WAVE LIMITS AND GENERALIZED HILBERT TRANSFORMS

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ABSTRACT

Let $\mathcal{H}_i = L_2(-\infty,\infty;N_i)$, where N_i is a Hilbert space ($i = 1,2$). Define the operator L by $Lf(x) = xf(x)$, and let χ_I be the characteristic function of I. We examine bounded linear operators $T:\mathcal{H}_1 \to \mathcal{H}_2$ which satisfy some or all of the following conditions:

(1) There exists a complex-valued function $K_{fg}(x,y)$ on $\mathcal{H}_1 \times \mathcal{H}_2 \times R^2$ such that $K_{fg} \in L_1(I \times J)$, and $(T\chi_I f, \chi_J g) = \int_{I\times J} K_{fg}$ for disjoint compact intervals I and J.

(2) $(T_0 f, g) = \lim_{\varepsilon\to 0+} \int_{X_\varepsilon} K_{fg}$ exists for $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$. $\{X_\varepsilon\}$ is a suitably chosen family of subregions of $\{(x,y):x\neq y\}$.

(3) $\text{s-lim}_{t\to\pm\infty} e^{itL}\, T\, e^{-itL}$ exists.

We show that if T satisfies 1 and 2, then $\chi_Z(T-T_0)\chi_Z$ is a multiplication operator for every bounded interval Z. Then T will satisfy 3 iff T_0 satisfies 3. We also obtain a representation for the limit 3. In case $N_1 = N_2$ = complex numbers, and $K(x,y)$ is the Fourier transform of an integrable function, then T defined by $(Tf,g) = \lim_{\varepsilon\to 0+} \int_{|x-y|\geq\varepsilon} dxdy\, \frac{K(x,y)}{x-y} f(x)\overline{g(y)}$ satisfies 1, 2 and 3.

The theory is applied to the situation V = symmetric operator, H = self-adjoint extension of L+V, and $H_0 = L$ in the space $\mathcal{H}_1$. Conditions analogous to 1, 2 and 3 are:

(1') Replace $(T\chi_I f, \chi_J g)$ in 1 by $(E(I)f, E_0(J)g)$.

(2') The same as 2.

(3') $\text{s-lim}_{t\to\pm\infty} \exp(itH)\exp(-itH_0)$ exists.

We show 1' is satisfied when V is a special Carleman operator, and 1', 2', 3' are satisfied when V is of trace class.

INTRODUCTION

In this thesis we study problems arising from a consideration of operator limits of the form $\underset{w}{\overset{s}{}}\text{-}\lim_{t\to\pm\infty} \exp(itH)\exp(-itH_0) = W_\pm(H,H_0)$, where H and H_0 are self-adjoint operators in a Hilbert space $\mathcal{H}$. We shall refer to $W_\pm(H,H_0)$ as the strong or weak, as the case may be, wave limits corresponding to the operators H and H_0. Since the limits need not exist on the whole space $\mathcal{H}$, it is of interest to find reasonable projections P and P_0 for which $\underset{w}{\overset{s}{}}\text{-}\lim_{t\to\pm\infty} P\exp(itH)\exp(-itH_0)P_0$ do exist. Then we denote them by $P\,W_\pm(H,H_0)P_0$.

During the last two decades there has been a lively interest in wave limit problems shown by both mathematicians and physicists. These limits arise in a branch of physics known as scattering theory. In elementary quantum mechanics, the state of a system is described by a vector ψ in a Hilbert space $\mathcal{H}$. In the Schrödinger model, it is further specified that ψ is a function of the time t and may vary in accordance with the differential equation $H\psi = i\frac{\partial\psi}{\partial t}$ which is called the Schrödinger equation.

The Hilbert space $\mathcal{H}$ usually consists of square-integrable vector-valued functions of position in 3k-space. The operator H is called the Hamiltonian. It is a symmetric operator usually of the form $H\psi = (-\Delta + V)\psi$, where $\Delta = \sum_{i=1}^{3k} \frac{\partial^2}{\partial x_i^2}$, and V is at most a first order symmetric differential operator whose coefficients are functions of position. The Hamiltonian characterises the system. It has been shown (13) that H is essentially self-adjoint under fairly general assumptions on V. This fact permits us to treat a good many physical problems, including that of scattering, in terms of the welldeveloped theory of unbounded self-adjoint operators. In particular, the solution to the Schrödinger equation can be written as $\psi(t) = \exp(-itH)\psi(0)$. The operators $\exp(-itH)$ $(-\infty < t < \infty)$, form a one-parameter group of unitary operators (8).

Scattering theory is concerned with the physical experiment in which we have an incoming beam of particles, an obstacle or 'scatterer' and an outgoing beam. We in fact need to imagine two situations, one in which the scatterer does interact with the beam, and one in which it does not. In the first case let us denote the state of the system by $\psi(t)$, and in the second by $\psi_0(t)$, corresponding to Hamiltonians H and H_0, respectively. Suppose, furthermore, that ψ and ψ_0 agree at some time τ in the remote past. Then $\psi(0) = \exp(i\tau H)\exp(-i\tau H_0)\psi_0(0)$. The wave limit $W_-(H,H_0)$ then gives a mapping of solutions of $H_0\psi_0 = i\frac{\partial\psi_0}{\partial t}$ into those of $H\psi = i\frac{\partial\psi}{\partial t}$ such that they 'agree at time $-\infty$'. Similarly $W_+(H,H_0)$ gives a mapping of solutions which agree at time $+\infty$. The scattering operator $S = W_+^* W_-$ maps each solution of $H_0\psi_0 = i\frac{\partial\psi_0}{\partial t}$ into another such solution and describes, in a sense, the total change caused by the scattering interaction in terms of solutions of this equation.

Scattering, when considered from this point of view, becomes a meaningful concept in comparing asymptotic properties of solutions of any two differential equations for which the solutions lie in the same space and the initial value problem has meaning. We mention, for example, a study of scattering for the wave equation by Lax and Phillips (21), and scattering for certain nonlinear equations by Browder and Strauss (3). There is a discussion of the physical basis for the formulation of scattering we have outlined here in a 1953 paper by Gell-Mann and Goldberger (7).

In order to apply mathematical analysis to scattering problems, it is necessary to specify more precisely the term wave limit and to make some assumptions about its properties. This was done by Jauch (11) in 1957 when he introduced the concept of scattering system. In 1959 Kuroda (18) gave some slight generalizations of Jauch's formulation. We describe here what is meant by a scattering system in this general sense.

Let H_1 and H_0 be two self-adjoint operators in a Hilbert space $\mathcal{H}$. Let P_1 and P_0 be the projections on the absolutely continuous subspaces of $\mathcal{H}$ with respect to H_1 and H_0 respectively. That is, $P_0\mathcal{H} = \{f : \|E^0_\mu f\| \text{ is an absolutely continuous function of } \mu\}$, where E^0_μ is the resolution of the identity corresponding to H_0. Then $(\mathcal{H}, H_1, H_0)$ is a scattering system if the following conditions are satisfied:

i) $W_\pm(H_1,H_0)\ P_0 = \operatorname*{s-lim}_{t\to\pm\infty} \exp(itH_1)\exp(-itH_0)\ P_0$ exist.

ii) $W_+(H_1,H_0)\ P_0\mathcal{H} = W_-(H_1,H_0)\ P_0\mathcal{H}$.

Even the single assumption that the strong limit W_+ exists implies a number of important consequences (18, Theorem 3.1) which we summarize here:

i) W_+ is a partial isometry with initial set $P_0\mathcal{H}$ and final set contained in $P_1\mathcal{H}$.

ii) $W_+\mathcal{H}$ reduces H_1 and the part of H_1 in $W_+\mathcal{H}$ is unitarily equivalent to the absolutely continuous part of H_0:

$$H_1P_1W_+ = W_+H_0P_0 , \quad W_+^*H_1P_1 = H_0P_0W_+^* .$$

If $(\mathcal{H}, H_1, H_0)$ is a scattering system, then the scattering operator $S = W_+^*W_-$ is a partial isometry whose restriction to $P_0\mathcal{H}$ is unitary. This fact is of importance to physicists (11, p. 152).

Mathematical efforts have been largely directed toward verifying conditions i) and ii) for scattering systems. We shall give a brief description of the techniques used and the results obtained below. Scattering theory, however, has not been the prime motivation for studying wave limits. They are a useful tool in studying conditions which guarantee the unitary equivalence of two self-adjoint operators H_1 and H_0, or more precisely, the unitary equivalence of H_1P_1 and H_0P_0 considered as self-adjoint operators on $P_1\mathcal{H}$ and $P_0\mathcal{H}$ respectively. The reason for this is that if the strong wave limit $W_+(H_1, H_0)P_0$ exists, and $W_+\mathcal{H} = P_1\mathcal{H}$, then H_1P_1 and H_0P_0 are unitarily equivalent (18, Theorem 3.2).

The unitary equivalence problem has, in fact, been investigated using two different approaches. The historically earlier one was used by Friedrichs (6) who constructs the unitary operator giving the equivalence by means of a perturbation method. Having established this equivalence, Friedrichs then proceeds to prove the existence of strong wave limits. The scattering theory approach, on the other hand, establishes strong wave limits first. This method usually yields equivalence of H_1P_1 and H_0P_0 under more general assumptions, but fails to say anything about the parts of H_1 and H_0 not in $P_1\mathcal{H}$ and $P_0\mathcal{H}$ respectively.

The starting point of the scattering theory approach is the equation

$$\exp(itH_1)\exp(-itH_0)\, u = u + i\int_0^t ds\, \exp(isH_1)\, V \exp(-isH_0)\, u,$$

where $u \in \mathcal{H}$, and $H_1 = H_0 + V$. In practice one first establishes the strong integrability of the integrand on $(-\infty,\infty)$ for all u in some manifold $\mathcal{M}$ dense in $P_0\mathcal{H}$. This, of course, implies the existence of strong wave limits on $\mathcal{M}$ and the extension to $P_0\mathcal{H}$ is straightforward.

We mention here some of the more significant results which have been obtained in this way. In 1957 Rosenblum (26) essentially proved the existence of the strong limit $P_1 W_{\pm}(H_1, H_0)P_0$ under the assumption that $H_1 = H_0 + V$, where V is of trace class. In the same year Kato (14) obtained the strong limit $W_{\pm}(H_1, H_0)P_0$ for V of finite rank as well as the fact that $W_{\pm}\mathcal{H} = P_1\mathcal{H}$. Kato subsequently extended his result (15) to operators V of trace class (27, p. 77) by means of an inequality which showed that for fixed H_0 $W_{\pm}(H_1, H_0)P_0$ is a continuous function of V, from the trace topology on the class of finite rank operators V to the strong operator topology. The extension to all V in the trace class is then clear.

It is interesting that the trace class is, in a sense, the largest class for which this result can hold if we deny ourselves the privilege of making assumptions about the relationship of V to H_0. Kuroda proved in 1958 (17) that, given any unitarily invariant cross-norm α which is not equivalent to the trace norm and $\varepsilon > 0$, then for any self-adjoint operator H_0 there exists a self-adjoint operator V with $\alpha(V) < \varepsilon$ such that $H_\varepsilon = H_0 + V$ has a pure point spectrum.

Nevertheless, the perturbations which appear in the physical scattering problems seldom, if ever, are of trace class. On the other hand, usually a great deal is known about the operator H_0. By taking this information into account, the existence of strong wave limits can still be proved (4), (12), (19). The literature on this topic is too vast to permit our giving a summary. We mention only a result by Kato (16) which includes many of these specialized theorems. It asserts, roughly speaking, that if $s\text{-}W_\pm(H_1, H_0)P_0$ exist, and ϕ is a real-valued function satisfying certain smoothness restrictions, then $s\text{-}W_\pm(\phi(H_1), \phi(H_0))P_0$ indeed exist.

Specialized techniques have also been devised for dealing with the case when H_0 is the Laplacian and V is a multiplication operator. There has been work done on eigenfunction expansions associated with the Schrödinger equation which yields slightly better existence theorems and gives representations for the wave limits. We mention in particular works by Povsner (24) and Ikebe (10). Recently representation theorems for the trace class have been obtained by Birman and Entina (1).

Although the existence of strong wave limits implies at least partial unitary equivalence, the opposite is not true. Jauch (11, p. 150) asks whether the condition that H_1P_1 and H_0P_0 be unitarily equivalent is sufficient to guarantee that $(\mathcal{H}, H_1, H_0)$ be a scattering system. The following example answers this question in the negative. Let $\mathcal{H} = L_2(E_2), H_0$ be a self-adjoint extension of $i\frac{\partial}{\partial x}$, and let H_1 be a self-adjoint extension of $i\frac{\partial}{\partial y}$. Then H_0 and H_1 have absolutely continuous spectra and are unitarily equivalent. Furthermore, $\exp(itH_1)f(x,y) = f(x-t,y)$, and $\exp(itH_0)f(x,y) = f(x,y-t)$. Hence $\exp(itH_1)\exp(-itH_0)f(x,y) = f(x-t,y+t)$, from which it follows that the weak wave limits are 0 and the strong ones do not exist. Consequently some other conditions are required.

The point of view taken in this thesis is that the unitary operator U giving the equivalence, that is $U^* H_0 U = H_1$, must be of a rather special form. The investigations of Friedrichs (6) on perturbations of the operator L in the Hilbert space $\mathcal{H} = L_2(-\infty,\infty)$, where $Lf(x) = xf(x)$, by an integral operator V satisfying certain Hölder conditions yield U of the form

$$Uf(x) = A(x)f(x) + \int dy\, \frac{B(x,y)}{x-y}\, f(y) \;. \qquad \text{(F)}$$

The integral must, of course, be interpreted as a principal value. The Friedrichs theory has since been extended to more general situations by means of the concept of a space of gentle operators introduced by Rejto (25). The generalization of F continues to play an important role. We might add that the gentle perturbation approach to the unitary equivalence problem deals with more restricted classes of perturbations than the wave operator approach, but has the advantage of giving information about the non-absolutely continuous parts of the spectrum.

In this work we show that a generalization of F is valid for a class of perturbations which includes the trace class. This class can be roughly described as those perturbations V for which we have

$$(E_1(I)f,\ E_0(J)g) = \int_{I \times J} d\mu d\lambda\ K_{fg}\ (\mu, \lambda)\ ,$$

where I and J are disjoint compact intervals, $K_{fg} \epsilon L_1(I \times J)$, and E_1, E_0 are the spectral measures corresponding to H_1, H_0 respectively. We also study the connection between F and weak and strong wave limits.

For purposes of analysis we need the theory of spectral representation for unbounded self-adjoint operators (5), (29). For the special case $H_1P_1 = H_1$ and $H_0P_0 = H_0$, it asserts essentially that $\mathcal{H}$ can be represented as a subspace $\mathcal{H}_i$ of $L_2(-\infty, \infty : N_i)$ in which H_i becomes multiplication by x, (i = 0,1). That is, there exist isometries $U_i : \mathcal{H} \to \mathcal{H}_i$, such that $U_iH_iU_i^* = L$, where $Lf(x) = xf(x)$.

The question of existence of wave limits is now decided by the existence of $\begin{smallmatrix} s \\ w \end{smallmatrix}\text{-}\lim\limits_{t\to\pm\infty} e^{itL}U_1U_0^*e^{-itL}$, where $U_1U_0^*$ is an isometry mapping $\mathcal{H}_0$ onto $\mathcal{H}_1$. Actually the isometric character of $U_1U_0^*$ does not seem to play a very special role insofar as the existence of these limits is concerned. More important is that $U_1U_0^*$ be of a form analogous to F.

Section I of the thesis deals with bounded operators $T : L_2(-\infty, \infty ; N_1) \to L_2(-\infty, \infty ; N_2)$ which possess a form analogous to F. We construct a large class of examples for which $\text{s-}\lim\limits_{t\to\pm\infty} e^{itL}Te^{-itL}$ exist and obtain a representation for these limits. In Section II we apply our theory to the operator $U_1U_0^*$.

SECTION I

Throughout this section, unless stated otherwise, $\mathcal{H}_i$ will denote the Bochner space $L_2(-\infty,\infty;N_i)$, where N_i is a Hilbert space with norm $|\cdot|_i$ and inner product $\langle\cdot,\cdot\rangle_i$, $(i = 1,2)$. The norm and the inner product for $\mathcal{H}_i$ will be denoted by $\|\cdot\|_i$ and $(\cdot,\cdot)_i$; that is, $(f,g)_i = \int dx \langle f(x), g(x)\rangle_i$, and $\|f\|_i^2 = \int dx \; |f(x)|_i^2$, where $f,g \in \mathcal{H}_i$, $(i = 1,2)$. Whenever there is no danger of confusion we may omit the indices on the norms and the inner products.

We shall consider linear operators $T : \mathcal{H}_1 \to \mathcal{H}_2$ and denote the norm of T by $\|T\| = \sup \{ \|Tf\|_2 : \|f\|_1 \leq 1\}$.

The symbol χ_S will denote the characteristic function of the set S. When S is Borel measurable we write $\chi_S f(x) = \chi_S(x)f(x)$. That is, we treat χ_S as a projection on the subspace of functions with support in S.

Our first step is to define a class $\mathcal{C}_1$ of bounded linear operators which will include those of type F.

Definition 1.1. $T \in \mathcal{C}_1$ if the following conditions are satisfied.

i) $T : \mathcal{H}_1 \to \mathcal{H}_2$ and $\|T\| < \infty$

ii) For each $f \in \mathcal{H}_1$ and each $g \in \mathcal{H}_2$, there exists a complex-valued function $K_{fg}(x,y)$ defined on R^2, such that for any two disjoint compact intervals I and J, $K_{fg} \in L_1(I \times J)$ and $(T\chi_I f, \chi_J g)_2 = \int_{I\times J} dxdy \, K_{fg}(x,y)$.

It is sometimes useful to have a characterization of $\mathcal{C}_1$ which does not demand us to display the kernel K_{fg}. Such an alternate characterization is contained in the following theorem. It is understood that T is a bounded linear operator.

<u>Theorem 1.2.</u> The condition $T \epsilon \mathcal{C}_1$ is equivalent to the statement:

Given disjoint compact intervals X and Y, $f \epsilon \mathcal{H}_1$, $g \epsilon \mathcal{H}_2$, and $\varepsilon > 0$, there exists $\delta > 0$ such that if I_k and J_k $(k = 1, \ldots, n)$ are disjoint subintervals of X and Y respectively which satisfy

$$\sum_{k=1}^{n} \ell(I_k)\ell(J_k) < \delta, \quad \text{then} \quad \sum_{k=1}^{n} |(Tf\chi_{I_k}, g\chi_{J_k})| < \varepsilon \quad . \qquad (1)$$

Proof. If $T \epsilon \mathcal{C}_1$, then the corresponding kernel $K_{fg} \epsilon L_1(X \times Y)$ and 1 follows immediately. Suppose statement 1 holds.

Let $f \epsilon \mathcal{H}_1$ and $g \epsilon \mathcal{H}_2$. We choose disjoint compact intervals X and Y and construct a finite complex measure μ_{fg} on the Lebesgue measurable subsets of $X \times Y$ which is absolutely continuous with respect to Lebesgue measure and satisfies $\mu_{fg}(I \times J) = (T\chi_I f, \chi_J g)$ where I and J are subintervals of X and Y respectively. We use the fact that the Lebesgue measurable subsets of $X \times Y$ form a complete metric space if we set $\rho(A, B) = \lambda((A-B) \cup (B-A))$, where λ denotes Lebesgue measure (22, p. 107). Furthermore, finite unions of rectangles $I \times J$ are dense in this space.

The next step is to define an additive function μ_{fg} on the semiring of rectangles by setting $\mu_{fg}(I \times J) = (T\chi_I f, \chi_J g)$. The additivity of μ_{fg} follows from the linearity of T and $(\cdot, \cdot)$. We extend μ_{fg} to the ring of finite disjoint unions of rectangles in a straightforward fashion. Statement 1 now asserts that μ_{fg} is a uniformly continuous function from this ring into the complex numbers. Since the elements of this ring are dense in our metric space, μ_{fg} can be uniquely extended by a well-known technique to the whole metric space.

The extension is a measure on the Lebesgue measurable subsets of $X \times Y$. Furthermore statement 1 implies that μ_{fg} is absolutely continuous with respect to Lebesgue measure since any set of measure zero can be covered by a countable union of rectangles of arbitrarily small total measure. Finally we remark that μ_{fg} is unique.

By the Radon-Nikodym theorem, there exists a function K_{fg} defined on $X \times Y$ and unique up to sets of measure zero such that $K_{fg} \in L_1(X \times Y)$ and $\mu_{fg}(I \times J) = \int_{I \times J} K_{fg}$.

If we have K^1_{fg} and K^2_{fg} obtained for rectangles $X_1 \times Y_1$ and $X_2 \times Y_2$ respectively the uniqueness of our construction implies $K^1_{fg} = K^2_{fg}$ a.e. on $(X_1 \cap X_2) \times (X_1 \cap Y_2) = (X_1 \times Y_1) \cap (X_2 \times Y_2)$. Hence, by an obvious patching procedure, a function K_{fg} can be defined in the whole plane, having the properties asserted in the theorem.

In the proof of the preceding theorem we have obtained the added bonus that if T satisfies statement 1, then the corresponding kernel K_{fg} is unique up to sets of measure zero. This implies:

Corollary 1.3. If $T \in \mathcal{C}_1$, then for fixed $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$, the corresponding kernel K_{fg} is unique up to sets of measure zero.

In the ensuing discussion we shall use some linearity properties of the kernel K_{fg} which are contained in the following lemma. We write $K(f,g;x,y)$ instead of $K_{fg}(x,y)$ for reasons of convenience.

Lemma 1.4. Let $T \in \mathcal{C}_1$ and let $K(f,g;x,y)$ be the corresponding kernel. Then the following statements are true:

i) For any f, h $\epsilon \mathcal{H}_1$, g, k $\epsilon \mathcal{H}_2$ and complex numbers α, β, we have

$$K(\alpha f+\beta h, g;x,y) = \alpha K(f,g;x,y) + \beta K(h,g;x,y) \text{ a.e., and}$$

$$K(f, \alpha g+\beta k;x,y) = \overline{\alpha} K(f,g;x,y) + \overline{\beta} K(f,k;x,y) \text{ a.e.}$$

ii) If $f(\cdot)\epsilon \mathcal{H}_1$, $g(\cdot)\epsilon \mathcal{H}_2$, and $\alpha(x)$, $\beta(y)$ are complex-valued functions such that also $\alpha(\cdot)f(\cdot)\epsilon \mathcal{H}_1$ and $\beta(\cdot)g(\cdot)\epsilon \mathcal{H}_2$, then

$$K(\alpha f, \beta g;x,y) = \alpha(x)\overline{\beta(y)}\, K(f,g;x,y) \text{ a.e.}$$

Proof. i) is a direct consequence of Corollary 1.3 and the corresponding linearity relations for $(T\chi_I f, \chi_J g)$

ii) Let X and Y be disjoint compact intervals and

$A_n = \{x : x\epsilon X, \ |\alpha(x)| \leq n\}$,

$B_n = \{y : y\epsilon Y, \ |\beta(y)| \leq n\}$ $\quad (n = 1, 2, \ldots)$.

Then $\{A_n\}$ and $\{B_n\}$ are expanding sequences of sets. If we choose a particular $x\epsilon X$ then $x\epsilon A_n$ for $n > |\alpha(x)|$. Consequently $\bigcup_n A_n = X$. Similarly $\bigcup_n B_n = Y$. It follows then that $\{A_n \times B_n\}$ is an expanding sequence of sets whose union is $X \times Y$. It is a simple calculation using i) to show that, when s_1 and s_2 are stepfunctions on X and Y respectively,

$$(Ts_1 f, s_2 g) = \int_{X\times Y} dxdy\, s_1(x)\, \overline{s_2(y)}\, K(f,g;x,y)$$

Then, using the bounded convergence theorem and the continuity of the inner product, we obtain the corresponding result for bounded measurable functions. Certainly $\chi_{A_n}\alpha(\cdot)$ and $\chi_{B_n}\beta(\cdot)$ $(n = 1, 2, \ldots)$ are bounded measurable functions. Consequently for all subintervals

I and J of X and Y respectively, we have for n = 1, 2, ...

$$(T\,\chi_{A_n}\,\alpha\,\chi_I f,\ \ \chi_{B_n}\,\beta\,\chi_J g)$$

$$= \int_{I\times J} dxdy\ \chi_{A_n}(x)\ \alpha(x)\ \overline{\chi_{B_n}(y)\beta(y)}\,K(f,g;x,y)$$

$$= \int_{I\times J} dxdy\ \chi_{A_n}(x)\ \chi_{B_n}(y)\ K(\alpha f,\beta g;x,y)$$

We conclude from Corollary 1.3 that

$$\chi_{A_n\times B_n}(x,y)\alpha(x)\overline{\beta(y)}\,K(f,g;x,y) = \chi_{A_n\times B_n}(x,y)K(\alpha f,\beta y;x,y)\ \text{a.e.}$$

Finally, using the fact that $A_n \times B_n \uparrow X \times Y$, we obtain

$$\alpha(x)\overline{\beta(y)}\,K(f,g;x,y) = K(\alpha f,\beta g;x,y)\ \text{a.e. on } X \times Y$$

Since the set $\{(x,y) : x \neq y\}$ can be written as a countable union of rectangles of the form $X \times Y$, it follows that the above equation holds a.e. in the plane.

For our purposes, the most interesting operators $T \in \mathcal{C}_1$ are those whose kernel $K_{fg}(x,y)$ is not integrable in a neighbourhood of the line x = y. We now define classes $\mathcal{C}_2$ and $\mathcal{C}_2'$ of operators T for which K_{fg} is integrable in a principal value sense.

Let X_ε and X'_ε be subsets of R^2 defined by

$$X_\varepsilon = \{(x,y) : |x-y| \geq \varepsilon\},\ \text{and}$$

$$X'_\varepsilon = \{(x,y) : |x-y| \geq \varepsilon\,\phi((x+y)/\varepsilon)\}\ (\varepsilon > 0),\ \text{where}$$

ϕ is the sawtooth function:

$$\phi(x) = \begin{cases} 2 + x & -1 \leq x < 0 \\ 2 - x & 0 \leq x < 1 \end{cases}, \quad \phi(2n+x) = \phi(x) \ (n = \pm 1, \pm 2, \ldots) .$$

See fig. 1, p. 45.

Fix a bounded interval Z and choose $T \in \mathcal{C}_1$ with corresponding kernel K_{fg}. We prove in Lemma 1.6 that T_ε and T'_ε defined by

$$(T_\varepsilon f,g) = \int_{Z^2 \cap X_\varepsilon} K_{fg} \text{ , and } (T'_\varepsilon f,g) = \int_{Z^2 \cap X'_\varepsilon} K_{fg} \qquad (2)$$

are bounded linear operators.

Then we define the classes

$$\mathcal{C}_2 (Z) = \{T : T \in \mathcal{C}_1, \ T_0 = \text{w-}\lim_{\varepsilon \to 0+} T_\varepsilon \text{ exists}\} \text{ , and}$$

$$\mathcal{C}'_2 (Z) = \{T : T \in \mathcal{C}_1, \ T'_0 = \text{w-}\lim_{\varepsilon \to 0+} T'_\varepsilon \text{ exists}\} \quad .$$

<u>Definition 1.5.</u>

i) $\mathcal{C}_2 = \{\cap \ \mathcal{C}_2(Z) : \text{all bounded intervals } Z\}$

ii) $\mathcal{C}'_2 = \{\cap \ \mathcal{C}'_2(Z) : \text{all bounded intervals } Z\}$

<u>Lemma 1.6.</u> The operators T_ε and T'_ε defined by equation 2 are bounded and linear. Furthermore,

$$\|T'_\varepsilon\| \leq 3 \|T\| \quad (\varepsilon > 0) \qquad (3)$$

Proof. We first prove 3. Let $I_k = [2k\varepsilon, 2(k+1)\varepsilon]$ and $J_k = [k\varepsilon, (k+1)\varepsilon]$ $(k = 0, \pm 1, \pm 2, \ldots)$. Choose $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$ with support in Z. Then

$$(Tf,g) = (T(\sum_k \chi_{J_k} f), \sum_\ell \chi_{J_\ell} g) = \sum_{k,\ell} (T\chi_{J_k} f, \chi_{J_\ell} g)$$

$$= \sum_k (T\chi_{I_k} f, \chi_{I_k} g)$$

$$+ [\ldots + (T\chi_{J_{-1}} f, \chi_{J_0} g) + (T\chi_{J_0} f, \chi_{J_{-1}} g) + (T\chi_{J_1} f, \chi_{J_2} g) + \ldots]$$

$$+ \int_{X'_\varepsilon} K_{fg} \qquad (4)$$

where we have used the assumption that

$$(T\chi_{J_k} f, \chi_{J_\ell} g) = \int_{J_k \times J_\ell} K_{fg}, \text{ whenever } J_k \cap J_\ell = \phi$$

(Definition 1.1, ii)). See fig. 1, p. 45.

We now solve equation 4 for $(T'_\varepsilon f, g) = \int_{X'_\varepsilon} K_{fg}$ and obtain the inequality

$$|(T'_\varepsilon f,g)| \leq \|T\| \; \|f\|_1 \; \|g\|_2 + \sum_k \|T\| \; \|\chi_{I_k} f\|_1 \; \|\chi_{I_k} g\|_2$$

$$+ \|T\| \left[\ldots + \|\chi_{J_{-1}} f\|_1 \|\chi_{J_0} g\|_2 + \|\chi_{J_0} f\|_1 \|\chi_{J_{-1}} g\|_2 + \|\chi_{J_1} f\|_1 \|\chi_{J_2} g\|_2 + \ldots\right]$$

$$\leq \|T\| \; \|f\|_1 \|g\|_2 + \|T\| \left(\sum_k \|\chi_{I_k} f\|_1^2\right)^{\frac{1}{2}} \left(\sum_k \|\chi_{I_k} g\|_2^2\right)^{\frac{1}{2}}$$

$$+ \|T\| \left(\sum_\ell \|\chi_{J_\ell} f\|_1^2\right)^{\frac{1}{2}} \left(\sum_\ell \|\chi_{J_\ell} g\|_2^2\right)^{\frac{1}{2}} .$$

Since $\sum_k \|\chi_{I_k} f\|_1^2 = \|f\|_1^2$, *and similarly for the other* sums, inequality 3 follows.

To prove that T_ε is bounded we construct a sequence of sets $X_{\varepsilon,n}$ $(n = 1, 2, \ldots)$ satisfying:

i) $X_{\varepsilon,n}$ is a finite union of disjoint subrectangles of Z^2.

ii) $X_{\varepsilon,n}$ is an expanding sequence whose union except for a null set is $Z^2 \cap X_\varepsilon$. See fig. 2, p. 45.

We define the operator $T_{\varepsilon,n}$ by $(T_{\varepsilon,n} f, g) = \int_{X_{\varepsilon,n}} K_{fg}$, and notice that it is bounded and linear since by i) above it can be written as a finite sum of operators of the form $\chi_J T \chi_I$. It is easy to see that $K_{fg} \in L_1(Z^2 \cap X_\varepsilon)$ and

$$\int_{X_{\varepsilon,n}} K_{fg} \to \int_{Z^2 \cap X_\varepsilon} K_{fg} \quad \text{as } n \to \infty .$$

Therefore $\text{w-}\lim_{n\to\infty} T_{\varepsilon,n} = T_\varepsilon$ and T_ε is bounded by the principle of uniform boundedness.

As a consequence of inequality 3 we obtain:

<u>Corollary 1.7</u>. $T \in \mathcal{C}_2'$ if and only if there exist linear manifolds M_i dense in $\mathcal{H}_i$ $(i = 1, 2)$ such that $h \in M_1$ and $k \in M_2$ implies

$$\lim_{\varepsilon \to 0+} \int_{Z^2 \cap X'_\varepsilon} K_{h,k} \text{ exists for all bounded intervals } Z.$$

At this point we make the additional assumption, to hold for the remainder of Section I, that $\mathcal{H}_i = L_2(-\infty, \infty; N_i)$ $(i = 1, 2)$ are separable.

Suppose $T \in \mathcal{C}_2(Z)$, and let $T_0 = \text{w-}\lim_{\varepsilon \to 0+} T_\varepsilon$. We show in Theorem 1.9 that $\chi_Z T \chi_Z - T_0$ is simply a multiplication operator. The proof is dependent upon the following lemma.

<u>Lemma 1.8</u>. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear transformation which satisfies

$$\text{support } (f(x)) \subset I \Rightarrow \text{support } (Tf(x)) \subset I$$

up to a set of measure zero where I is any compact interval and $f \in \mathcal{H}_1$.

Then for each $x \in (-\infty, \infty)$ there exists a bounded linear transformation $S(x)$ mapping N_1 into N_2 which satisfies

$$|S(x)| \leq \|T\| \text{ and } Tf(x) = S(x)\, f(x) \text{ x-a.e.}$$

Proof. Choose $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$. Then for any measurable set A $\chi_A f \in \mathcal{H}_1$, and $\nu(A) = (T\chi_A f, g)$ defines a finite complex measure on the measurable subsets of the real line. Furthermore, since $\nu(A) = 0$ whenever A is a null set, ν is absolutely continuous with respect to Lebesgue measure. To prove that ν is a measure let $\{E_n\}$ be a sequence of disjoint measurable sets whose union is A and let $A_n = \bigcup_1^n E_k$. It is clear that $\chi_{A_n} f \to \chi_A f$ strongly in $\mathcal{H}_1$. The result then follows from the equations:

$$\sum_{n=1}^{\infty} (T\chi_{E_n} f, g) = \lim_{n\to\infty} \sum_{k=1}^{n} (T\chi_{E_k} f, g)$$

$$= \lim_{n\to\infty} (T\chi_{A_n} f, g) = (T\chi_A f, g)\,.$$

We can now apply the Radon-Nikodym theorem to obtain a complex-valued integrable function $\phi(x)$, unique up to sets of measure zero, such that $\nu(A) = \int_A dx\,\phi(x)$ for measurable A.

It is important to notice the dependence of ν and ϕ on f and g. To this end let us write $\phi(x) = \phi(f, g; x)$. Let α, β be complex numbers, $f, h \in \mathcal{H}_1$, and $g, k \in \mathcal{H}_2$. We deduce from the corresponding relations for ν that:

$\phi(\alpha f+\beta h, g;x) = \alpha\phi(f,g;x) + \beta\phi(h,g;x)$, and

$\phi(f,\alpha g+\beta k;x) = \overline{\alpha}\phi(f,g;x) + \overline{\beta}\phi(f,k;x)$ x-a.e.

In order to prove that $|S(x)| \leqslant \|T\|$ we require the following sublemma: If $\alpha(x)$ is any bounded measurable complex-valued function, then $\phi(\alpha f, g;x) = \alpha(x)\phi(f,g;x)$ x-a.e.

The equation

$$(T\chi_A\chi_I f, g) = (T\chi_{I\cap A} f, g)$$

$$= \int_{I\cap A} dx\phi(f,g;x) = \int_A dx\,\chi_I(x)\phi(f,g;x)$$

shows that $\phi(\chi_I f, g;x) = \chi_I\phi(f,g;x)$ x-a.e. , which by linearity gives the result for stepfunctions. Let $\{\alpha_n\}$ $(n = 1, 2, \ldots)$ be a sequence of stepfunctions converging a.e. to α. Then $\alpha_n f \to \alpha f$ strongly in $\mathcal{H}_1$ implying $(T\alpha_n\chi_A f, g) \to (T\alpha\chi_A f, g)$ for measurable sets A. But $(T\alpha_n\chi_A f, g) = \int_A dx\alpha_n(x)\phi(f,g;x)$, which converges to $\int_A dx\alpha(x)\phi(f,g;x)$ by the dominated convergence theorem. It follows that

$$\phi(\alpha f, g;x) = \alpha(x)\phi(f,g;x) \text{ x-a.e.}$$

We now construct the operators S(x). Let $a \in N_1$ and $b \in N_2$. Then for any bounded interval I, $\chi_I a \in \mathcal{H}_1$ and $\chi_I b \in \mathcal{H}_2$, so we can construct $\phi(\chi_I a, \chi_I b;x)$. By our hypothesis that T preserves supports we have for every bounded interval J containing I, and every measurable subset A of I, the equation $(T\chi_A a, \chi_I b) = (T\chi_A a, \chi_J b)$. It follows that

$$\phi(\chi_I a,\ \chi_I b;x) = \phi(\chi_J a,\ \chi_J b;x) \text{ for a.e. } x \in I.$$

Thus there exists a function $S(a,b;x)$ which coincides x-a.e. with $\phi(\chi_I a,\ \chi_I b;x)$ on every interval I.

S must satisfy $|S(a,b;x)| \leq \|T\|\ |a|_1\ |b|_2$ x-a.e. For suppose it did not and the opposite inequality held on a set A of positive measure. Let $\overline{\alpha(x)} = \operatorname{sgn} S(a,b;x)$. Then $f = \alpha\chi_A a \in \mathcal{H}_1$ and $g = \chi_A b \in \mathcal{H}_2$. Furthermore, by our sublemma,

$$(Tf,g) = \int_A dx\ \phi(f,g;x) = \int_A dx\ \alpha(x)\ \phi(\chi_A a, \chi_A b;x)$$

$$= \int_A dx\, |S(a,b;x)|$$

$$> \|T\|\ |a|_1\ |b|_2 \int_A dx = \|T\|\ \|f\|_1\ \|g\|_2\ ,$$

which is a contradiction.

Since N_1 and N_2 are separable, they contain countable dense linear manifolds over the complex rationals, M_1 and M_2 respectively. Each of the following equations holds x-a.e. for a given complex rational r and given $a, a' \in M_1$ and $b, b' \in M_2$:

$$|S(a,b;x)| \leq \|T\|\ |a|_1\ |b|_2\ ,$$

$$S(ra,b;x) = r\, S(a,b;x)\ ,$$

$$S(a,rb;x) = \overline{r}\, S(a,b;x)\ ,$$

$$S(a+a',b;x) = S(a,b;x) + S(a',b;x), \text{ and}$$

$$S(a,b+b';x) = S(a,b;x) + S(a,b';x)\ .$$

Our assumptions on the countability of M_1 and M_2 allow us to construct one exceptional set E of measure zero such that all of the above equations hold on its complement E'.

For each $x \in E'$, $\langle S_0(x)a, b\rangle_2 = S(a,b;x)$ defines a bounded linear operator from M_1 into N_2 with norm $\leqslant \|T\|$. Let S(x) be the extension of $S_0(x)$ to $N_1 = \overline{M_1}$.

Our *final step is to show that* $Tf(x) = S(x)f(x)$ x-a.e., which we accomplish by proving that given any $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$, then

$$(Tf, g) = \int dx \langle S(x)f(x), g(x)\rangle_2 = \int dx\, S(f(x), g(x); x)$$

We prove the result first for stepfunctions $f = \sum_k \chi_{I_k} a_k$ and $g = \sum_\ell \chi_{J_\ell} b_\ell$, where $a_k \in N_1$, $b_\ell \in N_2$, the I_k are disjoint bounded intervals, and the J_ℓ are disjoint bounded intervals. Then

$$(Tf, g) = \sum_{k,\ell} (T\chi_{I_k} a_k, \chi_{J_\ell} b_\ell)$$

$$= \sum_{k,\ell} (T\chi_{I_k \cap J_\ell} a_k, \chi_{J_\ell} b_\ell) = \sum_{k,\ell} \int_{I_k \cap J_\ell} dx\, S(a_k, b_\ell; x)$$

$$= \int dx \sum_{k,\ell} \chi_{I_k} \chi_{J_\ell} S(a_k, b_\ell; x)$$

$$= \int dx\, S(f(x), g(x); x)$$

Now let $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$. Choose sequences $\{f_n\}$ and $\{g_n\}$ $(n = 1, 2, \ldots)$ of stepfunctions tending strongly to f and g respectively. Then certainly $(Tf_n, g_n) \to (Tf, g)$. On the other hand, using the inequality $|S(x)| \leqslant \|T\|$, we obtain also that

$$\int dx \langle S(x)\, f_n(x), g_n(x)\rangle_2 \to \int dx \langle S(x)f(x), g(x)\rangle_2 ,$$

completing the proof of the lemma.

Theorem 1.9 . Let $T \in \mathcal{C}_2$. Then for each real x, there exists a bounded operator $S(x) : N_1 \to N_2$ such that

$$(\chi_Z T \chi_Z - T_0) f(x) = \chi_Z(x) S(x) f(x),$$

where Z is any bounded interval, $f \in \mathcal{H}_1$, and

$$(T_0 f, g) = \lim_{\varepsilon \to 0+} \int_{Z^2 \cap X_\varepsilon} K_{fg}.$$

Proof. Fix Z, and let I, J be disjoint compact subintervals of Z. We shall apply Lemma 1.8 to the operator $\chi_Z T \chi_Z - T_0$. Let $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$ with support $(f) \subset I$, support $(g) \subset J$. Then

$$(\chi_Z T \chi_Z f, g) = (Tf, g) = \int_{I \times J} K_{fg} = \int_{Z^2} K_{fg}$$

$$= \lim_{\varepsilon \to 0} \int_{Z^2 \cap X_\varepsilon} K_{fg} = (T_0 f, g) ,$$

where we have freely used Lemma 1.4 ii).

It follows that support $[(\chi_Z T \chi_Z - T_0) f] \subset$ support (f) . The remainder of the proof is straightforward.

For the class $\mathcal{C}_2'$ we can obtain a stronger result, since inequality 3 permits us to define the operator T_0' for all $f \in \mathcal{H}_1$, removing the restriction that f have compact support. We denote the extended operator also by T_0'.

Theorem 1.9'. Let $T \in \mathcal{C}_2'$. Then for each real x, there exists an operator $S'(x) : N_1 \to N_2$ such that

$$|S'(x)| \leqslant 4 \|T\|, \text{ and}$$

$$(T - T_0') f(x) = S'(x) f(x) \text{ x-a.e.},$$

for all $f \in \mathcal{H}_1$.

We define the operator L by $Lf(x) = xf(x)$ and use the same symbol L for both $f \in \mathcal{H}_1$ and $f \in \mathcal{H}_2$.

Theorem 1.10. Let $T \in \mathcal{C}_2'$. Then:

i) $\underset{w}{\overset{s}{}}\text{-}\lim_{t\to\pm\infty} e^{itL}\, Te^{-itL}$ exists if and only if

$\underset{w}{\overset{s}{}}\text{-}\lim_{t\to\pm\infty} e^{itL}\, T_0'\, e^{-itL}$ exists respectively.

ii) If one such limit exists then for each real x, there exist operators $A_\pm(x) : N_1 \to N_2$ such that

$$|A_\pm(x)| \leq \|T\| \text{ , and}$$

$$(\underset{w}{\overset{s}{}}\text{-}\lim_{t\to\pm\infty} e^{itL}\, T\, e^{-itL} f)\,(x) = A_\pm(x) f(x) \quad x\text{-a.e.}$$

Proof: i) follows immediately upon applying Theorem 1.9'.

To prove ii), let I, J be disjoint compact intervals and $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$ have supports in I, J respectively. Then

$$(e^{itL}\, T\, e^{-itL} f, g) = (T e^{-itL} f,\ e^{-itL} g)$$

$$= \int_{I\times J} dxdy\ e^{-it(x-y)}\, K_{fg}(x,y)\ ,$$

where K_{fg} is the kernel corresponding to T.

Since $K_{fg} \in L_1(I\times J)$, the limit as $t \to \pm\infty$ is 0 by the Riemann-Lebesgue lemma. Therefore

$$\text{support } (w\text{-}\lim_{t\to\pm\infty} e^{itL}\, T\, e^{-itL} f) \subset I.$$

Furthermore $\|e^{itL}\, T\, e^{-itL}\| = \|T\|$, and our result follows from Lemma 1.8.

Note. It is possible to prove an analogous theorem for the class

$\mathcal{C}_2$. We do not, however, require such a result in the sequel.

In the remainder of section I we construct as examples two subclasses of $\mathcal{C}_2$ whose members T have the property that

$$s - \lim_{t \to \pm\infty} e^{itL} T e^{-itL}$$

exist.

Our first example is contained in the following theorem for the case $\mathcal{H}_1 = \mathcal{H}_2 = L_2(-\infty, \infty)$.

Theorem 1.11. Let K(x, y) be the Fourier transform of an integrable function $\hat{K}(p, q)$. That is,

$$K(x, y) = \int dpdq\, e^{i(px+qy)} \hat{K}(p, q) .$$

In particular K is bounded and

$$(T_\varepsilon f, g) = \int_{|x-y| \geq \varepsilon} dxdy \frac{K(x, y)}{x-y} f(x) \overline{g(y)} \quad (\varepsilon > 0)$$

defines a bounded linear operator on $L_2(-\infty, \infty)$. Then the following assertions are true:

i) $\text{w-}\lim_{\varepsilon \to 0+} T_\varepsilon = T_0$ exists

ii) $\|T_0\| \leq \pi \int dpdq\, |\hat{K}(p, q)|$

iii) $\text{s-}\lim_{t \to \pm\infty} (e^{itL} T_0 e^{-itL} f)(x) = \mp \pi i K(x, x) f(x)$

iv) $\text{s-}\lim_{t \to \pm\infty} (e^{itL} T_0^* e^{-itL} f)(x) = \pm \pi i \overline{K(x, x)} f(x)$

where $e^{itL} f(x) = e^{itx} f(x)$, and iii) and iv) hold x-a.e.

Proof. We have

$$(T_\varepsilon f, g) = \int_{X_\varepsilon} dxdy \frac{f(x)\overline{g(y)}}{x-y} \int dpdq\, e^{i(px+qy)} \hat{K}(p,q)$$

$$= \int dpdq\, \hat{K}(p,q) \int_{X_\varepsilon} dxdy \frac{e^{ipx} f(x) e^{iqy}\overline{g(y)}}{x-y} \qquad (5)$$

since $\frac{f(x)\overline{g(y)}}{x-y}$ is integrable for $|x-y| \geq \varepsilon$.

$X_\varepsilon = \{(x,y) : |x-y| \geq \varepsilon\}$ as before.

Let us define the operator A_ε by

$$(A_\varepsilon f, g) = \int_{X_\varepsilon} dxdy \frac{f(x)\overline{g(y)}}{x-y} \qquad (6)$$

It is well-known from the theory of Hilbert transforms (5, p. 1044) that $\text{s-}\lim_{\varepsilon \to 0+} A_\varepsilon = A_0$ exists. Hence $\lim_{\varepsilon \to 0+} (A_\varepsilon f, g)$ exists and there must be a constant $\gamma > 0$ such that $|(A_\varepsilon f, g)| \leq \gamma \|f\| \, \|g\|$ (7)
Thus, by applying the dominated convergence theorem in 5 we conclude that

$$\lim_{\varepsilon \to 0+} (T_\varepsilon f, g) = \int dpdq\, \hat{K}(p,q)\, (A_0\, e^{ipL} f,\; e^{-iqL} g)$$

$$= (T_0 f, g), \qquad (8)$$

which proves assertion i). Assertion ii) follows from the fact $\|A_0\| = \pi$.

Let $\hat{f}$ denote the Fourier transform of f, that is,

$$\hat{f}(\eta) = \lim_{M \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-M}^{M} e^{-i\eta x} f(x) dx \quad ,$$

where the limit is taken in the L_2 sense. We know from the theory of Hilbert transforms that

$$(A_0 f, g) = \pi i \int d\eta \, \mathrm{sgn}(\eta) \, \hat{f}(\eta) \, \overline{\hat{g}(\eta)} \tag{9}$$

Hence

$$(T_0 e^{itL} f, \; e^{itL} g) = \pi i \int dp dq \, \hat{K}(p,q) \int d\eta \, \mathrm{sgn}(\eta+t) \hat{f}(\eta - p) \, \overline{\hat{g}(\eta+q)} \; , \tag{10}$$

where we have freely made use of the fact that the Fourier transform of $e^{iax} f(x)$ is $\hat{f}(x-a)$. Observe that if we fix $\hat{f}$ of compact support and restrict p to a bounded interval I, there exists $M > 0$ such that $|t| > M$ implies

$$\int d\eta \, \mathrm{sgn}(\eta+t) \, \hat{f}(\eta-p) \, \overline{\hat{g}(\eta+q)} = \pm \int d\eta \, \hat{f}(\eta-p) \, \overline{\hat{g}(\eta+q)}$$

independently of q or $\hat{g}$. That is,

$$\lim_{t \to \pm\infty} \int d\eta \, \mathrm{sgn}(\eta+t) \, \hat{f}(\eta-p) \, \overline{\hat{g}(\eta+q)} = \pm \int d\eta \, \hat{f}(\eta-p) \, \overline{\hat{g}(\eta+q)}$$

uniformly for fixed $\hat{f}$ of compact support and $p \in I$, $-\infty < q < \infty$, and $\|\hat{g}\| \leq 1$. Since functions of compact support are dense in L_2, the restriction that $\hat{f}$ has compact support can be removed.

Suppose that $\hat{K}(p,q)$ vanishes for $p \notin I$. Then, using 10 and the fact that the Fourier transform is L_2-norm preserving, we conclude that $\lim_{t \to \pm\infty} (T_0 e^{itL} f, \; e^{itL} g)$ exists uniformly for fixed f and $\|g\| \leq 1$, and equals

$$\pm \pi i \int dp\, dq\, \hat{K}(p,q) \int d\eta\, \hat{f}(\eta - p)\, \hat{g}(\eta+q)$$

That is, $\text{s-}\lim_{t \to \pm\infty} e^{itL}\, T_0\, e^{-itL} f$ exists (11)

We now remove the restriction that $\hat{K}(p,q)$ vanish for $p \notin I$. Define for $n = 1, 2, \ldots$

$$\hat{K}_n(p,q) = \begin{cases} \hat{K}(p,q) & p \in [-n, n] \\ 0 & \text{otherwise} \end{cases}$$

Then $\lim_{n \to \infty} \int dpdq\, |\hat{K}(p,q) - \hat{K}_n(p,q)| = 0$.
Corresponding to $\hat{K}_n(p,q)$ we can define the operator $T_0^{(n)}$. Assertion ii) gives

$$\lim_{n \to \infty} \| T_0 - T_0^{(n)} \| = 0 .$$

This result allows us to extend 11 to all integrable functions $\hat{K}(p,q)$.

Using well-known properties of the Fourier transform we obtain

$$\int d\eta\, \hat{f}(\eta - p)\, \overline{\hat{g}(\eta+q)} = \int dx\, e^{ipx} f(x)\, e^{iqx}\, \overline{g(x)} ,$$

from which it follows that

$$\lim_{t \to \pm\infty} (T_0\, e^{-itL} f,\ e^{-itL} g)$$

$$= \mp \pi i \int dpdq\, \hat{K}(p,q) \int dx\, e^{ipx}\, e^{iqx} f(x)\, \overline{g(x)}$$

$$= \mp \pi i \int dx\, f(x)\, \overline{g(x)} \int dpdq\, \hat{K}(p,q)\, e^{i(px+qx)}$$

$$= \mp \pi i \int dx\, f(x)\, \overline{g(x)}\, K(x,x) .$$

This completes the proof of assertion iii). The proof of iv) is achieved by simply replacing conditions of the form 'the limit is uniform for fixed $\hat{f}$ and $\|\hat{g}\| \leq 1$' with 'the limit is uniform for fixed $\hat{g}$ and $\|\hat{f}\| \leq 1$' and making other minor changes in the preceding proof.

Corollary 1.12. If f and g are square integrable functions, then the following statements are true:

$$\text{i) } \lim_{\varepsilon \to 0+} \int_{X_\varepsilon} dxdy \, \frac{f(y)\, \overline{g(x)}}{x-y} \text{ exists}$$

uniformly for fixed f and $\|g\| \leq 1$

$$\text{ii) } \lim_{t \to \pm\infty} \lim_{\varepsilon \to 0} \int_{X_\varepsilon} dxdy \, e^{i(x-y)t} \frac{f(y)\overline{g(x)}}{x-y}$$

$$= \pm \pi i \int dx \, f(x)\, \overline{g(x)}, \text{ where}$$

the t-limit is uniform for fixed f and $\|g\| \leq 1$.

Proof: i) is an immediate consequence of Hilbert transform theory.

The proof of ii) is a minor modification of the argument given on page 25.

It is interesting that the class of operators T_0 defined in Theorem 1.10 form a Banach space if we set

$$\|T_0\|_* = \int dpdq \, |\tilde{K}(p,q)| \, .$$

The verification of the axioms is straightforward.

Our second example deals with the problem of extending Corollary 1.12 to general Bochner spaces $\mathcal{H}_i = L_2(-\infty,\infty;N_i)$ $(i=1,2)$.

Example 1.13. Consider functions $h : R \to N_1$ and $k : R \to N_2$, satisfying

$$\sup_y |h(y)|_1 \leq 1 \text{ and } \sup_x |k(x)|_2 \leq 1 \quad . \tag{12}$$

Then, if $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$, $\langle f(y), h(y)\rangle_1$ and $\langle k(x), g(x)\rangle_2 \in L_2(-\infty,\infty)$ and

$$\int dy\ |\langle f(y),\ h(y)\rangle_1|^2 \leq \|f\|_1\ ,\ \int dx\ |\langle k(x), g(x)\rangle_2|^2 \leq \|g\|_2\ .$$

We define $T_\varepsilon : \mathcal{H}_1 \to \mathcal{H}_2$ $(\varepsilon > 0)$ by

$$(T_\varepsilon f, g) = \int_{X_\varepsilon} dxdy \frac{1}{x-y} \langle f(y), h(y)\rangle_1 \langle k(x), g(x)\rangle_2 \tag{13}$$

Let $T_0 = \underset{\varepsilon \to 0+}{\text{s-lim}}\ T_\varepsilon$, which we know exists by Corollary 1.12 i). Furthermore, by equation 7 we have

$$\|T_\varepsilon\| \leq \gamma \quad (\varepsilon \geq 0) \tag{14}$$

Lastly, Corollary 1.12 ii) implies

$$\underset{t \to \pm\infty}{\text{s-lim}}\ e^{itL}\ T_0\ e^{-itL} \text{ exists, and}$$

$$\lim_{t \to \pm\infty} (e^{itL}\ T_0\ e^{-itL} f, g) = \pm \pi i \int dx\ \langle f(x), h(x)\rangle_1 \langle k(x), g(x)\rangle_2 \tag{15}$$

We make a further extension as follows. Choose h_i, k_i $(i = 1, 2, \ldots)$ as in 12 and real numbers $\alpha_i > 0$ satisfying $\sum_{i=1}^{\infty} \alpha_i < \infty$ (16)

Define $T_{\varepsilon,i} : \mathcal{H}_1 \to \mathcal{H}_2$ as in 13, and put $T_{0,i} = \text{s-}\lim_{\varepsilon\to 0+} T_{\varepsilon,i}$ $(i=1,2,\ldots)$.

Let $K_{fg}(x,y) = \frac{1}{x-y} \sum_{i=1}^{\infty} \alpha_i \langle f(y), h_i(y)\rangle_1 \langle k_i(x), g(x)\rangle_2$

and define $T_\varepsilon : \mathcal{H}_1 \to \mathcal{H}_2$ by

$(T_\varepsilon f, g) = \int_{X_\varepsilon} dxdy\, K_{fg}(x,y)$; that is $T_\varepsilon = \sum_{i=1}^{\infty} \alpha_i T_{\varepsilon,i}$. It follows from 13, 14, and 16 that

$$T_0 = \text{s-}\lim_{\varepsilon\to 0+} T_\varepsilon \text{ exists and equals } \sum_{i=1}^{\infty} \alpha_i T_{0,i}.$$

This means in particular that $T_0 \in \mathcal{C}_2$.

Now, combining 14, 15, and 16 we deduce that

$$\text{s-}\lim_{t\to\pm\infty} e^{itL} T_0 e^{-itL} \text{ exists, and that}$$

$$\lim_{t\to\pm\infty} (e^{itL} T_0 e^{-itL} f, g) = \pm\pi i \int dx \sum_{i=1}^{\infty} \alpha_i \langle f(x), h_i(x)\rangle_1 \langle k_i(x), g(x)\rangle_2 .$$

Remark 1.14. We have shown that the operators T_0 constructed in Theorem 1.11 and example 1.13 belong to $\mathcal{C}_2$. They also belong to $\mathcal{C}_2'$.

For the proof, we need only modify Theorem 1.11 and Corollary 1.12 by changing the regions of integration from X_ε to X'_ε (see fig. 1, p. 45).

We proceed by choosing smooth functions f and g which are dense in $L_2(-\infty,\infty)$, and show that

$$\lim_{\varepsilon\to 0+} \int_{X_\varepsilon} dxdy \frac{K(x,y)}{x-y} f(x)\overline{g(y)} = \lim_{\varepsilon\to 0+} \int_{X'_\varepsilon} dxdy \frac{K(x,y)}{x-y} f(x)\overline{g(y)} .$$

We use the fact that $K(x,y)$ is continuous. The extension to $L_2(-\infty,\infty)$ is immediate. Corollary 1.12 is modified in an analogous fashion.

SECTION II

Let $H = \int \lambda dE_\lambda$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$. The absolutely continuous subspace $\mathcal{H}_{ac}$ corresponding to H consists of those $f \in \mathcal{H}$ for which $\|E_\lambda f\|^2$ is an absolutely continuous function of λ. That $\mathcal{H}_{ac}$ is a linear manifold is immediate. We show that $\mathcal{H}_{ac}$ is closed by proving $\|f\|^2 = \int d\lambda \frac{d}{d\lambda} \|E_\lambda f\|^2$, given that $\|f_n\|^2 = \int d\lambda \frac{d}{d\lambda} \|E_\lambda f_n\|^2$ and $\|f-f_n\|^2 \to 0$ $(n = 1, 2, \dots)$. The proof follows from the fact that $\|f_n\| \to \|f\|^2$, and

$$\left| \int d\lambda \frac{d}{d\lambda} \|E_\lambda f\|^2 - \int d\lambda \frac{d}{d\lambda} \|E_\lambda f_n\|^2 \right| = \left| \int d\lambda \frac{d}{d\lambda} [(f-f_n, E_\lambda f) + (E_\lambda f_n, f-f_n)] \right|$$

$$\leq \|f-f_n\| \ \|f\| + \|f_n\| \ \|f-f_n\| .$$

Let P be the projection on $\mathcal{H}_{ac}$ and let $E(\cdot)$ denote the spectral measure corresponding to H, ($E(A) = \int_A dE_\lambda$, A Borel measurable). If $f \in \mathcal{H}_{ac}$ is also in the domain of H, then

$$\|E(A)Hf\|^2 \leq \int_A \lambda^2 d\ \|E_\lambda f\|^2 < \infty$$

for all Borel measurable sets A. We conclude $Hf \in \mathcal{H}_{ac}$ and $HP = PH$ is a self-adjoint operator in $\mathcal{H}_{ac}$ (or in $\mathcal{H}$).

The theory of spectral representation for self-adjoint operators (5, p. 1205) states that $\mathcal{H}$ can be represented as a Hilbert space of the form $\sum_\alpha \oplus L_2(\mu_\alpha)$ in which H becomes the simple multiplication operator. More specifically, there exists a family $\{f_\alpha\}$ of elements of $\mathcal{H}$ and a linear isometry U mapping $\mathcal{H}$ onto $\sum_\alpha \oplus L_2(\mu_\alpha)$, where $\mu_\alpha(A) = \int_A d\|E_\lambda f_\alpha\|^2$, such that $UHU^* = L$, $\left[Lg(\lambda) = \lambda g(\lambda),\ g \in \sum_\alpha \oplus L_2(\mu_\alpha)\right]$. $\{f_\alpha\}$ is a maximal family with the property that

$\alpha \neq \alpha'$ implies $(E(A)f_\alpha, f_{\alpha'}) = 0$ for all Borel measurable sets A.

Applying the representation theory to the restriction of H to $\mathcal{H}_{ac}$, we find that the measures μ_α are all absolutely continuous with respect to Lebesgue measure. This means that $\sum_\alpha \oplus L_2(\mu_\alpha)$ can be represented as a subspace of $L_2(-\infty, \infty; N)$, where N is the Hilbert space of complex-valued functions b_α on the index set $\{\alpha\}$, satisfying $\sum_\alpha |b_\alpha|^2 < \infty$. The mapping we have in mind is given by $b_\alpha(\lambda) = g_\alpha(\lambda)\left(\frac{d\mu_\alpha}{d\lambda}\right)^{\frac{1}{2}}$. Since $\frac{d\mu_\alpha}{d\lambda}$ may vanish, for each λ only a subspace N_λ of N figures in the mapping. It is evident that $\mathcal{H}_{ac}$ can be represented as a subspace $\int d\lambda N_\lambda$ of $L_2(-\infty, \infty; N)$ in which HP becomes the simple multiplication operator.

Let H_i $(i = 0,1)$ be a self-adjoint operator in $\mathcal{H}$ and let P_i be the projection on its absolutely continuous subspace. Let $\mathcal{H}_i = \int d\lambda N_{i,\lambda}$ be a representation space for $H_i P_i$ and $U_i : P_i \mathcal{H} \to \mathcal{H}_i$ be the corresponding isometry. We extend U_i to a partial isometry mapping $\mathcal{H}$ into $L_2(-\infty, \infty; N_i)$ by letting components perpendicular to $P_i\mathcal{H}$ be mapped into 0. Then $U_1 U_0^*$ maps $L_2(-\infty, \infty; N_0)$ into $L_2(-\infty, \infty; N_1)$ and has norm ≤ 1 although it may fail to be even a partial isometry. Furthermore $P_1 \exp(itH_1)\exp(-itH_0)P_0 = U_1^* e^{itL} U_1 U_0^* e^{-itL} U_0$, and we can apply the theory of Section I to the operator $U_1 U_0^*$.

We first restate the criterion for membership in the class $\mathcal{C}_1$ in terms of the spectral measures $E_i(\cdot)$ corresponding to H_i $(i = 0,1)$. Let $h \in L_2(-\infty, \infty; N_0)$ and $k \in L_2(-\infty, \infty; N_1)$. Then for Borel sets A and B $(U_1 U_0^* \chi_A h, \chi_B k) = (U_0^* \chi_A h, U_1^* \chi_B k) = (E_0(A) U_0^* h, E_1(B) U_1^* k)$. We conclude:

$U_1 U_0^* \in \mathcal{C}_1 \Longleftrightarrow$ for each $f, g \in \mathcal{H}$, there exists a complex-valued function $K_{fg}(x,y)$ defined on R^2 such that $K_{fg} \in L_1(I \times J)$ and

$$(E_0(I)P_0 f,\ E_1(J)P_1 g) = \int_{I\times J} dxdy\ K_{fg}(x,y) \tag{17}$$

for every pair of disjoint compact intervals I, J.

In case $H_0 P_0 = H_0$, there is no loss of generality in letting $H_0 = L$ in the Hilbert space $\int d\lambda N_\lambda$, where for each λ, N_λ is a subspace of N. We shall deal with the case in which H_1 is a self-adjoint extension of $H_0 + V_1$, and V_1 is a symmetric operator in $\int d\lambda N_\lambda$. Since $L_2(-\infty,\infty;N) = (\int d\lambda N_\lambda) \oplus (\int d\lambda N_\lambda)^\perp$, $H = H_1 \oplus L'$ is a self-adjoint extension of $L + V$, where L' is the restriction of L to $(\int d\lambda N_\lambda)^\perp$ and $V = V_1 \oplus O$. Hence conclusions concerning K_{fg} for the operators H and L apply also to H_1 and H_0 if we restrict f and g to the subspace $\int d\lambda N_\lambda$.

We assume for the remainder of this section that $\mathcal{H} = L_2(-\infty,\infty;N)$ and L is the simple multiplication operator in $\mathcal{H}$. In keeping with Section I, we denote the norm and inner product for $\mathcal{H}$ by $\|\cdot\|$ and $(\cdot,\cdot)$, and for N by $|\cdot|$ and $\langle\cdot,\cdot\rangle$.

Lemmas 2.1 and 2.2 provide the background for Theorem 2.3 which is the first important result of the section.

<u>Lemma 2.1</u>. Let H be a self-adjoint operator on the Hilbert space $\mathcal{H}$. Let $R_\lambda = (\lambda I - H)^{-1}$ be the resolvent of H and E_μ the corresponding resolution of the identity. Then the limits

$$\lim_{\varepsilon\to 0+} (R_{\mu \pm i\varepsilon} f, g) \text{ exist } \mu\text{-a.e.}$$

for each $f, g \in \mathcal{H}$.

Denote these limits by $(R_{\mu \pm i0} f, g)$. If we assume in addition that f (or g) is in the absolutely continuous subspace $\mathcal{H}_{ac}$ of H, then for any interval J we have

$$(E(J)f, g) = \frac{1}{2\pi i} \int_J d\mu \, [(R_{\mu - i0} f, g) - (R_{\mu + i0} f, g)]$$

Proof. We begin with the well-known equation (5, p. 1196)

$$(R_\lambda f, g) = \int \frac{d_\nu (E_\nu f, g)}{\lambda - \nu}, \quad \operatorname{Im} \lambda \neq 0.$$

Observe that

$$4(E_\nu f, g) = \|E_\nu (f+g)\|^2 - \|E_\nu (f-g)\|^2 + i(\|E_\nu (f+ig)\|^2 - \|E_\nu (f-ig)\|^2) \tag{18}$$

and that each term on the right is an increasing function of ν.

The integral $(\lambda = \mu + i\varepsilon)$

$$\int \frac{d_\nu \|E_\nu h\|^2}{\lambda - \nu} = \int \frac{(\mu - \nu) d_\nu \|E_\nu h\|^2}{(\mu - \nu)^2 + \varepsilon^2} - i\varepsilon \int \frac{d_\nu \|E_\nu h\|^2}{(\mu - \nu)^2 + \varepsilon^2}$$

defines an analytic function $\phi(\lambda)$ in the upper half plane having negative imaginary part. Hence $\psi(\lambda) = \frac{1}{i - \phi(\lambda)}$ defines an analytic function whose modulus is bounded by one. It is known (23) that under these conditions $\lim_{\varepsilon \to 0+} \psi(\mu + i\varepsilon)$ exists μ-a.e. and equals zero only on a null set. From this we conclude that $\lim_{\varepsilon \to 0+} \phi(\mu + i\varepsilon)$ exists μ-a.e. and that $\lim_{\varepsilon \to 0+} (R_{\mu + i\varepsilon} f, g)$ exists μ-a.e. because of 18.

To prove the second assertion of the lemma we argue that if f or g $\in \mathcal{H}_{ac}$, then $(E_\nu f, g)$ is an absolutely continuous function of ν. Let $\rho(\nu)$ be its Radon-Nikodym derivative. Then

$$(R_{\mu-i0}f, g) - (R_{\mu+i0}f, g) = \lim_{\varepsilon \to 0+} 2i \int \frac{\varepsilon \, d_\nu (E_\nu f, g)}{(\mu-\nu)^2 + \varepsilon^2}$$

$$= \lim_{\varepsilon \to 0} 2i \int \frac{\varepsilon \, \rho(\nu) d\nu}{(\mu-\nu)^2+\varepsilon^2} = 2\pi i \, \rho(\mu) \quad \mu\text{-a.e.} \; ,$$

where the last equality is obtained by using a standard argument (9, Chapter 8) based upon the fact that ρ is integrable and $\frac{1}{\pi} \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2}$ is an approximate identity. Furthermore we obtain

$$(E(J)f, g) = \int_J d\mu \; \rho(\mu) \; ,$$

which completes the proof.

Lemma 2.2. Let E_μ be a resolution of the identity in the Hilbert space $\mathcal{H}$. Let v(x) be a strongly measurable function from the real numbers into $\mathcal{H}$ such that $\int_J dx \; \|v(x)\| < \infty$ for every finite interval J. Then for each x:

i) $\frac{d}{d\mu} (E_\mu f, v(x)) = w(\mu, x)$ exists μ-a.e.

ii) $\left|\frac{d}{d\mu} (E_\mu f, v(x))\right| \leqslant \left(\frac{d}{d\mu} \; \|E_\mu f\|^2\right)^{\frac{1}{2}} \left(\frac{d}{d\mu} \; \|E_\mu v(x)\|^2\right)^{\frac{1}{2}}$ μ-a.e.

iii) $\int_J dx \int \quad d\mu \; |w(\mu, x)| \leqslant \|f\| \int_J dx \; \|v(x)\|$

Proof. Assertion i) follows from equation 18 with v(x) replacing g.

To prove ii), observe that

$$|(E_\nu f, v(x)) - (E_\mu f, v(x))| = |((E_\nu - E_\mu)f, v(x))|$$

$$= |(E_\nu - E_\mu)f, (E_\nu - E_\mu)v(x))| \leqslant \|(E_\nu - E_\mu)f\| \; \|(E_\nu - E_\mu)v(x)\|$$

$$= \left| \|E_\nu f\|^2 - \|E_\mu f\|^2 \right|^{\frac{1}{2}} \left| \|E_\nu v(x)\|^2 - \|E_\mu v(x)\|^2 \right|^{\frac{1}{2}} .$$

Now, whenever μ is a point where all three derivatives in ii) exist, we can divide the above inequalities by $|\nu-\mu|$ and take the limit as $\nu\to\mu$ which yields ii).

The following application of Schwarz's inequality yields iii)

$$\int_J dx \int d\mu\ |w(\mu,x)| \leq \int_J dx \int d\mu \left(\frac{d}{d\mu}\ \|E_\mu f\|^2\right)^{\frac{1}{2}}\left(\frac{d}{d\mu}\ \|E_\mu v(x)\|^2\right)^{\frac{1}{2}}$$

$$\leq \int_J dx\left(\int d\mu\ \frac{d}{d\mu}\ \|E_\mu f\|^2\right)^{\frac{1}{2}}\left(\int d\mu\ \frac{d}{d\mu}\ \|E_\mu v(x)\|^2\right)^{\frac{1}{2}}$$

$$\leq \int_J dx\ \|f\|\ \|v(x)\| < \infty\ .$$

In Theorem 2.3 we remain consistent with our previous notation by letting E_0, E be the spectral measures for H_0, H and $\mathcal{H}_{ac}$ be the absolutely continuous subspace corresponding to H. The proof is dependent upon the two lemmas which follow it.

Theorem 2.3. Let V be a symmetric linear operator in $\mathcal{H}$ whose domain includes that of $H_0 = L$. Suppose that for each f in the domain of V and each $g \in \mathcal{H}$, $\langle Vf(x), g(x)\rangle = (f, v_g(x))$ x-a.e., where $v_g(\cdot)$ is a locally strongly integrable $\mathcal{H}$-valued function on the real line. Furthermore, assume H = L+V is self-adjoint.

Then for $f \in \mathcal{H}_{ac}$, $g \in \mathcal{H}$ and disjoint compact intervals I, J

$$(E(I)f,\ E_0(J)g) = \int_{I\times J} d\mu\ dx\ \frac{w_{fg}(\mu,x)}{\mu-x} \tag{19}$$

where $w_{fg}(\mu,x) = \frac{d}{d\mu}\,(E_\mu f, v_g(x))$.

Proof. Since H_0 and H are self-adjoint, the resolvents $R_\lambda^0 = (\lambda I-H_0)^{-1}$ and $R_\lambda = (\lambda I-H)^{-1}$ exist and are bounded operators on $\mathcal{H}$ for Im $\lambda \neq 0$.

Since H_0 and H have common domain we have the resolvent equation $R_\lambda = R_\lambda^0 + R_\lambda^0 V R_\lambda$.

It is clear that $R_\lambda^0 f(x) = \frac{f(x)}{\lambda - x}$, and $E_0(J)f(x) = \chi_J(x)f(x)$. Hence, for $\mathrm{Im}\, \lambda \neq 0$, $(R_\lambda f, E_0(J)g) = \int_J dx \langle R_\lambda f(x), g(x)\rangle$

$$= \int_J dx \left\langle \frac{f(x)}{\lambda - x} + \frac{VR_\lambda f(x)}{\lambda - x}, g(x)\right\rangle$$

$$= \int_J dx \left\{ \frac{\langle f(x), g(x)\rangle}{\lambda - x} + \frac{\langle VR_\lambda f(x), g(x)\rangle}{\lambda - x} \right\}$$

$$= \int_J dx \frac{\langle f(x), g(x)\rangle}{\lambda - x} + \int_J dx \frac{(R_\lambda f, v_g(x))}{\lambda - x} \tag{20}$$

If μ is a positive distance from J,

$$\lim_{\varepsilon \to 0+} \int_J dx \frac{\langle f(x), g(x)\rangle}{\mu \pm i\varepsilon - x} = \int_J dx \frac{\langle f(x), g(x)\rangle}{\mu - x} . \tag{21}$$

By Lemma 2.1, $(R_{\mu \pm i0} f, E_0(J)g)$ exists μ-a.e. Thus we can conclude from 20 and 21 that

$$\lim_{\varepsilon \to 0+} \int_J dx \frac{(R_{\mu \pm i\varepsilon} f, v_g(x))}{\mu \pm i\varepsilon - x} \quad \text{exists } \mu\text{-a.e.},$$

and

$$(R_{\mu - i0} f, E_0(J)g) - (R_{\mu + i0} f, E_0(J)g)$$

$$= \lim_{\varepsilon \to 0+} \int_J dx \left\{ \frac{(R_{\mu - i\varepsilon} f, v_g(x))}{\mu - i\varepsilon - x} - \frac{(R_{\mu + i\varepsilon} f, v_g(x))}{\mu + i\varepsilon - x} \right\} \tag{22}$$

To compute the last limit we begin with the equation

$$(R_{\mu\pm i\varepsilon} f, v_g(x)) = \int \frac{d_\nu (E_\nu f, v_g(x))}{\mu \pm i\varepsilon - \nu}$$

$$= \int \frac{\mu - \nu}{(\mu-\nu)^2+\varepsilon^2} \, d_\nu (E_\nu f, v_g(x)) \mp i \int \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2} \, d_\nu (E_\nu f, v_g(x)) .$$

Then

$$\int_J dx \left\{ \frac{(R_{\mu-i\varepsilon} f, v_g(x))}{\mu - i\varepsilon - x} - \frac{(R_{\mu+i\varepsilon} f, v_g(x))}{\mu + i\varepsilon - x} \right\}$$

$$= \int_J dx \left(\frac{1}{\mu - i\varepsilon - x} - \frac{1}{\mu + i\varepsilon - x} \right) \int \frac{\mu - \nu}{(\mu-\nu)^2+\varepsilon^2} \, d_\nu (E_\nu f, v_g(x))$$

$$+ i \int_J \left(\frac{1}{\mu - i\varepsilon - x} + \frac{1}{\mu + i\varepsilon - x} \right) \int \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2} \, d_\nu (E_\nu f, v_g(x)) . \tag{23}$$

If we take the limit as $\varepsilon \to 0+$ in 23, the first term on the right tends to 0 by Lemma 2.4, and the second term tends to $\int_J dx \frac{2\pi i}{\mu - x} \frac{d}{d\mu} (E_\mu f, v_g(x))$ μ-a.e. by Lemma 2.5.

Thus 22 becomes

$$(R_{\mu - i0} f, E_0(J)g) - (R_{\mu + i0} f, E_0(J)g) = 2\pi i \int_J dx \frac{w_{fg}(\mu, x)}{\mu - x} \quad \mu\text{-a.e.},$$

as long as μ remains a positive distance from J. This condition is certainly satisfied if $\mu \in I$. Then Lemma 2.1 yields

$$(E(I)f, \; E_0(J)g) = \int_I d\mu \int_J dx \frac{w_{fg}(\mu, x)}{\mu - x} \tag{24}$$

By Lemma 2.2, $w_{fg}(\mu,x)$ is locally integrable and therefore $\frac{w_{fg}(\mu,x)}{\mu-x}$ is integrable over $I \times J$. Consequently the right side of 24 can be written as a double integral yielding 19.

<u>Lemma 2.4.</u> Let E_μ be a resolution of the identity in $\mathcal{H}$ with corresponding absolutely continuous subspace $\mathcal{H}_{ac}$. Let v(x) be a locally strongly integrable function from the real numbers into $\mathcal{H}$.

Then, for $f \in \mathcal{H}_{ac}$ and μ a positive distance from the bounded interval J,

$$\lim_{\varepsilon \to 0+} \int dx \frac{\varepsilon}{(\mu-x)^2+\varepsilon^2} \int \frac{\mu-\nu}{(\mu-\nu)^2+\varepsilon^2} d_\nu (E_\nu f, v(x)) = 0$$

Proof. Since $f \in \mathcal{H}_{ac}$, $(E_\mu f, v(x))$ is an absolutely continuous function of μ and $\frac{d}{d\mu}(E_\mu f, v(x)) = w(\mu,x)$ exists μ-a.e. By Lemma 2.2,

$$\int_J dx \int d\nu \ |w(\nu,x)| < \infty$$

Hence $\displaystyle\int_J dx \frac{\varepsilon}{(\mu-x)^2+\varepsilon^2} \int \frac{\mu-\nu}{(\mu-\nu)^2+\varepsilon^2} d_\nu (E_\nu f, v(x))$

$$= \int_J dx \int d\nu \frac{\varepsilon(\mu-\nu)}{[(\mu-x)^2+\varepsilon^2][(\mu-\nu)^2+\varepsilon^2]} w(\nu,x) .$$

Observe that the integrand in the last integral approaches zero as $\varepsilon \to 0$ for each choice of μ, ν, and x. Hence, if we could show that the integrand is dominated by an integrable function of (ν, x) which does not depend on ε, our result would follow by the dominated convergence theorem.

It is clear that $\left|\frac{\varepsilon(\mu-\nu)}{(\mu-\nu)^2+\varepsilon^2}\right| \leq \frac{1}{2}$.

Consequently $\left| \frac{\varepsilon(\mu-\nu)\, w(\nu,x)}{[(\mu-x)^2+\varepsilon^2][(\mu-\nu)^2+\varepsilon^2]} \right| \leqslant \frac{|w(\nu,x)|}{2\,\inf\{|\mu-x|;x\epsilon J\}}$

which completes the proof.

<u>Lemma 2.5.</u> Under the assumptions of Lemma 2.4,

$$\lim_{\varepsilon\to 0+} \int_J dx \frac{\mu-x}{(\mu-x)^2+\varepsilon^2} \int \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2}\, d_\nu (E_\nu f, v(x))$$

$$= \int_J dx \frac{\pi}{\mu-x} \frac{d}{d\mu} (E_\mu f, v(x)) \quad \mu\text{-a.e.}$$

Proof. Since $f \epsilon \mathcal{H}_{ac}$, $(E_\mu f, v(x))$ is an absolutely continuous function of μ. Therefore $\frac{d}{d\mu}(E_\mu f, v(x)) = w(\mu,x)$ exists μ-a.e., and

$$\int \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2}\, d_\nu (E_\nu f, v(x)) = \int d\nu \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2}\, w(\nu,x) \quad .$$

Then

$$\left| \int_J dx \frac{\mu-x}{(\mu-x)^2+\varepsilon^2} \int d\nu \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2}\, w(\nu,x) - \int_J dx \frac{\pi}{\mu-x}\, w(\mu,x) \right|$$

$$\leqslant \int_J dx \left| \frac{\mu-x}{(\mu-x)^2+\varepsilon^2} - \frac{1}{\mu-x} \right| \int d\nu \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2} |w(\nu,x)|$$

$$+ \left| \int_J dx \frac{1}{\mu-x} \int d\nu \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2}\, w(\nu,x) - \int_J dx \frac{\pi}{\mu-x}\, w(\mu,x) \right|$$

We now use the result of Lemma 2.2, that $\int_J dx \int d\nu\ |w(\nu,x)| < \infty$, to obtain

$$\int_J dx \left| \frac{\mu-x}{(\mu-x)^2+\varepsilon^2} - \frac{1}{\mu-x} \right| \int d\nu \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2} |w(\nu,x)|$$

$$= \int_J dx \int d\nu \frac{\varepsilon^2}{[(\mu-x)^2+\varepsilon^2] \, |\mu-x|} \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2} |w(\nu,x)|$$

$$\leqslant \frac{\varepsilon}{[\inf\{|\mu-x| : x\in J\}]^3} \int_J dx \int d\nu \, |w(\nu,x)|$$

which tends to zero as $\varepsilon \to 0+$.

Next we show that μ-a.e.,

$$\lim_{\varepsilon\to 0+} \int_J dx \frac{1}{\mu-x} \int d\nu \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2} w(\nu,x) = \int_J dx \frac{\pi}{\mu-x} w(\mu,x) \qquad (25)$$

Let I be any bounded interval. Since $\int d\nu \int_I dx |w(\nu,x)| < \infty$ and $\frac{1}{\pi} \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2}$ is an approximate identity, it follows that

$$\lim_{\varepsilon\to 0+} \int_I dx \int d\nu \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2} w(\nu,x) = \pi \int_I dx \, w(\mu,x) \quad \mu\text{-a.e.}$$

Hence if $s(x)$ is a stepfunction on J, we have

$$\lim_{\varepsilon\to 0+} \int_J dx \, s(x) \int d\nu \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2} w(\nu,x) = \pi \int_J dx \, s(x) \, w(\mu,x) \ \mu\text{-a.e.} \qquad (26)$$

Denote the expression whose limit we are taking by $F(s,\varepsilon,\mu)$. Then

$$|F(s,\varepsilon,\mu)| \leqslant \sup_x |s(x)| \int d\nu \frac{\varepsilon}{(\mu-\nu)^2+\varepsilon^2} \int_J dx \, |w(\nu,x)| ,$$

so that μ-a.e. there exists $M_\mu > 0$ such that

$$|F(s,\varepsilon,\mu)| \leqslant M_\mu \sup_x |s(x)| \tag{27}$$

With the help of 27, 26 can be extended by means of an elementary technique to the case where s is a uniform limit of stepfunctions. Thus 25 is proved.

Theorem 2.3 and statement 17 of this section together imply that the representation isometries U corresponding to HP and $U_0 = I$ corresponding to H_0 satisfy $UU_0^* \in \mathcal{C}_1$. Our next theorem shows that whenever V is of trace class, $UU_0^* \in \mathcal{C}_2'$.

Theorem 2.6. Suppose V is given by $Vf = \sum_{i=1}^{\infty} (f,h_i)h_i$, where $h_i \in \mathcal{H}$ $(i = 1, 2, \ldots)$ and

$$\sum_{i=1}^{\infty} \|h_i\|^2 < \infty \tag{28}$$

Then V satisfies the hypothesis of Theorem 2.3 with $v_g(x) = \sum_{i=1}^{\infty} h_i \langle h_i(x), g(x)\rangle$. In addition, if H, E_μ, and $w_{fg}(\mu, x)$ have the same meaning as in Theorem 2.3,

$$\lim_{\varepsilon \to 0+} \int_{X_\varepsilon'} d\mu \, dx \, \frac{w_{fg}(\mu, x)}{\mu - x} \quad \text{exists}$$

for each $f \in \mathcal{H}_{ac}$ and $g \in \mathcal{H}$. X_ε' is defined on p. 13.

Proof. Since $\|Vf\| \leqslant \sum_{i=1}^{\infty} |(f,h_i)| \, \|h_i\| \leqslant \|f\| \sum_{i=1}^{\infty} \|h_i\|^2$, and

$$(Vf, g) = (\sum_{i=1}^{\infty} (f,h_i)h_i, g) = \sum_{i=1}^{\infty} (f,h_i)(h_i, g) = (f, Vg),$$

V is bounded and self-adjoint. Therefore H is self-adjoint and has the same domain as L.

From 28 we deduce that $\sum_{i=1}^{\infty} |h_i(x)|^2 < \infty$ x-a.e., and then that

$$\sum_{i=1}^{\infty} |(f,h_i)| \, |h_i(x)| \leq \|f\| \sum_{i=1}^{\infty} \|h_i\| \; |h_i(x)| \leq \|f\| \Big(\sum_{i=1}^{\infty} \|h_i\|^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{\infty} |h_i(x)|^2\Big)^{\frac{1}{2}} < \infty$$

x-a.e.

Also,

$$\int dx \sum_{i=1}^{\infty} \|h_i\| \; |\langle h_i(x), g(x)\rangle| \leq \sum_{i=1}^{\infty} \|h_i\| \int dx \, |\langle h_i(x), g(x)\rangle|$$

$$\leq \sum_{i=1}^{\infty} \|h_i\|^2 \|g\| \; < \infty ,$$

which means that $v_g(x) = \sum_{i=1}^{\infty} h_i \langle h_i(x), g(x)\rangle$ converges strongly x-a.e.

Hence we are justified in writing

$$\langle Vf(x), g(x)\rangle = \langle \sum_{i=1}^{\infty} (f,h_i)\, h_i(x), g(x)\rangle$$

$$= \sum_{i=1}^{\infty} (f,h_i) \langle h_i(x), g(x)\rangle = (f, v_g(x)) \qquad \text{x-a.e.}$$

As in Theorem 2.3, we have

$$w_{fg}(\mu, x) = \frac{d}{d\mu} \sum_{i=1}^{\infty} (E_\mu f, h_i) \langle h_i(x), g(x)\rangle \; .$$

The differentiation can be carried out termwise if the series of derivatives is termwise integrable. This follows from 28 and

$$\int d\mu \; dx \; \Big|\frac{d}{d\mu} (E_\mu f, h_i)\Big| \; |\langle h_i(x), g(x)\rangle| = \int d\mu \, \Big|\frac{d}{d\mu}(E_\mu f, h_i)\Big| \int dx \, |\langle h_i(x), g(x)\rangle|$$

$$\leq \|f\| \; \|g\| \; \|h_i\|^2 \; .$$

In view of Corollary 1.7 it is sufficient to prove 29 for a dense set of f and g. Let $M > 0$ and define:

$$A_M = \{f : f \in \mathcal{H}_{ac}, \text{ and } \frac{d}{d\mu} \|E_\mu f\|^2 \leq M \quad (-\infty < \mu < \infty)\},$$

$$B_M = \{g : g \in \mathcal{H}, \text{ and } |g(x)|^2 \leq M \quad (-\infty < x < \infty)\} \quad .$$

It is clear that $\bigcup_M B_M$ is dense in $\mathcal{H}$, and since $\|f\|^2 = \int d\mu \frac{d}{d\mu} \|E_\mu f\|^2$, it follows that $\bigcup_M A_M$ is dense in $\mathcal{H}_{ac}$.

Hence suppose $f \in A_M$, $g \in B_M$. Then

$$\int d\mu \left|\frac{d}{d\mu} (E_\mu f, h_i)\right|^2 \leq M \|h_i\|^2 \quad \text{(Lemma 2.2, ii), and}$$

$$\int dx \left|\langle h_i(x), g(x)\rangle\right|^2 \leq M \|h_i\|^2, \quad (i = 1, 2, \ldots).$$

We conclude from Corollary 1.12 i) and Remark 1.14 that there exists $\gamma > 0$ such that

$$\left|\int_{X'_\varepsilon} d\mu dx \frac{1}{\mu - x} \frac{d}{d\mu} (E_\mu f, h_i) \langle h_i(x), g(x)\rangle\right| \leq \gamma M \|h_i\|^2 \quad (\varepsilon > 0,\ i = 1, 2, \ldots), \tag{29}$$

and that the limit of the integral on the left exists as $\varepsilon \to 0+$. Combining these facts with 28, we deduce that

$$\lim_{\varepsilon \to 0+} \int_{X'_\varepsilon} d\mu \frac{w_{fg}(\mu, x)}{\mu - x} \quad \text{exists and equals}$$

$$\sum_{i=1}^{\infty} \lim_{\varepsilon \to 0+} \int_{X'_\varepsilon} d\mu dx \frac{d}{d\mu} (E_\mu f, h_i) \langle h_i(x), g(x)\rangle \quad .$$

Corollary 2.7. Under the assumptions of Theorem 2.6

$$\lim_{t \to \pm\infty} \lim_{\varepsilon \to 0+} \int_{X'_\varepsilon} d\mu dx\, e^{i(\mu - x)t} \frac{w_{fg}(\mu, x)}{\mu - x} \quad \text{exists and equals}$$

$$\pm \pi i \sum_{j=1}^{\infty} \int dx \frac{d}{dx} (E_x f, h_j) \langle h_j(x), g(x) \rangle$$

Proof. We obtain from Corollary 1.12 ii) and Remark 1.14 that

$$\lim_{t\to\pm\infty} \lim_{\varepsilon\to 0+} \int_{X'_\varepsilon} d\mu \; dx \; e^{i(\mu-x)t} \frac{1}{\mu-x} \frac{d}{d\mu} (E_\mu f, h_j) \langle h_j(x), g(x) \rangle$$

$$= \pm \pi i \int dx \frac{d}{dx} (E_x f, h_j) \langle h_j(x), g(x) \rangle \qquad (j = 1, 2, \ldots) .$$

The proof then follows with the help of 28 and 29.

Note. Corollary 2.7 and Theorem 1.10 i) together imply that $\text{w-lim}_{t\to\pm\infty} e^{itL} UU_0^* e^{-itL}$ exist, which in turn implies the existence of

$$\text{w-lim}_{t\to\pm\infty} P \exp(itH) \exp(-itH_0).$$

It is known, however, from Rosenblum (26) and Kato (15), that strong limits exist in the last case. Armed with this fact, we reason in the reverse order to conclude that the t-limit in Corollary 2.7 is uniform for fixed g and $\|f\| \leqslant 1$. We are unable to obtain this result directly from our theory, possibly because we lack details concerning the nature of E_μ.

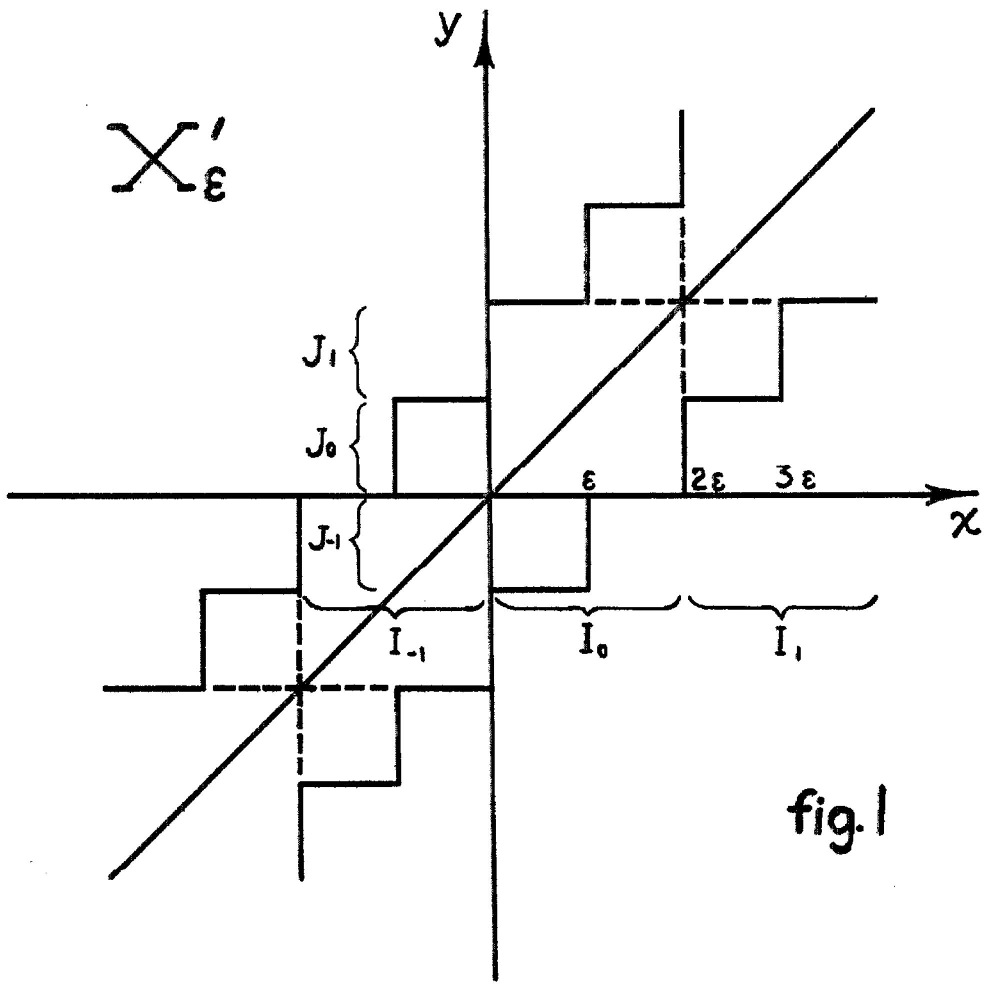

fig. 1

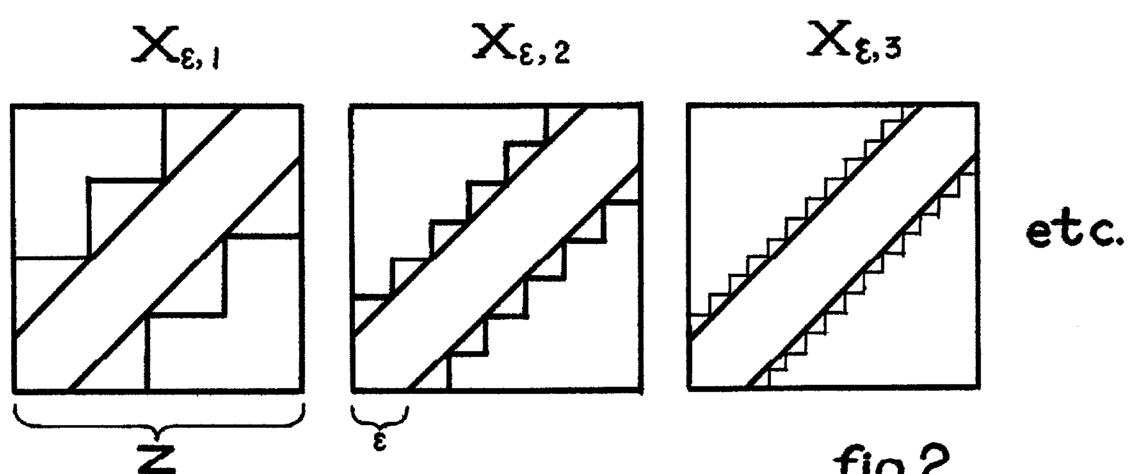

fig. 2

REFERENCES

(1) Birman, M. S., and Entina, S. B., A stationary approach in the abstract theory of scattering, Dokl. Akad. Nauk SSSR, Tom 155, 1964, pp. 506-508. = Soviet Math. Doklady, Vol. 5, 1964, pp. 432-435.

(2) de Branges, L., Perturbations of self-adjoint transformations, Amer. J. Math., Vol. 84, 1962, pp. 543-560.

(3) Browder, F. E., and Strauss, W. A., Scattering for nonlinear wave equations, Pacific J. Math., Vol. 13, 1963, pp. 23-43.

(4) Cook, J. M., Convergence to the Møller wave-matrix, J. Math. and Phys., Vol. 36, 1957, pp. 81-87.

(5) Dunford, N., and Schwartz, J., Linear Operators, Part II, Interscience Press, 1958.

(6) Friedrichs, K. O., On the perturbation of continuous spectra, Comm. Pure Appl. Math., Vol. 4, 1948, pp. 361-406.

(7) Gell-Mann, M., and Goldberger, M., Phys. Rev. 96, 1142 (1954).

(8) Hille, E., Functional Analysis and Semi-groups, New York, 1957.

(9) Hoffman, K., Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs, N. J., 1962.

(10) Ikebe, T., Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory, Arch. Rat. Mech. Anal., Vol. 5, 1960, pp. 1-34.

(11) Jauch, J. M., Theory of the scattering operator, Helvitica Physica, Vol. 31, 1958, pp. 127-158.

(12) Jauch, J. M., and Zinnes, I. I., Nuovo Cimento, Vol. 11, 1959.

(13) Kato, T., Fundamental properties of Hamiltonian operators of Schrödinger type, Trans. Amer. Math. Soc., Vol. 70, 1951, pp. 195-211.

(14) Kato, T., On finite dimensional perturbations of self-adjoint operators, J. Math. Soc. Japan, Vol. 9, 1957, pp. 239-249.

(15) Kato, T., Perturbation of continuous spectra by trace class operators, Proc. Japan Acad., Vol. 33, 1957, pp. 260-264.

(16) Kato, T., Wave operators and unitary equivalence, University of California, Berkeley, Technical report No. 11, 1963.

(17) Kuroda, S. T., On a theorem of Weyl-von Neumann, Proc. Japan Acad., Vol. 34, 1958, pp. 11-15.

(18) Kuroda, S. T., On the existence and unitary property of the scattering operator, Nuovo Cimento, Vol. 12, 1959, pp. 431-454.

(19) Kuroda, S. T., Perturbation of continuous spectra by unbounded operators. I, J. Math. Soc. Japan, Vol. 11, 1959, pp. 247-262.

(20) Kuroda, S. T., Finite dimensional perturbation and a representation of the scattering operator, University of California, Berkeley, Report No. CP-2, 1962.

(21) Lax, P. D., and Phillips, R. S., Scattering theory, Bull. Amer. Math. Soc., Vol. 70, 1964, pp. 130-142.

(22) Munroe, M. E., Introduction to Measure and Integration, Addison-Wesley, 1958.

(23) Nevanlinna, R., Eindeutige Analytische Funktionen, Berlin, 1936.

(24) Povsner, A. Ya., On the expansion of an arbitrary function in terms of eigenfunctions of the operator $-\Delta u+cu$, Mat. Sbornik, Vol. 32, 1953, pp. 109-156.

(25) Rejto, P., On gentle perturbations, I, Comm. Pure Appl. Math., Vol. 16, 1963, pp. 279-303.

(26) Rosenblum, M., Perturbation of the continuous spectrum and unitary equivalence, Pacific J. Math., Vol. 7, 1957, pp. 997-1010.

(27) Schatten, R., A Theory of Cross Spaces, Ann. Math. Studies, Princeton, 1950.

(28) Segal, I., Non-linear semi-groups, Ann. of Math., Vol. 78, 1963, pp. 339-364.

(29) Stone, M. H., Linear Transformations in Hilbert space, Amer. Math. Soc. Coll. Pub. Vol. XV, 1932.

(30) Zaanen, A. C., Linear Analysis, Interscience, New York, 1953.