I. THE TRANSIENT BOUNDARY LAYER PRODUCED BY A SINK ON A PLANE WALL

II. FLOW OF DUSTY GASES

Thesis by

Shakkottai P. Parthasarathy

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ABSTRACT

Part I

The solution for the problem of the transient boundary layers generated by a sink on a plane wall is obtained by an integral method. The incompressible flow is similar and the similarity solutions are obtained for the two dimensional and axisymmetric cases. The velocity layer reaches a steady state and the thermal layer does not. For large times, when the thermal layer is much thicker than the velocity layer, a solution for the temperature field is obtained ignoring the velocity layer. With some approximations to the flow near the sink, similar solutions for compressible flow are also obtained.

Part IIa

By using the integrated equations of motion, the development of a laminar, two-dimensional, dusty jet issuing from a slit is considered. The solutions are simple in the limits $\tau \rightarrow 0$ and $\tau \rightarrow \infty$, where τ is the particle relaxation time. For arbitrary τ, a numerical example is given. With some assumptions, the turbulent dusty jet is also considered.

Part IIb

There are three parameters in the problem of steady motion of a dusty gas around a sphere. These are the Reynolds number R, particle parameter σ and the mass concentration of dust f_∞. Solutions are obtained by the perturbation method by expanding in terms of R with

σ or σ/R fixed, in the limit $R \rightarrow 0$. Solutions are also obtained for the limit R tending to infinity with $f << 1$. In both cases critical values of σ exist, below which the sphere does not capture dust. The efficiency of capture as a function of σ is calculated in both cases.

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PART I

THE TRANSIENT BOUNDARY LAYER PRODUCED BY A SINK ON A PLANE WALL

1. INTRODUCTION

The problem arises in connection with the sampling of the fluid in contact with the end wall of a shock tube after the shock has been reflected away. The problem of normal shock reflection from a cold wall has been considered before by Goldsworthy [1]. The solution of this shows that for times such that the distance of the reflected shock wave from the end wall is much greater than the thickness of the thermal boundary layer, the dynamics is equivalent to that due to a cold wall being put in contact with a hot fluid of infinite extent. The sampling problem is posed in the following manner. A steady sink flow (line sink or point sink) is present in a hot fluid. At time $t = 0$, a plane wall is introduced into the fluid such that the sink is on the wall. The velocity and thermal boundary layers that develop on the wall are to be described.

The actual flow configuration, however, is like the one shown in Fig. 1.1. This shows the reflected shock wave moving away from the wall with velocity U_s, the disturbance produced by the sink moving radially outwards with speed a equal to the speed of sound in the hot gas and the boundary layers of velocity and temperature growing on the cold wall. Outside the region of influence of the sink, the thermal boundary layer is described by the Goldsworthy solution. Its thickness is of order $\sqrt{\nu t}$ where ν is the kinematic viscosity of the hot gas. The boundary layers on the wall for $x < at$ are expected to be described by the problem of a cold wall being suddenly introduced in a steady sink flow.

In the sections to follow these problems will be considered.

a) Line sink in incompressible flow

b) Point sink in incompressible flow

c) Line sink in compressible flow

d) Point sink in compressible flow.

The problem a) is considered in some detail in sections 2 and 3. The other three cases are very similar to a) and are therefore treated only briefly in sections 4, 5 and 6.

The effect of compressibility is small when the flow velocities are small compared to the velocity of sound. For any sink strength it is clear that close to the sink, the approximation of incompressibility breaks down. The flow close to the sink will not be radial and the viscous transonic flow near the sink is quite complicated. This difficult problem is not considered in this thesis.

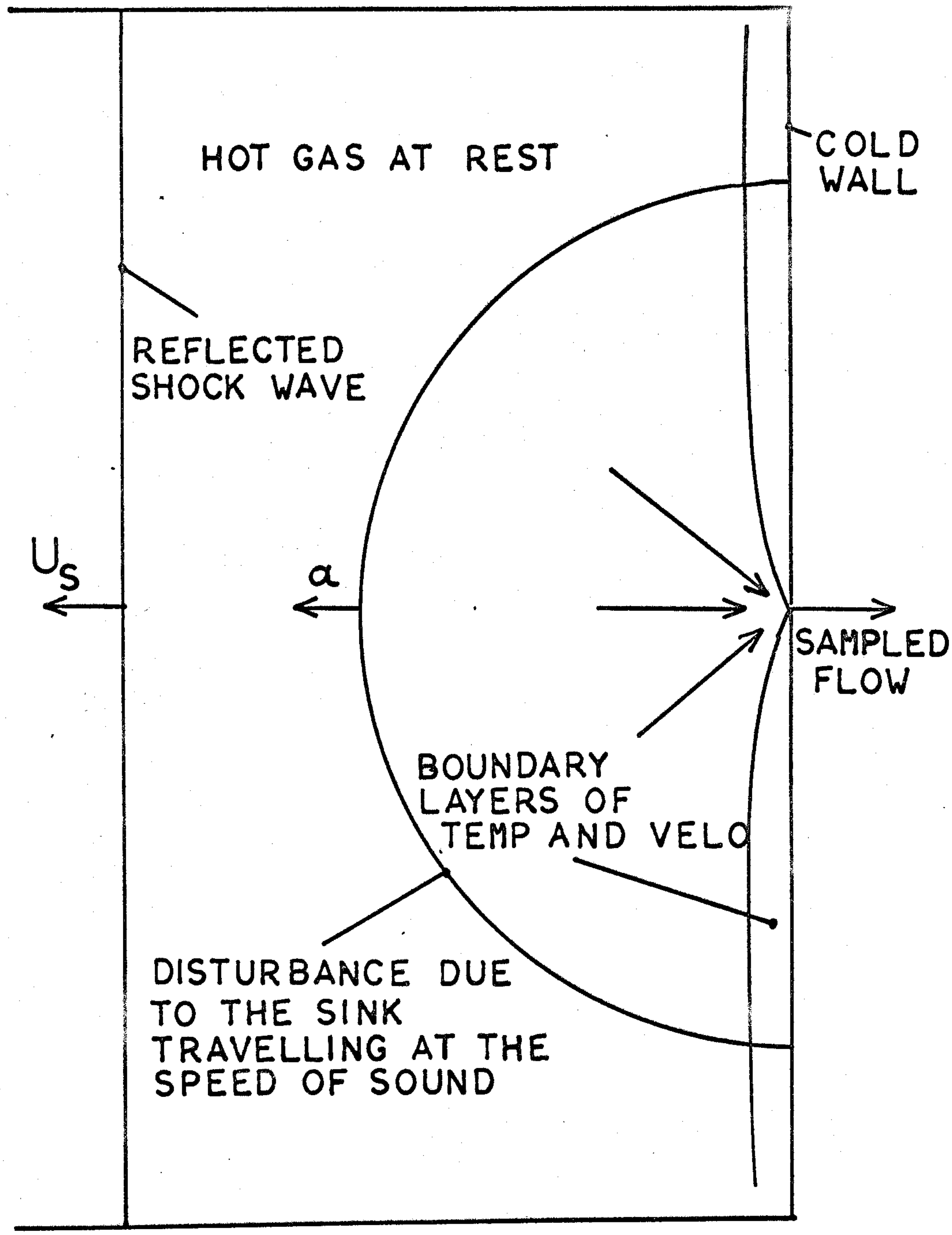

HOT GAS AT REST
COLD WALL
REFLECTED SHOCK WAVE
U_S
a
SAMPLED FLOW
BOUNDARY LAYERS OF TEMP AND VELO
DISTURBANCE DUE TO THE SINK TRAVELLING AT THE SPEED OF SOUND

FIG 1.1

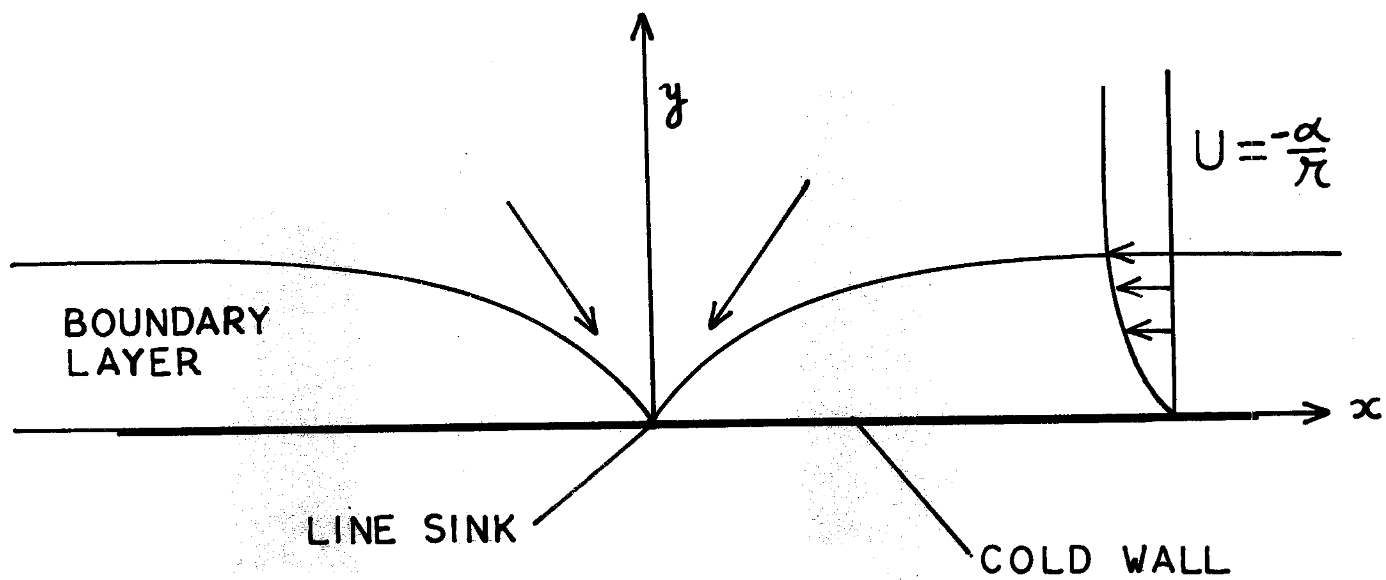


FIG 2.1

2. LINE SINK IN INCOMPRESSIBLE FLOW

There is a line sink on the wall at 0 and the coordinates x and y are chosen along the wall and perpendicular to it as shown in Fig. 2.1. The boundary layer equations for incompressible flow are the following.

The x momentum equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{2.1}$$

The equation of continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.2}$$

The energy equation:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \chi\frac{\partial^2 T}{\partial y^2} = \nu\frac{\partial^2 T}{\partial y^2} \tag{2.3}$$

Here u,v are the x and y components of velocity respectively, T is the temperature at any point (x,y), U is the velocity outside the boundary layer, ν is the kinematic viscosity of the fluid and χ is the thermometric conductivity of the fluid. It is assumed that the Prandtl number is equal to unity in which case $\nu = \chi$. $U = -\alpha/r$ where α is the sink strength (a constant greater than zero) and r is the radial distance from the sink. The velocity outside the boundary layer is therefore $U = -\alpha/x$. For incompressible flow the momentum equation is uncoupled from the energy equation. The boundary conditions are

$$\begin{aligned} &u(x,0) = v(x,0) = 0 \quad \text{for all } t \\ &T(x,0) = T_w = \text{constant} \\ &u(x,y) = -\alpha/x \quad \text{at } t=0 \\ &u(x,\infty) = -\alpha/x \quad \text{for all } t \\ &T(x,\infty) = T_0 \quad \text{for all } t \end{aligned} \tag{2.4}$$

Dimensional analysis of this problem indicates the following choice of variables. u/U and T/T_0 are functions of x, y, t and the parameters ν and α. From these the three dimensionless combinations y/x, $\alpha t/x^2$ and ν/α can be made. It is convenient instead to take $\eta = \frac{y}{x}\sqrt{\frac{\alpha}{\nu}}$ and $\tau = +\alpha t/x^2$ as the independent variables with ν/α as a fixed parameter. Thus

$$\frac{u}{U} = \text{function}\ (\eta, \tau;\ \nu/\alpha)$$

$$\frac{T}{T_0} = \text{function}\ (\eta, \tau;\ \nu/\alpha) \tag{2.5}$$

A stream function ψ is defined by

$$u = -\frac{\partial\psi}{\partial y}\ , \qquad v = \frac{\partial\psi}{\partial x} \tag{2.6}$$

Equations (2.1), (2.2) and (2.3) give

$$-\frac{\partial^2\psi}{\partial y\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} + \frac{\alpha^2}{x^3} = -\nu\frac{\partial^3\psi}{\partial y^3} \tag{2.7}$$

$$-\frac{\partial T}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = -\nu\frac{\partial^2 T}{\partial y^2} \tag{2.8}$$

These are rewritten with η and τ as independent variables using

$$\eta = y/x\sqrt{\alpha/\nu}\ , \qquad \tau = \frac{\alpha t}{x^2}\ ; \quad \psi = \sqrt{\nu\alpha}\ f(\eta, \nu)$$

$$\frac{\partial}{\partial t} = \frac{\tau}{t}\frac{\partial}{\partial\tau}$$

$$\frac{\partial}{\partial x} = -\frac{\eta}{x}\frac{\partial}{\partial\eta} - \frac{2\tau}{x}\frac{\partial}{\partial\tau}$$

$$\frac{\partial}{\partial y} = \frac{\eta}{y}\frac{\partial}{\partial\eta} \tag{2.9}$$

to get

$$\frac{\partial^2 f}{\partial\eta\,\partial\tau} + 2\tau\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\,\partial\tau} - \frac{\partial f}{\partial\tau}\frac{\partial^2 f}{\partial\eta^2}\right) + \left(\frac{\partial f}{\partial\eta}\right)^2 - 1 = \frac{\partial^3 f}{\partial\eta^3} \tag{2.10}$$

and

$$\frac{\partial T}{\partial\tau} + 2\tau\left(\frac{\partial f}{\partial\eta}\frac{\partial T}{\partial\tau} - \frac{\partial f}{\partial\tau}\frac{\partial T}{\partial\eta}\right) = \frac{\partial^2 T}{\partial\eta^2} \tag{2.11}$$

The boundary conditions for f are found now.

$$u = -\frac{\partial\psi}{\partial y} = -\frac{\alpha}{x}\frac{\partial f}{\partial\eta} \tag{2.12}$$

$$v = \frac{\partial\psi}{\partial x} = -\sqrt{\frac{\nu\alpha}{x}}\left(\eta\,\frac{\partial f}{\partial\eta} + 2\tau\frac{\partial f}{\partial\tau}\right) \tag{2.13}$$

Therefore the equations (2.4) lead to

$$\begin{aligned}
&f(0,\tau) = 0 \quad \text{for all } \tau\\
&f(\eta,0) = \eta\\
&f(\eta,\tau) \to \eta \quad \text{as } \eta \to \infty \quad \text{for all } \tau\\
&T(0,\tau) = T_w \quad \text{for all } \tau\\
&T(\infty,\tau) = T_0 \quad \text{for all } \tau\\
&T(\eta,0) = T_0 \quad \text{at } \tau = 0 \quad \text{for } \eta > 0
\end{aligned} \tag{2.14}$$

The equations (2.10) and (2.11) with boundary conditions (2.14) are solved approximately by the integral method of Karman and Pohlhausen. A good approximation to the profiles of velocity and temperature is the following.

$$f = \delta F(\eta/\delta)$$

so that

$$f_\eta = F'(\eta/\delta) = \text{erf}\,\eta/\delta \tag{2.15}$$

where

$$\delta = \delta(\tau) \tag{2.16}$$

and

$$T = T_w + (T_0 - T_w)\,\mathrm{erf}\,\eta/\Delta \tag{2.17}$$

where

$$\Delta = \Delta(\tau) \tag{2.18}$$

$\delta(\tau)$ and $\Delta(\tau)$ will be found by satisfying the integrals of (2.10) and (2.11) across the boundary layers. Integration of (2.10) from $\eta = 0$ to $\eta = \infty$ gives

$$\frac{d\delta}{dx}\left[\int_0^\infty F''(z)z\,dz + \tau\int_0^\infty 2FF''\,dz\right] + \delta\int_0^\infty \left(1 - (F')^2\right)dz = \frac{F''(0)}{\delta} \tag{2.19}$$

After some algebra this reduces to

$$\frac{d\delta}{d\tau}\left(\frac{1}{\sqrt{\pi}} + \frac{2\tau}{\sqrt{\pi}}(\sqrt{2}-1)\right) + \delta\sqrt{\frac{2}{\pi}} = \frac{2}{\delta\sqrt{\pi}}$$

whose solution is

$$\delta^2 = \sqrt{2}\left[1 - \frac{1}{\left[1 + (\sqrt{2}-1)2\tau\right]^{\sqrt{2}/(\sqrt{2}-1)}}\right] \tag{2.20}$$

Integration of (2.11) gives

$$\frac{d\Delta}{d\tau}\left(1 + \frac{2\tau}{\sqrt{1+\delta^2/\Delta^2}}\right) + 2\tau\frac{d\delta}{d\tau}\left(\frac{1}{\sqrt{1+\Delta^2/\delta^2}} - 1\right) = \frac{2}{\Delta} \tag{2.21}$$

It is seen from (2.21) that as $\tau \to 0$, $\Delta^2 \sim 4\tau$ and as $\tau \to \infty$, $\delta^2 \to \sqrt{2}$ and $\Delta^2 \sim 2\ln\tau$. This shows that the temperature boundary layer does not reach a steady state whereas the velocity boundary layer does. The equation (2.21) is integrated numerically and is presented in Fig. 2.2.

SOME NUMERICAL ESTIMATES.

Let ℓ, the length scale of the velocity boundary layer be defined by

$$\frac{\ell}{x}\sqrt{\frac{\alpha}{\nu}}\;\frac{1}{\delta(\tau)} = 1 \tag{2.22}$$

and L, the length scale of the thermal boundary layer by

$$\frac{L}{x}\sqrt{\frac{\alpha}{\nu}}\;\frac{1}{\Delta(\tau)} = 1 \tag{2.23}$$

The nature of the solution is now illustrated with some typical values. Let ρ_0, the pressure in the fluid far away from the sink be 1/100 atmospheres. Let T_0, the temperature far from the wall be 300 degrees Kelvin. $\nu = 13.3\ \text{cm}^2/\text{sec}$. Let the width of the slit be 0.4 mm. Let the flow through the slit be into a vacuum. At the slit the flow is sonic and the mass flow through the slit is therefore $da^*\rho^* = \rho_0 \pi r \frac{\alpha}{r} = \rho_0 \pi \alpha$, where a^* is the speed of sound at the throat and ρ^* is the density at the throat. Therefore

$$\alpha = \frac{d}{\pi}\left(\frac{a^*}{a_0}\right)\left(\frac{\rho^*}{\rho_0}\right) a_0 = 0.579\,\frac{d}{\pi}\,a_0$$

where a_0 and ρ_0 are the speed of sound and density at stagnation conditions. Here a_0 is 331 m/sec. From the above α becomes 240 cm^2/sec. Using these values of α and ν in (2.22) and (2.23) we get

$$\ell = 0.236\ x\ \delta\left(\frac{240t}{x^2}\right) \tag{2.24}$$

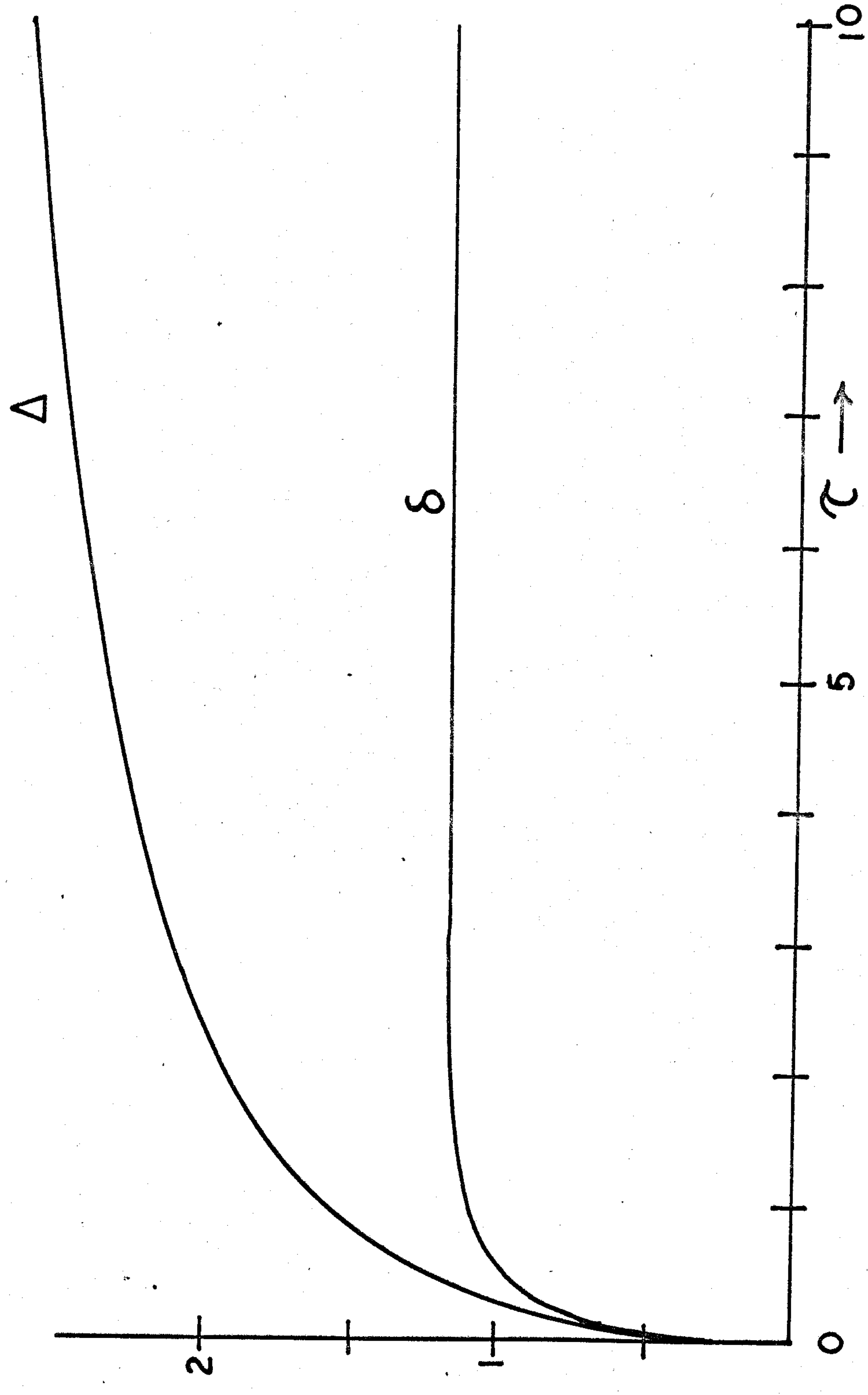


FIG 2.2

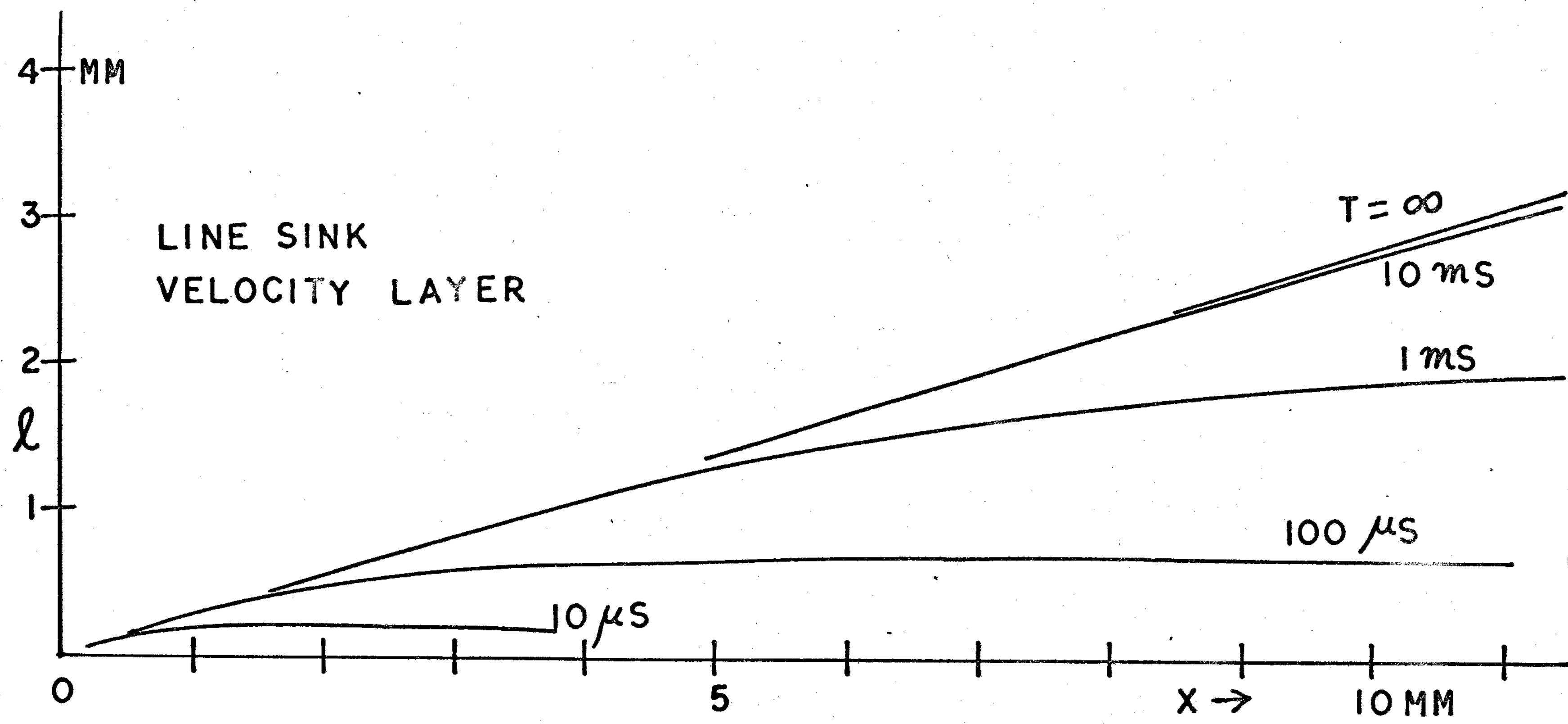

LINE SINK
VELOCITY LAYER
4
3
2
1
MM
l
0
5
X →
10 MM
T = ∞
10 ms
1 ms
100 μs
10 μs

FIG 2.3

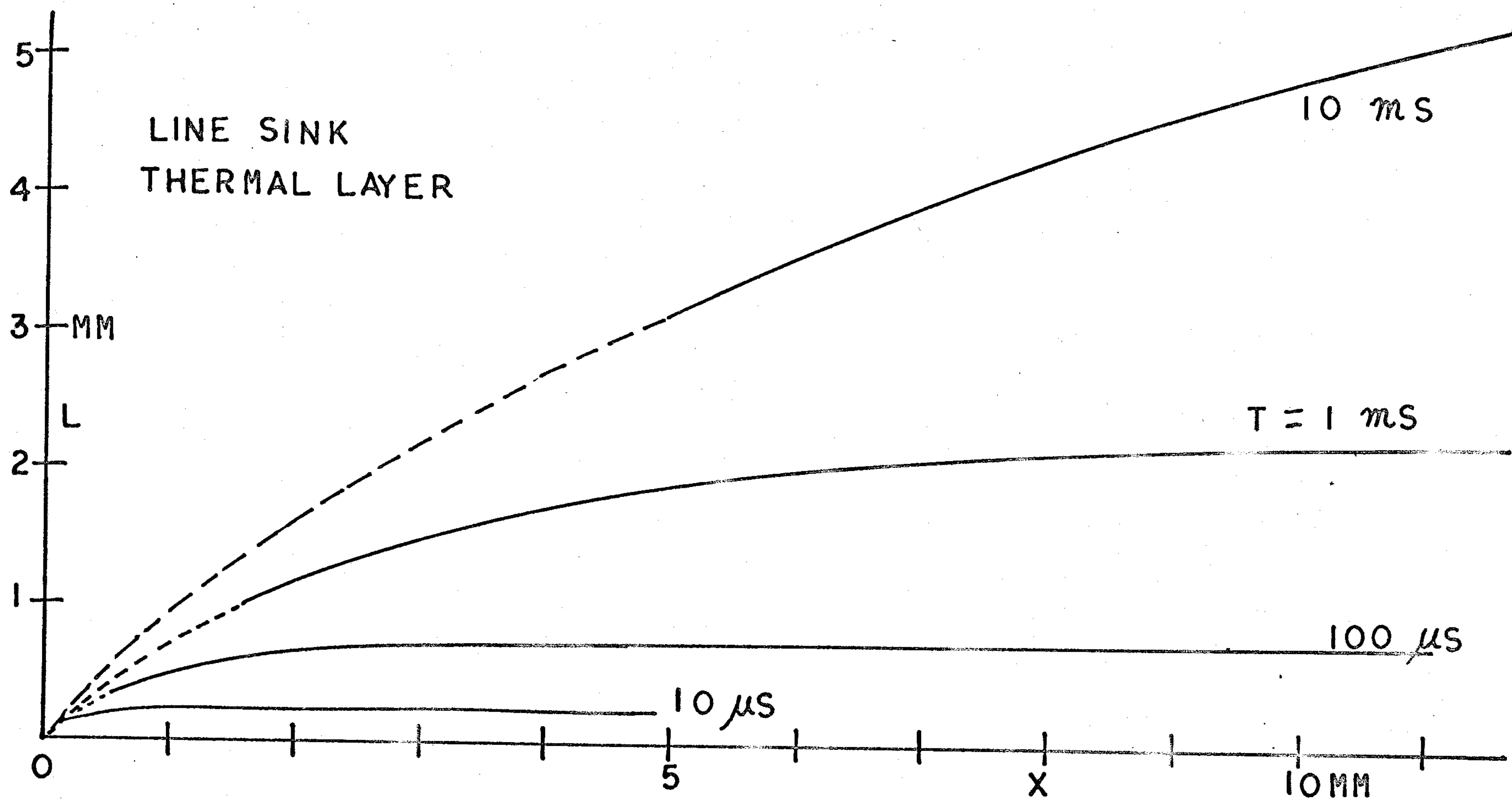

LINE SINK
THERMAL LAYER
5
4
3
MM
L
2
1
0
5
X
10MM
10 ms
T = 1 ms
100 μs
10 μs

FIG 2.4

and

$$L = 0.236\ x\ \Delta\left(\frac{240t}{x^2}\right) \tag{2.25}$$

The plots of ℓ and L for the four values of time $t = 10$ microsec, 100 microsec, 1 millisec and 10 millisec are shown in Figs. 2.3 and 2.4. The steady state for ℓ is also shown in Fig. 2.3.

DISCUSSION.

It is seen that for large x (i.e. $\frac{\alpha t}{x^2} << 1$) both δ and Δ are approximately equal to $2\sqrt{\tau}$ so that

$$\ell = L = x\sqrt{\frac{\nu}{\alpha}} \cdot 2\sqrt{\frac{\alpha t}{x^2}} = 2\sqrt{\nu t}\ .$$

For small x (i.e. $\frac{\alpha t}{x^2} >> 1$) δ tends to $2^{\frac{1}{4}}$, so that $= x\sqrt{\frac{\nu}{\alpha}} \cdot 2^{\frac{1}{4}}$. The velocity boundary layer close to the sink approaches the steady state more quickly than the regions far from it. The time scale of the velocity boundary layer to approach the steady state is x^2/α. L, the thickness of the thermal boundary layer for large τ is approximately (the dotted portion in Fig.2.4)

$$L \simeq 0.236\ x\sqrt{2\ \ln\left(\frac{\alpha t}{x^2}\right)} \tag{2.26}$$

which shows that the boundary layer approximation (viz $L << x$) breaks down close to the sink. The solution for large τ should therefore be reconsidered. The analysis of the thermal layer (assumed thick) can be made ignoring the velocity boundary layer which has become steady. This will be considered later.

The fact that the velocity boundary layer reaches a steady state whereas the thermal layer does not can be seen from the following simple considerations. Let us assume that a steady state for the thermal layer exists. The flux of heat across the thermal layer at any station x should be equal to the flux of heat into the portion of the cold wall from x to infinity. That is

$$\int_0^{\infty} u(T-T_0)\,dy = \nu \int_x^{\infty} \left(\frac{\partial(T-T_0)}{\partial y}\right)_{y=0} dx. \qquad (2.27)$$

Differentiating both sides with respect to x we get

$$\frac{d}{dx}\int^{\infty} u(T-T_0)\,dy = -\nu\left(\frac{\partial(T-T_0)}{\partial y}\right)_{y=0} \qquad (2.28)$$

Putting the orders of magnitude of u and $(T-T_0)$ here, we get approximately

$$\frac{d}{dx}\left(-\frac{\alpha}{x}\cdot(T_w-T_0)\cdot L\right) = -\nu\cdot\frac{(T_0-T_w)}{L} \qquad (2.29)$$

where L is the thickness of the thermal layer. On integration (2.29) gives

$$\frac{1}{2}\,\frac{L^2}{x^2} = \frac{\nu}{\alpha}\quad \ln\left(\frac{x_0}{x}\right) \qquad (2.30)$$

where x_0 is a constant. This solution is impossible because for $x > x_0$, L becomes imaginary. It can also be seen that the velocity distribution $U = -\alpha/x$ just fails to produce a steady thermal boundary layer. In fact for $U = -\alpha/x^m$ where $m > 1$, a steady state is possible because instead of (2.29) we have

$$\frac{d}{dx}\left(-\frac{\alpha}{x^m}(T_w-T_0)\,L\right) = \nu\,\frac{T_w-T_0}{L}$$

or

$$\frac{L}{x^m}\,\frac{d}{dx}\left(\frac{L}{x^m}\right) = -\frac{\nu}{\alpha}\,\frac{1}{x^m}$$

which gives on integration

$$\frac{1}{2}\frac{L^2}{x^{2m}} = \frac{\nu}{\alpha}\frac{x^{1-m}}{m-1} + \text{constant}$$

This solution is acceptable for the constant ≥ 0. It may be noted that as $x \to \infty$, $\frac{L}{x} \sim x^{m-1}$ if the constant is non zero and $\frac{L}{x} \sim x^{\frac{m-1}{2}}$ if the constant is zero. This shows that far from the sink the boundary layer approximation breaks down. This is a common feature of most similarity solutions, a familiar example of which is the Blasius solution for the flat plate where the boundary layer approximation fails at the leading edge.

Now let us consider the velocity boundary layer. In two dimensional incompressible flow the vorticity ω (the z component of the vorticity vector) satisfies the same equation as the temperature. So instead of (2.27) and (2.28) we get

$$\int_0^\infty u\omega\, dy = \nu \int_x^\infty \left(\frac{\partial\omega}{\partial y}\right)_{y=0} dx \tag{2.31}$$

and

$$\frac{d}{dx}\int_0^\infty u\omega\, dy = -\nu \left(\frac{\partial\omega}{\partial y}\right)_{y=0} \tag{2.32}$$

Putting the orders of magnitude of $u(\sim -\frac{\alpha}{x})$ and $\omega(\sim \frac{\alpha}{x\ell})$ in (2.32) we have approximately

$$\frac{d}{dx}\left(-\frac{\alpha}{x}\cdot\frac{\alpha}{x\ell}\cdot\ell\right) = -\nu\left(\frac{\alpha}{x\ell}\cdot -\frac{1}{\ell}\right) \quad \text{or} \quad \ell = \sqrt{\frac{\nu}{2\alpha}}\; x$$

showing that a steady velocity boundary layer is possible. It may be seen easily that the velocity distribution $U = -\alpha/x^m$ with $m \geq 1$ also produces a steady state for the velocity boundary layer with

$$\frac{d}{dx}\left(-\frac{\alpha^2}{x^{2m}}\right) = \nu\frac{\alpha}{x^m}\frac{1}{\ell^2} \quad \text{or} \quad \ell \simeq \sqrt{\frac{\nu}{2m\alpha}}\; x^{m+1/2}.$$

3. LARGE-TIME TEMPERATURE FIELD IN INCOMPRESSIBLE FLOW WITH THE VELOCITY BOUNDARY LAYER IGNORED

It was seen in Section 2 that the velocity boundary layer reaches a steady state whereas the temperature field does not. The boundary layer approximation is inappropriate for large τ when the thermal layer is very thick. The temperature field is recalculated here without making the boundary layer approximation but ignoring altogether the velocity boundary layer. This approximation is expected to be reasonable for large times, viz times large compared to x^2/α, the time scale of the velocity boundary layer to approach the steady state.

Fig. 3.1 shows a line sink at 0 producing the velocity field

$$u_r = -\alpha/r \quad \text{and} \quad u_\theta = 0$$

where (r,θ) are the cylindrical polar coordinates of any point P. The equations describing the temperature field are

$$\frac{\partial T}{\partial t} - \frac{\alpha}{r}\frac{\partial T}{\partial r} = \nu \nabla^2 T \tag{3.1}$$

with the boundary conditions

$$\begin{aligned} &T = T_w \ \text{ at } \ \theta = 0, \ \pi \ \text{ for all } r \text{ and } t > 0 \\ &T = T_0 \ \text{ for all } r, \ \theta \ \text{ for } t = 0 \end{aligned} \tag{3.2}$$

The equation (3.1) is linear in T and it is sufficient to solve (3.1) with boundary conditions (3.3) instead.

$$\begin{aligned} &T = 0 \ \text{ at } \ \theta = 0, \ \pi \ \text{ for all } r, \ t > 0. \\ &T = 1 \ \text{ for all } r, \ \theta \ \text{ for } t = 0. \end{aligned} \tag{3.3}$$

Dimensional analysis shows that T is a function of $\varphi = \theta$ and $\tau = \alpha t/r^2$ with α/ν as a fixed parameter. In these variables the equation (3.1) becomes

$$4\tau^2 \frac{\partial^2 T}{\partial \tau^2} + \frac{\partial T}{\partial \tau}\left(4\tau - 2\frac{\alpha}{\nu}\tau - \frac{\alpha}{\nu}\right) + \frac{\partial^2 T}{\partial \varphi^2} = 0 \tag{3.4}$$

with the boundary conditions (3.5) as shown in Fig. 3.2, viz

$$\begin{aligned} &T = 0 \text{ at } \varphi = 0, \pi \text{ for all } t > 0 \\ &T = 1 \text{ at } \tau = 0 \text{ for all } \varphi. \end{aligned} \tag{3.5}$$

The equation (3.4) is solved by separating variables assuming

$$T = \sum A_n R_n(\tau) \sin n\varphi \quad \cdot \quad n = 1, 3, 5 \cdots \tag{3.6}$$

From (3.4) it follows that $R_n(\tau)$ satisfies

$$4\tau^2 \frac{d^2 R_n}{d\tau^2} + \frac{dR_n}{d\tau}\left(4\tau - \frac{2\alpha\tau}{\nu}\right) - n^2 R_n = 0 \tag{3.7}$$

with boundary conditions

$$R_n \to \text{constant as } \tau \to 0 \text{ and}$$

$$R_n \to 0 \text{ as } \tau \to \infty.$$

The solutions of equation (3.7) are confluent hypergeometric functions. The general confluent equation is

$$\begin{aligned} &w'' + w'\left[\frac{2A}{z} + 2f' + \frac{bh'}{h} - h' - \frac{h''}{h'}\right] \\ &+ w\left[\left(\frac{bh'}{h} - h' - \frac{h''}{h'}\right)\left(\frac{A}{z} + f'\right) + \frac{A(A-1)}{z^2} + \frac{2Af'}{z} + f'' + f'^2 - \frac{ah'^2}{h}\right] = 0 \end{aligned} \tag{3.8}$$

where the primes indicate differentiation with respect to z. This has

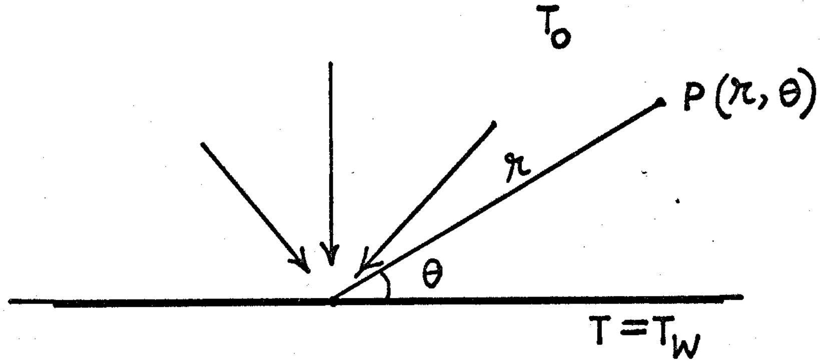
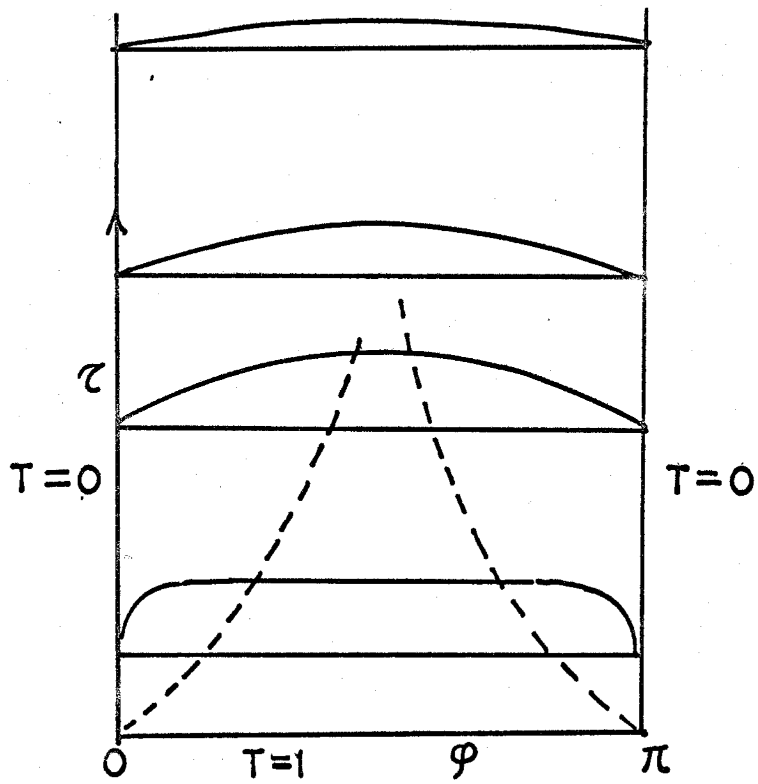


FIG 3.1



TEMP PROFILES AND THERMAL LAYER BOUNDARIES

FIG 3.2

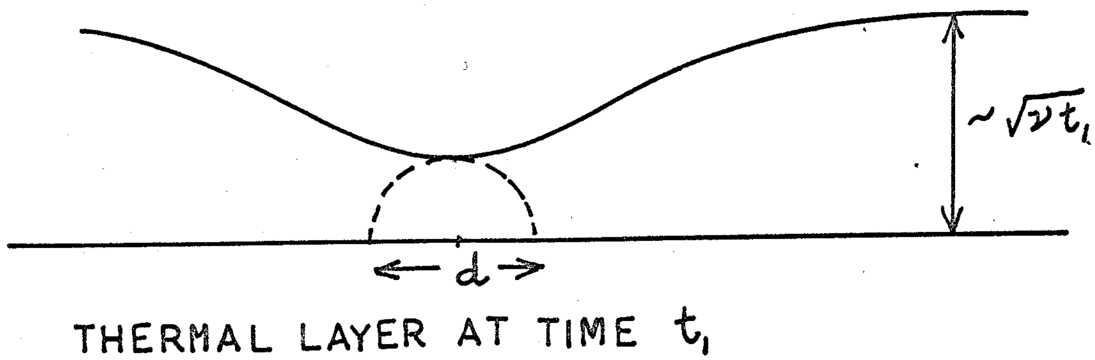


FIG 3.3

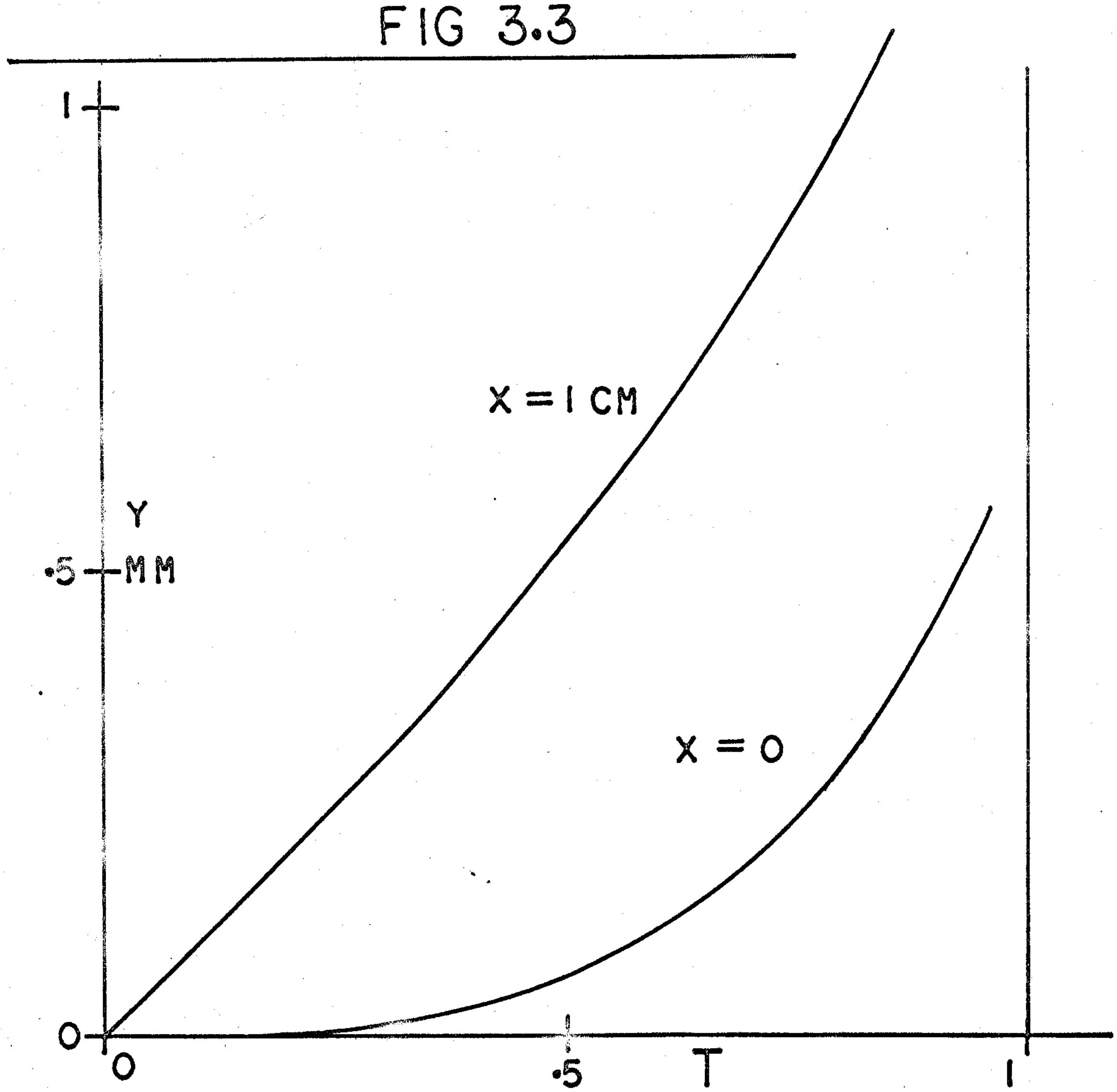


FIG 3.4

the independent solutions

$$z^{-A}\, e^{-f(z)}\, M(a, b, h(z)) \quad \text{and}$$
$$z^{-A}\, e^{-f(z)}\, U(a, b, h(z)) \qquad (3.9)$$

where the functions M and U are confluent hypergeometric functions. It is found that for equation (3.7)

$$\begin{aligned} a(n) &= 1 + \frac{\alpha}{4\nu} + \tfrac{1}{2}\sqrt{\frac{\alpha^2}{4\nu^2} + n^2} \\ b(n) &= 1 + \sqrt{\frac{\alpha^2}{4\nu^2} + n^2} \\ A(n) &= b - a = \tfrac{1}{2}\left(-\frac{\alpha}{2\nu} + \sqrt{\frac{\alpha^2}{4\nu^2} + n^2}\right) \\ f(\tau) &= h(\tau) = \alpha/4\nu\tau \end{aligned} \qquad (3.10)$$

Therefore the solutions of (3.7) are

$$\tau^{-\frac{1}{2}\left(-\frac{\alpha}{2\nu} + \sqrt{\frac{\alpha^2}{4\nu^2} + n^2}\right)} \cdot e^{-\frac{1}{4}\frac{\alpha}{\tau\cdot\nu}} \cdot M\left(1 + \frac{\alpha}{4\nu} + \tfrac{1}{2}\sqrt{\ },\ 1 + \sqrt{\ },\ \frac{\alpha}{4\nu\tau}\right) \qquad (3.11)$$

and

$$\tau^{-\frac{1}{2}\left(-\frac{\alpha}{2\nu} + \sqrt{\ }\right)} \cdot e^{-\frac{\alpha}{4\nu\tau}} \cdot U\left(1 + \frac{\alpha}{4\nu} + \tfrac{1}{2}\sqrt{\ },\ 1 + \sqrt{\ },\ \frac{\alpha}{4\nu\tau}\right) \qquad (3.12)$$

(3.12) does not satisfy the boundary conditions because as $\tau \to \infty$,

$$U \sim \frac{\Gamma(b-1)}{\Gamma(a)} \left(\frac{\alpha}{4\nu\tau}\right)^{1-b}$$

and (3.12)

$$\begin{aligned} &\sim \tau^{-A} \cdot e^{-\alpha/4\nu\tau} \cdot \frac{\Gamma(b-1)}{\Gamma(a)} \cdot \left(\frac{\alpha}{4\nu\tau}\right)^{1-b} \\ &\sim \tau^{-A-1+b} \\ &= \tau^{\frac{\alpha}{4\nu} + \frac{1}{2}\sqrt{\frac{\alpha^2}{4\nu^2} + n^2}} \to \infty \end{aligned}$$

On the other hand the solution (3.11) is well behaved. As $\tau \to 0$, the solution (3.11) tends to

$$\tau^{-A} \cdot e^{-\alpha/4\nu\tau} \cdot e^{\alpha/4\nu\tau} \cdot \left(\frac{\alpha}{4\nu\tau}\right)^{a-b} \cdot \frac{\Gamma(b)}{\Gamma(a)} = \frac{\Gamma(b)}{\Gamma(a)} \cdot \left(\frac{\alpha}{4\nu}\right)^{-A}$$

and as $\tau \to \infty$, $M(a, b, \frac{\alpha}{4\nu\tau}) \to 1$ and the solution (3.11) $\to 0$. The boundary condition at $\tau = 0$ gives

$$T(0,\varphi) = 1 = \frac{4}{\pi} \sum \frac{\sin n\varphi}{n} = \sum A_n \left(\frac{\alpha}{4\nu}\right)^{-A} \cdot \frac{\Gamma(b)}{\Gamma(a)} \sin n\varphi$$

so that

$$A_n = \frac{4}{\pi n} \cdot \frac{\Gamma(a)}{\Gamma(b)} \cdot \left(\frac{\alpha}{4\nu}\right)^{A} \tag{3.13}$$

The solution for T satisfying the boundary conditions (3.3) is therefore

$$T = \sum \frac{4}{\pi n} \cdot \frac{\Gamma(a)}{\Gamma(b)} \cdot \left(\frac{\alpha}{4\nu\tau}\right)^{A} \cdot e^{\frac{\alpha}{4\nu\tau}} \cdot \sin n\varphi \cdot M\left(a, b, \frac{\alpha}{4\nu\tau}\right) \qquad n = 1,\ 3,\ 5 \cdots \tag{3.14}$$

where a, b and A as functions of n are given by (3.10). The solution T_1 satisfying the boundary conditions (3.2) are therefore

$$T_1 = T_w + (T_0 - T_w)T \tag{3.15}$$

where T is given by (3.14).

AN APPROXIMATE SOLUTION WHEN $\frac{\alpha}{\nu} >> 1$

The convergence of the series in (3.14) is quite poor for small τ. When τ is small, the solution obtained in Section 2 can be used. An alternative approximate solution can be obtained easily as follows. Consider equation (3.4) for large $\frac{\alpha}{\nu}$. Define

$$\psi = \sqrt{\frac{\alpha}{\nu}}\ \varphi \tag{3.16}$$

In terms of variables ψ and τ, (3.4) becomes

$$4\tau^2 \frac{\partial^2 T}{\partial \tau^2} + \frac{\partial T}{\partial \tau}\left(4\tau - \frac{2\alpha}{\nu}\tau - \frac{\alpha}{\nu}\right) + \frac{\partial^2 T}{\partial \psi^2} \cdot \frac{\alpha}{\nu} = 0 \tag{3.17}$$

When $\frac{\alpha}{\nu} >> 1$, this is approximately

$$-\frac{\partial T}{\partial \tau} \cdot (1 + 2\tau) + \frac{\partial^2 T}{\partial \psi^2} = 0 \tag{3.18}$$

whose solution satisfying the boundary conditions (3.3) is

$$T = \operatorname{erf} \frac{\psi}{\sqrt{2 \ln(1+2\tau)}} = \operatorname{erf} \frac{\varphi}{\sqrt{2\nu/\alpha \ln(1+2\tau)}} \tag{3.19}$$

This describes the thermal layer on the wall $\varphi = 0$. The other wall $\varphi = \pi$ has a similar layer given by

$$T = \operatorname{erf} \frac{\pi - \varphi}{\sqrt{2\nu/\alpha \ln(1+2\tau)}}$$

The solution (3.19) is not uniformly valid in τ. This solution describes the thermal layer on the walls for $\tau <<$ the τ for which the thermal layers on the two walls $\varphi = 0$ and $\varphi = \pi$ meet, i.e. when T at $\varphi = \pi/2$ is significantly different from unity. (Fig. 3.2)

The solution (3.19) will now be compared with the exact solution (3.14). The right hand side of (3.19) can be expanded in Fourier series to get

$$T = \operatorname{erf} \frac{\varphi}{\sqrt{2\nu/\alpha \ln(1+2\tau)}} = \sum_{1,3,5\cdots} \frac{4}{\pi n} \cdot (1+2\tau)^{-\frac{\nu n^2}{2\alpha}} \cdot \sin n\varphi \tag{3.20}$$

valid when $2\nu/\alpha \ln(1+2\tau) << \frac{\pi}{2}$. For large z, M(a,b,z) has the asymptotic expansion

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)} e^{z} \cdot z^{a-b} \cdot \left(1 + \frac{(b-a)(1-a)}{z} + \cdots\right)$$

Using this in the exact solution (3.14), we get

$$T = \sum_{1,3,\cdots} \frac{4}{\pi n} \sin n\varphi \cdot \left(1 + \frac{(b-a)(1-a)}{\alpha/4\nu\tau} + \cdots\right)$$

$$= \sum \frac{4}{\pi n} \sin n\varphi \left(1 + (-\frac{n^2}{4})(\frac{4\nu\tau}{\alpha}) + \cdots\right)$$

$$= 1 - \sum \frac{4}{\pi n} \sin n\varphi \cdot \frac{\nu\tau n^2}{\alpha} \cdot + \cdots$$

which agrees with the approximate solution in (3.20) for small τ.

DISCUSSION:

The solution (3.19) is valid for small τ, i.e. a τ for which the thermal layers on the walls $\varphi = 0$ and $\varphi = \pi$ are not overlapping significantly. A τ-scale τ_1 is defined by

$$\left(2\frac{\nu}{\alpha} \ln(1+2\tau_1)\right)^{\frac{1}{2}} = \frac{\pi}{4}$$

i.e.

$$\tau_1 = \frac{1}{2}\left(e^{\frac{\alpha\pi^2}{32\nu}} - 1\right) \tag{3.21}$$

At $\tau = \tau_1$, $T(\tau_1, \frac{\pi}{2}) = \text{erf } 2 = 0.995$. Using the expression (3.19) the heat flux through the semicircle of diameter d centered at 0 is calculated. This is approximately equal to the heat flux out of a slit of width d, approximate because the velocity field near a finite slit is not given by the simple radial flow expression $u_r = -\frac{\alpha}{r}$; $u_\theta = 0$. Heat flux due to convection

$$= 2 \int_0^{\pi/2} \operatorname{erf} \frac{\varphi}{\left(\frac{2\nu}{\alpha}\ln(1+2\tau)\right)^{1/2}} \cdot \frac{\alpha}{r} \cdot r\, d\varphi$$

$$= \pi\alpha - 2\alpha \int_0^{\pi/2} \operatorname{erfc} \frac{\varphi}{(\quad)^{1/2}} \cdot d\varphi$$

$$\simeq \pi\alpha - 2\alpha \int_0^{\pi/2} \operatorname{erfc} \frac{\varphi}{(\quad)^{1/2}} \cdot d\varphi$$

$$= \pi\alpha - \frac{2\alpha}{\sqrt{\pi}} \left(\frac{2\nu}{\alpha}\ln(1+2\tau)\right)^{1/2} \tag{3.22}$$

This approximation is good for $\tau < \tau_1$. Heat flux due to conduction

$$= 2 \qquad \nu \frac{\partial T}{\partial r} r\, d\varphi$$

$$= 2\nu r \frac{\partial}{\partial r} \int_0^{\pi/2} T\, d\varphi$$

$$= 2\nu r \frac{\partial}{\partial r} \left(\pi - \frac{\partial}{\sqrt{\pi}} \left(\frac{2\nu}{\alpha}\ln(1+2\tau)\right)^{1/2}\right)$$

$$= \frac{8}{\sqrt{\pi}} \frac{\nu^2}{\alpha} \tau \Big/ (1+2\tau)\left(\frac{2\nu}{\alpha}\ln(1+2\tau)\right)^{1/2} \tag{3.23}$$

where $\tau = \alpha t/(\frac{1}{2}d)^2$. Let the total heat flux divided by $\pi\alpha$ be called $G(\frac{\nu}{\alpha}, \tau)$.

$$G(\frac{\nu}{\alpha}, \tau) = 1 - \frac{2}{\pi\sqrt{\pi}} \left(\frac{2\nu}{\alpha}\ln(1+2\tau)\right)^{1/2} + \frac{8\nu^2/\alpha^2\ \tau}{\pi\sqrt{\pi}\left(2\frac{\nu}{\alpha}\ln(1+2\tau)\right)^{1/2} \cdot (1+2\tau)} \tag{3.24}$$

The heat flux ratio at $\tau = \tau_1$ is

$$G(\frac{\nu}{\alpha}, \tau_1) = 1 - \frac{1}{2\sqrt{\pi}} + \frac{32(\nu/\alpha)^2}{\pi^2\sqrt{\pi}} \cdot \tfrac{1}{2}\left(1 - \exp - \frac{\pi^2}{32}\frac{\alpha}{\nu}\right) \tag{3.25}$$

The term containing $(\nu/\alpha)^2$ is due to conduction and is negligible for large $\alpha/\nu \cdot \tau_1 = \alpha t_1/(\frac{1}{2}d)^2$ defines a time scale t_1 at which time the thermal boundary layer just covers the region within the semicircle of diameter d centered at the sink. (Fig. 3. 3).

SOME NUMERICAL RESULTS FROM THE APPROXIMATE SOLUTION FOR LARGE α/ν;

Air at 0 degrees centigrade and a pressure of 1 atmosphere has a kinematic viscosity ν = 0.133 C.G.S units. α for a slit of width 0.4 mm, for a flow from a pressure of 1 atmosphere into vacuum is 238 cm^2/sec. For a pressure of p atmospheres, a temperature q times the standard temperature (300 degrees Kelvin) and a slit of width s times 0.4mm, ν and α are given by the following.

$$\alpha = 238\ s\sqrt{q} \tag{3.26}$$

$$\nu = 0.133\ q^2\sqrt{p} \tag{3.27}$$

The table below gives a few values of the characteristic time t_1.

Table 3.1

p	q	s	α C.G.S	ν C.G.S	α/ν	t_1 in sec
1	1	1	238	.133	1790	5. 25. 10^{-6}. exp 550.
.1	1	1	238	1.33	179	5. 25. 10^{-6}. exp 55.
.01	1	1	238	13.3	17.9	1. 28 10^{-3}.
.01	3	1	413	119.7	3.46	5.75 10^{-6}.

$G(\nu/\alpha, \tau)$ is initially equal to unity and at time t_1, $G(\nu/\alpha, \tau_1) = 1/2\sqrt{\pi} = .72$ for $\nu/\alpha << 1$. At time t_1 the thermal layer is considered to have just covered the slit. A significant amount of cooling has taken place by this time (i.e. by $1/2\sqrt{\pi}$ = 28%). The table above shows that t_1 is extremely sensitive to changes in the values of α and ν.

To find the heat flux for times large compared to t_1, again for large α/ν, the first term in the Fourier series in equation (3.20) may be used. Incidentally, for $\alpha/\nu >> 1$, the series expansion is the exact solution of equation (3.18) and is valid for all τ.

$$T \simeq \frac{4}{\pi}(1 + 2\tau)^{-\nu/2\alpha} \sin\varphi \quad \text{for} \quad \tau >> \tau_1$$

Heat flux by convection

$$= 2 \int_0^{\pi/2} \frac{4}{\pi}(1+2\tau)^{-\nu/2\alpha} \sin\varphi \, d\varphi \cdot \alpha$$

$$\simeq \frac{8\alpha}{\pi}(2\tau)^{-\nu/2\alpha}$$

Heat flux by conduction

$$= 2\nu r \frac{\partial}{\partial r} \int_0^{\pi/2} T \, d\varphi$$

$$\simeq 16 \frac{\nu^2}{\pi\alpha}(2\tau)^{-\alpha/2\nu} \simeq 0; \quad \alpha >> \nu.$$

Sum of the two fluxes

$$\simeq \frac{8\alpha}{\pi}(2\tau)^{-\alpha/2\nu} \quad \text{for} \quad \tau >> \tau_1 \tag{3.28}$$

In general when α/ν is not large compared to unity, the first term of equation (3.14) gives the following approximation for T, for large τ

$$T \simeq \frac{4}{\pi} \frac{\Gamma\left(1 + \frac{\alpha}{4\nu} + \frac{1}{2}\sqrt{\frac{\alpha^2}{4\nu^2} + 1}\right)}{\Gamma\left(1 + \sqrt{\frac{\alpha^2}{4\nu^2} + 1}\right)} \sin\varphi \cdot \left(\frac{\alpha}{4\nu\tau}\right)^{\frac{1}{2}\left[-\frac{\alpha}{2\nu} + \sqrt{\frac{\alpha^2}{4\nu^2} + 1}\right]}$$

This gives the heat flux for large τ

$$G(\frac{\nu}{\alpha},\tau) \simeq \frac{4}{\pi}\ \frac{\Gamma\left(1+\frac{\alpha}{4\nu}+\frac{1}{2}\sqrt{\frac{\alpha^2}{4\nu^2}+1}\right)}{\Gamma\left(1+\sqrt{\alpha^2/4\nu^2+1}\right)}\cdot\left(\frac{\alpha}{4\nu\tau}\right)^{\frac{1}{2}\left[-\frac{\alpha}{2\nu}+\sqrt{\frac{\alpha^2}{4\nu^2}+1}\right]}\cdot\left(\alpha+2\nu\left[-\frac{\alpha}{2\nu}+\sqrt{\frac{\alpha^2}{4\nu^2}+1}\right]\right) \tag{3.29}$$

which is only slightly different from (3.28).

To give an idea of the shapes of the temperature profiles, two profiles are shown in Fig. 3.4 for $\alpha t = .01\ cm^2$ (i.e. $t = 24.2$ microsec) at $x = 0$ and $x = 1$ cm. The constants α and ν refer to conditions on the last column of Table 3.1. On the centre line $\theta = \pi/2$, $r = y$ and $T \simeq \frac{4}{\pi}\left(1+\frac{.02}{y^2}\right)^{-1/6.92}$ for large τ (or small y). This profile has infinite slope at the origin which is consistent with (3.18). At $x = 1$ cm, $T = \operatorname{erf} 9.3y$ for small τ which in this case is .01. The profiles at other stations are expected to fall between the two shown in Fig. 3.4.

4. POINT SINK IN INCOMPRESSIBLE FLOW.

The analysis for this case follows closely the one in Section 2 for the line sink and will therefore be treated briefly. The equations of axisymmetric flow are

$$\frac{\partial(ux)}{\partial x} + \frac{\partial(vx)}{\partial y} = 0 \tag{4.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{4.2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = v\frac{\partial^2 T}{\partial y^2} \tag{4.3}$$

where x, φ and y are the cylindrical polar coordinates with the origin at the sink. A stream function ψ is defined by

$$-xu = \frac{\partial\psi}{\partial y} \qquad\qquad -xv = -\frac{\partial\psi}{\partial x}$$

U, the x-component of velocity outside the boundary layer = $-\alpha/x^2$ and $dp/dx = -\rho U\, dU/dx = 2\alpha^2/x^5$. In terms of the stream function (4.2) becomes

$$-\frac{1}{x}\frac{\partial^2\psi}{\partial y\partial t} + \frac{1}{x^2}\left(\frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\psi}{\partial y\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right) - \frac{1}{x^3}\left(\frac{\partial\psi}{\partial y}\right)^2 = -\frac{2\alpha^2}{x^5} - \frac{1}{x}\frac{\partial^3\psi}{\partial y^3} \tag{4.4}$$

Non dimensional coordinates η and τ are defined by

$$\eta = \frac{y}{x}\sqrt{\frac{\alpha}{2\nu x}} \qquad\qquad \tau = \frac{\alpha t}{x^3} \tag{4.5}$$

Dimensional analysis shows that u/U is a function of η and τ. This requires ψ to be of the form

$$\psi = \sqrt{2\alpha\nu x}\cdot f(\eta,\tau) \tag{4.6}$$

because then

$$u/U = -\frac{1}{x}\frac{\partial\psi}{\partial y}\Big/\frac{-\alpha}{x^2} = \frac{x}{\alpha}\sqrt{2\alpha\nu x}\cdot\frac{\partial f}{\partial\eta}\cdot\frac{1}{x}\sqrt{\frac{\alpha}{2\nu x}} = \frac{\partial f}{\partial\eta} \tag{4.6}$$

as it should be. (4.4) is rewritten in terms of η and τ as independent variables by using

$$\frac{\partial}{\partial x} = -\frac{3}{2}\frac{\eta}{x}\frac{\partial}{\partial\eta} - \frac{3\tau}{x}\frac{\partial}{\partial\tau}$$

$$\frac{\partial}{\partial y} = \frac{\eta}{y}\frac{\partial}{\partial\eta}$$

$$\frac{\partial}{\partial t} = \frac{\tau}{t}\frac{\partial}{\partial\tau}$$

to get

$$\frac{\partial^2 f}{\partial\tau\partial\eta} + 3\tau\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\,\partial\tau} - \frac{\partial^2 f}{\partial\eta^2}\frac{\partial f}{\partial\tau}\right) + 2\left(\frac{\partial f}{\partial\eta}\right)^2 + \frac{f}{2}\frac{\partial^2 f}{\partial\eta^2} = 2 + \frac{1}{2}\frac{\partial^3 f}{\partial\eta^3} \tag{4.7}$$

Similarly the energy equation (4.3) becomes

$$\frac{\partial T}{\partial\tau} + 3\tau\left(\frac{\partial f}{\partial\eta}\frac{\partial T}{\partial\tau} - \frac{\partial f}{\partial\tau}\frac{\partial T}{\partial\eta}\right) + \frac{\partial T}{\partial\eta}\frac{f}{2} = \frac{1}{2}\frac{\partial^2 T}{\partial\eta^2} \tag{4.8}$$

The boundary conditions are the same as for the line sink and are given by (2.14). As before profiles are assumed for f and T given by the equations (2.15) to (2.18). The integrated momentum equation now becomes

$$\frac{d\delta}{d\tau}\cdot\left[1 + 3\tau(\sqrt{2}-1)\right] + \frac{\delta}{2}\left[3\sqrt{2}+1\right] = \frac{1}{\delta}$$

whose solution is

$$\delta^2 = \frac{2}{(3\sqrt{2}+1)}\cdot\left\{1 - \left[\frac{1}{1+3\tau(\sqrt{2}-1)}\right]^{\frac{3\sqrt{2}+1}{3(\sqrt{2}-1}}\right\}. \tag{4.9}$$

The temperature equation gives instead of (2.21) the following.

$$\frac{d\Delta}{d\tau}\left(1+\frac{3\tau}{\sqrt{1+\delta^2/\Delta^2}}\right)+3\tau\frac{d\delta}{d\tau}\left(\frac{1}{\sqrt{1+\Delta^2/\delta^2}}-1\right)-\frac{\delta}{2}\left(\sqrt{1+\frac{\Delta^2}{\delta^2}}-1\right)=\frac{1}{\Delta} \tag{4.10}$$

It is seen that as $\tau \to 0$, $\Delta^2 = \delta^2 \sim 2\tau$. And as $\tau \to \infty$, $\delta^2 \to \frac{2}{3}\sqrt{2}+1$ and $\Delta \sim \tau^{\frac{1}{6}}$. The temperature layer does not reach a steady state whereas the velocity layer does. The equation (4.10) is integrated numerically and is presented graphically in Fig. 4.1. And Fig. 4.2 shows the graph of δ^2 as a function of τ.

SOME NUMERICAL RESULTS,

The length scales ℓ and L of the velocity and thermal layers respectively, are defined by

$$\ell = x \cdot \left(\frac{2\nu x}{\alpha 2}\right)^{\frac{1}{2}} \cdot \delta\left(\frac{\alpha t}{x^3}\right) \tag{4.11}$$

$$L = x \cdot \left(\frac{2\nu x}{\alpha}\right)^{\frac{1}{2}} \cdot \Delta\left(\frac{\alpha t}{x^3}\right) \tag{4.12}$$

The following numerical values are assumed. The flow is through a hole of diameter 0.7 mm, from a pressure of 1/100 atmos into vacuum. The temperature of the gas far from the wall is 300 degrees Kelvin. $\nu = 13.3$ C.G.S. units. The volume flow through the hole

$$= 2\pi r^2 \cdot \frac{\alpha}{r^2} = 2\pi\alpha$$

$$= \frac{\pi}{4}(.579)\, d^2 a_0$$

which gives $\alpha = 0.37$ C.G.S. units. With these numerical values

$$\ell = 1.68\, x^{\frac{3}{2}}\, \delta\left(\frac{9.37\, t}{x^3}\right) \tag{4.13}$$

$$L = 1.68\, x^{\frac{3}{2}}\, \Delta\left(\frac{9.37\, t}{x^3}\right) \tag{4.14}$$

The plots of ℓ and L for the four values of t = 10 microsec, 100 microsec, 1 millisec and 10 millisec and the steady state of ℓ are shown in the Figures 4.3 and 4.4.

For large τ, $\Delta \sim \tau^{1/6}$ and therefore, $L \sim x^{3/2}\left(\frac{9.37t}{x^3}\right)^{1/6}$, i.e.

$$L \sim xt^{1/6} \quad \text{as} \quad x \to 0 \tag{4.15}$$

The plots in Fig. 4.3 which correspond to $\tau < 10$ (refer to Fig. 4.1) are shown in dark lines and the data for $\tau > 10$ are those corresponding to the asymptotic linear relationship given by (4.15).

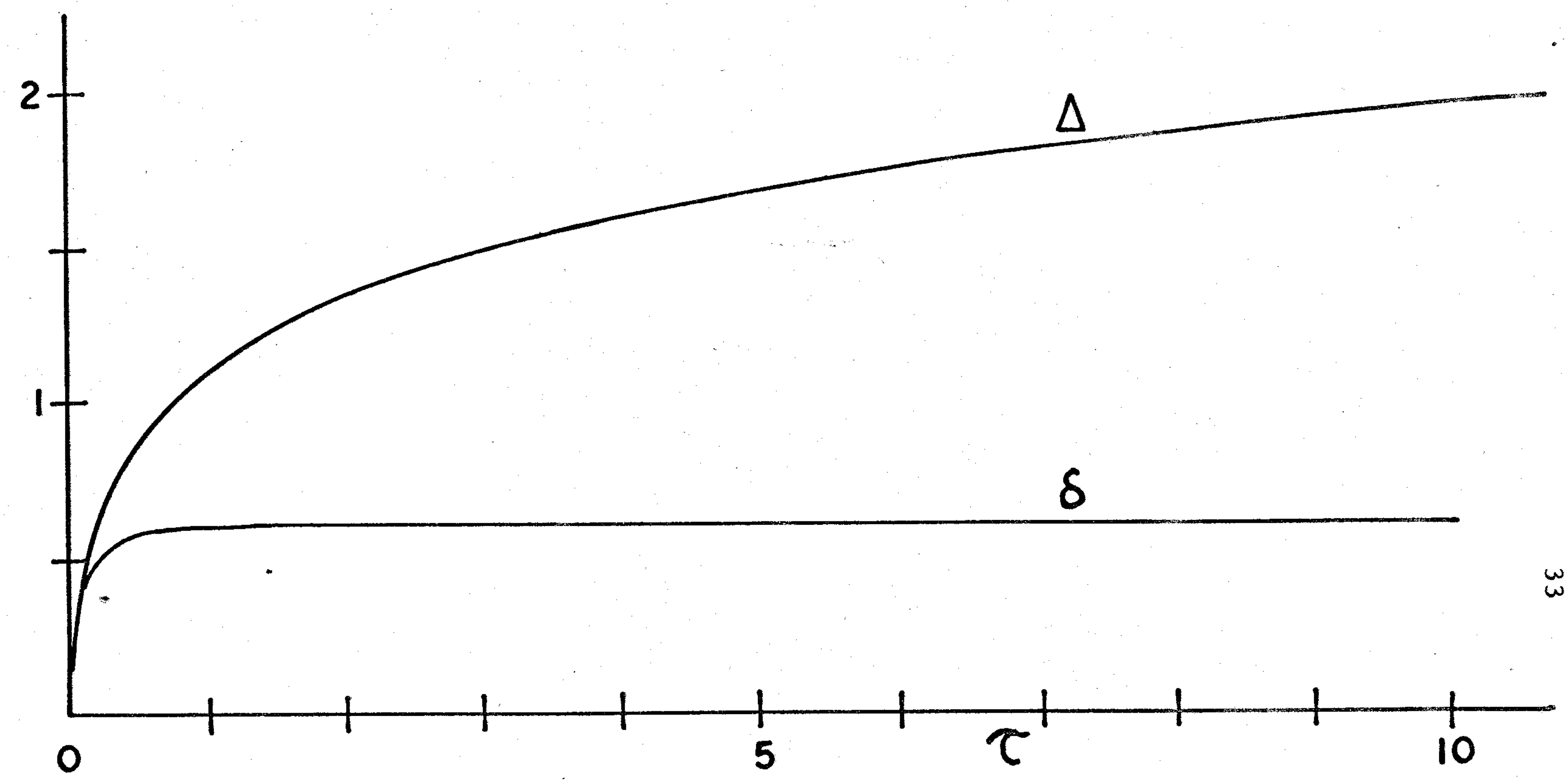


FIG 4.1

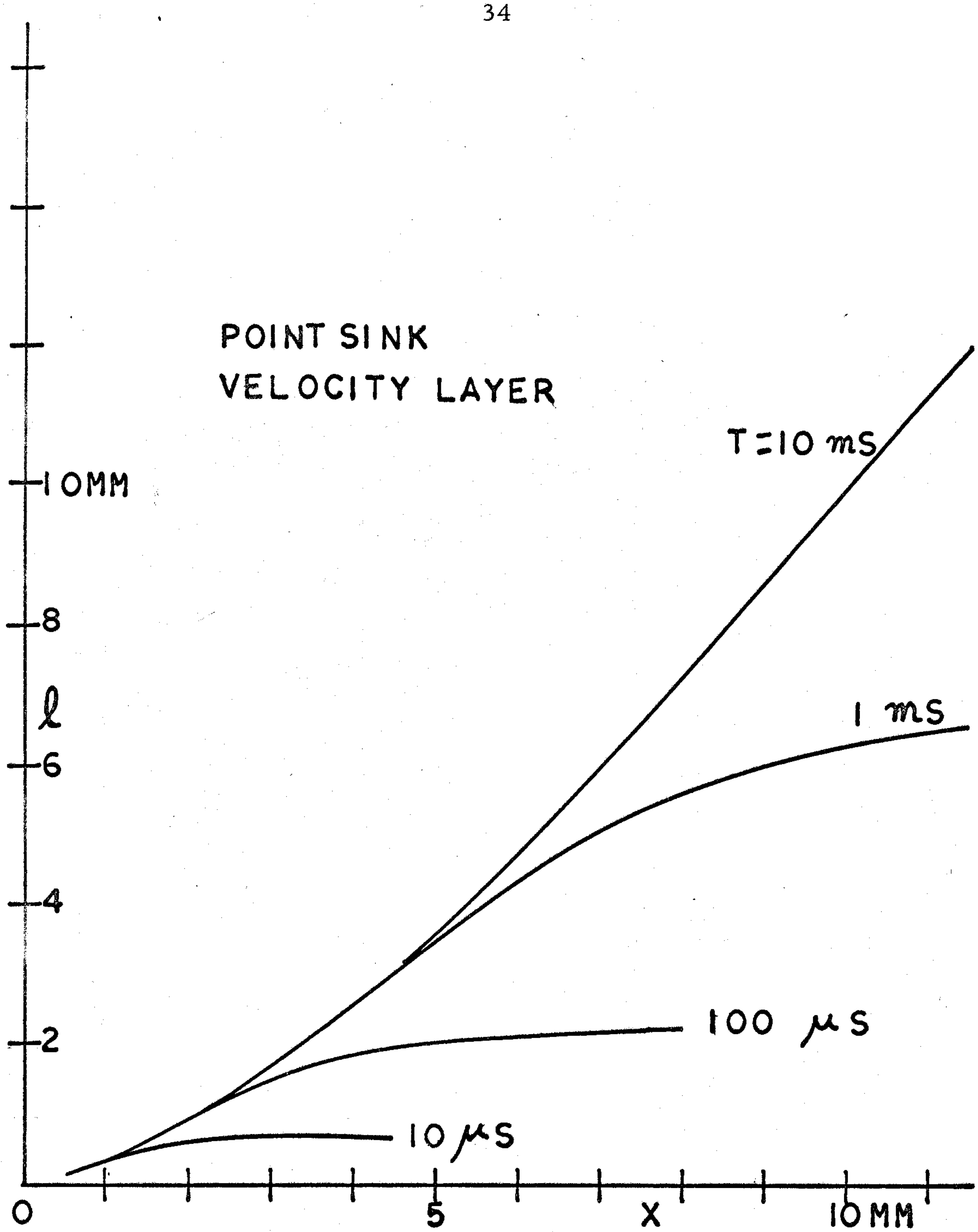

POINT SINK
VELOCITY LAYER
T=10 ms
1 ms
100 μs
10 μs
10MM
8
ℓ
6
4
2
0
5
X
10MM

FIG 4.2

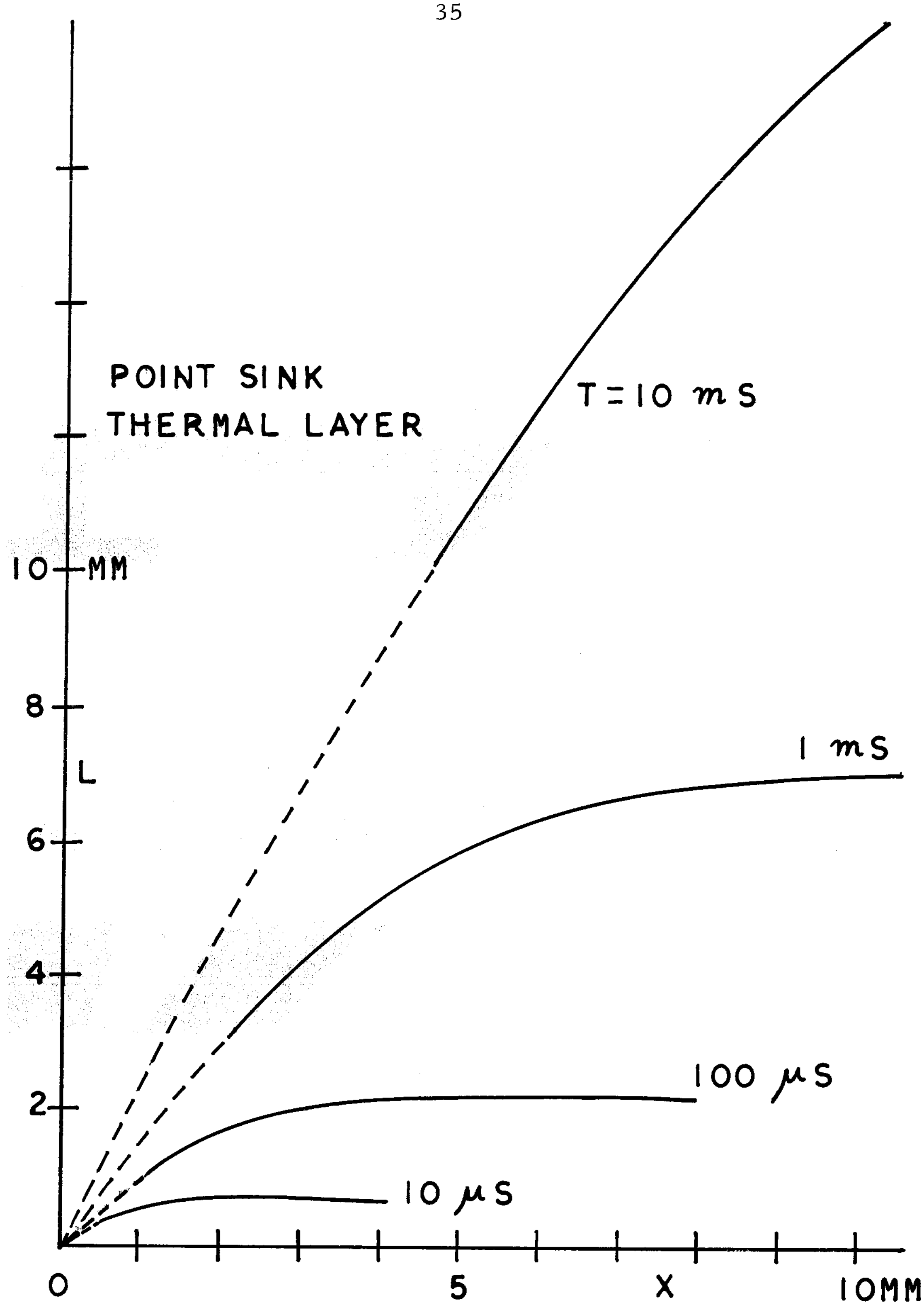

POINT SINK
THERMAL LAYER
T=10 ms
1 ms
100 μS
10 μS
10
MM
8
L
6
4
2
0
5
X
10MM

FIG 4.3

5. LINE SINK IN COMPRESSIBLE FLOW,

This analysis too, follows the one in Section 2 closely. The boundary layer equations of compressible, two-dimensional flow are

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{5.1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\right) = -\,\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu\,\frac{\partial u}{\partial y}\right) \tag{5.2}$$

$$\rho\left(\frac{\partial j}{\partial t} + u\,\frac{\partial j}{\partial x} + v\,\frac{\partial j}{\partial y}\right) = \frac{\partial p}{\partial t} + \frac{\partial}{\partial y}\left(\mu\,\frac{\partial j}{\partial y}\right) + \frac{1-Pr}{Pr}\,\frac{\partial}{\partial y}\left(\mu\,\frac{\partial h}{\partial y}\right) \tag{5.3}$$

$$= \frac{\partial}{\partial y}\left(\mu\,\frac{\partial j}{\partial y}\right) \quad \text{assuming} \quad Pr = 1$$

where

$$j = h + \tfrac{1}{2}u^2 \tag{5.4}$$

and

$$\frac{\mu}{\mu_0} = \frac{h}{h_0} \tag{5.5}$$

and the equation of state of a perfect gas is

$$p = \frac{\gamma-1}{\gamma}\,\rho h \tag{5.6}$$

Here h denotes the enthalpy per unit mass, j denotes the total enthalpy per unit mass and γ is the ratio of specific heats. The subscripts o and e refer to standard conditions and conditions outside the boundary layer respectively. In the boundary layer equations $p = p_e$.

These equations are converted to incompressible form by using the Illingworth-Stewartson transformation by using the new independent variables

$$X = \int_{\text{const}}^{x} \frac{p_e}{p_0} \cdot \frac{a_e}{a_0} \cdot dx$$

$$Y = \frac{a_e}{a_0} \int_0^y \frac{\rho}{\rho_0} \cdot dy$$

$$T = t \tag{5.7}$$

In (5.7), a denotes the velocity of sound. A stream function is defined by the following.

$$u = -\frac{\rho_0}{\rho} \frac{\partial\psi}{\partial y} = -\frac{a_e}{a_0} \frac{\partial\psi}{\partial y}$$

$$v = -\frac{\rho_0}{\rho}\left(\frac{a_0}{a_e}\frac{\partial Y}{\partial t} - \frac{\partial\psi}{\partial x}\right)$$

$$= -\frac{\rho_0}{\rho}\left(\frac{a_0}{a_e}\frac{\partial Y}{\partial t} - \frac{p_e}{p_0}\frac{a_e}{a_0}\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial Y} \cdot \frac{\partial Y}{\partial x}\right) \tag{5.8}$$

The momentum equation (5.2) becomes

$$-\left(\frac{a_0}{a_e}\right)^2 \frac{p_0}{p_e} \frac{\partial^2\psi}{\partial T \partial Y} + \frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial Y \partial X} - \frac{\partial\psi}{\partial X}\frac{\partial^2\psi}{\partial Y^2} - \frac{j}{j_0} a_0 M_e \cdot \frac{d}{dX}(a_0 M_e) = -\nu_0 \frac{\partial^3\psi}{\partial Y^3} \tag{5.9}$$

and the energy equation (5.3) gives

$$-\left(\frac{a_0}{a_e}\right)^2 \frac{p_0}{p_e}\frac{\partial j}{\partial T} + \frac{\partial\psi}{\partial Y} \cdot \frac{\partial j}{\partial X} - \frac{\partial\psi}{\partial X}\frac{\partial j}{\partial Y} = -\nu_0 \frac{\partial^2 j}{\partial Y^2} \tag{5.10}$$

The quantities p_e, M_e and a_e for the sink flow are found now.

Let us consider the steady radial flow outside the boundary layer. The equation of continuity gives

$$\rho_e q r = -A$$

where q is the radial velocity towards the sink and A is the constant mass flow into the sink. And the Bernoulli equation gives

$$\frac{\gamma-1}{2\gamma}\, q^2 + \frac{p_e}{\rho_e} = \frac{p_\infty}{\rho_\infty}$$

i.e.

$$\frac{\gamma-1}{2\gamma}\, \frac{A^2}{\rho_e^2 r^2} + \frac{p_e}{\rho_e} = \frac{p_\infty}{\rho_\infty}$$

For isentropic flow p_e and ρ_e are given by

$$\frac{\rho_\infty}{\rho_e} = \left(1 + \frac{\gamma-1}{2}\, M_e^2\right)^{1/\gamma-1}$$

$$\frac{p_e}{p_\infty} = \left(\frac{\rho_e}{\rho_\infty}\right)^{\gamma}$$

Therefore

$$\frac{A}{\gamma\rho_\infty^2\, r^2} \cdot \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{\frac{\gamma+1}{\gamma-1}} = \frac{p_\infty}{\rho_\infty}\, M_e^2 \qquad (5.11)$$

From (5.7)

$$dX = \frac{a_e}{a_0} \cdot \frac{p_e}{p_0} \cdot dX = \frac{a_e}{a_\infty} \cdot \frac{p_e}{p_\infty}\, dX$$

the reference conditions being taken to be those at infinity. Using the fact that

$$p \sim a^{2\gamma/\gamma-1}$$

for isentropic flow, we have

$$dX = \left(1 + \frac{\gamma-1}{2}\, M_e^2\right)^{-\frac{3\gamma-1}{2\gamma-2}} dX \qquad (5.12)$$

Let us consider a monatomic gas. $\gamma = 5/3$ for this case. The relation between M_e and $x(r = x$ just outside the boundary layer) is, from (5.11),

$$\left(1 + \frac{M_e^2}{3}\right)^4 = \left(M_e\, \frac{16}{9}\, \frac{x}{x_1}\right)^2 \qquad (5.13)$$

where x_1 is the position at which $M_e = 1$. On differentiation this gives

$$\frac{16}{9}\frac{dx}{x_1} = \left(1 + \frac{M_e^2}{3}\right)\left(1 - \frac{1}{M_e^2}\right) dM_e$$

and from (5.12)

$$dX = \left(1 + \frac{M_e^2}{3}\right)^{-2} \left(1 - \frac{1}{M_e^2}\right) \frac{9x_1}{16}\, dM_e$$

On integration this gives

$$\frac{16}{27} \cdot \frac{x}{x_1} = \frac{1+M_e^2}{3M_e + M_e^3} - \frac{1}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}}{M_e} - \frac{1}{2} + \frac{\pi}{3\sqrt{3}} \qquad (5.14)$$

A graph of $16\,X/27\,x_1$ vs $1/M_e$ is shown in Fig. 5.1. It is seen from this graph that the straight line approximation

$$\frac{1}{M_e} = 2.25 + 3\left(\frac{16X}{27x_1}\right) \qquad (5.15)$$

is reasonably good. It is noted here that the flow near the slit when $M_e \simeq 1$ will not be radial and therefore taking the exact equation (5.13) instead of the straight line approximation is probably not much of an improvement. On the other hand, the exact relationship (5.13) makes the problem much more complex because it destroys similarity. We shall assume (5.14) to be true.

The quantity $\left(\frac{a_0}{a_e}\right)^2 \frac{p_0}{p_e} = \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{\frac{2\gamma-1}{\gamma-1}}$ and is equal to $\left(1 + \frac{M_e^2}{3}\right)^{7/2}$ for a monatomic gas. The graph of $\xi = \left(\frac{a_0}{a_e}\right)^2 \frac{p_0}{p_e}$ is plotted against $16X/27x_1$ in Fig. 5.2. $\xi = 1$ is a reasonable approximation except very close to $X = 0$. We assume this to be true for the reasons given in the previous paragraph.

The relation between X and x is found by using (5.13) and (5.14). It is seen that the straight line approximation (from Fig. 5.3)

$$x = -1.3\, x_1 + x$$

is very good.

(5.15) can be written as

$$M_e = \frac{9}{16}\, x_1 \Big/ \, X + 1.265\, x_1$$

Let $\alpha = 9x_1 a_0/16$ and $\overline{X} = x + 1.265\, x_1$. Then $a_0 M_e = \alpha/\overline{X}$ where a_0 is the velocity of sound at infinity.

With these approximations the equations (5.9) and (5.10) become

$$-\frac{\partial^2\psi}{\partial Y \partial T} + \left(\frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial Y \partial \overline{X}} - \frac{\partial^2\psi}{\partial Y^2}\frac{\partial\psi}{\partial \overline{X}}\right) + \frac{j}{j_0}\frac{\alpha^2}{\overline{X}^3} = -\nu_0 \frac{\partial^3\psi}{\partial Y^3} \qquad (5.16)$$

$$-\frac{\partial j}{\partial T} + \left(\frac{\partial\psi}{\partial Y}\frac{\partial j}{\partial \overline{X}} - \frac{\partial\psi}{\partial \overline{X}}\frac{\partial j}{\partial Y}\right) = -\nu_0 \frac{\partial^2 j}{\partial Y^2} \qquad (5.17)$$

The equations of momentum and energy are coupled now. As before, non dimensional variables η and τ are defined by

$$\eta = \frac{Y}{\overline{X}}\sqrt{\frac{\alpha}{\nu_0}} \qquad \tau = \frac{\alpha t}{\overline{X}^2} \qquad (5.18)$$

Also defining

$$\psi = \sqrt{\nu_0 \alpha}\; f(\eta, \tau)$$

$$\frac{j}{j_0} = g(\eta, \tau) \qquad (5.19)$$

and using

$$\frac{\partial}{\partial t} = \frac{\alpha}{\overline{X}^2}\frac{\partial}{\partial \tau}$$

$$\frac{\partial}{\partial Y} = \frac{1}{\overline{X}}\sqrt{\frac{\alpha}{\nu_0}}\frac{\partial}{\partial \eta} = \frac{\eta}{Y}\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial \overline{X}} = -\frac{\eta}{\overline{X}}\frac{\partial}{\partial \eta} - \frac{2\tau}{\overline{X}}\frac{\partial}{\partial \tau}$$

the equations (5.16) and (5.17) are converted to

$$\frac{\partial^2 f}{\partial\eta\,\partial\tau} + 2\tau\left(\frac{\partial f}{\partial\eta}\ \frac{\partial^2 f}{\partial\eta\,\partial\tau} - \frac{\partial^2 f}{\partial\eta^2}\ \frac{\partial f}{\partial\tau}\right) + \left(\frac{\partial f}{\partial\eta}\right)^2 - g = \frac{\partial^3 f}{\partial\eta^3} \tag{5.20}$$

$$\frac{\partial g}{\partial\tau} + 2\tau\left(\frac{\partial f}{\partial\eta}\frac{\partial g}{\partial\tau} - \frac{\partial f}{\partial\tau}\ \frac{\partial g}{\partial\eta}\right) = \frac{\partial^2 g}{\partial\eta^2} \tag{5.21}$$

Again the velocity ratio $\frac{u}{u_e} = \frac{\partial f}{\partial\eta}$ because

$$u = -\frac{a_e}{a_0}\frac{\partial\psi}{\partial Y} = -\frac{a_e}{a_0}\sqrt{\frac{\alpha}{\nu_0}}\ \frac{1}{X}\ \sqrt{\alpha\nu_0}\ \frac{\partial f}{\partial\eta}$$

$$= -\frac{a_e}{a_0}\ \frac{\alpha}{X}\ \frac{\partial f}{\partial\eta}$$

and

$$u_e = -a_e M_e$$

$$= -\frac{a_e}{a_0}\ a_0 M_e$$

$$= -\frac{a_e}{a_0}\ \frac{\alpha}{X}\ .$$

The boundary conditions on f are given by (2.14). For g they are

$$g = g_w \quad \text{at} \quad \eta = 0, \quad \text{all} \quad \tau > 0$$

$$g = 1 \quad \text{as} \quad \eta \to 0, \quad \text{all} \quad \tau$$

$$g = 1 \quad \text{all} \quad \eta > 0\,, \quad \tau = 0 \tag{5.22}$$

We assume

$$\frac{\partial f}{\partial\eta} = \operatorname{erf}\frac{\eta}{\delta(\tau)}$$

i.e
$$f = \delta(\tau)\cdot\int_0^{\eta/\delta} \operatorname{erf} x\,dx$$

$$j = j_w + (j_0 - j_w)\operatorname{erf}\frac{\eta}{\Delta(\tau)}$$

i.e
$$g = g_w + (1-g_w)\operatorname{erf}\frac{\eta}{\Delta(\tau)}$$

where
$$g_w = \frac{j_w}{j_0} = \frac{T_w}{T_0}\ . \tag{5.23}$$

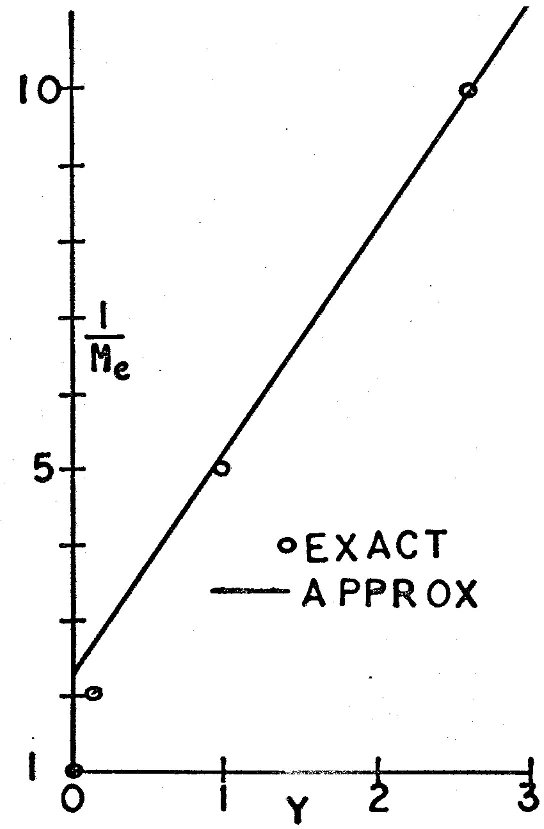
The integrated equations of momentum and energy are then

$$\frac{d\delta}{d\tau}\left[1 + 2(\sqrt{2} - 1)\tau\right] + \delta\sqrt{2} - (1-g_w)\frac{\Delta}{2} = \frac{2}{\delta} \qquad (5.24)$$

and

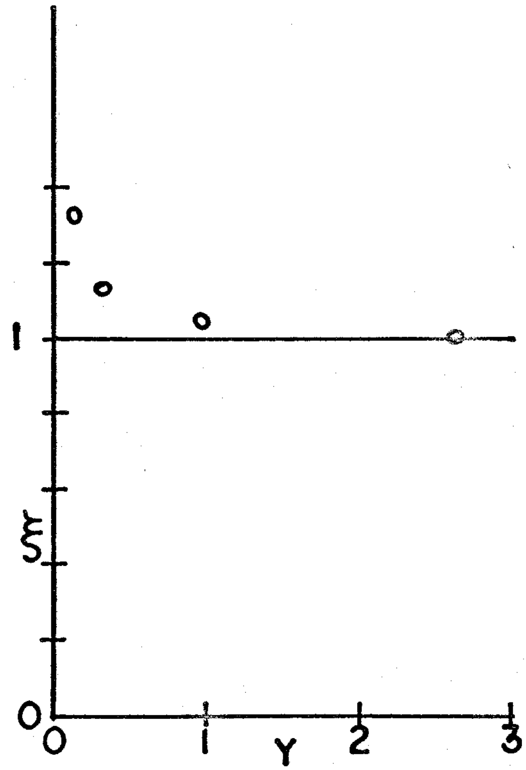
$$\frac{d\Delta}{d\tau}\left(1 + \frac{2\tau}{\sqrt{1+\delta^2/\Delta^2}}\right) + 2\tau\frac{d\delta}{d\tau}\,\frac{\delta/\Delta}{\sqrt{1+\delta^2/\Delta^2}} - 2\tau\frac{d\delta}{d\tau} = \frac{2}{\Delta} \qquad (5.25)$$

These are integrated numerically and presented as graphs in Figure 5.4 (for the particular case $g_w = \frac{1}{3}$). It is noted that in this case both the velocity and thermal layers do not reach a steady state. This is because the momentum and energy equations (5.24) and (5.25) are coupled except when g_w is equal to unity in which case there is no heat transfer at all. The thermal layer does not reach a steady state because of reasons given in Section 2 and therefore the velocity layer also does not.



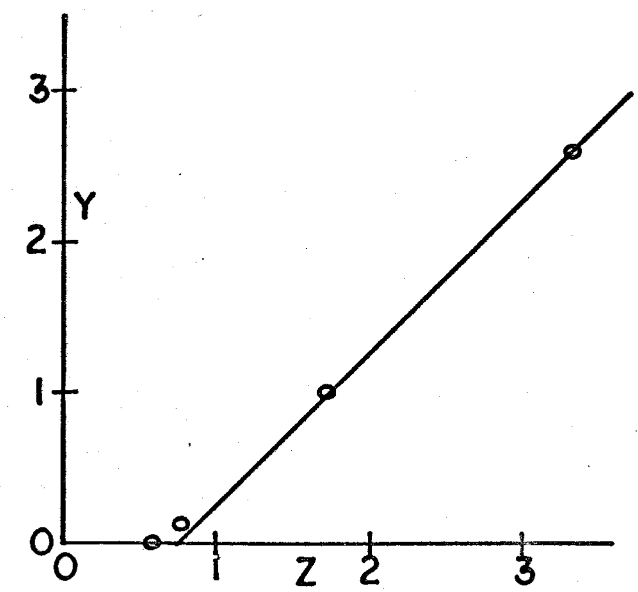
$$Y = \frac{16X}{27x_1}$$

FIG 5.1



$$\xi = \left(\frac{a_0}{a_e}\right)^2 \frac{P_0}{P_e}$$

FIG 5.2



$$Z = \frac{16x}{27x_1}$$

FIG 5.3

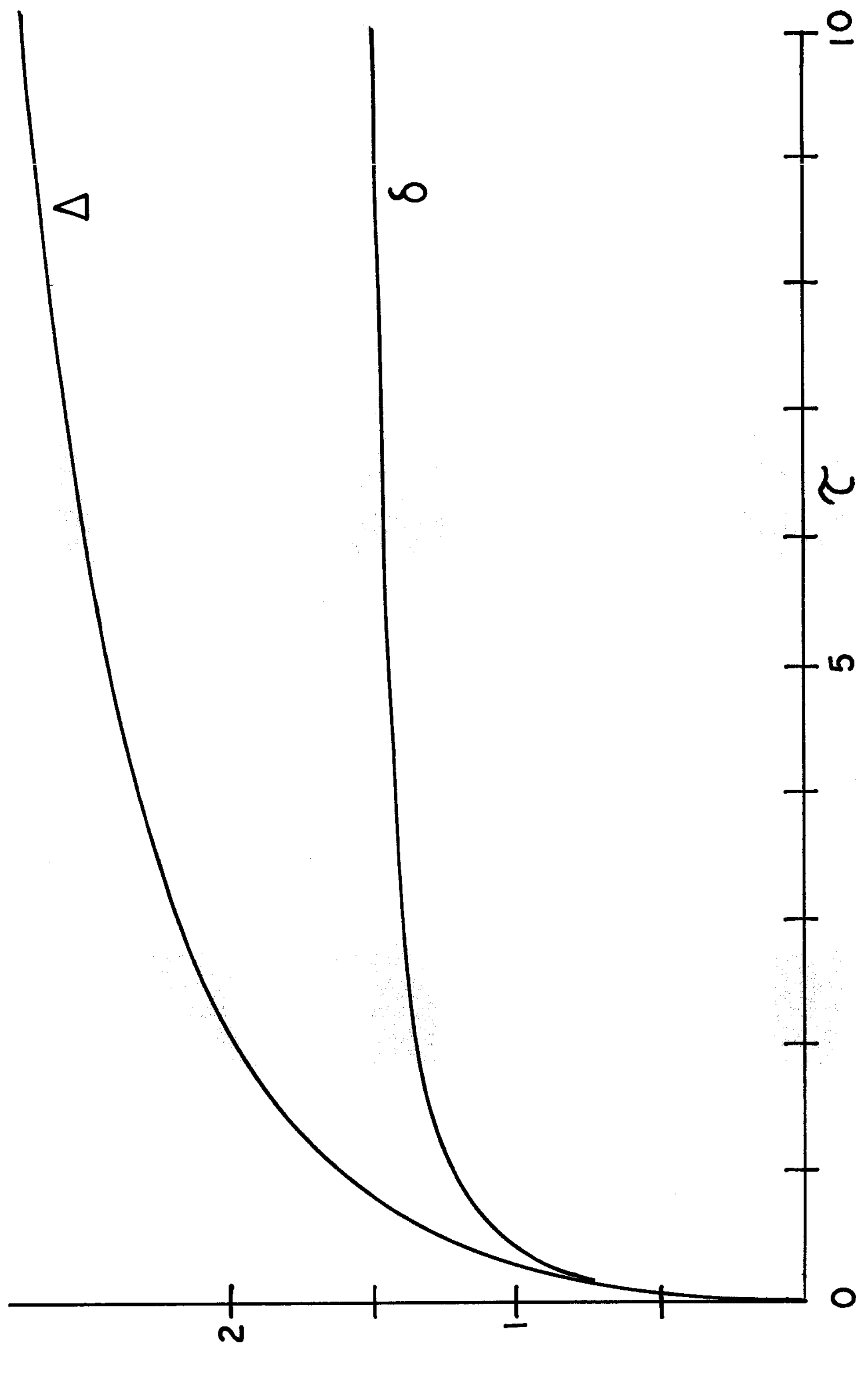


FIG 5.4

6. POINT SINK IN COMPRESSIBLE FLOW.

This is very similar to the analysis in Section 5. Only some of the equations are presented here and these are numbered so as to correspond with those in Section 5 for the line sink.

The equation of continuity for compressible, axisymmetric flowis

$$\frac{\partial \rho}{\partial t} + \frac{1}{x} \frac{\partial (x \rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \tag{6.1}$$

The equations of momentum and energy are the same as (5.2) and (5.3). The stream function is defined by

$$xu = - \frac{\rho_0}{\rho} \frac{\partial \psi}{\partial y}$$

$$xv = - \frac{\rho_0}{\rho} \left(x \frac{a_0}{a_e} \frac{\partial Y}{\partial t} - \frac{\partial \psi}{\partial x} \right) \tag{6.8}$$

In terms of ψ, the momentum equation is

$$- \frac{p_0}{p_e} \left(\frac{a_0}{a_e}\right)^2 \frac{1}{x} \frac{\partial^2 \psi}{\partial Y \partial T} + \frac{1}{x^2} \left(\frac{\partial \psi}{\partial Y} \frac{\partial^2 \psi}{\partial Y \partial X} - \frac{\partial \psi}{\partial X} \frac{\partial^2 \psi}{\partial Y^2} \right)$$

$$- \frac{1}{x^3} \frac{p_0 a_e}{p_e a_e} \left(\frac{\partial \psi}{\partial Y} \right)^2 - \frac{j}{j_0} (a_0 M_e) \frac{\partial}{\partial X} (a_0 M_e) = - \frac{\nu_0}{x} \frac{\partial^3 \psi}{\partial Y^3} \tag{6.9}$$

and the energy equation is

$$\left(\frac{a_0}{a_e}\right)^2 \frac{p_0}{p_e} x \frac{\partial j}{\partial T} - \left(\frac{\partial \psi}{\partial Y} \frac{\partial j}{\partial X} - \frac{\partial \psi}{\partial X} \frac{\partial j}{\partial Y} \right) = \nu_0 x \frac{\partial^2 j}{\partial Y^2} \ . \tag{6.10}$$

The relation between M_e and x, for a monotomic gas is

$$\left(1 + \frac{M_e^2}{3} \right)^4 = \left(M_e \frac{16}{9} \frac{x^2}{x_1^2} \right)^2 \tag{6.13}$$

x and M_e are related by

$$\frac{dX}{x_1} = \frac{3}{8} \left(1 + \frac{M_e^2}{3} \right)^{-3} \cdot \frac{1}{\sqrt{M_e}} \left(M_e - \frac{1}{M_e} \right) dM_e$$

which has been plotted in Fig. 6.1 after numerical integration. The following approximation is very good.

$$\frac{1}{\sqrt{M_e}} = \frac{4}{3x_1}(X + 1.05\, x_1) = \frac{\overline{X}}{\sqrt{\alpha}}$$

The plot of X vs x is shown in Fig. 6.2 and the approximation

$$x = X + 1.1\, x_1$$

is very good. As before $\left(\frac{a_0}{a_e}\right)^2 \frac{p_0}{p_e} \simeq 1.$

With these approximations (6.9) and (6.10) become

$$-\frac{1}{\overline{X}}\frac{\partial^2\psi}{\partial Y\partial T} + \frac{1}{\overline{X}^2}\left(\frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial Y\partial\overline{X}} - \frac{\partial\psi}{\partial\overline{X}}\frac{\partial^2\psi}{\partial Y^2}\right) - \frac{1}{\overline{X}^3}\left(\frac{\partial\psi}{\partial Y}\right)^2 + 2\frac{\alpha^2}{\overline{X}^5}\frac{j}{j_0} = -\frac{\nu_0}{\overline{X}}\frac{\partial^3\psi}{\partial Y^3}. \qquad (6.16)$$

$$\overline{X}\frac{\partial j}{\partial T} - \left(\frac{\partial\psi}{\partial Y}\frac{\partial j}{\partial X} - \frac{\partial\psi}{\partial X}\frac{\partial j}{\partial Y}\right) = \nu_0\,\overline{X}\frac{\partial^2 j}{\partial Y^2} \qquad (6.17)$$

Non dimensional variables are defined as before.

$$\eta = \frac{Y}{\overline{X}}\sqrt{\frac{\alpha}{2\nu\overline{X}}}$$

$$\tau = \alpha t/\overline{X}^3$$

$$\psi = \sqrt{2\alpha\nu\overline{X}}\;\; f(\eta, \tau)$$

$$\frac{j}{j_0} = g(\eta, \tau) \qquad (6.18)$$

Finally, the equations for f and g are

$$\frac{\partial^2 f}{\partial\tau\partial\eta} + 3\tau\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\partial\tau} - \frac{\partial^2 f}{\partial\eta^2}\frac{\partial f}{\partial\tau}\right)$$

$$+ 2\left(\frac{\partial f}{\partial\eta}\right)^2 + \frac{f}{2}\frac{\partial^2 f}{\partial\eta^2} = 2g + \frac{1}{2}\frac{\partial^3 f}{\partial\eta^3} \qquad (6.20)$$

$$\frac{\partial g}{\partial\tau} + 3\tau\left(\frac{\partial f}{\partial\eta}\frac{\partial g}{\partial\tau} - \frac{\partial f}{\partial\tau}\frac{\partial g}{\partial\eta}\right) + \frac{\partial g}{\partial\eta}\cdot\frac{f}{2} = \frac{1}{2}\frac{\partial^2 g}{\partial\eta^2} \qquad (6.21)$$

The integrated equations are

$$\frac{d\delta}{d\tau} \cdot \left(1 + 3\tau(\sqrt{2} - 1)\right) + \frac{\delta}{2}\,(3\sqrt{2} + 1) - (1 - g_w)\,\Delta = \frac{1}{\delta} \qquad (6.24)$$

$$\frac{d\Delta}{d\tau} \cdot \left(1 + \frac{3\tau}{\sqrt{1 + \delta^2/\Delta^2}}\right) + 3\tau\,\frac{d\delta}{d\tau} \cdot \left(\frac{1}{\sqrt{1 + \Delta^2/\delta^2}} - 1\right) - \frac{\delta}{2}\left(\sqrt{1 + \Delta^2/\delta^2} - 1\right) = \frac{1}{\Delta} \qquad (6.25)$$

These are integrated numerically and the graphs are presented in Figure 6.3 ($g_w = \frac{1}{3}$ as before). Both the velocity and thermal layers do not reach a steady state.

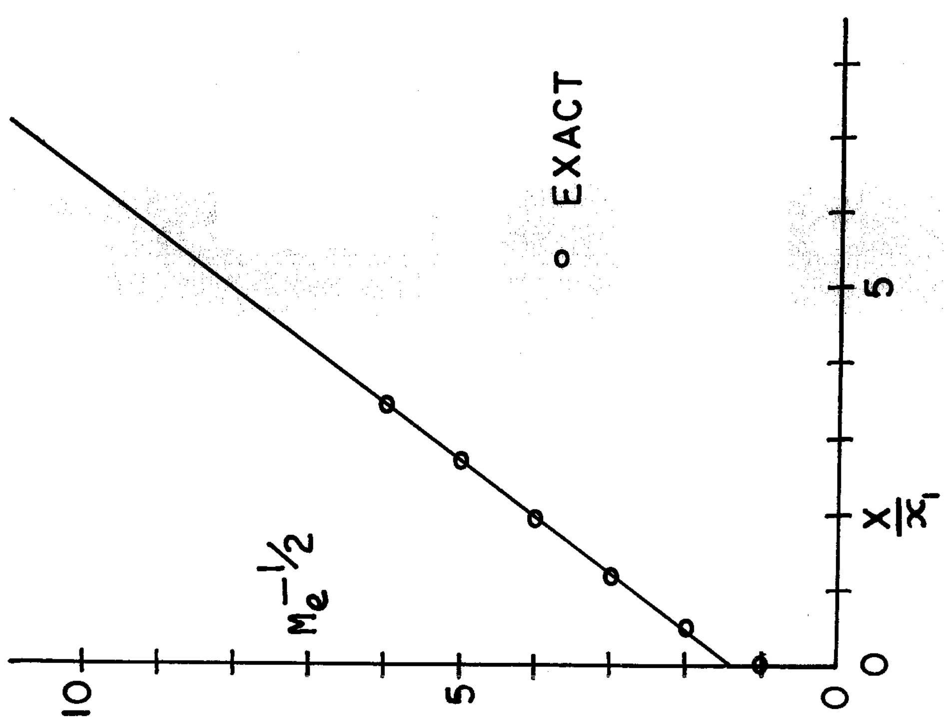


FIG 6.1

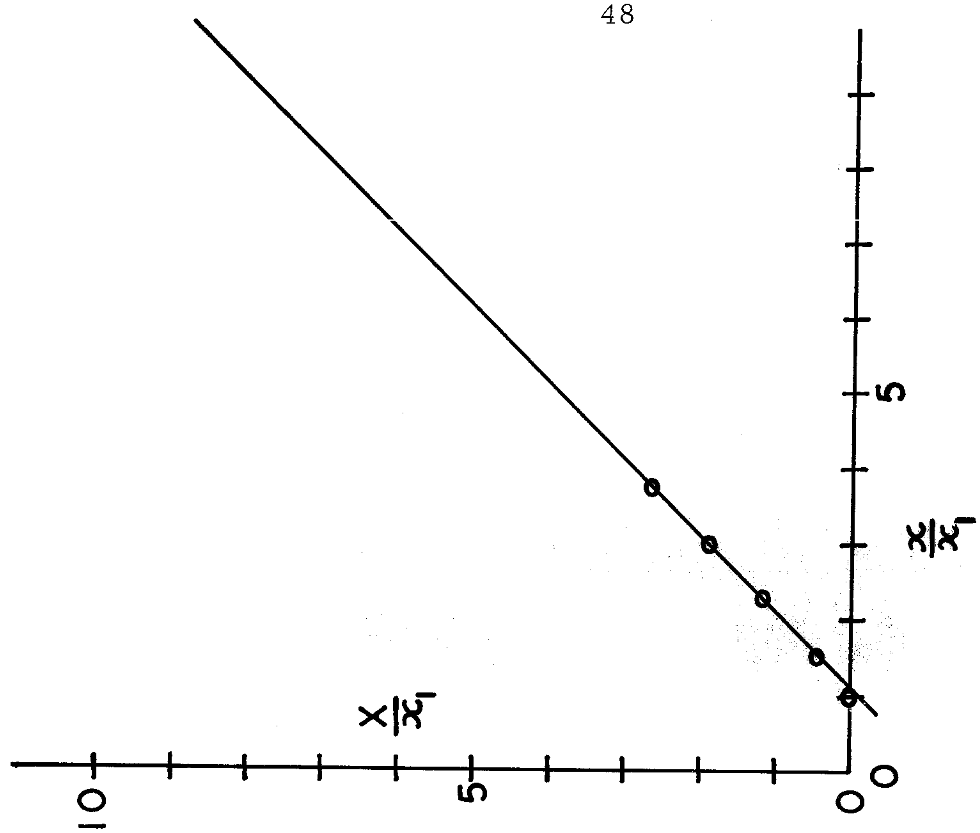


FIG 6.2

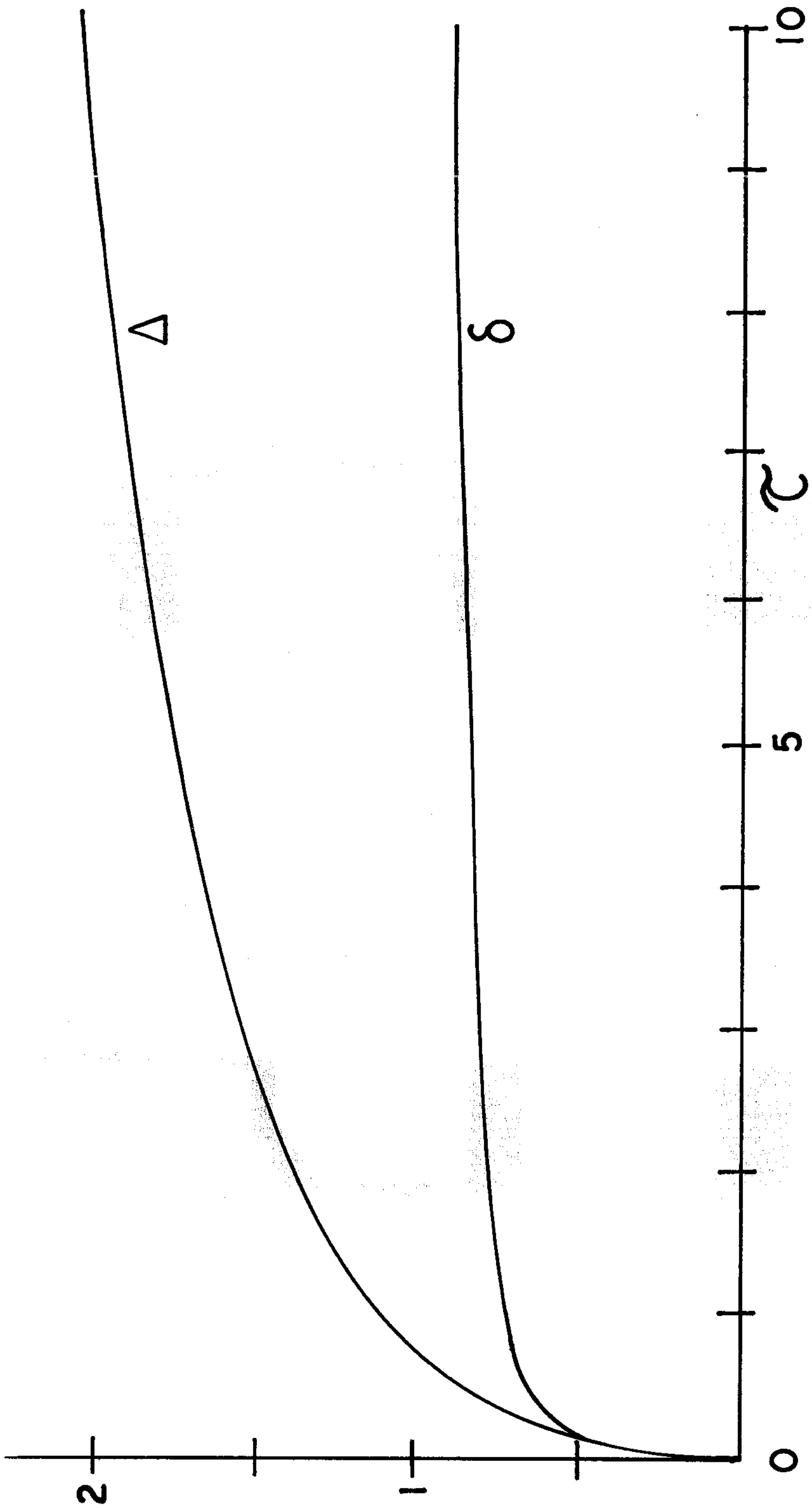


FIG 6.3

LIST OF SYMBOLS USED IN PART I.

A	constant mass flow into the sink
A_n	constants in the expansion for T
a	speed of sound
d	width of the slit or diameter of the hole
F	integral of the error function
$f(z)$	function appearing in the confluent hypergeometric equation
G	nondimensional heat flux through the slit
g	the ratio j/j_0
h	specific enthalpy
$h(z)$	function appearing in the hypergeometric equation
j	total specific enthalpy $= h + \frac{1}{2} u^2$
L	length scale of the thermal layer
l	length scale of the velocity layer
M	Mach number
M	confluent hypergeometric function
m	index in the velocity distribution $U = -\alpha/x^m$
n	integer
p	pressure
q	radial velocity
r	radial distance from the sink
$R_n(\tau)$	functions in the expansions for T
T	temperature
t	time
U	hypergeometric function
U	x component of velocity outside the boundary layer

u	x component of velocity
v	y component of velocity
w(z)	function appearing in the hypergeometric equation
$X, \overline{X}$	transformed x coordinates
x	x coordinate
Y	transformed y coordinate
y	y coordinate
z	independent variable in the hypergeometric equation
α	sink strength
γ	ratio of specific heats
$\Delta(\tau)$	thickness of the thermal layer
$\delta(\tau)$	thickness of the velocity layer
η	nondimensional variable $= \frac{y}{x}\sqrt{\frac{\alpha}{\nu}}$ for the line sink and $= \frac{y}{x}\cdot\sqrt{\frac{\alpha}{2\nu x}}$ for the point sink
μ	viscosity
ν	kinematic viscosity
ρ	density
θ	polar angle
φ	angle
τ	nondimensional variable $= \frac{\alpha t}{x^2}$ for the line sink and $= \frac{\alpha t}{x^3}$ for the point sink
ξ	$= \left(\frac{a_0}{a_e}\right)^2 \cdot \frac{p_0}{p_e}$

subscripts

0	standard conditions
∞	conditions at infinity
e	conditions outside the boundary layer
1	values at some fixed position

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PART IIa.

THE DUSTY JET

1. EQUATIONS OF MOTION OF A DUSTY GAS.

We follow Saffman's formulation of these equations [i]. The dust particles are supposed to be uniform in shape and size. Their velocity field is described by $\underline{U}(\underline{x},t)$ and the number density by $N(\underline{x},t)$. The bulk concentration (i.e. concentration by volume) of the dust is very small so that the net effect of the dust on the gas is equivalent to an extra force $KN(\underline{U}-\underline{u})$ where $\underline{u}$ is the velocity field of the gas and K is a constant, where it is also supposed that the Reynolds number of the relative motion of dust and gas is small compared to unity so that the force between dust and gas is proportional to the relative velocity. Then with small bulk concentration and neglect of compressibility of the gas, the equations of motion and continuity of the gas are

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right) = -\nabla p + \mu \nabla^2 \underline{u} + KN(\underline{U} - \underline{u}) \tag{1.1}$$

$$\nabla \cdot \underline{u} = 0 \tag{1.2}$$

where p is the pressure less the hydrostatic pressure, and ρ and μ are the density and viscosity of the clean gas. For spherical dust particles of radius a, $K = 6\pi a\mu$ by the Stokes drag formula.

Let $f = mN/\rho$ be the mass concentration of the dust. Let ρ be the density of the gas and ρ_1 the density of the material of the dust. Then the bulk concentration of dust which is the ratio of the volume occupied by the dust to the volume of clean gas in a given mass of dusty gas, is $f\rho/\rho_1$. For common materials ρ_1/ρ is of the order of several thousand, so that the mass concentration f may be of the order of unity with the bulk concentration remaining very small.

The force exerted on the dust by the gas is equal and opposite to the force exerted on the gas by the dust, so that the equation of motion of the dust is

$$mN\left(\frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \nabla \underline{U}\right) = mN\underline{g} + KN(\underline{u} - \underline{U}) \qquad (1.3$$

where m is the mass of a dust particle, N is the number of dust particles per unit volume of clean gas and $\underline{g}$ is the acceleration due to gravity. The buoyancy force is neglected since ρ/ρ_1 is small. The equation of continuity of the dust is

$$\frac{\partial N}{\partial t} + \mathrm{div}(N\underline{U}) = 0 \qquad (1.4)$$

Let $\tau = m/K$. This may be called the relaxation time of the dust particles. It is a measure of the time for the dust particles to adjust to changes in the gas velocity. For spherical particles of radius a,

$$\tau = m/K = \frac{4}{3}\pi a^3 \rho_1 / 6\pi a\mu$$

$$= \frac{2}{9}\frac{a^2}{\nu} \cdot \frac{\rho_1}{\rho} \qquad (1.5)$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the clean gas.

We consider a steady, two-dimensional dusty jet with its axis along x. The flow quantities do not vary with y and are functions of only x and z. With

$$u = (u,0,w) \quad \text{and}$$

$$U = (U,0,W)$$

the equations with the boundary layer approximation are then

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = \nu \frac{\partial^2 u}{\partial z^2} + \frac{f}{\tau} (U-u) \tag{1.6}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{1.7}$$

$$U \frac{\partial U}{\partial x} + W \frac{\partial U}{\partial z} = \frac{u-U}{\tau} \tag{1.8}$$

$$U \frac{\partial W}{\partial x} + W \frac{\partial W}{\partial z} = \frac{w-W}{\tau} \tag{1.9}$$

$$\frac{\partial NU}{\partial x} + \frac{\partial NW}{\partial z} = 0 \tag{1.10}$$

The equations (1.6 to 1.10) are the five equations which determine U, W, N, u and w. p = constant and g has been ignored in the equations.

Some typical values of the relaxation time are given below.

Material	Density gm/cc	a (microns)	τ
Coal dust in air	1.2	25	9.1 ms
Coal dust in air	1.2	50	36.4 ms
Sand in air	2	200	0.97 s
Cigarette smoke	1	.3	1.09 μs
Gold particles in olive oil	19.3	325	4.95 ms

2. THE JET WITH FINE DUST.

This is the limit $\tau \to 0$. Equations (1.8) and (1.9) show that in this limit $u = U$ and $w = W$. In the right hand side of (1.6), $\frac{f}{\tau}(U - u)$ is replaced by $f\left(U \frac{\partial U}{\partial x} + W \frac{\partial U}{\partial z}\right)$ using (1.8) and this is equal to $f\left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z}\right)$. Therefore the equations are now

$$(1 + f)\left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z}\right) = \nu \frac{\partial^2 u}{\partial z^2} \qquad (2.1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad (2.2)$$

$$u = U \qquad (2.3)$$

$$w = W \qquad (2.4)$$

$$\frac{\partial fU}{\partial x} + \frac{\partial fW}{\partial z} = 0 = u \frac{\partial f}{\partial x} + w \frac{\partial f}{\partial z}. \qquad (2.5)$$

The equations (2.1), (2.2) and (2.5) determine f, u and w as functions of x and z. Equations (2.3) and (2.4) show that there is no slip between the dust and gas in the limit $\tau \to 0$.

The flow configuration is shown in Fig. 2.1. The jet comes out from a slit of width d with a velocity u_0. The dust concentration is f_0 at $x = 0$. There is a region (the core) where the maximum velocity in the jet is still u_0. At some distance downstream the core disappears and the jet spreads with the maximum velocity decreasing with increasing x.

Von Mises transformation is natural to this problem. Here $X = x$ and ψ the stream function are taken as the independent variables. The equations of transformation are

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} - w\frac{\partial}{\partial \psi}$$

$$\frac{\partial}{\partial z} = u\frac{\partial}{\partial \psi} \qquad (2.6)$$

where $u = \frac{\partial \psi}{\partial z}$ and $w = -\frac{\partial \psi}{\partial x}$. The equation (2.5) becomes

$$u\left(\frac{\partial f}{\partial X} - w\frac{\partial f}{\partial \psi}\right) + wu\frac{\partial f}{\partial \psi} = 0$$

i.e.

$$\frac{\partial f}{\partial X} = 0$$

or

$$f = f(\psi) \qquad (2.7)$$

(2.1) becomes

$$(1+f)\,u\frac{\partial u}{\partial X} = \nu u\frac{\partial}{\partial \psi}\left(u\frac{\partial u}{\partial \psi}\right)$$

i.e.

$$(1+f)\quad \frac{\partial u}{\partial X} = \nu\frac{\partial}{\partial \psi}\left(u\frac{\partial u}{\partial \psi}\right) \qquad (2.8)$$

u and f are found from the two equations (2.7) and (2.8) as functions of X and ψ. From these u, w and f may be rewritten as functions of x and z. Initially (at $x = 0$) f has the value f_0 from $\psi = 0$ to $\psi = u_0\, d/2$ and is zero for ψ greater than $u_0\, d/2$. Therefore the solution for f is the rectangular profile shown in Fig. 2.2. With this value for f, the equation (2.8) is solved for u as a function of X and ψ.

First the region where the core is present is considered. The equation (2.8) will be solved by the integral method of Karman and Pohlhausen assuming for u the profile shown in Fig. 2.3. This shows that $u = u_0$ for $z \leq l_1$ or $\psi \leq u_0 l_1$ and u has the profile

$$\frac{u}{u_0} = 1 - \left(\frac{\psi - u_0\ell_1}{\Psi}\right)^2 = 1 - \left(\frac{\psi - \Phi}{\Psi}\right)^2 \qquad (2.9)$$

with velocity scale u_0 and stream function scale Ψ for $z > \ell_1$. This corresponds to the well known hyperbolic tangent profile for ψ found for the simple jet. (2.9) corresponds to the velocity profile in terms of z given by

$$\psi = u_0 \ell_1 + \Psi \tanh \cdot \frac{u_0(z-\ell_1)}{\Psi} \quad \text{for} \quad z > \ell_1$$

and

$$\psi = u_0 z \qquad \text{for} \qquad z \leqslant \ell_1 \tag{2.10}$$

(2.9) follows from (2.10) on differentiation with respect to z, i.e. using $\frac{\partial\psi}{\partial z}$. ~~Using the profile~~ (2.9), the two equations needed for finding Φ and Ψ are found from (2.8). Integration of (2.8) all the way across the jet gives

$$\int_0^{\Phi+\Psi} (1+f) \frac{\partial u}{\partial X} \, d\psi = \nu \left[u \frac{\partial u}{\partial \psi} \right]_0^{\Phi+\Psi} = 0 \, .$$

$\frac{\partial u}{\partial X} = 2u_0(\psi-\Phi)^2 \frac{\Psi_X}{\Psi^3} + 2u_0(\psi-\Phi)\frac{\Phi_X}{\Psi^2}$ outside the core where $\Psi_X = \frac{d\Psi}{dX}$ and $\Phi_X = \frac{d\Phi}{dX}$. In the core $\frac{\partial u}{\partial X} = 0$. Therefore

$$\int_\Phi^{\Phi+\Psi} (1+f) \left[2u_0(\psi-\Phi)^2 \frac{\Psi_X}{\Psi^3} + 2u_0(\psi-\Phi)\frac{\Phi_X}{\Psi^2} \right] d\psi = 0$$

i.e.
$$\frac{\Psi_X}{3} + \frac{\Phi_X}{2} + f_0 \left[\tfrac{1}{3}\left(\frac{u\,d}{2} - \Phi\right)^3 \frac{\Psi_X}{\Psi^3} + \tfrac{1}{2}\left(\frac{u_0 d}{2} - \Phi\right)^2 \frac{\Phi_X}{\Psi^2} \right] = 0 \tag{2.11}$$

Another equation may be obtained by taking a moment of (2.8) across the width of the jet but it is simpler to take instead the following. The integration is performed from $\psi = 0$ to $\psi = \Phi + \Psi/\sqrt{2}$ at which point $u = \frac{1}{2} u_0$. The dusty portion of the jet extends up to $\psi = u_0 d/2$ and the point $\psi = \Phi + \Psi/\sqrt{2}$ includes the whole of the dusty region. This integration gives the equation

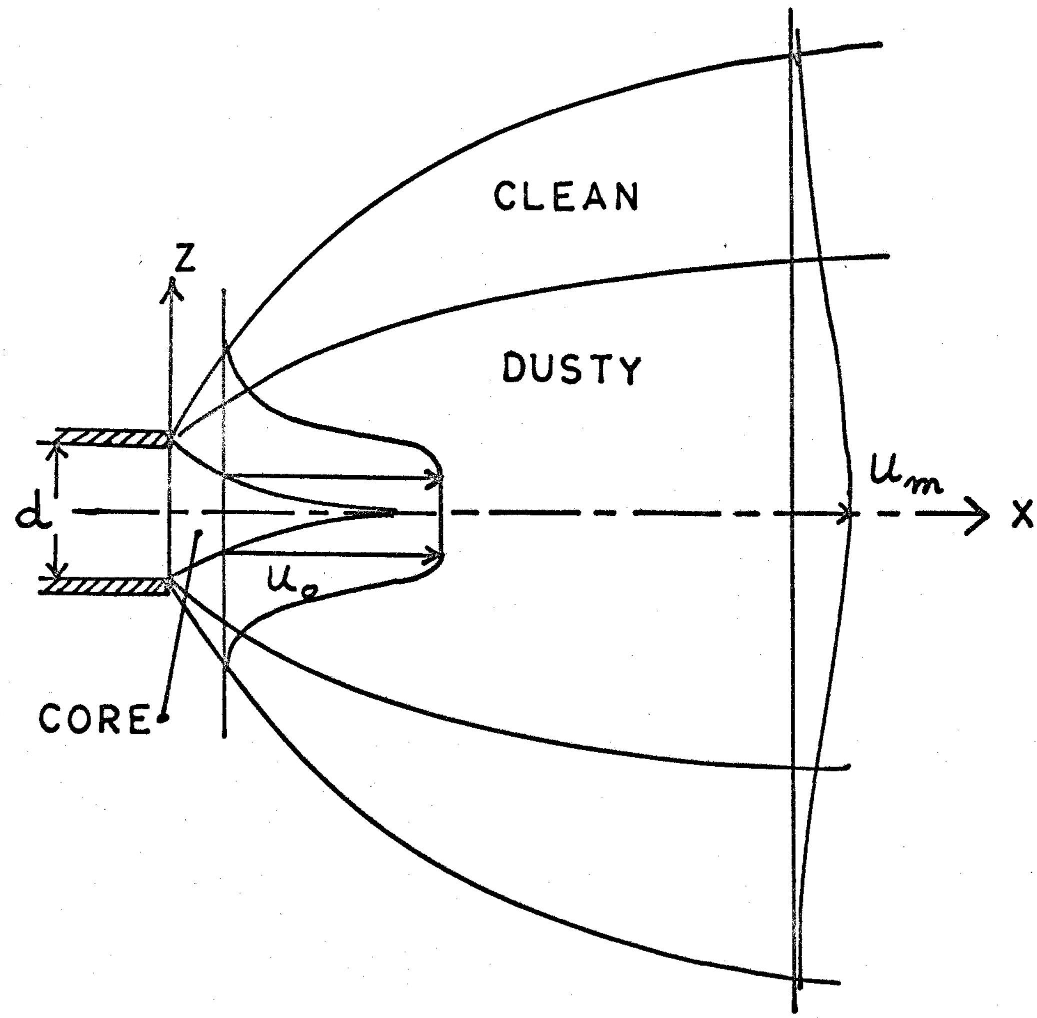

FIG 2.1

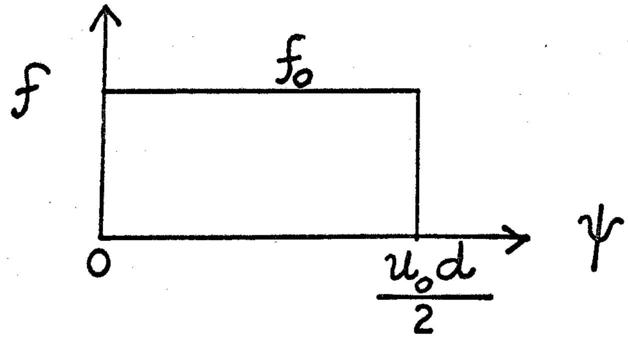

FIG 2.2

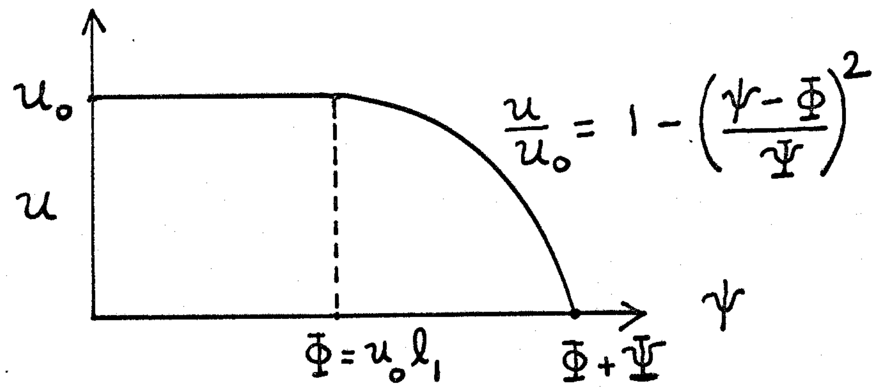
u_o
u
$\frac{u}{u_o} = 1 - \left(\frac{\psi - \Phi}{\Psi}\right)^2$
ψ
$\Phi = u_o l_1$
$\Phi + \Psi$

FIG 2.3

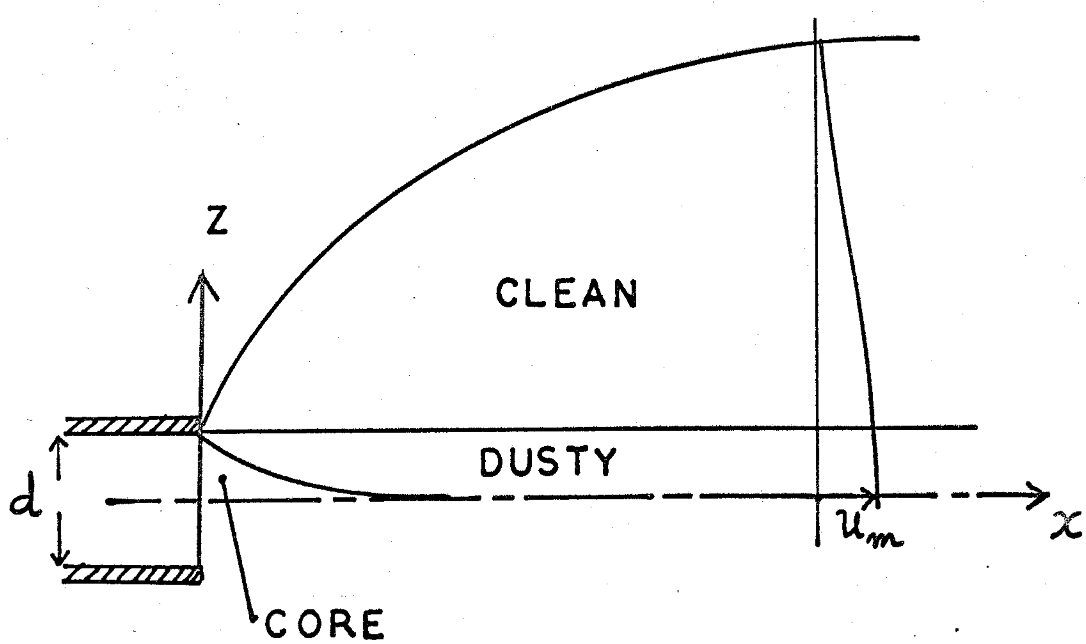
Z
CLEAN
DUSTY
d
u_m
x
CORE

FIG 3.1

$$2u_0\left(\frac{\Psi_X}{6\sqrt{2}} + \frac{\Phi_X}{4}\right) + 2u_0 f_0\left[\frac{1}{3}\left(\frac{u_0 d}{2} - \Phi\right)^3 \frac{\Psi_X}{\Psi^3} + \frac{1}{2}\left(\frac{u_0 d}{2} - \Phi\right)^2 \frac{\Phi_X}{\Psi^2}\right] = \nu \frac{u_0}{2}\left(\frac{-2u_0}{\sqrt{2}\Psi}\right). \quad (2.12)$$

From (2.11) and (2.12) we get

$$\frac{\Psi_X}{6\sqrt{2}} + \frac{\Phi_X}{4} - \left(\frac{\Psi_X}{3} + \frac{\Phi_X}{2}\right) = -\frac{\nu u_0}{2\sqrt{2}} \cdot \frac{1}{\Psi}$$

or

$$\Psi_X\left(\frac{1}{3} - \frac{1}{6\sqrt{2}}\right) + \frac{\Phi_X}{4} = \frac{\nu u_0}{2\sqrt{2}\,\Psi} \qquad (2.13)$$

(2.11) and (2.13) can be solved by iteration for small f_0. For $f_0 = 0$,

$$\frac{\Psi}{3} + \frac{\Phi}{2} = \text{constant} = \frac{1}{2}(u_0 \frac{d}{2})$$

Using this in (2.13),

$$\Psi_X\left(\frac{1}{3} - \frac{1}{6\sqrt{2}} - \frac{2}{3} \cdot \frac{1}{4}\right) = \frac{\nu u_0}{2\sqrt{2}\,\Psi}$$

or

$$2\Psi\Psi_X = \frac{6\nu u_0}{\sqrt{2} - 1}$$

i.e.

$$\Psi^2 = \frac{6\nu u_0}{\sqrt{2}-1} X \qquad (2.14)$$

and

$$\Phi = u_0 \frac{d}{2} - \frac{2}{3} \cdot \sqrt{\frac{6\nu u_0 X}{\sqrt{2} - 1}} \qquad (2.15)$$

For small f_0 we shall use the approximate relation (2.15) in the form

$$\Phi = u_0 \frac{d}{2} - \frac{2}{3}\Psi \qquad \text{in (2.11) to get}$$

$$\frac{\Psi_X}{3} + \frac{\Phi_X}{2} + f_0\left[\frac{1}{3} \cdot \frac{8}{27}\Psi_X + \frac{1}{2} \cdot \frac{4}{9}\Phi_X\right] = 0$$

$$\frac{\Psi_X}{3} + \frac{\Phi_X}{2} + f_0\left[\frac{8}{81}\Psi_X - \frac{4}{27}\Psi_X\right] = 0; \quad \frac{\Psi}{3} + \frac{\Phi}{2} - \frac{4}{81} f_0 \Psi = u_0 \frac{d}{4} .$$

Using this in (2.13) we get

$$\Psi_X\left(\frac{1}{3} - \frac{1}{6\sqrt{2}} + \frac{1}{4}\left[-\frac{2}{3} + \frac{8}{81} f_0\right]\right) = \frac{\nu u_0}{2\sqrt{2}\Psi}$$

which gives on integration

$$\Psi^2 = \frac{6\nu u_0 X}{\sqrt{2} - 1 + \frac{12}{81}\sqrt{2} f_0} \tag{2.16}$$

and

$$\Phi = \frac{u_0 d}{2} - \frac{2}{3}\left(1 - \frac{4f_0}{27}\right) \cdot \sqrt{\frac{6\nu u_0 X}{\left(\sqrt{2} - 1 + \frac{12}{81}\sqrt{2} f_0\right)}} \tag{2.17}$$

(2.16) and (2.17) represent the first approximation when f_0 is small. They are reasonable for $f_0 < 1$.

The position where the core vanishes (i.e. $\Phi = 0$) is given by

$$\frac{X_1}{d} = \frac{3}{32}\left(\frac{u_0 d}{\nu}\right) \cdot \frac{\sqrt{2} - 1 + \frac{12\sqrt{2}}{81} f_0}{\left(1 - \frac{4f_0}{27}\right)^2} \tag{2.18}$$

X_1/d is proportional to the jet Reynolds number.

SOLUTION FOR $X > X_1$.

For $X > X_1$ there is no core. The velocity scale u_m (the maximum velocity) and the stream function scale Ψ are to be found. The velocity profile is assumed to be

$$\frac{u}{u_m} = 1 - \left(\frac{\psi}{\Psi}\right)^2 \tag{2.19}$$

The equations are obtained as before by integrating (2.8)

$$\int_0^{\Psi} (1+f)\,\frac{\partial u}{\partial X}\,d\psi = \left[\nu\, u\,\frac{\partial u}{\partial \psi}\right]_0^{\Psi} = 0$$

Here

$$\frac{\partial u}{\partial X} = u_{mX}\left(1 - \frac{\psi^2}{\Psi^2}\right) + 2u_m\,\frac{\psi^2}{\Psi^3}\,\Psi_X$$

where

$$u_{mX} = \frac{du_m}{dX}\ , \qquad \Psi_X = \frac{d\Psi}{dX}\ .$$

Therefore

$$u_{mX}\cdot\tfrac{2}{3}\,\Psi + \tfrac{2}{3}\,u_m\,\Psi_X + \int_0^{u_m d/2} f_0\,\frac{\partial u}{\partial X}\,d\psi = 0$$

i.e.

$$\tfrac{2}{3}(u_m\Psi)_X + f_0\left[u_{mX}\cdot u_0\frac{d}{2} - \frac{u_{mX}}{3\Psi^2}\cdot\left(u_0\frac{d}{2}\right)^3 + \frac{2u_m\Psi_X}{3\Psi^3}\left(\frac{u_0 d}{2}\right)^3\right] = 0 \quad (2.20)$$

The other equation is

$$\int_0^{\Psi/\sqrt{2}} (1+f)\,\frac{\partial u}{\partial X}\,d\psi = \nu\,\frac{u_m}{2}\cdot\left(-\sqrt{2}\,\frac{u_m}{\Psi}\right)$$

or

$$u_{mX}\left(\frac{\Psi}{\sqrt{2}} - \frac{\Psi}{6\sqrt{2}}\right) + 2\,\frac{u_m}{6\sqrt{2}}\,\Psi_X + f_0\left[u_{mX}\,\frac{u_0 d}{2} + \frac{2}{3}\,\frac{\Psi_X}{\Psi^3}\left(\frac{u_0 d}{2}\right)^3 - \frac{u_{mX}}{3\Psi^2}\left(\frac{u_0 d}{2}\right)^3\right] = \frac{-u_m^2\,\nu}{\sqrt{2}}$$

(2.21)

(2.20) and (2.21) are solved by iteration as before. For $f_0 = 0$, (2.20) gives

$$u_m\,\Psi = u_0\Psi_1 \quad (2.22)$$

where Ψ_1 is the value of Ψ at $X = X_1$. From (2.21)

$$u_{mX} \frac{5}{6\sqrt{2}} \Psi + \frac{1}{3\sqrt{2}} u_m \Psi_X = -\frac{u_m^2 \nu}{\sqrt{2}\,\Psi}$$

Eliminating u_m we get

$$\Psi^2 \Psi_X = 2\nu u_0 \Psi_1$$

or

$$\frac{\Psi^3}{3} = 2\nu u_0 \Psi_1 X + \text{constant}$$

Using the initial condition,

$$\Psi^3 - \Psi_1^3 = 6\nu u_0 \Psi_1 \cdot (X - X_1) \tag{2.23}$$

(2.22) and (2.23) constitute the zeroth approximation. Now (2.20) can be integrated to yield

$$\frac{2}{3} u_m \Psi + f_0 \left[u_m \frac{u_0 d}{2} + \frac{1}{3}\left(\frac{u_0 d}{2}\right)^3 \left(-\frac{u_m}{\Psi^2}\right)\right]$$
$$= \frac{2}{3} u_0 \Psi_1 + f_0 \left[u_0 \left(\frac{u_0 d}{2}\right) + \frac{1}{3}\left(\frac{u_0 d}{2}\right)^3 \left(-\frac{u_0}{\Psi_1^2}\right)\right]$$
$$= A \quad \text{say} \tag{2.24}$$

(2.20) minus (2.21) is

$$\frac{2}{3} \frac{d}{dX}(u_m \Psi) - \left(u_{mX} \frac{5}{6\sqrt{2}} \Psi + \frac{1}{3\sqrt{2}} u_m \Psi_X\right) = \frac{\sqrt{2}\, u_m^2 \nu}{2\Psi} \tag{2.25}$$

The zeroth approximation is used to eliminate u_m in (2.24) (only in the term containing f_0 which is small). (2.24) gives for small f_0

$$\frac{1}{3} u_m \Psi + f_0 \left[\frac{u_0 \Psi_1}{\Psi} \left(\frac{u_0 d}{2}\right) + \frac{1}{3}\left(\frac{u_0 d}{2}\right)^3 \cdot \left(-\frac{u_0 \Psi_1}{\Psi} \cdot \frac{1}{\Psi^2}\right)\right] = A$$

This u_m is used in (2.25) to get a better approximation for Ψ. Let

$$C = u_0 \Psi_1 \cdot \frac{u_0 d}{2}$$

$$D = \frac{1}{3}\left(\frac{n_0 d}{2}\right)^3 u_0 \Psi_1$$

Then

$$\frac{2}{3} u_m \Psi + f_0\left(\frac{C}{\Psi} - \frac{D}{\Psi^3}\right) = A .$$

(2.25) becomes

$$\left(\frac{2}{3} - \frac{5}{6\sqrt{2}}\right) \frac{d}{d_X}\left(-\frac{3}{2} f_0\left(\frac{C}{\Psi} - \frac{D}{\Psi^3}\right)\right) + \frac{1}{2\sqrt{2}} \cdot \left(\frac{3}{2}\frac{A}{\Psi} - \frac{3}{2} f_0\left(\frac{C}{\Psi^2} - \frac{D}{\Psi^4}\right)\right) \Psi_X$$
$$= \frac{\sqrt{2}}{2} \frac{\nu}{\Psi}\left(\frac{3}{2}\frac{A}{\Psi} - \frac{3}{2} f_0\left(\frac{C}{\Psi^2} - \frac{D}{\Psi^4}\right)\right)^2 .$$

When $\Psi >> \Psi_1$, this may be integrated in closed form because we have, approximately

$$\left[\left(\frac{2}{3} - \frac{5}{6\sqrt{2}}\right) f_0 \frac{C}{\Psi^2} + \frac{1}{2\sqrt{2}}\left(\frac{A}{\Psi} - f_0 \frac{C}{\Psi^2}\right)\right] \Psi^3 \cdot \left(1 + 2f_0 \frac{C}{A\Psi}\right) d\Psi$$
$$= \frac{\sqrt{2}}{2} \nu \cdot \frac{3}{2} A^2 \, dX$$

which gives on integration

$$\frac{(\Psi^3 - \Psi_1^3)}{9A\nu} + \frac{(2\sqrt{2}-1)}{9} \cdot \frac{f_0}{A^2} \cdot \frac{u_0^2 \Psi_1 d}{2\nu} (\Psi^2 - \Psi_1^2) = (X - X_1) \qquad (2.26)$$

where the initial condition $\Psi(X_1) = \Psi_1$ has also been used. u_m is given by (2.24) and (2.26).

The approximate solution for Ψ, Φ and u_m is given by the equations (2.16) and (2.17) for $X \leq X_1$ and by (2.24) and (2.26) for $X > X_1$, $\Psi > \Psi_1$. The length scale in the core region ℓ_1 is known as a function of X. Another length scale ℓ_2 is defined as the value of z for which $\frac{u}{u_m}$ is $\frac{1}{2}$. i.e. $\frac{\psi}{\Psi} = \frac{1}{\sqrt{2}} = \tanh \frac{u_m \ell_2}{\Psi}$.

or

$$\ell_2 = \tanh^{-1}\left(\frac{1}{\sqrt{2}}\right) \cdot \frac{\Psi}{u_m}$$

where Ψ and u_m are known as functions of X. The boundary of the dust is given by ℓ_3, the value of z for which ψ is $\frac{u_0 d}{2}$. For $X < X_1$, ℓ_3 is given by

$$\frac{u_0 d}{2} = \Phi + \Psi \tanh \frac{u_0(\ell_3 - \ell_1)}{\Psi}$$

and for $x > X_1$ by

$$\frac{u_0 d}{2} = \Psi \tanh u_m \frac{\ell_3}{\Psi} .$$

The three length scales and the velocity scale u_m constitute the solution of the dusty jet problem.

3. THE JET WITH COARSE DUST.

This is the limit τ tending to infinity which may be obtained by letting a tend to infinity. τ and f are proportional to a^2 and a^3 respectively and therefore tend to infinity. The ratio $f/\tau = 6\pi a\mu N$ and if aN is held constant as a tends to infinity, f/τ can be of order unity. We now have the equations

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} + \frac{f}{\tau}(U - u) \tag{3.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{3.2}$$

$$U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial z} = 0 \tag{3.3}$$

$$U\frac{\partial W}{\partial x} + W\frac{\partial W}{\partial z} = 0 \tag{3.4}$$

$$\frac{\partial fU}{\partial x} + \frac{\partial fW}{\partial z} = 0 \tag{3.5}$$

The three equations (3.3), (3.4) and (3.5) give the solution

$$U = \text{constant}$$

$$W = \text{constant}\ , \quad \text{on a characteristic}$$

$$dx/dz = U/W.$$

Initially $U = u_0$ and $W = 0$, so that the solution becomes

$$U = u_0$$

$$W = 0, \quad \text{on } z = \text{constant}.$$

From (3.5) it follows

$$f = \text{constant} = f_0 \quad \text{for} \quad z \leqslant d/2.$$

Each dust particle moves along the x direction with its initial momentum. When f is large, most of the momentum is carried by the dust. The term $f(U-u)/\tau$ in (3.1) represents the momentum transferred to the gas from the dust. Because of the large inertia, the dust suffers a loss of only a small part of its initial momentum but this amount makes a significant contribution to the momentum contained in the gas. For large x, the dust and gas will move together and the solution will tend to the fine-dust limit. The solution for coarse dust is not uniformly valid for large x.

The solution follows the method of Section 2. By using the Von Mises transformation, we get

$$\frac{\partial u}{\partial X} = \nu \frac{\partial}{\partial \psi}\left(u \frac{\partial u}{\partial \psi}\right) + \frac{f}{\tau}\left(\frac{u_0}{u} - 1\right) \tag{3.6}$$

$$U = u_0 \tag{3.7}$$

$$W = 0 \tag{3.8}$$

$$f = f_0 \quad \text{for} \quad z \leqslant d/2 \tag{3.9}$$

In the region where a core exists, the two integrated equations of motion are

$$2u_0\left(\frac{\Psi_X}{3} + \frac{\Phi_X}{2}\right) = \frac{f}{\tau}\left[-\left(\frac{u_0 d}{2} - \Phi\right) + \frac{\Psi}{2}\,\ell n\left|\frac{\Psi + \frac{u_0 d}{2} - \Phi}{\Psi - \frac{u_0 d}{2} + \Phi}\right|\right] \tag{3.10}$$

and

$$\Psi_X\left(\frac{1}{3} - \frac{1}{6\sqrt{2}}\right) + \frac{\Phi_X}{4} = \frac{\nu u_0}{2\sqrt{2}\,\Psi} \tag{3.11}$$

The equation (3.10) corresponds to (2.11) and (3.11) is the same as (2.13) The solution of (3.10) and (3.11) is obtained by iteration for small f/τ.

The zeroth approximation is given by (2.14) and (2.15). The first approximation is

$$2u_0\left(\frac{\Psi_X}{3} + \frac{\Phi_X}{2}\right) = \frac{f}{\tau}\left[-\tfrac{2}{3}\Psi + \frac{\Psi}{2}\ln 5\right] \tag{3.12}$$

Using this in (3.16) we get

$$A\Psi_X^2 + B\Psi^2 = \frac{\nu u_0}{2\sqrt{2}}$$

where

$$A = \frac{\sqrt{2}-1}{12\sqrt{2}}$$

$$B = \frac{f}{4u_0\tau}\left(\frac{\ln 5}{2} - \frac{2}{3}\right)$$

The solution is

$$\Psi^2 = \frac{\nu u_0}{2\sqrt{2}} \cdot \frac{1}{B}\left(1 - \exp(-\frac{B}{A}X)\right) \tag{3.13}$$

(3.12) can be integrated using the zeroth approximation for Ψ on the right hand side to get

$$2u_0\left(\frac{\Psi}{3} + \frac{\Phi}{2}\right) = \frac{f}{\tau}\left(\frac{\ln 5}{2} - \frac{2}{3}\right)\sqrt{\frac{6\nu u_0}{\sqrt{2}-1}} \cdot \frac{2}{3}x^{3/2} + \frac{u_0^2 d}{2}\ . \tag{3.14}$$

(3.13) and (3.14) give the first approximation for $X < X_1$ where a core exists. X_1, the position at which the core disappears is found from (3.14) by iteration. The solution corresponding to (2.18) is

$$\frac{X_1}{d} = \frac{3}{32}(\sqrt{2}-1)R_d + \frac{f}{\sigma}(\ln 5 - \tfrac{4}{3})\left(\frac{3}{32}(\sqrt{2}-1)R_d\right)^2 \tag{3.15}$$

where $R_d = u_0 d/\nu$ and $\sigma = u_0\tau/d$ is the particle parameter.

SOLUTION FOR $X > X_1$.

The integrated equations are now

$$\frac{2}{3}\frac{d}{dX}(u_m \Psi) = \frac{f}{\tau}\left[\frac{u_0}{u_m}\cdot\frac{\Psi}{2}\,\ell n\left(\frac{\Psi+\frac{u_0 d}{2}}{\Psi-\frac{u_0 d}{2}}\right) - \frac{u_0 d}{2}\right] \qquad (3.16)$$

and (2.25). These are to be solved with the boundary condition $\Psi = \Psi_1$ at $X = X_1$. The solution for large X (corresponding to (2.24) and (2.26) is as follows. (3.16) leads to

$$\left(u_m \Psi - u_1 \Psi_1\right) = D\left(X^{4/3} - X_1^{4/3}\right) - E(X - X_1)$$

where

$$D = \frac{9}{16}\,\frac{f}{\tau}\,u_0 d\,\frac{(6\,\nu u_0 \Psi_1)^{1/3}}{\Psi_1}$$

and

$$E = \frac{3}{4}\,\frac{f}{\tau} u_0 d \qquad (3.17)$$

For Ψ we get the differential equation

$$\frac{d\Psi}{dX} = 2\nu\,\frac{(C + DX^{4/3} - DX)}{\Psi^2} - 2\sqrt{2}\,\Psi\,\frac{\left(\frac{2}{3} - \frac{5}{6\sqrt{2}}\right)\left(\frac{4}{3}DX^{\frac{1}{3}} - E\right)}{\left(C + DX^{4/3} - EX\right)}$$

where C is given by $u_1\Psi_1 = C + DX_1^{4/3} - EX_1$. It appears that this can not be integrated in closed form. The asymptotic solution for large X is

$$\Psi^3 \sim \frac{2\nu D X^{7/3}}{\frac{7}{9} + \frac{8}{3}\sqrt{2}\left(\frac{2}{3} - \frac{5}{6\sqrt{2}}\right)} \qquad (3.18)$$

provided that f_0 is not zero. If there is no dust, instead of (3.18) we get

$$\Psi^3 - \Psi^3{}_1 = 6\,\nu C(X - X_1).$$

4. THE DUSTY JET FOR ARBITRARY τ.

Physically it is clear that initially the jet starts off as one with coarse dust and finally for large x, approaches the fine dust case. The flow configuration is expected to be the one shown in Fig. 4.1. The velocity profiles of the gas and dust in the various regions are shown here in solid and dotted lines, respectively. In core 1, both gas and dust retain the maximum velocity u_0 over a part of the profile. In core 2, the maximum velocity of the gas is less than u_0 but the dust still retains the initial velocity u_0. Farther downstream, there are no cores. We consider only this region to avoid a lot of algebra associated with the regions where cores exist. The initial conditions needed for this solution are obtained by using the coarse dust solution for a distance equal to $u_0\tau$, the distance travelled by a dust particle in one relaxation time.

The equations for this case are (1.6) to (1.10) (or (2.5)). An additional stream function for dust is defined by

$$fU = \frac{\partial\psi_d}{\partial z}$$

$$fW = -\frac{\partial\psi_d}{\partial z} \qquad (4.1)$$

We now have the three equations of momentum, (1.6), (1.8) and (1.9) to determine ψ, ψ_d and f as functions of x and z. The two equations of continuity are satisfied automatically. The velocity profile for the gas is assumed to be given by (2.19) as in the case of fine dust. The stream function for the dust is assumed to be the one in Fig. 4.2.

i.e.

$$\psi_d = \alpha\psi \quad \text{for} \quad 0 \leqslant \psi \leqslant \Psi_d$$

$$\psi_d = \alpha\Psi_d \quad \text{for} \quad \Psi_d \leqslant \psi \leqslant \Psi \qquad (4.2)$$

where α and Ψ_d are functions of x. When $x \to \infty$, the slip velocity tends to zero; i.e.

$$\frac{1}{f_\infty} \frac{\partial\psi_d}{\partial z} \to \frac{\partial\psi}{\partial z} \quad \text{or} \quad \psi_d = \alpha_\infty \psi \to f_\infty \psi \quad \text{i.e.} \quad \alpha_\infty = f_\infty .$$

For f, the flat profile shown in Fig. 4.3 is assumed.

$$\begin{aligned} f &= \text{constant for} \quad 0 \leqslant \psi \leqslant \Psi_d \\ &= 0 \qquad \text{for} \qquad \psi > \Psi_d \end{aligned} \tag{4.3}$$

f is a function of x only.

Von Mises transformation is used as before. The equations of transformation are given by (2.6). From (4.1),

$$fU = \frac{\partial\psi_d}{\partial z} = u \frac{\partial\psi_d}{\partial\psi}$$

$$fW = -\frac{\partial\psi_d}{\partial x} = -\left(\frac{\partial\psi_d}{\partial x} - w \frac{\partial\psi_d}{\partial\psi}\right.$$

(1.6), the x-momentum equation becomes

$$\frac{\partial u}{\partial X} = \nu \frac{\partial}{\partial\psi}\left(u \frac{\partial u}{\partial\psi}\right) + \frac{1}{\tau}\left(\frac{\partial\psi_d}{\partial\psi} - f\right) \tag{4.4}$$

(1.8) and (1.9), the x and z momentum equations of dust lead to

$$\frac{\partial\psi_d}{\partial\psi} \cdot \frac{\partial}{\partial X}\left(\frac{u}{f}\frac{\partial\psi_d}{\partial\psi}\right) - \frac{\partial\psi_d}{\partial X} \cdot \frac{\partial}{\partial\psi}\left(\frac{u}{f}\frac{\partial\psi_d}{\partial\psi}\right) = \frac{f - \frac{\partial\psi_d}{\partial\psi}}{\tau}. \tag{4.5}$$

$$\frac{u}{f} \cdot \left\{ \frac{\partial\psi_d}{\partial\psi} \frac{\partial}{\partial X}\left(-\frac{1}{f}\frac{\partial\psi_d}{\partial X} + \frac{w}{f}\frac{\partial\psi_d}{\partial\psi}\right) - \frac{\partial\psi_d}{\partial X}\frac{\partial}{\partial\psi}\left(-\frac{1}{f}\frac{\partial\psi_d}{\partial X} + \frac{w}{f}\frac{\partial\psi_d}{\partial\psi}\right)\right\}$$
$$= \frac{1}{\tau}\left\{ w - \left(-\frac{1}{f}\frac{\partial\psi_d}{\partial X} + \frac{w}{f}\frac{\partial\psi_d}{\partial\psi}\right)\right\} \tag{4.6}$$

These equations are solved approximately by the integral method. The profiles for u and w are given by

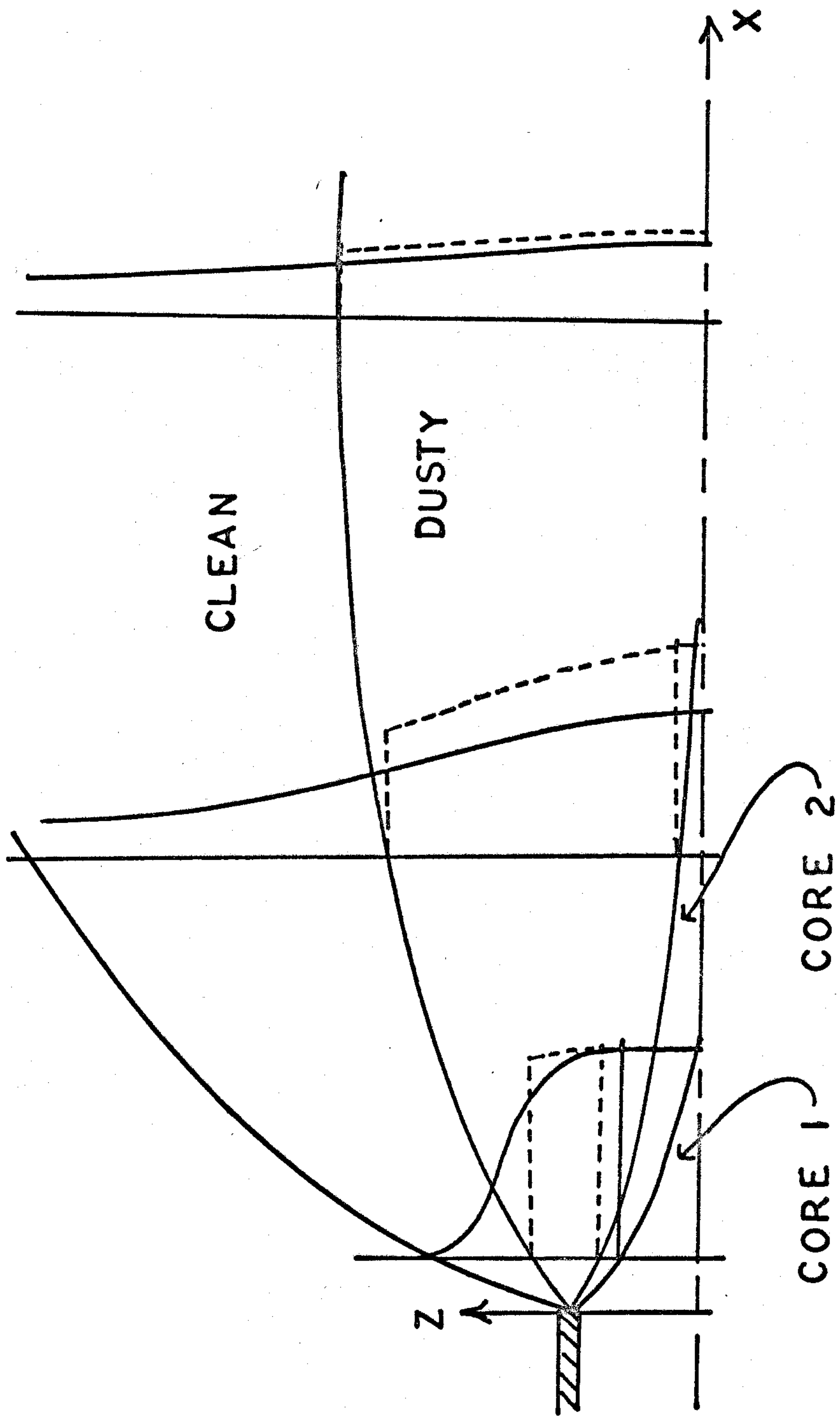

FIG 4.1

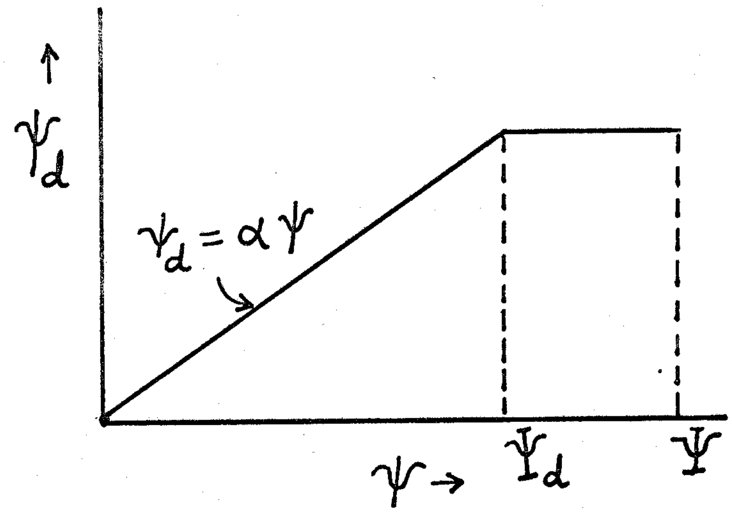
ψ_d
$\psi_d = \alpha\psi$
$\psi \rightarrow$
Ψ_d
Ψ

FIG 4.2

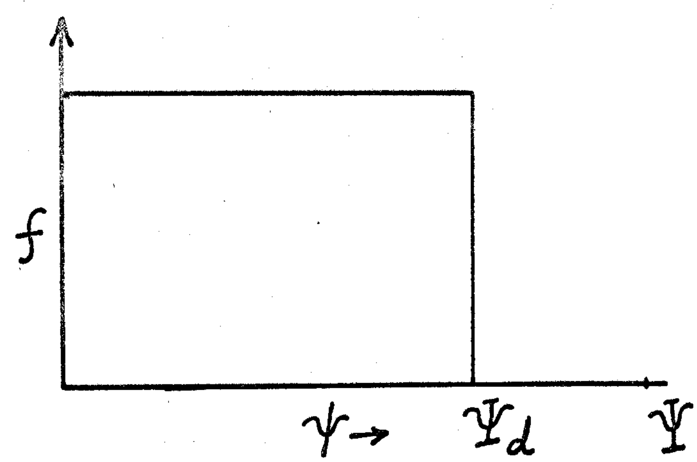
f
$\psi \rightarrow$
Ψ_d
Ψ

FIG 4.3

$$\frac{u}{u_m} = 1 - \psi^2/\Psi^2$$

and

$$w = -\frac{\partial\psi}{\partial x} = -\frac{\partial}{\partial x}\left(\Psi \tanh\frac{u_m z}{\Psi}\right)$$

The five equations needed for finding Ψ, Ψ_d, f, u_m and α as functions of X are derived now. Equation (4.4) is integrated from $\psi = 0$ to $\psi = \Psi$ to get

$$\frac{2}{3}\frac{d}{dX}(u_m\Psi) = \frac{\alpha - f}{\tau}\Psi_d \tag{4.7}$$

and from $\psi = 0$ to $\psi = \Psi/\sqrt{2}$ to get

$$u_{mX}\cdot\frac{5}{6\sqrt{2}}\Psi + \frac{1}{3\sqrt{2}}u_m\Psi_X = -\frac{1}{\sqrt{2}}\frac{\nu u_m^2}{\Psi} + \frac{\alpha - f}{\tau}\cdot\Psi_d \tag{4.8}$$

In using the two momentum equations for the dust (4.5) and (4.6) we make an additional approximation which simplifies the algebra considerably. For large X the dusty region occupies only a small part in the centre of the jet and so the stream function in this region $\psi = \Psi \tanh\frac{u_m z}{\Psi}$ is approximated by $\psi \simeq u_m z$ and $u = \frac{\partial\psi}{\partial z} \simeq u_m$ and $w = -\frac{\partial\psi}{\partial x} \simeq -u_{mX}\cdot z = -\frac{u_{mX}}{u_m}\cdot\psi$. (4.5) is integrated across the dust layer to get

$$\frac{d}{dX}\left(\frac{u_m\alpha}{f}\right) = \frac{\frac{f}{\alpha} - 1}{\tau} \tag{4.10}$$

and similarly from (4.6) we get

$$\frac{u_m}{f}\left\{-\alpha\frac{d}{dX}\left(\frac{(\alpha u_m)_X}{fu_m}\right) + \alpha_X\frac{(\alpha u_m)_X}{fu_m}\right\} = \frac{1}{\tau}\left\{\frac{-u_{mX}}{u_m} + \frac{(\alpha u_m)_X}{fu_m}\right\} \tag{4.11}$$

In the above equations the subscript X means d/dX. Finally, $\alpha\Psi_d$ is the value of the dust stream function at the edge of the dust layer and is equal to the total mass flux of the dust. This is a constant equal to

$f_0 u_0 d/2$ where f_0 and u_0 are the mass concentration and dust velocity at the slit.

i.e. $$\alpha \Psi_d = f_0 u_0 d/2 = B, \quad \text{say} \tag{4.12}$$

The equations (4.7), (4.8), (4.10), (4.11) and (4.12) are the five equations needed for finding Ψ, Ψ_d, f, u_m and α.

The asymptotic solution of equations (4.7), (4.8) (4.10), (4.11) and (4.12), for large x, can be found easily as follows. Let

$$f = f_\infty + g(x)$$

$$\alpha = \alpha_\infty + h(x)$$

where g and h are $<< f_\infty = g_\infty$. The simplified equations are

$$\frac{2}{3} u_m \Psi + B u_m = \text{constant} = \frac{2}{3} C, \quad \text{say}$$

$$\frac{2}{3} (u_m \Psi)_X - \left\{ u_{mX} \frac{5}{6\sqrt{2}} \Psi + \frac{1}{3\sqrt{2}} u_m \Psi_X \right\} = \frac{1}{\sqrt{2}} \frac{u_m^2 \nu}{\Psi}$$

$$u_{mX} = \frac{g-h}{\tau}$$

$$-u_m \frac{d}{dX}\left(\frac{u_{mX}}{u_m}\right) = \frac{1}{\tau} \quad \frac{(hu_m)_X}{u_m} - \frac{g u_{mX}}{u_m}$$

For $\psi >> B$, the solution of these equations is

$$\Psi = (6 \nu C X)^{\frac{1}{3}}$$

$$u_m = C(6 \nu C X)^{-\frac{1}{3}}$$

$$f = f_\infty - \nu C^2 \tau f_\infty (6 \nu C X)^{-\frac{4}{3}}$$

$$\alpha = f_\infty + \nu C^2 \tau f_\infty (6 \nu C X)^{-\frac{4}{3}} \tag{4.13}$$

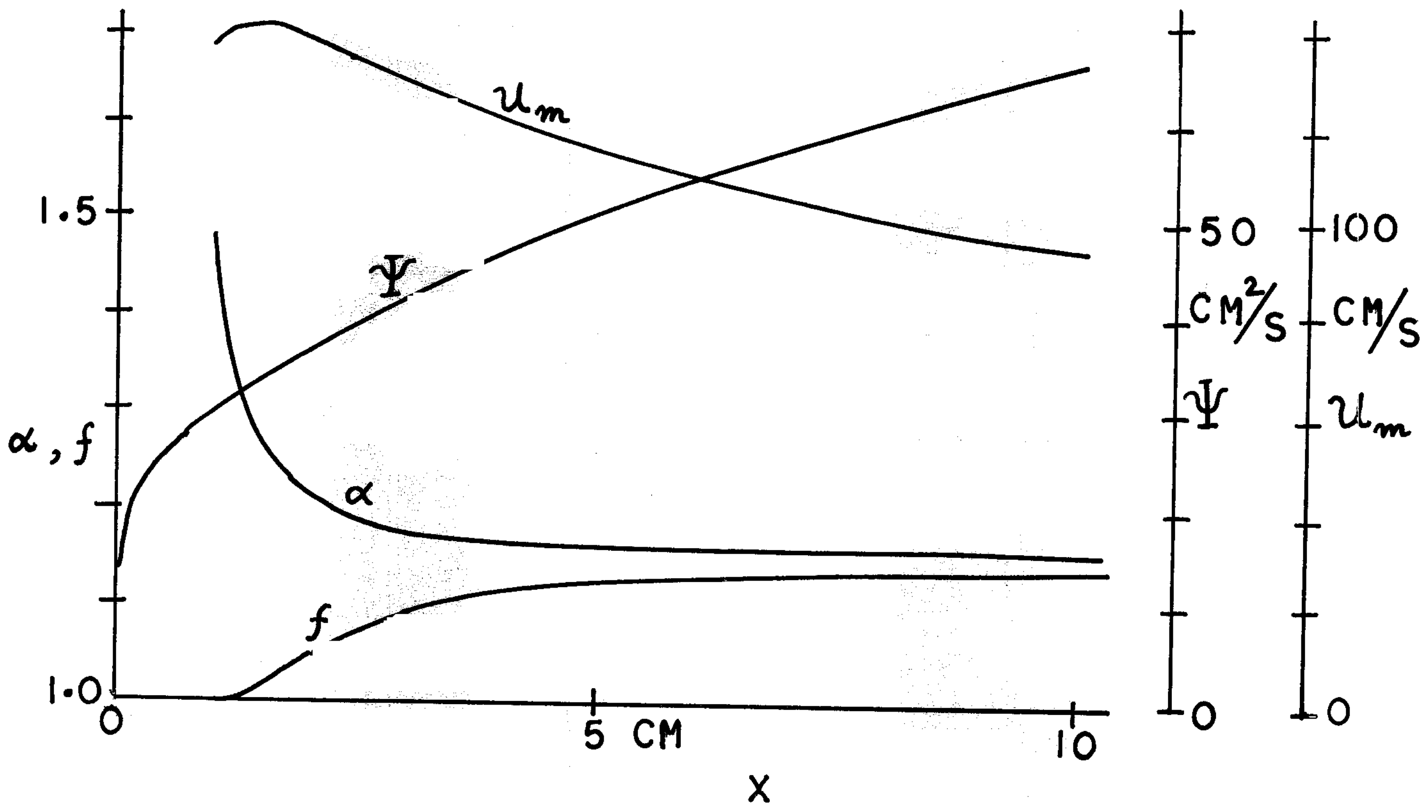

FIG 4.4

Illustrative example:

The following conditions should be satisfied:

1) $R_d \leq 30$, for laminar flow to exist.

2) $g\tau/u_0 << 1$, for the effect of gravity to be negligible.

3) $u_0\tau/d >> 1$, to ensure that the dust is coarse enough.

4) $f/\rho_p << 1$, for the volume concentration to be small.

Gold dust in olive oil is a reasonable choice. We choose

$$u_0 = 200 \text{ cm/sec}$$

$$d = 1.38 \text{ mm}$$

$$\nu = .92 \text{ cm}^2/\text{sec}$$

$$\rho = .92 \text{ gm/cm}^3$$

$$\rho_p = 19.3 \text{ gm/cm}^3$$

$$d_p = .65 \text{ mm}$$

$$f_0 = 1$$

$$\tau = 4.95 \text{ ms} \quad \text{from equation (1.5).}$$

With these values the four conditions look like this -

1) $R_d = 30$

2) $.024 << 1$

3) $7.2 >> 1$

4) $1/19.3 << 1$

For small x, when the slip velocity between the dust and oil is large, the assumption of Stokes drag is not satisfactory. The slip velocity decreases rapidly as x increases and the drag over most of the flow obeys the Stokes formula. The relaxation distance defined as the product of u_0 and the relaxation time is 200 cm/sec × 4.95 ms = 9.9 mm.

The calculation is done in the following manner. It is assumed that the dust does not spread for a distance equal to one relaxation distance. In this region u_m and Ψ are calculated by using the results of Section 3. For $x < u_0 \tau$ the dust concentration is taken to be unity. At $x = u_0 \tau$, $\alpha/f = u_0/u_m = 200/136 = 1.47$ i.e. $\alpha = 1.47$. And $df/dx = 0$, at $x = u_0 \tau$.

From this point onwards, the differential equations (4.7), (4.8), (4.10) and (4.11) are solved numerically and the solution is plotted in Fig. 4.1. The excess velocity of the dust particles decreases rapidly and approaches zero. The initial rise in u_m of the oil is due to the fact that the momentum lost by the dust particles is gained by the oil. In this simplified calculation it has been assumed that for one relaxation length, the dust keeps its momentum and for $x > u_0 \tau$, begins sharing its momentum with the oil. A calculation taking into account the two cores etc. suggested in Fig. 4.1, will not show such a rise in u_m. It is seen that the dust concentration increases with x and approaches the value 1.15 at infinity. The equation of continuity for the dust is $f l_d u_d$ = constant, where l_d is the scale of the dusty portion of the jet, u_d the maximum velocity of the dust and f is the mass concentration. The behaviour of f depends on how the product $l_d u_d$ behaves as x increases. The dusty portion is confined to a small region near the axis. The dust is decelerating faster than the oil but spreading along z at a smaller rate than the oil because of the slip. Therefore, its concentration increases. Actually in the region $x < u_0 \tau$, the dust will be decelerating less rapidly than the oil (we assumed it to have a constant velocity in this calculation) and spread at about the same rate as the oil and this could show a decrease of f in the beginning. The extent of this region leaving out the two cores is likely to be small. The change in f as x goes from 0 to infinity is rather small.

5. THE TURBULENT DUSTY JET

The equations of motion are

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = - \frac{\nabla p}{\rho} + \nu \nabla^2 \underline{u} + \frac{f}{\tau} (\underline{U} - \underline{u}) \tag{5.1}$$

$$\nabla \cdot \underline{u} = 0 \tag{5.2}$$

$$\frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \nabla \underline{U} = \frac{(\underline{u} - \underline{U})}{\tau} \tag{5.3}$$

$$\frac{\partial f}{\partial t} + \nabla \cdot (f \underline{U}) = 0 \tag{5.4}$$

where $\underline{u}$ and $\underline{U}$ are the instantaneous velocities of the gas and dust respectively. Stokes law has been assumed to be valid. In what follows, mean quantities will be indicated by an over-bar and fluctuations by a prime. We consider the case when the dust appears to be fine for the mean motion and coarse for the turbulent motion. The conditions for the validity of this will be considered soon. These assumptions mean

$$\overline{\underline{u}} = \overline{\underline{U}}; \quad f = \overline{f} \quad \text{and} \quad f' = 0 \tag{5.5}$$

For the mean motion, we get from (5.1) and (5.3)

$$\overline{\underline{u} \cdot \nabla \underline{u}} = -\nabla \overline{p} + \nu \nabla^2 \overline{\underline{u}} + \overline{f}\, \overline{\underline{u}} \cdot \nabla \overline{\underline{u}} \tag{5.6}$$

$$\nabla \cdot \overline{\underline{u}} = 0 \tag{5.7}$$

and from (5.4)

$$\overline{\underline{U}} \cdot \nabla \overline{f} = 0 \tag{5.8}$$

In component form, (5.6) gives

$$\left(\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{w}\,\frac{\partial \overline{u}}{\partial z}\right)\left(1+\overline{f}\right) + \left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'w'}}{\partial z}\right) = -\frac{\partial \overline{p}}{\partial x} + \nu\,\nabla^2\,\overline{u} \qquad (5.9)$$

and

$$\left(\overline{u}\,\frac{\partial \overline{w}}{\partial x} + \overline{w}\,\frac{\partial \overline{w}}{\partial z}\right)\left(1+\overline{f}\right) + \left(\frac{\partial \overline{u'w'}}{\partial x} + \frac{\partial \overline{w'^2}}{\partial z}\right) = -\frac{\partial \overline{p}}{\partial z} + \nu\,\nabla^2\,\overline{w} \qquad (5.10)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

Following Townsend [2], the orders of magnitude of the various terms c can be estimated in terms of the length scale of the jet, the velocity scale u_m and the scale of variation along x, viz L. For free jets, (5.10) shows p to be constant. Hence (5.9) simplifies to

$$\left(\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{w}\,\frac{\partial \overline{u}}{\partial z}\right)\left(1+\overline{f}\right) + \frac{\partial \overline{u'w'}}{\partial z} = \nu\,\frac{\partial^2 \overline{u}}{\partial z^2}$$

For flows at high Reynolds numbers, ν tends to zero and the final equations are

$$\left(\overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{w}\,\frac{\partial \overline{u}}{\partial z}\right)\left(1+\overline{f}\right) + \frac{\partial \overline{u'w'}}{\partial z} = 0 \qquad (5.11)$$

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{w}}{\partial z} = 0 \qquad (5.12)$$

$$\overline{u}\,\frac{\partial \overline{f}}{\partial x} + \overline{w}\,\frac{\partial \overline{f}}{\partial z} = 0 \qquad (5.13)$$

or f = constant on stream lines.

Before proceeding further, the validity of the various assumptions is discussed.

1) Stokes law: The slip velocity is of order $\overline{u}_m$. Stokes law is applicable if $\overline{u}_m\, d_p/\nu$ is less than unity. This assumption improves as x increases.

2) The dust appears to be coarse for the turbulent motion: This will be so if the time scale of turbulent fluctuations is small compared to the relaxation time. i.e., $\ell/\overline{u}_m << \tau$, where ℓ is the length scale of the jet. Because ℓ increases with x, this assumption gets worse as x increases.

3) The dust appears fine for the mean motion: i.e. the time scale of the mean motion $>> \tau$. Or $x/\overline{u}_m >> \tau$.

2) and 3) together require

$$\frac{\tau \overline{u}_m}{x} << 1 << \overline{u}_m \frac{\tau}{\ell}$$

and such a range might be narrow in practice.

Next an assumption about the shear stress is necessary. For jets it is well known that the assumption of a constant (across the jet) eddy viscosity is satisfactory. This means

$$\begin{aligned} -\overline{u'w'} &= \nu_T \frac{\partial \overline{u}}{\partial z} \\ &= K_1 \overline{u}_m \ell \frac{\partial \overline{u}}{\partial z} \\ &= K_2 \Psi \frac{\partial \overline{u}}{\partial z} \end{aligned} \tag{5.14}$$

Here u_m and ℓ are functions of x. Ψ is the maximum value of the stream function at the position x.

As in Section 2, Von Mises transformation simplifies (5.11) to

$$\left(1 + \overline{f}\right) \frac{\partial \overline{u}}{\partial x} = K_2 \Psi \frac{\partial}{\partial \psi} \left(\overline{u} \frac{\partial \overline{u}}{\partial \psi}\right)$$

with

$$\begin{aligned} \overline{f} &= \text{constant} = f_0 \quad \text{for} \quad 0 \leqslant \psi \leqslant u_0 \frac{d}{2} \\ &= 0 \quad \text{for} \quad \gamma > u_0 \frac{d}{2} \end{aligned} \tag{5.15}$$

The analysis is very similar to that of Section 2. The relevant equations are briefly as follows.

The velocity profile in the region with no core is assumed to be

$$\frac{\bar{u}}{\bar{u}_m} = 1\ 0\ (\frac{\psi}{\Psi})^2 \qquad (5.16)$$

The two equations obtained from the momentum equation are

$$\frac{2}{3}(\bar{u}_m\Psi)_X + f_0\left[\bar{u}_{nX}\cdot\frac{u_0 d}{2} - \frac{\bar{u}_{mX}}{3\Psi^2}\left(\frac{u_0 d}{2}\right)^3 + \frac{2}{3}\frac{\bar{u}_m}{\Psi^3}\Psi_X\left(\frac{u_0 d}{2}\right)^3\right] = 0 \qquad (5.17)$$

$$\bar{u}_{mX}\cdot\frac{5}{6\sqrt{2}}\Psi + \frac{\bar{u}_m}{3\sqrt{2}}\Psi_X + f_0\left[\cdots\right] = -\frac{\bar{u}_m^2}{\sqrt{2}}\frac{K_2\Psi}{\Psi}$$

$$= -\frac{\bar{u}_m^2}{\sqrt{2}}K_2 \qquad (5.18)$$

These correspond to (2.20) and (2.21) of the laminar jet. The approximate solutions for small $\bar{f}$ are the following.

$$\frac{2}{3}\bar{u}_m\Psi + f_0\bar{u}_m\frac{u_0 d}{2} = A \qquad (5.19)$$

$$\frac{\Psi^2 - \Psi_1^2}{6AK_2} + \frac{2\sqrt{2}-1}{9}\cdot\frac{f_0}{K_2}\cdot\frac{u_0\Psi_1}{A^2}\cdot\frac{u_0 d}{2}\cdot(\Psi-\Psi_1)$$

$$= (X-X_1) \qquad (5.20)$$

LIST OF SYMBOLS USED IN PART IIa.

A	constant in equation (5.19)
a	radius of a dust particle
d	width of the slit
g	acceleration due to gravity
$g(x)$	function defined in Section 4
$h(x)$	function defined in Section 4
K	constant in Stokes drag formula
K_2	constant in the eddy viscosity relation
l_1, l_2, l_3	length scales of the dusty jet
m	mass of a dust particle
N	number density of dust
p	pressure
$\underline{u}, u$	vector velocity of the fluid and its x component
w	z component of velocity of the fluid
$\underline{U}, U$	vector velocity of dust and its x component
W	z component of velocity of dust
x, z	Cartesian coordinates
X	$= x$, Von Mises transformation
α	defined by $\psi_d = \alpha\psi$ Fig. 4.2
μ	viscosity
ν	kinematic viscosity
ρ	density of the fluid
ρ_1	density of dust
Φ, ψ, Ψ	defined in Fig. 2.3, stream functions
τ	relaxation time

Subscripts

m	maximum value in the jet
0	at the slit
∞	at infinity
overbar	mean value
prime	fluctuation

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PART II b.

THE STEADY FLOW OF A DUSTY GAS AROUND A SPHERE

1. INTRODUCTION.

The equations of motion for steady flow of a dusty gas are as before [1]

$$\underline{\bar{u}} \cdot \bar{\nabla} \underline{\bar{u}} = - \frac{\bar{\nabla} \bar{p}}{\rho} + \nu \bar{\nabla}^2 \underline{\bar{u}} + \frac{f}{\tau} (\underline{\bar{v}} - \underline{\bar{u}})$$

$$\bar{\nabla} \cdot \underline{\bar{u}} = 0$$

$$\underline{\bar{v}} \cdot \bar{\nabla} \bar{v} = \frac{\underline{\bar{u}} - \underline{\bar{v}}}{\tau}$$

$$\bar{\nabla} \cdot (f \underline{\bar{v}}) = 0 \qquad (1.1)$$

where

$\underline{\bar{u}}$ = velocity of the gas at position $\underline{\bar{x}}$

$\underline{\bar{v}}$ = velocity of dust at position $\underline{\bar{x}}$

$\bar{p}$ = pressure at position $\underline{\bar{x}}$

ρ = density of the gas

ν = kinematic viscosity of the gas = η/ρ

τ = relaxation time = $m/6\pi\eta a_p$

a_p = radius of the dust particle

f = mass concentration of the dust at position $\underline{\bar{x}}$

The flow is over a sphere of radius a with the velocity of the dusty gas at infinity equal to U. Using a and U as scales of length and velocity, the equations of motion may be made nondimensional by defining

$$\underline{u} = \underline{\bar{u}}/U$$

$$\underline{v} = \underline{\bar{v}}/U$$

$$p = \bar{p}/\rho U^2$$

$$\sigma = \tau U/a = \frac{2}{9}\cdot\left(\frac{a_p}{a}\right)^2 \cdot \left(\frac{\rho_p}{\rho}\right)\cdot R = \alpha R$$

$$R = aU/\nu = \text{Reynolds number}$$

where ρ_p is the density of the material of the dust. The nondimensional position coordinate is defined by

$$\underline{x} = \bar{\underline{x}}/a.$$

The equations (1.1) become

$$\underline{u}\cdot\nabla\underline{u} = -\nabla p + \frac{\nabla^2\underline{u}}{R} + \frac{f}{\sigma}(\underline{v} - \underline{u}) \tag{1.2}$$

$$\nabla\cdot\underline{u} = 0 \tag{1.3}$$

$$\underline{v}\cdot\nabla\underline{v} = \frac{\underline{u} - \underline{v}}{\sigma} \tag{1.4}$$

$$\nabla\cdot(f\underline{v}) = 0 \tag{1.5}$$

The boundary conditions are that $\underline{u}$ and $\underline{v} \to i$ and $f \to f_\infty$ at upstream infinity, and $\underline{u} = 0$ on $r = 1$. If any dust hits the sphere, it is assumed that it sticks to the sphere. If $\underline{v}$ is directed away from the sphere, it is assumed that the sphere is not a source of dust (i.e. $f = 0$ on the surface of the sphere).

The particle parameter can vary widely depending on $\frac{a_p}{a}$ and $\frac{\rho_p}{\rho}$. For example, for sand in air and water, α is 4.44 and .0044 respectively for $\frac{a_p}{a} = .1$.

The problem is characterized by three parameters f, σ and R. Various limits of this flow will be considered in what follows.

2. FLOW AT LOW REYNOLDS NUMBER WITH $\alpha \sim 1$.

The solution in the limit $R \to 0$, α fixed, is found by the method of matched asymptotic expansions. The inner and outer variables are called Stokes and Oseen variables respectively. The equations of motion in Stokes variables are (1.2) to (1.5) of Section 1. The Oseen variables are

$$
\begin{aligned}
r' &= rR \\
\theta' &= \theta \\
\nabla' &= \nabla / R \\
u' &= u \\
f' &= f \\
p' &= p
\end{aligned}
\qquad (2.1)
$$

and the equations of motion in the Oseen variables are

$$R\underline{u}' \cdot \nabla'\underline{u}' = -R\nabla' p' + R\nabla'^2\underline{u}' - f'R\underline{v}' \cdot \nabla'\underline{v}' \qquad (2.2)$$

$$R\nabla' \cdot \underline{u}' = 0 \qquad (2.3)$$

$$R\underline{v}' \cdot \nabla\underline{v}' = (\underline{u}' - \underline{v}')/\alpha R \qquad (2.4)$$

$$R\nabla' \cdot (f'\underline{v}') = 0 \qquad (2.5)$$

The equation of continuity (1.3) may be eliminated by defining a stream function ψ by

$$
\begin{aligned}
u_r &= \frac{1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta} \\
u_\theta &= -\frac{1}{r\sin\theta} \frac{\partial\psi}{\partial r}
\end{aligned}
\qquad (2.6)
$$

Similarly the Oseen stream ψ' is defined to eliminate (2.3) by

$$u'_r = \frac{1}{r'^2 \sin\theta'} \frac{\partial \psi'}{\partial \theta'}$$

$$u'_\theta = - \frac{1}{r' \sin\theta'} \frac{\partial \psi'}{\partial r'} \tag{2.7}$$

It is seen that

$$\psi' = R^2 \psi$$

The Stokes expansion is assumed in the form

$$\underline{u} = \underline{u}_0 + R\underline{u}_1 + \cdots$$

$$\underline{v} = \underline{v}_0 + R\underline{v}_1 + \cdots$$

$$p = p_0/R + p_1 + \cdots$$

$$f = f_0 + R f_1 + \cdots$$

$$\psi = \psi_0 + R\psi_1 + \cdots \tag{2.8}$$

In this expansion the independent variables are r and θ. Further terms will be determined as the matching proceeds.

The Oseen expansion is assumed in the form

$$\underline{u}' = \underline{u}_0' + R\underline{u}_1' + \cdots$$

$$\underline{v}' = \underline{v}_0' + R\underline{v}_1' + \cdots$$

$$p' = p_0' + R p_1' + \cdots$$

$$f' = f_0' + R f_1' + \cdots$$

$$\psi' = \frac{\psi_0'}{R^2} + \frac{\psi_1'}{R} + \cdots \tag{2.9}$$

Further terms will be found as the expansion proceeds.

(2.9) is substituted into the set of equations (2.2) to (2.5) to get the equations for the various orders in the Reynolds number.

THE ZEROTH ORDER OSEEN SOLUTION.

These equations are

$$\underline{u}_0' \cdot \nabla' \underline{u}_0' = -\nabla' p_0' + \nabla'^2 \underline{u}_0' + f_0' \underline{v}_0' \cdot \nabla' \underline{v}_0'$$

$$\nabla' \cdot \underline{u}_0' = 0$$

$$\underline{u}_0' = \underline{v}_0'$$

$$\nabla' \cdot (f_0' \underline{v}_0') = 0 \qquad (2.10)$$

The solution satisfying the outer boundary condition at infinity is simply

$$\underline{u}_0' = \underline{v}_0' = \underline{i}$$

$f_0' = f_\infty$ = constant, and the stream function

$$\psi_0' = r'^2 \sin^2\theta'/2 \qquad (2.11)$$

THE ZEROTH ORDER STOKES SOLUTION.

These equations are obtained by substituting (2.8) in (2.1) to (2.5) and retaining terms of leading order in the Reynolds number. These are

$$-\nabla p_0 + \nabla^2 \underline{u}_0 = 0$$

$$\nabla \cdot \underline{u}_0 = 0$$

$$\underline{u}_0 = \underline{v}_0$$

$$\nabla \cdot (f_0 \underline{v}_0) = 0 \qquad (2.12)$$

The boundary condition is that $\underline{u}_0 = 0$ on $r = 1$. The solution is the familiar Stokes solution

$$\psi_0 = \tfrac{1}{4}(2r^2 - 3r + \frac{1}{r}) \sin^2\theta$$

$$f_0 = \text{constant} = f_\infty \qquad (2.13)$$

$\underline{u}_0 = \underline{v}_0$ is obtained from the above stream function ψ_0.

THE FIRST ORDER OSEEN SOLUTION.

The first order equations are

$$i \cdot \nabla' \underline{u}_1'(1+f_0') = -\nabla' p_1' + \nabla'^2 \underline{u}_1'$$

$$\nabla' \cdot \underline{u}_1' = 0$$

$$\underline{u}_1' = \underline{v}_1'$$

$$\nabla' \cdot (f_1' \underline{v}_0' + f_0' \underline{v}_1') = 0 \qquad (2.14)$$

The last of (2.14) gives

$$\frac{\partial f_1'}{\partial x} = 0$$

Therefore $f_1' = \text{constant} = 0$, to satisfy the boundary condition at infinity. In terms of the stream function, the first two equations of (2.14) give

$$\left\{D^2 + (1+f_0)\left(-\cos\theta' \frac{\partial}{\partial r'} + \frac{\sin\theta'}{r'} \frac{\partial}{\partial \theta'}\right)\right\} D^2 \psi_1' = 0$$

where

$$D^2 = \frac{\partial^2}{\partial r'^2} + \frac{\sin\theta'}{r'^2} \frac{\partial}{\partial \theta'}\left(\frac{1}{\sin\theta'} \frac{\partial}{\partial \theta'}\right) \qquad (2.15)$$

Except for the factor $(1+f_0)$, this is the same as the equation for the flow of a uniform fluid about a sphere. This solution is well known and may be taken from Van Dyke's book [2]. If the variables are changed to

$$\rho = r'/(1+f_0)$$

and $\vartheta = \theta'$, (2.15) becomes

$$\left\{ D_1^2 + \left(-\cos\vartheta \frac{\partial}{\partial\rho} + \frac{\sin\vartheta}{\rho} \frac{\partial}{\partial\vartheta} \right) \right\} D_1^2 \psi_1' = 0$$

where

$$D_1^2 = \frac{\partial^2}{\partial\rho^2} + \frac{\sin\vartheta}{\rho^2} \frac{\partial}{\partial\vartheta} \left(\frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \right) \qquad (2.16)$$

and ψ_1' is a function of ρ and ϑ. The particular integral of (2.16) which gives zero velocity at infinity and is least singular at the origin is (Van Dyke's equation 8.28)

$$\psi_1' = -2c_2' (1+\cos\vartheta) \cdot \left[1 - e^{-\frac{\rho}{2}(1-\cos\vartheta)} \right]$$

where c_2' is a constant. The Oseen expansion to two terms written in Stokes variables is

$$\psi_1' = \frac{r^2 \sin^2\theta}{2} - 2 \frac{c_2'}{R} (1+\cos\theta) \left[1 - e^{-R\frac{r}{2}(1+f_0)(1-\cos\theta)} \right]$$

which becomes in the limit $R \to 0$, becomes

$$\psi_1' = \frac{r^2 \sin^2\theta}{2} + 2c_2' \sin^2\theta \cdot r(1+f_0) + \cdots$$

and this should match the Stokes solution (2.13). Therefore

$$c_2' = \frac{3}{4} / (1+f_0) \text{ and}$$

$$\psi_1' = -\frac{3}{2} \frac{(1+\cos\theta')}{(1+f_0)} \cdot \left[1 - e^{-\frac{r'}{2}(1+f_0)(1-\cos\theta')} \right]$$

and

$$f_1' = 0 \qquad (2.17)$$

THE FIRST ORDER STOKES SOLUTION.

The equations are

$$\underline{u}_0 \cdot \nabla \underline{u}_0 (1+f_0) = -\nabla p_1 + \nabla^2 \underline{u}_1$$

$$\nabla \cdot \underline{u}_1 = 0$$

$$\frac{\underline{u}_1 - \underline{v}_1}{\alpha} = \underline{u}_0 \cdot \nabla \underline{u}_0$$

$$\nabla \cdot (f_1 \underline{v}_0 + f_0 \underline{v}_1) = 0 \qquad (2.18)$$

The last two of (2.18) lead to

$$\nabla \cdot \Big(f_1 \underline{u}_0 + f_0 (\underline{u}_1 - \alpha \underline{u}_0 \cdot \nabla \underline{u}_0)\Big) = 0$$

or

$$\underline{u}_0 \cdot \nabla f_1 - f_0 \alpha \nabla \cdot (\underline{u}_0 \cdot \nabla \underline{u}_0) = 0 \qquad (2.19)$$

The first equation of (2.18) can be written in terms of the stream function ψ_1 to get

$$D^2 \psi_1 = -\frac{9}{4} (1+f_0) \cdot \left(\frac{2}{r^2} - \frac{3}{r^3} + \frac{1}{r^5}\right) \sin^2\theta \cos\theta$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\right)$$

The solution (from Van Dyke's book) is

$$\psi_1 = c_1 (2r^2 - 3r + \frac{1}{r}) \sin^2\theta - \frac{3}{32}\left(2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right) \sin^2\theta \cos\theta \cdot (1+f_0)$$

where c_1 is a constant. The constant c_1 is found by matching $\psi_0 + R\psi_1$ and $\frac{\psi_0'}{R^2} + \frac{\psi_1'}{R}$ to two terms in the Reynolds number expansion.

The result is

$$c_1 = 3(1+f_0)/32$$

The first order solution for ψ_1 is therefore

$$\psi_1 = (1+f_0)\left[\frac{3}{32}(2r^2-3r+\frac{1}{r})\sin^2\theta - \frac{3}{32}\left(2r^2-3r+1-\frac{1}{r}+\frac{1}{r^2}\right)\sin^2\theta\cos\theta\right] \tag{2.20}$$

Next we consider (2.19), the equation for $f_1(r,\theta)$. In this $\underline{u}_0$ is known from the zeroth order stream function in (2.13). This is

$$\underline{u}_0 = (1-\frac{1}{r})\,i + \frac{r^2-1}{4}\,\nabla\left(\frac{x}{r^3}\right)$$

On substituting for $\underline{u}_0$, (2.19) becomes

$$\left(1-\frac{1}{r}-\frac{r^2-1}{2r^3}\right)\cos\theta\cdot\frac{\partial f_1}{\partial r} - \left(1-\frac{1}{r}+\frac{r^2-1}{4r^3}\right)\frac{\sin\theta}{r}\frac{\partial f_1}{\partial\theta}$$

$$= f_0\alpha\left\{-\frac{9}{4}\frac{(r^2-1)\sin^2\theta}{r^6} + \frac{9}{8}\frac{(r^2-1)^2}{r^8}(1+2\cos^2\theta)\right\} \tag{2.21}$$

The boundary condition is that f_1 tends to zero at infinity. (2.21) is a first order partial differential equation which may be reduced to the characteristic form

$$\frac{dr}{d\lambda} = \left(1-\frac{1}{r}-\frac{r^2-1}{2r^3}\right)\cos\theta$$

$$\frac{d\theta}{d\lambda} = -\left(1-\frac{1}{r}+\frac{r^2-1}{r^3}\right)\frac{\sin\theta}{r}$$

$$\frac{df_1}{d\lambda} = f_0\,\alpha\left[-\frac{9}{4}\frac{(r^2-1)\sin^2\theta}{r^6} + \frac{9}{8}\frac{(r^2-1)^2}{r^8}(1+2\cos^2\theta)\right] \tag{2.22}$$

where λ is the parameter defining the characteristic. The first two of (2.22) give

$$\frac{dr}{d\theta} = \frac{\left(1 - \frac{1}{r} - \frac{r^2-1}{2r^2}\right)\cos\theta}{-\left(1 - \frac{1}{r} + \frac{r^2-1}{4r^3}\right)\frac{\sin\theta}{r}}$$

and this integrates to

$$\psi = \frac{\sin^2\theta}{4r}\left(2r^3 - 3r^2 + 1\right) \tag{2.23}$$

The curves $\psi_0 =$ constant are the characteristics. Obviously they are the zero order stream lines. On a stream line the variation of f_1 is obtained by using the first and third of (2.22) to get

$$\frac{df_1}{dr} = \frac{f_0\alpha\left[-\frac{9}{4}(r^2-1)\frac{\sin^2\theta}{r^6} + \frac{9}{8}\frac{(r^2-1)^2}{r^8}(1+2\cos^2\theta)\right]}{\left(1 - \frac{1}{r} + \frac{1-r^2}{2r^3}\right)\cos\theta}$$

When θ is eliminated in the above by using (2.23), we get

$$\frac{df_1}{dr} = \frac{f_0\alpha\left[-\frac{9}{4}\frac{(r^2-1)}{r^6}\cdot\frac{(2r^2-1)}{r^2}\cdot\frac{4r\psi_0}{(1+2r)(r-1)^2} + \frac{27}{8}\frac{(r^2-1)^2}{r^8}\right]}{\pm\left(1 - \frac{1}{r} - \frac{r^2-1}{2r^3}\right)\cdot\sqrt{1 - \frac{4r\psi_0}{(1+2r)(r-1)^2}}} \tag{2.24}$$

Of the + or - sign, the sign to be chosen is the sign of $\cos\theta$.

First f_1 on the stream line $\psi_0 = 0$ will be found. On $\theta = \pi$,

$$\begin{aligned}\frac{df_1}{dr} &= f_0\alpha\left[\frac{27}{8}\frac{(r^2-1)^2}{r^8}\right] \Big/ -\left(1 - \frac{1}{r} - \frac{r^2-1}{2r^3}\right) \\ &= -\frac{27}{4}\frac{(r+1)^2 f_0\alpha}{r^5(2r+1)} \\ &= -\frac{27}{4} f_0\alpha\left\{\frac{1}{r^5} + \frac{1}{r^3} - \frac{2}{r^2} + \frac{4}{r} - \frac{8}{1+2r}\right\}\end{aligned}$$

This is integrated to get

$$f_1(r,\pi) = -\frac{27}{4} f_0\alpha \left(-\frac{1}{4r^4} - \frac{1}{2r^2} + \frac{2}{r} + 4\ln\frac{2r}{2r+1}\right) \qquad (2.25)$$

This satisfies the boundary condition $f_1(\infty,\pi) = 0$. At the front stagnation point,

$$f_1(1,\pi) = \frac{27}{4} f_0\alpha \left(\frac{5}{4} + 4\ \ln\frac{2}{3}\right) = 2 \cdot 51\ f_0\alpha$$

f_1 on the sphere is calculated next. At $r=1$, both sides of (2.21) vanish. Therefore on taking $\frac{\partial}{\partial r}$ of both sides of (2.21) and then putting $r=1$, we get

$$-\frac{3}{2}\sin\theta\ \frac{\partial f_1}{\partial\theta} = -\frac{9}{2} f_0\alpha\ \sin^2\theta$$

which gives on integration

$$f_1(1,\theta) = f_1(1,\pi) - 3f_0\alpha\ (1+\cos\theta) \qquad (2.26)$$

From (2.26) we get at $\theta=0$,

$$f_1(1,0) = f_1(1,\pi) - 6f_0\alpha$$

$$= -3 \cdot 49\ f_0\alpha .$$

The solution on the axis $\theta = 0$, $r \geqslant 1$, is given by the following.

$$f_1(r,0) = \frac{27}{4} f_0\alpha\left(-\frac{1}{4r^4} - \frac{1}{2r^2} + \frac{2}{r} + 4\ln\frac{2r}{2r+1}\right) + 2f_1(1,\pi) - 6f_0\alpha \qquad (2.27)$$

This satisfies at $r=1$,

$$f_1(1,0) = -f_1(1,\pi) + 2f_1(1,\pi) - 6f_0\alpha = -3 \cdot 49\, f_0\alpha$$

The equations (2.25), (2.26) and (2.27) complete the solution for $f_1(r,\theta)$ on the stream line $\psi_0 = 0$. From (2.27) we see that

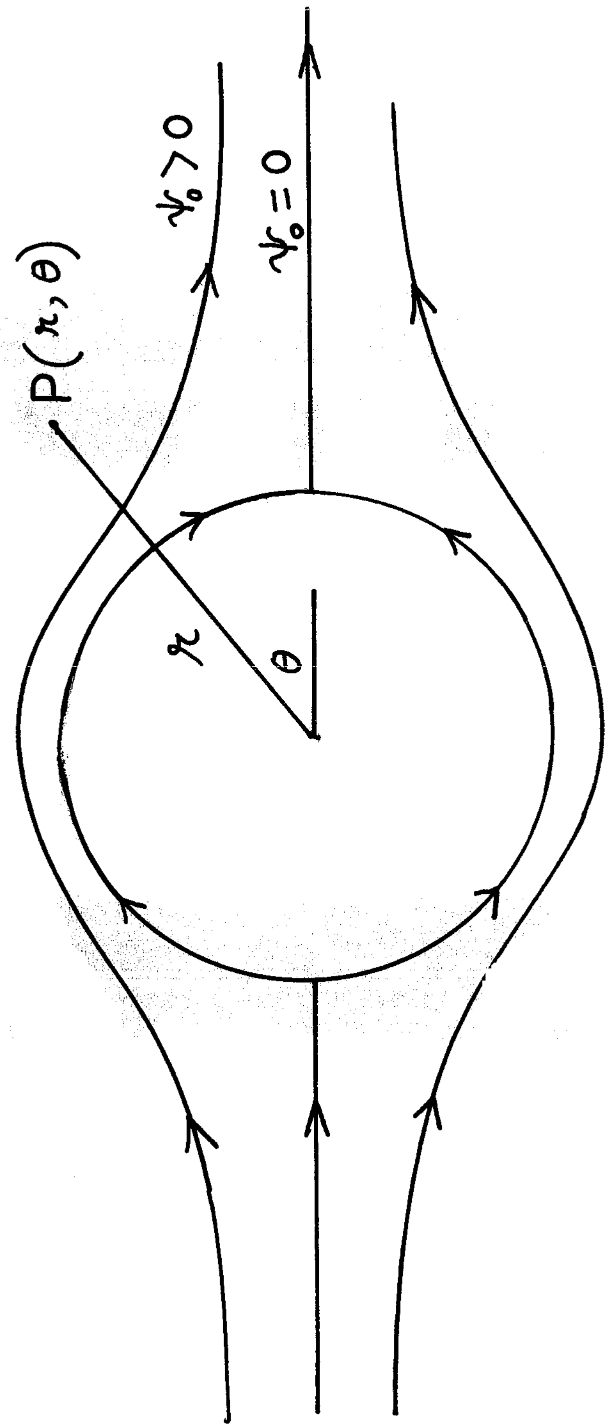
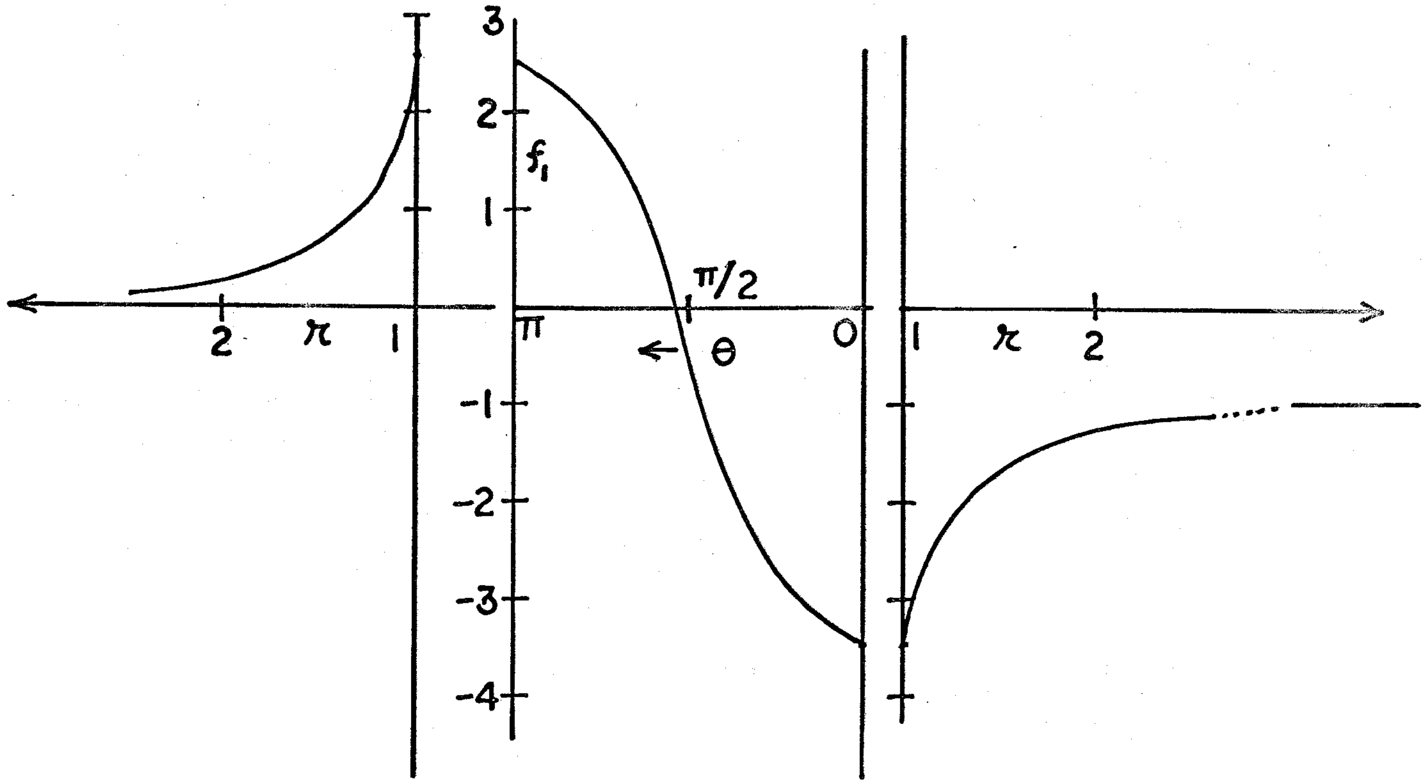


FIG 2.1



f_1 ON THE ZERO STREAM LINE

FIG 2.2

$$f_1(\infty,0) = 2f_1(1,\pi) - 6f_0\alpha = -.98\ f_0\alpha$$

Fig. 2.2 shows the variation of f_1 on the zero stream line.

To get more information about f_1, we consider the case when ψ_0 is large. The stream lines are straight lines given by

$$\psi_0 \simeq \frac{r^2}{2} \sin^2\theta \tag{2.28}$$

On a stream line $\frac{df_1}{d\theta}$ is obtained by using the second and third equations of (2.22). This is

$$\frac{df_1}{d\theta} = \frac{f_0\alpha\left[-\frac{9}{4}\frac{(r^2-1)}{r^6}\sin^2\theta + \frac{9}{8}\frac{(r^2-1)^2}{r^8}(1+2\cos^2\theta)\right]}{-\left(1 - \frac{1}{r} + \frac{r^2-1}{4r^3}\right)\frac{\sin\theta}{r}} \tag{2.29}$$

Because $r >> 1$ when $\psi_0 >> 1$, (2.29) simplifies to

$$\frac{df_1}{d\theta} \simeq \frac{f_0\alpha}{-\sin\theta}\left(-\frac{9}{4}\sin^2\theta + \frac{9}{8} + \frac{9}{4}\cos^2\theta\right)\cdot\frac{1}{r^3}$$

$$\simeq \frac{f_0\alpha}{\left(\sqrt{2\psi_0}\right)^3}\left(\frac{9}{2}\sin^4\theta - \frac{27}{8}\sin^2\theta\right)$$

On integration this gives

$$f_1 = \frac{9}{64}\frac{f_0\alpha}{\left(\sqrt{2\psi_0}\right)^3}\cdot(\sin 4\theta - 2\sin 2\theta)$$

$$= -\frac{9}{8}\frac{f_0\alpha}{\left(\sqrt{2\psi_0}\right)^3}\left(\cos\theta\sin^3\theta\right.$$

$$= -\frac{9}{8} f_0\alpha.\frac{\cos\theta}{r^3} \tag{2.30}$$

The above solution satisfies the boundary condition $f_1(\infty,\pi) = 0$. We see an increase of f_1 on the portion from $\theta = \pi$ to $\frac{\pi}{2}$ and a decrease from

$\theta = \frac{\pi}{2}$ to 0. At downstream infinity, f_1 goes to zero as $\frac{1}{r^3}$.

The solution for arbitrary ψ_0 can be done only numerically and this has not been attempted. The sketch in Fig. 2.3 shows the expected distribution of dust at downstream infinity. Here regions of positive f_1 must exist to satisfy conservation of dust. It is observed that the sphere does not catch any dust because the velocity of dust on the sphere is $\underline{v}_0 = \underline{u}_0 = 0$. There is only a redistribution of dust around the sphere. The sketch shows a trajectory of dust near the axis. Such a particle is deflected sideways as it goes around the sphere and it finally ends up in a streamline with a larger ψ_0. This causes a loss of f_1 near the axis and a gain farther away from the axis at downstream infinity.

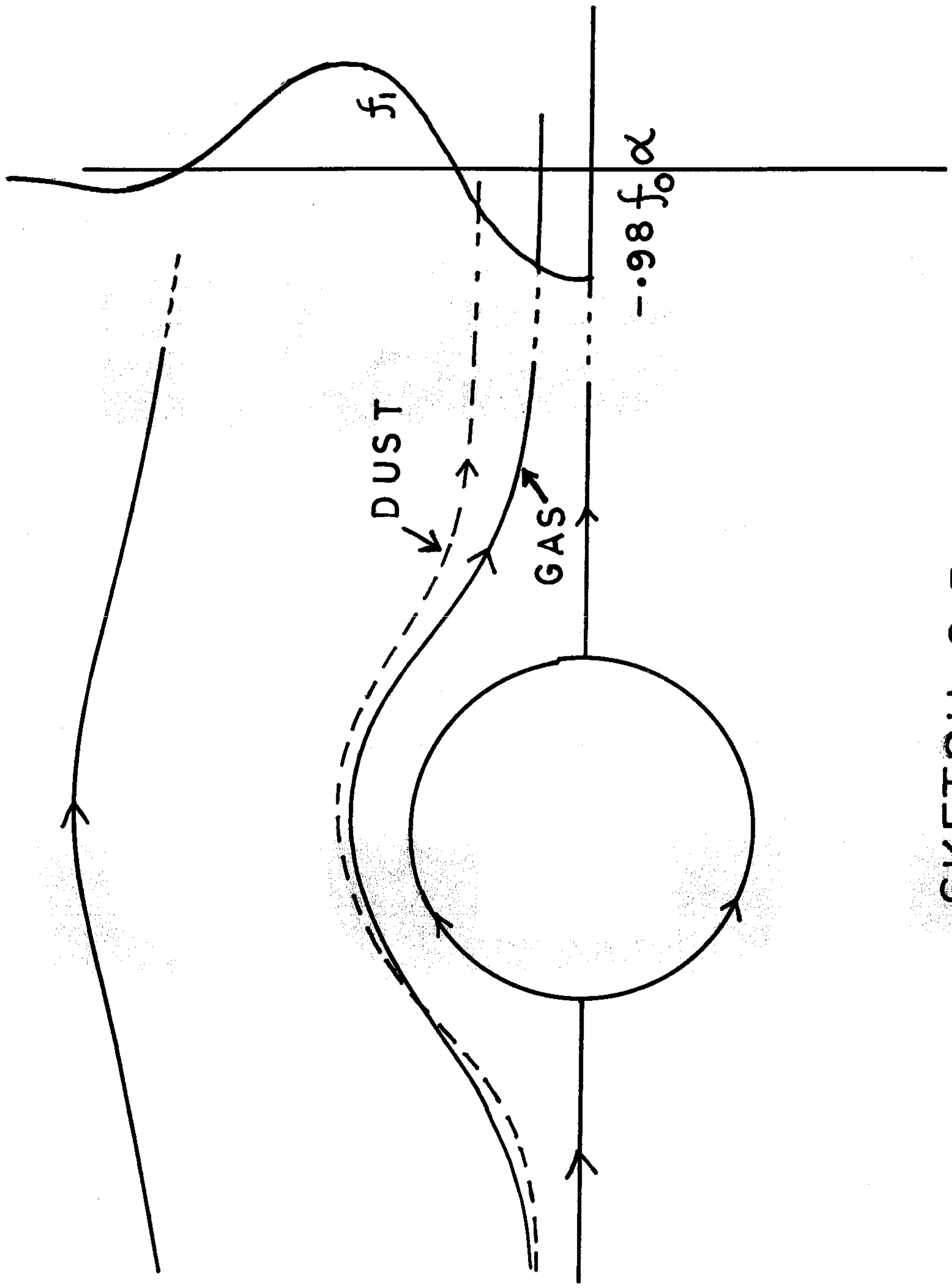
The formula (2.20) for ψ_1 and the formulas (2.25), (2.26), (2.27) and (2.30) for $f_1(r,\theta)$ constitute the first order solutions. The solution for f_1 is, however, incomplete and is known only for $\psi_0 >> 1$ and for $\psi_0 = 0$.

HIGHER ORDER SOLUTIONS.

It has been shown by Proudman and Pearson [3] that for the flow of a uniform fluid about a sphere, the next term in the Stokes expansion for ψ is of order $R^2 \ln R$ and the next term in the Oseen expansion for ψ' is unity. These results are true for the present case, also. These results are obtained easily by using the asymptotic matching principle.

The second order Oseen equations (i.e. of order unity in R) are the following

$$
\begin{aligned}
f_2' &= 0 \\
\underline{u}_2' &= \underline{v}_2' \\
i \cdot \nabla' \underline{u}_2' (1+f_0') &= {}_2\nabla' p_2' + \nabla'^2 \underline{u}_2' - (1+f_0')\underline{u}_1' \cdot \nabla' \underline{u}_1' \\
\nabla' \cdot \underline{u}_2' &= 0
\end{aligned}
\qquad (2.31)
$$



SKETCH 2.3

The boundary conditions are that $\underline{u}_2'$, $\underline{v}_2'$ and $p_2' \to 0$ at infinity and they match the Stokes solution near the sphere.

The next order Stokes equations (i.e. of order $R^2 \ln R$) are the following

$$0 = -\nabla p_2 + \nabla^2 \underline{u}_2$$

$$\nabla \cdot \underline{u}_2 = 0$$

$$f_2 = \text{constant}$$

$$\underline{u}_2 = \underline{v}_2 \tag{2.32}$$

The boundary conditions are that $\underline{u}_2 = 0$ at $r = 1$ and the solutions match the Oseen expansion as $r \to \infty$. (2.32) are nothing but the set of simple Stokes equations and the solution is a multiple of the zero order Stokes solution. This is

$$\psi_2 = \frac{9}{160}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta$$

The constant here has been chosen such that this term together with the next Stokes term (of order R^2) contains no term of order $\ln R$ in matching the Oseen expansion [3].

The order unity Oseen equations are hard to solve and this has not been attempted here.

The solution thus far can be summarized now. The Oseen stream function

$$\psi'(r',\theta') = \frac{r'^2 \sin^2\theta}{2} - \frac{3}{2} R \frac{(1+\cos\theta')}{(1+f_0)}\left[1 - e^{-\frac{r'}{2}(1+f_0)(1-\cos\theta')}\right] + O(R^2) \tag{2.33}$$

and the Stokes stream function

$$\psi(r,\theta) = \tfrac{1}{4}\left(2r^2-3r+\frac{1}{r}\right)\sin^2\theta + R(1+f_0)\left\{\frac{3}{32}(2r^2-3r+\frac{1}{r})\sin^2\theta\right.$$

$$\left. - \frac{3}{32}\,(2r^2-3r+1-\frac{1}{r}+\frac{1}{r^2})\sin^2\theta\cos\theta\right\}$$

$$+\frac{9}{160}\,R^2\ln R\,(2r^2-3r+\frac{1}{r})\sin^2\theta$$

$$+\ O(R^2) \qquad (2.34)$$

where $\psi'(r',\theta') = R^2\psi(r,\theta)$, $\theta' = \theta$ and $r' = Rr$. The dust concentration is given by the expansions

$$f' = f_0 + R(0) + R^2(0) + o(R^2)$$

and

$$f = f_0 + R f_1(r,\theta) + R^2\ln R(0) + O(R^2) \qquad (2.35)$$

where $f_1(r,\theta)$ is given by the equations (2.25), (2.26), (2.27) and (2.30). The velocity components are obtained by differentiating the stream function i.e. by using the formulas (2.6) and (2.7). The Stokes expansion for p is given by

$$p(r,\theta) = -\frac{1}{R}\left(\frac{3}{2}\frac{\cos\theta}{r^2}\right) + \frac{(1+f_0)}{16}\left\{9(2\cos^2\theta-1) - \frac{7}{r}(3\cos^2\theta-1)\right.$$

$$\left. + \frac{3}{r^2}(6\cos^2\theta-1) - \frac{1}{2r^4}(3\cos^2\theta+1) - \frac{9\cos\theta}{r^2}\right\}$$

$$+\frac{9}{40}\left(-\frac{3}{2}\frac{\cos\theta}{r^2}\right)R\ln R + O(R) \qquad (2.36)$$

This expression differs from the corresponding expression of Kaplun and Lagerstrom [4] by the factor $(1+f_0)$ in the second term. The drag coefficient for the flow of a uniform fluid about a sphere given by Proudman and Pearson [3] is

$$C_D = \text{Drag}/\rho U^2 a^2 = \frac{6\pi}{R}\left[1 + \frac{3}{8}R + \frac{9}{40}R^2 \ln R + O(R^2)\right]$$

An observation of (2.36) and the expression (2.34) for $\psi(r,\theta)$ shows that the drag coefficient in the present case is

$$C_D = \text{Drag}/\rho U^2 a^2 = \frac{6\pi}{R}\left[1 + \frac{3}{8}(1+f_0)R + \frac{9}{40}R^2 \ln R + O(R^2)\right] \tag{2.37}$$

It is seen that the leading term for the drag is not influenced by dust. In the limit R tending to zero, the gas and dust move together and the whole behaves like a uniform fluid of density $\rho(1+f_0)$. But the viscosity of the fluid is still η, the viscosity of the clean gas. This is so because the effective viscosity of a suspension is given by the Einstein formula [5]

$$\eta = \eta_0\left(1 + \frac{5}{2}\phi + \cdots\right)$$

where ϕ, the volume concentration is assumed to be zero in the present case.

3. VISCOUS FLOW. $R \to 0$, σ FIXED.

This is the limit when the particles are very massive. The non-dimensional equations of motion are the set of equation (1.2) to (1.5) with the boundary conditions given in Section 1. Expansions in R are considered as in Section 2. (This is not a physical limit. $\alpha \not\gg 1$ in practice. Page 90).

ZERO ORDER OSEEN SOLUTION.

This is quite simply

$$f_0' = \text{constant} = f_\infty$$

$$\underline{u}_0' = \underline{v}_0' = i \qquad (3.1)$$

ZERO ORDER STOKES SOLUTION.

As in Section 2, the stream function

$$\psi_0 = \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta \qquad (3.2)$$

and the corresponding velocity

$$\underline{u}_0 = \left(1 - \frac{1}{r}\right)i + \frac{r^2-1}{4}\nabla\left(\frac{x}{r^3}\right) \qquad (3.3)$$

where $r^2 = x^2 + y^2 + z^2$. The dust velocity and concentration are to be found from the zero order Stokes equations

$$\underline{v}_0 \cdot \nabla \underline{v}_0 = \frac{\underline{u}_0 - \underline{v}_0}{\sigma} \qquad (3.4)$$

$$\nabla \cdot (f_0 \underline{v}_0) = 0 \qquad (3.5)$$

We shall consider (3.4) and (3.5) presently.

FIRST ORDER OSEEN SOLUTION.

The equations are

$$\underline{u}_1' = \underline{v}_1'$$

$$i \cdot \nabla' \underline{u}_1'(1+f_0') = -\nabla' p_1' + \nabla'^2 \underline{u}_1'$$

$$\nabla' \cdot \underline{u}_1' = 0$$

$$\nabla' \cdot (f_0' \underline{u}_1' + \underline{f}_1' i) = 0 \qquad (3.6)$$

for which the solution is $f_1' = 0$ and the stream function

$$\psi_1' = -\frac{3(1+\cos\theta')}{2(1+f_\infty)} \cdot \left[1 - e^{-\frac{r'}{2}(1+f_\infty)(1-\cos\theta')}\right] \qquad (3.7)$$

The Oseen flow field is not influenced by the dust to this order even when the dust particles are massive. (Refer to (2.14) on page 94.)

Further order solutions are hard to find.

Now we consider the equation (3.4) for $\underline{v}_0$. This is to be found with the $\underline{u}_0$ given by (3.3). The equation (3.4) is in fact the equation of motion in nondimensional form for a single dust particle moving in the Stokes flow field. This equation can not be solved in closed form and therefore some approximate solutions are obtained.

First we consider the case when $\sigma >> 1$. The particle trajectory is then close to the straight line y = constant, z = constant and $\underline{v}_0 = i$. Therefore $\underline{v}_0$ can be found from the linearized equation

$$i \cdot \nabla \underline{v}_0 = \frac{\underline{u}_0 - \underline{v}_0}{\sigma}$$

or

$$\underline{v}_0 + \sigma \frac{\partial}{\partial x} \underline{v}_0 = \underline{u}_0 \ . \qquad (3.8)$$

The solution satisfying the boundary condition $\underline{v}_0 \to i$ at $\underline{x} \to \infty$ is

$$\underline{v}_0 = \frac{e^{-x/\sigma}}{\sigma} \int_{-\infty}^{x} \underline{u}_0(x', y, z)\, e^{x'/\sigma}\, dx' \tag{3.9}$$

From (3.9) it is easily seen that

$$\nabla \cdot \underline{v}_0 = \frac{e^{-x/\sigma}}{\sigma} \int_{-\infty}^{x} \nabla \cdot \underline{u}_0'\, e^{x'/\sigma}\, dx'$$

and consequently, from (3.5), it follows that f_0 = constant = f_∞.

Because $\underline{v}_0$ also satisfies the condition of incompressibility, a stream function for dust, viz ψ_d may be defined by

$$v_{0r} = \frac{1}{r^2 \sin\theta} \frac{\partial \psi_d}{\partial \theta}$$

$$v_{0\theta} = -\frac{1}{r \sin\theta} \frac{\partial \psi_d}{\partial r}$$

and (3.9) leads to

$$\psi_d = \frac{e^{-x/\sigma}}{\sigma} \int_{-\infty}^{x} \psi(x', y, z)\, e^{x'/\sigma}\, dx'$$

$$= \frac{e^{-x/\sigma}}{\sigma} \int_{-\infty}^{x} \tfrac{1}{4}\left(2 - \frac{3}{r'} + \frac{1}{r'^3}\right)\left(y^2 + z^2\right) e^{x'/\sigma} dx' \tag{3.10}$$

where

$$r'^2 = x'^2 + y^2 + z^2.$$

On the plane of symmetry $z = 0$ and with the substitution $x = -w$ we get

$$\psi_d(w, y; \sigma) = \frac{e^{w/\sigma}}{\sigma} \frac{y^2}{4} \int_{w}^{\infty} \left(2 - \frac{3}{\sqrt{1+w'^2}} + \frac{1}{(1+w'^2)^{3/2}}\right) e^{-w'/\sigma} dw'$$

The various integrals are as follows.

$$\int_0^\infty 2e^{-w'/\sigma}\, dw' = 2\sigma$$

$$\int_0^\infty \frac{3}{\sqrt{1+w'^2}}\, e^{-w'/\sigma}\, dw' = \frac{\pi}{2}\left[H_0(\tfrac{1}{\sigma}) - Y_0(\tfrac{1}{\sigma})\right]$$

$$= \left(\frac{1}{\sigma} - \frac{1}{9}\frac{1}{\sigma^3} + \cdots\right)$$

$$- \left[\left(\ln\frac{1}{2\sigma} + \gamma\right) + \cdots\right]$$

where H_0 and Y_0 are the Struve and Bessel functions respectively. γ is the Euler's constant equal to 0.5772.

$$\int_0^\infty \frac{1}{(1+w'^2)^{3/2}}\, e^{-w'/\sigma}\, dw' = 1 - \frac{1}{\sigma} + \cdots$$

Therefore

$$\psi_d(0,1;\sigma) \simeq \tfrac{1}{2} - \frac{3\ln\sigma}{4\sigma} + \frac{0.163}{\sigma} \tag{3.12}$$

The capture cross section is defined by $\frac{\pi}{2}y_\infty^2$ where y_∞ is the distance from the axis at infinity of the last trajectory that hits the sphere. y_∞ for the ψ_d given in (3.12) is such that

$$\frac{\pi}{2}y_\infty^2 = \pi\left(\tfrac{1}{2} - \frac{3\ln\sigma}{4\sigma} + \frac{0\cdot 163}{\sigma}\right)$$

= cross section. The ratio of this cross section to the frontal area of the sphere ($= \frac{\pi}{2}$) is

$$\Sigma = 1 - \frac{3}{2}\frac{\ln\sigma}{\sigma} + \frac{0\cdot 326}{\sigma} \tag{3.13}$$

Next we consider the case when $\sigma << 1$. Suppose $\underline{v}_0$ is expanded in the form

$$\underline{v}_0 = \underline{u}_0 + \sigma\underline{v}_0^{(1)} + \sigma^2\underline{v}_0^{(2)} + \cdots$$

Then the equation (3.4) leads to the set of equations

$$-\underline{v}_0^{(1)} = \underline{u}_0 \cdot \nabla \underline{u}_0$$

$$-\underline{v}_0^{(2)} = \underline{u}_0 \cdot \nabla \underline{v}_0^{(1)} + \underline{v}_0^{(1)} \cdot \nabla \underline{u}_0$$

$$-\underline{v}_0^{(3)} = \underline{u}_0 \cdot \nabla \underline{v}_0^{(2)} + \underline{v}_0^{(1)} \cdot \nabla \underline{v}_0^{(1)} + \underline{v}_0^{(2)} \cdot \nabla \underline{u}_0$$

etc.

It is seen immediately that $\underline{v}_0^{(1)}$ is zero on $r=1$ because $\underline{u}_0 = 0$ there. The next equation shows $\underline{v}_0^{(2)}$ to be zero and so on. Thus $\underline{v}_0 = 0$ on the sphere in the region of validity of the expansion. On the other hand for large σ, the dust particles do strike the sphere. We therefore expect a critical value of σ to exist above which there is capture of dust and below which there is no capture.

To explore this further, we consider the equation of motion for the particles that travel on the axis. Let $\underline{v} = vi$ and $\underline{u} = ui$. Therefore

$$-v\frac{dv}{dr} = \frac{u-v}{\sigma} \qquad \text{or}$$

$$\frac{dv}{dr} = \frac{2vr^3 - (2r^3 - 3r^2 + 1)}{2\sigma v r^3} \tag{3.14}$$

The boundary condition is that $v=1$ at $r \to \infty$. The point $r=1$ is a singular point. The nature of this singularity is investigated now. Let $r=1+h$. Then (3.14) becomes

$$\frac{dv}{dr} = \frac{2v(1+h)^3 - (2h^3 + 3h^2)}{2\sigma v(1+h)^3} \tag{3.15}$$

and the singular point is now at $v=0$, $h=0$. A parameter t is introduced to write the above in the form

$$\frac{dv}{dt} = 2v(1+h)^3 - (3h^2 + 2h^3)$$

$$\frac{dh}{dt} = 2\sigma v(1+h)^3$$

It is seen that the tangents to the trajectories at the origin are the straight lines $v=0$ and $h=\sigma v$. Let $h=\zeta v$. The equations in terms of ζ and v are the following.

$$\frac{dv}{dt} = 2v(1+\zeta v)^3 - (3\zeta^2 v^2 + 2\zeta^3 v^3)$$

$$\frac{d\zeta}{dt} = 2\sigma(1+\zeta v)^3 - \zeta\left[2(1+\zeta v)^3 - (3\zeta^2 v + 2\zeta^3 v^2)\right]$$

Retaining only the leading terms, they are

$$\frac{dv}{dt} = 2v$$

$$\frac{d\zeta}{dt} = 2\sigma - 2\zeta$$

which shows that the singular point $\zeta = \sigma$, $v = 0$ is a saddle point as shown in Fig. 3.1. There are only two trajectories passing through this point and they are given by $\zeta=\sigma$. Correspondingly, there are only trajectories in the h, v plane tangential to the straight line $h=\sigma v$. Consider the trajectories tangential to $v=0$. Let $v=wh$ and the equations in terms of w and h, retaining only the leading terms are

$$\frac{dw}{dt} = 2w - 3h$$

$$\frac{dh}{dt} = 2\sigma wh$$

Therefore the trajectories in the w,h plane are as shown in Fig. 3.2. The corresponding picture in the v,h plane is as shown in Fig. 3.3.

Qualitatively there are three possibilities for the solution of (3.15) satisfying the boundary condition $v=1$ at infinity. These are sketched in Fig. 3.4 together with some neighboring trajectories for $\sigma \lesseqgtr \sigma_{crit}$. The dotted line shows the solution $v=u=(3h^2+2h^3)/2(1+h)^3$ which is appropriate to the case $\sigma = 0$.

The critical value of σ_{crit} was obtained by Langmuir and Blodgett [6], who computed trajectories of particles in the flow field of a cylinder and a sphere for the case of viscous as well as ideal flow around them. They solved the set of equations

$$\sigma v_x \frac{dv_x}{dx} = -(v_x - u_x) \cdot \frac{C_D R}{24}$$

$$\sigma v_y \frac{dv_y}{dy} = -(v_y - u_y) \cdot \frac{C_D R}{24}$$

$$\frac{dx}{dt} = -v_x$$

$$\frac{dy}{dt} = v_y$$

$$\left(\frac{R}{R_U}\right)^2 = (v_x - u_x)^2 + (v_y - u_y)^2 \tag{3.16}$$

R_U is the Reynolds number based on the diameter of the sphere and the free stream velocity. $C_D R / 24$ would be equal to unity when the law of force between the particle and fluid is the Stokes law. C_D is taken at the local Reynolds number formed from the relative velocity. We observe here that $\underline{v} \cdot \nabla \underline{v}$ has been erroneously written down in these equations. However, the calculation of σ_{crit} is correct because the missing term in the first equation is zero on the axis. The second equation is identically zero in this case. The value of σ_{crit} obtained by Langmuir and Blodgett is 1.214 for the case of the sphere.

The plot of the efficiency of capture Σ as a function of σ is shown in Fig. 3.5. The asymptotic relation (3.13) is drawn in dark line. The dotted line passing through the value $\sigma_{crit} = 1.214$ is due to the computations of Fonda and Herne, referred to in the book by Richardson on page 31 [7]. The details of computations are not given in this book. This result of Fonda and Herne seems to be the accepted result in the literature on this subject. It is seen from the Fig. 3.5 that the asymptotic result is satisfactory for σ greater than 20. There do not seem to be any experimental data on the capture of particles at low Reynolds numbers with which a comparison can be made.

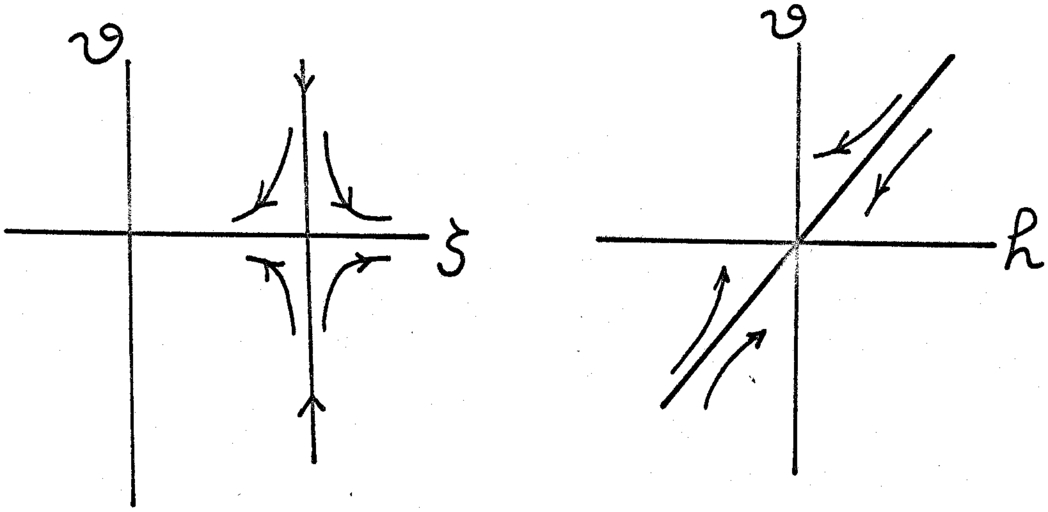


FIG 3.1

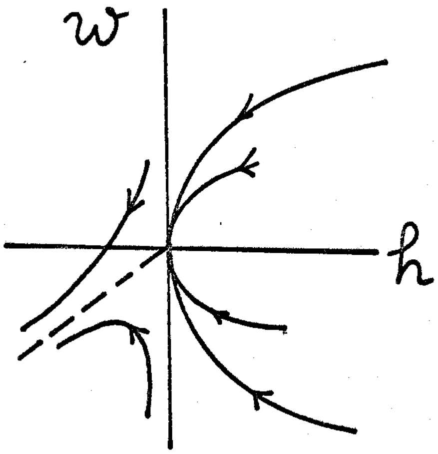


FIG 3.2

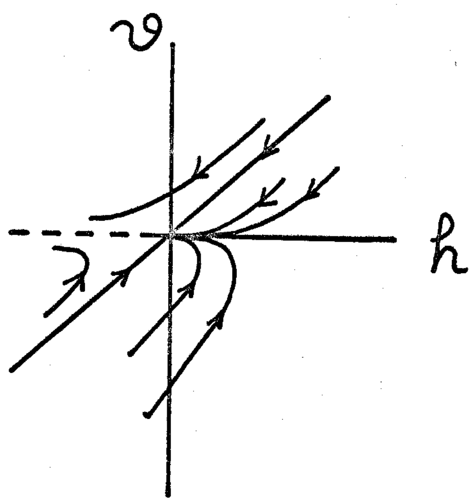


FIG 3.3

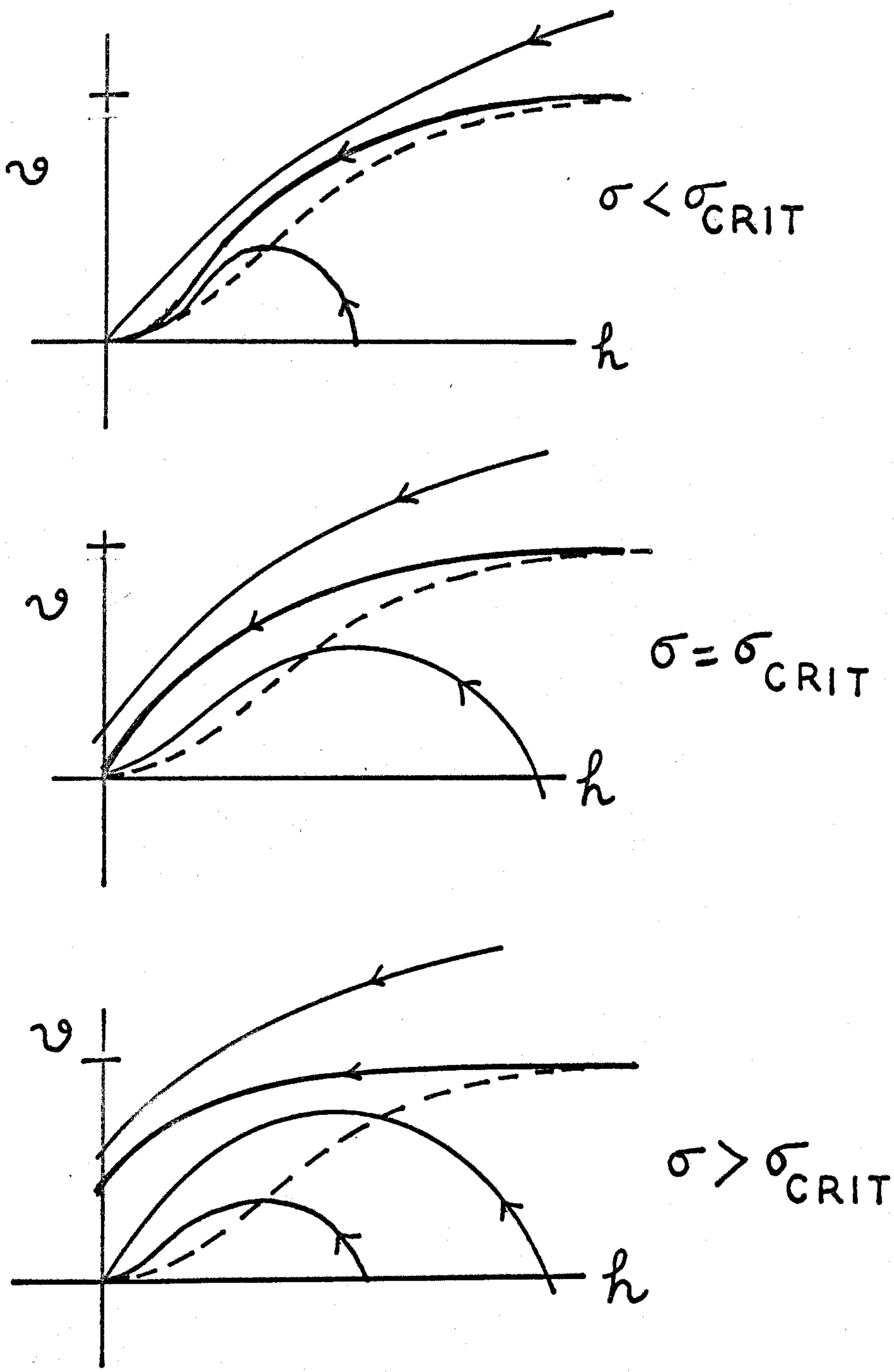

ϑ
$\sigma < \sigma_{CRIT}$
h
ϑ
$\sigma = \sigma_{CRIT}$
h
ϑ
$\sigma > \sigma_{CRIT}$
h

FIG 3.4

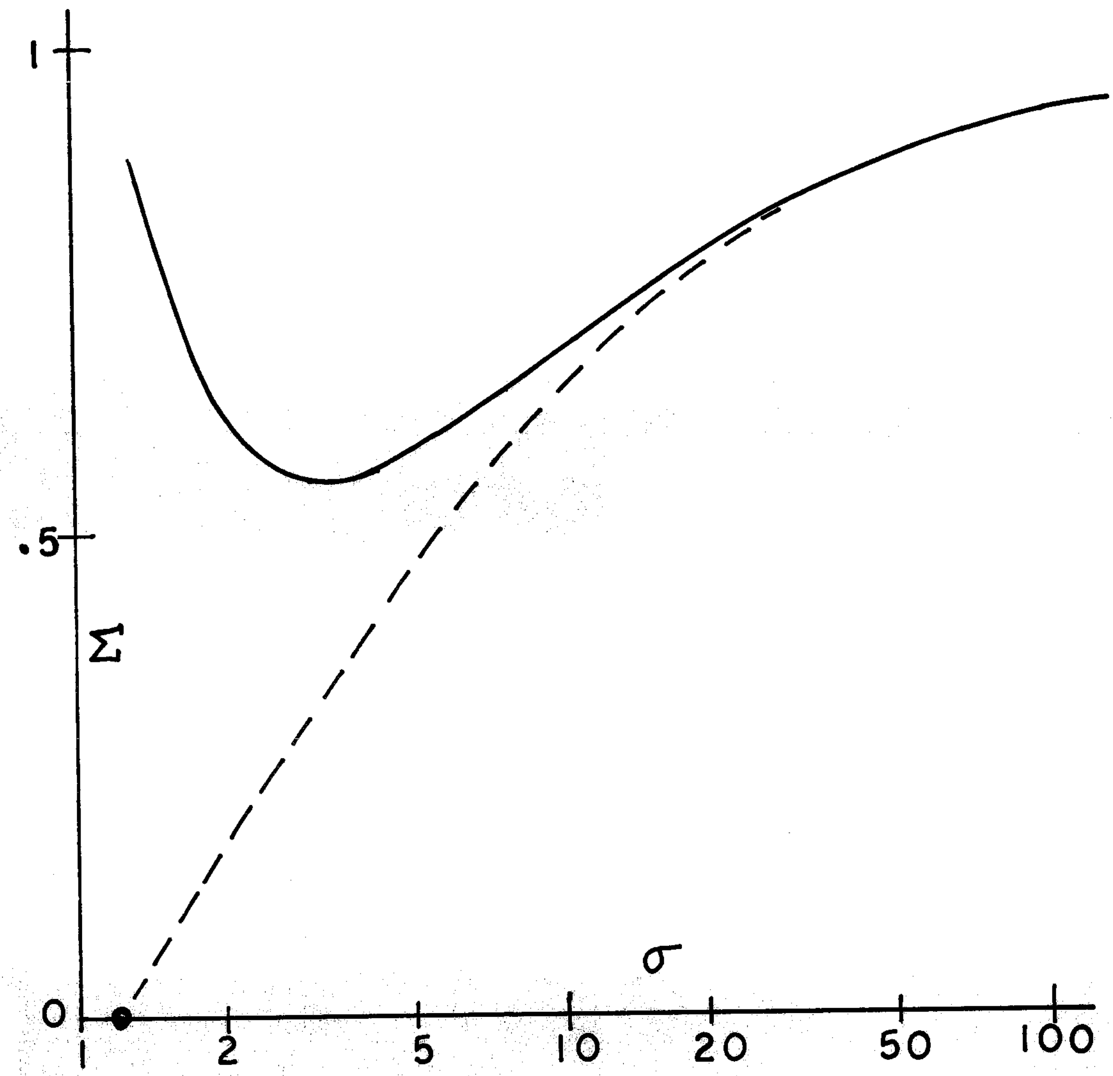


FIG 3.5

4. HIGH REYNOLDS NUMBER. $\sigma >> 1$.

The next limit to be considered is when R tends to infinity. The equations of motion in nondimensional form are

$$\underline{u} \cdot \nabla \underline{u} = -\nabla p - f \underline{v} \cdot \nabla \underline{v}$$

$$\nabla \cdot \underline{u} = 0$$

$$\underline{v} \cdot \nabla \underline{v} = \frac{\underline{u} - \underline{v}}{\sigma}$$

$$\nabla \cdot (f \underline{v}) = 0 \tag{4.1}$$

The boundary conditions are that $\underline{u} = \underline{v} = i$, $f = f_\infty$, $p = p_\infty$ at upstream infinity. We consider the limit $\sigma \rightarrow \infty$, f fixed. This is the limit when the particles are very massive and to keep f fixed, their number density tends to zero as σ tends to infinity. An expansion in powers of $\frac{1}{\sigma}$ is carried out by writing

$$\underline{v} = \underline{v}_0 + \frac{\underline{v}_2}{\sigma^2} + \cdots$$

$$f = f_0 + \frac{f_1}{\sigma} + \cdots \tag{4.2}$$

etc. The equations to the various orders and their solutions are as follows.

ZEROTH ORDER SOLUTION.

The equations are

$$\underline{u}_0 \cdot \nabla \underline{u}_0 = -\nabla p_0 - f_0 \, \underline{v}_0 \cdot \nabla \underline{v}_0$$
$$\nabla \cdot \underline{u}_0 = 0$$
$$\underline{v}_0 \cdot \nabla \underline{v}_0 = 0$$
$$\nabla \cdot (f_0 \, \underline{v}_0) = 0 \qquad (4.3)$$

The solution is very simple. $\underline{u}_0$ is the potential flow solution given by

$$\underline{u}_0 = \nabla \phi_0 = \nabla\left[\left(r + \frac{1}{2r^2}\right)\cos\theta\right]$$
$$\underline{v}_0 = i \qquad (4.4)$$

$f = f_\infty$ everywhere except in the wake where it is zero as sketched in Fig. 4.1. The pressure is given by the Bernoulli equation

$$p_0 + \tfrac{1}{2}(1 + f_0)\underline{u}_0^2 = p_\infty + \frac{(1 + f_\infty)}{2}$$

FIRST ORDER SOLUTION.

The equations are

$$\underline{u}_0 \cdot \nabla \underline{u}_1 + \underline{u}_1 \cdot \nabla \underline{u}_0 = -\nabla p_1 - f_0(\underline{u}_0 - \underline{v}_0) \qquad (4.5)$$

$$\underline{v}_0 \cdot \nabla \underline{v}_1 + \underline{v}_1 \cdot \nabla \underline{v}_0 = \underline{u}_0 - \underline{v}_0 \qquad (4.6)$$

$$\nabla \cdot \underline{u}_1 = 0 \qquad (4.7)$$

$$\nabla \cdot (f_0 \underline{v}_1 + f_1 \underline{v}_0) = 0 \qquad (4.8)$$

The boundary conditions are that all the first order quantities vanish at infinity. We consider (4.6) first. Because $\underline{v}_0 = i$ and $\underline{u}_0 = \nabla\phi_0$,

$i \cdot \nabla \underline{v}_1 = \underline{u}_0 - i$, whose solution is

$$\underline{v}_1 = \frac{\underline{x}}{2r^3} \tag{4.9}$$

It is seen from (4.9) that $\nabla \cdot \underline{v}_1 = 0$. Therefore from (4.8) we have

$$f_1 = \text{constant} = 0.$$

Because f_1 is zero, the velocity $\underline{v}_1$ in (4.9) can be obtained from a stream function defined by

$$v_{0r} + \frac{1}{\sigma} v_{1r} = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\psi_{0d} + \frac{1}{\sigma}\psi_{1d}\right)$$

$$v_{0\theta} + \frac{1}{\sigma} v_{1\theta} = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(\psi_{0d} + \frac{1}{\sigma}\psi_{1d}\right)$$

i.e.

$$\psi_{0d} + \frac{\psi_{1d}}{\sigma} = \frac{r^2\sin^2\theta}{2} - \frac{1+\cos\theta}{2\sigma} \tag{4.10}$$

The efficiency of capture can be calculated now (Fig. 4.2). The point at which the dust velocity is tangential to the sphere is given by

$$\left(i + \frac{\underline{x}}{2\sigma r^2}\right) \cdot \frac{\underline{x}}{r} = 0$$

where $r = 1$. i.e.

$$\cos\theta_1 + \frac{1}{2\sigma} = 0$$

$$\theta_1 = \cos^{-1}\left(-\frac{1}{2\sigma}\right)$$

Therefore the distance from the axis at infinity of the grazing stream line is given by

$$\tfrac{1}{2} y_\infty^2 = \psi_d(1,\theta) = \frac{\sin^2\theta_1}{2} - \frac{1+\cos\theta_1}{2\sigma}$$

$$= \tfrac{1}{2} - \frac{1}{2\sigma} + \frac{1}{8\sigma^2}$$

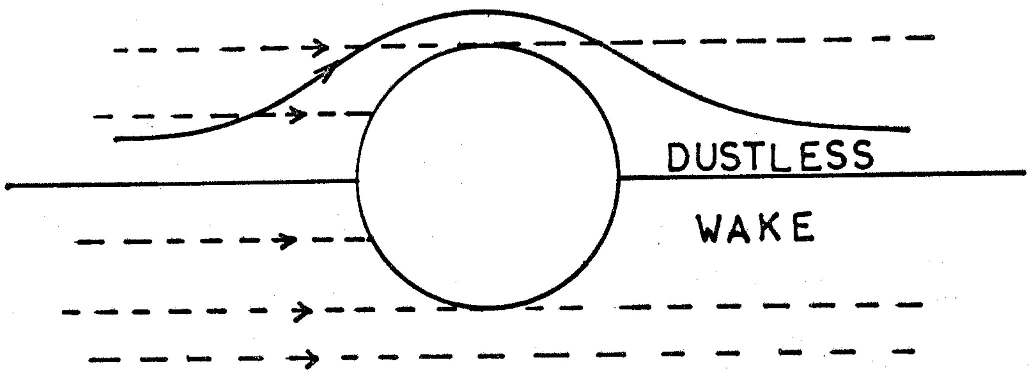
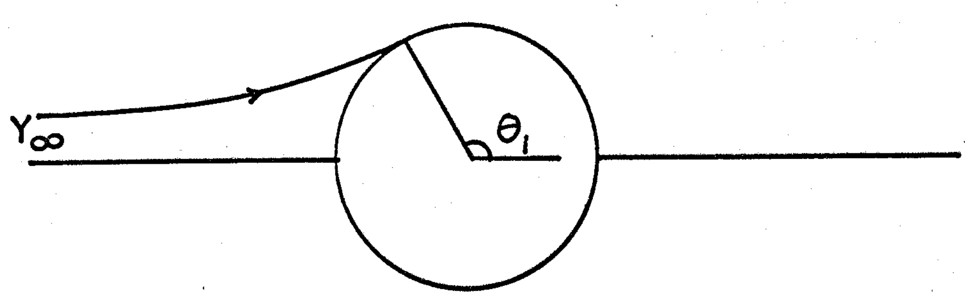


FIG 4.1



THE GRAZING DUST-STREAM LINE

FIG 4.2

The efficiency of capture is

$$\Sigma = 1 - \frac{1}{\sigma} + \frac{1}{4\sigma^2} \tag{4.11}$$

This will be plotted in the next section.

The solution of (4.5) and (4.7) to find $\underline{u}_1$ and p_1 with $\underline{v}_0$ given by (4.9) seems to be too difficult. We shall proceed to the case of $\sigma << 1$.

5. HIGH REYNOLDS NUMBER. $\sigma << 1$. $f << 1$.

First the case of f small of order σ^3 is considered. In this case, the inner expansion for f can be found to three terms. Later, some comments are made about the nature of the solutions when f is of order 1, σ or σ^2.

The equations of motion and boundary conditions are the same as in (4.1). An expansion in powers of σ is considered in the form

$$
\begin{aligned}
\underline{u} &= \underline{u}_0 + \sigma \underline{u}_1 + \cdots \\
\underline{v} &= \underline{v}_0 + \sigma \underline{v}_1 + \cdots \\
p &= p_0 + \sigma p_1 + \cdots \\
f &= \sigma^3 f_3 + \sigma^4 f_4 + \cdots
\end{aligned}
\tag{5.1}
$$

The independent variables are r and θ.

THE ZEROTH ORDER SOLUTION.

The equations are

$$
\begin{aligned}
\underline{u}_0 \cdot \nabla \underline{u}_0 &= -\nabla p_0 \\
\nabla \cdot \underline{u}_0 &= 0 \\
\underline{u}_0 &= \underline{v}_0 \\
\nabla \cdot (\underline{v}_0 f_3) &= 0
\end{aligned}
$$

with boundary conditions $\underline{u}_0 = \underline{v}_0 = i$, $f_3 = f_\infty$ at infinity. The solution is

$$
\begin{aligned}
\underline{u}_0 = \underline{v}_0 = \nabla \phi_0 &= \phi \left[\left(r + \frac{1}{2r^2} \right) \cos\theta \right] \\
f_3 &= \text{constant} = f_\infty \\
p_0 + \tfrac{1}{2} \underline{u}_0^2 &= p_\infty + \tfrac{1}{2}
\end{aligned}
\tag{5.2}
$$

~~The radial and tangential~~ components of $\underline{u}_0$ are

$$u_{0r} = \left(1 - \frac{1}{r^3}\right) \cos\theta$$

$$u_{0\theta} = \left(1 + \frac{1}{2r^3}\right) \sin\theta \tag{5.3}$$

THE FIRST ORDER SOLUTION.

The equations are

$$\underline{u}_0 \cdot \nabla \underline{u}_1 + \underline{u}_1 \cdot \nabla \underline{u}_0 = -\nabla p_1 \tag{5.4}$$

$$\nabla \cdot \underline{u}_1 = 0 \tag{5.5}$$

$$\underline{v}_0 \cdot \nabla \underline{v}_0 = \underline{u}_1 - \underline{v}_1 \tag{5.6}$$

$$\nabla \cdot (f_3 \underline{v}_1 + f_4 \underline{v}_0) = 0 \tag{5.7}$$

with the boundary conditions $\underline{u}_1$, $\underline{v}_1$, p_1 and $f_4 \to 0$ at ∞. The solution is

$$\underline{u}_1 = 0$$

$$p_1 = 0$$

From (5.6)

$$\underline{v}_1 = -\underline{u}_0 \cdot \nabla \underline{u}_0 = -\tfrac{1}{2} \nabla \left[\left(1 - \frac{1}{r^3}\right)^2 \cos^2\theta + \left(1 + \frac{1}{2r^3}\right)^2 \sin^2\theta\right] \tag{5.8}$$

From (5.7)

$$\underline{u}_0 \cdot \nabla f_4 + f_3 \nabla \cdot \underline{v}_1 = 0$$

i.e.

$$\underline{u}_0 \cdot \nabla f_4 = f_3 \nabla \cdot (\underline{u}_0 \cdot \nabla \underline{u}_0) = f_3 \nabla^2 \left(\frac{\underline{u}_0^2}{2}\right) \tag{5.9}$$

On substituting for $\underline{u}_0$, (5.9) gives

$$\left(1 - \frac{1}{r^3}\right) \cos\theta \frac{\partial f_4}{\partial r} - \frac{1}{r}\left(1 + \frac{1}{2r^3}\right) \sin\theta \frac{\partial f_4}{\partial \theta} = \frac{9}{2} f_3 \cdot \frac{1+2\cos^2\theta}{r^8}$$

This equation was obtained by Michael [8] and its solution on the axis $\theta = \pi$ given by him is

$$f_4 = \frac{27f_3}{2}\left\{\frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \tan^{-1}\frac{2r+1}{\sqrt{3}}\right) - \frac{1}{6}\ln\left[\frac{(r-1)^2}{r^2+r+1}\right] - \frac{1}{r} - \frac{1}{4r^2}\right\} \qquad (5.11)$$

On the sphere as r tends to unity, he found

$$f_4 \to -\frac{9}{2} f_3 \left\{ \ln(r-1) + \frac{8}{3}\cos^2\frac{\theta}{2} + {}_4\ln\sin\frac{\theta}{2} - \frac{\pi}{2\sqrt{3}} - \frac{\ln 3}{2} + \frac{15}{4}\right\} \qquad (5.12)$$

f_4 is seen to be singular on $r = 1$ for all θ. Michael also computed the distribution of f_4 on other stream lines near the sphere. At downstream infinity, f_4 is positive on all stream lines and because the sphere is not a source of dust, one suspects the presence of a dust free layer near the sphere. Because this solution shows a singularity on the sphere, the region near the sphere is examined afresh by considering the problem in the inner variables defined by

$$x = \frac{r-1}{\sigma}$$

$$\theta' = \theta$$

Primes denote the independent variables which are functions of x and $\theta' = \theta$.

The equations of motion in the inner variables are the following.

$$\frac{u'_r}{\sigma}\frac{\partial u'_r}{\partial x} + \frac{u'_\theta}{1+\sigma x}\frac{\partial u'_r}{\partial \theta} - \frac{u'^2_\theta}{1+\sigma x} = -\frac{1}{\sigma}\frac{\partial p'}{\partial x} - f'\left\{\frac{v'_r}{\sigma}\frac{\partial v'_r}{\partial x} + \cdots\right\} \qquad (5.13)$$

$$\frac{u'_r}{\sigma}\frac{\partial u'_\theta}{\partial x}+\frac{u'_\theta}{1+\sigma x}\frac{\partial u'_\theta}{\partial \theta}+\frac{u'_r u'_\theta}{1+\sigma x} = -\frac{1}{1+\sigma x}\frac{\partial p'}{\partial \theta} - f'\left\{\frac{v'_r}{\sigma}\frac{\partial v'_\theta}{\partial x}+\frac{v'_\theta}{1+\sigma x}\frac{\partial v'_\theta}{\partial \theta}+\frac{v'_r v'_\theta}{1+\sigma x}\right\} \quad (5.14)$$

$$\frac{1}{\sigma}\frac{\partial u'_r}{\partial x}+\frac{1}{1+\sigma x}\frac{\partial u'_\theta}{\partial \theta}+\frac{\partial u'_r}{1+\sigma x}+\frac{u'_\theta \cot\theta}{1+\sigma x} = 0 \quad (5.15)$$

$$\frac{v'_r}{\sigma}\frac{\partial v'_r}{\partial x}+\frac{v'_\theta}{1+\sigma x}\frac{\partial v'_r}{\partial \theta}-\frac{v'^2_\theta}{1+\sigma x} = \frac{u'_r - v'_r}{\sigma} \quad (5.16)$$

$$\frac{v'_r}{\sigma}\frac{\partial v'_\theta}{\partial x}+\frac{v'_\theta}{1+\sigma x}\frac{\partial v'_\theta}{\partial \theta}+\frac{v'_r v'_\theta}{1+\sigma x} = \frac{u'_\theta - v'_\theta}{\sigma} \quad (5.17)$$

$$\frac{1}{\sigma}\frac{\partial}{\partial x}(f'v'_r)+\frac{1}{1+\sigma x}\frac{\partial f'v'_\theta}{\partial \theta}+\frac{2f'v'_r}{1+\sigma x}+\frac{f'v'_\theta \cot\theta}{1+\sigma x} = 0 \quad (5.18)$$

The boundary conditions are that the inner solutions match the outer ones and satisfy the inner boundary conditions on $r=1$.

The outer solution $\underline{u}$ is expanded in inner variables to give

$$u_r = 3\sigma x\cos\theta - 6\sigma^2 x^2\cos\theta + O(\sigma^3)$$

$$u_\theta = -\frac{3}{2}\sin\theta + \frac{3\sigma x}{2}\sin\theta - 3\sigma^2 x^2\sin\theta + O(\sigma^3) \quad (5.19)$$

The inner solution is the same to this order because f' is of order σ^3. i.e.

$$u'_r = u_r$$
$$u'_\theta = u_\theta$$

The dust velocity $\underline{v}'$ is found next. The leading order equations for $\underline{v}'$ found from (5.16) and (5.17) are

$$v'_{or}\frac{\partial v'_{or}}{\partial x} = -v'_{or} \quad (5.20)$$

$$v'_{or}\frac{\partial v'_{o\theta}}{\partial x} = -\frac{3}{2}\sin\theta - v'_\theta \quad (5.21)$$

From (5.20), $v'_{0r} = 0$ or $-x +$ function (θ). The outer solution for $\underline{v}$ is

$$\underline{v} = \underline{u}_0 - \sigma \underline{u}_0 \cdot \nabla \underline{u}_0 + O(\sigma^2)$$

i.e.

$$v_r = \left(1 - \frac{1}{r^3}\right)\cos\theta - \sigma\left\{\left(1 - \frac{1}{r^3}\right)\frac{3\cos^2\theta}{r^4} - \left(1 + \frac{1}{2r^3}\right)\frac{3\sin^2\theta}{2r^4}\right\} + O(\sigma^2) \tag{5.22}$$

$$v_\theta = -\left(1 + \frac{1}{2r^3}\right)\sin\theta + \frac{\sigma}{r}\left\{\left(1 - \frac{1}{r^3}\right)^2 - \left(1 + \frac{1}{2r^3}\right)^2\right\}\sin\theta\cos\theta + O(\sigma^2) \tag{5.23}$$

Expressed in inner variables, these give

$$v_r = \left[3\sigma x\cos\theta - 6\sigma^2 x^2\cos\theta + O(\sigma^3)\right]$$

$$- \sigma\left[3\sigma x \cdot 3\cos^2\theta + O(\sigma^2) - \left(\frac{3}{2} - \frac{3}{2}\sigma x + O(\sigma^2)\right)\left(\frac{3\sin^2\theta}{2}\right)\left(1 - 4\sigma x + O(\sigma^2)\right)\right]$$

$$+ O(\sigma^2)$$

and

$$v_\theta = -\left(\frac{3}{2} - \frac{3}{2}\sigma x\right)\sin\theta - \frac{9}{4}\sin\theta\cos\theta\,\sigma + O(\sigma^2)$$

There is no term of order unity in v'_r. The solution of (5.20) and (5.21) is therefore

$$v'_{0r} = 0$$

$$v'_{0\theta} = -\frac{3}{2}\sin\theta \tag{5.24}$$

The order σ equations for v' are

$$-v'^2_{0\theta} = u'_{1r} - v'_{1r}$$

i.e. $$-\frac{9}{4}\sin^2\theta = 3x\cos\theta - v'_{1r} \quad v'_{1r} = \left(3x\cos\theta + \frac{9}{4}\sin^2\theta\right) \tag{5.25}$$

and

$$v'_{0\theta} \frac{\partial v'_{0\theta}}{\partial \theta} = u'_{1\theta} - v'_{1\theta}$$

i.e. $$\frac{9}{4} \sin\theta \cos\theta = \frac{3x}{2} \sin\theta - v'_{1\theta} \quad v'_{1\theta} = \left(\frac{3x}{2}\sin\theta - \frac{9}{4}\sin\theta\cos\theta\right) \tag{5.26}$$

The radial equation of order σ^2 is

$$v'_{1r} \frac{\partial v'_{1r}}{\partial x} + v'_{0\theta} \frac{\partial v'_{1r}}{\partial \theta} - 2v'_{0\theta} v'_{1\theta} + x v'^2_{0\theta} = u'_{2r} - v'_{2r}$$

and its solution is

$$v'_{2r} = -6x^2 \cos\theta - 9x + \frac{27}{4}\sin^2\theta\cos\theta - \frac{9}{4}x\sin^2\theta \tag{5.27}$$

The equation for the θ component is

$$v'_{1r} \frac{\partial v'_{1\theta}}{\partial x} + v'_{0\theta} \frac{\partial v'_{1\theta}}{\partial \theta} + v'_{1\theta} \frac{\partial v'_{0\theta}}{\partial \theta} - x\, v'_{0\theta} \frac{\partial v'_{0\theta}}{\partial \theta} + v'_{1r} v'_{0\theta} = u'_{2\theta} - v'_{2\theta}$$

and its solution is

$$v'_{2\theta} = -3x^2 \sin\theta + \frac{27}{4}x\sin\theta\cos\theta - \frac{27}{8}(2\cos^2\theta - \sin^2\theta)\sin\theta \tag{5.28}$$

Collecting these together, the solution $\underline{v}'$ which matches (5.22) and (5.23) is

$$v'_r = \sigma\left(3x\cos\theta + \frac{9}{4}\sin^2\theta\right) + \sigma^2\left(-6x^2\cos\theta - 9x + \frac{27}{4}\sin^2\theta\cos\theta - \frac{9x}{4}\sin^2\theta\right) + O(\sigma^3) \tag{5.27}$$

$$v'_\theta = -\frac{3}{2}\sin\theta + \sigma\left(\frac{3x}{2}\sin\theta - \frac{9}{4}\sin\theta\cos\theta\right) + \sigma^2\left(-3x^2\sin\theta + \frac{27}{4}x\sin\theta\cos\theta - \frac{27}{4}\cos^2\theta\sin\theta + \frac{27}{8}\sin^3\theta\right) + O(\sigma^3) \tag{5.28}$$

The dust concentration f' has an inner expansion given by

$$f' = \sigma^3 f'_3 + \sigma^3 \ln\sigma \, f'_4 + \sigma^4 f'_5 + O(\sigma^5)$$

The equations for the various orders are obtained from (5.18) by substituting for $\underline{v}'$ given in (5.27) and (5.28). These are

$$\left(3x\cos\theta + \frac{9}{4}\sin^2\theta\right)\frac{\partial f'_3}{\partial x} - \frac{3}{2}\sin\theta\frac{\partial f'_3}{\partial\theta} = 0 \qquad (5.29)$$

$$\left(3x\cos\theta + \frac{9}{}\sin^2\theta\right)\frac{\partial f'_5}{\partial x} - \frac{3}{2}\sin\theta\frac{\partial f'_5}{\partial\theta} = \frac{9}{2}f'_3\left(1+2\cos^2\theta\right) \qquad (5.31)$$

The above are first order partial differential equations and are solved by the characteristic method. The equation for the characteristics is

$$\frac{dx}{d\theta} = \frac{3x\cos\theta + \frac{9}{4}\sin^2\theta}{-\frac{3}{2}\sin\theta}$$

Let

$$x = y/\sin^2\theta$$

Then

$$\frac{dy}{d\theta} = -\frac{3}{2}\sin^3\theta$$

$$y = -\frac{3}{2}\left(\frac{\cos^3\theta}{3} - \cos\theta + \text{const}\right)$$

or

$$x = -\frac{3}{2\sin^2\theta}\cdot\left(\frac{\cos^3\theta}{3} - \cos\theta + \text{const}\right) \qquad (5.32)$$

In the x,θ plane, the characteristics are as shown in Fig. 5.1. $\theta = \pi$ is the front stagnation point. The characteristic which passes through this has the constant equal to $-\frac{2}{3}$. f'_3 is constant on a characteristic. Because the sphere is not a source of dust, $f'_3 = 0$ on all characteristics below the one that passes through $\theta = \pi$, the separation line. Above this separation line, f'_3 must match the outer solution (5.11). The outer solution when expanded in x and θ gives

$$f = \sigma^3 f_3 - \frac{9}{2} f_3 \sigma^4 \ln \sigma - \frac{9}{2} f_3 \sigma^4 \ln x + \cdots$$

Therefore $f'_3 = f_3 = f_\infty$ above the separation line and $f'_4 = -9f_3/2$ above the separation line. Below the separation line they are both zero.

For f'_5, on a characteristic given by (5.32)

$$\frac{df'_5}{d\theta} = \frac{9}{2} f'_3 \left(1 + 2\cos^2\theta\right) \Big/ -\frac{3}{2} \sin\theta$$

which gives on integration

$$f'_5 = -3 f'_3 \left[-\frac{3}{2} \ln \frac{1+\cos\theta}{1-\cos\theta} + 2\cos\theta + \alpha \right]$$

where α is a constant on each characteristic. Therefore the solution for f'_5 is given by

$$f'_5 = -3f_\infty \left[-\frac{3}{2} \ln \frac{1+\cos\theta}{1-\cos\theta} + 2\cos\theta \right] + F(\xi)$$

where

$$\xi = x\sin^2\theta + \frac{3}{2}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)$$

On a characteristic ξ = constant. The function F is found by matching as follows. If F is chosen to be

$$-\frac{9}{2} f_\infty \ln (A + \xi)$$

where A = constant, we get for large x,

$$f_5' \simeq \frac{9}{2} f_\infty \ln \frac{1+\cos\theta}{1-\cos\theta} \quad \frac{1}{A + x\sin^2\theta + \frac{3}{2}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)}$$

$$\simeq \frac{9}{2} f_\infty \ln \frac{1+\cos\theta}{1-\cos\theta} \frac{1}{x\sin^2\theta} = -\frac{9}{2} f_\infty \ln 4x\sin^4\frac{\theta}{2}$$

and on $\theta = \pi$, $f_5' \simeq -\frac{9}{2} f_\infty \ln x$ which matches (5.12). Therefore

$$f_5' = -6 f_\infty \cos\theta + \frac{9}{2} f_\infty \ln \frac{(1+\cos\theta)}{(1-\cos\theta)(A+\xi)}$$

where

$$\xi = x\sin^2\theta + \frac{3}{2}\left(\frac{\cos^3\theta}{3} - \cos\theta\right) \qquad (5.33)$$

On the separation line $\xi = 1$. $\xi > 1$ for the characteristics above the separation line and $\xi < 1$ for those below it. Therefore $A + \xi$ vanishes on a characteristic above or below the separation line depending on whether A is less than or greater than -1. If $A > -1$, there will be no singularity in the region of interest, viz above the separation line. However, the logarithmic singularity at the front stagnation point still exists.

In the physical plane, the characteristics and the separation line look like those sketched in Fig. 5.2. When σ is small there is no capture of dust. The centrifugal force on the dust particles deflects them away from the sphere. The flux of dust near the stagnation point also tends to zero because although the dust concentration goes to infinity like $\log x$, the dust velocity goes to zero linearly in x.

The constant A is to be found by making use of the conservation of dust. That is, the flux of dust through a plane perpendicular to the axis at down stream infinity should be equal to $\sigma^3 f_\infty$ times the area of the plane, as the area tends to infinity. This can not be done in the present case because the outer solution for the dust concentration is not known. The solutions (5.11) and (5.12) do not give any information of the distribution of f far from the axis.

When $f \sim 1$, f is expanded in the form

$$f = f_0 + \sigma f_1 + O(\sigma^2)$$

and the equation for f_1 obtained as before. This is the same as (5.9). i.e.

$$\underline{u}_0 \cdot \nabla f_1 = f_0 \nabla^2 \left(\frac{\underline{u}^2{}_0}{2}\right)$$

This was numerically integrated by Michael to get the distribution of f_1 near the sphere. It is found that f_1 increases as the sphere is approached and is positive on all stream lines (of the gas) at infinity downstream. The influence of f on $\underline{u}$ was not found because of the extensive numerical work it would have involved. In this case the zeroth order outer equations are

$$\underline{u}_0 \cdot \nabla \underline{u}_0 = -\nabla p_0 + f_0 \underline{v}_0 \cdot \nabla \underline{v}_0$$
$$\nabla \cdot \underline{u}_0 = 0$$
$$\underline{u}_0 = \underline{v}_0$$
$$\nabla \cdot (f_0 \underline{v}_0) =$$

Therefore $f_0 = \text{constant} = \dot{f}_\infty$. $\underline{u}_0$ is again given by the potential in (5.2). p_0 is merely changed by a factor $(1+f_\infty)$. The first order outer equations are

$$\underline{u}_0 \cdot \nabla \underline{u}_1 + \underline{u}_1 \cdot \nabla \underline{u}_0 = -\nabla p_1 + f_0(\underline{v}_0 \cdot \nabla \underline{v}_1 + \underline{v}_1 \cdot \nabla \underline{v}_0) + f_1 \underline{v}_0 \cdot \nabla \underline{v}_0$$

$$\nabla \cdot \underline{u}_1 = 0$$

$$\underline{v}_0 \cdot \nabla \underline{v}_0 = \underline{u}_1 - \underline{v}_1$$

$$\nabla \cdot (f_0 \underline{v}_1 + f_1 \underline{v}_0) = 0$$

with $\underline{u}_1$, $\underline{v}_1$, p_1 and f_1 tending to zero at infinity. Only the first of the above set is different now and the simple solution $\underline{u}_1 = 0$ and $p_1 = 0$ is no longer true. Because f_1 can not be found, $\underline{u}_1$ also can not be found. The outer solution $\underline{u}$ expanded in inner variables suffers a change of order σ. (equation (5.19)). So also does $\underline{v}$. v'_{1r} is no longer $(3x\cos\theta + \frac{9}{4}\sin^2\theta)$ as in (5.27). Similarly the equation for f'_1 is not (5.29). The inner solution for f' can not be found. The separation line (if it exists) is also altered.

Similarly it is seen that if $f = \sigma f_\infty + \sigma^2 f_2 + \cdots$, $f' = \sigma f_\infty + ? + \cdots$; if $f = \sigma^2 f_\infty + \sigma^3 f_3 + \cdots$, $f' = \sigma^2 f_\infty - \frac{9}{2}\sigma^3 \ln\sigma f_\infty + ? + \cdots$; and if $f = \sigma^3 f_\infty + \sigma^4 f_4 + \cdots$, f' is given by $\sigma^3 f_\infty - \frac{9}{2} f_\infty \sigma^4 \ln\sigma + \sigma^5 f'_5 + \cdots$ where f'_5 is given by (5.33). In all cases, $f' = 0$ inside the separation line.

The next section will be devoted to the case of $\sigma \sim 1$ and $f << 1$.

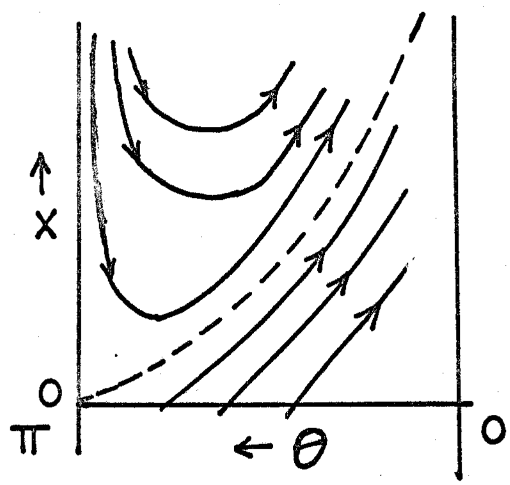


FIG 5.1

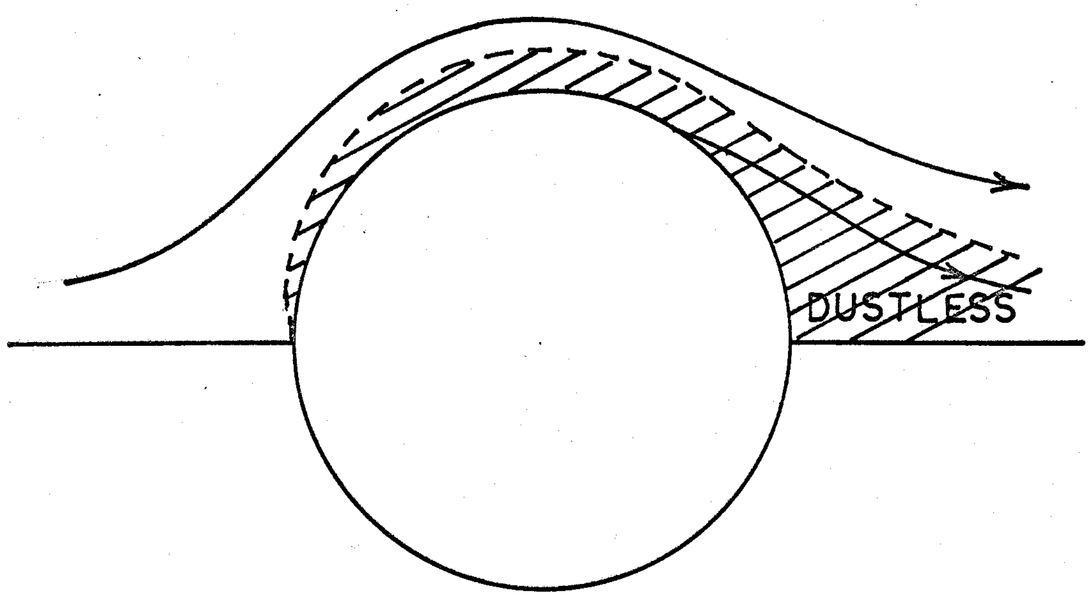


FIG 5.2

6. HIGH REYNOLDS NUMBER. $\sigma \sim 1$. $f << 1$

In this section the motion of a single dust particle moving in the gas flow field will be considered. The equations of motion in nondimensional form in Cartesian coordinates in the meridional plane are

$$\sigma \frac{dv_x}{dt} = u_x - v_x$$

$$\sigma \frac{dv_y}{dt} = u_y - v_y$$

$$\frac{dx}{dt} = v_x$$

$$\frac{dy}{dt} = v_y \qquad (6.1)$$

where t, the time is the independent variable. The velocity components of the gas are given by

$$u_x = 1 + \frac{y^2 - 2x^2}{2(x^2+y^2)^{5/2}}$$

$$u_y = - \frac{3xy}{2(x^2+y^2)^{5/2}} \qquad (6.2)$$

The origin of the coordinate system is the centre of the sphere.

The points $v_x = 0$, $v_y = 0$, $y = 0$, $x = +$ or -1 are the two stationary points of the system of equations (6.1). Trajectories in the neighborhood of the front stagnation point are now considered.

Let

$$x = -1 - \xi$$

$$y = \eta$$

$$v_x = w_x$$

$$v_y = w_y$$

The equations (6.1) linearized near the point (-1, 0) become

$$\frac{d}{dt}\begin{pmatrix} \xi \\ \eta \\ w_x \\ w_y \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3}{\sigma} & 0 & -\frac{1}{\sigma} & 0 \\ 0 & \frac{3}{2\sigma} & 0 & -\frac{1}{\sigma} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ w_x \\ w_y \end{pmatrix} \tag{6.3}$$

This is a matrix equation of the form

$$\frac{dz}{dt} = Bz$$

Assuming $z \sim e^{kt}$, $Bz = kz$ and therefore the determinant $|B - kI| = 0$. The four roots are

$$k = \frac{-1 \pm \sqrt{1-12\sigma}}{2\sigma}, \quad \frac{-1 \pm \sqrt{1+6\sigma}}{2\sigma} \tag{6.4}$$

When $\sigma < 1/12$, there are three negative roots and one positive root. When $\sigma > 1/12$, two are complex with negative real parts. In this case there are three with negative real parts and one positive root. For all σ, the point $\xi = 0$, $\eta = 0$ is a saddle point. The four eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ \frac{-1+\sqrt{1-12\sigma}}{2\sigma} \\ 0 \end{pmatrix} \quad \mathrm{Exp} \quad \frac{-1+\sqrt{1-12\sigma}}{2\sigma} \cdot t \tag{6.5}$$

$$\begin{pmatrix} 1 \\ 0 \\ \dfrac{1+\sqrt{1-2\sigma}}{2\sigma} \\ 0 \end{pmatrix} \quad \text{Exp} \quad \frac{-1-\sqrt{1-12\sigma}}{2\sigma}\, t \tag{6.6}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{-1+\sqrt{1+6\sigma}}{2\sigma} \end{pmatrix} \quad \text{Exp} \quad \frac{-1+\sqrt{1+6\sigma}}{2\sigma} \tag{6.7}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{-1-\sqrt{1+6\sigma}}{2\sigma} \end{pmatrix} \quad \text{Exp} \quad \frac{-1-\sqrt{1+6\sigma}}{2\sigma}\, t \tag{6.8}$$

It is seen that (6.5) and (6.6) involve only ξ and w_x, (6.7) and (6.8) involve only η and w_y. The eigenvector (6.7) is a growing exponential whereas the rest of them are decaying. It can be seen that if the particle has at $t = 0$, the same velocity as the gas, it will never cross the axis $\xi = 0$ if σ is less than 1/12 and will do so otherwise. Thus $\sigma = 1/12$ is a critical value. The sphere will collect dust when $\sigma > 1/12$ and will not if $\sigma < 1/12$. The existence of such a critical value is well known.

G. I. Taylor first found such a critical value $(= \frac{1}{4})$ for the motion of particles in the flow field near a two dimensional stagnation point. [9]

The trajectories are sketched in Fig. 6.1 for the two cases $\sigma \lessgtr 1/12$, in a manner consistent with the proper behavior at the stationary points. For $\sigma < 1/12$, the picture is the same as that obtained in Section 5. The trajectories inside the sphere are of no interest.

It would be of interest to get an idea of the efficiency of capture for σ greater than $1/12$. The asymptotic result for $\sigma >> 1$ of Section 4 provides part of the answer. A very crude analysis of the case when σ is close to $1/12$ is the following.

(6.1) can be written in the form

$$\sigma \frac{d^2x}{dt^2} + \frac{dx}{dt} - \left(1 - \frac{y^2 - 2x^2}{2(x^2+y^2)^{5/2}}\right) = 0 \qquad (6.9)$$

$$\sigma \frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{3xy}{2(x^2+y^2)^{5/2}} = 0 \qquad (6.10)$$

Consider the transformation to new coordinates z and defined by

$$x + \frac{1}{2} = \frac{1}{2}\left[\left(\frac{z}{\eta+1}\right)^2 - 9\eta + 1)^2\right]$$

$$y = z \qquad (6.10a)$$

The coordinate η is constant on parabolas, with $\eta = 0$ being the one that touches the sphere at the stagnation point. (6.9) and (6.10) can be written in terms of η and z and linearized to get

$$\sigma \frac{d^2\eta}{dt^2} + \frac{d\eta}{dt} + 3\eta = 0$$

$$\sigma \frac{d^2z}{dt^2} + \frac{dz}{dt} - \frac{3}{2} z = 0 \qquad (6.11)$$

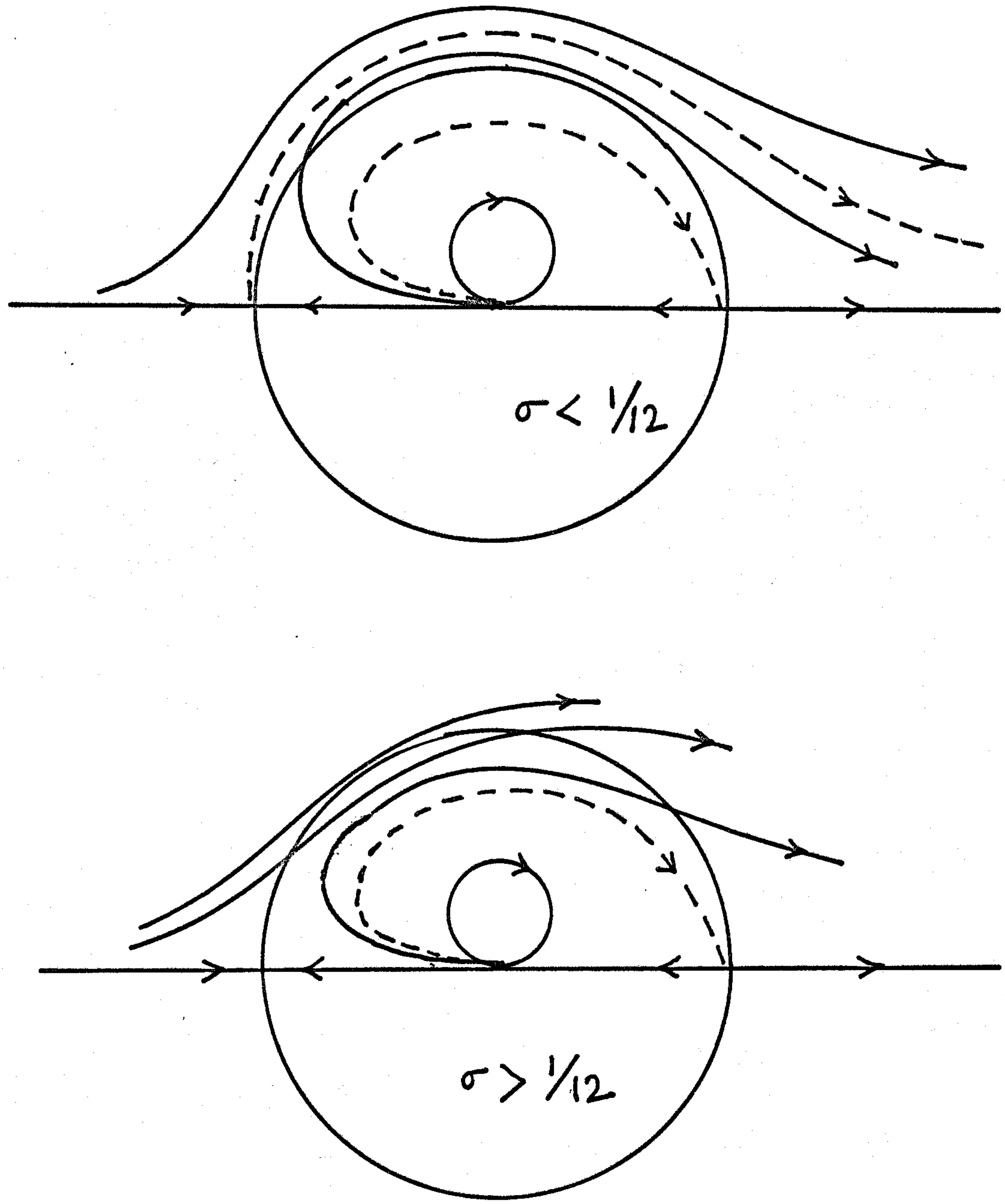


FIG 6.1

The regions of small η and z, where this linearization is expected to be reasonable is shown shaded in Fig. 6.2. These equations can be solved easily with the initial conditions

$$t = 0, \qquad \eta = 1 \qquad \frac{d\eta}{dt} = -3$$

$$z = 1 \qquad \frac{dz}{dt} = 0$$

Because the problem is linear, any other trajectory is obtained by multiplying by suitable constants. The solution is

$$\eta = \frac{1-6\sigma}{\sqrt{12\sigma-1}} e^{-t/2\sigma} \sin\frac{\sqrt{12\sigma-1}}{2\sigma} \cdot t + e^{-t/2\sigma} \cos\frac{\sqrt{12\sigma-1}}{2\sigma} t \tag{6.12}$$

$$z = \frac{\sqrt{1+6\sigma}-1}{2\sqrt{1+6\sigma}} e^{-\left(\frac{1+\sqrt{1+6\sigma}}{2\sigma}\right)t} + \frac{1+\sqrt{1+6\sigma}}{2\sqrt{1+6\sigma}} e^{-\left(\frac{1-\sqrt{1+6\sigma}}{2\sigma}\right)t} \tag{6.13}$$

and the time τ at which it hits $\eta = 0$ is given by

$$\tan\frac{\sqrt{12\sigma-1}}{2\sigma}\tau = \frac{\sqrt{12\sigma-1}}{6\sigma-1} \tag{6.14}$$

Fig. 6.3 is a plot of $\tan\alpha$ vs. $\alpha = \frac{\sqrt{12\sigma-1}}{2\sigma}\tau$. The locus of the solution point of (6.14) moves on the curves as shown here. When $\sigma = 1/12$, $\alpha = \tau$. As σ increases, α decreases to zero. In particular, when σ is near 1/12,

$$\tan\frac{\sqrt{12\sigma-1}}{2\sigma}\tau \doteq \frac{\sqrt{12\sigma-1}}{2\sigma}\tau - \pi = \frac{\sqrt{12\sigma-1}}{6\sigma-1}$$

i.e.

$$\tau \doteq \frac{2\sigma}{6\sigma-1} + \frac{2\sigma\pi}{\sqrt{12\sigma-1}} \doteq \frac{2\sigma\pi}{\sqrt{12\sigma-1}}$$

From (6.13) z at time τ is approximately

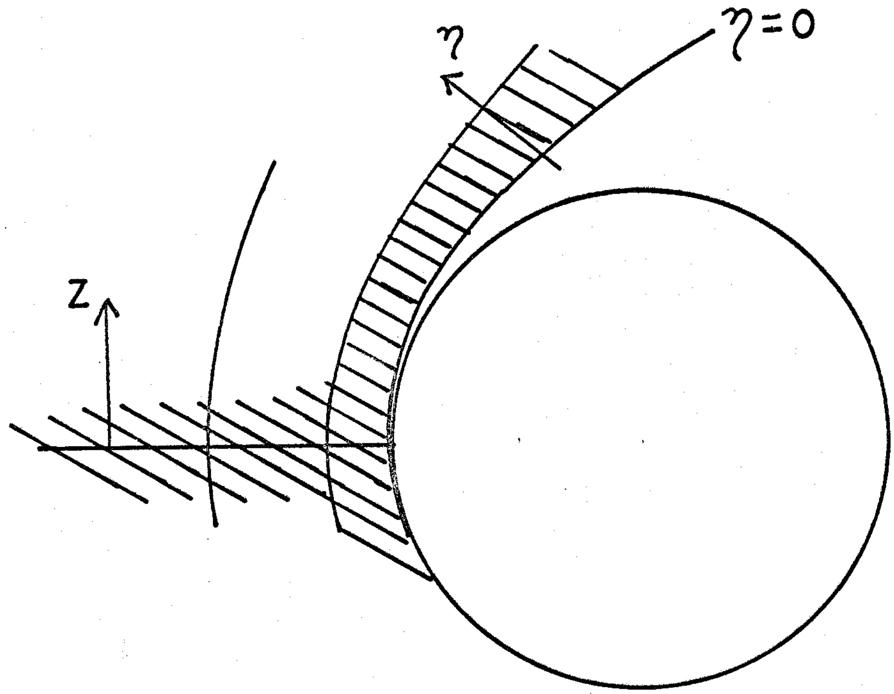


FIG 6.2

$$\frac{1+\sqrt{1+6\sigma}}{2\sqrt{1+6\sigma}} \; e^{\frac{(-1+\sqrt{1+6\sigma})\,\pi}{\sqrt{12\sigma-1}}} \tag{6.15}$$

The efficiency of capture Σ can be calculated if it is assumed that the last trajectory to hit the sphere is the one with $z = 1$. In fact, it will always be less than $z = 1$ as can be seen from the sketch 6.4. Σ is the reciprocal of the square of $z(\tau)$ in (6.15). Therefore

$$\Sigma = \frac{4\,(1+6\sigma)}{\left(1+\sqrt{1+6\sigma}\right)^2} \; \exp \frac{-2\pi(\sqrt{1+6\sigma}\;-1}{\sqrt{12\sigma-1}} \tag{6.16}$$

The log-plot of Σ vs. σ is shown in Fig. 6.5. The formula (6.16) for σ near 1/12 and the formula (4.11) for $\sigma >> 1$ are shown here. The dotted curve is the curve expected to be valid in the range in between. Experimental data of Walton and Woolcock are also plotted here. (These are given on page 129 in the book by Richardson [7]. These experiments were done by allowing particles of methylene dye to impinge on a water drop suspended by a glass capillary in a wind tunnel. The amount of dust collected is measured by the intensity of coloration of the drop. The Reynolds numbers for this data are in the range 300 to 900. The agreement with this data seems to be satisfactory.

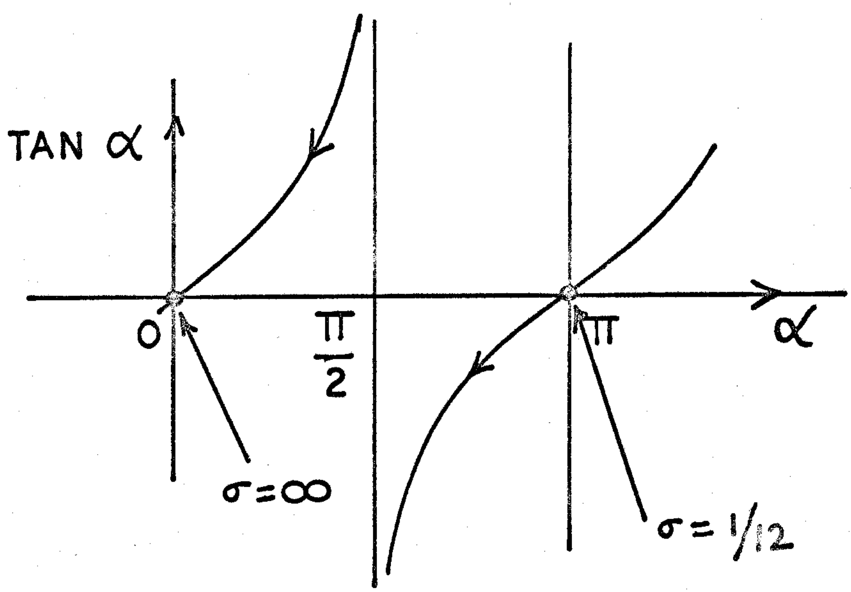


FIG 6.3

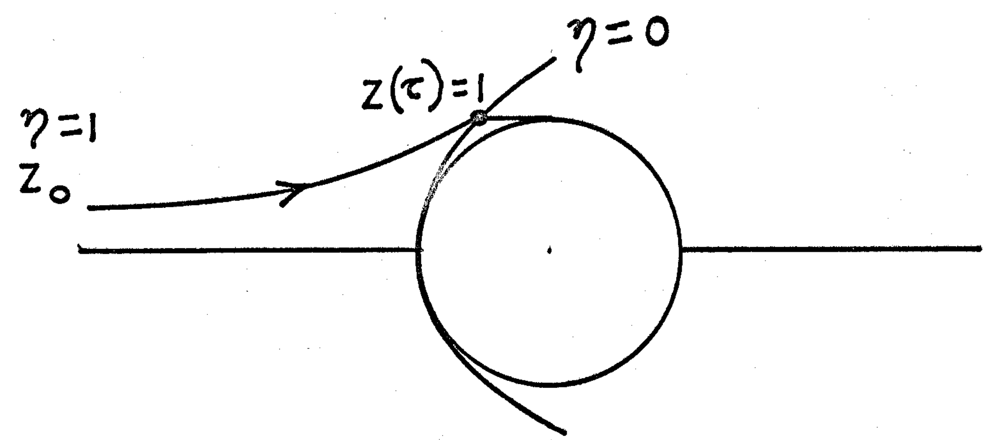


FIG 6.4

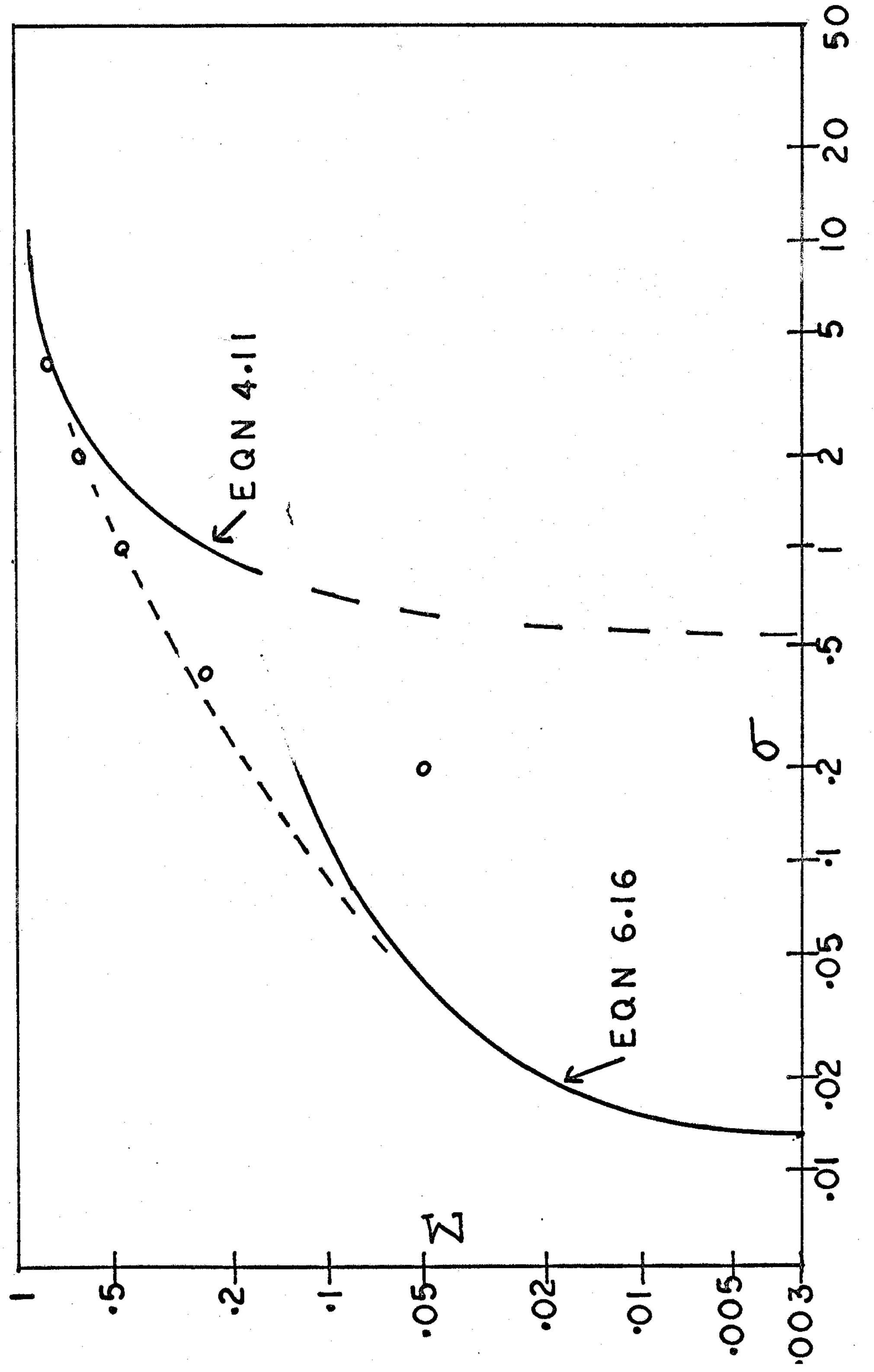


FIG 6.5

LIST OF SYMBOLS USED IN PART IIb.

a	radius of the sphere
a_p	radius of a dust particle
f	mass concentration of dust
i	unit vector in the x direction
p	pressure
r	polar coordinate in the meridional plane
R	Reynolds number
$\underline{u}$	velocity of gas
$\underline{v}$	velocity of dust
w	velocity defined in Section 6
$\underline{x}$	position vector
y	Cartesian coordinate
z	$= y$, in Section 6
α	$= \sigma/R$; also $= \frac{\sqrt{12\sigma-1}}{6\sigma-1}$ in (6.14)
ψ	stream function
η	parabolic coordinate defined in (6.10a)
μ	viscosity
ν	kinematic viscosity
σ	particle parameter
Σ	efficiency of capture
θ	angle
τ	relaxation time
ξ	coordinate defined in Section 6

Subscripts

$_0$	reference condition
∞	at infinity
r, θ	components along r and θ
x, y	components along x and y
$_{1,2,3 \ldots}$	orders in an expansion
$\sim$	dimensional quantity
prime	Oseen variables

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