Monopole Operators and Mirror Symmetry in Three-Dimensional Gauge Theories

Thesis by

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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

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Pasadena, California

2004
(Defended March 31, 2004)

Abstract

Many gauge theories in three dimensions flow to interacting conformal field theories in the infrared. We define a new class of local operators in these conformal field theories that are not polynomial in the fundamental fields and create topological disorder. They can be regarded as higher-dimensional analogs of twist and winding-state operators in free 2-D CFTs. We call them monopole operators for reasons explained in the text. The importance of monopole operators is that in the Higgs phase, they create Abrikosov-Nielsen-Olesen vortices. We study properties of these operators in three-dimensional gauge theories using large N_f expansion. For non-supersymmetric gauge theories we show that monopole operators belong to representations of the conformal group whose primaries have dimension of order N_f. We demonstrate that these monopole operators transform non-trivially under the flavor symmetry group.

We also consider topology-changing operators in the infrared limits of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric QED as well as $\mathcal{N} = 4$ $SU(2)$ gauge theory in three dimensions. Using large N_f expansion and operator-state isomorphism of the resulting superconformal field theories, we construct monopole operators that are primaries of short representation of the superconformal algebra and compute their charges under the global symmetries. Predictions of three-dimensional mirror symmetry for the quantum numbers of these monopole operators are verified. Furthermore, we argue that some of our large-N_f results are exact. This implies, in particular, that certain monopole operators in $\mathcal{N} = 4$ $d = 3$ SQED with $N_f = 1$ are free fields. This amounts to a proof of 3-D mirror symmetry in these special cases.

Acknowledgments

I want to express my gratitude to all the people who motivated me at various stages of my education. I am greatly indebted to Prof. Igor Tyutin, my advisor at Moscow Institute of Physics and Technology, for being a true mentor and a role model as a scientist. I have also learned a lot from Profs. Philip Argyres, S. -H. Henry Tye, and Eanna Flanagan during my enrollment at Cornell University.

Being a graduate student at Caltech, I have met many extraordinary people. I would like to thank all members of the Caltech theory group for providing friendly and stimulating environment. I have been fortunate to interact with Profs. John Schwarz, Edward Witten, John Preskill, Anton Kapustin, Steve Gubser, Hirosi Ooguri, and Mark Wise. I am especially thankful to my advisor Prof. Anton Kapustin for his guidance, motivation, and numerous illuminating meetings.

I have also benefited from discussions with graduate students Calin Ciocarlie, Jongwon Park, Alexander Putilin, Xinkai Wu, Yi Li, Takuya Okuda, Jonathan Tannenhauser, Harlan Robins, Peter Lee, Arkadash Ozakin, and post-docs Iouri Chepelev, Yuji Okawa, Andrei Mikhailov and Kirk Kaminsky.

Finally, I would like to thank my parents, Alexander and Nelly, as well as my brother Maxim for their unconditional support and encouragements throughout the years of my education. I am thankful to my grandfather Efim Britvin for having faith in me and sharing his wisdom and life experience. I am also much obliged to Julia Shifman for moral support and inspiration.

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Chapter 1

Introduction

One of the most fascinating problems in quantum field theory is understanding non-perturbative equivalences ("dualities") between superficially very different theories. A classic example is the quantum equivalence of the massive Thirring and sine-Gordon models [1]. The sine-Gordon model has topological solitons (kinks), and it can be shown that a certain local operator, which creates a kink, satisfies the equations of motion of the massive Thirring model [2].

More recently, a number of dualities has been conjectured for supersymmetric gauge theories in three and four dimensions. The earliest proposal of this kind is the S-duality of $\mathcal{N} = 4$ $d = 4$ super-Yang-Mills theory [3, 4, 5]. A decade and a half later, N. Seiberg proposed a dual description for the four-dimensional conformal field theory (CFT) that arises as the infrared (IR) limit of $\mathcal{N} = 1$ $d = 4$ super-QCD [6]. The dual theory is again the infrared limit of an $\mathcal{N} = 1$ $d = 4$ gauge theory. This proposal generated tremendous excitement, and soon many other candidate dualities have been found (see Refs. [7, 8] for a review). Later it was realized that many field-theoretic dualities arise from string theory dualities.

It is believed that many of these conjectural dualities have the same origin as the sine-Gordon/Thirring duality, i.e., they arise from "rewriting" a theory in terms of new fields that create topological disorder. But so far nobody has managed to prove a non-trivial higher-dimensional duality along the lines of Ref. [2]. The main reason for this is that the conjectured dualities in higher dimensions typically involve non-abelian gauge theories and are vastly more complicated than the sine-Gordon/Thirring duality. Usually, it is not even clear which solitons are "responsible" for the duality.

Until now, all dualities in dimensions higher than two remain conjectural, and the physical reasons for their existence are not completely understood.

A non-perturbative duality in three dimensions, known as 3-D mirror symmetry, has been proposed by K. Intriligator and N. Seiberg [9], and later studied by a number of authors [10]-[27]. The mirror symmetry predicts quantum equivalence of two different theories in the IR limit. In this regime a supersymmetric gauge theory is described by a strongly coupled superconformal field theory. The duality exchanges masses and Fayet-Iliopoulos terms as well as the Coulomb and Higgs branches implying that electrically charged particles in one theory correspond to the magnetically charged objects (monopoles) in the other. Also, since the Higgs branch does not receive quantum corrections and the Coulomb branch does, mirror symmetry exchanges classical effects in one theory with quantum effects in the dual theory. Many aspects of the three-dimensional mirror symmetry have a string theory origin.

Mirror symmetry in three dimensions has a number of special features that make it more amenable to study than other higher-dimensional dualities. First of all, mirror symmetry makes sense for abelian gauge theories, for which the complications due to the presence of unphysical degrees of freedom are not so severe. Second, it is known how to construct a mirror theory (in fact, many mirror theories [19]) for any abelian gauge theory [13, 19]. The mirror is always an abelian gauge theory, but usually with a different gauge group. Third, all mirror pairs can be derived from a certain "basic" mirror pair by formal manipulations [19]. This basic example identifies the infrared limit of $N_f = 1$ $\mathcal{N} = 4$ $d = 3$ SQED with a *free* theory of a twisted hypermultiplet. To prove this basic example of mirror symmetry, one only needs to construct a twisted hypermultiplet field out of the fields of $\mathcal{N} = 4$ SQED and show that it is free. Fourth, it is known what the relevant topological soliton is in this case: it is none other than the Abrikosov-Nielsen-Olesen vortex [15].

There are several related difficulties that one encounters in dimensions higher than two. First of all, interesting higher-dimensional dualities involve gauge theories. This implies that in order to write down an operator describing the dual degrees of freedom, one has to work in an enlarged state space which includes the unphysical degrees of freedom of both the original and the dual gauge fields. It is not known how to construct such an enlarged space. Fortunately, there are non-trivial examples of dualities in three dimensions [9] for some of which the dual theory has a trivial gauge group. In this case one can hope to construct the operators describing the dual degrees of freedom directly in the state space of the original gauge theory.

The second difficulty is that it is hard to construct topological disorder operators in interacting fields theories. For example, it is believed that three-dimensional mirror symmetry arises when one rewrites three-dimensional supersymmetric QED in terms of local operators that create Abrikosov-Nielsen-Olesen vortices [15]. This means that such operators are monopoles. However, it is not clear how to define monopole operators in SQED. A proposal in this direction was made by A. Kapustin and M. J. Strassler in Ref. [19], but it was only partially successful.

1.1 Gauge theory dynamics in three dimensions

The dimension of the gauge coupling e in 3-D gauge theory is $1/2$ in units of energy and, hence, gauge interactions are super-renormalizable. In the ultraviolet (UV) limit we have a free theory of abelian gauge fields and neutral matter fields. In fact, no renormalization of the gauge interactions is required in UV regime. Contrariwise, in the infrared regime the gauge theory is strongly coupled and is described by an interacting conformal field theory.

The difficulty of dealing with strongly coupled theories can be avoided by considering a limit of large number of flavors N_f. For large N_f fluctuations of the gauge field are suppressed and, to leading order in $1/N_f$, it can be treated as a classical background.

In the case of non-supersymmetric gauge theory with N_f flavors of charged matter it is natural to assume that the low-energy limit of a theory is described by a non-trivial CFT. In Refs. [28]-[30] it was demonstrated that three-dimensional gauge theories have severe perturbative infrared divergences due to logarithms of the coupling constant. In Refs. [31]-[32] it was shown that for three-dimensional QED and QCD, the $1/N_f$ expansion can be defined in such a way that the infrared divergences are absent in each order of the expansion and the theory has an IR fixed point.

The physics of three-dimensional non-supersymmetric gauge theories at finite N_f remains controversial. The conventional approach is to study a system of truncated Schwinger-Dyson equations and look for symmetry-breaking solutions. For simplicity, let us focus on the case of zero Chern-Simons coupling and even N_f. It has been claimed that in QED at finite N_f, flavor symmetry and parity are spontaneously broken by a dynamical mass for the fermions and the infrared limit is a theory of free photons [33]. The majority of such studies indicate that this happens for N_f smaller than a certain critical value of order 6 or 7 (see, for example, Refs. [34]-[38]). There are also claims that dynamical mass generation takes place for all N_f but is exponentially small for large N_f and therefore invisible in the $1/N_f$ expansion [33, 39, 40]. In QCD at large N_f the non-abelian interactions of gluons are suppressed and, the dynamics of the theory becomes similar to that of an abelian theory [32, 41]. It must be stressed that the results of such studies depend on the way one truncates an infinite system of Schwinger-Dyson equations, a procedure that cannot be fully justified. Lattice simulations of three-dimensional QED and QCD have been inconclusive so far.

The phase transition takes place at finite N_f and does not affect the dynamics at large N_f, which is studied in this manuscript. However, it indicates that the $1/N_f$ expansion has a finite radius of convergence. Note also that in the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric cases the situation is better, in the sense that one can argue for the existence of a non-trivial CFT at the origin of the quantum moduli space for all N_f. There is no evidence of the phase transitions at finite N_f. Hence, it is possible that the $1/N_f$ expansion is convergent all the way down to $N_f = 1$.

In the remainder of this introductory chapter we construct twist operators in two-dimensional conformal field theory.

In Chapter 2 we will consider topological disorder operators in three-dimensional QED with N_f flavors of fermions [42]. This theory is believed to flow to an interacting conformal fixed point for large enough N_f. The theory is not supersymmetric and is not expected to possess a simple dual. Nevertheless, it is a useful exercise to define monopole operators in this simple model and learn how to work with them. Besides, monopole operators are rather interesting objects even in the abelian non-supersymmetric case. First of all, these are the first examples of local operators in a three-dimensional CFT that are not polynomial in the fundamental fields. Thus, our construction can be regarded as a generalization of the vertex operator construction from free two-dimensional CFT to an interacting three-dimensional CFT. Second, we show that because of fermionic zero modes the monopole operators transform in a non-trivial representation of the flavor group, whose size depends on the Chern-Simons coupling.

We study monopole operators in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SQED [43] in Chapter 3. More precisely, we construct monopole operators in three-dimensional SCFTs that are the infrared limits of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SQEDs. We focus on operators that live in short multiplets of the superconformal algebra. The dimensions of primaries of such multiplets saturate a BPS-like bound, so that operators in short multiplets are referred to as BPS or chiral primary operators. Mirror symmetry makes predictions about the spectrum and other properties of BPS operators, including those with non-zero vortex charge. In Ref. [15] some of these predictions have been verified on the Coulomb branch of $\mathcal{N} = 2$ SQED, where the infrared theory is free.

In Chapter 4 the analysis is extended to the non-abelian gauge theories. We consider monopole operators in the IR limit of $SU(N_c)$ non-supersymmetric Yang-Mills theories as well as $\mathcal{N} = 4$ $SU(2)$ supersymmetric Yang-Mills models with a large number of flavors [44]. The conformal weight of a generic monopole operator in non-supersymmetric gauge theory is irrational. On the other hand, supersymmetric gauge theories have monopole operators that are superconformal chiral primaries. The conformal dimensions of such operators are uniquely determined by their R-symmetry representations. The R-symmetry group of $\mathcal{N} = 4$ supersymmetric theory is given by $SU(2) \times SU(2)$ and the conformal dimensions of the chiral primary operators are integral. The mirror symmetry predicts the spectrum and quantum numbers of chiral primary operators including the ones with magnetic charges. We use the $1/N_f$ expansion and the operator-state isomorphism of the resulting conformal field theories to study transformation properties of monopole operators under the global symmetries and verify the mirror symmetry predictions.

1.2 Twist operator in two dimensions

In this section we review a twist operator in a free scalar theory in two dimensions [45]. The twist operator is an example of an operator that is not polynomial in fundamental fields and creates topological disorder. Consider the conformally invariant action

$$S[X] = \frac{1}{2\pi} \int dz d\bar{z} \partial X \bar{\partial} X,$$

where $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$. The equation of motion $\delta S/\delta X = 0$ implies that X is given by a sum of holomorphic and anti-holomorphic functions:

$$X(z, \bar{z}) = \frac{1}{2} \left(X_L(z) + X_R(\bar{z}) \right).$$

The correlator $\langle X(z, \bar{z}) X(w, \bar{w}) \rangle$ is logarithmically divergent indicating that X is not a conformal field. On the other hand, its derivatives satisfy the operator product expansion (OPE)

$$\partial X_L(z)\partial X_L(w) \sim -\frac{1}{(z-w)^2} + \ldots,$$

and the finite part gives the stress-energy tensor

$$T(z) = -\frac{1}{2}\lim_{w\to z}\left(\partial X_L(z)\partial X_L(w) + \frac{1}{(z-w)^2}\right). \tag{1.1}$$

The OPE with the stress-energy tensor confirms that $\partial X_L(z)$ has conformal weight $(1,0)$:

$$T(z)\partial X_L(w) \sim \frac{1}{z-w}\partial X_L(w) + \ldots$$

The mode expansion of ∂X_L has the form

$$\partial X_L(z) = -i\sum_n \alpha_n z^{-n-1}, \quad [\alpha_n, \alpha_m] = n\delta_{n,-m}, \tag{1.2}$$

where summation over integer n corresponds to the non-twisted sector of the theory: ∂X_L is a single-valued operator in the complex plane, while summation over half-integer n introduces a branch cut and corresponds to a twisted sector denoted A. In the latter case, the operator ∂X is anti-periodic if analytically continued along a closed contour around the origin. It is worth mentioning that the second term in the defining equation (1.1) reflects renormalization of the stress-energy tensor and ensures that vacuum state in the non-twisted sector of a theory has vanishing energy. It is important that *exactly* the same expression for the stress-energy tensor is also used in a twisted sector of a theory. Thus, the renormalization procedure corresponds to the normal-ordering prescription in the non-twisted sector only.

A holomorphic twist operator $O(w)$ satisfies

$$\langle F_1\left[X_L(z_1)\right]\ldots F_k\left[X_L(z_k)\right]\rangle_A \equiv \left\langle O^+(\infty)F_1\left[X_L(z_1)\right]\ldots F_k\left[X_L(z_k)\right]O(0)\right\rangle, \tag{1.3}$$

for any local operators $F_1\left[X_L(z_1)\right]$, …, $F_k\left[X_L(z_k)\right]$, and the left-hand side of Eq.(1.3) is evaluated using the mode expansion (1.2) with half-integer n. Therefore, for a twist operator we have

$$\partial X_L(z) O(w) \sim \frac{1}{(z-w)^{1/2}} K(w) + \ldots,$$

with some local operator $K(w)$. In a twisted sector we have

$$\langle \partial X_L(z) \partial X_L(w) \rangle_A = -\frac{1}{2} \frac{\sqrt{z/w} + \sqrt{w/z}}{(z-w)^2}.$$

Therefore, the expectation value of the stress-energy tensor in the twisted sector is given by

$$\langle T(z) \rangle_A = \frac{1}{16 z^2},$$

implying that $O(z)$ is a conformal primary operator with conformal weight $\left(\frac{1}{16}, 0\right)$:

$$T(z) O(w) \sim \frac{1}{16\,(z-w)^2} O(w) + \ldots$$

An anti-holomorphic twist operator is defined in a similar way, and a twist operator that acts on both X_L and X_R is given by a product of the holomorphic and anti-holomorphic twist operators.

Chapter 2

Monopole operators in three-dimensional conformal field theory

2.1 Review of three-dimensional QED

The action of three-dimensional QED in the Euclidean space is given by

$$L_{QED} = \int d^3x \left(\frac{1}{4e^2} F_{ij} F^{ij} + i \sum_{s=1}^{N_f} \psi^{+s} \left(\sigma \cdot D \right) \psi^s \right),$$

where V is the $U(1)$ gauge field, $F = dV$ is the field-strength 2-form, D is the corresponding covariant derivative, and ψ^s is a complex two-component spinor. In three dimensions one can add to the action a Chern-Simons term

$$L_{CS} = \frac{i\kappa}{4\pi} \int d^3x \; \epsilon^{ijk} V_i \partial_j V_k.$$

Such a term breaks parity invariance of the theory. We will assume that the gauge group is compact, i.e., $U(1)$ rather than $\mathbb{R}$. Naively, this requires the Chern-Simons coupling κ to be an integer, to avoid global anomalies. The real story is slightly more complicated. When N_f is odd, the fermionic path integral is anomalous. The anomaly is the same as the anomaly due to a Chern-Simons term with $\kappa = 1/2$. Thus cancellation of global anomalies requires

$$\kappa - \frac{N_f}{2} \in \mathbb{Z}.$$

In particular, for odd N_f the Chern-Simons coupling must be non-zero, and parity is broken. This is known as parity anomaly [46].

In the limit $N_f \to \infty$ the infrared theory becomes weakly coupled, and conformal dimensions of all fields can be computed order by order in $1/N_f$. For example, the IR dimension of ψ^s is canonical, (i.e., the same as the UV dimension), up to corrections of order $1/N_f$.

More interestingly, the IR dimension of F_{ij} is 2 to all orders in $1/N_f$. To understand why this is the case, consider a current

$$J^i = \frac{1}{4\pi}\epsilon^{ijk}F_{jk}.$$

It is identically conserved by virtue of the Bianchi identity. A priori, this current could either be a primary field, or a descendant of the primary field. In the UV, the latter possibility is realized, since we can write

$$J^i = \partial^i \sigma, \tag{2.1}$$

where σ is a free scalar field. The scalar σ is usually referred to as the dual photon. It has dimension $1/2$ (as befits a free scalar in three dimensions), while J^i and F_{ij} have dimension $3/2$. On the other hand, in the IR an equation like Eq. (2.1) cannot hold. Indeed, Eq. (2.1) implies that F_{ij} obeys the free Maxwell equation, which clashes with the assumption that there are massless charged particles in the infrared. (We assume here that the fermions do not get a mass due to some non-perturbative effect, see the discussion in section 1.1.) This strongly suggests that in the IR limit J^i is a primary field.

It is well known that in a unitary 3-D CFT a conserved primary current has dimension 2. Hence the IR dimension of J^i and F_{ij} is 2. This conclusion can also be reached by directly studying the perturbative expansion in powers of $1/N_f$ [31].

Note that the difference between the UV and IR dimensions of F is of order 1, and therefore the IR fixed point is far from the UV fixed point, even in the limit $N_f \to \infty$. In this respect, the situation is very different from the Banks-Zaks-type theories in four-dimensions [47], where the IR dimensions of *all* operators are very close to their UV dimensions.

2.2 Defining monopole operators

2.2.1 A preliminary definition

As mentioned above, three-dimensional QED possesses an interesting conserved current, the dual of the field strength:

$$J^i = \frac{1}{4\pi}\epsilon^{ijk}F_{jk}.$$

Its conservation is equivalent to the Bianchi identity $dF = 0$. The corresponding charge is called the vortex charge, because in the Higgs phase it is carried by the Abrikosov-Nielsen-Olesen (ANO) vortices. The vortex charge is integral if the gauge field V is a well-defined connection on a $U(1)$ principal bundle. Loosely speaking, we would like to construct a vortex-creating operator. But in an interacting conformal field theory, it does not make sense to say that an operator is creating a particle. A vortex-creating operator will be defined as an operator with a unit vortex charge. This means that the OPE of such an operator with J^i has the form

$$J^i(x)O(0) \sim \frac{1}{4\pi}\frac{x^i}{|x|^3}O(0) + \text{less singular terms}.$$

Such operators can be organized in the representations of the conformal group. In a unitary theory local operators must transform according to lowest-weight representations, i.e., those representations in which the dimension of operators is bounded from below. The operator with the lowest dimension is called a conformal primary. It is standard to label a representation by the spin and dimension of its primary. Our problem can be formulated as follows: determine the spin, dimension, and other quantum numbers of primaries with a given vortex charge.

In the path integral language, an insertion of an operator with vortex charge q at a point p is equivalent to integrating over gauge fields which have a singularity at $x = p$ such that the magnetic flux through a 2-sphere surrounding $x = p$ is q. To be consistent, one must regard charged matter fields as sections of a non-trivial line bundle on the punctured R^3. Thus an insertion of a vortex-creating operator causes a change in the topology of fields near the insertion point. In what follows we will use the terms "vortex-creating operator" and "monopole operator" interchangeably.

This way of defining topological disorder operators is familiar from 2-D CFT. For example, a twist operator for a free boson in 2-D reviewed in section 1.2 is defined by the condition that the field changes sign as one goes around the insertion point [45]. Another example is provided by the theory of a periodic free boson in two dimensions. This theory has winding states, and the corresponding operators create a logarithmic singularity for the boson field. Thus our monopole operators can be regarded as three-dimensional analogues of twist operators or winding-state operators.

In the two-dimensional case one can loosely say that a winding-state operator creates a kink. The precise meaning of this statement is the following. Consider a perturbation of the free boson theory by a periodic potential, say, a sine-Gordon potential. The resulting massive theory has multiple vacua and topological excitations (kinks) interpolating between neighboring vacua. The operator which carries winding number one has non-zero matrix elements between the vacuum and the one-kink state.

Similarly, one can loosely say that a monopole operator creates an ANO vortex. To make this statement precise, one has to go to the Higgs phase (for example, by adding charged scalars with an appropriate potential). In the Higgs phase, the magnetic flux emanating from the insertion point of the monopole operator is squeezed into a thin tube. This tube is the world-line of a vortex.

2.2.2 A more precise definition

The definition of monopole operators given above is not yet complete. In effect, we have defined an insertion of a monopole operator by requiring that the gauge field strength have a particular singularity at the insertion point. However, we did not specify the behavior of the matter fields near the insertion point. In fact, we expect that there are many operators which carry the same vortex charge, and they differ precisely by the behavior of fields at the insertion point.

Another difficulty is that the IR theory is strongly coupled, and it seems hard to compute correlators involving monopole operators.

The first difficulty can be circumvented using radial quantization. It is a general feature of CFT in any dimension that local operators are in one-to-one correspondence with states in the Hilbert space of the radially quantized theory. This follows from the fact that one can use a conformal transformation to map an insertion point of an operator to infinity. In this way one trades a local operator for an incoming or outgoing state. In the case of monopole operators, such a mapping takes an operator with vortex charge q to a state on $\mathbf{S}^2 \times R$ with a magnetic flux q through $\mathbf{S}^2$. Here R is regarded as the time direction. Classifying states of a CFT on $\mathbf{S}^2 \times R$ with a given vortex charge is certainly a well-defined problem. Furthermore, the radially quantized picture is the most convenient one for computing correlators which involve two monopole operators with opposite vortex charges and an arbitrary number of ordinary operators.

By mapping the insertion of a monopole operator to an ingoing state and the insertion of an anti-monopole operator to an outgoing state, one reduces the problem to computing a particular matrix element of a product of several ordinary operators. A particularly important special case is the three-point function which involves a monopole operator, an anti-monopole operator, and a conserved current. Knowledge of such correlators allows one to read off the quantum numbers of a monopole operator. For example, in order to determine the dimension of an operator, one has to compute the expectation value of the stress-energy tensor in the corresponding state. This approach is familiar from 2-D CFT, where it is used to compute the quantum numbers of twist operators (see, e.g., Ref. [45]).

Of course, if one desires to compute four-point functions of monopole operators, mapping two of the insertion points to infinity does not help very much. In the case of 2-D CFT, one has to use tricks special to the theory in question in order to compute four-point functions of topological disorder operators. In this chapter, we will focus on studying the OPE of monopole operators with conserved currents.

The second difficulty can be avoided by working in the large N_f limit. It is a general feature of this limit that the gauge field does not fluctuate, and can be treated classically [28, 29, 31]. This can be seen as follows. The infrared limit in 3-D QED is simply the limit $e \to \infty$. This is literally true, because no renormalization of the Lagrangian is required. Thus one can simply drop the kinetic term for the gauge field. Integrating out the fermions then gives an effective action for the gauge field of order N_f. For example, when expanded around a trivial background, this action looks like

$$N_f \int \left(F_{ij}\, \Box^{-1/2} F^{ij} + \text{higher} - \text{order terms}\right) d^3x.$$

Thus the effective Planck constant is of order $1/N_f$, and in the large N_f limit the size of gauge-field fluctuations is order $1/N_f$. Moreover, if we absorb a factor of $N_f^{1/2}$ into F, we see that self-interactions of F are suppressed in the large N_f limit. In other words, $N_f^{1/2} F$ is a Gaussian field in the large N_f limit. It is this line of reasoning that allows one to show that the infrared CFT is weakly coupled in the large N_f limit.

The argument also applies to CFT on $\mathbf{S}^2 \times R$ with a flux. Thus we can regard the gauge field as a classical background. It is very plausible that the saddle point of the effective action for F on $\mathbf{S}^2 \times R$ is rotationally symmetric. Therefore we can assume that the classical background is simply a constant magnetic flux on $\mathbf{S}^2$.

The above discussion reduced our problem to computations with free fermions on $\mathbf{S}^2 \times R$ in the presence of a constant magnetic flux. Finding the dimension of a monopole operator is equivalent to computing the Casimir energy of free fermions on $\mathbf{S}^2$ with a flux. It is a priori clear that this energy scales like N_f. There are corrections to this result, which can be computed by taking into account the fluctuations of the gauge field. However, such effects are suppressed by powers of $1/N_f$.

The above discussion contains a gap as regards gauge invariance of monopole operators. Gauge-invariance of a local operator is equivalent to gauge-invariance of the corresponding state in the radially quantized picture. In other words, the state must satisfy the Gauss law. The Gauss law in QED on $\mathbf{S}^2 \times R$ comes from varying the action with respect to the "time-like" component of the gauge field A. In the limit $e \to \infty$ it simply reads

$$k(x)|\Phi\rangle = 0,$$

where

$$k(x) = \sum_s \psi^{+s}(x)\psi^s(x)$$

is the electric charge density operator. In particular, the total electric charge of a gauge-invariant state must be zero. The latter is a standard fact about gauge theory on a compact space, valid irrespective of the value of e. The definition of the electric charge operator involves normal-ordering ambiguities, which will be dealt with below. Note also that the inclusion of the Chern-Simons term in the action modifies the Gauss law constraint into

$$\left(k(x) + \frac{\kappa}{4\pi}\epsilon^{ij}F_{ij}(x)\right)|\Phi\rangle = 0.$$

In particular, the total electric charge of the matter modes must be equal to $-\kappa$ times the vortex charge. In this way (and only in this way) the Chern-Simons term will affect the physics at large N_f.

2.3 Properties of monopole operators

2.3.1 Radial quantization in the presence of a flux

As explained in the previous section, at large N_f all properties of monopole operators can be deduced from studying free fermions on $\mathbf{S}^2 \times R$ in a constant background magnetic flux. In this subsection we summarize the properties of this system, with detailed derivations relegated to the Appendix A.

The spectrum of the Dirac Hamiltonian on $\mathbf{S}^2 \times R$ with q units of magnetic flux is given by

$$E_p = \pm\sqrt{p^2 + p|q|}, \qquad p = 0, 1, 2, \ldots.$$

The degeneracy of the p-th eigenvalue is $2j_p + 1$, where

$$j_p = \frac{1}{2}(|q| - 1) + p.$$

These $2j_p + 1$ states transform as an irreducible representation of the rotation group $SU(2)_{rot}$.

The presence of q states with zero energy is particularly important. The existence of at least q zero modes is dictated by the Atiyah-Singer index theorem applied to the Dirac operator on $\mathbf{S}^2$ coupled to the magnetic field. In the case of a unit magnetic flux ($|q| = 1$), we have a single fermionic zero mode with zero spin. Thus a spinor is converted into a scalar due to the non-trivial topology of the magnetic monopole. This scalar-spinor transmutation is well known in other contexts; in particular it plays an important role in the conjectured S-duality of $\mathcal{N} = 4$ $d = 4$ supersymmetric Yang-

Mills theories. For general q, the fermionic zero modes transform in an irreducible representation of $SU(2)_{rot}$ with spin $j = (|q|-1)/2$. We will discuss in detail the case when $q = \pm 1$, and then comment on the higher-q case.

Let us denote the fermionic creation and annihilation operators by c^{+s}_{pm} and c^{s}_{pm} respectively, where $s = 1, \ldots, N_f$ is the flavor index, $p = 1, 2, \ldots,$ labels the energy eigenspaces as above, and $m = -j_p, -j_p + 1, \ldots, j_p$, labels the states within the p-th energy eigenspace. The fermion annihilation operators corresponding to $p = 0$ will be denoted simply by c^s_0. The Hilbert space of the theory is the tensor product of the Hilbert space of zero modes and the Hilbert space of all other modes. The latter is simply a fermionic Fock space with a unique vacuum $|vac\rangle_+$ which satisfies

$$c^s_{pm}|vac\rangle_+ = 0, \quad p > 0, \forall s, m.$$

This vacuum state is rotationally invariant. The Hilbert space of zero modes is also a Fock space of dimension 2^{N_f}, with the vacuum vector which we denote $|vac\rangle_0$. It is spanned by the vectors

$$|vac\rangle_0,\ c^{+s}_0|vac\rangle_0,\ c^{+s_1}_0 c^{+s_2}_0|vac\rangle_0,\ \ldots,\ c^{+s_1}_0 c^{+s_2}_0 \ldots c^{+s_{N_f}}_0|vac\rangle_0.$$

All these states are degenerate in energy, and none is a preferred vacuum. Since the zero modes have spin zero, all the ground states are rotationally invariant. We conclude that the radially-quantized theory of free fermions has a 2^{N_f}-fold degenerate ground state.

However, we still need to impose the Gauss law constraint. The charge density operator receives contributions from both zero and non-zero modes. The part due to non-zero modes can be defined using the obvious normal-ordering prescription. If we put all non-zero modes in the vacuum state, then the charge density due to non-zero modes vanishes. It remains to analyze the contribution from zero modes. Naively, it seems that the Fock vacuum must be assigned zero electric charge. Then the states obtained by acting on the vacuum with zero mode creation operators have positive charge and must be rejected. However, due to normal-ordering ambiguities, the situation is more interesting.

As stressed above, the Fock vacuum for the zero modes is not that special. The completely filled state appears to be an equally good candidate for a state with vanishing electric charge. The two just differ by a change in the normal ordering prescription. A statement which is independent of the normal-ordering prescription is that the electric charge of the filled state exceeds the charge of the vacuum by N_f. If one wants to be "democratic", one has to assign charge $-\frac{1}{2}N_f$ to the vacuum and charge $\frac{1}{2}N_f$ to the filled state. A similar symmetric charge assignment has been advocated by Jackiw and Rebbi in their pioneering study of fermions bound to solitons, on the grounds on charge-conjugation symmetry [48].

The precise argument for the symmetric charge assignment goes as follows. Charge conjugation maps a monopole to an anti-monopole and by itself does not tell us anything. But CP transformation maps a monopole to itself. If we want to quantize in a CP-invariant manner, we must assign opposite electric charges to states related by CP. Since CP takes annihilation operators into creation operators, the filled state and the vacuum are related by CP, and their electric charges must be opposite.

The invocation of CP invariance assumes that the theory we started with is CP-invariant. This means that the symmetric charge assignment is valid for a vanishing Chern-Simons coupling. But we know that turning on the Chern-Simons coupling κ is equivalent to shifting the electric charge by κ times the vortex charge. Therefore we conclude that in the presence of the Chern-Simons coupling the Fock vacuum has electric charge

$$-\frac{N_f}{2}+\kappa,$$

while the filled state has charge

$$\frac{N_f}{2}+\kappa.$$

Note that because of the parity anomaly, the electric charge is always integer-valued, whether N_f is even or odd. This a manifestation of the close relationship between the existence of parity anomaly and the induced vacuum charge [49].

Now we can analyze the consequences of the Gauss law constraint. If all non-zero modes are in their ground state, the constraint simply says that the total electric charge of the state must be zero. For $\kappa = 0$ it implies that a physical state is obtained by acting with $N_f/2$ zero modes on the vacuum. The number of such states is

$$\begin{pmatrix} N_f \\ \frac{1}{2}N_f \end{pmatrix},$$

and they transform as an anti-symmetric tensor of $SU(N_f)$ with $N_f/2$ indices. Note that cancellation of global anomalies requires N_f to be even when $\kappa = 0$, so this result makes sense. For κ between $-N_f/2$ and $N_f/2$ the physical states are obtained by acting with $N_f/2-\kappa$ zero modes on the vacuum. The corresponding states transform as an anti-symmetric tensor of $SU(N_f)$ with $N_f/2-\kappa$ indices. Again global anomaly cancellation ensures that $N_f/2-\kappa$ is an integer. For $|\kappa| > \frac{N_f}{2}$ there are no gauge-invariant states with unit vortex charge and all non-zero modes in their ground state. If one does not assume that positive-energy modes are in their ground state, then one can construct many other states which satisfy the Gauss law and have unit vortex charge. However, such states will have higher energy than the ones discussed above.

Now let us consider the more complicated case of $q = 2$. For simplicity we will set the Chern-Simons coupling to zero and take N_f to be even. In the case $q = 2$ each fermion has two zero modes which transform as a spin-$\frac{1}{2}$ representation of $SU(2)_{rot}$. Reasoning based on CP-invariance tell us that the Fock vacuum has electric charge $-N_f$. Physical states must have zero electric charge and are obtained by acting with N_f zero modes (out of a total number of $2N_f$) on the vacuum. But physical states must also be annihilated by the electric charge density operator. This is not automatic anymore, because the fermionic zero modes are not rotationally invariant. A short computation shows that the electric charge density operator for the zero modes $k_0(x)$ has a piece which transforms as a singlet of $SU(2)_{rot}$ and a piece which transforms as a triplet of $SU(2)_{rot}$. The former is simply the average of $k_0(x)$ over the sphere and is proportional to the total electric charge. The spin-triplet piece of $k_0(x)$ is proportional to the total spin, simply because this is the only spin-triplet one can make out of two spin-1/2 fermions. Thus the Gauss law constraint is equivalent to the requirement that the total electric charge as well as the total spin be zero.

For example, for $N_f = 2$, there are six states with zero total electric charge, which are obtained by acting on the Fock vacuum with two zero modes out of the available four. Three of these states transform as a vector of $SU(2)_{rot}$ and as a singlet of the flavor group $SU(2)_{flavor}$ and do not satisfy the Gauss law constraint. The remaining three transform as a singlet of $SU(2)_{rot}$ and as a triplet of $SU(2)_{flavor}$. These three states are gauge-invariant. Note that in this case the gauge-invariant states transform as an irreducible representation of the flavor group. For $N_f > 2$ this is no longer true, as one can easily check.

2.3.2 Quantum numbers of the monopole operators

In this section we determine the quantum numbers of the simplest monopole operators, the ones with the lowest conformal dimension for a given vortex charge. On general grounds, such an operator lives in a lowest-weight representation of the conformal group, and its conformal dimension is defined as the conformal dimension of the lowest-weight vector, or, if we pass to the radially quantized picture, as the energy of the corresponding state.

Let us begin with the case $q = 1$. As explained above, gauge-invariant states with lowest energy are obtained by putting all non-zero modes in their ground states and acting by $N_f/2 - \kappa$ zero mode creation operators on the vacuum. Obviously such states transform as an anti-symmetric representation of $SU(N_f)$ with $N_f/2 - \kappa$ indices. It is interesting to note that the usual gauge-invariant operators which are polynomials in the fundamental fields transform trivially under the center of $SU(N_f)$. Indeed, free fermions have flavor symmetry group $U(N_f)$, and since we are gauging its $U(1)$ subgroup, the flavor symmetry of QED appears to be $U(N_f)/U(1) = PU(N_f) = SU(N_f)/\mathbb{Z}_{N_f}$. But monopole operators transform non-trivially under $\mathbb{Z}_{N_f}$ (except for $\kappa = \pm N_f/2$). A very similar effect occurs in $\mathcal{N} = 2$ $d = 4$ supersymmetric QCD, where all perturbative states transform as tensor representations of the flavor group $SO(2N_f)$, while magnetically charge states transform as spinors [50].

Other quantum numbers of interest are spin and conformal dimension. Since the Fock vacuum and the zero modes are rotationally invariant, the spin of our monopole operator is zero. The dimension is proportional to the energy of the state. As usual, the definition of the energy is plagued by ordering ambiguities. However, we have a simple cure: we can normalize the energy by requiring that the unit operator have zero dimension. This means that the energy of the ground state on $\mathbf{S}^2$ with zero magnetic flux is defined to be zero. The energy of any other state can be defined by introducing a UV regulator, subtracting the regularized energy of the state corresponding to the unit operator, and then removing the regulator. This procedure gives a finite answer, which is not sensitive to the precise choice of the regulator, provided the regulator preserves the symmetries of the problem.

In order to make precise the relation between the Casimir energy and the dimension, recall that the OPE of a spin-zero primary field and the stress-energy tensor reads:

$$T_{ij}(x)O(y) \sim \frac{h}{8\pi}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\frac{1}{|x-y|}\right)O(y) + \ldots,$$

where h is the conformal dimension. If the stress-energy tensor of free fermions is defined by

$$T_{ij} = -\frac{i}{4}\sum_s \psi^{+s}\left(\sigma_i D_j + \sigma_j D_i\right)\psi^s + \frac{i}{4}\sum_s \left(\left(D_j\psi^s\right)^+\sigma_i + \left(D_i\psi^s\right)^+\sigma_j\right)\psi^s,$$

then $h_\psi = h_{\psi^+} = 1$, the standard normalization. This implies that in the radially-quantized picture the expectation value of the stress-energy tensor in the state $|O\rangle$ is given by

$$\langle T_{ij} dx^i \otimes dx^j \rangle_O = \frac{h}{4\pi} \left(d\tau^2 - \frac{1}{2}(d\theta^2 + \sin\theta^2 d\varphi^2) \right).$$

Thus h is simply the energy of $|O\rangle$ with respect to the Killing vector $\frac{\partial}{\partial \tau}$. In our case, this means that the conformal dimension of the monopole operator is the Casimir energy of N_f free fermions on $\mathbf{S}^2$ with a magnetic flux. This Casimir energy for any q is computed in the Appendix A. For $q = 1$ the result is

$$h_1 = N_f \cdot 0.265\ldots.$$

By charge-conjugation symmetry, the monopole operator with $q = -1$ has the same conformal dimension and spin and transforms in the conjugate representation of the flavor group $SU(N_f)$.

It is easy to extend the discussion to $q = \pm 2$. As explained in the previous section, the Gauss law constraint is equivalent to the requirement of zero spin and zero electric charge. The states with zero electric charge are obtained by acting with N_f zero modes (out of total number of $2N_f$ zero modes) on the Fock vacuum. These states transform as an anti-symmetric tensor of $SU(2N_f)$ with N_f indices. Gauge-invariant states are obtained by decomposing this representation with respect to the $SU(2)_{rot} \times SU(N_f)$ subgroup and separating out $SU(2)_{rot}$-singlets. In general, gauge-invariant states transform as a reducible representation of $SU(N_f)$. One can easily show that the dimension of this reducible representation is

$$\begin{pmatrix} \frac{1}{2}N_f^2 + N_f - 1 \\ \frac{1}{2}N_f \end{pmatrix}$$

The conformal dimension of the corresponding monopole operators is the Casimir energy of free fermions in a background magnetic field. Numerically, it is given by

$$h_2 = N_f \cdot 0.673\ldots.$$

It is interesting to note that $2h_1 < h_2$ (at least for large N_f). Therefore the OPE of two monopole operators with $q = 1$ and the lowest conformal dimension contains only terms with positive powers of $|x_1 - x_2|$.

In the next chapters we will consider monopole operators in 3-D gauge theories with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetry.

Chapter 3

Monopole operators and mirror symmetry in three-dimensional SQED

In the previous chapter, we showed how to define vortex-creating (or monopole) operators in the infrared limit of 3-D abelian gauge theories. The main tools used were radial quantization and large-N_f expansion. The only example considered was non-supersymmetric QED. In that theory monopole operators have irrational dimensions at large N_f and do not satisfy any simple equation of motion. In present chapter we will consider monopole operators in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SQEDs.

3.1 Monopole operators in three-dimensional $\mathcal{N} = 2$ SQED

3.1.1 Review of $\mathcal{N} = 2$ SQED and $\mathcal{N} = 2$ mirror symmetry

$\mathcal{N} = 2$ $d = 3$ SQED can be obtained by the dimensional reduction of $\mathcal{N} = 1$ $d = 4$ SQED. The supersymmetry algebra contains a complex spinor supercharge $\mathbb{Q}_\alpha$ and its complex-conjugate $\bar{\mathbb{Q}}_\alpha$. (In three dimensions it is not necessary to distinguish dotted and undotted indices on spinors. The two-dimensional spinor representation of the 3-

D Lorentz group $SO(1,2)$ is real.) The field content is the following: a vector multiplet with gauge group $U(1)$, N_f chiral multiplets of charge 1 and N_f chiral multiplets of charge -1. We will use $\mathcal{N}=2$ superspace to describe these fields. General superfields are functions of $x \in R^{2,1}$, a complex spinor θ^α, and its complex-conjugate $\bar{\theta}^\alpha$. The vector multiplet is described by a real superfield $V(x,\theta,\bar{\theta})$ satisfying $V^+ = V$. The corresponding field-strength multiplet is $\Sigma = \epsilon^{\alpha\beta} D_\alpha \bar{D}_\beta V$, where

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^i_{\alpha\beta}\bar{\theta}^\beta \frac{\partial}{\partial x^i}, \quad \bar{D}_\alpha = -\frac{\partial}{\partial\bar{\theta}^\alpha} - i\theta^\beta \sigma^i_{\beta\alpha}\frac{\partial}{\partial x^i}.$$

The lowest component of Σ is a real scalar χ, while its top component is the gauge field-strength F_{ij}. The vector multiplet also contains a complex spinor λ_α (photino). A chiral multiplet is described by a superfield $Q(x,\theta,\bar{\theta})$ satisfying the chirality constraint:

$$\bar{D}_\alpha Q = 0.$$

It contains a complex scalar A, a complex spinor ψ_α, and a complex auxiliary field F. We will denote the superfields describing charge 1 matter multiplets by Q^s, $s = 1,\ldots,N_f$, and the superfields describing charge -1 matter multiplets by $\tilde{Q}^s$, $s = 1,\ldots,N_f$. Then the action takes the form

$$S_{N=2} = \int d^3x\, d^4\theta \left\{ \frac{1}{4e^2}\Sigma^+\Sigma + \sum_{s=1}^{N_f} \left(Q^{+s} e^{2V} Q^s + \tilde{Q}^{+s} e^{-2V} \tilde{Q}^s \right) \right\}.$$

Besides being supersymmetric, this action has a global $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_N$ symmetry. The action of $SU(N_f) \times SU(N_f)$ is obvious (it is a remnant of the chiral flavor symmetry of $\mathcal{N} = 1$ $d = 4$ SQED). Under $U(1)_B$ the fields Q^s and $\tilde{Q}^s$ have charges 1, while V transforms trivially. Finally, there is an R-symmetry $U(1)_N$

under which the fields transform as follows:

$$Q^s(x,\theta,\bar\theta) \mapsto Q^s\left(x, e^{i\alpha}\theta, e^{-i\alpha}\bar\theta\right),$$
$$\tilde Q^s(x,\theta,\bar\theta) \mapsto \tilde Q^s\left(x, e^{i\alpha}\theta, e^{-i\alpha}\bar\theta\right),$$
$$V(x,\theta,\bar\theta) \mapsto V\left(x, e^{i\alpha}\theta, e^{-i\alpha}\bar\theta\right).$$

There is one other conserved current:

$$J^i = \frac{1}{4\pi}\epsilon^{ijk}F_{jk}.$$

Its conservation is equivalent to the Bianchi identity. We will call the corresponding charge the vortex charge, and the corresponding symmetry $U(1)_J$ symmetry. All the fundamental fields have zero vortex charge; our task will be to construct operators with non-zero vortex charge and compute their quantum numbers. Operators with non-zero vortex charge will be called monopole operators.

One can add an $\mathcal{N}=2$ Chern-Simons term to the action of $\mathcal{N}=2$ SQED. However, the theory is consistent without it, and we will limit ourselves to the case of vanishing Chern-Simons coupling.

$\mathcal{N}=2$ $d=3$ SQED is super-renormalizable and becomes free in the ultraviolet limit. In the infrared it flows to an interacting superconformal field theory (SCFT). Note that the action needs no counter-terms, if one uses a regularization preserving all the symmetries. Thus the infrared limit is equivalent to the limit $e\to\infty$.

In general, the infrared CFT is strongly coupled and quite hard to study. A simplification arises in the large N_f limit, where the infrared theory becomes approximately Gaussian. The reason for this is the same as in the non-supersymmetric case considered in the previous chapter. At leading order in the large N_f expansion, the matter fields retain their UV dimensions. The dimension of the gauge field strength multiplet Σ is 1 to all orders in $1/N_f$ expansion. This can be traced to the fact that the dual of the gauge field strength is an identically conserved current, as well as a primary field in the infrared SCFT.[1] A well-known theorem states that in a unitary CFT in d dimensions a conserved primary current has dimension $d-1$. Since the gauge field strength occurs as the top component of Σ, while θ and $\bar{\theta}$ have dimensions $-1/2$, the photino has infrared dimension $3/2$ and the lowest component χ has dimension 1.

The IR dimensions of Q and $\tilde{Q}$ can be computed order by order in $1/N_f$ expansion, but the exact answer for all N_f is unknown. The only other thing we know about these dimensions is that they are equal to the R-charges of Q and $\tilde{Q}$. This is a consequence of the fact that Q and $\tilde{Q}$ live in short representation of the superconformal algebra, and therefore their scaling dimensions are constrained by unitarity.[2] However, the R-current in question is not necessarily the one discussed above. Rather, it is some unknown linear combination of the $U(1)_N$ and $U(1)_B$ currents. We will call it the "infrared" R-current, to avoid confusion with $U(1)_N$ current defined above. In the large N_f limit it is easy to see that the infrared R-charge is

$$R_{IR} = N + B\left(\frac{1}{2} + O\left(\frac{1}{N_f}\right)\right),$$

where N and B are the charges corresponding to $U(1)_N$ and $U(1)_B$. For N_f of order 1 we do not know the coefficient in front of B, and so cannot easily determine the infrared dimensions of Q and $\tilde{Q}$.

For $N_f = 1$ mirror symmetry comes to our rescue. The statement of 3-D mirror symmetry in this case is that the IR limit of $\mathcal{N} = 2$ SQED is the same as the IR limit of another $\mathcal{N} = 2$ gauge theory. This other gauge theory has a gauge group $U(1)^{N_f}/U(1)_{diag}$, and $3N_f$ chiral matter multiplets $X^s, \tilde{X}^s, S^s$, $s = 1, \ldots, N_f$. The

[1]In the UV the dual of the field strength is not a primary, but a descendant of a scalar known as the dual photon.

[2]Strictly speaking, it is the dimension of gauge-invariant chiral primaries like $Q\tilde{Q}$ that is constrained by unitarity to be equal to the R-charge. However, since Q and $\tilde{Q}$ are chiral superfields, the dimension and R-charge of $Q\tilde{Q}$ is twice the dimension and R-charge of Q and $\tilde{Q}$, and the claimed result follows.

action of the mirror theory has the form

$$S_{dual} = \int d^3x\, d^4\theta \sum_{s=1}^{N_f} \left\{ \frac{1}{4e^2}\Sigma^{+s}\Sigma^s + \frac{1}{e^2}S^{+s}S^s + X^{+s}e^{2V^s-2V^{s-1}}X^s + \tilde{X}^{+s}e^{-2V^s+2V^{s-1}}\tilde{X}^s \right\} + \left(i\sqrt{2}\int d^3x\, d^2\theta \sum_{s=1}^{N_f} X^s\tilde{X}^sS^s + h.c. \right),$$

where the gauge multiplets satisfy the constraints

$$V^0 = V^{N_f}, \quad \sum_{s=1}^{N_f} V^s = 0. \tag{3.1}$$

Note that the chiral fields S^s are neutral with respect to the gauge group and couple to the rest of the theory only through a superpotential.

The mirror theory also flows to a strongly coupled SCFT in the infrared limit $e \to \infty$, and in general the mirror description does not help to compute the IR scaling dimensions in the original theory. However, the case $N_f = 1$ is very special: the mirror gauge group becomes trivial, and the mirror theory reduces to the Wess-Zumino model in three dimensions with the action

$$S_{WZ} = \int d^3x\, d^4\theta \left(X^+X + \tilde{X}^+\tilde{X} + S^+S\right) + \left(i\sqrt{2}\int d^3x\, d^2\theta\, X\tilde{X}S + h.c.\right).$$

This theory has "accidental" S_3 symmetry permuting $X, \tilde{X}$, and S, which allows one to determine their infrared R-charges. Indeed, since in the infrared limit the superpotential term must have R-charge 2, the R-charges of X,$\tilde{X}$ and S must be 2/3. The mirror map identifies S with the operator $Q\tilde{Q}$ in the original theory [15]. Thus we infer that for $N_f = 1$ Q and $\tilde{Q}$ have infrared R-charge 1/3. Comparing with large-N_f results, we see that the infrared R-charge has a non-trivial dependence on N_f.

Let us describe in more detail the matching of global symmetries between the original and mirror theories following Ref. [15]. The symmetry $U(1)_B$ of the original theory is mapped to the symmetry under which all S^s have charge 2, while X^s and $\tilde{X}^s$ have charges -1. The symmetry $U(1)_J$ is mapped to the $U(1)$ symmetry under which all X^s have charge $1/N_f$, all $\tilde{X}^s$ have charge $-1/N_f$, while S^s are uncharged. The R-symmetry $U(1)_N$ maps to an R-symmetry under which all X^s and $\tilde{X}^s$ have charge 1 and S^s are uncharged. The mapping of non-abelian symmetries is not well understood. It is only known that that the currents corresponding to the Cartan subalgebra of the diagonal $SU(N_f)$ are mapped to the $N_f - 1$ $U(1)_J$ currents of the mirror theory.

3.1.2 Monopole operators in $\mathcal{N} = 2$ SQED at large N_f

Our strategy for studying monopole operators will be the same as in Chapter 2. In any 3-D conformal field theory, there is a one-to-one map between local operators on R^3 and normalizable states of the same theory on $S^2 \times R$. Therefore we will look for states with non-zero vortex charge on $S^2 \times R$. In other words, we will be studying $\mathcal{N} = 2$ SQED on $S^2 \times R$ in the presence of a magnetic flux on S^2. Since our goal is to check the predictions of mirror symmetry, we will require that the states be annihilated by half of the supercharges; then the corresponding local operators will live in short representations of the superconformal algebra. The low-energy limit of $\mathcal{N} = 2$ SQED is an interacting SCFT, so in order to make computations possible, we will require N_f to be large. This has the effect of making the CFT weakly coupled. In particular, in the large N_f limit the fluctuations of the gauge field and its superpartners are suppressed, and one can treat them as a classical background. In other words, at leading order in $1/N_f$ we end up with free chiral superfields coupled to an appropriate background vector superfield. We will discuss how one can go beyond the large-N_f approximation in section 3.3.

The states on $S^2 \times R$ of interest to us are in some sense BPS-saturated, since they are annihilated by half of the supercharges. But in contrast to the situation in flat space, here the supercharges do not commute with the Hamiltonian $\mathbb{H}$ which generates translations on R. Indeed, since the Hamiltonian on $S^2 \times R$ is the same as the dilatation generator on R^3, and supercharges have dimension 1/2, it follows that the supercharges obey

$$[\mathbb{Q}_\alpha, \mathbb{H}] = -\frac{1}{2}\mathbb{Q}_\alpha, \quad [\bar{\mathbb{Q}}_\alpha, \mathbb{H}] = -\frac{1}{2}\bar{\mathbb{Q}}_\alpha. \tag{3.2}$$

Thus, although $\mathbb{Q}_\alpha$ and $\bar{\mathbb{Q}}_\alpha$ are conserved, they do not commute with the Hamiltonian $\mathbb{H}$. The reason is that supersymmetry transformations and, hence, supercharges on $S^2 \times R$ have explicit τ-dependence of the form $\exp(-\tau/2)$ which is consistent with Eq.(3.2).

The superconformal algebra arising in the IR limit has generators $\mathbb{S}_\alpha$ and $\bar{\mathbb{S}}_\alpha$ which are superpartners of the special conformal transformations $\mathbb{K}$:

$$[\mathbb{K}, \mathbb{Q}_\alpha] \sim \bar{\mathbb{S}}_\alpha, \quad \left[\mathbb{K}, \bar{\mathbb{Q}}_\alpha\right] \sim \mathbb{S}_\alpha.$$

Note also that in the radial quantization approach $\mathbb{Q}_\alpha$ and $\bar{\mathbb{Q}}_\alpha$ are no longer Hermitian conjugate of each other. Rather, their Hermitian conjugates are superconformal boosts $\mathbb{S}_\alpha$ and $\bar{\mathbb{S}}_\alpha$, which have dimension $-1/2$:

$$\mathbb{S}_\alpha = \mathbb{Q}_\alpha^+, \quad \bar{\mathbb{S}}_\alpha = \bar{\mathbb{Q}}_\alpha^+, \tag{3.3}$$

$$[\mathbb{S}_\alpha, \mathbb{H}] = \frac{1}{2}\mathbb{S}_\alpha, \quad [\bar{\mathbb{S}}_\alpha, \mathbb{H}] = \frac{1}{2}\bar{\mathbb{S}}_\alpha.$$

For the same reasons as in Chapter 2, in the large N_f limit the energy E of the states with non-zero vortex charge is of order N_f. By unitarity, for scalar states E is bounded from below by the R-charge R_{IR}. Furthermore, we will see below that in the limit $N_f \to \infty$ R_{IR} is also of order N_f, while the combination $E - R_{IR}$ stays finite for all the states we encounter. A similar limit in $d = 4$ SCFTs recently gained some prominence in connection with AdS/CFT correspondence [51]. But unlike Ref. [51], we take the number of flavors, rather than the number of colors, to infinity.

First let us determine which classical background on $S^2 \times R$ we need to consider. As in Chapter 2, we have a gauge field on $S^2 \times R$ with a magnetic flux q. Assuming rotational invariance of the large-N_f saddle point, this implies that we have a constant magnetic field on S^2. The only other bosonic field in the $\mathcal{N} = 2$ vector multiplet is the real scalar χ. It is determined by the condition of the vanishing of the photino variation under half of the SUSY transformations. This will ensure that the monopole operator we are constructing is a chiral primary.

It is convenient to work out the photino variations on R^3, and then make a conformal transformation to $S^2 \times R$. Photino variations in Euclidean $\mathcal{N} = 2$ SQED on R^3 have the form

$$
\begin{aligned}
\delta\lambda &= i\left(-\sigma^i\partial_i\chi - \frac{1}{2}\epsilon^{ijk}\sigma^k F_{ij} + D\right)\xi, \\
\delta\bar{\lambda} &= i\bar{\xi}\left(-\sigma^i\partial_i\chi + \frac{1}{2}\epsilon^{ijk}\sigma^k F_{ij} - D\right),
\end{aligned}
$$

where ξ and $\bar{\xi}$ are complex spinors which parameterize SUSY variations. (In Euclidean signature, they are not related by complex conjugation.) Since we are setting the background values of the matter fields to zero, the D-term can be dropped. Half-BPS states are annihilated by $\bar{\xi}_\alpha\bar{\mathbb{Q}}^\alpha$ for any $\bar{\xi}$ and therefore must satisfy

$$
F_{ij} = \epsilon_{ijk}\partial^k\chi.
$$

Hence the scalar background on R^3 is

$$
\chi = -\frac{q}{2r},
$$

where q is the vortex charge (the magnetic charge of the Dirac monopole on R^3). Unsurprisingly, supersymmetry requires the bosonic field configuration to be an abelian BPS monopole. Recalling that χ has dimension 1 in the infrared, we infer that on S^2 the scalar background is simply a constant:

$$\chi = -\frac{q}{2}.$$

Similarly, an anti-BPS state is annihilated by $\xi_\alpha \mathbb{Q}^\alpha$ for any ξ, and therefore the scalar field on S^2 is

$$\chi = \frac{q}{2}.$$

Having fixed the classical background, we are ready to compute the spectrum of matter field fluctuations. The details of the computation are explained in the Appendix B. The results are as follows. The energy spectra of charged scalars are the same for both A^s and $\tilde{A}^s$, do not depend on whether one is dealing with a BPS or an anti-BPS configuration, and are given by

$$E = E_p^+ = \left(\frac{|q|-1}{2} + p\right), \quad p = 1, 2, \ldots,$$
$$E = E_p^- = -\left(\frac{|q|-1}{2} + p\right), \quad p = 1, 2, \ldots.$$

The degeneracy of the p^{th} eigenvalue is $2|E_p|$, and the corresponding eigenfunctions transform as an irreducible representation of the rotation group $SU(2)_{rot}$. The spectrum is symmetric with respect to $E \to -E$. The energy spectra of charged spinors are the same for ψ^s and $\tilde{\psi}^s$ and are given by

$$E = E_p^+ = \frac{|q|}{2} + p, \quad p = 1, 2, \ldots,$$
$$E = E_p^- = -\frac{|q|}{2} - p, \quad p = 1, 2, \ldots,$$
$$E = E_0 = \mp\frac{|q|}{2}.$$

Here the upper (lower) sign refers to the BPS (anti-BPS) configuration[3]. The eigenspace with eigenvalue E has degeneracy $2|E|$ and furnishes an irreducible representation of $SU(2)_{rot}$.

Comparing the fermionic energy spectrum with the results of Chapter 2, we see that the inclusion of the scalar χ causes dramatic changes in the spectrum of fermions. First, there are no zero modes. Second, the spectrum is not symmetric with respect to $E \to -E$.

The absence of zero modes, either in the scalar or in the spinor sector, means that for a fixed magnetic flux the state of lowest energy is unique. We will call it the vacuum state. By construction, it is an (anti-)BPS state, and we would like to determine its quantum numbers. It is clear that the vacuum state is rotationally invariant, so its spin is zero. It is also a flavor singlet. The other quantum numbers of interest are the energy (which is the same as the conformal dimension of the corresponding local operator) and the $U(1)_B$ and $U(1)_N$ charges. Vacuum energy and charge are plagued by normal-ordering ambiguities, as usual, but as in Chapter 2 we can deal with them by requiring the state corresponding to the unit operator (i.e., the vacuum with zero magnetic flux) to have zero energy and charges.

The asymmetry of the fermionic energy spectra leads to a subtlety in the computation. Suppose we use point-splitting regularization to define vacuum energy and charges. Then one gets different results after renormalization depending on the ordering of operators ψ and $\bar{\psi}$. For example, consider two definitions of the $U(1)_N$

[3]We will use this convention throughout the manuscript.

charge

$$
\begin{aligned}
N(\tau) = \lim_{\beta\to 0+} &\left[\int_{S^2} -\bar{\psi}\left(\tau+\frac{\beta}{2}\right)\sigma_\tau\psi\left(\tau-\frac{\beta}{2}\right) - \bar{\tilde{\psi}}\left(\tau+\frac{\beta}{2}\right)\sigma_\tau\tilde{\psi}\left(\tau-\frac{\beta}{2}\right)\right. \\
&\left. -C(\beta)\right], \\
N'(\tau) = \lim_{\beta\to 0+} &\left[\int_{S^2} \psi\left(\tau+\frac{\beta}{2}\right)\sigma_\tau\bar{\psi}(\tau-\frac{\beta}{2}) + \tilde{\psi}\left(\tau+\frac{\beta}{2}\right)\sigma_\tau\bar{\tilde{\psi}}\left(\tau-\frac{\beta}{2}\right)\right. \\
&\left. -C'(\beta)\right],
\end{aligned}
$$

where τ is the time coordinate on $S^2 \times R$, and $C(\beta)$ and $C'(\beta)$ are c-numbers defined as the $U(1)_N$ charge of the vacuum with $q = 0$ regularized by means of appropriate point-splitting. One can easily see that these two definitions are equivalent only if the fermion spectra are symmetric with respect to zero; otherwise they differ by a c-number which depends on q. This ambiguity can be removed by requiring that the regularization procedure preserve charge-conjugation symmetry. This mandates using expressions symmetrized with respect to ψ and $\bar{\psi}$ (and $\tilde{\psi}$ and $\bar{\tilde{\psi}}$):

$$
\bar{\psi}O\psi \to \frac{1}{2}\bar{\psi}O\psi - \frac{1}{2}\psi O^T\bar{\psi}, \tag{3.4}
$$

where O is some operator independent of the fields. Thus we will define the $U(1)_N$ charge as the average of $N(\tau)$ and $N'(\tau)$. The same applies to the $U(1)_B$ charge and the energy operator.

As an illustration, let us compute the $U(1)_N$ charge of the vacuum for arbitrary q. The above definition yields the following regularized $U(1)_N$ charge:

$$
N_{reg}(\beta) = N_f \sum_E 2|E|\,\mathrm{sign}(E)\,e^{-\beta|E|}. \tag{3.5}
$$

Here the summation extends over the fermion energy spectrum, and we took into account that ψ and $\tilde{\psi}$ have the same energy spectra and $U(1)_N$ charge and contribute equally to $N_{reg}(\beta)$. The regularized charge of the unit operator is identically zero, since the spectrum is symmetric for $q = 0$. For non-zero vortex charge the spectrum is symmetric except for a single eigenvalue E_0. Thus the renormalized charge is equal to

$$N_{vac} = \pm \lim_{\beta \to 0+} N_f |q| = \pm N_f |q|,$$

where the upper (lower) sign refers to the BPS (anti-BPS) state. Since the spectrum of scalars is symmetric, only spinors will contribute to the $U(1)_B$ charge of the vacuum, and an identical argument gives

$$B_{vac} = \mp N_f |q|.$$

A similar computation performed in Appendix B, Eq.(B-1), gives the vacuum energy:

$$E = \frac{|q| N_f}{2}.$$

This is the same as the scaling dimension of the corresponding monopole operator. We note that vacuum energy of a true vacuum state, i.e., vacuum with zero vortex charge q, vanishes identically and does not require any renormalization. Standard argumentation given for supersymmetric theories in 3-D Minkowski space is that the vacuum state is invariant under supersymmetry transformations generated by $\mathbb{Q}$ and $\bar{\mathbb{Q}}$. Hence, an anticommutator $\left\{\mathbb{Q}, \bar{\mathbb{Q}}\right\} \sim \mathbb{P}_i$ and identity $\mathbb{Q}^+ = \bar{\mathbb{Q}}$ imply that the vacuum state has vanishing energy. In the radial quantization picture, however, generators of translations $\mathbb{P}_0$ in Minkowski space does not play a role of the Hamiltonian and hermitian conjugation operation is realized differently (3.3). The relevant anticommutators in the radially quantized theory are $\{\mathbb{Q}, \mathbb{S}\}$ and $\left\{\bar{\mathbb{Q}}, \bar{\mathbb{S}}\right\}$. Unitarity constraints applied to these anticommutators imply that a rotationally invariant state annihilated by all supercharges must have vanishing energy [52].

Recall that at large N_f the R-charge which is the superpartner of the Hamiltonian is given by

$$R_{IR} = N + \frac{1}{2}B.$$

It is easy to see from the above results that $E = \pm R_{IR}$ for our "vacuum" states. This is a satisfying result, since in a unitary 3-D CFT the scaling dimension of any (anti-) chiral primary must be equal to (minus) its R-charge.

As expected, the energy and the R-charge of the vacuum are of order N_f. Other states can be obtained by acting on the vacuum with a finite number of creation operators for the charged fields. If the number of creation operators is kept fixed in the limit of large N_f, then both E and R_{IR} tend to infinity, with $E - R_{IR}$ kept finite. Thus the limit we are considering is qualitatively similar to the PP-wave limit of $\mathcal{N} = 4$ $d = 4$ SYM theory considered in Ref. [51]. But since we are taking the number of flavors, rather than the number of colors, to infinity, the physics is rather different. For example, in Ref. [51] the combination R^2/N_c is kept fixed and can be an arbitrary positive real number (it is the effective string coupling in the dual string theory). The analogous quantity in our case is $2R_{IR}/N_f = |q|$, the vortex charge, which is quantized.

One issue which we have not mentioned yet is gauge-invariance. In order for the operator to be gauge-invariant, the corresponding state must satisfy the Gauss law constraint. In the limit $e \to \infty$ this is equivalent to requiring that the state be annihilated by the electric charge density operator. For the vacuum state, this is automatic. For excited states, the Gauss law constraint is a non-trivial requirement.

We have identified above a scalar state on $S^2 \times R$ which is a chiral primary. What about its superpartners? The key point is to realize that the classical field configuration we are considering breaks some of the symmetries of the CFT. In such a situation, one must enlarge the Hilbert space by extra variables ("zero modes") which correspond to the broken generators. In other words, the semi-classical Hilbert space is obtained by tensoring the "naive" Hilbert space by the space of functions on the coset G/H, where G is the symmetry group of the theory, and H is the invariance subgroup of the classical configuration. This observation plays an important role in the quantization of solitons. For example, if we are dealing with a soliton in a Poincaré-invariant theory which breaks translational symmetry to nothing, but preserves rotational symmetry, the zero mode Hilbert space is

$$ISO(d-1,1)/SO(d-1,1) = R^{d-1,1}.$$

Poincaré group acts on the space of functions on $R^{d-1,1}$ in the usual manner. Furthermore, if a soliton breaks some of supersymmetries, there will be fermionic zero modes, and the bosonic coset must be replaced by an appropriate supercoset.

In our case, the symmetry of theory is described by the $\mathcal{N}=2$ $d=3$ super-Poincaré group.[4] For the BPS state, the invariance subgroup is generated by rotations and the complex supercharge $\bar{\mathbb{Q}}_\alpha$. Thus the zero mode Hilbert space will consist of functions on the supercoset

$$\frac{\{\mathbb{M}_{ij}, \mathbb{P}_i, \mathbb{Q}_\alpha, \bar{\mathbb{Q}}_\alpha\}}{\{\mathbb{M}_{ij}, \bar{\mathbb{Q}}_\alpha\}},$$

where $\mathbb{M}_{ij}$ and $\mathbb{P}_i$ are the rotation and translation generators on R^3, respectively, and $\{A, B, \ldots\}$ denotes the super-group with Lie super-algebra spanned by $A, B, \ldots$. Functions on this supercoset are nothing but $\mathcal{N}=2$ $d=3$ chiral superfields [53]. Thus the usual rules of semi-classical quantization lead to the conclusion that the BPS monopole operator is described by a chiral superfield. Similarly, an anti-BPS monopole operator will be described by an anti-chiral superfield. In particular, $\mathcal{N}=2$ auxiliary fields are automatically incorporated. (Note that at large N_f our monopole operators are not expected to satisfy any closed equation of motion. On the other hand, auxiliary fields can be eliminated only on-shell. This suggests that any descrip-

[4] We may forget about $U(1)_N$, $U(1)_B$, and the flavor symmetry, since they are left unbroken by our field configuration. Furthermore, although conformal and superconformal boosts do not preserve our field configuration, they can be ignored, since these symmetry generators cannot be exponentiated to well-defined symmetry transformations on R^3.

tion of monopole operators without auxiliary fields would be rather cumbersome.)

3.1.3 A comparison with the predictions of $\mathcal{N} = 2$ mirror symmetry

As explained above, under mirror symmetry the vortex charge is mapped to $1/N_f$ times the charge which "counts" the number of X's minus the number of $\tilde{X}$'s. Thus the obvious gauge-invariant chiral primaries with vortex charge ± 1 are

$$V_+ = X^1 X^2 \ldots X^{N_f}, \quad V_- = \tilde{X}^1 \tilde{X}^2 \ldots \tilde{X}^{N_f}.$$

Using the matching of global symmetries explained above, we see that both V_+ and V_- are singlets under $SU(N_f) \times SU(N_f)$ flavor symmetry, have $U(1)_B$ charge $-N_f$ and $U(1)_N$ charge N_f. Comparing this with the previous subsection, we see that V_+ has the same quantum numbers as the BPS state with $q = 1$ that we have found, while $V_-^\dagger$ has the same quantum numbers as the anti-BPS state with $q = 1$. This agreement provides a non-trivial check of $\mathcal{N} = 2$ mirror symmetry.

Our computation of the charges was performed in the large-N_f limit, but mirror symmetry predicts that the result remains true for N_f of order 1. Can we understand this apparent lack of $1/N_f$ corrections to $U(1)_N$ and $U(1)_B$ charges? The answer is yes: $U(1)_N$ and $U(1)_B$ charges are not corrected at any order in $1/N_f$ expansion because they can be determined by quasi-topological considerations (L^2 index theorem on $S^2 \times R$). This will be discussed in more detail in section 3.3.

3.2 Monopole operators in three-dimensional $\mathcal{N} = 4$ SQED

3.2.1 Review of $\mathcal{N} = 4$ SQED and $\mathcal{N} = 4$ mirror symmetry

$\mathcal{N} = 4$ $d = 3$ SQED is the dimensional reduction of $\mathcal{N} = 2$ $d = 4$ SQED. The supersymmetry algebra includes two complex spinor supercharges $\mathbb{Q}^I_\alpha$, $I = 1, 2$ and their complex conjugates. In Minkowski signature, the spinor representation is real, so we may also say that we have four real spinor supercharges. If we regard $\mathcal{N} = 4$ SQED as an $\mathcal{N} = 2$ $d = 3$ gauge theory, then it contains, besides the fields of $\mathcal{N} = 2$ SQED, a chiral superfield Φ. This superfield is neutral and together with the $\mathcal{N} = 2$ vector multiplet V forms an $\mathcal{N} = 4$ vector multiplet. The chiral superfields Q^s and $\tilde{Q}^{+s}$ combine into an $\mathcal{N} = 4$ hypermultiplet. The action of $\mathcal{N} = 4$ SQED is the sum of the action of $\mathcal{N} = 2$ SQED, the usual kinetic term for Φ, and a superpotential term

$$i\sqrt{2} \int d^3x\, d^2\theta \sum_{s=1}^{N_f} Q^s \Phi \tilde{Q}^s + h.c.$$

The flavor symmetry of this theory is $SU(N_f)$. In addition, there is an important R-symmetry $SU(2)_R \times SU(2)_N$. In the $\mathcal{N} = 2$ superfield formalism used above, only its maximal torus $U(1)^2$ is manifest. The lowest components of Q and $\tilde{Q}^+$ are singlets under $SU(2)_N$ and transform as a doublet under $SU(2)_R$. The complex scalar Φ in the chiral multiplet and the real scalar χ in the $\mathcal{N} = 2$ vector multiplet transform as a triplet of $SU(2)_N$ and are singlets of $SU(2)_R$. The transformation properties of other fields can be inferred from these using the fact that the four real spinor supercharges of $\mathcal{N} = 4$ SQED transform in the $(2, 2)$ representation of $SU(2)_R \times SU(2)_N$.

Although there is a complete symmetry between $SU(2)_R$ and $SU(2)_N$ at the level of superalgebra, the transformation properties of fields do not respect this symmetry. Therefore one can define twisted vector multiplets and twisted hypermultiplets for which the roles of $SU(2)_N$ and $SU(2)_R$ are reversed. $\mathcal{N} = 4$ SQED contains only "ordinary" vector and hypermultiplets, while its mirror (see below) contains only twisted multiplets. There are interesting $\mathcal{N} = 4$ theories in 3-D which include both kinds of multiplets [19, 54], but in this chapter we will only consider the traditional ones, which can be obtained by dimensional reduction from $\mathcal{N} = 2$ $d = 4$ theories.

In order to make contact with our discussion of $\mathcal{N} = 2$ SQED, we will denote the global $U(1)$ symmetry under which Q and $\tilde{Q}$ have charge 1 and Φ has charge -2 by $U(1)_B$, and we will denote an R-symmetry under which Q and $\tilde{Q}$ are neutral and Φ has charge 2 by $U(1)_N$. It is easy to see that $U(1)_N$ is a maximal torus of $SU(2)_N$, while the generator of $U(1)_B$ is a linear combination of the generators of $SU(2)_N$ and $SU(2)_R$. The generator of the maximal torus of $SU(2)_R$ can be taken as

$$R = N + B.$$

$\mathcal{N} = 4$ SQED is free in the UV and flows to an interacting SCFT in the IR. The infrared dimensions of fields in short multiplets of the superconformal algebra are determined by their spin and transformation properties under $SU(2)_R \times SU(2)_N$. This is easily seen in the harmonic superspace formalism, where the compatibility of constraints on the superfields leads to relations between the dimension and the R-spins [53]. For gauge-invariant operators, one can alternatively use arguments based on unitarity (see, e.g., Ref. [52]).

Perhaps the easiest way to work out the relation between the IR dimension and $SU(2)_R \times SU(2)_N$ quantum numbers is to regard $\mathcal{N} = 4$ SQED as a special kind of $\mathcal{N} = 2$ theory. That is, it is an $\mathcal{N} = 2$ gauge theory which has, besides a manifest complex supercharge, a non-manifest one. It is easy to see that the combination $N + \frac{1}{2}B$ is the generator of the $U(1)$ subgroup of $SU(2)_N \times SU(2)_R$ with respect to which the manifest supercharge has charge 1, while the non-manifest supercharge has charge 0. In the IR limit, the corresponding current is in the same multiplet as the stress-energy tensor (because all $SU(2)_R \times SU(2)_N$ currents are), and therefore the dimension of chiral primary states must be equal to their charges with respect to $N + \frac{1}{2}B$. (Note that in the case of $\mathcal{N} = 2$ SQED this was true only in the large-N_f limit.) In particular, the IR dimensions of Q^s and $\tilde{Q}^s$ are 1/2, and the IR dimension of Φ and χ is 1.

According to Ref. [9], the mirror theory for $\mathcal{N} = 4$ SQED is a (twisted) $\mathcal{N} = 4$ $d = 3$ gauge theory with gauge group $U(1)^{N_f}/U(1)_{diag}$ and N_f (twisted) hypermultiplets $(X^s, \tilde{X}^s)$. The matter multiplets transform under the gauge group as follows:

$$X^s \to X^s e^{i(\alpha^s - \alpha^{s-1})}, \quad \tilde{X}^s \to \tilde{X}^s e^{-i(\alpha^s - \alpha^{s-1})}, \quad s = 1, \ldots, N_f,$$

where we set $\alpha^0 = \alpha^{N_f}$. The action of the mirror theory is

$$S_{dual} = \int d^3x\, d^4\theta \sum_{s=1}^{N_f} \left\{ \frac{1}{4e^2} \Sigma^{+s}\Sigma^s + \frac{1}{e^2} S^{+s} S^s + X^{+s} e^{2V^s - 2V^{s-1}} X^s + \tilde{X}^{+s} e^{-2V^s + 2V^{s-1}} \tilde{X}^s \right\}$$
$$+ \left(i\sqrt{2} \int d^3x\, d^2\theta \sum_{s=1}^{N_f} X^s \tilde{X}^s (S^s - S^{s-1}) + h.c. \right).$$

Here $\mathcal{N} = 2$ vector multiplets V^s satisfy the constraints Eq. (3.1), $\mathcal{N} = 2$ chiral multiplets S^s satisfy similar constraints

$$S^0 = S^{N_f}, \quad \sum_{s=1}^{N_f} S^s = 0,$$

and each pair (V^s, S^s) forms a (twisted) $\mathcal{N} = 4$ vector multiplet.

The global symmetries are matched as follows. The R-symmetries are trivially identified. The vortex current of $\mathcal{N} = 4$ SQED is mapped to $1/N_f$ times the Noether current corresponding to the following global $U(1)$ symmetry:

$$X^s \to e^{i\alpha} X^s, \quad \tilde{X}^s \to e^{-i\alpha} \tilde{X}^s, \quad s = 1, \ldots, N_f.$$

The currents corresponding to the maximal torus of $SU(N_f)$ flavor symmetry of $\mathcal{N}=4$ SQED are mapped to the vortex currents

$$2\pi\, J^s = *F^s,\ s = 1, \ldots, N_f, \quad \sum_{s=1}^{N_f} J^s = 0,$$

where F^s is the field-strength of the s^{th} gauge field. The mapping of the rest of $SU(N_f)$ currents is not well understood.

3.2.2 Monopole operators in $\mathcal{N}=4$ SQED at large N_f

To begin with, we can regard $\mathcal{N}=4$ SQED as a rather special $\mathcal{N}=2$ gauge theory, and look for BPS and anti-BPS monopole operators in this theory. This amounts to focusing on a particular $\mathcal{N}=2$ subalgebra of the $\mathcal{N}=4$ superalgebra. Different choices of an $\mathcal{N}=2$ subalgebra are all related by an $SU(2)_N$ transformation, so we do not loose anything by doing this.

From this point of view, our problem is almost exactly the same as in the case of $\mathcal{N}=2$ SQED. The only difference between the two is the presence of the chiral superfield Φ. But in the large N_f limit it becomes non-dynamical, and the $\mathcal{N}=2$ BPS condition requires the background value of Φ to be zero. This implies that the radial quantization of the matter fields $Q^s, \tilde{Q}^s$ proceeds in *exactly* the same way as in the $\mathcal{N}=2$ case and yields the same answer for the spectrum and properties of BPS and anti-BPS states. Namely, for any vortex charge q we have a single BPS and a single anti-BPS states, with charges

$$N = \pm|q|N_f, \quad B = \mp|q|N_f,$$

and energy $E = |q|N_f/2$.

An interesting new element in the $\mathcal{N}=4$ case is the way short multiplets of $\mathcal{N}=2$ superconformal symmetry fit into a short multiplet of $\mathcal{N}=4$ superconformal symmetry. Recall that we have made a certain choice of $\mathcal{N}=2$ subalgebra of the $\mathcal{N}=4$ superalgebra. This choice is preserved by the $U(1)_N$ symmetry, but not by the $SU(2)_N$ symmetry. Thus we have an $SU(2)/U(1)\simeq\mathbb{CP}^1$ worth of BPS conditions. Applying an $SU(2)_N$ rotation to the BPS state found above, we obtain a half-BPS state for every point on $\mathbb{CP}^1$. These half-BPS states fit into a line bundle $\mathcal{L}$ over $\mathbb{CP}^1$. Similarly, applying $SU(2)_N$ transformations to the anti-BPS state, we obtain another line bundle on $\mathbb{CP}^1$ which is obviously the complex conjugate of $\mathcal{L}$.

The $\mathbb{CP}^1$ which parameterizes different choices of the $\mathcal{N}=2$ subalgebra has a very clear meaning in the large N_f limit. Namely, we chose the scalar background on $S^2\times R$ to be $\Phi=0,\chi=\frac{q}{2}$, but obviously any $SU(2)_N$ transform of this is also a half-BPS configuration. The manifold of possible scalar backgrounds is a 2-sphere given by

$$|\Phi|^2+\chi^2=\left(\frac{q}{2}\right)^2.$$

The BPS state we are interested in is the Fock vacuum of charged matter fields on $S^2\times R$ in a *fixed* background. As we vary the background values of Φ and χ, we obtain a bundle of Fock vacua on $S^2\sim\mathbb{CP}^1$. This bundle can be non-trivial because of Berry's phase [55, 56].

Now we can easily see how $\mathcal{N}=4$ superconformal symmetry is realized in our formalism. As argued above, we need to enlarge our Hilbert space by the Hilbert space of zero modes, which arise because the classical background breaks some of the symmetries of the theory. Compared to the $\mathcal{N}=2$ case, we have additional bosonic zero modes coming from the breaking of R-symmetry from $SU(2)_N$ down to $U(1)_N$. Thus our fields will depend on coordinates on $R^3\times\mathbb{CP}^1$. As for fermionic zero modes, in the BPS case they are generated by a complex spinor supercharge which depends on the coordinates on $\mathbb{CP}^1$ as follows:

$$\mathbb{Q}_\alpha=\sum_{I=1,2}u_I\mathbb{Q}^I_\alpha.$$

Here $u_1, u_2 \in \mathbb{C}$ are homogeneous coordinates on $\mathbb{CP}^1$, and $\mathbb{Q}^I_\alpha$, $I = 1, 2$ are a pair of complex spinor supercharges which transform as a doublet of $SU(2)_N$. Therefore monopole operators will be described by "functions" on the supermanifold

$$S(R^3) \boxtimes \mathcal{O}(1),$$

where $S(R^3)$ is the trivial spinor bundle on R^3 (with fiber coordinates regarded as Grassmann-odd), while $\mathcal{O}(1)$ is the tautological line bundle on $\mathbb{CP}^1$. We put the word "functions" in quotes, because, as explained above, we may need to consider sections of non-trivial line bundles on $\mathbb{CP}^1$ instead of functions.

This supermanifold is known as the *analytic superspace* [53, 57, 58] (see also section 3 of Ref. [59]). It is a chiral version of the so-called harmonic superspace. It is well known that "functions" on the analytic superspace (analytic superfields) furnish short representations of the superconformal algebra with eight supercharges [53]. We conclude that in the large-N_f limit BPS monopole operators are described by $\mathcal{N} = 4$ $d = 3$ analytic superfields. Needless to say, anti-BPS monopole operators are described by anti-analytic superfields which are complex-conjugates of the analytic ones.

It remains to pin down the topology of the bundle $\mathcal{L}$ over $\mathbb{CP}^1$. Since this is a line bundle, its topology is completely characterized by the first Chern class. A "cheap" way to find the Chern class is to note that the scaling dimension of an analytic superfield (more precisely, of its scalar component) is equal to half the Chern number of the corresponding line bundle. (The Chern number is the value of the first Chern class on the fundamental homology class of $\mathbb{CP}^1$.) This follows from the way superconformal algebra is represented on analytic superfields [53]. We already know the dimension of our BPS state, and therefore infer that the Chern number of $\mathcal{L}$ is equal to $N_f|q|$.

We can also determine the Chern number directly, by computing the curvature of the Berry connection for the bundle of Fock vacua. In the present case, the computation is almost trivial, since the Hamiltonians at different points of $\mathbb{CP}^1$ are related by an $SU(2)_N$ transformation. In particular, it is sufficient to compute the curvature at any point on $\mathbb{CP}^1$. For example, we can identify $\mathbb{CP}^1$ with a unit sphere in R^3 with coordinates (x, y, z) and compute the curvature at the "North Pole," which has Euclidean coordinates $(0, 0, 1)$. (The abstract coordinates (x, y, z) can be identified with $(\mathrm{Re}\,\Phi, \mathrm{Im}\,\Phi, \chi)$.) Using $SU(2)_N$ invariance, we easily see that the Fock vacuum at the point (x, y, z) with $z \simeq 1$, $x, y \ll 1$ is given by

$$|x, y, z\rangle = \exp\left(i\left(\frac{x}{z}N_x - \frac{y}{z}N_y\right) + O(x^2 + y^2)\right)|0, 0, 1\rangle.$$

Here N_x and N_y are the generators of $SU(2)_N$ rotations about x and y axes. Therefore the curvature of the Berry connection at the point $(0, 0, 1)$ is

$$\begin{aligned}\mathcal{F} = i\left(d|x, y, z\rangle, \wedge d|x, y, z\rangle\right) = i dx \wedge dy \langle 0, 0, 1|[N_y, N_x]|0, 0, 1\rangle \\ = dx \wedge dy \langle 0, 0, 1|N_z|0, 0, 1\rangle.\end{aligned}$$

Now we recall that the vacuum at $(0, 0, 1)$ is an eigenstate of N_z with eigenvalue $\pm N_f|n|/2$ (one needs to remember that $\mathcal{N} = 2N_z$). Taking into account that $\mathcal{F}$ is an $SU(2)_N$-invariant 2-form on $\mathbb{CP}^1$, we conclude that it is given by

$$\mathcal{F} = \pm\frac{1}{2}N_f|q|\,\Omega,$$

where Ω is the volume form on the unit 2-sphere. It follows that the Chern number of the Fock vacuum bundle is

$$c_1 = \frac{1}{2\pi}\int_{S^2}\mathcal{F} = \pm N_f|q|,$$

where the upper (lower) sign refers to $\mathcal{L}$ (resp. $\mathcal{L}^*$). The result agrees with the indirect argument given above.

3.2.3 A comparison with the predictions of $\mathcal{N} = 4$ mirror symmetry

Chiral primaries in the mirror theory with vortex number ± 1 are exactly the same as in the $\mathcal{N} = 2$ case:

$$V_+ = X^1 X^2 \dots X^{N_f}, \quad V_- = \tilde{X}^1 \tilde{X}^2 \dots \tilde{X}^{N_f}.$$

Their $U(1)_N$ and $U(1)_B$ quantum numbers match those computed in the original theory using radial quantization and large-N_f expansion. This provides a check of $\mathcal{N} = 4$ mirror symmetry at the origin of the moduli space. We can also translate this into the language of analytic superfields. Then a hypermultiplet $(X^s, \tilde{X}^{+s})$ is described by an analytic superfield X^s whose Chern number is 1. The analytic superfield which is gauge-invariant and carries vortex charge 1 is given by

$$\mathsf{X}^1 \mathsf{X}^2 \dots \mathsf{X}^{N_f}.$$

It has Chern number N_f and corresponds to the BPS multiplet constructed in the previous section, while its complex conjugate corresponds to the anti-BPS multiplet.

Mirror symmetry also predicts a certain interesting relation in the chiral ring of the IR limit of $\mathcal{N} = 4$ SQED. Consider the product of V_+ and V_-:

$$V_+ V_- = (X^1 \tilde{X}^1)(X^2 \tilde{X}^2) \dots (X^{N_f} \tilde{X}^{N_f}).$$

Using the equation of motion for S^s, it is easy to see that the operators $(X^s \tilde{X}^s)$ for different s are equal modulo descendants. Furthermore, mirror symmetry maps any of these operators to Φ modulo descendants [15]. Thus we infer that modulo descendants we have a relation in the chiral ring:

$$V_+V_- \sim \Phi^{N_f}. \tag{3.6}$$

Can we understand this relation in terms of $\mathcal{N} = 4$ SQED? Indeed we can!

To begin with, it is easy to see that the operator Φ^{N_f} is the only chiral operator whose quantum numbers match those of V_+V_- and which could appear in the OPE of V_+ and V_-. Thus it is sufficient to demonstrate that it appears with a non-zero coefficient. To this end, we need to compute the 3-point function of V_+, V_-, and $(\Phi^+)^{N_f}$. In the radial quantization approach, we need to show that the matrix element

$$\langle V_-^+|\left(\Phi^+\right)^{N_f}|V_+\rangle$$

is non-zero.

Now we recall that the state corresponding to V_+ has magnetic flux $+1$ and scalar VEV $\chi = -\frac{1}{2}$, while the state corresponding to V_- has magnetic flux -1 and $\chi = \frac{1}{2}$. Hermitian conjugation reverses the sign of the magnetic flux and leaves the VEV of χ unchanged. It follows that the path integral which computes the matrix element of any operator between $\langle V_-^+|$ and $|V_+\rangle$ must be performed over field configurations such that the magnetic flux is equal to 1, while the scalar χ asymptotes to $-1/2$ at $\tau = -\infty$ and $1/2$ at $\tau = +\infty$. Thus we are dealing with a kink on $S^2 \times R$.

Next, we note that the Dirac operator on $S^2 \times R$ coupled to such a background may very well have normalizable zero modes. If this is the case, then in order to get a non-zero matrix element one needs to insert an operator which has the right quantum numbers to absorb the zero modes. For example, one can insert a product of all fermionic fields which possess a zero mode. Another possibility, which is more relevant for us, is to insert some bosonic fields which interact with fermions and can absorb the zero modes. In our case, the action contains a complex scalar Φ which has

Yukawa interactions of the form

$$\int d^3x\, \Phi \sum_{s=1}^{N_f} \psi^s \tilde{\psi}^s.$$

Thus if each ψ and each $\tilde{\psi}$ has a single normalizable zero mode, then we can get a non-zero result for the matrix element if we insert precisely N_f powers of Φ^+.

To complete the argument it remains to show that the Dirac operator for both ψ and $\tilde{\psi}$ has a single zero mode. The Atiyah-Patodi-Singer theorem says in this case that the L^2 index of the Dirac operator is

$$ind(D) = \frac{1}{2}(\eta(H_-) - \eta(H_+)),$$

where $\eta(H_\pm)$ denotes the η-invariant of the asymptotic Dirac Hamiltonian at $\tau \to \pm\infty$. We also made use of the fact that neither H_+ nor H_- have zero modes (see section 3.1). Now we recall that we have computed the η-invariants already: according to Eq. (3.5), $\eta(H_-)$ and $\eta(H_+)$ coincide with the $U(1)_N$ charges of the BPS and anti-BPS vacua, respectively, divided by N_f. This implies that the index of the Dirac operator is equal to 1, for both ψ and $\tilde{\psi}$, and therefore both ψ and $\tilde{\psi}$ have a single zero mode.

3.3 Beyond the large-N_f limit

3.3.1 Non-renormalization theorems for the anomalous charges

We have seen that mirror symmetry makes certain predictions about the quantum numbers of BPS monopole operators, and that our large-N_f computations confirm these predictions. But mirror symmetry also suggests that large-N_f results for $U(1)_B$ and $U(1)_N$ charges remain valid for all N_f, all the way down to $N_f = 1$. In this subsection we provide an explanation for this without appealing to mirror symmetry.

We show that the values of $U(1)_N$ and $U(1)_B$ charges for monopole operators are fixed by the L^2 index theorem for the Dirac operator on $S^2 \times R$ and therefore cannot receive $1/N_f$ corrections.

The argument is very simple. For concreteness, consider the monopole operators $V_\pm$ which have vortex charge $q = \pm 1$. These operators are related by charge conjugation and thus have the same $U(1)_N$ charge, which we denote N_V. To determine N_V, we need to consider the transition amplitude on $S^2 \times R$ from the state corresponding to V_+ to the state corresponding to V_-^+: if it violates the $U(1)_N$ charge by m, then $N_V = -m/2$. Since ψ and $\tilde{\psi}$ have $N = -1$, the charge is violated by $-2N_f$ times the index of the Dirac operator on $S^2 \times R$. The index of the Dirac operator in the present case has only boundary contributions (η-invariants), which depend on the asymptotics of the gauge field and the scalar χ. When these asymptotics are given by the large-N_f saddle points, the index was evaluated in section 3.2 with the result $ind(D) = 1$. Furthermore, in the large-N_f expansion fluctuations about the saddle point are treated using perturbation theory. Hence to all orders in $1/N_f$ expansion the transition amplitude from V_+ to V_-^+ will violate $U(1)_N$ charge by $-2N_f$. This implies that the $U(1)_N$ charge of $V_\pm$ is equal to N_f to all orders in $1/N_f$ expansion. An identical argument can be made for $U(1)_B$.

One may ask if it is possible to dispense with the crutch of $1/N_f$ expansion altogether. Naively, there is no problem: we consider the path integral for $\mathcal{N} = 4$ or $\mathcal{N} = 2$ SQED with $e = \infty$ and use the APS index theorem to infer the charges of $V_\pm$. However, this argument is only formal, because we do not know how to make sense of this path integral without using $1/N_f$ expansion. In particular, this leads to difficulties with the evaluation of the index: we cannot compute the η-invariants without knowing the precise asymptotic form of the background, but the asymptotic conditions put constraints only on the total magnetic flux through S^2 and the average value of χ at $\tau = \pm\infty$. (We remind that the L^2-index of a Dirac operator on a non-compact manifold is only a quasi-topological quantity, which can change if the asymptotic behavior of the fields is changed.) The index has a definite value only if we choose some particular asymptotics for the gauge field and χ.

3.3.2 A derivation of the basic $\mathcal{N} = 4$ mirror symmetry

It is plausible that the point $N_f = 1$ is within the radius of convergence of $1/N_f$ expansion. Singularities in an expansion parameter usually signal some sort of phase transition, and in the case of $\mathcal{N} = 4$ SQED we do not expect any drastic change of behavior as one decreases N_f.

With this assumption, we can prove the basic example of $\mathcal{N} = 4$ mirror symmetry, namely, that the IR limit of $\mathcal{N} = 4$ SQED with $N_f = 1$ is dual to the theory of a free twisted hypermultiplet. The proof is quite straightforward. As explained above, the $U(1)_N$ charge of the chiral field $V_\pm$ is equal to N_f to all orders in $1/N_f$ expansion, while its $U(1)_B$ charge is equal to $-N_f$. This implies that the IR dimension of V_+ is equal to $N_f/2$ to all orders in $1/N_f$ expansion (see section 3.2). Assuming that $1/N_f$ expansion converges at $N_f = 1$, this implies that for $N_f = 1$ the IR dimension of $V_\pm$ is $1/2$. In a unitary 3-D CFT, a scalar of dimension $1/2$ must be free [52]. Then, by virtue of supersymmetry, the $\mathcal{N} = 2$ superfields $V_\pm$ are free chiral superfields with $N = 1$ and $B = -1$, or, equivalently, the pair (V_+, V_-^+) is a free twisted hypermultiplet.

The above argument shows that the IR limit of $\mathcal{N} = 4$ SQED contains a free sector generated by the action of free fields $V_\pm$ on the vacuum. But this sector also contains all the states generated by Φ and its superpartners. Indeed, the product of V_+ and V_- is a chiral field which has zero vortex charge and $N = 2$, $B = -2$. It is easy to see that the only such field is Φ. In addition, since V_+ and V_- are independent free fields, their product is non-zero. Thus we must have $V_+V_- \sim \Phi$ (we have seen above how a more general relation Eq. (3.6) can be demonstrated in the large-N_f limit). We conclude that the sector of the IR limit of $\mathcal{N} = 4$ SQED generated by Φ and its superpartners is contained in the charge-0 sector of the theory of a free twisted hypermultiplet. This is precisely the statement of mirror symmetry in this particular case.

In the next chapter we extend the analysis to non-abelian gauge theories and non-abelian 3-D mirror symmetry.

Chapter 4

Monopole operators in three-dimensional non-abelian gauge theories

4.1 Monopole operators in three-dimensional non-supersymmetric $SU(N_c)$ gauge theories

4.1.1 IR limit of $SU(N_c)$ gauge theories

Consider a three-dimensional Euclidean Yang-Mills action for N_f flavors of matter fermions in the fundamental representation of the gauge group $SU(N_c)$ with generators $\{T^\alpha\}$[1], $(\alpha = 1, \ldots, N_c^2 - 1)$:

$$S = \int d^3x \left(\frac{1}{4e^2} \mathrm{Tr}\; V_{ij} V^{ij} + i \sum_{s=1}^{N_f} \psi^{+s} \vec{\sigma} \left(\vec{\nabla} + i\vec{V} \right) \psi^s \right), \tag{4.1}$$

where ψ are complex two-component spinors, $\vec{V} = \vec{V}^\alpha T^\alpha$ is gauge potential with a field-strength $V_{ij} = \left(\partial_i V_j^\alpha - \partial_j V_i^\alpha - f^{\alpha\beta\gamma} V_i^\beta V_j^\gamma \right) T^\alpha$, $f^{\alpha\beta\gamma}$ are the structure constants. To avoid a parity anomaly, Ref. [46], we choose N_f to be even. Action (4.1) is invariant under the flavor symmetry $U(N_f)_{flavor}$.

[1] We use $\mathrm{Tr}(T^\alpha T^\beta) = \frac{1}{2}\delta^{\alpha\beta}$ normalization.

There are two ways to classify monopoles in non-abelian theories. A dynamical description of monopoles in terms of weight vectors of the dual of (unbroken) gauge group was developed by Goddard, Nuyts, and Olive (GNO) in Ref. [60]; topological classification in terms of π_1 was suggested by Lubkin in Ref. [61], (see also Ref. [62] for a review). It is well known that in $R^{1,3}$ the dynamical (GNO) monopoles with vanishing topological charges are unstable in the small coupling limit. We will study the dynamical monopoles in the IR limit of (4.1). The theory is free in the UV limit ($\frac{e^2}{\Lambda} \to 0$, where Λ is a renormalization scale) and is strongly coupled in the IR ($\frac{e^2}{\Lambda} \to \infty$). In the strong coupling regime the dominant contribution to the gauge field effective action is given by the matter fields and stability analysis of GNO monopoles performed at weak coupling is no longer applicable. Since matter fields belong to the fundamental representation, the effective gauge group is given by $SU(N_c)$. The corresponding π_1 is trivial and all dynamical monopoles have vanishing topological charges. The GNO monopoles of $SU(N_c)$ are given by

$$V^N = H(1-\cos\theta)d\varphi, \quad V^S = -H(1+\cos\theta)d\varphi, \tag{4.2}$$

where V^N and V^S correspond to gauge potentials on upper and lower hemispheres respectively. H is a constant traceless hermitian $N_c \times N_c$ matrix, which can be assumed to be diagonal. On the equator V^N and V^S are transformed into each other by a gauge transformation with a group element $\exp(2iH\varphi)$. This transformation is single-valued if

$$H = \frac{1}{2}\mathrm{diag}(q_1, q_2, \ldots, q_{N_c}) \tag{4.3}$$

with integers q_a, $(a = 1, .., N_c)$,

$$\sum_{a=1}^{N_c} q_a = 0. \tag{4.4}$$

Consider a path integral over matter and gauge fields on the punctured R^3. Integration over the gauge fields asymptotically approaching (4.2) at the removed point of R^3

corresponds to an insertion of a monopole operator with magnetic charge H. To complete definition of the monopole operator we have to specify the behavior of matter fields at the insertion point. Thus monopole operators with a given magnetic charge are classified by the behavior of the matter fields near the singularity. In the IR limit the theory (4.1) flows to the interacting conformal field theory. In three-dimensional CFT operators on R^3 are in one-to-one correspondence with normalizable states on $S^2 \times R$. Namely, insertion of a monopole operator in the origin of R^3 corresponds to a certain in-going state in the radially quantized theory on $S^2 \times R$. Hamiltonian of the radially quantized theory is identical to the dilatation operator on R^3. In unitary CFT all physical operators including those creating topological disorders are classified by the lowest-weight irreducible representations labelled by the primary operators. We will say that topological disorder operator is a primary monopole operator, if such an operator has the lowest conformal weight among the monopole operators with a given magnetic charge H. Since conformal transformations do not affect the magnetic charge, the primary monopole operators are conformal primaries. Our task is to determine spin, conformal weight and other quantum numbers of primary monopole operators.

In the IR limit kinetic term for the gauge field can be neglected and integration over matter fields produces effective action for the gauge field proportional to N_f. Although IR theory is strongly coupled, the effective Planck constant is given by $1/N_f$ and in the large N_f limit the CFT becomes weakly coupled. It is natural to assume that saddle point of the gauge field effective action is invariant under rotations and corresponds to the GNO monopole. Since fluctuations of the gauge field are suppressed, it can be treated as a classical background. Thus, in the large N_f limit we have matter fermions moving in a presence of the GNO monopole. Therefore, a primary monopole operator is mapped to a Fock vacuum for matter fields moving in a monopole background on $S^2 \times R$. Conformal weight of the primary monopole operator is equal to Casimir energy of the corresponding vacuum state relative to the vacuum state with vanishing monopole charge.

4.1.2 Radial quantization

Let us implement the procedure outlined in the previous section. Namely, we consider CFT which appears in the IR limit of the theory (4.1). We neglect the kinetic term of a gauge field, introduce a radial time variable $\tau = \ln r$ and perform the Weyl rescaling to obtain metric on $S^2 \times R$:

$$ds^2 = d\tau^2 + d\theta^2 + \sin^2\theta d\varphi^2.$$

Since a gauge potential of the GNO monopole (4.2) with H given by Eqs.(4.3)-(4.4) is diagonal in color indices we may use results of Chapter 3 for fermionic energy spectra on $S^2 \times R$. We conclude that for each ψ_a^s, where $s = 1, \ldots, N_f$ and $a = 1, \ldots, N_c$ are flavor and color indices respectively, the energy spectrum is given by

$$E_n = \pm\sqrt{|q_a|n + n^2}, \quad n = 1, 2, \ldots$$

Each energy mode has a degeneracy $2|E_n|$ and spin $j = |E_n| - \frac{1}{2}$. In addition, there are $|q_a|$ zero-energy modes which transform as an irreducible representation of the rotation group $SU(2)_{rot}$ with spin $j = \frac{1}{2}(|q_a| - 1)$. In the large N_f limit leading contribution to the conformal weight $h_{\{q\}}$ of the GNO $SU(N_c)$ monopole is given by

$$h_{\{q\}} = N_f \sum_{a=1}^{N_c} \left(\frac{1}{6}\sqrt{1 + |q_a|}\,(|q_a| - 2) + \right.$$

$$\left. + 4\mathrm{Im} \int_0^\infty dt \left[\left(it + \frac{|q_a|}{2} + 1 \right) \sqrt{\left(it + \frac{|q_a|}{2} + 1 \right)^2 - \frac{q_a^2}{4}} \right] \frac{1}{e^{2\pi t} - 1} \right),$$

where branch of the square root under the integral is the one which is positive on the positive real axis.

Let us specialize in the case of GNO monopole with minimum magnetic charge:

$$H = \frac{1}{2}(1, -1, 0, \dots, 0), \tag{4.5}$$

and denote the fermionic non-zero energy mode annihilation operators by a^s_{akm}, b^s_{akm}, where k labels the energy level, and m accounts for a degeneracy. Fermionic zero-energy modes have vanishing spin and are present for ψ^s_1 and ψ^s_2 only. The corresponding annihilation operators we denote as c^s_1 and c^s_2. The Fock space of the theory is the tensor product of the zero-mode Fock space and the Fock space of all other modes. The latter is simply a fermionic Fock space with a unique rotationally invariant vacuum $|vac\rangle_+$ which is annihilated by all annihilation operators corresponding to excitations with non-zero energies. The Fock space of zero modes has a vacuum vector which we denote $|vac\rangle_0$. Consider a Fock space of states obtained by acting with creation operators on a state $|vac\rangle \equiv |vac\rangle_0 \bigotimes |vac\rangle_+$, which is annihilated by all the annihilation operators. Those elements of the Fock space which satisfy the Gauss law constraints form the physical Fock space.

The background (4.2) with H given by Eq.(4.5) breaks gauge group $G = SU(N_c)$ to $\bar{G} = U(1)$ for $N_c = 2$ and $\bar{G} = SU(N_c - 2) \times U(1) \times U(1)$ for $N_c > 2$, where generators of the two $U(1)$ groups are given by $(1, -1, 0, \dots, 0)$ and $(2 - N_c, 2 - N_c, 2, \dots, 2)$. Let $\bar{T}^{\bar{\alpha}}$ be generators of $\bar{G}$. In quantum theory we impose Gauss law constraints on physical states. In the IR limit it implies that they are annihilated by the charge density operators $k^{\bar{\alpha}}$. Consider charges $Q^{\bar{\alpha}}$ obtained by integration of $k^{\bar{\alpha}}$ over S^2. The most general form of the corresponding quantum operators is

$$Q^{\bar{\alpha}} = Q^{\bar{\alpha}}_+ + Q^{\bar{\alpha}}_0,$$

where $Q^{\bar{\alpha}}_0$ denote all terms that act within a zero-mode Fock space and $Q^{\bar{\alpha}}_+$ are assumed to be normal-ordered. Using explicit form of zero-energy solutions we find

$$Q_0^{\bar{\alpha}} = c_1^{+s} \bar{T}_{11}^{\bar{\alpha}} c_1^s + c_2^{+s} \bar{T}_{22}^{\bar{\alpha}} c_2^s + n^{\bar{\alpha}},$$

where C-numbers $n^{\bar{\alpha}}$ account for operator-ordering ambiguities. Since the zero modes are rotationally invariant, the Gauss law constraints in the zero-mode Fock space are translated into requirements that the states are annihilated by $Q_0^{\bar{\alpha}}$.

In the case of $N_c = 2$ we have

$$Q_0 = \frac{1}{2} \left(c_1^{+s} c_1^s - c_2^{+s} c_2^s \right) + n.$$

The zero-mode space in spanned by the 2^{2N_f} states

$$|vac\rangle_0, \quad c_1^{+s_1} |vac\rangle_0, \quad c_2^{+s_1} |vac\rangle_0, \quad \ldots, \qquad c_1^{+s_1} \ldots c_1^{+s_{N_f}} c_2^{+s_1} \ldots c_2^{+s_{N_f}} |vac\rangle_0.$$

A zero-mode vacuum state as well as completely filled state have Q_0-charge given by n. Since the monopole background is invariant under CP symmetry, we require CP-invariance of the Q_0 spectrum. Therefore, $n = 0$ and we have the following physical zero-mode vacuum states transforming as scalars under $SU(2)_{rot}$

$$|vac\rangle_0, \quad c_1^{+s_1} \ldots c_1^{+s_l} c_2^{+p_1} \ldots c_2^{+p_l} |vac\rangle_0, \qquad l = 1, \ldots, N_f.$$

Each set of the physical vacuum states labelled by l transforms as a product of two rank-l antisymmetric tensor representations under $U(N_f)_{flavor}$.

For $N_c > 2$ we choose $\bar{T}^1$ and $\bar{T}^2$ to be generators of the two $U(1)$ groups so that the only zero-mode contributions are

$$Q_0^1 = \frac{1}{2} \left(c_1^{+s} c_1^s - c_2^{+s} c_2^s \right) + n^1, \quad Q_0^2 = -\frac{1}{2} \sqrt{\frac{N_c - 2}{N_c}} \left(c_1^{+s} c_1^s + c_2^{+s} c_2^s \right) + n^2.$$

In this case CP-invariance gives $n^1 = 0$ and $n^2 = \frac{1}{2}\sqrt{\frac{N_c-2}{N_c}}N_f$. Therefore, we have $\binom{N_f}{\frac{1}{2}N_f}^2$ physical vacuum states

$$c_1^{+s_1}\dots c_1^{+s_{N_f/2}}c_2^{+p_1}\dots c_2^{+p_{N_f/2}}|vac\rangle_0,$$

transforming as scalars under $SU(2)_{rot}$ and as a product of two rank-$N_f/2$ antisymmetric tensor representations of $U(N_f)_{flavor}$.

4.2 Monopole operators and mirror symmetry in three-dimensional $\mathcal{N}=4$ $SU(2)$ gauge theories

4.2.1 IR limit of $\mathcal{N}=4$ $SU(2)$ gauge theory

Consider three-dimensional Euclidean $\mathcal{N}=4$ supersymmetric theory of vector multiplet $\mathcal{V}$ in the adjoint representation of the gauge group $SU(2)_{gauge}$ and N_f matter hypermultiplets $\mathcal{Q}^s$, $(s = 1,\dots,N_f)$, transforming under the fundamental representation. Decompositions of $\mathcal{N}=4$ multiplets into $\mathcal{N}=2$ multiplets are given in the following table

$\mathcal{N}=4$	$\mathcal{N}=2$
Vector multiplet $\mathcal{V}$	Vector multiplet $V = (V_i, \chi, \lambda, \bar{\lambda}, D)$, Chiral multiplet $\Phi = (\phi, \eta, K)$.
Hypermultiplet $\mathcal{Q}$	Chiral multiplets $Q = (A, \psi, F)$, $\tilde{Q} = (\tilde{A}, \tilde{\psi}, \tilde{F})$,

where V_i is a vector field in the adjoint representation of the gauge group, χ and ϕ are real and complex adjoint scalars respectively; λ, $\bar{\lambda}$, and η are the gluinos, whereas fields D and K are auxiliary. The action in terms of three-dimensional $\mathcal{N} = 2$ superspace formalism is given in the Appendix C. Scalar A ($\tilde{A}$), spinor ψ ($\tilde{\psi}$), and auxiliary field F ($\tilde{F}$) transform according to (anti-)fundamental representation of the gauge group:

$$Q \to e^{i\omega^\alpha T^\alpha} Q, \quad \tilde{Q} \to \tilde{Q} e^{-i\omega^\alpha T^\alpha},$$

under the gauge transformation with parameters $\omega^\alpha(x)$. Since all representations of $SU(2)$ are pseudo-real, we may define a chiral superfield

$$\Psi^a = \frac{1}{\sqrt{2}} \begin{pmatrix} Q^a - \epsilon^{ab}\tilde{Q}_b \\ i\left[Q^a + \epsilon^{ab}\tilde{Q}_b\right] \end{pmatrix},$$

where ϵ^{ab} is antisymmetric tensor with $\epsilon^{12} = 1$. Therefore kinetic term for a hypermultiplet has the form

$$\int d^2\theta d^2\bar{\theta} \sum_{I=1}^{2N_f} \Psi_I^+ e^{2V} \Psi_I,$$

where we used the identities

$$\epsilon_{ab} T^{\alpha b}_{c} \epsilon^{cd} = -(T^{\alpha a}_{d})^T, \quad \epsilon_{ab}\epsilon^{bc} = \delta^c_a.$$

The superpotential is

$$W = i\sqrt{2} \sum_{s=1}^{N_f} \tilde{Q}^s \Phi Q^s = \frac{i}{\sqrt{2}} \sum_{I=1}^{2N_f} \Psi^a_I \epsilon_{ab} \Phi^b_c \Psi^c_I.$$

The kinetic term is invariant under $SU(2N_f)$ flavor symmetry. The superpotential, however, is invariant under $SO(2N_f)$ subgroup only.

$4N_f - 6$ dimensional[2] Higgs branch is labelled by the mesons $M_{IJ} = \Psi^a_I \epsilon_{ab} \Psi^b_J$.

[2]Moduli space dimensions are assumed to be complex.

Using an identity

$$\epsilon^{I_1 \dots I_{2N_f}} \Psi^a_{I_1} \Psi^b_{I_2} \Psi^c_{I_3} \Psi^d_{I_4} = 0,$$

we obtain the constraints $\epsilon^{I_1 \dots I_{2N_f}} M_{I_1 I_2} M_{I_3 I_4} = 0$. The F-flatness condition implies $M^2_{IJ} = 0$.

On the Coulomb branch adjoint scalars χ and Φ can have non-vanishing expectation values. Let us make a gauge transformation to obtain $\chi = \chi^{(3)} T^3$. Dualizing a photon $V^{(3)} = *d\sigma^{(3)}$ we construct a chiral superfield $\Upsilon = \chi^{(3)} + i\sigma^{(3)} + \dots$ Potential energy density for scalars χ and Φ is given by $U = U_1 + U_2$, with

$$U_1 \sim \mathrm{Tr}\left(\left[\Phi, \Phi^+\right]\right)^2, \quad U_2 \sim \mathrm{Tr}\left(\chi^2\right) \mathrm{Tr}\left(\Phi^+ \Phi\right) - \left|\mathrm{Tr}\left(\chi\Phi\right)\right|^2.$$

Vanishing of the potential gives $\Phi = \Phi^{(3)} T^3$. Residual gauge symmetries are $U(1)_{gauge}$ generated by T^3 and Weyl subgroup Z_2 acting by $(\Upsilon, \Phi^{(3)}) \to (-\Upsilon, -\Phi^{(3)})$. Moreover, we have $\Upsilon \sim \Upsilon + 4\pi e^2 i$. Let us introduce a pair of operators Y_+ and Y_- corresponding to positive and negative expectation values of $\chi^{(3)}$ respectively. For large positive (negative) $\chi^{(3)}$ we have $Y_+ \sim e^{\Upsilon/(2e^2)}$ ($Y_- \sim e^{-\Upsilon/(2e^2)}$). We emphasize that none of the Y_+, Y_- is gauge invariant. In fact, $Y_+ \leftrightarrow Y_-$ under the Weyl subgroup Z_2. The gauge invariant coordinates on the Coulomb branch are

$$u = i(Y_+ - Y_-)\Phi^{(3)}, \quad v = (Y_+ + Y_-), \qquad w = \left(\Phi^{(3)}\right)^2. \tag{4.6}$$

In a semiclassical limit we have an equation

$$u^2 + v^2 w = 0. \tag{4.7}$$

Since the Coulomb branch receives quantum corrections we expect modification of the Eq.(4.7).

Three-dimensional $\mathcal{N} = 4$ theory has an R-symmetry group given by $SU(2)_R \times$

$SU(2)_N$. There are $SU(2)_R$ and $SU(2)_N$ gluino doublets, scalars A (A^+) and $\tilde{A}^+$ ($\tilde{A}$) make a doublet of $SU(2)_R$ and are singlets of $SU(2)_N$, spinors ψ ($\bar{\psi}$) and $\bar{\tilde{\psi}}$ ($\tilde{\psi}$) transform as a doublet of $SU(2)_N$ and singlets of $SU(2)_R$. Scalars χ, ϕ, and ϕ^+ form a triplet of $SU(2)_N$ and are neutral under $SU(2)_R$. In three-dimensional $\mathcal{N} = 2$ superspace formalism only the maximal torus $U(1) \times U(1)$ of the R-symmetry is manifest. Let us introduce a set of manifest R-symmetries denoted as $U(1)_N$, $U(1)_B$, and $U(1)_R$ with the corresponding charges given in the table

	N	B	R
Q	0	1	1/2
$\tilde{Q}$	0	1	1/2
Φ	2	-2	1

It is easy to see that B-charge of the Grassmannian coordinates of the $\mathcal{N} = 2$ superspace is zero and $R = N + \frac{1}{2}B$. The supercharge which is manifest in $\mathcal{N} = 2$ superspace formalism has R-charge one, whereas a non-manifest supercharge has vanishing R-charge.

Let us consider topological disorder operators which belong to $\mathcal{N} = 4$ (anti-)BPS multiplets. In the IR limit the theory flows to the interacting superconformal field theory and (anti-)BPS representations are labelled by the (anti-)chiral primary operators. The conformal dimensions of (anti-)chiral primary operators are smaller than those of other operators in the same representation and are determined by their spin and R-symmetry representations [53, 52]. We define an (anti-)BPS primary monopole operator as a topological disorder operator which is an (anti-)chiral operator with the lowest conformal weight among the (anti-)chiral topological disorder operators with a given magnetic charge H. It follows that (anti-)BPS primary monopole operators are (anti-)chiral conformal primaries. Using arguments similar to those presented in subsection 4.1.1 we conclude that in the large N_f limit we have matter fields in a background of the (anti-)BPS monopole. Our goal will be to determine the quantum numbers of (anti-)BPS primary monopole operators in the limit of large N_f.

Now we will identify (anti-)BPS backgrounds corresponding to (anti-)BPS GNO monopoles in $\mathcal{N}=4$ supersymmetric gauge theory. Background values of $\vec{V}^\alpha$, ϕ^α, $\phi^{*\alpha}$, and χ^α preserve some of the manifest $\mathcal{N}=2$ supersymmetry parameterized by $\xi, \bar{\xi}$ iff they satisfy the equations

$$\delta\lambda^\alpha = -i\left(\sigma^i\left(\partial_i\chi^\alpha + f^{\alpha\beta\gamma}\chi^\beta V_i^\gamma\right) + \frac{1}{2}\epsilon^{ijk}\sigma^k V_{ij}^\alpha - D^\alpha\right)\xi = 0, \tag{4.8}$$

$$\delta\bar{\lambda}^\alpha = -i\bar{\xi}\left(\sigma^i\left(\partial_i\chi^\alpha + f^{\alpha\beta\gamma}\chi^\beta V_i^\gamma\right) - \frac{1}{2}\epsilon^{ijk}\sigma^k V_{ij}^\alpha + D^\alpha\right) = 0, \tag{4.9}$$

$$\delta\eta^\alpha = \sqrt{2}\left(f^{\alpha\beta\gamma}\chi^\beta\phi^\gamma + i\sigma^i\left(\partial_i\phi^\alpha + f^{\alpha\beta\gamma}\phi^\beta V_i^\gamma\right)\right)\bar{\xi} + \sqrt{2}\xi K^\alpha = 0, \tag{4.10}$$

$$\delta\bar{\eta}^\alpha = -\sqrt{2}\xi\left(f^{\alpha\beta\gamma}\chi^\beta\phi^{*\gamma} + i\sigma^i\left(\partial_i\phi^{*\alpha} + f^{\alpha\beta\gamma}\phi^{*\beta}V_i^\gamma\right)\right) + \sqrt{2}\bar{\xi}K^{*\alpha} = 0. \tag{4.11}$$

The other set of supersymmetry transformations is obtained from (4.8)-(4.11) by the replacements $\lambda \to \eta$, $\eta \to -\lambda$. Consider a background with $\Phi = 0$. Let us set $D^\alpha = 0$ and introduce $E_i^\alpha = -\partial_i\chi^\alpha - f^{\alpha\beta\gamma}\chi^\beta V_i^\gamma$, $B^{\alpha i} = \frac{1}{2}\epsilon^{ijk}V_{jk}^\alpha$. Equations (4.8)-(4.11) imply

$$(\vec{E}^\alpha - \vec{B}^\alpha)\vec{\sigma}\xi = 0, \quad \bar{\xi}(\vec{E}^\alpha + \vec{B}^\alpha)\vec{\sigma} = 0.$$

For $\vec{B} = \vec{B}^\alpha T^\alpha = \frac{H}{r^3}\vec{r}$ we have the following backgrounds, each preserving half of the manifest $\mathcal{N}=2$ supersymmetry:

(i) BPS background

$$\vec{E}^\alpha = -\vec{B}^\alpha, \quad \forall\bar{\xi}, \quad \xi = 0,$$

(ii) anti-BPS background

$$\vec{E}^\alpha = \vec{B}^\alpha, \quad \forall\xi, \quad \bar{\xi} = 0.$$

We choose $\chi = \chi^\alpha T^\alpha = \mp H/r$ with $H = qT^3 = \frac{1}{2}(q, -q)$, where upper (lower) sign corresponds to the (anti-)BPS monopole backgrounds. These backgrounds are invariant under $SU(2)_R$ symmetry, break $\mathcal{N}=4$ to $\mathcal{N}=2$ supersymmetry, $SU(2)_{gauge}$

group to $U(1)_{gauge}$ subgroup, and $SU(2)_N$ to $U(1)_N$. We mention that contrary to monopoles in $U(1)$ gauge theory, $SU(2)$ monopoles specified by H and $-H$ are gauge equivalent.

4.2.2 Dual theory

The dual theory is a twisted $\mathcal{N} = 4$, $\left[U(2)^{N_f-3} \times U(1)^4\right]/U(1)_{diag}$ gauge theory based on the Dynkin diagram of $SO(2N_f)$ group. The fields include $N_f - 3$ $U(2)$ vector superfields which are made of $\mathcal{N} = 2$ $U(1)$ vector superfields U_l and neutral chiral superfields T_l, $SU(2)$ vector superfields T_l and adjoint chiral superfields S_l, $l = 1, \ldots, N_f - 3$. Also, there are four additional $U(1)$ vector superfields which consist of $\mathcal{N} = 2$ vector superfields U_{N_f-2},..., U_{N_f+1} and neutral chiral superfields T_{N_f-2},..., T_{N_f+1}. Factorization of the diagonal $U(1)$ implies the constraints

$$\sum_{p=1}^{N_f+1} U_p = 0, \quad \sum_{p=0}^{N_f+1} T_p = 0.$$

Matter fields include twisted $N_f - 4$ matter hypermultiplets made of $\mathcal{N} = 2$ chiral multiplets q_r and $\tilde{q}_r$, transforming as

$$q_r \longrightarrow U(2)_{r+1} q_r U(2)_r^+, \quad \tilde{q}_r \longrightarrow U(2)_r \tilde{q}_r U(2)_{r+1}^+, \qquad r = 1, \ldots, N_f - 4.$$

We also have four additional twisted matter hypermultiplets which decompose with respect to $\mathcal{N} = 2$ as chiral superfields $\left(X_1, \tilde{X}_1\right)$, ..., $\left(X_4, \tilde{X}_4\right)$. X_1 $\left(\tilde{X}_1\right)$ has charge $+1$ (-1) under $U(1)_{N_f-2}$ and transforms according to (anti-)fundamental representation of $U(2)_1$; X_2 $\left(\tilde{X}_2\right)$ has $U(1)_{N_f-1}$ charge $+1$ (-1) and is belongs to (anti-)fundamental representation of $U(2)_{N_f-3}$; X_3 $\left(\tilde{X}_3\right)$ has a charge $+1$ (-1) under $U(1)_{N_f}$ and transforms according to (anti-)fundamental representation of $U(2)_1$; X_4 $\left(\tilde{X}_4\right)$ has a charge $+1$ (-1) under $U(1)_{N_f+1}$ and is transformed according to (anti-)fundamental representation of $U(2)_{N_f-3}$. Superpotential is given by

$$W = i\sqrt{2}\left\{ \tilde{X}_1 \left(T_1 + S_1 - T_{N_f-2}\right) X_1 + \tilde{X}_2 \left(T_{N_f-3} + S_{N_f-3} - T_{N_f-1}\right) X_2 + \right.$$

$$+\tilde{X}_3 \left(T_1 + S_1 - T_{N_f}\right) X_3 + \tilde{X}_4 \left(T_{N_f-3} + S_{N_f-3} - T_{N_f+1}\right) X_4 +$$

$$\left. + \sum_{r=1}^{N_f-4} \tilde{q}_r \left(S_{r+1} + T_{r+1} - S_r - T_r\right) q_r \right\}.$$

The two-dimensional Higgs branch doesn't receive quantum corrections and is given by a hyper-Kahler quotient parameterized by x, y, and z subject to a constraint

$$x^2 + y^2 z = z^{N_f-1}. \tag{4.12}$$

Explicit form of these coordinates is given in Ref. [63]:

$$z = -X_1^{a_1} \tilde{X}_{3|a_1} X_3^{b_1} \tilde{X}_{1|b_1}, \tag{4.13}$$

and (for even N_f)

$$\begin{aligned} x &= 2X_1^{a_1} q_{1|a_1}^{a_2} \ldots q_{N_f-4|a_{N_f-4}}^{a_{N_f-3}} \tilde{X}_{2|a_{N_f-3}} X_2^{b_{N_f-3}} \tilde{q}_{N_f-4|b_{N_f-3}}^{b_{N_f-4}} \ldots \tilde{q}_{1|b_2}^{b_1} \tilde{X}_{3|b_1} X_3^{c_1} \tilde{X}_{1|c_1}, \quad (4.14)\\ y &= 2X_3^{a_1} q_{1|a_1}^{a_2} \ldots q_{N_f-4|a_{N_f-4}}^{a_{N_f-3}} \tilde{X}_{2|a_{N_f-3}} X_2^{b_{N_f-3}} \tilde{q}_{N_f-4|b_{N_f-3}}^{b_{N_f-4}} \ldots \tilde{q}_{1|b_2}^{b_1} \tilde{X}_{3|b_1} + (-z)^{N_f/2-1}. \end{aligned}$$

$4N_f - 6$ dimensional Coulomb branch is parameterized by $N_f + 1$ dual $U(1)$ photons $V_{\pm|r}$ (for a given r, $V_{+|r}$ and $V_{-|r}$ are used as coordinates on two distinct patches) subject to the constraints

$$\prod_r V_{+|r} = \prod_r V_{-|r} = 1,$$

N_f independent chirals T, $2N_f - 6$ independent coordinates analogous to the ones given in Eq.(4.6).

4.2.3 Mirror symmetry

Since mirror symmetry exchanges mass and Fayet-Iliopoulos terms, we identify N_f complex mass terms $\tilde{Q}^s Q^s$ (no sum over s) with N_f independent chirals T. Therefore chirals T and S have baryon charge 2 whereas baryon charges of X, $\tilde{X}$, q, and $\tilde{q}$ are -1. Baryon charges of x, y, and z are $2-2N_f$, $4-2N_f$, and -4 respectively which can be deduced both from the defining equations (4.13)-(4.14) and from the hyper-Kahler quotient equation (4.12). Likewise, T and S have vanishing $U(1)_N$ charges, whereas X, $\tilde{X}$, q, and $\tilde{q}$ have a charge $+1$. Finally, $U(1)_N$ charges of x, y, and z are $2N_f - 2$, $2N_f - 4$, and 4 respectively. We also have $R(x) = N_f - 1$, $R(y) = N_f - 2$, as well as $R(z) = 2$.

It follows that charges of z are independent of N_f and coincide with that of $w = 2 \text{ Tr } \Phi^2$. Also comparing Eq.(4.7) with Eq.(4.12) we obtain an identification

$$u \sim x, \quad v \sim y, \qquad w \sim z.$$

Thus, the mirror symmetry predicts the following charges for operators defined in Eq.(4.6)

	N	B	R
u	$2N_f - 2$	$2 - 2N_f$	$N_f - 1$
v	$2N_f - 4$	$4 - 2N_f$	$N_f - 2$
w	4	-4	2

Since x, y, and z are chiral primary operators which are polynomials of the electrically charged fields, operators u, v, and w are also chiral primaries and describe the sector with nontrivial magnetic charge. As explained in Chapter 3, the conformal dimension of $\mathcal{N} = 4$ (anti-)chiral primary operator equals (minus) the corresponding $U(1)_R$ charge.

4.2.4 Quantum numbers

Quantum numbers of the (anti-)BPS primary monopole state receive contributions from both matter hypermultiplet $\mathcal{Q}$ and vector multiplet $\mathcal{V}$. The former is proportional to N_f and is dominant in the large N_f limit, whereas the latter gives correction of the form $O(1)$. Let us determine the matter contributions first.

Energy spectra of matter fields in (anti-)BPS backgrounds are given in the Appendix C. Since matter fermionic particles and antiparticles have different energy spectra we adopt the "symmetric " ordering for the bilinear fermionic observables defined in Eq.(3.4). This procedure gives

$$E^{\text{Fermions}}_{\text{Casimir}} = \frac{1}{2}\left(\sum E^- - \sum E^+\right) - \text{"the same"}|_{q=0},$$

where E^+, E^- are positive and negative energies respectively. To define the formal sums appearing in this section we use

$$\sum E \to \sum E e^{-\beta|E|}$$

regularization and take $\beta \to 0$ limit at the end of calculations. Matter bosonic particles and antiparticles have identical energy spectra and standard prescription can be used. In our model the matter contribution to the Casimir energy is equal to $h_{\mathcal{Q}} = N_f|q|$ for both BPS and anti-BPS monopole backgrounds. For matter part of the R-charge operator we have

$$R_{\mathcal{Q}} = \frac{1}{4}\Big[\sum(a^+_\psi a_\psi - a_\psi a^+_\psi) + \sum(b_\psi b^+_\psi - b^+_\psi b_\psi) + \sum(a^+_{\tilde\psi} a_{\tilde\psi} - a_{\tilde\psi} a^+_{\tilde\psi}) +$$

$$\sum(b_{\tilde\psi} b^+_{\tilde\psi} - b^+_{\tilde\psi} b_{\tilde\psi}) - \sum(a^+_A a_A + b_A b^+_A) - \sum(a^+_{\tilde A} a_{\tilde A} + b_{\tilde A} b^+_{\tilde A})\Big] + const,$$

where a^+ (b^+) and a (b) denote the corresponding (anti-)particle creation and annihilation operators respectively. To fix a constant we define a vacuum state with zero magnetic charge $|0\rangle$ to have vanishing R-charge. It follows that

$$\langle R_{\mathcal{Q}}\rangle_q = \left(\sum_{E_\psi^-}|E_\psi^-| - \sum_{E_\psi^+}E_\psi^+ + \sum_{E_A^+}E_A^+ - \sum_{E_A^-}|E_A^-|\right) -'' \text{the same}''|_{q=0}.$$

As a result we have $\langle R_{\mathcal{Q}}\rangle = \pm h_{\mathcal{Q}}$. For N and B charges similar calculations give $\langle N_{\mathcal{Q}}\rangle_q = -\langle B_{\mathcal{Q}}\rangle_q = \pm 2N_f|q|$.

Now we will consider the vector multiplet contribution to the quantum numbers of the vacuum state. Relevant charges are summarized in the table

	N	B	R
λ	1	0	1
$\bar{\lambda}$	-1	0	-1
η	1	-2	0
$\bar{\eta}$	-1	2	0
ϕ	2	-2	1
ϕ^*	-2	2	-1
χ	0	0	0

Integration over the hypermultiplet $\mathcal{Q}$ produces an induced action for the vector multiplet $S_{Ind}[\mathcal{V}]$ proportional to N_f. Let us assume that supersymmetric monopole configuration minimizes vector multiplet effective action in the IR region. Changing $\mathcal{V} \to \mathcal{V}^{mon} + \hat{\mathcal{V}}/\sqrt{N_f}$ gives

$$S_{Ind}[\mathcal{V}] = S_{Ind}^{(2)}[\hat{\mathcal{V}}] + O\left(\frac{1}{\sqrt{N_f}}\right),$$

where $S_{Ind}^{(2)}[\hat{\mathcal{V}}]$ is quadratic in $\hat{\mathcal{V}}$ and independent from N_f. The full effective action for $\hat{\mathcal{V}}$ is

$$S_{Eff}[\mathcal{V}] = \frac{S_0[\mathcal{V}]}{e^2} + S_{Ind}[\mathcal{V}] = N_f \frac{S_0[\mathcal{V}^{mon}]}{\hat{e}^2} + \frac{S_0^{(2)}[\hat{\mathcal{V}}]}{\hat{e}^2} + S_{Ind}^{(2)}[\hat{\mathcal{V}}] + O\left(\frac{1}{\sqrt{N_f}}\right),$$

where S_0 is original action for a vector superfield and $\hat{e}^2 = e^2 N_f$. Linear term proportional to $\frac{\delta S_0}{\delta \mathcal{V}}\left[\mathcal{V}^{mon}\right]\hat{\mathcal{V}}$ vanishes because supersymmetric field configuration $\mathcal{V}^{mon}$ automatically minimizes the action S_0.

In the Euclidean space we have $\mathbb{Q}^+ = \mathbb{S}$ and $\bar{\mathbb{Q}}^+ = \bar{\mathbb{S}}$, hence special conformal transformations generated by $\bar{\mathbb{S}}$ ($\mathbb{S}$) leave the (anti-)BPS background invariant. The (anti-)BPS background breaks some of the global symmetries, and the full Hilbert space of states is given by a tensor product of the physical Fock space constructed from a vacuum state and a space of superfunctions on the appropriate supercoset. In Ref. [52] it was shown that unitarity condition applied to the anticommutator $\{\bar{\mathbb{Q}}, \bar{\mathbb{S}}\}$ in the $\mathcal{N} = 2$ supersymmetric theory implies that the conformal weight h and infrared R-charge R_{IR} of any state satisfy $h \geq R_{IR}$. Also, it follows from the anticommutator $\{\mathbb{Q}, \mathbb{S}\}$ that $h \geq -R_{IR}$ in a unitary theory. As explained in section 3.2, the infrared R-charge R_{IR} coincides with $U(1)_R$ charge R. Thus for the physical Fock space constructed from the (anti-)BPS vacuum, the unbroken subalgebra of the three-dimensional superconformal algebra implies that (minus) R-charge of a state can not exceed its conformal dimension:

$$h \geq \pm R, \tag{4.15}$$

with the lower bound saturated by the (anti-)chiral primary operators with vanishing spin.

Let us focus on the gluino contribution to the R-charge. We have two sets of gluinos $\hat{\lambda}$ and $\hat{\eta}$. Since $U(1)_R$ symmetry acts trivially on $\hat{\eta}$, the only contribution comes from $\hat\lambda$. Relevant quadratic terms in the effective action have the form (in R^3):

$$S_0^{(2)}[\hat\lambda,\hat{\bar\lambda}] = \int dx \left(i\hat{\bar\lambda}_+ \left[\vec\sigma\left(\vec\nabla - i\vec V^{mon}\right) \pm \frac{q}{r}\right]\hat\lambda_+ + i\hat{\bar\lambda}_- \left[\vec\sigma\left(\vec\nabla + i\vec V\right) \mp \frac{q}{r}\right]\hat\lambda_- + i\hat{\bar\lambda}^3\vec\sigma\vec\nabla\hat\lambda^3 \right),$$

$$S_{Ind}^{(2)}[\hat\lambda,\hat{\bar\lambda}] = \int dxdy \left(\hat{\bar\lambda}_+(x)O^{(+)}(x,y)\hat\lambda_+(y) + \hat{\bar\lambda}_-(x)O^{(-)}(x,y)\hat\lambda_-(y) \right.$$
$$\left. + \hat{\bar\lambda}^3(x)O^{(3)}(x,y)\hat\lambda^3(y) \right),$$

with $\hat\lambda_+ = (\hat\lambda^1 + i\hat\lambda^2)/\sqrt2$, $\hat\lambda_- = (\hat\lambda^1 - i\hat\lambda^2)/\sqrt2$, and

$$O^{(+)}(x,y) \sim \left\langle \bar\psi_1(x)\psi_1(y)\right\rangle \left\langle A_2(x)A_2^+(y)\right\rangle, \quad O^{(-)}(x,y) \sim \left\langle \bar\psi_2(x)\psi_2(y)\right\rangle \left\langle A_1(x)A_1^+(y)\right\rangle,$$

$$O^{(3)}(x,y) \sim \left\langle \bar\psi_1(x)\psi_1(y)\right\rangle \left\langle A_1(x)A_1^+(y)\right\rangle + \left\langle \bar\psi_2(x)\psi_2(y)\right\rangle \left\langle A_2(x)A_2^+(y)\right\rangle,$$

where we used the identities

$$\left\langle \bar{\tilde\psi}_1(x)\tilde\psi_1(y)\right\rangle = \left\langle \bar\psi_2(x)\psi_2(y)\right\rangle, \quad \left\langle \bar{\tilde\psi}_2(x)\tilde\psi_2(y)\right\rangle = \left\langle \bar\psi_1(x)\psi_1(y)\right\rangle,$$

$$\left\langle \tilde A_1(x)\tilde A_1^+(y)\right\rangle = \left\langle A_2(x)A_2^+(y)\right\rangle, \quad \left\langle \tilde A_2(x)\tilde A_2^+(y)\right\rangle = \left\langle A_1(x)A_1^+(y)\right\rangle.$$

R-charge contribution of $\hat\lambda_+$ and $\hat{\bar\lambda}_+$ can be expressed in terms of η-invariant of the Hamiltonian associated with $O^{(+)}$. If $\hat\lambda_+$ has zero-energy modes in the Fock space, it may lead to ambiguities in the R-charge computation. Let us show that such modes are not present. Induced action equation of motion $\delta S_{Ind}^{(2)}/\delta\hat{\bar\lambda}_+ = 0$ has the form

$$\int dy O^{(+)}(x,y)\hat\lambda_+(y) = 0. \tag{4.16}$$

Transforming to $S^2\times R$ and assuming $\hat\lambda_+$ independent of τ, we obtain

$$\int d\tau_y d\varphi_y d\theta_y O^{(+)}(\varphi_x,\theta_x,\tau_x;\varphi_y,\theta_y,\tau_y)\hat\lambda_+(\varphi_y,\theta_y) = 0. \tag{4.17}$$

If Eq.(4.17) has a non-trivial solution corresponding to an operator acting in the Fock space, $SU(2)_R$ symmetry implies that $\hat{\eta}_+ = (\hat{\eta}_1 + i\hat{\eta}_2)/\sqrt{2}$ also has a zero-energy mode. Then it follows from the supersymmetry transformation

$$\delta\hat{\phi}^*_+ = \sqrt{2}\bar{\xi}\hat{\bar{\eta}}_+ e^{-\tau/2},$$

that $\hat{\phi}_+$ has a mode with energy $-1/2$ in the Fock space associated with BPS monopole background. Let us denote the corresponding creation operator as $b^{+\{|E^-|=1/2\}}_{\hat{\phi}_+}$. Using the explicit form of the matter field energy modes it is straightforward to check that

$$O^{(-)} = O^{(+)}|_{\varphi_x \to -\varphi_x, \varphi_y \to -\varphi_y},$$

which implies that there is a zero-energy solution for $\hat{\lambda}_-$ as well. Hence, $\hat{\eta}_-$ has zero-energy mode and $\hat{\phi}_-$ has a mode with energy $-1/2$ which we denote as $b^{+\{|E^-|=1/2\}}_{\hat{\phi}_-}$. The product $b^{+\{|E^-|=1/2\}}_{\hat{\phi}_+} b^{+\{|E^-|=1/2\}}_{\hat{\phi}_-}$ is $U(1)_{gauge}$ invariant operator which has R-charge 2 and energy (conformal dimension) 1. Repeated action of this operator on any physical state with definite R-charge and conformal dimension will finally give a state with R-charge greater than the conformal dimension which violates the unitarity bound (4.15). Thus we conclude that $\hat{\lambda}_+$ does not have zero-energy modes in the Fock space constructed from the BPS vacuum. For anti-BPS monopole background similar analysis gives analogous conclusion.

Action $S^{(2)}_{Eff}$ acquires explicit τ-dependence on $S^2 \times R$ as a reminiscence of the fact that theory is not conformal invariant for $0 < \hat{e}^2 < \infty$. Let us define $g^2 = e^\tau \hat{e}^2$ and consider the resulting theory $S^{(2)}_g$, which can be viewed as conformal invariant deformation of $S^{(2)}_{Ind}$ with constant g being a deformation parameter. Let $\{E_k(g)\}$ be the energy spectrum of $\hat{\lambda}_+$, then the R-charge contribution is

$$< R_{\hat{\lambda}_+, \hat{\bar{\lambda}}_+} >_q = \lim_{\beta \to 0} \left(Z(q,\beta) - Z(0,\beta) \right), \quad Z(q,\beta) = \frac{1}{2} \sum_k \text{sign}\left[E_k(g) \right] e^{-\beta |E_k(g)|}. \tag{4.18}$$

Since $Z(q,\beta)$ is proportional to η-invariant, $< R_{\hat{\lambda}_+,\hat{\bar{\lambda}}_+} >_q$ is expected to be independent from g and, hence, can be computed in the region of small g. To make this argument rigorous it is necessary to show that $\hat{\lambda}_+$ does not have zero-energy modes for all values of the constant g. We hope to return to this problem in the future. If g is small the induced action terms can be ignored and we have gluinos moving in a monopole background $\mathcal{V}^{mon}$. The Hamiltonian eigen-value equation has a form

$$\left(\mathcal{H}_{\psi_2}|_{q\to 2q} - \frac{1}{2}\right)\hat{\lambda}_+ = E\hat{\lambda}_+,$$

where $\mathcal{H}_{\psi_2}$ is Hamiltonian for the matter field ψ_2. Using energy spectrum of ψ_2 given in the Appendix C, we find that the spectrum of $\hat{\lambda}_+$ in the limit of small g is given by $(n = 1, 2, \dots)$

$$E = -|q| - n - \frac{1}{2}, \quad \mp|q| - \frac{1}{2}, \qquad |q| + n - \frac{1}{2},$$

where each energy level has degeneracy $|2E + 1|$. We mention that energy level $E = \mp|q| - \frac{1}{2}$ has degeneracy $2|q|$ and is not present if $q = 0$. Using Eq.(4.18) in the small g region we obtain $< R_{\hat{\lambda}_+,\hat{\bar{\lambda}}_+} >_q = \mp|q|$.

Similar analysis can be implemented for $\hat{\lambda}_-$ and $\hat{\lambda}^3$. R-charge contribution of $\hat{\lambda}_-$ is identical to that of $\hat{\lambda}_+$, whereas in the small g limit $\hat{\lambda}^3$ is moving in the trivial $(\mathcal{V} = 0)$ background and does not contribute to the R-charge. Besides gluinos $\hat{\lambda}$, the only vector multiplet fields charged under the $U(1)_R$ symmetry are scalars ϕ and ϕ^*. Analogous calculations show that they do not contribute to the $O(1)$ terms of the R-charge. Therefore, in the large N_f limit, we have

$$\langle R\rangle_q = \pm\left(N_f - 2\right)|q|.$$

For the N-charge we have $\langle N_{\mathcal{V}}\rangle_q =< N_{\hat{\lambda},\hat{\bar{\lambda}}} >_q + < N_{\hat{\eta},\hat{\bar{\eta}}} >_q = 2 < N_{\hat{\lambda},\hat{\bar{\lambda}}} >_q$, where we used invariance of the (anti-)BPS background under $SU(2)_R$:

$$S^{(2)}_{Ind}[\hat{\eta}, \hat{\bar{\eta}}] = S^{(2)}_{Ind}[\hat{\lambda}, \hat{\bar{\lambda}}]\Big|_{\hat{\lambda}\to\hat{\eta},\hat{\bar{\lambda}}\to\hat{\bar{\eta}}}.$$

Calculations similar to those for the R-charge give $\langle N_{\mathcal{V}}\rangle_q = \mp 4|q|$, which implies

$$\langle N\rangle_q = \pm\left(2N_f - 4\right)|q|, \quad \langle B\rangle_q = \pm(4 - 2N_f)|q|.$$

4.2.5 Comparison with the mirror symmetry predictions

Mirror symmetry implications for the quantum numbers of w (see Eq.(4.6)) are trivially satisfied. Thus we conclude that $w \sim z$. Let us consider a physical state $|vac\rangle_q$. It is the lowest energy state and, hence, a superconformal primary. (Anti-)BPS background is annihilated by a supercharge $\bar{\mathbb{Q}}$ ($\mathbb{Q}$), therefore, a state $\bar{\mathbb{Q}}\,|vac\rangle_q$ $\left(\mathbb{Q}\,|vac\rangle_q\right)$ belongs to the Fock space associated with $|vac\rangle_q$. Since supercharge $\bar{\mathbb{Q}}$ ($\mathbb{Q}$) raises energy by 1/2 and has $U(1)_R$ charge (minus) one, we find that there is no such a physical state in the Fock space. Therefore, a state $|vac\rangle_q$ is annihilated by $\bar{\mathbb{Q}}$ ($\mathbb{Q}$) and corresponds to the insertion of the (anti-)chiral primary operator at the origin of R^3. Thus, conformal weight of $|vac\rangle_q$ equals its R-charge, i.e., $\pm\left(N_f - 2\right)|q|$. Background $\Phi = 0$ also corresponds to the (anti-)BPS monopoles in $\mathcal{N} = 2$ $SU(2)$ gauge theory which can be obtained by giving mass to the adjoint chiral field Φ. Therefore, matter contribution to the quantum numbers of the $\mathcal{N} = 2$ (anti-)BPS chiral monopoles is the same as in the $\mathcal{N} = 4$ theory. We also note that chiral primaries u and w are present for $\mathcal{N} = 4$ only and absent in $\mathcal{N} = 2$ theory. This observation implies

$$|vac\rangle^{\text{BPS}}_{|q|=1} \propto v(0)\,|0\rangle\,, \quad |vac\rangle^{\text{anti-BPS}}_{|q|=1} \propto v^{+}(0)\,|0\rangle\,.$$

Identity of $|vac\rangle^{\text{BPS}}_{|q|=1}$ and y quantum numbers gives $v \sim y$.

To obtain another (anti-)chiral primary state in the physical Fock space, we must act on a state $|vac\rangle^{\text{(anti-)BPS}}_{|q|=1}$ with an $U(1)_{gauge}$ invariant operator f such that it raises energy by $R(f)$ $(-R(f))$. It is easy to see that f can not be made of matter fields only. Indeed, the most general expression for (anti-)chiral primary $f(\mathcal{Q})\left|vac\right\rangle_q$ would be a superposition of gauge invariant states of the form $\left(a^+_{\mathcal{Q}}\right)^m \left(b^+_{\mathcal{Q}}\right)^p \left|vac\right\rangle_q$ with some non-negative integers m and p. However,

$$E\left(a^+_{\mathcal{Q}}\right) > \pm R\left(a^+_{\mathcal{Q}}\right), \quad E\left(b^+_{\mathcal{Q}}\right) > \pm R\left(b^+_{\mathcal{Q}}\right),$$

and the state $\left(a^+_{\mathcal{Q}}\right)^m \left(b^+_{\mathcal{Q}}\right)^p \left|vac\right\rangle_q$ is not an (anti-)chiral primary, unless $m = p = 0$.

Now we consider energy spectra of fields which belong to the vector multiplet. In the IR limit the only terms in the vector multiplet effective action are those induced by integration over the matter hypermultiplets. Let us show that gluinos $\hat{\eta}$, $\hat{\bar{\eta}}$, $\hat{\lambda}$, $\hat{\bar{\lambda}}$ do not have (anti-)chiral primary creation operators in the Fock space associated with the (anti-)BPS background. It follows from Eq.(4.15) that such modes can not be present in $\hat{\bar{\eta}}$ and $\hat{\bar{\lambda}}$, ($\hat{\eta}$ and $\hat{\lambda}$). Since R-charge of $\hat{\eta}$ vanishes, the (anti-)chiral primary creation operator corresponds to a mode with zero energy. It was shown in section 4.2.4 that $\hat{\eta}_+$ and $\hat{\eta}_-$, ($\hat{\bar{\eta}}_+$ and $\hat{\bar{\eta}}_-$), do not have such zero-energy modes. If a gauge invariant field $\hat{\eta}^{(3)}$ ($\hat{\bar{\eta}}^{(3)}$) has a creation operator with zero energy, then $SU(2)_R$ symmetry implies existence of $b^{+\{E=0\}}_{\hat{\lambda}^{(3)}}$, $(a^{+\{E=0\}}_{\hat{\lambda}^{(3)}})$, which is incompatible with Eq.(4.15). If present, an (anti-)chiral primary creation operator of $\hat{\lambda}^\alpha$ ($\hat{\bar{\lambda}}^\alpha$) has the form $b^{+\{|E^-|=1\}}_{\hat{\lambda}^\alpha}$, $(a^{+\{|E^-|=1\}}_{\hat{\lambda}^\alpha})$. Then $SU(2)_R$ symmetry ensures existence of $b^{+\{|E^-|=1\}}_{\hat{\eta}^\alpha}$, $(a^{+\{|E^-|=1\}}_{\hat{\eta}^\alpha})$. The supersymmetry transformation $\delta\hat{\phi}^{*\alpha} = \sqrt{2}\bar{\xi}\hat{\bar{\eta}}^\alpha e^{-\tau/2}$, $(\delta\hat{\phi}^\alpha = \sqrt{2}\xi\hat{\eta}^\alpha e^{\tau/2})$, implies a presence of $\hat{\phi}^\alpha$ mode with energy $|E^-| = 3/2$: $\hat{\phi}^\alpha \sim e^{3\tau/2}$, $\hat{\phi}^{*\alpha} \sim e^{-3\tau/2}$. Such modes should annihilate the right-hand side of $S^2 \times R$ counterpart of Eq.(4.10), (Eq.(4.11)), for all $\bar{\xi}$ (ξ) in the (anti-)BPS monopole background to ensure that an operator $b^{+\{|E^-|=1\}}_{\hat{\eta}^\alpha}$, $(a^{+\{|E^-|=1\}}_{\hat{\eta}^\alpha})$ is annihilated by $\bar{\mathbb{Q}}$ ($\mathbb{Q}$). It is easy to see that it can not be the case. Thus we conclude that gluinos do not have (anti-)chiral primary creation operators in the Fock space. Similar arguments reveal that it is true for $\hat{\chi}^\alpha$ and $\hat{V}^\alpha_i$ as well. Thus $\hat{\phi}$ and $\hat{\phi}^*$ are the only fields which could have such modes.

It follows from Eq.(4.15) that energy spectrum of $\hat{\phi}^{\alpha}$ satisfies $|E^-| \geq R\left(\hat{\phi}^{\alpha}\right) = 1$, $\left(E^+ \geq -R\left(\hat{\phi}^{*\alpha}\right) = 1\right)$. The (anti-)BPS background under consideration has vanishing expectation values of $U(1)_{gauge}$ invariant fields $\phi^{(3)}$ and $\phi^{*(3)}$. However, as it follows from Eqs.(4.8)-(4.11), setting $\phi^{(3)} = c$, $\left(\phi^{*(3)} = c\right)^3$, with constant c in R^3 leaves the (anti-)BPS background invariant under $\bar{\mathbb{Q}}$ ($\mathbb{Q}$). Therefore, the action $S_{Eff}[\mathcal{V}]$ is stationary on these field configurations. Since the constant c is arbitrary, quadratic part of $S_{Eff}[\hat{\mathcal{V}}]$ is stationary as well. In the IR limit it implies existence of the creation operator $b^{+\{|E^-|=1\}}_{\hat{\phi}^{(3)}}$ $\left(a^{+\{E^+=1\}}_{\hat{\phi}^{(3)}}\right)$, corresponding to the spinless mode of $\hat{\phi}^{(3)}$ $\left(\hat{\phi}^{*(3)}\right)$ on $S^2 \times R$. In the (anti-)BPS background any creation operator of $\hat{\phi}^{(3)}$ $\left(\hat{\phi}^{*(3)}\right)$ corresponding to a mode with energy $|E^-| = 1$ ($E^+ = 1$) saturates the unitarity bound given by Eq.(4.15). Hence, this mode has vanishing spin and is given by $const \times e^{\tau}$ on $S^2 \times R$. Thus the (anti-)chiral primary mode of $\hat{\phi}^{(3)}$ $\left(\hat{\phi}^{*(3)}\right)$ in the (anti-)BPS background is unique. Acting with the corresponding creation operators on the state $|vac\rangle^{\text{(anti-)BPS}}_{|q|=1}$ we obtain chiral primaries with the quantum numbers identical to those predicted for u (u^+). We have

$$b^{+\{|E^-|=1\}}_{\hat{\phi}^{(3)}} |vac\rangle^{\text{BPS}}_{|q|=1} \propto u(0)\,|0\rangle\,, \quad a^{+\{E^+=1\}}_{\hat{\phi}^{(3)}} |vac\rangle^{\text{anti-BPS}}_{|q|=1} \propto u^+(0)\,|0\rangle\,.$$

The BPS background breaks the Weyl subgroup Z_2 spontaneously and Z_2 invariance of the physical states is not required. However, it might be instructive to construct Z_2 invariant (anti-)chiral primary states by "integrating" the physical states over Z_2. Let us introduce a pair of gauge equivalent states

$$|vac\rangle^{\text{BPS}}_{q=1} \propto Y_+\,|0\rangle\,, \quad |vac\rangle^{\text{BPS}}_{q=-1} \propto Y_-\,|0\rangle\,.$$

Then,

$$v(0)\,|0\rangle \propto |vac\rangle^{\text{BPS}}_{q=1} + |vac\rangle^{\text{BPS}}_{q=-1}\,, \quad u(0)\,|0\rangle \propto \phi^{(3)}\left(|vac\rangle^{\text{BPS}}_{q=1} - |vac\rangle^{\text{BPS}}_{q=-1}\right).$$

[3] It also implies setting $\phi^{*(3)}$ $\left(\phi^{(3)}\right)$ to c^*/r^2 in R^3.

Similar construction can be made for the anti-BPS primary monopole operators as well.

We summarize results of this chapter as follows. In the case of $\mathcal{N} = 4$ $SU(2)$ gauge theory, the mirror symmetry predicts existence of two (anti-)chiral primary monopole operators corresponding to the (anti-)chiral primary operators x (x^+) and y (y^+) in the dual theory. The (anti-)chiral primary operator dual to y (y^+) exists in $\mathcal{N} = 2$ theory as well, whereas existence of the (anti-)chiral primary dual to x (x^+) is a special feature of $\mathcal{N} = 4$ theory. Using the radial quantization we have shown that a state $|vac\rangle_{|q|=1}^{\text{(anti-)BPS}}$ corresponds to the insertion of the (anti-)chiral primary monopole operator which is dual to the operator y (y^+) in the large N_f limit. We demonstrated that there is unique (anti-)chiral primary monopole operator with quantum numbers matching those of x (x^+). However we note that the relation in the chiral ring implied by Eq.(4.12) remains obscure.

It might be interesting to generalize our analysis for the monopole operators in supersymmetric $SU(N_c)$ gauge theories with $N_c > 2$.

Chapter 5

Conclusion

The idea that vortex-creation operators can be studied in the large N_f limit has been proposed previously in Ref. [19]. The approach taken there was to integrate out the matter fields, and then perform a duality transformation on the effective action for the gauge field. Then the vortex-creation operator is defined as the exponential of the dual photon. One drawback of this approach is that it is easy to miss fermionic zero modes, and consequently to misidentify the quantum numbers of the vortex-creating operator. It is preferable to keep the matter fields, and identify a vortex-creating operator by the property that its insertion causes a change of the gauge field topology. As we have seen above, this definition can be made precise by using radial quantization and the large N_f expansion.

We have constructed local operators in an interacting 3-D CFT, which carry vortex charge and therefore create Abrikosov-Nielsen-Olesen vortices in the Higgs phase. It was demonstrated that, for large N_f, conformal dimensions of these operators have leading terms of the order N_f. In the non-supersymmetric case, we showed that a monopole operator with the lowest possible dimension among the operators with unit vortex charge has zero spin and transforms in a non-trivial representation of the flavor group.

We have also studied certain monopole operators in 3-D SCFTs that arise in the IR limit of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ three-dimensional SQED as well as $\mathcal{N} = 4$

$SU(2)$ gauge theory. We constructed monopole operators that are conformal primary operators in short representations of the superconformal algebra. Certain predictions of three-dimensional mirror symmetry have been verified directly at the origin of the moduli space, where the IR theory is an interacting SCFT. Namely, we have shown that the chiral primary monopole operators are scalars under the $SU(2)_{rot}$ and transform trivially under the flavor symmetry group. Transformation properties under the global symmetries have been computed in the large N_f limit providing a new nontrivial verification of three-dimensional mirror symmetry.

In many cases one can go further and argue that certain results derived at large N_f remain valid even for N_f of order one. For example, the monopole operators in SQED have "anomalous" transformation laws under global symmetries, whose form is fixed by quasi-topological considerations (the Atiyah-Patodi-Singer index theorem). This implies that the global charges of monopole operators do not receive corrections at any order in the $1/N_f$ expansion. Furthermore, since our monopole operators belong to short representations of the superconformal algebra, their scaling dimensions are determined by their transformation law under R-symmetry. In the case of $\mathcal{N} = 4$ SQED and $\mathcal{N} = 4$ $SU(2)$ SYM, where it is easy to identify the relevant R-symmetry, this allows us to determine the exact scaling dimensions of monopole operators for all N_f. Our main assumption is that the $1/N_f$ expansion has a large enough domain of convergence. If we assume that $N_f = 1$ is within the convergence radius of this expansion, it can be concluded that a certain monopole operator in $\mathcal{N} = 4$ SQED with $N_f = 1$ is a (twisted) hypermultiplet whose lowest component is a scalar of dimension $1/2$. In a unitary theory, this is only possible if the hypermultiplet is free. Thus we are able to show that for $N_f = 1$ certain monopole operators satisfy free equations of motion. This is essentially the statement of mirror symmetry in this particular case.

Our computations were performed at the origin of the moduli space. Therefore, the agreement between our results and the predictions of mirror symmetry is a new check of this duality. In the case of SQED, we have been able to verify certain interesting relations in the chiral ring that follow from mirror symmetry. In the approach of Ref. [15], the origin of these relations was obscure.

Our main motivation for studying vortex-creation operators was the hope that this would enable us to give a constructive proof of 3-D mirror symmetry. We feel that these results go some way towards making the 3-D mirror symmetry conjecture into a theorem (on the physicist level of rigor). On the other hand, much remains to be done before it can be claimed that three-dimensional mirror symmetry is understood. First, it would be desirable to construct monopole operators directly, using the Hamiltonian formalism on $\mathbb{R}^3$, rather than by identifying the corresponding states on $S^2 \times \mathbb{R}$. Mandelstam's construction of soliton-creating operators in the sine-Gordon theory [2] serves as a model in this respect. Second, it would be interesting to find the mirror of more complicated observables in $\mathcal{N} = 4$ SQED. Third, mirror symmetry predicts that many 3-D gauge theories have "accidental" symmetries in the infrared limit [9, 17]. It appears possible to understand the origin of these symmetries using the methods presented in this manuscript. Fourth, for $N_f > 1$ the mirror theory of $\mathcal{N} = 4$ SQED is a gauge theory, and one would like to have a conceptual understanding of the origin of the dual gauge group. Although all abelian mirror pairs can be derived from the "basic" one, the derivation is rather formal and does not shed much light on this question.

It is natural to wonder if our approach to the construction of topological disorder operators has an analogue in four dimensions. In three dimensions, we defined the vortex charge of a local operator as the first Chern class of the gauge bundle evaluated on an $\mathbf{S}^2$ surrounding the insertion point. In four dimensions, we have $\mathbf{S}^3$ instead of $\mathbf{S}^2$, and since characteristic classes of vector bundles are even-dimensional, it appears impossible to define a similar topological charge for local operators. On the other hand, a B-field on an $\mathbf{S}^3$ can have non-trivial topology, since its field-strength is a 3-form. Thus, if there were an interacting 4-D CFT involving a B-field, one could define local operators which create topological disorder. In order for this to work, the field-strength 3-form must have dimension 3, so that its dual is a conserved primary current. Note that in the theory of a free B-field, the field-strength has dimension 2. In this case the dual current, although conserved, is not a primary, but a gradient of a free scalar. Thus in order to define a conformally-invariant topological charge, the 4-D CFT *must* be interacting. Unfortunately, no such theory is known at present. Perhaps there exists a duality-symmetric reformulation of $\mathcal{N} = 4$ $d = 4$ supersymmetric Yang-Mills theory which involves B-fields, and in which both W-bosons and dual W-bosons are described by topological disorder operators.

Appendix A

Radial quantization of 3-D QED in the IR limit. Monopole harmonics.

To solve for the energy spectrum of free fermions on $\mathbf{S}^2$ with a magnetic flux, we will use the fact that this system is related by a conformal transformation to the Dirac equation in R^3 in the monopole background. This allows us to use the machinery of "monopole harmonics" developed by Wu and Yang [64].

The three-dimensional Dirac operator on flat R^3 is given by

$$iD = -\vec{\sigma} \cdot \vec{\pi},$$

where σ_1, σ_2, and σ_3 are the Pauli matrices, and $\vec{\pi} = -i\vec{\nabla} + \vec{V}$ being the momentum operator. Following Ref. [64], let us define the generalized orbital angular momentum operator as

$$\vec{L} = \vec{r} \times \vec{\pi} - \frac{q\vec{r}}{2r} \tag{A-1}$$

with q being the vortex charge. It is straightforward to check that $\vec{L}$ defined this way satisfies the angular momentum algebra:

$$[L_j, x_k] = i\epsilon_{jkm}x_m,$$
$$[L_j, \pi_k] = i\epsilon_{jkm}\pi_m,$$
$$[L_j, L_k] = i\epsilon_{jkm}L_m.$$

Let us define the total angular momentum as

$$\vec{J} = \vec{L} + \frac{\vec{\sigma}}{2}$$

and take $r, \vec{L}^2, \vec{J}^2$, and J_3 as a complete set of observables (it is easy to check that they commute and are all self-adjoint with respect to the usual inner product). We also have $\left[\vec{J}, iD\right] = 0$, but $\left[\vec{L}^2, iD\right] \neq 0$. The simultaneous eigenfunctions of $\vec{L}^2$ and $\vec{L}_3$ are given by the monopole harmonics $Y_{q,l,m}(\theta, \varphi)$ which were constructed in Ref. [64]:

$$\vec{L}^2 Y_{q,l,m} = l(l+1)Y_{q,l,m}, \quad L_3 Y_{q,l,m} = mY_{q,l,m},$$
$$l = \frac{|q|}{2}, \frac{|q|}{2} + 1, \frac{|q|}{2} + 2, \ldots, \quad m = -l, \ldots, l.$$

The simultaneous eigenfunctions of $\left\{\vec{L}^2, \vec{J}^2, J_3\right\}$ will be denoted by Φ_{ljm_j}

$$\vec{L}^2 \Phi_{ljm_j} = l(l+1)\Phi_{ljm_j},$$
$$\vec{J}^2 \Phi_{ljm_j} = j(j+1)\Phi_{ljm_j},$$
$$J_3 \Phi_{ljm_j} = m_j \Phi_{ljm_j},$$

and are given by

$$\Phi_{ljm_j} = \begin{pmatrix} \sqrt{\frac{l+m+1}{2l+1}} Y_{q,l,m} \\ \sqrt{\frac{l-m}{2l+1}} Y_{q,l,m+1} \end{pmatrix} \text{ for } j = l + \frac{1}{2},\ (m_j = m + \frac{1}{2}),$$

$$\Phi_{ljm_j} = \begin{pmatrix} -\sqrt{\frac{l-m}{2l+1}} Y_{q,l,m} \\ \sqrt{\frac{l+m+1}{2l+1}} Y_{q,l,m+1} \end{pmatrix} \text{ for } j = l - \frac{1}{2},\ (l \neq 0, m_j = m + \frac{1}{2}).$$

We can summarize the possible value of l, j, m_j as follows:

- $j = \frac{|q|-1}{2}, \frac{|q|+1}{2}, \frac{|q|+3}{2}, \frac{|q|+5}{2}, \ldots$
(for $q = 0,\ j = \frac{|q|-1}{2}$ is not allowed);
- if $j = \frac{|q|-1}{2}$, then $l = j + \frac{1}{2} = \frac{|q|}{2}$, otherwise $l = j \pm \frac{1}{2}$;
- $m_j = -j, -(j-1), \ldots, j-1, j.$

A wave-function can be expanded as

$$\psi(\vec{r}) = \sum_{l,j,m_j} R_{ljm_j}(r) \Phi_{ljm_j}(\theta, \varphi),$$

where Φ_{ljm_j} are two-component spinors and R_{ljm_j} are scalars. Now we are ready to express iD in terms of the generalized angular momentum (A-1). Using

$$\left(\vec{\sigma} \cdot \vec{G}\right) \left(\vec{\sigma} \cdot \vec{K}\right) = \vec{G} \cdot \vec{K} + i\vec{\sigma} \cdot \left(\vec{G} \times \vec{K}\right)$$

for any $\vec{G}$ and $\vec{K}$ that commute with $\vec{\sigma}$, we can show that

$$\sigma_r(iD) = i\frac{\partial}{\partial r} - i\frac{1}{r}\vec{\sigma} \cdot \vec{L} - iq\frac{\sigma_r}{2r},$$

where $\sigma_r = \vec{\sigma} \cdot \vec{r}/r$. Now using ${\sigma_r}^2 = 1$, we obtain

$$\begin{aligned} iD &= \sigma_r \sigma_r (iD) = i\sigma_r \frac{\partial}{\partial r} - i\frac{\sigma_r}{r}\vec{\sigma}\cdot\vec{L} - iq\frac{1}{2r} \\ &= i\sigma_r \frac{\partial}{\partial r} - i\frac{\sigma_r}{r}(\vec{J}^2 - \vec{L}^2 - \frac{3}{4}) - iq\frac{1}{2r}. \end{aligned}$$

Thus the Dirac Lagrangian on R^3 in the presence of a monopole can be written as

$$\mathcal{L}_{R^3}[\psi^+, \psi] = \frac{i}{r}\psi^+\sigma_r \left(r\frac{\partial}{\partial r} - (\vec{J}^2 - \vec{L}^2 - \frac{3}{4}) - \frac{q}{2}\sigma_r \right) \psi.$$

Setting $r = e^\tau$ and performing a Weyl rescaling

$$g_{\mu\nu} \to e^{-2\tau} g_{\mu\nu}, \quad \psi, \psi^+ \to e^{-\tau}\psi, e^{-\tau}\psi^+, \quad \vec{V} \to \vec{V},$$

we obtain the Lagrangian on $\mathbf{S}^2 \times R$:

$$\mathcal{L}_{\mathbf{S}^2\times R}[\psi^+, \psi] = i\psi^+\sigma_r \left(\frac{\partial}{\partial \tau} - (\vec{J}^2 - \vec{L}^2 + \frac{1}{4}) - \frac{q}{2}\sigma_r \right) \psi. \tag{A-2}$$

Note that the norm

$$\int_{\mathbf{S}^2} r^2 \, d\Omega \, \psi^+ \sigma_r \psi$$

on R^3 is transformed to the norm

$$\int_{\mathbf{S}^2} \psi^+ \sigma_r \psi$$

on $\mathbf{S}^2 \times R$. Taking into account the above results, the Euclidean equation of motion for ψ following from the Lagrangian (A-2) is

$$\begin{aligned} &\frac{dR_{ljm_j}(\tau)}{d\tau} - \left(j(j+1) - l(l+1) + \frac{1}{4} \right) R_{ljm_j}(\tau) \\ &- \sum_{l'j'm'_j} qR_{l'j'm'_j}(\tau)\langle ljm_j|\sigma_r|l'j'm'_j\rangle = 0, \end{aligned} \tag{A-3}$$

where $\langle ljm_j|\sigma_r|l'j'm'_j\rangle$ denotes $\int d\Omega\phi^\dagger_{ljm_j}\sigma_r\phi_{l'j'm'_j}$. The identity $\left[\vec{J},\sigma_r\right]=0$ ensures that

$$\langle ljm_j|\sigma_r|l'j'm'_j\rangle = \delta_{jj'}\delta_{m_jm'_j}\langle ljm_j|\sigma_r|l'jm_j\rangle,$$

and thus the equation (A-3) has the form

$$\frac{dR_{ljm_j}(\tau)}{d\tau} - \left(j(j+1)-l(l+1)+\frac{1}{4}\right)R_{ljm_j}(\tau)$$
$$-\sum_{l'} qR_{l'jm_j}(\tau)\langle ljm_j|\sigma_r|l'jm_j\rangle = 0.$$

Let us suppress the j, m_j indices, and denote $R_{(l=j-\frac{1}{2})jm_j}$ as R^a, $|l=j-\frac{1}{2}, jm_j\rangle$ as $|a\rangle$, $R_{(l=j+\frac{1}{2})jm_j}$ as R^b, $|l=j+\frac{1}{2}, jm_j\rangle$ as $|b\rangle$, $\langle a|\sigma_r|a\rangle$ as σ_{aa}, $\langle a|\sigma_r|b\rangle$ as σ_{ab}, $\langle b|\sigma_r|a\rangle$ as σ_{ba}, and $\langle b|\sigma_r|b\rangle$ as σ_{bb}. Then for any given j, m_j, we have two coupled first-order differential equations:

$$\frac{dR^a(\tau)}{d\tau} = \left(j+\frac{1}{2}\right)R^a(\tau) + \frac{q}{2}(\sigma_{aa}R^a(\tau)+\sigma_{ab}R^b(\tau)),$$

$$\frac{dR^b(\tau)}{d\tau} = -\left(j+\frac{1}{2}\right)R^b(\tau) + \frac{q}{2}(\sigma_{bb}R^b(\tau)+\sigma_{ba}R^a(\tau)).$$

A straightforward calculation of the matrix elements σ_{aa}, σ_{ab}, and σ_{bb} gives

$$\sigma_{aa} = \frac{-q}{2j+1},\ \sigma_{bb} = \frac{q}{2j+1},\ \sigma_{ab} = -\sqrt{1-\left(\frac{q}{2j+1}\right)^2},$$

and $\sigma_{ba} = \sigma^*_{ab} = \sigma_{ab}$.

The energy spectrum can be read off from the behavior of the solutions as a function of τ: a solution with energy E behaves as $e^{-E\tau}$. The results are as follows.

Case (i): $q=0$.

The two equations decouple, and we find

$$R^a(\tau) = C^a e^{(j+\frac{1}{2})\tau}, \quad R^b(\tau) = C^b e^{-(j+\frac{1}{2})\tau},$$

where C^a and C^b are integration constants, and $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots$ There are no zero-energy solutions.

Case (ii): $q \neq 0$, $j = \frac{|q|-1}{2}$.

In this case, the first equation is absent, and the second equation gives

$$R^b(\tau) = C,$$

with an arbitrary constant C. This solution has zero energy and degeneracy $2j+1 = |q|$.

Case (iii): $q \neq 0$ *and* $j = \frac{|q|-1}{2} + p$, $p = 1, 2, \ldots$

In this case, we have

$$\begin{aligned} R^a(\tau) &= qC_1 e^{\frac{\tau}{2}\sqrt{(2j+1)^2-q^2}} + qC_2 e^{-\frac{\tau}{2}\sqrt{(2j+1)^2-q^2}}, \\ R^b(\tau) &= \left[\sqrt{(2j+1)^2-q^2} - (2j+1)\right] C_1 e^{\frac{\tau}{2}\sqrt{(2j+1)^2-q^2}} \\ &+ \left[\sqrt{(2j+1)^2-q^2} + (2j+1)\right] C_2 e^{-\frac{\tau}{2}\sqrt{(2j+1)^2-q^2}}, \end{aligned}$$

where C_1 and C_2 are integration constants. The corresponding energies are

$$E_p = \pm\frac{1}{2}\sqrt{(2j+1)^2 - q^2} = \pm\sqrt{|q|p + p^2},$$

with degeneracies $2j + 1 = |q| + 2p$. Note that the spectrum is symmetric under $q \to -q$. The regularized Casimir energy is given by

$$E_{reg}(\beta) = -N_f \sum_{p=0}^{\infty} (2p + |q|)\sqrt{p^2 + p|q|}\, e^{-\beta\sqrt{p^2+p|q|}}.$$

We renormalize it by requiring that the Casimir energy of the vacuum with $q = 0$ be zero. That is, we subtract from the above sum a similar sum with $q = 0$, and then take the limit $\beta \to 0$. Using the Abel-Plana summation formula

$$\sum_{k=0}^{\infty} F(k) = \frac{1}{2}F(0) + \int_0^{\infty} dx\, F(x) + i \int_0^{\infty} dt\, \frac{F(it) - F(-it)}{e^{2\pi t} - 1},$$

we obtain a finite answer for the Casimir energy:

$$E_{Casimir} = N_f \frac{1}{6}\sqrt{1+|q|}(|q|-2) + \\ 2N_f \,\mathrm{Im} \int_0^{\infty} dt \left((it + \frac{|q|}{2} + 1)\sqrt{(it + \frac{|q|}{2} + 1)^2 - \frac{q^2}{4}} \right) \frac{1}{e^{2\pi t} - 1}.$$

The integral cannot be expressed in terms of elementary functions, but can be easily evaluated numerically for any q.

Appendix B

Radial quantization of 3-D $\mathcal{N} = 2$ SQED in the IR limit

We start with the Lagrangian of $\mathcal{N} = 1$ $d = 4$ SQED in the conventions of Wess and Bagger [65] and perform a Wick rotation to Euclidean signature:

$$\mathcal{L}_{R^4} = -\mathcal{L}_{R^{1,3}}|_{x^0=-it}\,, \quad V_0|_{R^{1,3}} = i\chi|_{R^4}\,,$$

where V_0 is the time-like coordinate of the $U(1)$ connection. Then we require that all fields be independent of the Euclidean time t. This procedure gives the Lagrangian for $\mathcal{N} = 2$ $d = 3$ SQED on Euclidean R^3:

$$\begin{aligned}\mathcal{L} =& i\bar{\psi}\vec{\sigma}(\vec{\nabla} + i\vec{V})\psi + i\chi\bar{\psi}\psi + i\bar{\tilde{\psi}}\vec{\sigma}(\vec{\nabla} - i\vec{V})\tilde{\psi} - i\chi\bar{\tilde{\psi}}\tilde{\psi} + \chi^2\left(AA^* + \tilde{A}\tilde{A}^*\right)\\ &+ ([\vec{\nabla} + i\vec{V}]A)([\vec{\nabla} - i\vec{V}]A^*) + ([\vec{\nabla} - i\vec{V}]\tilde{A})([\vec{\nabla} + i\vec{V}]\tilde{A}^*)\\ &- D(AA^* - \tilde{A}\tilde{A}^*) + i\sqrt{2}(A\bar{\psi}\bar{\lambda} - A^*\psi\lambda - \tilde{A}\bar{\tilde{\psi}}\bar{\lambda} + \tilde{A}^*\tilde{\psi}\lambda) + O\left(\frac{1}{e^2}\right).\end{aligned}$$

In the infrared limit $e \to \infty$ the kinetic terms for the vector multiplet can be ignored. Note also that in the $e \to \infty$ limit the equation of motion for D enforces the vanishing of D-terms.

To go from R^3 to $S^2 \times R$, we perform a Weyl rescaling of the Euclidean metric $ds^2 = dr^2 + r^2 d\Omega^2$ by a factor $1/r^2$. If we set $r = e^\tau$, then τ is an affine parameter on

R. The component fields of Q must be rescaled as follows:

$$\psi \to e^{-\tau}\psi, \quad \bar{\psi} \to e^{-\tau}\bar{\psi}, \quad A \to e^{-\frac{\tau}{2}}A, \quad A^* \to e^{-\frac{\tau}{2}}A^*.$$

The component fields of $\tilde{Q}$ transform in a similar way. The bosonic fields in the vector multiplet transform as follows:

$$\chi \to e^{-\tau}\chi, \quad \vec{V} \to \vec{V}.$$

To find the one-particle energy spectra for charged fields, we use the procedure and notations of Appendix A. The Lagrangian for ψ and $\bar{\psi}$ in the background of the (anti-)BPS monopole on R^3 has the following form

$$\mathcal{L}_{S^2\times R}[\psi,\bar{\psi}] = i\bar{\psi}\sigma_r\left[\frac{\partial}{\partial\tau} - \left(\vec{J}^2 - \vec{L}^2 + \frac{1}{4}\right) - \frac{q}{2}\sigma_r \mp \frac{q}{2}\sigma_r\right]\psi,$$

where the upper (lower) sign corresponds to a BPS (anti-BPS) monopole. A solution with energy E has the form $\psi \sim e^{-E\tau}$, $\bar{\psi} \sim e^{E\tau}$. The above Lagrangian is the same as (A-2), except for the last term in brackets. We will not repeat the diagonalization procedure and simply quote the resulting energy spectra for ψ and $\tilde{\psi}$:

$$-\frac{|q|}{2} - p, \quad \mp\frac{|q|}{2}, \quad \frac{|q|}{2} + p,$$

where $p = 1, 2, \ldots$. Whereas the energy spectra for $\bar{\psi}$ and $\bar{\tilde{\psi}}$ is

$$-\frac{|q|}{2} - p, \quad \pm\frac{|q|}{2}, \qquad , \frac{|q|}{2} + p.$$

Each energy-level has spin $j = |E| - 1/2$ and degeneracy $2j + 1 = 2|E|$. The Lagrangian for A, A^* is

$$\mathcal{L}_{S^2\times R}[A, A^*] = [(\vec{\nabla}_a + i\vec{V}_a)A][(\vec{\nabla}_b - i\vec{V}_b)A^*]g^{ab} + \frac{1}{4}AA^* + \chi^2 AA^*.$$

The corresponding equation of motion for A has the from

$$\frac{d^2}{d\tau^2}A = \left(\vec{L}^2 + \frac{1}{4}\right)A,$$

where $\vec{L}$ is the generalized angular momentum defined in Eq.(A-1). Using the known spectrum of $\vec{L}^2$, we find the energy spectra for A and $\tilde{A}$:

$$-\frac{|q|-1}{2} - p, \qquad \frac{|q|-1}{2} + p, \qquad p = 1, 2, \ldots .$$

The degeneracy of each eigenvalue is again $2|E|$, and each eigenspace is an irreducible representation of the rotation group with spin $j = |E| - 1/2$.

Now we are ready to compute the Casimir energy of the (anti-)BPS vacuum state with q units of vortex charge

$$E = N_f \lim_{\beta\to 0}\left\{\left[-2\sum_{n=0}^{\infty}\left(\frac{|q|}{2}+n+1\right)^2 e^{-\beta(|q|/2+n+1)} - 2\sum_{n=0}^{\infty}\left(\frac{|q|}{2}+n\right)^2 e^{-\beta(|q|/2+n)} + \right.\right.$$

$$\left.\left. +4\sum_{n=0}^{\infty}\left(\frac{|q|}{2}+n+1/2\right)^2 e^{-\beta(|q|/2+n+1/2)}\right] - \text{“the same”}|_{q=0}\right\}.$$

The previous expression can be brought to the form

$$E = 2N_f \lim_{\beta\to 0}\frac{\partial^2}{\partial\beta^2}\left\{\left(1-e^{-\beta|q|/2}\right)\tanh\frac{\beta}{4}\right\} = \frac{N_f|q|}{2}. \tag{B-1}$$

Note that the Casimir energies of the BPS and (anti-)BPS vacuum states with identical vortex charges are equal.

Appendix C

Radial quantization of 3-D $\mathcal{N} = 4$ $SU(2)$ gauge theory in the IR limit

We begin with $\mathcal{N} = 2$ Lagrangian in four-dimensional Minkowski space in the notations Ref. [65] for the vector multiplet $\mathcal{V}$ in the adjoint representation of $SU(2)$ and hypermultiplets $\mathcal{Q}^s$ in the fundamental representation of the gauge group

$$\mathcal{L}_{\mathcal{V}}^{R^{3,1}} = \frac{1}{8e^2}\left(\int d^2\theta \mathrm{Tr}\left(W^\alpha W_\alpha\right) + h.c.\right) + \frac{1}{e^2}\int d^2\theta d^2\bar{\theta}\mathrm{Tr}(\Phi^+ e^{2V}\Phi),$$

$$\mathcal{L}_{\mathcal{Q}}^{R^{3,1}} = \int d^2\theta d^2\bar{\theta}\sum_{s=1}^{N_f}\left(Q^{s+}e^{2V}Q^s + \tilde{Q}^s e^{-2V}\tilde{Q}^{s+}\right) + \left(\int d^2\theta W + h.c.\right),$$

where a superpotential $W = i\sqrt{2}\sum_{s=1}^{N_f}\tilde{Q}^s\Phi Q^s$. Let us perform the Wick rotation to R^4

$$\mathcal{L}_{R^4} = -\mathcal{L}_{R^{3,1}}|_{x^0=-it}, \quad V_0^\alpha|_{R^{3,1}} = i\chi^\alpha|_{R^4},$$

and assume that all fields are independent of the Euclidean time t. This procedure gives $\mathcal{N} = 4$ supersymmetric Lagrangian in three-dimensional Euclidean space:

$$\mathcal{L}_{\mathcal{Q}}^{R^3} = i\bar{\psi}\vec{\sigma}\left(\vec{\nabla} + i\vec{V}\right)\psi + i\bar{\psi}\chi\psi + \left(\left[\vec{\nabla} + i\vec{V}\right]A\right)^+\left(\left[\vec{\nabla} + i\vec{V}\right]A\right) + A^+\chi^2 A +$$

$$+ i\sqrt{2}\left(\bar{\psi}\bar{\lambda}A - A^+\lambda\psi\right) - F^+F - A^+DA + i\bar{\tilde{\psi}}\vec{\sigma}(\vec{\nabla} - i\vec{V}^T)\tilde{\psi} - i\bar{\tilde{\psi}}\chi^T\tilde{\psi} +$$

$$+\left(\left[\vec{\nabla}+i\vec{V}\right]\tilde{A}^{+}\right)^{+}\left(\left[\vec{\nabla}+i\vec{V}\right]\tilde{A}^{+}\right)+\tilde{A}\chi^{2}\tilde{A}^{+}-\tilde{F}\tilde{F}^{+}+\tilde{A}D\tilde{A}^{+}-$$

$$-i\sqrt{2}\left(\tilde{A}\bar{\lambda}\bar{\tilde{\psi}}-\tilde{\psi}\lambda\tilde{A}^{+}\right)+i\sqrt{2}\left[\tilde{F}\phi A+\tilde{\psi}\phi\psi+\tilde{A}\phi F+\tilde{\psi}\eta A+\tilde{A}\eta\psi+\tilde{A}KA-h.c.\right],$$

where summation over flavor indices is implied and summation over color indices is performed in the order of multiplication, e.g.,

$$\tilde{F}\phi A\equiv\tilde{F}_{B}\phi^{B}{}_{C}A^{C}.$$

To obtain a theory on $S^2\times R$ we perform the Weyl rescaling $g_{ij}\rightarrow r^2 g_{ij}$ and introduce $\tau=\ln r$. The matter fields transform as

$$\left(\psi,\bar{\psi},\tilde{\psi},\bar{\tilde{\psi}}\right)\rightarrow e^{-\tau}\left(\psi,\bar{\psi},\tilde{\psi},\bar{\tilde{\psi}}\right),\quad\left(A,A^{+},\tilde{A},\tilde{A}^{+}\right)\rightarrow e^{-\frac{\tau}{2}}\left(A,A^{+},\tilde{A},\tilde{A}^{+}\right).$$

For fields in the vector multiplet we have

$$\left(\chi,\phi,\phi^{+}\right)\rightarrow e^{-\tau}\left(\chi,\phi,\phi^{+}\right),\quad\vec{V}\rightarrow\vec{V},\qquad\left(\lambda,\bar{\lambda},\eta,\bar{\eta}\right)\rightarrow e^{-\frac{3}{2}\tau}\left(\lambda,\bar{\lambda},\eta,\bar{\eta}\right).$$

The (anti-)BPS background is diagonal in color indices and, therefore, we may use results of Appendix B for matter energy spectra in a background of $U(1)$ monopole. Solutions with energy E have the form $Q,\tilde{Q}\sim e^{-E\tau}$, whereas $Q^+,\tilde{Q}^+\sim e^{E\tau}$. To summarize, we have ($n=1,2,\dots$):

$$E=-\frac{|q|}{2}-n,\quad\mp\frac{|q|}{2},\qquad\frac{|q|}{2}+n,\tag{C-1}$$

for ψ^s_a, $\tilde{\psi}^s_a$ and

$$E=-\frac{|q|}{2}-n,\quad\pm\frac{|q|}{2},\qquad\frac{|q|}{2}+n,$$

for $\bar{\psi}^s_a$ and $\bar{\tilde{\psi}}^s_a$. Scalar fields A^s_a, $\tilde{A}^s_a$, A^{+s}_a, and $\tilde{A}^{+s}_a$ have

$$E=-\frac{|q|-1}{2}-n,\quad\frac{|q|-1}{2}+n.$$

Each energy level with energy E has a spin $j = |E| - 1/2$ and a degeneracy $2|E|$. We notice that fermionic energy spectra are not invariant under $E \to -E$. The fact that A and $\tilde{A}^+$ have identical energy spectra is consistent with the action of $SU(2)_R$ symmetry. On the other hand fields ψ and $\bar{\tilde{\psi}}$ have different energy spectra which conforms with the breaking of $SU(2)_N$ symmetry to a $U(1)_N$ subgroup which doesn't mix these fermionic fields.

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