LOW THRUST TRAJECTORIES USING THE TWO VARIABLE ASYMPTOTIC EXPANSION METHOD

Thesis by

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To PAT

An island called freedom;

And a stolen moment from the

Impossible Dream!

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ABSTRACT

An approximate analytic solution is derived in this thesis for the variables which describe a heliocentric low-thrust trajectory. The two-variable asymptotic expansion procedure is used. It is assumed that the thrust acceleration varies as the inverse of the distance to the central body raised to an arbitrary power α. Thus the value of $\alpha = 1.4$ will represent a solar-electric propulsion system, and the value $\alpha = 0$ will represent a nuclear-electric system. It is also assumed that the mass of the spacecraft remains constant and that the direction of the thrust vector is arbitrary but remains constant. The results are compared to numerical integrations and to other integrating low-thrust programs.

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I. INTRODUCTION

Recent interest in low-thrust planetary missions created a need for trajectory computer programs to be used in mission analysis. The first level of such software was to develop an approximate, rapid and economical program to be used as a first cut in searching for opportunities to perform specific missions. Wesseling[42] derived an approximate solution to a low-thrust trajectory using the two-variable asymptotic expansion procedure for the case where the thrust varies as $1/r^2$, i.e., approximating a solar-electric spacecraft. In actuality the thrust in a solar-electric system was found to vary as $1/r^{1.4}$. In this analysis the general case where thrust varies as $1/r^{\alpha}$ is considered. Thus $\alpha = 1.4$ will reflect a solar-electric system, $\alpha = 0$ will reflect a nuclear-electric system and $\alpha = 2$ will parallel Wesseling's solution.

II. A SURVEY OF LOW THRUST TRAJECTORY STUDIES

With the scientific exploration of the solar system becoming a reality, the fields of mission analysis and space flight mechanics are receiving tremendous attention, and an avalanche of studies, analysis and proposals has been created. These studies concern themselves with those portions of celestial mechanics, aerodynamics, ballistics and the theory of rocket propulsion that bear upon the orbits or navigation of artificial satellites and other vehicles beyond the earth's denser atmosphere. More specifically, mission analysis is a term representing studies that determine the most optimum parameters of a specific mission, such as payload, launch date, flight time etc., within the constraints of a given system such as launch vehicle, science desires, cost and others. One of the major elements of mission analysis is knowledge of the space trajectory, which also plays a vital role in other fields of study such as space navigation and guidance.

Basically, space trajectories may be divided into three main categories:

a. Ballistic Trajectories. So far, all lunar and interplanetary missions that have been undertaken, such as the Rangers, Mariners, Lunar Orbiters, Surveyors and Pioneers, have used ballistic trajectories. Such a trajectory implies that the spacecraft is injected into a transfer trajectory to the target planet by the firing of a short-burn rocket, after which the spacecraft coasts under the gravitational influence of the planets to its destination. Additional similar rocket firings are performed for other maneuvers such as mid-course corrections, insertion into an orbit around a target planet etc. In mission analysis and trajectory design, these short-burn rockets are considered as velocity increments applied impulsively. This allows the use of the basic conic equations as a first approximation to the determination of a trajectory. In an actual mission study however, very sophisticated and complex computer programs are used, where the equations of motion are solved by numerical integration methods, and special perturbations are represented by constantly improving mathematical models. The most important asset that ballistic trajectories now have is that they are in a highly developed state, thus making a space mission possible at a minimum cost. Both the 1969 and 1971 Mariner Mars missions will follow ballistic trajectories. The main disadvantage of ballistic trajectories comes about in the exploration of far away planets. For example, a direct flight to Neptune requires a minimum velocity of approximately 56,000 fps with a 30-year flight time(1). This obviously creates great problems.

b. Low-Thrust Trajectories. With the shortcomings of ballistic trajectories for deep space exploration, spacecraft with low-thrust devices received extensive investigation and research. Low-thrust space exploration missions were found to be considerably more complex than ballistic missions. This complexity arises in all the interrelated fields of propulsion technology, trajectory analysis and optimization, navigation, and guidance. The main advantage of low-thrust trajectories is the remarkable increase in the delivered payload weight.

c. Gravity Assisted Trajectories. The most recent innovation in the exploration of far away planets is the use of what is now termed "gravity assisted" trajectories. These are trajectories that use the gravitational field of a planet to accelerate a spacecraft towards more distant planets. Swing-bys past Mars, Venus, and Jupiter were investigated initially by Minovitch[2]. Using Jupiter for specific missions in the 1970's was further investigated by Flandro[3] who proposed the now famous "Grand Tour Mission" for 1978 which swings by Jupiter, Saturn, Uranus and Neptune. The remarkable advantage of gravity assisted trajectories is the reduction in flight time. For example, a mission to Neptune requiring a 30-year flight time by a ballistic trajectory, would have an 8 year flight time with a Jupiter gravity assist[3]. The difficulty associated with the use of such a technique is with the accuracy requirements of the on-board navigation and guidance equipment required to ensure accurately controlled swing-bys to the proximity of far away planets. It can be seen therefore that each of the above mentioned categories for interplanetary trajectories has advantages and disadvantages. For the exploration of far away planets the use of low-thrust or gravity assisted trajectories is almost imperative. Stewart[1] discusses these new possibilities for solar system exploration. It should be noted that both low-thrust missions and gravity assisted missions are basically ballistic in nature and that the low-thrust or gravity assistance is used to increase payload and decrease flight time. For example, an optimized solar-electric low-thrust mission to Jupiter presented by

Flandro and Barber[4] uses an optimized path where the complete trajectory is split into coast phases and thrusting phases. Flandro[5] even proposes solar-electric low-thrust missions to Jupiter with swing-by continuation to the outer planets, thus using all the mentioned categories in a single mission.

The above breakdown was presented to show how low-thrust interplanetary trajectories fit in the broad spectrum of space exploration.

Thrusted space trajectories have also brought about extensive research in the field of propulsion, which of course affects trajectory studies since the behavior of the thrust must be known in order to solve the equations of motion. To obtain the high exhaust velocities needed for most planetary missions, it is evident that processes basically different from the simple heating of a propellant stream by chemical reactions or by solid-element heat transfer must be employed. This brought about the use of electric propulsion. By definition, electric propulsion is the acceleration of gases by electrical heating and/or by electric and magnetic body forces. Three concepts thus present themselves: electrothermal propulsion, where the propellant gas is heated electrically, then expanded in a suitable nozzle; electrostatic propulsion, where the propellant is accelerated by direct application of electric body forces to ionized particles; and electromagnetic propulsion, where an ionized propellant stream is accelerated by interactions of external and internal magnetic fields with electric currents driven through the system.

More specifically, the electrical rocket engine has three important subsystems: the energy source, the power converter and the electrical thrust device. The energy source required to generate electrical power can be either chemical, solar or nuclear. Chemical sources usually produce specific energies too low to be used effectively in space missions. Solar energy, already demonstrated as an energy source in numerous space missions, may be used. Although relatively limitless in amount, solar energy is limited in rate, and decreases as the distance from the sun increases. Nuclear energy sources include nuclear fission reactors and radioisotopes. Although still mostly in the development stages, they can produce large amounts of energy at the high rates required for extended space missions. There exists a wide variety of power conversion methods that may be used to generate electric power from these sources. However, they can be classified into four areas. Power can be generated mechanically, thermoelectrically, thermionically or by magnetohydrodynamical principles. The electrical thrust device, more commonly called the thrust chamber can be either electrothermodynamic (arc or plasma jet), electrostatic (ion or colloid), or magnetohydrodynamic.

It can be seen, therefore, that the choice of a proper propulsion system in a spacecraft on an interplanetary mission is a fairly complex task. Stewart[6] presents an excellent evaluation of space propulsion systems in the light of certain celestial mechanical problems. The thrust behavior will greatly vary depending on the choice of the space propulsion system. This fact is instrumental in the motivation that led to the present analysis as will be seen later.

Specifically now, the analytical work done so far in the analysis of low-thrust trajectories will be presented so as to show the ground work that led to the present study.

Investigators have generally used two approaches in examining low-thrust trajectories. These approaches are not alternatives to one another, but are independent and justifiable pursuits. The first approach which is also historically the first, was to attempt to solve the equations of motion of the spacecraft analytically and thus determine its path. This involved the solution of non-linear differential equations using generally some kind of perturbation scheme. The second approach was to solve the boundary value optimization problem (where the boundary values are the initial and final position and velocity of the spacecraft for a specified mission), and determine the optimal thrust programs to maximize certain mission parameters. This generally involved the use of the calculus of variations and quite extensive numerical techniques. The present study deals primarily with the first approach to the problem.

Tsien(7) in 1952 was the first to present a solution to the equations of motion of a space ship taking off from a satellite orbit by the use of low-thrust. His original intent was to settle a difference of opinion that existed about the magnitude of the thrust required for a space ship to take off from a satellite orbit. Some felt that, since the gravitational attraction in a satellite orbit is completely balanced by the centrifugal force, and the vehicle thus being essentially in a weightless mode, it would only take a minute thrust (of the order of 1/3000 g's) for it to take off. Others, on the other hand, such as W. Von Braun, believed a much larger acceleration, of the order of 1/2 g was necessary. It became important to settle this dispute since each point of view supported a specific propulsion system. Small thrust favored electric propulsion whereas higher thrusts favored chemical rockets. Tsien computed the mass ratio or the characteristic velocity for the take-off of a space ship from a satellite orbit (circular) for the two cases of radial thrust and circumferential thrust. He shows that in both cases an increase of the required mass ratio and the characteristic velocity is obtained when the acceleration is reduced. He also shows that the circumferential thrust is much more efficient in that the required mass ratio is much less than for the radial thrust.

It was after Tsien published his results that investigators split into the above mentioned two approaches to the problem of low-thrust trajectories. The optimization approach started here as an attempt to answer the question that since Tsien had shown that circumferential thrust was so much better than radial thrust, would such a thrust program be the optimum program in terms of achieving maximum payload? Lawden(8) had previously presented a solution to the problem of transferring a rocket between two points of space by consumption of the minimum quantity of fuel. In his paper he first used energy considerations to show that by aligning the direction of the thrust with the tangent to the trajectory, the rate of increase of the total energy with respect to the rocket's mass is made as large as possible. Since the object of an escape maneuver is to raise the total energy as rapidly as possible, then such a program of tangential thrust will be an economical one. This explains why Tsien's circumferential thrust case is more economical than his radial thrust case, since for a rocket spiraling outward from a circular orbit, the tangent of its trajectory will rarely differ in direction from that of the perpendicular to the central radius by more than a few degrees. However, Lawden further shows that even though the circumferential and tangential thrust cases are good, neither one is the absolute optimal solution. The reason is that it may be advantageous during the early period of the thrust to purposely direct the thrust out of alignment with the tangent to the trajectory with the object of acquiring a high velocity in the early stages of the maneuver, so that in the later stages when thrust and tangent are made coincident, the velocity takes a larger value than might otherwise have been the case, and hence the rate of increase of the energy is further augmented, with the augmentation so pronounced that it more than makes up for the reduction in the rate of increase of the energy accepted in the early stages. Lawden then proves this conjecture using the method of variation of parameters and numerical methods. In a later paper(9), Lawden further shows that for an optimal escape employing a small thrust during the major portion of the escape trajectory, the direction of thrust must bisect the angle between the direction of motion and the perpendicular to the radius, but ultimately the thrust direction must be aligned with the tangent at the instant of escape, as required by his theory.

It should be noted that the field of optimization of rocket trajectories had been investigated initially for chemical rockets, the basic problem being to determine the thrust program that would permit a vertically ascending rocket to reach maximum altitude. These and similar investigations primarily used the calculus of variations in their analysis. Low-thrust propulsion systems however, as mentioned before, derive their power from a separate power supply and are therefore power-limited. Optimization of low-thrust power-limited trajectories was originally presented by Irving[10]. One of his important contributions was the determination that for any given power supply mass the terminal rocket mass and consequently the payload mass may be maximized by minimizing (subject to the initial and terminal conditions imposed on the trajectory) the integral

$$\int_o^T a^2(t)\, dt$$

where a(t) = the acceleration due to thrust = thrust/mass. Note that in optimizing chemical rocket trajectories, the equivalent free-space velocity, $\int a\, dt$ (the speed which the rocket would reach in the absence of gravity and drag) is usually minimized, and not the integral of the square of the acceleration as found by Irving. The minimization of $\int_o^T a^2\, dt$ since then has become the standard way to design optimum low-thrust trajectories. Subsequently, trajectory engineers, such as Melbourne(11), Sturms(12) and many others, have used Irving's method to numerically evaluate trajectories for missions to various planets. No further discussion will be made on the optimization approach to the low-thrust problem; for further information reference(13) lists approximately 65 computer programs for low-thrust optimized trajectories with descriptions of the optimization processes.

The first approach to the low-thrust problem, that of solving the equations of motion analytically, involves the theory of non-linear mechanics and specifically the solution of quasi-linear differential equations. Tsien's initial analytic approach was relatively simple. In the first case of purely radial thrust, a closed form solution was obtained in terms of elliptical integrals of the first and second kind. For his second case of purely circumferential thrust, he found an approximate first order solution by regarding the acceleration d^2r/dt^2 as small compared to the centrifugal acceleration and neglecting it.

Benney(14) repeated Tsien's analysis for a different coordinate system. In lieu of using the equations of motion in the radial and circumferential directions, he used them in the tangential and normal directions. He compares the tangential thrust case to Tsien's circumferential thrust case and shows that the mass ratio is less for tangential thrust, thus concurring with Lawden's(8) results. He again finds an approximate first order solution by neglecting the d^2r/dt^2 acceleration term in the equation of motion.

The exact solution obtained by Tsien for the purely radial acceleration exhibited an interesting phenomenon. He found that when ϵ, the non-dimensional thrust factor (the vehicles acceleration divided by the planets gravitational acceleration), is exactly 1/8, the mass ratio becomes infinite. The reason, he explains, is that at this value of acceleration there is a radial position where the thrust force is equal to the gravitational attraction and no further increase in the energy of the vehicle can occur. This interesting result was further investigated by Dobrowolski(15) and later by Copeland(16). Copeland, by examining the equation for the radial velocity, determined certain roots for the radius where the radial velocity will be zero, which brought out four different types of trajectories corresponding to: $\epsilon < 0$, $0 < \epsilon < 1/8$, $\epsilon = 1/8$, and $\epsilon > 1/8$. He then examines these cases for trajectories in heliocentric space, exhibiting graphs of their particular trajectories. Dobrowolski on the other hand makes use of the elliptic integral solution for this radial case to get formulas for the rate of precession of the line of apsides. It is interesting to note that in Copeland's Fig. 2 he shows that the line of apsides advances rather than regresses. This contradiction was later settled as will be discussed subsequently.

Perkins(17) performed a stepwise integration of the classical equations of motion for the case of constant tangential thrust. He first integrated a reduced form of the equation for radial acceleration in which the radial velocity $\dot{r}$ was neglected. A plot of altitude versus time showed a steady oscillation about a mean path (straight line in the r-t plane). When the complete equation for radial acceleration was used, including $\dot{r}$, it was found that the results involved an exponential damping coefficient, and that these oscillations actually damp out as the trajectory progresses. The mean path in this case was not a straight line in the r-t plane but one where the mean altitude increased at an increasing rate. The oscillations were such that the mean path and the oscillatory path intersected every half revolution in the spiral. Perkins explained the physical reasons for these oscillations as follows: "The application of a small thrust along the flight path increases its centrifugal acceleration over that of the gravity acceleration; this causes the vehicle to move outward away from the planet, and the resulting conversion of kinetic energy to potential energy will reduce the vehicle's speed and consequently the centrifugal acceleration below the opposing gravity acceleration. This will cause a low-amplitude oscillation in velocity and altitude about a mean trajectory, which will damp out in time."

The graphical results of Moeckel[18], who also numerically integrated the equations of motions of trajectories with constant tangential thrust, also show an initial slight oscillation in the velocity versus time plots. Carl Sauer (private communication) states that these oscillations were also discovered by himself and Melbourne in their early low-thrust optimization work, but that for optimization purposes these oscillations were unimportant, and so they used the mean path, which they obtained by energy considerations.

All the above investigators used Tsien's direct method of approach. The basic equations of motion being non-linear second order differential equations attracted the use of certain non-linear techniques. Lass and Lorell[19] were the first to attempt a solution to these equations in other than the direct method. They used the method of Kryloff and Bogoliuboff[20] which was previously used for handling problems in non-linear mechanics. The key point of this method is the replacement of a slowly varing derivative by its average over one cycle of the independent variable where the dependent variables are considered constant in the averaging process. Lass and Lorell applied it to the two cases considered by Tsien: the purely radial direction and the transverse or circumferential direction. For the radial thrust case they showed that the line of apsides advances, thus confirming Copeland's[16] results. They furthermore determined the error made by Dobrowolski[15] which when corrected also shows that the line of apsides advances rather than regresses. It should be noted that the variations of the elements of the orbit is an important product of all averaging techniques used in this problem, as will be noted later.

The averaging method of Krylov-Bogoliuboff proved to be very fruitful when applied to various celestial mechanics problems. Following Lass and Lorell, Shapiro[21] applied it to the case of tangential thrust, which was first solved by Benney using Tsien's direct approach. Shapiro's interesting results showed that initially circular orbits remain circular to the first approximation while elliptical orbits become less elliptic when the thrust is in the direction of motion.

The oscillations about the mean spiral path of a vehicle under the influence of low-thrust constant tangential acceleration discovered by Perkins were then investigated by Zee[22] using the Krylov-Bogoliuboff method. In his paper, Zee derives analytic expressions for both the oscillations and the damping effect on these oscillations.

King-Hele[23] presented a different approach for investigating the variations of the elements of an initially elliptic orbit with low eccentricity under the influence of a small constant tangential thrust. He uses Lagrange's planetary equations as given by Moulton[24] which give the resulting changes in the orbital elements (semi major axis and eccentricity) due to an application of a force per unit mass applied in the tangential direction. His basic assumption, besides low eccentricity, is that the change in the orbit during one revolution is small. He thus obtains an approximate expression for the variation of the semi-major axis with eccentricity and their variation with time. He finds that the eccentricity is approximately inversely proportional to the square root of the semi-major axis and that it decreases almost linearly with time.

So far all investigators had used the special case of constant thrust acceleration, where the differential equations of motion are of the autonomous type. Since acceleration is equal to thrust force divided by mass, the variation of mass must be considered negligible in order that constant thrust signify constant thrust acceleration. For the actual case of constant thrust therefore, the thrust acceleration varies with time. Zee[25] repeats his previous analysis[22] for this variable mass case, where the equations now become nonautonomous. In this analysis he uses the stroboscopic method in non-linear mechanics as given by Minorsky[26], and brings out again the damped oscillatory nature of the spiral. Very recently Zee[27] presented an improved first order solution to both of his previous papers. Zee's solution to the variable mass case was further expounded by Cohen[28] with slight improvements.

Russian investigators had also followed a parallel path in their analysis of low-thrust trajectories. Although most of their papers were never translated to English, it can nevertheless be noted from their abstracts that their investigations were very similar to those referred to in this section. For example a translated paper by Evtushenko[29] presents an analysis of the influence of a small tangential acceleration on the motion of a satellite using the method of averaging. He presents a solution that is valid to the first approximation for any elliptic orbit and valid to the second approximation for the case of a circular orbit. He examines the behavior of the orbital elements by studing the phase plane.

The thrust vector so far had been considered only in the radial, circumferential or tangential direction. Johnson and Stumpf[30] consider the case of low-thrust constant acceleration where the thrust vector is maintained at a constant but arbitrarily chosen angle with respect to the radius vector. Their analysis yields perturbation solutions for departure from an initially circular orbit.

In order to discuss the works of the next few investigators, one must look at the basic mathematical problem in more detail. In Tsien's original approximate solution for the circumferential thrust case, he neglects the acceleration d^2r/dt^2 as compared to the centrifugal acceleration, thus reducing the equation of motion in the radial direction (see for example, Eq. 10) from a third order equation to a first order equation. In so doing only the initial condition $r = r_0$ can be satisfied, whereas the initial conditions $dr/dt = 0$ and $d^2r/dt^2 = 0$ cannot be satisfied. The error introduced by not satisfying these conditions is of $0(\epsilon)$ for dr/dt and of $0(\epsilon^2)$ for d^2r/dt^2, where ϵ is the ratio of the thrust acceleration to the gravitational acceleration. Thus Tsien's approximate solution can be viewed as the zero order solution of an asymptotic expansion in powers of ϵ, with a slow time ϵt as the independent variable. The question now arises about the possibility of obtaining higher order solutions. The higher order approximations however, also yield equations of the first order and cannot satisfy all the initial conditions. Also, for this problem, a straightforward perturbation expansion in powers of ϵ using the fast time t as the independent variable is found to be invalid for large times because of the appearance of secular terms such as t sin t or t cos t (see Section V). One can immediately note now the analogy between this problem and Prandtl's boundary layer problem or the oscillator with vanishing mass[31], and thus the theory of inner and outer expansions devised by Kaplun and Lagerstrom[32] shows up as a possible way to solve the problem. Ting and Brofman[33] attempted this approach using Tsien's solution with the slow time ϵt as independent variable corresponding to the zero order outer solution, and an expansion in fast time t as independent variable corresponding to an inner solution which fulfills the three initial conditions. However, they determine that the process of matching the limit of the inner solution as $t \rightarrow \infty$ to that of the outer solution as $\epsilon t \rightarrow 0$ works for the zero order solution, but breaks down for the first order solution. The reason is due to a term sin t in the inner solution which does not possess a limit as t approaches infinity. In order to extend Tsien's solution to a higher order, Ting and Brofman therefore had to use another approach. They first convert the equations of motion into an integrodifferential equation for the radius in terms of t. They then split the expansion for the radius into an oscillatory function of the fast time t and a non-oscillatory function of the slow time ϵt and then use the Bogoliubov-Mitropolsky perturbation method[34] which uses an expansion procedure combining the features of singular perturbations and the method of averaging.

Nayfeh[35] proceeding along the same line of reasoning, investigated two alternative methods to solve this problem. The first method attempted was the method of straining of coordinates known as PLK method (Poincaré-Lighthill-Kuo) as given by Tsien[36]. This involves expansions in terms of one stretched variable and thus Nayfeh shows that it is inapplicable to this problem because of the existence of two distinct times, a fast time t and a slow time ϵt. Nayfeh's second approach used his derivative-expansion method[37]. He compares the results he obtains by the use of this method to those of Ting and Brofman and to numerical solutions of the basic equations. He shows that for large values of t, his expansion is closer to the numerical integration than Ting and Brofman's whereas for moderate values of t the reverse is true. Both the investigations of Nayfeh and of Ting and Brofman concerned themselves with the case where the initial orbit was circular.

Shi and Eckstein[38] investigated the case of arbitrary eccentricity using the two variable asymptotic expansion procedure. This method is discussed in detail by Kevorkian[46] and is the method used in this analysis. Later, Brofman[39] and Shi and Eckstein[40] extend their initial analysis to the case of variable mass. Shi and Eckstein[41] also develop an approximate solution where the singularity near the escape point, which predicts infinite radial distance at finite time, does not appear.

The above presentation attempts to give the history of the investigations performed on the analysis of the motion of a spacecraft taking off from an initial orbit by low thrust. To summarize, the following points should be made:

The problem is basically twofold. One aspect is the mathematical approach for solving two non-linear ordinary differential equations. The other aspect concerns itself primarily with the problem of mission design, specifically the design of optimum interplanetary low-thrust trajectories using numerical integration techniques and optimization processes. As each field was further developed and expanded, the two approaches became almost completely independent fields of study. At the start of this section an attempt was made to show that basically the study of low-thrust trajectories was a tool to be used in the design and planning of space exploration missions. From this point of view one can look at the researchers in optimization techniques of low-thrust missions as those who remained closer and concerned themselves more intimately with the real problem of mission design. It would be unfair, however, to consider the applied mathematicians attempting to solve the basic equations as being divorced from the main objective. Such analysis has two great advantages. In the first place an analytic investigation into the basic equations of motion gives great insight into the general behavior of such trajectories, as for example, the determination of the behavior of the osculating orbital elements as shown above. Secondly, the computing time needed to obtain numerical results from the approximate closed form solutions is appreciably less than the time needed to perform numerical integrations of the equations of motion, thus a computer program which gives approximate results but is simple and fast would be an extremely useful tool for the design of low-thrust trajectories if the approximate results are accurate enough. Thus, an analytic treatment of the problem is certainly justifiable as a part of mission analysis. Wesseling[42] is probably the only investigator whose analysis was specifically directed toward becoming a tool for the mission analyst. All the studies mentioned in this introduction considered the case of constant thrust or constant thrust acceleration, i. e., approximately a nuclear-electric propulsion system. However, the mission studies presently under serious consideration for outer planet exploration in the 1970's have solar-electric propulsion systems (e. g., reference 43). For this system, the magnitude of the thrust is a function of the distance to the sun since the solar power decreases as the distance to the sun increases. Wesseling[(42)] considers, as an approximation, the case where the thrust acceleration varies as the inverse of the square of the distance to the sun in a heliocentric trajectory. He uses the two variable asymptotic expansion procedure and obtains the first three terms of the asymptotic series. A very interesting result obtained for such a thrust behavior is that the eccentricity of the osculating conic increases, whereas, it had been found that for constant thrust acceleration it decreases. This exhibits the fact that the behavior of the trajectory in general (such as the variation of the elements) seemed to depend on the behavior of the thrust acceleration. Furthermore, it was found that for a specific solar-electric propulsion system[(45)], the thrust acceleration varied as the inverse of the distance to the sun to the power 1. 7. Later refinements showed that this power is actually 1. 4. It is concluded therefore, that the thrust behavior can vary greatly depending on the propulsion system used and that that in turn will effect the behavior of the trajectory. The need therefore became apparent to study and investigate the trajectories of a low-thrust vehicle where the thrust can take different histories.

This study investigates the general case where the thrust acceleration can vary as the inverse of the distance to the sun raised to some arbitrary power α. Thus, the value $\alpha = 0$ will signify constant thrust acceleration, $\alpha = 2$ will parallel Wesseling's analysis, and $\alpha = 1.4$ will reflect the solar-electric propulsion system presently under development for a proposed low-thrust solar electric flyby mission to Jupiter in 1975.

The three dimensional equations of motion are first derived starting with Lagrange's equations. Certain changes of variables make it possible to uncouple the out of plane equation of motion under specific conditions, thus making it possible to solve for the third dimension once the reduced two-dimensional problem is solved. The appearance of two time scales and the small parameter ϵ induces the use of the two variable asymptotic expansion procedure already demonstrated to be a powerful tool by Wesseling(42) and by Shi and Eckstein(38). In Section IV, this procedure is applied to the differential equations of motion. In Section V, the first approximation solutions for the reduced two dimensional problem are obtained for various values of α when no assumption of small eccentricities or initially circular orbits are made. The variations of the elements of the orbit under different conditions of thrust direction and values of α are presented and discussed. A second approximation valid for small eccentricities is obtained in Section VI. In Section VII, the out of plane equation of motion is analyzed and discussed. Finally, comparisons to numerical integrations and to Sauer's integrating program are presented.

III. DERIVATION OF THE EQUATIONS OF MOTION

a. Lagrange's Equations of Motion. Consider a space vehicle of mass m moving in interplanetary space under the influence of the gravitational attraction of a central body (considered as a point mass), and subject to a thrust force F. The kinetic energy of the vehicle is given by

$$T = \frac{1}{2} m \sum_i v_i^2 \tag{1}$$

where, in rectangular cartesian coordinates:

$$\sum_i v_i^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

= the square of the resultant velocity of the vehicle.

In spherical polar coordinates, r, ϕ and ψ (see Fig. 1), we have the relations:

$$\left.\begin{aligned} x &= r \cos\psi \cos\phi \\ y &= r \cos\psi \sin\phi \\ z &= r \sin\psi \end{aligned}\right\} \tag{2}$$

their time derivatives are therefore

$$\left.\begin{aligned} \dot{x} &= \dot{r} \cos\psi \cos\phi - r\dot{\psi} \sin\psi \cos\phi - r\dot{\phi} \cos\psi \sin\phi \\ \dot{y} &= \dot{r} \cos\psi \sin\phi - r\dot{\psi} \sin\psi \sin\phi + r\dot{\phi} \cos\psi \cos\phi \\ \dot{z} &= \dot{r} \sin\psi + r\dot{\psi} \cos\psi \end{aligned}\right\} \tag{3}$$

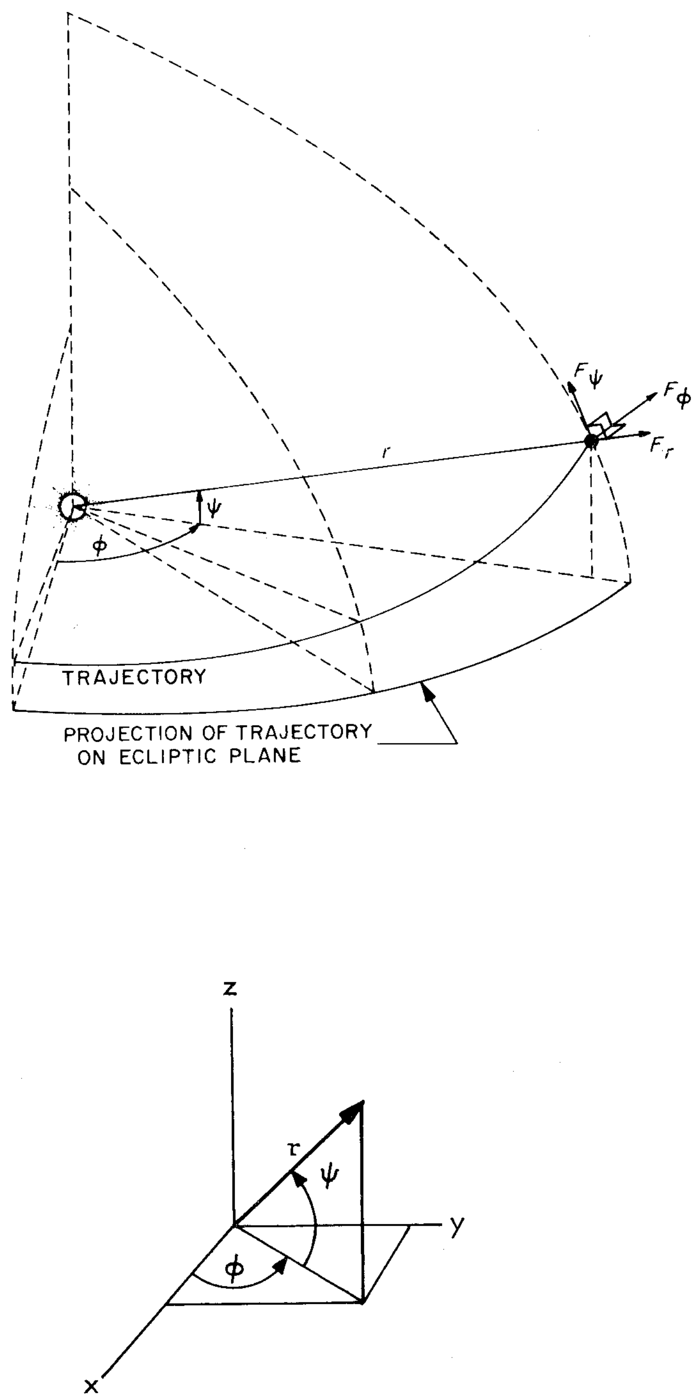

Fig. 1. Definition of the coordinate system (r, ϕ, ψ)

where the dot indicates the derivative with respect to time. Squaring and summing, the square of the velocity becomes:

$$\sum_i v_i^2 = \dot{r}^2 + r^2 \cos^2 \psi \dot{\phi}^2 + r^2 \dot{\psi}^2$$

so that the kinetic energy may now be written as

$$T = \frac{1}{2} m \left[\dot{r}^2 + r^2 \cos^2 \psi \dot{\phi}^2 + r^2 \dot{\psi}^2 \right] \tag{4}$$

The equations of motion of the space vehicle may now be obtained using Lagrange's equations.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \tag{5}$$

Note in numerous books the term "Lagrange's equations" usually is reserved for the case where the system is conservative, i.e., when the forces are derivable from a scalar potential function. In a general sense however, Eq. (5) are often called Lagrange's equation of motion where

q_j = any curvilinear coordinates

T = kinetic energy expressed as a function of q_j and $\dot{q}_j$

Q_j = the components of the "generalized force" defined as

$$Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} \tag{6}$$

To obtain the equations of motion in spherical polar coordinates, Eq. (4) is differentiated and substituted in Eq. (5):

Thus

$$\frac{\partial T}{\partial r} = mr(\dot{\phi}^2 \cos^2 \psi + \dot{\psi}^2)$$

$$\frac{\partial T}{\partial \phi} = 0$$

$$\frac{\partial T}{\partial \psi} = -mr^2\dot{\phi}^2 \cos \psi \sin \psi$$

and

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial T}{\partial \dot{\phi}} = mr^2 \dot{\phi} \cos^2 \psi \qquad \frac{\partial T}{\partial \dot{\psi}} = mr^2 \dot{\psi}$$

so that the equations of motion become

$$\left.\begin{aligned} \frac{d}{dt}[m\dot{r}] - mr[\dot{\phi}^2 \cos^2\psi + \dot{\psi}^2] &= Q_r \\ \frac{d}{dt}[mr^2\dot{\phi} \cos^2\psi] &= Q_\phi \\ \frac{d}{dt}[mr^2 \dot{\psi}] + mr^2\dot{\phi}^2 \cos \psi \sin \psi &= Q_\psi \end{aligned}\right\} \quad (7)$$

The components of the generalized forces are

$$Q_r = F_r - \frac{GMm}{r^2}$$

$$Q_\phi = r \cos \psi F_\phi$$

$$Q_\psi = rF_\psi$$

where

F_r = radial component of the thrust

F_ϕ = transverse (or circumferential) component of the thrust

F_ψ = out of plane component of the thrust

G = universal gravitational constant

M = mass of the central body

The complete equations of motion are therefore:

$$\left.\begin{aligned} \frac{d}{dt}(m\dot{r}) - mr(\dot{\phi}^2 \cos^2\psi + \dot{\psi}^2) &= F_r - \frac{GMm}{r^2} \\ \frac{d}{dt}(mr^2 \cos^2\psi\dot{\phi}) &= r \cos \psi F_\phi \\ \frac{d}{dt}(mr^2\dot{\psi}) + mr^2 \cos\psi \sin\psi\dot{\phi}^2 &= rF_\psi \end{aligned}\right\} \quad (8)$$

It will now be assumed that the mass of the space vehicle m remains constant, i. e., $\dot{m}$ is negligible, the equations reduce to:

$$\left.\begin{aligned} \frac{d^2 r}{dt^2} - r\left[\left(\frac{d\phi}{dt}\right)^2 \cos^2\psi + \left(\frac{d\psi}{dt}\right)^2\right] &= \frac{F_r}{m} - \frac{GM}{r^2} \\ \frac{d}{dt}\left[r^2 \cos^2\psi \left(\frac{d\phi}{dt}\right)\right] &= r \cos\psi \frac{F_\phi}{m} \\ \frac{d}{dt}\left[r^2 \left(\frac{d\psi}{dt}\right)\right] + r^2 \cos\psi \sin\psi \left(\frac{d\phi}{dt}\right)^2 &= r\frac{F_\psi}{m} \end{aligned}\right\} \quad (9)$$

b. <u>Choice of Dependent and Independent Variables.</u> The above system of differential equations are highly coupled non-linear differential equations. In order to reduce the coupling effect, it will be assumed that the spacecraft will remain in the vicinity of the ecliptic plane. It will therefore be assumed that the out of plane angle ψ is initially small and will remain small throughout the domain of validity of the solution. Substituting therefore ψ for $\sin\psi$ and 1 for $\cos\psi$, the equations of motion become:

$$\left.\begin{aligned} \frac{d^2 r}{dt^2} - r\left[\left(\frac{d\phi}{dt}\right)^2 + \left(\frac{d\psi}{dt}\right)^2\right] &= \frac{F_r}{m} - \frac{GM}{r^2} \\ \frac{d}{dt}\left(r^2 \frac{d\phi}{dt}\right) &= r\,\frac{F_\phi}{m} \\ \frac{d}{dt}\left(r^2 \frac{d\psi}{dt}\right) + r^2\psi\left(\frac{d\phi}{dt}\right)^2 &= r\,\frac{F_\psi}{m} \end{aligned}\right\} \qquad (10)$$

To further simplify those differential equations, the following transformation is performed:

Let

$$u = \frac{1}{r} \qquad \text{and} \qquad k = \frac{1}{H_\phi^2}$$

where

$$H_\phi = \text{angular momentum} = r^2\, d\phi/dt$$

Using u and k as the dependent variables and ϕ as the independent variable (instead of t), the equations of motion may be written after some manipulation as:

$$\left.\begin{aligned} \frac{d^2u}{d\phi^2} + u\left[1 + \left(\frac{d\psi}{d\phi}\right)^2\right] &= GMk - \frac{k}{u^2}\frac{F_r}{m} - \frac{k}{u^3}\frac{du}{d\phi}\frac{F_\phi}{m} \\ \frac{dk}{d\phi} &= -2\frac{k^2}{u^3}\frac{F_\phi}{m} \\ \frac{d^2\psi}{d\phi^2} - \frac{1}{2k}\frac{dk}{d\phi}\frac{d\psi}{d\phi} + \psi &= \frac{k}{u^3}\frac{F_\psi}{m} \end{aligned}\right\} \qquad (11)$$

This is the same choice of variables used by Wesseling[42] and differs slightly from Shi and Eckstein[38] who used $k = 1/H\phi$.

c. Power Law Behavior of the Thrust. A general power law behavior for the thrust will now be assumed, i.e., thrust proportional to the inverse of the distance to the central body raised to some arbitrary power α. The thrust components therefore may be written as:

$$\left.\begin{aligned} \frac{F_\phi}{m} &= u^\alpha Q\eta \\ \frac{F_r}{m} &= u^\alpha Q\zeta \\ \frac{F_\psi}{m} &= u^\alpha Q\xi \end{aligned}\right\} \qquad (12)$$

where

$$\eta^2 + \zeta^2 + \xi^2 = 1$$

and the constant of proportionality Q is

$$Q = \frac{F(o)}{m} u^{-\alpha}(o) \tag{13}$$

It is also assumed that η, ζ and ξ are constants i.e., the thrust vector will maintain the same orientation with respect to the central body. Substituting into the equations of motion:

$$\left.\begin{aligned} \frac{d^2u}{d\phi^2} + u\left[1 + \left(\frac{d\psi}{d\phi}\right)^2\right] &= GMk - ku^{(\alpha-2)}Q\zeta - ku^{(\alpha-3)}\frac{du}{d\phi}Q\eta \\ \frac{dk}{d\phi} &= -2k^2u^{(\alpha-3)}Q\eta \\ \frac{d^2\psi}{d\phi^2} - \frac{1}{2k}\frac{dk}{d\phi}\frac{d\psi}{d\phi} + \psi &= ku^{(\alpha-3)}Q\xi \end{aligned}\right\} \tag{14}$$

d. <u>Non-dimensionalizing, the Existence of Two Time Scales and the Small Parameter ϵ</u>. It is convenient to make these equations dimensionless. The initial distance to the sun r_o is the only unit of length that is available. The two accelerations GM/r_o^2 and Q/r_o^α give two time units, namely

$$T_1 = \sqrt{\frac{r_o^3}{GM}} \qquad T_2 = \sqrt{\frac{r_o^{\alpha+1}}{Q}}$$

It will now be assumed that the thrust is very small compared to the gravitational attraction of the sun. This signifies that

$$T_1 << T_2$$

Introducing the dimensionless variables

$$u^* = ur_o \qquad \text{and} \qquad t^* = \frac{t}{T_1}$$

and letting

$$\epsilon = \frac{Qr_o^{(2-\alpha)}}{GM}$$

the equations of motion reduce to (dropping the asterisk):

$$\left.\begin{aligned} \frac{d^2u}{d\phi^2} + u\left[1 + \left(\frac{d\psi}{d\phi}\right)^2\right] &= k - \epsilon u^{(\alpha-2)} k \left[\zeta + \eta \frac{1}{u}\frac{du}{d\phi}\right] \\ \frac{dk}{d\phi} &= -2\epsilon\eta u^{(\alpha-3)} k^2 \\ \frac{d^2\psi}{d\phi^2} + \psi &= \epsilon u^{(\alpha-3)} k \left[\xi - \eta \frac{d\psi}{d\phi}\right] \end{aligned}\right\} \qquad (15)$$

where for $T_1 << T_2$ we will have $\epsilon << 1$.

e. Uncoupling the Out of Plane Equation of Motion. It is noted that if the term $(d\psi/d\phi)^2$ is neglected in Eq. (15a) the u and k

Eq. (15a and b) will uncouple from the equation for ψ (15c). This means that the assumption $(d\psi/d\phi)^2 << 1$ will have to be made. The significance of this assumption will be apparent later when the ψ equation is solved. With this assumption, the u and k equation become:

$$\frac{d^2u}{d\phi^2} + u = k - \epsilon u^{(\alpha-2)} k \left[\zeta + \eta \frac{1}{u}\frac{du}{d\phi}\right]$$

$$\frac{dk}{d\phi} = -2\epsilon\eta u^{(\alpha-3)} k^2 \qquad (16)$$

They represent the "projection" of the trajectory on the ecliptic plane (Fig. 1). k and u can now be solved independently of ψ, and once they are determined, the ψ equation may be solved to give the elevation above the ecliptic plane.

IV. THE TWO VARIABLE ASYMPTOTIC EXPANSION PROCEDURE

A general discussion of the two variable expansion procedure is given by Kevorkian in Ref. (46). The applicability of the method manifests itself in problems that are characterized by the presence of a small force or disturbance which is active for a long time, and where the physical phenomena described by the equations reflect themselves in the occurrence of two-time scales.

It was shown in the derivation of the equations of motion of a space vehicle under low thrust (small force) that two time scales exhibit themselves. The gravitational pull of the sun causes a change in the space vehicle's position in a characteristic time T_1, whereas the low thrust force causes changes in position in a characteristic time T_2, where $T_2 >> T_1$.

Kevorkian shows that for several differential equations of the non-linear oscillator type, the two variable procedure gives a uniformly valid solution. However in this case, complete uniformity for all times cannot be achieved. Wesseling (Ref. 42) has shown, however, that the domain of validity using the two variable procedure is considerably larger than the initially valid series which breaks down for large values of ϕ (showing that the perturbation problem is singular).

Introducing the two time variables:

ϕ = fast time variable, associated with variables that vary appreciably in time T_1 (Poincaré variable)

$\tilde{\phi}$ = $\epsilon\phi$ = slow time variable, associated with variables which change only after time T_2 (like the elements of the osculating conic).

The two variables ϕ and $\tilde{\phi}$ in this procedure are treated as distinct independent variables

u, k and ψ are now expressed as explicit functions of ϕ and $\tilde{\phi}$ in the following asymptotic form:

$$u = U(\phi, \tilde{\phi}, \epsilon) = U^{(o)}(\phi, \tilde{\phi}) + \epsilon U^{(1)}(\phi, \tilde{\phi}) + \epsilon^2 U^{(2)}(\phi, \tilde{\phi}) + O(\epsilon^3)$$

$$k = K(\phi, \tilde{\phi}, \epsilon) = K^{(o)}(\phi, \tilde{\phi}) + \epsilon K^{(1)}(\phi, \tilde{\phi}) + \epsilon^2 K^{(2)}(\phi, \tilde{\phi}) + O(\epsilon^3)$$

$$\psi = \Psi(\phi, \tilde{\phi}, \epsilon) = \Psi^{(o)}(\phi, \tilde{\phi}) + \epsilon \Psi^{(1)}(\phi, \tilde{\phi}) + \epsilon^2 \Psi^{(2)}(\phi, \tilde{\phi}) + O(\epsilon^3)$$

Substituting these expansions into the equations of motion, it is noted that since ϕ and $\tilde{\phi}$ are treated as distinct variables, the total derivative with respect to ϕ is

$$\frac{d}{d\phi} = \frac{\partial}{\partial\phi} + \epsilon \frac{\partial}{\partial\tilde{\phi}}$$

After the substitution, equal powers of ϵ are equated. Let subscript 1 signify the partial derivative with respect to ϕ, and subscript 2 signify the partial derivative with respect to $\tilde{\phi}$.

For the uncoupled u and k equations (Eq. 16) (the projection of the trajectory on the ecliptic plane), the following set of differential equations are obtained:

O(1):

$$\left.\begin{aligned} K_1^{(o)} &= 0 \\ U_{11}^{(o)} + U^{(o)} &= K^{(o)} \end{aligned}\right\} \qquad (17)$$

O(ε):

$$\left.\begin{aligned} K_1^{(1)} &= -K_2^{(o)} - 2\eta U^{(o)\alpha-3} K^{(o)2} \\ U_{11}^{(1)} + U^{(1)} &= -2U_{12}^{(o)} + K^{(1)} - \zeta U^{(o)\alpha-2} K^{(o)} - \eta U^{(o)\alpha-3} K^{(o)} U_1^{(o)} \end{aligned}\right\} \qquad (18)$$

O(ε²):

$$\left.\begin{aligned} K_1^{(2)} &= -K_2^{(1)} - 2\eta(\alpha - 3) K^{(o)2} U^{(o)\alpha-4} U^{(1)} - 4\eta U^{(o)\alpha-3} K^{(o)} K^{(1)} \\ U_{11}^{(2)} + U^{(2)} &= -2U_{12}^{(1)} - U_{22}^{(o)} + K^{(2)} - \zeta U^{(o)\alpha-2} \left[K^{(1)} + (\alpha - 2)\frac{U^{(1)}K^{(o)}}{U^{(o)}}\right] \\ &\quad - \eta U^{(o)\alpha-3}\left[K^{(1)}U_1^{(o)} + K^{(o)}U_2^{(o)} + K^{(o)}U_1^{(1)} \right. \\ &\quad \left. + (\alpha - 3)\frac{U^{(1)}K^{(o)}U_1^{(o)}}{U^{(o)}}\right] \end{aligned}\right\} \qquad (19)$$

For the Ψ equation (out of plane angle), the following set of differential equations are obtained:

O(1):

$$\Psi_{11}^{(o)} + \Psi^{(o)} = 0 \tag{20}$$

O(ϵ):

$$\Psi_{11}^{(1)} + \Psi^{(1)} = -2\Psi_{12}^{(o)} + \xi K^{(o)} U^{(o)\alpha-3} - \eta K^{(o)} U^{(o)\alpha-3} \Psi_1^{(o)} \tag{21}$$

O(ϵ^2):

$$\Psi_{11}^{(2)} + \Psi^{(2)} = -2\Psi_{12}^{(1)} - \Psi_{22}^{(o)} + \xi U^{(o)\alpha-3}\left[K^{(1)} + (\alpha - 3)\frac{U^{(1)}K^{(o)}}{U^{(o)}}\right]$$

$$- \eta U^{(o)\alpha-3}\left[K^{(o)}\Psi_2^{(o)} + K^{(o)}\Psi_1^{(1)} + K^{(1)}\Psi_1^{(o)}\right.$$

$$\left. + (\alpha - 3)\Psi_1^{(o)}\frac{U^{(1)}K^{(o)}}{U^{(o)}}\right] \tag{22}$$

The set of differential equations for u and k will first be solved (Section V). Once this is done, the Ψ equations can then be solved separately (Section VI).

V. FIRST APPROXIMATION SOLUTIONS OF THE TRAJECTORY PROJECTION ON THE ECLIPTIC PLANE FOR ALL ECCENTRICITIES

The solution to the first set of differential Eq. (17)(O(1)) may be written as

$$\left.\begin{aligned} K^{(o)} &= f^{(o)}(\tilde{\phi}) \\ U^{(o)} &= f^{(o)}(\tilde{\phi})\,[1 + e(\tilde{\phi}) \cos\,(\phi - \omega(\tilde{\phi}))] \end{aligned}\right\} \quad (23)$$

It is noted that if $f^{(o)}$, e and ω were constants in Eq. (23), they would represent the unperturbed motion of a satellite in an elliptic orbit, where:

e = eccentricity of the ellipse.

ω = argument of periapsis $[(\phi - \omega)$ = true anomaly$]$.

$f^{(o)}$ = the inverse of the semi-latus rectum

However, because of the introduction of two variables, it is found that $f^{(o)}$, e and ω can be functions of the slow variable $\tilde{\phi}$. This means then that even though the solutions (23) are the first approximation solutions and the thrust term does not appear explicitly, it will be felt once the values of $f^{(o)}$, e and ω are determined as functions of $\tilde{\phi}$, since we would have an osculating conic and thus a perturbed orbit.

The functions $f^{(o)}$, e and ω are determined by imposing the "bondedness conditions" on $U^{(1)}$ and $K^{(1)}$. This is an important feature of the two-variable method discussed in References 46 and 42. It simply means that secular terms appearing in the solution of $U^{(1)}$ and $K^{(1)}$ will be set to zero, thus determining the variables that are functions of $\tilde{\phi}$ appearing in the solution of $U^{(o)}$ and $K^{(o)}$. The same thing is done in determining higher order terms, i.e., variables that are functions of $\tilde{\phi}$ only appearing in the solutions to $U^{(1)}$ and $K^{(1)}$ are determined by the boundedness conditions on the solutions of $U^{(2)}$ and $K^{(2)}$; and so on. This shows how the first approximation reflects the effect of the thrust since for determining $f^{(o)}$, e and ω, the second approximation has to be investigated inasmuch as to determine the unbounded terms in its solution.

Substituting now the solutions of $U^{(o)}$ and $K^{(o)}$ on the right hand side of the differential equation for $K^{(1)}$ [Eq. (18a)],

$$K_1^{(1)} = -\frac{\partial f^{(o)}}{\partial \tilde{\phi}} - 2\eta f^{(o)\alpha-1} \left[1 + e \cos (\phi - \omega)\right]^{\alpha-3}$$

Direct integration gives for $K^{(1)}$:

$$K^{(1)} = -\frac{\partial f^{(o)}}{\partial \tilde{\phi}} \phi - 2\eta f^{(o)\alpha-1} \int_0^{\phi} \left[1 + e \cos (\phi' - \omega)\right]^{\alpha-3} d\phi' + f^{(1)}(\tilde{\phi}) \qquad (24)$$

It is now required to determine the unbounded terms in the solution of $K^{(1)}$ [Eq. (24)]. The first term is obviously unbounded since it grows linearly with ϕ. The second term is not so obvious. However things are simplified if the following is noted: if a function is "bounded," then its integral from 0 to 2π must be zero, if it is "unbounded" then its integral from 0 to 2π is <u>not</u> zero and the value of the unbounded part becomes the value of that integral divided by 2π.

Thus the unbounded part of $\int_0^{\phi}[1 + e \cos (\phi' - \omega)]^{\alpha-3}\, d\phi'$ is $1/2\pi \int_0^{2\pi} [1 + e \cos (\phi - \omega)]^{\alpha-3}\, d\phi$.

Setting the unbounded terms to zero therefore in Eq. (24) gives the first boundedness condition

$$\frac{df^{(o)}}{d\tilde{\phi}} = - \frac{\eta}{\pi} f^{(o)\alpha-1} \int_0^{2\pi} [1 + e \cos (\phi - \omega)]^{\alpha-3}\, d\phi$$

or

$$\frac{df^{(o)}}{d\tilde{\phi}} = -2\eta f^{(o)\alpha-1} (1 - e^2)^{\frac{\alpha-3}{2}} P_{(\alpha-3)}\left(\frac{1}{\sqrt{1 - e^2}}\right) \tag{25}$$

where

$$P_{(\alpha-3)} = P_n = \text{Legendre polynomial of the first kind.}$$

Note n has to be an integer, thus Eq. (25) is good only for integer values of α. Equation (25) represents the behavior of $f^{(o)}$, and its solution would give the variation of $f^{(o)}$ as a function of $\tilde{\phi}$.

The integral

$$\int_0^{\phi} [1 + e \cos (\phi' - \omega)]^{\alpha-3}\, d\phi'$$

appearing in Eq. (24) needs to be solved in order to obtain the complete solution to $K^{(1)}$ and to continue to solve Eq. (19b) to determine

$U^{(1)}$. No solution to this integral exists for general α. There are two ways however to deal with it:

(a) Specify a value for α, and obtain the value of the integral for specific values of α using integration tables or modifications thereof.

(b) Expanding the integrand in a binomial series, neglect a certain order of e (the eccentricity) and integrate term by term keeping α general.

Method (a) will give solutions to specific values of α for all eccentricities and will be done in this section. Method (b) will give solutions to general α but for linearized eccentricities and will be investigated in Section VI.

The values of α chosen are 0, 1, 1.5, 2 and 3. The value $\alpha = 0$ signifies that the thrust remains constant and is thus equivalent to a nuclear powered thrust engine. For solar powered engines it might be assumed that thrust varies as $1/r^2$, however recent studies of solar powered ion engines show a thrust behavior varying as $1/r^{1.4}$. The values $\alpha = 1.5$ and 2 were therefore chosen. $\alpha = 1$ and 3 are included to complete a reasonable range for α.

It should be noted from Eq. (23), that the first approximation is independent of α, i.e., α does not appear in these equations. However in evaluating the values of e, ω and $\dot{f}^{(o)}$, α does come in causing the first approximation solutions to represent the effect of the thrust.

Constant Thrust Case ($\alpha = 0$)

Substituting the value $\alpha = 0$ in Eq. (25), we get for the first boundedness condition:

$$\frac{df^{(o)}}{d\tilde{\phi}} = \frac{-\eta(e^2 + 2)}{f^{(o)}(1 - e^2)^{5/2}} \tag{26}$$

where for $\alpha = 0$, $n = -3$ and we note that the Legendre polynomial

$$P_{-n-1} = P_n \qquad \text{i.e.,} \qquad P_{-3} = P_2$$

thus,

$$P_2\left(\frac{1}{\sqrt{1 - e^2}}\right) = \frac{1}{2}\left[\frac{3}{(1 - e^2)} - 1\right]$$

In order to compute the second and third boundedness conditions, to determine $e(\tilde{\phi})$ and $\omega(\tilde{\phi})$, we need to solve Eq. (18b) for $U^{(1)}$ and set the unbounded terms in its solution to zero. For $\alpha = 0$, the differential equation for $U^{(1)}$ becomes:

$$U_{11}^{(1)} + U^{(1)} = K^{(1)} - 2U_{12}^{(o)} - \zeta\frac{K^{(o)}}{U^{(o)2}} - \eta\frac{K^{(o)}U_1^{(o)}}{U^{(o)3}} \tag{27}$$

This second order total differential equation is of the type:

$$\frac{d^2U^{(1)}}{d\phi^2} + U^{(1)} = H(\phi)$$

where $H(\phi)$ is a function of $U^{(o)}$, $K^{(o)}$ and $K^{(1)}$ and their derivatives, and may therefore be determined.

The homogeneous solution to this second order total differential equation is

$$U^{(1)}_{H.S.} = A(\tilde{\phi}) \cos \phi + B(\tilde{\phi}) \sin \phi$$

The particular integral can be obtained by the use of Green's function. Note if G is the Greens function then

$$\frac{d^2G}{d\phi^2} + G = 0 \quad \text{with boundary conditions} \quad \begin{matrix} G(0) = 0 \\ G'(0) = 1 \end{matrix}$$

will give $G(\phi) = \sin \phi$.

Thus the particular solution $\int_o^\phi H(\phi')G(\phi - \phi')\, d\phi'$ becomes:

$$U^{(1)}_{P.S.} = \int_o^\phi H(\phi') \sin (\phi - \phi')\, d\phi'$$

where $H(\phi)$ is the right-hand side of the differential equation. Since $H(\phi)$ involves circular functions of $(\phi - \omega)$, it becomes very convenient to put the particular solution in the following form:

$$U^{(1)}_{P.S.} = \sin (\phi - \omega) \int_o^\phi H(\phi') \cos (\phi' - \omega)\, d\phi'$$

$$- \cos (\phi - \omega) \int_o^\phi H(\phi') \sin (\phi' - \omega)\, d\phi' \qquad (28)$$

For the case $\alpha = 0$, $H(\phi)$ is

$$H(\phi) = K^{(1)} - 2U_{12}^{(o)} - \zeta \frac{K^{(o)}}{U^{(o)2}} - \eta \frac{K^{(o)}U_1^{(o)}}{U^{(o)3}}$$

$$= \left(2e \frac{\partial f^{(o)}}{\partial \tilde{\phi}} + 2f \frac{\partial e}{\partial \tilde{\phi}}\right) \sin(\phi - \omega) - 2ef^{(o)} \frac{\partial \omega}{\partial \tilde{\phi}} \cos(\phi - \omega)$$

$$- \frac{\zeta}{f^{(o)}[1 + e \cos(\phi - \omega)]^2} + \frac{\eta}{f^{(o)}[1 + e \cos(\phi - \omega)]^3}$$

$$- \frac{\partial f^{(o)}}{\partial \tilde{\phi}} \phi - \frac{2\eta}{f^{(o)}} \int_o^{\phi} \frac{d\phi'}{[1 + e \cos(\phi' - \omega)]^3} + f^{(1)}(\tilde{\phi}) \qquad (29)$$

Substituting Eq. (29) into (28) and integrating will give the particular solution for $U^{(1)}$. In this solution we get terms that grow as $\phi \sin(\phi - \omega)$ and $\phi \cos(\phi - \omega)$. Disallowing this we get the second and third boundedness conditions:

$$\frac{d\omega}{d\phi} = \frac{\zeta}{f^{(o)2}(1 - e^2)^{3/2}} \qquad (30)$$

and

$$\frac{de}{d\phi} = -\frac{3}{2} \frac{\eta e}{f^{(o)2}(1 - e^2)^{3/2}} \qquad (31)$$

Equations (26), (30) and (31) govern the behavior of $f^{(o)}$, e and ω to first order. By manipulating them algebraically, we can get ω, $f^{(o)}$ and $\tilde{\phi}$ as functions of e by integration. These relations are:

$$\omega = \omega_o - \frac{2}{3}\frac{\zeta}{\eta} \log \left(\frac{e}{e_o}\right) \tag{32}$$

$$f^{(o)} = f_o^{(o)} \left[\frac{\left(1 - e_o^2\right)}{(1 - e^2)} \left(\frac{e}{e_o}\right)^{4/3}\right] \tag{33}$$

$$\tilde{\phi} = \frac{2f_0^{(o)2}\left(1 - e_o^2\right)^2}{3\eta e_o^{8/3}} \int_{e_o}^{e} \frac{e^{5/3}\, de}{\sqrt{1 - e^2}} \tag{34}$$

where $f_o^{(o)}$, e_o and ω_o are the initial values of $f^{(o)}$, e and ω.

Thrust Varying as $1/r$ ($\alpha = 1$)

Repeating the same procedure as above for $\alpha = 1$ we get the boundedness conditions:

$$\frac{df^{(o)}}{d\tilde{\phi}} = \frac{-2\eta}{(1 - e^2)^{3/2}} \tag{35}$$

$$\frac{d\omega}{d\tilde{\phi}} = \frac{\zeta}{f^{(o)}}\left[\frac{1 - \sqrt{1 - e^2}}{e^2\sqrt{1 - e^2}}\right] \tag{36}$$

$$\frac{de}{d\tilde{\phi}} = \frac{-\eta}{f^{(o)}}\left[\frac{1 - \sqrt{1 - e^2}}{e\sqrt{1 - e^2}}\right] \tag{37}$$

The solutions are:

$$\omega = \omega_o - \frac{\zeta}{\eta} \log \left(\frac{e}{e_o}\right) \tag{38}$$

$$f^{(o)} = f_o^{(o)} \left[\frac{\left(1 - e_o^2\right)}{\left(1 - \sqrt{1 - e_o^2}\right)^2} \right] \frac{\left(1 - \sqrt{1 - e^2}\right)^2}{(1 - e^2)} \tag{39}$$

$$\tilde{\Phi} = - \frac{f_o^{(o)}}{\eta} \frac{\left(1 - e_o^2\right)}{\left(1 - \sqrt{1 - e_o^2}\right)^2} \left\{ \left[\frac{2}{3} \left(e_o^2 + 2\right) \sqrt{1 - e_o^2} + \frac{1}{2} e_o^2 \right] - \left[\frac{2}{3} (e^2 + 2) \sqrt{1 - e^2} + \frac{1}{2} e^2 \right] \right\} \tag{40}$$

Note that the very cumbersome algebra and calculus are not shown for simplicity. The main integrations are given in the appendix and were obtained with the help of Reference 47.

Thrust Varying as $1/r^{3/2}$ ($\alpha = 3/2$)

As mentioned in Section II the thrust force in a solar electric spacecraft varies like $r^{-1.4}$. For this reason it was decided to solve the developed equations for $\alpha = 1.5$. It also turns out that this gives a special case where $de/d\tilde{\Phi} = 0$ regardless of the magnitude of the initial eccentricity. Substituting $\alpha = 1.5$, the integrals give rise to elliptic functions of the first and second kind (see appendix). Again to simplify the algebra, since we only need the unbounded part of the integrals, we integrate from 0 to 2π. If the value of this integral is zero, then the function is bounded. If the value is non-zero, then the unbounded part is the value of that integral divided by 2π. In doing this, the elliptic functions are reduced to 'complete elliptic functions' of the first and second kind.

The resulting differential equations for the boundedness conditions are:

$$\frac{df^{(o)}}{d\tilde{\phi}} = \frac{-4\eta}{\pi} \sqrt{f^{(o)}} \frac{E(k)}{(1-e)\sqrt{1+e}} \tag{41}$$

$$\frac{d\omega}{d\tilde{\phi}} = \frac{-2\zeta}{\pi\sqrt{f^{(o)}}} \frac{1}{e\sqrt{1+e}} \left[K(k) - \frac{1+e}{e}\left(K(k) - E(k)\right)\right] \tag{42}$$

$$\frac{de}{d\tilde{\phi}} = 0 \tag{43}$$

where

$K(k)$ = complete elliptic integral of the first kind

$E(k)$ = complete elliptic integral of the second kind

k = modulus = $\sqrt{2e/(1+e)}$

These equations may be integrated directly since the eccentricity remains constant, and we get:

$$e = e_o \tag{44}$$

$$f^{(o)} = \left[\sqrt{f_o^{(o)}} - \frac{2\eta E(k)}{\pi(1-e_o)\sqrt{1+e_o}}\tilde{\phi}\right]^2 \tag{45}$$

$$\omega = \omega_o + \frac{\zeta}{\eta}\frac{(1-e_o)}{e_o E(k)}\left[K(k) - \frac{1+e_o}{e_o}(K(k)-E(k))\right]\log\left[\sqrt{f_o^{(o)}} - \frac{2\eta E(k)}{\pi(1-e_o)\sqrt{1+e_o}}\tilde{\phi}\right] \tag{46}$$

Thrust Varying as $1/r^2$ $(\alpha = 2)$

For this case the boundedness conditions become:

$$\frac{df^{(o)}}{d\tilde{\phi}} = \frac{-2\eta f^{(o)}}{(1-e^2)^{1/2}} \tag{47}$$

$$\frac{d\omega}{d\tilde{\phi}} = 0 \tag{48}$$

$$\frac{de}{d\tilde{\phi}} = \eta\left[\frac{\sqrt{1-e^2} - (1-e^2)}{e\sqrt{1-e^2}}\right] \tag{49}$$

The solutions are:

$$\omega = \omega_o \tag{50}$$

$$f^{(o)} = f_o^{(o)}\left[\frac{\left(1-\sqrt{1-e_o^2}\right)}{\left(1-\sqrt{1-e^2}\right)}\right]^2 \tag{51}$$

$$\tilde{\phi} = \frac{1}{\eta}\left[\log\frac{\left(\sqrt{1-e^2}-1\right)}{\left(\sqrt{1-e_o^2}-1\right)} + \sqrt{1-e^2} - \sqrt{1-e_o^2}\right] \tag{52}$$

Thrust Varying as $1/r^3$ ($\alpha = 3$)

This case is interesting because it turns out that the boundedness conditions remain the same for the case of linearized eccentricity as will be shown later. For $\alpha = 3$, the boundedness conditions become:

$$\frac{df^{(o)}}{d\tilde{\phi}} = -2\eta f^{(o)2} \tag{53}$$

$$\frac{d\omega}{d\tilde{\phi}} = -\zeta \frac{f^{(o)}}{2} \tag{54}$$

$$\frac{de}{d\tilde{\phi}} = \frac{3}{2} \eta e f^{(o)} \tag{55}$$

direct integration gives the solutions:

$$f^{(o)} = f_o^{(o)} \left[1 + 2\eta f_o^{(o)} \tilde{\phi}\right]^{-1} \tag{56}$$

$$\omega = \omega_o - \frac{1}{4} \frac{\zeta}{\eta} \log \left[1 + 2\eta f_o^{(o)} \tilde{\phi}\right] \tag{57}$$

$$e = e_o \left[1 + 2\eta f_o^{(o)} \tilde{\phi}\right]^{3/4} \tag{58}$$

Discussion of Results

Equations (32) to (58) represent the behavior of eccentricity (e), argument of periapsis (ω), and the inverse of the semi-latus rectum ($f^{(o)}$)

with respect to $\tilde{\phi}$ the slowly varying angle ($\tilde{\phi} = \epsilon\phi$), to first order. Note that the angle ϕ is related to the time by the variable k:

$$k = \frac{1}{\left(r^2 \frac{d\phi}{dt}\right)^2}$$

i. e., time can be obtained from ϕ by the relation

$$t = \int_0^{\phi} \frac{k^{\frac{1}{2}}}{u^2}\, d\phi$$

These behaviors are plotted in Figs. 2 and 3. Figure 2 is for the case of tranverse thrust ($\eta = 1$, $\zeta = 0$) and Fig. 3 is for the case where thrust is inclined 45° ($\eta = 1/\sqrt{2}$, $\zeta = 1/\sqrt{2}$). The same plots are repeated in Figs. 4 and 5 for the linearized eccentricity case developed in Section VI. Figures 6 and 7 show the difference between the linearized and non-linearized case for eccentricity vs $\tilde{\phi}$.

Some general remarks:

Behavior of Eccentricity

For the case where thrust has a tangential component (Figs. 2 and 3) (η positive non-zero) the rate of change of eccentricity $de/d\tilde{\phi}$ is:

$$\frac{de}{d\tilde{\phi}} \begin{cases} < 0 \text{ for } 0 \leq \alpha < 1.5 \\ = 0 \text{ for } \alpha = 1.5 \\ > 0 \text{ for } \alpha > 1.5 \end{cases}$$

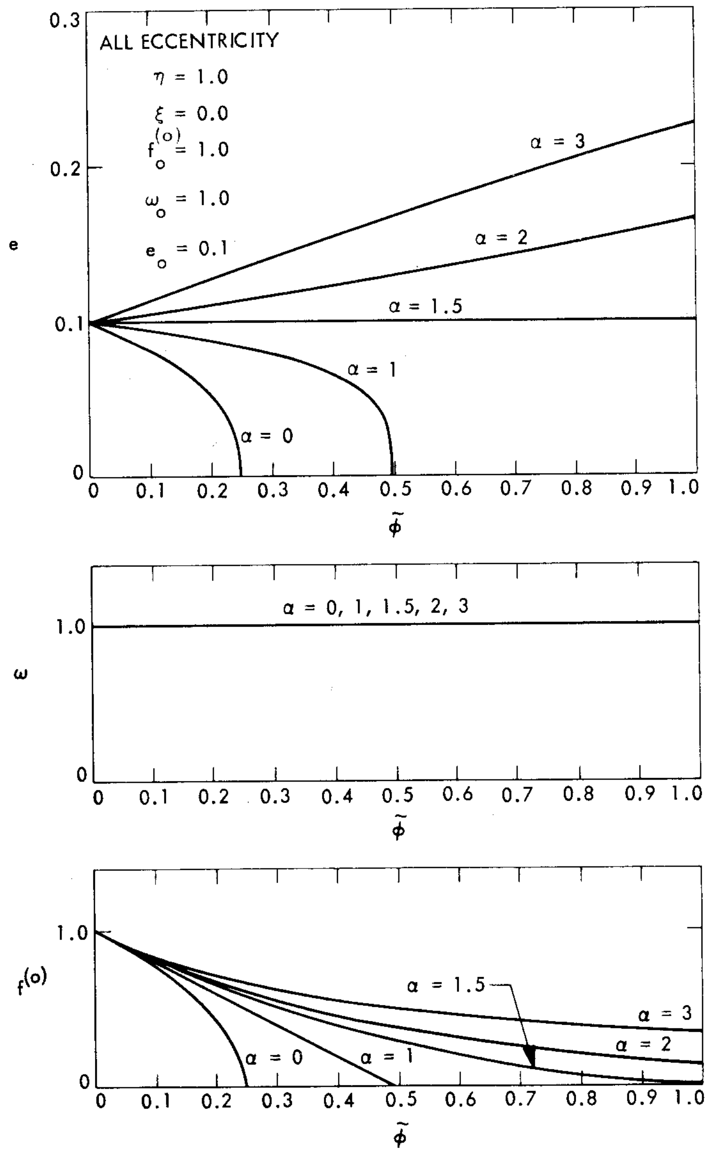

Fig. 2. Variations of e, ω, $f^{(o)}$- transverse thrust direction, all eccentricity

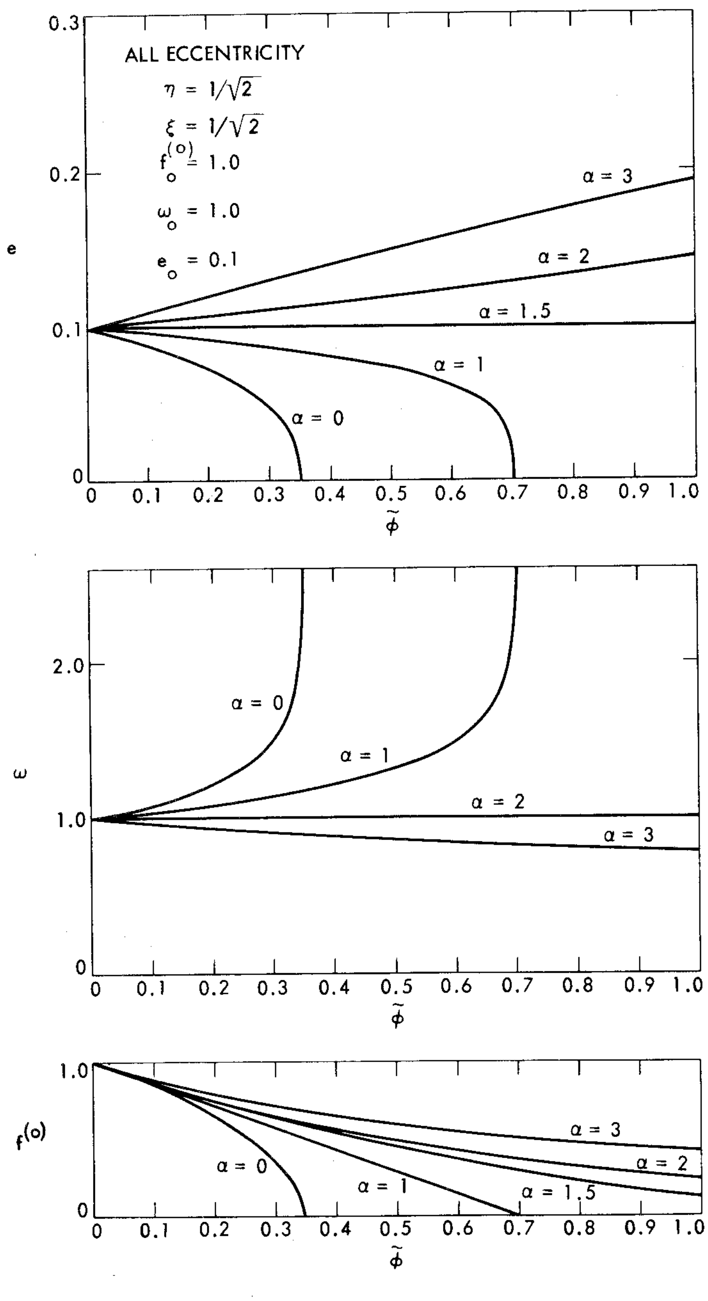

Fig. 3. Variations of e, ω, $f^{(o)}$ - 45° thrust direction, all eccentricity

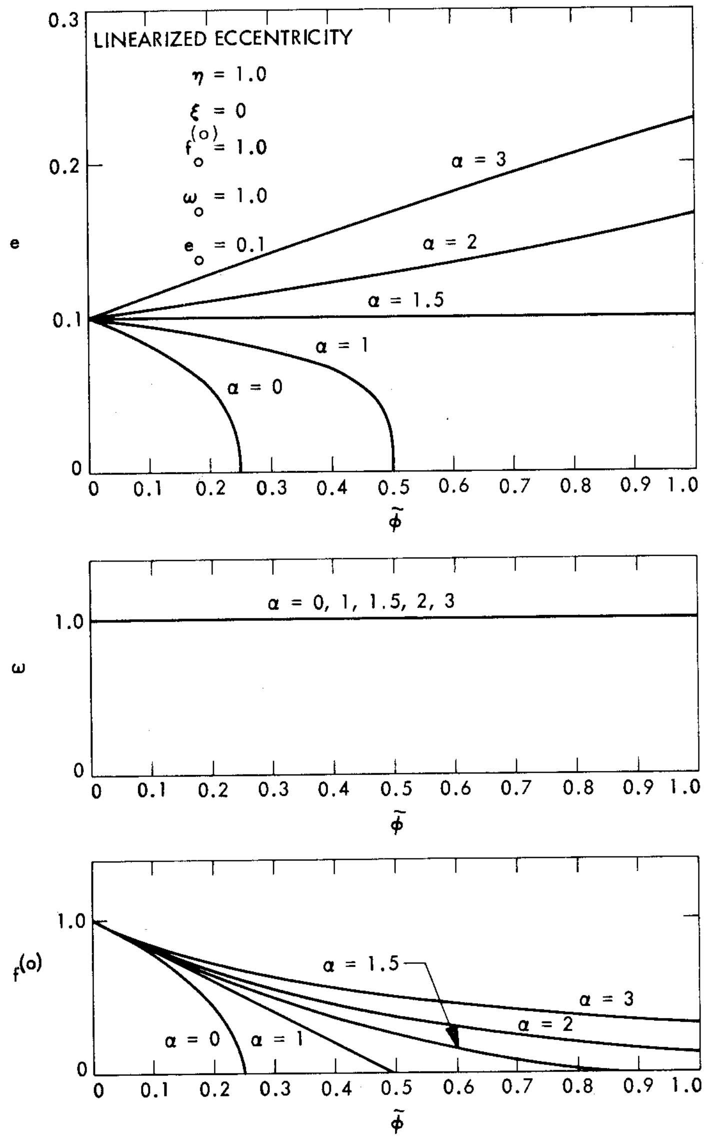

Fig. 4. Variations of e, ω and $f^{(o)}$ - transverse thrust direction, linearized eccentricity

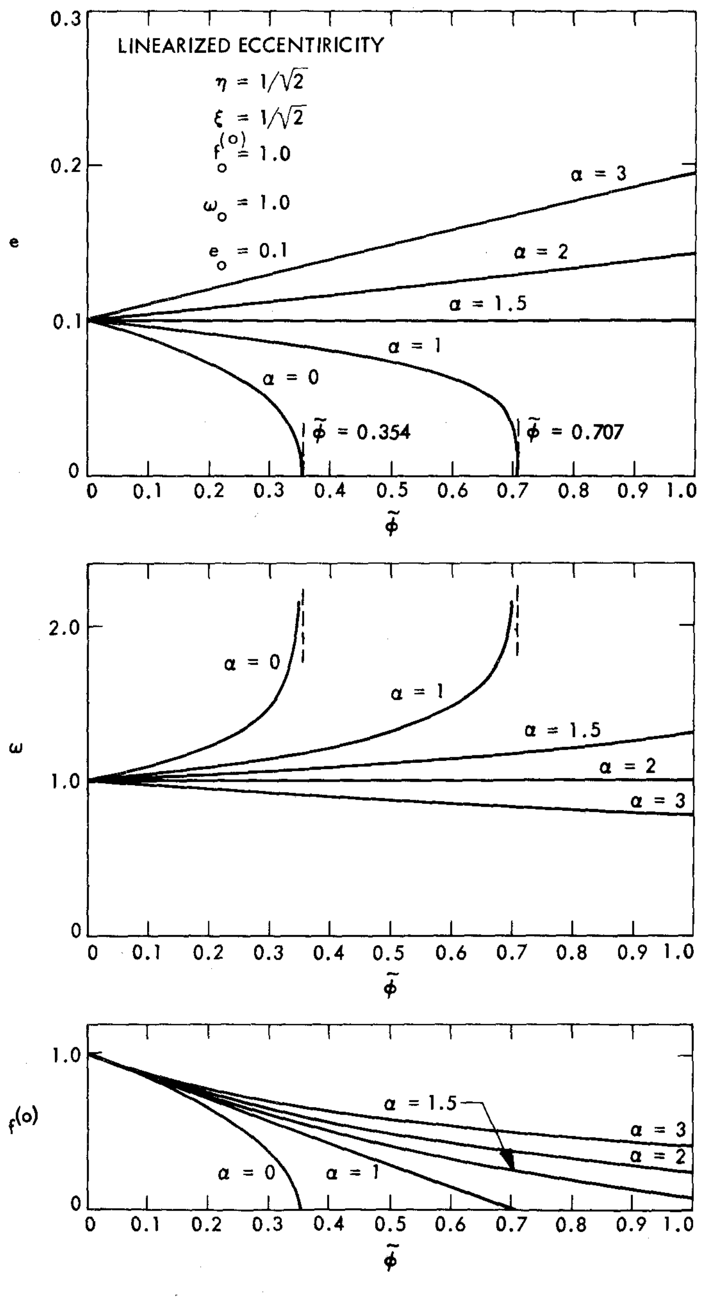

Fig. 5. Variations of e, ω and $f^{(o)}$ - 45° thrust direction, linearized eccentricity

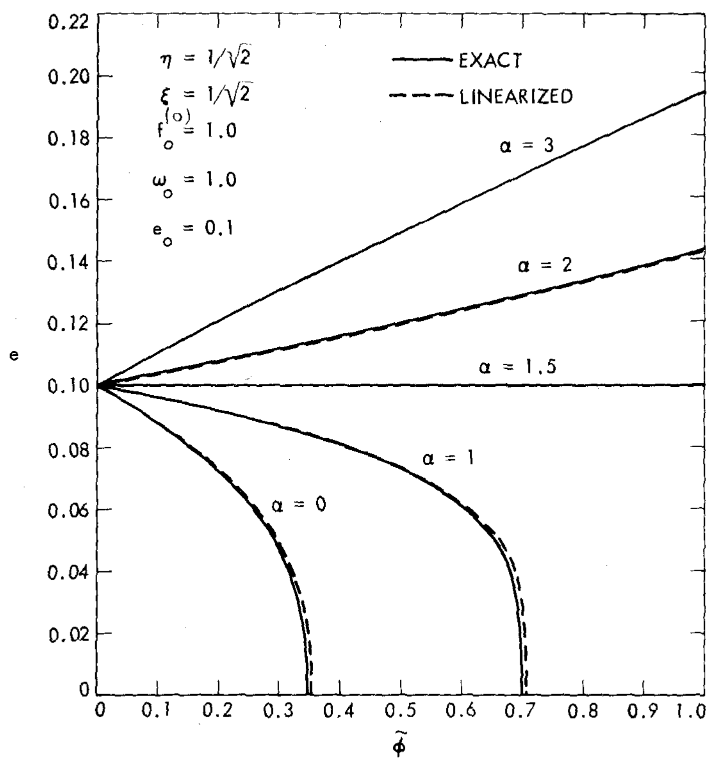

Fig. 6. Difference between linearized and non-linearized cases for $e_o = 0.1$

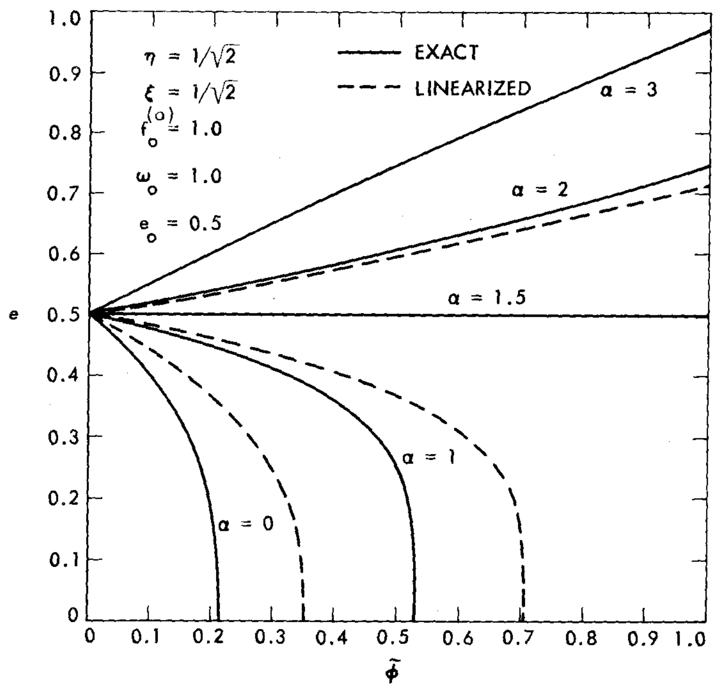

Fig. 7. Difference between linearized and non-linearized cases for $e_o = 0.5$

This means that for $\alpha = 1.5$, the vehicle follows a path on which eccentricity does not change ($e = e_o$), regardless of the value of e_o ($de/d\tilde{\Phi} = 0$ for both linearized and non-linearized eccentricities), to first order.

For $\alpha < 1.5$, eccentricity decreases with time; this essentially means that the orbit tends to become more circular. On the other hand if $\alpha > 1.5$, $de/d\phi$ is positive, i.e., the orbit tends to become more elliptic. This can be readily explained by considering impulsive tangential thrusts. An impulsive thrust performed tangentially at the periapsis of the initial ellipse will tend to increase the eccentricity whereas, if it is performed at the apoapsis, it will tend to decrease the eccentricity. Thus for constant thrust ($\alpha = 0$) it is clear the vehicle spends more time near apoapsis so that the net effect will be to decrease the eccentricity. Once thrust depends inversely on the radius, this effect diminishes until $\alpha = 1.5$, where the effects cancel each other exactly and eccentricity does not change. For $\alpha > 1.5$, even though the vehicle spends more time near apoapsis, the large radius gives rise to a small thrust enough so that the net effect is to increase the eccentricity.

We also note from the equations that if the initial eccentricity is zero (circular), the vehicle will follow a spiral on which eccentricity remains zero, regardless of the value of α.

For the case where the thrust is radial ($\zeta = 1$, $\eta = 0$), $de/d\tilde{\Phi} = 0$ and eccentricity does not change regardless of α and the initial eccentricity e_o.

For $\alpha < 1.5$, we have decreasing eccentricity, which means that at a specific value of $\tilde{\Phi}$, eccentricity becomes zero (Figs. 2, 3). This discontinuity simply means that the vehicle reaches escape, since we note that at these values $f^{(o)} \to 0$, i.e., the radius becomes infinite.

When eccentricity is linearized (Figs. 4 and 5) the behavior of e, $f^{(o)}$ and ω is still the same. Figure 6 shows the difference between the linearized and non-linearized cases for the behavior of eccentricity when $e_o = 0.1$. We note that for $\alpha = 1.5$ $de/d\tilde{\Phi} = 0$ for both linearized and non-linearized cases. Also we note that for $\alpha = 3$, the boundedness conditions are the same for both linearized and non-linearized cases. In Fig. 6, therefore, we note that for $\alpha = 1.5$ and 3, there is no difference between linearized and exact cases. For $\alpha = 0$, 1 and 2 the difference is found to be very small. This difference becomes larger if the initial eccentricity e_o is larger. Figure 7 is the same as Fig. 6 except the initial eccentricity $e_o = 0.5$.

By comparing Fig. 6 to Fig. 7, we note that for $\alpha < 1.5$, the eccentricity reaches zero faster when the initial eccentricity is higher. This means that the more eccentric the initial orbit, the faster it tends to become circular! This can be explained by remembering that for $e \to 0$, $f^{(o)} \to 0$ and the vehicle reaches escape. This means, therefore, that the more eccentric the initial orbit, the faster the vehicle reaches escape.

Behavior of Argument of Periapsis (ω)

For the case where thrust has a radial component (ζ positive non-zero), the rate of change of ω is noted to be:

$$\frac{d\omega}{d\tilde{\Phi}} \begin{cases} > 0 \text{ for } 0 \leq \alpha < 2 \\ = 0 \text{ for } \alpha = 2 \\ < 0 \text{ for } \alpha > 2 \end{cases}$$

The argument of periapsis can be thought of as the rotation of the ellipse. Thus for $\alpha = 2$, there is no rotation, for $\alpha < 2$ rotation is positive and for $\alpha > 2$ rotation is negative. Note that for $\alpha < 2$, $\omega \rightarrow \infty$ at the values for which $f^{(o)} \rightarrow 0$, i. e., escape. Again, if we compare linearized and non-linearized cases (Figs. 3 and 5) the difference is very small when $e_o = 0.1$.

We also note that for the case of transverse thrust ($\eta = 1$, $\zeta = 0$), (Figs. 2 and 4), $d\omega/d\tilde{\Phi} = 0$, i. e., ω remains constant (no rotation) regardless of the values of α, e_o or ω_o.

Behavior of the Inverse Semi-Latus Rectum ($f^{(o)}$)

For the case where thrust has a tangential component (η positive, non-zero), the rate of change of $f^{(o)}$ is always negative, i. e., $f^{(o)}$ always decreases. This of course signifies that r has to increase, i. e., ascent from the initial orbit, which are the only cases considered here.

Again for $\alpha < 2$, $f^{(o)} \rightarrow 0$ at the escape points (Figs. 2 and 3). Comparing the linearized to the exact case (Figs. 2 and 4) we note that again the difference is very small.

From the above analysis we note the interesting case that for radial thrust ($\eta = 0$, $\zeta = 1$) and for the $\alpha = 2$ case, we get

$$\frac{de}{d\tilde{\phi}} = \frac{df^{(o)}}{d\tilde{\phi}} = \frac{d\omega}{d\tilde{\phi}} = 0$$

This means that the orbit doesn't change! To explain this we note that the foregoing analysis is only carried out to first order. Also Wesseling (Ref. 42) has shown that for $\alpha = 2$ and radial thrust, the radius of the trajectory after one-revolution changes by an amount of $0(\epsilon)$, which supports the above results.

VI. LINEARIZING THE ECCENTRICITY

a. <u>An Order of Magnitude Analysis</u>. The original expansions for u and k are:

$$u = U^{(o)} + \epsilon U^{(1)} + \epsilon^2 U^{(2)} + \cdots \tag{59}$$

$$k = K^{(o)} + \epsilon K^{(1)} + \epsilon^2 K^{(2)} + \cdots \tag{60}$$

The solutions to $U^{(o)}$ and $K^{(o)}$ were found to be:

$$U^{(o)} = f^{(o)}[1 + e \cos(\phi - \omega)] \tag{61}$$

$$K^{(o)} = f^{(o)} \tag{62}$$

The values of $f^{(o)}$, e and ω are found by imposing boundedness conditions on $U^{(1)}$ and $K^{(1)}$.

For example, $f^{(o)}$ is found from the boundedness condition on $K^{(1)}$, which was:

$$-\frac{df^{(o)}}{d\tilde{\phi}} = \text{unbounded part of } 2\eta f^{(o)\alpha - 1} \int_o^{\phi} [1 + e \cos(\phi - \omega)]^{\alpha - 3}\, d\phi \tag{63}$$

It is here that the assumption of linearizing must be used, by expanding the integrand in a binomial series:

$$[1 + e \cos(\phi - \omega)]^{\alpha - 3} = 1 + (\alpha - 3) e \cos(\phi - \omega) + \frac{(\alpha - 3)(\alpha - 4)}{2!} e^2 \cos^2(\phi - \omega)$$

$$+ \frac{(\alpha - 3)(\alpha - 4)(\alpha - 5)}{3!} e^3 \cos^3(\phi - \omega) + \cdots \tag{64}$$

Integrating term by term w. r. t. ϕ:

$$\int_0^{\phi} [1 + e\cos(\phi - \omega)]^{\alpha - 3} = \phi + (\alpha - 3)e[\sin(\phi - \omega) + \sin\omega]$$

$$+ \frac{(\alpha - 3)(\alpha - 4)}{2!} e^2 \left[\frac{\phi}{2} + \frac{\sin 2(\phi - \omega)}{4} + \frac{\sin 2\omega}{4}\right] + O(e^3) \tag{65}$$

we note that

$$\int_0^{\phi} \cos\phi \, d\phi \qquad \text{is bounded}$$

$$\int_0^{\phi} \cos^2\phi \, d\phi \qquad \text{has one unbounded term}$$

$$\int_0^{\phi} \cos^3\phi \, d\phi \qquad \text{is bounded}$$

$$\int_0^{\phi} \cos^4\phi \, d\phi \qquad \text{has one unbounded term}$$

and so on. Therefore, the boundedness condition on $K^{(1)}$ is:

$$\frac{df^{(o)}}{d\tilde{\phi}} = -2\eta f^{(o)\alpha - 1}\left[1 + \frac{(\alpha - 3)(\alpha - 4)}{4} e^2 + O(e^4) + O(e^6) + \cdots\right] \tag{66}$$

Now linearizing in eccentricity, we assume $e^2 \ll e$ and thus

$$\frac{df^{(o)}}{d\tilde{\phi}} = -2\eta f^{(o)\alpha-1} \tag{67}$$

Thus in calculating $f^{(o)}$, terms of $O(e^2)$ are neglected.

This means, therefore, that Eq. (62) can be written as

$$K^{(o)} = K_1^{(o)} + eK_2^{(o)} + e^2K_3^{(o)} + \cdots \tag{68}$$

where in this case (note subscripts here do not imply differentiation)

$$K_1^{(o)} = f^{(o)}$$

$$K_2^{(o)} = 0 \qquad \text{and} \qquad e^2K_3^{(o)}$$

and higher order terms were neglected! The same linearization is done in determining e and ω and thus Eq. (61) can be written as:

$$U^{(o)} = U_1^{(o)} + eU_2^{(o)} + e^2U_3^{(o)} + \cdots \tag{69}$$

where

$$U_1^{(o)} = f^{(o)}$$

$$U_2^{(o)} = f^{(o)} \cos (\phi - \omega)$$

and $e^2U_3^{(o)}$ and higher order terms are neglected.

We would like to look now at Eqs. (59) and (60) in the light of this approximation. Thus express $U^{(1)}$, $U^{(2)}$, $\cdots$, and $K^{(1)}$, $K^{(2)} \ldots$ as terms of O(1), O(e), $O(e^2) \cdots$ as in Eqs. (68) and (69).

Examining first Eq. (59) we have:

$$
\begin{aligned}
u = \quad & \left[U_1^{(o)} + eU_2^{(o)} + e^2 U_3^{(o)} + \cdots\right] \\
& + \epsilon\left[U_1^{(1)} + eU_2^{(1)} + e^2 U_3^{(1)} + \cdots\right] \\
& + \epsilon^2\left[U_1^{(2)} + eU_2^{(2)} + e^2 U_3^{(2)} + \cdots\right] \\
& + \epsilon^3\left[U_1^{(3)} + eU_2^{(3)} + e^2 U_3^{(3)} + \cdots\right] + \cdots
\end{aligned}
\tag{70}
$$

In general when one linearizes the eccentricity (cfe Ref. 38), the term of $O(e^2)$ in $U^{(o)}$ is neglected. Suppose now that the term of O(e) in $U^{(1)}$ and the term of O(1) in $U^{(3)}$ are neglected as we look at higher order ϵ terms. We would then have:

$$
U = \left[U_1^{(o)} + eU_2^{(o)}\right] + \epsilon U_1^{(1)} \tag{71}
$$

We now must examine this approximation to make sure that the neglected terms are very much smaller than the retained terms.

The worst case is that where a term of $O(e^2)$ was neglected and a term of $O(\epsilon)$ was retained. This requires that

$$
\boxed{e^2 \ll \epsilon} \tag{72}
$$

Thus the domain of validity in this approximation is one in which the eccentricity is much smaller than the $\sqrt{\epsilon}$ or, i.e., $e = o(\epsilon^{1/2})$.

Carrying the approximation one step ahead, i.e., if we were to include terms of $O(e^2)$, $O(\epsilon e)$ and $O(\epsilon^2)$, we will have:

$$u = \left[U_1^{(o)} + eU_2^{(o)} + e^2U_3^{(o)}\right] + \epsilon\left[U_1^{(1)} + eU_2^{(1)}\right] + \epsilon^2U_1^{(2)} \tag{73}$$

Examining this now for terms neglected and retained we get for the worst case:

$$O(e^3) \ll O(\epsilon e) \qquad \text{and} \qquad O(\epsilon e^2) \ll O(\epsilon^2)$$

which both signify that

$$\boxed{e^2 \ll \epsilon}$$

Thus, the domain of validity remains the same. However, the accuracy (i.e., comparison to numerical integration in the same domain of $e = o(\epsilon^{1/2})$) will be much better since the approximation is carried out to order ϵ^2.

The above applies also to Eq. (60).

In this section, the analysis will be carried out to $O(\epsilon)$, i.e., the values of $K^{(o)}$ and $U^{(o)}$ will be determined to $O(e)$, $K^{(1)}$ and $U^{(1)}$ to $O(1)$. Note that $K^{(1)}$ and $U^{(1)}$ will contain constants that must be determined by boundedness conditions on $U^{(2)}$ and $K^{(2)}$ carried out to $O(1)$.

b. <u>First Approximation Solution for General α.</u> The solution to the first set of differential equations (O(1)) was found to be:

$$\left.\begin{aligned} K^{(o)} &= f^{(o)} \\ U^{(o)} &= f^{(o)}[1 + e\cos(\phi - \omega)] \end{aligned}\right\} \tag{74}$$

To determine $f^{(o)}$, e and ω we must solve for $K^{(1)}$ and $U^{(1)}$ (by solving $O(\epsilon)$ equations) and imposing boundedness conditions on them. Note $f^{(o)}$, e and ω are functions of the slow variable $\tilde{\Phi}$ only.

Substituting (74) into (18a) we get:

$$K_1^{(1)} = -\frac{\partial f^{(o)}}{\partial\tilde{\Phi}} - 2\eta f^{(o)\alpha-1}[1 + e\cos(\phi - \omega)]^{\alpha-3} \tag{75}$$

We will now assume small eccentricity and neglect terms of $O(e^2)$ as discussed in the previous section. Therefore,

$$[1 + e\cos(\phi - \omega)]^{\alpha-3} = 1 + (\alpha - 3)\, e\cos(\phi - \omega) \tag{76}$$

and Eq. (75) becomes:

$$K_1^{(1)} = -\frac{\partial f^{(o)}}{\partial\tilde{\Phi}} - 2\eta f^{(o)\alpha-1} - 2\eta(\alpha - 3)f^{(o)\alpha-1}\, e\cos(\phi - \omega) \tag{77}$$

integrating gives:

$$K^{(1)} = \left[-\frac{\partial f^{(o)}}{\partial\tilde{\Phi}} - 2\eta f^{(o)\alpha-1}\right]\phi - 2\eta(\alpha - 3)f^{(o)\alpha-1}e[\sin(\phi - \omega)]_o^{\phi} + f^{(1)}(\tilde{\Phi}) \tag{78}$$

The first boundedness condition is therefore:

$$\frac{df^{(o)}}{d\tilde{\phi}} = -2\eta f^{(o)\alpha-1} \tag{79}$$

whose solution is:

$$f^{(o)2-\alpha} = f_o^{(o)2-\alpha}\left[1 - \frac{2(2-\alpha)}{f_o^{(o)2-\alpha}}\eta\tilde{\phi}\right] \qquad \text{for } \alpha \neq 2 \tag{80}$$

$$f^{(o)} = f_o^{(o)} \exp\left[-2\eta\tilde{\phi}\right] \qquad \text{for } \alpha = 2 \tag{81}$$

Equation (78) for $K^{(1)}$ will become:

$$K^{(1)} = f^{(1)} - 2(\alpha - 3)\eta e f^{(o)\alpha-1}\left[\sin(\phi - \omega) + \sin\omega\right] \tag{82}$$

If we neglect terms of O(e) in the expression for $K^{(1)}$, this simply becomes

$$K^{(1)} = f^{(1)} \tag{83}$$

where $f^{(1)}$ will have to be determined by the boundedness conditions on $K^{(2)}$.

Solving (18b) for $U^{(1)}$ we have:

$$U_{11}^{(1)} + U^{(1)} = H(\phi) \tag{84}$$

where $H(\phi)$ is a function of $K^{(o)}$, $U^{(o)}$ and $K^{(1)}$ and their derivatives. Note that we must obtain $U^{(1)}$ to O(e) because the boundedness condition will give us e and ω and thus $U^{(o)}$ which has to be carried out to O(e). Thus Eq. (82) for $K^{(1)}$ and not Eq. (83) must be used. We neglect, however, terms of $O(e^2)$. $H(\phi)$ therefore is:

$$H(\phi) = 2\frac{\partial f^{(o)}}{\partial\tilde{\phi}} e \sin(\phi - \omega) + 2\frac{\partial e}{\partial\tilde{\phi}} f^{(o)} \sin(\phi - \omega) - 2\frac{\partial\omega}{\partial\tilde{\phi}} e f^{(o)} \cos(\phi - \omega)$$

$$+ f^{(1)} - 2(\alpha - 3)\eta f^{(o)\alpha - 1} e\left[\sin(\phi - \omega) + \sin\omega\right] - \zeta f^{(o)\alpha - 1}$$

$$- (\alpha - 2)\zeta f^{(o)\alpha - 1} e \cos(\phi - \omega) + \eta e f^{(o)\alpha - 1} \sin(\phi - \omega) \qquad (85)$$

The homogeneous solution of $U^{(1)}$ is:

$$U_H^{(1)} = A(\tilde{\phi}) \sin(\phi - \omega) + B(\tilde{\phi}) \cos(\phi - \omega) \qquad (86)$$

The particular solution is obtained by the Green's function as in Section V.

$$U_P^{(1)} = \sin(\phi - \omega) \int_o^\phi H(\phi') \cos(\phi' - \omega)\, d\phi'$$

$$- \cos(\phi - \omega) \int_o^\phi H(\phi') \sin(\phi' - \omega)\, d\phi'$$

Substituting Eq. (85) and integrating, we get

$$U_P^{(1)} = \sin(\phi - \omega)\Bigg\{ e \frac{\partial f^{(o)}}{\partial \tilde{\phi}} \Big[-\cos^2\phi' \Big]_{-\omega}^{\phi-\omega} + f^{(o)} \frac{\partial e}{\partial \tilde{\phi}} \Big[-\cos^2\phi' \Big]_{-\omega}^{\phi-\omega}$$

$$- e f^{(o)} \frac{\partial \omega}{\partial \tilde{\phi}} \Big[\phi' + \frac{1}{2}\sin 2\phi' \Big]_{-\omega}^{\phi-\omega} + f^{(1)} \Big[\sin(\phi - \omega) \Big]_{o}^{\phi}$$

$$+ (\alpha - 3)\eta f^{(o)\alpha - 1}\, e \Big[(\cos^2\phi')_{-\omega}^{\phi-\omega} - \sin\omega\, (\sin\phi')_{-\omega}^{\phi-\omega} \Big]$$

$$- \zeta f^{(o)\alpha - 1} \Big[\sin\phi' \Big]_{-\omega}^{\phi-\omega} - \left(\frac{\alpha - 2}{2} \right) \zeta f^{(o)\alpha - 1}\, e \Big[\phi' + \frac{1}{2}\sin 2\phi' \Big]_{-\omega}^{\phi-\omega}$$

$$- \frac{1}{2}\eta e f^{(o)\alpha - 1} \Big[\cos^2\phi' \Big]_{-\omega}^{\phi-\omega} \Bigg\} - \cos(\phi - \omega)\Bigg\{ e \frac{\partial f^{(o)}}{\partial \phi} \Big[\phi' - \frac{1}{2}\sin 2\phi' \Big]_{-\omega}^{\phi-\omega}$$

$$+ f^{(o)} \frac{\partial e}{\partial \phi} \Big[\phi' - \frac{1}{2}\sin 2\phi' \Big]_{\omega}^{\phi-\omega} + e f^{(o)} \frac{\partial \omega}{\partial \tilde{\phi}} \Big[\cos^2\phi' \Big]_{-\omega}^{\phi-\omega} - f^{(1)} \Big[\cos(\phi - \omega) \Big]_{o}^{\phi}$$

$$- (\alpha - 3)\eta f^{(o)\alpha - 1}\, e \Big[\phi' - \frac{1}{2}\sin 2\phi' \Big]_{-\omega}^{\phi-\omega}$$

$$+ 2(\alpha - 3)\eta f^{(o)\alpha - 1}\, e \sin\omega \Big[\cos\phi' \Big]_{-\omega}^{\phi-\omega} + \zeta f^{(o)\alpha - 1} \Big[\cos\phi' \Big]_{-\omega}^{\phi-\omega}$$

$$+ \left(\frac{\alpha - 2}{2} \right) \zeta f^{(o)\alpha - 1}\, e \Big[\cos^2\phi' \Big]_{-\omega}^{\phi-\omega} + \frac{1}{2}\eta e f^{(o)\alpha - 1} \Big[\phi' - \frac{1}{2}\sin 2\phi' \Big]_{-\omega}^{\phi-\omega} \Bigg\} \tag{87}$$

Setting the terms proportional to $\phi \sin(\phi - \omega)$ to zero gives us the second boundedness condition:

$$\frac{d\omega}{d\tilde{\Phi}} = -\left(\frac{\alpha - 2}{2}\right) \zeta f^{(o)\alpha-2} \tag{88}$$

Setting the terms proportional to $\phi \cos(\phi - \omega)$ to zero gives us the third boundedness condition:

$$\frac{de}{d\tilde{\Phi}} = \left(\alpha - \frac{3}{2}\right) \eta e f^{(o)\alpha-2} \tag{89}$$

The solution to Eq. (88) is:

$$\left.\begin{aligned} \omega &= \omega_o - \frac{1}{4}\frac{\zeta}{\eta} \log\left[1 - \frac{2(2-\alpha)}{f_o^{(o)2-\alpha}} \eta\tilde{\Phi}\right], && \alpha \neq 2 \\ &= \omega_o, && \alpha = 2 \end{aligned}\right\} \tag{90}$$

The solution to (89) is:

$$\left.\begin{aligned} e &= e_o\left[1 - \frac{2(2-\alpha)}{f_o^{(o)2-\alpha}} \eta\tilde{\Phi}\right]^{\frac{3-2\alpha}{4(2-\alpha)}}, && \alpha \neq 2 \\ &= e_o \exp\left[\frac{1}{2}\eta\tilde{\Phi}\right], && \alpha = 2 \end{aligned}\right\} \tag{91}$$

Equations (80), (81), (90) and (91) show the behavior of $f^{(o)}$, e and ω to first order in the linearized eccentricity case. These are plotted in Figs. 4 to 7 and are discussed in Section V.

The complete solution to $U^{(1)}$ neglecting terms of O(e) is therefore

$$U^{(1)} = (1 - \cos\phi)\left[f^{(1)} - \zeta f^{(o)\alpha-1}\right] + A \sin(\phi - \omega) + B\cos(\phi - \omega) \tag{92}$$

Again A and B will be determined by the boundedness conditions on $U^{(2)}$.

c. Second Approximation Solution. So far we have determined $U^{(o)}$ and $K^{(o)}$ completely (by determining e, $f^{(o)}$ and ω) to O(e). We now need to determine $f^{(1)}$, A and B by imposing boundedness conditions on $K^{(2)}$ and $U^{(2)}$. Since $U^{(1)}$ and $K^{(1)}$ need to be calculated only to O(1) (and thus $f^{(1)}$, A and B need only to be obtained to O(1)), we will neglect terms of O(e) in determining $K^{(2)}$ and $U^{(2)}$.

Solving for $K^{(2)}$ [Eq. (19a)] and neglecting terms of O(e) we get:

$$K^{(2)} = -\frac{\partial f^{(1)}}{\partial\bar{\phi}}\phi - 2(\alpha - 3)\eta f^{(o)\alpha-2}\left[f^{(1)} - \zeta f^{(o)\alpha-1}\right](\phi - \sin\phi)$$

$$- 2(\alpha - 3)\eta f^{(o)\alpha-2}\left[A\cos(\phi - \omega) - A\cos\omega - B\sin(\phi - \omega) - B\sin\omega\right]$$

$$- 4\eta f^{(o)\alpha-2} f^{(1)}\phi + f^{(2)} \tag{93}$$

Setting the terms proportional to φ equal to zero we get the boundedness condition that determines $f^{(1)}$:

$$\frac{df^{(1)}}{d\bar{\phi}} + 2(\alpha - 1)\eta f^{(o)\alpha-2}\eta f^{(1)} = 2(\alpha - 3)\zeta\eta f^{(o)2\alpha-3} \tag{94}$$

and $K^{(2)}$ becomes:

$$K^{(2)} = 2(\alpha - 3)\eta f^{(o)\alpha-2} \sin\phi \, [f^{(1)} - \zeta f^{(o)\alpha-1}]$$

$$- 2(\alpha - 3)\eta f^{(o)\alpha-2}[A(\cos(\phi - \omega) - \cos\omega) - B(\sin(\phi - \omega) + \sin\omega)]$$

$$+ f^{(2)} \tag{95}$$

Solving for $U^{(2)}$ [Eq. (19b)] and neglecting terms of O(e), we get:

$$U_{11}^{(2)} + U^{(2)} = H(\phi)$$

where

$$H(\phi) = -2\left[\frac{\partial f^{(1)}}{\partial\tilde{\phi}} + 2(\alpha - 1)\zeta\eta f^{(o)2\alpha-3}\right]\sin\phi - 2\frac{\partial A}{\partial\tilde{\phi}}\cos(\phi - \omega)$$

$$+ 2\frac{\partial B}{\partial\tilde{\phi}}\sin(\phi - \omega) - 2\frac{\partial\omega}{\partial\tilde{\phi}}[A\sin(\phi - \omega) + B\cos(\phi - \omega)] - \frac{\partial^2 f^{(o)}}{\partial\tilde{\phi}^2}$$

$$+ 2(\alpha - 3)\eta f^{(o)\alpha-2}\sin\phi[f^{(1)} - \zeta f^{(o)\alpha-1}]$$

$$- 2(\alpha - 3)\eta f^{(o)\alpha-2}[A(\cos(\phi - \omega) - \cos\omega) - B(\sin(\phi - \omega) + \sin\omega)]$$

$$+ f^{(2)} - \zeta f^{(1)} f^{(o)\alpha-2} - (\alpha - 2)\zeta f^{(o)\alpha-2}\Big[f^{(1)} - \zeta f^{(o)\alpha-1}$$

$$- (f^{(1)} - \zeta f^{(o)\alpha-1})\cos\phi + A\sin(\phi - \omega) + B\cos(\phi - \omega)\Big]$$

$$- \eta f^{(o)\alpha-2}\frac{\partial f^{(o)}}{\partial\tilde{\phi}} - \eta f^{(o)\alpha-2}\,[(f^{(1)} - \zeta f^{(o)\alpha-1})\sin\phi$$

$$+ A\cos(\phi - \omega) - B\sin(\phi - \omega)] \tag{96}$$

Equation (96) can be written in the form:

$$H(\phi) = R(\tilde{\phi}) \cos \phi + S(\tilde{\phi}) \sin \phi + T(\tilde{\phi})$$

where:

$$R(\tilde{\phi}) = -2 \frac{\partial A}{\partial \tilde{\phi}} \cos \omega - 2 \frac{\partial B}{\partial \tilde{\phi}} \sin \omega - (2\alpha - 5)\eta f^{(o)\alpha-2} A \cos \omega$$

$$- (2\alpha - 5)\eta f^{(o)\alpha-2} B \sin \omega + (\alpha - 2)\zeta f^{(o)\alpha-2} f^{(1)} - (\alpha - 2)\zeta^2 f^{(o)2\alpha-3} \tag{97}$$

$$S(\tilde{\phi}) = -2 \frac{\partial A}{\partial \tilde{\phi}} \sin \omega + 2 \frac{\partial B}{\partial \tilde{\phi}} \cos \omega - (2\alpha - 5)\eta f^{(o)\alpha-2} A \sin \omega$$

$$+ (2\alpha - 5)\eta f^{(o)\alpha-2} B \cos \omega + (6\alpha - 11)\eta f^{(o)\alpha-2} f^{(1)} - (10\alpha - 23)\zeta\eta f^{(o)2\alpha-3} \tag{98}$$

$$T(\tilde{\phi}) = -4(\alpha - 1)\eta^2 f^{(o)2\alpha-3} + 2(\alpha - 3)\eta f^{(o)\alpha-2}(A \cos \omega + B \sin \omega)$$

$$- (\alpha - 1)\zeta f^{(o)\alpha-2} f^{(1)} + (\alpha - 2)\zeta^2 f^{(o)2\alpha-3} + 2\eta^2 f^{(o)2\alpha-3} + f^{(2)} \tag{99}$$

Solving by the Green's function as before, and setting unbounded terms to zero we get

$$R(\tilde{\phi}) = 0 \qquad S(\tilde{\phi}) = 0$$

In order to reduce these to differential equations for A and B, we form the two equations

$$R(\tilde{\phi}) \cos \omega + S(\tilde{\phi}) \sin \omega = 0 \quad \text{and} \quad R(\tilde{\phi}) \sin \omega - S(\tilde{\phi}) \cos \omega = 0$$

These give us the differential equations for A and B:

$$\frac{dA}{d\tilde{\Phi}} + \left(\frac{2\alpha - 5}{2}\right) \eta f^{(o)\alpha-2} A = X(\tilde{\Phi}) \tag{100}$$

$$\frac{dB}{d\tilde{\Phi}} + \left(\frac{2\alpha - 5}{2}\right) \eta f^{(o)\alpha-2} B = Y(\tilde{\Phi}) \tag{101}$$

where:

$$X(\tilde{\Phi}) = \left[\frac{(\alpha - 2)}{2} \zeta f^{(o)\alpha-2} f^{(1)} - \frac{(\alpha - 2)}{2} \zeta^2 f^{(o)2\alpha-3}\right] \cos\omega$$

$$+ \left[\left(\frac{6\alpha - 11}{2}\right) \eta f^{(o)\alpha-2} f^{(1)} - \left(\frac{10\alpha - 23}{2}\right) \zeta\eta f^{(o)2\alpha-3}\right] \sin\omega$$

$$Y(\tilde{\Phi}) = \left[\frac{\alpha - 2}{2} \zeta f^{(o)\alpha-2} f^{(1)} - \left(\frac{\alpha - 2}{2}\right) \zeta^2 f^{(o)2\alpha-3}\right] \sin\omega$$

$$- \left[\left(\frac{6\alpha - 11}{2}\right) \eta f^{(o)\alpha-2} f^{(1)} - \left(\frac{10\alpha - 23}{2}\right) \zeta\eta f^{(o)2\alpha-3}\right] \cos\omega$$

Equations (94), (100) and (101) are the boundedness conditions and their solution will give $f^{(1)}$, A and B.

Solving for $f^{(1)}$ Eq. (94) we have:

$$\frac{df^{(1)}}{d\tilde{\Phi}} + 2(\alpha - 1)\eta f^{(o)\alpha-2} f^{(1)} = 2(\alpha - 3)\zeta\eta f^{(o)2\alpha-3} \tag{102}$$

The homogeneous solution is:

$$f_H^{(1)} = a f^{(o)\alpha-1} \tag{103}$$

To obtain the particular solution we let

$$f_P^{(1)} = f_H^{(1)} \cdot g$$

substituting in (102):

$$f_H^{(1)} \frac{dg}{d\tilde{\phi}} + g \frac{df_H^{(1)}}{d\tilde{\phi}} = -2(\alpha - 1)\eta f^{(o)\alpha-2} f_H^{(1)} \cdot g + 2(\alpha - 3)\zeta\eta f^{(o)2\alpha-3}$$

or

$$\frac{dg}{d\tilde{\phi}} = 2(\alpha - 3)\zeta\eta \frac{f^{(o)\alpha-2}}{a} \qquad (104)$$

substituting the values of $f^{(o)}$ we get

$$g = \frac{\alpha - 3}{\alpha - 2} \zeta \frac{1}{a} \ln\left[1 - 2(2 - \alpha)\eta f_o^{(o)\alpha-2} \tilde{\phi}\right] \qquad \alpha \neq 2 \qquad (105)$$

and

$$g = -\frac{2\zeta\eta}{a} \tilde{\phi} \qquad \alpha = 2 \qquad (106)$$

and thus the particular solution becomes:

$$f_P^{(1)} = \frac{\alpha - 3}{\alpha - 2} \zeta f^{(o)\alpha-1} \ln\left[1 - 2(2 - \alpha)\eta f_o^{(o)\alpha-2} \tilde{\phi}\right] \qquad \alpha \neq 2 \qquad (107)$$

$$f_P^{(1)} = -2\zeta\eta \frac{1}{f_o^{(o)}} \tilde{\phi} \exp(2\eta\tilde{\phi}) \qquad \alpha = 2 \qquad (108)$$

and the complete solution for $f^{(1)}$ becomes:

$$f^{(1)} = f^{(o)\alpha-1}\left[a + \frac{(\alpha - 3)}{(\alpha - 2)} \zeta \ln \left(1 - 2(2 - \alpha)\eta f_o^{(o)\alpha-2} \tilde{\phi}\right)\right] \qquad \alpha \neq 2 \qquad (109)$$

$$f^{(1)} = \frac{1}{a} \exp (2\eta\tilde{\phi})\left[a - 2\zeta\eta\tilde{\phi}\right] \qquad \alpha = 2 \qquad (110)$$

Equation (109) can be written in terms of ω [see Eq. (90)]

$$f^{(1)} = f^{(o)\alpha-1}\left[a - \frac{4(\alpha - 3)}{(\alpha - 2)} \eta (\omega - \omega_o)\right], \qquad \alpha \neq 2 \qquad (111)$$

where a = constant of integration = $f_o^{(1)} f_o^{(0)(1-\alpha)}$ and $f_o^{(1)} = f^{(1)}$ at $\phi = 0$.

In a similar fashion we can solve Eqs. (100) and (101) for A and B. The solutions are: for $\alpha \neq 2$:

$$A = a_o f^{(o)(2\alpha-5)/4} \left[1 + g\right]$$

$$B = b_o f^{(o)(2\alpha-5)/4} \left[1 + h\right]$$

where:

$$g = \frac{M}{A_o}\bar{\alpha} + \frac{N}{A_o}\bar{\beta} - \frac{2(\alpha - 3)\zeta\eta}{A_o}\bar{\gamma} - \frac{2(\alpha - 3)(6\alpha - 11)\eta^2}{(\alpha - 2)A_o}\bar{\delta}$$

$$h = \frac{M}{A_o}\bar{\beta} - \frac{N}{A_o}\bar{\alpha} - \frac{2(\alpha - 3)\zeta\eta}{A_o}\bar{\delta} + \frac{2(\alpha - 3)(6\alpha - 11)\eta^2}{(\alpha - 2)A_o}\bar{\gamma}$$

where

$$M = \left(\frac{\alpha - 2}{2}\right) \zeta a + 2(\alpha - 3)\zeta\eta\omega_o - \left(\frac{\alpha - 2}{2}\right) \zeta^2$$

$$N = \left(\frac{6\alpha - 11}{2}\right) \eta a - \left(\frac{10\alpha - 23}{2}\right) \zeta\eta + \frac{2(\alpha - 3)(6\alpha - 11)}{(\alpha - 2)} \eta^2 \omega_o$$

and

$$\overline{\alpha} = \frac{K}{1 + \mu^2} \left[e^{-\mu\omega} (\sin\omega - \mu \cos\omega)\right]_{\omega_o}^{\omega(\tilde{\phi})}$$

$$\overline{\beta} = \frac{K}{1 + \mu^2} \left[e^{-\mu\omega} (-\mu \sin\omega - \cos\omega)\right]_{\omega_o}^{\omega(\tilde{\phi})}$$

$$\overline{\gamma} = \frac{K}{1 + \mu^2} \left[\omega e^{-\mu\omega} (\sin\omega - \mu \cos\omega) + \frac{e^{-\mu\omega}}{1 + \mu^2} ((1 - \mu^2)\cos\omega + 2\mu \sin\omega)\right]_{\omega_o}^{\omega(\tilde{\phi})}$$

$$\overline{\delta} = \frac{K}{1 + \mu^2} \left[\omega e^{-\mu\omega} (-\mu \sin\omega - \cos\omega) + \frac{e^{-\mu\omega}}{1 + \mu^2} ((1 - \mu^2)\sin\omega - 2\mu \cos\omega)\right]_{\omega_o}^{\omega(\tilde{\phi})}$$

where:

$$K = f_o^{(o)} \frac{\gamma e^{\mu\omega_o}}{d} \qquad d = 2(2 - \alpha)\eta f_o^{(o)\alpha - 2}$$

$$\mu = \gamma(1 + c) \qquad \gamma = \frac{4\eta}{\zeta}$$

$$\omega(\tilde{\phi}) = \omega_o - \frac{1}{\gamma} \log(1 - \tilde{\phi} d) \qquad c = \frac{(6\alpha - 7)}{4(2 - \alpha)}$$

a_o = constant of integration = $A_o f_o^{(o)(5-2\alpha)/4}$ and $A_o = A$ at $\phi = 0$

b_o = constant of integration = $B_o f_o^{(o)(5-2\alpha)/4}$ and $B_o = B$ at $\phi = 0$

for $\alpha = 2$, the differential equations for A and B [Eqs. (100), (101)] become:

$$\frac{dA}{d\tilde{\phi}} - \frac{1}{2}\eta A = \left(\frac{1}{2}\eta f^{(1)} + \frac{3}{2}\zeta\eta f^{(o)2\alpha-3}\right)\sin\omega_o$$

$$\frac{dB}{d\tilde{\phi}} - \frac{1}{2}\eta B = \left(-\frac{1}{2}\eta f^{(1)} - \frac{3}{2}\zeta\eta f^{(o)2\alpha-3}\right)\cos\omega_o$$

where now

$$f^{(o)2\alpha-3} = f_o^{(o)}\, e^{-2\eta\tilde{\phi}} \qquad\qquad f^{(1)} = \frac{1}{f_o^{(o)}}\left[a - 2\zeta\eta\tilde{\phi}\right] e^{2\eta\tilde{\phi}}$$

Therefore we have:

$$\frac{dA}{d\tilde{\phi}} - \frac{1}{2}\eta A = \left[\frac{1}{2}\frac{\eta a}{f_o^{(o)}}\, e^{2\eta\tilde{\phi}} - \frac{\zeta\eta^2}{f_o^{(o)}}\,\tilde{\phi}\, e^{2\eta\tilde{\phi}} + \frac{3}{2}\zeta\eta f_o^{(o)}\, e^{-2\eta\tilde{\phi}}\right]\sin\omega_o$$

$$\frac{dB}{d\tilde{\phi}} - \frac{1}{2}\eta B = \left[-\frac{1}{2}\frac{\eta a}{f_o^{(o)}}\, e^{2\eta\tilde{\phi}} + \frac{\zeta\eta^2}{f_o^{(o)}}\,\tilde{\phi}\, e^{2\eta\tilde{\phi}} - \frac{3}{2}\zeta\eta f_o^{(o)}\, e^{-2\eta\tilde{\phi}}\right]\cos\omega_o$$

Their solutions are:

$$A = C_o e^{\frac{\eta\tilde{\phi}}{2}} + C_1 e^{-2\eta\tilde{\phi}} + C_2 e^{2\eta\tilde{\phi}} + C_3 \tilde{\phi} e^{2\eta\tilde{\phi}}$$

$$B = D_o e^{\frac{\eta\tilde{\phi}}{2}} + D_1 e^{-2\eta\tilde{\phi}} + D_2 e^{2\eta\tilde{\phi}} + D_3 \tilde{\phi} e^{2\eta\tilde{\phi}}$$

where

$$C_1 = \frac{3}{5} \zeta f_o^{(o)} \sin \omega_o \qquad D_1 = -\frac{3}{5} \zeta f_o^{(o)} \cos \omega_o$$

$$C_2 = -\frac{1}{f_o^{(o)}} \left(4\zeta + \frac{1}{3} f_o^{(1)}\right) \sin \omega_o \qquad D_2 = \frac{1}{f_o^{(o)}} \left(4\zeta + \frac{1}{3} f_o^{(1)}\right) \cos \omega_o$$

$$C_3 = \frac{2}{3} \frac{\zeta\eta}{f_o^{(o)}} \sin \omega_o \qquad D_3 = -\frac{2}{3} \frac{\zeta\eta}{f_o^{(o)}} \cos \omega_o$$

and

C_o = constant of integration = $A_o - C_1 - C_2$ where $A_o = A$ at $\phi = 0$

D_o = constant of integration = $B_o - D_1 - D_2$ where $B_o = B$ at $\phi = 0$

Initial Conditions

The initial conditions for a standard satellite problem is:

at $t = 0$:

$$\phi = \phi_o$$
$$\dot{\phi} = \dot{\phi}_o$$
$$r = r_o$$
$$\dot{r} = \dot{r}_o$$

Due to our choice of variables, the initial conditions in this case will be, at $\phi = 0$

$$t(o) = 0$$

$$u(o) = u_o = 1$$

$$\frac{du}{d\phi}(o) = \left.\frac{du}{d\phi}\right|_o$$

$$k(o) = k_o$$

now

$$u = U^{(o)} + \epsilon U^{(1)} + \cdots$$

$$k = K^{(o)} + \epsilon K^{(1)} + \cdots$$

let

$$u(o) = C_1 \qquad \frac{du}{d\phi}(o) = C_2 \qquad k(o) = C_3$$

therefore

$$u(o) = f_o^{(o)}\left[1 + e_o \cos\omega_o\right] + \epsilon\left[B_0 \cos\omega_o - A_o \sin\omega_o\right] + O(\epsilon^2) = C_1$$

$$k(o) = f_o^{(o)} + \epsilon f_o^{(1)} + O(\epsilon^2) = C_3$$

$$\frac{du}{d\phi}(o) = \left[U_1^{(o)} + \epsilon\left(U_2^{(o)} + U_1^{(1)}\right) + O(\epsilon^2)\right]_{\phi=0}$$

$$= f_o^{(o)} e_o \sin\omega_o + \epsilon\left[\left(-2\eta f_o^{(o)(\alpha-1)}\right)(1 + e_o \cos\omega_o)\right.$$

$$+ \left(\alpha - \frac{3}{2}\right)\eta e_o f_o^{(o)\alpha-1} \cos\omega_o - \left(\frac{\alpha - 2}{2}\right)\zeta f_o^{(o)\alpha-1} e_o \sin\omega_o$$

$$\left. + A_o \cos\omega_o + B_o \sin\omega_o\right] + O(\epsilon^2) = C_2$$

Equating powers of ϵ:

$$C_1 = f_o^{(o)}\left[1 + e_o \cos\omega_o\right] = 1$$

$$O = B_o \cos\omega_o - A_o \sin\omega_o$$

$$C_3 = f_o^{(o)}$$

$$O = f_o^{(1)}$$

$$C_2 = f_o^{(o)} e_o \sin\omega_o$$

$$O = \left(-2\eta f_o^{(o)\alpha-1}\right)(1 + e_o \cos\omega_o) + \left(\alpha - \frac{3}{2}\right)\eta e_o f_o^{(o)(\alpha-1)} \cos\omega_o$$

$$- \left(\frac{\alpha - 2}{2}\right)\zeta f_o^{(o)\alpha-1} e_o \sin\omega_o + A_o \cos\omega_o + B_o \sin\omega_o$$

from the above six equations, the initial conditions may be determined, i.e., given C_1, C_2, C_3 we can determine $f_o^{(o)}, e_o, \omega_o, f_o^{(1)}, A_o$ and B_o.

VII. THE OUT OF PLANE EQUATION OF MOTION

In this section the differential equations governing the angular variation out of the ecliptic will be investigated [Eqs. (20), (21) and (22)].

The solution to Eq. (20) is:

$$\Psi^{(o)} = A_1(\tilde{\phi}) \cos \phi + A_2(\tilde{\phi}) \sin \phi \tag{112}$$

substituting in (21) we get

$$\Psi_{11}^{(1)} + \Psi^{(1)} = 2 \frac{dA_1}{d\tilde{\phi}} \sin \phi - 2 \frac{dA_2}{d\tilde{\phi}} \cos \phi$$

$$+ \xi f^{(o)\alpha-2} [1 + e \cos (\phi - \omega)]^{\alpha-3}$$

$$+ \eta f^{(o)\alpha-2} [1 + e \cos (\phi - \omega)]^{\alpha-3} [A_1 \sin \phi - A_2 \cos \phi] \tag{113}$$

We now need to determine the unbounded parts in the solution of $\Psi^{(1)}$ In order to facilitate the understanding of Eq. (113), let us look at a simple harmonic oscillator of the type:

$$\frac{d^2 y}{dt^2} + y = f(t)$$

Let

$$f(t) = a_o + a_1 \cos t + b_1 \sin t + a_2 \cos 2t$$

$$+ b_2 \sin 2t + \cdots + a_n \cos nt + b_n \sin nt$$

We note that the homogeneous solution to the differential equation is

$$y = a \cos t + b \sin t$$

It is clear that any forcing function (any term in the expression for f(t)) that has the same frequency as the free oscillations, would cause resonance, i.e., unbounded response. Therefore in order that the response be bounded, a_1 and b_1 in the expression for the forcing function f(t) must be zero.

With this in mind therefore we can examine Eq. (113). Let

$$f^{(o)\alpha-2}[1 + e \cos(\phi - \omega)]^{\alpha-3} = a_o + a_1 \sin \phi + b_1 \cos \phi + a_2 \sin 2\phi + b_2 \cos 2\phi + \cdots \tag{114}$$

where the a's and b's are the coefficients that can be obtained by the proper expansion of the L. H. S.

Substituting now in Eq. (113) we get

$$\Psi_{11}^{(1)} + \Psi^{(1)} = 2 \frac{dA_1}{d\tilde{\phi}} \sin \phi - 2 \frac{dA_2}{d\tilde{\phi}} \cos \phi + \xi[a_o + a_1 \sin \phi + b_1 \cos \phi + a_2 \sin 2\phi + \cdots] + \eta[a_o + a_1 \sin \phi + b_1 \cos \phi + a_2 \sin 2\phi + \cdots][A_1 \sin \phi - A_2 \cos \phi] \tag{115}$$

Now, as we have seen, in order to set the unbounded terms to zero, we must let the coefficients of cos ϕ and sin ϕ to zero. Therefore the boundedness conditions that must be satisfied are:

$$\left.\begin{aligned} 2\frac{dA_1}{d\tilde{\phi}} + \eta a_o A_1 + \xi a_1 &= 0 \\ 2\frac{dA_2}{d\tilde{\phi}} + \eta a_o A_2 - \xi b_1 &= 0 \end{aligned}\right\} \tag{116}$$

We now have to go back and recall that in Section III when the u and k equations were uncoupled from the ψ equation, the assumption was made that $d\psi/d\phi = O(\epsilon)$.

Now

$$\psi = \Psi^{(o)} + \epsilon\Psi^{(1)} + \ldots$$

therefore

$$\frac{d\psi}{d\phi} = \Psi_1^{(o)} + \epsilon\left(\Psi_2^{(o)} + \Psi_1^{(1)}\right) + O(\epsilon^2)$$

Then if $d\psi/d\phi = O(\epsilon)$ we must have $\Psi_1^{(o)} = 0$, i.e.,

$$\Psi^{(o)} = \Psi^{(o)}(\tilde{\phi})$$

but we found that

$$\Psi^{(o)} = A_1(\tilde{\phi}) \cos\phi + A_2(\tilde{\phi}) \sin\phi$$

Therefore in order to satisfy the assumption that $d\psi/d\phi$ must be of $O(\epsilon)$ and not larger, we must have

$$A_1 = A_2 = 0$$

Looking at Eq. (116) we see that $A_1 = A_2 = 0$ can only be satisfied under the following two conditions:

(a) If $a_1 = b_1 = 0$. From Eq. (114) we note that letting $a_1 = b_1 = 0$ signifies linearizing in eccentricity!

(b) If ξ is of a higher order, i.e., $O(\epsilon)$ smaller than η and ζ so that it wouldn't have appeared at all in the differential equation for $\Psi^{(1)}$ [Eq. (113)].

The above results can be easily explained by physical interpretation. Consider a three dimensional orbit that is slightly out of the ecliptic. The assumption was made that ψ, the out of ecliptic angle, must be small and remain small. Now if the orbit has a low eccentricity, then the torques created by the out of plane thrust tend to almost cancel each other since the orbit is near circular. Thus if it is initially small it will remain small. However if the orbit is highly eccentric, the spacecraft spends much more time on one side than the other and thus the torques do not equalize each other, and therefore even if ψ is initially small it will grow to be very large. Of course if the thrust component in the ψ direction is one order of magnitude less than the components in the other two directions, it is clear that ψ will not grow to be large.

VIII. COMPARISON WITH NUMERICAL INTEGRATIONS AND TO THE MELBOURNE-SAUER INTEGRATING PROGRAM

In Section V, the first approximation solutions of the trajectory projection on the ecliptic plane were obtained without linearizing in eccentricity. These solutions were determined for values of α of 0, 1, 1.5, 2 and 3. In order to determine how close these solutions were to the numerical integration of the equations of motion, Fig. 8 was plotted. The figure shows the trajectories for the case where $\alpha = 2$, $\zeta = \eta = 1/\sqrt{2}$, $e_o = 0.10$, $\omega_o = 0$, $f_o^{(o)} = 1.0$ and $\epsilon = 0.05$.

The second approximation solutions for linearized eccentricity obtained in Section VI were plotted in Fig. 9. Three curves show the trajectories for $\alpha = 0$, $\alpha = 1.4$ and $\alpha = 1.9$. The other three curves show Wesseling's case with a thrust that varies like $1/r^2$, and two versions of the Melbourne-Sauer integrating program that consider variable mass, however one has a thrust varying like $1/r^2$ and the other shows the exact behavior of the thrust. These three curves therefore show the errors introduced by the assumptions of constant mass and thrust varying as $1/r^2$. In order to run a case similar to the Melbourne-Sauer program, the initial conditions used were $e_o = 0.1225$, $\omega_o = -27.155°$, and $f_o^{(o)} = 0.89087$. Again $\epsilon = 0.05$ in these cases and the thrust direction was at 45° to the radial direction, i.e., $\zeta = \eta = 1/\sqrt{2}$.

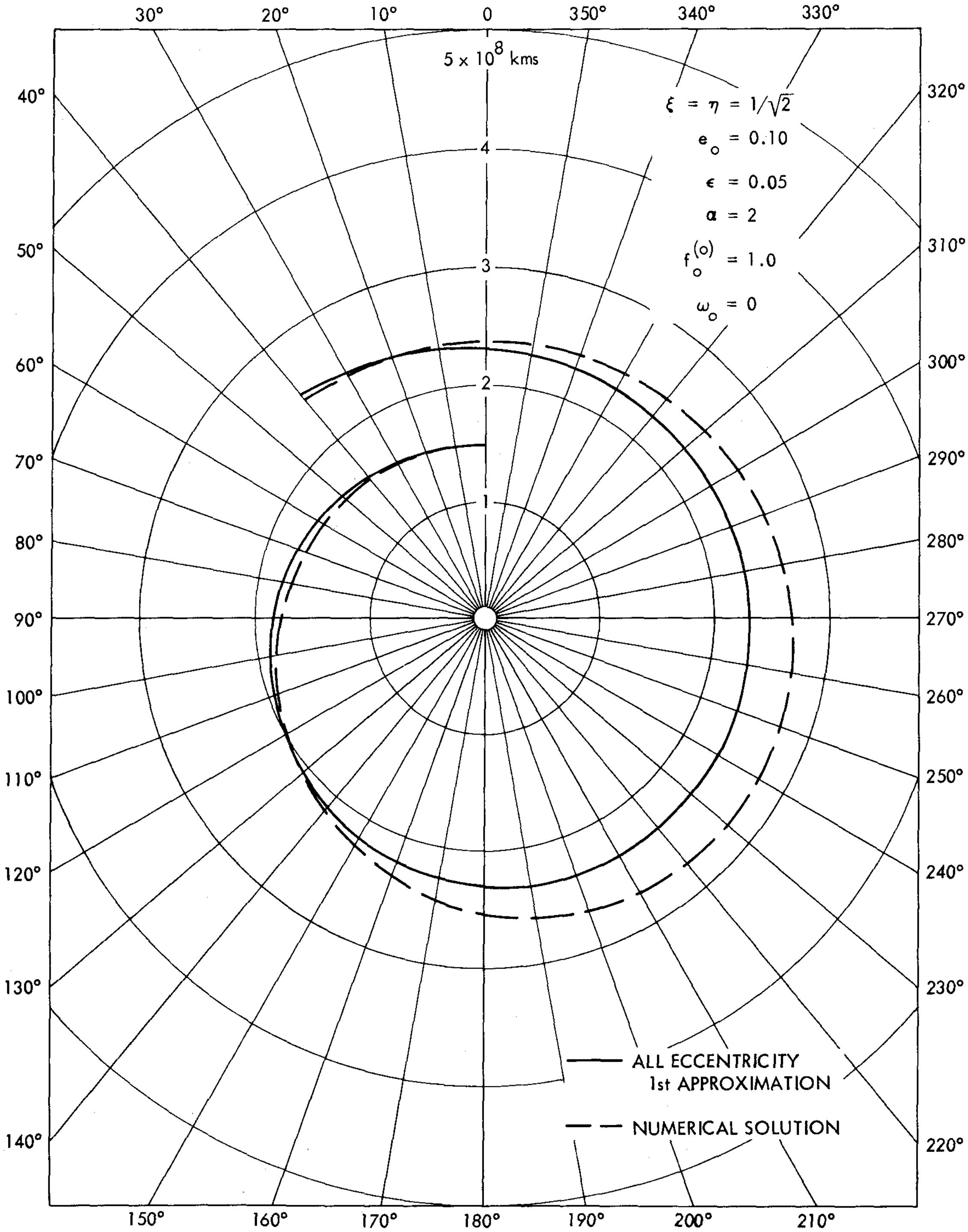

Fig. 8. Comparison of the all eccentricity trajectory to the numerical integration of the equations of motion

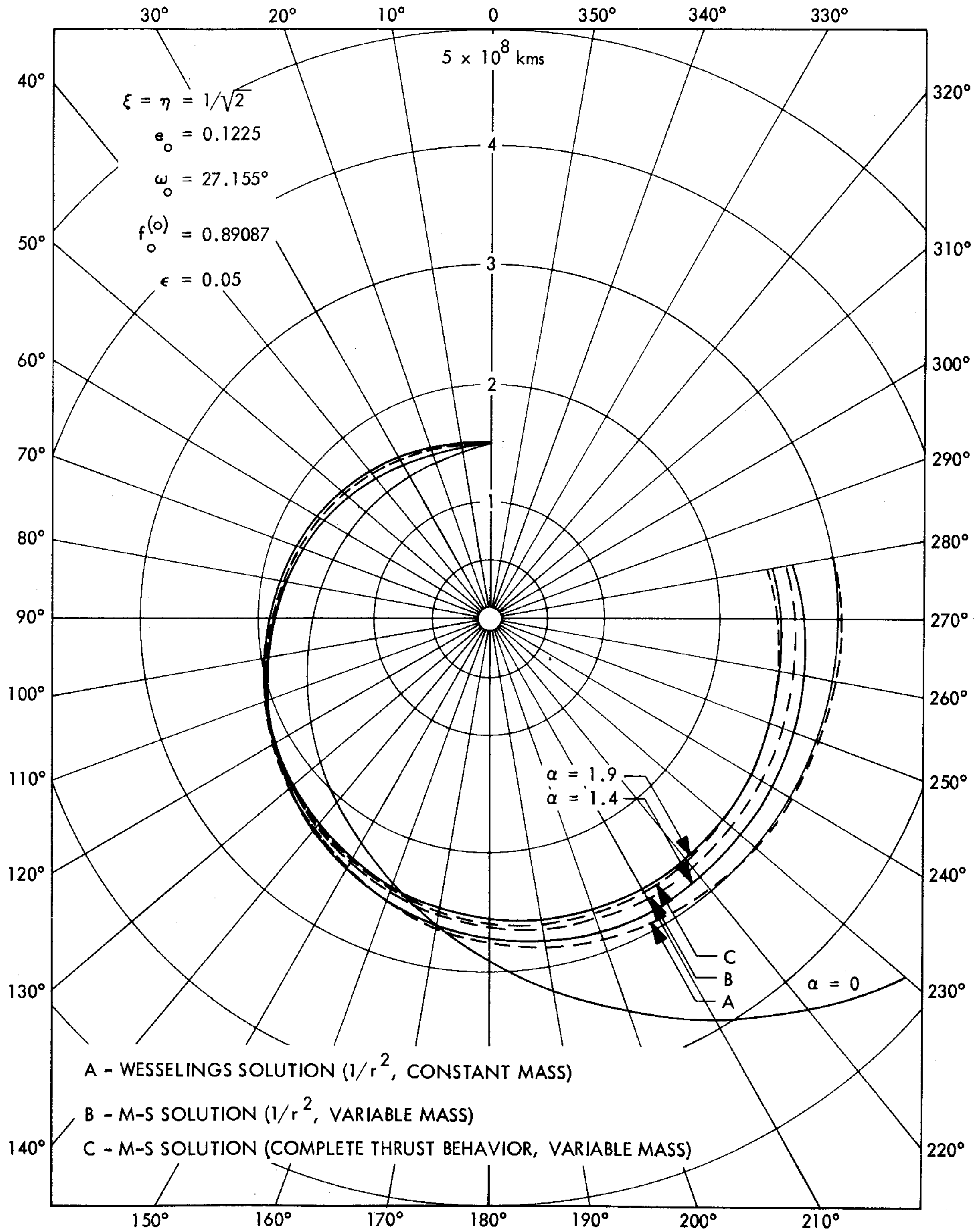

Fig. 9. Comparison of the linearized eccentricity trajectories to Wesseling's solution and to the Melbourne-Sauer program

By comparing the different trajectories in Fig. 9, the following is noted:

(a) The assumption that thrust varies as $1/r^2$ is more severe than the constant mass assumption.

(b) The $\alpha = 1.4$ case takes care of the thrust behavior effect but the constant mass assumption is still apparent.

At present the Jet Propulsion Laboratory is including the analysis performed in this thesis in their ASTRAL[48] computer program which uses the Wesseling solution to determine optimized low-thrust solar electric interplanetary trajectories.

APPENDIX A

TABLE OF SPECIAL INTEGRATIONS

1. $$\int_0^{\phi} \sin(\phi' - \omega)\cos(\phi' - \omega)\, d\phi' = \left[-\frac{1}{2}\cos^2\phi'\right]_{-\omega}^{\phi-\omega}$$

2. $$\int_0^{\phi} \sin^2(\phi' - \omega)\, d\phi' = \left[\frac{\phi'}{2} - \frac{\sin 2\phi'}{4}\right]_{-\omega}^{\phi-\omega}$$

3. $$\int_0^{\phi} \cos^2(\phi' - \omega)\, d\phi' = \left[\frac{\phi'}{2} + \frac{\sin 2\phi'}{4}\right]_{-\omega}^{\phi-\omega}$$

4. $$\int_0^{\phi} \phi' \cos(\phi' - \omega)\, d\phi' = \cos(\phi - \omega) + \phi \sin(\phi - \omega) - \cos\omega$$

5. $$\int_0^{\phi} \phi' \sin(\phi' - \omega)\, d\phi' = \sin(\phi - \omega) - \phi\cos(\phi - \omega) + \sin\omega$$

6. $$\int_0^{\phi} \frac{d\phi'}{[1 + e\cos(\phi' - \omega)]^3} = \left[-e\sin\phi' \frac{(4 - e^2) + 3e\cos\phi'}{2(1 - e^2)^2(1 + e\cos\phi')^2}\right.$$

$$\left. + \frac{2 + e^2}{(1 - e^2)^{5/2}} \tan^{-1}\sqrt{\frac{1 - e}{1 + e}} \tan\frac{\phi'}{2}\right]_{-\omega}^{\phi-\omega}$$

7. $$\int_0^{\phi} \frac{\cos(\phi' - \omega)\, d\phi'}{[1 + e \cos(\phi' - \omega)]^2} = \left[\frac{-2e}{(1 - e^2)^{3/2}} \tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}} \tan\frac{\phi'}{2}\right) + \frac{1}{1 - e^2}\, \frac{\sin\phi'}{(1 + e\cos\phi')}\right]_{-\omega}^{\phi-\omega}$$

8. $$\int_0^{\phi} \frac{\sin(\phi' - \omega)\cos(\phi' - \omega)\, d\phi'}{[1 + e\cos(\phi' - \omega)]^3} = \left[\frac{1}{e^2(1 + e\cos\phi')} - \frac{1}{2e^2(1 + e\cos\phi')^2}\right]_{-\omega}^{\phi-\omega}$$

9. $$\int_0^{\phi} \frac{\sin(\phi' - \omega)\, d\phi'}{[1 + e\cos(\phi' - \omega)]^2} = \left[\frac{1}{e(1 + e\cos\phi')}\right]_{-\omega}^{\phi-\omega}$$

10. $$\int_0^{\phi} \frac{\sin^2(\phi' - \omega)\, d\phi'}{[1 + e\cos(\phi' - \omega)]^3} = \left[\frac{\sin\phi'}{2e(1 + e\cos\phi')^2} - \frac{1}{2e(1 - e^2)}\, \frac{\sin\phi'}{(1 + e\cos\phi')} + \frac{1}{(1 - e^2)^{3/2}} \tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}} \tan\frac{\phi'}{2}\right)\right]_{-\omega}^{\phi-\omega}$$

11. $$\int_0^{\phi}\left\{\int_0^{\phi'}\frac{d\phi_1}{[1+e\cos(\phi_1-\omega)]^3}\right\}\cos(\phi'-\omega)\,d\phi' =$$

$$-e\left[\sin^2\phi'\,\frac{(4-e^2)+3e\cos\phi'}{2(1-e^2)^2(1+e\cos\phi')^2}\right.$$

$$\left.+\frac{(2+e^2)\sin\phi'}{(1-e^2)^{5/2}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\phi'}{2}\right)-\frac{1}{2e(1+e\cos\phi')^2}\right]_{-\omega}^{\phi-\omega}$$

12. $$\int_0^{\phi}\left\{\int_0^{\phi'}\frac{d\phi_1}{[1+e\cos(\phi_1-\omega)]^3}\right\}\sin(\phi'-\omega)\,d\phi' =$$

$$\left[e\sin\phi'\cos\phi'\,\frac{(4-e^2)+3e\cos\phi'}{2\left(1-e^2\right)^2(1+e\cos\phi')^2}+\frac{\sin\phi'}{2(1-e^2)(1+e\cos\phi')^2}\right.$$

$$-\frac{(2+e^2)\cos\phi'}{(1-e^2)^{5/2}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\phi'}{2}\right)+\frac{e^2\sin\phi'}{(1-e^2)^2(1+e\cos\phi')}$$

$$\left.-\frac{3e}{(1-e^2)^{5/2}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\phi'}{2}\right)+\frac{\sin\phi'}{2(1-e^2)^2(1+e\cos\phi')}\right]_{-\omega}^{\phi-\omega}$$

13. $$\int_0^{\phi}\frac{d\phi'}{[1+e\cos(\phi'-\omega)]^2}=\left[\frac{2}{(1-e^2)^{3/2}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\phi'}{2}\right)\right.$$

$$\left.-\frac{e}{(1-e^2)}\,\frac{\sin\phi'}{(1+e\cos\phi')}\right]_{-\omega}^{\phi-\omega}$$

14. $$\int_0^{\phi} \frac{\cos(\phi' - \omega)\, d\phi'}{1 + e\cos(\phi' - \omega)} = \left[\frac{\phi'}{e} - \frac{2}{e\sqrt{1 - e^2}} \tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}} \tan\frac{\phi'}{2}\right)\right]_{-\omega}^{\phi - \omega}$$

15. $$\int_0^{\phi} \frac{\sin(\phi' - \omega)\cos(\phi' - \omega)\, d\phi'}{[1 + e\cos(\phi' - \omega)]^2} = \left[\frac{-1}{e^2(1 + e\cos\phi')} - \frac{1}{e^2}\log(1 + e\cos\phi')\right]_{-\omega}^{\phi - \omega}$$

16. $$\int_0^{\phi} \frac{\sin(\phi' - \omega)\, d\phi'}{1 + e\cos(\phi' - \omega)} = \left[-\frac{1}{e}\log(1 + e\cos\phi')\right]_{-\omega}^{\phi - \omega}$$

17. $$\int_0^{\phi} \frac{\sin^2(\phi' - \omega)\, d\phi'}{[1 + e\cos(\phi' - \omega)]^2} = \left[\frac{\sin\phi'}{e(1 + e\cos\phi')} - \frac{\phi'}{e^2} + \frac{2}{e^2\sqrt{1 - e^2}} \tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}} \tan\frac{\phi'}{2}\right)\right]_{-\omega}^{\phi - \omega}$$

18. $$\int_0^{\phi}\left\{\int_0^{\phi'} \frac{d\phi_1}{[1 + e\cos(\phi' - \omega)]^2}\right\}\cos(\phi' - \omega)\, d\phi' =$$

$$\left[\frac{2\sin\phi'}{(1 - e^2)^{3/2}} \tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}} \tan\frac{\phi'}{2}\right) - \frac{e\sin^2\phi'}{(1 - e^2)(1 + e\cos\phi')} - \frac{1}{e(1 + e\cos\phi')}\right]_{-\omega}^{\phi - \omega}$$

19. $$\int_0^{\phi}\left\{\int_0^{\phi'} \frac{d\phi_1}{\left[1 + e\cos\left(\phi_1 - \omega\right)\right]^2}\right\}\sin(\phi' - \omega)\, d\phi' =$$

$$\left[\frac{-2\cos\phi'}{(1 - e^2)^{3/2}}\tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\tan\frac{\phi'}{2}\right)\right.$$

$$+ \frac{e\sin\phi'\cos\phi'}{(1 - e^2)(1 + e\cos\phi')} - \frac{2e}{(1 - e^2)^{3/2}}\tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\tan\frac{\phi'}{2}\right)$$

$$\left. + \frac{1}{(1 - e^2)}\;\frac{\sin\phi'}{(1 + e\cos\phi')}\right]_{-\omega}^{\phi - \omega}$$

20. $$\int_0^{\phi}\frac{d\phi'}{1 + e\cos(\phi' - \omega)} = \left[\frac{2}{\sqrt{1 - e^2}}\tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\tan\frac{\phi'}{2}\right)\right]_{-\omega}^{\phi - \omega}$$

21. $$\int_0^{\phi}\frac{\sin(\phi' - \omega)\cos(\phi' - \omega)\, d\phi'}{1 + e\cos(\phi' - \omega)} = \left[\frac{1}{e^2}\log(1 + e\cos\phi') - \frac{\cos\phi'}{e}\right]_{-\omega}^{\phi - \omega}$$

22. $$\int_0^{\phi}\frac{\sin^2(\phi' - \omega)\, d\phi'}{1 + e\cos(\phi' - \omega)} = \left[\frac{2}{\sqrt{1 - e^2}}\;\frac{e^2 - 1}{e^2}\;\tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\tan\frac{\phi'}{2}\right)\right.$$

$$\left. + \frac{\phi'}{e^2} - \frac{1}{e}\sin\phi'\right]_{-\omega}^{\phi - \omega}$$

23. $$\int_0^{\phi}\left\{\int_0^{\phi'} \frac{d\phi_1}{1 + e\cos(\phi_1 - \omega)}\right\}\cos(\phi' - \omega)\, d\phi' =$$

$$\left[\frac{1}{e}\log(1 + e\cos\phi') + \frac{2\sin\phi'}{\sqrt{1 - e^2}}\tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\tan\frac{\phi'}{2}\right)\right]_{-\omega}^{\phi - \omega}$$

24. $$\int_0^{\phi}\left\{\int_0^{\phi'} \frac{d\phi_1}{1 + e\cos(\phi_1 - \omega)}\right\}\sin(\phi' - \omega)\, d\phi' =$$

$$\left[\frac{-2\cos\phi'}{\sqrt{1 - e^2}}\tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\tan\frac{\phi'}{2}\right) + \frac{\phi'}{e}\right.$$

$$\left. - \frac{2}{e\sqrt{1 - e^2}}\tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\tan\frac{\phi'}{2}\right)\right]_{-\omega}^{\phi - \omega}$$

25. $$\int_0^{\phi}\cos(\phi' - \omega)\left[1 + e\cos(\phi' - \omega)\right] d\phi' =$$

$$\left[\sin\phi' + \phi'\frac{e}{2} + \frac{e}{4}\sin 2\phi'\right]_{-\omega}^{\phi - \omega}$$

26. $$\int_0^{\phi}\sin(\phi' - \omega)\left[1 + e\cos(\phi' - \omega)\right] d\phi' = \left[-\cos\phi' - \frac{e}{2}\cos^2\phi'\right]_{-\omega}^{\phi - \omega}$$

Note: For the following integrals

$$k = \sqrt{\frac{2e}{1 + e}} \qquad \Delta = \sqrt{1 - k^2 \sin^2 \phi'}$$

$$k' = \sqrt{1 - k^2} = \sqrt{\frac{1 - e}{1 + e}}$$

27. $$\int_0^{2\pi} \frac{d\phi'}{\left(\sqrt{1 + e \cos \phi}\right)^3} = \frac{2}{(1 + e)^{3/2}} \int_0^{\pi} \frac{d\phi'}{\left(\sqrt{1 - k^2 \sin^3 \phi'}\right)^3}$$

$$= \frac{4E(k)}{(1 + e)^{3/2} k'^{2}}$$

28. $$\int_0^{2\pi} \frac{\cos \phi' \, d\phi'}{\sqrt{1 + e \cos \phi'}} = \frac{2}{\sqrt{1 + e}} \int_0^{\pi} \frac{d\phi'}{\Delta} - \frac{4}{\sqrt{1 + e}} \int_0^{\pi} \frac{\sin^2 \phi' \, d\phi'}{\Delta}$$

$$= \frac{4}{\sqrt{1 + e}} \left[K(k) - \frac{(1 + e)}{e} \left(K(k) - E(k) \right) \right]$$

29. $$\int_0^{2\pi} \frac{\sin^2 \phi' \, d\phi'}{\left(\sqrt{1 + e \cos \phi}\right)^3} = \frac{8}{(1 + e)^{3/2}} \int_0^{\pi} \frac{\sin^2 \phi' \cos^2 \phi' \, d\phi'}{\Delta^3}$$

$$= \frac{8}{(1 + e)^{3/2}} \left[\frac{2(2 - k^2)}{k^4} K(k) - \frac{4}{k^4} E(k) \right]$$

30. $$\int_0^{2\pi}\left\{\int_0^{\phi'} \frac{d\phi_1}{\left(\sqrt{1 + e\cos\phi_1}\right)^3}\right\}\sin\phi'\, d\phi' =$$

$$\frac{4}{(1+e)^{3/2}}\left[\frac{E(k)}{1-k^2} - \frac{2E(k)}{k'^2k^2} + \frac{2K(k)}{k^2}\right]$$

31. $$\int_0^{2\pi} \frac{\sin\phi'\, d\phi'}{\sqrt{1 + e\cos\phi'}} = 0$$

32. $$\int_0^{2\pi} \frac{\sin\phi'\cos\phi'\, d\phi'}{\left(\sqrt{1 + e\cos\phi'}\right)^3} = 0$$

33. $$\int_0^{2\pi}\left\{\int_0^{\phi'} \frac{d\phi_1}{\left(\sqrt{1 + e\cos\phi_1}\right)^3}\right\}\cos\phi'\, d\phi' = 0$$

Note: K(k) = Complete elliptic integral of the first kind

E(k) = Complete elliptic integral of the second kind

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