ANALYTICAL APPROXIMATIONS TO THE SOLUTIONS OF THE EQUATIONS OF MOTION IN EARTH-MOON SPACE

Thesis by

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ABSTRACT

Two methods of obtaining approximate solutions of the equations of motion in the Earth-Moon space are derived. The first method - asymptotic expansions of the solutions of the equations of motion - is a power series expansion of the solutions in powers of the inverse maximum velocity $\left[\left(\frac{1}{U}\right)^n\right]$. A comparison of the results of numerical integration with the asymptotic expansions is presented, which shows the range of applicability of this method.

The second method is similar to the small perturbation approach. In this method the zeroth order solution is a Keplerian orbit about the Earth (the Moon's effect being neglected). The first order solution corrects for the lunar gravity effects on the zeroth order trajectory. To demonstrate the computational difficulties involved in the application of this method, a straight line Keplerian trajectory was used as the zeroth order solution. Several applications of the solutions are discussed.

CONTENTS

I. INTRODUCTION

Interest in the problem of the motion of a small body in earth-moon space has increased greatly within recent years.

The complexity of the problem does not permit an all-encompassing analytical solution; therefore, the possible approaches left are (a) exact numerical solutions of particular cases, (b) analytical solutions of simplified models, and (c) analytical approximate solutions for certain ranges of the parameters involved. Numerous examples of the first approach can be found in the literature. Several examples of the second approach can also be found. In this thesis an attempt is made to use the third approach.

The advantages of having analytical solutions are well known and appreciated. In addition to the general advantages of relating initial and final conditions, good analytical approximations could:

A. Provide a method for checking the accuracy of numerical integration schemes.

B. Provide a convenient method for determining thrust maneuvers (impulsive and continuous) in the earth-moon space.

II. EARTH-MOON SYSTEM MODEL

For the sake of simplicity, and in order to bring out the salient points of the problem, a simplified model of the earth-moon space was used in this study. The earth and the moon are assumed to be spherical bodies gravitating like point masses isolated in space and revolving in circles about their common center of mass. (It should be noted that any improvement in this model will limit the applicability of the results to certain periodic times.)

The equations of motion of a vehicle in the above model of earth-moon space can be written in a coordinate system with its origin at the earth moon center of mass and rotating at the same rate as the earth and the moon from an initial position chosen so that the earth and the moon always lie on the X-axis, as follows (1):

$$\ddot{x} - 2\omega\dot{y} - \omega^2 x = -K\left\{\frac{1-\mu}{r_e^3} + \frac{\mu}{r_m^3}\right\}(x + \mu d) + K\frac{\mu}{r_m^3}d + T_x \qquad [1(a)]$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -K\left\{\frac{1-\mu}{r_e^3} + \frac{\mu}{r_m^3}\right\}y + T_y \qquad [1(b)]$$

$$\ddot{z} = -K\left\{\frac{1-\mu}{r_e^3} + \frac{\mu}{r_m^3}\right\}z + T_z \qquad [1(c)]$$

where

$$r_e = \sqrt{(x + \mu d)^2 + y^2 + z^2} \quad ; \quad r_m = \sqrt{(x + \mu d - d)^2 + y^2 + z^2}$$

$$\mu = \frac{m_m}{m_m + m_e} = \frac{1}{82.45} \quad ; \quad K = G(m_e + m_m) = 4.035187\cdot 10^{20}\ \frac{cm^3}{sec^2}$$

$$\omega = 2.6616995 \cdot 10^{-6} \frac{\text{rad}}{\text{sec}} \quad ; \quad d = 3.847527 \cdot 10^{10} \text{ cm}$$

$$T_x, T_y, T_z = \text{thrust acceleration components}$$

No analytical solution is known for the system described by equations 1(a-c); however, for vehicles moving at very high velocities in the earth-moon space, certain analytical approximations give results with good accuracy.

III. APPROXIMATE SOLUTIONS FOR VEHICLES MOVING AT VERY HIGH VELOCITIES

For the sake of definiteness, we will consider cases of launchings from the moon to the earth. Considering the full range of firing velocities from the moon, it is observed first that at an infinitely large firing velocity† neither the earth nor the moon has any effect on the straight-line trajectory; however, as the firing velocity is reduced, the earth's gravitational field will influence the trajectory. The moon's gravitational field has no appreciable effect until the velocity is reduced much further. Thus, for the range of velocities between infinity and where the moon's gravitational field has an appreciable effect, the trajectories will be well approximated by hyperbolas with the earth's c.g. as the focus.

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†By firing velocity is meant the total velocity of the vehicle $\vec{V}$ after launch.

The gradually increasing effect of the earth's gravitational field with the decrease in firing velocity is demonstrated in figure 1, which shows the ratio of atmospheric entry velocity (V_h) to the firing velocity (V_f) vs firing velocity for the range of firing velocities approximated by hyperbolas about the earth's c.g.

As the firing velocity is decreased, the moon's effect becomes more noticeable and, for firing velocities of about 21,000 ft/sec, the difference between the firing velocity calculated by neglecting the moon and the actual firing velocity is about 10%. For firing velocities lower than 21,000 ft/sec, the effect of the moon's gravitational field on the firing velocity is increasing rapidly. Plotted in figure 2 is the actual firing velocity vs the ratio of the actual firing velocity at the moon, to the firing velocity neglecting the moon, resulting in the same atmospheric entry velocity.

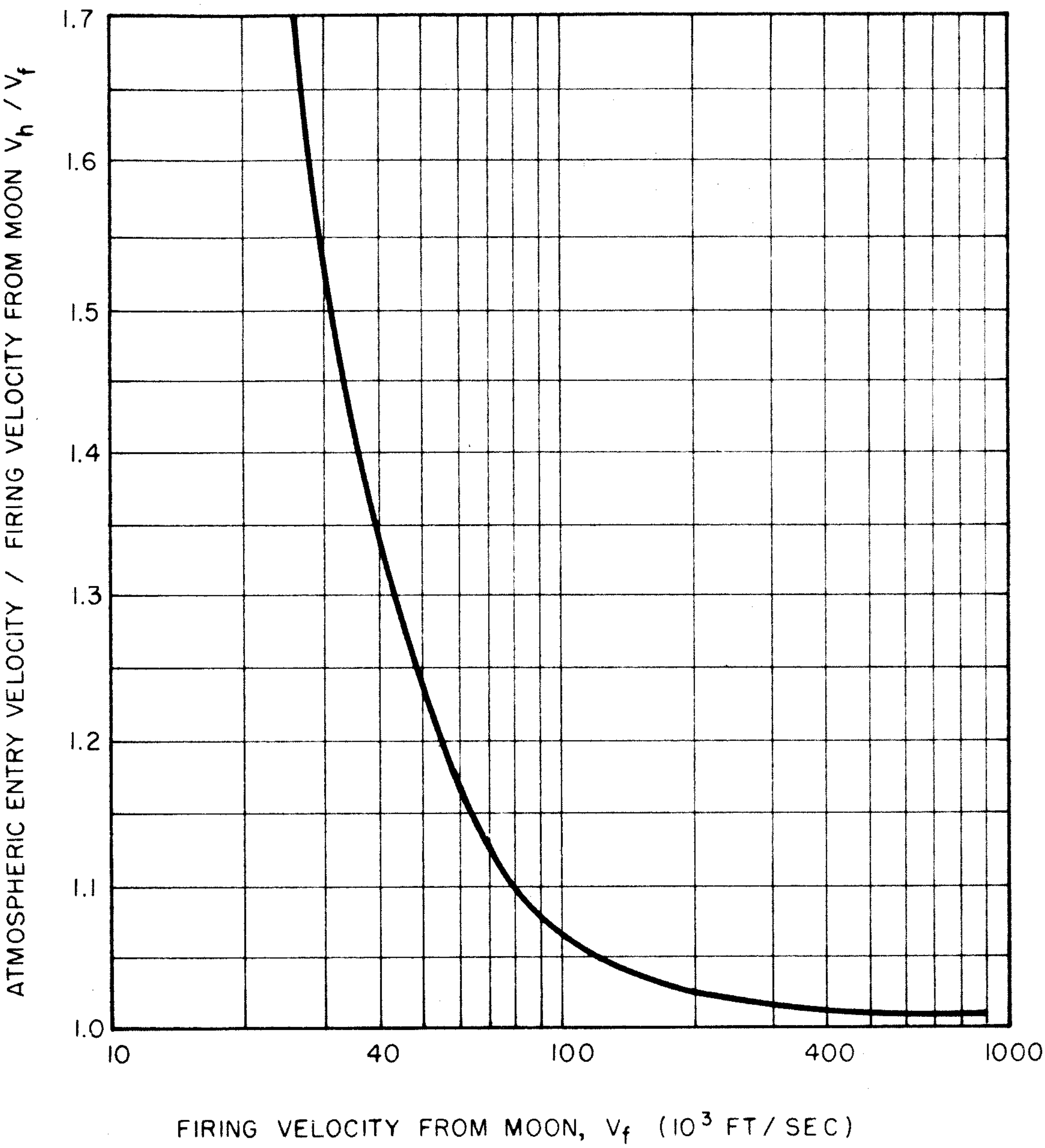
It can be summarized, on the basis of figures 1 and 2, that

A. For velocities lower than approximately 25,000 ft/sec, solutions of equations 1(a-c) are required, and

B. The velocity (firing or impact) is the basic parameter determining the nature of the motion.

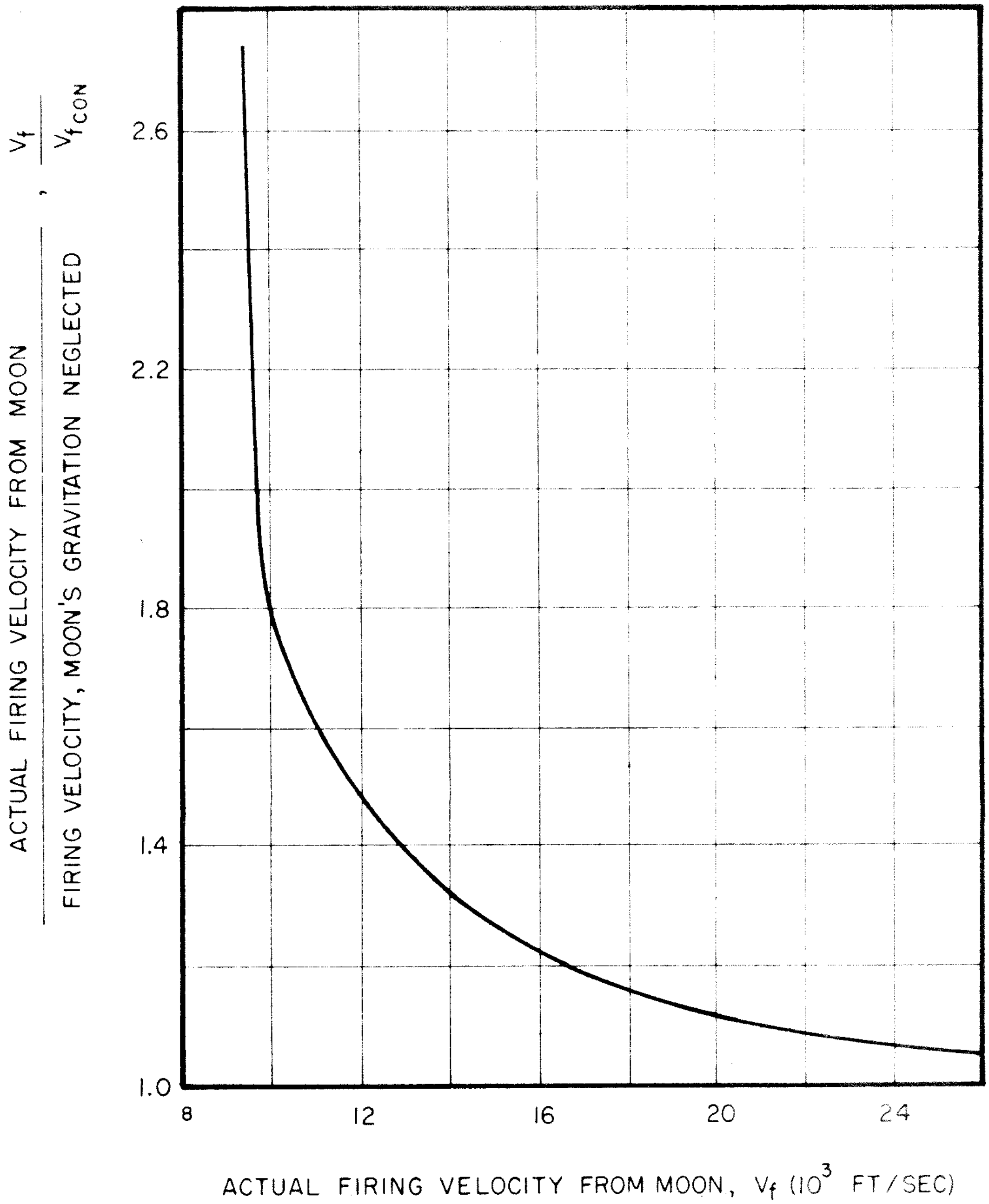
IV. DERIVATION OF THE ASYMPTOTIC EXPANSION OF THE SOLUTIONS OF THE EQUATIONS OF MOTION IN EARTH-MOON SPACE

The equations of motion 1(a-c) are first non-dimensionalized by using the following characteristic quantities:



EFFECT OF EARTH'S GRAVITATIONAL FIELD ON MOON EARTH TRAJECTORIES FOR LARGE FIRING VELOCITIES FROM MOON

Figure 1



DEVIATION OF MOON-EARTH TRAJECTORIES FROM CONICS ABOUT EARTH'S C.G. WITH FIRING VELOCITY FROM MOON

Figure 2

Length - d - distance between the centers of the earth and the moon

Velocity - U - the "vacuum" velocity at the surface of the earth†

The non-dimensional variables resulting in the simplest set of differential equations are:

$$x^* = \frac{x}{d} \; ; \; y^* = \frac{y}{d} \; ; \; t^* = t\,\frac{U}{d} \; ; \; \omega^* = \omega \left[\frac{d^3}{K(1-\mu)}\right]^{1/2} \; ; \; T^* = T\,\frac{d^2}{K(1-\mu)}$$

Introducing these variables into the D.E. results in

$$\ddot{x}^* - 2\omega^* \eta \dot{y}^* - \omega^{*2} x^* \eta^2 = -\eta^2 \left\{ \left[\frac{1}{r_e^{*3}} + \frac{\mu}{1-\mu}\frac{1}{r_m^{*3}}\right](x^* + \mu) - \frac{\mu}{1-\mu}\frac{1}{r_m^{*3}} - T_x^* \right\} \qquad [2(a)]$$

$$\ddot{y}^* + 2\omega^* \eta \dot{x}^* - \omega^{*2} y\, \eta^2 = -\eta^2 \left\{ \left[\frac{1}{r_e^{*3}} + \frac{\mu}{1-\mu}\frac{1}{r_m^{*3}}\right] y^* - T_y^* \right\} \qquad [2(b)]$$

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†In the rotating system of coordinates the conserved quantity resulting from the Jacobi Integral is the total energy minus ω times the angular momentum: $c = \omega^2(x^2 + y^2) + \frac{2K(1-\mu)}{r_e} + \frac{2K\mu}{r_m} - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. This constant is evaluated using initial conditions. The first three terms are then evaluated for any point on the earth and subtracted from c. The square root of this quantity is defined as U. The variation of U with position on the earth is a fraction of 1 ft/sec.

$$\ddot{z}^* = - \eta^2 \left\{ \left[\frac{1}{r_e^{*3}} + \frac{\mu}{1 - \mu} \frac{1}{r_m^{*3}} \right] z^* - T_z^* \right\} \qquad [2(c)]$$

where $\eta = \left[\frac{K(1 - \mu)}{d\, U^2} \right]^{1/2}$ is a non-dimensional parameter. This choice of normalization introduces a somewhat restricting ordering of terms; this restriction did not prove serious. For high velocities η becomes small, so that for these cases an expansion of the form

$$f = f_o(t^*) + \sum_{j=1}^{\infty} \eta^j f_j(t^*) \qquad [3]$$

is valid. Substituting these expressions in the equation 2 and applying the limit process for $\eta \longrightarrow 0$ $(U \longrightarrow \infty)$ results in the following equations to order zero:

$$\ddot{x}^*_o = 0 \qquad [3(a)]$$

$$\ddot{y}^*_o = 0 \qquad [3(b)]$$

$$\ddot{z}^*_o = 0 \qquad [3(c)]$$

the general solution of which is

$$x^*_o = u^*_i\, t^* + x^*_i \qquad [4(a)]$$

$$y^*_o = v^*_i\, t^* + y^*_i \qquad [4(b)]$$

$$z^*_o = w^*_i\, t^* + z^*_i \qquad [4(c)]$$

where x_i^*, y_i^*, z_i^* are the components of the initial radius vector and

$$u_i^* = \left(\frac{dx^*}{dt^*}\right)_i , \quad v_i^* = \left(\frac{dy^*}{dt^*}\right)_i , \quad w_i^* = \left(\frac{dz^*}{dt^*}\right)_i$$

This result is to be expected at the limit of infinite velocities ($\eta = 0$). To the first order one has the equations

$$\ddot{x}_1^* = 2\omega^* \, v_i^* \qquad [5(a)]$$

$$\ddot{y}_1^* = - \, 2\omega^* \, u_i^* \qquad [5(b)]$$

$$\ddot{z}_1^* = 0 \qquad [5(c)]$$

To order 1 the solution satisfying the initial conditions is given by

$$x^* = (x_i^* + u_i^* \, t^*) + \eta \, (\omega^* \, v_i^* \, t^{*2}) \qquad [6(a)]$$

$$y^* = (y_i^* + v_i^* \, t^*) + \eta \, (- \, \omega^* \, u_i^* \, t^{*2}) \qquad [6(b)]$$

$$z^* = (z_i^* + w_i^* \, t^*) \qquad [6(c)]$$

The equations to order 2 are

$$\ddot{x}_2^* - 2\omega^* \dot{y}_1^* - \omega^{*2} x_o^* = - \left[\frac{1}{r_{e_o}^{*3}} + \left(\frac{\mu}{1-\mu}\right) \frac{1}{r_{m_o}^{*3}}\right] (x_o^* + \mu) + \left(\frac{\mu}{1-\mu}\right) \frac{1}{r_{m_o}^{*3}} + T_x^* \qquad [7(a)]$$

$$\ddot{y}_2^* + 2\omega^* \dot{x}_1^* - \omega^{*2} y_o^* = - \left[\frac{1}{r_{e_o}^{*3}} + \left(\frac{\mu}{1 - \mu}\right) \frac{1}{r_{m_o}^{*3}}\right] y_o^* + T_y^* \qquad [7(b)]$$

$$\ddot{z}^*_2 = - \left[\frac{1}{r^{*3}_{e_o}} + \left(\frac{\mu}{1 - \mu} \right) \frac{1}{r^{*3}_{m_o}} \right] z^*_o + T^*_z \qquad [7(c)]$$

Substituting the results from the zeroth and first-order solution one obtains

$$\ddot{x}^*_2 = 2\omega^* (- 2\omega^* u^*_i t^*) + \omega^{*2} (x^*_i + u^*_i t^*)$$

$$- \left\{ \frac{x^*_i + u^*_i t^* + \mu}{\left[(x^*_i+\mu+u^*_i t^*)^2 + (y^*_i+v^*_i t^*)^2 + (z^*_i+w^*_i t^*)^2\right]^{3/2}} \right.$$

$$+ \frac{\mu}{1-\mu} \frac{x^*_i + u^*_i t^* + \mu}{\left[(x^*_i+\mu-1+u^*_i t^*)^2 + (y^*_i+v^*_i t^*)^2 + (z^*_i+w^*_i t^*)^2\right]^{3/2}}$$

$$\left. - \frac{\mu}{1-\mu} \frac{1}{\left[(x^*_i+\mu-1+u^*_i t^*)^2 + (y^*_i+v^*_i t^*)^2 + (z^*_i+w^*_i t^*)^2\right]^{3/2}} - T^*_x \right\} \qquad [8(a)]$$

$$\ddot{y}^*_2 = - 2\omega^* (2\omega^* v^*_i t^*) + \omega^{*2} (y^*_i + v^*_i t^*)$$

$$- \left\{ \frac{y^*_i + v^*_i t^*}{\left[(x^*_i+\mu+u^*_i t^*)^2 + (y^*_i+v^*_i t^*)^2 + (z^*_i+w^*_i t^*)^2\right]^{3/2}} \right.$$

$$\left. + \frac{\mu}{1-\mu} \frac{y^*_i + v^*_i t^*}{\left[(x^*_i+\mu-1+u^*_i t^*)^2 + (y^*_i+v^*_i t^*)^2 + (z^*_i+w^*_i t^*)^2\right]^{3/2}} - T^*_y \right\} \qquad [8(b)]$$

$$\ddot{z}^*_2 = - \left\{ \frac{z^*_i + w^*_i\ t^*}{\left[(x^*_i+\mu+u^*_i\ t^*)^2 + (y^*_i+v^*_i\ t^*)^2 + (z^*_i+w^*_i\ t^*)^2\right]^{3/2}} \right.$$

$$\left. + \frac{\mu}{1-\mu} \frac{z^*_i + w^*_i\ t^*}{\left[(x^*_i+\mu-1+u^*_i\ t^*)^2 + (y^*_i+v^*_i\ t^*)^2 + (z^*_i+w^*_i\ t^*)^2\right]^{3/2}} - T^*_z \right\} \qquad [8(c)]$$

The solution to order 2 is therefore given by

$$x^*(t) = (x^*_i+u^*_i\ t^*) + \eta\omega^* v^*_i\ t^{*2} + \eta^2 \left[- \frac{1}{2}\,\omega^{*2} u^*_i\ t^{*3} + \frac{1}{2}\,\omega^{*2} x^*_i\ t^{*2} \right.$$

$$\left. - I_x(1) - \frac{\mu}{1-\mu}\, I_x(2) + \frac{\mu}{1-\mu}\, I_x(3) + \int_0^{t^*}\int_0^{t} T^*_x\ dt\ dt \right] \qquad [9(a)]$$

$$y^*(t) = (y^*_i+v^*_i\ t^*) - \eta\omega^* u^*_i\ t^{*2} + \eta^2 \left[- \frac{1}{2}\,\omega^{*2} v^*_i\ t^{*3} + \frac{1}{2}\,\omega^{*2} y^*_i\ t^{*2} \right.$$

$$\left. - I_y(1) - \frac{\mu}{1-\mu}\, I_y(2) + \int_0^{t^*}\int_0^{t} T^*_y\ dt\ dt \right] \qquad [9(b)]$$

$$z^*(t) = (z^*_i+w^*_i\ t^*) + \eta^2 \left[- I_z(1) - \frac{\mu}{1-\mu}\, I_z(2) + \int_0^{t^*}\int_0^{t} T^*_z\ dt\ dt \right] \qquad [9(c)]$$

The components of the velocity vector are given by

$$\frac{dx^*}{dt^*} = u_i^* + \eta(2\omega^* v_i^* \; t^*) + \eta^2 \left[- \frac{3}{2} \omega^{*2} u_i^* \; t^{*2} + \omega^{*2} x_i^* \; t^* \right.$$

$$\left. - \frac{d\, I_x(1)}{d\, t^*} - \frac{\mu}{1-\mu} \frac{d\, I_x(2)}{d\, t^*} + \frac{\mu}{1-\mu} \frac{d\, I_x(3)}{d\, t^*} + \int_0^{t^*} T_x^* \, dt \right] \qquad [9(d)]$$

$$\frac{dy^*}{dt^*} = v_i^* - \eta(2\omega^* u_i^* \; t^*) + \eta^2 \left[- \frac{3}{2} \omega^{*2} v_i^* \; t^{*2} + \omega^{*2} y_i^* \; t^* \right.$$

$$\left. - \frac{d\, I_y(1)}{d\, t^*} - \frac{\mu}{1-\mu} \frac{d\, I_y(2)}{d\, t^*} + \int_0^{t^*} T_y^* \, dt \right] \qquad [9(e)]$$

$$\frac{dz^*}{dt^*} = w_i^* + \eta^2 \left[- \frac{d\, I_z(1)}{d\, t^*} - \frac{\mu}{1-\mu} \frac{d\, I_z(2)}{d\, t^*} + \int_0^{t^*} T_z^* \, dt \right] \qquad [9(f)]$$

where

$$I = \frac{2}{4ac - b^2} \left\{ \frac{2aq - bp}{a} \left[(at^2 + bt + c)^{1/2} - c^{1/2} \right] - \frac{bq - 2cp}{c^{1/2}} \, t \right.$$

$$\left. - \frac{p}{2a^{3/2}} (4ac - b^2) \left[\ln \frac{b + 2at + 2a^{1/2} (at^2 + bt + c)^{1/2}}{b + 2(ac)^{1/2}} \right] \right\}$$

$$\frac{dI}{dt} = \frac{2}{4ac - b^2} \left[\frac{(2aq - bp)t^* + (bq - 2cp)}{(at^{*2} + bt^* + c)^{1/2}} - \frac{bq - 2cp}{c^{1/2}} \right]$$

The values of p, q, a, b, c in I_x, I_y, I_z are tabulated on the next page.

Integral	p	q	a	b	c
$I_x(1)$	u_i^*	$x_i^*+\mu$	$u_i^{*2}+v_i^{*2}+w_i^{*2}$	$2\left[u_i^*(x_i^* + \mu) + v_i^* y_i^* + z_i^* w_i^*\right]$	$\left[(x_i^* + \mu)^2 + y_i^{*2} + z_i^{*2}\right]$
$I_x(2)$	u_i^*	$x_i^*+\mu$	$u_i^{*2}+v_i^{*2}+w_i^{*2}$	$2\left[u_i^*(x_i^* + \mu - 1) + v_i^* y_i^* + z_i^* w_i^*\right]$	$\left[(x_i^* + \mu - 1)^2 + y_i^{*2} + z_i^{*2}\right]$
$I_x(3)$	0	1	$u_i^{*2}+v_i^{*2}+w_i^{*2}$	$2\left[u_i^*(x_i^* + \mu - 1) + v_i^* y_i^* + z_i^* w_i^*\right]$	$\left[(x_i^* + \mu - 1)^2 + y_i^{*2} + z_i^{*2}\right]$
$I_y(1)$	v_i^*	y_i^*	$u_i^{*2}+v_i^{*2}+w_i^{*2}$	$2\left[u_i^*(x_i^* + \mu) + v_i^* y_i^* + z_i^* w_i^*\right]$	$\left[(x_i^* + \mu)^2 + y_i^{*2} + z_i^{*2}\right]$
$I_y(2)$	v_i^*	y_i^*	$u_i^{*2}+v_i^{*2}+w_i^{*2}$	$2\left[u_i^*(x_i^* + \mu - 1) + v_i^* y_i^* + z_i^* w_i^*\right]$	$\left[(x_i^* + \mu - 1)^2 + y_i^{*2} + z_i^{*2}\right]$
$I_z(1)$	w_i^*	z_i^*	$u_i^{*2}+v_i^{*2}+w_i^{*2}$	$2\left[u_i^*(x_i^* + \mu) + v_i^* y_i^* + z_i^* w_i^*\right]$	$\left[(x_i^* + \mu)^2 + y_i^{*2} + z_i^{*2}\right]$
$I_z(2)$	w_i^*	z_i^*	$u_i^{*2}+v_i^{*2}+w_i^{*2}$	$2\left[u_i^*(x_i^* + \mu - 1) + v_i^* y_i^* + z_i^* w_i^*\right]$	$\left[(x_i^* + \mu - 1)^2 + y_i^{*2} + z_i^{*2}\right]$

Since the first three terms of the expansion include all the physical effects, namely: inertial, Coriolis, centrifugal, gravitational and thrusting it was felt desirable to compare results using this method with numerical integration. Although a discrepancy is inherent in this solution due to calculation of all but the inertial effect on a "wrong" trajectory, if this discrepancy turned out to be of the same order as the errors introduced by simplifying assumptions on the earth-moon model, the results will still be useful. Furthermore, some "contingency" devices could be used to improve the results. One such device is the use of stepwise evaluation similar to stepwise integration, the advantage of this method being a more accurate evaluation of centrifugal and gravitational effects.

In order to make the comparison, several moon-to-earth trajectories were run on the IBM 704 computer. The trajectories chosen were selected to cover the full range of interest of firing velocities from the moon. The initial conditions were taken from reference (2). A comparison of the results of the numerical integrations with the three-term expansion is given in figures 3, 4, and 5. As seen from these results, the 3-term asymptotic expansion gives good results for the high firing velocity even without the use of the stepwise evaluation. The accuracy of the asymptotic expansion is degraded as the firing velocity is reduced, particularly near the earth where the curvature of the trajectory is considerable. In practical applications the last difficulty can be aleviated by the use of Keplerian orbits about the earth with initial conditions taken as far as half way between the earth and the moon. For example, in Case 2, the eccentricity (evaluated by neglecting the moon) at the half-way point between the earth and the moon is 1.0499 while at impact the eccentricity is 1.0496.

Days	Method	x (miles)	y (miles)	z (miles)
	Nu. Int. (Δt=.001)	211,558.5	-3,592.036	238.990
.05	Asy. Exp.(Δt=.05)	211,546.2	-3,593.255	239.072
	Nu. Int. (Δt=.001)	188,040.6	-6,631.539	477.165
.1	Asy. Exp.(Δt=.1)	188,012.0	-6,634.301	477.367
	Nu. Int. (Δt=.001)	140,760.6	-11,075.86	952.726
.2	Asy. Exp.(Δt=.2)	140,696.5	-11,080.51	953.158
	" " (Δt=.1)	140,715.1	-11,021.73	952.537
	Nu. Int. (Δt=.001)	93,049.63	-13,310.26	1,425.417
.3	Asy. Exp.(Δt=.3)	92,935.04	-13,311.23	1,425.784
	" " (Δt=.1)	92,981.26	-13,336.67	1,425.003
	Nu. Int. (Δt=.001)	44,483.67	-13,202.28	1,882.034
.4	Asy. Exp.(Δt=.4)	44,218.47	-13,134.44	1,878.459
	" " (Δt=.1)	44,391.77	-13,259.18	1,881.698
	Nu. Int. (Δt=.001)	3,024.379	-10,311.12	2,054.298
.48	Asy. Exp.(Δt=.48)	3,126.558	-9,746.04	2,099.225
	" " (Δt=.01)	3,087.239	-10,483.53	2,069.436

Comparison between Numerical Integration (Runge - Kutta) and the 3-term Asymptotic Expansion.

Case 1: "Vacuum" velocity at the surface of the earth = 46,484 ft/sec=U

Initial Conditions:

x_i = 235,082.87 miles $\qquad$ $y_i = z_i = 0$

u_i = -29,570 ft/sec $\qquad$ v_i = -4,783.8 ft/sec

w_i = 296 ft/sec

Figure 3

Days	Method	x (miles)	y (miles)	z (miles)
	Nu. Int. (Δt=.001)	231,884.4	-565.177	101.791
.01	Asy. Exp.(Δt=.01)	231,882.2	-565.352	101.822
	Nu. Int. (Δt=.001)	203,844.27	-4,901.684	999.022
.1	Asy. Exp.(Δt=.1)	203,766.83	-4,910.362	1,000.816
	" " (Δt=.01)	203,814.1	-4,897.727	999.727
	Nu. Int. (Δt=.001)	141,082.3	-10,315.03	2,979.446
.3	Asy. Exp.(Δt=.3)	140,796.54	-10,334.732	2,985.839
	" " (Δt=.01)	140,988.3	-10,298.13	2,981.660
	Nu. Int. (Δt=.001)	76,924.02	-9,793.21	4,917.868
.5	Asy. Exp.(Δt=.5)	76,248.03	-9,751.12	4,915.340
	" " (Δt=.01)	76,764.45	-9,759.93	4,921.442
	Nu. Int. (Δt=.001)	5,784.93	-2,062.98	5,719.413
.7	Asy. Exp.(Δt=.7)	4,518.57	+1,331.59	5,637.831
	" " (Δt=.01)	5,485.59	-2,009.93	5,696.935
	Nu. Int. (Δt=.001)	803.393	-1,187.579	5,066.576
.71	Asy. Exp.(Δt=.71)	578.523	+2,715.458	5,511.463
	" " (Δt=.01)	466.758	-1,127.436	5,009.153

Comparison between Numerical Integration (Runge - Kutta) and the 3-term Asymptotic Expansion

Case 2: "Vacuum" velocity at the surface of the earth = 41,210 ft/sec=U

Initial Conditions:

x_i = 235,082.87 miles $\qquad$ $y_i = z_i = 0$

u_i = -20,340.1 ft/sec $\qquad$ v_i = -3,541.9 ft/sec

w_i = 629.7 ft/sec

Figure 4

Days	Method	x (miles)	y (miles)	z (miles)
	Nu. Int. (Δt=.001)	234,497.7	-390.300	36.910
.005	Asy. Exp.(Δt=.005)	234,496.8	-390.396	36.924
	Nu. Int. (Δt=.001)	196,483.9	-25,279.01	2,789.921
.5	Asy. Exp.(Δt=.5)	190,336.2	-25,142.24	3,105.048
	" " (Δt=.01)	195,301.3	-25,557.03	2,835.669
	Nu. Int. (Δt=.001)	153,845.6	-39,659.67	5,365.726
1.0	Asy. Exp.(Δt=1.0)	137,812.5	-31,558.46	6,004.490
	" " (Δt=.01)	151,283.3	-39,966.26	5,460.906
	Nu. Int. (Δt=.001)	104,642.4	-42,066.75	7,663.783
1.5	Asy. Exp.(Δt=1.5)	76,572.8	-36,767.92	7,738.524
	" " (Δt=.01)	100,466.7	-41,972.24	7,789.638
	Nu. Int. (Δt=.001)	44,119.83	-26,650.32	8,724.632
2.0	Asy. Exp.(Δt=2.0)	16,681.03	+4,102.02	7,030.494
	" " (Δt=.01)	37,208.78	-24,867.89	8,648.342
	Nu. Int. (Δt=.001)	-3,491.612	822.468	-1,430.927
2.24	Asy. Exp.(Δt=2.24)	-6,331.503	34,746.924	+5,924.168
	" " (Δt=.01)	-2,768.496	895.049	-30,829.10

Comparison between Numerical Integration (Runge - Kutta) and the 3 term Asymptotic Expansion

Case 3: "Vacuum" velocity at the surface of the earth = 36,743 ft/sec=U

Initial Conditions:

x_i = 235,082.87 miles $\qquad$ $y_i = z_i = 0$

u_i = -7,959.01 ft/sec $\qquad$ v_i = -4,845.4 ft/sec

w_i = 457.4 ft/sec

Figure 5

Since one source of error in the three term expansion is the computation of the gravity losses on the zeroth order solution, it was felt that an additional term, which will include corrections to the gravity losses based on the first order solution might improve the solution. The equations for the 3rd order term are:

$$\ddot{x}^*_3 - 2\omega^*\dot{y}^*_2 - \omega^* x^*_1 = -\left\{ \frac{1}{r^{*3}_{e_o}} \left[-3(x^*_o+\mu) \frac{(x^*_o+\mu)x^*_1+y^*_o y^*_1+z^*_o z^*_1}{r^{*2}_{e_o}} + x^*_1 \right] \right.$$

$$\left. + \frac{\mu}{1-\mu} \frac{1}{r^{*3}_{m_o}} \left[-3(x^*_o+\mu-2) \frac{(x^*_o+\mu-1)x^*_1+y^*_o y^*_1+z^*_o z^*_1}{r^{*2}_{m_o}} + x^*_1 \right] \right\} \qquad [10(a)]$$

$$\ddot{y}^*_3 + 2\omega^*\dot{x}^*_2 - \omega^* y^*_1 = -\left\{ \frac{1}{r^{*3}_{e_o}} \left[-3y^*_o \frac{(x^*_o+\mu)x^*_1+y^*_o y^*_1+z^*_o z^*_1}{r^{*2}_{e_o}} + y^*_1 \right] \right.$$

$$\left. + \frac{\mu}{1-\mu} \frac{1}{r^{*3}_{m_o}} \left[-3y^*_o \frac{(x^*_o+\mu-1)x^*_1+y^*_o y^*_1+z^*_o z^*_1}{r^{*2}_{m_o}} + y^*_1 \right] \right\} \qquad [10(b)]$$

$$\ddot{z}^*_3 = -\left\{ \frac{1}{r^{*3}_{e_o}} \left[-3z^*_o \frac{(x_o+\mu)x^*_1+y^*_o y^*_1+z^*_o z^*_1}{r^{*2}_{m_o}} + z^*_1 \right] \right.$$

$$\left. + \frac{\mu}{1-\mu} \frac{1}{r^{*3}_{m_o}} \left[-3z^*_o \frac{(x^*_o+\mu-1)x^*_o+y^*_o y^*_1+z^*_o z^*_1}{r^{*2}_{m_o}} + z^*_1 \right] \right\} \qquad [10(c)]$$

Substitution of the zeroth, first and second order solutions in Equation 10 and integration yields

$$x_3^* = \frac{1}{6}\,\omega^{*3} v_i^* t^{*4} + \frac{1}{3}\,\omega^{*3} y_i^* t^{*3} - 2\omega^* \int_0^{t^*} I_y(1)dt - 2\omega^* \int_0^{t^*} \frac{\mu}{1-\mu} I_y(2)dt$$

$$+ 3\omega^* \left[J_x(1) - \frac{v_i^*}{3} K_x(1) + \frac{\mu}{1-\mu} J_x(2) - \frac{\mu}{1-\mu}\frac{v_i^*}{3} K_x(2) \right]$$

$$+ \int_0^{t^*}\int_0^{t}\int_0^{t} T_y^*\, dtdtdt \qquad [11(a)]$$

$$y_3^* = \frac{1}{6}\omega^{*3} u_i^* t^{*4} - \frac{1}{3}\omega^{*3} x_i^* t^{*3} + 2\omega^* \int_0^{t^*} I_x(1)dt + 2\omega^* \frac{\mu}{1-\mu}\int_0^{t^*} I_x(2)dt$$

$$- 2\omega^* \frac{\mu}{1-\mu}\int_0^{t^*} I_x(3)dt + 3\omega^* \left[J_y(1) + \frac{u_i^*}{3} K_y(1) + \frac{\mu}{1-\mu} J_y(2) \right.$$

$$\left. + \frac{u_i^*}{3}\frac{\mu}{1-\mu} K_y(2) \right] + \int_0^{t^*}\int_0^{t}\int_0^{t} T_x^*\, dtdtdt \qquad [11(b)]$$

$$z_3^* = 3\omega^* \left[J_z(1) + \frac{\mu}{1-\mu} J_z(2) \right] \qquad [11(c)]$$

where

$$\int_0^{t^*} I\,dt = \frac{a(aq-b_1p)t^* + (ab_1q+2acp-3b_1^2p)}{2a^2\Delta}\left(at^{*2} + 2b_1t^* + c\right)^{1/2} + \frac{cp-b_1q}{2\Delta c^{1/2}}\,t^{*2}$$

$$- \frac{(aq-b_1p)c^{1/2}}{a\Delta}\,t^* - \frac{2apt^* + (3b_1p-aq)}{2a^{5/2}}\ln\left|\frac{a^{1/2}(at^{*2}+2b_1t^*+c)^{1/2}+at^*+b_1}{(ac)^{1/2} + b_1}\right|$$

$$- \frac{(ab_1q + 2acp - 3b_1^2p)c^{1/2}}{2a^2\Delta}$$

$$K = \int_0^{t^*}\int_0^{t} \frac{t^2\,dt\,dt}{(at^2+2b_1t+c)^{3/2}} = \frac{3b_1^2-2ac}{a^2\Delta}\left[(at^{*2}+2b_1t^*+c)^{1/2}-c^{1/2}\right]$$

$$+ \frac{3b_1 + at^*}{a^{5/2}}\ln\left|\frac{a^{1/2}(at^{*2}+2b_1t^*+c)^{1/2}+at^*+b_1}{(ac)^{1/2} + b_1}\right| - \frac{b_1c^{1/2}}{a\Delta}\,t^*$$

$$J = \int_0^{t^*}\int_0^{t} \frac{(st+r)t^2\,dt\,dt}{(at^2+2b_1t+c)^{5/2}} = \frac{r}{3a\Delta}\left[\frac{2b_1t^*+c}{(at^{*2}+2b_1t^*+c)^{1/2}}\right.$$

$$\left. + \frac{ac+b_1^2}{\Delta}(at^{*2}+2b_1t^*+c)^{1/2} - \frac{2abc^{1/2}}{\Delta}\,t^* - \frac{2ac^{3/2}}{\Delta}\right]$$

$$+ \frac{s}{3a\Delta}\left\{\frac{(4b^2-5ab_1^2c+a^2c^2)t^*-2b_1c(ac-b^2)}{a\Delta(at^{*2}+2b_1t^*+c)^{1/2}} + \frac{b_1(b_1^2-3ac)}{a\Delta}(at^{*2}+2b_1t^*+c)^{1/2}\right.$$

$$\left. - \frac{3\Delta}{a^{3/2}}\ln\left|\frac{a^{1/2}(at^{*2}+2b_1t^*+c)^{1/2}+at^*+b_1}{(ac)^{1/2}+b_1}\right| + \frac{2ac^{3/2}}{\Delta}\,t + \frac{b_1(3b_1^2-5ac)c^{1/2}}{a\Delta}\right\}$$

and $b_1 = 2b \quad \Delta = ac-b_1^2$

The values of a,b,c,s,r for the J's and K's are given on the next page.

	s	r	a	b	c
$J_x(1)$	$(x_i^*+\mu)u_i^*v_i^*-y_i^*u_i^{*2}$	$(x_i^*+\mu)^2v_i^*-(x_i^*+\mu)y_i^*u_i^*$	$u_i^{*2}+v_i^{*2}+w_i^{*2}$	$2\left[(x_i^*+\mu)u_i^*+y_i^*v_i^*+z_i^*w_i^*\right]$	$(x_i^*+\mu)^2+y_i^{*2}+z_i^{*2}$
$J_y(1)$	$(x_i^*+\mu)v_i^{*2}-y_i^*u_i^*v_i^*$	$(x_i^*+\mu)y_i^*v_i^*-y_i^{*2}u_i^*$	"	"	"
$J_z(1)$	$(x_i^*+\mu)v_i^*w_i^*-y_i^*u_i^*w_i^*$	$(x_i^*+\mu)z_i^*v_i^*-y_i^*z_i^*u_i^*$	"	"	"
$K_x(1)$	0	0	"	"	"
$K_y(1)$	0	0	"	"	"
$J_x(2)$	$(x_i^*+\mu-1)u_i^*v_i^*-y_i^*u_i^{*2}$	$(x_i^*+\mu-1)(x_i^*+\mu-2)v_i^{*2}-(x_i^*+\mu-2)y_i^*u_i^*$	"	$2\left[(x_i^*+\mu-1)u_i^*+y_i^*v_i^*+z_i^*w_i^*\right]$	$(x_i^*+\mu-1)^2+y_i^{*2}+z_i^{*2}$
$J_y(2)$	$(x_i^*+\mu-1)v_i^{*2}-y_i^*u_i^*v_i^*$	$(x_i^*+\mu-1)(x_i^*+\mu-1)y_i^*v_i^*-y_i^{*2}u_i^*$	"	"	"
$J_z(2)$	$(x_i^*+\mu-1)v_i^*w_i^*-y_i^*u_i^*w_i^*$	$(x_i^*+\mu-1)(x_i^*+\mu-1)z_i^*v_i^*-y_i^*z_i^*u_i^*$	"	"	"
$K_x(2)$	0	0	"	"	"
$K_y(2)$	0	0	"	"	"

The corresponding velocity contributions of the 3rd order terms are:

$$u^*_3 = -\frac{2}{3}\omega^{*3}v^*_i t^{*3}+\omega^{*3}y_i t^{*2}-2\omega^* I_y(1)-2\omega^*\frac{\mu}{1-\mu}I_y(2)+3\omega^*\left[\frac{dJ_x(1)}{dt^*}-\frac{v^*_i}{3}\frac{dK_x(1)}{dt^*}+\frac{\mu}{1-\mu}\frac{dJ_x(2)}{dt^*}-\frac{\mu}{1-\mu}\frac{v^*_i}{3}\frac{dK_x(2)}{dt^*}\right]+\int_0^{t^*}\int_0^{t}T^*_y\,dt\,dt \qquad [11(d)]$$

$$v^*_3 = \frac{2}{3}\omega^{*3}u_i t^{*3}-\omega^{*3}x_i t^{*2}+2\omega^* I_x(1)+2\omega^*\frac{\mu}{1-\mu}I_x(2)-2\omega^*\frac{\mu}{1-\mu}I_x(3)$$

$$+3\omega^*\left[\frac{dJ_y(1)}{dt^*}+\frac{u_i}{3}\frac{dK_y}{dt^*}+\frac{\mu}{1-\mu}\frac{dJ_y(2)}{dt^*}+\frac{u^*_i}{3}\frac{\mu}{1-\mu}\frac{dK_y(2)}{dt^*}\right]+\int_0^{t^*}\int_0^{t}T^*_x\,dt\,dt \qquad [11(e)]$$

$$w^*_3 = 3\omega^*\left[J_z(1)+\frac{\mu}{1-\mu}J_z(2)\right] \qquad [11(f)]$$

where

$$\frac{dJ}{dt^*} = \frac{r}{3a\Delta}\left\{\frac{(2b_1^2-ac)t^*+cb_1}{(at^{*2}+2b_1t^*+c)^{3/2}}+\left(\frac{ac+b_1^2}{\Delta}\right)\left[\frac{at^*+b_1}{(at^{*2}+2b_1t^*+c)^{1/2}}\right]-\frac{2ab_1c^{1/2}}{\Delta}\right\}$$

$$+\frac{S}{3a\Delta}\left\{\frac{(2b_1^2-ac)t^{*2}+c\,b_1t^*}{(at^{*2}+2b_1t^*+c)^{3/2}}-\left(\frac{ac+b_1^2}{\Delta}\right)\left[\frac{at^{*2}+b_1t^*}{(at^{*2}+2b_1t^*+c)^{1/2}}\right]\right.$$

$$\left.-\frac{2b_1t^*+c}{(at^{*2}+2b_1t^*+c)^{1/2}}+\frac{ac+b_1^2}{\Delta}(at^{*2}+2b_1t^*+c)^{1/2}-\frac{2ac^{3/2}}{\Delta}\right\}$$

$$\frac{dK}{dt^*} = \frac{(2b_1^2-ac)t^*+cb_1}{a\Delta(at^{*2}+2b_1t^*+c)^{1/2}}-\frac{bc^{1/2}}{a\Delta}+\frac{1}{a^{3/2}}\ln\left|\frac{a^{1/2}(at^{*2}+2b_1t^*+c)^{1/2}+at^*+b_1}{(ac)^{1/2}+b_1}\right|$$

The results of the 4 term asymptotic expansion for the same cases as in figures 3 through 5 are shown in figures 6 through 8. Although these results are closer to the results of the numerical integration than the three term expansion, the particular asymptotic expansion used seems to impose limitations on the accuracy achievable.

One source of the difficulty lies in the singular nature of the present asymptotic expansion. The singularity being at infinite launch velocities ($\eta \rightarrow 0$). The present solution gives the expected behaviour (straight line constant velocity motion) only at the limit. For smaller velocities the rotation of the coordinates distorts this trajectory even when gravitational terms are omitted. The discrepancies observed seem to be caused in a great part by this rotation of the coordinates. When this effect is small (i.e., short flight times), the effect is small. As the time of flight is extended the discrepancy increases, as expected. The reason for the success of the stepwise evaluation can also be explained in a similar fashion.

Since no physical reasons exist for this singularity, it should be possible to find a similar asymptotic expansion that approaches the limit of $\eta \rightarrow 0$ uniformly.† Such an expansion is derived next.

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†In this context uniformity means reduction of the solution to a straight line when the gravity terms in the differential equation are neglected.

Days	Method	x (miles)	y (miles)	z (miles)
	Nu. Int. (Δt=.001)	211,558.5	-3,592.036	238.990
.05	Asy. Exp.(Δt=.05)	211,552.2	-3,590.295	239.039
	Nu. Int. (Δt=.001)	188,040.6	-6,631.539	477.165
.1	Asy. Exp.(Δt=.1)	188,272.7	-6,627.542	477.269
	Nu. Int. (Δt=.001)	140,760.6	-11,075.86	952.726
.2	Asy. Exp.(Δt=.2)	140,733.4	-11,067.02	952.939
	Nu. Int. (Δt=.001)	93,049.63	-13,310.26	1,425.417
.3	Asy. Exp.(Δt=.3)	93,008.62	-13,296.65	1,425.738
	Nu. Int. (Δt=.001)	44,483.67	-13,202.28	1,882.034
.4	Asy. Exp.(Δt=.4)	44,427.80	-13,184.66	1,882.431
	Nu. Int. (Δt=.001)	3,024.379	-10,311.12	2,054.298
.48	Asy. Exp.(Δt=.48)	2,947.927	-10,290.56	2,053.672

Comparison between Numerical Integration (Runge - Kutta) and the 4-term Asymptotic Expansion

Case 1: "Vacuum" velocity at the surface of the earth = 46,484 ft/sec=U

Initial Conditions:

x_i = 235,082.87 miles $\qquad y_i = z_i = 0$

u_i = -29,570.0 ft/sec $\qquad v_i$ = -4,783.8 ft/sec

w_i = 296 ft/sec

Figure 6

Days	Method	x (miles)	y (miles)	z (miles)
	Nu. Int. (Δt=.001)	231,884.4	-565.177	101.791
.01	Asy. Exp.(Δt=.01)	231,882.9	-565.081	101.822
	Nu. Int. (Δt=.001)	203,844.2	-4,901.684	999.022
.1	Asy. Exp.(Δt=.1)	203,812.6	-4,897.801	999.744
	Nu. Int. (Δt=.001)	141,082.3	-10,315.03	2,979.446
.3	Asy. Exp.(Δt=.3)	140,983.9	-10,298.16	2,981.718
	Nu. Int. (Δt=.001)	76,924.02	-9,793.21	4,917.868
.5	Asy. Exp.(Δt=.5)	76,757.06	-9,759.64	4,921.534
	Nu. Int. (Δt=.001)	5,784.93	-2,062.98	5,719.413
.7	Asy. Exp.(Δt=.7)	5,471.41	-2,008.14	5,695.683
	Nu. Int. (Δt=.001)	803.39	-1,187.579	5,066.576
.71	Asy. Exp.(Δt=.71)	4,502.11	-1,124.897	5,005.665

Comparison between Numerical Integration (Runge - Kutta) and the 4-term Asymptotic Expansion

Case 2: "Vacuum" velocity at the surface of the earth = 41,210 ft/sec=U

Initial Conditions:

x_i = 235,082.87 miles $\qquad$ $y_i = z_i = 0$

u_i = -20,340.1 ft/sec $\qquad$ v_i = -3,541.9 ft/sec

w_i = 629.7 ft/sec

Figure 7

Days	Method	x (miles)	y (miles)	z (miles)
	Nu. Int. (Δt=.001)	234,497.7	-390.300	36.910
.005	Asy. Exp.(Δt=.005)	234,497.5	-390.306	36.910
	Nu. Int. (Δt=.001)	196,483.9	-25,279.01	2,789.921
.5	Asy. Exp.(Δt=.5)	195,276.6	-25,564.34	2,836.615
	Nu. Int. (Δt=.001)	153,845.6	-39,659.67	5,365.726
1.0	Asy. Exp.(Δt=1.0)	151,229.5	-39,975.53	5,462.895
	Nu. Int. (Δt=.001)	104,642.4	-42,066.75	7,663.783
1.5	Asy. Exp.(Δt=1.5)	100,378.9	-41,974.36	7,792.255
	Nu. Int. (Δt=.001)	44,119.82	-26,650.32	8,724.632
2.0	Asy. Exp.(Δt=2.0)	37,062.61	-24,831.85	8,646.127
	Nu. Int. (Δt=.001)	-3,491.612	+822.468	-1,430.927
2.24	Asy. Exp.(Δt=2.24)	-446.217	+17,082.49	-32,520.24

Comparison between Numerical Integration (Runge - Kutta) and the 4-term Asymptotic Expansion.

Case 3: "Vacuum" velocity at the surface of the earth = 36,743 ft/sec=U

Initial Conditions:

x_i = 235,082.87 miles $\qquad$ $y_i = z_i = 0$

u_i = -7,959.0 ft/sec $\qquad$ v_i = -4,845.4 ft/sec

w_i = 457.4 ft/sec

Figure 8

If the non-dimensional variables are chosen as follows

$$x^* = \frac{x}{d};\ y^* = \frac{y}{d};\ z^* = \frac{z}{d};\ t^* = t\,\frac{U}{d};\ \omega^+ = \omega\,\frac{d}{U};\ T^* = T\,\frac{d^2}{K(1-\mu)}$$

then introducing these variables into Equation 1 results in

$$\ddot{x}^* - 2\omega^+\dot{y}^* - \omega^{+^2} x^* = -\eta^2\left[\left(\frac{1}{r_e^{*3}} + \frac{\mu}{1-\mu}\,\frac{1}{r_m^{*3}}\right)(x^*+\mu) - \frac{\mu}{1-\mu}\,\frac{1}{r_m^{*3}} - T_x^*\right] \qquad [12(a)]$$

$$\ddot{y} + 2\omega^+\dot{x}^* - \omega^{+^2} y^* = -\eta^2\left[\left(\frac{1}{r_e^{*3}} + \frac{\mu}{1-\mu}\,\frac{1}{r_m^{*3}}\right) y^* - T_y^*\right] \qquad [12(b)]$$

$$\ddot{z}^* = -\eta^2\left[\left(\frac{1}{r_e^{*3}} + \frac{\mu}{1-\mu}\,\frac{1}{r_m^{*3}}\right) z^* - T_z^*\right] \qquad [12(c)]$$

where $\eta^2 = \frac{K(1-\mu)}{dU^2}$ is a non-dimensional parameter. For very high velocities η becomes small, so that for these cases an expansion of the form

$$f = f_o(t^*) + \sum_{j=1}^{\infty} \eta^2\, f_j(t^*)$$

is valid. Substituting these expressions in the differential equation and applying the limit process for $\eta \rightarrow o$ ($U \rightarrow \infty$) results in the following equations to order zero:

$$\ddot{x}_o^* - 2\omega^+\dot{y}_o^* - \omega^{+^2} x_o^* = 0 \qquad [13(a)]$$

$$\ddot{y}_o^* + 2\omega^+\dot{x}_o^* - \omega^{+^2} y_o^* = 0 \qquad [13(b)]$$

$$\ddot{z}^*_o = 0 \qquad [13(c)]$$

The general solution of which is

$$x^*_o = (x^*_i + u^*_i t^*) \cos \omega^+ t^* + (y^*_i + v^*_i t^*) \sin \omega^+ t^* \qquad [14(a)]$$

$$y^*_o = - (x^*_i + u^*_i t^*) \sin \omega^+ t^* + (y^*_i + v^*_i t^*) \cos \omega^+ t^* \qquad [14(b)]$$

$$z^*_o = (z^*_i + w^*_i t^*) \qquad [14(c)]$$

$$\left(\frac{dx^*}{dt^*}\right)_o = -\omega^+(x^*_i+u^*_i t^*)\sin\omega^+ t^*+u^*_i\cos\omega^+ t^*+\omega^+(y^*_i+v^*_i t^*)\cos\omega^+ t^*+v^*_i\sin\omega^+ t^* \qquad [14(d)]$$

$$\left(\frac{dy^*}{dt^*}\right)_o = -\omega^+(x^*_i+u^*_i t^*)\cos\omega^+ t^*-u^*_i\sin\omega^+ t^*-\omega^+(y^*_i+v^*_i t^*)\sin\omega^+ t^*+v^*_i\cos\omega^+ t^* \qquad [14(e)]$$

$$\left(\frac{dz^*}{dt^*}\right)_o = w^*_i \qquad [14(f)]$$

where $u^*_i = \left(\frac{dx^*}{dt^*}\right)_i - \omega^+ y^*_i$, $v^*_i = \left(\frac{dy^*}{dt^*}\right)_i + \omega^+ x^*_i$, and $w^*_i = \left(\frac{dz^*}{dt^*}\right)_i$ †

These results are to be expected at the limit of infinite velocity ($\eta \to \infty$). Furthermore, when gravitational effects are neglected the result is a straight line constant velocity motion in an inertial frame of reference.

To order η^2 one has the equations

$$\ddot{x}^*_1 - 2\omega^+\dot{y}^*_1 - \omega^{+2} x^*_1 = -\left[\left(\frac{1}{r^{*3}_{e_o}} - \frac{\mu}{1-\mu}\frac{1}{r^{*3}_{m_o}}\right)(x^*_o+\mu) - \frac{\mu}{1-\mu}\frac{1}{r^{*3}_{m_o}} - T^*_x\right] = f(t^*) \qquad [15(a)]$$

---

† Note the difference in definitions of u^*_i, v^*_i, and w^*_i with those used in Equations 4 et al. The different choice was made for reasons of convenience only. In this case (u^*_i, v^*_i, w^*_i) corresponds to the initial velocity in inertial space.

$$\ddot{y}^*_1 + 2\omega^+\dot{x}^*_1 - \omega^{+2} y^*_1 = -\left[\left(\frac{1}{r^{*3}_{e_o}} - \frac{\mu}{1-\mu}\frac{1}{r^{*3}_{m_o}}\right) y^*_o - T^*_y\right] = g(t^*) \qquad [15(b)]$$

$$\ddot{z}^*_1 = -\left[\left(\frac{1}{r^{*3}_{e_o}} - \frac{\mu}{1-\mu}\frac{1}{r^{*3}_{m_o}}\right) z^*_o - T^*_z\right] = h(t^*) \qquad [15(c)]$$

The initial conditions on (x^*_1, y^*_1, z^*_1) and on $(\dot{x}^*_1, \dot{y}^*_1, \dot{z}^*_1)$ for these equations vanish identically since the zeroth order solution fulfills the prescribed six initial conditions. Thus it is necessary to find particular integrals satisfying zero initial position and velocity conditions. One method of obtaining these integrals is as follows:

Equations 15(a), 15(b) are uncoupled using regular methods, resulting in:

$$\left(D^2-\omega^{+2}\right)^2 x^* + 4\omega^{+2} D^2 x^* = \left(D^2-\omega^{+2}\right) f(t^*) + 2\omega^+ Dg(t^*) = F(t^*) \qquad [16(a)]$$

$$\left(D^2-\omega^{+2}\right)^2 y^* + 4\omega^{+2} D^2 y^* = \left(D^2-\omega^{+2}\right) g(t^*) - 2\omega^+ Df(t^*) = G(t^*) \qquad [16(b)]$$

The solutions for $x^*_1(t^*)$, $y^*_1(t^*)$ can be obtained by the use of Laplace Transforms resulting in

$$x^*_1(t^*) = \int_0^{t^*} f(\tau)(t^*-\tau)\cos\omega^+(t^*-\tau)d\tau + \int_0^{t^*} g(\tau)(t^*-\tau)\sin\omega^+(t^*-\tau)d\tau \qquad [17(a)]$$

$$y^*_1(t^*) = \int_0^{t^*} g(\tau)(t^*-\tau)\cos\omega^+(t^*-\tau)d\tau - \int_0^{t^*} f(\tau)(t^*-\tau)\sin\omega^+(t^*-\tau)d\tau \qquad [17(b)]$$

The solution for $z^*_1(t^*)$ is

$$z^*_1(t^*) = \int_0^{t^*}\int_0^{t} h(\tau)\, d\tau \qquad [17(c)]$$

where the expressions for f, g, h are

$$f(t^*) = \frac{-\left\{\left[(x_i^*+u_i^*t^*)\cos\omega^+t^* + (y_i^*+v_i^*t^*)\sin\omega^+t^*\right] + \mu\right\}}{\left\{(x_i^*+u_i^*t^*)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2+\mu^2+2\mu\left[(x_i^*+u_i^*t^*)\cos\omega^+t^* + (y_i^*+v_i^*t^*)\sin\omega^+t^*\right]\right\}^{3/2}}$$

$$+ \frac{\mu}{1-\mu}\frac{1-\left\{\left[(x_i^*+u_i^*t^*)\cos\omega^+t^* + (y_i^*+v_i^*t^*)\sin\omega^+t^*\right] + \mu\right\}}{\left\{(x_i^*+u_i^*t^*)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2+(1-\mu)^2-2(1-\mu)\left[(x_i^*+u_i^*t^*)\cos\omega^+t^*+(y_i^*+v_i^*t^*)\sin\omega^+t^*\right]\right\}^{3/2}} + T_x^*$$

$$g(t^*) = \frac{-\left[-(x_i^*+u_i^*t^*)\sin\omega^+t^* + (y_i^*+v_i^*t^*)\cos\omega^+t^*\right]}{\left\{(x_i^*+u_i^*t^*)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2+\mu^2+2\mu\left[(x_i^*+u_i^*t^*)\cos\omega^+t^*+(y_i^*+v_i^*t^*)\sin\omega^+t^*\right]\right\}^{3/2}}$$

$$+ \frac{\mu}{1-\mu}\frac{-\left[-(x_i^*+u_i^*t^*)\sin\omega^+t^* + (y_i^*+v_i^*t^*)\cos\omega^+t^*\right]}{\left\{(x_i^*+u_i^*t^*)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2+(1-\mu)^2-2(1-\mu)\left[(x_i^*+u_i^*t^*)\cos\omega^+t^*+(y_i^*+v_i^*t^*)\sin\omega^+t^*\right]\right\}^{3/2}} + T_y^*$$

$$h(t^*) = \frac{-(z_i^*+w_i^*t^*)}{\left\{(x_i^*+u_i^*t^*)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2+\mu^2+2\mu\left[(x_i^*+u_i^*t^*)\cos\omega^+t^*+(y_i^*+v_i^*t^*)\sin\omega^+t^*\right]\right\}^{3/2}}$$

$$+ \frac{\mu}{1-\mu}\frac{-(z_i^*+w_i^*t^*)}{\left\{(x_i^*+u_i^*t^*)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2+(1-\mu)^2-2(1-\mu)\left[(x_i^*+u_i^*t^*)\cos\omega^+t^*+(y_i^*+v_i^*t^*)\sin\omega^+t^*\right]\right\}^{3/2}} + T_z^*$$

The complexity of the last expressions makes direct integration of the integrals in Equations 22 tedious at best. However, the resemblance of these integrals to the I integrals discussed previously enables a good approximation using the I integrals. By rearrangement and expansion in a series the expression for f becomes:

$$f(t^*) = \frac{-\left[(x_i^*+u_i^*t^*+\mu)\ \cos\omega^+t^*+(y_i^*+v_i^*t^*)\ \sin\omega^+t^*+\mu(1-\cos\omega^+t^*)\right]}{\left[(x_i^*+u_i^*t^*+\mu)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2\right]^{3/2}} \cdot$$

$$\left\{1-3\ \frac{\mu\left[(x_i^*+u_i^*t^*)(\cos\omega^+t^*-1)+(y_i^*+v_i^*t^*)\ \sin\omega^+t^*\right]}{(x_i^*+u_i^*t^*+\mu)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2}+\ \cdots\right\}$$

$$-\frac{\mu}{1-\mu}\ \frac{\left[(x_i^*+u_i^*t^*+\mu)\ \cos\omega^+t^*+(y_i^*+v_i^*t^*)\ \sin\omega^+t^*+\mu(1-\cos\omega^+t^*)\right]}{\left[(x_i^*+u_i^*t^*+\mu-1)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2\right]^{3/2}} \cdot$$

$$\left\{1+3\ \frac{(1-\mu)\left[(x_i^*+u_i^*t^*)(\cos\omega^+t^*-1)+(y_i^*+v_i^*t^*)\ \sin\omega^+t^*\right]}{(x_i^*+u_i^*t^*+\mu-1)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2}+\ \cdots\right\}$$

$$+\frac{\mu}{1-\mu}\ \frac{1}{\left[(x_i^*+u_i^*t^*+\mu-1)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2\right]^{3/2}} \cdot$$

$$\left\{1+3\ \frac{(1-\mu)\left[(x_i^*+u_i^*y^*)(\cos\omega^+t^*-1)+(y_i^*+v_i^*t^*)\ \sin\omega^+t^*\right]}{(x_i^*+u_i^*t^*+\mu-1)^2+(y_i^*+v_i^*t^*)^2+(z_i^*+w_i^*t^*)^2}+\ \cdots\right\}$$

$$+ \, O\left\{\left[(x_i^*+u_i^*t^*+\mu)^2 + (y_i^*+v_i^*t^*)^2 + (z_i^*+w_i^*t^*)^2\right]^{-4},\right.$$

$$\left.\left[(x_i^*+u_i^*t^*+\mu-1)^2 + (y_i^*+v_i^*t^*)^2 + (z_i^*+w_i^*t^*)^2\right]^{-4}\right\} + T_x^*$$

with similar expressions for g, h. If terms of order r_e^{-3}, r_m^{-3} are neglected compared to terms of order r_e^{-2}, r_m^{-2}, and the previous notation is used, f, g, and h become:

$$f(t^*) = -\frac{d^2I_x(1)}{dt^{*2}}\cos\omega^+t^* - \frac{d^2I_y(1)}{dt^{*2}}\sin\omega^+t^* - \frac{\mu(1-\cos\omega^+t^*)}{r_{e_o}^3} - \frac{\mu}{1-\mu}\frac{d^2I_x(2)}{dt^{*2}}\cos\omega^+t^*$$

$$-\frac{\mu}{1-\mu}\frac{d^2I_y(2)}{dt^{*2}}\sin\omega^+t^* - \frac{\mu^2}{1-\mu}\frac{1-\cos\omega^+t^*}{r_{m_o}^3} + \frac{\mu}{1-\mu}\frac{d^2I_x(3)}{dt^{*2}} + O\left[r_{e_o}^{-3},\ r_{e_o}^{-3}\right]$$

$$g(t^*) = \frac{d^2I_x(1)}{dt^{*2}}\sin\omega^+t^* - \frac{d^2I_y(1)}{dt^{*2}}\cos\omega^+t^* - \frac{\mu\sin\omega^+t^*}{r_{e_o}^3} + \frac{\mu}{1-\mu}\frac{d^2I_x(2)}{dt^{*2}}\sin\omega^+t^*$$

$$-\frac{\mu}{1-\mu}\frac{d^2I_y(2)}{dt^{*2}}\cos\omega^+t^* - \frac{\mu^2}{1-\mu}\frac{\sin\omega^+t^*}{r_{m_o}^3} + O\left[r_{e_o}^{-3},\ r_{m_o}^{-3}\right]$$

$$h(t^*) = -\frac{d^2I_z(1)}{dt^{*2}} - \frac{\mu}{1-\mu}\frac{d^2I_z(2)}{dt^{*2}} + O\left[r_{e_o}^{-3},\ r_{m_o}^{-3}\right]$$

It is now possible to evaluate the integrals in Equations 17 as follows:

$$\int_0^{t^*}\left[f(\tau)\cos\omega^+(t^*-\tau) + g(\tau)\sin\omega^+(t^*-\tau)\right](t^*-\tau)d\tau = -\left\{ I_x(1)\cos\omega^+t^* \right.$$

$$+ I_y(1)\sin\omega^+t^* + \frac{\mu}{1-\mu}\left[I_x(2)\cos\omega^+t^* + I_y(2)\sin\omega^+t^*\right] - \mu\left[I_o(1)\right.$$

$$\left. + \frac{\mu}{1-\mu} I_o(2)\right] \cos\omega^+t^* + \mu\int_0^{t^*}\left(\frac{1}{r_{e_o}^3} - \frac{1}{r_{m_o}^3}\right)(t^*-\tau)\cos\omega^+(t^*-\tau)d\tau\left.\right\} \qquad [18(a)]$$

$$\int_0^{t^*}\left[g(\tau)\cos\omega^+(t^*-\tau) - f(\tau)\sin\omega^+(t^*-\tau)\right](t^*-\tau)d\tau$$

$$= -\left\{ -I_x(1)\sin\omega^+t^* + I_y(1)\cos\omega^+t^* - \frac{\mu}{1-\mu}\left[I_x(2)\sin\omega^+t^*\right.\right.$$

$$\left. + I_y(2)\cos\omega^+t^*\right] + \mu\left[I_o(1) + \frac{\mu}{1+\mu} I_o(2)\right] \sin\omega^+t^*$$

$$\left. - \int_0^{t^*}\mu\left(\frac{1}{r_{e_o}^3} - \frac{1}{r_{m_o}^3}\right)(t^*-\tau)\sin\omega^+(t^*-\tau)d\tau \right\} \qquad [18(b)]$$

where I_x, I_y are as defined previously and

$$I_n(1) = \int_0^{t^*}\int_0^{t}\frac{\tau^n d\tau d\tau}{r_{e_o}^3} \qquad n=0,\ 1,\ 2\ \ldots$$

$$I_n(2) = \int_0^{t*} \int_0^{t} \frac{\tau^n d\tau d\tau}{r^3_{m_o}} \qquad n=0,\ 1,\ 2\ \ldots$$

by the use of the expansions

$$\sin\omega^+(t*-\tau) = \sin\omega^+t* - \omega^+\tau\cos\omega^+t* - \frac{1}{2}(\omega^+\tau)^2\sin\omega^+t* + \frac{1}{6}(\omega^+\tau)^3\cos\omega^+t* + \ldots$$

$$\cos\omega^+(t*-\tau) = \cos\omega^+t* - \omega^+\tau\sin\omega^+t* - \frac{1}{2}(\omega^+\tau)^2\cos\omega^+t* - \frac{1}{6}(\omega^+\tau)^3\sin\omega^+t* + \ldots$$

the integrals on the right hand side can be approximated as follows:

$$\int_0^{t*} \left(\frac{1}{r^3_{e_o}} - \frac{1}{r^3_{m_o}} \right) (t*-\tau)\cos\omega^+(t*-\tau)d\tau = \left[I_o(1) - I_o(2) \right] \cos\omega^+t*$$

$$+ \left[I_1(1) - I_1(2) \right] \omega^+\sin\omega^+t* - \frac{1}{2}\left[I_2(1) - I_2(2) \right] \omega^{+2}\cos\omega^+t* + \ldots$$

$$\int_0^{t*} \left(\frac{1}{r^3_{e_o}} - \frac{1}{r^3_{m_o}} \right) (t*-\tau)\sin\omega^+(t*-\tau)d\tau = \left[I_o(1) - I_o(2) \right] \sin\omega^+t*$$

$$- \left[I_1(1) - I_1(2) \right] \omega^+\cos\omega^+t* - \frac{1}{2}\left[I_2(1) - I_2(2) \right] \omega^{+2}\sin\omega^+t* + \ldots$$

For the expected transfer times, the terms containing I_2 in the above expansions will contribute a fraction of 1% of the contribution of the first terms. Since the contribution of the neglected terms in the expansion of f, g might be of the same order, additional terms should not be added.

Using these results, and the notation used in the previous asymptotic expansion, the two term present asymptotic expansion becomes:

$$x^*(t^*) = \Big[(x_i^*+u_i^*t^*)\cos\omega^+t^* + (y_i+v_it^*)\sin\omega^+t^*\Big]$$

$$+ \eta^2 \Big[-I_x(1)\cos\omega^+t^* + I_y(1)\sin\omega^+t^* + \frac{\mu}{1-\mu} I_x(2)\cos\omega^+t^*$$

$$+ \frac{\mu}{1-\mu} I_y(2)\sin\omega^+t^* - \frac{\mu}{1-\mu} I_o(2)\cos\omega^+t^* + \mu\Big(I_1(1)-I_1(2)\Big)\omega^+\sin\omega^+t^*$$

$$- \frac{\mu}{2}\Big(I_2(1) - I_2(2)\Big)\omega^{+^2} \cos\omega^+t^* + \dots \Big] + o\,(\eta^4) \qquad [19(a)]$$

$$y^*(t^*) = \Big[-(x_i^*+u_i^*t^*)\sin\omega^+t^* + (y_i+v_it^*)\cos\omega^+t^*\Big]$$

$$-\eta^2 \Big[-I_x(1)\sin\omega^+t^* + I_y(1)\cos\omega^+t^* - \frac{\mu}{1-\mu} I_x(2)\sin\omega^+t^*$$

$$+ \frac{\mu}{1-\mu} I_y(2)\cos\omega^+t^* + \frac{\mu}{1-\mu} I_o(2)\sin\omega^+t^* + \mu\Big(I_1(1)-I_1(2)\Big)\omega^+\cos\omega^+t^*$$

$$+ \frac{\mu}{2}\Big(I_2(1) - I_2(2)\Big) \omega^{+^2}\sin\omega^+t^* + \dots \Big] + o\,(\eta^4) \qquad [19(b)]$$

$$z^*(t^*) = (z_i^*+w_i^*t^*) + \eta^2\Big[-I_z(1) - \frac{\mu}{1-\mu} I_z(2) + \dots \Big] + o\,(\eta^4) \qquad [19(c)]$$

The results of an application of these solutions to the cases considered previously are presented in Figures 9 through 11. As expected, the x and y solutions are in closer agreement with the numerical integration then the previous method for short times. Near the earth the effect of the neglected terms in the expressions for f, g, and h becomes appreciable, to the extent of equalling or surpassing the terms retained, in some cases. The omission of these terms will cause a degradation in accuracy near the earth, particularly for the slower trajectories. This effect can be reduced by inclusion of more terms.

The physical reason for the degradation of accuracy of the last asymptotic expansion for long flight times is primarily due to the errors involved in computing gravity effects on the straight line constant velocity trajectory rather than on the actual trajectory. Although successive terms might eventually approach the right result, the complexity of deriving these terms suggests a different approach. Such an approach will be illustrated next. It should be noted again that for practical applications the present approach, coupled with the Keplerian orbit for the latter half of the trajectory will give satisfactory results.

Days	Method	x (miles)	y (miles)	z (miles)
	Nu. Int. (Δt=.001)	231,884.4	-565.177	101.791
.01	Asy. Exp.(Δt=.01)	231,883.7	-565.283	101.804
	Nu. Int. (Δt=.001)	203,844.2	-4,901.684	999.022
.1	Asy. Exp.(Δt=.1)	203,803.7	-4,907.665	1,000.242
	" " (Δt=.01)	203,833.4	-4,903.421	999.338
	Nu. Int. (Δt=.001)	141,082.5	-10,315.03	2,979.446
.3	Asy. Exp.(Δt=.3)	140,920.4	-10,330.01	2,983.741
	" " (Δt=.1)	140,950.9	-10,331.34	2,983.890
	" " (Δt=.01)	141,049.0	-10,318.63	2,980.411
	Nu. Int. (Δt=.001)	76,924.02	-9,793.21	4,917.868
.5	Asy. Exp.(Δt=.5)	76,448.32	-9,773.67	4,914.749
	" " (Δt=.1)	76,694.71	-9,810.32	4,925.360
	" " (Δt=.01)	76,867.32	-9,796.37	4,919.452
	Nu. Int. (Δt=.001)	5,784.93	-2,062.98	5,719.413
.7	Asy. Exp.(Δt=.7)	1,873.51	-785.11	4,267.114
	" " (Δt=.1)	5,644.54	-2,102.35	5,812.580
	" " (Δt=.01)	5,683.79	-2,057.02	5,713.840
	Nu. Int. (Δt=.001)	7,707.55	-1,187.579	5,066.576
.71	Asy. Exp.(Δt=.71)	-3,546.74	+580.098	2,807.909
	" " (Δt=.01)	6,993.71	-1,179.198	5,057.164

Comparison between Numerical Integration (Runge - Kutta) and the second Asymptotic Expansion (2 terms)

Case 2 (see figure 4 for initial conditions)

Figure 10

Days	Method	x (miles)	y (miles)	z (miles)
	Nu. Int. (Δt=.001)	234,497.7	-390.300	36.910
.005	Asy. Exp.(Δt=.005)	234,496.9	-390.311	36.920
	Nu. Int. (Δt=.001)	196,483.9	-25,279.01	2,789.921
.5	Asy. Exp.(Δt=.5)	194,378.0	-27,002.12	2,971.965
	" " (Δt=.1)	195,240.9	-26,442.99	2,912.013
	" " (Δt=.01)	197,412.7	-24,947.58	2,748.041
	Nu. Int. (Δt=.001)	153,845.6	-39,659.67	5,365.726
1.0	Asy. Exp.(Δt=1.0)	146,901.9	-42,182.91	5,685.100
	" " (Δt=.1)	151,052.6	-41,673.09	5,608.436
	Asy. Exp.(Δt=.01)	155,952.5	-39,170.37	5,275.325
	Nu. Int. (Δt=.001)	104,642.4	-42,066.76	7,663.783
1.5	Asy. Exp.(Δt=1.5)	86,128.4	-36,957.21	6,893.551
	" " (Δt=.1)	99,948.3	-44,374.50	8,001.947
	" " (Δt=.01)	108,128.3	-41,791.21	7,539.697
	Nu. Int. (Δt=.001)	44,119.82	-26,650.32	8,724.632
2.0	Asy. Exp.(Δt=2.0)	20,931.65	+27,005.32	-3,629.236
	" " (Δt=.1)	36,716.69	-27,634.90	8,925.200
	" " (Δt=.01)	49,755.74	-27,524.90	8,720.586
	Nu. Int. (Δt=.001)	-3,491.612	+822.468	-1,430.927
2.24	Asy. Exp.(Δt=2.24)	+6,932.498	+78,203.136	-13,939.685
	" " (Δt=.01)	+8,140.654	-7,014.860	+6,057.270

Comparison between Numerical Integration (Runge - Kutta) and the second Asymptotic Expansion (2 terms).

Case 3: (see figure 5 for initial conditions)

Figure 11

V. APPLICATION OF THE SMALL PERTURBATION APPROACH TO THE RESTRICTED THREE BODY PROBLEM

The limitations on accuracy of the asymptotic expansions near escape velocity indicate the necessity for a different approach for problems in this velocity range. Since the major source of error was due to the calculation of the gravity effect near the earth on the "wrong" trajectory, the obvious approach is to use the "right" trajectory in this region. One way of achieving this is described below.

In the following derivations, trajectories from the earth to the moon will be considered. Although the derivation of the expressions for the solution is straightforward and the procedure has a simple physical explanation, the algebra involved is exceedingly complicated. This point is demonstrated by the application of this method to near parabolic near straight line trajectories.

The equations of motion 1(a-c) are first non-dimensionalized by using the following characteristic properties:

Length - d - distance between the centers of the earth and the moon

Time - $\frac{1}{\omega}$ - angular velocity of the coordinate system

The non-dimensional variables then become

$$x^+ = \frac{x}{d};\ y^+ = \frac{y}{d};\ t^+ = t\omega;\ K^+ = K\,\frac{(1-\mu)}{d^3\omega^2};\ T^+ = T\,\frac{1}{d\omega^2}$$

Introducing these variables in the D.E. results in

$$\ddot{x}^+ - 2\dot{y}^+ - x^+ = -K^+\left[\frac{1}{r_e^{+3}} + \frac{\mu}{1-\mu}\,\frac{1}{r_m^{+3}}\right](x^+ + \mu) + K^+\,\frac{\mu}{1-\mu}\,\frac{1}{r_m^{+3}} + T_x^+ \qquad [20(a)]$$

$$\ddot{y}^{+} + 2\dot{x}^{+} - y^{+} = -K^{+}\left[\frac{1}{r_e^{+3}} + \frac{\mu}{1-\mu}\frac{1}{r_m^{+3}} \right] y^{+} + T_y^{+} \qquad [20(b)]$$

$$\ddot{z}^{+} = -K^{+}\left[\frac{1}{r_e^{+3}} + \frac{\mu}{1-\mu}\frac{1}{r_m^{+3}} \right] z^{+} + T_z^{+} \qquad [20(c)]$$

In the above equations $K^{+} = 1-\mu \doteq 1$. Since $\frac{\mu}{1-\mu} = \frac{1}{81.45}$ is a "small-parameter" in the problem, it should be possible to simplify the equations by expanding the spatial variables in a power series in this parameter, namely,

$$f^{+}(t^{+}) = f_o(t^{+}) + \sum_{n=1}^{\infty} \epsilon^n f_n(t^{+}) \qquad [(21)]$$

where

$$\epsilon = \frac{\mu}{1-\mu}$$

Substitution of these expansions in Equations 20(a-c) results in the following set of equations for the zeroth order terms (in principle K^{+} and $(x_o+\mu)$ should also be expanded; however, this added complication will add little to the accuracy and will distract from the simple physical picture).

$$\ddot{x}_o - 2\dot{y}_o - x_o = -K^{+}\frac{x_o+\mu}{r_{e_o}^3} + T_x \qquad [22(a)]$$

$$\ddot{y}_o + 2\dot{x}_o - y_o = -K^{+}\frac{y_o}{r_{e_o}^3} + T_y \qquad [22(b)]$$

$$\ddot{z}_o = -K^+ \frac{z_o}{r_{e_o}^3} + T_z \qquad [22(c)]$$

For the case of zero thrust the solution of Equations 22(a-c) is a conic section with the earth's center as focus (motion of a small body in the central gravitational field of the earth). Impulsive thrust can also be treated with the known methods. The equations for the first order terms are

$$\ddot{x}_1 - 2\dot{y}_1 - x_1 = -K^+ \left\{ \frac{x_1}{r_{e_o}^3} - 3\,\frac{x_o+\mu}{r_{e_o}^5}\left[(x_o+\mu)x_1 + y_o y_1 + z_o z_1\right] + \frac{x_o+\mu-1}{r_{m_o}^3} \right\} \qquad [23(a)]$$

$$\ddot{y}_1 + 2\dot{x}_1 - y_1 = -K^+ \left\{ \frac{y_1}{r_{e_o}^3} - 3\,\frac{y_o}{r_{e_o}^5}\left[(x_o+\mu)x_1 + y_o y_1 + z_o z_1\right] + \frac{y_o}{r_{m_o}^3} \right\} \qquad [23(b)]$$

$$\ddot{z}_1 = -K^+ \left\{ \frac{z_1}{r_{e_o}^3} - 3\,\frac{z_o}{r_{e_o}^5}\left[(x_o+\mu)x_1 + y_o y_1 + z_o z_1\right] + \frac{z_o}{r_{m_o}^3} \right\} \qquad [23(c)]$$

The right-hand side of Equations 23(a-c) contains terms of zeroth order and first order. Furthermore the zeroth order terms contain the factor $r_{m_o}^{-3}$ resulting in a large contribution near the moon. Thus, an iteration solution where the zeroth order terms in the variables only are retained initially, suggests itself. Once this solution is found, namely $x_{1_I}(t^+)$, $y_{1_I}(t^+)$, $z_{1_I}(t^+)$, it is inserted in the right-hand side of the equations and the next iteration obtained, until the desired accuracy is achieved.

Physically the procedure so far can be summarized as follows: (a) compute the trajectory about the earth, neglecting the moon's gravitational field, (b) compute the effect of the moon's gravitational field on the trajectory calculated in (a), and (c) compute corrections to this trajectory due to the earth's gravitational field with the moon's effect computed on the trajectory calculated in (a). The higher order terms should increase the accuracy by computing corrections to the gravitational effects of the earth and the moon on better approximations to the actual trajectories.

The equations for the first iterations are:

$$\ddot{x}_1 - 2\dot{y}_1 = -K^+ \frac{x_o+\mu-1}{r_{m_o}^3} = f(t^+) \qquad [24(a)]$$

$$\ddot{y}_1 + 2\dot{x}_1 = -K^+ \frac{y_o}{r_{m_o}^3} = g(t^+) \qquad [24(b)]$$

$$\ddot{z}_1 = -K^+ \frac{z_o}{r_{m_o}^3} \qquad [24(c)]$$

In Equations 24(a,b) x_1, y_1 on the left-hand side were omitted because terms of the same order were omitted on the right-hand side. The solution of Equation 24(c) is obtained directly as

$$z_{1_I}(t^+) = -K^+ \int_o^{t^*} \int_o^{t} \frac{z_o(\tau)}{r_{m_o}^3(\tau)}\, d\tau\, dt \qquad [(25)]$$

The solution of Equations 24(a,b) is obtained by first separating variables, giving

$$D^3 x_{1_I} + 4Dx_{1_I} = Df(t^+) + 2g(t^+) \qquad [26(a)]$$

$$D^3 y_{1_I} + 4Dy_{1_I} = Dg(t^+) - 2f(t^+) \qquad [26(b)]$$

Use of the same method as used in the previous case to obtain a particular solution results in:

$$x_{1_I}(t^+) = \frac{1}{2}\int_0^{t^+} \left[Df(\tau) + 2g(\tau)\right] \sin 2(t^+ - \tau)d\tau \qquad [27(a)]$$

$$y_{1_I}(t^+) = \frac{1}{2}\int_0^{t^+} \left[Dg(\tau) - 2f(\tau)\right] \sin 2(t^+ - \tau)d\tau \qquad [27(b)]$$

Integrating by parts and noting that x_{1_I}, $\dot{x}_{1_I}$, $\ddot{x}_{1_I}$, y_{1_I}, $\dot{y}_{1_I}$, $\ddot{y}_{1_I}$ vanish at $t^+=0$ (conditions used in the derivation of x_{1_I}, y_{1_I}), the following results are obtained

$$x_{1_I} = \int_0^{t^+} \left[f(\tau) \cos 2(t^+ - \tau) + g(\tau) \sin 2(t^+ - \tau) \right] d\tau \qquad [28(a)]$$

$$y_{1_I} = \int_0^{t^+} \left[-f(\tau) \sin 2(t^+ - \tau) + g(\tau) \cos 2(t^+ - \tau) \right] d\tau \qquad [28(b)]$$

The solution for the second iteration for z_1 is

$$z_{1_{II}}(t^+) = -K^+ \int_0^{t^+} \int_0^{t} \left\{ \frac{z_{1_I}(\tau)}{r_{e_o}^3(\tau)} - 3\,\frac{z_o(\tau)}{r_{e_o}^5(\tau)} \Big[(x_o(\tau)+\mu)x_{1_I}(\tau) \right.$$

$$\left. + \, y_o(\tau)y_1(\tau) + z_o(\tau)z_1(\tau) \Big] + \frac{z_o(\tau)}{r_{m_o}^3(\tau)} \right\} d\tau d\tau \qquad [(29)]$$

The equations for the second iteration for x_1, y_1 are

$$\ddot{x}_{1_{II}} - 2\dot{y}_{1_{II}} - x_{1_{II}} = -K^+ \left\{ \frac{x_{1_I}}{r_{e_o}^3} - 3\,\frac{(x_o+\mu)}{r_{e_o}^5} \Big[(x_o+\mu)x_{1_I} \right.$$

$$\left. + \, y_o y_{1_I} + z_o z_{1_I} \Big] + \frac{x_o+\mu-1}{r_{m_o}^3} \right\} = f_1(t^+) \qquad [30(a)]$$

$$\ddot{y}_{1_{II}} + 2\dot{x}_{1_{II}} - y_{1_{II}} = -K^+ \left\{ \frac{y_{1_I}}{r_{e_o}^3} - 3\,\frac{y_o}{r_{e_o}^5} \Big[(x_o+\mu)x_{1_I} \right.$$

$$\left. + \, y_o y_{1_I} + z_o z_{1_I} \Big] + \frac{y_o}{r_{m_o}^3} \right\} = g_1(t^+) \qquad [30(b)]$$

Equations 30(a,b) are of the same form as Equations 15(a,b) and therefore their solutions are

$$x_{1_{II}}(t^+) = \int_0^{t^+} f_1(\tau)(t^+-\tau)\cos(t^+-\tau)d\tau + \int_0^{t^+} g_1(\tau)(t^+-\tau)\sin(t^+-\tau)d\tau \qquad [31(a)]$$

$$y_{1_{II}}(t^+) = -\int_0^{t^+} f_1(\tau)(t^+-\tau)\sin(t^+-\tau)d\tau + \int_0^{t^+} g_1(\tau)(t^+-\tau)\cos(t^+-\tau)d\tau \qquad [31(b)]$$

The solution of the higher order iterations has the same form as Equations 29 and 31.

Evaluation of the integrals in Equations 26, 28, 29, and 31 involves inversion of Kepler's equation into an explicit function of t. Such an inversion is not known for the general case. However, for some special cases good approximations are possible. These cases are the near escape ($e \sim 1$) straight line (or almost straight line) trajectories. The derivation of the special cases is as follows:

$$v^2 = u_r^2 + u_\theta^2 \qquad [(32)]$$

$$2\ vdv = 2u_r \frac{du_r}{dt} dt + 2u_\theta \frac{du_\theta}{dt} d\theta \qquad [(33)]$$

for a central gravitational field and "near-escape" trajectories the only acceleration of importance is radial and equal to $-\frac{r_o^2\ g_o}{r^2}$ where r_o, g_o are the earth's radius and acceleration at the surface, respectively. Thus Equation 33 reduces to

$$\frac{dv}{dt} = - \frac{u_r}{v} \frac{r_o^{\,2}\, g_o}{r^2} = \sin \gamma \, \frac{r_o^{\,2}\, g_o}{r^2} \qquad [(34)]$$

where γ is the angle between the velocity vector and the local horizontal and is given by:

$$\gamma = \tan^{-1} \frac{\dot{r}}{r\dot{\theta}} \qquad [(35)]$$

Using the equation for conic sections

$$r^{\dagger} = \frac{ep}{1 + e \cos \theta}$$

where p is the distance from the focus to the corresponding directrix, Equation 35 can be written as

$$\gamma = \tan^{-1} \left(\frac{r \sin \theta}{p} \right) = \frac{\pi}{2} - \frac{p}{r \sin \theta} + \frac{p^3}{3(r \sin \theta)^3}$$

$$- \frac{p^5}{5(r \sin \theta)^5} + \ldots ; \quad \left(\frac{r \sin \theta}{p} > 1 \right) \qquad [(36)]$$

$$\sin \gamma = 1 - \frac{1}{2} \left(\frac{p}{r \sin \theta} \right)^2 \left(1 - \frac{p^2}{3(r \sin \theta)^2} + \frac{p^4}{5(r \sin \theta)^2} - \ldots \right)^2$$

$$+ \frac{1}{4!} \left(\frac{p}{r \sin \theta} \right)^4 \left(1 - \frac{p^2}{3(r \sin \theta)^2} + \frac{p^4}{5(r \sin \theta)^4} \ldots \right)^4 + \ldots \qquad [(37)]$$

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† This form was chosen because it is the same for the ellipse, parabola and hyperbola.

thus for the cases of small p (trajectories with perigees near the earth's center) and for all cases far away from the earth, $\sin \gamma \doteq 1$.

For these cases we therefore have

$$dv \doteq - \frac{r_o^{\,2}\, g_o}{r^2}\, dt \qquad [(38)]$$

The relationship for dv in terms of r can be derived as follows:

for e = 1 (parabolic)

$$v_{par}^2 = g_o\, r_o^{\,2} \left(\frac{2}{r}\right) \qquad [39(a)]$$

for e < 1 (elliptic)

$$v_{ell}^2 = g_o\, r_o^{\,2} \left(\frac{2}{r} - \frac{1}{a}\right) \qquad [39(b)]$$

for e > 1 (hyperbolic)

$$v_{hy}^2 = g_o\, r_o^{\,2} \left(\frac{2}{r} + \frac{1}{a}\right) \qquad [39(c)]$$

By introducing the factor k defined by

$$2 \left| 1 - k^2 \right| = \frac{r_o}{a} \qquad [(40)]$$

with the properties $e = 1 \Rightarrow k = 1$, $e < 1 \Rightarrow k < 1$, $e > 1 \Rightarrow k > 1$, all equations can be combined into

$$v(r) = (2 r_o g_o)^{1/2} \left[(k^2 - 1) + \frac{r_o}{r} \right]^{1/2} \qquad [(41)]$$

The solution t = t(r) is obtained as follows:

$$2\, v dv = - (2 r_o g_o)\, (r_o)\, \frac{dr}{r^2} \qquad [(42)]$$

$$dv = \frac{r_o{}^2 \; g_o}{(2r_og_o)^{1/2}\left[(k^2 - 1) + \frac{r_o}{r}\right]^{1/2}} \frac{dr}{r^2} \qquad [(43)]$$

combining with Equation 38 results in

$$dt = \frac{r^{1/2}\, dr}{(2r_og_o)^{1/2}\left[\, r(k^2-1)+r_o\right]^{1/2}} = \left[\,(k^2-1)(2r_og_o)\right]^{-1/2} \frac{r^{1/2}\, dr}{\left[r + \frac{r_o}{k^2-1}\right]^{1/2}} \qquad [(44)]$$

$$t-t_o = \left[(k^2-1)(2r_og_o)\right]^{-1/2} \Bigg\{ r^{1/2}\left[\, r + \frac{r_o}{k^2-1}\right]^{1/2}$$

$$- r_i{}^{1/2}\left[\, r_i + \frac{r_o}{k^2-1}\right]^{1/2} - 2\,\frac{r_o}{k^2-1}\, \ln \frac{\left[r + \frac{r_o}{k^2-1}\right]^{1/2} + r^{1/2}}{\left[r_i + \frac{r_o}{k^2-1}\right]^{1/2} + r_i{}^{1/2}} \Bigg\} \qquad [(45)]$$

for the near escape cases ($e \doteq 1$) one can write

$$V(r) = (2g_or_o)^{1/2} \left(\frac{r_o}{r}\right)^{1/2} \left[\, 1 + \frac{1}{2}\left(\frac{k^2-1}{r_o}\right) r - \frac{1}{8}\left(\frac{k^2-1}{r_o}\right)^2 r^2 + \ldots\right] \qquad [(46)]$$

$$dV(r) = \left[- \frac{1}{2}\,(2g_o)^{1/2}\; r_o r^{-3/2} + \frac{1}{4}\,(k^2-1)\,(2g_o)^{1/2}\; r^{-1/2}\right.$$

$$\left. - \frac{3}{16}\,(k^2-1)^2\, \frac{(2g_o)^{1/2}}{r_o}\, r^{1/2} + \ldots\, \right] dr \qquad [(47)]$$

$$(t-t_i)\left(\frac{2r_o}{g_o}\right)^{-1/2} =$$

$$- \sum_{j=o}^{\infty} \frac{(j-1/2)}{(j+3/2)} \frac{1}{j!} \left[\prod_{\ell=o}^{j} \left(\frac{3}{2} - \ell\right) \right] (k^2-1)^j \left[\left(\frac{r}{r_o}\right)^{j+3/2} - \left(\frac{r_i}{r_o}\right)^{j+3/2} \right] \qquad [(48)]$$

where the convention $o! = 1$ and $\prod_{\ell=o}^{o} = 1$ should be used. Without loss of generality $t_i = o$ can be taken.

Letting $t\left(\frac{2r_o}{g_o}\right)^{-1/2} = T$, Equation 48 can be written as

$$T = \sum_{j=o}^{\infty} A_j \left[\left(\frac{r}{r_o}\right)^{j+3/2} - \left(\frac{r_i}{r_o}\right)^{j+3/2} \right] \qquad [(49)]$$

An inversion of this series gives

$$r(t) = r_o \sum_{n=1}^{\infty} a_n \left[3/2 \left(\frac{2g_o}{r_o}\right)^{1/2} \right]^{2n/3} \left[t \right.$$

$$\left. + 2\left(\frac{2g_o}{r_o}\right)^{-1/2} \sum_{j=o}^{\infty} A_j \left(\frac{r_i}{r_o}\right)^{j+3/2} \right]^{2n/3} = \sum_{n=1}^{\infty} \alpha_n (t-\beta_n)^{2n/3} \qquad [(50)]$$

where $a_1 = 1$, $a_2 = -\frac{2}{3}\frac{A_1}{A_o}$, $a_3 = -\left[\frac{2}{3}\frac{A_2}{A_o} - \left(\frac{A_1}{A_o}\right)^2\right]$ etc. ...

For the parabolic case (k=1), $A_o = \frac{1}{3}$ and $A_i = o$ for $i > o$, therefore

$$r(t)_{para} = \left[3/2\ (2g_o)^{1/2}\ r_o\right]^{2/3} \left\{t + \frac{2}{3}\left(\frac{2g_o}{r_o}\right)^{-1/2}\left[\frac{r_i}{r_o}\right]^{3/2}\right\}^{2/3} \qquad [(51)]$$

The last equation checks with direct inversion of the solution for the straight line parabolic solution.

To the degree of approximation used previously

$1-\mu-x^+ \doteq \rho_m$, and therefore $\rho_m = 1 - \frac{r}{d} = 1 - \sum_{n=1}^{\infty} \alpha_n' \ (t-\beta_n')^{2n/3}$

where $\alpha_n' = \frac{\alpha n}{\alpha}\ \omega^{-2/3n}$, $\beta_n' = \beta_n \omega$

The equation for f(t) can therefore be written as

$$f(t) = \frac{K^+\mu}{1-\mu} \frac{1}{\left[1 - \sum_{n=1}^{\infty} \alpha_n'(t-\beta_n')^{2n/3}\right]^2} = \frac{K^+\mu}{1-\mu}\left\{1+2\sum_{n=1}^{\infty} \alpha_n'(t-\beta_n')^{2n/3}\right.$$

$$\left. + 3\left[\sum_{n=1}^{\infty} \alpha_n'\ (t-\beta_n')^{2n/3}\right]^2 + \ldots\right\} \qquad [(52)]$$

The integrals of $f(t^+)$ in Equations 31(a,b) can be evaluated term by term as follows:

$$\int_0^t f(\tau)(t-\tau)\cos(t-\tau)d\tau \doteq \frac{K^+\mu}{1-\mu}\Bigg\{ (\cos t - t\sin t - 1)$$

$$+ \sum_{n=1}^{\infty} \frac{3}{2n+3}\alpha_n' \left[(t-\beta_n')^{\frac{2n}{3}+1} + \beta_n'^{\frac{2n}{3}+1} \right] \left[t\cos t \sum_{k=0}^{\infty} (-)^k \frac{\beta_n'^{2k}}{2k!} \right.$$

$$- \cos t \sum_{k=0}^{\infty} (-)^k \frac{\beta_n'^{2k+1}}{(2k)!} + t\sin t \sum_{k=0}^{\infty} (-)^{3k+2} \frac{\beta_n'^{2k+1}}{(2k+1)!}$$

$$\left. - \sin t \sum_{k=0}^{\infty} (-)^{3k+2} \frac{\beta_n'^{2k+2}}{(2k+1)!} \right] + \sum_{n=1}^{\infty} \frac{3}{2n+6}\alpha_n' \left[(t-\beta_n')^{\frac{2n}{3}+2} \right.$$

$$\left. + \beta_n'^{\frac{2n}{3}+2} \right] \left[t\cos t \ldots \right] + \ldots \Bigg\} \qquad [(53)]$$

$$\int_0^t f(\tau)(t-\tau)\sin(t-\tau)d\tau \doteq \frac{K^+\mu}{1-\mu}\Bigg\{ (\sin t - t\cos t)$$

$$+ \sum_{n=1}^{\infty} \frac{3}{2n+3}\alpha_n' \left[(t-\beta_n')^{\frac{2n}{3}+1} + \beta_n'^{\frac{2n}{3}+1} \right] \left[t\sin t \sum_{k=0}^{\infty} (-)^k \frac{\beta_n'^{2k}}{2k!} \right.$$

$$- \sin t \sum_{k=0}^{\infty} (-)^k \frac{\beta_n'^{2k+1}}{(2k)!} + t\cos t \sum_{k=0}^{\infty} (-)^{3k+1} \frac{\beta_n'^{2k+1}}{(2k+1)!}$$

$$\left. - \cos t \sum_{k=0}^{\infty} (-)^{3k+1} \frac{\beta_n'^{2k+2}}{(2k+1)!} \right] + \sum_{n=1}^{\infty} \frac{3}{2n+6}\alpha_n' \left[(t-\beta_n')^{\frac{2n}{3}+2} \right.$$

$$\left. + \beta_n'^{\frac{2n}{3}+2} \right] \left[t\sin t \ldots \right] \Bigg\} \qquad [(54)]$$

where

$$\alpha'_n = \frac{r_o}{d\omega^{2/3n}} \left[a_n \quad 3/2 \left(\frac{2g_o}{r_o} \right)^{1/2} \right]^{2n/3}$$

$$a_1 = 1, \; a_2 = - \frac{2}{3} \frac{A_1}{A_o}, \; a_3 = - \left[\frac{2}{3} \frac{A_1}{A_o} - \left(\frac{A_1}{A_o} \right)^2 \right] \cdots$$

$$A_i = \left(\frac{i-1/2}{i+3/2} \right) \frac{1}{i!} \prod_{L=0}^{i} \left(\frac{3}{2} - L \right) \left(\frac{k^2-1}{g_o} \right)^i$$

$$\beta'_n = 2\omega \left(\frac{2g_o}{r_o} \right)^{-1/2} \sum_{j=0}^{\infty} A_j \left(\frac{r_i}{r_o} \right)^{j+3/2}$$

The other terms are similarly integrated. It should be noted that for the parabolic case the summation over k in Equations 53 and 54 is eliminated, and the summations over n are convergent. A finite interval of convergence in excentricity about e=1 is assured by the $(k^2-1)^i$ in the expressions for A_i. The extent of this interval has not been established accurately. Although the solution presented is approximate (zeroth and first order solutions), and the assumptions that had to be made limit the solution to cases of a small perigee radius or else for the part of the trajectory far away from the gravitational center, the solution has in principle far greater applicability than might be expected. The reason for this being that the perturbation of the moon on the motion near the earth is negligible up to the "half"-point on the trajectory.

The complication of the resulting expressions after the integration make the practical use of this method questionable.

The inversion of Kepler's equation for the "near-escape" "near straight-line" trajectories (Equation 50) might prove useful in many other applications.

VI. APPLICATIONS OF THE APPROXIMATE ANALYTICAL SOLUTIONS OF THE RESTRICTED THREE BODY PROBLEM

A. TRAJECTORY ANALYSIS

The solutions presented offer a handy tool for trajectory analysis and selection for the numerous problems of restricted three bodies. Among these problems are:

1. Lunar trajectories (earth to moon and moon to earth)
2. Comet and meteor trajectories in the earth-moon space
3. The part of the earth to planet trajectories within the earth-moon space
4. The part of the earth to planet trajectories within the earth-sun space, and planet-sun space, using the rotating earth-sun and planet-sun systems, respectively.

The three solutions presented can be viewed as 1st, 2nd, and 3rd order solutions in accuracy and complexity. Thus for exploratory work the 1st solution is recommended. Using a small number of consecutive steps in the evaluation will result in improved accuracy in most cases. This method is not recommended for slow trajectories ($e \approx 1$). The second method will give more accurate results with direct evaluation and is recommended for most applications. The third method should give results comparable to numerical integration, at the expense of great complexity.

B. THRUST MANEUVERS

In many applications it is necessary to establish thrust maneuvers in order to accomplish a given mission. Thus establishing a lunar satellite by a purely ballistic trajectory from the earth is impossible. A thrust application is necessary in this case in order to achieve a given velocity and position vector. The determination of this thrust maneuver is a tedious task when no analytical solution is available. The solutions derived can all accommodate thrust maneuvers. However the simple 1st solution using 3 terms is most handy and sufficient for exploratory work in this area. The determination of thrust maneuvers using the 3-term first solution is as follows:

At a given time t^* it is desired that the vehicle have given position and velocity vectors. The three term solution without the thrust term give a solution that differs from the required one by

$\Delta u^*(t^*)$, $\Delta v^*(t^*)$, $\Delta w^*(t^*)$, $\Delta x(t^*)$, $\Delta y(t^*)$, $\Delta z^*(t^*)$.

The three term solution then gives the following results:

$$\left[\Delta u^* \ (t^*)\right]_T = \eta^2 \int_0^{t^*} T_x^* \, dt \qquad [55(a)]$$

$$\left[\Delta v^* \ (t^*)\right]_T = \eta^2 \int_0^{t^*} T_y^* \, dt \qquad [55(b)]$$

$$\left[\Delta w^* \ (t^*)\right]_T = \eta^2 \int_0^{t^*} T_z^* \, dt \qquad [55(c)]$$

$$\left[\Delta x^* \ (t^*)\right]_T = \eta^2 \int_0^{t^*} \int_0^{t} T_x^* \, dt \, dt \qquad [55(d)]$$

$$\left[\Delta y^* \ (t^*)\right]_T = \eta^2 \int_0^{t^*} \int_0^{t} T_y^* \, dt \, dt \qquad [55(e)]$$

$$\left[\Delta z^* \ (t^*)\right]_T = \eta^2 \int_0^{t^*} \int_0^{t} T_z^* \, dt \, dt \qquad [55(f)]$$

Considering first the case of impulsive thrusting, it is seen that to the present order of approximation the solution is simply given by impulsing with Δv velocity increments given by $\frac{\Delta u^*(t^*)}{\eta^2}$, $\frac{\Delta v^*(t^*)}{\eta^2}$, $\frac{\Delta w^*(t^*)}{\eta^2}$ at times given by $\frac{\Delta x^*(t^*)}{\Delta u^*(t^*)}$, $\frac{\Delta y^*(t^*)}{\Delta v^*(t^*)}$, $\frac{\Delta z^*(t^*)}{\Delta w^*(t^*)}$, respectively.

Next consider the case of constant-acceleration thrusting where a range of acceleration is available; for this case

$$\left[\Delta u^* \ (t^*)\right] = \eta^2 \ T_x^* \ \Delta t^* \qquad [56(a)]$$

$$\left[\Delta x^* \ (t^*)\right] = \eta^2 \ T_x^* \ \frac{(\Delta t^*)^2}{2} \qquad [56(b)]$$

$$\therefore\ (\Delta t^*) = 2\,\frac{\Delta x^*\ (t^*)}{\Delta u^*\ (t^*)}\ ;\quad T^*_x = \frac{1}{2}\,\frac{\left[\Delta u^*\ (t^*)\right]^2}{\Delta x^*\ (t^*)}\,\frac{1}{\eta^2}$$

The implications of the last results are obvious. Although for other cases the solutions are not as obvious, Equations 55(a-f), provide a comparatively simple tool for establishing required thrust maneuvers.

C. CORRECTIONS TO THE FIRING VECTOR FOR TRAJECTORIES FROM THE MOON TO THE EARTH

One of the problems arising in trajectory work is the determination of corrections to the firing vector in order to achieve desired corrections in the terminal point, namely it is desirable to determine Δu_i, Δv_i, Δw_i, so that the desired Δx_t, Δy_t, Δz_t, Δu_t, Δv_t, Δw_t will be achieved (t-terminal). As an example consider attempting to determine the firing vector from the moon to impact at a given point on the earth. After a first trial determination is made (say by numerical integration, or stepwise expansion) an impact error results. From the known impact error it is necessary to change the firing vector to achieve an impact at the desired point. In this case only Δx_t, Δy_t, Δz_t are of interest. On the other hand if it is desired to achieve a given orbit around the earth (moon) then the terminal velocity is also important.

A correction up to order η^2 can be determined easily using the second asymptotic expansion. From Equations 14 one obtains

$$\Delta x^*_t = (t^*_t \cos\omega^+ t^*_t)\Delta u^*_i + (t^*_t \sin\omega^+ t^*_t)\Delta v^*_i \qquad [57(a)]$$

$$\Delta y^*_t = -\,(t^*_t \sin\omega^+ t^*_t)\Delta u^*_i + (t^*_t \cos\omega^+ t^*_t)\Delta v^*_i \qquad [57(b)]$$

$$\Delta z^*_t = t^*_t\ \Delta w^*_i \qquad [57(c)]$$

$$\Delta\left(\frac{dx^*}{dt^*}\right)_t = -\,(\omega^+ t^*_t \sin\omega^+ t^*_t)\Delta u^*_i + (\cos\omega^+ t^*_t)\Delta u^*_i$$

$$+\,(\omega^+ t^*_t \cos\omega^+ t^*_t)\Delta v^*_i + (\sin\omega^+ t^*_t)\Delta v^*_i \qquad [57(d)]$$

$$\Delta\left(\frac{dy^*}{dt^*}\right)_t = -\,(\omega^+ t^*_t \cos\omega^+ t^*_t)\Delta u^*_i - (\sin\omega^+ t^*_t)\Delta u^*_i$$

$$-\,(\omega^+ t^*_t \sin\omega^+ t^*_t)\Delta v^*_i + (\cos\omega^+ t^*_t)\Delta v^*_i \qquad [57(e)]$$

$$\Delta\left(\frac{dz^*}{dt^*}\right)_t = \Delta w^*_i \qquad [57(f)]$$

The problems of correcting for either the terminal velocity or position vector can be handled easily. For convenience the equations are inverted here. For correction of errors in the terminal position vector, the corrections to the firing vector are

$$\Delta u^*_i = \frac{1}{\Delta_1}\left[\,(t^*_t \cos\omega^+ t^*_t)\Delta x^*_t - (t^*_t \sin\omega^+ t^*)\Delta y^*_t\,\right] \qquad [58(a)]$$

$$\Delta v^*_i = \frac{1}{\Delta_1}\left[\,(t^*_t \cos\omega^+ t^*_t)\Delta y^*_t + (t^*_t \sin\omega^+ t^*)\Delta x^*_t\,\right] \qquad [58(b)]$$

$$\Delta w^*_i = \frac{1}{t^*_t}\,\Delta z^*_t$$

where

$$\Delta_1 = \begin{vmatrix} t_t^* \cos\omega^+ t_t^* & t_t^* \sin\omega^+ t_t^* \\ - t_t^* \sin\omega^+ t_t^* & t_t^* \cos\omega^+ t_t^* \end{vmatrix} = t_t^{*2}$$

For correction of errors in the terminal velocity vector the corrections to the firing vector are

$$\Delta u_i^* = \frac{1}{\Delta_2} \left[\left(\cos\omega^+ t_t^* - \omega^+ t_t^* \sin\omega^+ t_t^* \right) \Delta \left(\frac{dx^*}{dt^*} \right)_t - \left(\sin\omega^+ t_t^* + \omega^+ t^* \cos\omega^+ t_t^* \right) \Delta \left(\frac{dy^*}{dt^*} \right)_t \right] \qquad [59(a)]$$

$$\Delta v_i^* = \frac{1}{\Delta_2} \left[\left(\cos\omega^+ t_t^* - \omega^+ t_t^* \sin\omega^+ t_t^* \right) \Delta \left(\frac{dy^*}{dt^*} \right)_t - \left(- \sin\omega^+ t_t^* - \omega^+ t^* \cos\omega^+ t_t^* \right) \Delta \left(\frac{dx^*}{dt^*} \right)_t \right] \qquad [59(b)]$$

$$\Delta w_i^* = \Delta \left(\frac{dz^*}{dt^*} \right)_t \qquad [59(c)]$$

where

$$\Delta_2 = \begin{vmatrix} - \omega^+ t_t^* \sin\omega^+ t_t^* + \cos\omega^+ t_t^* & \omega^+ t_t^* \cos\omega^+ t_t^* + \sin\omega^+ t_t^* \\ - \omega^+ t_t^* \cos\omega^+ t_t^* - \sin\omega^+ t_t^* & - \omega^+ t_t^* \sin\omega^+ t_t^* + \cos\omega^+ t_t^* \end{vmatrix} = \omega^{+2} t_t^{*2} + 1$$

When correcting for errors in terminal position vector the resulting change in the terminal velocity vector can be obtained by substitution of the results of Equations 58 in Equations 57 similarly for the case of corrections for the velocity vector.

Higher order corrections can also be obtained using Equations 24. However the algebraic complications are great.

The problem of correcting simultaneously for terminal position and velocity vector errors is more complicated. Since the equations of motion for this case are of 6th order, in general only 6 constants can be specified, these, then completely determine the trajectory. The natural 6 constants are the initial position and velocity vectors, however in principle any six constants will determine a unique solution. It is clear therefore that specifying final position and velocity vector and initial position vector is overspecifying the problem and will in general not have a solution.

One seemingly "overspecified" case for which solutions can always be found will be discussed here because of its practical importance. This case is characterized by a given terminal position and velocity vector at a given instant in time and a given selenographical location on the moon as the firing point. Provided certain constraints are met (proper Jacobi constant and velocity vector orientation) the proper firing vector at the moon can be found. In the simple planar case the solution is easily obtained by integrating backwards from the terminal position in a non-rotating earth centered coordinate system and adjusting the position of the moon at the beginning of the integration until the required initial position on the moon is reached. The reason why this is possible is due to the fact that the initial position vector was not specified in the same coordinate system as the final position and velocity vectors. Had this been done, no solution would in general be found. Thus the seemingly "overspecified" case is not truly overspecified, but rather a special case of a constrained problem, where the constraint results in an apparent overspecification.

In the rotating system used in the rest of this thesis there is no possibility of adjusting the position of the moon at the beginning of the integration so that the same problem could not be solved by the method discussed previously.

It is easily seen that the corresponding problem is a specification of the final position and velocity vectors to within a rigid rotation about the origin of the rotating coordinate system.

SYMBOLS

d	distance between the centers of the earth and the moon
G	gravitational constant in Newton's law of gravity
K	$G(m_e + m_m)$
r_e	distance from the center of the earth
r_m	distance from the center of the moon
t	time
T_x, T_y, T_z	thrust acceleration components
U	reference "vacuum" velocity at the surface of the earth
u, v, w	components of the vehicle's velocity vector in the instantaneous directions of x, y, z, respectively
x, y, z	coordinates of the r.h. rotating coordinate system with the origin at the earth-moon center of mass, the X-axis passing the moon's c.g. and the Z-axis perpendicular to the moon's orbital plane

Greek

ω	moon's orbital velocity
μ	$\frac{m_m}{m_m + m_e}$
η	$\left(\frac{K(1 - \mu)}{d\, U^2}\right)^{1/2}$

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APPENDIX

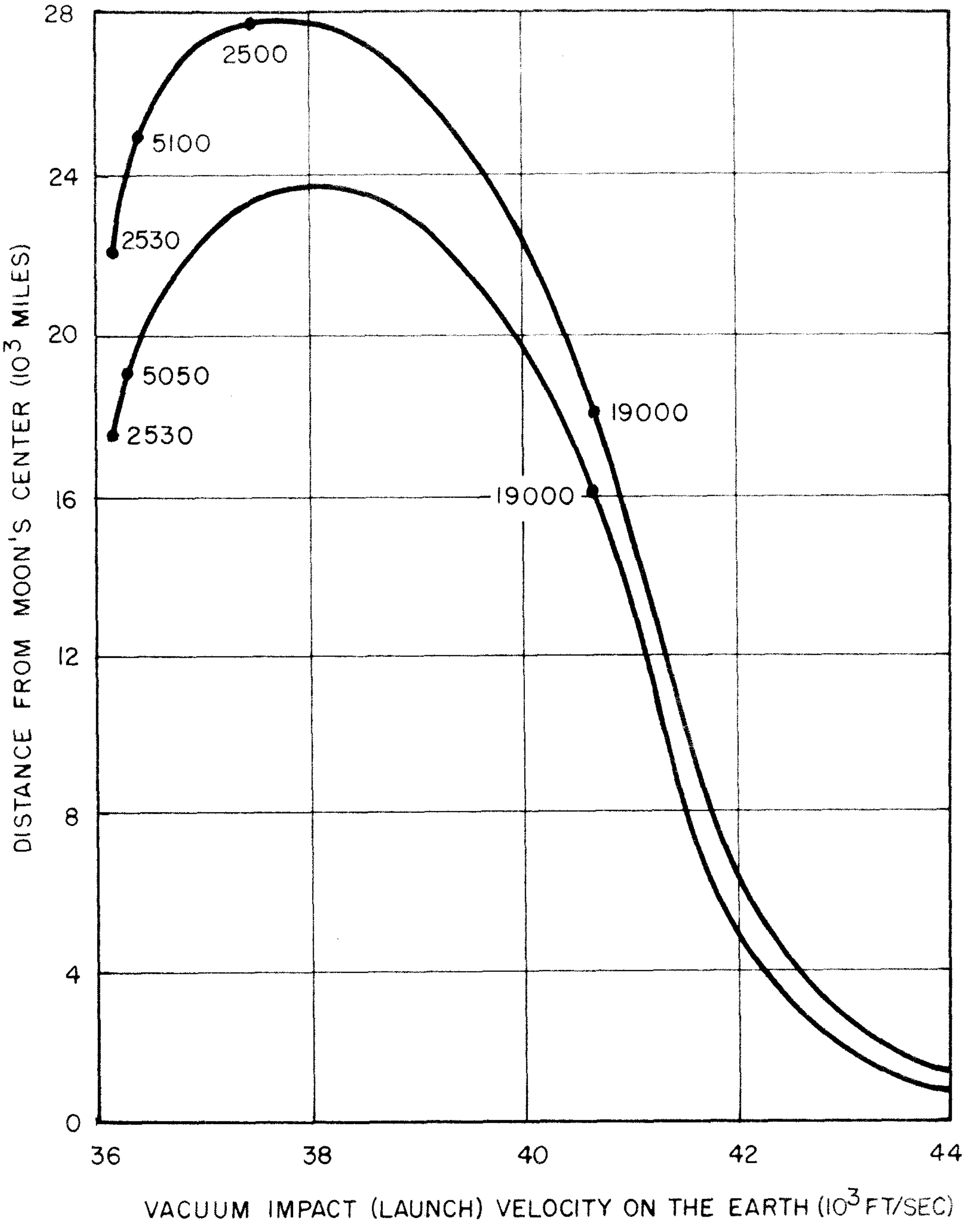
The use of the simplified model of the earth-moon space enabled an analysis applicable to all chronological times. In reality however the rotation of the earth and the moon about the baricenter is not circular. The motion can however be approximated by circular motion for the short periods involved in transfer trajectories (as opposed to satellite trajectories). The variation in the distance between the centers of the earth and moon has to be accounted for. Past experience has shown that the major contribution of this difference is in changing the time of flight. This change can be accounted for by dividing the distance difference $\Delta r_{earth\text{-}moon}$ by the minimum velocity along the chosen trajectory. Figure 13 shows the location of this point as a function of the vacuum impact (launch) velocity on the earth.

Following is an approximate relationship for establishing the minimum velocity

$$V_{min} = \left[V^2_{impact} + 41{,}860{,}900 \left(\frac{32.2 \cdot 3958.885}{D_f - r_{m\ min}} - 31.5903 \right) \right]^{1/2} \qquad [(A1)]$$

where $r_{m\ min}$ is the distance of the point of minimum velocity from the moon (figure 13) and D_f is the instantaneous earth-moon distance at firing.

Since astronomical data is generally given in geocentric coordinates, it is necessary to transform the initial conditions from the



[NUMBERS ON GRAPH INDICATE ACTUAL VELOCITY OF VEHICLE AT GIVEN DISTANCE (FT/SEC)]

DISTANCE OF POINT OF MINIMUM VELOCITY FROM MOON VS "VACUUM" IMPACT (LAUNCH) VELOCITY ON THE EARTH

Figure 13

coordinate system used in the analysis to the geocentric coordinate system. The geocentric coordinate system chosen is a non-rotating system with the positive Z-axis pointing to the true north, and the positive X-axis passing through the launching point at launch. The required transformation for the point on the moon nearest to the earth (the point used in the examples) is given by

$$\underline{\bar{X}} = (D_f - 1079.93) \cos \theta_{md} \text{ (miles)} \qquad [A2(a)]$$

$$\underline{\bar{Y}} = 0 \qquad [A2(b)]$$

$$\underline{\bar{Z}} = (D_f - 1079.93) \sin \theta_{md} \text{ (miles)} \qquad [A2(c)]$$

$$\underline{\dot{\bar{X}}} = (u \cos A + v \sin A) \cos B - \left[(-u \sin A + v \cos A) \cos C - w \sin C \sin B\right] \qquad [A2(d)]$$

$$\underline{\dot{\bar{Y}}} = (u \cos A + v \sin A) \sin B + \left[(-u \sin A + v \cos A) \cos C - w \sin C \cos B\right] \qquad [A2(e)]$$

$$\underline{\dot{\bar{Z}}} = (-u \sin A + v \cos A) \sin C + w \cos C \qquad [A2(f)]$$

where

$$\cos A = \cos(\theta_{ln} - \theta_{mra}) \cos \theta_{md}$$

$$B = \theta_{ln} - \theta_{mra}$$

$$\tan C = \frac{\tan \theta_{md}}{\sin(\theta_{lm} - \theta_{mra})}$$

θ_{md} - Moon's declination (given in Ephemeris)

θ_{lm} - Longitude of moon's line of nodes (given in Ephemeris)

θ_{mra} - Moon's right ascention (given in Ephemeris)