A NEW THEORY FOR WINGS OF SMALL ASPECT RATIO

Thesis by

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SUMMARY:

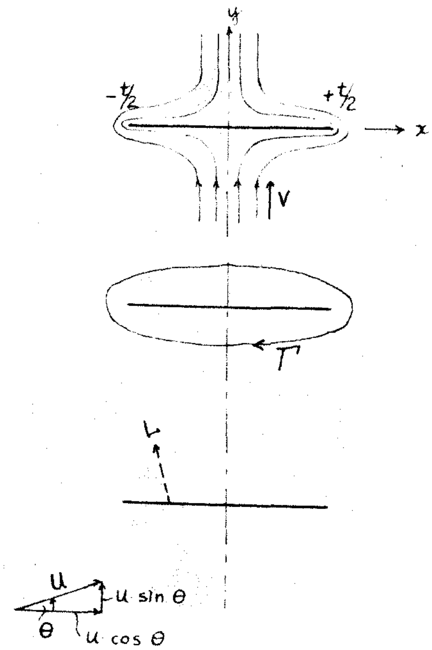
Experiments on wings have shown that a very different kind of flow takes place for very small aspect ratios than for large aspect ratios. The lift curve continues up to about 45° before stalling occurs. During this range it has a concave curvature upward rather than downward as the lifting line or lifting surface theories predict. No theoretical explanation of this effect has yet been given since it was generally supposed to be a stalling phenomenon and thus not adaptable to perfect fluid theories. The present paper shows that this curvature effect is due to the fact that the trailing vortices leave at an angle α to the plate. For the limiting case of a plate with finite span and infinite chord it is shown that the bound vorticity and induced downwash are constant across the span, and the trailing vortices leave the wing at the half-angle of attack, $\alpha = \frac{\theta}{2}$. These results are carried over into the assumptions for the analysis of the finite rectangular flat plate of very small aspect ratio. A surface distribution of vorticity over the plate is assumed, constant across the span, and varying according to the formula $\gamma = \gamma_o \sqrt{\frac{t/2 - x}{t/2 + x}}$ along the chord. Straight trailing vortices are assumed leaving the plate at an undetermined angle α. The boundary condition assumed is that

the mean value of the induced velocity along the center line of the span is equal to the normal component of the free-stream velocity. This determines the constant γ_o and thus the normal force coefficient C_N as a function of θ. The parameter α is still undetermined; however, its limits are given. For very small aspect ratios $\alpha = \frac{\theta}{2}$, for large aspect ratios it approaches θ. Winter's experiments on a wing of aspect ratio $\mathcal{R} = \frac{1}{30}$ are checked very closely by this theory assuming $\alpha = \frac{\theta}{2}$. At larger aspect ratios up to about $\mathcal{R} = 1$ the experimental curves lie between the theoretically predicted curves corresponding to $\alpha = \frac{\theta}{2}$ and $\alpha = \theta$, moving toward the latter limit at $\mathcal{R} = 1$.

A. INTRODUCTION:

The theory for flow about a wing of large aspect ratio ($AR = \frac{span}{chord} = \frac{b}{t}$) has been worked out quite completely by Prandtl and his pupils. This is the flow which is of principal interest for the main lifting surfaces of present day airplanes. However, for some unconventional airplane wings and especially for the control surfaces of dirigibles the lifting surfaces have generally a very small aspect ratio, and the present theory is inadequate to give either the magnitude or the distribution of the forces correctly. It seems advisable therefore to extend the range of validity of this theory.

Let us consider the present status of the problem: The limiting case of a wing of large aspect ratio is one of infinite aspect ratio. The flow past such a wing with a constant profile is two-dimensional and thus enormously simpler than that in the general case.
In the case of infinite span it is possible to use the method of conformal transformation to calculate the flow past airfoils of arbitrary shapes. The simplest example, which however, already gives the characteristic laws of lift is that of a flat plate, and since in this investigation we are principally concerned with the influence of aspect ratio we shall restrict ourselves to flow past a flat-plate wing. The complex potential function w for flow past an infinitely wide flat plate without circulation is easily shown to be



$$w = \phi + i\psi = -i\,V\cdot\sqrt{z^2 - \left(\frac{t}{2}\right)^2}$$

where ϕ = potential function

ψ = stream function

V = normal velocity to plate at infinity

t = chord of plate

z = $x + iy$ = complex coordinate of plate

This gives for the potential on the plate $y = 0$

$$\phi = V\sqrt{\left(\frac{t}{2}\right)^2 - x^2}$$

and for the velocity along the plate

$$\frac{\partial\phi}{\partial x} = V\cdot\frac{\frac{x}{t/2}}{\sqrt{1-\left(\frac{x}{t/2}\right)^2}}$$

This velocity is infinite at $x = \pm\frac{t}{2}$ i.e. at the leading and trailing edge of the airfoil. The infinite velocity at the leading edge will vanish if we consider a finite radius of curvature of the leading edge. In order to avoid infinite velocities at the trailing edge we must superimpose a circulation about the wing and adjust its strength until the infinite velocities at the trailing edge are cancelled. This gives a circulation

$$\Gamma = \pi V t$$

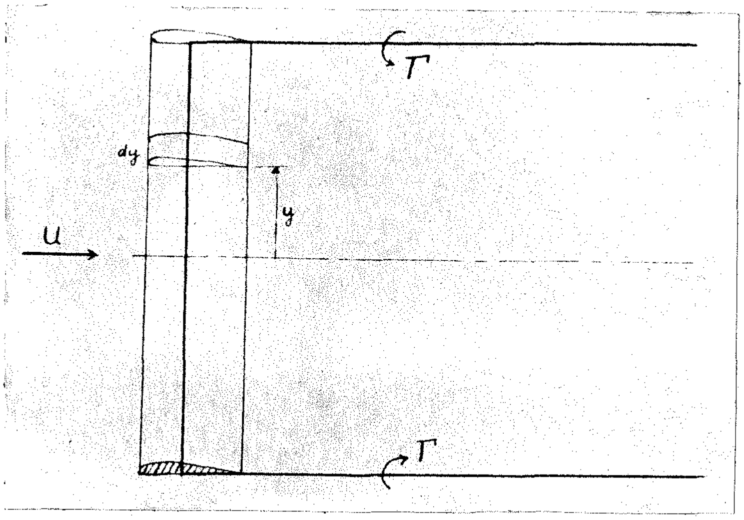
If the velocity $V = U \cdot \sin\theta$ i.e. is the normal component of the free stream velocity, then we get a resultant lift-force normal to the resultant velocity U given by

$$L = \rho U \Gamma = \rho U \cdot \pi U \sin\theta \cdot t = 2\pi \sin\theta \cdot \tfrac{1}{2} \rho U^2 \cdot t$$

and a lift-coefficient

$$C_L = \frac{L}{\frac{1}{2} \rho U^2 \cdot t} = 2\pi \sin\theta$$

The next logical extension of the theory was to consider a wing of finite but large aspect ratio. Conditions over the greater part of the wing must be similar to the two-dimensional case, i.e. we have a circulation Γ about the wing near the middle, and the velocity of the free-stream normal to the wing is approximately $V = U \cdot \sin\theta$. The circulation can be thought of as originating from a bound vortex lying in the wing. By the Helmholtz vortex theorems the vortex cannot end at the tips of the wing. It therefore separates from the wing and follows the fluid particles downstream. We have thus as the simplest equivalent of the finite wing of large aspect ratio a single horse-shoe shaped vortex system. These trailing vortices correspond physically to a flow from the lower side of the wing around the tips to the upper side due to the ex~~s~~isting pressure difference. The effect of these



trailing vortices is now to induce down-velocities w all along the wing. Thus the effective velocity normal to the wing is now not $V = U \cdot \sin\theta$ but V-w where the induced velocity w is obtained from the Biot-Savart Law.

$$w = \frac{\Gamma}{4\pi}\left[\frac{1}{b/2 + y} + \frac{1}{b/2 - y}\right]$$

The effective angle of attack of the wing is thus reduced from $\frac{V}{U}$ to $\frac{V}{U} - \frac{w}{U}$, and correspondingly the lift-coefficient is reduced to

$$C_L = 2\pi \cdot \sin\left(\theta - \frac{w}{U}\right)$$

The above assumption of a uniform vortex running across the wing is evidently not applicable anymore since the induced velocities and hence the lift now vary across the span. The next step was therefore to assume that the line vortex lying in the wing had a variable intensity $\Gamma(y)$ across the span and was thus shedding trailing vortices $\sim \frac{d\Gamma}{dy}$ continuously. This left a vortex sheet in the wake behind the wing. The intensity of the vortex distribution was then obtained as the solution of the integral equation

$$\Theta_{\text{effective}} = \Theta_{\text{geometrical}} - \frac{w}{U}$$

or

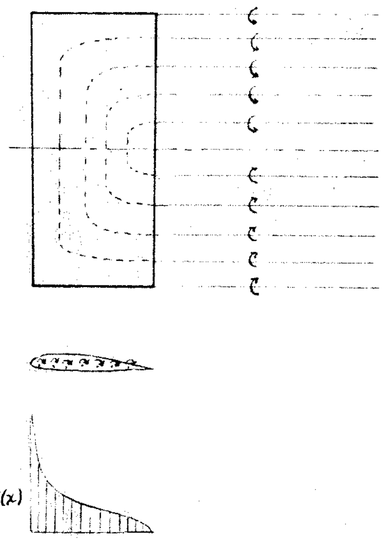
$$\frac{\Gamma(y)}{\pi \eta V t} = \Theta_{\text{geometrical}} - \frac{1}{4\pi U}\int_{-b/2}^{+b/2} \frac{d\Gamma(y')}{dy'} \cdot \frac{dy'}{y-y'}$$

The lift coefficient of a strip of width dy is given then as before as $C_L' = 2\pi \cdot \sin \theta_{eff.}$; the total lift is obtained by integrating these strips over the span.

The wing theory in the above form is ordinarily used, and for $A\!\!R > 3$ or 4 it gives quite good agreement with experiment when a multiplicative correction factor η is introduced in the formula for the lift coefficient, i.e. if we write $C_L' = 2\pi\eta \cdot \sin \theta_{eff.}$. The value of η lies ordinarily between about .85 to .95. It is a correction factor for the reduction of the circulation about the wing due to viscosity.

If we consider the approximations involved in the above theory, namely that we can replace the wing by a single lifting line it becomes evident that this cannot hold for <u>small aspect ratios</u>. It becomes necessary to consider the actual distribution of vorticity along the chord of the wing. This problem was first solved by

Birnbaum[1] for the two-dimensional case, and then extended by Blenk[2] to the wing of finite span.



Blenk's Vorticity Distribution

Blenk did not use any efficiency factor η in his calculations. In spite of this fact when he tried to compare his theoretical lifts with experiment he found that they were still too low. He ascribed this to having chosen the mean camber line of this wing of finite thickness through the center line of the profile. He found that by shifting this up to 2/3 height between the upper and lower side of the profile he could get good agreement between theory and experiment for aspect ratios down to 1 - at least for the initial tangent of the lift curve. His curves show that there is a curvature of the lift-curves which is not given by his theory. Concerning this he says "Offenbar spielt aber hier noch ein anderer in der Theorie bisher unberücksichtigt gebliebener Einfluss herein, der die Krümmung der Kurven verursacht und besonders bei kleinen Seitenverhältnissen von Bedeutung zu sein scheint. Diesen Einfluss mit den bisherigen Mitteln der Tragflügeltheorie zu erfassen, dürfte wohl nicht möglich sein, da diese im wesentlichen eine lineare Theorie ist."

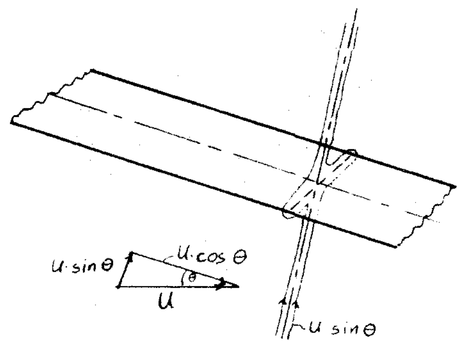
Further experiments by Flügel(3) and Zimmermann(4) and especially the extensive investigation by Winter(5) indicate that some additional influences must be entering which the theory has so far neglected. All of these investigators believed with Blenk that these additional influences could not be treated theoretically since they were thought to be essentially stalling phenomena.

It is the purpose of the present thesis to investigate this question and to try to formulate the outline of a theory which holds for the region of aspect ratios from 1 to 0.

For the wings of large aspect ratio the starting point of the analysis was the wing of infinite aspect ratio. Similarly it is suggestive to start the present investigation with the opposite limiting case, namely the wing of $AR = 0$, that is finite span and infinite chord. For very small aspect ratios, then, we would expect a flow of similar nature about the middle part of the chord with possibly some deviations as we approach the leading and trailing edges of the wing. As we approach aspect ratios of the order of 1 the theory should agree with the Prandtl-Blenk theories.

B. WING OF INFINITE CHORD - ZERO ASPECT RATIO

Consider a plate of span b and infinite chord moving through a fluid with the velocity U at an angle of attack θ By the principle of relative motion we get the same conditions if we assume the plate fixed and the fluid moving. An equivalent picture is then obtained by resolving the flow U into two components, namely

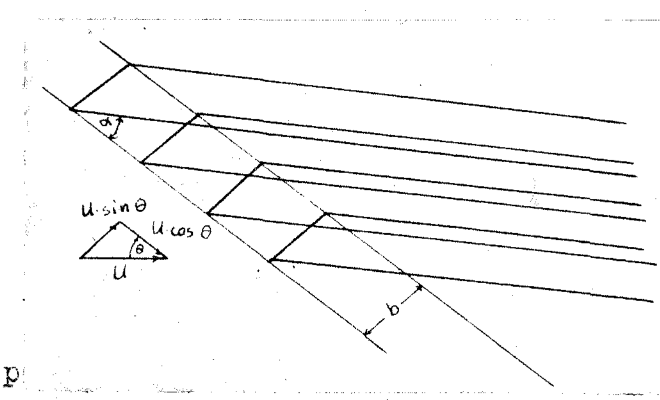


(1) A uniform rectilinear flow with velocity $U \cdot \cos\theta$ parallel to the plate, and

(2) A flow normal to the plate with velocity $w = U \sin\theta$ at minus infinity.

The normal flow (2) must be of a type such that there is no flow through the plate. The potential flow which we discussed on page 5 satisfies this condition. It gives a potential

$$\phi = \mathcal{R}_e \left\{ -i\, U \cdot \sin\theta \cdot \sqrt{z^2 - \left(\frac{b}{2}\right)^2} \right\}$$



We shall designate this flow henceforth as the "displacement flow" w_1. This flow, however, will actually not occur because the fluid separates at the sharp edge and gives rise to the formation of vortices.

The vortices formed at the sharp edge of the plate will be carried in some way down-stream with the fluid particles. It would be very difficult if not impossible to find an exact solution of this complicated motion. However, it can be assumed that a good approximation can be obtained if the vortices are replaced by a straight vortex lines lying in the two vertical planes through the edges of the plate. This flow-picture suggests that our plate is more nearly equivalent to an infinite lattice of airfoils of finite span. As we let the gap between the airfoils tend to zero, we get a distribution of bound vortices of strength γ_1 per unit length along the plate with trailing vortices leaving at some angle with respect to the plate. The angle α at which these vortices leave is determined by the condition that the vortex lines follow the fluid particles, at least as far as the components in the plane of the vortex sheet are concerned. If we now determine the strength of the vortices γ_1 by the condition that the normal component of the induced velocity due to the vortices just cancels the component U sin θ of the uniform flow through the plate, we have another type of normal flow which satisfies the boundary conditions at the plate. It is made up of a uniform flow $U \cdot \sin\theta$ perpendicular to the plate plus the normal flow due to the vortex system. We shall designate this flow henceforth as the "induced flow" w_2.

Because of the two solutions there is a degree of indeterminacy left in our problem, for theoretically there are an infinite number of combinations of the two flow types which satisfy the condition of no flow through the plate. Both types satisfy the condition that the velocity of the undisturbed flow is not changed by the presence of the plate at minus infinity. At infinity behind the plate the displacement flow does not give any finite contribution to the velocity. The pure induced type of flow gives at infinity in the wake twice the value of the induced velocity at the plate, i.e. $2U \sin \theta$ normal to the plate. For every combination of the two types of flow the flow at infinity behind the plate varies between no deflection for the pure displacement flow, to a deflection twice that immediately behind the plate for the case of pure induced type of flow.

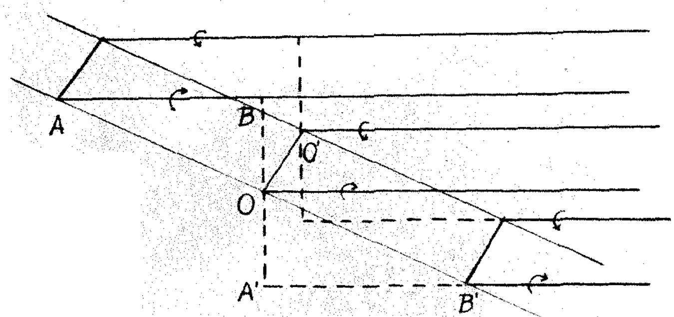
The pure displacement flow which does not result in any forces acting on the moving plate can be disregarded. We know that it physically cannot take place and in any case vortices are formed which come off the sides of the plate. In the appendix a combination between displacement and induced flow is discussed, which is characterized by the condition that the flow in the wake at infinity is parallel to the plane of the wing. This is the sort of effect one might expect from an infinite lattice. In this case the two flow types contribute equal portions to the component $U \sin \theta$ normal to the plate at minus infinity. We find as a result of the calculation however, that the resultant normal force is zero for this combined flow the same as for the pure displacement type flow. Moreover, this combination of induced-type flow and displacement flow was arrived at by the condition that the flow at infinity behind the plate be parallel to the plate. No such condition will characterize the wing of small aspect ratio even if it has a very long chord, i.e. there is a discontinuity between the wing with truly infinite chord and the wing with very large but finite chord. We are thus led to consider the pure induced type of flow which we know holds in the ordinary range of aspect ratios.

We shall consider flow of the pure induced type past a wing of very small aspect ratio ($k \doteq 0$). The wing is assumed to have a real leading and trailing edge near plus and minus infinity, so that when we come to calculate the forces we may assume the region possesses the same Bernoulli constants inside and outside the vortex wake. As far as the calculation of the down-wash in the vicinity of the middle portion of the wing is concerned, however, this is essentially the same as for the infinite wing. This analysis leads to the result that the normal force coefficient is

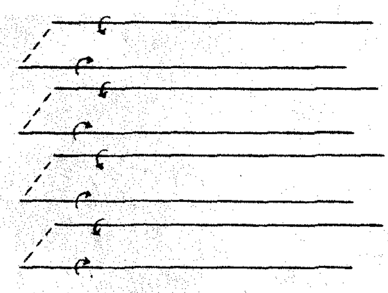
$C_N = 2 \sin^2\theta \cdot \dfrac{2}{1+\cos\theta}$. The extrapolated experimental value was given by Flügel and Winter as $C_N = 2 \sin^2\theta$. These two results do not differ very much within the working range of the angles of attack. Moreover, the curve of C_N vs. θ for the smallest value of aspect ratio (k = 1/30), on which the extrapolated value is based, is checked very closely by the present theory for the finite wing.

Let us consider now the velocity field for the vortex system corresponding to the above case, i.e. that of pure induced flow. We assume a uniform distribution of bound vortices of length b placed along the span of the plate. The strength per unit length of the plate is γ_1 . From the ends of the bound vortices straight trailing vortices pass off at an angle α to the plate. The induced velocity due to this system of vortices at any point on the plate can be written down at once, for evidently the contributions of the bound vortices normal to the plate cancel at every point. The induced velocity tangential to the plate is $+\frac{\gamma_1}{2}$ on the upper surface, and $-\frac{\gamma_1}{2}$ on the lower surface.

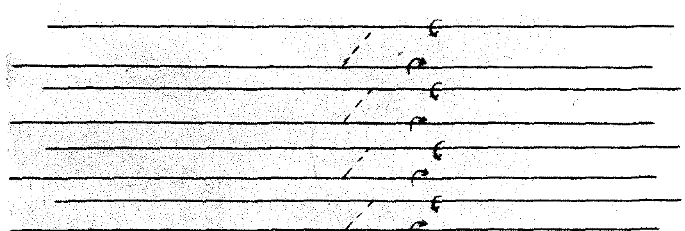
The contribution of the trailing vortices is evidently the same as if they were arranged as starting from a vertical plane through the origin, and had a strength per unit height of $\frac{\gamma_1}{\sin\alpha}$. For, the contribution normal to the wing of a vortex segment AB is exactly the same as if it were placed along A'B'. Thus we get the equivalent trailing vortex system shown, consisting of two semi-infinite walls of vortices. The down-wash due to this semi-infinite system of vortices at any point between them is just half that of a doubly infinite system.



Actual Vortex System



Equivalent Vortex System



Doubly Infinite Wall of Vortices

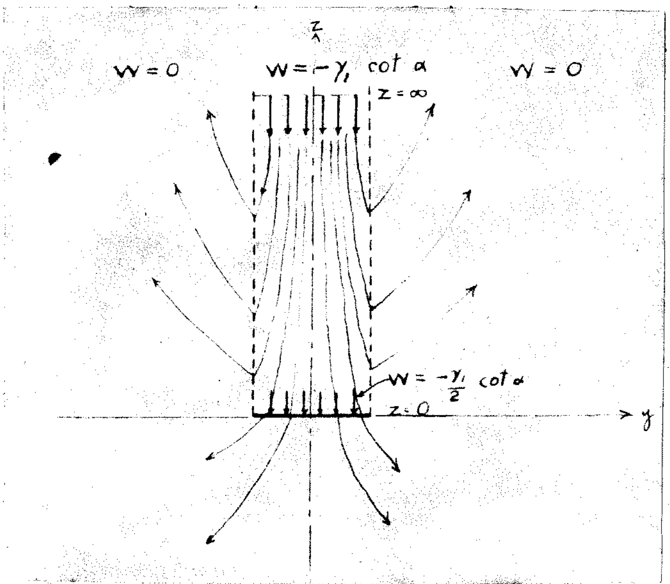
This latter has a uniform down-wash $\frac{\gamma_1}{\sin\alpha}$ everywhere between the two walls of vortices, and zero outside. Thus the trailing vortex system of our wing has a uniform down-wash at every point of the span. This down-wash is perpendicular to the vortices, and thus the component of the induced velocity along the plate is $u = \frac{\gamma_1}{2 \cdot \sin\alpha} \cdot \sin\alpha = \frac{\gamma_1}{2}$, the component normal to the plate is $w = \frac{\gamma_1}{2} \cdot \cot\alpha$

The velocity field due to the above vortex system in a cross-section perpendicular to the wing is then as shown below. The calculations in the appendix on page 65 give the following results for the induced velocities.

$$w_i = \frac{-\gamma_1 \cdot \cot\alpha}{2\pi} \cdot \left[\pi + \tan^{-1} \frac{z}{b/2 + y} + \tan^{-1} \frac{z}{b/2 - y} \right] \quad \text{for } |y| < \frac{b}{2}$$

$$w_o = \frac{-\gamma_1 \cdot \cot\alpha}{2\pi} \cdot \left[\tan^{-1} \frac{z}{|y| + b/2} - \tan^{-1} \frac{z}{|y| - b/2} \right] \quad \text{for } |y| > \frac{b}{2}$$

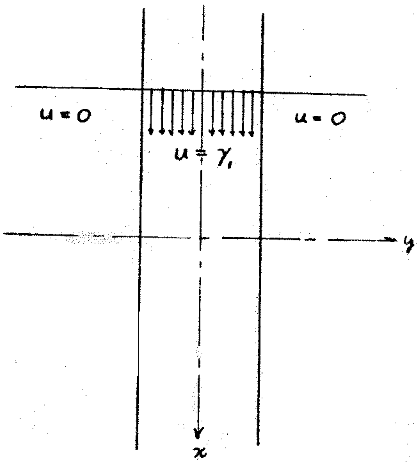
$$v = \frac{\gamma_1 \cdot \cot\alpha}{2\pi} \cdot \log \frac{\sqrt{(b/2 - y)^2 + z^2}}{\sqrt{(b/2 + y)^2 + z^2}}$$



Thus the velocity normal to the plate is

at $z = \infty$	$\lvert y\rvert < \frac{b}{2}$	$w = \gamma_1 \cdot \cot\alpha$
	$\lvert y\rvert > \frac{b}{2}$	$w = 0$
at $z = 0$	$\lvert y\rvert < \frac{b}{2}$	$w = \frac{\gamma_1 \cdot \cot\alpha}{2}$
	$\lvert y\rvert > \frac{b}{2}$	$w = 0$

In any cross-section parallel to the plane of the wing we have the following velocity field



for z > 0

$$|y| < b/2 \qquad u = \gamma_1$$

$$|y| > b/2 \qquad u = 0$$

for z < 0

$$|y| < b/2 \qquad u = 0$$

$$|y| > b/2 \qquad u = 0$$

Now if in order to satisfy our boundary condition of no flow through the plate we superimpose a uniform flow $U \cdot \sin\theta$ normal to the wing we have

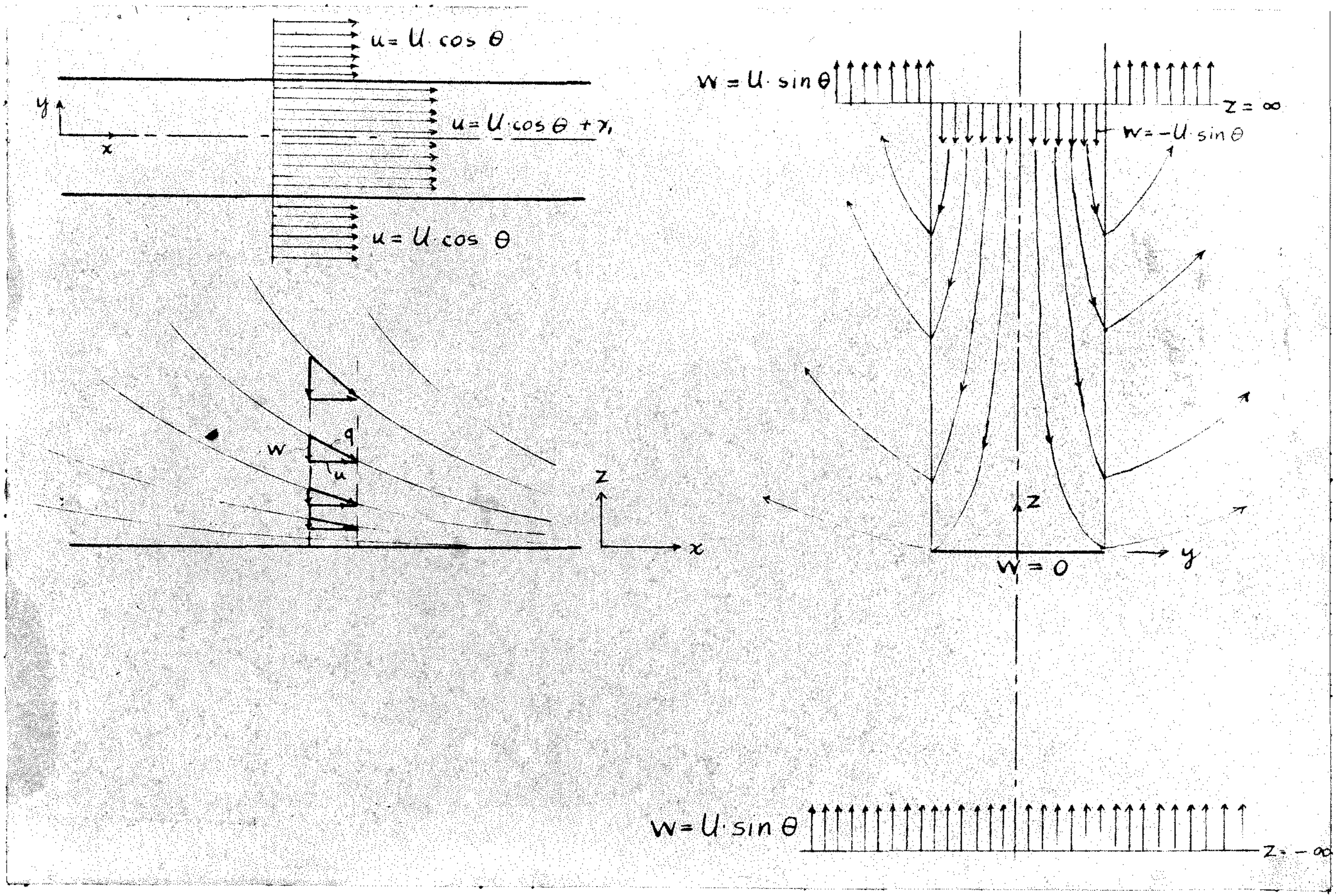
$$U \cdot \sin\theta = \frac{\gamma_1}{2} \cdot \cot\alpha$$

or

$$\gamma_1 = 2U \cdot \sin\theta \cdot \tan\alpha$$

There is, of course, also another component of the flow U parallel to the wing, namely the uniform flow $U \cos\theta$. The resultant velocity field of vortex flow and uniform flow U at an angle θ gives us then the resultant flow picture shown below.

In the shadow of the plate the u-component of the velocity is constant and equal to $U \cdot \cos\theta + \gamma_1$. The w-component of the velocity varies from $w = -U \cdot \sin\theta$ at $z = \infty$ to $w = 0$ at $z = 0$. There is thus a constant influx of fluid from $z = \infty$ which flows out over the sides of the vortex sheet as shown below.



The angle α is determined by the condition that the vortex lines follow the stream-lines. (This does not apply to the components perpendicular to the vortex sheet. These components are small except in the immediate neighbourhood of the plate.) In the wake near to the plate we find $w \doteq 0$ inside the vortex system and w = U sin outside. The vortices thus move with a mean velocity $\tilde{w} = U \cdot \frac{\sin\theta}{2}$. At a great distance behind the plate the w-velocities inside and outside the vortex sheet are equal and opposite and thus the mean w-velocity of the vortices is zero. The mean w-velocity at any point is obtained as

$$\bar{w} = \frac{w_i + w_o}{2} + U \sin\theta = \frac{-\gamma_1 \cdot \cot\alpha}{2\pi} \cdot \left[\frac{\pi + 2 \tan^{-1}\frac{z}{b}}{2} \right] + U \cdot \sin\theta$$

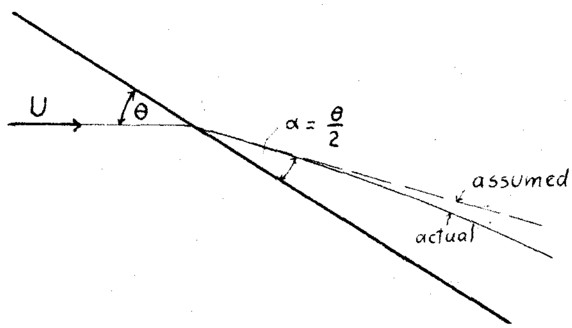
$$\bar{w} = -\frac{U \cdot \sin\theta}{2} \left[1 + \frac{2}{\pi} \cdot \tan^{-1}\frac{z}{b} \right] + U \cdot \sin\theta$$

$$\bar{w} = \frac{U \cdot \sin\theta}{2} \cdot \left[1 - \frac{2}{\pi} \cdot \tan^{-1}\frac{z}{b} \right]$$

Similarly the velocity $u = U \cos\theta + \gamma_1$ inside the vortex system, and $u = U \cdot \cos\theta$ outside. This holds true for all distances z above the plate. Therefore the mean velocity is $\bar{u} = U \cdot \cos\theta + \frac{\gamma_1}{2}$. The mean angle α at which the vortices leave is thus

$$\tan\alpha = \frac{\bar{w}}{\bar{u}} = \frac{U \cdot \frac{\sin\theta}{2} \cdot \left[1 - \frac{2}{\pi} \cdot \tan^{-1}\frac{z}{b} \right]}{U \cdot \cos\theta + U \cdot \sin\theta \cdot \tan\alpha}$$

This gives at $z = 0$ an inclination $\alpha = \frac{\theta}{2}$ and at $z = \infty$, $\alpha = 0$. The trailing vortices thus are not actually straight lines but lie along curved paths. This contradiction to our original assumptions arises out of the fact that we have in part violated the Helmholtz condition that the vortices follow the fluid particles.



It will be remembered that the Prandtl wing theory which assumes the trailing vortices to be carried along the direction of the undisturbed flow similarly violates the Helmholtz conditions. It thus seems justifiable to retain the assumption that our trailing vortices follow straight lines at some effective inclination α with respect to the plate. For this effective inclination we shall choose $\alpha = \frac{\theta}{2}$, the inclination near z = 0 since the vortices near the plate are the most effective in inducing the normal velocity at the plate.

The normal force acting on the wing can now be determined from the formula for the force on a vortex. We consider our wing of infinite chord as the limiting case of the wing with finite chord and so this formula is applicable.* It gives

$$N = (U \cdot \cos\theta + u_p) \cdot \gamma_1 \cdot b$$

$$N = \rho \left(U \cdot \cos\theta + U - U \cdot \cos\theta\right) \cdot 2U \cdot (1 - \cos\theta) \cdot b$$

$$C_N = \frac{N}{\frac{1}{2}\rho U^2 \cdot b} = 4(1 - \cos\theta)$$

$$C_N = 2 \cdot \sin^2\theta \cdot \frac{2}{1 + \cos\theta}$$

The important results of the analysis of the wing with infinite chord k ≐ 0 are the following:

(1) The bound vortices have a constant strength across the span. Such a distribution gives a constant down-wash across the span. When considering finite

* If we consider the wing as truly infinite, the Bernoulli constants within and outside the vortex wake are different. As a result zero normal force would again be obtained.

wings of very small aspect ratio it should therefore be a very good approximation to assume a uniform strength of the bound vortices and to calculate the induced velocities say at the center of the span, considering this as a constant over the span.

(2) The angle at which the vortices leave is approximately $\alpha = \frac{\theta}{2}$

(3) The normal force coefficient is

$$C_N = 2 \sin^2\theta \cdot \frac{2}{1 + \cos\theta}$$

C. RECTANGULAR WING OF FINITE CHORD - SMALL ASPECT RATIO:

1. Assumptions and Method of Calculation:

The wing of finite chord with small aspect ratio will now be treated upon making certain assumptions suggested by the analysis of the wing of infinite chord. These assumptions are:

(1) The lift is constant across the span. This corresponds to bound vortices of constant strength across the span.

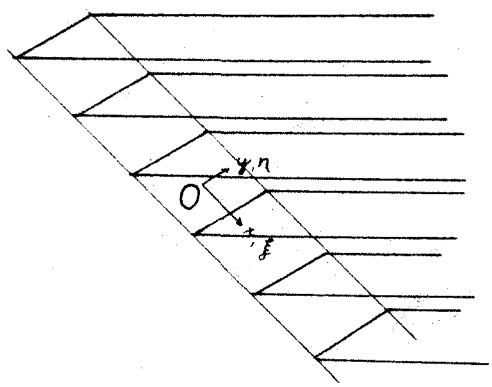
(2) The down-wash is constant across the span. Therefore we shall assume it to be equal to the value at the middle.

(3) There is no displacement flow. We assume the induced velocity normal to the wing to cancel the normal component of the free stream-velocity, i.e. $U \sin \theta$

(4) The vortices leave at some angle α to the wing.

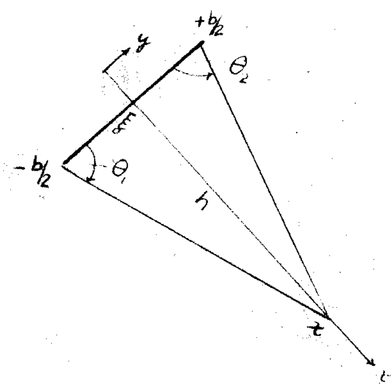
Our first problem is to calculate the induced velocities normal to the wing along the center line of the wing $y = 0$ at any point $\underline{x}$ along the chord. We have therefore again a similar system of horse-shoe vortices as for the wing of infinite chord except that now the intensity of the distribution of the vortices along the chord is variable and is given by $\gamma(\xi)$.

Let x and y be the variables along the chord and span respectively, and ξ and η the current variables in the same directions. Let us place the origin O in the center of the wing. Then we obtain the induced velocities normal to the wing w_{z_1} and w_{z_2} due to the bound and trailing vortices respectively as follows:



2. Normal Component of Induced Velocity due to Bound Vortices

A single bound vortex gives us



$$w_z = \frac{\gamma}{4\pi} \cdot \frac{1}{h} \cdot \left(\cos\theta_1 + \cos\theta_2\right)$$

$$w_z = \frac{\gamma}{4\pi} \cdot \frac{1}{(x-\xi)} \cdot \frac{2\,\frac{b}{2}}{\sqrt{\left(\frac{b}{2}\right)^2 + (x-\xi)^2}}$$

Let the origin O be at the middle of the wing, then we have to integrate the effect of such vortices from $\xi = -\frac{t}{2}$ to $\xi = +\frac{t}{2}$.

$$w_{z_1} = \frac{1}{2\pi}\int_{-\frac{t}{2}}^{+\frac{t}{2}} \frac{\gamma(\xi)}{x-\xi}\,\frac{b/2 \cdot d\xi}{\sqrt{\left(\frac{b}{2}\right)^2 + (x-\xi)^2}}$$

Introducing dimensionless coordinates $\frac{x}{t/2} \longrightarrow x$, $\frac{\xi}{t/2} \longrightarrow \xi$

$\frac{b}{t} \longrightarrow \mathcal{R}$ = *aspect ratio*

$$\boxed{w_{z_1} = \frac{\mathcal{R}}{2\pi} \int_{-1}^{+1} \frac{\gamma(\xi)}{(x-\xi)} \, \frac{d\xi}{\sqrt{\mathcal{R}^2 + (x-\xi)^2}}}$$

3. <u>Normal Component of Induced Velocity due to Trailing Vortices:</u>

A single trailing vortex gives

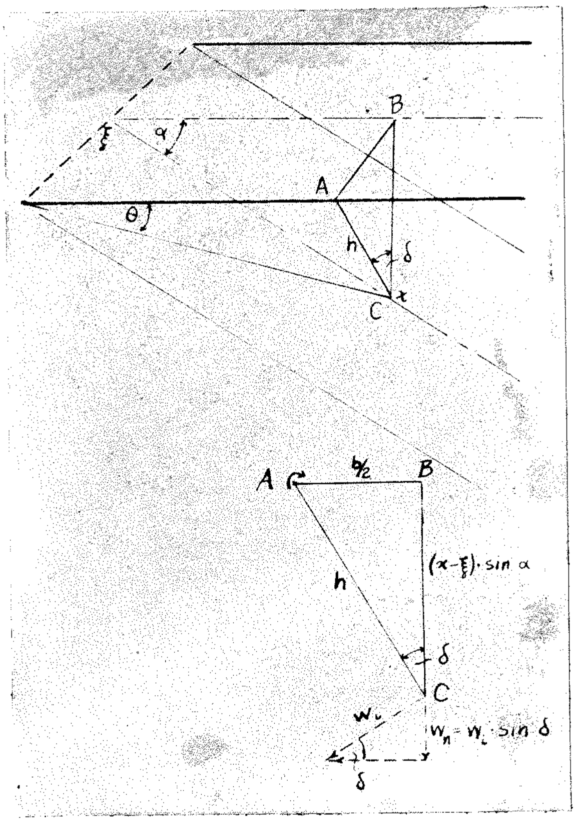
$$w_z = (w_i \cdot \sin\delta) \cdot \cos\alpha$$

$$= \frac{\gamma}{4\pi} \cdot \frac{1}{h} \cdot (\cos\theta + 1) \cdot \sin\delta \cdot \cos\alpha$$

where $\sin\delta = \frac{b/2}{h}$

$$h^2 = (b/2)^2 + (x-\xi)^2 \cdot \sin^2\alpha$$

$$\cos\theta_1 = \frac{(x-\xi) \cdot \cos\alpha}{\sqrt{(b/2)^2 + (x-\xi)^2}}$$



As the integral of the trailing vortices from both ends of the span we get:

$$w_{z_2} = \frac{\cos\alpha}{2\pi} \int_{-\frac{t}{2}}^{+\frac{t}{2}} \frac{\gamma(\xi) \cdot d\xi}{(b/2)^2 + (x-\xi)^2 \cdot \sin^2\alpha} \cdot \left[\frac{(x-\xi) \cdot \cos\alpha}{\sqrt{(b/2)^2 + (x-\xi)^2}} + 1 \right]$$

And again introducing dimensionless coordinates

$$\boxed{w_{z_2} = \frac{k \cdot \cos\alpha}{2\pi} \int_{-1}^{+1} \frac{\gamma(\xi) \cdot d\xi}{k^2 + (x-\xi)^2 \cdot \sin^2\alpha} \cdot \left[\frac{(x-\xi) \cdot \cos\alpha}{\sqrt{k^2 + (x-\xi)^2}} + 1 \right]}$$

4. Derivation and Solution of Integral Equation:

Expressing the fact that the induced velocity must equal the normal component of the free stream-velocity we get an integral equation for the determination of $\gamma(\xi)$:

$$w_{z_1}(x) + w_{z_2}(x) = U \cdot \sin\theta$$

or

$$\frac{k}{2\pi} \int_{-1}^{+1} \frac{\gamma(\xi)}{(x-\xi)} \frac{d\xi}{\sqrt{k^2 + (x-\xi)^2}} + \frac{k}{2\pi} \int_{-1}^{+1} \frac{\gamma(\xi) \cdot \cos\alpha \cdot d\xi}{k^2 + (x-\xi)^2 \cdot \sin^2\alpha} + \frac{k}{2\pi} \int_{-1}^{+1} \frac{\gamma(\xi) \cdot \cos^2\alpha \cdot d\xi \cdot (x-\xi)}{\left[k^2 + (x-\xi)^2 \sin^2\alpha\right] \sqrt{k^2 + (x-\xi)^2}}$$

$$= U \cdot \sin\theta$$

In order to get the pressure distribution over the wing it is necessary to solve this integral equation for $\gamma(\xi)$. In the present investigation we shall restrict ourselves to finding the total force alone. This makes it necessary to satisfy the integral equation only in the mean. We assume some reasonable distribution for the vorticity over the chord such as $\gamma(\xi) = \gamma_0 \cdot \sqrt{\frac{t/2 - \xi}{t/2 + \xi}}$ and then use the integral equation to determine the constant γ_0.

Expressing the fact that the mean value of the induced velocity over the chord is equal to the normal component of the free-stream velocity we get

$$\frac{1}{2}\int_{-1}^{+1}\left(w_{z_1}(x) + w_{z_2}(x)\right)\cdot dx = U\cdot \sin\theta$$

or interchanging the order of integration of x and ξ

$$\frac{k}{4\pi}\int_{-1}^{+1}\gamma(\xi)\cdot d\xi\underbrace{\int_{-1}^{+1}\frac{dx}{(x-\xi)\sqrt{k^2+(x-\xi)^2}}}_{I_1(\xi)} + \frac{k}{4\pi}\int_{-1}^{+1}\gamma(\xi)\cdot\cos\alpha\cdot d\xi\underbrace{\int_{-1}^{+1}\frac{dx}{k^2+(x-\xi)^2\cdot\sin^2\alpha}}_{I_2(\xi)}$$

$$+\frac{k}{4\pi}\int_{-1}^{+1}\gamma(\xi)\cdot\cos^2\alpha\cdot d\xi\underbrace{\int_{-1}^{+1}\frac{(x-\xi)\cdot dx}{\left[k^2+(x-\xi)^2\cdot\sin^2\alpha\right]\sqrt{k^2+(x-\xi)^2}}}_{I_3(\xi)} = U\cdot\sin\theta$$

where

$$I_1(\xi)=\int_{-1}^{+1}\frac{dx}{(x-\xi)\sqrt{k^2+(x-\xi)^2}} = \frac{1}{k}\log\left[\frac{k+\sqrt{k^2+(1+\xi)^2}}{k+\sqrt{k^2+(1-\xi)^2}}\cdot\frac{1-\xi}{1+\xi}\right]$$

$$I_2(\xi)=\int_{-1}^{+1}\frac{dx}{k^2+(x-\xi)^2\cdot\sin^2\alpha} = \frac{1}{k\cdot\sin\alpha}\left[\tan^{-1}\left\{\frac{(1+\xi)\cdot\sin\alpha}{k}\right\}+\tan^{-1}\left\{\frac{(1-\xi)\cdot\sin\alpha}{k}\right\}\right]$$

$$I_3(\xi)=\int_{-1}^{+1}\frac{(x-\xi)\cdot dx}{\left[k^2+(x-\xi)^2\cdot\sin^2\alpha\right]\sqrt{k^2+(x-\xi)^2}} = \frac{-1}{k\cdot\sin\alpha\cdot\cos\alpha}\left[\tan^{-1}\left\{\frac{\tan\alpha}{k}\sqrt{k^2+(1+\xi)^2}\right\}\right.$$

$$\left.-\tan^{-1}\left\{\frac{\tan\alpha}{k}\sqrt{k^2+(1-\xi)^2}\right\}\right]$$

Now substituting these values in the above equation and letting $\gamma = \gamma_0 \sqrt{\dfrac{1-\xi}{1+\xi}}$

we get

$$\underbrace{\frac{\gamma_0}{4\pi}\int_{-1}^{+1}\sqrt{\frac{1-\xi}{1+\xi}}\cdot\left\{\log\frac{k+\sqrt{k^2+(1+\xi)^2}}{k+\sqrt{k^2+(1-\xi)^2}} - \log\frac{(1+\xi)}{(1-\xi)}\right\}\cdot d\xi}_{F_1(k)}$$

$$+\ \frac{\gamma_0\cdot\cos\alpha}{4\pi\cdot\sin\alpha}\underbrace{\int_{-1}^{+1}\sqrt{\frac{(1-\xi)}{(1+\xi)}}\cdot\left\{\tan^{-1}\left[\frac{(1+\xi)\cdot\sin\alpha}{k}\right]+\tan^{-1}\left[\frac{(1-\xi)\cdot\sin\alpha}{k}\right]\right\}\cdot d\xi}_{F_2(k,\alpha)}$$

$$-\ \frac{\gamma_0\cdot\cos\alpha}{4\pi\cdot\sin\alpha}\underbrace{\int_{-1}^{+1}\sqrt{\frac{1-\xi}{1+\xi}}\left\{\tan^{-1}\left[\frac{\tan\alpha}{k}\sqrt{k^2+(1+\xi)^2}\right]-\tan^{-1}\left[\frac{\tan\alpha}{k}\sqrt{k^2+(1-\xi)^2}\right]\right\}\cdot d\xi}_{F_3(k,\alpha)}$$

$$= \ U\cdot\sin\theta$$

i.e. solving for γ_0 :

$$\gamma_0 = \frac{4\pi\cdot U\cdot\sin\theta}{F_1(k) + \cot\alpha\cdot F_2(k,\alpha) - \cot\alpha\cdot F_3(k,\alpha)}$$

The problem is thus reduced to solving the three integrals $F_1(k)$, $F_2(k,\alpha)$ and $F_3(k,\alpha)$.

5. <u>Discussion of $F_1(k)$</u> $$F_1(k) = \int_{-1}^{+1} \sqrt{\frac{1-\xi}{1+\xi}} \left\{ \log \frac{k+\sqrt{k^2+(1+\xi)^2}}{k+\sqrt{k^2+(1-\xi)^2}} - \log \frac{1+\xi}{1-\xi} \right\} d\xi$$

The second of the above integrals is integrable by elementary methods. However, the first cannot be integrated by any of the elementary methods nor does it adapt itself to solution by contour integration or by a differential equation. Some method of expansion was thus the only alternative. The first attempt was to expand the radical in the domains $(1 \pm \xi) < k$ and $(1 \pm \xi) > k$ and then expand the logarithm and integrate. This was carried out. However, the result was very long and the convergence of the series rather doubtful. A better method of expansion was found to be the following: As a first approximation since $k \ll 1$ we can write (letting $u = 1 + \xi$)

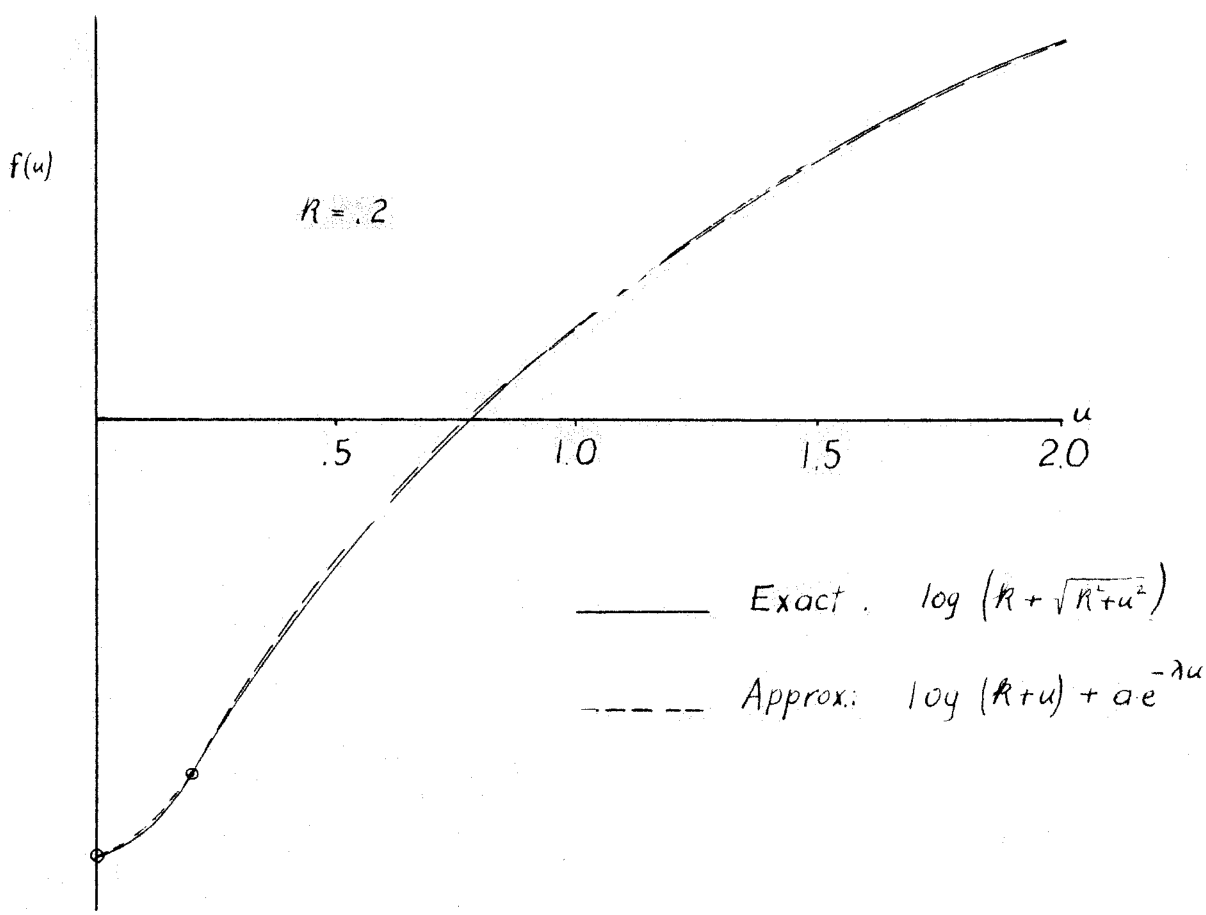
$$\log\left\{k + \sqrt{k^2+u^2}\right\} \doteq \log\left\{k+u\right\}$$

This approximation is good if $k \ll u$ over most of the domain, but it does not hold near $u = 0$ i.e. $\xi = -1$ (the leading edge). Since the weight function $\gamma(\xi)$ becomes infinite in that vicinity, however, it is necessary to have a good approximation there. A better approximation can be obtained by letting

$$\log\left\{k + \sqrt{k^2+u^2}\right\} = \log\left(k+u\right) + a \cdot e^{-\lambda u}$$

where constants a and λ are determined to fit the difference curve of $\left[\log\left\{k+\sqrt{k^2+u^2}\right\} - \log\left(k+u\right)\right]$ at $u = 0$ and $u = k$. The exact and approximate formulae are plotted up below for the case k = .2 and show a very close fit indeed. It was found possible to integrate the new functions. The first came out in terms of elementary functions, the second in terms of the Bessel function with imaginary argument of the order of 1, $I_1(\lambda)$. The result of the analysis which is given in the appendix on page 77 is:

$$\boxed{F_1(k) = \frac{2\pi k}{k+2} + \frac{4\pi}{k+2}\cdot\sqrt{\frac{k}{k+2}}\;\frac{\sqrt{\frac{k}{k+2}}+2}{\left(\sqrt{\frac{k}{k+2}}+1\right)^2} + 4.35\,e^{-\frac{1.302}{k}}\cdot I_1\!\left(\frac{1.302}{k}\right)}$$



6. Discussion of $F_2(k,\alpha)$

$$F_2(k,\alpha) = \int_{-1}^{+1} \sqrt{\frac{1-\xi}{1+\xi}} \left\{ \tan^{-1}\left[\frac{(1+\xi)\cdot \sin\alpha}{k}\right] + \tan^{-1}\left[\frac{(1-\xi)\cdot \sin\alpha}{k}\right] \right\} \cdot d\xi$$

This integral can be evaluated exactly using the method of contour integration as shown in the appendix on page 81 . The result is

$$F_2(k,\alpha) = 2\pi \cdot \tan^{-1}\left\{\frac{2\sin\alpha}{k}\right\} - 4\pi \cdot \tan^{-1}\left\{ \frac{\sin\left(\frac{1}{2}\tan^{-1}\left[\frac{2\sin\alpha}{k}\right]\right)}{\sqrt[4]{1+\left(\frac{2\cdot\sin\alpha}{k}\right)^2} + \cos\left(\frac{1}{2}\tan^{-1}\left[\frac{2\sin\alpha}{k}\right]\right)} \right\}$$

This expression can be reduced so as to involve only one parameter, namely $\mu = \frac{2\sin\alpha}{k}$

Then

$$\boxed{F_2(k,\alpha) = 2\pi \cdot \tan^{-1}\mu - 4\pi \cdot \tan^{-1}\left\{ \frac{\sin\left(\frac{1}{2}\tan^{-1}\mu\right)}{\sqrt[4]{1+\mu^2} + \cos\left(\frac{1}{2}\tan^{-1}\mu\right)} \right\}}$$

7. Discussion of $F_3(k,\alpha)$.

$$F_3(k,\alpha) = \int_{-1}^{+1} \sqrt{\frac{1-\xi}{1+\xi}} \left\{ \tan^{-1}\left[\frac{\tan\alpha}{k}\sqrt{k^2+(1+\xi)^2}\right] - \tan^{-1}\left[\frac{\tan\alpha}{k}\sqrt{k^2+(1-\xi)^2}\right] \right\} d\xi$$

This integral like the first one cannot be exactly calculated by any of the conventional methods. However, it is again adaptable to the same method of expansion as $F_1(k)$ namely we write (letting $(1+\xi) = u$)

$$\tan^{-1}\left[\frac{\tan\alpha}{k}\sqrt{k^2+u^2}\right] = \tan^{-1}\left[\frac{\tan\alpha}{k}\cdot u\right] + a\, e^{-\lambda u}$$

the constants α and λ being again adjusted to fit the curves at u = 0 and u = k. The exact and approximate formulae again showed very close agreement when plotted for the case k = .2. The term involving $\tan^{-1}\left[\frac{\tan\alpha}{k}\cdot u\right]$ could again be evaluated by the method of contour integration similarly to $F_2(k)$. The term involving $ae^{-\lambda u}$ again came out in terms of the Bessel function $I_1(\lambda)$. The result of the analysis shown in the appendix on page 84 is

$$F_3(k,\alpha) = 4\pi\cdot\frac{k}{2\cdot\tan\alpha}\left[1 - \frac{\cos\left(\frac{1}{2}\tan^{-1}\frac{2\tan\alpha}{k}\right) + \frac{2\tan\alpha}{k}\cdot\sin\left(\frac{1}{2}\tan^{-1}\frac{2\tan\alpha}{k}\right)}{\sqrt[4]{1+\left(\frac{2\tan\alpha}{k}\right)^2}}\right] + 2\pi\alpha\cdot e^{-\lambda}\cdot I_1(\lambda)$$

where

$$\lambda = -\frac{1}{k}\log\left\{\frac{\tan^{-1}(\sqrt{2}\tan\alpha)}{\alpha} - 1\right\}$$

The first half of this can be reduced in terms of the parameter $\nu = \frac{2\tan\alpha}{k}$. Then

$$\boxed{F_3(k,\alpha) = 4\pi\left[\frac{1}{\nu} - \frac{1}{\nu}\frac{\cos\left(\frac{1}{2}\tan^{-1}\nu\right)}{\sqrt[4]{1+\nu^2}} - \frac{\sin\left(\frac{1}{2}\tan^{-1}\nu\right)}{\sqrt[4]{1+\nu^2}}\right] + 2\pi\alpha\cdot e^{-\lambda}\cdot I_1(\lambda)}$$

Now

$$\gamma_0 = \frac{4\pi U\cdot\sin\theta}{F_1(k) + \cot\alpha\cdot F_2(k,\alpha) - \cot\alpha\cdot F_3(k,\alpha)}$$

Since each term in the numerator and the denominator involves the factor 2π we shall divide through by it and obtain the new expressions letting

$$\frac{F_1(k)}{2\pi} = F_1'(k) \qquad \frac{F_2(k,\alpha)}{2\pi} = A(\mu) \qquad \frac{F_3(k,\alpha)}{2\pi} = -B(\nu) + \alpha \cdot C(\lambda)$$

where

$$B(\nu) = \left[-\frac{2}{\nu} + \frac{2}{\nu}\,\frac{\cos\left(\frac{1}{2}\tan^{-1}\nu\right)}{\sqrt[4]{1+\nu^2}} + \frac{2}{\nu}\,\frac{\sin\left(\frac{1}{2}\tan^{-1}\nu\right)}{\sqrt[4]{1+\nu^2}} \right]$$

$$C(\lambda) = e^{-\lambda} \cdot I_1(\lambda)$$

then

$$\boxed{\gamma_0 = \frac{2\,U \cdot \sin\theta}{F_1'(k) + \cot\alpha \cdot \left[A(\mu) + B(\nu) - \alpha \cdot C(\lambda)\right]}}$$

8. Calculation of Induced Velocity Tangential to Plate

In the ordinary wing theory which assumes small angles of attack the effect of the induced velocity tangential to the plate is neglected. Since the angles of attack which wings of small aspect ratio reach before stalling may run up to 45^0 it is necessary to consider this velocity in the calculation of the forces acting. The bound vortices induce a velocity tangential to the plate which is $+\frac{\gamma(x)}{2}$ at the top of the wing and $-\frac{\gamma(x)}{2}$ at the bottom. Their mean value at the wing is therefore zero. The trailing vortices, however, induce a velocity which has a component along the wing equal to $w_T = w_i \cdot \sin\alpha = w_z \cdot \tan\alpha$. The mean value of $\overline{w_T}$ over the chord is then given as

$$\overline{w_T} = \frac{\gamma_0}{4\pi}\left[F_2(k,\alpha) - F_3(k,\alpha)\right] = \frac{\gamma_0}{2}\left[A(\mu) + B(\nu) - \alpha \cdot C(\lambda)\right]$$

and since

$$\frac{\gamma_0}{2}\left[A(\mu) + B(\nu) - \alpha \cdot C(\lambda)\right] = U \cdot \sin\theta \cdot \tan\alpha - \frac{F_1'(k) \cdot \tan\alpha}{2} \cdot \gamma_0$$

$$\boxed{\bar{w}_T = U \cdot \sin\theta \cdot \tan\alpha - \frac{F_1'(k) \cdot \tan\alpha}{2} \cdot \gamma_0}$$

9. <u>Calculation of Normal Force</u>:

By applying the formula for the force on a vortex we can calculate the normal force <u>N</u> acting on the wing as

$$N = \rho \cdot \left(U \cdot \cos\theta + \bar{w}_T\right) \cdot \Gamma \cdot b$$

where

$$\Gamma = \int_{-t/2}^{t/2} \gamma(\xi) \cdot d\xi = \gamma_0 \int_{-t/2}^{+t/2} \sqrt{\frac{t/2 - \xi}{t/2 + \xi}}\, d\xi = \frac{\pi}{2} \cdot \gamma_0 \cdot t$$

$$w_T = U \cdot \sin\theta \cdot \tan\alpha - \frac{F_1'(k) \cdot \tan\alpha}{2} \cdot \gamma_0$$

$$\gamma_0 = \frac{2\, U \cdot \sin\theta}{F_1'(k) + \cot\alpha \cdot \left[A(\mu) + B(\nu) - \alpha \cdot C(\lambda)\right]}$$

$$\therefore N = \rho\left(U \cdot \cos\theta + U \cdot \sin\theta \cdot \tan\alpha - U \cdot \frac{F_1'(k) \cdot \tan\alpha}{2} \cdot \frac{\gamma_0}{U}\right) \cdot U \frac{\gamma_0}{U} \cdot \frac{\pi}{2} \cdot bt$$

$$C_N = \frac{N}{\frac{1}{2}\rho U^2 \cdot bt}$$

$$C_N = \pi \cdot \frac{\gamma_0}{U}\left[\cos\theta + \sin\theta\cdot\tan\alpha - \frac{F_1'(k)\cdot\tan\alpha}{2}\cdot\frac{\gamma_0}{U}\right]$$

If $\alpha = \frac{\theta}{2}$ $\qquad \tan\alpha = \dfrac{(1-\cos\theta)}{\sin\theta}$

$$\cos\theta + \tan\alpha\cdot\sin\theta = \cos\theta + \frac{(1-\cos\theta)}{\sin\theta}\cdot\sin\theta = 1$$

$$C_N = \pi\frac{\gamma_0}{U}\cdot\left[1 - \frac{\gamma_0}{U}\cdot\frac{F_1'(k)\cdot\tan\alpha}{2}\right]$$

If $\alpha = \theta$ $\qquad \cos\theta + \sin\theta\cdot\tan\alpha = \dfrac{\cos^2\theta + \sin^2\theta}{\cos\theta} = \dfrac{1}{\cos\theta}$

$$C_N = \pi\frac{\gamma_0}{U}\left[\frac{1}{\cos\theta} - \frac{\gamma_0}{U}\cdot\frac{F_1'(k)\cdot\tan\theta}{2}\right]$$

The term in $\left(\frac{\gamma_0}{U}\right)^2$ is small for small aspect ratios. Its value is less than 10% for $k < 1$ and $\theta < 40°$.

The angle α at which the vortices leave the plate is so far undetermined. We have found that for very small aspect ratios $k \doteq 0$ they leave at the half angle of attack, that is, $\alpha = \frac{\theta}{2}$. For large aspect ratios on the other hand, they leave at the full angle of attack, i.e. $\alpha = \theta$. For the intermediate range we can thus fix the limits $\frac{\theta}{2} \leq \alpha \leq \theta$. The true angle at which they leave is really determined by the Helmholtz vortex law that vortices follow the fluid particles. The application of this law involves, however, considerable mathematical difficulties.

10. Summary of Formulae and Auxiliary Graphs for Calculation.

Summing up the formulae for the calculation of the normal force coefficients we have:

$$\alpha = \alpha \qquad C_N = \pi \cdot \frac{\gamma_0}{U} \cdot \left[\cos\theta + \sin\theta \cdot \tan\alpha - \frac{F_1'(\kappa)\cdot \tan\alpha}{2} \cdot \frac{\gamma_0}{U} \right]$$

$$\alpha = \frac{\theta}{2} \qquad C_N = \pi \cdot \frac{\gamma_0}{U} \cdot \left[1 - \frac{\gamma_0}{U} \cdot \frac{F_1'(\kappa)\cdot \tan\alpha}{2} \right]$$

$$\alpha = \theta \qquad C_N = \pi \cdot \frac{\gamma_0}{U} \cdot \left[\frac{1}{\cos\theta} - \frac{\gamma_0}{U} \cdot \frac{F_1'(\kappa)\cdot \tan\alpha}{2} \right]$$

where

$$\frac{\gamma_0}{U} = \frac{2\sin\theta}{F_1'(\kappa) + \cot\alpha \cdot \left[A(\mu) + B(\nu) - \alpha \cdot C(\lambda) \right]}$$

$$F_1'(\kappa) = \frac{\kappa}{\kappa+2} + \frac{2}{\kappa+2} \cdot \sqrt{\frac{\kappa}{\kappa+2}} \; \frac{\sqrt{\frac{\kappa}{\kappa+2}} + 2}{\left(\sqrt{\frac{\kappa}{\kappa+2}} + 1 \right)^2} + \log_e 2 \cdot e^{-\frac{1.302}{\kappa}} \cdot I_1\left(\frac{1.302}{\kappa}\right) \qquad \text{Graph 1}$$

$$A(\mu) = \tan^{-1}\mu - 2 \tan^{-1} \left\{ \frac{\sin\left(\frac{1}{2}\tan^{-1}\mu\right)}{\sqrt[4]{1+\mu^2} + \cos\left(\frac{1}{2}\tan^{-1}\mu\right)} \right\} \qquad \text{Graphs 2+3}$$

$$B(\nu) = -\frac{2}{\nu} + \frac{2}{\nu} \frac{\cos\left(\frac{1}{2}\tan^{-1}\nu\right)}{\sqrt[4]{1+\nu^2}} + \frac{2\sin\left(\frac{1}{2}\tan^{-1}\nu\right)}{\sqrt[4]{1+\nu^2}} \qquad \text{Graph 4}$$

$$C(\lambda) = e^{-\lambda} I_1(\lambda) \qquad \text{Graph 5}$$

$$\lambda = -\frac{1}{k} \log\left\{ \frac{\tan^{-1}(\sqrt{2}\tan\alpha)}{\alpha} - 1 \right\} \qquad \text{Graph 6}$$

The five functions have been plotted in terms of the parameters $k, \alpha, \lambda, \mu, \nu$ in the graphs designated above. The parameters μ and ν have been defined as

$$\mu = \frac{2 \cdot \sin\alpha}{k}$$

$$\nu = \frac{2 \cdot \tan\alpha}{k}$$

For the limiting case $\underline{k = 0}$ we have $F_1'(k) = B(\nu) = C(\lambda) = 0$ $A(\mu) = \frac{\pi}{2}$, $\alpha = \frac{\theta}{2}$ and therefore $\frac{\gamma_0}{U} = \frac{2 \sin\theta}{\frac{\pi}{2} \cdot \cot\alpha} = \frac{4}{\pi} \cdot \frac{\sin^2\theta}{1 + \cos\theta}$

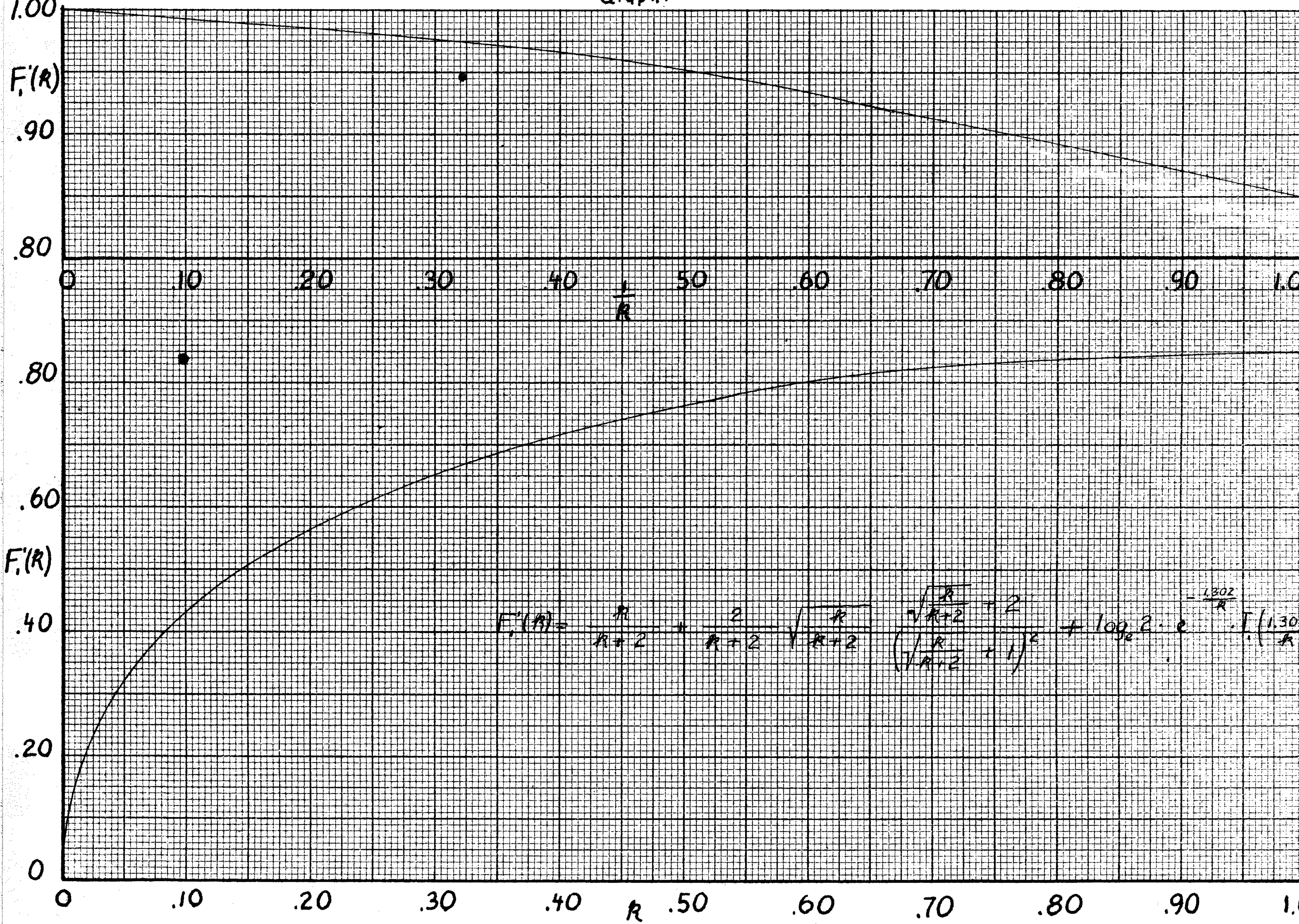
$$\boxed{C_N = \pi \frac{\gamma_0}{U} = 2 \sin^2\theta \cdot \frac{2}{1 + \cos\theta}}$$

which agrees with the previously derived result.

11. Comparison with experiments:

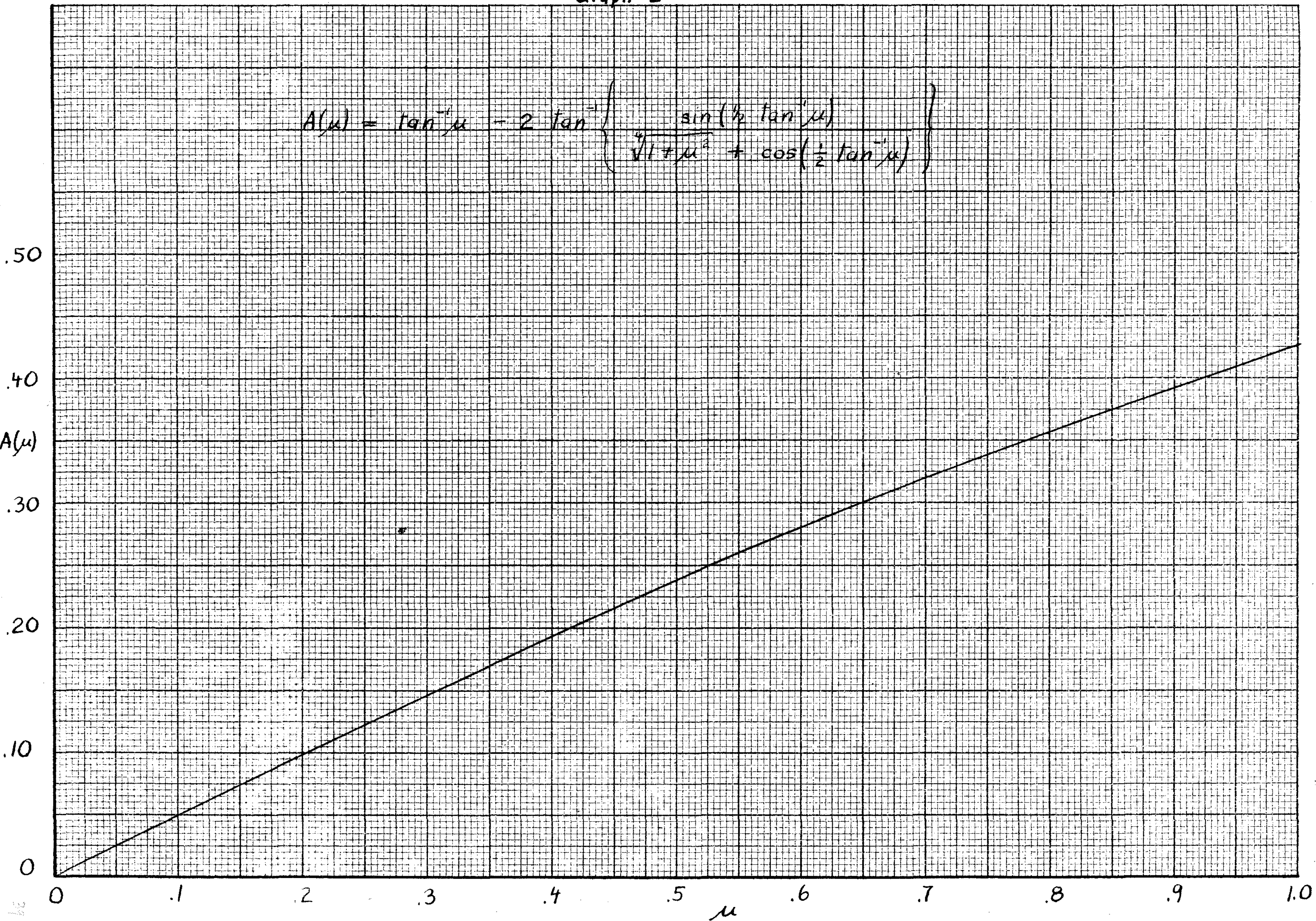
The most extensive investigations of the character of the flow about wings of small aspect ratios were made by H.Winter[5]) at Danzig. His pressure distribution measurements across the span for a square plate confirm quite closely the assumption made that the vorticity distribution is constant across the span rather than elliptical. This was derived in our theory for a wing of

Graph I



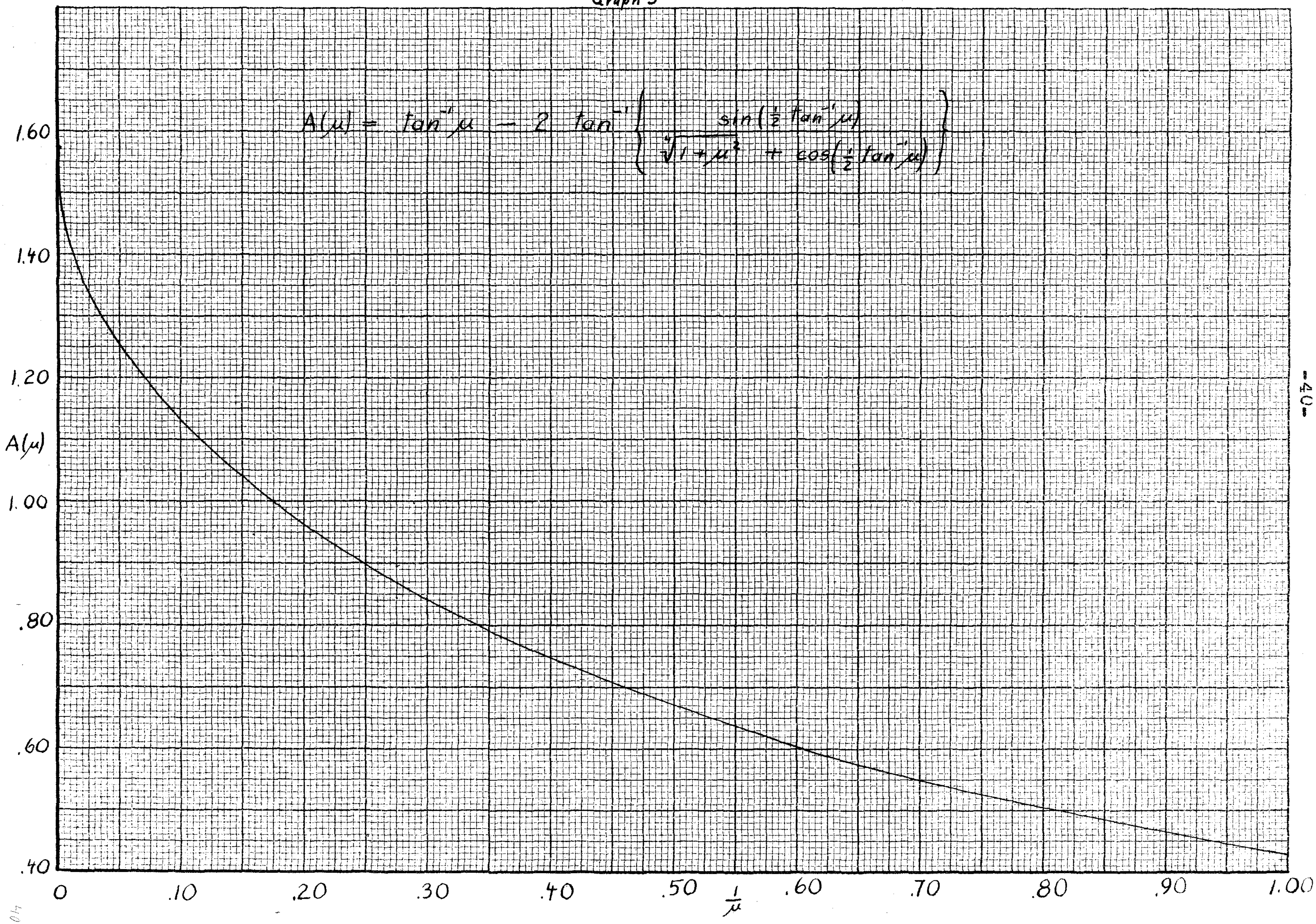
1.00
$F_1'(R)$
.90
.80
0
.10
.20
.30
.40
$\frac{1}{R}$
.50
.60
.70
.80
.90
1.00
.80
.60
$F_1'(R)$
.40
.20
0
$F_1'(R) = \frac{R}{R+2} + \frac{2}{R+2}\sqrt{\frac{R}{R+2}} \frac{\sqrt{\frac{R}{R+2}}+2}{\left(\sqrt{\frac{R}{R+2}}+1\right)^2} + \log_e 2 \cdot e^{-\frac{1.302}{R}} \cdot I_1\left(\frac{1.302}{R}\right)$
0
.10
.20
.30
.40
R
.50
.60
.70
.80
.90
1.00

Graph 2

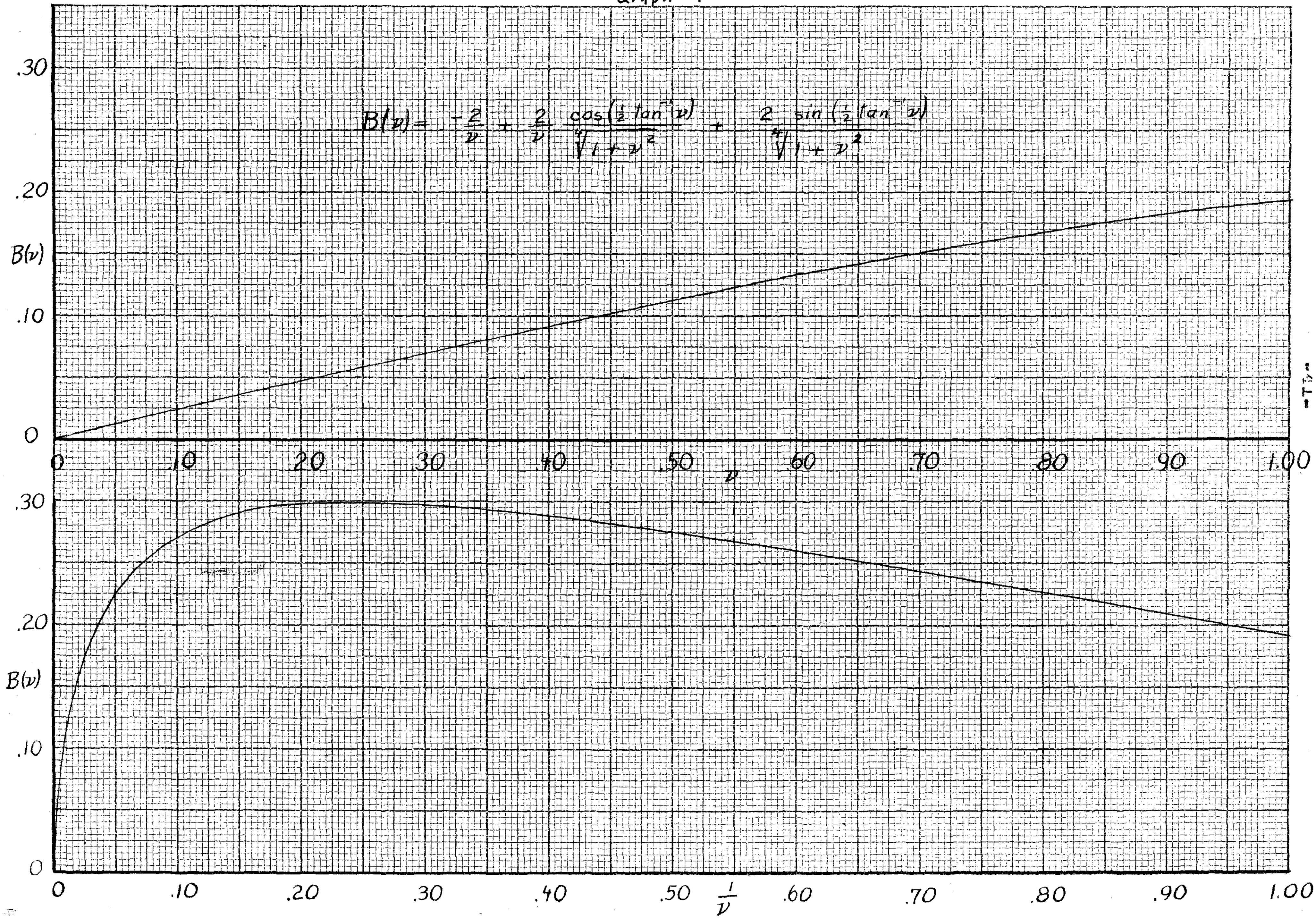


$A(\mu) = \tan^{-1}\mu - 2\tan^{-1}\left\{ \frac{\sin(\frac{1}{2}\tan^{-1}\mu)}{\sqrt[4]{1+\mu^2} + \cos(\frac{1}{2}\tan^{-1}\mu)} \right\}$
.50
.40
$A(\mu)$
.30
.20
.10
0
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
μ

Graph 3

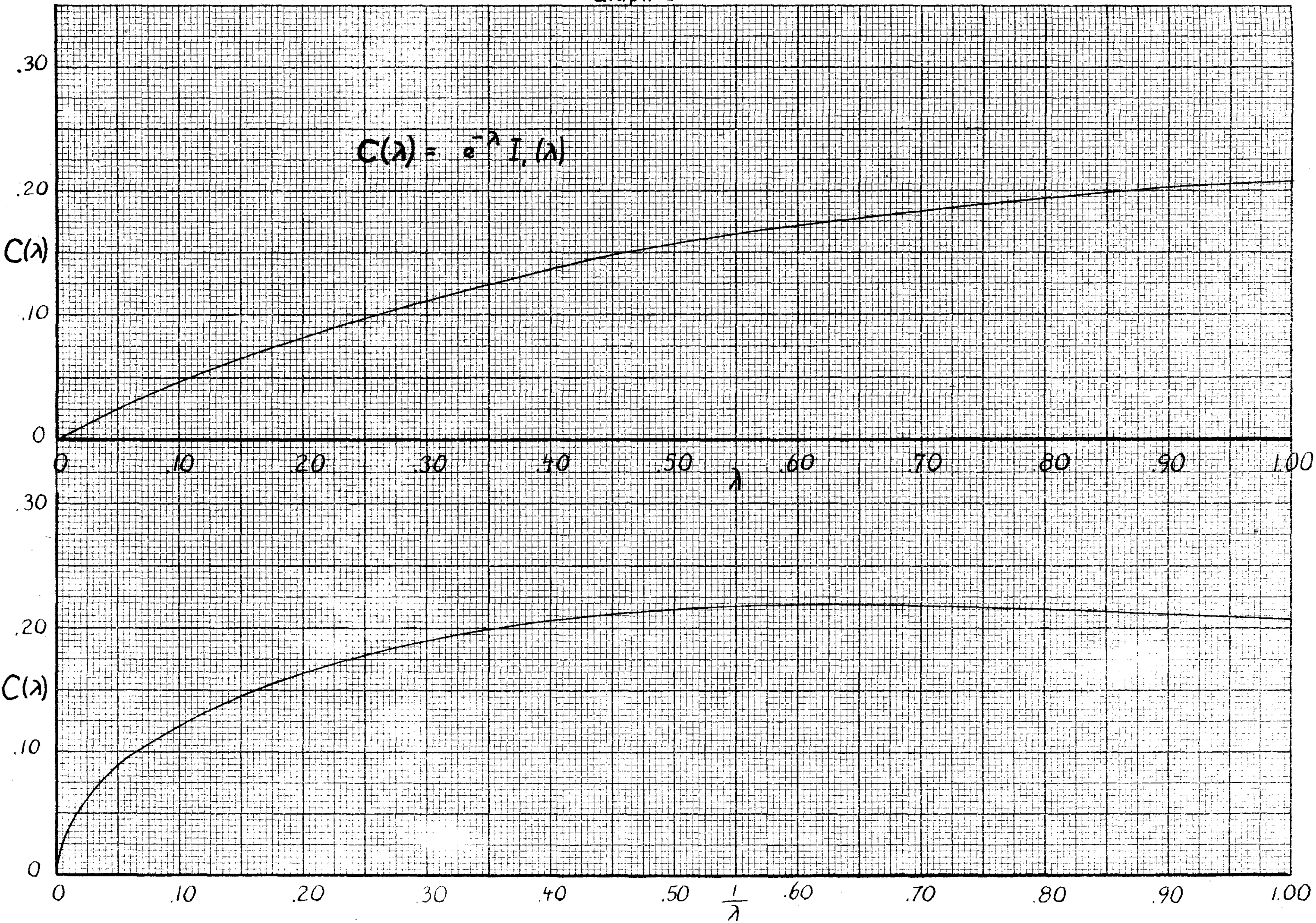

$A(\mu) = \tan^{-1}\mu - 2\tan^{-1}\left\{\dfrac{\sin\left(\frac{1}{2}\tan^{-1}\mu\right)}{\sqrt[4]{1+\mu^2} + \cos\left(\frac{1}{2}\tan^{-1}\mu\right)}\right\}$
1.60
1.40
1.20
$A(\mu)$
1.00
.80
.60
.40
0
.10
.20
.30
.40
.50
$\frac{1}{\mu}$
.60
.70
.80
.90
1.00

Graph 4



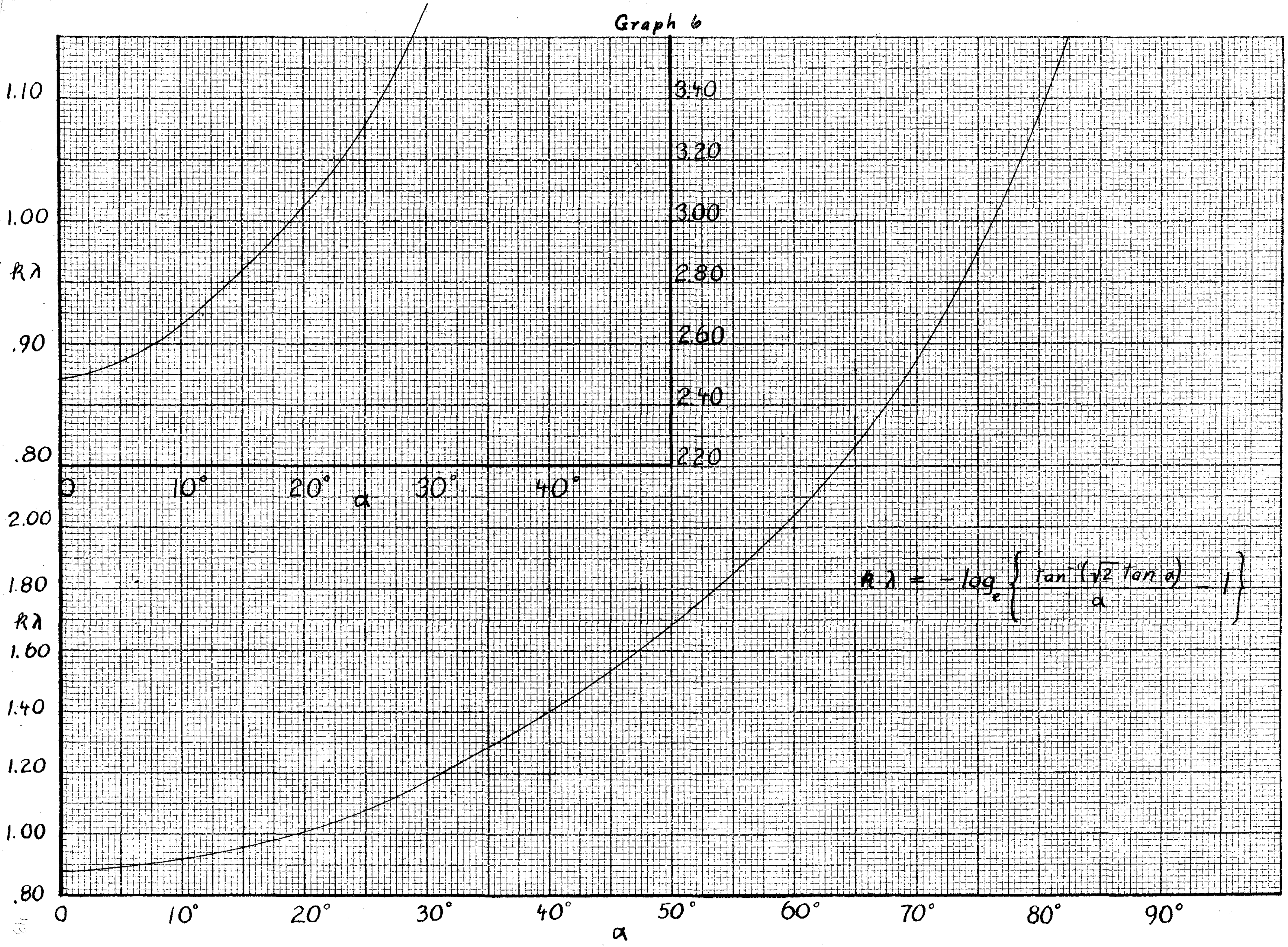
$B(\nu) = -\frac{2}{\nu} + \frac{2}{\nu}\frac{\cos\left(\frac{1}{2}\tan^{-1}\nu\right)}{\sqrt[4]{1+\nu^2}} + \frac{2\sin\left(\frac{1}{2}\tan^{-1}\nu\right)}{\sqrt[4]{1+\nu^2}}$
.30
.20
.10
0
$B(\nu)$
0 .10 .20 .30 .40 .50 .60 .70 .80 .90 1.00
ν
.30
.20
.10
0
$B(\nu)$
0 .10 .20 .30 .40 .50 .60 .70 .80 .90 1.00
$\frac{1}{\nu}$

Graph 5

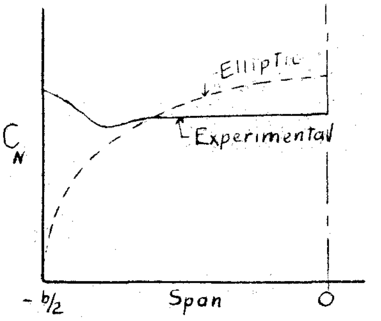


$C(\lambda) = e^{-\lambda} I_1(\lambda)$
C(λ)
.30
.20
.10
0
.10
.20
.30
.40
.50
.60
.70
.80
.90
1.00
λ
C(λ)
1/λ

Graph 6


1.10
1.00
$k\lambda$
.90
.80
3.40
3.20
3.00
2.80
2.60
2.40
2.20
0
10°
20°
α
30°
40°
2.00
1.80
$k\lambda$
1.60
1.40
1.20
1.00
.80
0
10°
20°
30°
40°
50°
60°
70°
80°
90°
α
$k\lambda = -\log_e \left\{ \frac{\tan^{-1}(\sqrt{2}\tan\alpha)}{\alpha} - 1 \right\}$

infinite chord and zero aspect ratio. The fact that this still holds quite well at k = 1 justifies the assumption for the entire range of aspect ratios considered, i.e. $0 < Æ < 1$



Similarly the assumption of a chord distribution $\gamma = \gamma_0 \sqrt{\frac{t/2 - x}{t/2 + x}}$ appears reasonable from his experiments. The actual pressure distributions showed very great suction peaks at the leading edge as postulated by the theory. In addition they showed, however, smaller peaks near the sides and at the trailing edge which have not been included in the theory. The measured pressure distributions indicated a considerable variation with the angle of attack. This has not been taken into account in our theory. It indicates that the true solution of our integral equation for $\gamma(\xi)$ is probably quite complicated. For this reason the assumption of a mean distribution function $\gamma = \gamma_0 \sqrt{\frac{t/2 - \xi}{t/2 + \xi}}$ is probably as good an approach as can be made theoretically.

Winter also studied the character of the flow by means of flow pictures. It is interesting to consider his description in some detail to see within what limits of aspect ratios and angles of attack the present theory could be expected to hold.

(1) Aspect Ratio $k > 2.0$

(a) Small angles of attack: Well known flow about airfoil of type postulated by Prandtl theory of lifting line.

(b) Large angles of attack to 90^o: Well defined separation takes place with vortices leaving parallel to undisturbed flow direction.

(2) Aspect Ratio between limits $0.1 < k < 2.0$

(a) Small angles of attack: Smooth flow along wing as above. With decreasing aspect ratios the region of angles of attack within which this flow takes place decreases.

(b) Larger angles of attack: Very strong flow over the sides of the wing with formation of tip vortices of the type postulated in the present theory. Over the middle part of the span the flow separates from the wing. With decreasing aspect ratio and increasing angles of attack the tip flow dominates the flow over the middle and a real separation of the main flow is prevented until very large angles are reached.

(c) Very large angles of attack up to 90^o: Well defined separation of the main flow takes place analagously as for 1 (b)

(3) Aspect ratio between limits $0 < k < 0.1$

(a) Very small angles of attack: A smooth type of flow along the wing of the type 1(a) and 2(a) takes place with the tip vortices clinging to the wing along the entire length. This type of flow is resticted to very small angles of attack. It also takes place over the forward region of the wing for larger angles, the region decreasing with decreasing aspect ratio.

(b) Larger angles of attack: The tip vortices follow the plate for a small distance, and then bend off rather sharply approximately in the direction of undisturbed flow. With increasing angle of attack the location of this point at which the vortices bend away moves forward lying, for instance at 40% of chord for $\theta = 16^{\circ}$. Our theoretical deduction that all of the vortices leave at the half-angle of attack gives about the same effect as that described above.

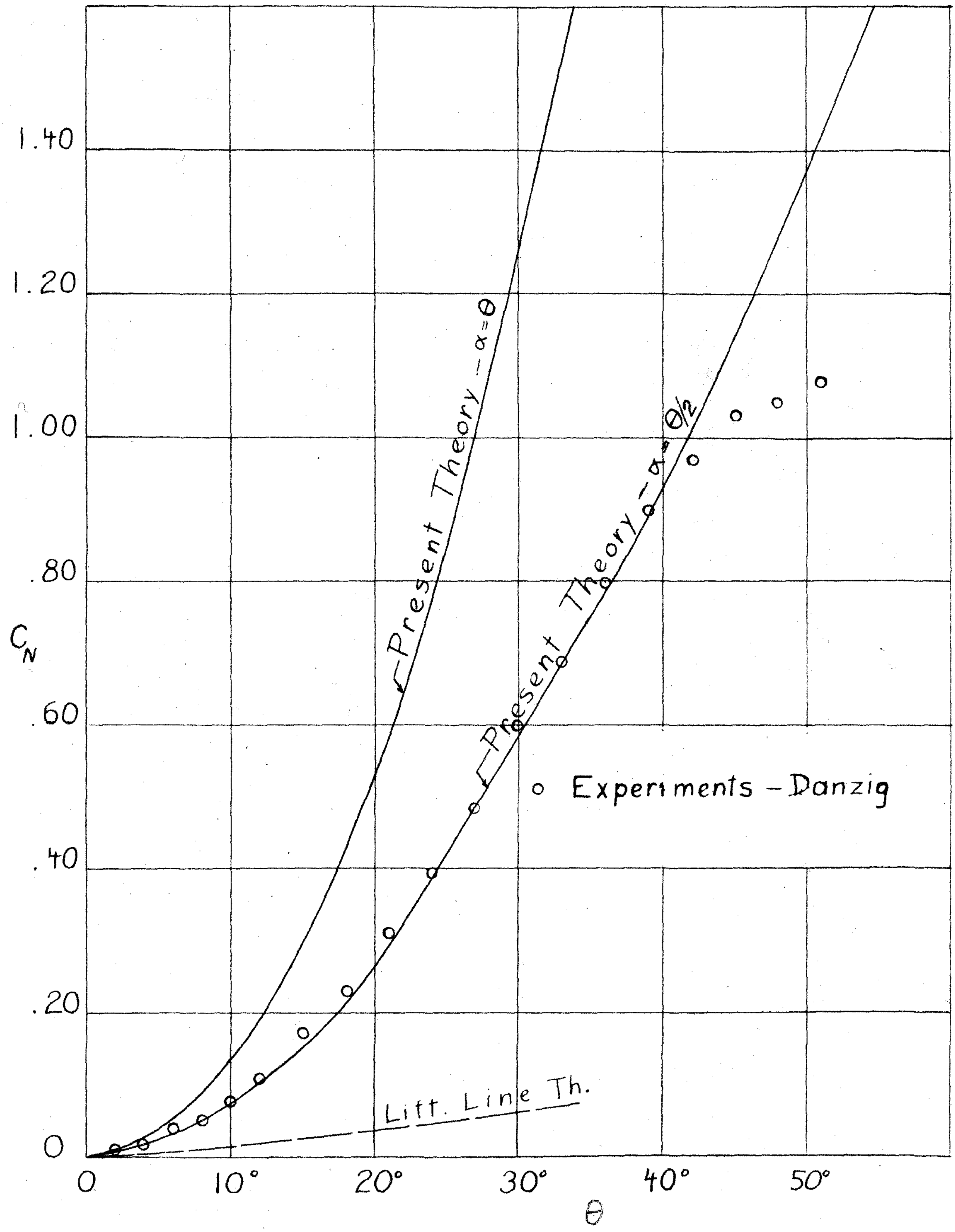
(c) Very large angles of attack to 90^{o}: Flow separation takes place at about 45^{o}. At large angles Winter believes a Kármán vortex street is formed in the wake.

Judging from the above **description of** the actual flow we would expect **our theory** to apply to the regions 2(b) and 3(b) described above. For small angles of attack we would expect the initial tangent to follow the Prandtl lifting line formula. A comparison of the experimental results with the calculated values of C_N vs. θ is given in the following series of graphs and shows that such is the case. The experimental points for $0 < k < 1$ lie between the limits given by the curves corresponding to $\alpha = \theta$ and $\alpha = \frac{\theta}{2}$., except possibly for the initial tangent. At k = 1/30 the theoretical curve for $\alpha = \frac{\theta}{2}$ follows the experimental curve very closely. For larger aspect ratios k = .134 and k = .20 the experimental points lie between $\alpha = \frac{\theta}{2}$ and $\alpha = \theta$ moving towards $\alpha = \theta$ as k increases. This tendency increases for k = .35, k = .50, and k = .66, and the initial tangent is somewhat smaller than the theory indicates following more nearly the Prandtl lifting line curve. At k = 1 and k = 1.25 the experimental points fall about half-way between our theoretical curves and the lifting line curves, while at k = 2 the Prandtl lifting line curve already gives the results quite closely. It should be noticed that in the above comparison no account is taken of the viscosity effect in reducing the actual circulation about the wing. This may amount to 10%-15% for wings of large aspect ratio. If we assume that a similar reduction is effective for wings of small aspect ratio the experimental points are brought into the region between the curves $\alpha = \theta$ and $\alpha = \frac{\theta}{2}$ for all aspect ratios below 2.

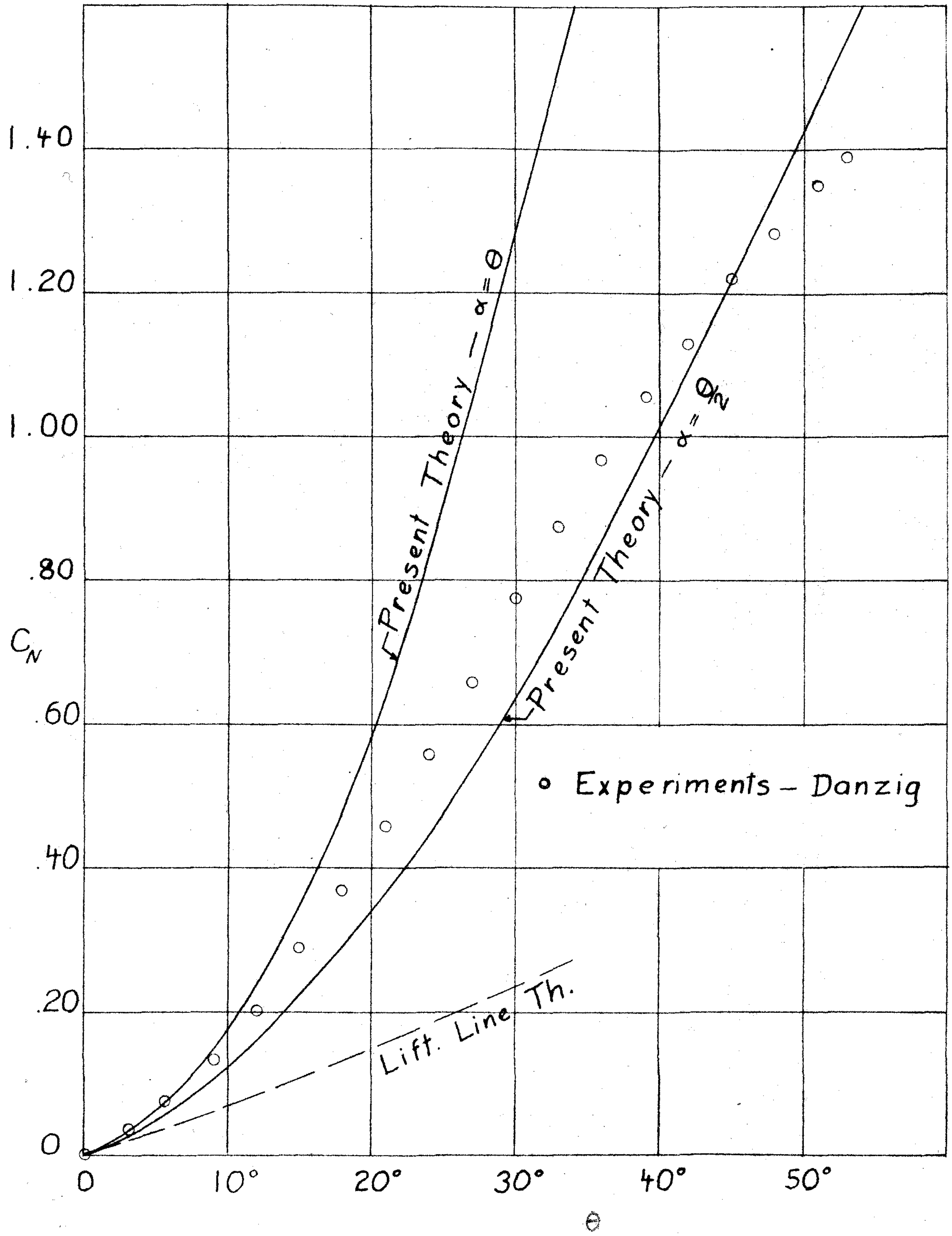
12. Conclusion:

From the above comparison with experiments it appears then as though the present theory succeeded in giving at least a qualitative explanation for the peculiar curvature of the curve of C_N vs. θ in the fact that the trailing vortices leave the plate at an angle α. The theory also agrees quite well quantitatively in the region of aspect ratios $k < 1$, which is the range intended to be covered by it. It has not yet been found possible to determine theoretically the exact curve C_N vs. θ because in general the angle α is unknown. However, the limits within which it should lie have been specified since $\frac{\theta}{2} < \alpha < \theta$.

In one limiting case, however, namely $k \doteq 0$ we do know the angle α from our theoretical considerations as $\alpha = \frac{\theta}{2}$. This angle must also hold approximately at extremely small aspect ratios such as $k = 1/30$ for which experiments have been made. A comparison between theory and experiment for this case showed very good agreement.

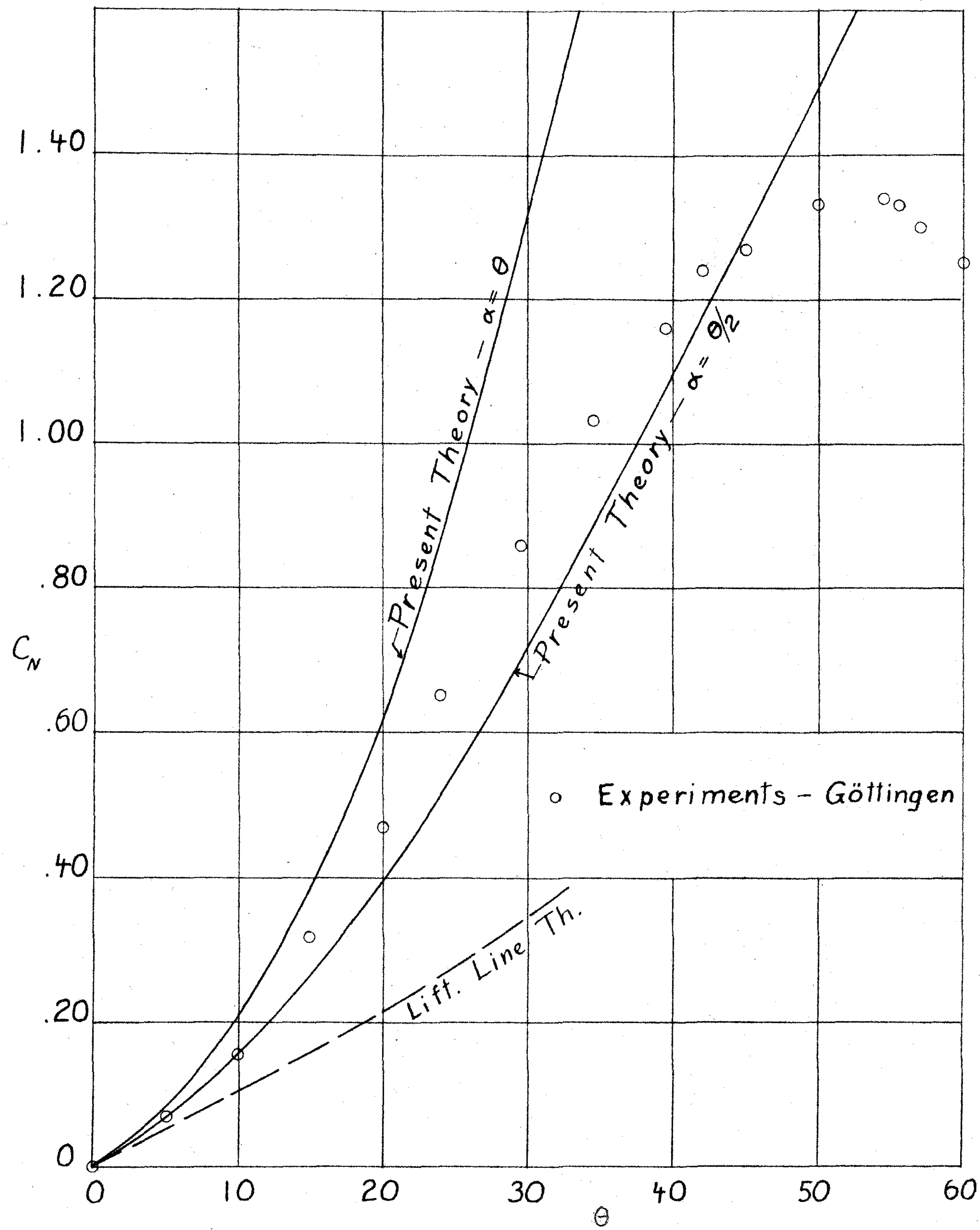

1.40
1.20
1.00
.80
.60
.40
.20
0
C_N
Present Theory $-\alpha=\theta$
Present Theory $-\alpha=\theta/2$
o Experiments - Danzig
Lift. Line Th.
0
10°
20°
30°
40°
50°
θ

$R = \frac{1}{30}$

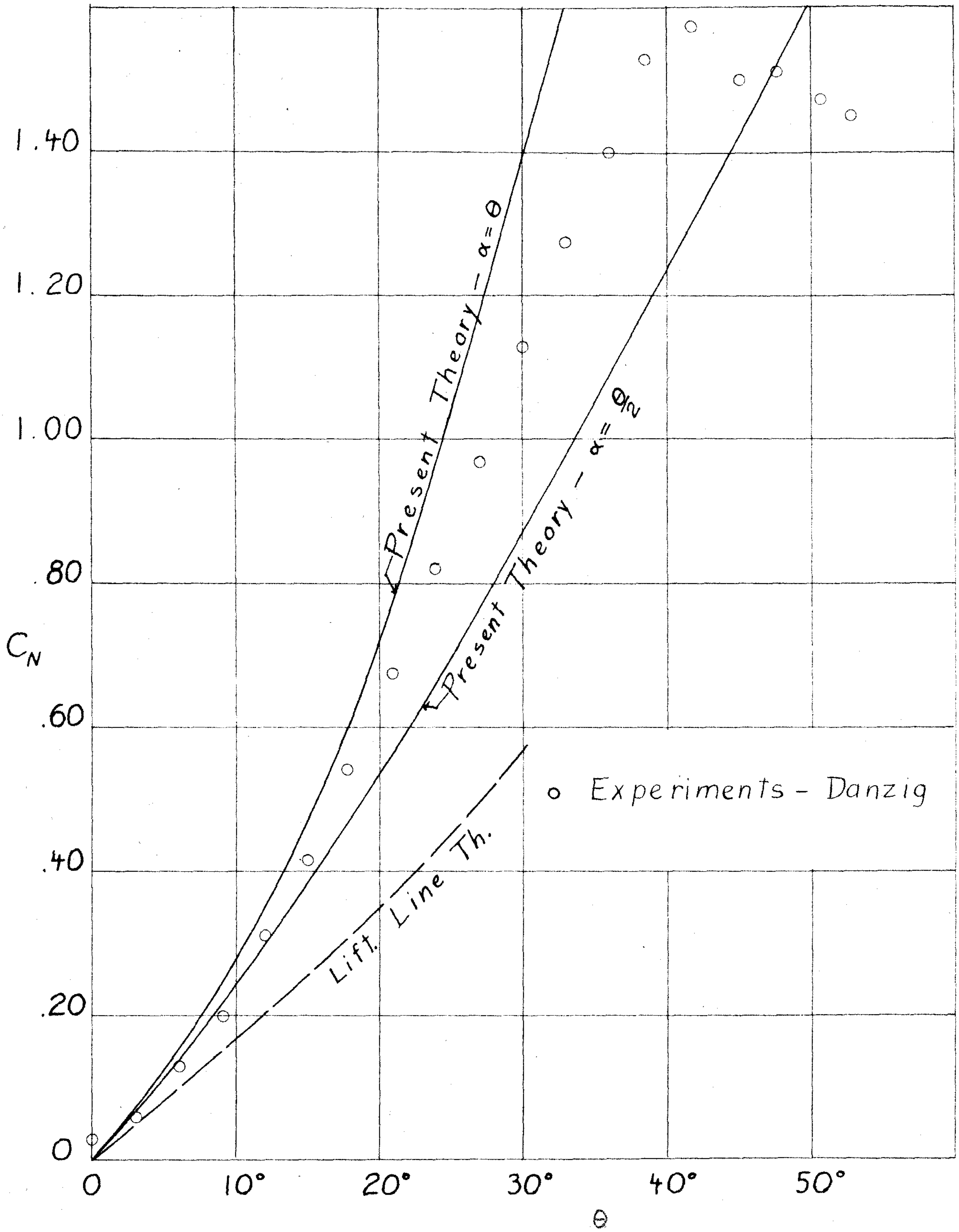


1.40
1.20
1.00
.80
.60
.40
.20
0
C_N
0
10°
20°
30°
40°
50°
θ
Present Theory — $\alpha = \theta$
Present Theory — $\alpha = \theta/2$
Lift. Line Th.
o Experiments – Danzig

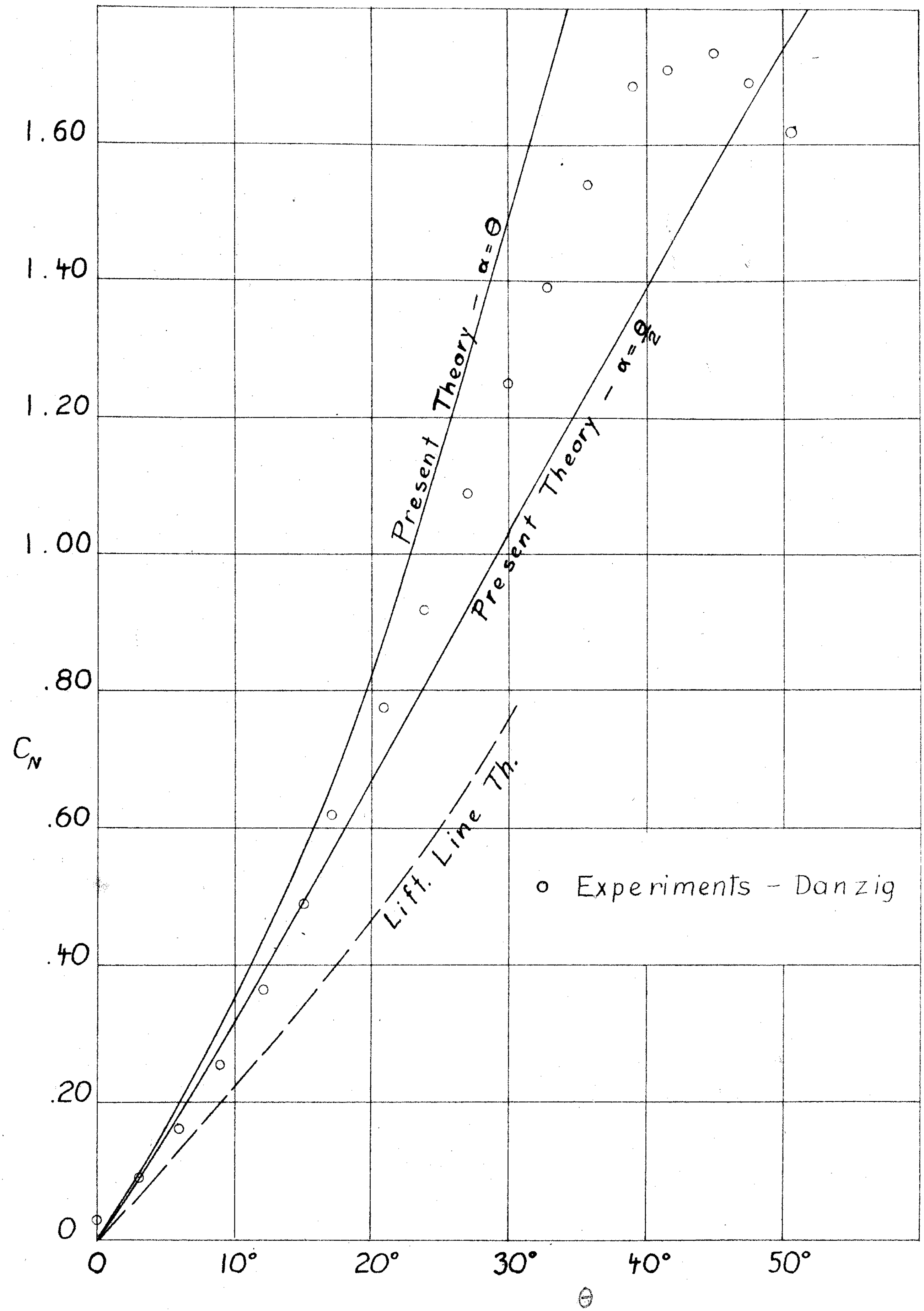
$k = .134$


1.40
1.20
1.00
.80
.60
.40
.20
0
C_N
0
10
20
30
40
50
60
θ
Present Theory — $\alpha = \theta$
Present Theory — $\alpha = \theta/2$
Lift. Line Th.
o Experiments - Göttingen

$\mathcal{R} = 0.2$

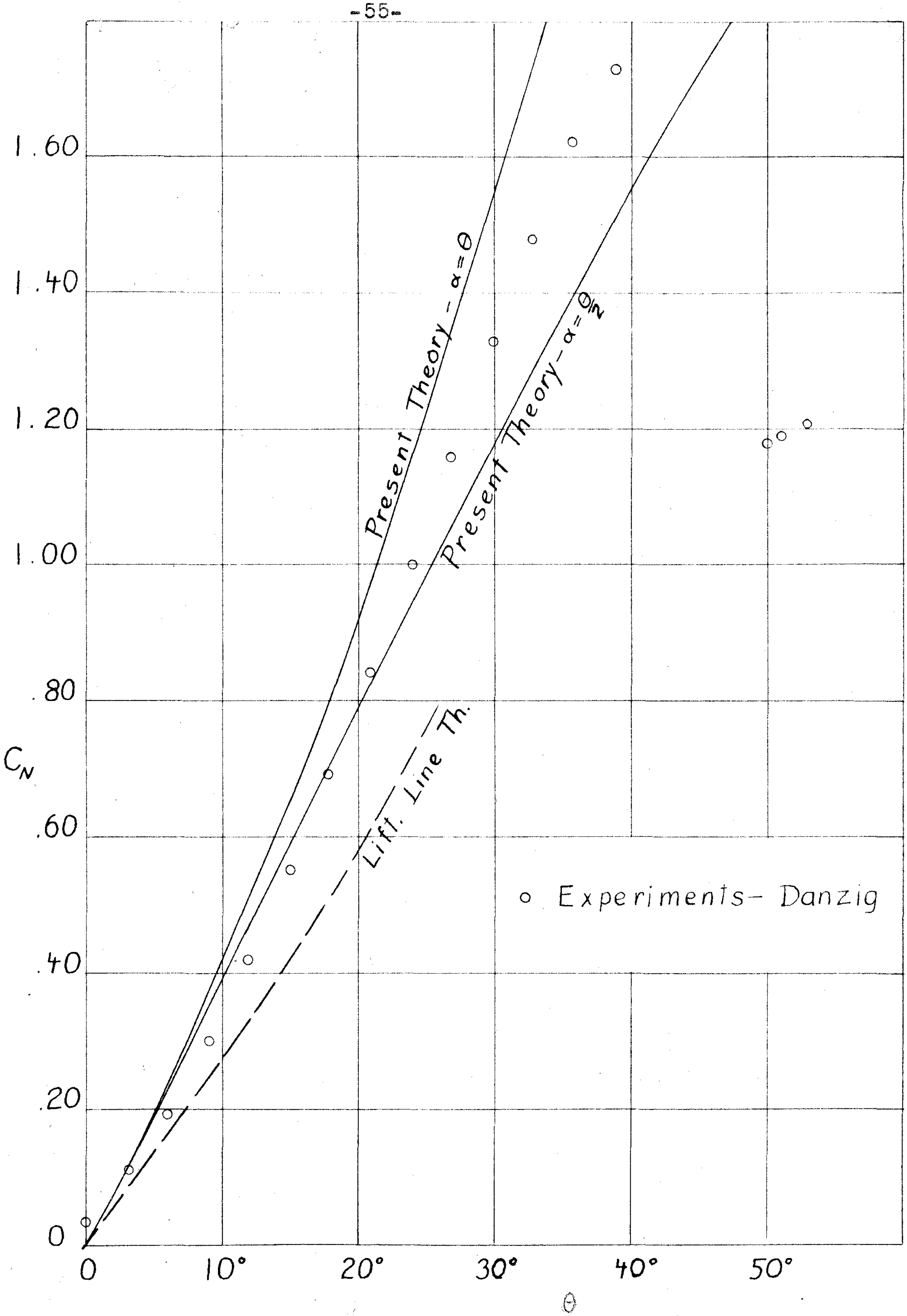

1.40
1.20
1.00
.80
.60
.40
.20
0
C_N
0
10°
20°
30°
40°
50°
θ
Present Theory – $\alpha = \theta$
Present Theory – $\alpha = \theta/2$
Lift. Line Th.
o Experiments – Danzig

$R = .35$

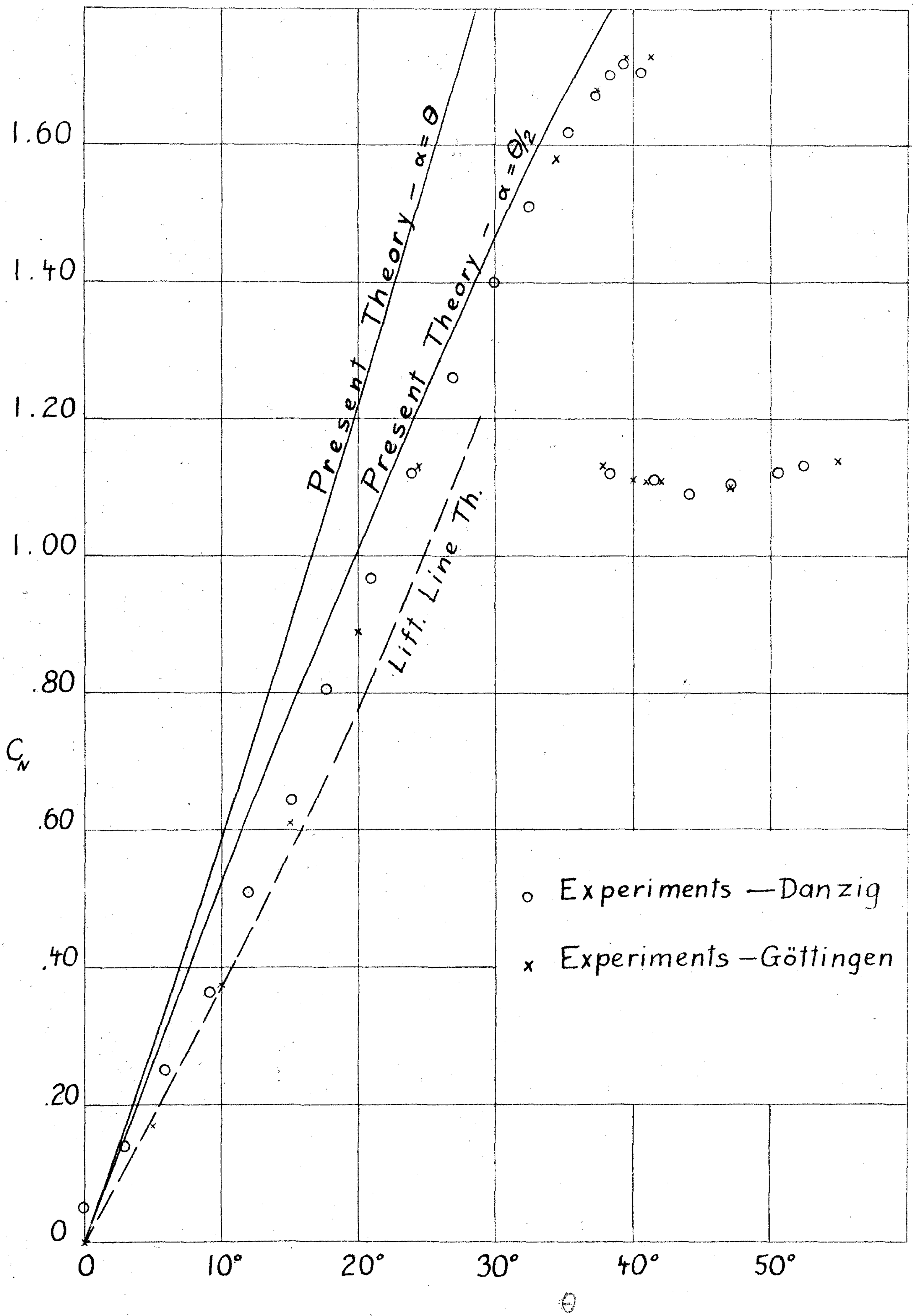


1.60
1.40
1.20
1.00
.80
.60
.40
.20
0
C_N
0
10°
20°
30°
40°
50°
Θ
Present Theory – $\alpha = \Theta$
Present Theory – $\alpha = \Theta/2$
Lift. Line Th.
o Experiments – Danzig

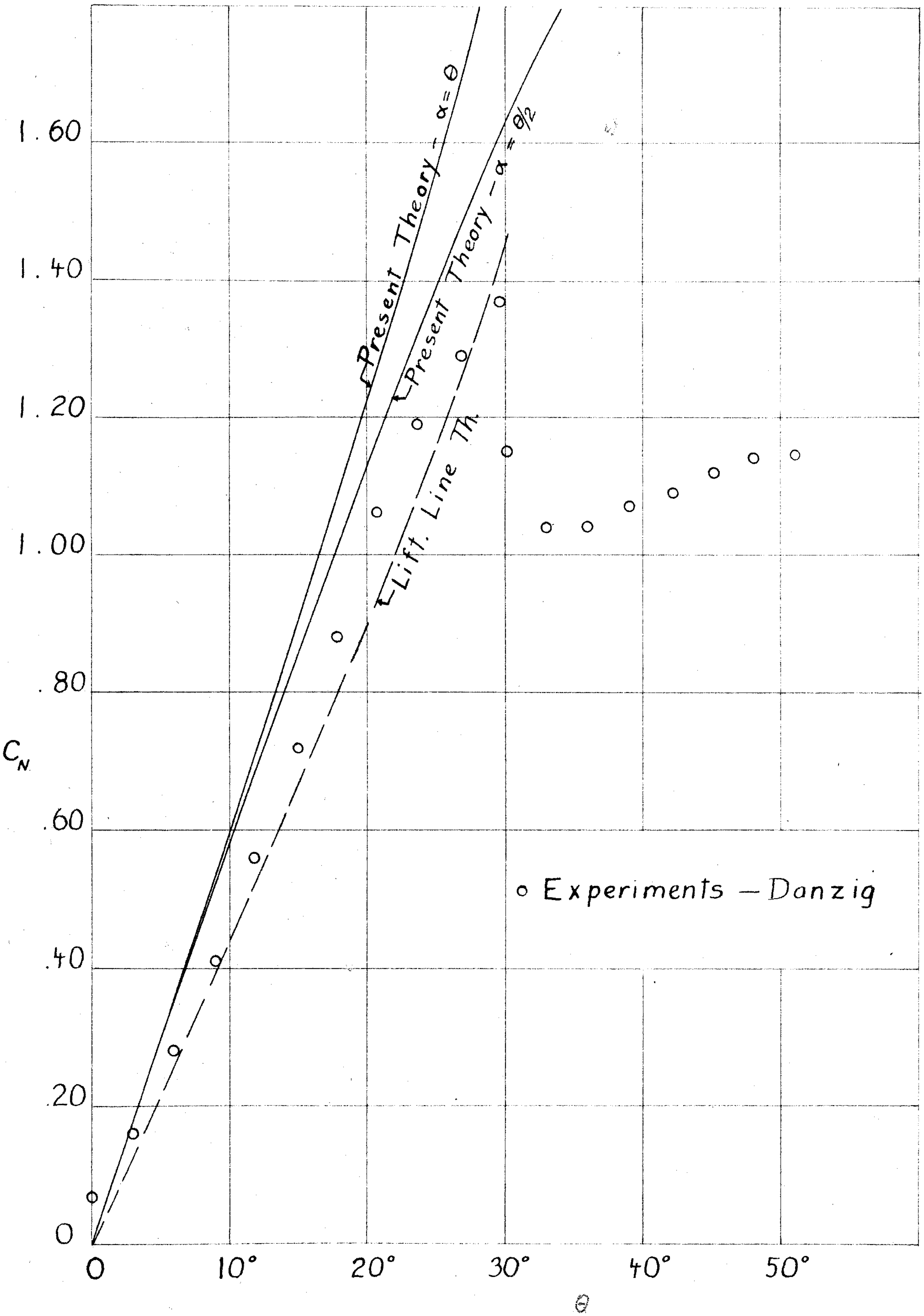
$k = 0.5$


1.60
1.40
1.20
1.00
.80
.60
.40
.20
0
C_N
0
10°
20°
30°
40°
50°
θ
Present Theory - $\alpha = \theta$
Present Theory - $\alpha = \frac{\theta}{2}$
Lift. Line Th.
○ Experiments- Danzig

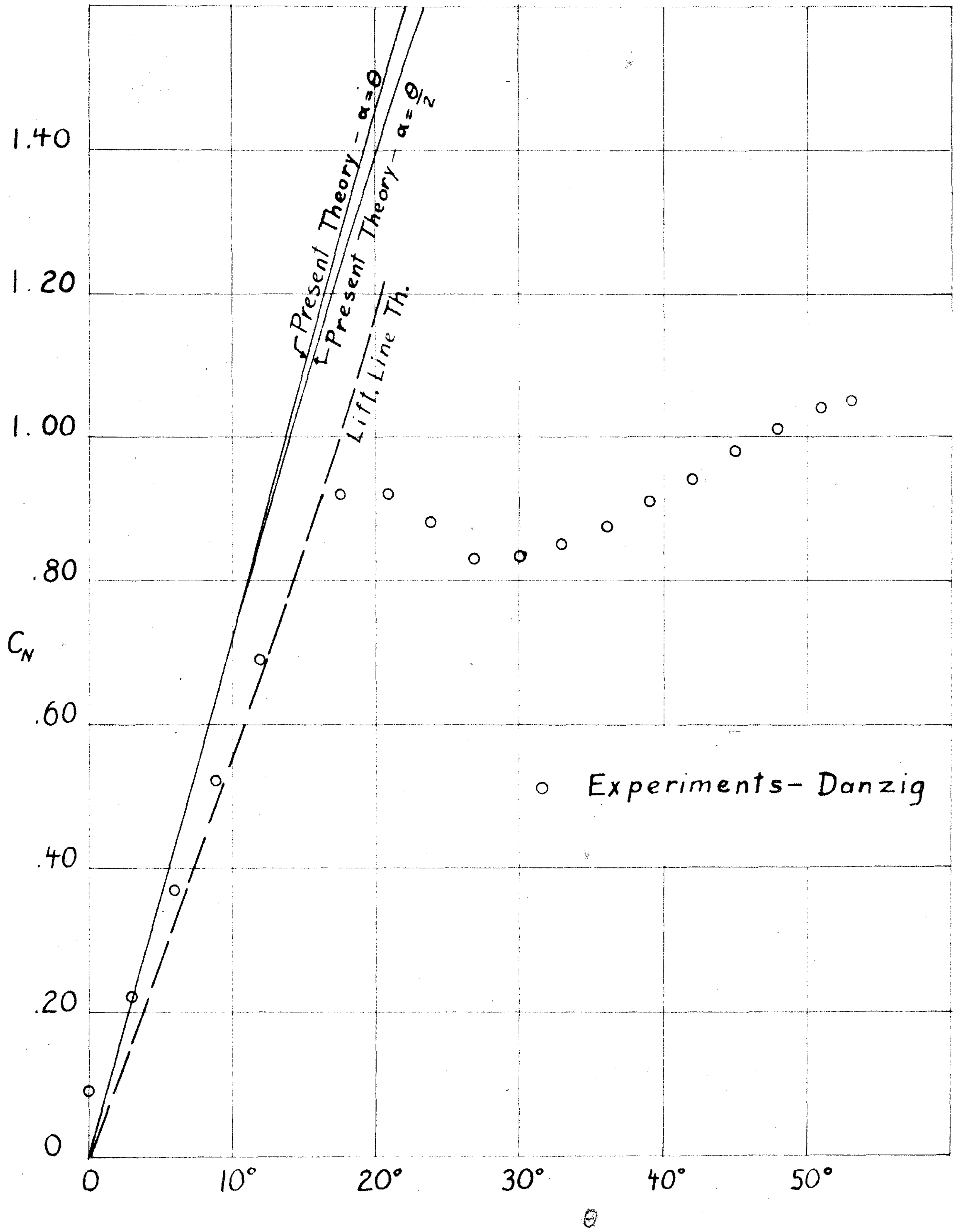
$\mathcal{R} = .66$


1.60
1.40
1.20
1.00
.80
.60
.40
.20
0
C_N
0
10°
20°
30°
40°
50°
θ
Present Theory – $\alpha = \theta$
Present Theory – $\alpha = \theta/2$
Lift. Line Th.
○ Experiments —Danzig
× Experiments —Göttingen

$A\!R = 1.00$


1.60
1.40
1.20
1.00
.80
.60
.40
.20
0
C_N
0
10°
20°
30°
40°
50°
θ
Present Theory – $\alpha = \theta$
Present Theory – $\alpha = \theta/2$
Lift. Line Th.
Experiments – Danzig

$R = 1.25$



1.40
1.20
1.00
.80
.60
.40
.20
0
C_N
Present Theory – $\alpha = \theta$
Present Theory – $\alpha = \frac{\theta}{2}$
Lift. Line Th.
Experiments – Danzig
0
10°
20°
30°
40°
50°
θ

$\mathcal{R} = 2.0$

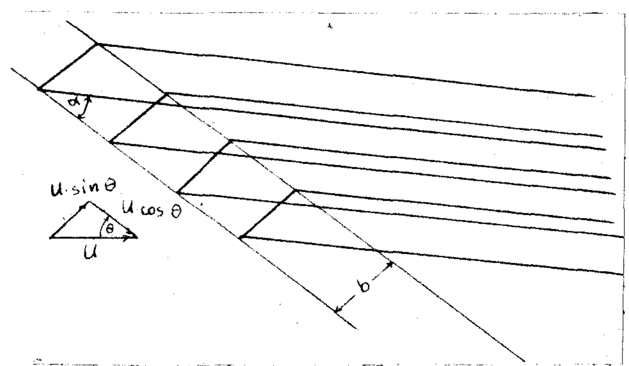
D. Appendix:

1) Wing of Infinite Chord - Zero Aspect Ratio - Assuming the Flow at Infinity Is Parallel to the Plate:

The flow past a wing of infinite chord treated in section B was considered to be of the pure induced type. This gives a flow at infinity in the wake behind the plate which is inclined at twice the angle as the plate itself and thus corresponds to a sort of inflow from above. A wing with a leading edge out at infinity might be expected to correspond to this type of flow. If we consider the plate as the limiting form of an infinite lattice, however, we should expect a flow at infinity which is just deflected parallel to the plate. This is the case which we shall consider now.

We shall again consider a similar vortex system as before consisting of

(a) A uniform distribution of bound vortices of length b placed spanwise along a plate. We shall denote their strength per unit length of the plate as γ_1.



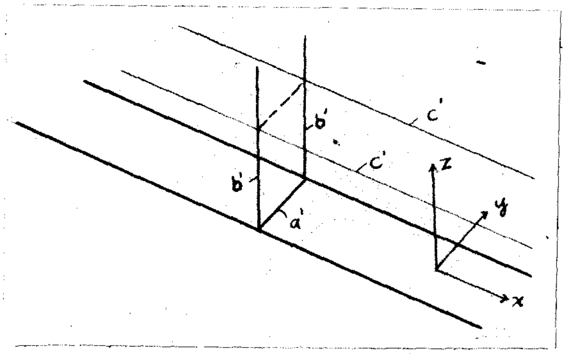
(b) Straight trailing vortices passing off from the bound vortices in the plate at an angle α. Their strength is thus $\gamma = \frac{\gamma_1}{\sin\alpha}$ per unit height measured in the direction normal to them.

It is possible to split the trailing vortex system (b) into two components: a component parallel to the plate $= \gamma \cdot \cos\alpha = \gamma_1 \cdot \cot\alpha$ and a component perpendicular to the plate $= \gamma \cdot \sin\alpha = \gamma_1$. Thus an equivalent vortex system to the above is

(a') A system of bound vortices of strength γ_1 per unit length.

(b') Their trailing vortices in the direction normal to the plate.

(c') A system of vortices of strength $\gamma_1 \cdot \cot\alpha$ running parallel to the plate, and lying above the ends of the span.



We can easily calculate the induced velocities due to the above vortices by an application of the Biot-Savart law. Let us use the same notation as before namely:

x = chord direction along plate

y = span direction along plate. Origin at center of plate

z = direction normal to plate

u = induced velocity along x

v = induced velocity along y

w = induced velocity along z

(a') Induced Velocities Due to Bound Vortices:

The induced velocity at a point x,y,z due to a single vortex γ at $x=\xi$ is

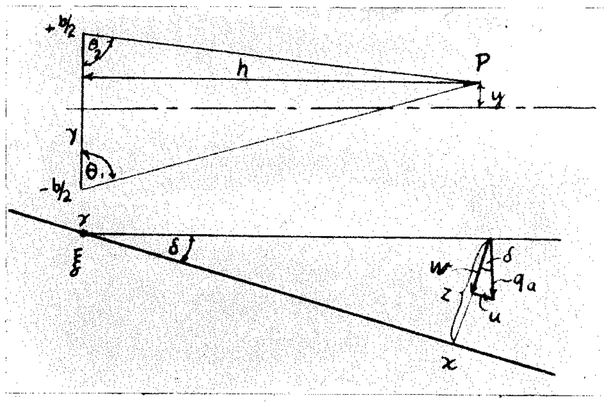
$$q_a = \frac{\gamma}{4\pi}\cdot\frac{1}{h}\left[\cos\theta_1 + \cos\theta_2\right]$$

and

$$u_a = q_a \cdot \sin\delta$$

$$v_a = 0$$

$$w_a = q_a \cdot \cos\delta$$



where

$$h = \sqrt{(x-\xi)^2 + z^2}$$

$$\cos\theta_1 = \frac{b/2 + y}{\sqrt{\left(\frac{b}{2}+y\right)^2 + h^2}} \qquad \sin\delta = \frac{z}{h}$$

$$\cos\theta_2 = \frac{b/2 - y}{\sqrt{\left(\frac{b}{2}-y\right)^2 + h^2}} \qquad \cos\delta = \frac{x-\xi}{h}$$

Due to all the bound vortices between $\xi=-\infty$ and $\xi=+\infty$

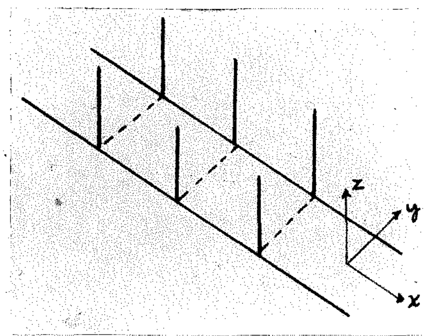
$$u_a = \frac{\gamma_1}{4\pi}\int_{\xi=-\infty}^{\xi=+\infty} \frac{z}{(x-\xi)^2 + z^2}\cdot\left[\frac{b/2 + y}{\sqrt{(b/2+y)^2 + (x-\xi)^2 + z^2}} + \frac{b/2 - y}{\sqrt{(b/2-y)^2 + (x-\xi)^2 + z^2}}\right]\cdot d\xi$$

$$\boxed{\begin{aligned} u_a &= \frac{\gamma_1}{2\pi}\left[\tan^{-1}\frac{(b/2 - y)}{z} + \tan^{-1}\frac{(b/2 + y)}{z}\right] \\ v_a &= 0 \\ w_a &= 0 \qquad \text{By symmetry} \end{aligned}}$$

(b') <u>Induced Velocities Due to Normal Component of Trailing Vortices:</u>

This system of vortices again only induced velocities in the direction x. We can write down the expression for <u>u</u> at once from the previous investigation by letting

$$\left.\begin{array}{lcl} b/2 - y & \longrightarrow & z \\ b/2 + y & \longrightarrow & \infty \\ z & \longrightarrow & b/2 \mp y \end{array}\right\} \text{ for vortex sheet at } y = \pm\, b/2$$



Due to vortex sheet at $y = b/2$

$$u_{b_1} = \frac{\gamma_1}{2\pi}\left[\tan^{-1}\frac{z}{b/2 - y} + \tan^{-1}\frac{\infty}{b/2 - y}\right]$$

$$u_{b_1} = \frac{\gamma_1}{2\pi}\left[\tan^{-1}\frac{z}{b/2 - y} + \frac{\pi}{2}\right] \qquad \text{if} \quad y < \frac{b}{2}$$

$$u_{b_1} = \frac{-\gamma_1}{2\pi}\left[\tan^{-1}\frac{z}{y - b/2} + \frac{\pi}{2}\right] \qquad \text{if} \quad y > \frac{b}{2}$$

Due to vortex sheet at $y = -b/2$

$$u_{b_2} = \frac{\gamma_1}{2\pi}\left[\tan^{-1}\frac{z}{b/2 + y} + \tan^{-1}\frac{\infty}{b/2 + y}\right]$$

$$u_{b_2} = \frac{\gamma_1}{2\pi}\left[\tan^{-1}\frac{z}{b/2 + y} + \frac{\pi}{2}\right] \qquad \text{if} \quad y > -b/2$$

$$u_{b_2} = -\frac{\gamma_1}{2\pi}\left[\tan^{-1}\frac{z}{b/2 + y} + \frac{\pi}{2}\right] \qquad \text{if} \quad y < -b/2$$

Thus

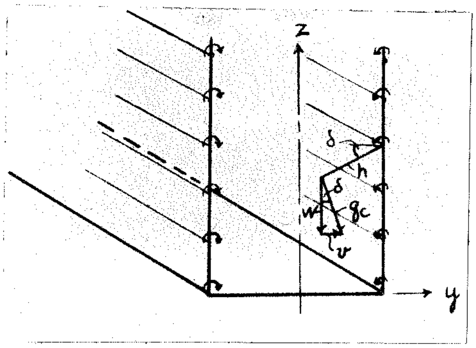
$$\boxed{\begin{array}{ll} \text{for } |y| < \frac{b}{2} & u_b = \frac{\gamma_1}{2\pi}\left[\tan^{-1}\frac{z}{b/2 + y} + \tan^{-1}\frac{z}{b/2 - y} + \pi\right] \\ \phantom{\text{for }} |y| > \frac{b}{2} & u_b = -\frac{\gamma_1}{2\pi}\left[-\tan^{-1}\frac{z}{b/2 + y} + \tan^{-1}\frac{z}{|y| - b/2}\right] \\ & v_b = 0 \\ & w_b = 0 \end{array}}$$

Notice that for

$$|y| < \frac{b}{2} \qquad u_a + u_b = \frac{\gamma_1}{2\pi}\left[\pi + \underbrace{\left(\tan^{-1}\frac{b/2-y}{z} + \tan^{-1}\frac{z}{b/2-y}\right)}_{\frac{\pi}{2}} + \underbrace{\left(\tan^{-1}\frac{b/2+y}{z} + \tan^{-1}\frac{z}{b/2+y}\right)}_{\frac{\pi}{2}}\right] = \gamma_1$$

$$|y| > \frac{b}{2} \qquad u_a + u_b = \frac{\gamma_1}{2\pi}\left[\underbrace{\left(\tan^{-1}\frac{b/2-y}{z} + \tan^{-1}\frac{z}{b/2-y}\right)}_{-\frac{\pi}{2}} + \underbrace{\left(\tan^{-1}\frac{b/2+y}{z} + \tan^{-1}\frac{z}{b/2+y}\right)}_{\frac{\pi}{2}}\right] = 0$$

(c') <u>Induced Velocities Due to Tangential Component of Trailing Vortices:</u>



This system of vortices induces velocities in the directions y and z. Due to a single vortex at $z = \xi$ we get an induced velocity at P(x,y,z):

$$q_c = \frac{\Gamma_0}{2\pi}\cdot\frac{1}{h} \qquad \text{where} \qquad \Gamma_0 = \gamma_0 \cdot \cot\alpha$$

$$u_c = 0 \qquad\qquad h = \sqrt{(b/2 - y)^2 + (z-\xi)^2}$$

$$v_c = q_c \cdot \sin\delta \qquad\qquad \sin\delta = \frac{z-\xi}{h}$$

$$w_c = -q_c \cdot \cos\delta \qquad\qquad \cos\delta = \frac{b/2 - y}{h}$$

Due to the two vortex sheets at $y = +\frac{b}{2}$ and $y = -\frac{b}{2}$

$$v_c = \frac{\gamma_1 \cdot \cot\alpha}{2\pi}\left\{-\int_0^\infty \frac{(z-\xi)\cdot d\xi}{(b/2-y)^2 + (z-\xi)^2} + \int_0^\infty \frac{(z-\xi)\cdot d\xi}{(b/2+y)^2 + (z-\xi)^2}\right\}$$

$$\boxed{v_c = -\frac{\gamma_1 \cdot \cot\alpha}{2\pi}\,\log\frac{\sqrt{(b/2-y)^2 + z^2}}{\sqrt{(b/2+y)^2 + z^2}}}$$

Due to the vortex sheet at $y = b/2$

$$w_{c_1} = \frac{-\gamma_1 \cdot \cot\alpha}{2\pi}\int_0^\infty \frac{(b/2 - y)\cdot d\xi}{(b/2 - y)^2 + (z-\xi)^2} = \frac{-\gamma_1 \cdot \cot\alpha}{2\pi}\cdot\left[\tan^{-1}\frac{\infty}{b/2 - y} + \tan^{-1}\frac{z}{b/2 - y}\right]$$

$$w_{c_1} = \frac{-\gamma_1 \cdot \cot\alpha}{2\pi}\left[\frac{\pi}{2} + \tan^{-1}\frac{z}{b/2 - y}\right] \qquad \text{for} \qquad y < \frac{b}{2}$$

$$w_{c_1} = \frac{\gamma_1 \cdot \cot\alpha}{2\pi}\left[\frac{\pi}{2} + \tan^{-1}\frac{z}{y - b/2}\right] \qquad y > \frac{b}{2}$$

Due to the vortex sheet at $y = -\frac{b}{2}$

$$w_{c_2} = \frac{-\gamma_1 \cdot \cot\alpha}{2\pi}\left[\frac{\pi}{2} + \tan^{-1}\frac{z}{b/2 + y}\right] \qquad \text{for} \qquad y > -b/2$$

$$w_{c_2} = \frac{\gamma_1 \cdot \cot\alpha}{2\pi}\left[\frac{\pi}{2} + \tan^{-1}\frac{-z}{b/2 + y}\right] \qquad y < -b/2$$

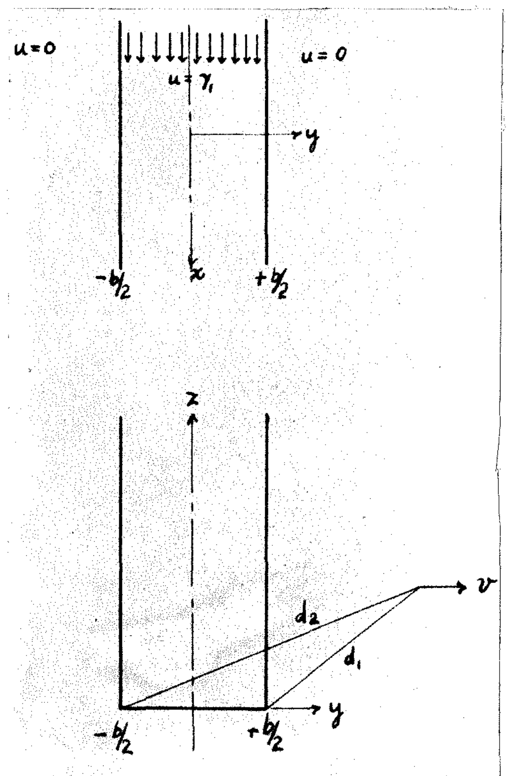
$$\therefore\ w_c = -\frac{\gamma_1 \cdot \cot\alpha}{2\pi}\left[\pi + \tan^{-1}\frac{z}{b/2 - y} + \tan^{-1}\frac{z}{b/2 + y}\right] \qquad \text{for } |y| < \frac{b}{2}$$

$$w_c = \frac{\gamma_1 \cdot \cot\alpha}{2\pi}\left[\tan^{-1}\frac{z}{|y| - b/2} - \tan^{-1}\frac{z}{|y| + b/2}\right] \qquad \text{for } |y| > \frac{b}{2}$$

The velocity field of the above potential motion thus consists of

(1) A uniform solid body flow $u = \gamma_1$ in the wake of the plate $|y| < \frac{b}{2}$, with u = 0 everywhere outside the wake $|y| > \frac{b}{2}$

(2) A flow in the y direction which is symmetrical about the xz-plane.



$$v = \frac{\gamma_1 \cdot \cot\alpha}{2\pi} \cdot \log \frac{\overbrace{\sqrt{(b/2-y)^2+z^2}}^{d_1}}{\underbrace{\sqrt{(b/2+y)^2+z^2}}_{d_2}}$$

$v = 0$ at $z = \infty$ for all y

$v = 0$ at $y = 0$ for all z

$v = \pm\infty$ at $z = 0$ $y = \pm\frac{b}{2}$

(3) A flow in the z-direction which is symmetrical about the xz-plane.

$$w_i = -\frac{\gamma_1 \cdot \cot\alpha}{2\pi} \cdot \left[\pi + \tan^{-1}\frac{z}{b/2+y} + \tan^{-1}\frac{z}{b/2-y}\right] \quad \text{for } |y| < \frac{b}{2}$$

$$w_o = -\frac{\gamma_1 \cdot \cot\alpha}{2\pi} \cdot \left[\tan^{-1}\frac{z}{|y|+b/2} - \tan^{-1}\frac{z}{|y|-b/2}\right] \quad \text{for } |y| > \frac{b}{2}$$

At $z = \infty$ $w = -\gamma_1 \cdot \cot\alpha$ $|y| < \frac{b}{2}$

$w = 0$ $|y| > \frac{b}{2}$

$z = 0$ $w = -\frac{\gamma_1 \cdot \cot\alpha}{2}$ $|y| < \frac{b}{2}$

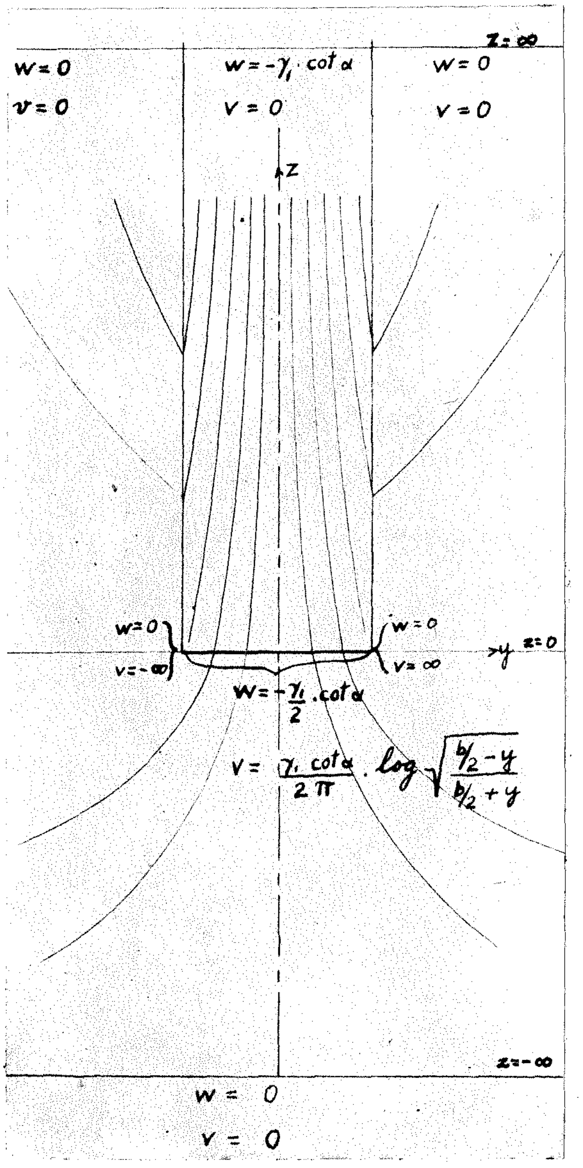
$w = 0$ $|y| > \frac{b}{2}$

$z = -\infty$ $w = 0$ all y

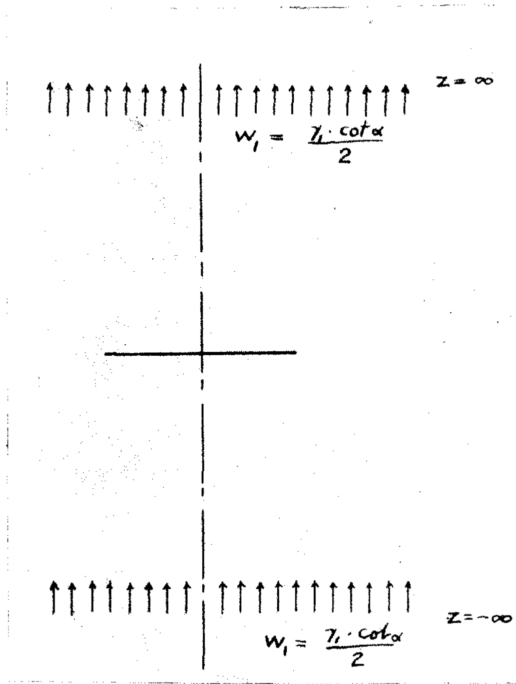
$z > 0$ $w = (-)$ down $|y| < \frac{b}{2}$

$w = (+)$ up $|y| > \frac{b}{2}$

$z < 0$ $w = (-)$ down all y



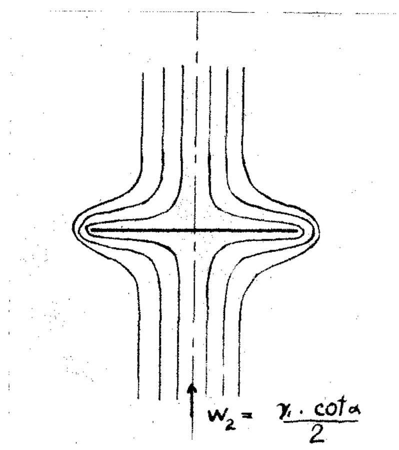
In order to satisfy the boundary conditions of our problem it is necessary to superimpose on the above flow



(1) A uniform flow of velocity $w_1 = \frac{\gamma_1 \cdot \cot\alpha}{2}$ to satisfy the condition of no flow through the plate. This leaves a flow $w = -\frac{\gamma_1 \cdot \cot\alpha}{2}$ at z = + ∞ and a flow $w = \frac{\gamma_1 \cdot \cot\alpha}{2}$ at z = - ∞ and zero flow through the plate.

(2) A displacement type flow around the plate with velocity at ∞, $w_2 = \frac{\gamma_1 \cdot \cot\alpha}{2}$

$$v - iw = -\frac{\gamma_1 \cdot \cot\alpha}{2} \cdot \frac{(y+iz)}{\sqrt{\left(\frac{b}{2}\right)^2 - (y+iz)^2}}$$

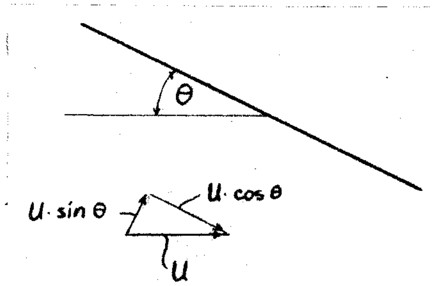


This gives us w = 0 at z = + ∞ as required by our boundary conditions and a flow $w = w_1 + w_2 = \gamma_1 \cdot \cot\alpha$ at z = - ∞

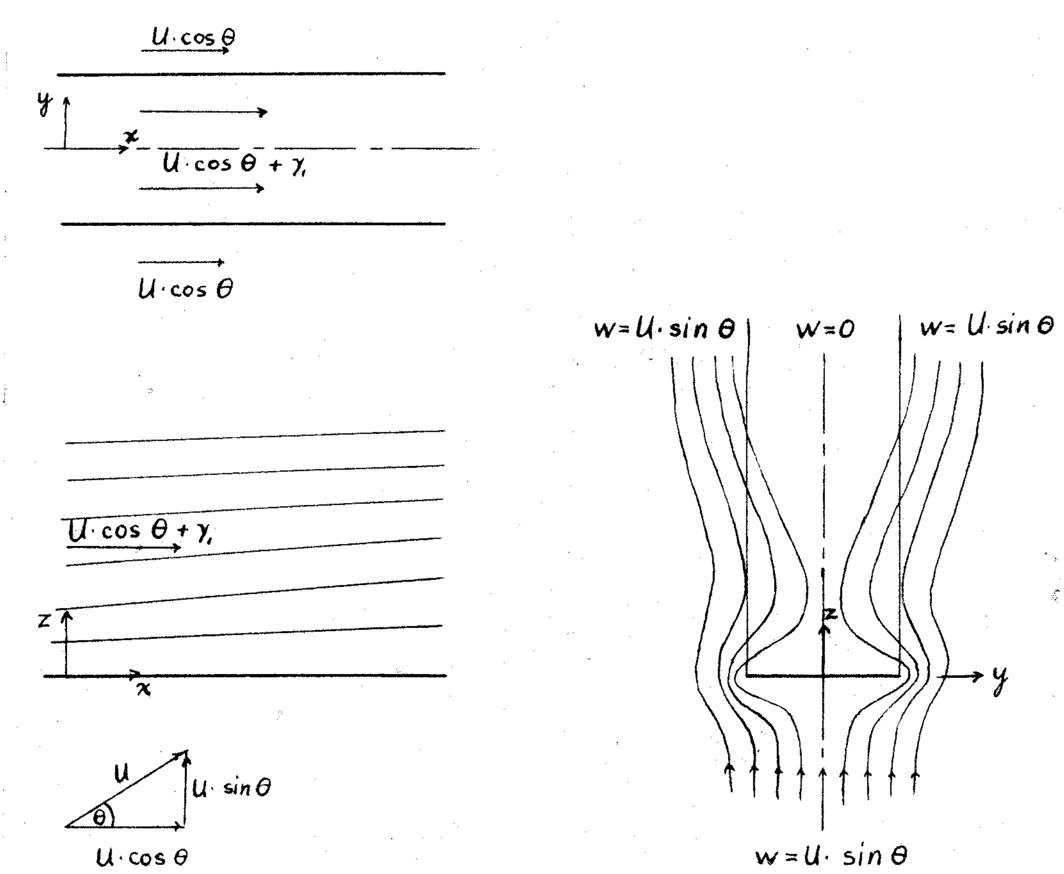
If we now consider the w-velocity at $z = -\infty$ to be the normal component of a velocity $\underline{U}$ i.e. $w = U \cdot \sin\theta$

Then

$$\boxed{\gamma_1 = U \sin\theta \cdot \tan\alpha}$$



It still remains to determine the angle α at which the vortices leave the plate. As before we shall assume that they remain in the planes passed through the edges of the plate but follow the mean u and w components of the flow in these planes. Let us consider the flow picture which we have below:



There is some flow v and w in the wake immediately behind the plate. Along y = 0, for instance, the w-velocity is

$$w\Big]_{y=0} = w_{induced} + w_{uniform} + w_{displacement}$$

$$= \left[-\frac{\gamma_1 \cdot \cot\alpha}{2} - \frac{\gamma_1 \cdot \cot\alpha}{\pi} \cdot \tan^{-1}\frac{z}{b/2} \right] + \left[\frac{\gamma_1 \cdot \cot\alpha}{2} \right]$$

$$+ \left[\frac{\gamma_1 \cdot \cot\alpha}{2} \cdot \frac{z}{\sqrt{(b/2)^2 + z^2}} \right]$$

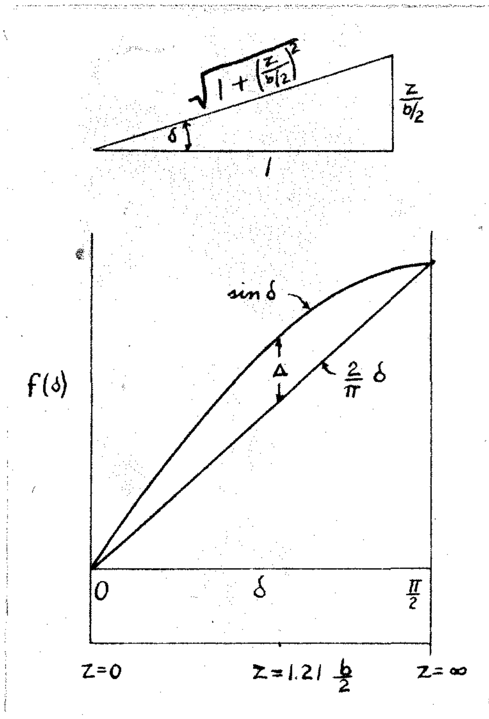
and since

$$\gamma_1 \cdot \cot\alpha = U \cdot \sin\theta$$

$$w\Big]_{y=0} = \frac{U \cdot \sin\theta}{2} \left[-\frac{2}{\pi} \cdot \tan^{-1}\frac{z}{b/2} + \frac{z/\frac{b}{2}}{\sqrt{\left(\frac{z}{b/2}\right)^2 + 1}} \right]$$

$$\therefore \quad w\Big]_{y=0} = \frac{U \cdot \sin\theta}{2} \Big[\underbrace{-\frac{2}{\pi}\,\delta + \sin\delta}_{+\Delta} \Big]$$

writing $\delta = \tan^{-1}\frac{z}{b/2}$



where $\Delta = \sin\delta - \frac{2}{\pi}\delta$ is a positive increment which vanishes at $z = 0$ and $z = \infty$. There is thus an influx over the vortex surface near the plate and the streamlines have an inclination with respect to the plate. This inclination which is zero right at the plate increases to a maximum at $\cos\delta = \frac{2}{\pi}$ which is $\frac{z}{b/2} = \sqrt{\frac{\pi^2}{4} - 1} = 1.21$

At that point $\delta = 50°$ and $w\Big]_{y=0} = .107\ U \cdot \sin\theta$.

For greater z the inclination decreases again approaching zero at $z = \infty$.

We can similarly determine the mean w-velocity along $y = \frac{b}{2}$. We have

$$\bar{w}\Big]_{y=\frac{b}{2}} = w_{induced} + w_{uniform} + w_{displacement}$$

where

$$w\Big]_{induced} = \frac{w_i + w_o}{2} = \frac{-\gamma_i \cdot \cot\alpha}{2\pi} \cdot \left[\frac{\pi + 2\tan^{-1}\frac{z}{b}}{2}\right]$$

$$= -U\cdot\sin\theta \cdot \left[\frac{1}{4} + \frac{1}{2\pi} \cdot \tan^{-1}\frac{z/\frac{b}{2}}{2}\right]$$

$$w\Big]_{uniform} = \frac{\gamma_i \cdot \cot\alpha}{2} = \frac{U\cdot\sin\theta}{2}$$

$$w\Big]_{displacement} = -\mathcal{I}\left(\frac{\partial}{\partial\xi}\left[\frac{-i\cdot U\cdot\sin\theta}{2} \cdot \sqrt{\xi^2 - \left(\frac{b}{2}\right)^2}\right]_{y=\frac{b}{2}}\right) \qquad \text{where } \xi = y + iz$$

$$= \frac{U\cdot\sin\theta}{2} \cdot \mathcal{R}e \frac{\xi}{\sqrt{\xi^2 - \left(\frac{b}{2}\right)^2}}$$

$$= \frac{U\cdot\sin\theta}{2} \cdot \mathcal{R}e \frac{1}{\sqrt{1 - \frac{(b/2)^2}{[b/2 + iz]^2}}} = \frac{U\cdot\sin\theta}{2} \cdot \mathcal{R}e \frac{1}{\sqrt{1 - \frac{1}{(1+i\eta)^2}}}$$

where $\eta = \frac{z}{b/2}$

$$\mathcal{R}e \frac{1}{\sqrt{1 - \frac{1}{(1+i\eta)^2}}} = \frac{(1+\eta^2)}{\sqrt[4]{4\eta^2 + (\eta^4 + 3\eta^2)^2}} \cdot \cos\left(\frac{1}{2}\tan^{-1}\frac{2}{\eta^3 + 3\eta}\right)$$

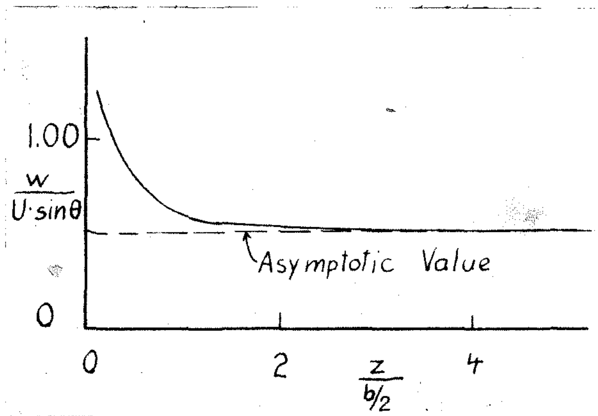
$$\therefore \bar{w}\Big]_{y=\frac{b}{2}} = U\cdot\sin\theta \cdot \left[\frac{1}{4} - \frac{1}{4}\,\frac{\tan^{-1}\frac{\eta}{2}}{\frac{\pi}{2}} + \frac{1}{2}\,\frac{(1+\eta^2)}{\sqrt[4]{4\eta^2 + (\eta^4 + 3\eta^2)^2}} \cdot \cos\left(\frac{1}{2}\tan^{-1}\frac{2}{\eta^3 + 3\eta}\right)\right]$$

This curve is plotted below. It shows that $\frac{\bar{w}}{U \cdot \sin\theta}$ approaches the asymptotic value 1/2 very rapidly. We shall use this value in calculating the effective angle. The mean value of the u-velocity at $y = b/2$ is $\bar{u} = U \cdot \cos\theta + \frac{\gamma_1}{2}$. Thus the effective angle of inclination of the vortices is

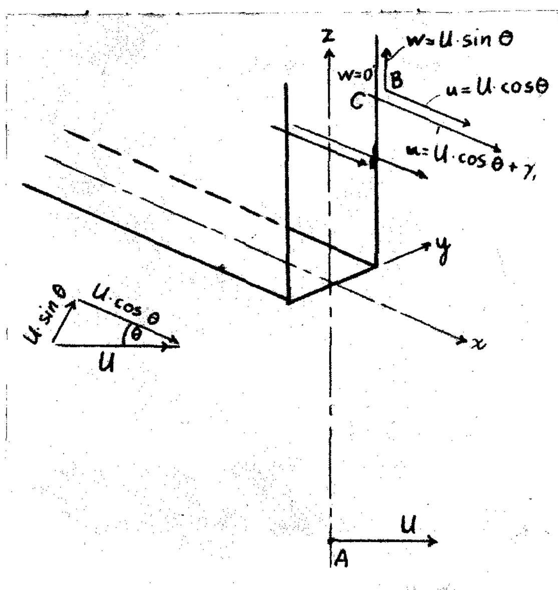
$$\alpha = \tan^{-1} \frac{\bar{w}}{\bar{u}} = \tan^{-1} \frac{\frac{U \cdot \sin\theta}{2}}{U \cdot \cos\theta + \frac{U \cdot \sin\theta \cdot \tan\alpha}{2}}$$

or

$$\boxed{\alpha = \frac{\theta}{2}}$$



In order to calculate the normal force acting on the wing it is now necessary to apply Bernoulli's equation. The Bernoulli constant H at a point A is the same as that at B, i.e.



$$H_1 = p + \tfrac{1}{2}\rho U^2$$

At a point just inside the vortex sheet the pressure is the same as outside and thus

$$H_2 = p + \tfrac{1}{2}\rho U^2$$

since $\quad \gamma_1 = U\cdot \sin\theta \cdot \tan\frac{\theta}{2} = U\cdot(1-\cos\theta)$

$$H_2 = p + \tfrac{1}{2}\rho U^2\cdot\left(\cos\theta + 1 - \cos\theta\right) = p + \tfrac{1}{2}\rho U^2 = H_1$$

The Bernoulli constants H_1 and H_2 are thus the same for the region inside and outside the vortex wake. The pressure difference Δp between upper and lower surfaces of the plate is thus

$$\Delta p = \tfrac{1}{2}\rho\left[u_u^2 + v_u^2 - u_\ell^2 - v_\ell^2\right]$$

$$u_\ell = U\cdot\cos\theta \qquad\qquad v_\ell = v_{induced} + v_{displacement}$$

$$u_u = U\cdot\cos\theta + \gamma_1 \qquad\qquad v_u = v_{induced} - v_{displacement}$$

where

$$\gamma_1 = U\cdot(1-\cos\theta)$$

$$v_{induced} = \frac{-U\cdot\sin\theta}{2\pi}\cdot\log\frac{b/2 - y}{b/2 + y}$$

$$v_{displacement} = \frac{U\cdot\sin\theta}{2}\,\frac{y/b/2}{\sqrt{1-\left(\frac{y}{b/2}\right)^2}}$$

The normal force N per unit length is then obtained as the integral of Δp over the span.

$$N = \int_{-\frac{b}{2}}^{+\frac{b}{2}} \Delta p \cdot dy = \frac{1}{2}\rho \int_{-b/2}^{+b/2} \left\{ \left[\underbrace{U \cdot \cos\theta + \gamma_1}_{U} \right]^2 - \left[U \cdot \cos\theta \right]^2 \right\} dy$$

$$+ \frac{1}{2}\rho \int_{-b/2}^{+b/2} \left\{ \left[v_{induced} - v_{displ.} \right]^2 - \left[v_{induced} + v_{displ.} \right]^2 \right\} dy$$

$$N = \frac{1}{2}\rho \int_{-b/2}^{+b/2} U^2 \cdot \sin^2\theta \cdot dy + \frac{1}{2}\rho \int_{-b/2}^{+b/2} \left[-4 \; v_{induced} \cdot v_{displ.} \right] \cdot dy$$

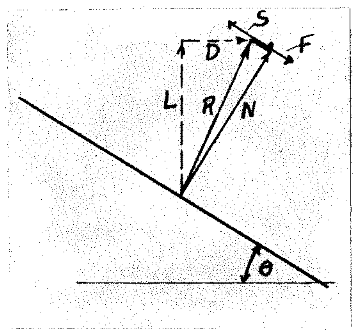
$$N = \frac{1}{2}\rho \; U^2 \cdot \sin^2\theta \cdot b \quad - \frac{1}{2}\rho U^2 \cdot \sin^2\theta \cdot b$$

$\therefore \; N = 0$ i.e. the normal force acting on the wing of truly infinite chord is zero for the combination of induced type flow and displacement type flow exactly as for either of these flows alone.

3. Conversion from Normal Force Coefficient to Lift and Drag Coefficient.

The principal force acting on a wing of small aspect ratio is the force N normal to the wing. For the region below the stall the only additional forces are the skin friction F acting to the rear and the suction force $\mathcal{S}$ acting forward on the wing. The magnitude of F can be estimated from the formula

$$F = C_f \cdot \tfrac{1}{2}\rho U^2 \cdot 2S$$



where C_f is the coefficient of skin friction for flat plates[6] which in the range of Reynold's numbers concerned is about $C_f \cong .003$. The suction force should be about the same as for the plane wing since we assumed the same formula for the vortex distribution as for the latter. Our vortex distribution was $\gamma = \gamma_0 \sqrt{\frac{t/2 - x}{t/2 + x}}$. For x near to $-t/2$ this reduces to $\gamma\Big]_{x \cong -t/2} = \frac{\gamma_0 \sqrt{t}}{\sqrt{t/2 + x}}$. Von Kármán[7] shows that the suctional force for such a distribution is

$$\mathcal{S} = \pi\rho \, \frac{\gamma_0^2 \, t b}{4} = \underbrace{\frac{\pi}{2} \cdot \left(\frac{\gamma_0}{U}\right)^2}_{C_s} \cdot \tfrac{1}{2}\rho U^2 \cdot bt$$

The lift and drag forces thus become,

$$L = N \cdot \cos\theta + (S-F)\cdot \sin\theta$$

$$D = N \cdot \sin\theta - (S-F)\cdot \cos\theta$$

Assuming

$$\alpha = \theta/2 \qquad \left(\frac{\gamma_0}{U}\right) \cong \frac{C_N}{\pi} \qquad \therefore\ C_s \cong \frac{C_N^2}{2\pi}$$

$$\alpha = \theta \qquad \left(\frac{\gamma_0}{U}\right) \cong \frac{C_N}{\pi}\cdot\cos\theta \qquad C_s \cong \frac{C_N^2}{2\pi}\cdot\cos\theta$$

The lift and drag coefficients become

for $\alpha = \frac{\theta}{2}$

$$C_L = C_N \cdot \cos\theta + \left(\frac{C_N^2}{2\pi} - .006\right)\cdot \sin\theta$$

$$C_D = C_N \cdot \sin\theta - \left(\frac{C_N^2}{2\pi} - .006\right)\cdot \cos\theta$$

for $\alpha = \theta$

$$C_L = C_N \cdot \cos\theta + \left(\frac{C_N^2}{2\pi}\cdot\cos\theta - .006\right)\cdot \sin\theta$$

$$C_D = C_N \cdot \sin\theta - \left(\frac{C_N^2}{2\pi}\cdot\cos\theta - .006\right)\cdot \cos\theta$$

The frictional force is negligible compared to the other terms and thus

for $\alpha = \frac{\theta}{2}$

$$C_L = C_N \cdot \cos\theta + \frac{C_N^2}{2\pi}\cdot \sin\theta$$

$$C_D = C_N \cdot \sin\theta - \frac{C_N^2}{2\pi}\cdot \cos\theta$$

for $\alpha = \theta$

$$C_L = C_N \cdot \cos\theta + \frac{C_N^2}{2\pi}\cdot \sin\theta\cdot\cos\theta$$

$$C_D = C_N \cdot \sin\theta - \frac{C_N^2}{2\pi}\cdot \cos^2\theta$$

By means of these formulas the theoretical curve of C_N vs. θ was transformed into curves of C_L vs. θ and C_L vs. C_D for the plate with aspect ratio $k = 1/30$ and $\alpha = \frac{\theta}{2}$. The result is shown in the graph on the next page.

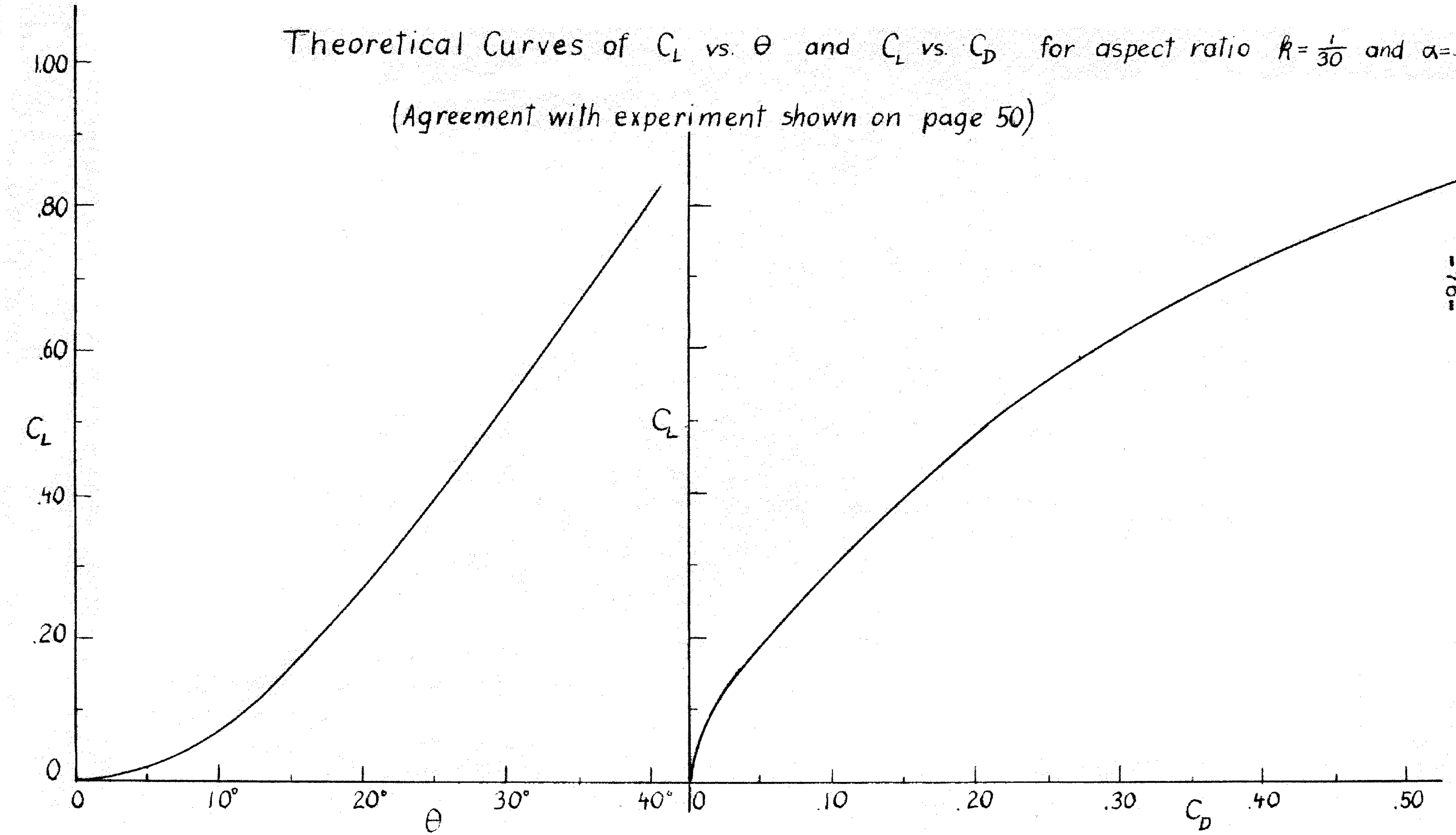
The angle of down-wash θ_i is defined as the angle between the resultant force and the direction perpendicular to the undisturbed flow. This can be written at once as

$$\theta_i = \theta - \frac{S-F}{N}$$

for $\alpha = \frac{\theta}{2}$ $$\theta_i = \theta - \frac{\frac{C_N^2}{2\pi} - .006}{C_N}$$

and for $\alpha = \theta$ $$\theta_i = \theta - \frac{\frac{C_N^2 \cdot \cos\alpha}{2\pi} - .006}{C_N}$$

respectively.



Theoretical Curves of C_L vs. θ and C_L vs. C_D for aspect ratio $R = \frac{1}{30}$ and $\alpha = \frac{\theta}{2}$

(Agreement with experiment shown on page 50)

3. Mathematical Part — Solution of Integrals

a.) $F_1(k)$:

$$F_1(k) = \int_{-1}^{+1} \sqrt{\frac{1-\xi}{1+\xi}} \left\{ \log \frac{k + \sqrt{k^2 + (1+\xi)^2}}{k + \sqrt{k^2 + (1-\xi)^2}} + \log \frac{1-\xi}{1+\xi} \right\} d\xi$$

Let $\xi = \cos 2\theta$, then

$$\int_{-1}^{+1} \sqrt{\frac{1-\xi}{1+\xi}} \log \frac{1-\xi}{1+\xi} \cdot d\xi = 8 \int_0^{\frac{\pi}{2}} \sin^2\theta \cdot \log \tan\theta \cdot d\theta$$

$$= 2\int_0^{\pi} (1-\cos\tau)\cdot\left\{\log \sin\tau - \log(1+\cos\tau)\right\}\cdot d\tau \qquad \theta = \frac{\tau}{2}$$

$$= \underbrace{2\int_0^{\pi} \log \sin\tau \cdot d\tau}_{-\pi\cdot\log 2} - \underbrace{2\int_0^{\pi} \cos\tau \cdot \log \sin\tau \cdot d\tau}_{0 \text{ by symmetry}} - \underbrace{2\int_0^{\pi} \log(1+\cos\tau)\cdot d\tau}_{+\pi\cdot\log 2}$$

$$+ 2\int_0^{\pi} \cos\tau \cdot \log(1+\cos\tau)\cdot d\tau$$

$$= 2\Big[\sin\tau \cdot \log(1+\cos\tau)\Big]_0^{\pi} + 2\int_0^{\pi} \frac{\sin^2\tau \cdot d\tau}{1+\cos\tau}$$

$$= 2\int_0^{\pi} (1-\cos\tau)\cdot d\tau = 2\pi$$

$$\therefore \boxed{\int_{-1}^{+1} \sqrt{\frac{1-\xi}{1+\xi}} \cdot \log \frac{(1-\xi)}{(1+\xi)} \cdot d\xi = 2\pi}$$

$$\int_{-1}^{+1}\sqrt{\frac{1-\xi}{1+\xi}}\cdot\log\ k+\sqrt{k^2+(1+\xi)^2}\cdot d\xi-\int_{-1}^{+1}\sqrt{\frac{1-\xi}{1+\xi}}\cdot\log\ k+\sqrt{k^2+(1-\xi)^2}\cdot d\xi$$

$$=\int_0^2\sqrt{\frac{2-u}{u}}\cdot\log\left\{k+\sqrt{k^2+u^2}\right\}\cdot du-\int_0^2\sqrt{\frac{u}{2-u}}\cdot\log\left\{k+\sqrt{k^2+u^2}\right\}\cdot du$$

$$=\int_0^2\frac{(2-2u)}{\sqrt{u}\cdot\sqrt{2-u}}\cdot\log\left\{k+\sqrt{k^2+u^2}\right\}\cdot du$$

Let $\log\left\{k+\sqrt{k^2+u^2}\right\} \doteq \log(k+u)+a\cdot e^{-\lambda u}$

At $u=0$ $\quad \log 2k = \log k + a$

$$\therefore\ \boxed{a=\log_e 2=.69315}$$

At $u=k$ $\quad \log\left\{k+\sqrt{k^2+k^2}\right\}=\log 2k+\log 2\cdot e^{-\lambda k}$

$$\therefore\ e^{-\lambda k}\cdot\log 2=\log\frac{1+\sqrt{2}}{2}$$

$$e^{-\lambda k}=\frac{\log\frac{1+\sqrt{2}}{2}}{\log 2}$$

$$\therefore\ \lambda=-\frac{1}{k}\log\frac{\log\frac{1+\sqrt{2}}{2}}{\log 2}$$

$$\boxed{\lambda=\frac{1.302}{k}}$$

$$\therefore \quad \log \mathscr{k} + \sqrt{\mathscr{k}^2+u^2} = \log(\mathscr{k}+u) + \log_e 2 \cdot e^{-\lambda u} \quad \text{with} \quad \lambda = \frac{1.302}{\mathscr{k}}$$

and
$$\int_{-1}^{+1} \sqrt{\frac{1-\xi}{1+\xi}} \left\{ \log \frac{\mathscr{k} + \sqrt{\mathscr{k}^2+(1+\xi)^2}}{\mathscr{k}+\sqrt{\mathscr{k}^2+(1-\xi)^2}} \right\} \cdot d\xi = 2\int_0^2 \frac{(1-u)\cdot \log(\mathscr{k}+u)\cdot du}{\sqrt{u}\cdot\sqrt{2-u}}$$

$$+ 2\cdot \log_e 2 \cdot \int_0^2 \frac{(1-u)\cdot e^{-\lambda u}\cdot du}{\sqrt{u}\cdot\sqrt{2-u}}$$

$$\int_0^2 \frac{(1-u).\ \log(\mathscr{k}+u)\cdot du}{\sqrt{u}\cdot\sqrt{2-u}} = 2\int_0^{\sqrt{2}} \frac{(1-v^2).\ \log(\mathscr{k}+v^2)\cdot dv}{\sqrt{2-v^2}}$$

$u = v^2$

$v = \sqrt{2}\cdot\sin\theta$

$\tau = 2\theta$

$$= 2\int_0^{\frac{\pi}{2}} (1-2\sin^2\theta)\cdot\log\left(\mathscr{k}+2\sin^2\theta\right).d\theta = \int_0^{\pi} \cos\tau\cdot\log\left(\mathscr{k}+1-\cos\tau\right)\cdot d\tau$$

$$= \Big[\underbrace{\sin\tau\cdot\log(\mathscr{k}+1-\cos\tau)}_{0}\Big]_0^{\pi} - \int_0^{\pi} \frac{\sin^2\tau\cdot d\tau}{\mathscr{k}+1-\cos\tau}$$

let $x = \tan\frac{\tau}{2}$

$b^2 = \frac{\mathscr{k}}{\mathscr{k}+2}$

$$= \frac{-8}{\mathscr{k}+2}\int_0^{\infty} \frac{x^2\cdot dx}{(1+x^2)^2\cdot\left(\frac{\mathscr{k}}{\mathscr{k}+2}+x^2\right)} = \frac{-8}{\mathscr{k}+2}\left[\int_0^{\infty}\frac{dx}{(1+x^2)^2} - b^2\int_0^{\infty}\frac{dx}{(1+x^2)^2\cdot(b^2+x^2)}\right]$$

The indefinite integral of the first gives:

$$\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \left[\frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1}x\right]_0^{\infty} = \frac{\pi}{4}$$

The definite integral: $\displaystyle\int_0^{\infty}\frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2}\,\frac{1}{ab\,(a+b)}$ (Edwards[8] p. 217)

Differentiating this with respect to $\underline{a}$, and then setting $a=1$

$$\int_0^{\infty}\frac{dx}{(1+x^2)^2(b^2+x^2)} = \frac{\pi}{4b}\cdot\frac{2+b}{(1+b)^2}$$

$$\therefore \quad \boxed{2\int_0^2 \frac{(1-u)\cdot\log(k+u)\cdot du}{\sqrt{u}\cdot\sqrt{2-u}} = \frac{-4\pi}{k+2} + \frac{4\pi}{k+2}\cdot\sqrt{\frac{k}{k+2}}\,\frac{\sqrt{\frac{k}{k+2}}+2}{\left(\sqrt{\frac{k}{k+2}}+1\right)^2}}$$

$$\int_0^2 \frac{(1-u)\cdot e^{-\lambda u}\cdot du}{\sqrt{u}\cdot\sqrt{2-u}} = 2\int_0^{\sqrt{2}} \frac{(1-v^2)\cdot e^{-\lambda v^2}\cdot dv}{\sqrt{2-v^2}} = 2\int_0^{\frac{\pi}{2}} (1-2\cdot\sin^2\theta)\cdot e^{-2\lambda\cdot\sin^2\theta}\cdot d\theta$$

$$= e^{-\lambda}\int_0^{\pi} \cos\tau\cdot e^{\lambda\cdot\cos\tau}\cdot d\tau = e^{-\lambda}\cdot\frac{\partial}{\partial\lambda}\int_0^{\pi} e^{\lambda\cdot\cos\tau}\cdot d\tau$$

$u = v^2$

$v = \sqrt{2}\cdot\sin\theta$

$\theta = \frac{\tau}{2}$

$x = \cos\tau$

But $\displaystyle\int_0^{\pi} e^{\lambda\cdot\cos\tau}\cdot d\tau = \int_{-1}^{+1}\frac{e^{\lambda x}\cdot dx}{\sqrt{1-x^2}} = \pi\cdot I_0(\lambda)$

Where $I_0(\lambda)$ is the Bessel Function with imaginary argument of order zero. (Macrobert[9)])

And since $\dfrac{\partial I_0(\lambda)}{\partial\lambda} = I_1(\lambda)$ Reference 10) p. 20

$$\therefore \quad \boxed{2\cdot\log 2\cdot\int_0^2 \frac{(1-u)\cdot e^{-\lambda u}\cdot du}{\sqrt{u}\cdot\sqrt{2-u}} = 2\pi\cdot\log 2\cdot e^{-\lambda}\cdot I_1(\lambda)}$$

$$\boxed{F_1(k) = \frac{2\pi k}{k+2} + \frac{4\pi}{k+2}\cdot\sqrt{\frac{k}{k+2}}\,\frac{\sqrt{\frac{k}{k+2}}+2}{\left(\sqrt{\frac{k}{k+2}}+1\right)^2} + 2\pi\cdot\log 2\cdot e^{-\frac{1.302}{k}}\cdot I_1\left(\frac{1.302}{k}\right)}$$

and $\displaystyle F_1'(k) = \frac{F_1(k)}{2\pi}$

b) $F_2(R,\alpha)$

$$F_2(R,\alpha)=\int_{-1}^{+1}\sqrt{\frac{1-\xi}{1+\xi}}\cdot\tan^{-1}\left\{\frac{(1+\xi)\cdot\sin\alpha}{R}\right\}\cdot d\xi+\int_{-1}^{+1}\sqrt{\frac{1-\xi}{1+\xi}}\cdot\tan^{-1}\left\{\frac{(1-\xi)\cdot\sin\alpha}{R}\right\}d\xi$$

Let $\dfrac{(1+\xi)\cdot\sin\alpha}{R}=u^2$ and $\dfrac{(1-\xi)\cdot\sin\alpha}{R}=u^2$ respectively

and $\dfrac{2\cdot\sin\alpha}{R}=a^2$

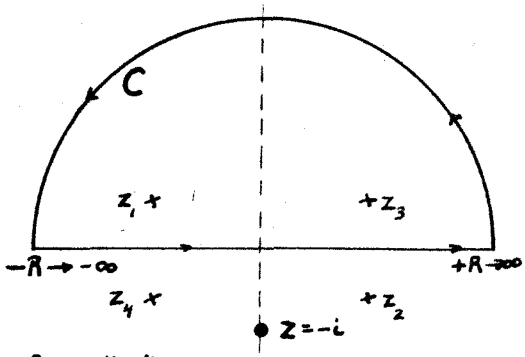
Then

$$F_2(R,\alpha)=4\int_0^a\frac{\tan^{-1}u^2\cdot du}{\sqrt{a^2-u^2}}=\left[4\tan^{-1}u^2\cdot\sin^{-1}\frac{u}{a}\right]_{u=0}^{u=a}-8\int_0^a\frac{u\cdot\sin^{-1}\frac{u}{a}\cdot du}{1+u^4}$$

$$=2\pi\cdot\tan^{-1}a^2-4a^2\int_0^{\frac{\pi}{2}}\frac{2t\cdot\sin t\cdot\cos t\cdot dt}{1+a^4\cdot\sin^4 t}\qquad u=a\cdot\sin t$$

$$=2\pi\cdot\tan^{-1}a^2-4a^2\int_{-\infty}^{\infty}\frac{x\cdot\arctan x\cdot dx}{(1+x^2)^2+a^4x^4}$$

Consider the complex integral $\displaystyle\int_C\frac{z\cdot\log(1-iz)\cdot dz}{(1+z^2)^2+a^4z^4}$ where $z=x+iy$ around the contour $\underline{C}$ shown below. The integrand has the following singularities:



(1) A branch-point at $z=-i$. This point lies outside of our contour $\underline{C}$, however, and thus has no effect.

(2) Four simple poles at the zeros of $(1+z^2)^2+a^4z^4$. It will be shown that two of these lie in the upper half-plane and two in the lower. Only the first two will contribute to the residues which we desire.

Thus

$$\int_C \frac{z \cdot \log(1-iz)\cdot dz}{(1+z^2)^2 + a^4 z^4} = \int_{-R}^{R} \frac{-ix \cdot \arctan x \cdot dx}{(1+x^2)^2 + a^4 x^4} + \int_0^{\pi} \frac{Re^{i\theta} \cdot \log(1-iRe^{i\theta}) \cdot iRe^{i\theta} \cdot d\theta}{(1+R^2 e^{2i\theta})^2 + a^4 R^4 e^{4i\theta}}$$

As the radius $\underline{R}$ of the semi-circle tends to ∞, it can be shown that the line-integral along the semi-circle vanishes as it is of order $\left(\frac{1}{R}\right)$. At the same time the integral along the real axis approaches the desired integral which extends from $x = -\infty$ to $x = +\infty$. By Cauchy's Theorem we then get

$$\int_{-\infty}^{\infty} \frac{x \cdot \arctan x \cdot dx}{(1+x^2)^2 + a^4 x^4} = 2\pi i \left[\operatorname{Res} f(z)\Big]_{z_1} + \operatorname{Res} f(z)\Big]_{z_3} \right]$$

The poles are the roots of $a^4 z^4 + (1+z^2)^2 = 0$

$$(1+z^2) = z^2 a^2 e^{\frac{\pi i}{2}} \quad \text{or} \quad (1+z^2) = z^2 a^2 e^{\frac{3\pi i}{2}}$$

$\therefore$ Either

$$z^2 = -\frac{(1+a^2 i)}{1+a^4} \quad \text{or} \quad z^2 = \frac{-(1-a^2 i)}{1+a^4}$$

$$z_1 = \frac{e^{i\left(\frac{\pi}{2} + \frac{1}{2}\arctan a^2\right)}}{\sqrt[4]{1+a^4}} \qquad z_3 = \frac{e^{i\left(\frac{\pi}{2} - \frac{1}{2}\arctan a^2\right)}}{\sqrt[4]{1+a^4}}$$

$$z_2 = \frac{e^{i\left(\frac{3\pi}{2} + \frac{1}{2}\arctan a^2\right)}}{\sqrt[4]{1+a^4}} \qquad z_4 = \frac{e^{i\left(\frac{3\pi}{2} - \frac{1}{2}\arctan a^2\right)}}{\sqrt[4]{1+a^4}}$$

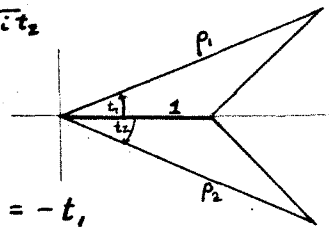
By $\underline{\arctan a^2}$ we refer to the principal value lying between 0 and $\frac{\pi}{2}$.
The residues at the poles $\underline{z_1}$ and $\underline{z_3}$ which lie within the contour are:

$$\operatorname{Res} f(z)\Big]_{z_1} = \lim_{z \to z_1} (z - z_1)\cdot f(z) = \frac{i}{4a^2} \cdot \log\left(1 + \frac{e^{\frac{i}{2}\cdot \tan^{-1} a^2}}{\sqrt[4]{1+a^4}}\right)$$

$$\operatorname{Res} f(z)\Big]_{z_3} = \lim_{z \to z_3} (z - z_3)\cdot f(z) = \frac{-i}{4a^2} \cdot \log\left(1 + \frac{e^{-\frac{i}{2}\cdot \tan^{-1} a^2}}{\sqrt[4]{1+a^4}}\right)$$

Therefore $$-i\int_{-\infty}^{\infty}\frac{x\cdot\arctan x\cdot dx}{(1+x^2)^2+a^4x^4}=\frac{-\pi}{2a^2}\log\frac{1+\dfrac{e^{\frac{i}{2}\tan^{-1}a^2}}{\sqrt[4]{1+a^4}}}{1+\dfrac{e^{-\frac{i}{2}\tan^{-1}a^2}}{\sqrt[4]{1+a^4}}}$$

$$=-\frac{\pi}{2a^2}\log\frac{\rho_1 e^{it_1}}{\rho_2 e^{it_2}}$$



But $$\log\frac{\rho_1 e^{it_1}}{\rho_2 e^{it_2}}=\log\frac{\rho_1}{\rho_2}+i\,(t_1-t_2)$$

By inspection of the vector diagram $\rho_1=\rho_2$ and $t_2=-t_1$

$$\therefore \log\frac{\rho_1 e^{it_1}}{\rho_2 e^{it_2}}=2i\,t_1=2i\cdot\tan^{-1}\left\{\frac{\sin\left(\frac{1}{2}\tan^{-1}a^2\right)}{\sqrt[4]{1+a^4}+\cos\left(\frac{1}{2}\tan^{-1}a^2\right)}\right\}$$

$$\therefore \int_{-\infty}^{\infty}\frac{x\cdot\arctan x\cdot dx}{(1+x^2)^2+a^4x^4}=\frac{\pi}{a^2}\cdot\tan^{-1}\left\{\frac{\sin\left(\frac{1}{2}\tan^{-1}a^2\right)}{\sqrt[4]{1+a^4}+\cos\left(\frac{1}{2}\tan^{-1}a^2\right)}\right\}$$

$$\boxed{\therefore F_2(k,\alpha)=2\pi\tan^{-1}\left\{\frac{2\sin\alpha}{k}\right\}-4\pi\tan^{-1}\left\{\frac{\sin\left(\frac{1}{2}\tan^{-1}\left[\frac{2\sin\alpha}{k}\right]\right)}{\sqrt[4]{1+\left(\frac{2\sin\alpha}{k}\right)^2}+\cos\left(\frac{1}{2}\tan^{-1}\left[\frac{2\sin\alpha}{k}\right]\right)}\right\}}$$

c. $F_3(R,\alpha)$

$$F_3(R,\alpha) = \int_{-1}^{+1} \sqrt{\frac{1-\xi}{1+\xi}} \cdot \left\{ \tan^{-1}\left[\frac{\tan\alpha}{R} \sqrt{R^2+(1+\xi)^2} \right] - \tan^{-1}\left[\frac{\tan\alpha}{R} \sqrt{R^2+(1-\xi)^2} \right] \right\} d\xi$$

$$= 2\int_0^{+2} \frac{(1-u)}{\sqrt{u}\,\sqrt{2-u}} \cdot \tan^{-1}\left\{ \frac{\tan\alpha}{R} \sqrt{R^2+u^2} \right\} \cdot du$$

Now let $\quad \tan^{-1}\left\{ \frac{\tan\alpha}{R} \sqrt{R^2+u^2} \right\} \doteq \tan^{-1}\left\{ \frac{\tan\alpha \cdot u}{R} \right\} + a \cdot e^{-\lambda u}$

At $u = 0 \qquad \tan^{-1}\{\tan\alpha\} = a \qquad$ or $\qquad \boxed{a = \alpha}$

At $u = R \qquad \tan^{-1}\{\tan\alpha \cdot \sqrt{2}\} = \alpha + \alpha \cdot e^{-\lambda R}$

$$e^{-\lambda R} = \frac{\tan^{-1}(\sqrt{2}\cdot\tan\alpha)}{\alpha} - 1$$

$$\boxed{\lambda = -\frac{1}{R} \cdot \log\left\{ \frac{\tan^{-1}(\sqrt{2}\cdot\tan\alpha)}{\alpha} - 1 \right\}}$$

Then

$$F_3(R,\alpha) = 2\int_0^2 \frac{(1-u)}{\sqrt{u}\cdot\sqrt{2-u}} \cdot \tan^{-1}\left\{ \frac{\tan\alpha \cdot u}{R} \right\} . du + 2\alpha \int_0^2 \frac{(1-u)\cdot e^{-\lambda u} \cdot du}{\sqrt{u}\cdot\sqrt{2-u}}$$

The first integral we shall obtain by a contour integration. The second integral we have already shown to be

$$\boxed{2\alpha \cdot \int_0^2 \frac{(1-u)\cdot e^{-\lambda u} \cdot du}{\sqrt{u}\cdot\sqrt{2-u}} = 2\pi \cdot \alpha \cdot e^{-\lambda} \cdot I_1(\lambda)}$$

$$\int_0^2 \frac{(1-u)\cdot \tan^{-1}\left\{\frac{\tan\alpha}{k}u\right\}\cdot du}{\sqrt{u}\cdot\sqrt{2-u}} = 2\int_0^a \frac{\left(1-\frac{2t^2}{a^2}\right)\cdot \tan^{-1}t^2\cdot dt}{\sqrt{a^2-t^2}}$$

$t^2 = \frac{\tan\alpha}{k}\cdot u$

$a^2 = \frac{2\tan\alpha}{k}$

$t = a\cdot\sin\theta$

$x = \tan\theta$

$$= 2\left[\underbrace{\frac{t}{a^2}\sqrt{a^2-t^2}\cdot\ \tan^{-1}t^2}_{0}\right] - \frac{4}{a^2}\int_0^a \frac{t^2\cdot\sqrt{a^2-t^2}\cdot dt}{1+t^4}$$

$$= -4\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cdot a\cos\theta\cdot a\cos\theta\cdot d\theta}{1+a^4\cdot\sin^4\theta} = -4a^2\int_0^\infty \frac{x^2}{1+x^2}\,\frac{dx}{(1+x^2)^2+a^4x^4}$$

$$= -2a^2\int_{-\infty}^\infty \frac{x^2}{1+x^2}\,\frac{dx}{(1+x^2)^2+a^4x^4}$$

Consider $\int_C \frac{z^2\cdot dz}{(1+z^2)\left[(1+z^2)^2+a^4z^4\right]}$ around the same contour $\underline{C}$ as before.

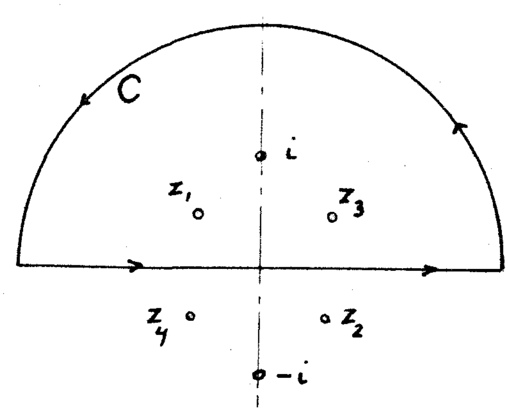
The poles of this function are at $z=+i$, $z=-i$, $z=z_1$, $z=z_2$, $z=z_3$ and $z=z_4$. The last four are the same as in the preceding case. Also the integral along the infinite semi-circle again vanishes. Along the real axis it reduces to our desired integral. By Cauchy's Theorem we then get

$$\int_{-\infty}^\infty \frac{x^2\cdot dx}{(1+x^2)\left[(1+x^2)^2+a^4x^4\right]} = 2\pi i\left[\operatorname{Res} f(z)\Big]_{z=i} + \operatorname{Res} f(z)\Big]_{z_1} + \operatorname{Res} f(z)\Big]_{z_3}\right]$$

since the only poles included by the contour are $z=i$, z_1 and z_3 where

$$z_1 = \frac{i\,e^{\frac{i}{2}\tan^{-1}a^2}}{\sqrt[4]{1+a^4}}$$

$$z_3 = \frac{i\,e^{-\frac{i}{2}\tan^{-1}a^2}}{\sqrt[4]{1+a^4}}$$



The residues at these poles are

$$\operatorname{Res} f(z)\Big]_{z=i} = \lim_{z\to i} (z-i)\cdot f(z) = \frac{i}{2a^4}$$

$$\operatorname{Res} f(z)\Big]_{z_1} = \lim_{z\to z_1} (z-z_1)\cdot f(z) = \frac{-i\cdot e^{\frac{i}{2}\tan^{-1}a^2}\cdot(1-ia^2)}{\sqrt[4]{1+a^4}\cdot 4a^4}$$

$$\operatorname{Res} f(z)\Big]_{z_3} = \lim_{z\to z_3} (z-z_3)\cdot f(z) = \frac{-i\cdot e^{-\frac{i}{2}\tan^{-1}a^2}\cdot(1+ia^2)}{\sqrt[4]{1+a^4}\cdot 4a^4}$$

Let $\delta = \dfrac{\tan^{-1}a^2}{2}$, then

$$\operatorname{Res} f(z)\Big]_i + \operatorname{Res} f(z)\Big]_{z_1} + \operatorname{Res} f(z)\Big]_{z_3} = \frac{i}{2a^4}\left[1 - \frac{(1-ia^2)\cdot e^{i\delta}}{2\cdot\sqrt[4]{1+a^4}} - \frac{(1+ia^2)\cdot e^{-i\delta}}{2\cdot\sqrt[4]{1+a^4}}\right]$$

$$= \frac{i}{2a^4}\left[1 - \frac{\cos\delta}{\sqrt[4]{1+a^4}} - \frac{a^2\cdot\sin\delta}{\sqrt[4]{1+a^4}}\right]$$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2\cdot dx}{(1+x^2)\cdot\left[(1+x^2)^2+a^4x^4\right]} = \frac{\pi}{a^4}\left[-1 + \frac{\cos(\frac{1}{2}\tan^{-1}a^2)}{\sqrt[4]{1+a^4}} + \frac{a^2\cdot\sin(\frac{1}{2}\tan^{-1}a^2)}{\sqrt[4]{1+a^4}}\right]$$

$$\boxed{2\int_0^2 \frac{(1-u)\cdot\tan^{-1}\left\{\frac{\tan\alpha}{k}u\right\}\cdot du}{\sqrt{u}\cdot\sqrt{2-u}} = \frac{4\pi}{a^2}\left[1 - \frac{\cos(\frac{1}{2}\tan^{-1}a^2)}{\sqrt[4]{1+a^4}} - \frac{a^2\cdot\sin(\frac{1}{2}\tan^{-1}a^2)}{\sqrt[4]{1+a^4}}\right]}$$

$$\therefore F_3(k,\alpha) = 4\pi\cdot\frac{k}{2\tan\alpha}\left[1 - \frac{\cos\left\{\frac{1}{2}\tan^{-1}\left[\frac{2\tan\alpha}{k}\right]\right\} + \frac{2\tan\alpha}{k}\cdot\sin\left\{\frac{1}{2}\tan^{-1}\left[\frac{2\tan\alpha}{k}\right]\right\}}{\sqrt[4]{1+\left(\frac{2\cdot\tan\alpha}{k}\right)^2}}\right]$$

$$+ 2\pi\alpha\cdot e^{-\lambda}\cdot I_1(\lambda)$$

where $\lambda = -\dfrac{1}{k}\log\left\{\dfrac{\tan^{-1}(\sqrt{2}\cdot\tan\alpha)}{\alpha} - 1\right\}$

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