INITIAL MOTION OF A ROCKET MOVING ON A STRETCHED CABLE

Thesis by

Warren E. Weinberg
Captain, U.S.M.C.

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SUMMARY

The practicability of using a stretched cable for a rocket launching device is dependent primarily upon the transverse movement of the rocket and the stresses involved in the system. This requires analysis of the effects of the moving rocket mass on the cable from the instant of contact, and the problem reduces to one of wave propagation.

The analysis here is restricted essentially to developing a procedure by which the initial motion of the mass can be calculated, and an expression is obtained which permits determination of the deflections. The problem is approached by first assuming that the mass exerts a constant transverse force on the cable. Admittedly, this is a simplifying assumption, and the result is not valid for the instant the mass hits the cable or for a short time thereafter because the inertia of the mass is not considered.

Next, the problem is solved by taking into account the dynamics of the mass, and the solution reveals that the path of the actual mass deviates from the force path by as much as twenty percent during the initial motion but soon returns to the force path. The mass does not exert its full force on the cable at the instant of contact, but comes down on the cable with full force a short time later. Since maximum stresses on the system occur at this time, this factor is an important result of the analysis. Further, it is shown that increased velocity of the mass increases the deviation from the force path, and increased mass lengthens the time of return to the force path.

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I. INTRODUCTION

The problem investigated here results from an original inquiry by the U. S. Navy as to the feasibility of using a stretched cable for a rocket launching device. Recent interest in the subject has extended greatly the range of possible applications, and since the analysis of the physical phenomena involved in a mass moving on a stretched cable is general in nature, it need not be restricted to the launching of rockets.

C. R. DePrima and F. E. Marble[1] have investigated the problem of a force moving on a stretched cable. Their results in the supercritical case (i.e. where the force travels faster than the speed of propagation of a disturbance on the cable) show that although the cable distorts under the influence of the force, the path of the force is unaffected and all distortions occur behind the moving force. In the subcritical case, the path of the force is affected but the initial conditions are unrealistic because a force and not a mass was considered. Of interest then is the subcritical case of a mass traveling with constant velocity across a stretched cable. By considering the subcritical case, the analysis will hold for slower and heavier masses, and the range of application can be extended.

For the purposes of this work, a portion of this phase of the problem is investigated: i.e. the initial motion of a mass traveling with constant subcritical velocity on a stretched cable. The procedure will be to derive the partial differential equation which describes the physical phenomena, solve the equation, and arrive at a result which can be used to determine the deflections along the path of the mass. Determination of the mass deflections will permit calculation of the stresses involved, which of course are of primary interest in the practical problem.

II. DEVELOPMENT OF THE PROBLEM

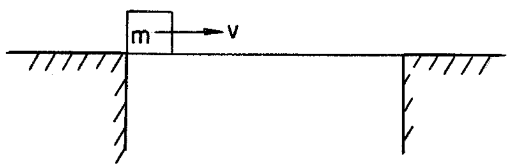

Fig. 1

For the system, assume a stretched string between two fixed points and consider a mass moving across it with constant velocity v (Fig. 1).

First, consider the partial differential equation for the transverse motion of a string distorted into some curve u(x, t), (Fig. 2). Application of Newton's second law leads to the wave equation in one dimension(2)

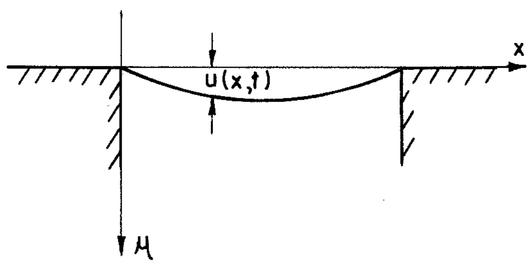

Fig. 2

$$\rho\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} \tag{1}$$

where ρ is mass per unit length of string, T is tension in string, and u(x, t) is the displacement of the string. The term $\rho\frac{\partial^2 u}{\partial t^2}$ is the inertia force due to string mass and $T\frac{\partial^2 u}{\partial x^2}$ is force due to tension in the string.

If an arbitrary force F(x, t) is applied to the string, Equation (1) becomes

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\frac{F}{T} \tag{2}$$

where $c^2 \equiv \frac{T}{\rho}$.

If the force is exerted by a mass m on the string at a definite point, x_1 distance from the origin (Fig. 3), and the mass has a displacement W(t), then the force F applied at the point of the mass may be calculated by applying Newton's laws to the mass.

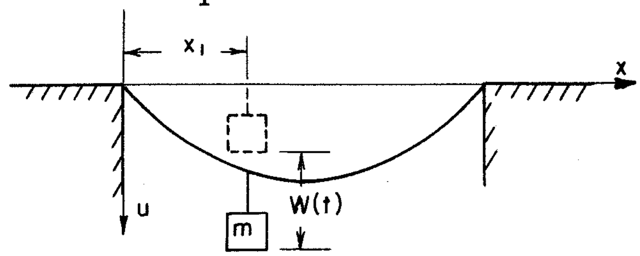

Fig. 3

$$m \frac{d^2W}{dt^2} = -F + mg$$

or

$$F = -m \left(\frac{d^2W}{dt^2} - g\right) \qquad (3)$$

where mg is the force due to gravity.

The acceleration of the mass may be expressed in terms of u, the displacement of the string. Consider the mass moving across the string with uniform velocity v. Taking an instantaneous look at a particular point P, the vertical velocity of the mass is $\frac{\partial u}{\partial t}$. However, the vertical velocity due to the velocity v along the string must also be considered. Looking at Fig. 4, (greatly exaggerated because θ actually is very small),

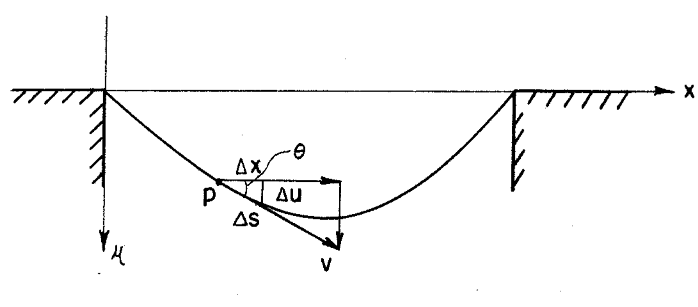

Fig. 4

Vertical component of v = v sin θ

and replacing sin θ by tan θ,

Vertical component of v = v tan θ = $v\frac{\Delta u}{\Delta x}$

Passing to the limit, $\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}$

and therefore

Vertical component of v = $v\frac{\partial u}{\partial x}$

Thus, the total vertical velocity in terms of u(x, t) is the sum of two terms: $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x}$. It was agreed before the vertical displacement of the mass at a fixed point is W(t). Therefore, if u is fixed at a point determined by x, t,

$$\frac{dW}{dt} = \left[\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x}\right]_{x,t}$$

$$\frac{d}{dt}\left(\frac{dW}{dt}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x}\right)\frac{dx}{dt} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x}\right)\frac{dt}{dt}$$

$$= v\frac{\partial^2 u}{\partial x\partial t} + v^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} + v\frac{\partial^2 u}{\partial x\partial t}$$

or

$$\frac{d^2W}{dt^2} = \left[\frac{\partial^2 u}{\partial t^2} + 2v\frac{\partial^2 u}{\partial x\partial t} + v^2\frac{\partial^2 u}{\partial x^2}\right]_{x,t}$$

and introducing this into Equation (3)

$$F = -m\left(\left.\frac{\partial^2 u}{\partial t^2}\right|_{x,t} + 2v\left.\frac{\partial^2 u}{\partial x\partial t}\right|_{x,t} + v^2\left.\frac{\partial^2 u}{\partial x^2}\right|_{x,t} - g\right) \qquad (4)$$

Since at t = 0, u = 0, and the mass moves with a constant velocity v, the position x of the mass is fixed by x = vt. Using this notation to denote a particular point and combining Equations (2) and (4) gives

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{m}{T}\left(\frac{\partial^2 u}{\partial t^2} + 2v\frac{\partial^2 u}{\partial x \partial t} + v^2\frac{\partial^2 u}{\partial x^2} - g\right)_{x=vt} \qquad (5)$$

For all values of x other than x = vt, Equation (5) gives the homogeneous wave equation, i. e.

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad \text{for } x \neq vt$$

and therefore it is convenient to introduce the Dirac delta function[3] defined as follows:

$$\delta(x-vt) = \begin{cases} 0 \text{ for } x \neq vt \\ \infty \text{ for } x = vt \end{cases} \qquad \int_{vt-\epsilon}^{vt+\epsilon} (x-vt)dx = 1$$

or saying the same thing

$$\int_{-\infty}^{+\infty} \delta(x - vt)\, dx = 1 \qquad (6)$$

Introducing this function, Equation (5) becomes

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{m}{T}\left(\frac{\partial^2 u}{\partial t^2} + 2v\frac{\partial^2 u}{\partial x \partial t} + v^2\frac{\partial^2 u}{\partial x^2} - g\right)\delta(x-vt) \qquad (7)$$

This is a linear inhomogeneous partial differential equation of the hyperbolic type which is complicated by the fact that the coefficients are variables. The problem at hand is to solve this equation, and a dimensionless form will be convenient.

Define the dimensionless parameters

$$\xi \equiv \frac{x}{L} \qquad v \equiv \frac{u}{L} \qquad \lambda \equiv \frac{mg}{\rho c^2}$$

$$\tau \equiv \frac{ct}{L} \qquad \mu \equiv \frac{v}{c} \qquad \beta \equiv \frac{c^2}{Lg}$$

where L is the length of the string between the two fixed points.

First, note $\int \delta(x-vt)dx = 1$ and since dx has the dimension of length, this implies the delta function has the dimension L^{-1}. This indicates $L\left[\delta(x-vt)\right]$ is dimensionless, and dimensionless notation for the delta function is $\delta(\xi-\mu\tau)$ *. Therefore $L\left[\delta(x-vt)\right]$ corresponds to $\delta(\xi-\mu\tau)$.

Multiplying the left side by $\frac{L^2}{L}$, the right side by $\frac{c^2L^3}{c^2L^2}$, and noting $T = \rho c^2$, Equation (7) can be written

$$\frac{\partial^2(\frac{u}{L})}{\partial^2(\frac{x}{L})^2} - \frac{\partial^2(\frac{u}{L})}{\partial(\frac{ct}{L})^2} = \frac{mg}{\rho c^2}\left\{\frac{c^2}{Lg}\left(\frac{\partial^2(\frac{u}{L})}{\partial(\frac{ct}{L})^2} + 2\frac{v}{c}\frac{\partial^2(\frac{u}{L})}{\partial(\frac{x}{L})\partial(\frac{ct}{L})} + (\frac{v}{c})^2\frac{\partial^2(\frac{u}{L})}{\partial(\frac{x}{L})^2}\right) - \frac{c^2L^2}{c^2L^2}\right\} L\,\delta(x-vt)$$

or

$$\frac{\partial^2 v}{\partial\xi^2} - \frac{\partial^2 v}{\partial\tau^2} = -\lambda\left\{1 - \frac{\beta}{\lambda}\left(\frac{\partial^2 v}{\partial\tau^2} + 2\mu\frac{\partial^2 v}{\partial\xi\,\partial\tau} + \mu^2\frac{\partial^2 v}{\partial\xi^2}\right)\right\}\delta(\xi-\mu\tau) \qquad (8)$$

It should be noted that the initial conditions imposed by the fixed end point of the stretched string are $v = \frac{\partial v}{\partial t} = 0$ at $\tau = 0$.

Two solutions will be considered in the analysis of the problem: first, the solution by assuming a constant transverse force on the string; and second, the solution of Equation (8) which takes into account the effects of a mass on the string. The constant transverse

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*$x = vt$ can be written $\frac{x}{L} = \frac{v}{c}\,\frac{ct}{L}$ which becomes $\xi = \mu\tau$.

force equation is obtained by making $\beta = 0$ in Equation (8) and is

$$\frac{\partial^2 v}{\partial \xi^2} - \frac{\partial^2 v}{\partial \tau^2} = -\lambda\, \delta(\xi - \mu\tau) \tag{9}$$

III. SOLUTION OF THE PROBLEM WITH CONSTANT TRANSVERSE FORCE

Courant[4] shows that the solution of Equation (9), with the given initial conditions, can be written

$$v(\xi, \tau) = \frac{\lambda}{2} \int_{t=0}^{\tau} \int_{x=\xi-(\tau-t)}^{\xi+(\tau-t)} \delta(x-\mu t)\, dxdt \qquad (10)$$

where ξ, τ are the dimensionless coordinates of the physical system, and x, t are used as variables of integration. It should be emphasized that the variables of integration x, t as used in Equation (10) are not the same as the x, t coordinates of the physical system used in the original development of the problem.

Since interest is focused on evaluating the solution on the path of the moving force defined by $\xi = \mu\tau$, $v(\xi, \tau)$ can be written $v(\mu\tau, \tau) = W_o(\tau)$ where $W_o(\tau)$ represents the displacement of the force. Equation (10) becomes

$$W_o(\tau) = \frac{\lambda}{2} \int_{t=0}^{\tau} \int_{x=\xi-(\tau-t)}^{\xi+(\tau-t)} \delta(x-\mu t)\, dxdt \qquad (11)$$

The evaluation of the integrals of Equation (11) can be accomplished readily by geometric interpretation and the use of characteristics. Since the slope of the characteristics in the dimensionless ξ-τ plane are ± 1 and the subcritical case ($v < c$) is to be considered, the diagram (Fig. 5) can be drawn. The path of the force, labeled F, is given by $\xi = \mu\tau$ and is drawn with a slope $1 < \frac{1}{\mu} < \infty$.

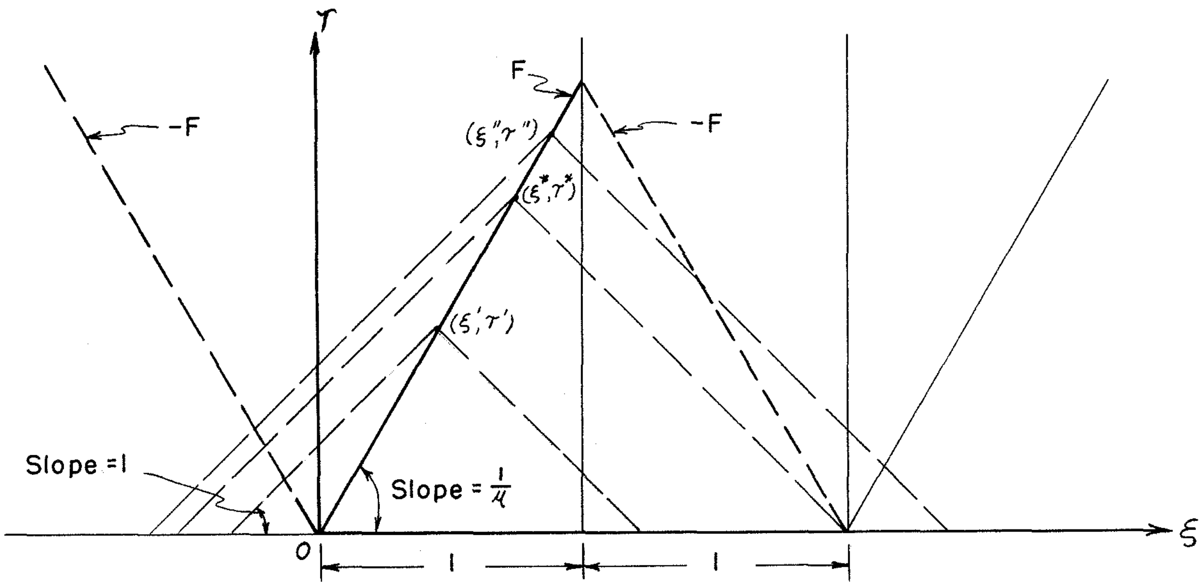

Fig. 5

The initial condition imposed by the physical fact that the string is fixed at x = 0 may be taken into account as follows. If the string is assumed infinite in length, then the effect of this fixed end is simulated by assuming an equal force starting at x = 0 and traveling in the negative direction with velocity v. Similarly, the boundary condition imposed by the string being fixed at x = L can be simulated by an equal force starting at x = 2L and traveling in the negative direction with velocity v. The paths of these fictitious forces are labeled -F. As will be shown, these paths have a negative effect in the geometric process for evaluating the integrals. Further, it should be noted that the distance L in the dimensionless plane is unity.

The arbitrary points (ξ , τ) are picked on the path of the force and triangles of integration (commonly referred to as domains of dependence) are defined by drawing characteristics through them. From Fig. 5, it immediately is evident that two cases must be considered if a solution is to be obtained along the entire length of the string. At the point (ξ^*, τ^*), the force encounters the wave which has been reflected from the fixed end, and the solution at point (ξ', τ'), good for $0 < \tau < \tau^*$, will be different than the solution at point (ξ'', τ''), good for $\tau^* < \tau < \frac{1}{\mu}$. For the purposes of this work, only the initial motion of the force is considered and therefore detailed analysis is presented for just the first case, i.e. where $0 < \tau < \tau^*$. Proceeding now to the evaluation of Equation (11), it will be convenient to refer to the simplified diagram, Fig. 6.

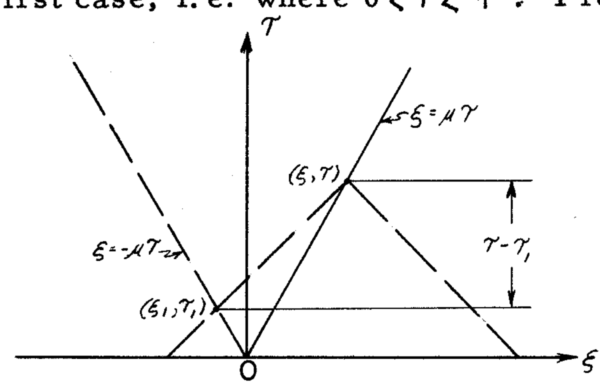

Fig. 6

From the definition of the delta function,

$$\int_{x = \xi - (\tau - t)}^{\xi + (\tau - t)} \delta(x - \mu t)\, dx = \begin{cases} 0 & \text{for } x \neq \mu t \\ 1 & \text{for } x = \mu t \\ -1 & \text{for } x = -\mu t \end{cases}$$

and since the value of the solution is dependent on the given values only in the interior of the triangle of integration, this integral yields a value of + 1 at, and only at, each point on the line $\xi = \mu\tau$, and a value of - 1 at, and only at, each point on the line $\xi = -\mu\tau$. This may be thought of as a series of positive unit impulses along $\xi = \mu\tau$ and negative unit impulses along $\xi = -\mu\tau$, the effects cancelling in the region 0 to τ_1. Now taking into consideration the summing process indicated by $\int_{\tau=0}^{\tau} dt$, the double integral of Equation (11) yields nothing more than $\tau - \tau_1$. It can be shown that

$$\tau - \tau_1 = \frac{\mu\tau + \xi}{1+\mu} = \frac{2\mu\tau}{1+\mu}$$

Therefore

$$W_o(\tau) = \frac{\lambda\mu\tau}{1+\mu} \qquad (12)$$

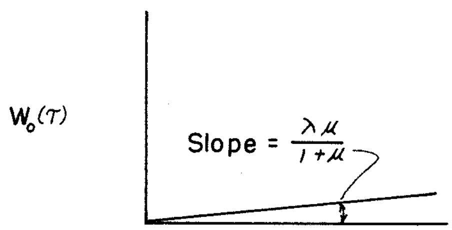

Fig. 7

It is evident that a plot of displacement versus time would give a straight line, arbitrarily drawn in Fig. 7. It should be noted that positive displacements are directed down or earthward in the physical sense.

Investigation of the physical results of this solution reveals that if a force $F = \frac{mg}{T}$ is moving along the solid foundation with constant velocity v at $t < 0$ and hits the string at $t = 0$, the string deflects instantaneously, and the force takes on a constant vertical velocity (Fig. 8). Therefore, the vertical velocity jumps from zero to some finite value and the acceleration is infinite at $t = 0$. This

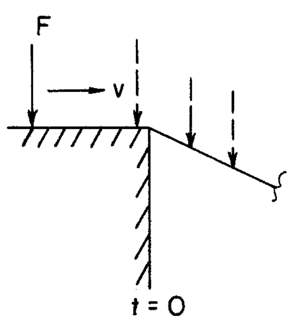

Fig. 8

of course is physically impossible.

The reason for such a result is obvious. Equation (11) gives the solution for a constant transverse force moving across the string and not a mass which is physically the case. That is, Equation (11) ignores the dynamics of a mass. Intuitively, it is evident that a mass moving along the solid foundation and hitting the string would give a path probably something like that of Fig. 9. The inertia of the mass would prevent in effect the infinite vertical acceleration. The solution of Equation (8) which takes into account the effects of a mass should correct this discrepancy.

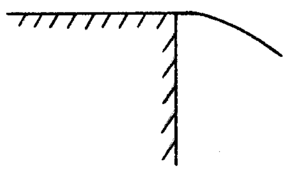

Fig. 9

IV. THE DIFFERENTIAL DIFFERENCE EQUATION FOR THE MOTION OF THE MASS

Rewrite the complete physical equation (i.e. Equation (8))

$$\frac{\partial^2 v}{\partial \xi^2} - \frac{\partial^2 v}{\partial \tau^2} = -\lambda \left\{ 1 - \frac{\beta}{\lambda} \left(\frac{\partial^2 v}{\partial \tau^2} + 2\mu \frac{\partial^2 v}{\partial \xi \partial \tau} + \mu^2 \frac{\partial^2 v}{\partial \xi^2} \right) \right\} \delta(\xi - \mu\tau) \tag{13}$$

The solution in integral form where x, t are the variables of integration is

$$v(\xi, \tau) = \frac{\lambda}{2} \int_{t=0}^{\tau} \int_{x=\xi-(\tau-t)}^{\xi+(\tau-t)} \left[1 - \frac{\beta}{\lambda} \left(\frac{\partial^2 v}{\partial t^2} + 2\mu \frac{\partial^2 v}{\partial x \partial t} + \mu^2 \frac{\partial^2 v}{\partial x^2} \right) \right] \delta(x - \mu t)\, dx\, dt \tag{14}$$

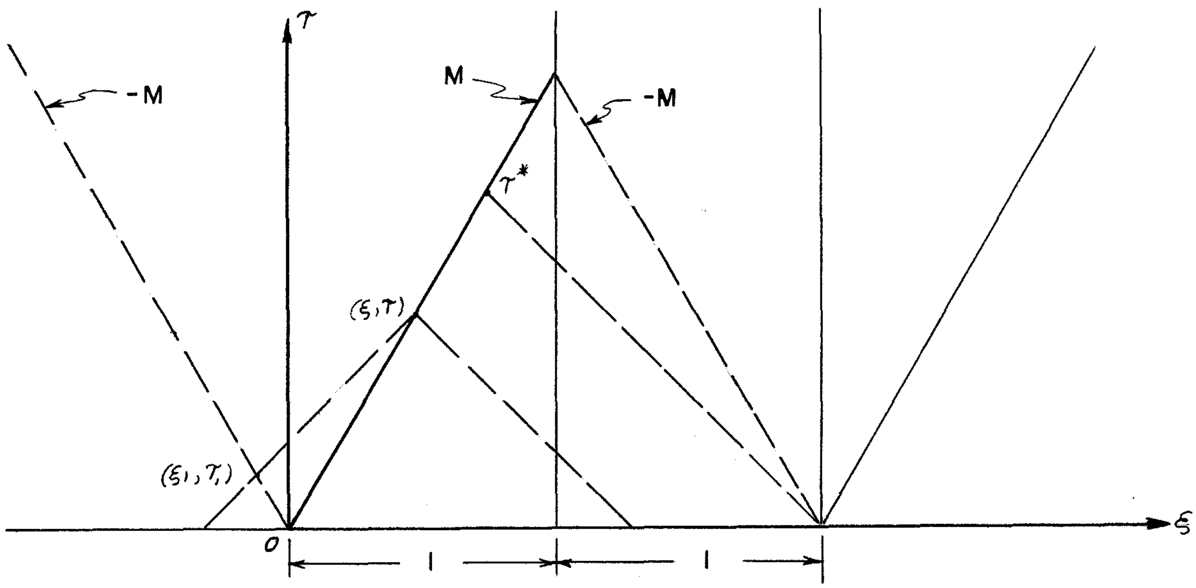

Fig. 10

It will be remembered that the integral with respect to x is zero except at each point on the path of the mass where it yields unity. Further, the values of this integral cancel in the region 0 to τ_1 (Fig. 10). It can be shown easily that

$$\tau_1 = \left(\frac{1-\mu}{1+\mu}\right)\tau \tag{15}$$

and since the path of the mass is defined by $\xi = \mu\tau$, Equation (14) can be written

$$v(\mu\tau, \tau) = \frac{\lambda}{2} \int_{t=\left(\frac{1-\mu}{1+\mu}\right)\tau}^{\tau} \left[1 - \frac{\beta}{\lambda}\left(\frac{\partial^2 v}{\partial t^2} + 2\mu\frac{\partial^2 v}{\partial x \partial t} + \mu^2 \frac{\partial^2 v}{\partial x^2}\right)\right] dt \tag{16}$$

Using the notation $W(\tau) \equiv v(\mu\tau, \tau)$, it is clear that Equation (16) becomes

$$2W(\tau) = \int_{t=a\tau}^{\tau} \left(\lambda - \beta \frac{d^2W(t)}{dt^2}\right) dt \tag{17}$$

where the system parameter $a \equiv \frac{1-\mu}{1+\mu}$.

Integrating Equation (17) gives

$$W(\tau) = \frac{\lambda}{2}(1-a)\tau - \frac{\beta}{2}\left[\frac{dW(\tau)}{d} - \frac{dW(a\tau)}{da\tau}\right] \quad \text{for } 0<\tau<\tau^* \tag{18}$$

This is the differential difference equation which can be used to connect the solutions at two values of time, $a\tau$ and τ. It should be noted that the validity of this equation is limited to $0 < \tau < \tau^*$.

In a similar manner, a differential difference equation can be developed for that portion of the string where $\tau^* < \tau < \frac{1}{\mu}$, (Fig. 11).

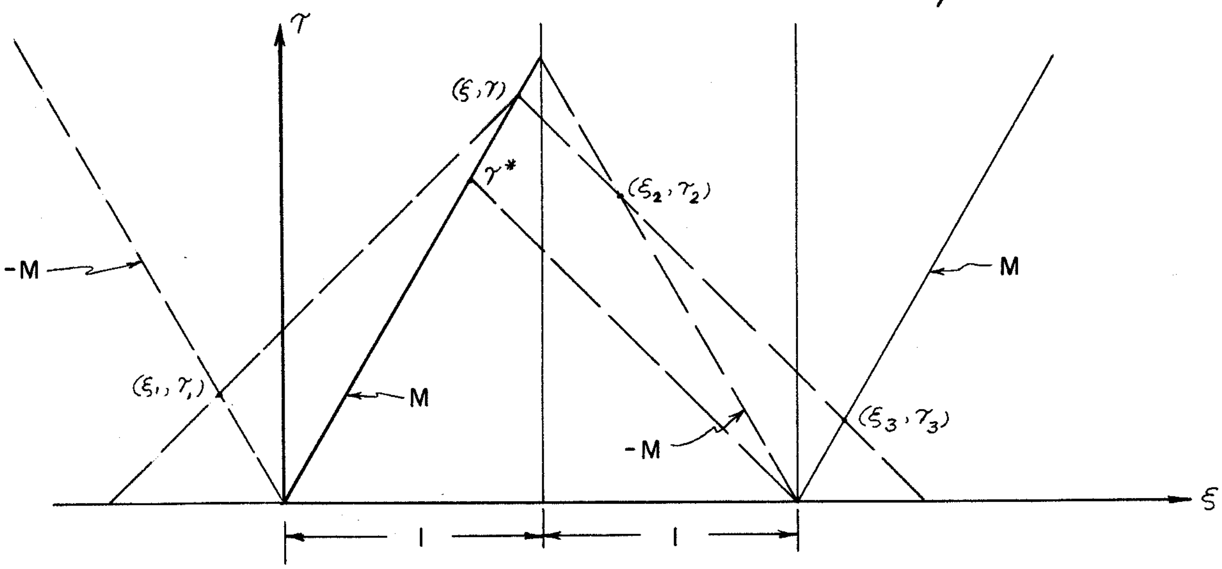

Fig. 11

It is simple to show

$$\tau_1 = \left(\frac{1-\mu}{1+\mu}\right)\tau \tag{19}$$

$$\tau_2 = \left(\frac{1+\mu}{1-\mu}\right)\tau - \frac{2}{1-\mu} \tag{20}$$

$$\tau_3 = \tau - \frac{2}{1+\mu} \tag{21}$$

Writing Equation (14) for this case, it is clear that

$$2W(\tau) = \int_{t=\tau_2}^{\tau} \left[\lambda - \beta \frac{d^2W(t)}{dt^2}\right] dt - \int_{t=\tau_3}^{\tau_1} \left[\lambda - \beta \frac{d^2W(t)}{dt^2}\right] \quad dt$$

Integrating and using Equations (19), (20) and (21) gives the differential difference equation good for $\tau^* < \tau < \frac{1}{\mu}$.

$$W(\tau) = \frac{2\lambda\mu(1-\mu\tau)}{1-\mu^2} - \frac{\beta}{2}\left[\frac{dW(\tau)}{d\tau} - \frac{dW(\tau_2)}{d\tau_2} - \frac{dW(\tau_1)}{d\tau_1} + \frac{dW(\tau_3)}{d\tau_3}\right] \tag{22}$$

This equation will permit evaluation of the displacements of the mass over the last portion of the string.

V. SOLUTION FOR THE INITIAL MOTION

Rewrite Equation (18)

$$W(\tau) = \frac{\;}{2}(1-\alpha)\tau - \frac{\beta}{2}\left[\frac{dW(\tau)}{d\tau} - \frac{dW(\alpha\tau)}{d\alpha\tau}\right] \tag{23}$$

This differential difference equation connects the value of the displacement of the mass $W(\tau)$ at two values of time within the limits $0 < \tau < \tau^*$. It should be noted that $\alpha < 1$ and consequently $\alpha^n\tau$ defines times less than τ. Bearing these facts in mind and using a stepwise procedure, this unique equation can be used to derive an expression for the initial motion of the mass. The number of steps taken will determine the accuracy of the result. Three steps will be sufficient for the purpose of illustrating the procedure.

By using Taylor's series expansion and expanding about zero, it is possible to obtain an expression for the solution which will include a point, say $\alpha^2\tau$, close to zero on the path of the mass (Fig. 12). If the next step is taken to point $\alpha\tau$, the solutions at the two points are connected by the differential difference equation, and by inserting the result at point $\alpha^2\tau$ into Equation (23) and integrating from $\alpha^2\tau$ to $\alpha\tau$, the solution at point $\alpha\tau$ is obtained. Repeating the

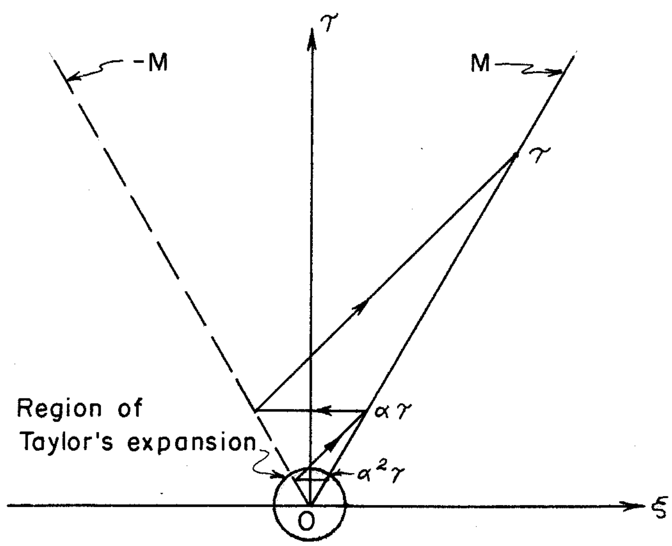

Fig. 12

process will yield the solution at τ . It is of interest to note that if steps of $a^n \tau$ are taken where n is an integer, each subsequent point is located geometrically in the ξ-τ plane by use of the characteristics as shown in Fig. 12. Proof of this fact is trivial. The actual mechanics of the procedure are carried out as follows.

Differentiating Equation (23)

$$\frac{dW(\tau)}{d\tau} = \frac{\lambda}{2}(1-a) - \frac{\beta}{2}\left[\frac{d^2W(\tau)}{d\tau^2} - a\,\frac{d^2W(a\tau)}{d(a\tau)^2}\right]$$

and as $\tau \rightarrow 0$, $W(\tau) = W(a\tau) = W(o)$. Therefore

$$W'(o) = \frac{\lambda}{2}(1-a) - \frac{\beta}{2}(1-a)\,W''(o)$$

$$W''(o) = \frac{-W'(o) + \frac{\lambda}{2}(1-a)}{\frac{\beta}{2}(1-a)}$$

Using these in the Taylor's series

$$W(\tau) = W'(o)\tau + \left[\frac{-W'(o) + \frac{\lambda}{2}(1-a)}{\beta(1-a)}\right]\tau^2 + \text{----}$$

and inserting a more convenient notation where

$$A \equiv W'(o) \qquad\qquad B \equiv \frac{-W'(o) + \frac{\lambda}{2}(1-a)}{(1-a)}$$

$$W(\tau) = A\tau + B\tau^2 + \text{-----} \tag{24}$$

It is evident that the accuracy could be increased by taking more terms of the series.

Taking the first step to $a^2\tau$, Equation (24) gives

$$W(a^2\tau) = A(a^2\tau) + B(a^2\tau)^2 \tag{25}$$

where $W(a^2\tau)$ is interpreted to mean the displacement at the point $a^2\tau$.

The next step to point $a\tau$ is taken by use of the differential difference equation. Inserting $a\tau$ for τ in Equation (23)

$$\frac{\beta}{2}\frac{dW(a\tau)}{d(a\tau)} + W(a\tau) = \frac{\lambda}{2}(1-a)(a\tau) + \frac{\beta}{2}\frac{dW(a^2\tau)}{d(a^2\tau)}$$

Using Equation (25) to obtain $\frac{dW(a^2\tau)}{d(a^2\tau)}$

$$\frac{\beta}{2}\frac{dW(a\tau)}{d(a\tau)} + W(a\tau) = \frac{\lambda}{2}(1-a)(a\tau) + \frac{\beta}{2}\left[A+2Ba(a\tau)\right]$$

This equation now is integrated over the step under consideration, i.e. from $a^2\tau$ to $a\tau$, to give the result

$$e^{\frac{2}{\beta}a\tau}W(a\tau) - e^{\frac{2}{\beta}a^2\tau}W(a^2\tau) = \left[\frac{\lambda}{2}e^{\frac{2}{\beta}a\tau}(a\tau) - \frac{\lambda\beta}{4}e^{\frac{2}{\beta}a\tau} - \frac{\lambda a}{2}e^{\frac{2}{\beta}a\tau}(a\tau) + \frac{\lambda a\beta}{4}e^{\frac{2}{\beta}a\tau} + A\frac{\beta}{2}e^{\frac{2}{\beta}a\tau} + Ba\beta e^{\frac{2}{\beta}a\tau}(a\tau) - \frac{Ba\beta^2}{2}e^{\frac{2}{\beta}a\tau}\right]_{a^2\tau}^{a\tau}$$

Inserting the limits and using Equation (25), the expression for $W(a\tau)$ is obtained.

Repeating the above process to the point τ leads to the final result

$$\left.\begin{aligned} W(\tau) &= (K_5\tau^2 + K_6\tau - K_7)\,e^{\frac{2}{\beta}(a^2-1)\tau} \\ &+ (K_8\tau^2 + K_9\tau - K_{10})\,e^{\frac{2}{\beta}(a^2-a)\tau} \\ &- (a^2K_8\tau^2 + aK_9\tau - K_{10})\,e^{\frac{2}{\beta}(a^3-a^2+a-1)\tau} \\ &+ (aK_{13}\tau + K_{11})\,e^{\frac{2}{\beta}(a-1)\tau} + K_{12}\tau + \frac{\beta}{2}K_{13} \end{aligned}\right\} \quad (26)$$

where

$$\left.\begin{aligned}
&K_1 \equiv \frac{\lambda}{2} - \frac{\lambda a}{2} + Ba\beta && K_2 \equiv \frac{(a-1)}{(a^2 - a + 1)} \\
&K_3 \equiv \frac{\beta a}{2(a^2 - a + 1)} && K_4 \equiv \frac{a^2}{(a^2 - a + 1)} \\
&K_5 \equiv Ba^4 && K_6 \equiv a^2(A - K_1) \\
&K_7 \equiv \frac{\beta}{2}(A - K_1) && K_8 \equiv Ba^4K_2 \\
&K_9 \equiv 2Ba^2K_3(1 - aK_2) + a^2K_2(A - K_1) && \\
&K_{10} \equiv B\beta K_3K_4(1 - aK_2) - K_3(1 - aK_2)(A - K_1) + \frac{\beta}{2}K_2(A - K_1) && \\
&K_{11} \equiv \frac{\beta}{2}(A - K_1) - B\left(\frac{\beta}{2}\right)^2 a && \\
&K_{12} \equiv \frac{\lambda}{2}(1 - a) && K_{13} \equiv B\beta a
\end{aligned}\right\} \quad (27)$$

By setting $\beta = 0$ and renumbering that $a \equiv \frac{1-\mu}{1+\mu}$, Equation (26) reduces to

$$W(\tau) = \frac{\lambda}{2}\left(1 - \frac{1-\mu}{1+\mu}\right)$$

or

$$W(\tau) = \frac{\lambda\mu}{1+\mu}\,\tau \qquad (28)$$

which is exactly Equation (12), the result of the simple analysis considering a constant transverse force.

The algebraic expression, Equation (26), can be used to solve for the displacements of the mass for $0 < \tau < \tau^*$. Selecting the fixed parameters to be

mg = 250 lbs.	c = 350 ft/sec.
L = 2500 ft.	ρ = 0.1866 slugs/ft.

λ and β can be determined, and calculations were carried out for the following representative conditions: (1) $\mu = \frac{1}{2}$, $W'(o) = 0$; (2) $\mu = \frac{1}{3}$, $W'(o) = 0$; (3) $\mu = \frac{1}{2}$, $W'(o) = \frac{2\lambda\mu}{1+\mu}$; (4) $\mu = \frac{1}{3}$, $W'(o) = \frac{2\lambda\mu}{1+\mu}$; and (5) $\mu = \frac{1}{2}$, $W'(o) = 0$ with mass increased tenfold. For convenient comparison, plots of $\frac{W(\tau)}{W_o(\tau)} - 1$ versus τ for conditions (1) and (2) are shown in Fig. 13. Condition (5) is shown in Fig. 14, and the difference in τ scales between the two figures should be noted. As should be the case, it was found that conditions (3) and (4) were symmetric to conditions (1) and (2), respectively, and therefore are not shown.

Reference to the plots shows that when the dynamics of the mass are considered by solving the complete physical equation, the discrepancy of infinite acceleration which appears in the result of the constant transverse force analysis is eliminated. In fact, the slope of the displacement curve, $W(\tau)$ versus τ, is zero at time $\tau = 0$. Further, the actual mass path $W(\tau)$ returns very rapidly to the force path $W_o(\tau)$.

Since $\mu = \frac{1}{3}$ is indicative of a greater horizontal velocity than $\mu = \frac{1}{2}$, other parameters being equal, the plots in Fig. 13 reveal that increased velocity gives the actual mass path a larger deviation from and slower return to the force path. Also, other parameters being equal, the plot in Fig. 14 indicates that an increase in mass decreases the deviation from the force path and increases greatly the time of return to it.

As was pointed out, greater accuracy can be attained in the solution by increasing the number of steps. The result by using five steps is given in the appendix.

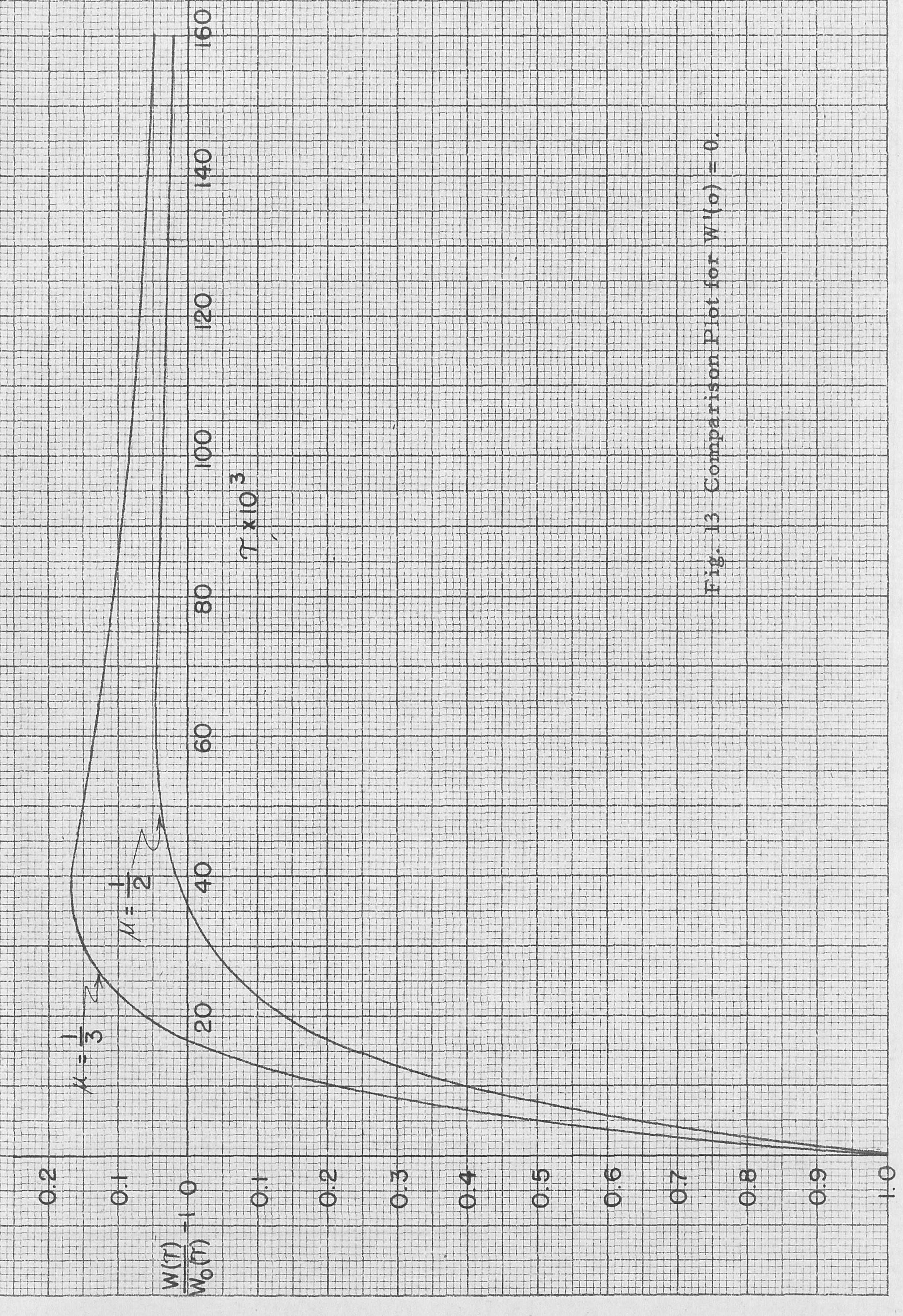

Fig. 13 Comparison Plot for $W'(o) = 0$.

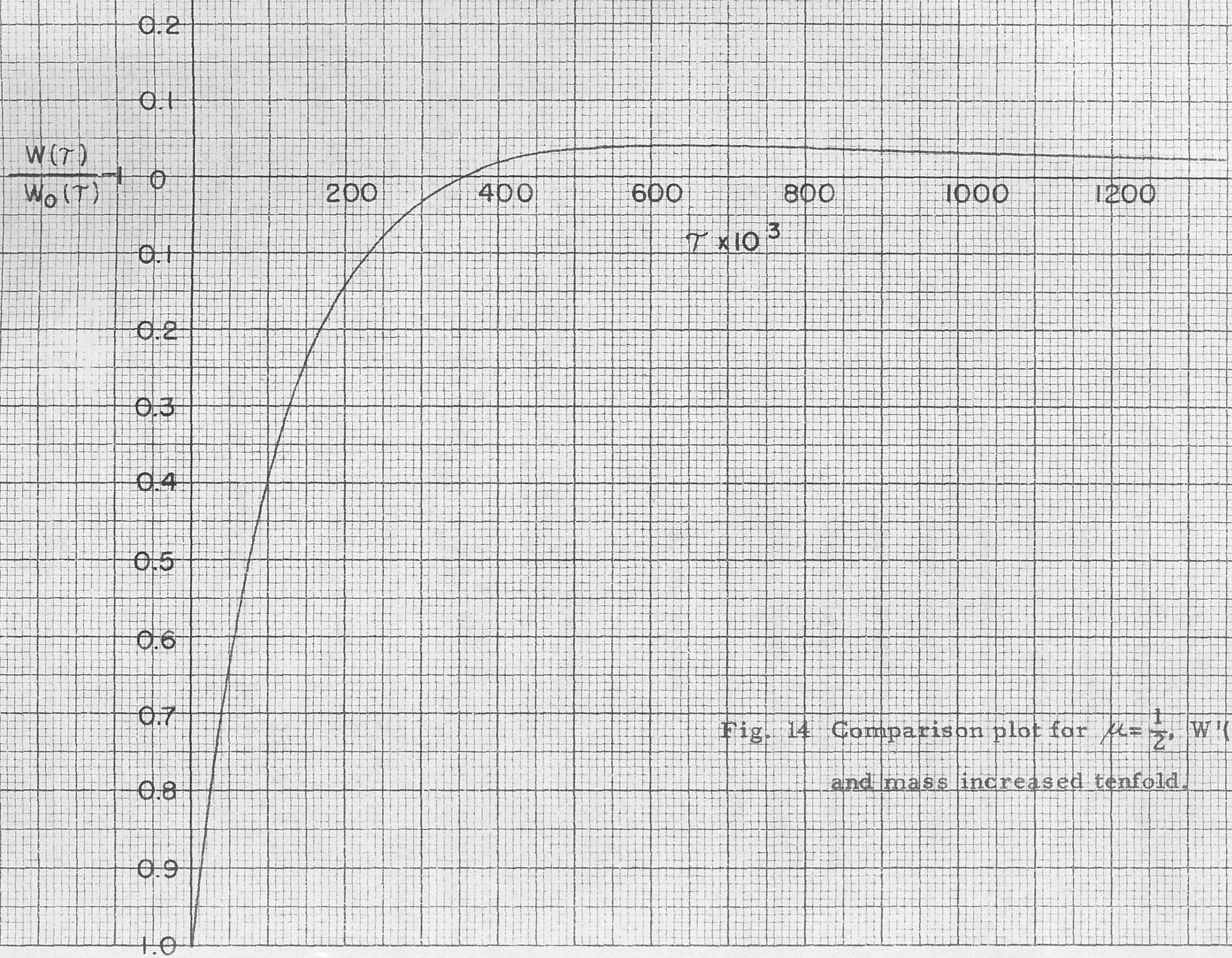

Fig. 14 Comparison plot for $\mu = \frac{1}{2}$, $W'(o) = 0$ and mass increased tenfold.

APPENDIX

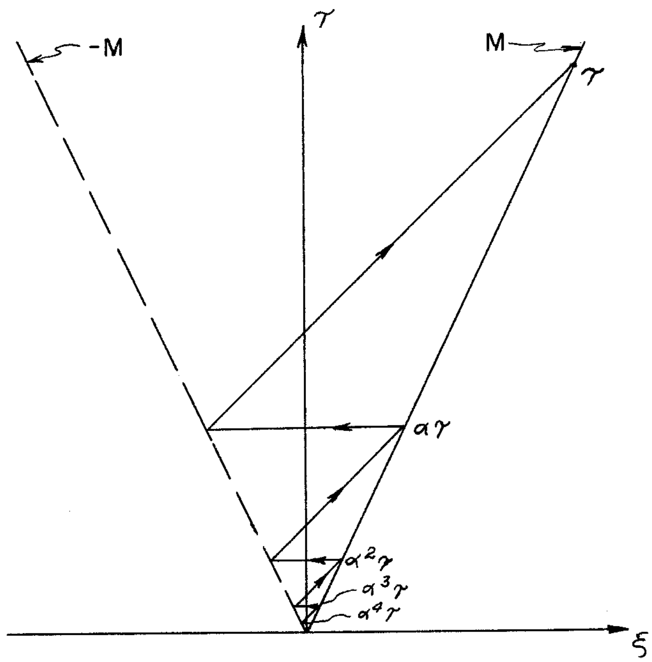

Fig. 15

A more accurate expression for the initial motion of the mass is obtained by increasing the number of differential steps taken. The mathematical procedure is the same as outlined in the text, simply being repeated more times. If five steps be taken as shown in Fig. 15, the result is

$$
\begin{aligned}
W(\tau) = {} & \frac{\beta}{2} K_7 \, e^{\frac{2}{\beta}(a^2-1)\tau} + \left[a^3 K_7 \tau + (K_4 - \frac{\beta}{2} K_7)\right] e^{\frac{2}{\beta}(a^3-1)\tau} \\
& + \left[B a^8 \tau^2 + a^3 K_1 \tau + K_3\right] e^{\frac{2}{\beta}(a^4-1)\tau} + K_{23} \, e^{\frac{2}{\beta}(a^2-a)\tau} \\
& + \left[K_{24} \tau + K_{25}\right] e^{\frac{2}{\beta}(a^3-a)\tau} + \left[K_{26} \tau^2 + K_{27} \tau - K_{28}\right] e^{\frac{2}{\beta}(a^4-a)\tau} \\
& + \left[K_{29} \tau + K_{30}\right] e^{\frac{2}{\beta}(a^3-a^2)\tau} + \left[K_{31} \tau^2 + K_{32} \tau - K_{33}\right] e^{\frac{2}{\beta}(a^4-a^2)\tau} \\
& + \left[K_{34} \tau^2 + K_{35} \tau - K_{36}\right] e^{\frac{2}{\beta}(a^4-a^3)\tau} \\
& + \left[a K_{18} \tau - (K_{23} - K_{21})\right] e^{\frac{2}{\beta}(a^3-a^2+a-1)\tau}
\end{aligned}
$$

$$+\left[a^2 K_{10}\tau^2 + a(K_{14} - K_{24})\tau - (K_{19} + K_{25})\right] e^{\frac{2}{\beta}(a^4 - a^2 + a - 1)\tau}$$

$$+\left[a^2 K_{11}\tau^2 + a(K_{16} - K_{29})\tau - (K_{20} + K_{30})\right] e^{\frac{2}{\beta}(a^4 - a^3 + a - 1)\tau}$$

$$+\left[a^4 K_5\tau^2 + aK_{13}\tau - (K_8 + K_{21})\right] e^{\frac{2}{\beta}(a^4 - a^3 + a^2 - 1)\tau}$$

$$+\left[K_{12}\tau^2 + K_{38}\tau - K_{39}\right] e^{\frac{2}{\beta}(a^4 - a^3 + a^2 - a)\tau}$$

$$-\left[K_{26}\tau^2 + aK_{27}\tau - K_{28}\right] e^{\frac{2}{\beta}(a^5 - a^2 + a - 1)\tau}$$

$$-\left[a^2 K_{31}\tau^2 + aK_{32}\tau - K_{33}\right] e^{\frac{2}{\beta}(a^5 - a^3 + a - 1)\tau}$$

$$-\left[a^4 K_{10}\tau^2 + a^2 K_{14}\tau - K_{19}\right] e^{\frac{2}{\beta}(a^5 - a^3 + a^2 - 1)\tau}$$

$$-\left[K_{40}\tau^2 + K_{41}\tau - K_{42}\right] e^{\frac{2}{\beta}(a^5 - a^3 + a^2 - a)\tau}$$

$$-\left[a^2 K_{34}\tau^2 - aK_{35}\tau - K_{36}\right] e^{\frac{2}{\beta}(a^5 - a^4 + a - 1)\tau}$$

$$-\left[a^4 K_{11}\tau^2 + a^2 K_{16}\tau - K_{20}\right] e^{\frac{2}{\beta}(a^5 - a^4 + a^2 - 1)\tau}$$

$$-\left[K_{37}\tau^2 + K_{43}\tau - K_{44}\right] e^{\frac{2}{\beta}(a^5 - a^4 + a^2 - a)\tau}$$

$$+\left[a^6 K_5\tau^2 - a^3 K_6\tau + K_8\right] e^{\frac{2}{\beta}(a^5 - a^4 + a^3 - 1)\tau}$$

$$-\left[K_{45}\tau^2 + K_{46}\tau - K_{47}\right] e^{\frac{2}{\beta}(a^5 - a^4 + a^3 - a)\tau}$$

$$- \left[K_{48} \tau^2 + K_{49} \tau - K_{50} \right] e^{\frac{2}{\beta}(a^5 - a^4 + a^3 - a^2)\tau}$$

$$- \left[2a^2 K_{12} \tau^2 + a(K_{17} + K_{38}) \tau - (K_{22} + K_{39}) \right] e^{\frac{2}{\beta}(a^5 - a^4 + a^3 - a^2 + a - 1)\tau}$$

$$+ \left[a^2 K_{40} \tau^2 + a K_{41} \tau - K_{42} \right] e^{\frac{2}{\beta}(a^6 - a^4 + a^3 - a^2 + a - 1)\tau}$$

$$+ \left[a^2 K_{37} \tau^2 + a K_{43} \tau - K_{44} \right] e^{\frac{2}{\beta}(a^6 - a^5 + a^3 - a^2 + a - 1)\tau}$$

$$+ \left[a^2 K_{45} \tau^2 + a K_{46} \tau - K_{47} \right] e^{\frac{2}{\beta}(a^6 - a^5 + a^4 - a^2 + a - 1)\tau}$$

$$+ \left[a^2 K_{48} \tau^2 + a K_{44} \tau - K_{50} \right] e^{\frac{2}{\beta}(a^6 - a^5 + a^4 - a^3 + a - 1)\tau}$$

$$+ \left[a^4 K_{12} \tau^2 + a^2 K_{17} \tau - K_{22} \right] e^{\frac{2}{\beta}(a^6 - a^5 + a^4 - a^3 + a^2 - 1)\tau}$$

$$+ \left[K_{51} \tau^2 + K_{52} \tau - K_{53} \right] e^{\frac{2}{\beta}(a^6 - a^5 + a^4 - a^3 + a^2 - a)\tau}$$

$$- \left[a^2 K_{51} \tau^2 - a K_{52} \tau - K_{53} \right] e^{\frac{2}{\beta}(a^7 - a^6 + a^5 - a^4 + a^3 - a^2 + a - 1)\tau}$$

$$+ K_9 \tau$$

where

$$K_1 \equiv Aa - \frac{\lambda a}{2} + \frac{\lambda a^2}{2} - Ba^2 \beta$$

$$K_2 \equiv \frac{\lambda}{2} - \frac{\lambda a}{2} + Ba\beta$$

$$K_3 \equiv \frac{\lambda \beta}{4} - \frac{\lambda a \beta}{4} - \frac{A\beta}{2} + \frac{Ba\beta^2}{2}$$

$$K_4 \equiv -\frac{\lambda\beta}{4} + \frac{\lambda\alpha\beta}{4} + \frac{A\beta}{2} - \frac{B\alpha\beta^2}{2}$$

$$K_5 \equiv \frac{B\alpha^4(\alpha-1)}{(\alpha^2-\alpha+1)}$$

$$K_6 \equiv \frac{B\beta\alpha^3}{(\alpha^2-\alpha+1)} - \frac{B\beta\alpha^4(\alpha-1)}{(\alpha^2-\alpha+1)^2} + \frac{K_1\alpha(\alpha-1)}{(\alpha^2-\alpha+1)}$$

$$K_7 \equiv B\alpha\beta$$

$$K_8 \equiv \frac{B\beta^2\alpha^3}{2(\alpha^2-\alpha+1)^2} - \frac{B\beta^2\alpha^4(\alpha-1)}{2(\alpha^2-\alpha+1)^3} - \frac{K_1\beta}{2(\alpha^2-\alpha+1)} + \frac{K_1\beta\,\alpha(\alpha-1)}{2(\alpha^2-\alpha+1)^2} - \frac{K_3(\alpha-1)}{(\alpha^2-\alpha+1)}$$

$$K_9 \equiv \frac{\lambda}{2} - \frac{\lambda\alpha}{2}$$

$$K_{10} \equiv \frac{B\alpha^6(\alpha^2-1)}{\alpha^3-\alpha+1}$$

$$K_{11} \equiv \frac{K_5\alpha^2(\alpha^2-\alpha)}{(\alpha^3-\alpha^2+1)}$$

$$K_{12} \equiv \frac{K_5\alpha^4(\alpha^3-\alpha^2+\alpha-1)}{(\alpha^4-\alpha^3+\alpha^2-\alpha+1)}$$

$$K_{13} \equiv K_6\alpha - \frac{K_6\alpha^3(\alpha-1)}{(\alpha^2-\alpha+1)}$$

$$K_{14} \equiv \frac{B\beta\alpha^5}{(\alpha^3-\alpha+1)} - \frac{B\beta\alpha^6(\alpha^2-1)}{(\alpha^3-\alpha+1)^2} + \frac{K_1\alpha^2(\alpha^2-1)}{(\alpha^3-\alpha+1)}$$

$$K_{15} \equiv \frac{K_5\beta\alpha}{(\alpha^3-\alpha^2+1)} - \frac{K_5\beta\alpha^2(\alpha^2-\alpha)}{(\alpha^3-\alpha^2+1)^2} + \frac{K_6\alpha(\alpha^2-\alpha)}{(\alpha^3-\alpha^2+1)}$$

$$K_{17} \equiv \frac{K_5\beta\alpha^3}{(\alpha^4-\alpha^3+\alpha^2-\alpha+1)} - \frac{K_5\beta\alpha^4(\alpha^3-\alpha^2+\alpha-1}{(\alpha^4-\alpha^3+\alpha^2-\alpha+1)^2} + \frac{K_6\alpha^2(\alpha^3-\alpha^2+\alpha-1)}{(\alpha^4-\alpha^3+\alpha^2-\alpha+1)}$$

$$K_{18} \equiv \frac{K_7 \alpha^2(\alpha -1)}{(\alpha^2-\alpha+1)}$$

$$K_{19} \equiv \frac{B\,\beta^2\alpha^5}{2(\alpha^3-\alpha+1)^2} - \frac{B\,\beta^2\alpha^6(\alpha^2-1)}{2(\alpha^3-\alpha+1)^3} - \frac{K_1\beta\,\alpha}{2(\alpha^3-\alpha+1)} + \frac{K_1\beta\,\alpha^2(\alpha-1)}{2(\alpha^3-\alpha+1)^2} - \frac{K_3(\alpha^2-1)}{(\alpha^3-\alpha+1)}$$

$$K_{20} \equiv \frac{K_5\,\beta^2\alpha}{2(\alpha^3-\alpha^2+1)^2} - \frac{K_5\beta^2\alpha^2(\alpha^2-\alpha)}{2(\alpha^3-\alpha^2+1)^3} - \frac{K_6\,\beta}{2(\alpha^3-\alpha^2+1)} + \frac{K_6\beta\,\alpha(\alpha^2-\alpha)}{2(\alpha^3-\alpha^2+1)^2} + \frac{K_8(\alpha^2-\alpha)}{(\alpha^3-\alpha^2+1)}$$

$$K_{21} \equiv \frac{K_7\,\beta\,\alpha}{2(\alpha^2-\alpha+1)} - \frac{K_7\,\beta\,\alpha^2(\alpha-1)}{2(\alpha^2-\alpha+1)^2} + \frac{(K_4 - \frac{\beta}{2}K_7)(\alpha-1)}{(\alpha^2-\alpha+1)}$$

$$K_{22} \equiv \frac{K_5\,\beta^2\alpha^3}{2(\alpha^4-\alpha^3+\alpha^2-\alpha+1)^2} - \frac{K_5\,\beta^2\alpha^4(\alpha^3-\alpha^2+\alpha-1)}{2(\alpha^4-\alpha^3+\alpha^2-\alpha+1)^3} - \frac{K_6\,\beta\,\alpha}{2(\alpha^4-\alpha^3+\alpha^2-\alpha+1)} + \frac{K_6\,\beta\,\alpha^2(\alpha^3-\alpha^2+\alpha-1)}{2(\alpha^4-\alpha^3+\alpha^2-\alpha+1)^2} + \frac{K_8(\alpha^3-\alpha^2+\alpha-1)}{(\alpha^4-\alpha^3+\alpha^2-\alpha+1)}$$

$$K_{23} \equiv \frac{K_7(\alpha-1)\beta}{2(\alpha^2-\alpha+1)}$$

$$K_{24} \equiv \frac{K_7\alpha^3(\alpha^2-1)}{(\alpha^3-\alpha+1)}$$

$$K_{25} \equiv \frac{K_7\,\beta\,\alpha^2}{2(\alpha^3-\alpha+1)} - \frac{K_7\,\beta\,\alpha^3(\alpha^2-1)}{2(\alpha^3-\alpha+1)^2} + \frac{K_4 - \frac{\beta}{2}K_7)(\alpha^2-1)}{(\alpha^3-\alpha+1)}$$

$$K_{26} \equiv \frac{B a^8 (a^3-1)}{(a^4-a+1)}$$

$$K_{27} \equiv \frac{B \beta a^7}{(a^4-a+1)} - \frac{B \beta a^8 (a^3-1)}{(a^4-a+1)^2} + \frac{K_1 a^3 (a^3-1)}{(a^4-a+1)}$$

$$K_{28} \equiv \frac{B \beta a^7}{2(a^4-a+1)^2} - \frac{B \beta^2 a^8 (a^3-1)}{2(a^4-a+1)^3} - \frac{K_1 \beta a^2}{2(a^4-a+1)}$$

$$+ \frac{K_1 \beta a^3 (a^3-1)}{2(a^4-a+1)^2} - \frac{K_3 (a^3-1)}{(a^4-a+1)}$$

$$K_{29} \equiv \frac{K_{18} a (a^2-a)}{(a^3-a^2+1)}$$

$$K_{30} \equiv \frac{K_{18} \beta}{2(a^3-a^2+1)} - \frac{K_{18} \beta a (a^2-a)}{2(a^3-a^2+1)^2} + \frac{K_{21} (a^2-a)}{(a^3-a^2+1)}$$

$$K_{31} \equiv \frac{K_{10} a^2 (a^3-a)}{(a^4-a^2+1)}$$

$$K_{32} \equiv \frac{K_{10} \beta a}{(a^4-a^2+1)} - \frac{K_{10} \beta a^2 (a^3-a)}{(a^4-a^2+1)^2} + \frac{K_{14} a (a^3-a)}{(a^4-a^2+1}$$

$$K_{33} \equiv \frac{K_{10} \beta^2 a}{2(a^4-a^2+1)^2} - \frac{K_{10} \beta^2 a^2 (a^3-a)}{2(a^4-a^2+1)^3} - \frac{K_{14} \beta}{2(a^4-a^2+1)}$$

$$+ \frac{K_{14} \beta a (a^3-a)}{2(a^4+a^2+1)^2} + \frac{K_{19} (a^3-a)}{(a^4-a^2+1}$$

$$K_{34} \equiv \frac{K_{11} a^2 (a^3-a^2)}{(a^4-a^3+1)}$$

$$K_{35} \equiv \frac{K_{11}\beta\alpha}{(\alpha^4-\alpha^3+1)} - \frac{K_{11}\beta\alpha^2(\alpha^3-\alpha^2)}{(\alpha^4-\alpha^3+1)^2} + \frac{K_{16}\alpha(\alpha^3-\alpha^2)}{(\alpha^4-\alpha^3+1)}$$

$$K_{36} \equiv \frac{K_{11}\beta^2\alpha}{2(\alpha^4-\alpha^3+1)^2} - \frac{K_{11}\beta^2\alpha^2(\alpha^3-\alpha^2)}{2(\alpha^4-\alpha^3+1)^3} - \frac{K_{16}\beta}{2(\alpha^4-\alpha^3+1)} + \frac{K_{16}\beta\alpha(\alpha^3-\alpha^2)}{2(\alpha^4-\alpha^3+1)^2} + \frac{K_{20}(\alpha^3-\alpha^2)}{(\alpha^4-\alpha^3+1)}$$

$$K_{37} \equiv \frac{K_{11}\alpha^4(\alpha^4-\alpha^3+\alpha-1)}{(\alpha^5-\alpha^4+\alpha^2-\alpha+1)}$$

$$K_{38} \equiv \frac{K_5\beta\alpha^3}{(\alpha^4+\alpha^3+\alpha^2-\alpha+1)} - \frac{K_5\beta\alpha^4(\alpha^3-\alpha^2+\alpha-1)}{(\alpha^4-\alpha^3+\alpha^2-\alpha+1)^2} + \frac{K_{13}\alpha(\alpha^3-\alpha^2+\alpha-1)}{(\alpha^4-\alpha^3+\alpha^2-\alpha+1)}$$

$$K_{39} \equiv \frac{K_5\beta^2\alpha^3}{2(\alpha^4-\alpha^3+\alpha^2-\alpha+1)^2} - \frac{K_5\beta^2\alpha^4(\alpha^3-\alpha^2+\alpha-1)}{2(\alpha^4-\alpha^3+\alpha^2-\alpha+1)^3} - \frac{K_{13}\beta}{2(\alpha^4-\alpha^3+\alpha^2-\alpha+1)} + \frac{K_{13}\beta\alpha(\alpha^3-\alpha^2+\alpha-1)}{2(\alpha^4-\alpha^3+\alpha^2-\alpha+1)^2} + \frac{K_8(\alpha^3-\alpha^2+\alpha-1)}{(\alpha^4-\alpha^3+\alpha^2-\alpha+1)} + \frac{K_{21}(\alpha^3-\alpha^2+\alpha-1)}{(\alpha^4-\alpha^3+\alpha^2-\alpha+1)}$$

$$K_{41} \equiv \frac{K_{10}\beta\alpha^3}{(\alpha^5-\alpha^3+\alpha^2-\alpha+1)} - \frac{K_{10}\beta\alpha^4(\alpha^4-\alpha^2+\alpha-1)}{(\alpha^5-\alpha^3+\alpha^2-\alpha+1)^2} + \frac{K_{14}\alpha^2(\alpha^4-\alpha^2+\alpha-1)}{(\alpha^5-\alpha^3+\alpha^2-\alpha+1)}$$

$$K_{40} \equiv \frac{K_{10}\alpha^4(\alpha^4-\alpha^2+\alpha-1)}{(\alpha^5-\alpha^3+\alpha^2-\alpha+1)}$$

$$K_{42} \equiv \frac{K_{10}\beta^2\alpha^3}{2(\alpha^5-\alpha^3+\alpha^2-\alpha+1)^2} - \frac{K_{10}\beta^2\alpha^4(\alpha^4-\alpha^2+\alpha-1)}{2(\alpha^5-\alpha^3+\alpha^2-\alpha+1)^3} - \frac{K_{14}\beta\alpha}{2(\alpha^5-\alpha^3+\alpha^2-\alpha+1)} + \frac{K_{14}\beta\alpha^2(\alpha^4-\alpha^2+\alpha-1)}{2(\alpha^5-\alpha^3+\alpha^2-\alpha+1)^2} + \frac{K_{19}(\alpha^4-\alpha^2+\alpha-1)}{(\alpha^5-\alpha^3+\alpha^2-\alpha+1)}$$

$$K_{43} \equiv \frac{K_{11}\beta\alpha^3}{(\alpha^5-\alpha^4+\alpha^2-\alpha+1)} - \frac{K_{11}\beta\alpha^4(\alpha^4-\alpha^3+\alpha-1)}{(\alpha^5-\alpha^4+\alpha^2-\alpha+1)^2} + \frac{K_{16}\alpha^2(\alpha^4-\alpha^3+\alpha-1)}{(\alpha^5-\alpha^4+\alpha^2-\alpha+1)}$$

$$K_{44} \equiv \frac{K_{11}\beta^2\alpha^3}{2(\alpha^5-\alpha^4+\alpha^2-\alpha+1)^2} - \frac{K_{11}\beta^2\alpha^4(\alpha^4-\alpha^3+\alpha-1)}{2(\alpha^5-\alpha^4+\alpha^2-\alpha+1)^3} - \frac{K_{16}\beta\alpha}{2(\alpha^5-\alpha^4+\alpha^2-\alpha+1)}$$

$$+ \frac{K_{16}\beta\alpha^2(\alpha^4-\alpha^3+\alpha-1)}{2(\alpha^5-\alpha^4+\alpha^2-\alpha+1)^2} + \frac{K_{20}(\alpha^4-\alpha^3+\alpha-1)}{(\alpha^5-\alpha^4+\alpha^2-\alpha+1)}$$

$$K_{45} \equiv \frac{K_5\alpha^6(\alpha^4-\alpha^3+\alpha^2-1)}{(\alpha^5-\alpha^4+\alpha^3-\alpha+1)}$$

$$K_{46} \equiv \frac{K_5\beta\alpha^5}{(\alpha^5-\alpha^4+\alpha^3-\alpha+1)} - \frac{K_5\beta\alpha^6(\alpha^4-\alpha^3+\alpha^2-1)}{(\alpha^5-\alpha^4+\alpha^3-\alpha+1)^2} + \frac{K_6\alpha^3(\alpha^4-\alpha^3+\alpha^2-1)}{(\alpha^5-\alpha^4+\alpha^3-\alpha+1)}$$

$$K_{47} \equiv \frac{K_5\beta^2\alpha^5}{2(\alpha^5-\alpha^4+\alpha^3-\alpha+1)^2} - \frac{K_5\beta^2\alpha^6(\alpha^4-\alpha^3+\alpha-1)}{2(\alpha^5-\alpha^4+\alpha^3-\alpha+1)^3} - \frac{K_6\beta\alpha^2}{2(\alpha^5-\alpha^4+\alpha^3-\alpha+1)}$$

$$+ \frac{K_6\beta\alpha^3(\alpha^4-\alpha^3+\alpha^2-1)}{2(\alpha^5-\alpha^4+\alpha^3-\alpha+1)^2} + \frac{K_8(\alpha^4-\alpha^3+\alpha^2-1)}{(\alpha^5-\alpha^4+\alpha^3-\alpha+1)}$$

$$K_{48} \equiv \frac{K_{12}\alpha^2(\alpha^4-\alpha^3+\alpha^2-\alpha)}{(\alpha^5-\alpha^4+\alpha^3-\alpha^2+1)}$$

$$K_{49} \equiv \frac{K_{12}\beta\alpha}{(\alpha^5-\alpha^4+\alpha^3-\alpha^2+1)} - \frac{K_{12}\beta\alpha^2(\alpha^4-\alpha^3+\alpha^2-\alpha)}{(\alpha^5-\alpha^4+\alpha^3-\alpha^2+1)^2} + \frac{K_{17}\alpha(\alpha^4-\alpha^3+\alpha^2-\alpha)}{(\alpha^5-\alpha^4+\alpha^3-\alpha^2+1)}$$

$$K_{50} \equiv \frac{K_{12}\beta^2\alpha}{2(\alpha^5-\alpha^4+\alpha^3-\alpha^2+1)^2} - \frac{K_{12}\beta^2\alpha^2(\alpha^4-\alpha^3+\alpha^2-\alpha)}{2(\alpha^5-\alpha^4+\alpha^3-\alpha^2+1)^3} - \frac{K_{17}\beta}{2(\alpha^5-\alpha^4+\alpha^3-\alpha^2+1)}$$

$$+\frac{K_{17}\beta\,\alpha(\alpha^4-\alpha^3+\alpha^2-\alpha)}{2(\alpha^5-\alpha^4+\alpha^3-\alpha^2+1)^2}+\frac{K_{22}(\alpha^4-\alpha^3+\alpha^2-\alpha)}{(\alpha^5-\alpha^4+\alpha^3-\alpha^2+1)}$$

$$K_{51}\equiv\frac{K_{12}\alpha^4(\alpha^5-\alpha^4+\alpha^3-\alpha^2+\alpha-1)}{(\alpha^6-\alpha^5+\alpha^4-\alpha^3+\alpha^2-\alpha+1)}$$

$$K_{52}\equiv\frac{K_{12}\beta\,\alpha^3}{(\alpha^6-\alpha^5+\alpha^4-\alpha^3+\alpha^2-\alpha+1)}-\frac{K_{12}\beta\,\alpha^4(\alpha^5-\alpha^4+\alpha^3-\alpha^2+\alpha-1)}{(\alpha^6-\alpha^5+\alpha^4-\alpha^3+\alpha^2-\alpha+1)^2}$$

$$+\frac{K_{17}\alpha^2(\alpha^5-\alpha^4+\alpha^3-\alpha^2+\alpha-1)}{(\alpha^6-\alpha^5+\alpha^4-\alpha^3+\alpha^2-\alpha+1)}$$

$$K_{53}\equiv\frac{K_{12}\beta^2\alpha^3}{2(\alpha^6-\alpha^5+\alpha^4-\alpha^3+\alpha^2-\alpha+1)^2}+\frac{K_{12}\beta^2\alpha^4(\alpha^5-\alpha^4+\alpha^3-\alpha^2+\alpha-1)}{2(\alpha^6-\alpha^5+\alpha^4-\alpha^3+\alpha^2-\alpha+1)^3}$$

$$-\frac{K_{17}\beta\,\alpha}{2(\alpha^6-\alpha^5+\alpha^4-\alpha^3+\alpha^2-\alpha+1)}+\frac{K_{17}\beta\,\alpha^2(\alpha^5-\alpha^4+\alpha^3-\alpha^2+\alpha-1)}{2(\alpha^6-\alpha^5+\alpha^4-\alpha^3+\alpha^2-\alpha+1)^2}$$

$$+\frac{K_{22}(\alpha^5-\alpha^4+\alpha^3-\alpha^2+\alpha-1)}{(\alpha^6-\alpha^5+\alpha^4-\alpha^3+\alpha^2-\alpha+1)}$$

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