THE MOTION OF A DISC AT ANGLE OF ATTACK
IN A RAPIDLY ROTATING FLUID

Thesis by

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ABSTRACT

The motion of a thin disc in a slightly viscous incompressible rotating fluid is studied. The axis of rotation is termed the vertical axis and the fluid and disc are in a container which is bounded by horizontal planes. Nonlinear inertia terms and unsteady effects are assumed small relative to the Coriolis acceleration and hence neglected. Of most importance is the fact that the disc is inclined to the container walls at an angle, α, which is not necessarily small. The angle is assumed to be large enough so that there are no closed geostrophic contours between the disc and the walls.

Since the equations of motion are linear, the motions in the six degrees of freedom are considered independently. In all cases, a Taylor column is present although, in all but one case, there is fluid flowing across the boundary of the column. The detailed structure of the shear column is examined for infinitesimal angle of incidence. It is shown that it is possible to solve for the geostrophic flow without actually doing the detailed solution for the shear column structure.

A static stability study is done and the disc is found to be unstable to small disturbances.

The motion of an elliptical plate at finite angle of attack for which the Taylor column is circular is studied. Using the techniques developed for infinitesimal α, an equation relating the geostrophic flow inside and outside the Taylor column is proposed. This equation is general enough to be used for arbitrary motion of any thin plate.

However, only the solution for horizontal translation in a specific direction of the elliptical plate is done.

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LIST OF SYMBOLS

a	=	disc radius
A_n, B_n, C_n, D_n	=	Fourier coefficients
$A(\theta), B(\theta)$	=	arbitrary functions in $\frac{1}{4}$-layer
$a(\theta), b(\theta), c(\theta)$	=	arbitrary functions in α-layer
E	=	Ekman number
$\underline{e}_\theta$	=	unit vector in circumferential direction
$f(x)$	=	arbitrary function in expression for pressure
$f(\theta)$	=	shorthand for $f(a\cos\theta)$
$\underline{F}$	=	force on disc
$\underline{F}_{buoy}$	=	buoyancy force on disc
F_{cor}, F_{cen}	=	Coriolis and centrifugal forces on disc
F_x, F_y, F_z	=	x, y and z components of $\underline{F}$
g	=	gravitational acceleration constant
h	=	distance between container walls
h_T, h_B	=	distances between disc and container walls
$H(\theta)$	=	Heaviside stepfunction
$\underline{i}, \underline{j}, \underline{k}$	=	unit vectors in x, y and z directions
$J(\theta)$	=	excess flux of fluid in α-layer
K	=	constant defined by equation (188)
$\underline{M}$	=	moment acting on disc
m	=	mass of the disc
$\underline{n}$	=	unit vector normal to disc
$\underline{n}_e$	=	unit vector normal to edge of elliptical plate
p	=	reduced pressure
p, p_T, p_B	=	constants defined by (161)

LIST OF SYMBOLS (Cont'd)

p	=	physical pressure
$\underline{Q}$	=	excess flux of fluid in Ekman layer
$Q_e(\theta)$	=	excess fluid flux from Ekman layer on disc into $\frac{1}{3}$-layer
Ro	=	Rossby number
r, θ, z	=	cylindrical polar coordinates
R	=	specified radial distance
S	=	constant defined by equation (122)
t	=	time
$\underline{t}$	=	unit vector parallel to elliptical plate
U, V, W	=	velocity components of disc in x, y, z directions
$\underline{u}'$	=	fluid velocity as seen by a rotating observer
$\underline{u}_w, \underline{u}_p$	=	velocity of container walls and disc
u, v, w	=	velocity components in cylindrical polar coordinates
u, v, w	=	velocity components in rectangular cartesian coordinates
$U(\theta), V(\theta)$	=	radial and circumferential velocity components at the edge of the Taylor column
$V_g(x)$	=	arbitrary function in expression for v inside the Taylor column
$w^{(n)}$	=	velocity component normal to disc
x, y, z	=	rectangular cartesian coordinates
$\bar{x}$	=	boundary layer coordinate
X_n	=	constants defined by (385)

LIST OF SYMBOLS (Cont'd)

α	=	angle of attack of disc
β	=	constant defined by equation (359)
γ	=	constant defined by equation (142)
δ	=	boundary layer thickness
δ_E	=	Ekman layer thickness
δ_T, δ_B	=	constants defined by equation (62)
Δ	=	change in elevation of disc across shear column
ϵ_x, ϵ_y, ϵ_z	=	constants defined by equations (242)-(244)
η	=	$\frac{1}{3}$-layer variable defined by (63)
κ	=	constant defined in equation (236)
λ'	=	constant defined by equation (80)
μ	=	viscosity of fluid
ν	=	kinematic viscosity of fluid
ξ, y, ζ	=	rectangular cartesian coordinate system fixed on elliptical plate
ξ	=	$\frac{1}{4}$-layer variable defined by (53)
ξ_I^*	=	α-layer variable defined by (62)
ξ^*	=	Gα-layer variable defined by (171)
ρ	=	density of fluid
ρ_d	=	density of disc
$\underline{\tau}$	=	viscous stress on disc
ψ	=	streamfunction
$\underline{\Omega}$	=	angular velocity of rotating coordinate frame

LIST OF SYMBOLS (Cont'd)

Subscripts and Superscripts

B, T	=	quantity below and above disc respectively
G	=	geostrophic quantity outside Taylor column
g	=	geostrophic quantity inside Taylor column
$G\alpha$	=	quantity in $G\alpha$-layer
I	=	dummy subscript representing either B or T
α	=	quantity in α-layer
θ	=	quantity in θ-layer
$\theta_{\frac{1}{4}}$	=	quantity in $\theta_{\frac{1}{4}}$-layer
$\frac{1}{3}$	=	quantity in $\frac{1}{3}$-layer
$\frac{1}{4}$	=	quantity in $\frac{1}{4}$-layer

1. INTRODUCTION

A container filled with a viscous incompressible fluid rotates at angular velocity, Ω, about what shall be termed its vertical axis. The container is bounded above and below by horizontal walls. Between the walls there is a "thin" disc of radius, a, inclined at an angle, α, to the horizontal walls. By "thin" is meant the thickness of the disc is small compared to the thickness of the Ekman Layers which are present on the surface of the disc.

The aim of this thesis is to examine the flow caused by "slow" motion of the disc in the limit of a "slightly" viscous or "rapidly" rotating fluid of kinematic viscosity, ν. The terminology, "slow" motion, means that the Rossby number, Ro, defined by

$$Ro = U/\Omega a$$

where U is the speed characteristic of the motion of the disc, is very small. "Slightly" viscous or "rapidly" rotating means that the Ekman number, E, defined by

$$E = \nu/\Omega a^2$$

is small.

The first restriction permits linearization of the equations of motion as it is a measure of the ratio of the nonlinear inertia terms to the Coriolis acceleration. This means each of the six motions corresponding to the six degrees of freedom can be studied independently and the results superposed. The second restriction permits the use of boundary layer techniques, i. e., singular perturbation theory.

In general, when a body moves slowly in a rapidly rotating fluid it carries a column of fluid along. Such a column of fluid is named a Taylor column after G. I. Taylor.[1] As shown by Moore and Saffman,[2, 3] it is not always true that the fluid in the Taylor column moves with the velocity of the body. In all but one of the cases considered here the velocity of the fluid inside the Taylor column is not the same as the velocity of the disc.

The motion ensuing from both vertical and horizontal translation of the disc with zero α has been studied by Moore and Saffman.[2, 3] For the former motion, the geostrophic flow is determined uniquely by the Ekman compatibility relation (or Ekman condition for short). The shear column is inserted to make the solution analytic across the boundary of the Taylor column and to complete the circulation of the fluid between the disc and the walls. Hence, the shear column can be said to be "passive". For the latter motion, the Ekman condition is not sufficient to determine the geostrophic flow. Appeal to the dynamics of the shear column is made to remove the indeterminateness of the geostrophic flow. In this case, the shear column can be said to be "active".

For the problems at hand, the geostrophic flow is again found to be indeterminate and the shear column active. However, an extra complication arises over what was encountered for Moore and Saffman's horizontal translation. When α is zero, the Ekman condition shows the geostrophic flow to be irrotational both inside and outside the Taylor column. This is no longer true when

$$\tan\alpha \gg E^{\frac{1}{2}}$$

However, the techniques first suggested by Stewartson[4] and later exploited by Moore and Saffman for analyzing the shear column are still applicable. Their implementation is more subtle here than in previous problems.

The physical meaning of this lower bound on α is the following. The Ekman layer thickness, δ_E, is in order of magnitude,

$$\delta_E \sim E^{\frac{1}{2}} a$$

At a fixed radial distance, R, from the center of the disc, as we move around in the circumferential direction, the elevation of the disc changes by an amount Δ, where

$$\Delta = R\tan\alpha$$

Hence,

$$\frac{\Delta}{\delta_E} \sim \frac{R}{a}\,\frac{\tan\alpha}{E^{\frac{1}{2}}} \sim \frac{\tan\alpha}{E^{\frac{1}{2}}} \gg 1$$

An important fact becomes obvious from this observation. Namely, there are no closed geostrophic contours. The meaning of this statement is that for geostropy to exist, there must be closed contours of constant total height which is an immediate consequence of the Taylor-Proudman theorem. This condition is clearly violated here. The physical implication is clarified by first noting that the geostrophic motion must not stretch the rigid body rotation vortex lines (Greenspan[5]). Such stretching can only occur in the Ekman layers and the shear column. Any relative motion of fluid from one elevation to another is hence prohibited as it would carry a vortex line whose length would have to experience a change of order Δ.

In Chapter 2, the detailed structure of the shear column for horizontal translation is developed for infinitesimal α. As α increases from zero the shear column structure evolves as follows:

(a) $\alpha = 0$: The x and y translations are identical by symmetry in this case. Moore and Saffman[3] have solved this problem. There is a flow through the Taylor column of constant velocity which is deflected at an angle 18.4° to the free stream direction;

(b) $\alpha \ll E^{\frac{1}{2}}$: To leading order, the solution will simply be the motion for vanishing α;

(c) $\alpha \sim E^{\frac{1}{2}}$: This is a transition range between the problem solved by Moore and Saffman where closed geostrophic contours exist and where they cease to exist. Closed geostrophic contours are still present so that the solution should still be essentially the solution for zero α;

(d) $E^{\frac{1}{2}} \ll \alpha \ll E^{\frac{1}{4}}$: The lower bound on α shows that closed geostrophic contours are not present. The upper bound insures the usual Stewartson sandwich structure of the shear column. It consists of an inner layer whose thickness is proportional to $E^{\frac{1}{3}}$ sandwiched by a fatter layer with thickness of order $E^{\frac{1}{4}}$. These layers are referred to as the $\frac{1}{3}$-layer and the $\frac{1}{4}$-layer respectively. It turns out that yet another, thicker, layer is required on the inside edge of the shear column to complete the solution. The thickness of this layer is proportional to $E^{\frac{1}{2}}\cot\alpha$ and is referred to as the $G\alpha$-layer;

(e) $\alpha \sim E^{\frac{1}{4}}$: The triple layer structure of (d) collapses to a double layer structure as the $G\alpha$-layer and $\frac{1}{4}$-layer merge. The solution bears some similarity to the solution in both case (d) and case (f) which is discussed next;

(f) $E^{\frac{1}{4}} \ll \alpha \ll E^{1/6}$: A new layer appears in place of the $\frac{1}{4}$-layer at the inner edge of the shear column. It is thinner than a $\frac{1}{4}$-layer having thickness $(E\cot\alpha)^{\frac{1}{3}}$ and is referred to as an α-layer. Unlike the $\frac{1}{4}$-layer, this layer is "aware" of the fact that the disc is tilted. The $\frac{1}{3}$-layer still sees the disc as being at zero angle of attack. A $\frac{1}{4}$-layer is still present at the outer edge of the shear column;

(g) $\alpha \sim E^{1/6}$ and $E^{1/6} \ll \alpha \ll 1$: The α-layer, $\frac{1}{3}$-layer, $\frac{1}{4}$-layer structure of (f) is maintained. However, for these ranges, the $\frac{1}{3}$-layer begins to feel the effect of the disc inclination.

This discussion indicates that little is to be learned from consideration of (b) while (a) has been done already. (c) might yield interesting results pertaining to the transition from the range where closed geostrophic contours exist to the range where they do not exist. However, since the aim of this thesis is to study flows in which closed geostrophic contours are absent, case (c) will not be considered.

The detailed solution for case (d) is given in section 2.4 including the solution for the shear column structure. The shear column structure for case (e) is sketched in section 2.5 but only the geostrophic flow is determined. Cases (f) and (g) are done in detail in 2. 3.

It is found that the solution for the geostrophic flow is the same for all cases considered. Thus, in section 2. 6, an equation applicable in all cases is derived in a form general enough to be applied to motions other than horizontal. It is argued, of course, that the shear column structure is essentially the same for all motions of the disc to justify the use of this equation. Further justification is given in section 3. 3.

The rising disc is considered in Chapter 3, utilizing the equation derived in Chapter 2 to determine the geostrophic flow without repeating the detailed shear column analysis.

Chapter 4 deals with the three rotations.

In all cases the force and moment on the disc are evaluated. These results are used in Chapter 5 to perform a static stability study. Motion of the disc is found to be unstable to small disturbances. If, however, the disc is free to translate but not rotate under the action of its own buoyancy and the centrifugal force, it is shown that the buoyancy force causes the disc to move horizontally! On the other hand, the centrifugal force causes the disc to rise!

Chapter 6 reveals some of the difficulties encountered in constructing a solution when α is finite. An equation relating the geostrophic flows inside and outside the Taylor column is proposed. This equation is utilized to solve for horizontal translation of an elliptical plate in a specified direction.

2. HORIZONTAL TRANSLATION FOR INFINITESIMAL ANGLE OF ATTACK

2.1 Statement of the Problem

The upper wall of the container is located at

$$z = h_T$$

and the lower wall is located at

$$z = -h_B$$

where z is the vertical coordinate. The center of the disc is the origin of a rectangular cartesian coordinate system as shown in Figure 1. The equation of the disc is

$$z = - x\tan\alpha \tag{1}$$

with

$$x^2 + y^2 \leqslant a^2$$

For convenience, let

$$h = h_T + h_B$$

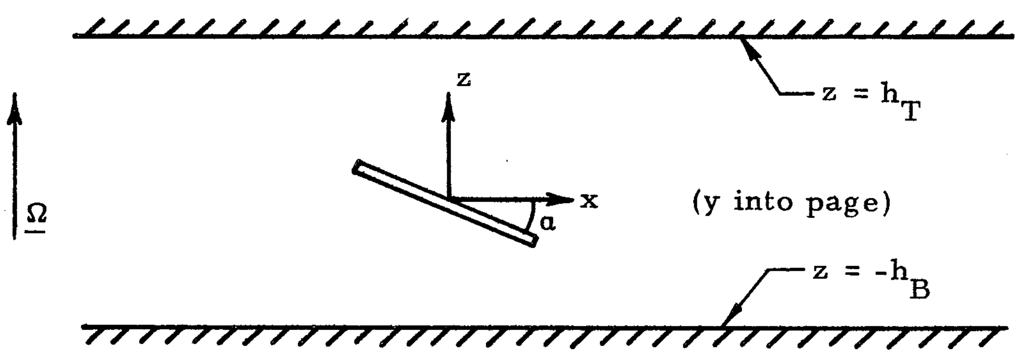

Figure 1. Basic Geometry of the Container and the Disc.

The full equations of motion written in a rotating coordinate frame are, for an incompressible fluid,

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + 2 \underline{\Omega} \times \underline{u} = -\nabla p + \nu \nabla^2 \underline{u} \tag{2}$$

$$\nabla \cdot \underline{u} = 0 \tag{3}$$

where p is the reduced pressure defined by

$$p = \frac{\mathcal{P}}{\rho} - \tfrac{1}{2}\Omega^2 r^2 \tag{4}$$

$\mathcal{P}$ being the physical pressure and r the radial distance from the origin in a horizontal plane.[1] Also,

$$\underline{\Omega} = \Omega \underline{k}$$

$\underline{k}$ is a unit vector in the vertical direction.

The problem is to determine the motion of the fluid due to the steady translation of the disc in a horizontal direction. If the speed of the disc is U, then

$$\frac{|\underline{u} \cdot \nabla \underline{u}|}{|\underline{\Omega} \times \underline{u}|} \sim \frac{U}{\Omega a} = Ro$$

Assuming that

$$Ro \ll 1 \tag{5}$$

and noting that the motion is steady, the first two terms in (2) can be

----

[1] It is assumed in (4) that the disc is at the center of the container. If this is not true, r must be altered accordingly. Since the disc is formally regarded as being of zero thickness, the calculation of forces on the disc is not affected by this term.

dropped so that

$$2\underline{\Omega} \times \underline{u} = -\nabla p + \nu\nabla^2\underline{u} \tag{6}$$

Since (6) is linear, horizontal translation in any direction can be studied by first studying translation in the x and y directions independently and then superposing the results. It is instructive to let the disc remain at rest and move the walls and the fluid at infinity with velocity

$$\underline{u}_w = U\underline{i} \tag{7}$$

for the x-translation and

$$\underline{u}_w = V\underline{j} \tag{8}$$

for the y-translation. $\underline{i}$ and $\underline{j}$ are unit vectors aligned with the x and y-axes respectively. Once the solution is obtained for these motions, a simple Galilean transformation will yield the solution for the disc advancing into an ambient fluid with the walls at rest.

2.2 The Geostrophic Flow

In the geostrophic regions, the viscous term in (6) is negligible so that in component form (3) and (6) reduce to

$$-2\Omega v = -\frac{\partial p}{\partial x} \tag{9}$$

$$2\Omega u = -\frac{\partial p}{\partial y} \tag{10}$$

$$0 = -\frac{\partial p}{\partial z} \tag{11}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{12}$$

where u, v and w are the velocity components in the x, y and z directions respectively. It is obvious from (9)-(12) that

$$\frac{\partial \underline{u}}{\partial z} = \underline{0}$$

which is the Taylor-Proudman theorem. The Ekman condition shows that on the walls (Greenspan[5])

$$w = -\tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \underline{k} \cdot \text{curl}\,(\underline{u}_G - \underline{u}_w) \quad \text{on} \quad z = h_T \qquad (13)$$

$$w = \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \underline{k} \cdot \text{curl}\,(\underline{u}_G - \underline{u}_w) \quad \text{on} \quad z = -h_B \qquad (14)$$

For an inclined surface with unit normal $\underline{n}$, the Ekman condition is

$$(\underline{u}_G - \underline{u}_w) \cdot \underline{n} = \pm \tfrac{1}{2} \left(\frac{\nu}{|\underline{\Omega} \cdot \underline{n}|}\right)^{\frac{1}{2}} \underline{n} \cdot \text{curl}\,(\underline{u}_G - \underline{u}_w) \qquad (15)$$

where + corresponds to being above the surface and - to being below the surface. From (1) it follows that on the disc

$$\underline{n} = \underline{i} \sin\alpha + \underline{k} \cos\alpha \qquad (16)$$

Hence, (15) becomes

$$w + u \tan\alpha = \pm \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] \quad \text{on} \quad z = 0^{\pm} \qquad (17)$$

where it has been noted that the velocity of the disc is zero and α is very small as compared to unity.

Introduce cylindrical polar coordinates (r, θ, z) with the velocity components becoming (u, v, w). For r greater than a, (13) and (14) coupled with the Taylor-Proudman theorem show that

$$\underline{k} \cdot \text{curl } (\underline{u}_G - \underline{u}_w) = \underline{k} \cdot \text{curl } \underline{u}_G = 0$$

However,

$$\nabla^2 p_G = 2\Omega \underline{k} \cdot \text{curl } \underline{u}_G$$

So, the pressure satisfies Laplace's equation and the general solution for p_G is

$$p_G(r, \theta) = A_o + B_o\theta + C_o \log r + D_o \theta \log r$$
$$+ \sum_{n=1}^{\infty} \left\{ [A_n (\tfrac{r}{a})^n + B_n (\tfrac{r}{a})^{-n}] \cos n\theta + [C_n (\tfrac{r}{a})^n + D_n (\tfrac{r}{a})^{-n}] \sin n\theta \right\} \tag{18}$$

To have a single-valued pressure, necessarily

$$B_o = D_o = 0$$

The constant A_o can be absorbed in p_G with no loss of generality and can hence be taken to be zero.

The asymptotic behavior of $p_G(r, \theta)$ as $r \to \infty$ is

$$p_\infty(r, \theta) \sim \begin{cases} -2\Omega\, Ur\sin\theta & \text{for x-translation} \\ 2\Omega\, Vr\cos\theta & \text{for y-translation} \end{cases} \tag{19}$$

The radial flux of fluid in the Ekman layers on the top and bottom walls can be shown (Greenspan[5]) to be

$$Q_r = -(\tfrac{\nu}{\Omega})^{\frac{1}{2}} \frac{C_o}{2\Omega}$$

This must be balanced by the radial flux of fluid in the geostrophic interior, Q_G. However, it is easily demonstrated that

$$Q_G = h \int_o^{2\pi} u_G(r,\theta) r d\theta = 0$$

which is valid to $O(E^{\frac{1}{2}})$. Hence, continuity will be violated unless

$$C_o = 0$$

Then, (18) reduces to

$$p_G(r,\theta) = p_\infty(r,\theta) + \sum_{n=1}^{\infty} [B_n \cos n\theta + D_n \sin n\theta] (\frac{a}{r})^n \tag{20}$$

For r less than a, the boundary conditions (13) and (14) read

$$w_g = -\frac{1}{2}(\frac{\nu}{\Omega})^{\frac{1}{2}} [\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}] \quad \text{on} \quad z = h_T \tag{21}$$

$$w_g = \frac{1}{2}(\frac{\nu}{\Omega})^{\frac{1}{2}} [\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}] \quad \text{on} \quad z = -h_B \tag{22}$$

Appealing to (21) and (22) for the scaling on w, (there is no loss of generality in taking u and v to be of the same order of magnitude) there follows

$$w_g \sim E^{\frac{1}{2}} u_g$$

Since by hypothesis

$$E^{\frac{1}{2}} << \tan\alpha$$

necessarily

$$w_g << u_g \tan\alpha$$

and similarly the right hand side of (17) is negligible. So, to leading order,

$$u_g = 0$$

From (12) there results

$$v_g = \begin{cases} V_g^T(x) & \text{for} \quad z > 0 \\ V_g^B(x) & \text{for} \quad z < 0 \end{cases}$$

and hence

$$p_g = \begin{cases} p_g^T(x) & \text{for} \quad z > 0 \\ p_g^B(x) & \text{for} \quad z < 0 \end{cases}$$

where

$$\frac{dp_g^I(x)}{dx} = 2\Omega V_g^I(x) \quad \text{for} \quad I = T, B$$

$V_g^T(x)$ and $V_g^B(x)$ are arbitrary functions of x. Thus, p_g is constant along lines of constant x so that at the inner edge of the shear column the pressure is an even function of θ. See Figure 2.

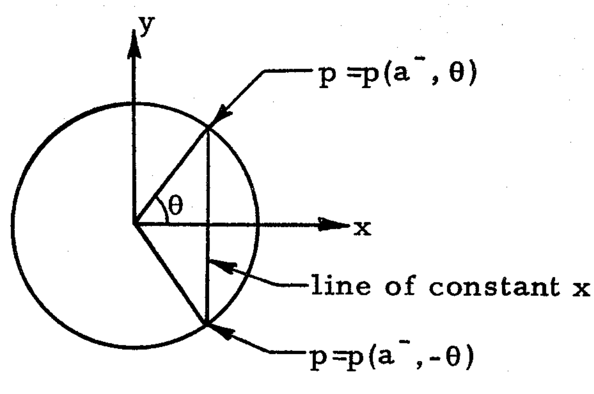

Figure 2. Geometrical Demonstration that the Pressure is an Even Function of θ.

Experience has shown, as will be verified a posteriori, that in vertical shear columns it is true that to leading order the radial momentum equation is

$$2\Omega v = \frac{\partial p}{\partial r}$$

Hence, if the thickness of the shear column is δ where

$$\delta << a$$

then the jumps in p and v are related in order of magnitude by

$$\Delta p \sim \frac{\delta}{a} \Delta v << \Delta v$$

where $\Delta(\ )$ denotes the jump in ( ) across the shear column. Since p and v are of the same orders of magnitude in the geostrophic regions this means that p is continuous to leading order across the shear column. Finally, noting that in the geostrophic regions

$$2\Omega u = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$

necessarily u is also continuous across the shear column.

At this point the order of magnitude of the geostrophic flow inside the Taylor column can be determined. If the flow is of order much larger than unity, then requiring continuity of the pressure across the shear column forces $p(a, \theta)$ to vanish. This is a consequence of the fact that the order of the flow outside the Taylor column is fixed at unity by conditions at infinity. But this means

$$p_g^T(x) = p_g^B(x) = 0$$

so that

$$V_g^T(x) = V_g^B(x) = 0$$

to all orders greater than one. Hence, the geostrophic flow inside the Taylor column is also of order unity. Furthermore, since p is independent of z outside the Taylor column, necessarily

$$p_g^T(x) = p_g^B(x) \equiv p_g(x)$$

and

$$V_g^T(x) = V_g^B(x) \equiv V_g(x)$$

Proceeding to next order, clearly, from (21) and (22)

$$w_g = \begin{cases} -\frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{dV_g(x)}{dx} & \text{for } z > 0 \\ \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{dV_g(x)}{dx} & \text{for } z < 0 \end{cases}$$

Inserting this expression into (17) there follows

$$u_g = \pm\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \frac{dV_g(x)}{dx} \quad \text{for } z \gtrless 0$$

Then, integrating to find the pressure there follows finally

$$u_g = \pm\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \frac{dV_g(x)}{dx} \tag{23}$$

$$v_g = V_g(x) + \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \left[\frac{df^{\frac{T}{B}}(x)}{dx} \mp y \frac{d^2V_g(x)}{dx^2}\right] \tag{24}$$

$$w_g = \mp \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{dV_g(x)}{dx} \tag{25}$$

$$p_g = p_g(x) + 2\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \left[f^{\frac{T}{B}}(x) \mp y \frac{dV_g(x)}{dx}\right] \tag{26}$$

where

$$\frac{dp_g(x)}{dx} = 2\Omega V_g(x) \qquad (27)$$

and $f^T(x)$, $f^B(x)$ are arbitrary functions of x. The superscripts T and B correspond to values above and below the disc respectively. When ambiguity in sign occurs in (23)-(26) the top sign corresponds to positive z and the bottom sign to negative z.

Finally, (20) can be simplified further by using the fact that the geostrophic pressure must be an even function of θ. The results are

$$p_G(r,\theta) = -2\Omega aU[\frac{r}{a} - \frac{a}{r}]\ \sin\theta + \sum_{n=1}^{\infty} B_n(\frac{a}{r})^n \cos n\theta \qquad (28)$$

$$u_G(r,\theta) = U[1-(\frac{a}{r})^2]\ \cos\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n(\frac{a}{r})^{n+1} \sin n\theta \qquad (29)$$

$$v_G(r,\theta) = -U[1+(\frac{a}{r})^2]\ \sin\theta - \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n(\frac{a}{r})^{n+1} \cos n\theta \qquad (30)$$

valid for x-translation. Similarly,

$$p_G(r,\theta) = 2\Omega Vr\cos\theta + \sum_{n=1}^{\infty} B_n(\frac{a}{r})^n \cos n\theta \qquad (31)$$

$$u_G(r,\theta) = V\sin\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n(\frac{a}{r})^{n+1} \sin n\theta \qquad (32)$$

$$v_G(r,\theta) = V\cos\theta - \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n(\frac{a}{r})^{n+1} \cos n\theta \qquad (33)$$

valid for y-translation.

Before going on to consideration of the dynamics of the shear column it is worthwhile to pause and discuss the implications of the results to date. Equations (23), (24) and (25) show that to leading order the flow is possibly rotational. Furthermore, since the rigid body rotation vortex lines cannot be stretched in the geostrophic regions there can be no flow to leading order in the x-direction which is borne out by (23). However, the vortex lines are permitted to translate across the disc in the y-direction as no stretching occurs with such motion.

As a final note, define the velocity components for the geostrophic flow outside the Taylor column evaluated at $r = a$ to be

$$U(\theta) = u_G(a, \theta) \qquad (34)$$

$$V(\theta) = v_G(a, \theta) \qquad (35)$$

2.3 Shear Column Structure for $E^{\frac{1}{4}} << \alpha << 1$

To determine the function $V_g(x)$ and the Fourier coefficients B_n, it is necessary to examine the dynamics of the shear column. Since there are two small parameters (α and E) in this problem it turns out that several distinct ranges must be considered as is usually the case in such problems. The order in which the ranges will be studied is:

(a) $E^{\frac{1}{4}} << \alpha << 1$;

(b) $E^{\frac{1}{2}} << \alpha << E^{\frac{1}{4}}$;

(c) $\alpha \sim E^{\frac{1}{4}}$

The reason this order is chosen is because (a) is the most physically interesting range. Also, a new boundary layer appears which has not been studied in great detail to date. In (b) a new layer also appears, but the range is not physically relevant. (c) is a transition regime and adds very little new information.

2.3.1 The Shear Column Equations

If the equations of motion are written in boundary layer coordinates as shown in Figure 3, the shear column appears to be locally plane and the approximate equations of motion become

$$-2\Omega v = -\frac{\partial p}{\partial \bar{x}} + \nu \frac{\partial^2 u}{\partial \bar{x}^2} \tag{36}$$

$$2\Omega u = -\frac{1}{a}\frac{\partial p}{\partial \theta} + \nu \frac{\partial^2 v}{\partial \bar{x}^2} \tag{37}$$

$$0 = -\frac{\partial p}{\partial z} + \nu \frac{\partial^2 w}{\partial \bar{x}^2} \tag{38}$$

$$\frac{\partial u}{\partial \bar{x}} + \frac{1}{a}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \tag{39}$$

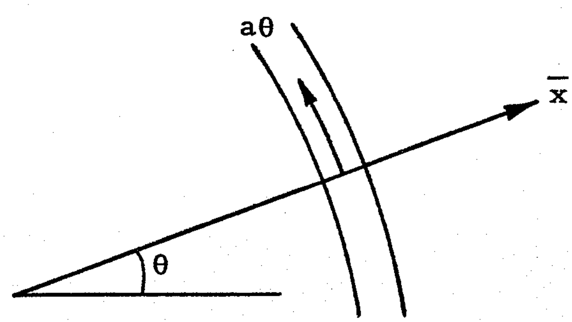

Figure 3. Boundary Layer Coordinates

where

$$\bar{x} \equiv r - a \tag{40}$$

It has been assumed that

$$\frac{\partial}{\partial \bar{x}} >> \frac{1}{a}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}$$

to arrive at these equations.

Now, from (39), if the thickness of the shear column is δ, then

$$u \sim \frac{\delta}{a} v$$

so that

$$\frac{\nu \dfrac{\partial^2 u}{\partial \bar{x}^2}}{\Omega v} \sim \frac{E}{(\delta/a)} << 1$$

provided

$$\delta >> Ea \tag{41}$$

Assuming (41) to be valid, (36) reduces to

$$-2\Omega v = -\frac{\partial p}{\partial \bar{x}} \tag{42}$$

Checking a posteriori shows that indeed (41) is satisfied. Equations (37), (38), (39) and (42) can be combined to yield the shear column equations:

$$\frac{\partial w}{\partial z} = -\frac{\nu}{2\Omega} \frac{\partial^3 v}{\partial \bar{x}^3} \tag{43}$$

$$\frac{\partial v}{\partial z} = \frac{\nu}{2\Omega} \frac{\partial^3 w}{\partial \bar{x}^3} \tag{44}$$

The Stewartson sandwich layer structure consists of an inner layer of thickness

$$\delta_{\frac{1}{3}} \sim E^{\frac{1}{3}}$$

called a $\frac{1}{3}$-layer, and a thicker surrounding layer of thickness

$$\delta_{\frac{1}{4}} \sim E^{\frac{1}{4}}$$

called a $\frac{1}{4}$-layer. Both of these layers are fat as compared to an Ekman layer so that the Ekman condition still applies in the shear column. That is, the Ekman layer appears as a sheet of zero thickness on the $\frac{1}{3}$-layer and $\frac{1}{4}$-layer scaling. In boundary layer coordinates, (13), (14) and (15) become:

$$w = -\frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{\partial v}{\partial \bar{x}} \qquad \text{on } z = h_T \tag{45}$$

$$w = (v\sin\theta - u\cos\theta)\tan\alpha \pm \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{\partial v}{\partial \bar{x}} \qquad \text{on } z = 0^{\pm} \tag{46}$$

$$w = \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{\partial v}{\partial \bar{x}} \qquad \text{on } z = -h_B \tag{47}$$

2.3.2 The $\frac{1}{4}$-Layer

On the outer edge of the shear column, there is a standard $\frac{1}{4}$-layer. The scaling on the vertical velocity component, $w_{\frac{1}{4}}$, follows from (45) and (47) which is

$$w_{\frac{1}{4}} \sim E^{\frac{1}{4}} v_{\frac{1}{4}}$$

So, (43) and (44) reduce to

$$\frac{\partial w_{\frac{1}{4}}}{\partial z} = -\frac{\nu}{2\Omega} \frac{\partial^3 v_{\frac{1}{4}}}{\partial \bar{x}^3} \tag{48}$$

$$\frac{\partial v_{\frac{1}{4}}}{\partial z} = 0 \tag{49}$$

Hence, noting that

$$\frac{\partial^2 w_{\frac{1}{4}}}{\partial z^2} = 0$$

the boundary conditions determine $w_{\frac{1}{4}}$. Therefore

$$w_{\frac{1}{4}} = \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \left[\frac{h_T - h_B}{2h} - \frac{z}{h} \frac{\partial v_{\frac{1}{4}}}{\partial \bar{x}}\right] \tag{50}$$

and the equation for $v_{\frac{1}{4}}$ is thus

$$\frac{\partial^3 v_{\frac{1}{4}}}{\partial \bar{x}^3} = \left(\frac{4\Omega}{\nu h^2}\right)^{\frac{1}{2}} \frac{\partial v_{\frac{1}{4}}}{\partial \bar{x}} \tag{51}$$

The solution is

$$v_{\frac{1}{4}} = A(\theta)e^{-p\xi} + B(\theta) \tag{52}$$

where

$$p^2 = 2/ah, \qquad \xi = \bar{x}/E^{\frac{1}{4}} \tag{53}$$

This layer is commonly referred to as being quasi-geostrophic as $u_{\frac{1}{4}}$ and $v_{\frac{1}{4}}$ are independent of z just as in the geostrophic region. However, $w_{\frac{1}{4}}$ is a linear function of z which is not true in the geostrophic interior.

2.3.3 The α-Layer

It is not possible to have a $\frac{1}{4}$-layer on the inner edge of the shear column as the first term on the right hand side of (46) is much larger than the second. So, if we appeal to this fact to establish the scaling for w then

$$w \sim v\tan\alpha$$

To determine the thickness of the layer use (43) to conclude that the thickness δ is

$$\delta \sim (E\cot\alpha)^{\frac{1}{3}}$$

This layer is fat as compared to a $\frac{1}{3}$-layer and thin as compared to a $\frac{1}{4}$-layer. In summary the relevant equations are

$$\frac{\partial w_\alpha}{\partial z} = -\frac{\nu}{2\Omega}\frac{\partial^3 v_\alpha}{\partial \bar{x}^3} \tag{54}$$

$$\frac{\partial v_\alpha}{\partial z} = 0 \tag{55}$$

subject to:

$$w_\alpha = 0 \qquad \text{on } z = -h_B, h_T \tag{56}$$

$$w_\alpha = (v_\alpha \sin\theta - U(\theta)\cos\theta)\tan\alpha \quad \text{on } z = 0^{\pm} \tag{57}$$

This layer will be referred to as the α-layer and quantities in this layer bear the subscript α. In (57), account has been taken of the fact that to first order,

$$u_\alpha(\bar{x}, \theta) = U(\theta)$$

The change in elevation of the disc across this layer is

$$\Delta_\alpha \sim E^{\frac{1}{3}}\,\alpha^{-\frac{1}{3}} \cdot \alpha \;= E^{\frac{1}{3}}\alpha^{\frac{2}{3}}$$

so that

$$\frac{\Delta_\alpha}{E^{\frac{1}{2}}} \sim \left(\frac{\alpha}{E^{\frac{1}{4}}}\right)^{\frac{2}{3}} >> 1$$

Hence, the α-layer is "aware" of the fact that the disc is tilted.

Pedlosky and Greenspan[6] find an α-layer in their work on the sliced cylinder problem.

Proceeding to the α-layer solution

$$w_a = \begin{cases} (v_a^T \sin\theta - U(\theta)\cos\theta)\tan\alpha\left[1 - \frac{z}{h_T}\right] \; ; \; \bar{x} < 0, \; z > 0 \\ \\ (v_a^B \sin\theta - U(\theta)\cos\theta)\tan\alpha\left[1 + \frac{z}{h_B}\right] \; ; \; \bar{x} < 0, \; z < 0 \end{cases} \quad (58)$$

Hence, the equations for v_a^I become

$$\frac{\partial^3 v_a^T}{\partial \bar{x}^3} = \frac{2\Omega \tan\alpha \sin\theta}{\nu h_T} (v_a^T - U(\theta)\cos\theta) \qquad \text{for } z > 0 \quad (59)$$

$$\frac{\partial^3 v_a^B}{\partial \bar{x}^3} = -\frac{2\Omega \tan\alpha \sin\theta}{\nu h_B} (v_a^B - U(\theta)\cot\theta) \qquad \text{for } z < 0 \quad (60)$$

The solution for v_a is:

$$v_a(\bar{x}, \theta) = U(\theta)\cot\theta + \begin{cases} a_T(\theta)\, e^{(\sin\theta)^{\frac{1}{3}} \xi_T^*} & \text{for } \theta > 0, \; z > 0 \\ e^{\frac{1}{2}|\sin\theta|^{\frac{1}{3}} \xi_T^*} \left[b_T(\theta)\cos \frac{\sqrt{3}}{2} |\sin\theta|^{\frac{1}{3}} \xi_T^* + c_T(\theta)\sin \frac{\sqrt{3}}{2} |\sin\theta|^{\frac{1}{3}} \xi_T^* \right] & \text{for } \theta < 0, \; z > 0 \\ e^{\frac{1}{2}(\sin\theta)^{\frac{1}{3}} \xi_B^*} \left[b_B(\theta)\cos \frac{\sqrt{3}}{2} (\sin\theta)^{\frac{1}{3}} \xi_B^* + c_B(\theta)\sin \frac{\sqrt{3}}{2} (\sin\theta)^{\frac{1}{3}} \xi_B^* \right] & \text{for } \theta > 0, \; z < 0 \\ a_B(\theta) e^{|\sin\theta|^{\frac{1}{3}} \xi_B^*} & \text{for } \theta < 0, \; z < 0 \end{cases}$$

(61)

where

$$\xi_I^* = \bar{x}/\delta_I \quad \text{and} \quad \delta_I = \left(\frac{\nu h_I}{2\Omega \tan \alpha}\right)^{\frac{1}{3}} \text{ for } I = T, B \qquad (62)$$

It is necessary to consider the dynamics of the $\frac{1}{3}$-layer to determine the relations between the functions $A(\theta)$, $B(\theta)$, $a_I(\theta)$, $b_I(\theta)$ and $c_I(\theta)$. Since the $\frac{1}{3}$-layer appears as a sheet of zero thickness on the $\frac{1}{4}$-layer and α-layer scaling, the relations will be termed jump relations.

2.3.4 The $\frac{1}{3}$-Layer and the Jump Relations

For a $\frac{1}{3}$-layer, equations (43) and (44) suffer no further approximation. The scaling in this layer is

$$w_{\frac{1}{3}} \sim v_{\frac{1}{3}}$$

Hence, the boundary conditions (45) and (47) show that on the walls

$$w_{\frac{1}{3}} = O(E^{1/6})$$

while (46) shows that on the disc

$$w_{\frac{1}{3}} = \max\left[\,O(E^{1/6}), O(\alpha)\right]$$

The ambiguity in the latter case is a consequence of the fact that the change in elevation of the disc across the $\frac{1}{3}$-layer is given by

$$\Delta_{\frac{1}{3}} \sim E^{\frac{1}{3}}\alpha$$

which means

$$\Delta_{\frac{1}{3}}/E^{\frac{1}{2}} \sim \alpha/E^{1/6}$$

If $\alpha << E^{1/6}$, the $\frac{1}{3}$-layer structure is unaffected by the disc's inclination. However, when $\alpha \sim E^{1/6}$ or $\alpha >> E^{1/6}$ the $\frac{1}{3}$-layer becomes "aware" of the fact that the disc is tilted and special care must be taken in constructing the solution.

It is not necessary to solve the $\frac{1}{3}$-layer equations to determine the unknown functions defined in (52) and (61) which in turn lead to the solution in the geostrophic regions from matching. Rather, jump relations across the $\frac{1}{3}$-layer can be obtained by exploiting the techniques first tried by Stewartson[4] and extended by Moore and Saffman.[2, 3] It will become necessary to extend the techniques a bit further to determine the solution.

To set up the method, define

$$\eta \equiv \bar{x} / E^{\frac{1}{3}} \tag{63}$$

Matching requires that

$$\lim_{\xi^* \to 0^-} v_\alpha(\xi^*, \theta, z) = \lim_{\eta \to -\infty} v_{\frac{1}{3}}(\eta, \theta, z)$$

and

$$\lim_{\xi \to 0^+} v_{\frac{1}{4}}(\xi, \theta, z) = \lim_{\eta \to +\infty} v_{\frac{1}{3}}(\eta, \theta, z)$$

Now, comparison of (53), (62) and (63) shows

$$\xi = E^{1/12}\eta \qquad \text{and} \qquad \xi_I^* = \left(\frac{2}{a^2 h_I}\right)^{\frac{1}{3}} (\tan\alpha)^{\frac{1}{3}} \eta$$

and hence

$$
v_{\frac{1}{3}} \sim \begin{cases}
[U(\theta)\cot\theta + a_T(\theta)] + \left(\frac{2}{h_T a^2}\tan\alpha\sin\theta\right)^{\frac{1}{3}} a_T(\theta)\eta + \ldots \quad \eta \to -\infty, \theta > 0, z > 0 \\
[U(\theta)\cot\theta + b_T(\theta)] + \frac{1}{2}\left|\frac{2}{hTa^2}\tan\alpha\sin\theta\right|^{\frac{1}{3}} (b_T(\theta) + \sqrt{3}\, c_T(\theta))\eta + \ldots \\
\qquad \eta \to -\infty, \theta < 0, \ z > 0 \\
[U(\theta)\cot\theta + a_B(\theta)] + \frac{1}{2}\left(\frac{2}{h_B a^2}\tan\alpha\sin\theta\right)^{\frac{1}{3}} (b_B(\theta) + \sqrt{3}\, c_B(\theta))\eta + \ldots \\
\qquad \eta \to -\infty, \theta > 0, \ z < 0 \\
[U(\theta)\cot\theta + a_B(\theta)] + \left|\frac{2}{h_B a^2}\tan\alpha\sin\theta\right|^{\frac{1}{3}} a_B(\theta)\eta + \ldots \quad \eta \to -\infty, \theta < 0, z < 0 \\
B(\theta) + A(\theta)[1 - E^{1/12} p\eta + \ldots] \qquad \eta \to +\infty
\end{cases} \tag{64}
$$

Equation (64) suggests that the structure of the solution of the $\frac{1}{3}$-layer is

$$v_{\frac{1}{3}} = v_o + v_1 + v_2 + \ldots \ldots \tag{65}$$

where

$$v_n \sim \eta^n \text{ as } |\eta| \to \infty \tag{66}$$

Also, writing

$$w_{\frac{1}{3}} = w_o + w_1 + w_2 + \ldots \ldots$$

where

$$w_n = O(v_n)$$

there results the following hierarchy of problems:

$$\left.\begin{aligned} \frac{\partial w_n}{\partial z} &= -\frac{\nu}{2\Omega}\frac{\partial^3 v_n}{\partial \bar{x}^3} \qquad (67) \\ \frac{\partial v_n}{\partial z} &= \frac{\nu}{2\Omega}\frac{\partial^3 w_n}{\partial \bar{x}^3} \qquad (68) \end{aligned}\right\} \text{ for } n = 0, 1, 2, \ldots\ldots$$

with

$$w_o = 0 \quad \text{on} \quad z = h_T,\ 0^{\pm},\ -h_B \tag{69}$$

Boundary conditions on higher order problems cannot be established until the order of the v_n has been established.

The $n = 0$ problem has been solved by Moore and Saffman.[2] The solution involves use of the Weiner-Hopf technique. There are an infinite number of solutions all of which are singular near the edge of the disc except for one. If the nature of the singularity in v_o were known, the unique solution could be chosen. However, to do this, it would be necessary to consider the flow in a thin "collar" around the edge of the disc of dimensions $E^{\frac{1}{2}} \times E^{\frac{1}{2}}$ where the $\frac{1}{3}$-layer and Ekman layer merge. This problem is extremely difficult and has not yet been solved. To circumvent this difficulty, Moore and Saffman have advanced an "hypothesis of minimum singularity" which turns out to be analogous to the Kutta condition of airfoil theory. The hypothesis is that the "radial pressure gradient be not larger inside the $\frac{1}{3}$-layer than it is just outside."

The consequence of invoking this hypothesis is that the $n = 0$ problem is not singular. The only solution to the $n = 0$ problem which is nonsingular is the solution:

$$w_o = 0 \quad \text{and} \quad v_o = v_o(\theta)$$

Hence, matching to the $\frac{1}{4}$-layer and α-layer solutions it must be the case that

$$a_T(\theta) = b_B(\theta) = A(\theta) + B(\theta) - U(\theta)\cot\theta \quad \text{for} \quad \theta > 0 \tag{70}$$

$$a_B(\theta) = b_T(\theta) = A(\theta) + B(\theta) - U(\theta)\cot\theta \quad \text{for } \theta < 0 \tag{71}$$

This is equivalent to saying that the circumferential velocity is continuous across the $\frac{1}{3}$-layer, i. e.,

$$v_\alpha^T(0,\theta) = v_\alpha^B(0,\theta) = v_{\frac{1}{4}}(0,\theta) \tag{72}$$

For the n = 1 problem, it is clear that

$$w_1 = 0 \quad \text{on} \quad z = -h_B, h_T \tag{73}$$

because the asymptotic form of $v_{\frac{1}{3}}$ as $\eta \to +\infty$ shows that at least

$$v_1 = O(E^{1/12}) >> O(E^{1/6})$$

Furthermore, at worst,

$$w_1 = (v_o \sin\theta - U(\theta)\cos\theta)\tan\alpha \text{ on } z = 0^{\pm} \tag{74}$$

Hence, taking account of (73) and (74),

$$\int_{-h_B}^{h_T} \frac{\partial w_1}{\partial z}\,dz = \int_{-h_B}^{0^-} \frac{\partial w_1}{\partial z}\,dz + \int_{0^+}^{h_T} \frac{\partial w_1}{\partial z}\,dz = 0 \tag{75}$$

for both η positive and negative. Using (63) and (67), equation (75) reduces to

$$\frac{\partial^3}{\partial \eta^3} \int_{-h_B}^{h_T} v_1 dz = 0$$

Thus,

$$\int_{-h_B}^{h_T} \frac{\partial v_1}{\partial \eta} dz = a(\theta) + b(\theta)\eta \tag{76}$$

and (66) shows that

$$b(\theta) = 0 \tag{77}$$

So, combining (64), (76) and (77) leads to the second jump relation, namely

$$[1 + \tfrac{1}{2}\lambda^2]\, a_T(\theta) + \frac{\sqrt{3}}{2}\lambda^2 c_B(\theta) = -2^{1/6}\left(\frac{h}{h_T}\right)^{\frac{2}{3}} \frac{\left(\frac{\nu}{\Omega h^2}\right)^{1/12}}{(\tan\alpha)^{1/3}} \frac{A(\theta)}{(\sin\theta)^{1/3}}$$

$$\text{for } \theta > 0 \tag{78}$$

$$[\tfrac{1}{2} + \lambda^2]\, a_B(\theta) + \frac{\sqrt{3}}{2} c_T(\theta) = -2^{1/6}\left(\frac{h}{h_T}\right)^{\frac{2}{3}} \frac{\left(\frac{\nu}{\Omega h^2}\right)^{1/12}}{(\tan\alpha)^{1/3}} \frac{A(\theta)}{|\sin\theta|^{1/3}}$$

$$\text{for } \theta < 0 \tag{79}$$

where

$$\lambda = \left(\frac{h_B}{h_T}\right)^{\frac{1}{3}} \tag{80}$$

and (70) and (71) have been used to eliminate $b_B(\theta)$ and $b_T(\theta)$. This condition has been referred to as the "continuity of total tangential shear stress" by Moore and Saffman.[2] It is easily shown to be the equivalent of

$$\int_{-h_B}^{0^-} \mu \frac{\partial v_\alpha^B}{\partial \bar{x}} (0,\theta)dz + \int_{0^+}^{h_T} \mu \frac{\partial v_\alpha^T}{\partial \bar{x}} (0,\theta)dz - \int_{-h_B}^{h_T} \mu \frac{\partial v_{\frac{1}{4}}(0,\theta)}{\partial \bar{x}} dz = 0 \tag{81}$$

The physical meaning of (81) is that angular momentum is conserved in the $\frac{1}{3}$-layer. (Moore and Saffman[2]).

Equations (78) and (79) fix the scaling on $a_I(\theta)$, $b_I(\theta)$ and $c_I(\theta)$. First, note that matching to the geostrophic flow for $r > a$,

$$\lim_{\xi \to \infty} v_{\frac{1}{4}}(\xi, \theta) = V(\theta) \tag{82}$$

and using (52) this means

$$B(\theta) = V(\theta) \tag{83}$$

Therefore, $B(\theta) = O(1)$ and certainly this is also true for $A(\theta)$. So,

$$a_I(\theta),\ b_I(\theta),\ c_I(\theta) = O(E^{1/12}\alpha^{-\frac{1}{3}})$$

and the flow in the α-layer will be given by

$$v_\alpha \sim U(\theta)\cot\theta + O(E^{1/12}\alpha^{-\frac{1}{3}}) \tag{84}$$

the correction term being small compared to the order one term. So, (70) and (71) simplify to the single equation

$$U(\theta)\cot\theta = V(\theta) + A(\theta) = v_{\frac{1}{4}}(0,\theta) \tag{85}$$

Going back to (74), this result shows that v_1 and w_1 are $O(E^{1/12})$ and

$$w_1 = 0 \qquad \text{on} \qquad z = 0^{\pm} \tag{86}$$

The solution to the n = 1 problem is again obtained by solving a Weiner-Hopf problem. The minimum singularity hypothesis permits v_1 to be singular and the nature of the singularity is discussed thoroughly by Moore and Saffman.[2] The most important point, however, is that the n = 1 problem here is identical to the n = 1 problem encountered for motion with zero α which firmly establishes the validity of (81).

The solution of the n = 2 problem is much more complicated as it involves a delta function singularity in v_2. Physically, one observes that there is a radial flux of fluid of order $E^{\frac{1}{2}}$ from the Ekman layers on the solid surfaces into the $\frac{1}{3}$-layer given by Greenspan[5] to be:

$$Q^I(\theta) = -\tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \left[u_\alpha^I(0,\theta) + v_\alpha^I(0,\theta)\right] \tag{87}$$

As argued in section 2. 2, the radial velocity is continuous across the shear column and in particular across the $\frac{1}{3}$-layer. Also, (72) shows that the circumferential velocity is continuous across the $\frac{1}{3}$-layer. Hence, the Ekman fluxes on the walls balance.

However, the Ekman flux from the disc must either enter the $\frac{1}{3}$-layer or circulate around the disc in the collar mentioned earlier. Conservation of mass would require velocity of order $E^{-\frac{1}{2}}$ which entails viscous dissipation of order unity. However, there is no mechanism to balance this dissipation so that the possibility of a collar is ruled out.

To escape from the Ekman layers on the disc, the fluid erupts as a vertical jet into the $\frac{1}{3}$-layer. Using (87), the net radial flux of fluid into the $\frac{1}{3}$-layer is

$$Q_e(\theta) = -\tfrac{1}{2}(\tfrac{\nu}{\Omega})^{\frac{1}{2}}[2U(\theta) + v_a^T(0,\theta) + v_a^B(0,\theta)] \tag{88}$$

This must be balanced by the net flux of fluid in the direction normal to the disc in the $\frac{1}{3}$-layer so that

$$Q_e(\theta) = \int_{-\infty}^{\infty} [w_{\frac{1}{3}}^{(n)}(\bar{x},\theta,0^+) - w_{\frac{1}{3}}^{(n)}(\bar{x},\theta,0^-)]\ d\bar{x} \tag{89}$$

where the integration is understood to be carried out on the scale of the $\frac{1}{3}$-layer given by (63). $w_{\frac{1}{3}}^{(n)}$ is given by (for $a<<1$):

$$w_{\frac{1}{3}}^{(n)} = w_{\frac{1}{3}} - (v_{\frac{1}{3}}\sin\theta - U(\theta)\cos\theta)\tan a \tag{90}$$

This is the way in which the $\frac{1}{3}$-layer expresses its "awareness" of the fact that the disc is tilted. It will be seen that the contribution from the second term in the right hand side of (90) is unimportant for $a<<E^{1/6}$. On the other hand, it is quite relevant for $a \sim E^{1/6}$ and $a>>E^{1/6}$. Integrating (43),

$$w_{\frac{1}{3}}(h_T) - w_{\frac{1}{3}}(0^+) = -\frac{\nu}{2\Omega}\int_0^{h_T} \frac{\partial^3 v_{\frac{1}{3}}}{\partial\bar{x}^3}\,dz$$

where $w_{\frac{1}{3}}(\tau)$ is shorthand for $w_{\frac{1}{3}}(\bar{x},\theta,\tau)$.
Similarly,

$$w_{\frac{1}{3}}(0^-) - w_{\frac{1}{3}}(-h_B) = -\frac{\nu}{2\Omega}\int_{-h_B}^{0^-} \frac{\partial^3 v_{\frac{1}{3}}}{\partial\bar{x}^3}\,dz$$

Then, using (45) and (47),

$$w_{\frac{1}{3}}(0^+) - w_{\frac{1}{3}}(0^-) = -\tfrac{1}{2}(\tfrac{\nu}{\Omega})^{\frac{1}{2}}\frac{\partial}{\partial\bar{x}}[v_{\frac{1}{3}}(h_T) + v_{\frac{1}{3}}(-h_B)] + \frac{\nu}{2\Omega}\frac{\partial}{\partial\bar{x}}\int_{-h_B}^{h_T}\frac{\partial^2 v_{\frac{1}{3}}}{\partial\bar{x}^2}\,dz$$

Hence, integrating and matching to the $\frac{1}{4}$-layer and α-layer solutions there follows:

$$\int_{-\infty}^{\infty} [w_{\frac{1}{3}}(0^+)-w_{\frac{1}{3}}(0^-)]\, d\bar{x} = \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} [v_\alpha^T(0,\theta)+v_\alpha^B(0,\theta)-2v_{\frac{1}{4}}(0,\theta)]$$

$$+ \frac{\nu}{2\Omega}\int_{-h_B}^{h_T} \left[\frac{\partial^2 v_{\frac{1}{4}}}{\partial\bar{x}^2}(0,\theta) - \frac{\partial^2 v_\alpha^I}{\partial\bar{x}^2}(0,\theta)\right] dz \tag{91}$$

By virtue of (72) the first term on the right hand side of (91) vanishes. Therefore,

$$\int_{-\infty}^{\infty} [w_{\frac{1}{3}}(0^+)-w_{\frac{1}{3}}(0^-)]\, d\bar{x} = \frac{\nu}{2\Omega}\int_{-h_B}^{h_T} \left[\frac{\partial^2 v_{\frac{1}{4}}}{\partial\bar{x}^2}(0,\theta) - \frac{\partial^2 v_\alpha^I}{\partial\bar{x}^2}(0,\theta)\right] dz \tag{92}$$

Inserting (90) into (89) and noting that

$$v_{\frac{1}{3}} = \frac{1}{2\Omega}\frac{\partial p_{\frac{1}{3}}}{\partial\bar{x}}$$

then

$$\int_{-\infty}^{\infty} [w_{\frac{1}{3}}^{(n)}(0^+)-w_{\frac{1}{3}}^{(n)}(0^-)]\, d\bar{x} = \int_{-\infty}^{\infty} [w_{\frac{1}{3}}(0^+)-w_{\frac{1}{3}}(0^-)]\, d\bar{x}$$

$$- \frac{\tan\alpha\sin\theta}{2\Omega}\int_{-\infty}^{\infty}\frac{1}{2\Omega}\frac{\partial}{\partial\bar{x}}(\Delta p_{\frac{1}{3}})\, d\bar{x}$$

where the notation $\Delta(\ \ )$ means the difference between ( ) evaluated at $z = 0^+$ and $z = 0^-$.

Hence

$$\int_{-\infty}^{\infty}[w_{\frac{1}{3}}^{(n)}(0^{+})-w_{\frac{1}{3}}^{(n)}(0^{-})]\,d\bar{x} = \int_{-\infty}^{\infty}[w_{\frac{1}{3}}(0^{+})-w_{\frac{1}{3}}(0^{-})]\,d\bar{x}$$

$$- \frac{\tan\alpha\sin\theta}{2\Omega}[\Delta p_{\frac{1}{3}}(\infty,\theta)-\Delta p_{\frac{1}{3}}(-\infty,\theta)] \qquad (93)$$

However, from matching

$$\Delta p_{\frac{1}{3}}(-\infty,\theta) = \Delta p_{\alpha}(0,\theta) \qquad (94)$$

and

$$\Delta p_{\frac{1}{3}}(\infty,\theta) = \Delta p_{\frac{1}{4}}(0,\theta) = 0 \qquad (95)$$

where the second equality in (95) follows from the fact that $p_{\frac{1}{4}}$ is independent of z. Using (93)-(95) in (89),

$$Q_e(\theta) = \int_{-\infty}^{\infty}[w_{\frac{1}{3}}(0^{+})-w_{\frac{1}{3}}(0^{-})]\,d\bar{x} + \tan\alpha\sin\theta\,\frac{\Delta p_{\alpha}(0,\theta)}{2\Omega}$$

which, when combined with (88) and (92) reduces to

$$-\tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}[2U(\theta)+v_{\alpha}^{T}(0,\theta)+v_{\alpha}^{B}(0,\theta)]$$

$$= \frac{\nu}{2\Omega}\int_{-h_B}^{h_T}\left[\frac{\partial^2 v_{\frac{1}{4}}(0,\theta)}{\partial\bar{x}^2} - \frac{\partial^2 v_{\alpha}^{I}}{\partial\bar{x}^2}(0,\theta)\right]dz$$

$$+ \tan\alpha\sin\theta\,\frac{\Delta p_{\alpha}(0,\theta)}{2\Omega} \qquad (96)$$

With the obvious changes in notation, this reduces to the result given by Moore and Saffman[2, 3] when α vanishes.

It is now possible to simplify the right hand side of (96) by integrating across the $\frac{1}{4}$-layer and the α-layer. From (51), there follows after integration

$$\left.\frac{\partial^2 v_{\frac{1}{4}}}{\partial \bar{x}^2}\right|_0^{\infty} = \left(\frac{4\Omega}{\nu h^2}\right)^{\frac{1}{2}} [v_{\frac{1}{4}}(\infty, \theta) - v_{\frac{1}{4}}(0, \theta)]$$

so that

$$\frac{\nu}{2\Omega} \frac{\partial^2 v_{\frac{1}{4}}}{\partial \bar{x}^2}(0, \theta) = \frac{1}{h}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} [v_{\frac{1}{4}}(0, \theta) - V(\theta)] \tag{97}$$

noting (82) plus the fact that

$$\frac{\partial^2 v_{\frac{1}{4}}}{\partial \bar{x}^2}(\bar{x}, \theta) \rightarrow 0 \qquad \text{as } \bar{x} \rightarrow \infty$$

Since $v_{\frac{1}{4}}$ is independent of z, (97) can be integrated directly to yield

$$\frac{\nu}{2\Omega} \int_{-h_B}^{h_T} \frac{\partial^2 v_{\frac{1}{4}}}{\partial \bar{x}^2}(0, \theta) dz = \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} [v_{\frac{1}{4}}(0, \theta) - V(\theta)] \tag{98}$$

Similarly, integrating (59) and (60):

$$\frac{\nu}{2\Omega} \frac{\partial^2 v_a^T}{\partial \bar{x}^2}(0, \theta) = \frac{\sin\theta \tan a}{h_T} \int_{-\infty}^{0} [v_a^T - U(\theta)\cot\theta]\, d\bar{x} \tag{99}$$

$$\frac{\nu}{2\Omega} \frac{\partial^2 v_a^T}{\partial \bar{x}^2}(0, \theta) = -\frac{\sin\theta \tan a}{h_B} \int_{-\infty}^{0} [v_a^B - U(\theta)\cot\theta]\, d\bar{x} \tag{100}$$

Then, integrating (99) and (100) over z:

$$\frac{\nu}{2\Omega} \int_{-h_B}^{h_T} \frac{\partial^2 v_a^I}{\partial \bar{x}^2}(0, \theta) dz = \sin\theta \tan a \int_{-\infty}^{0} [v_a^T - v_a^B]\, d\bar{x} \tag{101}$$

However,

$$v_a = \frac{1}{2\Omega} \frac{\partial p_a}{\partial \bar{x}}$$

so that (101) becomes (after matching to the geostrophic flow):

$$\frac{\nu}{2\Omega}\int_{-h_B}^{h_T} \frac{\partial^2 v_a^I}{\partial \bar{x}^2}(0,\theta)dz = \sin\theta\tan\alpha\left[\frac{\Delta p_a(0,\theta)}{2\Omega} - \frac{\Delta p_g(a,\theta)}{2\Omega}\right] \qquad (102)$$

It is necessary at this point to pause and determine the orders of magnitude of the various terms in (96). Reference to (84) and (101) show that

$$\frac{\nu}{2\Omega}\int_{-h_B}^{h_T} \frac{\partial^2 v_a^I}{\partial \bar{x}^2}(0,\theta)dz = O(E^{5/12}\alpha^{\frac{1}{3}})$$

while (98) shows that

$$\frac{\nu}{2\Omega}\int_{-h_B}^{h_T} \frac{\partial^2 v_{\frac{1}{4}}}{\partial \bar{x}^2}(0,\theta)dz = O(E^{\frac{1}{2}})$$

The left hand side of (96) is clearly of order $E^{\frac{1}{2}}$ also. Then, since

$$\frac{E^{5/12}\alpha^{\frac{1}{3}}}{E^{\frac{1}{2}}} = \left(\frac{\alpha}{E^{\frac{1}{4}}}\right)^{\frac{1}{3}} >> 1$$

equation (96) reduces to:

$$\frac{\nu}{2\Omega}\int_{-h_B}^{h_T} \frac{\partial^2 v_a^I}{\partial \bar{x}^2}(0,\theta)dz = \tan\alpha\sin\theta\,\frac{\Delta p_a(0,\theta)}{2\Omega} + O(E^{\frac{1}{2}}) \qquad (103)$$

so that using (102) there results finally

$$\Delta p_g(a,\theta) = O(E^{\frac{1}{2}}\cot\alpha) \qquad (104)$$

Furthermore, (103) can be simplified by noting that (72) is valid to $O(E^{1/12}\alpha^{-\frac{1}{3}})$. So,

$$\Delta v_a = v_a^T(0,\theta) - v_a^B(0,\theta) = o(E^{1/12}a^{-\frac{1}{3}})$$

which means that

$$\Delta p_a(0,\theta) = o(E^{5/12}a^{-\frac{2}{3}}) \tag{105}$$

Therefore, to leading order, (103) (and hence (96)) reduces to

$$\frac{\nu}{2\Omega}\int_{-h_B}^{h_T} \frac{\partial^2 v_a^I}{\partial \bar{x}^2}(0,\theta)dz = 0$$

Using (61), this adds the equations

$$[1 - \tfrac{1}{2}\lambda]\ a_T(\theta) + \frac{\sqrt{3}}{2}\lambda\ c_B(\theta) = 0 \qquad \text{for} \qquad \theta > 0 \tag{106}$$

$$[\tfrac{1}{2} - \lambda]\ a_B(\theta) - \frac{\sqrt{3}}{2}\ c_T(\theta) = 0 \qquad \text{for} \qquad \theta < 0 \tag{107}$$

where (70) and (71) have again been used to eliminate $b_B(\theta)$ and $b_T(\theta)$ with λ given by equation (80).

The physical meaning of this constraint can be seen by noting that

$$\frac{\Delta p_a(0,\theta)}{2\Omega} - \frac{\Delta p_g(a,\theta)}{2\Omega} = \int_{-\infty}^{0} [v_a^T - v_a^B]\ d\bar{x}$$

so that

$$\frac{1}{2\Omega}\frac{d}{d\theta}[\Delta p_a(0,\theta) - \Delta p_g(a,\theta)] = \frac{d}{d\theta}\int_{-\infty}^{0} [v_a^T - v_a^B]\ d\bar{x}$$

Now, to order $E^{5/12}a^{-\frac{2}{3}}$, u_a and p_a are related by

$$2\Omega u_a = -\frac{1}{a}\frac{\partial p_a}{\partial \theta}$$

which is also true in the geostrophic region. Hence,

$$\Delta u_g(a,\theta) - \Delta u_\alpha(0,\theta) = \frac{1}{a}\frac{d}{d\theta}\int_{-\infty}^{0} [v_\alpha^T - v_\alpha^B]\, d\bar{x}$$

or

$$[u_g^T(a,\theta) - u_\alpha^T(0,\theta)] - [u_g^B(a,\theta) - u_\alpha^B(0,\theta)] =$$

$$= \frac{1}{a}\frac{d}{d\theta}\int_{-\infty}^{0} \left\{[v_\alpha^T - U(\theta)\cot\theta] - [v_\alpha^B - U(\theta)\cot\theta]\right\} d\bar{x}$$

By virtue of (104) and (105) the left hand side vanishes so that

$$\frac{d}{d\theta}\int_{-\infty}^{0} [v_\alpha^T - U(\theta)\cot\theta]\, d\bar{x} = \frac{d}{d\theta}\int_{-\infty}^{0} [v_\alpha^B - U(\theta)\cot\theta]\, d\bar{x}$$

Of course, (104) and (105) also show that

$$u_g^T(a,\theta) = u_g^B(0,\theta)$$

and

$$u_\alpha^T(0,\theta) = u_\alpha^B(0,\theta)$$

Hence, it follows that (106) and (107) come from the requirement that

$$u_g(a,\theta) - u_\alpha(0,\theta) = \frac{1}{a}\frac{d}{d\theta}\int_{-\infty}^{0} [v_\alpha - U(\theta)\cot\theta]\, d\bar{x} \qquad (108)$$

independent of z. This merely insures that mass is conserved in the α-layer. To see this, define

$$J(\theta) \equiv \int_{-\infty}^{0} [v_\alpha - U(\theta)\cot\theta]\, d\bar{x} \qquad (109)$$

This is the excess flux of fluid swirling around the α-layer. It is present because of the oscillatory nature of v_α across the layer. Consider an infinitesimal volume element of unit height as shown in Figure 4.

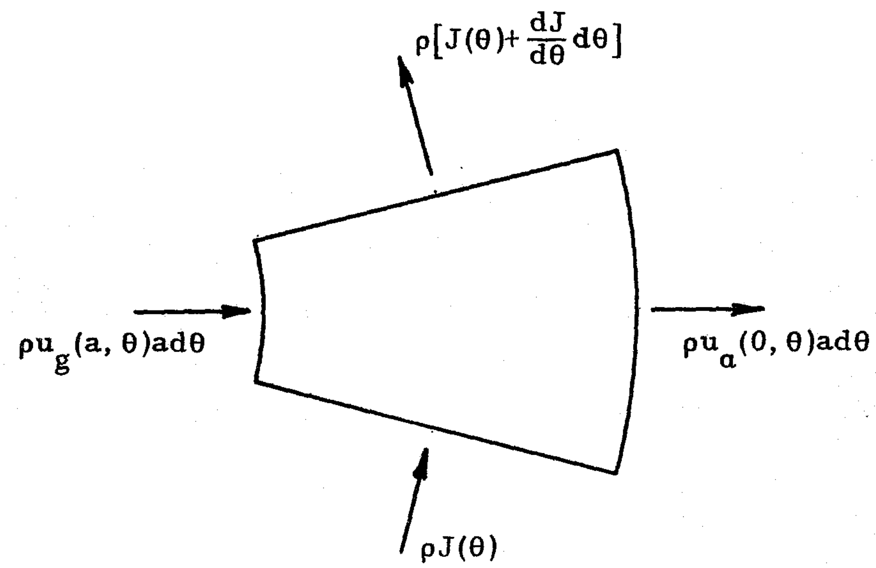

Figure 4. Volume Element and Fluxes of Fluid into and out of the Element.

Conservation of mass demands that

$$\rho u_g(a,\theta)ad\theta + \rho J(\theta) - \rho u_\alpha(0,\theta)ad\theta - \rho\left[J(\theta) + \frac{dJ}{d\theta}d\theta\right] = 0 .$$

or, simplifying

$$u_g(a,\theta) - u_\alpha(0,\theta) = \frac{1}{a}\frac{dJ(\theta)}{d\theta} \tag{110}$$

Noting (109), it is clear that (108) and (110) are identical.

It is possible to obtain another equation from (96) by going to next order. This is equivalent to writing

$$a_I(\theta) = E^{1/12}(\cot\alpha)^{\frac{1}{3}}\, a_I^{(0)}(\theta) + E^{1/6}(\cot\alpha)^{\frac{2}{3}} a_I^{(1)}(\theta) + \ldots\ldots$$

and similarly for $b_I(\theta)$ and $c_I(\theta)$. Then, equations (106) and (107) result for $a_I^{(0)}(\theta)$, etc. The problem of having to determine $a_I^{(1)}(\theta)$, etc. is circumvented by integrating across the α-layer to eliminate

$$\frac{\partial^2 v_\alpha}{\partial \bar{x}^2}\,(0,\theta)$$

in favor of $\Delta p_g(a,\theta)$ which is known from equation (26). Inserting (98) and (102) into (96) yields the final and most important jump relation:

$$\left[2v_{\frac{1}{4}}(0,\theta) - 2V(\theta) + 2U(\theta) + v_\alpha^T(0,\theta) + v_\alpha^B(0,\theta)\right] + 2\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} \tan\alpha\, \frac{\Delta p_g(a,\theta)}{2\Omega}\sin\theta = 0$$

and using (72) this reduces to:

$$2v_{\frac{1}{4}}(0,\theta) + U(\theta) - V(\theta) = -\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} \tan\alpha\, \frac{\Delta p_g(a,\theta)}{2\Omega}\sin\theta \qquad (111)$$

Finally, (61) shows that

$$v_g(a,\theta) = U(\theta)\cot\theta \qquad (112)$$

and from (85) this means

$$v_g(a,\theta) = v_{\frac{1}{4}}(0,\theta)$$

wherefore (111) becomes

$$2v_g(a,\theta) + U(\theta) - V(\theta) = -\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} \tan\alpha\, \frac{\Delta p_g(a,\theta)}{2\Omega}\sin\theta \qquad (113)$$

It is clear at this point how the second term on the right hand side of (90) influences the solution. Had it been neglected, the term involving $\Delta p_a(0, \theta)$ would appear in (113) and it would be of order $E^{\frac{1}{2}}\cot a$. But,

$$\frac{E^{\frac{1}{2}}\cot a}{E^{\frac{1}{3}}} = E^{1/6}\cot a$$

so that if $a << E^{1/6}$,

$$\Delta p_a(0, \theta) = \Delta p_{\frac{1}{4}}(0, \theta) + o(E^{\frac{1}{2}}\cot a)$$

or

$$\Delta p_a(0, \theta) = o(E^{\frac{1}{2}}\cot a)$$

On the other hand, for $a \sim E^{1/6}$ or $a >> E^{1/6}$, $\Delta p_a(0, \theta)$ will not, in fact, vanish to order $E^{\frac{1}{2}}\cot a$.

To evaluate $\Delta p_g(a, \theta)$ note that on the inner edge of the shear column

$$x = a\cos\theta$$

so that

$$\frac{d}{dx} = -\frac{1}{a\sin\theta}\frac{d}{d\theta}$$

Hence,

$$y\frac{dv_g(x)}{dx} = -\frac{dv_g(\theta)}{d\theta}$$

where $V_g(\theta)$ is shorthand for $V_g(a\cos\theta)$. Using the same shorthand for $p_g(x)$ and $f^I(x)$ there follows from equation (26)

$$\Delta p_g = p_g(a,\theta,0^+) - p_g(a,\theta,0^-) = 2\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha\left[f^T(\theta) - f^B(\theta) + 2\,\frac{dV_g(\theta)}{d\theta}\right] \tag{114}$$

But,

$$u_g(a,\theta) = V_g(\theta)\sin\theta \tag{115}$$

and

$$v_g(a,\theta) = V_g(\theta)\cos\theta \tag{116}$$

Combining equations (112), (114) and (116) and rearranging terms, (113) reduces to

$$2\,\frac{dU(\theta)}{d\theta} + U(\theta) - V(\theta) = -\left[f^T(\theta) - f^B(\theta)\right]\sin\theta \tag{117}$$

It is appropriate to mention here the result for the case when the upper surface is free. If the surface tension is large enough to keep the surface plane to within $O(E^{\frac{1}{2}})$, the free surface cannot sustain an Ekman layer. This is true because the geostrophic flow satisfies the condition that the tangential stress vanishes on a free surface. Hence, (13) must be replaced by

$$w = 0 \qquad \text{on} \qquad z = h_T \tag{13a}$$

The geostrophic flow is unaffected to order one so that (23)-(33) still hold to leading order. However, (114) must be replaced by

$$\Delta p_g(a,\theta) = 2\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha\left[f^T(\theta) - f^B(\theta) + \frac{3}{2}\,\frac{dV_g(\theta)}{d\theta}\right] \tag{114a}$$

Integrating across the shear column, it is found that (113) must be replaced by

$$\frac{3}{2} v_g(a,\theta) + U(\theta) - V(\theta) = -\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \tan\alpha \frac{\Delta p_g(a,\theta)}{2\Omega} \sin\theta \qquad (113a)$$

so that (117) must be replaced by

$$\frac{3}{2} \frac{dU(\theta)}{d\theta} + U(\theta) - V(\theta) = -\left[f^T(\theta) - f^B(\theta)\right] \sin\theta \qquad (117a)$$

Equations (70), (71), (78), (79), (80), (85), (106) and (107) are sufficient to determine all of the unknown quantities in (61) as functions of A(θ). Omitting the algebra for brevity, the results are:

$$a_T(\theta) = b_B(\theta) = -S \frac{A(\theta)}{(\sin\theta)^{\frac{1}{3}}} \qquad (118)$$

$$a_B(\theta) = b_T(\theta) = -S \frac{A(\theta)}{|\sin\theta|^{\frac{1}{3}}} \qquad (119)$$

$$c_B(\theta) = \frac{2-\lambda}{\sqrt{3}\,\lambda} S \frac{A(\theta)}{(\sin\theta)^{\frac{1}{3}}} \qquad (120)$$

$$c_T(\theta) = \frac{2\lambda - 1}{\sqrt{3}} S \frac{A(\theta)}{|\sin\theta|^{\frac{1}{3}}} \qquad (121)$$

where

$$S = \frac{2^{1/6} \left(\frac{h}{h_T}\right)^{\frac{2}{3}} \left(\frac{\nu}{\Omega h^2}\right)^{1/12}}{(\tan\alpha)^{\frac{1}{3}} (1 - \lambda + \lambda^2)} \qquad (122)$$

So, once A(θ) is known, the α-layer structure (and, of course, the $\frac{1}{4}$-layer structure) is known. But, from (85),

$$A(\theta) = U(\theta)\cot\theta - V(\theta) \qquad (123)$$

so that once the geostrophic flow is known, the shear column structure is known.

It will be demonstrated in the next two sections that equation (117) is sufficient to determine the solution in the geostrophic regions.

2.3.5 x-Translation Solution

Equation (117) has a very important property. Since $f^T(\theta)$ and $f^B(\theta)$ are both even functions of θ, the right hand side of (117) must be an odd function of θ. Using equations (29) and (30) in equations (34) and (35) respectively there follows:

$$U(\theta) = \frac{1}{2\Omega a} \sum_{n=1}^{\infty} nB_n \sin n\theta \tag{124}$$

$$V(\theta) = -2U\sin\theta - \frac{1}{2\Omega a} \sum_{n=1}^{\infty} nB_n \cos n\theta \tag{125}$$

Thus, (117) simplifies to:

$$2U\sin\theta + \frac{1}{2\Omega a} \sum_{n=1}^{\infty} n(2n+1)B_n \cos n\theta + \frac{1}{2\Omega a} \sum_{n=1}^{\infty} nB_n \sin n\theta$$
$$= -\left[f^T(\theta) - f^B(\theta)\right] \sin\theta \tag{126}$$

Observing that the right hand side of (126) is odd in θ, necessarily only terms in $\sin n\theta$ can appear on the left hand side. Therefore

$$B_n = 0 \quad \text{for} \quad \text{all} \quad n \geq 1 \tag{127}$$

It follows immediately that

$$f^T(\theta) - f^B(\theta) = -2U$$

and hence

$$f^T(x) - f^B(x) = -2U \tag{128}$$

Also,

$$V_g(\theta) = \frac{U(\theta)}{\sin\theta} = 0$$

Therefore,

$$V_g(x) = 0 \tag{129}$$

This shows that to order one there is no flow relative to the disc inside the Taylor column. The flow outside the Taylor column is potential flow of a uniform stream past a circular cylinder given by

$$u_G(r,\theta) = U[1 - (\frac{a}{r})^2] \cos\theta \tag{130}$$

$$v_G(r,\theta) = -U[1 + (\frac{a}{r})^2] \sin\theta \tag{131}$$

The structure of the shear column around an axisymmetric fat body was studied by Jacobs.[7] The fluid inside the Taylor column was found to be stagnant just as in this problem. Hence, when the disc moves in the x-direction it behaves as though it were a "fat" body.

To calculate the force and moment on the disc when the disc is moving at uniform speed, U, in the negative x-direction, it is necessary to perform a Galilean transformation. So, defining

$$\underline{\hat{u}} = \underline{u} - U\underline{i}$$

then

$$2\underline{\Omega} \times \underline{\hat{u}} = -\nabla\hat{p} + \nu\nabla^2 \underline{\hat{u}}$$

with

$$\hat{p} = p + 2\Omega Uy$$

The physical pressure follows from (4).

$$\mathcal{P} = \rho p + 2\rho\Omega Uy + \tfrac{1}{2}\rho\Omega^2 r^2 \tag{132}$$

Inserting equations (26), (128) and (129) into (132) there results:

$$\Delta \mathcal{P}(x, y) = -4\rho\Omega U\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \tag{133}$$

where

$$\Delta\mathcal{P}(x, y) \equiv \mathcal{P}(x, y, 0^+) - \mathcal{P}(x, y, 0^-)$$

The viscous stress on the disc is shown by Greenspan[5] to be (noting that $\alpha << 1$)

$$\frac{\underline{\tau}}{\mu} = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} \left[(\hat{\underline{u}}_g - \hat{\underline{u}}_p) \pm \underline{k} \times (\hat{\underline{u}}_g - \hat{\underline{u}}_p)\right] \quad \text{for } z \gtrless 0$$

Therefore,

$$\Delta \underline{\tau}(x, y) = 2\rho\Omega \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \underline{k} \times (\hat{\underline{u}}_g - \hat{\underline{u}}_p) \tag{134}$$

with

$$\Delta \underline{\tau}(x, y) \equiv \underline{\tau}(x, y, 0^+) - \underline{\tau}(x, y, 0^-)$$

However, for this flow,

$$\hat{\underline{u}}_g - \hat{\underline{u}}_p = o(1)$$

so that

$$\Delta\underline{\tau} = o(E^{\frac{1}{2}})$$

So, only the pressure term is relevant in the calculation of the force and moment. Integrating (133) over the disc

$$\underline{F} = -\iint_{disc} \Delta p\, \underline{n}\, dxdy$$

and therefore

$$\underline{F} = 4\pi\rho\Omega a^2 U\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha\, [\underline{i}\tan\alpha + \underline{k}] \qquad (135)$$

The moment on the disc is

$$\underline{M} = -\iint_{disc} \Delta p\, \underline{r} \times \underline{n}\, dxdy$$

or

$$\underline{M} \doteq 4\rho\Omega U\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [\underline{i}y - \underline{j}x + \underline{k}y\tan\alpha]\, dydx$$

and each term in the integrand is odd in either x or y so that the integral vanishes to leading order. Therefore

$$\underline{M} = o(E^{\frac{1}{2}} \cot\alpha) \qquad (136)$$

In summary, there is no net moment on the disc to $O(E^{\frac{1}{2}}\cot\alpha)$. There is a vertical force (lift) on the disc of order $E^{\frac{1}{2}}\cot\alpha$. The disc suffers a horizontal force (drag) of order $E^{\frac{1}{2}}$.

The α-layer and $\frac{1}{4}$-layer solutions can now be determined. In light of (127), equations (124) and (125) become

$$U(\theta) = 0 \tag{137}$$

$$V(\theta) = -2U\sin\theta \tag{138}$$

Thus, (83) and (123) show that

$$A(\theta) = 2U\sin\theta \tag{139}$$

and

$$B(\theta) = -2U\sin\theta \tag{140}$$

Inserting (139) into (118)-(121) completes the α-layer solution.

Conservation of mass in the α-layer indicated earlier that there is a flux of fluid of order $E^{5/12}\alpha^{-2/3}$ swirling around in the layer. Doing the integration in (109),

$$J(\theta) = -\gamma \frac{2^{1/6}h}{(1-\lambda+\lambda^2)} \left(\frac{h}{h_T}\right)^{\frac{1}{3}} \left(\frac{\nu}{\Omega h^2}\right)^{5/12} (\cot\alpha)^{\frac{2}{3}} \frac{A(\theta)}{|\sin\theta|^{\frac{2}{3}}} \tag{141}$$

where

$$\gamma = \begin{cases} 1 & \text{for} \quad \theta > 0 \\ \frac{2}{7}(1+2\lambda) & \text{for} \quad \theta < 0 \end{cases} \tag{142}$$

Hence, $J(\theta)$ will be continuous across $\theta = 0$ and $\theta = \pi$ while $J'(\theta)$ has a singularity of order

$$J'(\theta) \sim E^{5/12}\alpha^{-2/3}\theta^{-2/3} \quad \text{as} \quad \theta \to 0$$

and a similar singularity as $\theta \to \pi$. In fact, v_α will be continuous across $\theta = 0$ and $\theta = \pi$ while $\partial v_\alpha/\partial\theta$ goes to infinity like

$$\frac{\partial v_\alpha}{\partial \theta} \sim E^{1/12}\, \alpha^{-\frac{1}{3}}\theta^{-\frac{1}{3}} \quad \text{as } \theta \to 0$$

and similarly for $\theta \to \pi$. To resolve this problem, further layers in which derivatives with respect to θ become important must be inserted. These layers are again passive and their detailed structure will not be given. The appropriate equations are given in the Appendix.

As a final note, if the upper surface is free, (117a) shows that the geostrophic flow is unchanged.

2.3.6 y-Translation Solution

Combining equations (34) and (35) with equations (32) and (33) there follows:

$$U(\theta) = V\sin\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n \sin n\theta \tag{143}$$

$$V(\theta) = V\cos\theta - \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n \cos n\theta \tag{144}$$

Proceeding as in 2.3.5, (117) becomes

$$V\cos\theta + V\sin\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} n(2n+1)B_n \cos n\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n \sin n\theta = -\left[f^T(\theta) - f^B(\theta)\right]\sin\theta$$

so that necessarily

$$B_1 = -\tfrac{2}{3}\Omega a V, \quad B_n = 0 \text{ for all } n \geq 2 \tag{145}$$

Hence,

$$f^T(x) - f^B(x) = -\tfrac{2}{3}V \tag{146}$$

$$V_g(x) = \tfrac{2}{3}V \tag{147}$$

This means that there is a uniform flow in the y-direction inside the Taylor column. The uniform flow at infinity has suffered a deceleration from V to $\frac{2}{3}V$ in transit through the Taylor column. The potential flow outside the Taylor column is:

$$u_G(r,\theta) = V\left[1 - \tfrac{1}{3}\left(\tfrac{a}{r}\right)^2\right] \sin\theta \tag{148}$$

$$v_G(r,\theta) = V\left[1 + \tfrac{1}{3}\left(\tfrac{a}{r}\right)^2\right] \cos\theta \tag{149}$$

and the streamlines are plotted in Figure 5.

Performing a Galilean transformation to make the disc move into an ambient fluid with the container walls at rest, it must be true that

$$\underline{\hat{u}} = \underline{u} - V\,\underline{j}$$

and

$$\hat{p} = p - 2\Omega Vx$$

so that

$$\mathcal{P} = \rho p - 2\rho\Omega Vx + \tfrac{1}{2}\rho\Omega^2 r^2 \tag{150}$$

Therefore, from (26), (146) and (147):

$$\Delta\mathcal{P}(x,y) = -\frac{4}{3}\rho\Omega V\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha \tag{151}$$

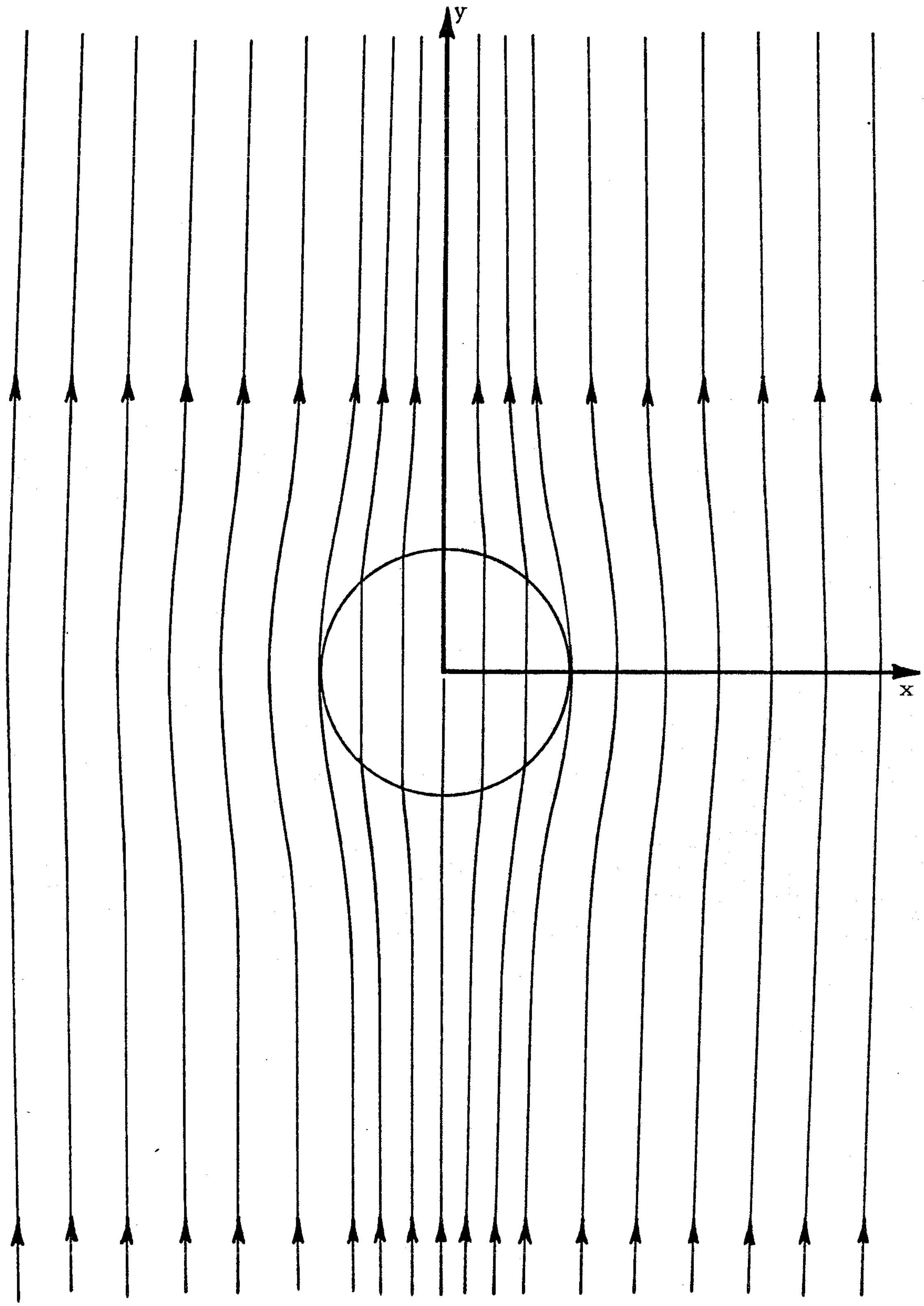

Figure 5. Geostrophic Flow for y-Translation. (Disc at Rest)

Therefore, using (134),

$$\Delta \underline{\tau} = -\frac{4}{3}\rho\Omega V\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \underline{i} \tag{152}$$

The force on the disc is

$$\underline{F} = -\int\int \Delta \mathcal{P} \; \underline{n}dxdy + \int\int \Delta \underline{\tau} \; dxdy$$

or

$$\underline{F} = \frac{4}{3}\pi\rho\Omega a^2 V\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \left[\underline{i}\tan\alpha + \underline{k}\right] - \frac{4}{3}\pi\rho\Omega a^2 V\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \underline{i}$$

so that

$$\underline{F} = \frac{4}{3}\pi\rho\Omega a^2 V\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \; \underline{k} \tag{153}$$

Hence, to leading order ($O(E^{\frac{1}{2}})$) there is no drag on the disc!

Just as in the x-translation solution, it is clear that the net moment on the disc involves integration of odd functions of x and y over the surface of the disc. Therefore,

$$\underline{M} = o(E^{\frac{1}{2}}\cot\alpha) \tag{154}$$

To complete the shear column structure note that:

$$U(\theta) = \tfrac{2}{3}V\sin\theta \tag{155}$$

and

$$V(\theta) = \frac{4}{3}V\cos\theta \tag{156}$$

So, (123) reduces to:

$$A(\theta) = -\frac{2}{3} V\cos\theta \tag{157}$$

while

$$B(\theta) = \frac{4}{3} V\cos\theta \tag{158}$$

From (141) and (157) it is seen that this motion involves an even stronger singularity. Here,

$$J(\theta) \sim E^{5/12}\, \alpha^{-\frac{2}{3}}\, \theta^{-\frac{2}{3}} \quad \text{as} \quad \theta \to 0$$

Also,

$$v_\alpha \sim E^{1/12}\, \alpha^{-\frac{1}{3}}\, \theta^{-\frac{1}{3}} \quad \text{as} \quad \theta \to 0$$

so that the velocity is also singular. The analysis in the Appendix is applicable for resolving the apparent singular behavior of these functions.

For a free surface on $z = h_T$, (146) and (147) must be replaced by

$$f^T(x) - f^B(x) = -\frac{4}{5} V \tag{146a}$$

$$V_g(x) = \frac{4}{5} V \tag{147a}$$

and the force on the disc becomes

$$\underline{F} = \frac{8}{5}\, \pi\, \rho\, \Omega\, a^2 V \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha\, \underline{k} \tag{153a}$$

The moment is still $o(E^{\frac{1}{2}}\cot\alpha)$.

2.3.7 Summary

For x-translation, there is no flow inside the Taylor column to order one and the flow outside is simple potential flow of a uniform stream past a circular cylinder. There is no net torque on the disc to $O(E^{\frac{1}{2}}\cot\alpha)$ while there is a lift (vertical force) to this order. The drag (horizontal force) is $O(E^{\frac{1}{2}})$.

For y-translation, there is a uniform flow in the y-direction inside the Taylor column. The speed of this flow is two thirds the value of the speed of the uniform flow at infinity. There is again no net moment to $O(E^{\frac{1}{2}}\cot\alpha)$ while there is a lift to this order. However, the plate suffers no drag to order $E^{\frac{1}{2}}$.

The shear column structure differs from the standard Stewartson sandwich layer structure. It is not possible to have a $\frac{1}{4}$-layer on the inner edge of the shear column. A new layer on the inner edge of the shear column appears. The thickness of the layer is proportional to $(E\cot\alpha)^{\frac{1}{3}}$ and it is referred to as the α-layer. The velocity behaves like an exponentially damped sinusoid across the layer giving rise to an excess flux of fluid of order $E^{5/12}\alpha^{-\frac{2}{3}}$ swirling around circumferentially. There is a $\frac{1}{4}$-layer on the outer edge of the shear column. Rather than solving the $\frac{1}{3}$-layer equations directly, the equations are integrated across the layer to determine jump conditions. The jump conditions across the $\frac{1}{3}$-layer are used to relate the α-layer and the $\frac{1}{4}$-layer. There are four jump conditions which are:

(a) the circumferential velocity is continuous;

(b) the total tangential shear stress is continuous;

(c) mass is conserved in the α-layer;

(d) mass is conserved in the $\frac{1}{3}$-layer.

As a final note, it must be pointed out that this analysis depends upon the radius of the disc being much larger than the thickness of the α-layer. This leads to the restriction

$$\frac{h}{a} << \frac{\alpha}{E}$$

Requiring the distance from the walls to the disc to be small compared to the thickness of the Ekman layers on the disc and walls means

$$\frac{h}{a} >> E^{\frac{1}{2}}$$

Also, for the Ekman layer and shear column structure to be unaffected by the nonlinear inertia terms it is necessary to require that

$$Ro << E^{\frac{1}{3}} (\frac{h}{a})^{\frac{1}{3}}$$

2.4 Shear Column Structure for $E^{\frac{1}{2}} << \alpha << E^{\frac{1}{4}}$

This range will be studied to complete the study of the entire range of α for which no closed geostrophic contours exist inside the Taylor column.

2.4.1 The $\frac{1}{4}$-Layers

For this range of α, the first term on the right hand side of (46) is negligible. So, the usual Stewartson sandwich layer structure holds, i.e., there is a $\frac{1}{3}$-layer sandwiched by $\frac{1}{4}$-layers both on the inside and outside edges of the shear column. Hence,

$$\frac{\partial^3 v_{\frac{1}{4}}}{\partial \bar{x}^3} = \begin{cases} \left(\frac{4\Omega}{\nu h_T^2}\right)^{\frac{1}{2}} \frac{\partial v_{\frac{1}{4}}^T}{\partial \bar{x}} & ; \quad \bar{x} < 0, \; z > 0 \\ \left(\frac{4\Omega}{\nu h_B^2}\right)^{\frac{1}{2}} \frac{\partial v_{\frac{1}{4}}^B}{\partial \bar{x}} & ; \quad \bar{x} < 0, \; z < 0 \\ \left(\frac{4\Omega}{\nu h^2}\right)^{\frac{1}{2}} \frac{\partial v_{\frac{1}{4}}}{\partial \bar{x}} & ; \quad \bar{x} > 0 \end{cases} \tag{159}$$

The solution is

$$v_{\frac{1}{4}} = \begin{cases} A_T(\theta)e^{p_T\xi} + B_T(\theta) & ; \quad \bar{x} < 0, \; z > 0 \\ A_B(\theta)e^{p_B\xi} + B_B(\theta) & ; \quad \bar{x} < 0, \; z < 0 \\ A(\theta)e^{-p\xi} + B(\theta) & ; \quad \bar{x} > 0 \end{cases} \tag{160}$$

where

$$p_I^2 = 2/ah_I, \quad p^2 = 2/ah, \quad \xi = \bar{x}/E^{\frac{1}{4}} \tag{161}$$

2. 4. 2 The Jump Relations

The first jump relation follows from continuity of v across the $\frac{1}{3}$-layer. That is, (72) is valid here with the obvious change in notation. Therefore

$$A_T(\theta) + B_T(\theta) = A_B(\theta) + B_B(\theta) = A(\theta) + B(\theta) \tag{162}$$

The second jump relation follows from continuity of total tangential shear stress, i. e., from (81). Using (160),

$$h_T^{\frac{1}{2}} A_T(\theta) + h_B^{\frac{1}{2}} A_B(\theta) = - h^{\frac{1}{2}} A(\theta) \tag{163}$$

The third jump relation is derived by requiring the radial Ekman flux on the disc to erupt as a vertical jet in the $\frac{1}{3}$-layer. However, in this range

$$\frac{E^{\frac{1}{2}}\cot\alpha}{E^{\frac{1}{3}}} = E^{1/6}\cot\alpha << 1$$

so that

$$\Delta p^{I}_{\frac{1}{4}}(0^-, \theta) = \Delta p_{\frac{1}{4}}(0^+, \theta) + o(E^{\frac{1}{2}}\cot\alpha)$$

which implies

$$\Delta p^{I}_{\frac{1}{4}}(0^-, \theta) = o(E^{\frac{1}{2}}\cot\alpha)$$

Consequently, (89) can be replaced by

$$Q_e(\theta) = \int_{-\infty}^{\infty} [w_{\frac{1}{3}}(0^+) - w_{\frac{1}{3}}(0^-)]\, d\bar{x}$$

Proceeding as in subsection 2. 3. 4 there follows

$$-\tfrac{1}{2}(\tfrac{\nu}{\Omega})^{\frac{1}{2}}[2U(\theta) + v^{T}_{\frac{1}{4}}(0, \theta) + v^{B}_{\frac{1}{4}}(0, \theta)] = \frac{\nu}{2\Omega}\int_{-h_B}^{h_T}[\frac{\partial^2 v_{\frac{1}{4}}}{\partial\bar{x}^2}(0, \theta) - \frac{\partial^2 v^{I}_{\frac{1}{4}}}{\partial\bar{x}^2}(0, \theta)]\, dz \qquad ; (164)$$

Integrating (159) for $\bar{x} < 0$,

$$\frac{\partial^2 v^{I}_{\frac{1}{4}}}{\partial\bar{x}^2}(0, \theta) = (\frac{4\Omega}{\nu h_I^2})^{\frac{1}{2}}[v^{I}_{\frac{1}{4}}(0, \theta) - v^{I}_{\frac{1}{4}}(-\infty, \theta)]$$

so that

$$\frac{\nu}{2\Omega}\int_{-h_B}^{h_T}\frac{\partial^2 v^I_{\frac{1}{4}}}{\partial \bar{x}^2}(0,\theta)dz = \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[v^T_{\frac{1}{4}}(0,\theta)+v^B_{\frac{1}{4}}(0,\theta)-v^T_{\frac{1}{4}}(-\infty,\theta)-v^B_{\frac{1}{4}}(-\infty,\theta)\right] \tag{165}$$

and the corresponding result for $\bar{x} > 0$ is still given by (98). So, inserting (98) and (165) into (164), grouping terms and using (72) leads to the third jump relation:

$$U(\theta) - V(\theta) + v^T_{\frac{1}{4}}(-\infty,\theta) + v^B_{\frac{1}{4}}(-\infty,\theta) = 0 \tag{166}$$

Equations (162), (163) and (166) are the required jump relations. It is not yet possible to proceed to the solution as consideration of yet another layer is required. The reason for this is explained in the next section.

2.4.3 The Gα-Layer

Up to this point, the shear column structure is identical to the structure for zero angle of attack. However, if the inner $\frac{1}{4}$-layer velocity is matched to the geostrophic velocity a contradiction arises. Equation (166) possesses no solution! To understand this apparent inconsistency, consider the following argument. Since the jump in v across the $\frac{1}{4}$-layer is of order one, the jump in the pressure must be

$$\Delta p_{\frac{1}{4}} \sim E^{\frac{1}{4}}$$

However,

$$\frac{E^{\frac{1}{2}}\cot\alpha}{E^{\frac{1}{4}}} = \frac{E^{\frac{1}{4}}}{\tan\alpha} >> 1$$

Thus, p must be continuous to $O(E^{\frac{1}{2}}\cot\alpha)$ across the $\frac{1}{4}$-layer (and of course it is also continuous across the $\frac{1}{3}$-layer to this order). To this order, $\Delta p_g(a, \theta)$ is given by (114). So, under the assumption that the standard Stewartson layer structure connects the geostrophic regions, necessarily Δp_g must vanish. This is true because p must match to the geostrophic pressure outside the Taylor column and outside the Taylor column p is independent of z. However, $f^B(\theta)$ and $f^T(\theta)$ are even in θ as is $V_g(\theta)$, so that $\frac{dV_g}{d\theta}(\theta)$ is odd in θ. The only way for Δp_g to vanish is to have

$$f^T(\theta) = f^B(\theta)$$

and

$$\frac{dV_g(\theta)}{d\theta} = 0$$

Then $V_g(\theta)$ (and hence $V_g(x)$) is constant so that, to leading order, the flow inside the Taylor column is irrotational. But, disregarding the extra knowledge that u vanishes inside the Taylor column, this problem is identical to the problem solved by Moore and Saffman.[3] The solution to that problem shows the free stream to suffer a deflection of 18.4° in which case u is not zero.

It must be concluded that an additional layer is required connecting the geostrophic flow inside the Taylor column to the $\frac{1}{4}$-layer. Furthermore, this layer must be such that a jump in p of order $E^{\frac{1}{2}}\cot\alpha$ can be accomplished. Anticipating a jump in v of order one across this layer then the thickness must be

$$\delta \sim \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha$$

Referring to the boundary conditions (45), (46) and (47) it is clear that

$$w_{G\alpha} \sim v_{G\alpha} \tan\alpha$$

so that (43) and (44) reduce to

$$\frac{\partial w_{G\alpha}}{\partial z} = 0 \qquad (167)$$

$$\frac{\partial v_{G\alpha}}{\partial z} = 0 \qquad (168)$$

where the subscript $G\alpha$ refers to quantities in this layer. The layer will be referred to as the $G\alpha$-layer. The boundary conditions (45), (46) and (47) are correct as they stand with the understanding that

$$u_{G\alpha}(\bar{x}, \theta) = U(\theta)$$

This layer, like the α-layer, appears in the sliced cylinder problem (Beardsley[8]).

Noting (167) and (168), it follows immediately that the relevant equations of motion are

$$\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \frac{\partial v_{G\alpha}}{\partial \bar{x}} \pm (v_{G\alpha} \sin\theta - U(\theta)\cos\theta) = 0 \text{ for } z \gtrless 0 \qquad (169)$$

The solution to (169) is:

$$v_{G\alpha} = U(\theta)\cot\theta + \begin{cases} V^T_{G\alpha}(\theta)H(-\theta)e^{-\xi^*\sin\theta}, & z > 0 \\ V^B_{G\alpha}(\theta)H(\theta)e^{\xi^*\sin\theta}, & z < 0 \end{cases} \qquad (170)$$

where $H(\theta)$ is the Heaviside stepfunction and

$$\xi^* = \frac{\bar{x}}{\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha} \qquad (171)$$

(170) shows that

$$v_{G\alpha} \to U(\theta)\cot\theta \qquad \text{as } \xi^* \to -\infty$$

which checks with (112).

Integrating (169) across the Gα-layer, there follows:

$$v^T_{G\alpha}(0,\theta) - v^T_{G\alpha}(-\infty,\theta) = -\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} \tan\alpha\sin\theta \int_{-\infty}^{0} \left[v^T_{G\alpha}(\bar{x},\theta) - U(\theta)\cot\theta\right] d\bar{x} \quad (172)$$

and

$$v^B_{G\alpha}(0,\theta) - v^B_{G\alpha}(-\infty,\theta) = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} \tan\alpha\sin\theta \int_{-\infty}^{0} \left[v^B_{G\alpha}(\bar{x},\theta) - U(\theta)\cot\theta\right] d\bar{x} \quad (173)$$

From matching to the $\frac{1}{4}$-layer,

$$v^I_{G\alpha}(0,\theta) = v^I_{\frac{1}{4}}(-\infty,\theta) \quad (174)$$

and similarly, matching to the geostrophic flow

$$v^T_{G\alpha}(-\infty,\theta) = v^B_{G\alpha}(-\infty,\theta) = v_g(a,\theta) \quad (175)$$

Adding equations (172) and (173) leaves

$$v^B_{\frac{1}{4}}(-\infty,\theta) + v^T_{\frac{1}{4}}(-\infty,\theta) = 2v_g(a,\theta) - \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} \tan\alpha\sin\theta \int_{-\infty}^{0} \left[v^T_{G\alpha} - v^B_{G\alpha}\right] d\bar{x} \quad (176)$$

Finally, from (42) and matching:

$$\int_{-\infty}^{0} \left[v^T_{G\alpha} - v^B_{G\alpha}\right] d\bar{x} = \frac{1}{2\Omega}\left[\Delta p_{\frac{1}{4}}(-\infty,\theta) - \Delta p_g(a,\theta)\right]$$

where

$$\Delta p(\bar{x},\theta) \equiv p(\bar{x},\theta,0^+) - p(\bar{x},\theta,0^-)$$

Then, since

$$\Delta p_{\frac{1}{4}}(-\infty, \theta) = o(E^{\frac{1}{2}} \cot\alpha)$$

equation (176) becomes

$$v^B_{\frac{1}{4}}(-\infty, \theta) + v^T_{\frac{1}{4}}(-\infty, \theta) = 2v_g(a, \theta) + (\frac{\Omega}{\nu})^{\frac{1}{2}} \tan\alpha \frac{\Delta p_g(a, \theta)}{2\Omega} \sin\theta \tag{177}$$

Inserting (177) into (166) leads to (113) which in turn reduces to (117). Similarly, it is easy to demonstrate that (117a) holds if the top surface is free.

This means that the same geostrophic flow prevails in this range as in the range

$$E^{\frac{1}{4}} << \alpha << 1$$

The shear column structure will be worked out in the next two sub-sections.

2. 4. 4 <u>x-Translation Solution</u>

For brevity, detailed calculation of the Gα-layer and $\frac{1}{4}$-layer solutions will not be given. Only important results are shown.

Equation (174) can be rewritten as

$$B_T(\theta) = U(\theta)\cot\theta + V^T_{G\alpha}(\theta)H(-\theta) \tag{178}$$

$$B_B(\theta) = U(\theta)\cot\theta + V^B_{G\alpha}(\theta)H(\theta) \tag{179}$$

Also, (166) means

$$B_B(\theta) + B_T(\theta) = V(\theta) - U(\theta) \tag{180}$$

The system of equations (83), (137), (138), (162), (163), (178), (179) and (180) can be solved to yield

$$V^B_{G\alpha}(\theta) = V^T_{G\alpha}(\theta) = -2U\sin\theta \tag{181}$$

$$B_B(\theta) = -2U\sin\theta H(\theta) \tag{182}$$

$$B_T(\theta) = -2U\sin\theta H(-\theta) \tag{183}$$

$$B(\theta) = -2U\sin\theta \tag{184}$$

$$A_B(\theta) = \frac{1}{K}\left\{-(h_T^{\frac{1}{2}} + h^{\frac{1}{2}})H(-\theta) + h_T^{\frac{1}{2}}H(\theta)\right\} 2U\sin\theta \tag{185}$$

$$A_T(\theta) = \frac{1}{K}\left\{h_B^{\frac{1}{2}}H(-\theta) - (h_B^{\frac{1}{2}} + h^{\frac{1}{2}})H(\theta)\right\} 2U\sin\theta \tag{186}$$

$$A(\theta) = \frac{1}{K}\left\{h_B^{\frac{1}{2}}H(-\theta) + h_T^{\frac{1}{2}}H(\theta)\right\} 2U\sin\theta \tag{187}$$

where

$$K \equiv h_B^{\frac{1}{2}} + h_T^{\frac{1}{2}} + h^{\frac{1}{2}} \tag{188}$$

2. 4. 5 <u>y-Translation Solution</u>

Equations (137) and (138) are replaced by (155) and (156). The other equations used in 2. 4. 4 are still valid so that the solution becomes:

$$V^B_{G\alpha}(\theta) = V^T_{G\alpha}(\theta) = -\tfrac{2}{3}V\sin\theta \tag{189}$$

$$B_B(\theta) = \tfrac{2}{3}V\left[\cos\theta - \sin\theta H(\theta)\right] \tag{190}$$

$$B_T(\theta) = \tfrac{2}{3}V\left[\cos\theta - \sin\theta H(-\theta)\right] \tag{191}$$

$$B(\theta) = \frac{4}{3}V\cos\theta \tag{192}$$

$$A_B(\theta) = \tfrac{2}{3}\frac{V}{K}\left\{h^{\frac{1}{2}}\cos\theta + \left[-h_T^{\frac{1}{2}}H(-\theta)+(h_T^{\frac{1}{2}}+h^{\frac{1}{2}})H(\theta)\right]\sin\theta\right\} \quad (193)$$

$$A_T(\theta) = \tfrac{2}{3}\frac{V}{K}\left\{h^{\frac{1}{2}}\cos\theta+\left[(h_B^{\frac{1}{2}}+h^{\frac{1}{2}})H(-\theta)-h_B^{\frac{1}{2}}H(\theta)\right]\sin\theta\right\} \quad (194)$$

$$A(\theta) = -\tfrac{2}{3}\frac{V}{K}\left\{(h_B^{\frac{1}{2}}+h_T^{\frac{1}{2}})\cos\theta + \left[h_T^{\frac{1}{2}}H(-\theta)+h_B^{\frac{1}{2}}H(\theta)\right]\sin\theta\right\} \quad (195)$$

with K defined in (188).

2.4.6 Summary

The flows in the geostrophic regions are the same as in the range $E^{\frac{1}{4}} << \alpha << 1$. This is, of course, true for both x and y-translation.

The shear column structure is essentially identical for both motions. There is a $\frac{1}{3}$-layer sandwiched by $\frac{1}{4}$-layers both on the inner and outer edges of the shear column. It becomes necessary to infer the existence of yet another, fatter, layer to complete the solution. This layer is located on the inner edge of the shear column. Its thickness is proportional to $E^{\frac{1}{2}}\cot\alpha$. It is referred to as the $G\alpha$-layer. Just as for the case when α vanishes only three jump conditions across the $\frac{1}{3}$-layer evolve:

(a) the $\frac{1}{4}$-layer circumferential velocity is continuous;

(b) the total tangential shear stress is continuous;

(c) mass is conserved in the $\frac{1}{3}$-layer.

2.5 Shear Column Structure For $\alpha \sim E^{\frac{1}{4}}$

This section is included for two reasons. One is for the sake of completeness in the study of the motion of the disc for

$$E^{\frac{1}{2}} << \alpha << 1$$

The other is to add further clarification of the technique employed to determine the geostrophic flow without formally solving the shear column equations. That is, the intent is to further enunciate the importance of (113).

When α is of the same order of magnitude as $E^{\frac{1}{4}}$, the $G\alpha$-layer, $\frac{1}{4}$-layer and α-layer merge. The equations for the $\frac{1}{4}$-layer in this range are again (48) and (49). However, (46) is valid as it stands since both terms are of order $E^{\frac{1}{4}}$. Using the same notation as in 2.4, it will still be true that

$$u_{\frac{1}{4}}(\bar{x}, \theta) = U(\theta)$$

Proceeding just as in section 2.4, the equations for the $\frac{1}{4}$-layer at the inner edge of the shear column become

$$\frac{\partial^3 v^T_{\frac{1}{4}}}{\partial \bar{x}^3} - \left(\frac{4\Omega}{\nu h_T^2}\right)^{\frac{1}{2}} \frac{\partial v^T_{\frac{1}{4}}}{\partial \bar{x}} - \frac{2\Omega \tan\alpha}{\nu h_T} (v^T_{\frac{1}{4}} \sin\theta - U(\theta)\cos\theta) = 0 \qquad (196)$$

$$\frac{\partial^3 v^B_{\frac{1}{4}}}{\partial \bar{x}^3} - \left(\frac{4\Omega}{\nu h_B^2}\right)^{\frac{1}{2}} \frac{\partial v^B_{\frac{1}{4}}}{\partial \bar{x}} + \frac{2\Omega \tan\alpha}{\nu h_B} (v^B_{\frac{1}{4}} \sin\theta - U(\theta)\cos\theta) = 0 \qquad (197)$$

The $\frac{1}{4}$-layer at the outer edge of the shear column remains unchanged.

There is little to be learned from doing the complete solution for the $\frac{1}{4}$-layer in this case. It is in most respects similar to the solution when

$$E^{\frac{1}{2}} << \alpha << E^{\frac{1}{4}}$$

It does however possess one feature which is not seen in a standard $\frac{1}{4}$-layer. Over a fixed range of θ, the circumferential velocity behaves like an exponentially damped sinusoid rather than strict exponential decay. This feature is reminiscent of the behavior of the α-layer. To achieve this behavior it is necessary for the $\frac{1}{4}$-layer solution to contain additional unknown functions of θ. This, of course, requires more equations to determine a solution. In these respects the formal solution will more closely resemble the work of section 2. 3. To see this, assume that

$$v^{I}_{\frac{1}{4}}(\bar{x},\theta) - U(\theta)\cot\theta = F(\theta)e^{\sigma\bar{x}}$$

Equations (196) and (197) show that

$$\sigma^3 - \left(\frac{4\Omega}{\nu h_I^2}\right)^{\frac{1}{2}}\sigma \mp \frac{2\Omega\tan\alpha\sin\theta}{\nu h_I} = 0$$

There will be one real root and two conjugate imaginary roots if

$$\sin\theta > \left(\frac{8}{27}\right)^2 \frac{\left(\frac{\nu}{\Omega h_I^2}\right)^{\frac{1}{4}}}{\tan\alpha} \equiv \sin\theta_t$$

which is reminiscent of the α-layer structure. There will be three real and unequal roots if the inequality is reversed. This is the nature of a conventional $\frac{1}{4}$-layer. Hence, the $\frac{1}{4}$-layer will appear as in Figure 6.

It is still possible to obtain the solution for the geostrophic regions by using the techniques developed in 2. 4. Note first that another consequence of the fact that

$$E^{\frac{1}{2}}\cot\alpha \sim E^{\frac{1}{4}}$$

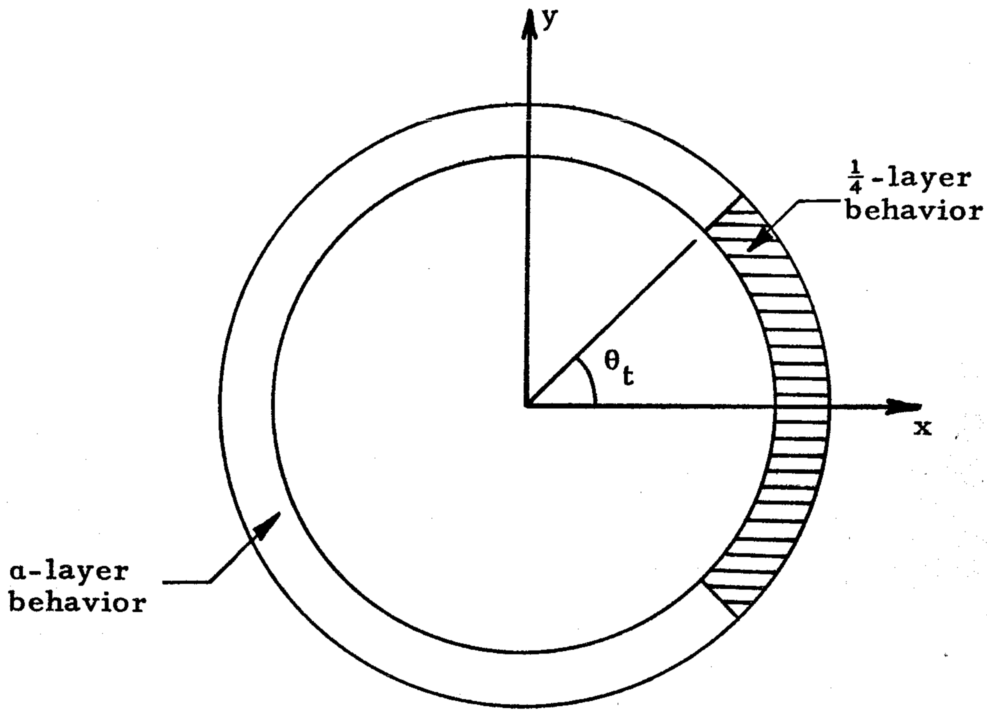

Figure 6. Schematic Representation of $\frac{1}{4}$-Layer Structure for $\alpha \sim E^{\frac{1}{4}}$. (not to scale)

is that a jump in the pressure of order $E^{\frac{1}{2}}\cot\alpha$ can now be accomplished by a $\frac{1}{4}$-layer. Certainly $v_{\frac{1}{4}}$ will again be continuous to order one across the $\frac{1}{3}$-layer so that (164) is valid. Integrating (196) and (197) across the $\frac{1}{4}$-layer leads to the expressions:

$$\frac{\nu}{2\Omega}\frac{\partial^2 v^T_{\frac{1}{4}}}{\partial \bar{x}^2}(0,\theta) = \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\frac{1}{h_T}\left[v^T_{\frac{1}{4}}(0,\theta) - v^T_{\frac{1}{4}}(-\infty,\theta)\right]$$

$$+ \frac{\tan\alpha}{2\Omega h_T}\left[p^T_{\frac{1}{4}}(0,\theta) - p^T_{\frac{1}{4}}(-\infty,\theta)\right]\sin\theta \qquad (198)$$

$$\frac{\nu}{2\Omega}\frac{\partial^2 v^B_{\frac{1}{4}}}{\partial \bar{x}^2}(0,\theta) = \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\frac{1}{h_B}\left[v^B_{\frac{1}{4}}(0,\theta) - v^B_{\frac{1}{4}}(-\infty,\theta)\right]$$

$$- \frac{\tan\alpha}{2\Omega h_B}\left[p^B_{\frac{1}{4}}(0,\theta) - p^B_{\frac{1}{4}}(-\infty,\theta)\right]\sin\theta \qquad (199)$$

so that

$$\frac{\nu}{2\Omega}\int_{-h_B}^{h_T} \frac{\partial^2 v_{\frac{1}{4}}^I}{\partial \bar{x}^2}\, dz = \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[v_{\frac{1}{4}}^T(0,\theta)+v_{\frac{1}{4}}^B(0,\theta)-v_{\frac{1}{4}}^T(-\infty,\theta)-v_{\frac{1}{4}}^B(-\infty,\theta)\right]$$

$$+\frac{\tan\alpha}{2\Omega}\left[\Delta p_{\frac{1}{4}}(0,\theta)-\Delta p_{\frac{1}{4}}(-\infty,\theta)\right]\sin\theta \qquad (200)$$

Since p is continuous to $O(E^{\frac{1}{2}}\cot\alpha)$ across the $\frac{1}{3}$-layer, matching to the $\frac{1}{4}$-layer on the outside edge of the shear column shows that

$$\Delta p_{\frac{1}{4}}(0,\theta) = 0$$

since p is independent of z in that layer. Equation (98) still holds and when used in conjunction with (164) and (200) there follows:

$$2U(\theta)+2\left[v_{\frac{1}{4}}^T(-\infty,\theta) + v_{\frac{1}{4}}^B(-\infty,\theta)-V(\theta)\right] = -\frac{\tan\alpha}{2\Omega}\Delta p_{\frac{1}{4}}(-\infty,\theta)\sin\theta \qquad (201)$$

This is the analog of equation (166). The only difference is that (166) is homogeneous. The fact that the right hand side of (201) is nonvanishing is a result of the extra unknown functions in the $\frac{1}{4}$-layer solution mentioned earlier. Matching to the geostrophic regions

$$v_{\frac{1}{4}}^T(-\infty,\theta) = v_{\frac{1}{4}}^B(-\infty,\theta) = v_g(a,\theta)$$

and

$$\Delta p_{\frac{1}{4}}(-\infty,\theta) = \Delta p_g(a,\theta)$$

Clearly equation (113) and hence equation (117) follow immediately from (201). This, of course, means that the results of subsections 2. 3. 5 and 2. 3. 6 pertaining to the geostrophic regions are valid when $\alpha \sim E^{\frac{1}{4}}$.

2.6 Discussion

The same geostrophic flow is valid for both x-translation and y-translation for all α in the range

$$E^{\frac{1}{2}} << \alpha << 1$$

The solutions are discussed thoroughly in subsections 2.3.5, 2.3.6 and 2.3.7.

The shear column structure varies according to the ratio of α to $E^{\frac{1}{4}}$. Discussion of the detailed structure is given at the ends of sections 2.3 and 2.4 and at the beginning of section 2.5. The results will not be repeated here.

Rather, a more important fact is relevant. In all ranges of α, equation (113) was shown to hold. Because of that, the geostrophic flow is invariant throughout the range of α. The question naturally arises as to whether or not (113) or an analog to (113) can be used to determine the geostrophic flow for the other four motions. If this is true, the detailed structure of the shear column needn't be worked out.

The answer to this question is in the affirmative. Justification of this claim is very simple. Regardless of the type of motion, the shear column equations will always be equations (43) and (44). The boundary conditions (46) will be changed according to whether or not the disc has a normal component of velocity. Also, if the curl of the velocity of the disc or the walls has a normal component, (45)-(47) must be altered accordingly. However, this only affects the structure of the outer layers (α-layer, $\frac{1}{4}$-layer, Gα-layer). It will still be true from the minimum singularity hypothesis that v experiences no jump across the $\frac{1}{3}$-layer. This means (92) will always be valid. Further simplification of (92) results from integrating across the outer layers. The only way in which the outer layers are influenced by the boundary conditions is in the asymptotic form of v as $|\bar{x}| \to \infty$. This behavior is unimportant in determining the jump condition relating the geostrophic flows inside and outside the Taylor column, e.g., see equation (101).

One change will be necessary. (88) is not completely general. If the unit outer normal to a surface is $\underline{n}$, and the surface velocity is $\underline{u}_p$, the excess flux in the Ekman layer will be given by

$$\underline{Q} = \tfrac{1}{2}\left(\frac{\nu}{|\underline{\Omega}\cdot\underline{n}|}\right)^{\frac{1}{2}} \underline{n} \times \left[(\underline{u}-\underline{u}_p) + \underline{n} \times (\underline{u}-\underline{u}_p)\right] \qquad (202)$$

where $\underline{u}$ is the fluid velocity outside the Ekman layer. In this problem $\underline{n}$ is given by (16). For α small, the radial component of (202) becomes simply:

$$Q_r = -\tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[u-u_p + v-v_p\right]$$

This means that (88) must be replaced by

$$Q_e(\theta) = -\tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[2u(\theta) + v_\alpha^T(0,\theta) + v_\alpha^B(0,\theta) - 2u_p - 2v_p\right] \qquad (203)$$

As will be shown in the following sections, whenever the disc has a nonvanishing vertical component of velocity, the geostrophic flow will be of order cotα rather than one. But this means that for these motions,

$$u_p,\ v_p \ll u_\alpha,\ v_\alpha$$

and hence (203) is identical to (88). Therefore, (113) will apply to all motions of the disc.

To clarify this discussion, the equations for the $G\alpha$, $\frac{1}{4}$ and α-layers will be derived for the rising disc in Chapter 3. The equations will then be integrated to demonstrate that indeed (113) still holds.

3. VERTICAL TRANSLATION FOR INFINITESIMAL α

3.1 Statement of the Problem

All relevant notation used in Chapter 2 will again be used. The disc will be assumed to move vertically with speed, W, so that

$$\underline{u}_p = W\underline{k} \tag{204}$$

The walls and the fluid at infinity remain at rest.

Anticipating that the geostrophic motion inside the Taylor column should be such that vortex lines are not stretched, one prediction can be made. Assume that at time t_o the center of the disc is located at $z = 0$. The equation of the disc is given by (1) so that

$$z(t_o) = -x(t_o)\tan\alpha$$

At a time $t_1 > t_o$, the disc will have moved vertically a distance

$$\Delta z = W(t_1 - t_o)$$

so that the equation of the disc at this time is

$$z(t_1) = -x(t_1)\tan\alpha + W(t_1 - t_o)$$

Certainly, a vortex line which is initially at a station x cannot remain there as it will undergo a change in length Δz. So requiring

$$z(t_1) = z(t_o)$$

necessarily,

$$\frac{x(t_1) - x(t_o)}{t_1 - t_o} = W\cot\alpha$$

Taking the limit $t_1 \to t_o$, the velocity of the vortex line is hence

$$u = W\cot\alpha \tag{205}$$

Since the vortex lines are carried with the fluid in the geostrophic regions, necessarily (205) represents the x-component of the fluid velocity.

Note finally that this argument yields no constraint on v. This is because the vortex lines are free to translate across the disc in the y-direction as no stretching occurs from such motion.

3.2 The Geostrophic Flow

Outside the Taylor column, equations (13) and (14) are applicable. Since $\underline{u}_w$ vanishes for this problem, necessarily

$$\underline{k} \cdot \operatorname{curl} \underline{u}_G = 0$$

so that there is again two-dimensional potential flow outside the Taylor column. Since

$$\underline{u}_G \to \underline{0} \qquad \text{as} \qquad r \to \infty$$

the solution for the pressure is:

$$p_G(r,\theta) = \sum_{n=1}^{\infty} [B_n \cos n\theta + D_n \sin n\theta] \left(\frac{a}{r}\right)^n \tag{206}$$

Inside the Taylor column, (13) and (14) are still valid. However, (15) and (204) show that

$$w_g + u_g \tan\alpha = W \pm \tfrac{1}{2} \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right] \text{ on } z = 0^{\pm} \tag{207}$$

Appealing to (13) and (14) for the scaling on w_g, there follows

$$w_g \sim E^{\frac{1}{2}} u_g$$

so that, to leading order, (207) reduces to

$$u_g = W\cot\alpha$$

as predicted in 3.1. The rest of the calculation goes through just as for horizontal translation. The final results are:

$$u_g = W\cot\alpha \pm \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot^2\alpha \frac{dV_g(x)}{dx} \tag{208}$$

$$v_g = V_g(x)\cot\alpha + \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot^2\alpha \left[\frac{df_B^T(x)}{dx} \mp y \frac{d^2V_g(x)}{dx^2} \right] \tag{209}$$

$$w_g = \mp \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \frac{dV_g(x)}{dx} \tag{210}$$

$$p_g = [p_g(x) - 2\Omega Wy]\cot\alpha + 2\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot^2_\alpha \left[f_B^T(x) \mp y \frac{dV_g(x)}{dx} \right] \tag{211}$$

where $p_g(x)$ and $V_g(x)$ are again related by (27).

Requiring that the pressure be continuous to first order across the shear column, there follows

$$D_1 = -2\Omega aW\cot\alpha, \quad D_n = 0 \text{ for all } n \geq 2$$

So, outside the Taylor column, the flow simplifies to

$$p_G(r,\theta) = -2\Omega aW\cot\alpha\left(\frac{a}{r}\right)\sin\theta + \sum_{n=1}^{\infty} B_n \left(\frac{a}{r}\right)^n \cos n\theta \tag{212}$$

$$u_G(r,\theta) = W\cot\alpha\left(\frac{a}{r}\right)^2\cos\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n\left(\frac{a}{r}\right)^{n+1}\sin n\theta \tag{213}$$

$$v_G(r,\theta) = W\cot\alpha\left(\frac{a}{r}\right)^2\sin\theta - \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n\left(\frac{a}{r}\right)^{n+1}\cos n\theta \tag{214}$$

Before going on to the next section it is worthwhile to determine $v_g(a, \theta)$. From (208) and (209), the values of $\mathcal{U}_g$ and $\mathcal{V}_g$ on $r = a$ are:

$$\mathcal{U}_g = W\cot\alpha$$

$$\mathcal{V}_g = V_g(\theta)\cot\alpha$$

so that

$$u_g(a, \theta) = \left[W\cos\theta + V_g(\theta)\sin\theta\right]\cot\alpha$$

$$v_g(a, \theta) = \left[-W\sin\theta + V_g(\theta)\cos\theta\right]\cot\alpha$$

However,

$$u_g(a, \theta) = U(\theta)$$

which implies that

$$V_g(\theta) = \frac{U(\theta)\tan\alpha - W\cos\theta}{\sin\theta} \tag{215}$$

and hence

$$v_g(a, \theta) = \frac{U(\theta)\cos\theta - W\cot\alpha}{\sin\theta} \tag{216}$$

3. 3 The Shear Column Equations

Although a formal solution for the shear column structure will not be given, the equations for the $G\alpha$, $\frac{1}{4}$ and α-layers will be derived. As mentioned in 2. 6, it is only necessary to show that (113) is still applicable. In the shear column, the boundary conditions on the disc will be:

$$w = W + (v\sin\theta - u\cos\theta)\tan\alpha \pm \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{\partial v}{\partial \bar{x}} \quad \text{on} \quad z = 0^{\pm} \tag{217}$$

and on the walls (45) and (47) still hold.

It is important to note that the last two terms in (203) are O(1) while the other terms are O(cotα). Hence, as hinted in section 2.6, (203) reduces to (88). Thus, it is only necessary to verify that

$$\int_{-\infty}^{\infty} \left[w_{\frac{1}{3}}(0^{+}) - w_{\frac{1}{3}}(0^{-})\right] d\bar{x}$$

can be evaluated in the same manner as in Chapter 2.

(a) Gα-Layer

For this layer, equations (167) and (168) apply. Thus

$$\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \frac{\partial v_{G\alpha}}{\partial \bar{x}} \pm \left[v_{G\alpha}\sin\theta - (U(\theta)\cos\theta - W\cot\alpha)\right] = 0 \quad \text{for } z \gtrless 0 \tag{218}$$

where it has been noted that

$$u_{G\alpha}(\bar{x}, \theta) = U(\theta)$$

The asymptotic behavior of $v_{G\alpha}$ as $\bar{x} \to -\infty$ is clearly

$$\lim_{\bar{x} \to -\infty} v_{G\alpha}(\bar{x}, \theta) = \frac{U(\theta)\cos\theta - W\cot\alpha}{\sin\theta} \tag{219}$$

But, matching demands that

$$\lim_{\bar{x} \to -\infty} v_{G\alpha}(\bar{x}, \theta) = v_g(a, \theta)$$

and (216) shows that (219) is correct. So, in fact, writing

$$\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \frac{\partial v_{G\alpha}}{\partial \bar{x}} \pm \left[v_{G\alpha} - v_g(a, \theta)\right] \sin\theta = 0 \quad \text{for} \quad z \gtrless 0 \tag{220}$$

then (169) and (218) are identical. Thus, it is obvious that integrating (220) and performing the necessary matching, etc., the result will be (177).

(b) $\frac{1}{4}$-Layer and α-Layer

To handle the entire range $E^{\frac{1}{2}} << \alpha << 1$, simply retain all terms in (217) and then look at limiting cases to study the various ranges independently. It will be true to first order that

$$u = U(\theta)$$

which means (217) can be rewritten as

$$w = \left[v\sin\theta-(U(\theta)\cos\theta-W\cot\alpha)\right]\tan\alpha \pm \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{\partial v}{\partial \bar{x}} \quad \text{on } z = 0^{\pm}$$

or, using (216),

$$w = \left[v-v_g(a,\theta)\right]\sin\theta\tan\alpha \pm \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{\partial v}{\partial \bar{x}} \quad \text{on } z = 0^{\pm} \tag{221}$$

Hence, the $\frac{1}{4}$-layer equations become

$$\frac{\partial^3 v_{\frac{1}{4}}^T}{\partial \bar{x}^3} - \left(\frac{4\Omega}{\nu h_T^2}\right)^{\frac{1}{2}} \frac{\partial v_{\frac{1}{4}}^T}{\partial \bar{x}} - \frac{2\Omega\tan\alpha}{\nu h_T}\left(v_{\frac{1}{4}}^T - v_g(a,\theta)\right)\sin\theta = 0 \text{ for } z > 0 \tag{222}$$

$$\frac{\partial^3 v_{\frac{1}{4}}^B}{\partial \bar{x}^3} - \left(\frac{4\Omega}{\nu h_B^2}\right)^{\frac{1}{2}} \frac{\partial v_{\frac{1}{4}}^B}{\partial \bar{x}} + \frac{2\Omega\tan\alpha}{\nu h_B}\left(v_{\frac{1}{4}}^B - v_g(a,\theta)\right)\sin\theta = 0 \text{ for } z < 0 \tag{223}$$

In the range $E^{\frac{1}{2}} << \alpha << E^{\frac{1}{4}}$, the last terms in (221) and (223) drop out so that (159) results. Hence, integrating across the $\frac{1}{4}$-layer, (166) follows. Matching to the Gα-layer (i.e., using (177)) clearly leads to (113).

In the range $a \sim E^{\frac{1}{4}}$, first replace $v_g(a, \theta)$ by $v^I_{\frac{1}{4}}(-\infty, \theta)$. Integrating across the $\frac{1}{4}$-layer leads immediately to (200) which in turn leads to (113).

For $E^{\frac{1}{4}} << a << 1$, the central terms of (222) and (223) drop out. Integrating over z shows that (101) is still valid. Hence, (113) is verified in this case also.

3.4 Solution for the Geostrophic Flow

Having demonstrated the validity of (113), the problem of finding the geostrophic flow is quite easy. $v_g(a, \theta)$ is given by (216). From (211) a short calculation reveals:

$$\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} \tan a \frac{\Delta p_g(a,\theta)}{2\Omega} = \cot a \left[f^T(\theta) - f^B(\theta) + 2\frac{dV_g(\theta)}{d\theta} \right] \qquad (224)$$

Insertion of (216) and (224) into (113) leaves:

$$U(\theta) - V(\theta) - 2W \cot a \sin\theta + 2 \cot a \frac{d}{d\theta}\left[V_g(\theta)\sin\theta\right]$$
$$= -\cot a \left[f^T(\theta) - f^B(\theta)\right] \sin\theta$$

Using (215), there results finally

$$2\frac{dU(\theta)}{d\theta} + U(\theta) - V(\theta) = -\cot a \left[f^T(\theta) - f^B(\theta)\right]\sin\theta \qquad (225)$$

With the exception of the cot a term on the right hand side (which could be removed by redefining U(θ) and V(θ) or $f^I(\theta)$), equation (225) is identical to (117). Finally, from (213) and (214),

$$U(\theta) = W \cot a \cos\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n \sin n\theta \qquad (226)$$

$$V(\theta) = W \cot a \sin\theta - \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n \cos n\theta \qquad (227)$$

Upon substitution of (226) and (227) into (225) and regrouping like terms, there follows:

$$-3W\cot\alpha\sin\theta + W\cot\alpha\cos\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} n(2n+1)B_n\cos n\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n\sin n\theta = -\cot\alpha\left[f^T(\theta)-f^B(\theta)\right]\sin\theta \qquad (228)$$

As before, the right hand side of (228) is an odd function of θ. Henceforth,

$$B_1 = -\tfrac{2}{3}\Omega aW\cot\alpha, \quad B_n = 0 \text{ for all } n \geq 2 \qquad (229)$$

Inserting (229) in (228) and noting the definition of $f^I(\theta)$,

$$f^T(x) - f^B(x) = \frac{10}{3}W \qquad (230)$$

In a similar way, there results

$$V_g(x) = -\tfrac{1}{3}W \qquad (231)$$

The geostrophic flow outside the Taylor column follows from substitution of (229) into (213) and (214). Therefore,

$$u_G(r,\theta) = W\cot\alpha\left[\cos\theta - \tfrac{1}{3}\sin\theta\right]\left(\frac{a}{r}\right)^2 \qquad (232)$$

$$v_G(r,\theta) = W\cot\alpha\left[\sin\theta + \tfrac{1}{3}\cos\theta\right]\left(\frac{a}{r}\right)^2 \qquad (233)$$

The geostrophic flow is shown in Figure 7. The magnitude of the constant velocity inside the Taylor column is

$$\frac{\sqrt{10}}{3}W\cot\alpha$$

and the flow is inclined to the x-axis at angle

$$-\tan^{-1}\tfrac{1}{3} \doteq -18.4^{o}$$

If the upper surface is free, the angle of inclination to the x-axis is

$$-\tan^{-1}\frac{2}{5} \doteq -21.8^{o}$$

As in the case of y-translation there is fluid flowing across the boundary of the Taylor column since

$$U(\theta) = W\cot\alpha\left[\cos\theta - \tfrac{1}{3}\sin\theta\right] \not\equiv 0 \tag{234}$$

The streamfunction for the geostrophic flow follows immediately from the relation

$$\psi = -\frac{1}{2\Omega}p + \text{constant} \tag{235}$$

and the streamlines outside the Taylor column are given by

$$(x - \tfrac{1}{3}\kappa)^2 + (y-\kappa)^2 = \frac{10}{9}\kappa^2 \tag{236}$$

where κ is an arbitrary constant such that

$$|\kappa| \geqslant \tfrac{1}{2} \tag{237}$$

The physical pressure follows from (4) so that using (211), (230) and (231) the result is

$$\Delta p = \frac{20}{3}\rho\Omega W\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot^2\alpha \tag{238}$$

The viscous stress is given by (134) noting that here

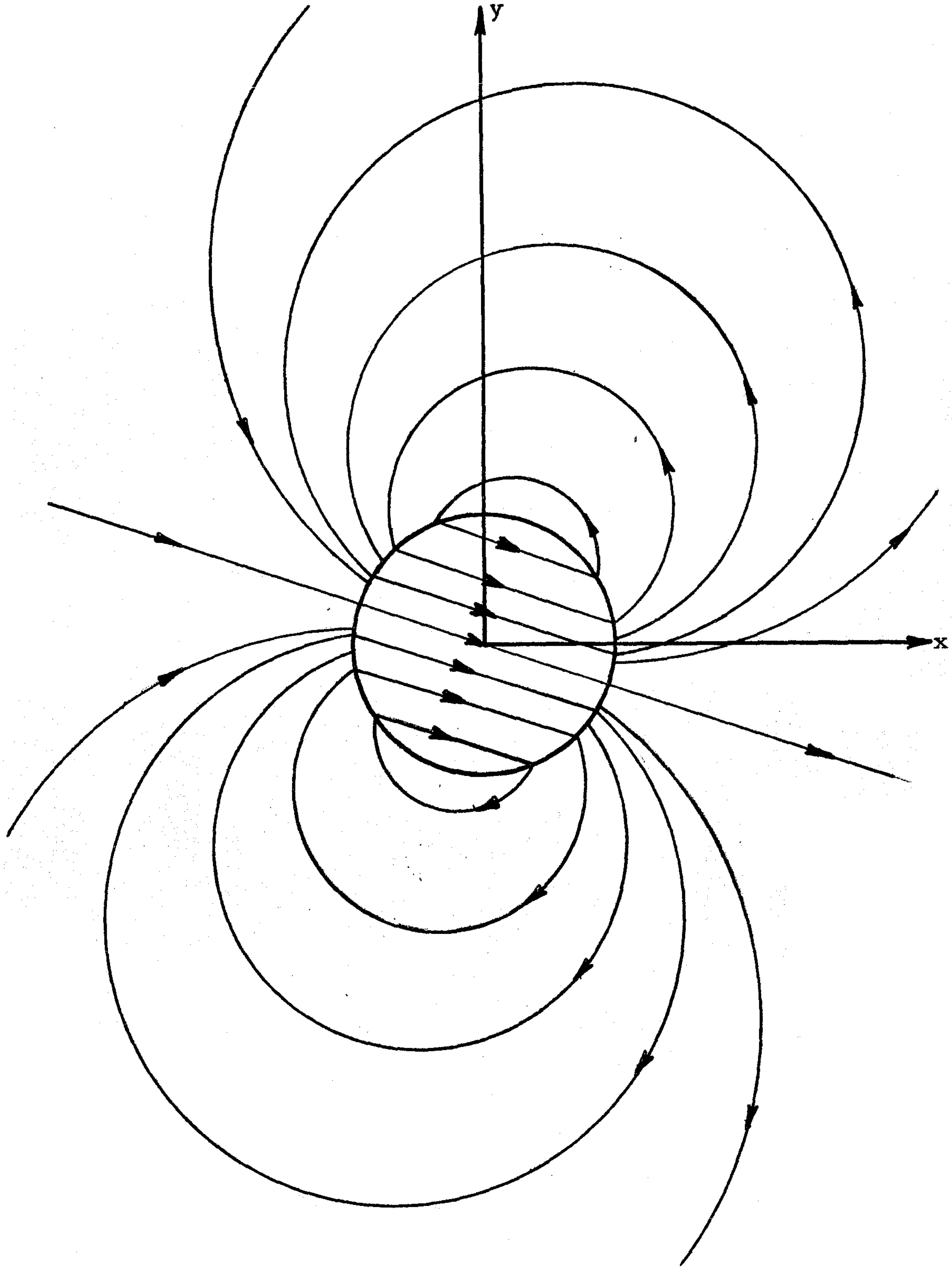

Figure 7. Geostrophic Flow for Vertical Translation

$$\underline{\hat{u}}_g = \underline{u}_g = W \cot\alpha \left[\underline{i} - \tfrac{1}{3} \underline{j} \right]$$

and

$$\underline{\hat{u}}_p = \underline{u}_p = W\underline{k}$$

then

$$\Delta \underline{\tau} = 2\rho\Omega W \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \left[\tfrac{1}{3} \underline{i} + \underline{j} \right] \tag{239}$$

Hence, the force on the disc is given by:

$$\underline{F} = -\tfrac{2}{3}\pi \rho\Omega a^2 W \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot^2\alpha \left[9\tan\alpha\, \underline{i} - 3\tan\alpha\, \underline{j} + 10\underline{k} \right] \tag{240}$$

The moment on the disc must be zero to order $E^{\frac{1}{2}} \cot^2\alpha$ as the difference in force acting on the top and bottom surfaces is independent of position. Hence:

$$\underline{M} = o(E^{\frac{1}{2}} \cot^2\alpha) \tag{241}$$

One feature of this motion is worthy of additional comment. In contrast to horizontal translation where the vertical force was of order $E^{\frac{1}{2}} \cot\alpha$ and hence small compared to one, vertical translation involves a vertical force component of order $E^{\frac{1}{2}} \cot^2\alpha$. For the range $E^{\frac{1}{2}} << \alpha << E^{\frac{1}{4}}$, this force is very much larger than order one. This is not too surprising, however, as Moore and Saffman[2] demonstrated that for vertical translation with zero α the vertical force is order $E^{-\frac{1}{2}}$.

3.5 Summary

The geostrophic flow is of order $\cot\alpha$. There is no torque on the disc to $O(E^{\frac{1}{2}}\cot^2\alpha)$. The drag (vertical force) is of order $E^{\frac{1}{2}}\cot^2\alpha$. There is also a horizontal force of order $E^{\frac{1}{2}}\cot\alpha$ inclined at an angle

$$-\tan^{-1}\tfrac{1}{3} \doteq -18.4^{\circ}$$

to the x-axis which is parallel to the direction of the uniform flow inside the Taylor column.

For the Ekman layer and shear column structure to be independent of Rossby number effects, a more rigid restriction than that needed for horizontal translation must be invoked. The restriction is

$$Ro \ll E^{\frac{1}{3}}\left(\frac{h}{a}\right)^{\frac{1}{3}}\tan\alpha$$

4. THE THREE ROTATIONS FOR INFINITESIMAL α

4.1 Statement of the Problems

The disc will be assumed to rotate about an axis not necessarily parallel to the z-axis. To study this motion it is again convenient to break the problem up into three components. Hence, x-rotation is the name given to the motion corresponding to

$$\underline{u}_p = \underline{r} \times (\epsilon_x \Omega \underline{i}) = \epsilon_x \Omega [z\underline{j} - y\underline{k}] \tag{242}$$

Similarly, for y-rotation, the disc velocity is given by

$$\underline{u}_p = \underline{r} \times (\epsilon_y \Omega \underline{j}) = \epsilon_y \Omega [-z\underline{i} + x\underline{k}] \tag{243}$$

In both cases, the disc is assumed to move while the walls and fluid at infinity remain stationary. Note that in (242) and (243) the value of z is given by (1). Hence, only the vertical component is relevant in these equations as the other components are of higher order.

For z-rotation, it is convenient to let the disc remain at rest and have the walls move with velocity

$$\underline{u}_w = - \epsilon_z \Omega r \, \underline{e}_\theta \tag{244}$$

as well as the fluid at infinity. $\underline{e}_\theta$ is a unit vector in the circumferential direction related to $\underline{i}$ and $\underline{j}$ by:

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta$$

It must be true that

$$\epsilon_x, \epsilon_y, \epsilon_z << 1$$

to justify neglect of the nonlinear terms.

4. 2 The Geostrophic Flows

The basic technique is essentially the same as for the three translations. A little care is required for z-rotation, but everything else goes through without any change from the previous analysis.

(a) x-Rotation

The Ekman condition shows the geostrophic flow outside the Taylor column to be a potential flow with no disturbance at infinity. Hence, $p_G(r,\theta)$ is given by (206).

Inside the Taylor column, (15) and (242) lead to:

$$w_g + u_g \tan\alpha = -\epsilon_x \Omega y \pm \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + 2\epsilon_x \Omega \tan\alpha\right] \text{ on } z = 0\pm \tag{245}$$

while (13) and (14) still hold. Hence, the geostrophic flow inside the Taylor column turns out to be:

$$u_g = -\epsilon_x \Omega y \cot\alpha \pm \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot^2\alpha \left[\frac{dV_g(x)}{dx} + \epsilon_x \Omega\right] \tag{246}$$

$$v_g = V_g(x)\cot\alpha + \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot^2\alpha \left[\frac{df^{T}_{B}(x)}{dx} \mp y \frac{d^2V_g(x)}{dx^2}\right] \tag{247}$$

$$w_g = \mp \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \left[\frac{dV_g(x)}{dx} + \epsilon_x \Omega\right] \tag{248}$$

$$p_g = \left[p_g(x) + \epsilon_x \Omega^2 y^2\right]\cot\alpha + 2\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot^2\alpha \left[f^{T}_{B}(x) \mp y\left(\frac{dV_g(x)}{dx} + \epsilon_x \Omega\right)\right] \tag{249}$$

with $p_g(x)$ and $V_g(x)$ related by (27).

Hence, from continuity of p across the shear column,

$$D_n = 0 \text{ for all } n \geqslant 1$$

Thus,

$$p_G(r,\theta) = \sum_{n=1}^{\infty} B_n \left(\frac{a}{r}\right)^n \cos n\theta \tag{250}$$

$$u_G(r,\theta) = \frac{1}{2\Omega a} \sum_{n=1}^{\infty} nB_n \left(\frac{a}{r}\right)^{n+1} \sin n\theta \tag{251}$$

$$v_G(r,\theta) = -\frac{1}{2\Omega a} \sum_{n=1}^{\infty} nB_n \left(\frac{a}{r}\right)^{n+1} \cos n\theta \tag{252}$$

(b) y-Rotation

The flow outside the Taylor column is again specified by writing the pressure in the form given by (206). The only difference between this case and case (a) is that (245) is replaced by

$$w_g + u_g \tan\alpha = \epsilon_y \Omega x \pm \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right] \text{ on } z = 0^{\pm} \tag{253}$$

The geostrophic flow inside the Taylor column is hence given by:

$$u_g = \epsilon_y \Omega x \cot\alpha \pm \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot^2\alpha \frac{dV_g(x)}{dx} \tag{254}$$

$$v_g = [V_g(x) - \epsilon_y \Omega y] \cot\alpha + \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot^2\alpha \left[\frac{df^{T}_{B}(x)}{dx} \mp y \frac{d^2V_g(x)}{dx^2}\right] \tag{255}$$

$$w_g = \mp \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \frac{dV_g(x)}{dx} \tag{256}$$

$$p_g = [p_g(x) - 2\epsilon_y \Omega^2 xy] \cot\alpha + 2\Omega \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot^2\alpha \left[f^{T}_{B}(x) \mp y \frac{dV_g(x)}{dx}\right] \tag{257}$$

with $p_g(x)$ and $V_g(x)$ related in the usual way.

Evaluating p_g on the edge of the shear column, it is seen that:

$$p_g(a,\theta) - p_g(a,-\theta) = -2\epsilon_y \Omega^2 a^2 \cot\alpha \sin 2\theta$$

and therefore,

$$D_1 = 0,\ D_2 = -\epsilon_y\ \Omega^2 a^2 \cot\alpha,\ D_n = 0 \quad \text{for all} \quad n \geq 3$$

Thus, the geostrophic flow outside the Taylor column is

$$p_G(r,\theta) = -\epsilon_y \Omega^2 a^2 \cot\alpha (\frac{a}{r})^2 \sin 2\theta + \sum_{n=1}^{\infty} B_n (\frac{a}{r})^n \cos n\theta \tag{258}$$

$$u_G(r,\theta) = \epsilon_y \Omega a \cot\alpha (\frac{a}{r})^3 \cos 2\theta + \frac{1}{2\Omega a} \sum_{n=1}^{\infty} n B_n (\frac{a}{r})^{n+1} \sin n\theta \tag{259}$$

$$v_G(r,\theta) = \epsilon_y\ \Omega a \cot\alpha (\frac{a}{r})^3 \sin 2\theta - \frac{1}{2\Omega a} \sum_{n=1}^{\infty} n B_n (\frac{a}{r})^{n+1} \cos n\theta \tag{260}$$

(c) <u>z-Rotation</u>

Since curl $\underline{u}_w$ does not vanish here, (13) and (14) show that

$$w = \mp \frac{1}{2} (\frac{\nu}{\Omega})^{\frac{1}{2}} [\underline{k} \cdot \text{curl}\ \underline{u}_G + 2\epsilon_z \Omega] \quad \text{on } z = \begin{matrix} h_T \\ -h_B \end{matrix}$$

Hence, from the Taylor-Proudman theorem,

$$\underline{k} \cdot \text{curl}\ \underline{u}_G = -2\epsilon_z \Omega$$

outside the Taylor column. Then writing

$$p_G(r,\theta) = -\epsilon_z \Omega^2 r^2 + \tilde{p}(r,\theta)$$

it follows that

$$\nabla^2 \tilde{p} = 0$$

So, the solution for $p_G(r,\theta)$ is:

$$p_G(r,\theta) = -\epsilon_z \Omega^2 r^2 + \sum_{n=1}^{\infty} [B_n \cos n\theta + D_n \sin n\theta] (\frac{a}{r})^n \tag{261}$$

Inside the Taylor column, the boundary conditions become

$$w_g = -\frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + 2\epsilon_z\Omega\right] \qquad \text{on } z = h_T \tag{262}$$

$$w_g + u_g \tan\alpha = \pm\frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right] \qquad \text{on } z = 0^{\pm} \tag{263}$$

$$w_g = \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + 2\epsilon_z\Omega\right] \qquad \text{on } z = -h_B \tag{264}$$

so that the geostrophic flow for $r < a$ is:

$$u_g = \pm\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha\left[\frac{dV_g(x)}{dx} + \epsilon_z\Omega\right] \tag{265}$$

$$v_g = V_g(x) + \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha\left[\frac{df^{\,T}_{\,B}(x)}{dx} \mp y\frac{d^2V_g(x)}{dx^2}\right] \tag{266}$$

$$w_g = \mp\frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[\frac{dV_g(x)}{dx} + 2\epsilon_z\Omega\right] \tag{267}$$

$$p_g = p_g(x) + 2\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha\left[f^{\,T}_{\,B}(x) \mp y\left(\frac{dV_g(x)}{dx} + \epsilon_z\Omega\right)\right] \tag{268}$$

From (268), p_g must be an even function of θ on $r = a$ so that

$D_n = 0$ for all $n \geq 1$

Therefore

$$p_G(r,\theta) = -\epsilon_z\Omega^2 r^2 + \sum_{n=1}^{\infty} B_n\left(\frac{a}{r}\right)^n \cos n\theta \tag{269}$$

$$u_G(r,\theta) = \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n\left(\frac{a}{r}\right)^{n+1}\sin n\theta \tag{270}$$

$$v_G(r,\theta) = -\epsilon_z\Omega r - \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n\left(\frac{a}{r}\right)^{n+1}\cos n\theta \tag{271}$$

For both x and y-rotation the flow is seen to be of order cota while for z-rotation it is order one. So, for the same reason as for vertical translation, the Ekman flux on the disc is given by (88) for x and y-rotation. Since the disc does not move for z-rotation, (88) is immediately correct as it is identical to (203). Hence, equation (113) applies directly to all three rotations.

4. 3 Solution for the Geostrophic Flows

The calculation of the geostrophic flows is routine and, hence, only important results will be shown.

(a) x-Rotation

The quantities needed in equation (113) are:

$$U(\theta) = \frac{1}{2\Omega a} \sum_{n=1}^{\infty} nB_n \sin n\theta \qquad (272)$$

$$V(\theta) = -\frac{1}{2\Omega a} \sum_{n=1}^{\infty} nB_n \cos n\theta \qquad (273)$$

$$v_g(a,\theta) = [V_g(\theta)\cos\theta + \epsilon_x \Omega a \sin^2\theta] \cot\alpha \qquad (274)$$

$$\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} \tan\alpha \, \frac{\Delta p_g(a,\theta)}{2\Omega} = \left[f^T(\theta) - f^B(\theta) + 2\frac{dV_g(\theta)}{d\theta} - 2\epsilon_x \Omega a \sin\theta\right] \cot\alpha \qquad (275)$$

$$V_g(\theta)\cot\alpha = \frac{U(\theta)}{\sin\theta} + \epsilon_x \Omega a \cot\alpha \cos\theta \qquad (276)$$

Insertion of (272)-(276) into (113) leads to:

$$2\frac{dU(\theta)}{d\theta} + U(\theta) - V(\theta) + 2\epsilon_x \Omega a \cot\alpha \cos 2\theta = -[f^T(\theta) - f^B(\theta)] \cot\alpha \sin\theta$$

or

$$2\epsilon_x \Omega a \cot\alpha \cos 2\theta + \frac{1}{2\Omega a} \sum_{n=1}^{\infty} n(2n+1)B_n \cos n\theta$$

$$+ \frac{1}{2\Omega a} \sum_{n=1}^{\infty} nB_n \sin n\theta = -\left[f^T(\theta) - f^B(\theta)\right] \cot\alpha \sin\theta \tag{277}$$

The solution of (277) is:

$$B_1 = 0, \; B_2 = -\frac{2}{5}\epsilon_x \Omega^2 a^2 \cot\alpha, \; B_n = 0 \text{ for all } n \geq 3 \tag{278}$$

Henceforth,

$$f^T(x) - f^B(x) = \frac{4}{5}\epsilon_x \Omega x \tag{279}$$

and

$$V_g(x) = \frac{1}{5}\epsilon_x \Omega x \tag{280}$$

Outside the Taylor column,

$$p_G(r, \theta) = -\frac{2}{5}\epsilon_x \Omega^2 a^2 \cot\alpha \left(\frac{a}{r}\right)^2 \cos 2\theta \tag{281}$$

$$u_G(r, \theta) = -\frac{2}{5}\epsilon_x \Omega a \cot\alpha \left(\frac{a}{r}\right)^3 \sin 2\theta \tag{282}$$

$$v_G(r, \theta) = \frac{2}{5}\epsilon_x \Omega a \cot\alpha \left(\frac{a}{r}\right)^3 \cos 2\theta \tag{283}$$

Finally,

$$p_g(x, y) = \epsilon_x \Omega^2 a^2 \left[\left(\frac{y}{a}\right)^2 + \frac{1}{5}\left(\frac{x}{a}\right)^2 - \frac{3}{5}\right] \cot\alpha \tag{284}$$

Noting (235), it is a simple matter to calculate the streamlines, and they are shown in Figure 8.

The jump in the physical pressure is given by:

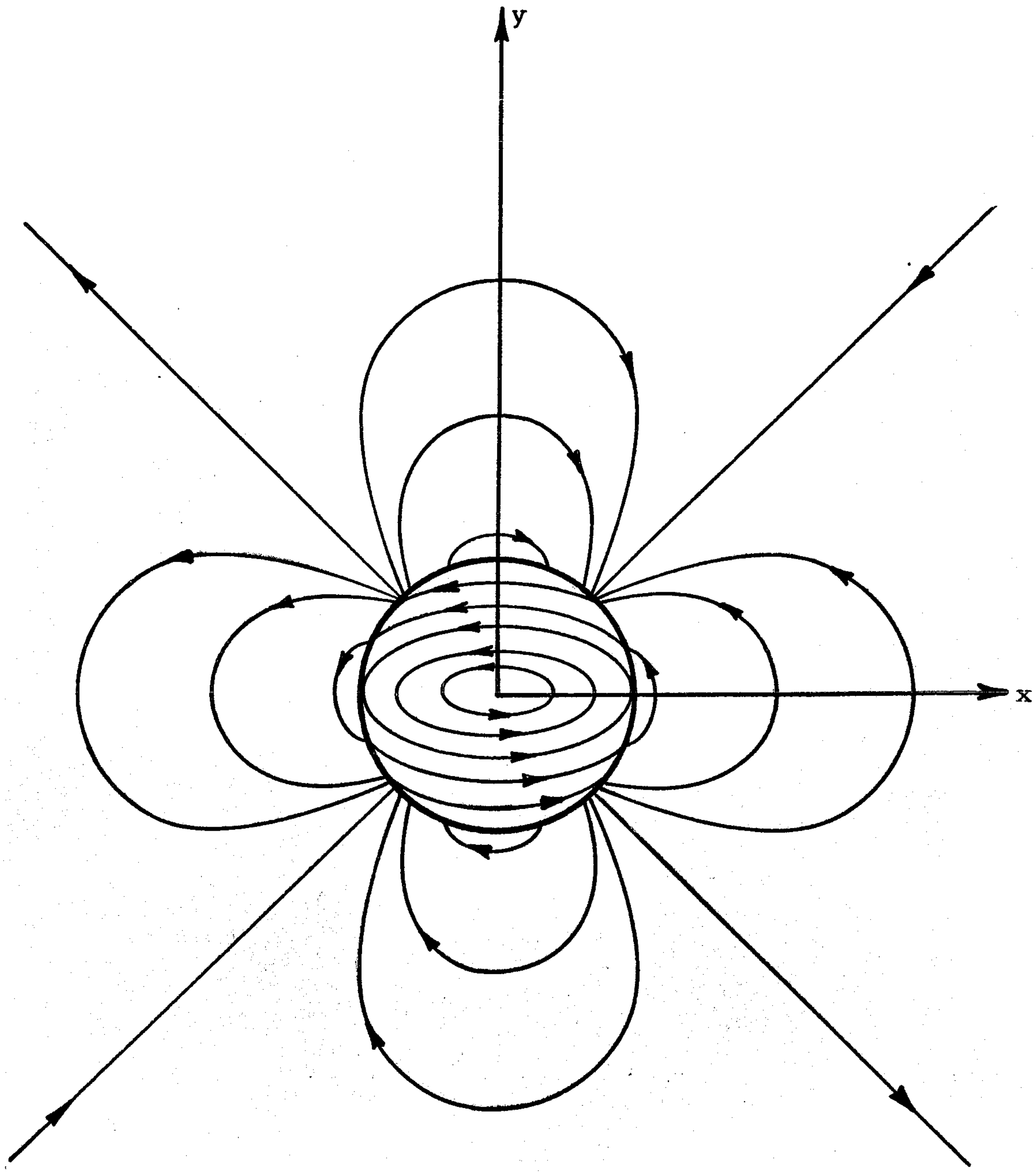

Figure 8. Geostrophic Flow for x-Rotation (Disc Moving)

$$\Delta p = \frac{8}{5}\rho\epsilon_x\Omega^2\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot^2\alpha\,[x-3y] \tag{285}$$

and the jump in the shear stress is:

$$\Delta \underline{\tau} = -2\rho\epsilon_x\Omega^2\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha\left[\frac{1}{5}x\,\underline{i} + y\,\underline{j}\right] \tag{286}$$

Thus, the force and moment on the disc are:

$$\underline{F} = o(E^{\frac{1}{2}}\cot^2\alpha) \tag{287}$$

$$\underline{M} = \frac{2}{5}\pi\rho\epsilon_x\Omega^2 a^4\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot^2\alpha\,[3\underline{i} + \underline{j} - 3\tan\alpha\underline{k}] \tag{288}$$

(b) y-Rotation

For this motion:

$$U(\theta) = \epsilon_y\Omega a\cot\alpha\cos 2\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n\sin n\theta \tag{289}$$

$$V(\theta) = \epsilon_y\Omega a\cot\alpha\sin 2\theta - \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n\cos n\theta \tag{290}$$

$$v_g(a,\theta) = [V_g(\theta)\cos\theta - \epsilon_y\Omega a\sin 2\theta]\cot\alpha \tag{291}$$

$$\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\tan\alpha\,\frac{\Delta p_g(a,\theta)}{2\Omega} = \left[f^T(\theta) - f^B(\theta) + 2\frac{dV_g(\theta)}{d\theta}\right]\cot\alpha \tag{292}$$

$$V_g(\theta)\cot\alpha = \frac{U(\theta)-\epsilon_y\Omega a\cot\alpha\cos 2\theta}{\sin\theta} \tag{293}$$

Hence, (113) reduces to:

$$2\frac{dU(\theta)}{d\theta} + U(\theta) - V(\theta) + 2\epsilon_y\Omega a\cot\alpha\sin 2\theta = -[f^T(\theta)-f^B(\theta)]\cot\alpha\sin\theta$$

or

$$\epsilon_y \Omega a \cot\alpha \left[\cos 2\theta - 3\sin 2\theta\right] + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} n(2n+1)B_n \cos n\theta$$

$$+ \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n \sin n\theta = -\left[f^T(\theta) - f^B(\theta)\right] \cot\alpha \sin\theta \qquad (294)$$

Inspection of (294) shows that necessarily:

$$B_1 = 0, \; B_2 = \frac{1}{5}\epsilon_y \Omega^2 a^2 \cot\alpha, \; B_n = 0 \text{ for all } n \geq 3 \qquad (295)$$

Therefore,

$$f^T(x) - f^B(x) = \frac{28}{5}\epsilon_y \Omega x \qquad (296)$$

and

$$V_g(x) = \frac{2}{5}\epsilon_y \Omega x \qquad (297)$$

For r greater than a

$$p_G(r, \theta) = -\epsilon_y \Omega^2 a^2 \cot\alpha \left[\sin 2\theta - \frac{1}{5}\cos 2\theta\right] \left(\frac{a}{r}\right)^2 \qquad (298)$$

$$u_G(r, \theta) = \epsilon_y \Omega a \cot\alpha \left[\cos 2\theta + \frac{1}{5}\sin 2\theta\right] \left(\frac{a}{r}\right)^3 \qquad (299)$$

$$v_G(r, \theta) = \epsilon_y \Omega a \cot\alpha \left[\sin 2\theta - \frac{1}{5}\cos 2\theta\right] \left(\frac{a}{r}\right)^3 \qquad (300)$$

The pressure inside the Taylor column is given by:

$$p_g(x, y) = 2\epsilon_y \Omega^2 a^2 \left[\frac{1}{5}\left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)\left(\frac{y}{a}\right) - \frac{1}{10}\right] \cot\alpha \qquad (301)$$

The streamlines are shown in Figure 9.

The jumps in the physical pressure and shear stress across the disc are

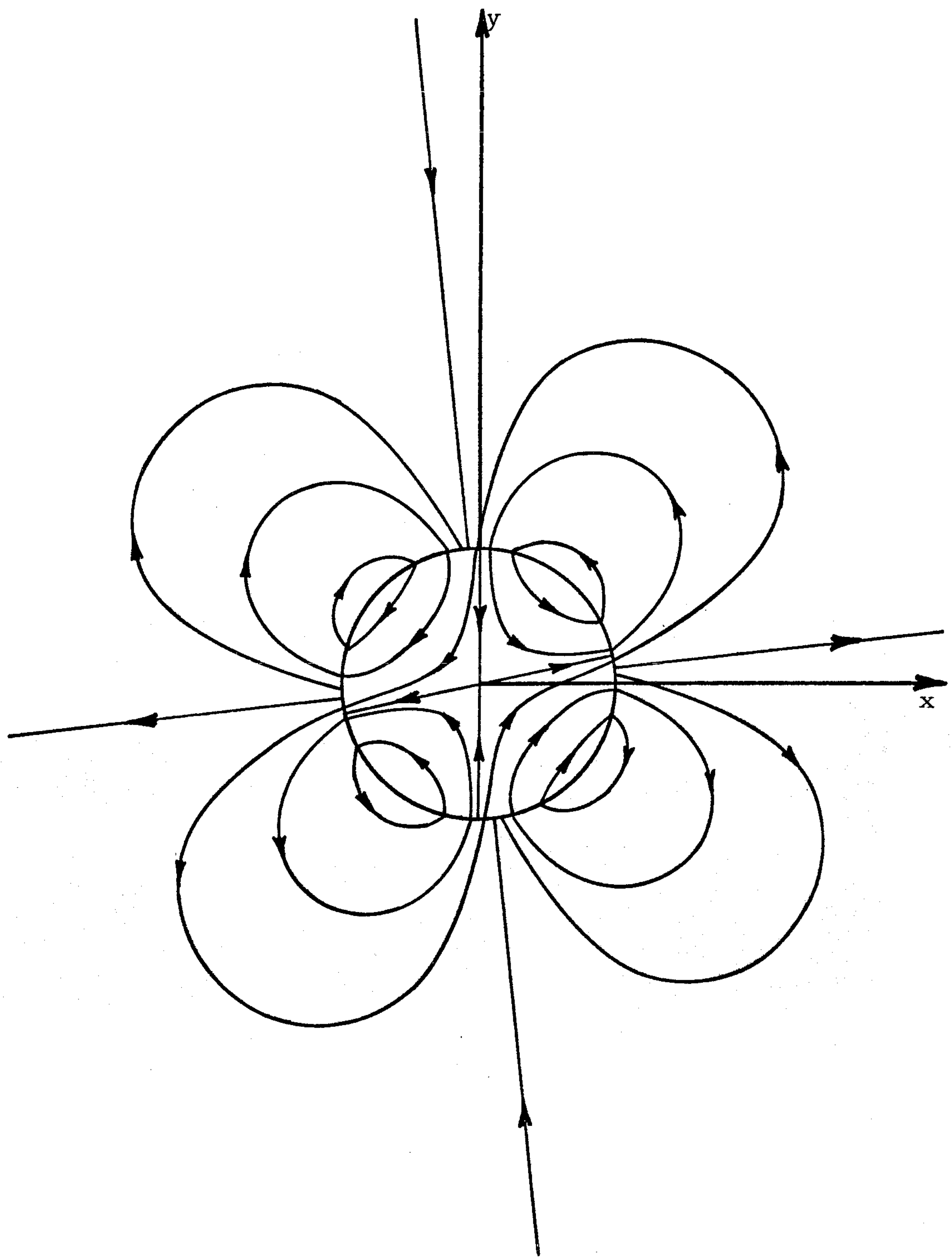

Figure 9. Geostrophic Flow for y-Rotation (Disc Moving)

$$\Delta p = \frac{8}{5}\,\rho\epsilon_y\Omega^2\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot^2\alpha\,[\,7\,x\,-y\,] \tag{302}$$

$$\Delta\underline{\tau} = 2\rho\epsilon_y\Omega^2\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha\,[\,x\underline{j} - \frac{2}{5}\,x\,\underline{i}\,] \tag{303}$$

The force and moment on the disc are therefore

$$\underline{F} = o(E^{\frac{1}{2}}\cot^2\alpha) \tag{304}$$

$$\underline{M} = \frac{1}{10}\,\pi\rho\epsilon_y\Omega^2 a^4\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot^2\alpha\,[\,4\,\underline{i} + 28\,\underline{j} + \tan\alpha\underline{k}\,] \tag{305}$$

(c) <u>z-Rotation</u>

The relevant quantities for use in (113) are:

$$U(\theta) = \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n \sin n\theta \tag{306}$$

$$V(\theta) = -\epsilon_z\Omega a - \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n \cos n\theta \tag{307}$$

$$v_g(a,\theta) = U(\theta)\cot\theta \tag{308}$$

$$\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\tan\alpha\,\frac{\Delta p_g(a,\theta)}{2\Omega} = f^T(\theta) - f^B(\theta) + 2\,\frac{dV_g(\theta)}{d\theta} - 2\epsilon_z\Omega a\sin\theta \tag{309}$$

$$V_g(\theta) = \frac{U(\theta)}{\sin\theta} \tag{310}$$

Hence, using (113):

$$2\,\frac{dU(\theta)}{d\theta} + U(\theta) - V(\theta) - 2\epsilon_z\Omega a\sin^2\theta = -\,[f^T(\theta) - f^B(\theta)]\,\sin\theta$$

and therefore,

$$-\epsilon_z\Omega a\cos 2\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} n(2n+1)B_n\cos n\theta + \frac{1}{2\Omega a}\sum_{n=1}^{\infty} nB_n\sin n\theta = -\,[\,f^T(\theta) - f^B(\theta)\,]\sin\theta \tag{311}$$

for which the solution is:

$$B_1 = 0,\ B_2 = \frac{1}{5}\epsilon_z\Omega^2 a^2,\ B_n = 0 \text{ for all } n \geq 3 \tag{312}$$

Thus,

$$f^T(x) - f^B(x) = -\frac{2}{5}\epsilon_z\Omega x \tag{313}$$

and

$$V_g(x) = \frac{2}{5}\epsilon_z\Omega x \tag{314}$$

Hence, in a frame of reference fixed on the disc, the streamlines are simply lines of constant x.

Outside the Taylor column,

$$p_G(r,\theta) = -\epsilon_z\Omega^2 r^2 + \frac{1}{5}\epsilon_z\Omega^2 a^2 (\frac{a}{r})^2 \cos 2\theta \tag{315}$$

$$u_G(r,\theta) = \frac{1}{5}\epsilon_z\Omega a (\frac{a}{r})^3 \sin 2\theta \tag{316}$$

$$v_G(r,\theta) = -\epsilon_z\Omega r - \frac{1}{5}\epsilon_z\Omega a (\frac{a}{r})^3 \cos 2\theta \tag{317}$$

To transform to a frame in which the disc rotates and the walls remain at rest define

$$\underline{\hat{u}} \equiv \underline{u} + \epsilon_z\Omega r\,\underline{e}_\theta \tag{318}$$

and

$$\hat{p} \equiv p + \epsilon_z\Omega^2 r^2 \tag{319}$$

Then,

$$\mathcal{P} = \rho p + \frac{1}{2}(1+2\epsilon_z)\rho\Omega^2 r^2$$

so that

$$\Delta p = -\frac{4}{5}\rho\epsilon_z\Omega^2\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha\left[x + 7y\right] \tag{320}$$

Also,

$$\Delta \underline{\tau} = -\frac{4}{5}\rho\epsilon_z\Omega^2\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} x\,\underline{i} \tag{321}$$

When (320) and (321) are integrated over the disc they integrate to zero. Therefore

$$\underline{F} = o(E^{\frac{1}{2}}\cot\alpha) \tag{322}$$

The moment is

$$\underline{M} = \frac{1}{5}\pi\rho\epsilon_z\Omega^2 a^4\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha\left[7\underline{i} - \underline{j} - 7\tan\alpha\underline{k}\right] \tag{323}$$

To complete the study of this motion, note that in the frame defined by (318), the geostrophic flow is:

$$\hat{u}_g = -\epsilon_z\Omega y \tag{324}$$

$$\hat{v}_g = \frac{7}{5}\epsilon_z\Omega x \tag{325}$$

$$\hat{p}_g = \frac{1}{5}\epsilon_z\Omega^2 a^2\left[7\left(\frac{x}{a}\right)^2 + 5\left(\frac{y}{a}\right)^2 - 6\right] \tag{326}$$

inside the Taylor column, and

$$\hat{u}_G = \frac{1}{5}\epsilon_z\Omega a\left(\frac{a}{r}\right)^3\sin 2\theta \tag{327}$$

$$\hat{v}_G = -\frac{1}{5}\epsilon_z\Omega a\left(\frac{a}{r}\right)^3\cos 2\theta \tag{328}$$

$$\hat{p}_G = \frac{1}{5}\epsilon_z\Omega^2 a^2\left(\frac{a}{r}\right)^2\cos 2\theta \tag{329}$$

outside the Taylor column.

The streamlines are shown in Figure 10.

4.4 Summary

For x and y-rotation, the geostrophic flow is of order $\cot\alpha$ while it is order one for z-rotation. In all three cases, the force on the disc has been shown to vanish to leading order. The moment about the vertical axis is much smaller than the moments about the horizontal axes, the ratio being $O(\tan\alpha)$. Of course, the moments are $O(E^{\frac{1}{2}}\cot^2\alpha)$ for x and y-rotation and $O(E^{\frac{1}{2}}\cot\alpha)$ for z-rotation. Finally, (288), (305) and (323) show that the moment is not aligned (even to leading order) with the direction of rotation of the disc.

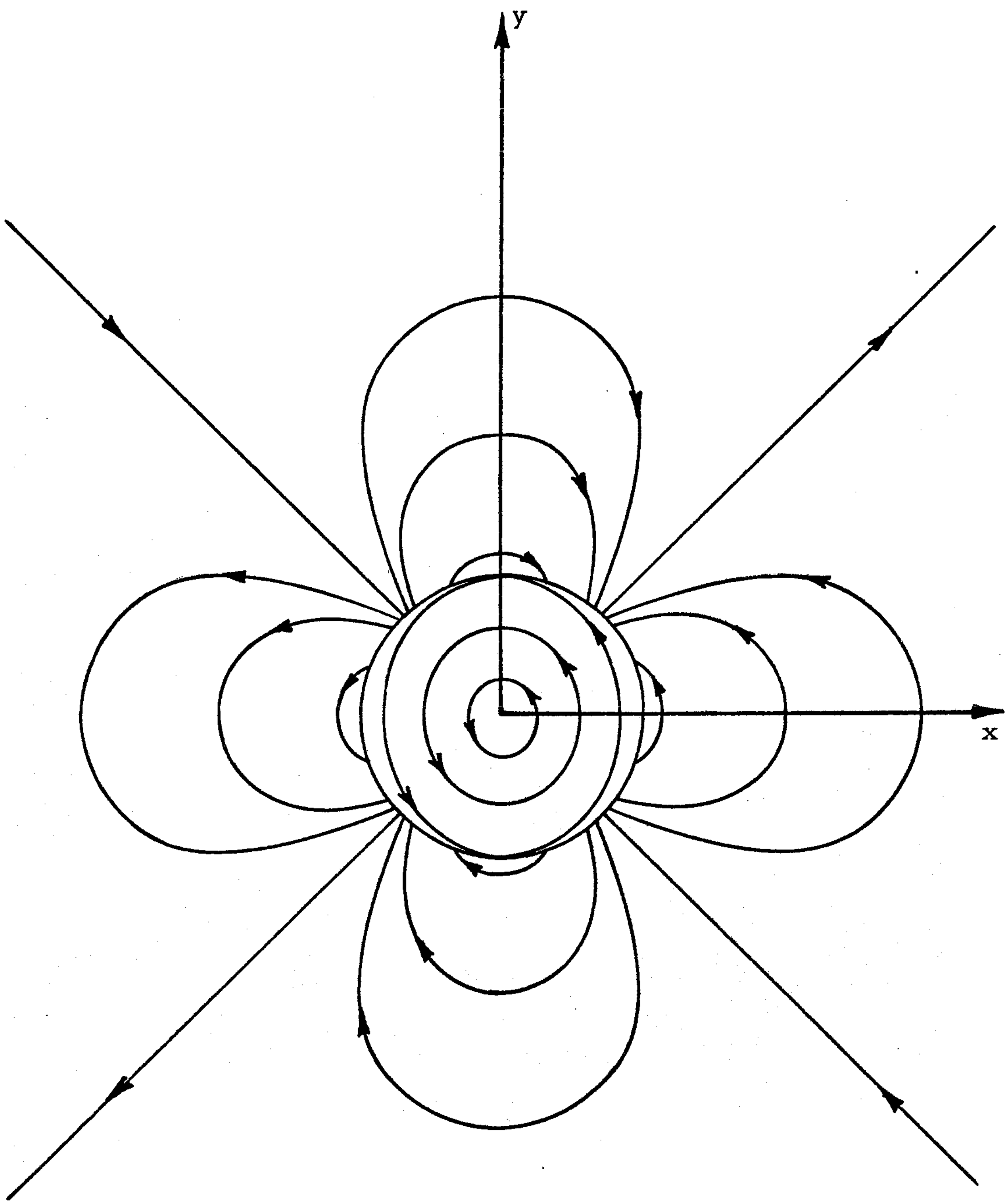

Figure 10. Geostrophic Flow for z-Rotation (Disc Moving).

5. STATIC STABILITY AND FREE MOTION OF THE DISC

5.1 The Stability Criterion

For all of the motions studied in Chapters 3 and 4, not just the nonlinear inertia term has been dropped. The unsteady term $\partial \underline{u}/\partial t$ has also been neglected so that the solutions are not truly steady state motions. Hence, it is not possible to perform the usual type of stability analysis by introducing $\partial \underline{u}/\partial t$ into the equations and checking to see how an initially small disturbance behaves as $t \rightarrow \infty$. Rather than this approach, something different must be done to see what happens to such a disturbance. This is the origin of the name "static stability."

One question which arises is that of what happens to the angle of attack of the disc due to a small disturbance. The answer is seen immediately from equation (305). The torque about the y-axis is in the same direction as the rotation rather than opposing the motion. Consequently, if a small disturbance is introduced which causes the disc to rotate about the y-axis, it will generate a torque given by (305). This torque will accelerate the change in angle of attack. Hence, the motion will be termed unstable to small disturbances.

5.2 Free Motion of the Disc

If the disc is free to translate but constrained so that it will not rotate about its center of gravity, it is acted on by three external forces. One force is the buoyancy force given by

$$\underline{F}_{buoy} = (\frac{\rho}{\rho_d} - 1)mg\underline{k}$$

where ρ_d and m are the density and mass of the disc respectively.

If the disc is very much less dense than the fluid,

$$\underline{F}_{buoy} \doteq \frac{\rho}{\rho_d} mg\underline{k} \tag{330}$$

The other two forces are the centrifugal and Coriolis forces. These forces arise by noting that for x-translation and y-translation the physical pressure is given by (132) and (150) respectively. In calculating $\Delta \mathcal{P}$, it was assumed that the disc had zero thickness. As a result of this assumption, it follows that

$$\Delta \mathcal{P} = \rho \Delta p$$

and the centrifugal and Coriolis terms cancel identically. For a disc with nonzero thickness the forces are easily shown to be

$$F_{cor} \sim \frac{\rho}{\rho_d} m\Omega U \tag{331}$$

$$F_{cen} \sim \frac{\rho}{\rho_d} m\Omega^2 R \tag{332}$$

if the disc is located at a radial distance R from the center of the container. Both of these forces will lie in the horizontal plane. The ratio of (331) to (332) is

$$\frac{F_{cor}}{F_{cen}} \sim Ro << 1$$

so that the Coriolis force need not be considered.

For the three translations, the ratio of the horizontal force component on the disc to the vertical force component is of order $\tan\alpha$. Hence, F_{cen} must be compared to $|\underline{F}_{buoy}|\tan\alpha$. The ratio is

$$\frac{F_{cen}}{|\underline{F}_{buoy}|\tan\alpha} \sim \frac{\Omega^2 R}{g} \cot\alpha \tag{333}$$

If it is true that

$$\frac{\Omega^2 R}{g} << \tan\alpha \tag{334}$$

the centrifugal force can also be neglected.

Assuming the disc moves with velocity

$$\underline{u}_p = U\,\underline{i} + V\,\underline{j} + W\tan\alpha\,\underline{k}$$

where U, V and W are of order one, (135), (153), (240) and (330) show that the force on the disc is

$$\underline{F} = F_x\,\underline{i} + F_y\,\underline{j} + F_z\,\underline{k}$$

where

$$F_x = -2\pi\rho\Omega a^2 \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} [2U + 3W]$$

$$F_y = 2\pi\rho\Omega a^2 \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} W$$

$$F_z = -4\pi\rho\Omega a^2 \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \cot\alpha \left[U + \tfrac{1}{3}V + \tfrac{5}{3}W\right] + \frac{\rho}{\rho_d} mg$$

with the angle of attack positive. Then, taking

$$\underline{F} = \underline{0}$$

there follows immediately:

$$U = W = 0 \tag{335}$$

$$V = \frac{3}{4}\frac{\rho}{\rho_d}\frac{mg}{\pi\rho\Omega a^2(\frac{\nu}{\Omega})^{\frac{1}{2}}\cot\alpha} \tag{336}$$

(335) and (336) show that under the action of the buoyancy force, the disc will slip to the side rather than rise! There is a restriction on the thickness, δ, of the disc implied by (336). Since

$$m \sim \pi\rho_d a^2\delta$$

(336) shows that

$$Ro = \frac{V}{\Omega a} \sim \frac{\delta}{(\frac{\nu}{\Omega})^{\frac{1}{2}}}\frac{g\tan\alpha}{\Omega^2 a}$$

and since

$$Ro << E^{\frac{1}{3}}(\frac{h}{a})^{\frac{1}{3}}$$

necessarily

$$\frac{\delta}{a} << E^{5/6}\cot\alpha\frac{\Omega^2 a}{g}\ (\frac{h}{a})^{\frac{1}{3}} \tag{337}$$

This result is not very interesting physically as it would be virtually impossible to satisfy (334) and (337).

If the inequality in (334) is reversed so that only the centrifugal force is relevant, the force in the y-direction becomes

$$F_y = 2\pi\rho\Omega a^2(\frac{\nu}{\Omega})^{\frac{1}{2}}\ W - \frac{\rho}{\rho_d}m\Omega^2 R\sin\theta \tag{338}$$

where θ is the angle shown in Figure 11. The center of the container is

$$X = Y = 0$$

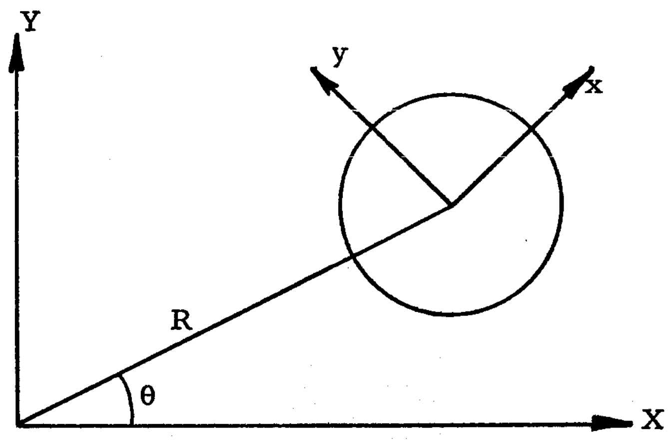

Figure 11. Reference Angle for Free Motion of the Disc

Requiring F_y to vanish shows that

$$W = \frac{\frac{\rho}{\rho_d} m\Omega^2 R\sin\theta}{2\pi\rho\Omega a^2 (\frac{\nu}{\Omega})^{\frac{1}{2}}} \tag{339}$$

Therefore, under the action of the centrifugal force with buoyancy negligible, the disc will rise! The restriction on δ implied by (339) is

$$\frac{\delta}{a} << \frac{a}{R} E^{5/6} (\frac{h}{a})^{\frac{1}{3}}$$

which is just as restrictive as (337). In any physical situation both forces will have to be accounted for. These limiting cases do, however, reveal the gyroscopic nature of motion in a rapidly rotating fluid.

6. MOTION OF THE DISC FOR FINITE ANGLE OF ATTACK

6.1 Difficulties Associated with a Formal Solution

The fundamental difficulty associated with solving for the motion of the fluid when α is of order one appears in the structure of the shear column. The α-layer merges with the $\frac{1}{3}$-layer whereas the $\frac{1}{4}$-layer on the outer edge of the shear column remains. Equations (43)-(47) become, on the $\frac{1}{3}$-layer scaling:

$$\frac{\partial w_{\frac{1}{3}}}{\partial z} = -\frac{\nu}{2\Omega}\frac{\partial^3 v_{\frac{1}{3}}}{\partial \bar{x}^3} \tag{340}$$

$$\frac{\partial v_{\frac{1}{3}}}{\partial z} = \frac{\nu}{2\Omega}\frac{\partial^3 w_{\frac{1}{3}}}{\partial \bar{x}^3} \tag{341}$$

subject to:

$$w_{\frac{1}{3}} = 0 \qquad \text{on } z = -h_B,\ h_T \tag{342}$$

and

$$w_{\frac{1}{3}} = (v_{\frac{1}{3}}\sin\theta - U(\theta)\cos\theta)\tan\alpha \qquad \text{on } z = -x\tan\alpha \tag{343}$$

where it has been noted that u is still continuous across the shear column. Matching now requires that

$$\lim_{\eta\to+\infty} v_{\frac{1}{3}}(\eta,\theta,z) = \lim_{\xi\to 0^+} v_{\frac{1}{4}}(\xi,\theta) \tag{344}$$

$$\lim_{\eta\to-\infty} v_{\frac{1}{3}}(\eta,\theta,z) = v_g(a,\theta) \tag{345}$$

This problem requires the solution of a Weiner-Hopf problem which is much more difficult than the ones considered by Moore and Saffman.[2] The complication results from the nonhomogeneous boundary condition (343). A solution will not be attempted here.

However, one solution can be picked out by inspection, namely

$$v_{\frac{1}{3}}(\eta, \theta, z) = U(\theta)\cot\theta \tag{346}$$

$$w_{\frac{1}{3}}(\eta, \theta, z) = 0 \tag{347}$$

This solution corresponds to the n = 0 problem considered in the infinitesimal α analysis. The physical meaning is that v is continuous across the $\frac{1}{3}$-layer. (345) and (346) are consistent since the problem posed here is for horizontal translation.

Another complication which arises is that for a disc, the Taylor column will be elliptical rather than circular. The way in which this affects the analysis will become clear in the following sections.

6.2 The Geostrophic Flow for x-Translation

Rather than study the motion of a disc, an elliptical plate will be considered. The equation of the plate is

$$\zeta = 0 \tag{348}$$

with

$$\xi^2 + \left(\frac{y}{\cos\alpha}\right)^2 \leq a^2/\cos^2\alpha \tag{349}$$

where

$$\xi = x\cos\alpha - z\sin\alpha \tag{350}$$

$$\zeta = x\sin\alpha + z\cos\alpha \tag{351}$$

That is, (ξ, y, ζ) are coordinates in a rectangular cartesian coordinate system fixed on the plate. The projection of (349) into the xy-plane is a circle of radius a. This, of course, means the Taylor column is a circular cylinder.

The walls and fluid at infinity will again be assumed to move with the velocity given by (7).

Boundary conditions for the geostrophic flow inside the Taylor column come from (13), (14) and (15) which can be written here as

$$w_g = -\tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right] \quad \text{on } z = h_T \tag{352}$$

$$w_g + u_g \tan\alpha = \pm\tfrac{1}{2}\left(\frac{\nu}{\Omega\cos\alpha}\right)^{\frac{1}{2}}\left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right] \quad \text{on } \zeta = 0\pm \tag{353}$$

$$w_g = \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[\frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right] \quad \text{on } z = -h_B \tag{354}$$

Calculation of the geostrophic flow goes through just as for infinitesimal α and there follows:

$$u_g = \pm\tfrac{1}{2}(1+\beta)\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha\,\frac{dV_g(x)}{dx} \tag{355}$$

$$v_g = V_g(x) + \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha\left[\frac{df^{T}_{B}(x)}{dx} \mp \frac{1+\beta}{2}\, y\,\frac{d^2V_g(x)}{dx^2}\right] \tag{356}$$

$$w_g = \mp\tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\frac{dV_g(x)}{dx} \tag{357}$$

$$p_g = p_g(x) + 2\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha\left[f^{T}_{B}(x) \mp \frac{1+\beta}{2}\, y\,\frac{dV_g(x)}{dx}\right] \tag{358}$$

where

$$\beta \equiv (\cos\alpha)^{-\frac{1}{2}} \tag{359}$$

while $p_g(x)$ and $V_g(x)$ are related by (27). Since the Taylor column is circular, the geostrophic flow outside the column is still given by (28)-(30).

6. 3 Equation Relating the Geostrophic Regions

It is possible to generate an equation relating the geostrophic regions by proceeding as in the previous sections. Certainly mass must be conserved in the $\frac{1}{3}$-layer so that (89) must still hold. It must be altered slightly to account for the inclination of the plate. That is, the integration must be performed in a direction parallel to the plate. Henceforth, (89) is replaced by

$$Q_e(\theta) = \int_{-\infty}^{\infty} \frac{w_{\frac{1}{3}}^{(n)}(0^+) - w_{\frac{1}{3}}^{(n)}(0^-)}{\cos\alpha} \, d\bar{x} \tag{360}$$

with the understanding that

$$w_{\frac{1}{3}}^{(n)}(0^{\pm}) = w_{\frac{1}{3}}^{(n)}(x, y, -x\tan\alpha + 0^{\pm}) \tag{361}$$

Now, $w_{\frac{1}{3}}^{(n)}$ is the velocity component normal to the plate and is given by

$$w_{\frac{1}{3}}^{(n)}/\cos\alpha = w_{\frac{1}{3}} - (v_{\frac{1}{3}}\sin\theta - U(\theta)\cos\theta)\tan\alpha$$

so that

$$\frac{\Delta w_{\frac{1}{3}}^{(n)}}{\cos\alpha} = \Delta w_{\frac{1}{3}} - \Delta v_{\frac{1}{3}}\sin\theta\tan\alpha$$

where $\Delta(\ )$ means the difference between ( ) evaluated at $\zeta = 0^+$ and $\zeta = 0^-$. Therefore,

$$\frac{\Delta w_{\frac{1}{3}}^{(n)}}{\cos\alpha} = \Delta w_{\frac{1}{3}} - \frac{\partial}{\partial\bar{x}}\left(\frac{\Delta p_{\frac{1}{3}}}{2\Omega}\right)\sin\theta\tan\alpha \tag{362}$$

Insertion of (362) into (360) leaves

$$Q_e(\theta) = \int_{-\infty}^{\infty} \Delta w_{\frac{1}{3}} d\bar{x} - \left[\Delta p_{\frac{1}{3}}(\infty,\theta) - \Delta p_{\frac{1}{3}}(-\infty,\theta)\right] \sin\theta \tan\alpha \qquad (363)$$

Integrating (340) and using the full boundary conditions (45) and (47) there follows:

$$\Delta w_{\frac{1}{3}} = \frac{\nu}{2\Omega} \int_{-h_B}^{h_T} \frac{\partial^3 v_{\frac{1}{3}}}{\partial \bar{x}^3} dz - \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{\partial}{\partial \bar{x}} \left[v_{\frac{1}{3}}(-h_B) + v_{\frac{1}{3}}(h_T)\right]$$

which means

$$\int_{-\infty}^{\infty} \Delta w_{\frac{1}{3}} d\bar{x} = \frac{\nu}{2\Omega} \int_{-h_B}^{h_T} \frac{\partial^2 v_{\frac{1}{3}}}{\partial \bar{x}^2} \Bigg|_{-\infty}^{+\infty} dz - \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \left[v_{\frac{1}{3}}(-h_B) + v_{\frac{1}{3}}(h_T)\right] \Bigg|_{-\infty}^{+\infty} \qquad (364)$$

The implication of (344), (345) and (346) is that the last term in (364) vanishes. Furthermore,

$$\lim_{\bar{x} \to -\infty} \frac{\partial^2 v_{\frac{1}{3}}}{\partial \bar{x}^2} = 0$$

in light of (345), i. e., the $\frac{1}{3}$-layer must match directly to the geostrophic flow inside the Taylor column. (364) thus reduces to

$$\int_{-\infty}^{\infty} \Delta w_{\frac{1}{3}} d\bar{x} = \frac{\nu}{2\Omega} \int_{-h_B}^{h_T} \frac{\partial^2 v_{\frac{1}{4}}}{\partial \bar{x}^2} (0,\theta) dz$$

Integrating across the $\frac{1}{4}$-layer yields (98) so that

$$\int_{-\infty}^{\infty} \Delta w_{\frac{1}{3}} d\bar{x} = \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \left[v_g(a,\theta) - V(\theta))\right] \qquad (365)$$

where it has been noted that here

$$v_{\frac{1}{4}}(0,\theta) = v_g(a,\theta)$$

as a consequence of (346). Finally, matching shows that

$$\Delta p_{\frac{1}{3}}(\infty, \theta) = \Delta p_{\frac{1}{4}}(0, \theta) = 0 \tag{366}$$

while

$$\Delta p_{\frac{1}{3}}(-\infty, \theta) = \Delta p_g(a, \theta) \tag{367}$$

Using (365)-(367) in (363) yields

$$Q_e(\theta) = (\frac{\nu}{\Omega})^{\frac{1}{2}} \left[v_g(a, \theta) - V(\theta)\right] + \tan\alpha \sin\theta \frac{\Delta p_g(a, \theta)}{2\Omega} \tag{368}$$

All that remains to be done is to evaluate $Q_e(\theta)$. It is at this point that geometrical difficulties become relevant. (202) yields the excess flux of fluid in the Ekman layers on the plate. $Q_e(\theta)$ is the component of $\underline{Q}^I$ entering the $\frac{1}{3}$-layer. Hence,

$$Q_e(\theta) = \underline{Q}^T \cdot \underline{n}_e + \underline{Q}^B \cdot \underline{n}_e \tag{369}$$

where $\underline{n}_e$ is a unit vector normal to the perimeter of the plate. (Equations (202), (368) and (369) are actually valid for any geometry provided v is interpreted as the velocity component parallel to the edge of the Taylor column.) For the geometry at hand,

$$\underline{n}_e = \frac{\xi \underline{t} + \frac{y}{\cos^2 \alpha} \underline{j}}{\sqrt{\xi^2 + y^2/\cos^4 \alpha}}$$

where $\underline{t}$ is a unit vector parallel to the plate given by

$$\underline{t} = \underline{i} \cos\alpha - \underline{k} \sin\alpha$$

Hence,

$$\underline{n}_e = \frac{x\cos\alpha\underline{t}+y\underline{j}}{\sqrt{x^2\cos^2\alpha+y^2}} \tag{370}$$

Note finally that $\underline{Q}$ is evaluated by using the geostrophic velocity inside the Taylor column.

6.4 x-Translation Solution

To leading order, (355)-(357) show that the flow inside the Taylor column is

$$\underline{u}_g = V_g(x)\,\underline{j}$$

wherefore

$$\underline{Q}^I = -\tfrac{1}{2}\beta(\tfrac{\nu}{\Omega})^{\frac{1}{2}}\left[\,V_g(\theta)\underline{t} + V_g(\theta)\underline{k}\,\right] \tag{371}$$

Also, on the edge of the Taylor column,

$$\underline{n}_e = \frac{\cos\theta\cos\alpha\underline{t}+\sin\theta\,\underline{j}}{\sqrt{1-\sin^2\alpha\sin^2\theta}} \tag{372}$$

Equations (369), (371) and (372) combine to yield:

$$Q_e(\theta) = -\beta(\tfrac{\nu}{\Omega})^{\frac{1}{2}}V_g(\theta)\,\frac{\sin\theta+\cos\theta\cos\alpha}{\sqrt{1-\sin^2\alpha\sin^2\theta}}$$

But, since

$$U(\theta) = V_g(\theta)\sin\theta \tag{373}$$

$$v_g(a,\theta) = V_g(\theta)\cos\theta \tag{374}$$

further manipulation leads to

$$Q_e(\theta) = -\beta\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{U(\theta)+v_g(a,\theta)\cos\alpha}{\sqrt{1-\sin^2\alpha\sin^2\theta}} \tag{375}$$

From (358) it is easily shown that

$$\Delta p_g(a,\theta) = 2\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\cot\alpha\left[f^T(\theta)-f^B(\theta)+(1+\beta)\frac{dV_g(\theta)}{d\theta}\right] \tag{376}$$

Inserting (373)-(376) into (368) and grouping like terms

$$(1+\beta)\frac{dU(\theta)}{d\theta} + \frac{\beta}{\sqrt{1-\sin^2\alpha\sin^2\theta}}U(\theta) - V(\theta)$$

$$-\beta U(\theta)\cot\theta\left\{1-\frac{\cos\alpha}{\sqrt{1-\sin^2\alpha\sin^2\theta}}\right\} = -\left[f^T(\theta)-f^B(\theta)\right]\sin\theta \tag{377}$$

For $\alpha \to 0$, (377) reduces to (117) so that at least consistency with a limiting case is confirmed.

To solve (377), note that $U(\theta)$ and $V(\theta)$ are given by (124) and (125). Since the right hand side of (377) is an odd function of θ, a solution exists only if

$$(1+\beta)\frac{dU(\theta)}{d\theta} - \left[V(\theta) + 2U\sin\theta\right]$$

$$-\beta U(\theta)\cot\theta\left\{1-\frac{\cos\alpha}{\sqrt{1-\sin^2\alpha\sin^2\theta}}\right\} = 0 \tag{378}$$

Substitution of (124) and (125) into (378) yields finally

$$\sum_{n=1}^{\infty} n\left[(1+\beta)n+1\right]B_n\cos n\theta = \beta\left\{1-\frac{\cos\alpha}{\sqrt{1-\sin^2\alpha\sin^2\theta}}\right\}\cot\theta\sum_{n=1}^{\infty} nB_n\sin n\theta \tag{379}$$

This equation is not nearly as pleasant as its counterpart for infinitesimal α. When $\alpha \to 0$, the right hand side vanishes so that all the B_n must vanish. A little more progress can be made by expanding

$$\left[1 - \frac{\cos\alpha}{\sqrt{1-\sin^2\alpha\sin^2\theta}}\right]\cot\theta = \sum_{n=1}^{\infty} A_n \sin n\theta \tag{380}$$

The A_n are quite difficult to obtain and involve complete elliptic integrals. No recurrence relation is obvious and the details of their calculation will not be given here.

Multiplication of the two Fourier series on the right hand side of (379) shows that

$$\sum_{n=1}^{\infty} n\left[(1+\beta)n+1\right] B_n \cos n\theta$$

$$= \tfrac{1}{2}\beta \sum_{n=1}^{\infty}\sum_{k=1}^{n} kB_k A_{n+1-k}\left[\cos(2k-n-1)\theta - \cos(n+1)\theta\right] \tag{381}$$

Multiplying (381) by cos $m\theta$ for $m = 0, 1, 2, \ldots\ldots$, integrating over θ from 0 to 2π and doing a large amount of algebra (which won't be reproduced here) the following set of equations emerges:

$$0 = \sum_{n=1}^{\infty} A_n X_n \tag{382}$$

$$(2+\beta)X_1 = \tfrac{1}{2}\beta\sum_{k=1}^{\infty}\left[A_{k+1}X_k + A_k X_{k+1}\right] \tag{383}$$

$$\left[(1+\beta)n+1\right] X_n = \tfrac{1}{2}\beta\sum_{k=1}^{\infty}\left[A_{k+n}X_k + A_k X_{k+n}\right] - \tfrac{1}{2}\beta\sum_{k=1}^{n-1} A_{n-k}X_k \quad \text{for } n \geqslant 2 \tag{384}$$

where

$$X_k \equiv kB_k \tag{385}$$

(382)-(384) constitute an infinite number of equations for the infinite number of unknown quantities, X_k. One solution to this system is

$$X_n = 0 \text{ for all } n \geqslant 1$$

and therefore

$$B_n = 0 \text{ for all } n \geqslant 1 \tag{386}$$

This means that

$$V_g(x) = 0$$

so that there is no flow to order one inside the Taylor column.

6.5 Discussion

The analysis for the motion of a thin plate when α is finite is far from complete. For motions other than x-translation it is no longer possible to pick out a solution to the infinite set of equations corresponding to (382)-(384). However, the most important part of this analysis is that a jump condition relating the geostrophic regions has been generated. This indicates that a solution for motion at finite angle of attack is in principle possible without having to analyze the detailed structure of the shear column.

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APPENDIX

It has been shown that the shear column structure is such that the circumferential velocity is not analytic near $\theta = 0$ and $\theta = \pi$. For the range

$$E^{\frac{1}{4}} \ll \alpha \ll 1$$

the jump in the α-layer swirl velocity is of order

$$\Delta v_\alpha \sim \begin{cases} E^{1/12}\alpha^{-1/3}\theta^{2/3} & \text{for x-translation} \\ E^{1/12}\alpha^{-1/3}\theta^{-1/3} & \text{for y-translation} \end{cases}$$

as $\theta \to 0$ and similarly for $\theta \to \pi$. v_α is continuous across $\theta = 0$ for x-translation. However, v_α is singular for y-translation so that some explanation is required.

The principle of minimum singularity demands that v_α not exceed order one. Now,

$$E^{1/12}\alpha^{-1/3}\theta^{-1/3} \sim 1$$

when

$$\theta \sim E^{1/4}\alpha^{-1} \tag{A1}$$

Hence, the α-layer solution must see $\theta = 0$ as given by (A1). It is noteworthy that when (A1) is valid,

$$\delta_\alpha \sim E^{1/4}$$

so that the α-layer has the same order thickness as a $\frac{1}{4}$-layer.

All of this suggests yet another layer of dimensions

$$\frac{r-a}{a} \sim E^{\frac{1}{4}}$$

$$\theta \sim E^{\frac{1}{4}} \cot\alpha$$

This is shown in Figure 12.

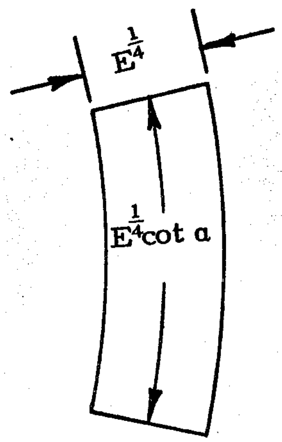

Figure 12. Dimensions of the θ-layer.

This layer will be referred to as the θ-layer. The relevant equations will be:

$$\frac{\partial w_\theta}{\partial z} = \frac{\nu}{2\Omega} \frac{\partial^3 v_\theta}{\partial \bar{x}^3} \quad \text{(A2)}$$

$$\frac{\partial v_\theta}{\partial z} = 0 \quad \text{(A3)}$$

Also, continuity shows that

$$u_\theta = U(\theta) + o(1)$$

as

$$\frac{\partial}{\partial \bar{x}} \sim E^{-\frac{1}{4}}$$

whilst

$$\frac{1}{a}\frac{\partial}{\partial\theta} \sim E^{-\frac{1}{4}}\tan\alpha << \frac{\partial}{\partial\bar{x}}$$

For y-translation,

$$U(\theta) = \tfrac{2}{3}V\sin\theta$$

However, it is clear that

$$U(\theta) \sim E^{\frac{1}{4}}\cot\alpha$$

so that

$$w_\theta \sim E^{\frac{1}{4}}$$

and the boundary conditions on w_θ become

$$w_\theta = -\tfrac{1}{2}\left(\frac{\nu}{2\Omega}\right)^{\frac{1}{2}} \frac{\partial v_\theta}{\partial\bar{x}} \qquad \text{on } z = h_T \tag{A4}$$

$$w_\theta = (v_\theta\sin\theta - u_\theta\cos\theta)\tan\alpha \pm \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{\partial v_\theta}{\partial\bar{x}} \qquad \text{on } z = 0^{\pm} \tag{A5}$$

$$w_\theta = \tfrac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{\partial v_\theta}{\partial\bar{x}} \qquad \text{on } z = -h_B \tag{A6}$$

The equations for v_θ hence become:

$$\frac{\partial^3 v_\theta}{\partial\bar{x}^3} - \left(\frac{4\Omega}{\nu h_T^2}\right)^{\frac{1}{2}} \frac{\partial v_\theta}{\partial\bar{x}} - \frac{2\Omega\tan\alpha}{\nu h_T}(v_\theta\sin\theta - u_\theta\cos\theta) = 0 \quad \text{for } z > 0 \tag{A7}$$

$$\frac{\partial^3 v_\theta}{\partial\bar{x}^3} - \left(\frac{4\Omega}{\nu h_B^2}\right)^{\frac{1}{2}} \frac{\partial v_\theta}{\partial\bar{x}} + \frac{2\Omega\tan\alpha}{\nu h_B}(v_\theta\sin\theta - u_\theta\cos\theta) = 0 \quad \text{for } z < 0 \tag{A8}$$

These are precisely the equations that arise when

$$a \sim E^{\frac{1}{4}}$$

in place of either the $\frac{1}{4}$-layer equations or the a-layer equations.

The jump in v_θ across the θ-layer can be determined as follows. From continuity,

$$\frac{1}{a}\frac{\partial v_\theta}{\partial \theta} = -\frac{\partial u_\theta}{\partial \bar{x}} - \frac{\partial w_\theta}{\partial z} \sim -\frac{\partial u_\theta}{\partial \bar{x}}$$

So,

$$\Delta v_\theta = \Delta v_a \sim -a\frac{\partial}{\partial \bar{x}}\int_{-\infty}^{\infty} u_\theta d\theta$$

or,

$$\Delta v_a \sim -\frac{a}{2\Omega}\frac{\partial}{\partial \bar{x}}\int_{-\infty}^{\infty}\left[-\frac{1}{a}\frac{\partial p_\theta}{\partial \theta} + \nu\frac{\partial^2 v_\theta}{\partial \bar{x}^2}\right] d\theta$$

$$\sim \frac{1}{2\Omega}\frac{\partial}{\partial \bar{x}}\Delta p_\theta - \frac{\nu a}{2\Omega}\int_{-\infty}^{\infty}\frac{\partial^3 v_\theta}{\partial \bar{x}^3} d\theta$$

and $\Delta p_\theta = \Delta p_\theta(\theta)$ to order one. Therefore,

$$\Delta v_a \sim -\frac{\nu a}{2\Omega}\int_{-\infty}^{\infty}\frac{\partial^3 v_\theta}{\partial \bar{x}^3} d\theta$$

For $v_\theta = O(1)$ inside the θ-layer, this means that

$$\Delta v_a \sim E\frac{1}{E^{3/4}} E^{\frac{1}{4}} \cot a = E^{\frac{1}{2}} \cot a$$

Hence, this would predict no jump in v_a to order one which is not consistent with what is known. This apparent contradiction is resolved by inserting a thinner layer of dimensions

$$E^{\frac{1}{4}} \times E^{\frac{1}{4}}$$

across which a jump in v_θ and hence v_a is accomplished. (See Figure 13.) This layer will be referred to as the $\theta_{\frac{1}{4}}$-layer.

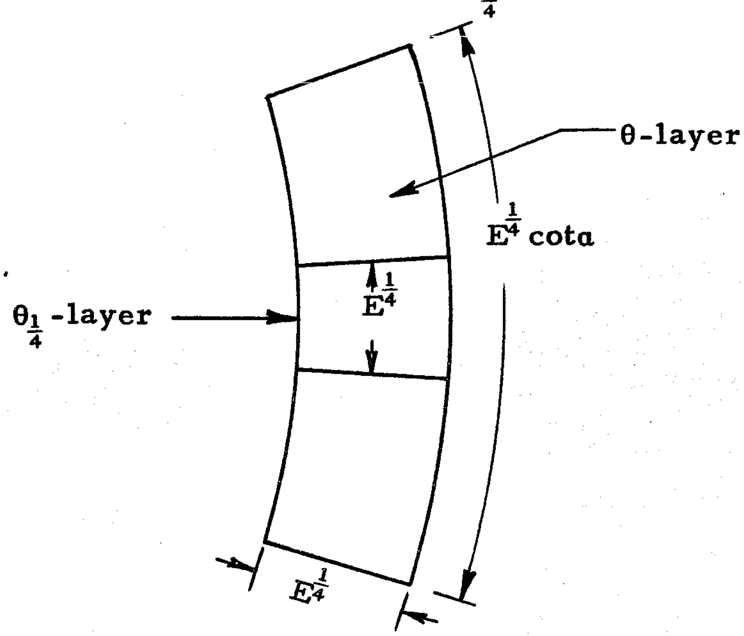

Figure 13. Sandwich Structure for θ and $\theta_{\frac{1}{4}}$-Layers

The relevant equations and boundary conditions become:

$$\frac{\partial w_{\theta_{\frac{1}{4}}}}{\partial z} = -\frac{\nu}{2\Omega}\left[\frac{\partial^3 v_{\theta_{\frac{1}{4}}}}{\partial \bar{x}^3} + \frac{1}{a^2}\frac{\partial^3 v_{\theta_{\frac{1}{4}}}}{\partial \theta^2 \partial \bar{x}}\right] \tag{A9}$$

$$\frac{\partial v_{\theta_{\frac{1}{4}}}}{\partial z} = 0 \tag{A10}$$

and,

$$w_{\theta_{\frac{1}{4}}} = -\frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[\frac{\partial v_{\theta_{\frac{1}{4}}}}{\partial \bar{x}} - \frac{1}{a}\frac{\partial u_{\theta_{\frac{1}{4}}}}{\partial \theta}\right] , \quad z = h_T \tag{A11}$$

$$w_{\theta_{\frac{1}{4}}} = \pm\frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[\frac{\partial v_{\theta_{\frac{1}{4}}}}{\partial \bar{x}} - \frac{1}{a}\frac{\partial u_{\theta_{\frac{1}{4}}}}{\partial \theta}\right] , \quad z = 0^{\pm} \tag{A12}$$

$$w_{\theta_{\frac{1}{4}}} = \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\left[\frac{\partial v_{\theta_{\frac{1}{4}}}}{\partial \bar{x}} - \frac{1}{a}\frac{\partial u_{\theta_{\frac{1}{4}}}}{\partial \theta}\right] , \quad z = -h_B \tag{A13}$$

Combining (A9)-(A13) there follows

$$\frac{\partial^3 v_{\theta_{\frac{1}{4}}}}{\partial \bar{x}^3} + \frac{1}{a^2}\frac{\partial^3 v_{\theta_{\frac{1}{4}}}}{\partial \theta^2 \partial \bar{x}} = \left(\frac{4\Omega}{\nu h_I^2}\right)^{\frac{1}{2}} \left[\frac{\partial v_{\theta_{\frac{1}{4}}}}{\partial \bar{x}} - \frac{1}{a}\frac{\partial u_{\theta_{\frac{1}{4}}}}{\partial \theta}\right]$$

However, to leading order the continuity equation is

$$\frac{\partial u_{\theta_{\frac{1}{4}}}}{\partial \bar{x}} + \frac{1}{a}\frac{\partial v_{\theta_{\frac{1}{4}}}}{\partial \theta} = 0$$

so that finally,

$$\frac{\partial^2}{\partial \bar{x}^2}\left[\frac{\partial^2 v_{\theta_{\frac{1}{4}}}}{\partial \bar{x}^2} + \frac{1}{a^2}\frac{\partial^2 v_{\theta_{\frac{1}{4}}}}{\partial \theta^2}\right] - \left(\frac{4\Omega}{\nu h_I^2}\right)^{\frac{1}{2}}\left[\frac{\partial^2 v_{\theta_{\frac{1}{4}}}}{\partial \bar{x}^2} + \frac{1}{a^2}\frac{\partial^2 v_{\theta_{\frac{1}{4}}}}{\partial \theta^2}\right] = 0 \tag{A14}$$

Defining the two-dimensional Laplacian as

$$\nabla^2 \equiv \frac{\partial^2}{\partial \bar{x}^2} + \frac{1}{a^2}\frac{\partial^2}{\partial \theta^2}$$

(A14) can be rewritten as

$$\nabla^2\left[\frac{\partial^2 v_{\theta_{\frac{1}{4}}}}{\partial \bar{x}^2} - \left(\frac{4\Omega}{\nu h_I^2}\right)^{\frac{1}{2}} v_{\theta_{\frac{1}{4}}}\right] = 0 \tag{A15}$$

The fact that this equation is of fourth order rather than third order as the α-layer, $\frac{1}{4}$-layer and θ-layer equations are means that further boundary conditions must be specified. In particular, all of the jump conditions across the $\frac{1}{3}$-layer and matching to the geostrophic interior are not sufficient to solve for $v_{\theta_{\frac{1}{4}}}$ uniquely. Certainly the additional boundary condition must be

$$v_{\theta_{\frac{1}{4}}}(\bar{x}, +\infty) - v_{\theta_{\frac{1}{4}}}(\bar{x}, -\infty) = \Delta v_a \tag{A16}$$

So, without obtaining the precise solution to (A15) subject to all the matching conditions used previously with the addition of (A16), it can at least be concluded that this is a well-posed problem. Hence, a jump in v_a of order one is possible.