THE EXISTENCE AND STABILITY OF PERIODIC MOTIONS IN FORCED NON-LINEAR OSCILLATIONS

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ABSTRACT

A general first order theory is presented for treating forced oscillations in multiple degree of freedom quasi-linear systems. It is shown that under suitable conditions ultraharmonic or subharmonic motion may exist in addition to the harmonic motion which a linearized theory would predict. A general study of the stability of such motions reveals that a sufficient condition for the instability, and consequent jump phenomena, of forced oscillations, is that the amplitude frequency response curves possess a vertical tangent. By considering some fairly general two degree of freedom systems it has been shown that a necessary and sufficient condition for stable forced oscillations in non-linear passive systems is that the amplitude lie outside the region enclosed by the loci of vertical tangency. For systems containing an energy source there is, in addition, a restriction on the magnitude of the non-linear damping force.

The general theory has also been applied to ultraharmonic and subharmonic motion in a one degree of freedom system having a cubic non-linearity in the restoring force. It has been shown here also, that a necessary and sufficient condition for stability is that the amplitude of forced oscillation lie outside the region enclosed by the loci of vertical tangency.

A study of the dependence of the motion on the initial conditions reveals that, while ultraharmonic and harmonic motions are relatively insensitive to the initial conditions, the existence of subharmonic motion can be achieved only for a rather restrictive set of initial conditions.

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1. INTRODUCTION

In essence all problems encountered in mechanics are non-linear in nature and the linearizations commonly used are only approximations, though in many cases such approximations are sufficiently good for all practical purposes. There are, however, some problems in which a linearized theory fails completely to explain an experimentally observed phenomenon. For this reason the engineer is sometimes forced to examine, in some detail, the behavior of non-linear systems. The purpose of this paper is to focus attention on those aspects of the problem in which non-linearity introduces essentially new phenomena, such as jump behavior, ultraharmonic or subharmonic motion, and frequency entrainment, phenomena which cannot be explained by a linear theory.

Forced oscillations are of considerable engineering interest and the subsequent discussion of non-linear systems will be restricted to the analysis of forced oscillations.

Historically, the earliest report of forced non-linear vibrations is undoubtedly Huygen's observations on the synchronous time-keeping of two clocks hung on the same wall. Almost two hundred years passed before any further contributions were made to the subject, most notable of nineteenth century contributions were the writings of Helmholtz and Rayleigh. The early part of this century saw Duffing's classic work on forced oscillations in a one degree of freedom mechanical system, but it was not until the advent of the thermionic electron tube that serious interest was shown by engineers and applied physicists in the general problems of non-linear mechanics. The work of Appleton in England and Van der Pol in Holland attracted considerable interest, particularly from the Russian school of physicists, and it is true to say that the most significant contributions of the last thirty years have been of Russian origin.

Perhaps the best known of these contributions is the work of Kryloff and Bogliuboff and this paper is chiefly concerned with the application of their method to the solution of forced oscillations in non-linear systems.

It is interesting to observe in passing that, while the single degree of freedom system has received considerable attention, the multi-periodic case has been almost entirely neglected despite the fact that the methods of solution are essentially the same in both cases.

As is well known, exact solutions of non-linear problems exist in only a few cases, and in any approximate solution one must be guided by the physics of the problem. The existence of subharmonics in non-linear systems is a well known experimental fact, but as is shown in this paper and in Ref. 1, the existence of subharmonics is dependent on the initial conditions. It is also a well known experimental fact that in systems with small non-linearity, the frequency bands over which one can obtain subharmonic or ultraharmonic motion are rather small compared with the separation between them. These experimental facts allow considerable simplification to be made in the analytic treatment of the problem for, instead of trying to solve the complete problem one can break the problem down into several parts, each of which can be solved with comparative ease.

2. GENERAL THEORY

2.0. Forced Oscillations in a "n" Degree of Freedom Non-Linear System.

Differential equations of the form:

$$\ddot{y}_i + w_i^2 y_i + \mu f_i(y_1, \ldots , y_n, \dot{y}_1, \ldots , \dot{y}_n)^* = \mu \Delta_i \cos(\Omega t + a_i)$$

$$i = 1,2, \ldots , n, \quad \mu \ll 1 \qquad (2.1)$$

in which all non-linearities, damping and coupling terms are grouped in the term $\mu f_i(\cdots)$, occur in many branches of physics and engineering.

An approximate solution, correct to the first order in μ, will be developed for the case where the non-linearities and coupling terms are small in comparison to the linear terms, and where the frequencies w_i differ from one another by an amount of first order in μ. The restrictions on the coupling terms and the frequencies can be relaxed if one is interested only in the stability close to the steady state solutions.

From the theory of differential equations it is known that equations such as (2.1) possess solutions $y_i(t)$ which are uniquely determined once the initial conditions are specified. In the case of a linear system there is only one set of periodic solutions after transients have died out. Non-linear systems possess the distinctive characteristic that various types of periodic solutions may exist depending on the initial conditions and the relative values of the natural frequencies and the forcing frequency.

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* Henceforth this term will be denoted by $(\cdots)$.

If the discussion is restricted to forced oscillations of a system having non-linearities in the restoring forces, then three essentially new phenomena are introduced:

a) Predominantly harmonic solutions--such solutions have a period equal to that of the forcing function. Harmonic non-linear solutions may exhibit jump phenomena--a feature which distinguishes them from linear harmonic solutions.

b) Predominantly ultraharmonic motion--under suitable conditions a system may exhibit ultraharmonic solutions, in which the main component of the motion occurs at some integral multiple of the frequency of the forcing function. Ultraharmonic motion exhibits jump phenomena similar to which occurs in a), the existence of such phenomena cannot be predicted on the basis of a linear theory.

c) Predominantly subharmonic motion--a non-linear system may under suitable conditions, exhibit subharmonic motion in which the main component of the motion has a frequency which is a rational fraction of the frequency of the forcing function.

If the system has non-linear damping such that steady self excited oscillations are possible, then an additional phenomenon is introduced.

d) Frequency entrainment--a non-linear self excited system in the presence of a forcing function may exhibit the phenomenon of frequency entrainment in which the oscillating system pulls into synchronism with the forcing function over narrow frequency bands--a phenomenon which is without parallel in linear systems.

A) Predominantly Harmonic Motion.

In the case of predominantly harmonic motion solutions of equation (2.1) may be taken of the form*

$$\begin{cases} y_i = A_i \cos x_i & (2.2a) \\ y_i = -\Omega A_i \sin x_i & (2.2b) \end{cases}$$

where $x_i = \Omega t + \phi_i$, A_i and ϕ_i are slowly varying functions of time. The implication of equation (2.2b) is that:

$$\dot{A}_i \cos x_i - \dot{\phi}_i A_i \sin x_i = 0. \qquad (2.3)$$

Substituting (2.2a) and (2.2b) into (2.1)

$$(w_i^2 - \Omega^2) A_i \cos x_i - \Omega \dot{A}_i \sin x_i - \Omega A_i \dot{\phi}_i \cos x_i$$

$$+ \mu f_i(\cdots) = \mu \Delta_i \cos(x_i + a_i - \phi_i) . \qquad (2.4)$$

Multiplying both sides of (2.4) by $\cos x_i$, (2.3) by $\Omega \sin x_i$, and adding,

$$(w_i^2 - \Omega^2) A_i \cos^2 x_i - \Omega A_i \dot{\phi}_i + \mu f_i(\cdots) \cos x_i$$

$$= \mu \Delta_i \cos(x_i + a_i - \phi_i) \cos x_i \qquad (2.5)$$

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* E. Trefftz [Math. Ann. 95, p. 307, 1925] points out that (in one degree of freedom) if a solution of (2.1) is stable, it must ultimately lead to a periodic solution whose period is equal to that of the forcing function, or to some integral multiple of that period.

Recently, D. Graffi has extended the existence proof to two degrees of freedom, and it seems reasonable to extend it to n degrees of freedom. [Math. Ann. 54, p262, 1951]

Since A_i and ϕ_i are slowly varying functions of time, they may be replaced by their average values over one cycle. Denoting the average values by barred superscript

$$\therefore \quad -2\Omega \bar{A}_i \dot{\bar{\phi}}_i + (w_i^2 - \Omega^2)\bar{A}_i + \frac{\mu}{\pi}\int_0^{2\pi} f_i(\cdots)\cos x_i \, dx_i$$

$$= \mu \Delta_i \cos(\bar{\phi}_i - \alpha_i) \qquad (2.6)$$

which may be written as

$$2\Omega \bar{A}_i \dot{\bar{\phi}}_i = h_i(\cdots) \qquad (2.7)$$

where

$$h_i(\cdots) = -(\Omega^2 - w_i^2)\bar{A}_i + \frac{\mu}{\pi}\int_0^{2\pi} f_i(\cdots)\cos x_i \, dx_i$$

$$-\mu \Delta_i \cos(\bar{\phi}_i - \alpha_i) \, . \qquad (2.8)$$

If now, equation (2.4) is multiplied by $\sin x_i$, (2.3) by $\Omega \cos x_i$, and the resulting equations subtracted, then

$$(w_i^2 - \Omega^2) A_i \sin x_i \cos x_i - \Omega \dot{A}_i + \mu f_i(\cdots) \sin x_i$$

$$= \mu \Delta_i \cos(x_i + \alpha_i - \phi_i) \sin x_i \, . \qquad (2.9)$$

Averaging over one cycle

$$-2\Omega \dot{\bar{A}}_i + \frac{\mu}{\pi}\int_0^{2\pi} f_i(\cdots)\sin x_i \, dx_i = \mu \Delta_i \sin(\bar{\phi}_i - \alpha_i) \quad (2.10)$$

which may be written:

$$2\Omega \dot{\bar{A}}_i = g_i(\cdots) \qquad (2.11)$$

where

$$g_i(\cdots) = \frac{\mu}{\pi}\int_0^{2\pi} f_i(\cdots)\sin x_i\, dx_i - \mu\Delta_i \sin(\bar{\phi}_i - \alpha_i) \quad (2.12)$$

The behavior of the system is therefore described by the 2n set of equations:

$$\left.\begin{aligned} 2\Omega \bar{A}_i \mathring{\bar{\phi}}_i &= h_i(\cdots) \\ 2\Omega \mathring{\bar{A}}_i &= g_i(\cdots). \end{aligned}\right\} \quad (2.13)$$

From equations (2.8) and (2.12) it will be observed that if the detuning is small, i.e. $|\Omega - w_i| = 0_1(\mu)$, then $h_i(\cdots)$ and $g_i(\cdots)$ are quantities of order μ, and the approximations made in the Kryloff change of variables are therefore justified since $\mathring{\bar{\phi}}_i$ and $\mathring{\bar{A}}_i$ are then of order μ.

Steady State.

The steady state forced oscillations are characterized by constant amplitude and phase, i.e., by $\mathring{\bar{A}}_i = \mathring{\bar{\phi}}_i = 0$. Hence the steady state solutions are determined by the equations

$$\left.\begin{aligned} h_i(\cdots) &= 0 \\ g_i(\cdots) &= 0. \end{aligned}\right\} \quad (2.14)$$

From this set of 2n equations, the steady state phases ${}_o\bar{\phi}_i$, and amplitudes ${}_o\bar{A}_i$ can be obtained.

Stability of Steady State Solutions.

The stability of the steady state solutions will be studied by analyzing the stability of equations (2.13). Let

$$\left.\begin{aligned} \bar{A}_i &= {}_o\bar{A}_i + \xi_i \\ \bar{\phi}_i &= {}_o\bar{\phi}_i + \eta_i \end{aligned}\right\} \qquad (2.15)$$

where ξ_i and η_i are small perturbations on the steady state amplitude and phase, ${}_o\bar{A}_i$, ${}_o\bar{\phi}_i$.

Substituting (2.15) into (2.13), the perturbation equations are:

$$\left.\begin{aligned} 2\Omega\, {}_o\bar{A}_i \overset{\circ}{\eta}_i &= \sum_{j=1}^{n} \frac{\partial h_i}{\partial \bar{A}_j} \xi_j + \sum_{j=1}^{n} \frac{\partial h_i}{\partial \bar{\phi}_j} \eta_j \\ 2\Omega\, \overset{\circ}{\xi}_i &= \sum_{j=1}^{n} \frac{\partial g_i}{\partial \bar{A}_j} \xi_j + \sum_{j=1}^{n} \frac{\partial g_i}{\partial \bar{\phi}_j} \eta_j \end{aligned}\right\} \qquad (2.16)$$

Assuming solutions of the form:

$$\left.\begin{aligned} \eta_i &\sim e^{\lambda t} \\ \xi_i &\sim e^{\lambda t} \end{aligned}\right\}$$

results in a set of 2n linear homogeneous equations, for non-trivial solutions, the determinant of the 2n equations should be zero, hence

$$\begin{vmatrix} \frac{\partial g_1}{\partial \bar{A}_1} - 2\Omega\lambda, & \frac{\partial g_1}{\partial \bar{A}_2}, & \dots, & \frac{\partial g_1}{\partial \bar{\phi}_1}, & \dots\dots\dots\dots, & \frac{\partial g_1}{\partial \bar{\phi}_n} \\ \frac{\partial g_2}{\partial \bar{A}_1}, & \frac{\partial g_2}{\partial \bar{A}_2} - 2\Omega\lambda, & \dots, & \frac{\partial g_2}{\partial \bar{\phi}_1} & \dots\dots\dots\dots, & \frac{\partial g_2}{\partial \bar{\phi}_n} \\ \frac{\partial h_1}{\partial \bar{A}_1}, & \dots\dots\dots\dots, & \dots, & \frac{\partial h_1}{\partial \bar{\phi}_1} - 2\Omega\lambda\bar{A}_1 & \dots, & \frac{\partial h_1}{\partial \bar{\phi}_n} \\ \frac{\partial h_n}{\partial \bar{A}_1}, & \frac{\partial h_n}{\partial \bar{A}_2}, \dots\dots, & \dots, & \frac{\partial h_n}{\partial \bar{\phi}_1}, \dots, & & \frac{\partial h_n}{\partial \bar{\phi}_n} - 2\Omega\lambda\bar{A}_n \end{vmatrix} = 0 \qquad (2.17)$$

Expanding out

$$a_{2n}(2\Omega\lambda)^{2n} + a_{2n-1}(2\Omega\lambda)^{2n-1} \dots\dots a_1(2\Omega\lambda) + a_o = 0 \qquad (2.18)$$

The nature of the roots of the characteristic equation can be determined by use of the Routh-Hurwitz criteria. In particular, a_o in equation (2.18) is given by

$$a_o = \begin{vmatrix} \frac{\partial g_1}{\partial \bar{A}_1}, & \frac{\partial g_1}{\partial \bar{A}_2}, & \dots\dots, & \frac{\partial g_1}{\partial \bar{\phi}_1}, & \dots, & \frac{\partial g_1}{\partial \bar{\phi}_n} \\ \frac{\partial g_2}{\partial \bar{A}_1}, & \frac{\partial g_2}{\partial \bar{A}_2}, & \dots\dots, & \frac{\partial g_2}{\partial \bar{\phi}_1}, & \dots, & \frac{\partial g_2}{\partial \bar{\phi}_n} \\ \vdots & \vdots & & & & \\ \frac{\partial h_1}{\partial \bar{A}_1}, & \frac{\partial h_1}{\partial \bar{A}_2}, & \dots\dots, & \frac{\partial h_1}{\partial \bar{\phi}_1}, & \dots, & \frac{\partial h_1}{\partial \bar{\phi}_n} \\ \vdots & \vdots & & & & \\ \frac{\partial h_n}{\partial \bar{A}_1}, & \frac{\partial h_n}{\partial \bar{A}_2}, & \dots\dots, & \frac{\partial h_n}{\partial \bar{\phi}_1}, & \dots, & \frac{\partial h_n}{\partial \bar{\phi}_n} \end{vmatrix} \qquad (2.19)$$

$$a_o = \frac{\partial(g_1,\ g_2,\ \dots\ ,\ g_n,\ \ h_1,\ \dots\ ,\ h_n)}{\partial(\bar{A}_1,\ \bar{A}_2,\ \dots\ ,\ A_n,\ \ \bar{\phi}_1,\ \dots\ ,\ \bar{\phi}_n)} \tag{2.20}$$

In equations (2.14)

$$\left.\begin{aligned} g_i(\cdots) &= 0 \\ h_i(\cdots) &= 0 \end{aligned}\right\}$$

but $\bar{A}_1,\ \bar{A}_2,\ \dots\ ,\ \bar{\phi}_1,\ \dots\ ,\ \bar{\phi}_n$, are all functions of frequency, Ω,

Thus

$$\frac{\partial \bar{A}_i}{\partial \Omega} = - \frac{\left\{ \dfrac{\partial(g_1,\ g_2,\dots,g_n,\ h_1,\ h_2,\ \dots\ ,\ h_n)}{\partial(\bar{A}_1,\ \bar{A}_2,\dots,\bar{A}_{i-1},\ \Omega\ ,\ \bar{A}_{i+1},\dots,\bar{\phi}_1,\dots,\bar{\phi}_n)} \right\}}{\left\{ \dfrac{\partial(g_1,\ g_2,\dots,\ g_n,\ h_1,\ h_2,\ \dots\ ,\ h_n)}{\partial(\bar{A}_1,\ \bar{A}_2,\dots,\ \bar{A}_n,\ \bar{\phi}_1,\ \bar{\phi}_2,\ \dots\ ,\ \bar{\phi}_m)} \right\}} \tag{2.21}$$

$$\therefore \quad \frac{\partial \bar{A}_i}{\partial \Omega} = - \left\{ \frac{\partial(g_1,\ g_2,\ \dots\ ,\ g_n,\ h_1,\ h_2,\ \dots\ ,\ h_n)}{\partial(\bar{A}_1,\ \bar{A}_2,\ \Omega\ ,\ \bar{A}_n,\ \bar{\phi}_1,\ \dots\dots\ ,\ \bar{\phi}_n)} \right\} \Big/ a_o \tag{2.22}$$

According to the Routh-Hurwitz criteria, a sufficient condition for the system to be unstable is that $a_o \leq 0$. Now $a_o = 0$ marks the transition from stability to instability but from equation (2.22) it will be observed that $\frac{\partial \bar{A}_i}{\partial \Omega} \to \infty$ as $a_o \to 0$, thus, at the boundary between stability and instability the amplitude/frequency response characteristic has a vertical tangent. Conversely if the amplitude/frequency response has a vertical tangent, a_o is zero and the system is on the boundary of stability.

B) <u>Ultraharmonic and Subharmonic Motion</u>.

Consider now the case of predominantly ultraharmonic or subharmonic motion.

Since ultraharmonics and subharmonics occur at frequencies well separated from those at which predominantly harmonic motion occurs, approximate solutions of equations (2.1) may be taken in the form:

$$\begin{cases} y_i = A_i \cos\Omega t + B_i \cos \gamma_i & (2.23a) \\ \dot{y}_i = -\Omega A_i \sin\Omega t - \frac{p}{g} B_i \sin \gamma_i \,.\Omega & (2.23b) \\ 0 = \dot{B}_i \cos \gamma_i - \dot{\beta}_i B_i \sin \gamma_i & (2.23c) \end{cases}$$

where, $A_i \cos\Omega t$ are the solutions of equations (2.1) in the absence of ultraharmonics or subharmonics

$$\gamma_i = \frac{p}{g}\Omega t + \beta_i,$$

B_i and β_i being slowly varying function of time, p and g are both integers. If $p = 1$ and $g = 2,3, \ldots , n$, $B_i = {}_{1/r} A_i$ is the amplitude of the r^{th} subharmonic. If $g = 1$, $p = r$, $r = 2,3,4,$ etc. then $B_i = {}_r A_i$ is the amplitude of the r^{th} ultraharmonic. For convenience in further work, let $\frac{p}{g}\Omega = \Omega'$. Substituting equations (2.2a) and (2.2b) into equation (2.1) gives:

$$(w_i^2 - \Omega^2) A_i \cos \Omega t + (w_i^2 - \Omega'^2) B_i \cos \gamma_i - \Omega' \dot{B}_i \sin \gamma_i$$

$$+\mu f_i(\cdot\cdot\cdot) - \Omega' B_i \dot{\beta}_i \cos \gamma_i = \mu \Delta_i \cos(\Omega t + \alpha_i) \,. \qquad (2.24)$$

Multiplying both sides of (2.24) by $\cos \gamma_i$, (2.3) by $\Omega' \sin \gamma_i$ and adding

$$(w_i^2 - \Omega'^2)\, B_i \cos^2 \gamma_i - \Omega' B_i \dot{\beta}_i + \mu f_i(\cdots) \cos \gamma_i$$

$$+ (w_i^2 - \Omega^2)\, A_i \cos \Omega t \cos \gamma_i = \mu \Delta_i \cos(\Omega t + a_i) \cos \gamma_i \tag{2.25}$$

Since B_i and β_i are slowly varying functions of time they may be replaced by their average values over one cycle. Denoting the average values by bar superscripts:

$$- 2\Omega' \bar{B}_i \dot{\bar{\beta}}_i + (w_i^2 - \Omega'^2)\, \bar{B}_i + \frac{\mu}{\pi} \int_0^T f_i(\cdots) \cos \gamma_i \, d\gamma_i = 0$$

The integral being evaluated over the lowest common period. (2.26)

Equation (2.26) may be written as

$$2\Omega' \bar{B}_i \dot{\bar{\beta}}_i = H_i(\cdots) \tag{2.27}$$

where $H_i(\cdots) = - (\Omega'^2 - w_i^2)\, \bar{B}_i + \frac{\mu}{\pi} \int_0^T f_i(\cdots) \cos \gamma_i \, d\gamma_i$

(2.28)

If now, equation (2.24) is multiplied by $\sin \gamma_i$, (2.23) by $\Omega' \cos \gamma_i$, and the resulting equations subtracted, then

$$(w_i^2 - \Omega'^2)\, B_i \sin \gamma_i \cos \gamma_i - \Omega' \dot{B}_i + \mu f_i(\cdots) \sin \gamma_i$$

$$+ (w_i^2 - \Omega^2)\, A_i \cos \Omega t \sin \gamma_i = \mu \Delta_i \cos(\Omega t + a_i) \sin \gamma_i \tag{2.29}$$

Averaging over one cycle

$$- 2\Omega' \dot{\bar{B}}_i + \frac{\mu}{\pi} \int_0^T f_i(\cdots) \sin \gamma_i \, d\gamma_i = 0 \tag{2.30}$$

which may be written

$$2\Omega' \dot{\bar{B}}_i = G_i(\cdots) \tag{2.31}$$

where

$$G_i(\cdots) = \frac{\mu}{\pi} \int_0^T f_i(\cdots) \sin \gamma_i \, d\gamma_i \tag{2.32}$$

The subharmonic or ultraharmonic behavior of the system is therefore described by the 2n set of equations:

$$\left.\begin{aligned} 2\Omega' \bar{B}_i \dot{\bar{\beta}}_i &= H_i(\cdots) \\ 2\Omega' \dot{\bar{B}}_i &= G_i(\cdots) \end{aligned}\right\} \tag{2.33}$$

From equations (2.28) and (2.32) it will be observed that if the detuning is small, i.e., $|w_i' - \Omega'| = O(\mu)$, then $H_i(\cdots)$ and $G_i(\cdots)$ are quantities of order μ, and the approximations made are therefore justified since $\dot{\bar{\beta}}_i$ and $\dot{\bar{B}}_i$ are quantities of order μ.

Steady State Solutions.

Steady state conditions are characterized by constant amplitude and phase, i.e. by $\dot{\bar{B}}_i = \dot{\bar{\beta}}_i = 0$.

Hence the steady state solutions are determined by the equation

$$\left.\begin{aligned} H_i(\cdots) &= 0 \\ G_i(\cdots) &= 0 \end{aligned}\right\} \tag{2.34}$$

From this set of 2n equations, the steady state phases ${}_o\bar{\beta}_i$, and amplitudes ${}_o\bar{B}_i$ can be calculated.

Stability of Steady State Solutions.

The stability of the steady state solutions will be studied by analyzing the stability of equations (2.33). Let

$$\left.\begin{aligned} \bar{B}_i &= {}_o\bar{B}_i + \xi_i \\ \bar{\beta}_i &= {}_o\bar{\beta}_i + \eta_i \end{aligned}\right\} \qquad (2.35)$$

where ξ_i and η_i are small perturbations on the steady state amplitude and phase ${}_o\bar{B}_i$, ${}_o\bar{\beta}_i$.

Substituting (2.35) into (2.33), the perturbation equations are:

$$\left.\begin{aligned} 2\Omega' \, {}_o\bar{B}_i \, \dot{\eta}_i &= \sum_{j=1}^{n} \frac{\partial H_i}{\partial \bar{B}_j} \xi_j + \sum_{j=1}^{n} \frac{\partial h_i}{\partial \bar{\beta}_j} \eta_j \\ 2\Omega' \, \dot{\xi}_i &= \sum_{j=1}^{n} \frac{\partial G_i}{\partial \bar{B}_j} \xi_j + \sum_{j=1}^{n} \frac{\partial G_i}{\partial \bar{\beta}_j} \eta_j \end{aligned}\right\} \qquad (2.36)$$

Assuming solutions of the form:

$$\eta_i \sim e^{\lambda t}$$

$$\xi_i \sim e^{\lambda t}$$

result in a set of 2n linear homogeneous equations, for non-trivial solutions, the determinant of the 2n equations should be zero, hence:

$$\begin{vmatrix} \frac{\partial G_1}{\partial \bar{B}_1} - 2\Omega'\lambda' , & \frac{\partial G_1}{\partial \bar{B}_2} , & & & \frac{\partial G_1}{\partial \bar{\beta}_n} \\ \frac{\partial G_2}{\partial \bar{B}_1} , & \frac{\partial G_2}{\partial \bar{B}_2} - 2\Omega'\lambda , & & & \frac{\partial G_2}{\partial \bar{\beta}_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial H_n}{\partial \bar{B}_1} , & \frac{\partial H_n}{\partial \bar{B}_n} , & \cdots\cdots , & \frac{\partial H_n}{\partial \bar{\beta}_{n-1}} , & \frac{\partial H_n}{\partial \bar{\beta}_n} - 2\Omega' \bar{B}_n \lambda \end{vmatrix} = 0$$

(2.37)

Expanding out

$$b_{2n}(2\Omega'\lambda)^{2n} + b_{2n-1}(2\Omega'\lambda)^{2n-1} \cdots\cdots b_1(2\Omega'\lambda) + b_o = 0$$

(2.38)

The nature of the roots of the characteristic equation can be determined by the use of the Routh-Hurwitz criteria.

In particular, b_o in equation (2.38) is given by

$$b_o = \frac{\partial(G_1, G_2, \dots, H_1, H_2, \dots, H_n)}{\partial(\bar{B}_1, \bar{B}_2, \dots, \bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n)} \qquad (2.39)$$

Now in the steady state equations (2.34)

$$\left\{ \begin{array}{l} G_i(\dots) = 0 \\ \\ H_i(\dots) = 0 \end{array} \right. \qquad \bar{B}_1, \bar{B}_2, \dots, \bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_n$$

are all functions of frequency Ω'.

Thus

$$\frac{\partial \bar{B}_i}{\partial \Omega} = - \frac{\left\{ \dfrac{\partial(G_1, G_2, \ldots, G_n, H_1, H_2, \ldots, H_n)}{\partial(\bar{B}_1, \bar{B}_2, \ldots, \bar{B}_{i-1}, \Omega, \bar{B}_{i+1}, \ldots, \bar{\beta}_1, \ldots, \bar{\beta}_n)} \right\}}{\left\{ \dfrac{\partial(G_1, G_2, \ldots, G_n, H_1, H_2, \ldots, H_n)}{\partial(\bar{B}_1, \bar{B}_2, \ldots, \bar{B}_n, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n)} \right\}} \tag{2.40}$$

$$\therefore \quad \frac{\partial \bar{B}_i}{\partial \Omega} = - \left\{ \frac{\partial(G_1, G_2, \ldots, G_n, H_1, H_n, \ldots, H_n)}{\partial(\bar{B}_1, \ldots, \bar{B}_{i-1}, \Omega, \bar{B}_{i+1}, \ldots, \bar{\beta}_1, \ldots, \bar{\beta}_n)} \right\} \Big/ b_o \tag{2.41}$$

The condition that the amplitude/frequency curve have a vertical tangent is clearly that $b_o = 0$. Hence loci of vertical tangency is given by

$$b_o = \frac{\partial(G_1, G_2, \ldots, G_n, H_1, H_2, \ldots, H_n)}{\partial(\bar{B}_1, \bar{B}_2, \ldots, \bar{B}_n, \bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n)} = 0 \tag{2.42}$$

According to the Routh-Hurwitz criteria, a sufficient condition for instability in the system is that $b_o \leq 0$. That is, $b_o = 0$ marks the transition from possible stability to definite instability, but $b_o = 0$ defines the loci of vertical tangency, thus, on one side of $b_o = 0$ the system may be stable, on the other side it is unstable. The general theory presented above gives a method of attacking a wide class of non-linear forced oscillation problems. It establishes the fact that a sufficient condition for instability of harmonic, ultra-harmonic or subharmonic motion is that the solutions lie inside the region enclosed by the loci of vertical tangency. Unfortunately it does not seem possible to establish both necessary and sufficient conditions for stability in general. However, by treating some fairly general two degree of freedom systems, it has been possible to show that for any small monotonically increasing non-linearity in the restoring forces a necessary and sufficient condition for the stability of harmonic oscillations is that solutions lie outside the region enclosed by the loci of vertical tangency. The problem of the existence and stability of ultraharmonics and subharmonics has been treated for a single degree of freedom system with a cubic non-linearity and again it has been possible to show that for stability, solutions must lie outside the region enclosed by the loci of vertical tangency.

3. APPLICATION OF GENERAL THEORY TO HARMONIC OSCILLATIONS IN FORCED NON-LINEAR MOTION.

The application of the general theory to harmonic oscillations will be made to two classes of systems:

1) Those systems having non-linearities in the restoring forces.

2) Those systems having non-linear damping.

In 1) the general theory will be applied to two simple two degree of freedom systems having one non-linear spring. The effect of linear damping and initial conditions will be studied for a simple one degree of freedom non-linear spring mass system.

In 2) the general theory will be applied to Van der Pol's equation in two degrees of freedom, in addition a study will be made of a system with hysteresis damping.

3. APPLICATION

Case 1. Consider the simple two degree of freedom system shown in Fig. 1, in which one of the end springs has a small non-linearity.

Equations of Motion.

$$\left.\begin{aligned} m\ddot{y}_1 + k_1 y_1 + k_1 F(y_1) + k_{12}(y_1 - y_2) &= 0 \\ m\ddot{y}_2 + k_1 y_2 + k_{12}(y_2 - y_1) &= P \cos \Omega t \end{aligned}\right\} \quad (3.1)$$

where $k_1 F(y_1)$, the non-linear part of the restoring force on the first mass is an odd function of y_1*.

Rearranging equation (3.1)

$$\left.\begin{aligned} \ddot{y}_1 + w_1^2 y_1 + w_1^2 F(y_1) + w_{12}^2(y_1 - y_2) &= 0 \\ \ddot{y}_2 + w_1^2 y_2 + w_{12}^2(y_2 - y_1) &= \frac{P}{m} \cos \Omega t \end{aligned}\right\} \quad (3.2)$$

where $w_1^2 = k_1/m$, $w_{12}^2 = k_{12}/m$. By comparison with equation (2.1), it will be seen that:

$$\begin{aligned} w_1^2 = w_1^2, \ \mu f_1(\cdots) &= w_1^2 F(y_1) + w_{12}^2(y_1 - y_2), \mu \Delta_1 = 0 \\ w_2^2 = w_1^2, \ \mu f_2(\cdots) &= w_{12}^2(y_2 - y_1), \mu \Delta_2 = P/m \end{aligned} \quad (3.3)$$

∴ from equation (2.8)

$$h_1(\cdots) = (w_1^2 - \Omega^2)\bar{A}_1 + \frac{1}{\pi}\int_0^{2\pi}\Big\{ w_1^2 F(A_1 \cos x_1) \cos x_1 + w_{12}^2(A_1 \cos x_1 - A_2 \cos x_2)\Big\} \cos x_1 \, dx$$

* The reason for choosing an odd function of y_1 is simply that jump phenomena and instabilities are first order effects, with an even function such phenomena are second order effects. See p.66-70, N.W. McLachlan, Ordinary Non-Linear Differential Equations.

FIGURE 1- SIMPLE TWO DEGREE OF FREEDOM NON-LINEAR SYSTEM

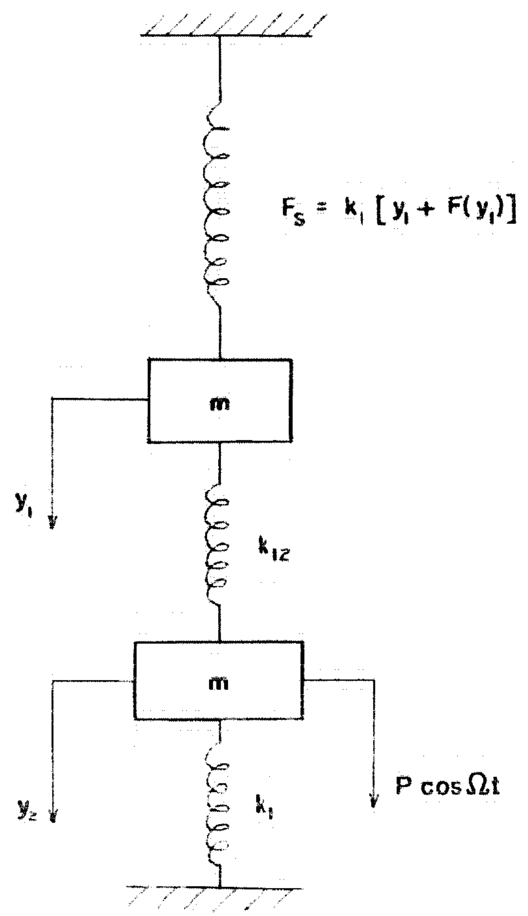
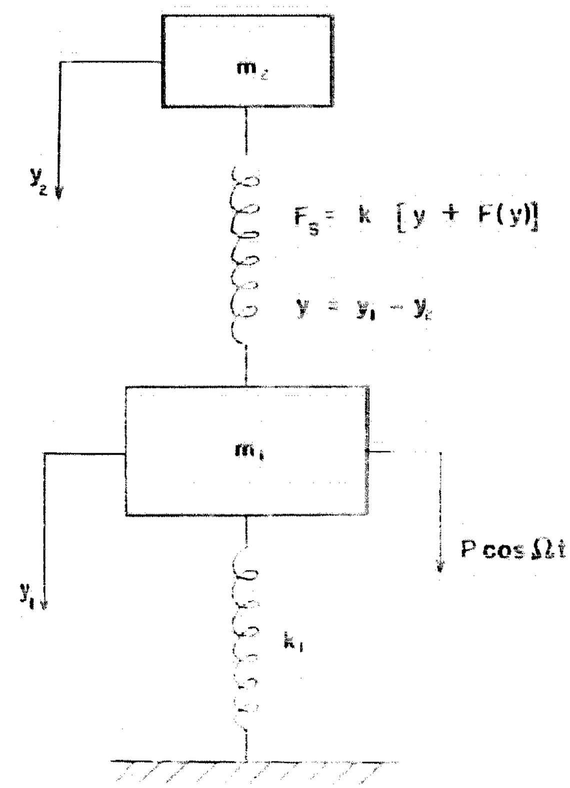


FIGURE 2- SIMPLE NON-LINEAR VIBRATION ABSORBER



$$\therefore \quad h_1 = (w_1^2 + w_{12}^2 - \Omega^2)\,\bar{A}_1 + w_1^2\, s(\bar{A}_1) - w_{12}^2 \bar{A}_2 \cos(\bar{\phi}_2 - \bar{\phi}_1) \qquad (3.4)$$

where

$$s(\bar{A}_1) = \frac{1}{\pi}\int_0^{2\pi} F(\bar{A}_1 \cos x_1) \cos x_1 \, dx_1 \qquad (3.5)$$

from equation (2.12)

$$g_1(\cdots) = \frac{1}{\pi}\int_0^{2\pi} \Big\{ w_1^2\, F(A_1 \cos x_1) \sin x_1 + w_{12}^2 (A_1 \cos x_1 - A_2 \cos x_2) \sin x_1 \Big\} dx_1$$

$$\therefore \qquad g_1 = w_{12}^2\, \bar{A}_2 \sin(\bar{\phi}_2 - \bar{\phi}_1) \qquad (3.6)$$

In the same way

$$h_2 = (w_1^2 - \Omega^2)\,\bar{A}_2 + \frac{1}{\pi}\int_0^{2\pi} w_{12}^2 (A_2 \cos x_2 - A_1 \cos x_1) \cos x_2 \, dx_2 - \frac{P}{m} \cos \bar{\phi}_2 .$$

$$\therefore \qquad h_2 = (w_1^2 + w_{12}^2 - \Omega^2)\,\bar{A}_2 - w_{12}^2\, \bar{A}_1 \cos(\bar{\phi}_2 - \bar{\phi}_1) - \frac{P}{m} \cos \bar{\phi}_n \qquad (3.7)$$

$$g_2 = \frac{1}{\pi}\int_0^{2\pi} w_{12}^2 (A_2 \cos x_2 - A_1 \cos x_1) \sin x_2 \, dx_2 - \frac{P}{m} \sin \bar{\phi}_2$$

$$\therefore \quad g_2 = w_{12}^2\, \bar{A}_1 \sin(\bar{\phi}_1 - \bar{\phi}_2) - \frac{P}{m} \sin \bar{\phi}_2 \qquad (3.8)$$

Steady State

The steady state equations are obtained, as shown in equation (2.14), by setting g_1, g_2, h_1 and h_2 equal to zero. Thus

$$\left.\begin{aligned}
&w_{12}^2\,\bar{A}_2\,\sin(\bar{\phi}_2-\bar{\phi}_1)=0\\
&w_{12}^2\,\bar{A}_1\,\sin(\bar{\phi}_1-\bar{\phi}_2)=\frac{P}{m}\sin\bar{\phi}_2\\
&(w_1^2+w_{12}^2-\Omega^2)\bar{A}_1+w_1^2\,s(\bar{A}_1)-w_{12}^2\,\bar{A}_2\cos(\bar{\phi}_2-\bar{\phi}_1)=0\\
&(w_1^2+w_{12}^2-\Omega^2)\bar{A}_2-w_{12}^2\,\bar{A}_1\cos(\bar{\phi}_2-\bar{\phi}_1)=\frac{P}{m}\cos\phi_2
\end{aligned}\right\}\qquad(3.9)$$

From the first two equations $\bar{\phi}_2 - \bar{\phi}_1 = 0, \pi$, $\phi_2 = 0, \pi$. If $\bar{A}_1, \bar{A}_2$ are allowed to take on either positive or negative values, then it is sufficient to consider the case where $\bar{\phi}_1 = \bar{\phi}_2 = 0$. Thus if $\bar{A}_2 \neq 0$

$$\left.\begin{aligned}
&(w_{12}^2+w_1^2-\Omega^2)\bar{A}_1+w_1^2\,s(\bar{A}_1)-w_{12}^2\,\bar{A}_2=0\\
&(w_{12}^2+w_1^2-\Omega^2)\bar{A}_2-w_{12}^2\,\bar{A}_1=\frac{P}{m}
\end{aligned}\right\}\qquad(3.10)$$

Or, eliminating $\bar{A}_2$ from equation (3.10)

$$[(w_{12}^2+w_1^2-\Omega^2)^2-w_{12}^4]\bar{A}_1+w_1^2(w_{12}^2+w_1^2-\Omega^2)\,s(\bar{A}_1)=w_{12}^2\,\frac{P}{m}\qquad(3.11)$$

<u>Loci of Vertical Tangency</u>.

If $\bar{A}_2$ is eliminated from equation (3.9), the amplitude equation occurs in exactly the same form as (3.11), but both sides of the equation occur squared. Using this form of the amplitude equation the loci of vertical tangency are obtained by differentiation with respect to $\bar{A}_1$, setting $\partial\Omega/\partial\bar{A}_1 = 0$.

If $\bar{A}_1 \neq 0$, the loci of vertical tangency are

$$\left\{(w_{12}^2 + w_1^2 - \Omega^2)^2 - w_{12}^4 + w_1^2(w_{12}^2 + w_1^2 - \Omega^2)\,\frac{s(\bar{A}_1)}{\bar{A}_1}\right\} \times$$

$$\left\{(w_{12}^2 + w_1^2 - \Omega^2)^2 - w_{12}^4 + w_1^2(w_{12}^2 + w_1^2 - \Omega^2)\,s'(\bar{A}_1)\right\} = 0 \qquad (3.12)$$

or

$$(w_{12}^2 + w_1^2 - \Omega^2)^2 - w_{12}^4 + w_1^2(w_{12}^2 + w_1^2 - \Omega^2)\,\frac{s(\bar{A}_1)}{\bar{A}_1} = 0$$

$$(w_{12}^2 + w_1^2 - \Omega^2)^2 - w_{12}^4 + w_1^2(w_{12}^2 + w_1^2 - \Omega^2)\,s'(\bar{A}_1) = 0$$

<u>Stability of Steady State Solutions.</u>

The nature of the stability of the steady state solutions will be determined by the roots of equation (2.17), i.e.,

$$\begin{vmatrix} -2\Omega\lambda\,, & w_{12}^2 \sin(\bar{\phi}_2 - \bar{\phi}_1), & -w_{12}^2\bar{A}_2 \cos(\bar{\phi}_2 - \bar{\phi}_1), & +\,w_{12}^2\bar{A}_2 \cos(\bar{\phi}_2 - \bar{\phi}_1) \\ w_{12}^2 \sin(\bar{\phi}_1 - \bar{\phi}_2), & -2\Omega\lambda\,, & w_{12}^2\bar{A}_1 \cos(\bar{\phi}_1 - \bar{\phi}_2), & -\,w_{12}^2\bar{A}_1 \cos(\bar{\phi}_1 - \bar{\phi}_2) - \frac{P}{m}\cos\bar{\phi}_2 \\ (w_{12}^2 + w_1^2 - \Omega^2) + w_1^2\,s'(\bar{A}_1), & -w_{12}^2 \cos(\bar{\phi}_2 - \bar{\phi}_1), & -w_{12}^2\bar{A}_2\sin(\bar{\phi}_2 - \bar{\phi}_1) - 2\Omega\bar{A}_1\lambda\,, & w_{12}^2\bar{A}_2 \sin(\bar{\phi}_2 - \bar{\phi}_1) \\ -w_{12}^2 \cos(\bar{\phi}_2 - \bar{\phi}_1), & (w_1^2 + w_{12}^2 - \Omega^2), & w_{12}^2\bar{A}_1 \sin(\bar{\phi}_1 - \bar{\phi}_2), & -\,w_{12}^2\bar{A}_1 \sin(\bar{\phi}_1 - \bar{\phi}_2) - \frac{P}{m}\sin\bar{\phi}_2 - 2\Omega\lambda\bar{A}_2 \end{vmatrix} = 0$$

(3.14)

making use of the steady state equations:

$$0 = \begin{vmatrix} -2\Omega\lambda, & 0, & -w_{12}^2\, {}_o\bar{A}_2, & w_{12}^2\, {}_o\bar{A}_2 \\ 0, & -2\Omega\lambda, & w_{12}^2\, {}_o\bar{A}_1, & -(w_1^2 + w_{12}^2 - \Omega^2)\, {}_o\bar{A}_2 \\ (w_{12}^2 + w_1^2 - \Omega^2) + w_1^2\, s'({}_o\bar{A}_1), & -w_{12}^2, & -2\Omega\, {}_o\bar{A}_1\lambda, & 0 \\ -w_{12}^2, & (w_1^2 + w_{12}^2 - \Omega^2), & 0, & -2\Omega\lambda\, {}_o\bar{A}_2 \end{vmatrix}$$

(3.15)

Expanding out and grouping terms

$$(2\Omega\lambda)^4\, {}_o\bar{A}_1\, {}_o\bar{A}_2 + (2\Omega\lambda)^2 \, [2w_{12}^4\, {}_o\bar{A}_1\, {}_o\bar{A}_2 + (w_1^2 + w_{12}^2 - \Omega^2)^2\, {}_o\bar{A}_1\, {}_o\bar{A}_2$$
$$+ w_{12}^2\, {}_o\bar{A}_2 \left\{ {}_o\bar{A}_2 (w_1^2 + w_{12}^2 - \Omega^2 + w_1^2\, s'({}_o\bar{A}_1)) \right\}]$$
$$+ w_{12}^2 \{ w_{12}^2\, \bar{A}_1 - (w_{12}^2 + w_1^2 - \Omega^2)\, {}_o\bar{A}_2 \}\, {}_o\bar{A}_2 [(w_{12}^2 + w_1^2 - \Omega^2)$$
$$\times \; (w_{12}^2 + w_1^2 - \Omega^2 + w_1^2\, s'({}_o\bar{A}_1)) - w_{12}^4] = 0$$

(3.16)

making use of equations (3.10) and (3.11), equation (3.16) can be written

$${}_o\bar{A}_1\, {}_o\bar{A}_2 \Big\{ (2\Omega\lambda)^4 + (2\Omega\lambda)^2 \Big[2w_{12}^4 + (w_1^2 + w_{12}^2 - \Omega^2)^2$$
$$+ (w_1^2 + w_{12}^2 - \Omega^2 + w_1^2 \; s'({}_o\bar{A}_1))(w_1^2 + w_{12}^2 - \Omega^2 - w_1^2 \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}) \Big]$$
$$+ \Big[(w_{12}^2 + w_1^2 - \Omega^2)(w_{12}^2 + w_1^2 - \Omega^2 + w_1^2\, s'({}_o\bar{A}_1)) - w_{12}^4 \Big]$$
$$\times \Big[(w_{12}^2 + w_1^2 - \Omega^2)(w_{12}^2 + w_1^2 - \Omega^2 + w_1^2 \frac{s_o(\bar{A}_1)}{{}_o\bar{A}_1}) - w_{12}^4 \Big] \Big\} = 0$$

(3.17)

If ${}_{o}\bar{A}_1 \, {}_{o}\bar{A}_2 \neq 0$, then

$$(2\Omega\lambda)^4 + b(2\Omega\lambda)^2 + c = 0 \qquad (3.18)$$

where

$$b = 2w_{12}^4 + (w_1^2 + w_{12}^2 - \Omega^2)^2 + (w_1^2 + w_{12}^2 - \Omega^2 + w_1^2 \, s'({}_{o}\bar{A}_1)) \times (w_1^2 + w_{12}^2 - \Omega^2 + w_1^2 \, \frac{s({}_{o}\bar{A}_1)}{{}_{o}\bar{A}_1})$$

$$c = \left[(w_{12}^2 + w_1^2 - \Omega^2)(w_{12}^2 + w_1^2 - \Omega^2 + w_1^2 \, s'({}_{o}\bar{A}_1)) - w_{12}^4\right] \times \left[(w_{12}^2 + w_1^2 - \Omega^2)(w_{12}^2 + w_1^2 - \Omega^2 + w_1^2 \frac{s({}_{o}\bar{A}_1)}{{}_{o}\bar{A}_1}) - w_{12}^4\right] \qquad (3.19)$$

The conditions for stability are

$$c \geq 0, \quad b \geq 0, \quad b^2 \geq 4c. \qquad (3.20)$$

Consider the first condition, $c \geq 0$; this implies that

$$\left[(w_{12}^2 + w_1^2 - \Omega^2)(w_{12}^2 + w_1^2 - \Omega^2 + w_1^2 \, s'(\bar{A}_1)) - w_{12}^4\right] \times \left[(w_{12}^2 + w_1^2 - \Omega^2)(w_{12}^2 + w_1^2 - \Omega^2 + w_1^2 \, \frac{s(\bar{A}_1)}{\bar{A}_1} - w_{12}^4\right] \geq 0$$

Comparison with equations (3.13) shows that this condition is simply a statement of the fact that $\bar{A}_1$ must lie outside the region enclosed by the loci of vertical tangency.

Second condition, $b \geq 0$; regarding $\underline{b}$ as a quadratic function in Ω^2, the condition that $\underline{b}$ shall always be positive, is that the discriminant be negative.

$$b = 2\Omega^4 - 2\Omega^2\left(2w_1^2 + 2w_{12}^2 + \frac{w_1^2}{2}\left(s'({}_o\bar{A}_1) + \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\right)\right)$$

$$+ (w_1^2 + w_{12}^2)^2 + (w_{12}^2 + w_1^2 + w_1^2\, s'({}_o\bar{A}_1))(w_{12}^2 + w_1^2 + w_1^2\,\frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}) + 2w_{12}^4$$

(3.21)

whose discriminant can be written as

$$\Delta = 2w_1^4\left(s'({}_o\bar{A}_1) - \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\right)^2 - \left\{16w_{12}^4 + w_1^4\left(s'({}_o\bar{A}_1) + \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\right)^2\right\}$$

(3.22)

Thus b will be positive if

$$\frac{16w_{12}^4}{w_1^4} + \left(s'({}_o\bar{A}_1) + \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\right)^2 \geq 2\left(s'({}_o\bar{A}_1) - \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\right)^2$$

(3.23)

This condition is automatically satisfied for a cubic or a fifth power non-linearity for any ${}_o\bar{A}_1$. For any monotonic non-linearity the condition will also be satisfied if such non-linearities are small.

Third requirement:

$$b^2 - 4c \geq 0$$

For convenience c may be written in the form

$$c = \left[(w_{12}^2 + w_1^2 - \Omega^2)^2 - w_{12}^4 + \frac{w_1^2}{2}(w_{12}^2 + w_1^2 - \Omega^2)\left(s'({}_o\bar{A}_1) + \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\right)\right]^2$$

$$- \frac{w_1^4}{4}(w_{12}^2 + w_1^2 - \Omega^2)^2\left(s'({}_o\bar{A}_1) - \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\right)^2 \qquad (3.24)$$

Thus the third requirement becomes

$$\Big[4(w_1^2 + w_{12}^2 - \Omega^2)^2 + w_1^4 \, s'({}_o\bar{A}_1) \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1} + 2w_1^2(w_1^2 + w_{12}^2 - \Omega^2) \times \Big(s'({}_o\bar{A}_1) + \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\Big) \Big] \times$$

$$\Big[4w_{12}^4 + w_1^4 \, s'({}_o\bar{A}_1) \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1} \Big] + w_1^4(w_{12}^2 + w_1^2 - \Omega^2)^2 \times \Big(s'({}_o\bar{A}_1) - \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\Big)^2 \geq 0 \qquad (3.25)$$

The first term is positive except in a small region

$$(w_1^2 + w_{12}^2) + \frac{w_1^2}{2} \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1} < \Omega^2 < (w_1^2 + w_{12}^2) + \frac{w_1^2}{2} s'({}_o\bar{A}_1) \qquad (3.26)$$

The first term has its minimum value at

$$\Omega^2 = w_1^2 + w_{12}^2 + \frac{w_1^2}{4} \Big(s'({}_o\bar{A}_1) + \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\Big) \qquad (3.27)$$

and is equal to

$$- \frac{w_1^4}{4} \Big(s'({}_o\bar{A}_1) - \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\Big)^2 \qquad (3.28)$$

$b^2 - 4c$ therefore has the value

$$\frac{w_1^4}{4}\Big(s'({}_o\bar{A}_1) - \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\Big)^2 \Big[- 4w_{12}^4 - w_1^4 \, s'({}_o\bar{A}_1) \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1} + \frac{4}{16} w_1^4 \Big(s'({}_o\bar{A}_1) + \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1}\Big)^2 \Big]$$

$$(b^2 - 4c)_{min} = \frac{1}{16} w_1^4 (s'({}_o\bar{A}_1) - \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1})^2$$

$$\times \left[-16 w_{12}^4 + w_1^4 (s'({}_o\bar{A}_1) - \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1})^2 \right] \qquad (3.29)$$

To insure that b is positive

$$16 w_{12}^4 \geq w_1^4 (s'({}_o\bar{A}_1) - \frac{s({}_o\bar{A}_1)}{{}_o\bar{A}_1})^2$$

$\therefore$ $(b^2 - 4c)_{min}$ is negative, at this point.

However, substitution into the steady state amplitude equations shows that the amplitude is single valued and of the same order of magnitude as the driving force, which is assumed to be small, of order μ. Therefore, $[(b^2 - 4c)_{min}]^{1/2}$ is very small, of order μ^2 and can be neglected in a first order theory.

Thus the condition for first order stability is that $c \geq 0$

(3.30)

This means that those solutions are stable which lie outside the region enclosed by the loci of vertical tangency.

Topological Discussion of Results.

The results of the above analysis may be summarized as follows: For the non-linear system described by equations (3.1), the amplitude is governed by equations (3.10), and (3.11), those solutions will be stable which lie outside the region enclosed by the loci of vertical tangency as defined by equations (3.13).

For the purposes of this discussion the amplitude equation (3.11) may be written

$$s({}_o\bar{A}_1) = \frac{w_{12}^2 \, \frac{P}{m}}{w_1^2(w_{12}^2 + w_1^2 - \Omega^2)} - \frac{(w_{12}^2 + w_1^2 - \Omega^2)^2 - w_{12}^4}{w_1^2(w_{12}^2 + w_1^2 - \Omega^2)} \, {}_o\bar{A}_1 \tag{3.31}$$

if $F(y_1)$ is a monotonically increasing odd function of y_1, then $s({}_o\bar{A}_1)$ is also a monotonically increasing function of ${}_o\bar{A}_1$. Four distinct regions may be recognized in equation (3.31).

<u>Region 1.</u>

$$w_{12}^2 + w_1^2 - \Omega^2 > w_{12}^2 \tag{3.32}$$

In this region no vertical tangents can occur, as is seen from equations (3.13). The single root of equation (3.31) is readily seen to be positive; see Fig. 3.

$$od = \frac{w_{12}^2 \, \frac{P}{m}}{w_1^2(w_{12}^2 + w_1^2 - \Omega^2)}$$

$$\text{Slope: } bdc = \text{arc tan } \frac{w_{12}^4 - (w_{12}^2 + w_1^2 - \Omega^2)^2}{(w_{12}^2 + w_1^2 - \Omega^2)^2} \tag{3.33}$$

<u>Region 2.</u>

$$0 < w_{12}^2 + w_1^2 - \Omega^2 < w_{12}^2 \tag{3.34}$$

There are three solutions, a, b, c, to the amplitude equation (3.31). Solution <u>a</u> lies inside the region enclosed by the loci of vertical

FIGURE 3— REGION I

a - STABLE

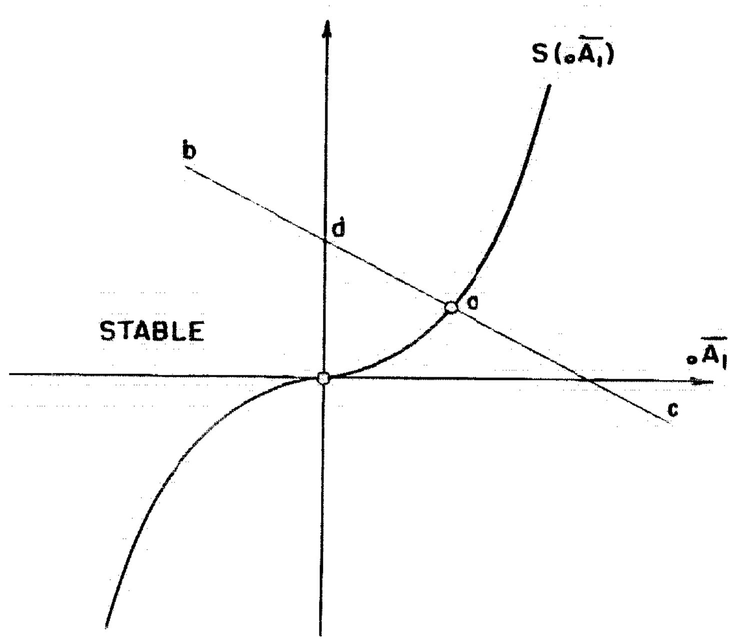
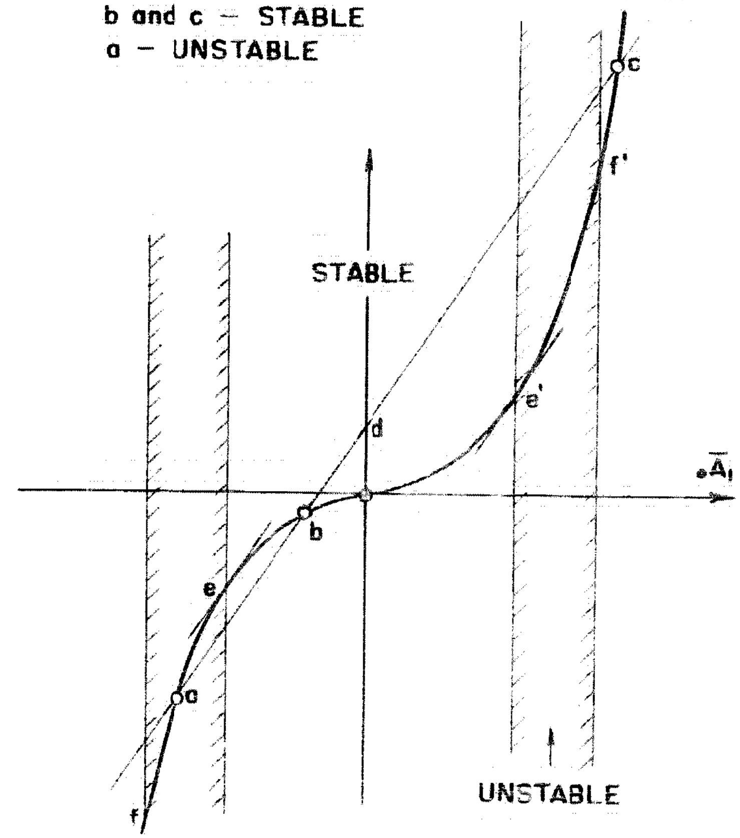


FIGURE 4— REGION 2

b and c — STABLE
a — UNSTABLE



tangency and is therefore unstable. Solutions b and c lie outside the region of vertical tangency and are therefore stable; see Fig. 4.

Region 3.

$$w_{12}^2 + w_1^2 - \Omega^2 < 0$$

$$(w_{12}^2 + w_1^2 - \Omega^2)^2 < w_{12}^4 \qquad (3.35)$$

There are no vertical tangents in this region. The single root of equation (3.31) is readily seen to be negative, and stable; see Fig 5.

Region 4.

$$w_{12}^2 + w_1^2 - \Omega^2 < 0$$

$$(w_{12}^2 + w_1^2 - \Omega^2)^2 > w_{12}^4 \qquad (3.36)$$

There are three solutions a, b, c, to the amplitude equation, two are positive, and one negative; see Fig. 6. Solutions a and b are stable.

Fig. 7 shows a typical response curve for the system of Fig. 1. There are two regions of instability, inside which vertical jumps will take place. Region 2 starts at the lower of the two linear natural frequencies and region 4 starts at the upper frequency. It will be observed that despite the presence of the non-linear spring, large amplitudes will result close to the lower natural frequency unless damping is present, this is quite different from what happens in one degree of freedom. The response of only one mass has been plotted, the response of the second mass is very similar in nature to that of the first mass.

FIGURE 5- REGION 3

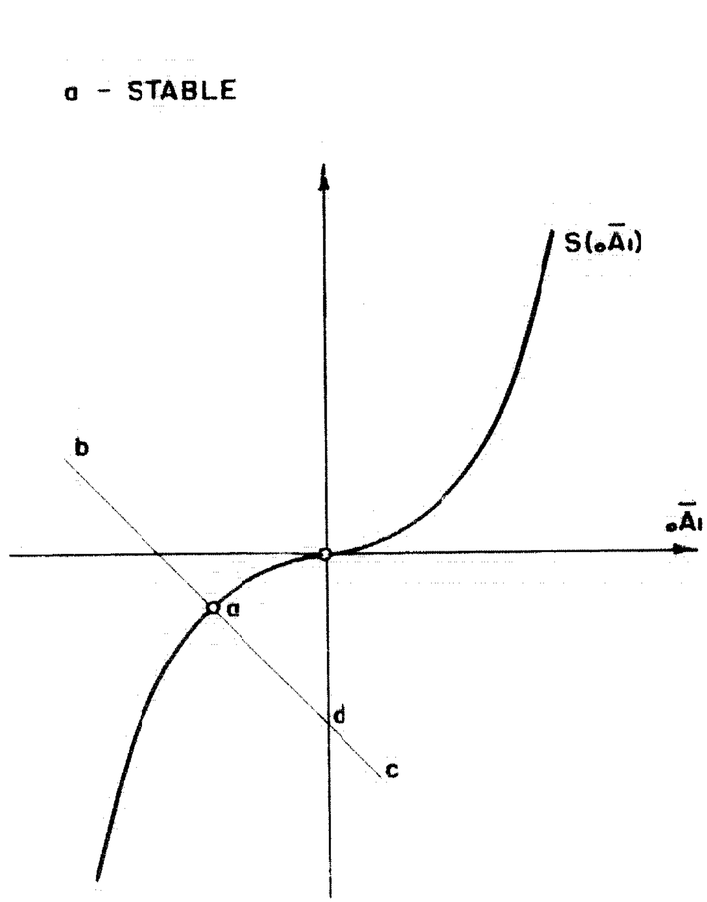


FIGURE 6- REGION 4

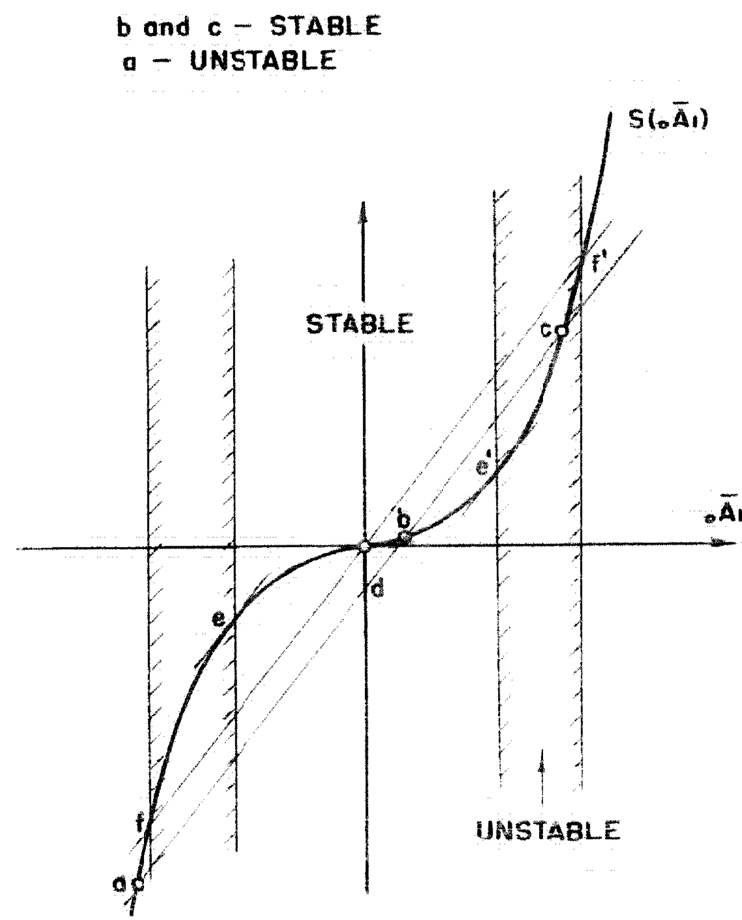
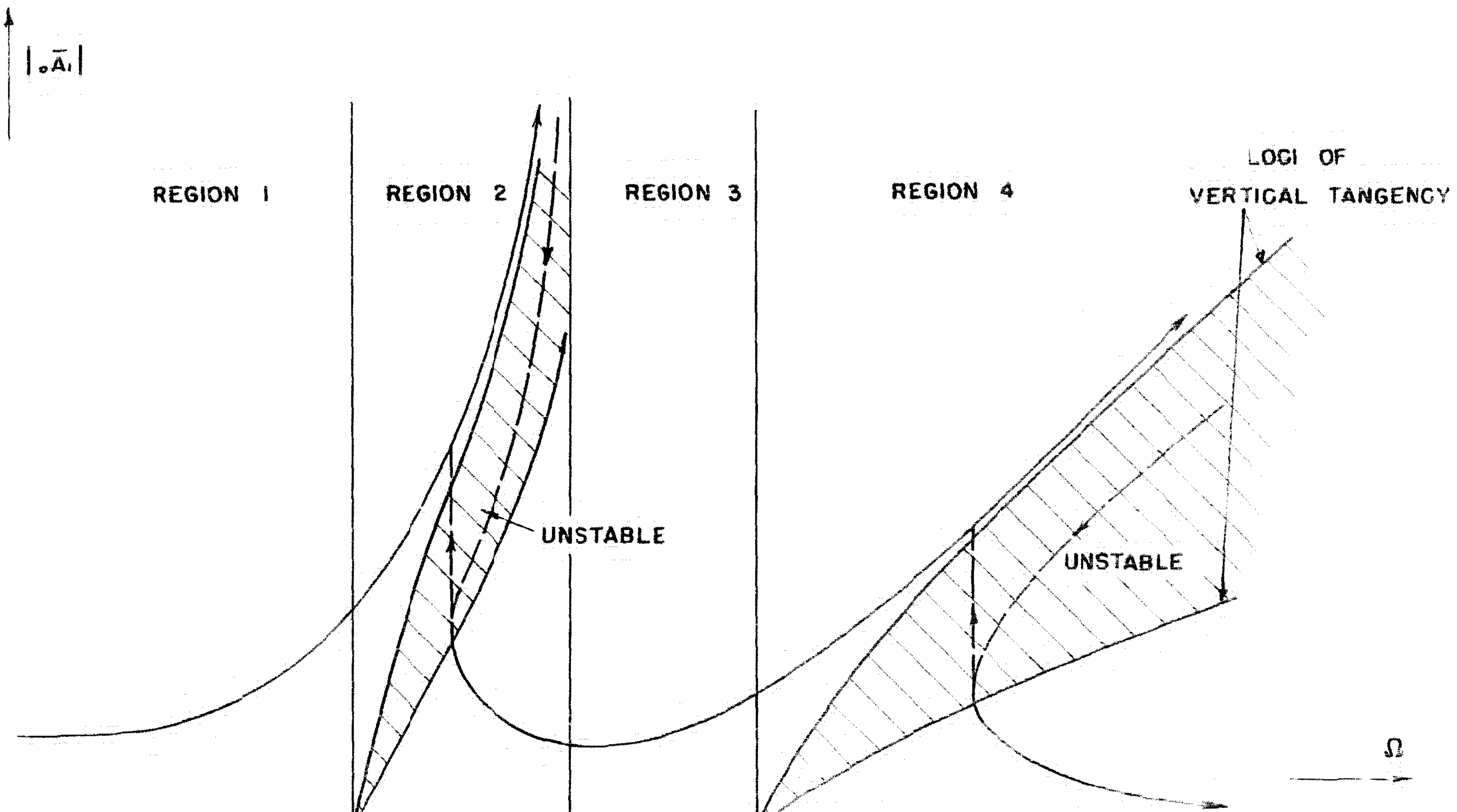


FIGURE 7— TYPICAL AMPLITUDE / FREQUENCY CHARACTERISTICS FOR SYSTEM IN FIGURE 1



Case 2.

Non-Linear Coupling Spring.

In recent papers L.A. Pipes[1] and R.E. Roberson[2] analyzed a simple non-linear vibration absorber. Such a system is an example of the use of a non-linear spring to couple two systems, and a detailed analysis is given below of a simple system, such as Pipes used, with particular emphasis on the stability of the steady state solutions, an aspect of the problem not previously considered.

Consider the simple system shown in Fig. 2, p. 20.

Equations of Motion.

$$\left.\begin{aligned} m_1\ddot{y}_1 + k_1y_1 + k(y + F(y)) &= P\cos\Omega t \\ m_2\ddot{y}_2 - k(y + F(y)) &= 0 \end{aligned}\right\} \quad (3.37)$$

where $F(y)$ is an odd function of y. Rearranging

$$\left.\begin{aligned} \ddot{y}_1 + w_1^2y_1 + w_{12}^2(y + F(y)) &= \frac{P}{m_1}\cos\Omega t \\ \ddot{y}_2 - w_2^2(y + F(y)) &= 0 \end{aligned}\right\} \quad (3.38)$$

where

$$w_1^2 = \frac{k_1}{m_1}, \quad w_{12}^2 = \frac{k}{m_1}, \quad w_2^2 = \frac{k}{m_2}.$$

Subtracting equations (3.38) and substituting $y = y_1 - y_2$

$$\left.\begin{aligned} \ddot{y}_1 + w_1^2y_1 + w_{12}^2(y + F(y)) &= \frac{P}{m_1}\cos\Omega t \\ \ddot{y} + (w_{12}^2 + w_2^2)\,y + (w_{12}^2 + w_2^2)F(y) + w_1^2y_1 &= \frac{P}{m_1}\cos\Omega t \end{aligned}\right\} \quad (3.39)$$

1. Analysis of a Non-linear Dynamic Vibration Absorber, L.A.Pipes, p. 515, J.A.M. Dec. 1953.

2. Synthesis of a Non-linear Vibration Absorber, R.E. Roberson, J.F.I. vol. 254, Sept. 1952.

Comparison with equation (2.1) shows that

$$w_1^2 = w_1^2, \ \mu f_1 = w_{12}^2(y + F(y)), \ \mu\Delta_1 = \frac{P}{m_1}$$

$$w_2^2 = w_2^2 + w_{12}^2, \ \mu f_2 = (w_{12}^2 + w_2^2)\ F(y) + w_1^2 y_1, \tag{3.40}$$

$$\mu\Delta_2 = \frac{P}{m_1}$$

Let $y = A \cos x$, $y_1 = A_1 \cos x_1$. Thus

$$h_1 = -(\Omega^2 - w_1^2)\bar{A}_1 + \frac{1}{\pi}\int_0^{2\pi} w_{12}^2(A \cos x + F(A \cos x)) \cos x_1 \, dx$$
$$- \frac{P}{m_1} \cos \phi_1$$

$$h_1 = -(\Omega^2 - w_1^2)\bar{A}_1 + w_{12}^2 \bar{A} \cos(\bar{\phi} - \bar{\phi}_1)$$
$$+ w_{12}^2 \, s(\bar{A}) \cos(\bar{\phi}_1 - \bar{\phi}) - \frac{P}{m_1} \cos \phi_1 \tag{3.41}$$

where $$s(\bar{A}) = \frac{1}{\pi}\int_0^{2\pi} F(A \cos x) \cos x \, dx \tag{3.42}$$

$$g_1 = \frac{1}{\pi}\int_0^{2\pi} w_{12}^2(A \cos x + F(A \cos x)) \sin x_1 \, dx - \frac{P}{m_1} \sin \phi_1$$

$$g_1 = w_{12}^2 \bar{A} \sin(\bar{\phi}_1 - \bar{\phi}) + s(\bar{A}) \sin(\bar{\phi}_1 - \bar{\phi}) - \frac{P}{m_1} \sin \phi_1$$

(3.43)

$$h_2 = -(\Omega^2 - w_2^2 - w_{12}^2)\bar{A} + \frac{1}{\pi}\int_0^{2\pi} \{w_1^2 A_1 \cos x_1 + (w_2^2 + w_{12}^2)\ F(A \cos x)\}$$
$$x \cos x \, dx - \frac{P}{m} \cos \phi$$

$$h_2 = -(\Omega^2 - w_2^2 - w_{12}^2)\bar{A} + w_1^2\,\bar{A}_1 \cos(\bar{\phi} - \bar{\phi}_1)$$

$$+ (w_2^2 + w_{12}^2)\; s(\bar{A}) - \frac{P}{m} \cos \bar{\phi} \qquad (3.44)$$

$$g_2 = \frac{1}{\pi} \int_0^{2\pi} [(w_2^2 + w_{12}^2)\; F(A \cos x) + w_1^2 A_1 \cos x_1] \sin x \, dx$$

$$- \frac{P}{m} \sin \bar{\phi}$$

$$g_2 = w_1^2\bar{A}_1 \sin(\bar{\phi} - \bar{\phi}_1) - \frac{P}{m} \sin \bar{\phi} \qquad (3.45)$$

Steady State Equations.

The steady state equations are obtained, as was shown in equation (2.14), by setting g_1, g_2, h_1, h_2 equal to zero. Thus

$$\left.\begin{aligned}
&-(\Omega^2 - w_1^2)\,{}_o\bar{A}_1 + w_{12}^2\; s({}_o\bar{A}_1) \cos({}_o\bar{\phi}_1 - {}_o\bar{\phi}) + w_{12}^2\;{}_o\bar{A} \cos({}_o\bar{\phi}_1 - {}_o\bar{\phi}) \\
&\qquad - \frac{P}{m_1} \cos {}_o\bar{\phi}_1 = 0 \\
&w_{12}^2\;{}_o\bar{A} \sin({}_o\bar{\phi}_1 - {}_o\bar{\phi}) + s({}_o\bar{A}) \sin({}_o\bar{\phi}_1 - {}_o\bar{\phi}) - \frac{P}{m_1} \sin {}_o\phi_1 = 0 \\
&-(\Omega^2 - w_2^2 - w_{12}^2)\,{}_o\bar{A} + w_1^2\;{}_o\bar{A}_1 \cos({}_o\bar{\phi} - {}_o\bar{\phi}_1) + (w_2^2 + w_{12}^2)\; s({}_o\bar{A}) \\
&\qquad - \frac{P}{m_1} \cos \phi = 0
\end{aligned}\right\} \qquad (3.46)$$

$$w_1^2\;\bar{A}_1 \sin(\bar{\phi} - \bar{\phi}_1) - \frac{P}{m} \sin \phi = 0$$

From the second and fourth equations ${}_o\bar{\phi}_1 = {}_o\bar{\phi} = 0$ or π, ${}_o\bar{A}_1 \neq 0$. If $\bar{A}$ and $\bar{A}_1$ are allowed to take on positive or negative values, it is sufficient to consider $_o\bar{\phi}_1 = {}_o\bar{\phi} = 0$.

$$w_1^2 \, {}_o\bar{A}_1 + (w_{12}^2 + w_2^2) \, s({}_o\bar{A}) + (w_{12}^2 + w_2^2 - \Omega^2)\bar{A} = \frac{P}{m_1} \tag{3.47}$$

$$\bar{A}_1(w_1^2 - \Omega^2) + w_{12}^2\bar{A} + w_{12}^2 \, s(\bar{A}) = \frac{P}{m_1} \tag{3.48}$$

Eliminating $\bar{A}_1$ between (3.47) and (3.48)

$$[s({}_o\bar{A}) + {}_o\bar{A}][\Omega^2(w_{12}^2 + w_2^2) - w_1^2 w_2^2] + {}_o\bar{A} \, \Omega^2(w_1^2 - \Omega^2) = \Omega^2 \frac{P}{m_1} \tag{3.49}$$

or eliminating straight from (3.46)

$$\left\{[s({}_o\bar{A}) + {}_o\bar{A}][\Omega^2(w_{12}^2 + w_2^2) - w_1^2 w_2^2] + {}_o\bar{A} \quad {}^2(w_1^2 - \Omega^2)\right\}^2 = \Omega^4 \left(\frac{P}{m_1}\right)^2 \tag{3.50}$$

Loci of Vertical Tangency.

Using equation (3.50), the loci of vertical tangency are readily obtained by differentiation with respect to $\bar{A}$, setting $\frac{\partial \Omega}{\partial \bar{A}} = 0$. Thus if $\bar{A} \neq 0$, the loci of vertical tangency are:

$$\left.\begin{aligned} &[s({}_o\bar{A})/{}_o\bar{A} + 1][\Omega^2(w_{12}^2 + w_2^2) - w_1^2 w_2^2] + \Omega^2(w_1^2 - \Omega^2) = 0 \\ &[s'({}_o\bar{A}) + 1][\Omega^2(w_{12}^2 + w_2^2) - w_1^2 w_2^2] + \Omega^2(w_1^2 - \Omega^2) = 0 \end{aligned}\right\} \tag{3.51}$$

Stability of Steady State Solutions.

The nature of the stability of the steady state solutions will be determined by the roots of equation (2.17), i.e.,

$$\begin{vmatrix} 0 - 2\Omega\lambda & w_{12}^2 \sin({}_o\bar{\phi}_1 - {}_o\bar{\phi}) + s'({}_o\bar{A}) \sin({}_o\bar{\phi}_1 - {}_o\bar{\phi}) & w_{12}^2\bar{A} \cos({}_o\bar{\phi}_1) - {}_o\bar{\phi}) + s(\bar{A}) \cos({}_o\bar{\phi}_1 - {}_o\bar{\phi}) - \frac{P}{m} \cos {}_o\bar{\phi}_1 & -w_{12}^2\bar{A} \cos({}_o\bar{\phi}_1 - {}_o\bar{\phi}) - S(\bar{A}) \cos({}_o\bar{\phi}_1 - {}_o\bar{\phi}) \\ w_1^2 \sin({}_o\bar{\phi} - {}_o\bar{\phi}_1) & 0 - 2\Omega\lambda & -w_1^2 \, {}_o\bar{A}_1 \cos({}_o\bar{\phi} - {}_o\bar{\phi}_1) & w_1^2 \, {}_o\bar{A}_1 \cos({}_o\bar{\phi} - {}_o\bar{A}_1) - \frac{P}{m} \cos {}_o\bar{\phi} \\ (w_1^2 - \Omega^2) & w_{12}^2 \cos({}_o\bar{\phi} - {}_o\bar{\phi}_1) + w_{12}^2 s'({}_o\bar{A}) \times \cos({}_o\bar{\phi}_1 - {}_o\bar{\phi}) & w_{12}^2\bar{A} \sin(\bar{\phi} - \bar{\phi}_1) - w_{12}^2 s({}_o\bar{A}) \sin(\bar{\phi}_1 - \bar{\phi}) - \frac{P}{m} \sin {}_o\bar{\phi}_1 - 2\Omega \, {}_o\bar{A}_1 \lambda & -w_{12}^2\bar{A} \sin({}_o\bar{\phi} - {}_o\bar{\phi}_1) + w_{12}^2 s({}_o\bar{A}) \sin({}_o\bar{\phi} - {}_o\bar{\phi}_1) \\ w_1^2 \cos(\bar{\phi} - \bar{\phi}_1) & (w_2^2 + w_{12}^2 - \Omega^2) + (w_2^2 + w_{12}^2) \times s'({}_o\bar{A}) & w_1^2 \, {}_o\bar{A}_1 \sin({}_o\bar{\phi} - {}_o\bar{\phi}_1) & -w_1^2 \bar{A}_1 \sin({}_o\bar{\phi} - {}_o\bar{\phi}_1) + \frac{P}{m} \sin {}_o\bar{\phi} - 2\Omega\lambda \, {}_o\bar{A} \end{vmatrix} = 0 \qquad (3.52)$$

Using the steady state conditions ${}_o\bar{\emptyset} = {}_o\bar{\emptyset}_1 = 0$, (3.52) becomes

$$\begin{vmatrix} -2\Omega\lambda, & 0, & w_{12}^2({}_o\bar{A} + s({}_o\bar{A})) - \frac{P}{m}, & -w_{12}^2({}_o\bar{A} + s({}_o\bar{A})) \\ 0, & -2\Omega\lambda, & -w_1^2\,{}_o\bar{A}, & w_1^2\,{}_o\bar{A}_1 - \frac{P}{m} \\ (w_1^2 - \Omega^2), & w_{12}^2(1 + s'({}_o\bar{A})), & -2\Omega\,{}_o\bar{A}_1\lambda, & 0 \\ w_1^2, & (w_2^2 + w_{12}^2)(s'({}_o\bar{A}) + 1) - \Omega^2, & 0, & -2\Omega\,{}_o\bar{A}\,\lambda \end{vmatrix} = 0 \tag{3.53}$$

Expanding the determinant and grouping terms

$$\begin{aligned}
&{}_o\bar{A}_1\,{}_o\bar{A}[2\Omega\lambda]^4 + [2\Omega\lambda]^2[\frac{P}{m_1}\,{}_o\bar{A}\,w_{12}^2(1 + s'({}_o\bar{A})) + \frac{P}{m_1}(w_1^2 - \Omega^2) \\
&\quad + (w_1^2\,{}_o\bar{A}_1 - \frac{P}{m_1})\{[\Omega^2 - w_2^2(1 + s'({}_o\bar{A}))]\ \bar{A}_1 \\
&\quad + w_{12}^2(s'({}_o\bar{A}) + 1)({}_o\bar{A} - {}_o\bar{A}_1)\} + w_{12}^2\ \ s({}_o\bar{A}) + {}_o\bar{A}\quad {}_o\bar{A}_1 w_1^2 - {}_o\bar{A}(w_1^2 - \Omega^2)\ ] \\
&\quad + \frac{P}{m_1}(w_1^2\,{}_o\bar{A}_1 - \frac{P}{m_1})\{\Omega^2\,w_{12}^2(1 + s'({}_o\bar{A})) - w_2^2(1 + s'({}_o\bar{A}))(w_1^2 - \Omega^2) \\
&\quad + \Omega^2(w_1^2 - \Omega^2)\} = 0
\end{aligned} \tag{3.54}$$

Using the steady state equations (3.46) and (3.47), equation (3.54) can be written in the form:

$$ {}_o\bar{A}_1\,{}_o\bar{A}\quad [2\Omega\lambda]^4 + b[2\Omega\lambda]^2 + c \quad = 0 \tag{3.55}$$

where

$$
\left.
\begin{aligned}
c &= [(s'({}_o\bar{A}) + 1)(\Omega^2(w_{12}^2 + w_2^2) - w_1^2 w_2^2) + \Omega^2(w_1^2 - \Omega^2)] \\
&\quad \times [(\frac{s({}_o\bar{A})}{{}_o\bar{A}} + 1)(\Omega^2(w_{12}^2 + w_2^2) - w_1^2 w_2^2) + \Omega^2(w_1^2 - \Omega^2)] \\
b &= (w_1^2 - \Omega^2)^2 + \Omega^4 - (w_{12}^2 + w_2^2)(2 + s'({}_o\bar{A}) + \frac{s({}_o\bar{A})}{{}_o\bar{A}})\Omega^2 \\
&\quad + (w_{12}^2 + w_2^2)^2 (s'({}_o\bar{A}) + 1)(\frac{s({}_o\bar{A})}{{}_o\bar{A}} + 1) + w_{12}^2 w_1^2 \{2 + s'({}_o\bar{A}) \\
&\qquad + \frac{s({}_o\bar{A})}{\bar{A}}\}
\end{aligned}
\right\} \tag{3.56}
$$

If $\bar{A}_1 \bar{A} \neq 0$, the conditions for stability are:

$$c \geq 0, \quad b \geq 0, \quad b^2 - 4c \geq 0.$$

<u>First Condition</u>. $c \geq 0$.

This condition, as will readily be seen by inspection of equation (3.51), is simply the requirement that the solution lie outside the region enclosed by the loci of vertical tangency.

<u>Second Condition</u>. $b \geq 0$.

Treating b as a quadratic function in Ω^2, the condition that $\underline{b}$ shall be positive is the requirement that the discriminant of the quadratic be negative. Now

$$
\begin{aligned}
\Delta = {} & (w_2^2 + w_{12}^2)^2 \, (s'({}_o\bar{A}) - \frac{s({}_o\bar{A})}{{}_o\bar{A}})^2 \\
& - 4 \left\{ w_1^4 + w_1^2(w_{12}^2 - w_2^2)(2 + s'({}_o\bar{A}) + \frac{s({}_o\bar{A})}{{}_o\bar{A}}) \right. \\
& \left. \qquad + (w_{12}^2 + w_2^2)^2 \, (1 + s'({}_o\bar{A}))(1 + \frac{s({}_o\bar{A})}{{}_o\bar{A}}) \right\} \\
\leq {} & 0
\end{aligned}
$$

If $w_1 = w_2$

$$\Delta = (w_2^2 + w_{12}^2)^2(s'({}_o\bar{A}) - \frac{s({}_o\bar{A})}{{}_o\bar{A}})^2 + 4 \Big\{ w_{12}^2(4w_1^2 + w_{12}^2)$$

$$+ w_{12}^2(3w_1^2 + w_{12}^2)(s'({}_o\bar{A}) + \frac{s({}_o\bar{A})}{{}_o\bar{A}}) + (w_{12}^2 + w_2^2)^2 \ s'({}_o\bar{A}) \ \frac{s({}_o\bar{A})}{{}_o\bar{A}} \Big\} \tag{3.57}$$

for a cubic or fifth power non-linearity the condition that $b > 0$ is automatically satisfied; for any other type of non-linearity the condition can be satisfied by restricting the non-linearity to small values.

<u>Third condition</u>. $b^2 - 4c \geq 0$.

$$(b^2 - 4c) = \Big\{ (w_1^2 - \Omega^2)^2 + \Omega^4 - (w_{12}^2 + w_2^2)(2 + s'({}_o\bar{A}) + \frac{s({}_o\bar{A})}{{}_o\bar{A}})\Omega^2$$

$$w_1^2 w_{12}^2(\frac{s({}_o\bar{A})}{{}_o\bar{A}} + s'({}_o\bar{A}) + 2) + (w_{12}^2 + w_2^2)^2(1 + s'({}_o\bar{A})(1 + \frac{s({}_o\bar{A})}{{}_o\bar{A}}) \Big\}^2$$

$$- 4 \Big\{ \tfrac{1}{2}(2 + s'({}_o\bar{A}) + \frac{s({}_o\bar{A})}{{}_o\bar{A}})(\Omega^2(w_{12}^2 + w_2^2) - w_1^2 w_2^2) + \Omega^2(w_1^2 - \ ^2) \Big\}^2$$

$$+ (s'({}_o\bar{A}) - \frac{s({}_o\bar{A})}{{}_o\bar{A}})^2 \Big\{ \Omega^2(w_{12}^2 + w_2^2) - w_1^2 \, w_2^2 \Big\}^2$$

$$(b^2 - 4c) = \Big\{ (w_1^2 - 2\Omega^2)^2 + (w_{12}^2 + w_2^2)^2 (1 + s'(_o\bar{A}) + \frac{s(_o\bar{A})}{\bar{A}})$$

$$+ (w_{12}^2 + w_2^2)(w_1^2 - 2\Omega^2)(2 + s'(_o\bar{A}) + \frac{s(_o\bar{A})}{_o\bar{A}}) \Big\}$$

$$\times \Big\{ w_1^4 + (w_{12}^2 + w_2^2)^2 (1 + S'(_o\bar{A}))(1 + \frac{s(_o\bar{A})}{_o\bar{A}})$$

$$+ w_1^2 (w_{12}^2 - w_2^2)(\frac{s(_o\bar{A})}{_o\bar{A}} + s'(_o\bar{A}))$$

$$+ (s' - \frac{s}{_o\bar{A}})^2 (\Omega^2 (w_{12}^2 + w_2^2) - w_1^2 w_2^2)^2 \Big\} \qquad (3.58)$$

This is positive except in a small neighborhood around

$$\Omega^2 = \frac{1}{2}(w_1^2 + w_{12}^2 + w_2^2) + \frac{1}{4}(w_{12}^2 + w_2^2)(s'(_o\bar{A}) + \frac{s(_o\bar{A})}{_o\bar{A}})$$

where it may become negative, the minimum value being

$$b^2 - 4c = \Big[(w_{12}^2 + w_2^2)^2 (s'(_o\bar{A}) - \frac{s(_o\bar{A})}{_o\bar{A}})^2$$

$$- 16 w_{12}^2 w_2^2 \Big] \frac{w_1^4}{16} (s'(_o\bar{A}) - \frac{s(_o\bar{A})}{_o\bar{A}})^2 \qquad (3.59)$$

which may be negative, however, in the neighborhood of this point, the amplitude $_o\bar{A}$, is single valued and of the same order of magnitude as the applied force, which is assumed to be small. Hence $(b^2 - 4c)^{1/2}$ is of the order of μ^2 at this point and can be neglected in a first order theory. Thus, on the assumption of small non-linearity and small driving force, the steady state solutions given by equation (3.50) will be stable provided such solutions lie outside the region enclosed by the loci of vertical tangency as defined by equations (3.50).

Discussion of Results:

As in case 1, the results of this analysis lend themselves to simple topological discussion. It can again be shown that there are four regions of interest

Region 1.

$$\left.\begin{aligned} &\Omega^2(w_1^2 + w_{12}^2 + w_2^2 - \Omega^2) - w_1^2\, w_2^2 < 0 \\ &\Omega^2(w_{12}^2 + w_2^2) - w_{12}^2\, w_2^2 < 0 \end{aligned}\right\} \qquad (3.60)$$

in this region $\bar{A}$ is single valued and is stable, hence $\bar{A}_1$ and $\bar{A}_2$ are single valued and stable.

Region 2.

$$\left.\begin{aligned} &\Omega^2(w_1^2 + w_{12}^2 + w_2^2 - \Omega^2) - w_{12}^2\, w_2^2 > 0 \\ &\Omega^2(w_{12}^2 + w_2^2) - w_{12}^2 < 0 \end{aligned}\right\} \qquad (3.61)$$

in this region $\bar{A}$ is triple valued, two values are stable and one is unstable. From this it follows that there are two stable amplitudes for $\bar{A}_1$ and $\bar{A}_2$.

Region 3.

$$\left.\begin{aligned} &\Omega^2(w_1^2 + w_{12}^2 - \Omega^2) - w_{12}^2\, w_2^2 > 0 \\ &\Omega^2(w_{12}^2 + w_2^2) - w_{12}^2\, w_2^2 > 0 \end{aligned}\right\} \qquad (3.62)$$

in this region $\bar{A}$ is again single valued and stable, as are $\bar{A}_1$ and $\bar{A}_2$.

Region 4.

$$\left.\begin{aligned} &\Omega^2(w_1^2 + w_{12}^2 - \Omega^2) - w_{12}^2 w_2^2 < 0 \\ &\Omega^2(w_{12}^2 + w_2^2) - w_{12}^2 w_2^2 > 0 \end{aligned}\right\} \qquad (3.63)$$

in this region $\bar{A}$ is again triple valued, as are $\bar{A}_1$ and $\bar{A}_2$; two values being stable and one unstable.

The points of greatest interest in cases 1 and 2 are

1) If there is no damping present; jump phenomena and instabilities can occur in suitable frequency ranges and for small monotonically increasing non-linearities these phenomena are independent of the exact nature of the non-linearities.

2) Despite the presence of the non-linear spring it is still possible to obtain very large amplitudes unless some damping is present.

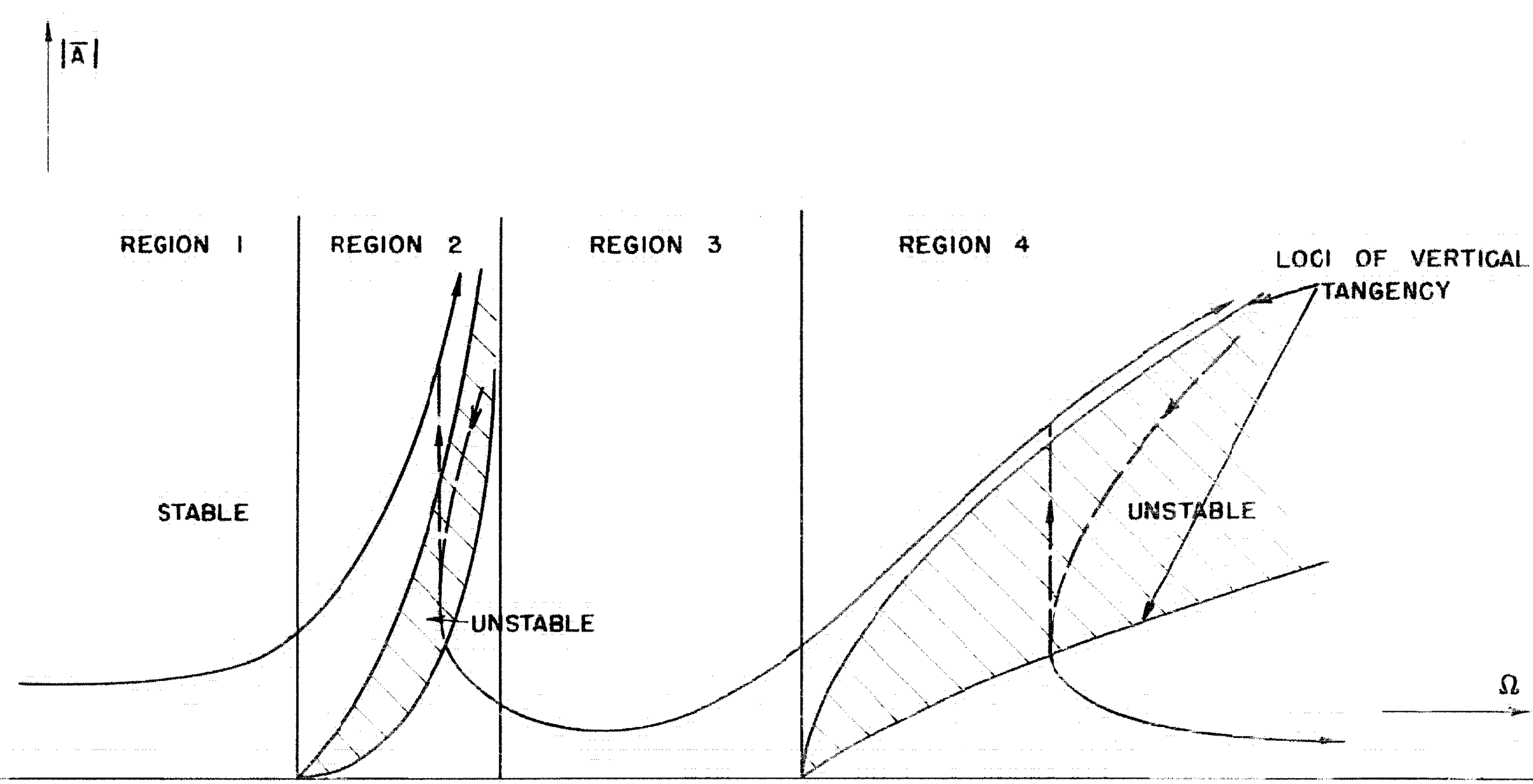
Effect of Damping on Stability.

In the foregoing analysis the effect of damping was neglected, for the present it will suffice to consider the effect of damping on the stability in one degree of freedom. Consider the simple non-linear system, with damping, described by the equation

$$\ddot{y} + w_1^2 y + \mu(2D\dot{y} + F(y)) = \mu\,\Delta\cos\Omega t \qquad (3.64)$$

Comparison with equation (2.1) shows that:

FIGURE 8 – TYPICAL AMPLITUDE / FREQUENCY RESPONSE OF NON-LINEAR VIBRATION ABSORBER



$w_1^2 = w_1^2, \qquad f(y) = \mu(2Dy + F(y)), \quad \text{etc.}$

$$\therefore \quad h_1(\cdots) = -(\Omega^2 - w_1^2)\bar{A} + \frac{\mu}{\pi}\int_0^{2\pi}\left\{-2D \quad A \sin x + F(A \cos x)\right\}$$

$$x \cos x \, dx - \mu \Delta \cos \bar{\phi}_1 \tag{3.65}$$

$$h_1(\cdots) = -(\Omega^2 - w_1^2)\bar{A} + \mu s(\bar{A}) - \mu\Delta \cos \bar{\phi}_1 \tag{3.66}$$

Similarly

$$g_1(\cdots) = -\mu 2D\Omega\bar{A} - \mu\Delta \sin \bar{\phi}_1 \tag{3.67}$$

Steady State Equations.

The steady state equations will be obtained by setting $g_1(\cdots)$ and $h_1(\cdots)$ equal to zero. Thus

$$\left.\begin{aligned} -\mu 2D\Omega_o\bar{A} &= \mu\Delta \sin {}_o\bar{\phi}_1 \\ (w^2 - \Omega^2)_o\bar{A} + \mu s({}_o\bar{A}) &= \mu\Delta \cos {}_o\bar{\phi}_1 \end{aligned}\right\} \tag{3.68}$$

Squaring and adding

$$[(w^2 - \Omega^2)\bar{A}_o + \mu s(\bar{A}_o)]^2 + 4\Omega^2 \bar{A}_o^2(\mu D)^2 = (\mu\Delta)^2 \tag{3.69a}$$

Loci of Vertical Tangency.

As in previous examples the loci of vertical tangency can be obtained by differentiation of equation (3.69) with respect to $\bar{A}$, setting $\frac{\partial\Omega}{\partial\bar{A}} = 0$.

Thus if $\bar{A} \neq 0$, the loci of vertical tangency are:

$$\left[(w^2 - \Omega^2) + \mu s'(\bar{A}_o)\right]\left[(w^2 - \Omega^2) + \mu \frac{s(\bar{A}_o)}{\bar{A}_o}\right] + 4\Omega^2(\mu D)^2 = 0 \tag{3.70}$$

It will be observed that if $D = 0$, equation (3.70) reduces to the same form as equations (3.13) and (3.51).

Stability of Steady State Solutions.

The nature of the stability of the steady state solutions will be decided by the roots of equation (2.17). For this system (2.17) becomes

$$\begin{vmatrix} (-\mu 2D\Omega - 2\Omega\lambda), & -\mu\Delta \cos {}_o\bar{\phi}_1 \\ -(\Omega^2 - w_1^2) + \mu s'(\bar{A}_o), & +\mu\Delta \sin {}_o\bar{\phi}_1 - 2\Omega\lambda\bar{A}_o \end{vmatrix} = 0 \tag{3.71}$$

now

$$\mu\,\Delta \cos {}_o\bar{\phi}_1 = \bar{A}_o(w^2 - \Omega^2)$$

$$-\mu\,\Delta \sin {}_o\bar{\phi}_1 = 2\mu\,\bar{A}_o\,D\,\Omega$$

$$\therefore \quad (2\mu\Omega D + 2\Omega\lambda)^2 + ((w^2 - \Omega^2) + \mu s'(\bar{A}_o))\left\{(w^2 - \Omega^2) + \mu\frac{s(\bar{A}_o)}{\bar{A}_o}\right\} = 0$$

i.e.,

$$2\Omega\lambda = -2\mu\Omega D \pm i\sqrt{\left\{\left[(w^2 - \Omega^2) + \mu s'(\bar{A}_o)\right]\left[(w^2 - \Omega^2) + \mu\frac{s(\bar{A}_o)}{\bar{A}_o}\right]\right\}} \tag{3.72}$$

The system will certainly be stable if

$$[(w^2 - \Omega^2) + \mu s'(\bar{A}_o)][(w^2 - \Omega^2) + \mu \frac{s(\bar{A}_o)}{\bar{A}_o}] \geq 0$$

If

$$[(w^2 - \Omega^2) + \mu s'(\bar{A}_o)][(w^2 - \Omega^2) + \mu \frac{s(\bar{A}_o)}{\bar{A}_o}] < 0$$

the system will be unstable, if

$$4\Omega^2(\mu D)^2 + [(w^2 - \Omega^2) + \mu s'(\bar{A}_o)][(w^2 - \Omega^2) + \mu \frac{S(\bar{A}_o)}{\bar{A}_o}] < 0 \tag{3.73}$$

Comparison with (3.70) shows that this condition is simply the requirement that the solution of (3.69) lie inside the region enclosed by the loci of vertical tangency. Conversely, for stability, solutions should lie outside the region.

A simple topological discussion, similar to that used in case 1, shows that for a hard spring, independently of the exact form of $F(y)$, triple solutions are to be expected if $\Omega > w_1$, two of these solutions will be stable and one will be unstable.

The sketch in Fig. 9 shows the general nature of the amplitude-frequency characteristics for the system. For small values of damping D and relatively large values of Δ, jump phenomena and instability can still occur, but if the damping is large, or the driving force is small, no such phenomena will occur.

The general nature of the effects of damping carry over into the multiperiodic case, and it is again found that for stability, the

FIGURE 9 - TYPICAL AMPLITUDE/FREQUENCY RESPONSE FOR HARMONIC OSCILLATIONS WITH DAMPING

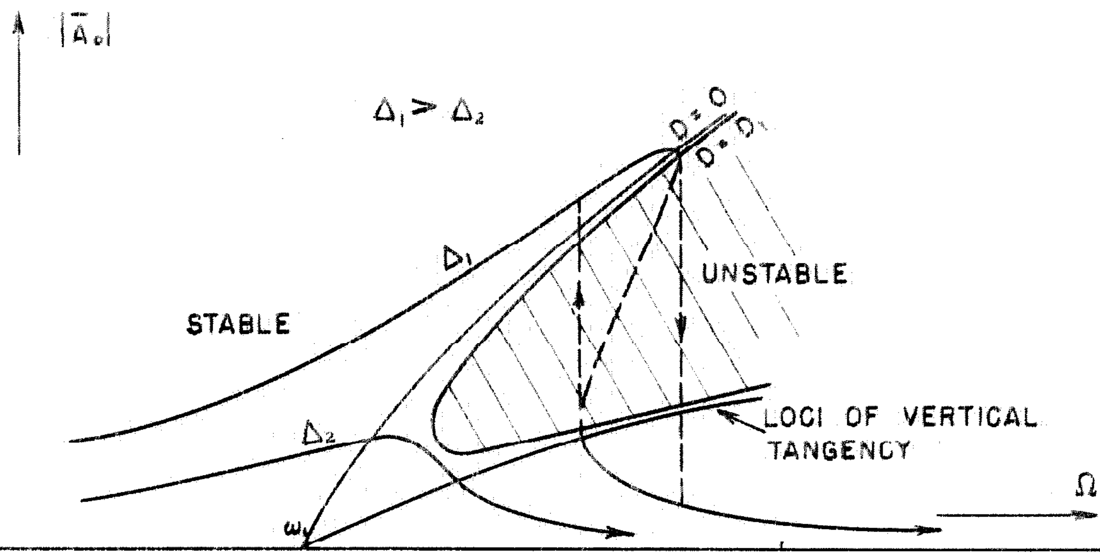
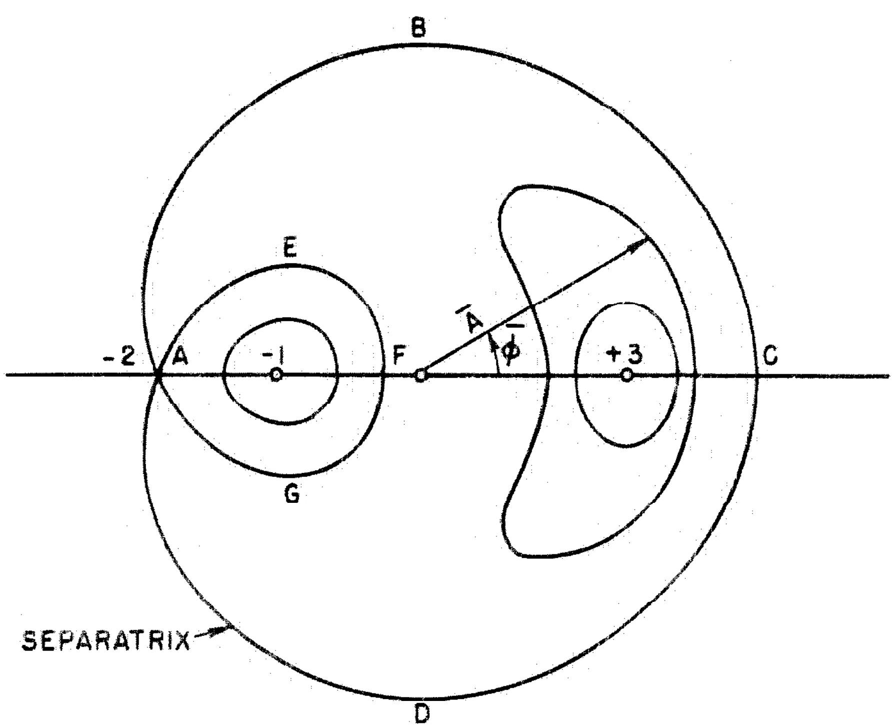


FIGURE 10 - TYPICAL INTEGRAL CURVES FOR HARMONIC OSCILLATIONS WITH THREE SINGULARITIES



solutions of the steady state equation should lie outside the region enclosed by the loci of vertical tangency.

<u>Effect of Initial Conditions on the Steady State Solutions.</u>

In discussing cases one and two, it was pointed out that within certain frequency bands two stable solutions could be found which satisfied the amplitude equation. The stability analysis was performed close to these solutions and it was shown that if one of these solutions exists then it is stable; this does not answer the question: 'When will such a solution exist?' In this section the dependence of the steady state solutions on the initial conditions will be studied, for simplicity the discussion will be restricted to a system with one degree of freedom, however, the results of this discussion are qualitatively true for a system of any order.

The method of solution will follow that of Andranow and Witt, but bears the same relationship to their method as does the Kryloff method to the Van der Pol method.

From equations (2.7) and (2.11) and equations (3.66) and (3.67) of the last section

$$\begin{aligned} -2\Omega\mathring{\bar{A}} &= \mu 2\Omega\bar{A}D + \mu\Delta\sin\phi \\ -2\Omega\bar{A}\mathring{\bar{\phi}} &= (\Omega^2 - w^2)\bar{A} - \mu s(\bar{A}) + \mu\Delta\cos\phi \end{aligned} \qquad (3.74)$$

$$\therefore \quad \frac{d\bar{A}}{d\bar{\phi}} = \frac{\mu 2\Omega\bar{A}^2 D + \mu\bar{A}\sin\phi\,\Delta}{(\Omega^2 - w^2)\bar{A} - \mu s(\bar{A}) + \mu\Delta\cos\phi} \qquad (3.75)$$

For the present, neglect damping, i.e., D = 0.

Singular points.

The singular points, neglecting damping, will be given by:

$$\mu \bar{A} \Delta \sin \bar{\phi} = 0, \quad (\Omega^2 - w^2)\bar{A} - \mu s(\bar{A}) + \mu \Delta \cos \bar{\phi} \tag{3.76}$$

and if ${}_o\bar{A} \neq 0$, then $\sin \bar{\phi} = 0$

$$[(\Omega^2 - w^2)\,{}_o\bar{A} - \mu s({}_o\bar{A})]^2 = (\mu f)^2 \tag{3.77}$$

Nature of Singular Points.

The nature of the singular points can easily be obtained from the perturbation equations

$$2\Omega\lambda = \pm i \sqrt{\left((w^2 - \Omega^2) + \mu s'(\bar{A}_o)\right)\left((w^2 - \Omega^2) + \mu s(\bar{A}_o)/\bar{A}_o\right)} \tag{3.78}$$

Phase Trajectories.

If damping is neglected, equation (3.75) may be written

$$d\bar{A} \left\{(\Omega^2 - w^2)\bar{A} - \mu s(\bar{A}) + \mu \Delta \cos \bar{\phi}\right\} - (\mu \bar{A} \Delta \sin \bar{\phi})d\bar{\phi} = 0.$$

But this is the exact differential of

$$\tfrac{1}{2}(\Omega^2 - w^2)\bar{A}^2 - \mu \int s(\bar{A})d\bar{A} + \mu \bar{A} \Delta \cos \bar{\phi} = \text{const} \tag{3.79}$$

Example.

If the non-linear function is a cubic, then $s(\bar{A}) = \frac{3}{4}\bar{A}^3$,

$$\therefore \quad \frac{d\bar{A}}{d\bar{\phi}} = \frac{\mu \Delta \sin \bar{\phi}}{(\Omega^2 - w^2)\bar{A} - \mu \frac{3}{4} \bar{A}^3 + \mu \Delta \cos \bar{\phi}} \tag{3.80}$$

Let

$$\left.\begin{aligned} \frac{4}{3} \Delta &= 6 \\ \frac{4}{3} (\Omega^2 - w^2) &= 7\mu \end{aligned}\right\} \tag{3.81}$$

For this case, the singular points are given by

$$\begin{aligned} &\sin \bar{\phi} = 0 \\ &7\bar{A} - \bar{A}^3 + 6 = 0 \end{aligned} \tag{3.82}$$

i.e. $(\bar{A} + 1)(\bar{A} + 2)(\bar{A} - 3) = 0$

i.e. ${}_o\bar{A} = -1, -2, 3.$

Stability.

The stability depends on the sign of

$$[w^2 - \Omega^2 + \mu s'({}_o\bar{A})][w^2 - \Omega^2 + \mu s({}_o\bar{A})/{}_o\bar{A}]$$

$$= [\tfrac{3}{4}\mu]^2 [-6 + 3\bar{A}^2] [-6 + \bar{A}^2]$$

if positive, system is stable,

if negative, system is unstable.

${}_o\bar{A} = -1$	stable	- vortex point no. 1	
${}_o\bar{A} = -2$	unstable	- saddle point no. 2	(3.83)
${}_o\bar{A} = 3$	stable	- vortex point no. 3	

Phase Trajectories.

The equation of the trajectories is

$$\frac{1}{2}(\Omega^2 - w^2)\bar{A}^2 - \mu \frac{3}{16}\bar{A}^4 + \mu\bar{A}\,\Delta\cos\bar{\phi} = \text{const},$$

i.e., $\bar{A}^4 - 14\,\bar{A}^2 - 24\,\bar{A}\cos\bar{\phi} = \text{const.}$

Now

$$\bar{A}^2 = r^2 = x^2 + y^2$$

$$\bar{A}\cos\bar{\phi} = x$$

$\therefore$ the equation of the trajectory is

$$r^4 - 14\,r^2 - 24\,x = \text{constant} \tag{3.84}$$

Equation of the Separatrix.

The separatrix passes through the saddle point, hence the constant is determined by the condition that

$$r = 2, \quad x = -2.$$

Hence the separatrix is defined by the equation

$$r^4 - 14\ \ ^2 - 24\,x = 8 \tag{3.85}$$

Fig.10 shows the separatrix and several typical integral curves.

For stable forced oscillations the initial conditions must lie inside the region ABCD, if the initial conditions lie inside the region AEFG, oscillation will be about vortex point one, but if they lie inside ABCDGFE, the oscillations will be about vortex point number three.

Effect of Damping.

Qualitatively, the effect of introducing a small amount of damping into the system will be to convert the vortex points into stable focal points and to prevent closure of the separatrix. If the initial conditions lie within the region AEFG, the oscillations will spiral down to focal point number one, and if they lie inside the other region, they will spiral down to focal point number two.

For a qualitative treatment of the case with damping see ref. 1. In this paper Hayashi treats the case of a system with a cubic non-linearity with damping - but without the linear term in the stiffness. The results of his analysis can be made applicable to our problem by writing the equation (1.3) of his paper in the form

$$\frac{d^2V}{dt^2} + \mu k\frac{dV}{dt} + \mu V^3 + (1 - \mu)V = B \cos \tau \qquad (3.86)$$

By substituting

$$V(\tau) = x(\tau) \sin \tau + y(\tau) \cos \tau \qquad (3.87)$$

into the differential equation and carrying out the Van der Pol procedure, one readily obtains

$$\left.\begin{aligned} \frac{dx}{\mu d\tau} &= \frac{1}{2}[B - kx + y - \frac{3}{4}(x^2 + y^2)y] = X(x,y) \qquad (3.88) \\ \frac{dy}{\mu d\tau} &= \frac{1}{2}[- x - ky + \frac{3}{4}(x^2 + y^2)x] = Y(x,y) \qquad (3.89) \end{aligned}\right\}$$

With the exception of the replacement of τ by $\mu\tau$, this is exactly the same as equation (1.5) of Hayashi's analysis, and so the integral curves obtained by him are directly applicable to the present problem.

The result of this analysis is to show that in the region where two stable solutions are possible, the existence of a particular solution depends only on the choice of initial conditions.

B. Forced Oscillations in a Two Degree of Freedom System With Non-Linear Damping.

Case 1. Consider the simple two degree of freedom system shown in Fig. 11.

Equations of Motion.

$$\left.\begin{aligned} L\ddot{y}_1 + \frac{1}{c}\,y_1 + \frac{1}{c_{12}}\,(y_1 - y_2) + f(y_1)\dot{y}_1 &= 0 \\ L\ddot{y}_2 + \frac{1}{c}\,y_2 + \frac{1}{c_{12}}\,(y_2 - y_1) &= P\cos\Omega t \end{aligned}\right\} \qquad (3.90)$$

Dividing through by L and letting

$$\frac{1}{Lc} = w_1^2$$

$$\frac{1}{Lc_{12}} = w_{12}^2$$

$$\left.\begin{aligned} \ddot{y}_1 + w_1^2 y_1 + w_{12}^2(y_1 - y_2) + \frac{1}{L_1}\,f(y_1)\dot{y}_1 &= 0 \\ \ddot{y}_2 + w_1^2 y_2 + w_{12}^2(y_2 - y_1) &= \frac{P}{L}\cos\Omega t \end{aligned}\right\} \qquad (3.91)$$

Comparison with equation (2.1) shows that

$$\begin{aligned} w_1^2 &= w_1^2, \quad \mu f_1 = w_{12}^2(y_1 - y_2) + \frac{1}{L}\,f(y_1)\dot{y}_1, \quad \mu\Delta_1 = 0 \\ w_2^2 &= w_1^2, \quad \mu f_2 = w_{12}^2(y_2 - y_1), \qquad\qquad \mu\Delta_2 = \frac{P}{L} \end{aligned} \qquad (3.92)$$

FIGURE 11 — SIMPLE TWO DEGREE OF FREEDOM SYSTEM WITH NON-LINEAR DAMPING

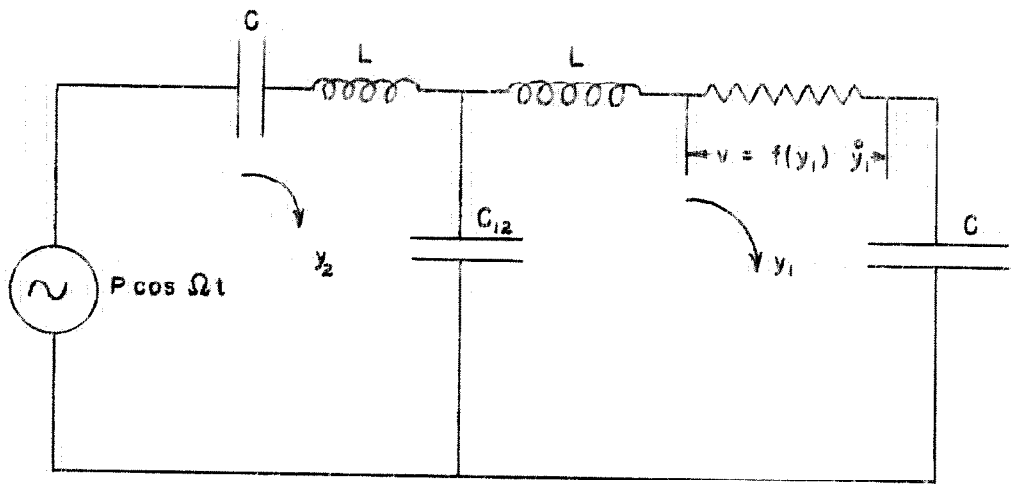
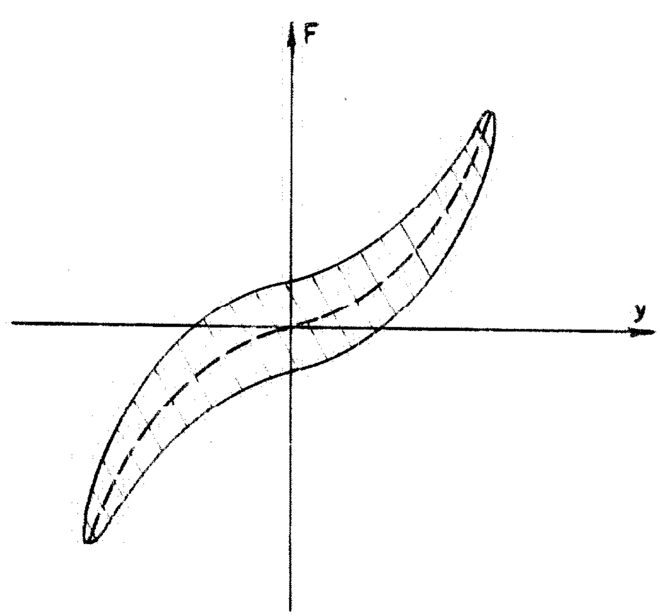


FIGURE 12 — HYSTERESIS DAMPING



thus

$$h_1 = (w_1^2 - \Omega^2)\bar{A}_1 + \frac{1}{\pi} \int_0^{2\pi} w_{12}^2(A_1 \cos x_1 - A_2 \cos x_2) \cos x_1$$

$$+ \frac{1}{L} f(A_1 \cos x_1)\ (-\Omega A_1 \sin x_1) \cos x_1\ dx$$

if $f(y)$ is an even function of y, then

$$h_1 = (w_1^2 + w_{12}^2 - \Omega^2)\bar{A}_1 - w_{12}^2\ \bar{A}_2 \cos(\bar{\phi}_2 - \bar{\phi}_1) \tag{3.93}$$

$$h_2 = (w_1^2 - \Omega^2)\bar{A}_2 + \frac{1}{\pi} \int_0^{2\pi} w_{12}^2(\bar{A}_2 \cos x_2 - \bar{A}_1 \cos x_1) \cos x_2\ dx$$

$$- \frac{P}{L} \cos \bar{\phi}_2$$

$$h_2 = (w_1^2 + w_{12}^2 - \Omega^2)\bar{A}_2 - w_{12}^2\ \bar{A}_1 \cos(\bar{\phi}_1 - \bar{\phi}_2) - \frac{P}{L} \cos \bar{\phi}_2 \tag{3.94}$$

$$g_1 = \frac{1}{\pi} \int_0^{2\pi} \Big\{ w_{12}^2(A_1 \cos x_1 - A_2 \cos x_2)$$

$$+ \frac{1}{L} f(A_1 \cos x_1)(-\Omega A_1 \sin x_1)\Big\} \sin x_1\ dx$$

$$\therefore \quad g_1 = -\ w_{12}^2\ \bar{A}_2 \sin(\bar{\phi}_1 - \bar{\phi}_2) + \Omega\, G(\bar{A}_1) \tag{3.95}$$

where

$$\Omega\, G(\bar{A}_1) = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{L} f(A_1 \cos x_1)(-\Omega A_1 \sin x_1)\ dx_1 \tag{3.96}$$

$$g_2 = \frac{1}{\pi} \int_0^{2\pi} w_{12}^2 (A_2 \cos x_2 - A_1 \cos x_1) \sin x_2 \, dx_2 - \frac{P}{L} \sin \bar{\phi}_2$$

$$\therefore \quad g_2 = - w^2_{12} \bar{A}_1 \sin(\bar{\phi}_2 - \bar{\phi}_1) - \frac{P}{L} \sin \bar{\phi}_2 \qquad (3.97)$$

Steady State equations.

The steady state conditions are obtained by setting h_i, g_i equal to zero.

$$\left.\begin{aligned}
& w_{12}^2 \, {}_o\bar{A}_2 \sin({}_o\bar{\phi}_2 - {}_o\bar{\phi}_1) + \Omega G({}_o\bar{A}_1) = 0 \\
& {}_o\bar{A}_1 (w_{12}^2 + w_1^2 - \Omega^2) - w_{12}^2 \, {}_o\bar{A}_1 \cos({}_o\bar{\phi}_1 - {}_o\bar{\phi}_2) = 0 \\
& {}_o\bar{A}_1 \, w_{12}^2 \sin(\bar{\phi}_1 - \bar{\phi}_2) - \frac{P}{L} \sin {}_o\bar{\phi}_2 = 0 \\
& {}_o\bar{A}_2 (w_{12}^2 + w_1^2 - \Omega^2) - {}_o\bar{A}_1 \, w_{12}^2 \cos({}_o\bar{\phi}_2 - {}_o\bar{\phi}_1) = \frac{P}{L} \cos {}_o\bar{\phi}_2
\end{aligned}\right\} \qquad (3.98)$$

Let $\Psi = (w_{12}^2 + w_1^2 - \Omega^2)$; eliminating ${}_o\bar{\phi}_2$ and ${}_o\bar{\phi}_1$ gives

$$w_{12}^4 \, {}_o\bar{A}_2 = \Omega^2 G^2 + \Psi^2 \, {}_o\bar{A}_1^{\,2} \qquad (3.99)$$

and

$$(\Psi^2 - w_{12}^4)^2 \, {}_o\bar{A}_1^{\,2} + \Psi^2 G^2 \Omega^2 = w_{12}^4 (\tfrac{P}{L})^2 \qquad (3.100)$$

Loci of Vertical Tangency.

The loci of vertical tangency can be obtained from equation (3.100) by differentiating with respect to $\bar{A}_1$ and setting $\frac{\partial \Omega}{\partial {}_o\bar{A}_1} = 0$.

Thus if ${}_o\bar{A}_1 \neq 0$

$$(\Psi^2 - w_{12}^2) + \Psi^2 \Omega^2 \frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} G'({}_o\bar{A}_1) = 0 \qquad (3.101)$$

gives the loci of vertical tangency.

<u>Stability of Steady State Solutions.</u>

As in previous cases the stability of the system depends on the roots of equation (2.17). For the system under analysis this equation is

$$\begin{vmatrix} \Omega\, G'({}_o\bar{A}_1) - 2\Omega\lambda, & -w_{12}^2\ {}_o\bar{A}_2 \sin({}_o\bar{\phi}_1 - {}_o\bar{\phi}_2), & -w_{12}^2\ {}_o\bar{A}_2 \cos({}_o\bar{\phi}_1 - {}_o\bar{\phi}_2), & +w_{12}^2\ {}_o\bar{A}_2 \cos({}_o\bar{\phi}_1 - {}_o\bar{\phi}_2) \\ -w_{12}^2 \sin({}_o\bar{\phi}_2 - {}_o\bar{\phi}_1), & 0 - 2\Omega\lambda, & w_{12}^2\ {}_o\bar{A}_1 \cos({}_o\bar{\phi}_2 - {}_o\bar{\phi}_1), & -w_{12}^2\ {}_o\bar{A}_1 \cos({}_o\bar{\phi}_2 - {}_o\bar{\phi}_1) - \frac{P}{L}\cos {}_o\bar{\phi}_2 \\ \Psi, & -w_{12}^2 \cos({}_o\bar{\phi}_2 - {}_o\bar{\phi}_1), & -w_{12}^2\ {}_o\bar{A}_2 \sin({}_o\bar{\phi}_2 - {}_o\bar{\phi}_1) - 2\,{}_o\bar{A}_1 \Omega\lambda, & +w_{12}^2\ {}_o\bar{A}_2 \sin({}_o\bar{\phi}_2 - {}_o\bar{\phi}_1) \\ -w_{12}^2 \cos({}_o\bar{\phi}_1 - {}_o\bar{\phi}_2), & \Psi, & w_{12}^2\ {}_o\bar{A}_1 \sin({}_o\bar{\phi}_1 - {}_o\bar{\phi}_2), & -w_{12}^2\ {}_o\bar{A}_2 \sin({}_o\bar{\phi}_1 - {}_o\bar{\phi}_2) + \frac{P}{L}\sin {}_o\bar{\phi}_2 - 2\Omega\,{}_o\bar{A}_2\lambda \end{vmatrix} = 0$$

(3.102)

Using the steady state equations this may be written

$$\begin{vmatrix} \Omega G' - 2\Omega\lambda, & -\dfrac{G}{{}_o\bar{A}_2}, & -\Psi, & 0 \\ \dfrac{\Omega G}{{}_o\bar{A}_2}, & -2\Omega\lambda, & \dfrac{{}_o\bar{A}_1}{{}_o\bar{A}_2}\Psi, & \dfrac{{}_o\bar{A}_1^{\,2} - {}_o\bar{A}_2^{\,2}}{{}_o\bar{A}_1\,{}_o\bar{A}_2}\Psi \\ \Psi, & -\Psi\dfrac{{}_o\bar{A}_1}{{}_o\bar{A}_2}, & \left(\dfrac{\Omega G}{{}_o\bar{A}_1} - 2\Omega\lambda\right), & -2\Omega\lambda \\ -\dfrac{{}_o\bar{A}_1}{{}_o\bar{A}_2}\Psi, & \Psi, & \dfrac{\Omega G}{{}_o\bar{A}_2}, & \dfrac{\Omega G}{{}_o\bar{A}_2} - \dfrac{{}_o\bar{A}_2}{{}_o\bar{A}_1}2\Omega\lambda \end{vmatrix} \times {}_o\bar{A}_1{}^2 = 0$$

(3.103)

Expanding out and grouping terms the characteristic equation becomes

$${}_o\bar{A}_1{}^2 \Bigg\{ \frac{{}_o\bar{A}_2}{{}_o\bar{A}_1}(2\Omega\lambda)^4 + (2\Omega\lambda)^3\left[-\frac{{}_o\bar{A}_2}{{}_o\bar{A}_1}\,\Omega\,\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} - \Omega G'({}_o\bar{A}_1)\frac{{}_o\bar{A}_2}{{}_o\bar{A}_1} \right]$$

$$+ (2\Omega\lambda)^2\Bigg[\frac{{}_o\bar{A}_2}{{}_o\bar{A}_1}(w_{12}^4 + \Psi^2) + \frac{{}_o\bar{A}_2}{{}_o\bar{A}_1}\,\Omega^2\,G'({}_o\bar{A}_1)\,\frac{G}{{}_o\bar{A}_1}({}_o\bar{A}_1)$$

$$+ \frac{\Omega^2 G^2({}_o\bar{A}_1)}{{}_o\bar{A}_1\,{}_o\bar{A}_2} + \left(\frac{{}_o\bar{A}_2}{{}_o\bar{A}_1} + \frac{{}_o\bar{A}_1}{{}_o\bar{A}_2} \right)\Psi^2 \Bigg] + (2\Omega\lambda)\Bigg[-\Psi^2\,\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1}\,\frac{{}_o\bar{A}_2}{{}_o\bar{A}_1}$$

$$- \frac{{}_o\bar{A}_2}{{}_o\bar{A}_1}(w_{12}^4 + \Psi^2)\,\Omega\,G'({}_o\bar{A}_1) + \frac{\Omega G}{{}_o\bar{A}_2}\left(-\frac{\Omega^2 G^2}{{}_o\bar{A}_1{}^2} + 2\Psi^2 \right) - 3\Psi^2\,\frac{\Omega G({}_o\bar{A}_1)}{{}_o\bar{A}_2} \Bigg]$$

+

$$+ \left[\Psi^2 \frac{{}_o\bar{A}_2}{{}_o\bar{A}_1} \Omega^2 G'({}_o\bar{A}_1) \frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + \Omega^2 \frac{G^2({}_o\bar{A}_1)}{{}_o\bar{A}_2} \left(\Omega^2 \frac{G^2({}_o\bar{A}_1)}{{}_o\bar{A}_1\ {}_o\bar{A}_2} - \frac{\Omega G({}_o\bar{A}_1)\Psi^2}{{}_o\bar{A}_1} \right) \right.$$

$$\left. + \Psi^2 \Omega^2 \frac{{}_o\bar{A}_1}{{}_o\bar{A}_2{}^2} \quad G^2({}_o\bar{A}_1) + \frac{({}_o\bar{A}_1{}^2 - {}_o\bar{A}_2)^2}{{}_o\bar{A}_1\ {}_o\bar{A}_2{}^3} \quad \Psi^4 \right] = 0 \qquad (3.104)$$

Making use of the steady state equations (3.104) can be reduced to

$$\begin{aligned} {}_o\bar{A}_1\ {}_o\bar{A}_2 \Big\{ & (2\Omega\lambda)^4 - (2\Omega\lambda)^3 \Omega \left[\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + G'({}_o\bar{A}_1) \right] \\ & + (2\Omega\lambda)^2 \left[2(w_{12}^4 + \Psi^2) + \Omega^2 G'({}_o\bar{A}_1) \frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} \right] \\ & - (2\Omega^2\lambda) \left[(w_{12}^4 + \Psi^2)\left(\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + G'({}_o\bar{A}_1) \right) \right] \\ & + \left[(\Psi^2 - w_{12}^4)^2 + \Psi^2 \Omega^2 G'({}_o\bar{A}_1) \frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} \right] = 0 \end{aligned}$$

(3.105)

if ${}_o\bar{A}_1\ {}_o\bar{A}_2 \neq 0$, this equation is of the form:

$$a_4 h^4 + a_3 h^3 + a_2 h^2 + a_1 h + a_o = 0$$

where $h = 2\Omega\lambda$

The Routh-Hurwitz criterion is

1) all coefficients positive

2) $a_1 a_2 > a_o a_3$

3) $a_1 a_2 a_3 > a_1^2 a_4 + a_3^2 a_o$.

For the given equation

$$\left.\begin{aligned}
a_4 &= 1 \\
a_3 &= -\left(\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + G'({}_o\bar{A}_1) \right) \Omega \\
a_2 &= 2(w_{12}^4 + \Psi^2) + \Omega^2 \frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} \; G'({}_o\bar{A}_1) \\
a_1 &= -(w_{12}^4 + \Psi^2) \, \Omega \left(\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + G'({}_o\bar{A}_1) \right) \\
a_o &= (\Psi^2 - w_{12}^4)^2 + \Psi^2 \Omega^2 \, G' \, \frac{G}{{}_o\bar{A}}
\end{aligned}\right\} \qquad (3.106)$$

<u>First condition</u>. $a_3 > 0$; $\left(G({}_o\bar{A}_1)/{}_o\bar{A}_1 + G'({}_o\bar{A}_1)\right) < 0$

$a_2 > 0$ i.e., $2(w_{12}^4 + \Psi^2 + \Omega^2 G'({}_o\bar{A}_1) \, [G({}_o\bar{A}_1)/{}_o\bar{A}_1] > 0$

this can be satisfied for small non-linearities. $a_1 > 0$, if $a_3 > 0$ a_1 is also positive; $a_o > 0$, i.e.,

$$(\Psi^2 - w_{12}^4)^2 = \Psi^2 \Omega^2 \, G'({}_o\bar{A}_1) \, \frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} > 0$$

It will be noted that this issimply the requirement that the solution $\bar{A}_1$ lie outside the region enclosed by the loci of vertical tangency as defined by equation (3.101).

<u>Condition 2.</u> $a_1 \, a_2 > a_o \, a_3$

$$\text{i.e., } -(w_{12}^4 + \Psi^2)\,\Omega\,\left(\frac{G}{{}_o\bar{A}_1} + G'\right)\left[\,2(w_{12}^4 + \Psi^2) + \Omega^2 G'({}_o\bar{A}_1)\,\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1}\,\right]$$

$$> -\,\Omega\left[\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + G'({}_o\bar{A}_1)\right]\left[\,(\Psi^2 - w_{12}^4)^2 + \Psi^2\Omega^2 G'({}_o\bar{A}_1)\,\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1}\,\right]$$

$$\text{i.e., } 2(w_{12}^4 + \Psi^2)^2 + \Omega^2(w_{12}^4 + \Psi^2)\,G'({}_o\bar{A}_1)\,\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1}$$

$$> (\Psi^2 - w_{12}^4)^2 + \Psi^2\,\Omega^2\,G'({}_o\bar{A}_1)\,\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1}$$

$$\therefore \quad (w_{12}^4 + \Psi^2)^2 + 4w_{12}^4\,\Psi^2 + \Omega^2\,w_{12}^4\,G'({}_o\bar{A}_1)\,\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} > 0$$

$\therefore$ For small non-linearities, therefore, $a_1 \, a_2 > a_o \, a_3$

<u>Condition 3.</u> $a_1 \, a_2 \, a_3 > a_1^2 \, a_4 + a_3^2 \, a_o$

$$\text{i.e., } -(w_{12}^4 + \Psi^2)\,\Omega\left\{\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + G'({}_o\bar{A}_1)\right\}\left[\,2(w_{12}^4 + \Psi^2)\right.$$

$$\left. + \Omega^2\,G'({}_o\bar{A}_1)\,\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1}\,\right]\left[\,-\,\Omega\left(\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + G'({}_o\bar{A}_1)\right)\right]$$

$$> (\Psi^2 + w_{12}^4)\,\Omega^2\left(\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + G'({}_o\bar{A}_1)\right)^2 + \Omega^2\left(\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + G'({}_o\bar{A}_1)\right)^2$$

$$\times\left[\,2(w_{12}^4 + \Psi^2) + \Omega^2\,\Psi^2\,G'({}_o\bar{A}_1)\,\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1}\,\right]$$

If condition $\left(\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + G'({}_o\bar{A}_1)\right) \neq 0$, then condition 3 becomes

$$(w_{12}^4 + \Psi^2)\left[2(w_{12}^4 + \Psi^2) + \Omega^2 G'({}_o\bar{A}_1)\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1}\right]$$

$$> (\Psi^2 + w_{12}^4)^2 + (w_{12}^4 - \Psi^2)^2 + \Psi^2 \Omega^2 G'({}_o\bar{A}_1)\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1}$$

$$\therefore 4w_{12}^4 \Psi^2 + \Omega^2 w_{12}^4 G'({}_o\bar{A}_1)\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} > 0$$

for small non-linearities this condition can be satisfied, except, perhaps closeto $\Psi = 0$ where it may become negative. Examination of the amplitude equation (3.100), shows that at $\Psi = 0$, $\bar{A}_1$ is small, of the order of $\frac{P}{L}$, which was assumed small; hence condition 3 will be satisfied up to the second power in μ. Thus for small non-linear damping all stability conditions will be satisfied, to the first order in small quantities, provided

$$1) \qquad \frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} + G'({}_o\bar{A}_1) < 0 \qquad (3.106)$$

and

$$2) \qquad (w_{12}^4 - \Psi^2)^2 + \Omega^2 \Psi^2 G'({}_o\bar{A}_1)\frac{G({}_o\bar{A}_1)}{{}_o\bar{A}_1} > 0$$

The second condition, as already pointed out, is simply the requirement that the solution $\bar{A}_1$ lie outside the region enclosed by the loci of vertical tangency.

Two cases may be distinguished.

a) If $G({}_o\bar{A}_1)$ is a monotonically increasing or decreasing function, the system will be unconditionally stable, or unstable, depending on the sign associated with $G({}_o\bar{A}_1)$.

b) If $G({}_o\bar{A}_1)$ is a function which changes sign, a much more interesting situation arises; to illustrate this, consider the damping function:

$$f(y, y_1) = (-b + c3y_1^2)\,\mathring{y}_1 \qquad (3.107)$$

for this case

$$G({}_o\bar{A}_1) = (b - \frac{3c}{4}\,\bar{A}_1^{\,2})\bar{A}_1 \qquad (3.108)$$

Substituting equation (3.108) into the amplitude equation (3.100) gives

$$(\Psi^2 - w_{12}^4)^2\,\bar{A}_1^{\,2} + \Psi^2\,\Omega^2(b - \frac{3c}{4}\,\bar{A}_1^{\,2})^2\,\bar{A}_1^{\,2} = w_{12}^4(\frac{P}{L})^2$$

which may be written in the form:

$$\left(\frac{\Psi^2 - w_{12}^4}{\Psi\Omega b}\right)^2 \bar{A}_1^{\,2}\,\frac{3c}{4b} + (1 - \frac{3c}{4b}\,\bar{A}_1^{\,2})^2\,\bar{A}_1^{\,2}\,\frac{3c}{4b} = \frac{w_{12}^4\,(\frac{P}{L})^2}{\Psi^2\,\Omega^2}\,\frac{3c}{4b^3} \qquad \Psi \neq 0 \qquad (3.109)$$

Let

$$\left.\begin{array}{ll} \dfrac{\Psi^2 - w_{12}^4}{\Psi\ \ b} = X; & \dfrac{3c}{4b^3}\,\dfrac{w_{12}^4(\frac{P}{L})^2}{\Psi^2\ \ ^2} = E \\ \dfrac{3c}{4b}\,\bar{A}_1^{\,2} = Y; & \end{array}\right\} \qquad (3.110)$$

Equation (3.109) can be expressed as

$$X^2Y + (1 - Y)^2Y = E \tag{3.111}$$

This equation is exactly of the same form as Van der Pol derived for the forced oscillations of a vacuum type oscillator.

For Stability.

$$G'(\bar{A}_1) + \frac{G(\bar{A}_1)}{\bar{A}_1} = (b - \frac{9c}{4}\bar{A}_1^{\,2}) + (b - \frac{3c}{4}\bar{A}_1^{\,2}) < 0$$

$$= 2b - 3c\,\bar{A}_1^{\,2} < 0.$$

Threshold amplitude $\bar{A}_1 = \frac{1}{\sqrt{2}}\sqrt{\frac{4b}{3c}} = \frac{1}{\sqrt{2}}\bar{A}_o$ where $\bar{A}_o$ is the amplitude of free oscialltions. The other condition for stability is the solutions should lie outside the region enclosed by the loci of vertical tangency, as defined by

$$(\Psi^2 - w_{12}^4)^2 + \Psi^2\Omega^2(b - \frac{9c}{4}\bar{A}_1^{\,2})(b - \frac{3c}{4}\bar{A}_1^{\,2}) = 0$$

i.e., $X^2 + (1 - 3Y)(1 - Y) = 0$ (3.112)

i.e., the ellipse $\frac{X^2}{(\frac{1}{\sqrt{3}})^2} + \frac{(Y - \frac{2}{3})^2}{(\frac{1}{3})^2} = 1$ (3.113)

For a good ~~copy~~ graph of these functions, see McLachlan, p. 82, Chapter IV.

Since

$$X = \frac{\Psi^2 - w_{12}^4}{\Psi\Omega b} = \frac{(\Omega^2 - w_{12}^2 - w_1^2)^2 - w_{12}^2}{(\Omega^2 - w_{12}^2 - w_1^2)\Omega b}$$

X will have the same value for two different values of Ω.

From the form of the equations it is easy to see that the amplitude $({}_o\bar{A}_1)$ is of the general nature shown in Fig. 13 . The shaded regions are unstable in the sense that the motion is a mixture of forced and free oscillations. The solid curves indicate the regions where stable forced oscillations exist, free oscillations being completely suppressed. Points at which the solid curves emerge from the shaded zones, represent points at which free oscillations and beat phenomena disappear and are replaced by pure forced oscillations.

Case 2. Hysteresis Damping.

As a second example of non-linear damping, consider the steady state of vibration of a mass m moving under the action of a restoring force kF with hysteresis characteristics and excited by a simple harmonic force $P \cos \Omega t$. While the following analysis is applicable to any number of degrees of freedom, it will be restricted in this case to a single degree of freedom for simplicity.

Equations of Motion:

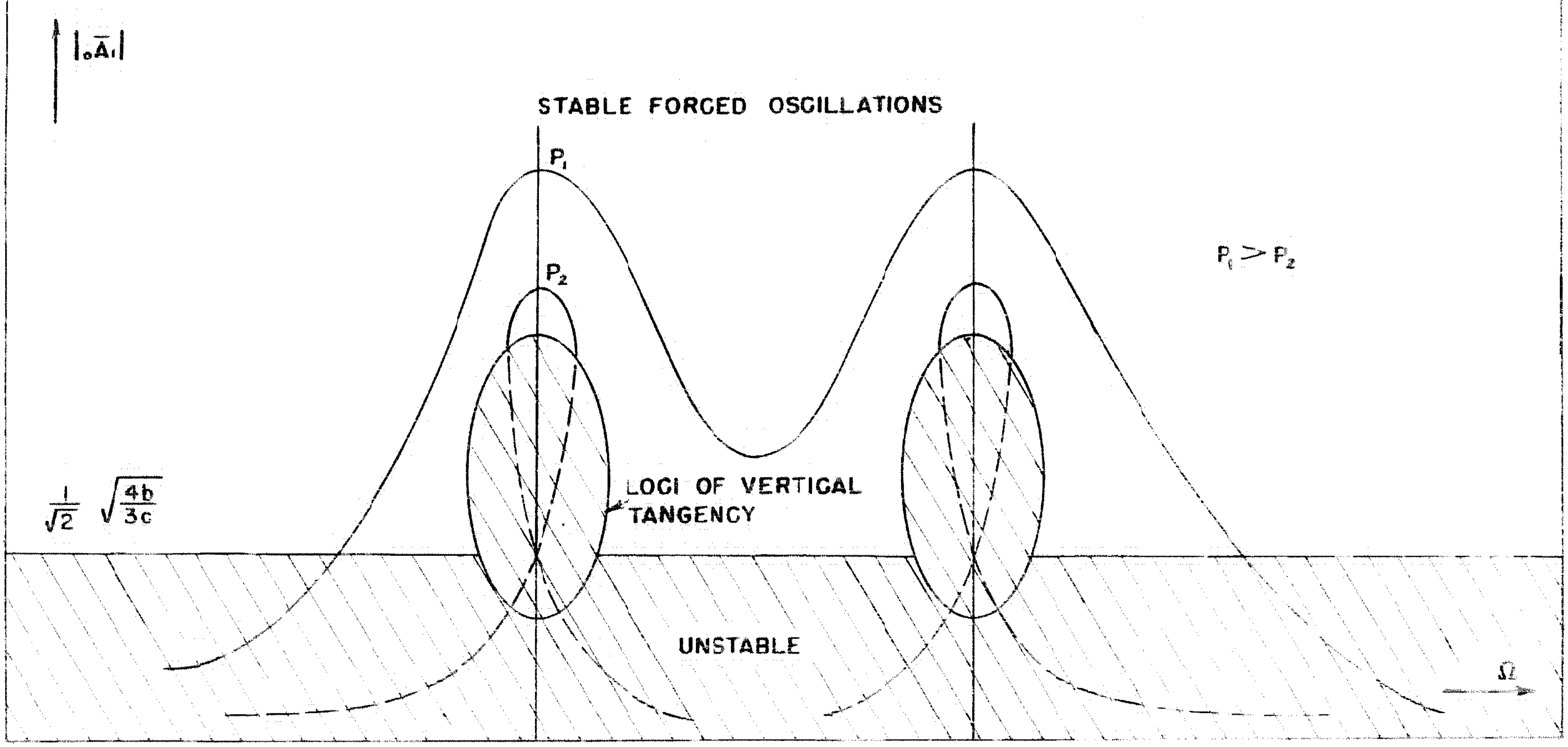
$$m\ddot{y} + kF(y) = P \cos \Omega t \qquad (3.114)$$

or rearranging slightly

$$\ddot{y} + w_1^2 y + w_1^2 (F(y) - y) = \frac{P}{m} \cos \Omega t \qquad (3.115)$$

This is now in the form required for the general theory.

FIGURE 13 — GENERAL CHARACTERISTICS OF VAN DER POL'S EQUATION IN TWO DEGREES OF FREEDOM



Using the general theory

$$g = \frac{w_1^2}{\pi}\int_0^{2\pi} (F(A\cos x) - A\cos x)\sin x\,dx - \frac{P}{m}\sin \bar{\emptyset}_1 \tag{3.116}$$

$$h = (w_1^2 - \Omega^2)\bar{A}_1 + \frac{w_1^2}{\pi}\int_0^{2\pi} (F(A\cos x) - A\cos x)\cos x\,dx - \frac{P}{m}\cos \bar{\emptyset}_1 \tag{3.117}$$

For a group of simple non-linearities, the restoring force $F(y)$ can be written

$$F(y) = f(A\cos(x + \beta))$$

where β is a phase shift which may depend on the frequency and the amplitude, but for several simple cases may be assumed constant.

Case 1. Classical problem, 'linear spring' with hysteresis damping

$$F(y) = k(A\cos(x + \beta))$$

$$g = \frac{w_1^2}{\pi}\int_0^{2\pi} (A\cos(x + \beta) - A\cos x)\sin x\,dx - \frac{P}{m}\sin \bar{\emptyset}_1$$

$$g = - w_1^2\,\bar{A}\sin\beta - \frac{P}{m}\sin \bar{\emptyset}_1 \tag{3.118}$$

$$h = (w_1^2 - \Omega^2)\bar{A}_1 + \frac{w_1^2}{\pi}\int_0^{2\pi} (A\cos(x + \beta) - A\cos x)\cos x\,dx - \frac{P}{m}\cos \bar{\emptyset}_1$$

$$h = -\Omega^2\,\bar{A}_1 + w_1^2\,\bar{A}_1\cos\beta - \frac{P}{m}\cos \bar{\emptyset}_1 \tag{3.119}$$

Steady State Equations.

The steady state equations are obtained by setting h = g = 0

$$\therefore \quad \bar{A}_1(-\Omega^2 + w_1^2 \cos\beta) = \frac{P}{m}\cos\bar{\phi}_1$$

$$\bar{A}_1 w^2 \sin\beta = -\frac{P}{m}\sin\bar{\phi}_1 \qquad (3.120)$$

eliminating $\bar{\phi}_1$

$$\bar{A} = \frac{\frac{P}{m}}{\sqrt{\left[w^4 - 2\Omega^2 w^2 \cos\beta + \Omega^4\right]}}$$

which can be written in the form

$$\bar{A} = \frac{\delta_{st}}{\sqrt{\left\{1 - 2\left(\frac{\Omega}{w_1}\right)^2 \cos\beta + \left(\frac{\Omega}{w_1}\right)^4\right\}}} \qquad (3.121)$$

Similarly eliminating $\bar{A}$

$$\tan\bar{\phi}_1 = \frac{-\sin\beta}{\cos\beta - \left(\frac{\Omega}{w_1}\right)^2} \qquad (3.122)$$

Equations (3.121) and (3.122) are identical with those obtained by Myklestad (J.A.M., p. 284, Vol. 19, Sept. 1952).

Stability.

The perturbation equations lead to the characteristic equation

$$(w^2 \sin\beta + 2\Omega\lambda)^2 = -(\Omega^2 - w^2\cos\beta)^2$$

i.e., $$2\Omega\lambda = - w^2 \sin\beta \pm i(\Omega^2 - w^2 \cos\beta) \qquad (3.123)$$

For equation (3.123) it will be seen that for stability $\sin\beta > 0$, which implies that for small β, $\beta > 0$.

- - - - - - - - - - - - -

The whirling of a vertical shaft offers a very nice example of hysteresis damping, and the effect of the phase angle β on the stability of the system. For rotational speeds below critical, it can be shown that β is positive (see p. 340 of Edition II of Mechanical Vibrations by J.P. Den Hartog) and so the system is stable. At speeds above the critical, β becomes negative, and so the system is unstable.

Non-Linear Spring with Hysteresis.

As a second example of a system with hysteresis damping, take the spring characteristic

$$F(y) = k(A\cos(x+\beta)) + \mu k(A\cos(x+\beta))^3$$

for this spring force

$$g = - w_1^2 \bar{A} \sin\beta - \frac{P}{m}\sin\bar{\phi} - \mu\frac{3}{4} w_1^2 \bar{A}_1^3 \sin\beta \qquad (3.124)$$

$$h = - \Omega^2 \bar{A}_1 + w_1^2 \bar{A}\cos\beta + \mu\frac{3}{4} w_1^2 \bar{A}^3 \cos\beta - \frac{P}{m}\cos\bar{\phi}_1 \qquad (3.125)$$

Steady State Equations.

Setting $g = h = 0$ gives

$$w_1^2(\bar{A}_1 + \mu\frac{3}{4}\bar{A}_1^3)\cos\beta - \Omega^2 \bar{A}_1 = \frac{P}{m}\cos\bar{\phi}_1$$

$$(3.126)$$

$$\sin\beta\,(\bar{A}_1 + \mu\frac{3}{4}\bar{A}_1^3)\, w_1^2 = - \frac{P}{m}\sin\bar{\phi}_1$$

Eliminating $\bar{\phi}_1$

$$w_1^4(\bar{A}_1 + \mu\frac{3}{4}\bar{A}_1^{\,3})^2 + \Omega^4\bar{A}_1^{\,2} - 2\Omega^2 w_1^2(\bar{A}_1 + \mu\frac{3}{4}\bar{A}_1^{\,3})\bar{A}_1\cos\beta = (\frac{P}{m})^2 \tag{3.127}$$

and

$$\tan\bar{\phi}_1 = -\frac{\sin\beta(\bar{A}_1 + \frac{3}{4}\bar{A}_1^{\,2})\,w_1^2}{w_1^2(\bar{A}_1 + \mu\frac{3}{4}\bar{A}_1^{\,3})\cos\beta - \Omega^2\bar{A}_1} \tag{3.128}$$

The loci of vertical tangency is given by

$$w_1^4(1 + \mu\frac{3}{4}\bar{A}_1^{\,2})(1 + \mu\frac{9}{4}\bar{A}_1^{\,2}) + \Omega^4 - \Omega^2 w_1^2\left[2 + 3\mu\bar{A}_1^{\,2}\right]\cos\beta = 0 \tag{3.129}$$

<u>Stability</u>.

Following the standard procedure leads to the characteristic equation

$$(2\Omega\lambda)^2 + 2(2\Omega\lambda)\,w^2(2 + 3\mu\bar{A}^2)\sin\beta + \sin^2\beta\,(1 + \mu\frac{3}{4}\bar{A}^2)(1 + \mu\frac{9}{4}\bar{A}^2) + (\Omega^2 - w^2(1 + \frac{9}{4}\mu\bar{A}^2)\cos\beta)(\Omega^2 - w^2(1 + \frac{3}{4}\mu\bar{A}^2)\cos\beta) = 0$$

i.e.,

$$2\Omega\lambda = -w^2(2 + 3\mu \bar{A}^2)\sin\beta$$
$$+ i\sqrt{\left\{ w_1^4(1 + \mu\tfrac{3}{4}\bar{A}^2)(1 + \mu\tfrac{9}{4}\bar{A}^2) + \Omega^4 \right.}$$
$$\overline{-\Omega^2 w_1^2(2 + 3\mu \bar{A}_1^2)\cos\beta - w^4(2 + 3\mu\bar{A}^2)\sin^2\beta} \tag{3.130}$$

For stability $w^2(2 + 3\mu\bar{A}^2)\sin\beta \geq 0$

$$\left\{ w_1^4(1 + \mu\tfrac{3}{4}\bar{A}^2)(1 + \mu\tfrac{9}{4}\bar{A}^2) - \Omega^2 w_1(2 + 3\mu\bar{A}_1^2)\cos\beta + \Omega^4 \right\} \geq 0$$

It will be observed that this condition is simply the requirement that $\bar{A}$ lie outside the region enclosed by the loci of vertical tangency, as given by equation (3.129). Thus for stability , $\mu > 0$

1) $\sin\beta \geq 0$ and for small β this requires that $\beta \geq 0$

2) $\bar{A}$ lies outside the region enclosed by the loci of vertical tangency.

From the few examples given above, it will be seen that the Kryloff Bogluiboff analysis is a very powerful method for obtaining the first order solutions of non-linear problems involving forced oscillations. Simple extensions of the method have enabled us to study the stability of forced oscillations and to prove analytically that points of vertical tangency in the amplitude/frequency response correspond with points of instability, a fact long recognized experimentally.

The present treatment is far from being exhaustive, and it is hoped that numerous other problems will be treated by this simple yet elegant technique.

4. APPLICATION

4.0. The Existence and Stability of Ultraharmonics and Subharmonics in Forced Non-Linear Oscillations.

In order to illustrate the application of the general theory to the existence and stability of ultraharmonic and subharmonic oscillations, consider the forced oscillations of a single degree of freedom system with a small cubic non-linearity in the restoring force.

$$\frac{d^2y}{dt^2} + k\frac{dy}{dt} + w_1^2 y + \mu y^3 = P\cos\Omega t$$

A) Ultraharmonic Oscillations

Under suitable conditions, a non-linear system may exhibit marked ultraharmonic oscillations in which the main component of motion has a frequency which is an integral multiple of the frequency of the driving force. In the case of a cubic non-linearity, marked ultraharmonic behavior has been observed experimentally when the frequency of the driving force was reduced to about one third of the natural frequency of the system.

In the spirit of Trefftz, one is led to try a solution of the form:

$$y = A_1\cos\Omega t + A_3\cos(3\Omega t + \phi_3) \qquad (4.1)$$

where $\Omega \approx \frac{1}{3}w_1$, hence $A_1 \doteq \dfrac{P}{w_1^2 - \Omega^2} = \dfrac{\frac{9}{8}P}{w_1^2}$.

A_3 and ϕ_3 are slowly varying functions of time. For convenience let $\Omega' = 3\Omega$, then $\Omega' \approx w_1$

Comparison with equation (2.1) shows that

$$\left.\begin{aligned} \mu f_1(\ldots) &= \mu y^3 + k\dot{y} \\ \mu \Delta_1 &= p \end{aligned}\right\} \qquad \left.\begin{aligned} \beta &= \phi_3 \\ B &= A_3 \end{aligned}\right\} \tag{4.2}$$

Hence substituting into equation (2.28)

$$H_1(\ldots) = -(\Omega'^2 - w_1^2)\,\bar{A}_3 + \frac{\mu}{\pi}\int_0^{6\pi}\Big[\mu(A_1\cos\Omega t + A_3\cos\gamma_1)^3$$
$$- k(A_1\Omega\sin wt + \Omega' A_3\sin\gamma_1)\Big]\cos\gamma_1\,\frac{d\gamma_1}{\mu}$$
$$= (w_1^2 - \Omega'^2)\,\bar{A}_3 + \mu\left(\frac{3}{2}\bar{A}_1^2\bar{A}_3 + \frac{3}{4}\bar{A}_3^3 + \frac{\bar{A}_1^3}{4}\cos\bar{\phi}_3\right)$$

Hence substituting into equation (2.27)

$$2\Omega'\,\bar{A}_3\,\mathring{\bar{\phi}}_3 = (w_1^2 - \Omega'^2)\,\bar{A}_3 + \mu\left(\frac{3}{2}\bar{A}_1^2\bar{A}_3 + \frac{3}{4}\bar{A}_3^3 + \frac{\bar{A}_1^3}{4}\cos\bar{\phi}_3\right) \tag{4.3}$$

Similarly substituting into equations (2.32) and (2.31)

$$2\Omega'\,\mathring{\bar{A}}_3 + k\Omega'\,\bar{A}_3 = \mu\frac{\bar{A}_1^3}{4}\sin\bar{\phi}_3 \tag{4.4}$$

<u>Steady State Solutions</u>

The steady state ultraharmonic solution corresponds with

$$\frac{d\bar{A}_3}{dt} = \frac{d\bar{\phi}_3}{dt} = 0$$

which leads to

$$\left.\begin{aligned} K\Omega'\,{}_o\bar{A}_3 &= \mu\frac{{}_o\bar{A}_1^{\,3}}{4}\ \sin {}_o\bar{\phi}_3 \\ {}_oA_3(w_1^2-\Omega'^2)+\mu\left(\tfrac{3}{2}\,{}_o\bar{A}_1^{\,2}\,{}_o\bar{A}_3+\tfrac{3}{4}\,{}_o\bar{A}_3^{\,3}\right) &= -\mu\frac{{}_o\bar{A}_1^{\,3}}{4}\cos {}_o\bar{\phi}_3 \end{aligned}\right\} \tag{4.5}$$

Eliminating ${}_o\bar{\phi}_3$

$$\left(\mu\frac{{}_o\bar{A}_1^{\,3}}{4}\right)^2 = (K\Omega')^2\,{}_o\bar{A}_3^{\,2} + \left[(w_1^2-\Omega'^2)\,{}_o\bar{A}_3+\mu\left(\tfrac{3}{2}\,{}_o\bar{A}_1^{\,2}\,{}_o\bar{A}_3+\tfrac{3}{4}\,{}_o\bar{A}_3^{\,3}\right)\right]^2 \tag{4.6}$$

and

$$\sin {}_o\bar{\phi}_3 = \frac{\pm K\Omega'}{\sqrt{\left\{(K\Omega')^2+\left[w_1^2-\Omega'^2+\mu\left(\tfrac{3}{2}\,{}_o\bar{A}_1^{\,2}+\tfrac{3}{4}\,{}_o\bar{A}_3^{\,2}\right)\right]^2\right\}}} \tag{4.7}$$

provided

$${}_o\bar{A}_3 \neq 0$$

<u>Loci of Vertical Tangency</u>.

Viewed as a function of ${}_o\bar{A}_3$, the loci of vertical tangency can be obtained from equation (4.6), they are

$$(K\Omega')^2 + \left[w_1^2-\Omega'^2+\mu\left(\tfrac{3}{2}\,{}_o\bar{A}_1^{\,2}+\tfrac{3}{4}\,{}_o\bar{A}_3^{\,2}\right)\right]\left[w_1^2-\Omega'^2+\mu\left(\tfrac{3}{2}\,{}_o\bar{A}_1^{\,2}+\tfrac{9}{4}\,{}_o\bar{A}_3^{\,2}\right)\right] = 0 \tag{4.8}$$

if $K \ll 1\, w_1^2$,

the loci are approximately

$$\left.\begin{aligned} w_1^2 - \Omega'^2 + \mu\left(\tfrac{3}{2}\,{}_0\bar{A}_1^{\,2} + \tfrac{3}{4}\,{}_0\bar{A}_3^{\,2}\right) &= 0 \\ w_1^2 - \Omega'^2 + \mu\left(\tfrac{3}{2}\,{}_0\bar{A}_1^{\,2} + \tfrac{9}{4}\,{}_0\bar{A}_3^{\,2}\right) &= 0 \end{aligned}\right\} \qquad (4.9)$$

As in the case of harmonic oscillations, it will be observed that on the upper locus $\sin {}_0\bar{\phi}_3 = \pm 1$, ${}_0\bar{\phi}_3 = \frac{\pi}{2}, \frac{3\pi}{2}$, etc.

<u>Stability of Steady State Solutions</u>.

Let

$$\left.\begin{aligned} \bar{A}_3 &= {}_0\bar{A}_3 + \xi \\ \bar{\phi}_3 &= {}_0\bar{\phi}_3 + \eta \end{aligned}\right\} \qquad (4.10)$$

Writing perturbations on equations (4.3) and (4.4)

$$\left.\begin{aligned} 2\Omega'\mathring{\xi} + K\Omega'\xi &= \mu\frac{{}_0\bar{A}_1^{\,3}}{4}\cos {}_0\bar{\phi}_3\,\eta \\ 2\Omega'\,{}_0\bar{A}_3\,\mathring{\eta} &= \left\{(w_1^2 - \Omega'^2) + \mu\left(\tfrac{3}{2}\,{}_0\bar{A}_1^{\,2} + \tfrac{9}{4}\,{}_0\bar{A}_3^{\,2}\right)\right\}\xi \\ &\quad - \mu\frac{{}_0\bar{A}_1^{\,3}}{4}\sin {}_0\bar{\phi}_3\,\eta \end{aligned}\right\} \qquad (4.11)$$

Assuming solutions of the form $e^{\lambda t}$, and substituting for $\cos {}_0\bar{\phi}_3$ and $\sin {}_0\bar{\phi}_3$ from equation (4.5)

$$\begin{cases} (2\Omega'\lambda + K\Omega')\xi = \left\{(\Omega'^2 - w_1^2)\,{}_o\bar{A}_3 - \mu\,{}_o\bar{A}_3(\frac{3}{2}\,{}_o\bar{A}_1^{\,2} + \frac{3}{4}\,{}_o\bar{A}_3^{\,2})\right\}\eta \\ \left\{(w_1^2 - \Omega'^2) + \mu(\frac{3}{2}\,{}_o\bar{A}_1^{\,2} + \frac{9}{4}\,{}_o\bar{A}_3^{\,2})\right\}\xi = {}_o\bar{A}_3\left[2\Omega'\lambda + K\Omega'\right]\eta \end{cases} \tag{4.12}$$

which leads to the characteristic equation

$$(2\Omega'\lambda)^2 + (2\Omega'\lambda)(K\Omega') + \Big\{(K\Omega')^2 + [(w_1^2 - \Omega'^2) + \mu(\tfrac{3}{2}\,{}_o\bar{A}_1^{\,2} + \tfrac{9}{4}\,{}_o\bar{A}_3^{\,2})]$$
$$\times [(w_1^2 - \Omega'^2) + \mu(\tfrac{3}{2}\,{}_o\bar{A}_1^{\,2} + \tfrac{3}{4}\,{}_o\bar{A}_3^{\,2})]$$
$$= 0 \tag{4.13}$$

For stability; $K \geq 0$

and

$$(K\Omega')^2 + [(w_1^2 - \Omega'^2) + \mu(\tfrac{3}{2}\,{}_o\bar{A}_1^{\,2} + \tfrac{9}{4}\,{}_o\bar{A}_3^{\,2})]\,[w_1^2 - \Omega'^2)$$
$$+ \mu(\tfrac{3}{2}\,{}_o\bar{A}_1^{\,2} + \tfrac{3}{4}\,{}_o\bar{A}_3^{\,2})] \geq 0 \tag{4.14}$$

Comparison with equation (4.8) shows that the second condition for stability is simply the requirement that ${}_o\bar{A}_3$ lie outside the region enclosed by the loci of vertical tangency.

If $K \ll 1$ it will be observed from equation (4.9) that for $\bar{A}_3$ real

$$\Omega'^2 \geq w_1^2 + \mu\frac{3}{2}\,\bar{A}_1^{\,2} \tag{4.15}$$

Figure 14 shows a typical Amplitude Frequency curve for ultraharmonic of order 3.

FIGURE 14 – TYPICAL AMPLITUDE/FREQUENCY RESPONSE FOR ULTRA HARMONIC MOTION OF ORDER THREE

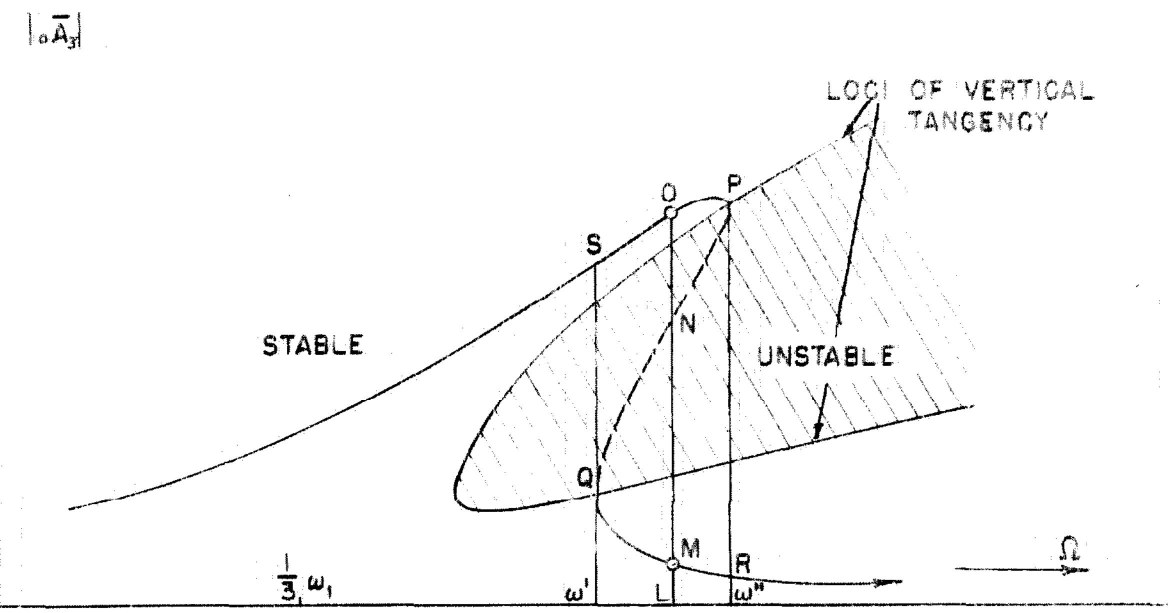
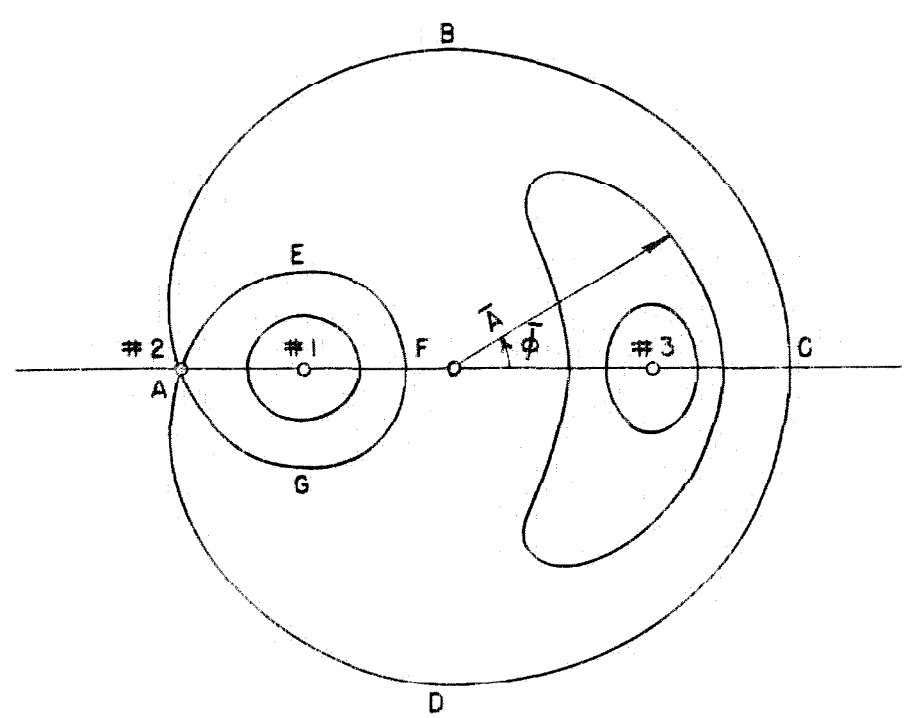


FIGURE 15 – TYPICAL INTEGRAL CURVES FOR ULTRAHARMONIC MOTION WITH THREE SINGULARITIES



The general features of the amplitude frequency response are identical with those for harmonic oscillations. At points P and Q where ${}_o\bar{A}_3$ enters the unstable region, vertical jumps occur, exactly as in the harmonic case. With the exception of a small region close to $\frac{1}{3} w_1$, the amplitude ${}_o\bar{A}_3$ is small compared to ${}_o\bar{A}_1$, and hence may be neglected.

<u>Effect of Initial Conditions on Ultraharmonic Solutions.</u>

The effect of the initial conditions on the ultraharmonic motion can best be studied by means of the integral curves for the motion. To obtain the integral curves for the motion equations (4.3) and (4.4) can be written in the form

$$\frac{d\bar{A}_3}{d\bar{\phi}_3} = \frac{\mu \frac{\bar{A}_1^{\,3}}{4} \bar{A}_3 \sin \bar{\phi}_3 - K\Omega' \bar{A}_3^{\,2}}{(w_1^2 - \Omega'^2)\bar{A}_3 + \mu(\frac{3}{2}\bar{A}_1^{\,2}\bar{A}_3 + \frac{3}{4}\bar{A}_3^{\,3} + \frac{\bar{A}_1^{\,3}}{4}\cos\bar{\phi}_3)} \qquad (4.16)$$

if damping is neglected (4.16) becomes

$$\frac{d\bar{A}_3}{d\bar{\phi}_3} = \frac{\mu \frac{\bar{A}_1^{\,3}}{4} \bar{A}_3 \sin \bar{\phi}_3}{(w_1^2 + \mu\frac{3}{2}\bar{A}_1^{\,2} - \Omega'^2)\bar{A}_3 + \mu\frac{3}{4}\bar{A}_3^{\,3} + \mu\frac{\bar{A}_1^{\,3}}{4}\cos\bar{\phi}_3} \qquad (4.17)$$

Comparison of equation (4.17) with equation (3.80) shows that if

$$\left.\begin{aligned} &\bar{A}_3 \text{ is replaced by } \bar{A}_1 \\ &\bar{\phi}_3 \text{ is replaced by } \bar{\phi}_1 \\ &\mu \frac{\bar{A}_1^{\,3}}{4} \text{ is replaced by } \mu\,\Delta \\ &w_1^2 + \mu \tfrac{3}{2} \bar{A}_1^{\,2} \text{ is replaced by } w_1^2 \\ &\Omega'^2 \text{ is replaced by } \Omega^2 \end{aligned}\right\} \qquad (4.18)$$

then the two equations are identical in form, and the remarks made about the harmonic motion will be equally true for the ultraharmonic motion.

If the forcing frequency Ω lies in the range $w' < \Omega < w''$ the amplitude equation (4.6) has three solutions, and if damping is neglected the three singular points are:

$$\left.\begin{aligned} &\text{No. 1} \quad {}_o\bar{A}_3 = \text{LM - stable - center point}, \quad {}_o\bar{\phi}_3 = \pi \\ &\text{No. 2} \quad {}_o\bar{A}_3 = \text{LN - unstable - saddle point}, \quad {}_o\bar{\phi}_3 = \pi \\ &\text{No. 3} \quad {}_o\bar{A}_3 = \text{LO - stable - center point}, \quad {}_o\bar{\phi}_3 = 0 \end{aligned}\right\} \qquad (4.19)$$

Equation (4.17) can be integrated directly, to give

$$\bar{A}_3^{\,2}\left(w_1^2 + \mu \frac{3}{2} \bar{A}_1^{\,2} - \Omega'^2\right) + \mu \frac{3}{8} \bar{A}_3^{\,4} + \mu \frac{\bar{A}_1^{\,3} \bar{A}_3}{2} \cos \bar{\phi}_3 = \text{const} \qquad (4.20)$$

Figure 15 shows some typical integral curves for the case of ultraharmonic motion with three singularities.

For stable ultraharmonic oscillations, the initial conditions must lie inside region A B C D. If the initial conditions lie inside region A E F G oscillation will be about center No. 1, and if the initial conditions lie inside A B C D G F E the oscillations will be about center No. 3. The effect of introducing a small amount of damping into the problem will be to change the center into stable focal points and to prevent closure of the separatrix. It will still be true, however, that the separatrix will define two regions such that, if the initial conditions lie within one region the oscillations will spiral down to focal point No. 1, and if they lie inside the other region the oscillations will spiral down to focal point No. 3.

Thus the particular value of the steady state ultraharmonic will be completely determined once the initial conditions have been specified.

B) Subharmonic Motion.

Under suitable conditions a non-linear system may exhibit subharmonic oscillations, in which the main component of the motion has a period which is an integral multiple of the period of the forcing function. In the case of a cubic non-linearity, marked subharmonic, behavior has been observed experimentally when the frequency of the forcing function is increased to about three times the natural frequency of the system.

Following Mandelstam and Papolexi*, one is led to try a solution of the form

$$y = A_1 \cos \Omega t + A_{1/3} \cos \left(\frac{\Omega}{3} t + \phi_{1/3}\right) \qquad (4.21)$$

---

* L. Mandelstam and N. Papolexi, Z. Physik 73, 227 (1932).

where

$$\begin{cases} \Omega \approx 3w_1, \text{ hence } A_1 \approx \dfrac{P}{w_1^2 - \Omega^2} \approx -\dfrac{P}{8\,w_1^2} . \\ A_{1/3} \text{ and } \phi_{1/3} \text{ are slowly varying functions of time.} \end{cases}$$

Let

$$\Omega'' = \frac{\Omega}{3} \approx w_1 \tag{4.22}$$

Comparison with equation (2.1) shows that

$$\left.\begin{aligned} \mu f_1(\cdots) &= \mu y^3 + k\mathring{y} \\ \mu \Delta_1 &= P \end{aligned}\right\} \qquad \begin{aligned} \beta &= \phi_{1/3} \\ B &= A_{1/3} \end{aligned}$$

Hence, substitution into equations (2.2?) and (2.27) yields the first of equations (4.23). Similarly substitution in equations (2.32) and (2.31) yields the second of equations (4.23)

$$\left.\begin{aligned} 2\Omega''\mathring{A}_{1/3} + K\Omega'' A_{1/3} &= \mu\frac{3}{4}\bar{A}_{1/3}^{2}\ \bar{A}_{1/3}^{2}\ \bar{A}_1 \sin 3\bar{\phi}_{1/3} \\ 2\Omega'' A_{1/3}\mathring{\phi}_{1/3} &= (w_1^2 + \frac{3}{2}\mu\bar{A}_1^{\,2} - \Omega''^2)\,\bar{A}_{1/3} \\ &\quad + \mu\frac{3}{4}(\bar{A}_{1/3}^{3} + \bar{A}_{1/3}^{2}\ \bar{A}_1 \cos 3\bar{\phi}_{1/3}) \end{aligned}\right\} \tag{4.23}$$

<u>Steady State Subharmonic Motion</u>

The steady state subharmonic motion will be obtained from equation (4.23) by setting $\mathring{\bar{A}}_{1/3} = \mathring{\bar{\phi}}_{1/3} = 0$, whence

$$\begin{cases} K\Omega'' \, {}_o\bar{A}_{1/3} = \mu \frac{3}{4} \, {}_o\bar{A}_{1/3}^2 \, {}_o\bar{A}_1 \sin 3 \, {}_o\bar{\phi}_{1/3} \\ (w_1^2 + \frac{3}{2}\mu \, {}_o\bar{A}_1^{\,2} - \Omega''^2) {}_o\bar{A}_{1/3} + \mu \frac{3}{4} \, {}_o\bar{A}_{1/3}^3 = -\mu \frac{3}{4} \, {}_o\bar{A}_{1/3}^2 \, {}_o\bar{A}_1 \cos 3 \, {}_o\bar{\phi}_{1/3} \end{cases} \qquad (4.24)$$

eliminating ${}_o\bar{\phi}_{1/3}$ gives

$$(\tfrac{3}{4}\mu \, {}_o\bar{A}_{1/3}^2 \, {}_o\bar{A}_1)^2 = [(w_1^2 + \tfrac{3}{2}\mu \, {}_o\bar{A}_1^{\,2} - \Omega''^2) \, {}_o\bar{A}_{1/3} + \mu \tfrac{3}{4} \, {}_o\bar{A}_{1/3}^3]^2 + [K\Omega'' \, {}_o\bar{A}_{1/3}]^2 \qquad (4.25)$$

and if ${}_o\bar{A}_{1/3} \neq 0$

$$\sin 3 \, {}_o\bar{\phi}_{1/3} = \frac{\pm K\Omega''}{\sqrt{(K\Omega'')^2 + [(w_1^2 + \frac{3}{2}\mu \, {}_o\bar{A}_1^{\,2} - \Omega''^2) + \mu \frac{3}{4} \, {}_o\bar{A}_{1/3}^2]^2}} \qquad (4.26)$$

Loci of Vertical Tangency.

Viewed as a function of $\bar{A}_{1/3}$, the loci of vertical tangency are obtained from (4.25) by differentiation, they are

$$\bar{A}_{1/3} \Big\{ [(w_1^2 + \tfrac{3}{2}\mu \bar{A}_1^{\,2} - \Omega''^2) + \mu \tfrac{3}{4} \bar{A}_{1/3}^2] \, [(w_1^2 + \tfrac{3}{2}\mu \bar{A}_1^{\,2} - \Omega''^2) + \tfrac{9}{4} A_{1/3}^2] - 2(\tfrac{3}{4}\mu \bar{A}_{1/3} \bar{A}_1)^2 \Big\} = 0$$

From which $A_{1/3} = 0$, or

$$(K\Omega'')^2 + [w_1^2 + \tfrac{3}{2}\mu \bar{A}_1^{\,2} - \Omega''^2]^2 - (\mu \tfrac{3}{4} \bar{A}_{1/3}^2)^2 = 0 \qquad (4.27)$$

if K << 1 the second locus is given approximately by

$$\frac{3}{4}\ \bar{A}_{1/3}^{2} = \pm\ (w_1^2 + \frac{3}{2}\mu \bar{A}_1^{\ 2} - \Omega''^2)$$

Using the negative sign, it will be observed that if ${}_oA_{1/3}$ lies on the upper branch of the loci then

$$\sin 3\ {}_o\bar{\phi}_{1/3} = \pm 1\ ,\ \text{i.e.,}\ 3\ {}_o\phi_{1/3} = \frac{\pi}{2}\ ,\ \frac{3\pi}{2}\ ,\ \text{etc.} \qquad (4.28)$$

Stability of Steady State Solutions

Let

$$\bar{A}_{1/3} = {}_o\bar{A}_{1/3} + \xi$$

$$\bar{\phi}_{1/3} = {}_o\bar{\phi}_{1/3} + \eta$$

then writing perturbations on equation (4.24)

$$2\Omega''\mathring{\xi} + K\Omega''\xi = \frac{3}{2}\mu\ {}_o\bar{A}_{1/3}\ {}_o\bar{A}_1 \sin 3\ {}_o\phi_{1/3}\ \xi$$

$$+ \frac{9}{4}\mu\ {}_o\bar{A}_{1/3}^{2}\ {}_o\bar{A}_1 \cos 3\ {}_o\bar{\phi}_{1/3}\ \eta \qquad (4.29)$$

$$2\Omega''\ {}_o\bar{A}_{1/3}\mathring{\eta} = (w_1^2 - \Omega''^2 + \frac{3}{2}\mu\ {}_o\bar{A}_1^{\ 2})\xi - \mu\ \frac{9}{4}\ {}_o\bar{A}_{1/3}^{2}\ {}_o\bar{A}_1 \sin 3\ {}_o\bar{\phi}_{1/3}\ \eta$$

$$+ \mu\frac{3}{4}\ (3\ {}_o\bar{A}_{1/3}^{2} + 2\ {}_o\bar{A}_{1/2}\ {}_o\bar{A}_1 \cos 3\ {}_o\phi_{1/3})\ \xi \qquad (4.30)$$

Assuming solutions of the form $e^{\lambda t}$ and substituting for $\sin 3\ {}_o\phi_{1/3}$ and $\cos 3\ {}_o\phi_{1/3}$ from (4.24), gives the characteristic equation

$$(2\Omega''\lambda)^2 + (2\Omega''\lambda)\ (2K\Omega'') - 3(K\Omega)^2$$

$$+ 3\left\{ (\tfrac{3}{4}\mu\ {}_o\bar{A}_{1/3}^2)^2 - (w_1^2 - \Omega''^2 + \tfrac{3}{2}\mu\ {}_o\bar{A}_1^2)^2 \right\} = 0 \qquad (4.31)$$

For stability $K \geq 0$

$$(\tfrac{3}{4}\mu\ {}_oA_{1/3}^2)^2 - (w_1^2 - \Omega''^2 + \tfrac{3}{2}\mu\ {}_o\bar{A}_1^2)^2 - (K\Omega'')^2 \geq 0 \qquad (4.32)$$

It will be observed that this condition is simply the requirement that ${}_o\bar{A}_{1/3}$ lie outside the region enclosed by the loci of vertical tangency, as defined by equation (4.27).

From equations (4.29) and (4.30) it can also be shown that for stability

$$\bar{A}_1\ {}_o\bar{A}_{1/2} \cos 3\ {}_o\bar{\phi}_{1/3} \leq 0$$

but since $\bar{A}_1$ is negative and ${}_o\bar{A}_{1/3}$ is positive, it follows that $\cos 3\ {}_o\bar{\phi}_{1/3}$ is positive. Thus, those values of ${}_o\bar{\phi}_{1/3}$ are admissible, for stable subharmonics, for which $\sin 3\ {}_o\bar{\phi}_{1/3}$ is given by equation (4.26) and for which $\cos 3\ {}_o\bar{\phi}_{1/3}$ is positive.

If the damping is small, then except close to points of instability $\sin 3\ {}_o\bar{\phi}_{1/3} = 0$, from which

$${}_o\bar{\phi}_{1/3} = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Since $\cos 3\ {}_o\bar{\phi}_{1/3}$ must also be positive, ${}_o\bar{\phi}_{1/3}$ may take on only the values $0, 2\pi/3, 4\pi/3$.

Thus there exists three subharmonics with amplitude ${}_o\bar{A}_{1/3}$ but differing in phase by 120^o .

From equation (4.25) it will be observed that if ${}_o\bar{A}_{1/3} \neq 0$

$$\left(\frac{3}{4}\mu\, {}_o\bar{A}_{1/3}\, \bar{A}_1\right)^2 = (K\Omega'')^2 + \left[\left(w_1^2 + \frac{3}{2}\mu\, {}_o\bar{A}_1{}^2 - \Omega''^2\right) + \mu\frac{3}{4}\, {}_oA_{1/3}^2\right]^2$$

from which

$$\frac{3}{4}\quad {}_o\bar{A}_{1/3}^2 = -\left(w_1^2 + \frac{9}{8}\mu\, {}_o\bar{A}_1{}^2 - \Omega''^2\right) \pm \sqrt{\left\{\left(w_1^2 + \frac{9}{8}\mu\, {}_oA_1^2 - \Omega''^2\right)^2 - (K\Omega'')^2 - \left(w_1^2 + \frac{3}{2}\mu\, {}_oA_1^2 - \Omega''^2\right)^2\right\}}$$

$$= -\left(w_1^2 + \frac{9}{8}\mu\, {}_o\bar{A}_1{}^2 - \Omega''^2\right) \pm \sqrt{\left\{2\left(w_1^2 + \frac{21}{16}\mu\, {}_oA_1^2 - \Omega''^2\right) \times \left(-\frac{3}{8}\mu\, {}_o\bar{A}_1{}^2\right) - (K\Omega'')^2\right\}} \qquad (4.34)$$

for real ${}_o\bar{A}_{1/3}$

$$-\frac{3}{4}\mu\, {}_o\bar{A}_1{}^2\left(w_1^2 + \frac{21}{16}\mu\, {}_o\bar{A}_1{}^2 - \Omega''^2\right) - (K\Omega'')^2 \geq 0 .$$

This requires that

$$\Omega''^2 \geq \left(w_1^2 + \frac{21}{16}\mu\, {}_oA_1^2\right) \Big/ \left(1 - \frac{4}{3\mu\, {}_o\bar{A}_1{}^2}\,(K)^2\right) \qquad (4.35)$$

This places both a lower and an upper bound on the frequency, for

$${}_o\bar{A}_1 = -\frac{P}{w_1^2 - 9\Omega''^2}$$

If K << 1 the lower bound is

$$\Omega''^2 \geq w_1^2 + \frac{21}{16}\mu\, {}_oA_1^2 \tag{4.36}$$

The general features of amplitude frequency response for subharmonic of order 1/3 are indicated in Fig. 16. For any frequency between w' and w", three solutions exist to the amplitude equation, the two solutions corresponding to $|{}_o\bar{A}_{1/3}| = 0$, and LN, being stable while the third solution $|{}_oA_{1/3}| =$ LM is unstable. At points A and B, where the upper branch of the amplitude curve enters the region of instability, vertical jumps occur, both downwards. It will be observed that the general nature of subharmonics differ considerably from either harmonic or ultraharmonic motion.

Effect of Initial Conditions on Subharmonic of Order One Third.

To get a more complete picture of subharmonics it is necessary to examine the dependence of the steady state amplitude on the initial conditions. The easiest way to do this is by studying the integral curves for the motion, to this end equations (4.23) are written in the form:

$$\frac{d\bar{A}_{1/3}}{d\bar{\phi}_{1/3}} = \frac{\mu\ \frac{3}{4}\bar{A}_{1/3}^{3}\bar{A}_1 \sin 3\bar{\phi}_{1/3} - K\Omega''\bar{A}_{1/3}^{2}}{(w_1^2 + \frac{3}{2}\mu\bar{A}_1^{2} - \Omega''^2)\bar{A}_{1/3} + \mu\frac{3}{4}(\bar{A}_{1/3}^{3} + \bar{A}_{1/3}^{2}\bar{A}_1\cos 3\bar{\phi}_{1/3})} \tag{4.37}$$

If K = 0 then

FIGURE 16– TYPICAL AMPLITUDE/FREQUENCY RESPONSE FOR SUBHARMONICS OF ORDER ONE-THIRD

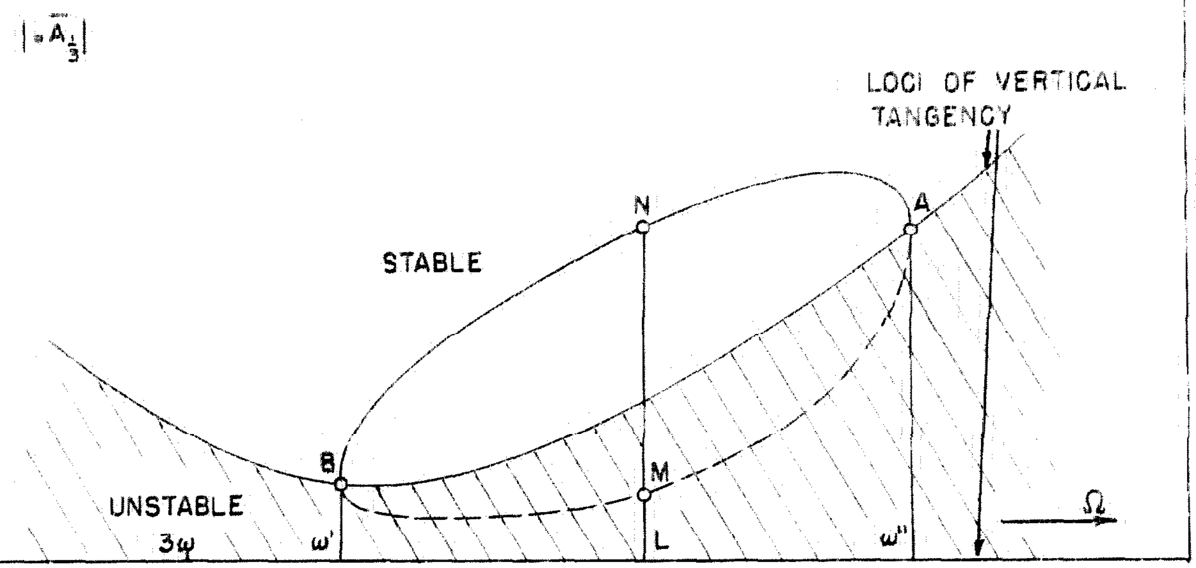
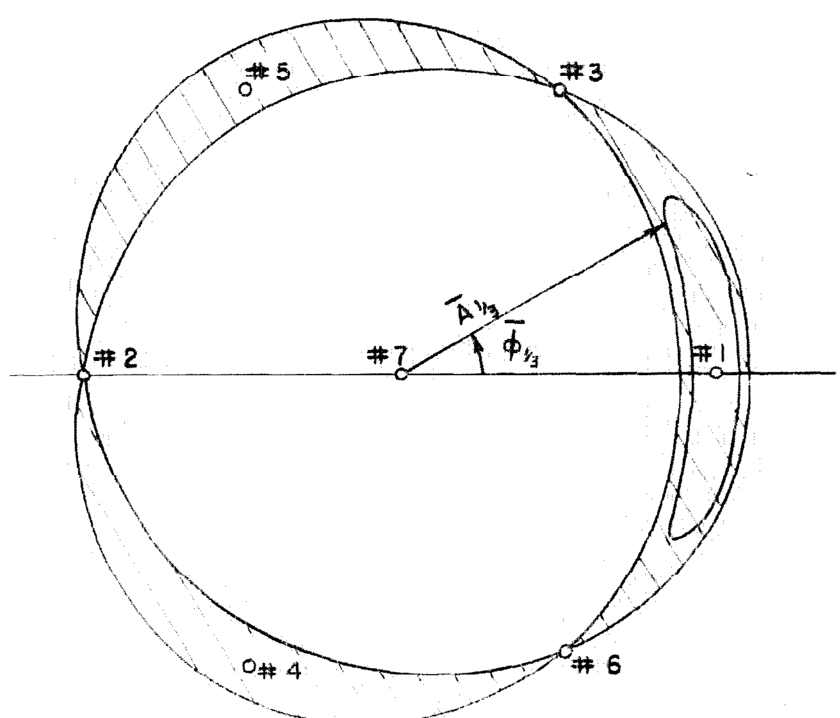


FIGURE 17– TYPICAL INTEGRAL CURVES FOR SUBHARMONIC OSCILLATIONS OF ORDER ONE-THIRD



$$\frac{d\bar{A}_{1/3}}{d\bar{\phi}_{1/3}} = \frac{\mu \frac{3}{4} \bar{A}_{1/3}^{3} \bar{A}_1 \sin 3 \bar{\phi}_{1/3}}{(w_1^2 + \frac{3}{2}\mu \bar{A}_1^{2} - \Omega''^2)\bar{A}_{1/3} + \mu \frac{3}{4}(\bar{A}_{1/3}^{3} + \bar{A}_{1/3}^{2} \bar{A}_1 \cos 3 \bar{\phi}_{1/3})} \quad (4.38)$$

If $w' < \Omega < w''$ in Fig. 17 then the amplitude ${}_o\bar{A}_{1/3}$ has three solutions, 0, LM, and LN while ${}_o\bar{\phi}_{1/3}$ has the values $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$. The nature of the singular points can be determined from equations (4.32) and (4.33).

Classification of Singular Points.

	${}_o\bar{A}_{1/3}$	${}_o\bar{\phi}_{1/3}$	Classification
No. 1	LN	0	center - stable
No. 2	LM	π	saddle point - unstable
No. 3	LM	$\frac{\pi}{3}$	saddle point - unstable
No. 4	LN	$\frac{4\pi}{3}$	center - stable
No. 5	LN	$\frac{2\pi}{3}$	center - stable
No. 6	LM	$\frac{5\pi}{3}$	saddle point - unstable
No. 7	0	indeterminate	center

Singularity No. 7 is phase unstable, but amplitude stable.

If damping is neglected equation (4.38) can be integrated to give

$$(w_1^2 + \frac{3}{2}\mu \bar{A}_1^{2} - \Omega''^2)\frac{1}{2}\bar{A}_{1/3}^{2} + \mu \frac{3}{4}(\frac{1}{4}\bar{A}_{1/3}^{4} + \frac{1}{3}\bar{A}_{1/3}^{3} A_1 \cos 3 \bar{\phi}_{1/3}) = \text{const} \quad (4.40)$$

Figure 17 shows the general feature of the integral curves for subharmonic motion of order one third, when damping is neglected. If the initial condition lies inside the shaded regions stable subharmonic motion will be obtained, outside these regions stable subharmonics cannot be obtained. This is quite different in nature from the case of harmonic and ultraharmonic motion in the fact that the second zone has shrunk down to zero.

Effect of Damping.

Qualitatively, the effect of adding a small amount of damping to the system will be to transform the centers into stable focal points and to prevent closure of the separatrix. However, it will still be true that the separatrix will divide the phase plane into four regions, inside three of which stable subharmonic will be obtained. For a quantitative discussion of a system with a cubic non-linearity with no linear term; see ref. 1.

5. CONCLUSIONS

The general first order theory presented in section 2 gives a method for treating forced oscillations in any multiperiodic quasi-linear system, it also gives a method for determining the stability of such motions, and in particular shows that points on the amplitude/frequency response curve having vertical tangents are likely to be points of instability. Detailed analysis of some fairly general systems revealed that a necessary and sufficient condition for the stability of forced harmonic oscillations is that solutions lie outside the region enclosed by the loci of vertical tangency. Thus points on the amplitude/frequency response curve having vertical tangents are indeed points of instability, a fact first observed experimentally by Appleton and Van der Pol in their classic work on the synchronization of a triode oscillator. Ultraharmonic and subharmonic oscillations were found to have the same type of stability criteria as harmonic oscillations; detailed analysis in this case was restricted to the discussion of a specific system.

From the general theory and from the detailed applications of the general theory the following conclusions can be drawn:

1) Harmonic Oscillations.

In a quasi-linear system with small damping, predominantly harmonic oscillation will exist close to the linear natural frequencies of the system, these oscillations will be stable provided the amplitude lies outside the region enclosed by the loci of vertical tangency, thus jump phenomena may result close to the natural frequencies of the system.

If the non-linearities are small, the amplitudes of oscillation may well be comparable in magnitude to those in a linear system; the effect of the non-linearity simply causing a shift in the resonances of the system. Unlike a linear system operating at constant exciting frequency, the steady state amplitudes in a non-linear system are dependent on the initial conditions. Fig. 10 shows such dependence in the case of a one degree of freedom system with zero damping.

2) Ultraharmonic oscillations.

Close to $\frac{1}{n}$ times the linear natural frequencies of the system, predominantly ultraharmonic motion of order n may exist. The stability of such motion again requires that the amplitude lie outside the region enclosed by the loci of vertical tangency. Detailed analysis of a one degree of freedom system with a cubic non-linearity shows that close to one third of the linear natural frequency marked ultraharmonic motion results, especially if the damping in the system is low, however, the amplitude of such motion is generally much smaller than that of the harmonic oscillation which occurs close to the linear resonant frequency. As in the case of harmonic forced oscillations, the steady state amplitude of ultraharmonic oscillation depends on the initial conditions.

3) Subharmonic Oscillation.

Close to n times the linear natural frequencies of the system, predominantly subharmonic motion of order $\frac{1}{n}$ may exist. The stability of such motion again requires that the amplitude lie outside the region enclosed by the loci of vertical tangency. Detailed analysis of a single degree of freedom system with a cubic non-linearity in the restoring force reveals that close to three times the linear natural frequency, marked subharmonic motion results, especially if the damping is small. Indeed, a study of the effect of damping, shows that subharmonics can occur only if the damping is small. The existence of subharmonics is very sensitive to the choice of initial conditions, which may explain why there has been so much misunderstanding on the subject of subharmonics.

Engineering Implications of Small Non-Linearities in Dynamic Systems.

As the detailed analysis of forced harmonic oscillations amply illustrates, the differences between linear and non-linear systems are differences at large, and are not dependent on the exact nature of the non-linearity. The existence of ultraharmonics, subharmonics and jump behavior depends on the existence of vertical tangents in the amplitude frequency response of the system and this does not depend on the exact nature of, the non-linearity. The introduction of damping into a non-linear system restricts the regions of vertical tangency and may place them outside the normal range of operation. In general it may be said that small non-linearities become important only if the damping in the system is small, and there exists the possibility of resonance.

Since most engineering problems involve components which are at best only quasi-linear, it might be thought that non-linear behavior should be of more frequent occurrence. The reason why this is not the case is that dynamic systems are usually designed for stability, and for this reason, are usually quite well damped. In such systems the loci of vertical tangency lie well outside the normal range of operation of the system, and the normal behavior of the system can be predicted, with sufficient accuracy, by a linearized theory. This is particularly true of control systems where the relatively large amounts of damping necessary for stability, effectively neutralize the effects of any small non-linearities. Here however, large scale non-linearities are very important, unfortunately large scale non-linearities do not lend themselves readily to analytic treatment, and little can be said in general about large non-linear effects.

Small non-linearities become very important in those systems where the damping is naturally small and where the introduction of further damping is undesirable if not impossible. There are many references to subharmonics and jump behavior in systems ranging from aircraft structures to three phase induction motors, where, from the nature of the system, the natural damping is small. Such effects are usually regarded as parasitic, and the engineer is usually concerned with means of controlling or preventing them. As in the case of linear systems, non-linear effects can be reduced or suppressed either by detuning or by increasing the damping. In most cases a small increase in the damping is sufficient to suppress subharmonic motion. Increasing the damping is not quite as effective in suppressing harmonic jump behavior; however, it is usually a better remedy than detuning the system.

There is no reason to regard non-linear effects as being entirely undesirable. In the past many ingenious uses have been made of non-linear effects, for example, the use of subharmonic frequency entrainment for frequency demultiplication in such diverse systems as oscilloscope time bases and quartz standard clocks, or the use of jump behavior to make a rugged, non-electronic flip-flop. It is felt that non-linear mechanics is still in its infancy and that the next decade will see very rapid strides both in the analysis and the synthesis of non-linear systems. This is especially true as applied to control problems, where the optimization of a given system requires the use of non-linear elements in the control computers (18,19,20,21). In theory, at any rate, it is possible to synthesize a control system to meet any desired set of integral criteria, and the problem of analysis of non-linearities is thereby avoided since the synthesized system is the best possible system that will meet the specified requirements. However, engineering design is dictated to a large extent by economic factors, and at present the cost of a synthesized control system is out of all proportion to the job it does, for this reason we are still faced with the problem of analyzing non-linear systems, which if not as good as the ideal system, are at least realizable economically.

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