}_i)} \cdots g_{\Omega(\mathbf{r}_i)-2} g_{\Omega(\mathbf{r}_i)-1} \equiv \prod_{\ell=\Omega(\mathbf{r}_i)-1}^{\Omega(\mathbf{r}_i)-n_b(\mathbf{r}_i)} g_\ell \in \bar{H} \tag{3.15}$$

The product is the product in the group $\bar{H}$. Eqn. 3.15 defines the ordered product $\prod$, which is taken to order terms right to left.

The operator that conjugates the vortices braided by the moving vortex is

$$\Lambda(\mathbf{r}_i, \mathbf{r}_j) = \prod_{m=\Omega(\mathbf{r}_i)-n_b(\mathbf{r}_i)-1}^{\Omega(\mathbf{r}_i)-n_b(\mathbf{r}_i)-n_a(\mathbf{r}_j)-1} \Lambda_m(k(\mathbf{r}_i)g_{\Omega(\mathbf{r}_i)}k(\mathbf{r}_i)^{-1}) \tag{3.16}$$

In terms of these operators, the Hamiltonian is [1]

$$H = -t\sum_{\langle i,j\rangle}\sum_a (c^\dagger_{k(\mathbf{r}_i)ak(\mathbf{r}_i)^{-1}}(\mathbf{r}_j)\Lambda(\mathbf{r}_i,\mathbf{r}_j)c_a(\mathbf{r}_i) + h.c.) \tag{3.17}$$

An alternate form of the Hamiltonian may be constructed using the braid operators. In terms of the ordered product, define two braiding operators $B_1(\mathbf{r}_i)$ and $B_2(\mathbf{r}_i, \mathbf{r}_j)$:

$$B_1(\mathbf{r}_i) = \prod_{\ell=\Omega(\mathbf{r}_i)-1}^{\Omega(\mathbf{r}_i - n_b(\mathbf{r}_i)} R_\ell \tag{3.18}$$

$$B_2(\mathbf{r}_i, \mathbf{r}_j) = \prod_{m=\Omega(\mathbf{r}_i)-n_b(\mathbf{r}_i)-1}^{\Omega(\mathbf{r}_i)-n_b(\mathbf{r}_i)-n_a(\mathbf{r}_i)-1} R_m^{-1} \tag{3.19}$$

One can now consider the braiding process to be a sequence in which one removes a particle from the position part of the state, applies the appropriate braiding operators to the internal state, and then creates the particle in the new site in the position space.

$$H = -t\sum_{\langle i,j\rangle}(a^\dagger(\mathbf{r}_j)B_2(\mathbf{r}_i,\mathbf{r}_j)B_1(\mathbf{r}_i)a(\mathbf{r}_i) + h.c.) \tag{3.20}$$

3.4 Translation Symmetry on the Cylinder

We periodically identify the lattice in one dimension (call it the x direction) and erect hard walls in the other direction, so the configuration space is a cylinder. Typically, one would expect that the translation invariance in the x direction enables one to

[1] It is implicit that k and Λ are the identity for up and down hops, *i.e.*, for hops in the same column.

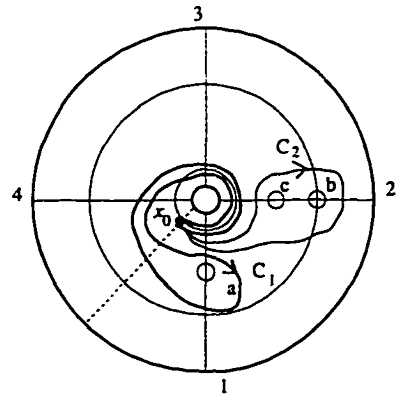

Figure 3.2: Three particles on a 3×4 lattice: the dashed line is the International Dateline.

block-diagonalize the Hamiltonian by a discrete Fourier transform in x, *i.e.*, by going to momentum space in the x direction. The spatial symmetry decomposes the multi-vortex space into blocks labeled by the wave vector k_x, where k_x ranges over the Brillouin Zone for the lattice. It turns out that the translation symmetry is modified for non-abelian vortices, as we shall now elucidate.

Topologically, the cylinder is homeomorphic to an annulus. The columns of the lattice are spokes on the annulus, and the rows are concentric circles. We chose a basepoint x_0 for the flux definition, and select a column as a starting point, numbering the columns from 1 to N_c. Fig. 3.2 shows the case of three particles on a 3×4 lattice. The total flux of this state is $\Phi_t = abc$, by our patching convention.

The particle with flux a is in column 1, and by hopping one unit clockwise it can reach column 4. In doing so, it goes from being the *first* vortex by our patching convention to the *last*, so apparently

$$|a, b, c\rangle \mapsto |b, c, a\rangle \tag{3.21}$$

But because these are non-abelian particles, in general $abc \neq bca$, so it would appear that the total flux changes. Since the total flux is conserved, this is not possible.

This apparent dilemma is resolved by taking care to use only the paths prescribed by the patching convention to measure flux. The standard paths to define single particle fluxes reach only as far as column N_c, they are not allowed to wrap back around to column 1. Consider a particle with flux a, the first particle in the internal state, sitting in column 1. The combined flux of the rest of the particles is $a^{-1}\Phi_t$. Take the path C_2 from the basepoint to encircle all of the particles except the first, and take C_1 to be the path encircling particle 1 that *deforms* to the proper path *after particle 1 has jumped from the first to the last column, becoming the last particle.* Paths C_1 and C_2 are indicated in Fig. 3.2. The flux bounded by C_1 is equal to the flux that the hopping particle will have after it jumps from the first to the last particle.

Let B be a path that encircles the central hole of the annulus in a clockwise sense: by considering the gauge strings of the vortices (*cf.* Sec. 3.1) we see that the flux linked by B is the total flux Φ_t. Let C_0 be a *standard* path that encloses vortex a, before the hopping. Then it is easy to see that, under homotopy,

$$C_1 \cong B^{-1}C_0B \tag{3.22}$$

In other words, the flux linked by C_1 is $\Phi_t^{-1}a\Phi_t$, so the hopping from column 1 to column N_c yields

$$|a,b,c\rangle \mapsto |b,c,\Phi_t^{-1}a\Phi_t\rangle \tag{3.23}$$

and the total flux is preserved.

Obviously, it is possible for the last particle to hop across the "International Dateline"[2](a line separating column 1 and column N_c) and become the first particle. The operator F that performs the last-to-first rotation on the state, conserving total flux, may be expressed in terms of the braid operators:

$$F = R_1R_2\dots R_{n-1} \tag{3.24}$$

$$F|a,b,\dots,k\rangle = |\Phi_t k\Phi_t^{-1},a,b,\dots\rangle \tag{3.25}$$

The first-to-last rotation is performed by F^{-1}.

Because acting on a state with F is equivalent to a braiding, F is trivial on any bosonic state. It is simply a rotation on trivial flux sectors.

[2]Credit for coining this term goes to John Preskill.

To analyze the effect of the International Dateline on the translational symmetry of the vortex gas, it is instructive to consider a simple system: indistinguishable hard-core bosons on a one-dimensional ring with N sites. Define the operator T which translates every particle one site to the right; if T commutes with the Hamiltonian, there is a translation symmetry that is isomorphic to Z_N. The Brillouin Zone consists of the wave vectors

$$k_j = j\frac{2\pi}{N} - \pi, \quad 1 \le j \le N \tag{3.26}$$

The projection operator that projects onto the j-th momentum state is

$$\mathcal{P}_j = \sum_{m=0}^{N-1} e^{imk_j} T^m \tag{3.27}$$

If we consider the same space, with indistinguishable vortices instead of identical bosons, we must consider a different translation operator $\tilde{T}$, which translates every particle to the right one lattice spacing *and* applies the operator F to the state every time a particle hops across the International Dateline. Applying $\tilde{T}$ N times leads to a gauge transformation by the total flux of the state:

$$\tilde{T}^N|\psi\rangle = \mathcal{C}(\Phi_t)|\psi\rangle \tag{3.28}$$

If the total flux is non-trivial, and the order of the total flux in the global gauge group is p, then we see that the translation symmetry group is now isomorphic to $Z_{(p\cdot N)}$.

For example, consider the sectors of the S_3 vortex model with an odd number of particles, so that the total flux is a two-cycle and the global gauge group is Z_2 generated by the total flux. The translation symmetry group is Z_{2N}, with Brillouin Zone wave vectors

$$k_j = j\frac{\pi}{N} - \pi, \quad 1 \le j \le 2N \tag{3.29}$$

On the uncharged sector, $\mathcal{C}(\Phi_t)$ acts trivially, so under the translation symmetry, the states transform as the Z_N subgroup of Z_{2N}, corresponding to wave vectors with j odd or even, depending on whether N is odd or even. (Eqn. 3.28 shows that the effective translational symmetry group will always be Z_N on an uncharged sector). On the charged sector, the translation symmetry will be Z_{2N}; furthermore, the states in this sector will have momenta k_j with j even or odd, depending on N odd or even. The other projection operators will annihilate the charged states.

An immediate consequence is that there are no translationally invariant states in charged sectors with Z_2 global gauge group, *i.e.*, there are no zero-momentum states in these sectors. An analogous situation may be found in an ordinary conducting cylinder. If there is a magnetic flux through the cylinder that is not an integer multiple of the flux quantum, the momentum in the angular coordinate (in the direction that is periodically identified) is shifted by some fractional amount. For the vortices on a cylinder, the International Dateline is a branch cut which alters the boundary conditions in the x direction, shifting the momentum spectrum[3].

3.5 Ground State Structure

Having constructed a Hamiltonian with the required braiding properties, it is now possible to numerically calculate the ground state for small numbers of particles on a finite lattice. The numerical techniques used are described in Appendix C.

Actually, one can predict the qualitative structure for various vortex sectors by simply considering the braiding and translation properties of the system. For any sector that contains a bosonic subspace, we can expect that the bosonic states will have the lowest g.s. energy. There should be a finite gap between the bosonic subspace and the non-abelian subspace. The ground state energies should be zero-momentum ($k_x = 0$) states, *unless* the sector does not contain zero-momentum states, as is the case for non-trivial Z_2 charge sectors.

Consider first the sector $(3, (31),^1 \Gamma(Z_2)\ )$ on a 2×3 lattice. As discussed in Sec. 3.4, the translation symmetry group for this (uncharged) sector is isomorphic to

[3] I would like to thank John Preskill for pointing out this analogy.

Z_3, with Brillouin Zone vectors $k_x = 0, \pm\frac{2\pi}{3}$. There are two bosonic states in this sector, so the ground state energy (with $k_x = 0$) is doubly degenerate, and equal to -5.60 ± 0.005 (in some units). The lowest energy in the non-abelian subspace (corresponding to the three-dimensional irreducible representation of the A_4 sector group) for $k_x = 0$ is -4.56 ± 0.01 (in the same units), illustrating the anticipated gap between the bosonic and non-abelian subspaces.

The ground state degeneracy of the uncharged sector is somewhat mysterious. It is easily shown that a set of degenerate states of a Hamiltonian forms a representation of the symmetry group of that Hamiltonian. In the usual case, the representation is irreducible, so that the degeneracy can be clearly interpreted in terms of the symmetry of the physical system. When the representation is *reducible*, the degeneracy is termed accidental, and it is usually lifted by an arbitrarily small perturbation to the Hamiltonian. In the case of the sector $(3, (31),^1 \Gamma)$, the two-dimensional ground state subspace is reducible, splitting into two trivial representations of the sector group. Additionally, we note that this degeneracy is very robust: it is unaffected by any perturbation that acts only on the position space, and moreover it is unaffected by any perturbation involving products of the braid operators (since these operators act trivially on bosonic states). Furthermore, this degeneracy is pervasive, as shown by Lemma 3.

When accidental degeneracies occur with such regularity, one usually suspects that there is a larger, "hidden" symmetry group of the Hamiltonian[4]. There may be a hidden symmetry of Eqn. 3.20, such that the distinct bosonic states in sectors where the g.s. degeneracy occurs form irreducible representations under the larger symmetry group.

Moving on to the non-trivial charge $(3, (31),^1 \Gamma(Z_2)\ )$ sector, we note from Sec. 3.4 that the translation group is Z_6, with odd-j representations realized on the charged sector, so the "effective" Brillouin Zone is $k_x = \pi, \pm\frac{\pi}{3}$. Also, in Sec. 2.3, we discovered that under the sector group, the charged sector splits into two automorphic irreducible

[4]The classic example of this is the explanation of the 'extra' degeneracy of the hydrogen atom spectrum in terms of the $SU(2) \times SU(2)$ symmetry of the Hamiltonian, corresponding classically to the conservation of the Runge-Lenz vector in a $1/r$ potential.

n	Φ_t	E_{bose}	$E_{\text{non-ab}}$	$\Delta E/n$
3	(31)	-5.606 ± 0.005	-4.56 ± 0.01	0.35
4	e	-8.801 ± 0.003	-6.91 ± 0.04	0.47
4	(123)	-8.801 ± 0.003	-7.99 ± 0.03	0.20
5	(31)	-8.800 ± 0.003	-8.17 ± 0.06	0.13

Table 3.1: Uncharged-sector g.s. energies.

representations, so there will again be at least a two-fold ground state degeneracy. It turns out that the actual ground state energy of -3.8 ± 0.1 occurs for $k_x = \pi$, so the ground state is indeed doubly degenerate.

For the trivial flux sectors of the four vortices on a 3×3 lattice, the translation symmetry group is always isomorphic to Z_3, and the lowest energies will be found in the translationally invariant (zero-momentum) blocks. The g.s. energy for four bosons on a 3×3 lattice is -8.801 ± 0.003, which is the g.s. energy for both the uncharged and the non-abelian charged (${}^3\Gamma(S_3)$) sectors, both of which are doubly degenerate. The g.s. energy in the ${}^2\Gamma(S_3)$ sector, which doesn't contain a bose state, is -4.23 ± 0.13. The g.s. energies of the non-abelian subspaces of the uncharged and non-abelian charged sectors are -6.91 ± 0.04 and -7.97 ± 0.12, respectively.

The uncharged, $\Phi_t = (123)$ sector has one bosonic state, and the g.s. energy of the non-abelian subspace of this sector is -7.99 ± 0.03. Since this is the same energy as the non-abelian charged trivial flux sector (within numerical uncertainties), it is consistent with the conjecture that the ${}^3\Gamma(S_3)$ sector is isomorphic to two copies of the uncharged, three-cycle flux sector.

In Table 3.1, the bosonic and non-abelian ground state energies for the uncharged three through five vortex sectors are listed. The last column gives the per-particle energy gap, where $\Delta E \equiv E_{\text{non-ab}} - E_{\text{bose}}$.

Chapter 4 Summary and Conclusions

A general description of the behavior of large systems of indistinguishable non-abelian vortices remains an extremely difficult problem. However, we have made incremental progress in discerning some features, at least in one representative model.

The braid group essentially acts as a finite group on sectors with a finite number of particles. The generators of the associated finite group, which is in a many-to-one homomorphism with the corresponding braid group, necessarily obey the Yang-Baxter relations, plus additional defining relations related to the underlying physical model. One of these additional relations, Lemma 2 , is a necessary condition for the sector group to be finite (whether it is sufficient in general is unknown).

The size of the sector group grows very rapidly in the number of particles. The dimension of the vortex sectors also grows exponentially, and one would expect that the dimensions of the irreducible representations of the braid group contained in the sectors will also grow rapidly. Thus excited states should be highly degenerate.

However, this huge degeneracy doesn't seem to occur in the bosonic subspaces that appear in the uncharged and non-abelian charged sectors of the S_3 vortex model, at least for reasonable numbers of particles. For the Hamiltonian studied in this thesis, a kinetic term encoding the braiding properties of the vortices, the bosonic subspaces (where they occur) will contain the ground states of the lattice gas, so the ground state degeneracy doesn't seem to be growing at a large rate.

However, in a number of subspaces, the ground state is (at least) doubly degenerate. These degeneracies don't result from any obvious symmetry of the Hamiltonian, so they hint at some deeper symmetry.

One expects, for the kinetic Hamiltonian, that the ground states of the uncharged and non-abelian charged sectors will be a Bose gas, with an energy gap separating the Bose states from the non-abelian states. This is not the case for charge states associated with non-trivial one-dimensional representations of the global gauge group.

since Bose states do not occur in these sectors.

The calculation of the ground state energies is technically difficult. We define a patching convention for the multi-vortex states, and (at least partially) decompose the internal space of a sector into invariant subspaces under the sector group. The periodic boundary conditions enable us to exploit the translational symmetry of the Hamiltonian, but one must be careful to properly account for the "International Dateline." The ground state energies are calculated iteratively, using a modified Lanczos technique. The dimension of the total Hilbert space (internal and position) grows extremely rapidly with particle number, slowing convergence drastically. We have succeeded in calculating g.s. energies for up to five vortices, and the calculations appear to confirm expectations.

Future Directions

One obvious extension to the work in this thesis would be to analyze other relatively simple models, in an attempt to discern universal features of the non-abelian many body problem. In addition, we can consider more complicated models, such as models with charged vortices, or stable vortices from several different conjugacy classes.

Flux quantization is an effective probe for pairing and collective behavior[9]. Our geometry allows for the introduction of an external flux, which essentially amount to a non-trivial boundary condition for vortices traversing the cylinder. However, we can only choose external fluxes from a discrete set. One might expect that vortices would become pair-correlated in such a way that they commute with the external flux, and that sectors with an odd number of vortices would be frustrated. We have written code to calculate g.s. energies with an external flux, but have not yet achieved reasonable convergence for more than three vortices.

Appendix A Proofs of the Lemmas in Section 2.5

Lemma 1 *For an odd number n of vortices, the dimensions of the sectors are*

$$\dim(n,(31),^1\Gamma(Z_2)) = \frac{1}{2}(3^n+1)$$
$$\dim(n,(31),^2\Gamma(Z_2)) = \frac{1}{2}(3^n-1)$$

For an even vortex number n, the sectors with trivial total flux have dimensions

$$\dim(n,e,^1\Gamma(S_3)) = \frac{1}{2}(3^{n-2}+1)$$
$$\dim(n,e,^2\Gamma(S_3)) = \frac{1}{2}(3^{n-2}-1)$$
$$\dim(n,e,^3\Gamma(S_3)) = 2(3^{n-2})$$

The sectors with even vortex number n, three-cycle flux have dimensions

$$\dim(n,(123),^1\Gamma(Z_3)) = \dim(n,(123),^2\Gamma(Z_3))$$
$$= \dim(n,(123),^3\Gamma(Z_3)) = 3^{n-2}$$

Proof: Consider the case n odd. There are 3^n states of n two-cycle vortices. For odd n, these states have a total flux which is a two-cycle, so there are 3^{n-1} states with $\Phi_t = (31)$. We may construct the matrices that represent the well-defined global gauge group,. $\{e,(31)\}$. Since distinct two-cycles of S_3 do not commute, the only state which is invariant under conjugation by the total flux (31) is the state $|(31),(31),\ldots,(31)\rangle$. There is only one non-zero element on the diagonal of the matrix representative of (31), so we can compute the characters of the representation of the Z_2 global gauge group on this space, namely,

$$\chi(e) = 3^{n-1}, \quad \chi((31)) = 1$$

The orthogonality theorem for group characters states that the number of times n_p the p-th irreducible representation of a group G appears in a given representation is

$$n_p = \frac{1}{n_G} \sum_{g \in G} \chi_p^*(g)\chi(g) \tag{A.1}$$

where n_G is the order of the group, the χ_P are the characters of the p-th irreducible representation, and the χ are the characters of the reducible representation. Using the character table of Z_2 and the characters of the vortex sector above, one immediately arrives at the first two equations in Lemma 1. All of the other formulae are obtained by an analogous calculation.

Lemma 2 *In any n vortex sector with trivial total charge, and in any sector with trivial total flux and arbitrary charge, the generators of the sector group obey the relation*

$$(R_1 R_2 \dots R_{n-1})^n = e$$

In any sector with n vortices, the generators of the sector group must obey

$$(R_1 R_2 \dots R_{n-1})^{n \cdot n_G} = e$$

where n_G is the order of the global gauge group for that sector.

Proof: Consider the operator $(R_1 R_2 \dots R_{n-1})$ acting on an n-vortex state:

$$(R_1 R_2 \dots R_{n-1})|a, b, \dots, k\rangle = |\Phi_t k \Phi_t^{-1}, a, b, \dots\rangle$$

where $\Phi_t = ab\ldots k$. Applying this operator n times yields

$$(R_1 R_2 \ldots R_{n-1})^n |a, b, \ldots, k\rangle = |\Phi_t a \Phi_t^{-1}, \Phi_t b \Phi_t^{-1}, \ldots, \Phi_t k \Phi_t^{-1}\rangle$$

This is a global gauge transformation of the state by the total flux. Since the sector has zero total charge, the operator must act trivially on any state.
The second relation is a consequence of the fact that the order of an element of a finite group must divide the order of the group.

Lemma 3 *In any sector whose global gauge group is a cyclic group generated by the total flux, there are no bosonic states in sectors with nontrivial charge.*

Proof: Lemma 2 shows that a global gauge transformation by the total flux can always be achieved by a braiding. In a sector where the global gauge group is generated by the total flux, all global gauge transforms are equivalent to a braiding. Therefore any state that is trivial under all braidings is in an uncharged sector.

Lemma 4 *In any d-dimensional uncharged sector of the S_3 vortex model, the sector group is a subgroup (not necessarily proper) of the alternating group on d objects, that is to say,*

$$K(n, \Phi_t, {}^1\Gamma) \subseteq A_d \quad d = \dim(n, \Phi_t, {}^1\Gamma)$$

Proof:

Consider an initial basis of the n-vortex space with given total flux, consisting of states which have a definite flux at every point, *i.e.*, states of the form $|a, b, \ldots, k\rangle$, $\Phi_t = ab\ldots k$. The operator that projects onto the uncharged subspace is

$$\mathcal{P} = \sum_{h \in N(\Phi_t)} \mathcal{C}(h) \tag{A.2}$$

This operator adds together all of the states of the initial basis that are related by a global gauge transformation, with equal weight. Label the linearly independent states obtained by applying the projection operator to the initial basis $|1\rangle$, $|2\rangle$, ..., $|d\rangle$. Because the braid operators commute with global gauge transformations, all of the braid operators simply permute the d uncharged states. The relation $R_i^3 = e$ for the braid operators of the S_3 model implies that the braid operators must be products of one-cycles and disjoint three-cycles, therefore they must be even permutations, hence $K(n, \Phi_t, {}^1\Gamma) \subseteq A_d$.

Lemma 5 *In any sector whose global gauge group is a cyclic group Z_N generated by the total flux, the space of any non-trivially charged sector is a Z_N projective representation of the sector group of the corresponding uncharged sector.*

Proof: The group Z_N has N inequivalent irreducible representations. The projection operator for the j-th irreducible representation is

$$\mathcal{P}_j = \sum_{l=0}^{N-1} e^{ilk_j} \mathcal{C}(\Phi_t^l), \quad k_j = \tfrac{(j-1)\pi}{N}, \quad 1 \leq j \leq N \tag{A.3}$$

where we define $\Phi_t^0 = e$. Applying the projection operator to the initial flux eigenstate basis, one obtains d linearly independent vectors as a basis for the j-th charge state, labeled $|1\rangle$, ..., $|d\rangle$. For any $j > 1$, it is clearly possible to put these states in a one-to-one correspondence with the uncharged $j = 1$ states, in which each charged state contains the same flux eigenstate basis members as the corresponding uncharged state, but weighted by phase factors that are N-th roots of unity. The braid operators on the charged sector connect the same states as the corresponding operator on the uncharged sector, modulo these phases. The unremovability of the phases for the charged sectors follows from Lemma 2

Theorem 1 *The representation space, with respect to the sector group, of the uncharged, $\Phi_t = (31)$, n-vortex sector (n odd) is isomorphic to the sector group representation space of the uncharged, $\Phi_t = e$, $(n+1)$- vortex sector: that is to say, the corresponding sector groups are isomorphic:*

$$K(n, (31), {}^1\Gamma(Z_2)) \cong K(n+1, e, {}^1\Gamma(S_3)) \quad n \text{ odd}$$

and the representations (typically reducible) of the sector group on these sections are equivalent. A corresponding isomorphism exists between the charged, $\Phi_t = (31)$, *n-vortex representation space and the non-trivial one dimensional charged* $({}^2\Gamma(S_3))$, $\Phi_t = e$ $(n+1)$*-vortex space.*

Proof: Let the states $|i\rangle$, $1 \leq i \leq d$ be the basis states for the sector $(n, (31), {}^1\Gamma(Z_2))$, and the operators R_i, $1 \leq i \leq n-1$ the braid operators calculated using this basis. From the states $|i\rangle$, obtain the states $|i'\rangle$ by applying the operator $\mathcal{C}((23))$, and the states $|i''\rangle$ by applying the operator $\mathcal{C}((12))$,

$$|i'\rangle = \mathcal{C}((23))|i\rangle \tag{A.4}$$

$$|i''\rangle = \mathcal{C}((12))|i\rangle \tag{A.5}$$

It should be clear that the $|i'\rangle$ and $|i''\rangle$ are bases for the uncharged, n-vortex sectors with total flux (12) and (23), respectively, and that the braid operators on each of these sectors are identical to the braid operators on the (31) flux sector. Now construct a basis for the uncharged, $(n+1)$-vortex trivial flux sector by the following procedure: take a state $|i\rangle$, append a flux (31), take a state $|i'\rangle$, append a flux (12), take a state $|i''\rangle$, append a flux (23), and add the states together:

$$|e_i\rangle = |i,(31)\rangle + |i',(12)\rangle + |i'',(23)\rangle \tag{A.6}$$

It is obvious from this construction that

(i) the d states $|e_i\rangle$ form a basis for the sector $(n+1, e, {}^1\Gamma(S_3))$ (*cf.* Lemma 1), and

(ii) the first $n-1$ braid operators R_i on this sector are identical to the braid operators on the sector $(n, (31), {}^1\Gamma(Z_2))$.

Because there is one additional vortex in the trivial flux sector, there is one more braid operator, R_n. In order for the sector groups to be isomorphic, this extra operator must be 'redundant,' *i.e.*, it must be possible to obtain this operator by taking products of the first $n-1$ braid operators. This is always possible, as we shall now see. Lemma 2. applied to the sector $(n, (31), {}^1\Gamma(Z_2))$, says that

$$(R_1 R_2 \dots R_{n-1})^n = e \tag{A.7}$$

By construction , this is also true in the $n+1$ vortex sector. Lemma 2 applied to $(n+1, e, {}^1\Gamma(S_3))$ tells us that

$$(R_1 R_2 \dots R_n)^{n+1} = e \tag{A.8}$$

Using the Yang-Baxter relations Eqn. 2.4, it is possible to establish, by induction, the formula

$$R_n R_{n-1} \dots R_{n-i}(R_1 R_2 \dots R_n) = (R_1 R_2 \dots R_{n-1}) R_n \dots R_{n-i-1} \tag{A.9}$$

Applying Eqn. A.9 repeatedly to Eqn. A.8 yields

$$\begin{aligned}(R_1 \dots R_{n-1})^n R_n R_{n-1} \dots R_1 R_1 R_2 \dots R_n \\ = R_n \dots R_1 R_1 \dots R_n = e \quad \text{(by Eqn. A.7)}\end{aligned} \tag{A.10}$$

A straightforward group manipulation puts this in the form

$$R_n = R_n^{-1} R_{n-1}^{-1} \dots R_1^{-1} R_1^{-1} \dots R_{n-1}^{-1} \tag{A.11}$$

Now we multiply both sides from the left with R_n^2, and use the S_3 model defining relation Eqn. 2.8:

$$\begin{aligned}R_n^3 = e = R_n (R_{n-1} \dots R_1 R_1 \dots R_{n-1})^{-1} \\ R_n = R_{n-1} \dots R_1 R_1 \dots R_{n-1}\end{aligned} \tag{A.12}$$

The group generated by $R_1, \dots, R_n$ is the same as the group generated by $R_1, \dots, R_{n-1}$,

and the theorem is proved. The statement concerning the charged sectors follows from an analogous construction, applied to the charged n-vortex $\Phi_t = (31)$ sector.
Eqn. A.12 for $n = 3$ is

$$R_3 = R_2 R_1^2 R_2 \tag{A.13}$$

Using the defining relations, it is possible to show that $R_2 R_1^2 R_2 = R_1$ on the three-vortex uncharged subspace, providing a check on Eqn. 2.29.

Lemma 6 *Let bose$(n, \Phi_t, {}^{\alpha}\Gamma)$ denote the bosonic subspace of the given sector. For any uncharged sector with an odd number of vortices, and for any uncharged sector with an even number of vortices and trivial total flux,*

$$\dim(\mathrm{bose}(n, (31), {}^{1}\Gamma)) \geq 2 \quad n \text{ odd},$$
$$\dim(\mathrm{bose}(n, e, {}^{1}\Gamma)) \geq 2 \quad n \text{ even},$$
$$\dim(\mathrm{bose}(n, e, {}^{3}\Gamma)) \geq 2$$

For an uncharged sector with an even number of vortices and three-cycle total flux,

$$\dim(\mathrm{bose}(n, (123), {}^{1}\Gamma)) \geq 1 \quad n \text{ even}.$$

Proof: The proof is by construction. An uncharged sector with an even number of vortices and three-cycle total flux is spanned by states labeled $|1\rangle, \ldots, |d\rangle$, and the braid operators simply permute the states. The state

$$|B\rangle = |1\rangle + \ldots + |d\rangle \tag{A.14}$$

On uncharged odd vortex sectors with $\Phi_t = (31)$, there is a state

$$|B_0\rangle = |(31), (31), \ldots, (31)\rangle \tag{A.15}$$

which is bosonic, hence it decouples from the other $d-1$ states, which can be combined as in Eqn. A.14 to form another bosonic state. On uncharged even vortex sectors with

$\Phi_t = e$, the 'extra' bosonic state is

$$|B_0\rangle = |(12,(12),\ldots,(12)\rangle + |(23),(23),\ldots,(23)\rangle + |(31),(31),\ldots,(31)\rangle \qquad \text{(A.16)}$$

The (unnormalized) projection operator for the $^3\Gamma$ sectors with $\Phi_t = e$ is

$$\mathcal{P}_3 = 2\mathcal{C}(e) - \mathcal{C}((123)) - \mathcal{C}((321))$$

Applying this in turn to the states $|(12),(12),\ldots,(12)\rangle$, $|(23),(23),\ldots,(23)\rangle$, and $|(31),(31),\ldots,(31)\rangle$ yields the bosonic states

$$|B_1\rangle = 2|(12),(12),\ldots,(12)\rangle - |(23),(23),\ldots,(23)\rangle - |(31),(31),\ldots,(31)\rangle$$
$$|B_2\rangle = 2|(23),(23),\ldots,(23)\rangle - |(12),(12),\ldots,(12)\rangle - |(31),(31),\ldots,(31)\rangle$$
$$|B_3\rangle = 2|(31),(31),\ldots,(31)\rangle - |(12),(12),\ldots,(12)\rangle - |(23),(23),\ldots,(23)\rangle$$

But, *e.g.*, $|B_1\rangle + |B_2\rangle = -|B_3\rangle$, so these states represent only two linearly independent bosonic states.

Appendix B Numerical Calculation of Subspaces and Sector Groups

For more than three or four vortices, hand calculation of braid matrices, charge sectors, etc. become impractical. This Appendix describes the numerical techniques used to facilitate the process.

B.0.1 Singular Value Decomposition

The Singular Value Decomposition (SVD) is an extremely effective techniqe for computing orthonormal bases of subspaces. It is based on the fact that any real, $m \times n$ matrix A can be factored in the form

$$A = U\Sigma V^T \tag{B.1}$$

where U is an $m \times n$ column-orthogonal matrix, Σ is an $n \times n$ diagonal matrix with non-negative values on the diagonal, and V is an $n \times n$ orthogonal matrix (for proof, see Ref. [18]). The columns of V whose corresponding values on the diagonal of Σ are zero form an orthonormal basis for the nullspace of A, and the rows of U whose corresponding diagonal values in Σ are nonzero form an orthonormal basis for the range of A. The code I wrote to perform the SVD is essentially a translation of the Algol routine in Ref. [17] to ANSI/ISO C, incorporating a few elements of the routine in Ref. [29].

B.0.2 Sector Group Calculations

It is straightforward to encode the multiplication table for a finite group. Given an $n_G \times n_G$ array (n_G is the order of the group) representing the multiplication table of a finite group (*e.g.*, S_3) and a set of basis elements, (*e.g.*, the two-cycle class), it is then not difficult to find all sets of n elements of the basis set that combine to give some specified total flux.

The states thus generated are not in a definite charge sector, i.e., they are in a reducible representation of the centralizer of the total flux. If the uncharged subspace is desired, one can specify the non-trivial elements of the global gauge group; the program selects a state and conjugates it by all non-trivial elements of the global gauge group, generating the uncharged states. The braid operators can now be calculated on the uncharged sector. By Lemma 4, the braid operators on the uncharged sector are permutations of the states, so the code represents the R_i as permutations.

If a non-trivial charge sector is required, the code makes use of the projection operator for irreducible representations,

$$\mathcal{P}_p = \frac{d_p}{n_G} \sum_{g \in N(\phi_t)} \chi_p^*(g) T(g) \tag{B.2}$$

Here d_p is the dimension of the p-th representation, the χ_p are the characters of the representation, and $T(g)$ is the matrix representing g in the reducible representation. To find the subspace for a given charge state, the appropriate projection operator is calculated, and then passed to the SVD routine. The rows of the resulting U (*cf.* Sec. B.0.1) whose singular values are non-zero are selected as a basis for the given charge sector, the columns of V whose singular values vanish form a basis for the orthogonal complement. Using these vectors. an orthogonal transformation matrix S is constructed, which block-diagonalizes the braid matrices, one block corresponding to the desired charge sector.

After the braid operators on a given charge sector are calculated, the sector group may be computed in a brute force manner by starting with the braid operators, systematically multiplying the group elements with each other, storing new elements as they are created, until the group closes. In this way, a multiplication table for the sector group is generated. After the multiplication table is complete, the classes can be computed. The elements may be stored and manipulated either as matrices in the representation of the group on the sector space, or, if the sector is uncharged, they can be represented as permutations. The former possibility has the advantage that the characters of the sector group representation may be computed, but it is much more costly in terms of storage requirements and computation time, so it is impractical for sectors with more than four vortices.

B.0.3 Calculation of the Bosonic Subspace

Once the braid matrices for a given sector are computed, it is possible (and desirable) to extract the bosonic subspace. A state is bosonic if and only if it is invariant under all of the braid operators of the sector. Consequently, the bosonic subspace of a given sector $\mathrm{bose}(n, \Phi_t, {}^{\alpha}\Gamma(N(\Phi_t)))$ can be determined as

$$\mathrm{bose}(n, \Phi_t, {}^{\alpha}\Gamma) = \bigcap_{i=1}^{n-1} \mathrm{null}(R_i - \mathcal{I}) \tag{B.3}$$

Here $\mathcal{I}$ is the identity matrix, and null() denotes the nullspace of the given matrix.

The SVD is useful to extract the bosonic subspace. The $n-1$ braid matrices are 'stacked' into one larger matrix (whose dimensions are $(n-1)d \times d$, if the braid matrices are $d \times d$), which is passed to the SVD routine. Any element of the nullspace of this large matrix is in the intersection of the nullspaces of all of the braid matrices [1]. As in the calculation of charge sectors, a basis for the bosonic subspace and its orthogonal complement is built from the U and V matrices returned from the SVD routine, which are in turn used to construct an orthogonal transformation that block-diagonalizes the matrices.

[1] If one only desires to find the nullspace intersection of two matrices there is a more efficient algorithm [18, pp. 602-603], but for more than two matrices, the procedure above calculates the bosonic space in one fell swoop

Appendix C The Modified Lanczos Method

The modified Lanczos method is an efficient iterative algorithm to find the ground state of a given Hamiltonian[12]. The algorithm is as follows: given a trial state ψ_0, calculate the expectation value of the Hamiltonian H;

$$\epsilon_0 = \langle\psi_0|H|\psi_0\rangle \tag{C.1}$$

Let the state $\tilde{\psi}_0$ measure the difference between the trial state and an eigenstate of H;

$$H\psi_0 - \epsilon_0\psi_0 \equiv b\tilde{\psi}_0 \tag{C.2}$$

where b is chosen so that $\tilde{\psi}_0$ is properly normalized.

From the trial state ψ_0 and the state $\tilde{\psi}_0$ (which is orthogonal to ψ_0), construct the state ψ_1:

$$\psi_1 = \frac{\psi_0 + \alpha\tilde{\psi}_0}{\sqrt{1+\alpha^2}} \tag{C.3}$$

where α is a variational parameter. Calculating the expectation value of H in the state ψ_1 as a function of α, one finds the value of α that minimizes this expectation value. The state ψ_1 is a closer approximation to the true ground state of H, and $\epsilon_1 = \langle\psi_1|H|\psi_1\rangle$ is an improved estimate of the ground state energy. This method can be iterated, with ψ_1 serving as the trial state for the next iteration.

To summarize the formulae for one iteration (all expectation values are taken with respect to the trial state ψ_0):

$$b = \sqrt{\langle H^2\rangle - \langle H\rangle^2} \tag{C.4}$$

$$f = \frac{\langle H^3\rangle - 3\langle H\rangle\langle H^2\rangle + 2\langle H\rangle^3}{2b^3} \tag{C.5}$$

$$\alpha = f - \sqrt{f^2 + 1} \tag{C.6}$$

$$\epsilon_1 = \langle H \rangle + \alpha b \tag{C.7}$$

and the state ψ_1 is given by Eqn. C.3.

C.1 Useful Techniques

This section describes the actual implementation of the Lanczos method, adapting standard techniques[15, 31, 19].

A basis for the position states is developed as follows. Since every site is occupied by either one or no vortices, the position states may be represented as binary integers[15]. If there are N_r rows and N_c columns, the lowest-order N_c bits represent the first row, the next N_c bits the second row, and so on. For example, with three vortices on a 2×3 lattice, the state

$$\left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{matrix} \right\rangle \tag{C.8}$$

is represented by the number $(1 + 2) + (32) = 35$.

The internal states are also represented by a number: the smallest number of high-order bits required by the dimension of the internal space are reserved for this purpose. In this way, the internal and position states are combined to give a single, unique integer.

The (modified) translation symmetry of the lattice vortices is used to block-diagonalize the Hamiltonian. To begin, consider the case of a trivial flux sector, so that the translation symmetry in the x direction is unmodified, and we need only consider the position part of the state. All of the position eigenstates that can be transformed into one another by shifting every particle to the left or right a certain number of sites will be combined into states with definite total x-momentum. For example, the translationally invariant (zero-momentum) state associated with Eqn. C.8

is

$$\frac{1}{\sqrt{3}}\left(\left|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 1\end{matrix}\right\rangle + \left|\begin{matrix}0 & 1 & 0\\ 1 & 0 & 1\end{matrix}\right\rangle + \left|\begin{matrix}0 & 0 & 1\\ 1 & 1 & 0\end{matrix}\right\rangle\right) \tag{C.9}$$

Clearly, it is only necessary to store one of the terms of the state to uniquely specify the state. By an arbitrary convention, we store the position state whose binary representation is a maximum as a representative of the entire translationally invariant state. For example, the entire state Eqn. C.9 would be represented by the number 35.

Higher momentum states are represented analogously. To generate a state with x-momentum k_j, apply the projection operator

$$\mathcal{P}_j = \sum_{m=0}^{N_c-1} e^{-imk_j}\tilde{T}^m \tag{C.10}$$

to the position eigenstate that is the representative (*i.e.*, the maximum integer) for the momentum state. Note that by our phase convention, the coefficient of the representative state is real.

The operators in the Hamiltonian correspond to bitwise operations on the integer representatives. The integers generated by moving bits around won't necessarily be proper representatives of the new states generated; every time a state is generated, it is rotated to produce the maximum possible number, which is stored as the representative for the state. The code also keeps track of how many shifts are necessary to produce the representative; for non-zero momentum, the state must be multiplied by a factor $\exp(isk_j)$, for s shifts, due to our phase convention.

New complications arise if the sector doesn't have trivial total flux, due to the modification of the translational symmetry discussed in Sec. 3.4. Now the operator F must be applied to the internal state every time every time a particle is rotated through the International Dateline. For example, consider the uncharged sector of the S_3 three vortex problem, in which one finds three internal states that transform into each other under the action of F. Labeling the internal states $|1\rangle$, $|2\rangle$, and $|3\rangle$, F acts as the three-cycle (123). The translationally invariant state analogous to Eqn. C.9 is

$$\frac{1}{\sqrt{3}}\left(|1\rangle \otimes \left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{matrix} \right\rangle + |2\rangle \otimes \left| \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \right\rangle + |3\rangle \otimes \left| \begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{matrix} \right\rangle\right) \qquad \text{(C.11)}$$

Only the first term is stored as a representative. If the bit in the top left corner jumps one space to the left, the state generated is

$$|3\rangle \otimes \left| \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} \right\rangle \qquad \text{(C.12)}$$

The internal state changes because the first vortex becomes last, so the operator F^{-1} acts on the state. The new state is not a valid representative, it must be rotated twice to the left to be maximized. In doing so, another particle hops to the left across the International Dateline, so F^{-1} acts again. The representative of the new state generated by the upper left bit hopping is finally determined to be

$$|2\rangle \otimes \left| \begin{matrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{matrix} \right\rangle \qquad \text{(C.13)}$$

Had the state been in a higher momentum block, the phase $\exp(2ik_j)$ would have been generated.

In calculating expectation values, care must be taken in evaluating the norms of the states[15]. For example, the norm of the state represented by Eqn. C.12 is 3, but the norm of the state represented (in position space) by

$$\left| \begin{matrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{matrix} \right\rangle \qquad \text{(C.14)}$$

is $3^2 = 9$ in a trivial flux sector, or 3 again if in an uncharged sector whose states are not invariant under F.

In a non-trivial charge sector, further complications arise. For example, as noted in Sec. 3.4, in a non-trivial Z_2 charge sector, the translation group is Z_{2N_c}. The non-vanishing states are the odd-momentum states, and there is a further factor of $(2)^2$ in the norm.

C.2 Implementation Details

All of the code described in this appendix and App. B was written in standard ANSI/ISO C. Much of it was originally developed on a Sun Sparcstation 10, and has been ported to a Pentium 166MHz. based PC running Linux 2.0.29 and Windows 95, and to a dual-CPU Sun UltraSparc.

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