A TEN CHANNEL STATISTICAL ANALYZER
FOR USE
IN TURBULENCE RESEARCH

Thesis by

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ABSTRACT

An experimental investigation of a stationary random process will involve the measurement of the mean values and probability distribution of a random function of time. This thesis describes an automatic ten channel "statistical analyzer" for measuring the probability distribution function of a continuous or discontinuous function of time that can be represented by a suitable electric voltage. The "statistical analyzer" is based upon a system of pulse amplitude modulation, followed by pulse height selection and pulse counting. An equation is developed by means of which the mean values can be computed from the distribution function. Finally, sample applications from the field of turbulence research are given.

I. INTRODUCTION

An experimental investigation of a stationary random process will involve the measurement of the mean values and probability distribution of a random function of time. Let $I(t)$ represent any physical variable, for example an air velocity, which is a function of time. The n^{th} mean power of $I(t)$ is defined by,

$$\overline{I^n} = \frac{1}{T} \int_0^T I^n(t)\, dt \tag{1}$$

If the statistical structure of the process is stationary with respect to time, the value of this integral will approach a constant value for sufficiently large T.

Measurement of a specified mean value may be done by using a device which either (1) instantaneously forms $I^n(t)$ and then proceeds to average these values over an interval of time, or (2) directly produces the mean of a definite power of I, as for example a thermocouple meter that measures the time mean square of the electric current passing through it.

A third method for obtaining mean values, which is essentially different from the two previously mentioned, is by measuring the probability distribution of $I(t)$. The mean powers of I are then computed from the moments of the probability distribution curve. In principle <u>all</u> the mean values of a function may be obtained from the probability distribution, whereas the first two methods are restricted to the measurement of a specified mean power.

It was in consideration of this universality of the probability distribution method for obtaining mean values that led to the design and construction of the "statistical analyzer" described in this thesis for automatically measuring the probability distribution of a random function of time.

This project was originally initiated as part of a general program of investigation of turbulent air flows and the examples cited will be drawn from this field. However, the apparatus described in this report may be used in the statistical investigation of any continuous or discontinuous random function of time that can be represented by a suitable electric voltage.

II. THE DISTRIBUTION FUNCTION

A time dependent function, I(t), has a probability density $\rho(\eta)$, where $\rho(\eta)\,d\eta$ is the probability of finding I with a value between η and $\eta + d\eta$. The mean values of I can be obtained from the moments of $\rho(\eta)$ as follows:

$$\overline{I^n} = \int_{-\infty}^{\infty} \eta^n \rho(\eta)\, d\eta , \quad \left(\int_{-\infty}^{\infty} \rho(\eta)\, d\eta = 1 \right) \tag{2}$$

An instrument designed to measure $\rho(\eta)$ has to be able to distinguish between three possible states of the input signal, i.e., whether I lies above, below, or in the interval η to $\eta + d\eta$. Combinations of gate circuits and coincidence circuits have been devised for performing this function (Ref. 1), but they are generally quite complicated and their accuracy is limited because of hysteresis and drift in their discrimination levels. There is, however, a more fundamental objection to this type of device and that is due to the fact that such a device must of necessity use a finite interval or gate width as an approximation to the infinitesimal interval $d\eta$, and therefore measurements of $\rho(\eta)$ so taken must be corrected for this error.

It is possible to eliminate the error due to a finite gate width entirely and in addition simplify the necessary selection operations of the measuring device. This is done by working with the <u>integrated</u> probability density. The integrated probability density, $F(\eta)$, which we will call the <u>distribution function</u>, is defined as the probability of finding I(t) with a value greater than η, and is related to the probability density as follows:

$$F(\eta) = \int_{\eta}^{\infty} p(z)\, dz \qquad (3)$$

$$F(-\infty) = 1 \,, \; F(+\infty) = 0$$

Therefore,

$$dF = -p(\eta)\, d\eta \qquad (4)$$

Thus, an instrument designed to measure $F(\eta)$ has to distinguish between but two states of the input signal, i.e., whether $I(t)$ is above or below the value η ; and since we are no longer concerned with measuring the length of time that $I(t)$ spends in an infinitesimal interval $d\eta$, there is no longer an error due to a finite gap width.

Because of the simplification which has been effected in the selection functions of our measuring device, it is not necessary to use grid control vacuum tubes for discrimination purposes, for it is apparent that a diode will perform all the selection operations we now need. A diode is a non-linear resistor which ideally has but two possible values--infinite or zero, depending on whether the cathode is positive or negative with respect to the anode. We can therefore use a diode as an electronic switch to open or close a circuit depending on whether the input signal is greater or less than the bias voltage on the diode. Since a diode does not have any control grids that have to be properly biased, diode circuitry is much simpler than that of grid type vacuum tubes. In addition, in comparison with grid control tubes, vacuum tube diodes are more stable, have sharper selection characteristics, and exhibit very little hysteresis (Ref. 2).

It now remains to be shown how mean values may be obtained from the distribution function. The first step is to combine Equation (2)

$$\overline{I^n} = \int_{-\infty}^{\infty} \eta^n \rho(\eta)\, d\eta$$

and Equation (4)

$$dF = -\rho(\eta)\, d\eta$$

to obtain the relation that

$$\overline{I^n} = \int_0^1 \eta^n \, dF(\eta) \tag{5}$$

The integral in Equation (5) is in an unsatisfactory form for evaluation by numerical methods, such as Simpson's rule, because the integrand becomes infinite at the limits. This difficulty can be overcome by transforming this integral with respect to the F axis into an integral with respect to the η axis. This is accomplished by first dividing the integral in Equation (5) into two parts, one on either side of the $\eta = 0$ axis. This gives

$$\overline{I^n} = \int_0^a \eta^n \, dF + \int_a^1 \eta^n \, dF$$

$$= \int_0^a \eta^n \, dF - \int_{1-a}^{0} \eta^n \, d(1-F)$$

where $F(0) = a$.

Next, integrate by parts

$$\overline{I^n} = \eta^n F \Big|_{\eta=\infty}^{\eta=0} - \eta^n (1-F) \Big|_{\eta=0}^{\eta=-\infty}$$

$$+ n \left\{ \int_0^{\infty} \eta^{n-1} F \, d\eta + \int_0^{-\infty} \eta^{n-1} (1-F) \, d\eta \right\}$$

The first two terms vanish at the limits if, as η approaches infinity, F(η) approaches the value zero exponentially or faster. This requirement will always be satisfied if the mean value is finite. We thus obtain the following important equation

$$\overline{I^n} = n \left\{ \int_0^{\infty} \eta^{n-1} F \, d\eta + \int_0^{-\infty} \eta^{n-1} (1-F) \, d\eta \right\} \tag{6}$$

The integral in Equation (6) is in a form suitable for numerical integration and in practice is evaluated by an application of Simpson's Rule. Comparison of Equation (2) with Equation (6) brings out the interesting relationship that the n^{th} moment of the probability density is proportional to the (n-1) moment of the distribution function.

A typical distribution function is sketched in Figure (1), where the shaded area corresponds to the region of integration in a computation of mean values from Equation (6).

III. COUNTING METHODS

The length of time required to measure the probability of an event with a given accuracy is inversely proportional to the probability of occurrence of that event. Therefore, a "statistical analyzer" should be able to record continuously for arbitrarily long time intervals in order that very small probabilities may be accurately measured.

This requirement precludes the use of the common electrical analog methods for summing time intervals, such as by measuring the charge accumulated on a condenser that is being charged by a known current. Because of leakage of charge, such devices are usually not sufficiently accurate if the duration of the run exceeds five minutes.

In order to accurately sum over long time intervals digital techniques must be employed. That is, time will be measured by counting the number of occurrences of an event which is being repeated regularly at equal intervals of time. The measured time will then be given by

$$t = N\Delta t$$

where, N = total number of counts

Δt = constant time interval between events.

Because a digital counter can retain its information for an indefinite period of time without deterioration of its reading, the maximum length of time over which a "statistical analyzer" employing counting techniques can measure is limited only by the available number capacity of the counters and by the stability of the equipment used in conjunction with the analyzer.

In practice the event counted is a standard pulse that is generated periodically by a master oscillator.

IV. SAMPLING METHODS

In the following discussion it will be assumed that the random input to the "statistical analyzer" is a fluctuating electric voltage, denoted by V(t).

Let us suppose we have a pulse generator that produces a sequence of equidistant, constant amplitude, narrow pulses (Fig. 2). We wish to measure the distribution function of V(t) by counting the number of pulses generated during the time that V(t) exceeds a given amplitude.

One way to do this is to feed V(t) into an "amplitude discriminator" which serves to close a circuit connecting the pulse generator to a group of pulse counters whenever V(t) is above the discrimination level. It is easily shown that this method is equivalent to a process whereby V(t) is sampled periodically at the pulse frequency and a sample is counted with unit weight if its amplitude is above the discrimination level, and a sample is not counted if its amplitude is below the discrimination level.

The argument is as follows: the pulse generator produces a pulse at a time t_p, where

$$t_p = n\Delta t \quad , \quad n = 0, 1, 2, 3, \ldots..$$

and $\quad \Delta t$ = time interval between pulses .

Whether or not a pulse is counted depends on the amplitude of $V(t_p)$. But since a pulse doesn't exist except at time t_p as defined above, the amplitude of V(t) during the time interval between pulses has no effect on the total number of counts. Thus, in effect, the amplitude of V(t) is measured only at the time of occurrence of a generated pulse, and the equivalence to the sampling method stated above is proved.

This consideration leads to a second method for measuring distribution functions: samples of V(t) are taken periodically and these discrete samples are fed into an amplitude discriminator. By this method, the original input signal, which was a continuous function of time, is transformed into a discrete function of time consisting of a sequence of pulses, each pulse having an amplitude proportional to that of the original signal, V(t), for $t = t_p$. The function of the amplitude discriminator is to pass that portion of a pulse which exceeds the selection level. This selected portion of the pulse is then amplified to a standard amplitude and fed into a pulse counter. Thus, a sample of the original signal is either counted once or is not counted at all, depending on whether the amplitude of the sample is higher or lower than the discriminator selection level.

It is this second method of operation, which we shall call the pulse amplitude modulation method, that was actually used for the design of the "statistical analyzer".

It is appropriate here to point out the analogy between the pulse amplitude modulation method for obtaining the probability distribution of a continuous function and the methods used in the field of nuclear physics for measuring particle energy distributions. The nuclear physicist uses an amplitude discriminator or "pulse height analyzer" in conjunction with an ionization chamber to obtain detailed information on the probability of occurrence of ionization pulses of various amplitudes. During the past few years many such "pulse analyzers" have been described in the literature (Refs. 1, 3, 4, 5). However, they are not suitable for our needs because of their low maximum counting rate

-- on the order of 500 counts per minute, whereas we will want to count at a rate of at least 100,000 counts per minute in order to reduce the measuring time as much as possible.

V. DESCRIPTION OF APPARATUS

The "statistical analyzer" consists of a pulse amplitude modulator, amplitude discriminators, and pulse counters. There are ten "channels" so that ten points on a distribution function curve may be obtained simultaneously.

The physical layout of the equipment consists of two relay racks, each containing five banks of counters, and a small table top rack containing the modulator and discriminator chassis, a square wave generator, and a power supply.

1. Pulse Amplitude Modulator

Due to the high sampling rate desired, electro-mechanical "choppers" are ruled out, and for this reason an electronic pulse amplitude modulator was devised. The modulator circuit is given in Figure (6), and its operation will now be described.

A sequence of unmodulated pulses (Fig. 2) is obtained by differentiating and rectifying the output of a square wave generator. The pulse frequency may be varied and may be as high as 10,000 pulses per second. The pulses are characterized by a rise time of about 2 microseconds and a short peak duration. These features are shown in the oscillogram of Figure (3).

The random input signal V(t) can be obtained from any source capable of providing a signal of approximately one volt peak to peak amplitude across a load of 500,000 ohms. The frequency range of the input signal may be between 1/2 to 50,000 cycles per second. In the field of measurement of turbulent air velocities, for which this machine was originally designed, the input signal is obtained from a hot wire anemometer-amplifier combination.

The input signal and the unmodulated pulse train are combined in a differential amplifier (Ref. 6), yielding an output that is the result of adding the pulses to the signal. This is illustrated in the oscillogram of Figure (4) for a sine wave input signal.

The output of the differential amplifier is then rectified so that only the upper portion of the differential signal remains, which yields at the modulator output a sequence of pulses whose amplitude is proportional to that of the original input signal V(t). This is illustrated in the oscillogram of Figure (5), which is that part of Figure (4) which lies above line A-A.

The maximum output of the modulator is 90 volts. D.C. restoration (Ref. 6) is used at the modulator output to establish the base of the pulses as a zero volts reference for measuring the pulse amplitude.

We have, by this process of pulse amplitude modulation, converted a continuous function of time into a discrete function of time whose distribution function may be measured by counting methods.

2. Pulse Amplitude Discriminators

The modulator output is fed into ten amplitude discriminator channels, each of which is biased at a different level. A typical discriminator circuit is shown in Figure (7).

Biased vacuum tube diodes (6AL5) are used as amplitude discriminators. The discriminators pass only that portion of a pulse that is above the bias level. For example, if a modulated pulse that has a 51 volt amplitude is fed into a discriminator biased at a 50 volt level, a 1 volt pulse will be obtained at the discriminator output.

The discriminator output is fed into an amplifier which has enough gain so that discriminator outputs of as low as .05 volts may be counted. This corresponds to a voltage that is less than one per cent of the bias spread between adjacent discriminator levels.

The diode bias voltages are obtained from dry cells supplying a total of 90 volts with taps at 1-1/2 volt intervals. However, it is planned to replace the batteries with a chain of precision resistors in series with a regulated power supply.

It is sometimes desirable to shift the discriminator levels so that points between the original ten may be obtained. This may be done very conveniently by changing the voltage tap to which the diode bias ground is attached. This serves to raise or lower all the discriminator levels equally with respect to ground while maintaining their original spacing. Of course, if it is desired, an individual bias level can be changed by merely shifting its bias battery connection to another voltage tap.

The diode heaters are operated at 3.9 volts by means of a 6 volt storage battery and a 1 ohm adjustable rheostat in series with the heaters. Operation of the diodes at this low value of heater voltage serves to reduce their quiescent plate current and minimizes differences in selection characteristics among the various diodes.

3. Pulse Counters

Each pulse amplitude discriminator amplifier feeds a bank of pulse counters consisting of a chain of from one to four commercial electronic decade scalars followed by a five-place impulse type mechanical register.

The decade scalars operate on the "flip-flop" principle (Ref. 2) and indicate their count by means of neon lights. The mechanical register is coupled to the last decade scalar by means of a cathode-follower and power-amplifier stage. Every tenth count of the last decade scalar trips the mechanical register. Thus, a maximum of 10^9 counts may be stored in each channel.

4. Operating Methods

It is possible to use all ten channels for measuring the distribution function of either the entire input signal or one-half of the input signal.

If it is desired to measure the entire signal, the initial height of the unmodulated pulse is adjusted to a value such that the channel that is biased midway between the highest and the lowest level just begins to count. This channel will then correspond to the zero level of the input signal, and amplitudes above and below this zero level can be measured.

For the measurement of only one-half the signal at a time, the initial amplitude of the unmodulated pulse train is adjusted to a value such that the least biased channel just begins to count. Thus, since the zero level of the input signal corresponds to the least biased channel, only positive values of the signal will be measured. By inverting the input signal, the other half of the signal may be measured. In this manner twenty points on the distribution curve may be obtained by making two sets of measurements, one set for each half of the input signal.

A run is started and stopped by means of a manually operated toggle switch in the circuit connecting the modulator output to the amplitude discriminators. The runs are timed by means of a stopwatch. The timing accuracy is better than 1/10 second, which for a
1 run of one minute is less than 1/2 per cent error.

VI. COMPUTATION OF MEAN VALUES

The mean powers of a function are obtained as the moments of the distribution function by means of Equation (6). A numerical procedure based upon Simpson's rule has been devised for computing mean values from the data obtained from the "statistical analyzer". Only the actual experimental points are used in the computations. As a check on the accuracy of Simpson's rule for computing mean values numerically, the fourth moment of a Gaussian probability distribution curve was computed by using Simpson's rule with nine points. The result obtained was found to agree within one per cent of the theoretical value.

The readings obtained with the "statistical analyzer" are normalized by dividing each reading by the number of counts corresponding to the total time duration of the run. The validity of this normalizing procedure may be shown as follows:

Let $P'(\eta), F'(\eta) \sim$ the actual readings obtained on the "statistical analyzer".

$N_0 \sim$ total count corresponding to the time duration of the run.

then,

$$\overline{I^n} = \frac{\int_{-\infty}^{\infty} \eta^n P'(\eta)\, d\eta}{\int_{-\infty}^{\infty} P'(\eta)\, d\eta}$$

Substituting Equations (5) and (6) into this expression gives,

$$\overline{I^n} = \frac{n\left\{\int_0^{\infty} \eta^{n-1} F' d\eta + \int_0^{-\infty} \eta^{n-1}(1-F')\, d\eta\right\}}{\int_0^{N_0} dF'(\eta)}$$

therefore,

$$\overline{I^n} = n\left\{\int_0^{\infty} \eta^{n-1} \frac{F'}{N_0} d\eta + \int_0^{-\infty} \eta^{n-1}\left(1-\frac{F'}{N_0}\right) d\eta\right\} \qquad (7)$$

which is the expression desired.

In this connection it has been found convenient to make each run a minute in duration and to use a pulse rate of 100,000 counts per minute. Thus, $N_o = 10^5$ and the experimental readings are normalized merely by dividing each reading by 10^5.

The mean values are usually presented in the form of dimensionless ratios. The most common ones and their definitions are the following:

$$C \equiv \frac{\sqrt{\overline{I^2}}}{\overline{|I|}} \qquad (8)$$

$$S \equiv \frac{\overline{I^3}}{(\overline{I^2})^{3/2}}, \quad \text{SKEWNESS} \qquad (9)$$

$$F_1 \equiv \frac{\overline{I^4}}{(\overline{I^2})^{2}}, \quad \text{FLATNESS} \qquad (10)$$

The values of these factors for the case of a Gaussian probability distribution are, $C = \sqrt{\frac{\pi}{2}}$

$S = 0$

$F_1 = 3$

It is not possible to set the initial "zero" pulse amplitude at some known arbitrary value with the accuracy required for computing odd moments (see Sect. VII). Therefore the initial "zero" pulse height is set only approximately to some convenient value, and the exact position of the zero axis is later computed on the basis that the average (D. C.) value of the input signal V(t) is zero because a condenser at the modulator input blocks any steady component of voltage present in the input signal.

The statement that the average value of the signal is zero is equivalent to setting the 1st mean value equal to zero. Therefore, from Equation (6), the $\eta = 0$ axis is defined as that axis about which the shaded areas in Figure (1) are equal. This formulation is used as the basis of a small perturbation method by means of which the position of the zero axis is readily computed to the required accuracy (Appendix I).

VII. DISCUSSION OF ERRORS

The most important factor limiting the accuracy of amplitude selection is the lack of a perfect break in the diode characteristic since there is no discontinuity in slope but only an exponentially shaped transition region. By operating the diode heaters at a reduced voltage the sharpness of the break becomes more pronounced and variation among different tubes is reduced. With this method of operation the indeterminancy of the diode characteristic break is approximately .05 volts, a value that is less than one per cent of the usual bias interval between channels.

It is desirable that the amplifiers associated with the diode selectors have gain characteristics as nearly alike as possible. Therefore each amplifier is provided with a trimmer gain control. The uniformity in selection characteristics of the various "channels" is checked by biasing each "channel" to the same level and checking the agreement between the readings obtained when a random voltage is used as an input signal to the modulator. With no attempt to trim the amplifier gain controls the readings will agree within about one per cent. However, by trimming the gain of each amplifier agreement of readings to within 1/2 per cent for random inputs is easily obtained. Unless a tube needs replacement, adjustment of the amplifier gain controls need not be done more often than about once every two months.

At present the bias levels are set by means of 7-1/2 volt dry batteries having taps at 1-1/2 volt intervals. The spacing between adjacent channels is uniform to an accuracy of 1-1/2 per cent when the bias levels are set at 7-1/2 volt intervals. The drift in battery voltage over a period of two months has been negligible. However, it is planned to replace these batteries with a chain of precision resistors in series with a stabilized power supply.

When computing odd mean powers, i.e., skewness factors, a small difference between large numbers results and the result is very sensitive to the position of the $\eta = 0$ axis. The error in the computed skewness factor due to a small shift in the zero axis was estimated by calculating the moments about an axis shifted small amounts from the true $\eta = 0$ axis. For the specific case of the distribution function curve for the first derivative of a turbulent velocity, it was found that in order to have an error of less than 3 per cent in the computed skewness factor, the position of the zero axis has to be within 1/2 per cent of the bias interval from the true $\eta = 0$ axis.

Because of this stringent requirement, the position of the zero axis is computed rather than attempting to set the initial pulse height to an exact arbitrary value. The position of the zero is readily computed to an accuracy of $\pm .1$ per cent of the bias difference between adjacent "channels".

As a check on the repeatability of the readings obtained with the "statistical analyzer", consecutive one minute runs were taken with a sine wave input to the modulator. The reading of each "channel" was found to be repeatable to within $\pm .1$ per cent of the average for the various runs.

Measurement of the probability distribution of the first derivative of a turbulent air velocity gave the following repeatability for the mean values computed from data obtained from consecutive one minute runs:

$\overline{\lvert V \rvert}$	$\pm$ 1 %
$\overline{V^2}$	$\pm$ 2 %
$\overline{V^3}$	$\pm$ 3 %
$\overline{V^4}$	$\pm$ 4 %

Another way to determine the accuracy of the "statistical analyzer" is to take measurements of a random signal having a known probability distribution. The most convenient way to do this is to use a stable source of random noise that is known to have a Gaussian probability distribution. Unfortunately, such a source of random noise was not available at the time of this writing.

VIII. SAMPLE APPLICATIONS

A. Sine Wave

The distribution function for a sine wave (190 cps) was measured with the "statistical analyzer". The sine wave signal was obtained from a Hewlett-Packard audio oscillator.

Measurements of only one half the sine wave signal were taken, and the bias level was shifted once in order to obtain more points on the curve. The experimental results along with the theoretical distribution curve for a sine wave are shown in Figure (8). The theoretical distribution function for a sine wave is an arcosine and the experimental points are seen to follow this curve quite closely.

B. Isotropic Turbulence

Figure (9) shows a sample of a distribution function for the axial air velocity as measured in isotropic turbulence behind a grid. The curve is seen to be quite symmetrical. The mean value factors computed from this distribution are given in Table I. These values show that the distribution is fairly close to a Gaussian, except that the tails of the curve are slightly longer than for a Gaussian distribution.

Figure (10) presents the results of measurement of the distribution function of the velocity derivative $\frac{du}{dt}$. This probability distribution differs appreciably from the Gaussian because it is asymmetric and the central portion of the curve is steeper than that of a Gaussian. The mean value factors are given in Table I.

Figure (11) shows the distribution function curve obtained for the second derivative of the turbulent velocity. This distribution is almost symmetrical, but the flatness factor is much higher than that obtained for a Gaussian distribution.

Table I

	S	F_1	C
u	.006	2.73	$.980\sqrt{\frac{\pi}{2}}$
$\frac{du}{dt}$	-.439	3.80	$.984\sqrt{\frac{\pi}{2}}$
$\frac{d^2u}{dt^2}$	.031	4.59	$1.04\sqrt{\frac{\pi}{2}}$

R_M = 11,000 Mesh size = 1.68 cm. x/M = 50

The values in Table I agree fairly closely with the results obtained by Batchelor and Townsend (Refs. 7, 8).

C. Vortex Street

At low Reynold's numbers, the flow velocity behind a cylinder fluctuates with a definite periodicity. Numerous measurements of the probability distribution of the velocity were taken at different positions in the cylinder wake. These results are presented in detail in Ref. (9), and we will discuss only one example here.

Figure (12) shows the distribution function obtained in a vortex street at a Reynold's number of 100. Examination of the hot-wire signal on an oscilloscope screen showed a very stable pattern that was almost triangular in shape except for a rounding of the peaks. If the signal were truly triangular, the distribution function would be a straight line, and Figure (12) shows that the distribution is, indeed, quite linear except at the outer edges where it curves inward toward zero. This curvature at the ends is due to the rounding of the peaks in the velocity fluctuations which tends to give the velocity pattern some characteristics of a sine wave. In this respect, it is instructive to compare Figure (12), for the vortex street, with Figure (8), which is for a sine wave.

IX. CONCLUSIONS

Computation of the mean values of a stationary random function from the probability distribution is superior to other methods in that all the mean powers can be obtained from one set of data.

The use of counting methods in obtaining probability distributions enables small probabilities to be accurately measured, and by the use of many "channels" a substantial reduction in the time required for the measurement of distributions has been achieved. As an example, the measurement of a complete probability distribution for a turbulent air velocity can now be done in less than five minutes, whereas the previously used method of point by point measurement required approximately an hour's time. Thus, the required stability of the associated measuring equipment has been considerably reduced.

APPENDIX I. CALCULATION OF POSITION OF $\eta = 0$ AXIS

Figure (13) is to be used in conjunction with this discussion. The distance, ϵ , of the zero axis from channel 5 is found by the following method.

For zero average value of the input signal, we have,

$$A_1 + a_1 = A_2 + a_2$$

$$a_2 - a_1 = A_1 - A_2$$

A_1 and A_2 are determined from the data by use of Simpson's rule.

The curve between channels 5 and 6 can usually be accurately approximated by a straight line. Therefore, a_1 and a_2 are computed by using the trapezoidal rule as follows:

$$a_1 = \frac{1}{2}(h_1 + \ell_1)\epsilon, \quad a_2 = \frac{1}{2}(h_2 + \ell_2)(1-\epsilon)$$

$$2(a_2 - a_1) = h_2 + \ell_2 - (h_1 + h_2)\epsilon - (\ell_1 + \ell_2)\epsilon$$

$$= h_2 + \ell_2 - (h_1 + h_2 + 1)\epsilon$$

But, $\ell_1 = h_1 + [1 - h_2 - h_1]\epsilon$, $\ell_2 = 1 - \ell_1$

Substituting, we get

$$2(a_2 - a_1) = h_2 + 1 - h_1 + (h_1 + h_2 - 1)\epsilon - (h_1 + h_2)\epsilon - \epsilon$$

$$2(A_1 - A_2) = h_2 - h_1 + 1 - 2\epsilon$$

Therefore,

$$\epsilon = \frac{h_2 - h_1 + 1}{2} - (A_1 - A_2) \tag{11}$$

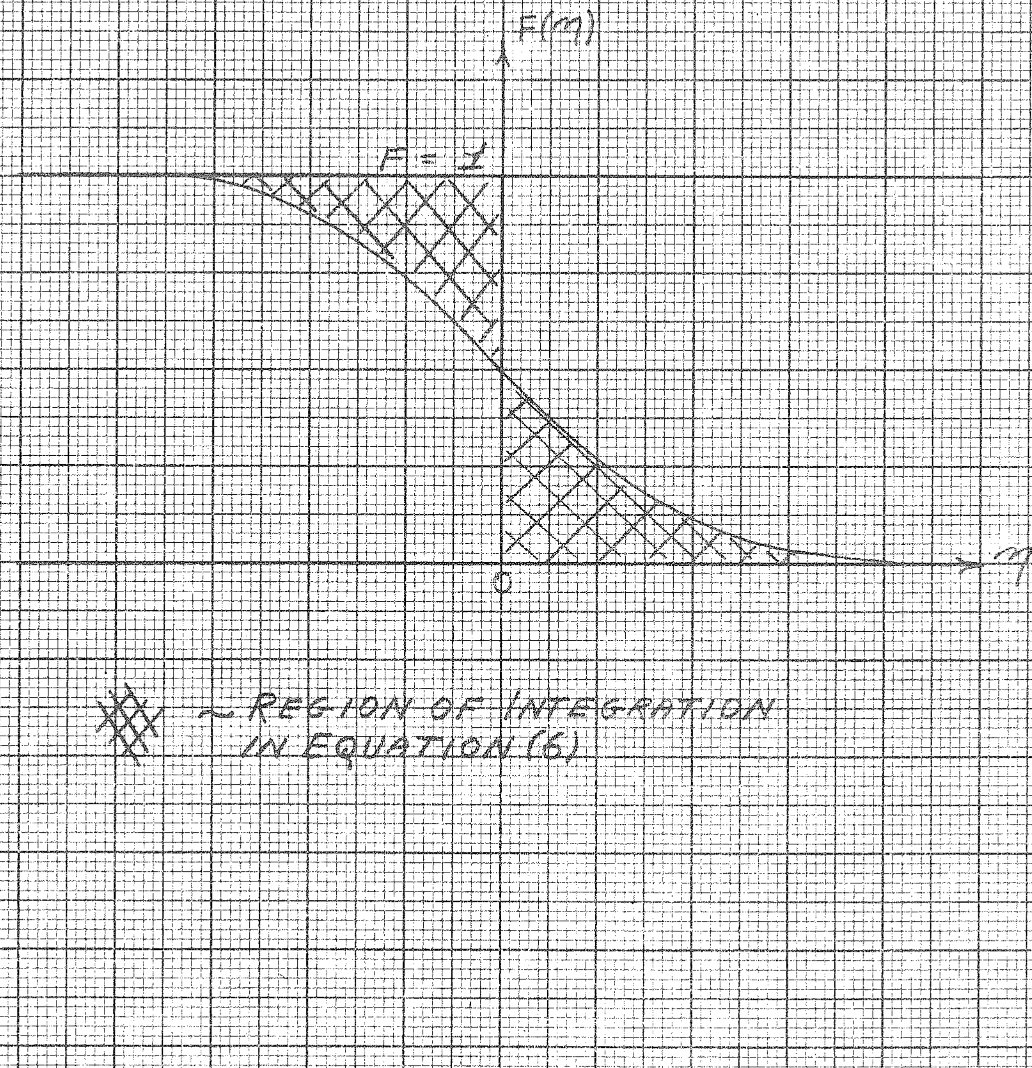
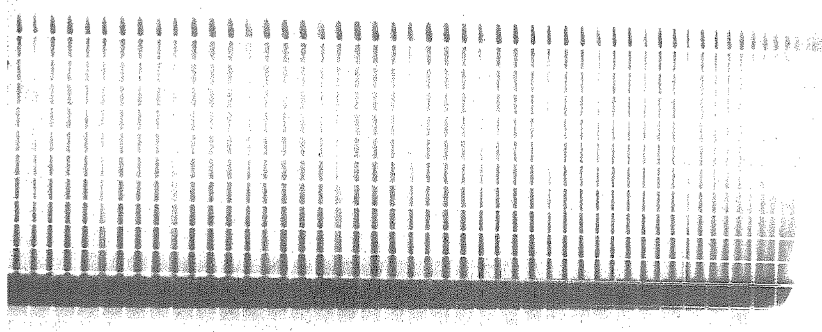


FIG. 1 : TYPICAL DISTRIBUTION FUNCTION



2000 PULSES/SEC.

FIG 2 : UNMODULATED PULSES

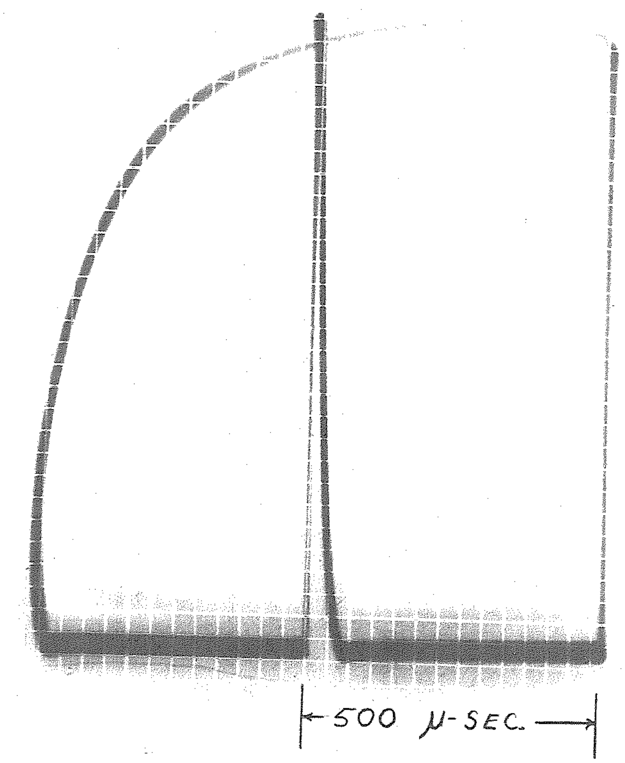


FIG. 3: VIEW OF A SINGLE PULSE

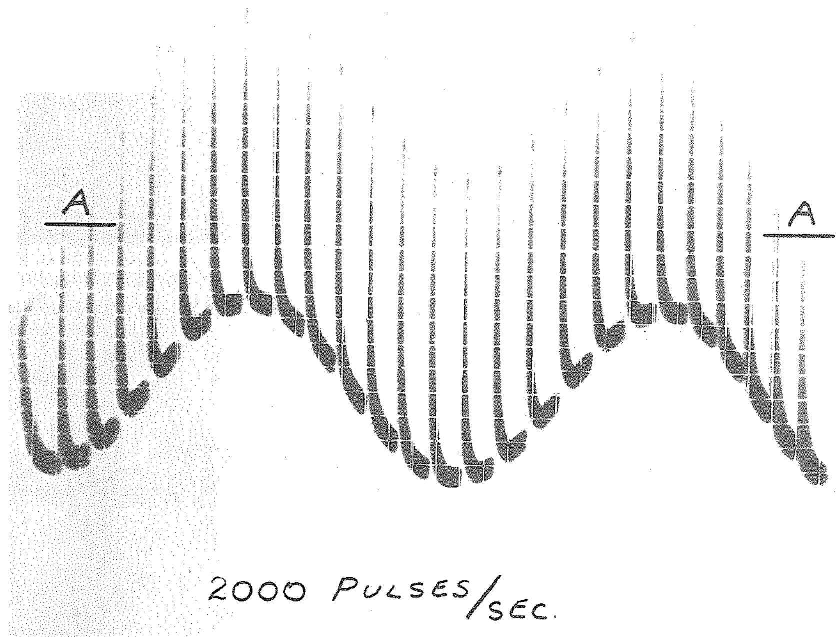
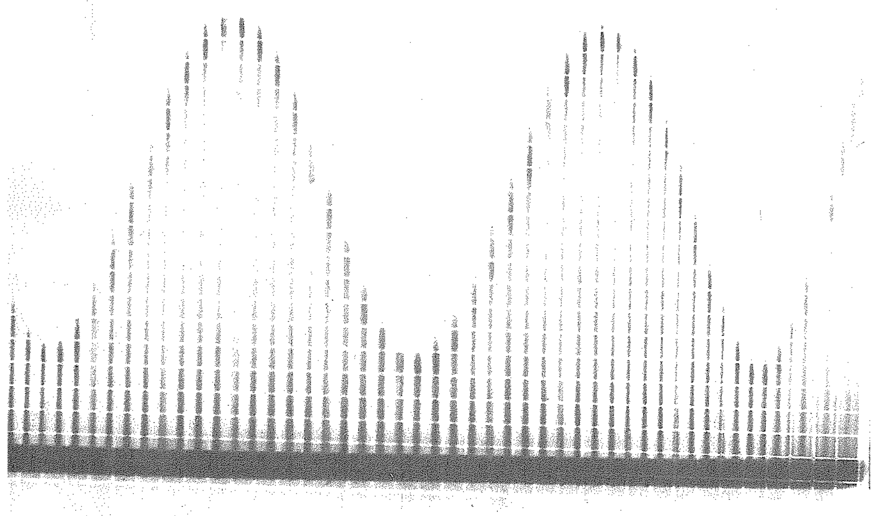


FIG. 4 : DIFFERENTIAL AMPLIFIER OUTPUT
FOR A SINE WAVE INPUT



2000 PULSES/SEC.

FIG. 5 : MODULATOR OUTPUT FOR A SINE WAVE INPUT

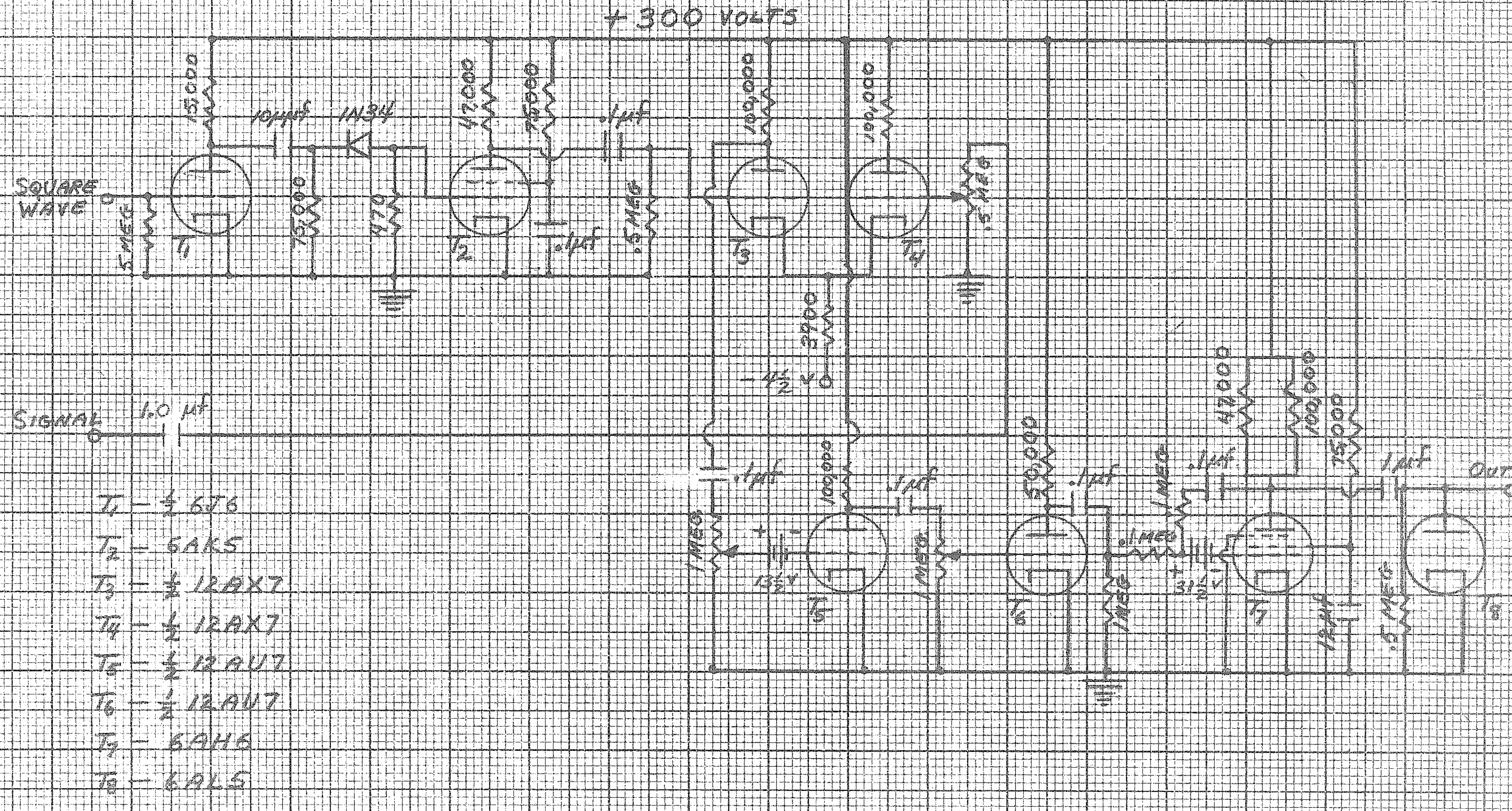


FIG 6: PULSE AMPLITUDE MODULATOR

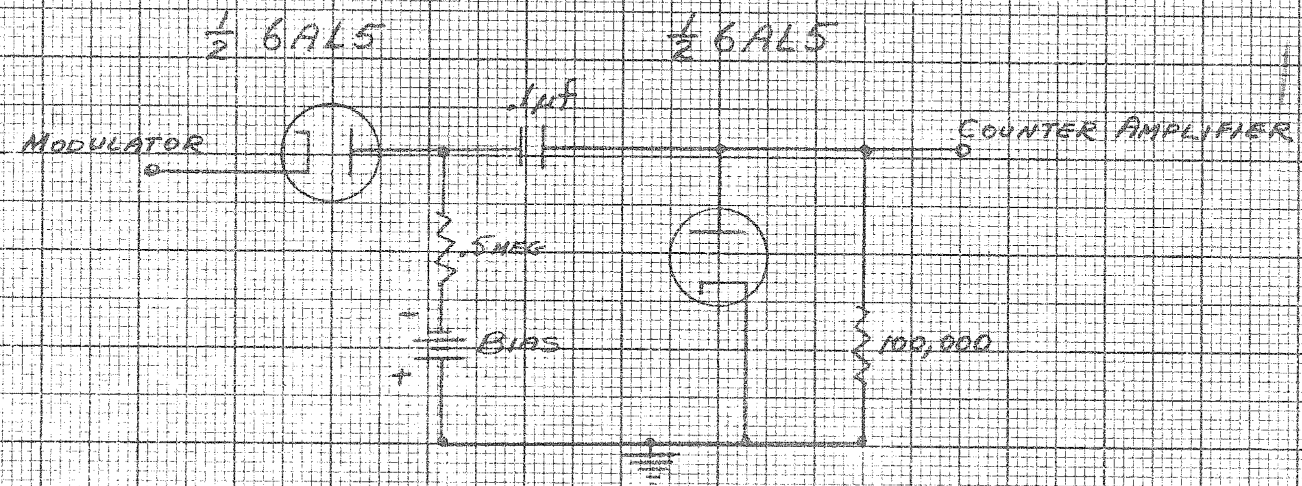


FIG. 7 : PULSE HEIGHT DISCRIMINATOR

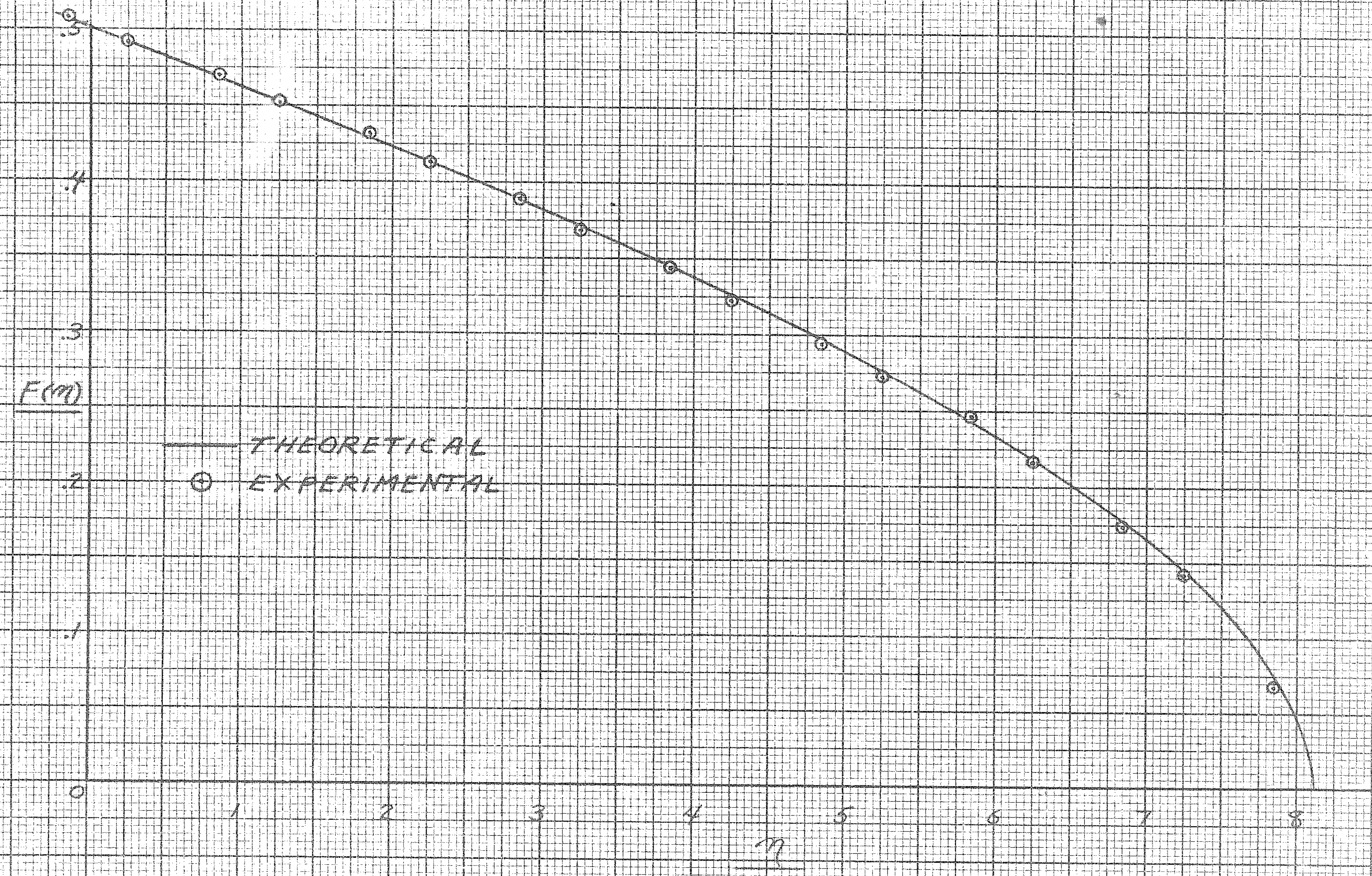


FIG. 8 : DISTRIBUTION FUNCTION FOR A SINE WAVE

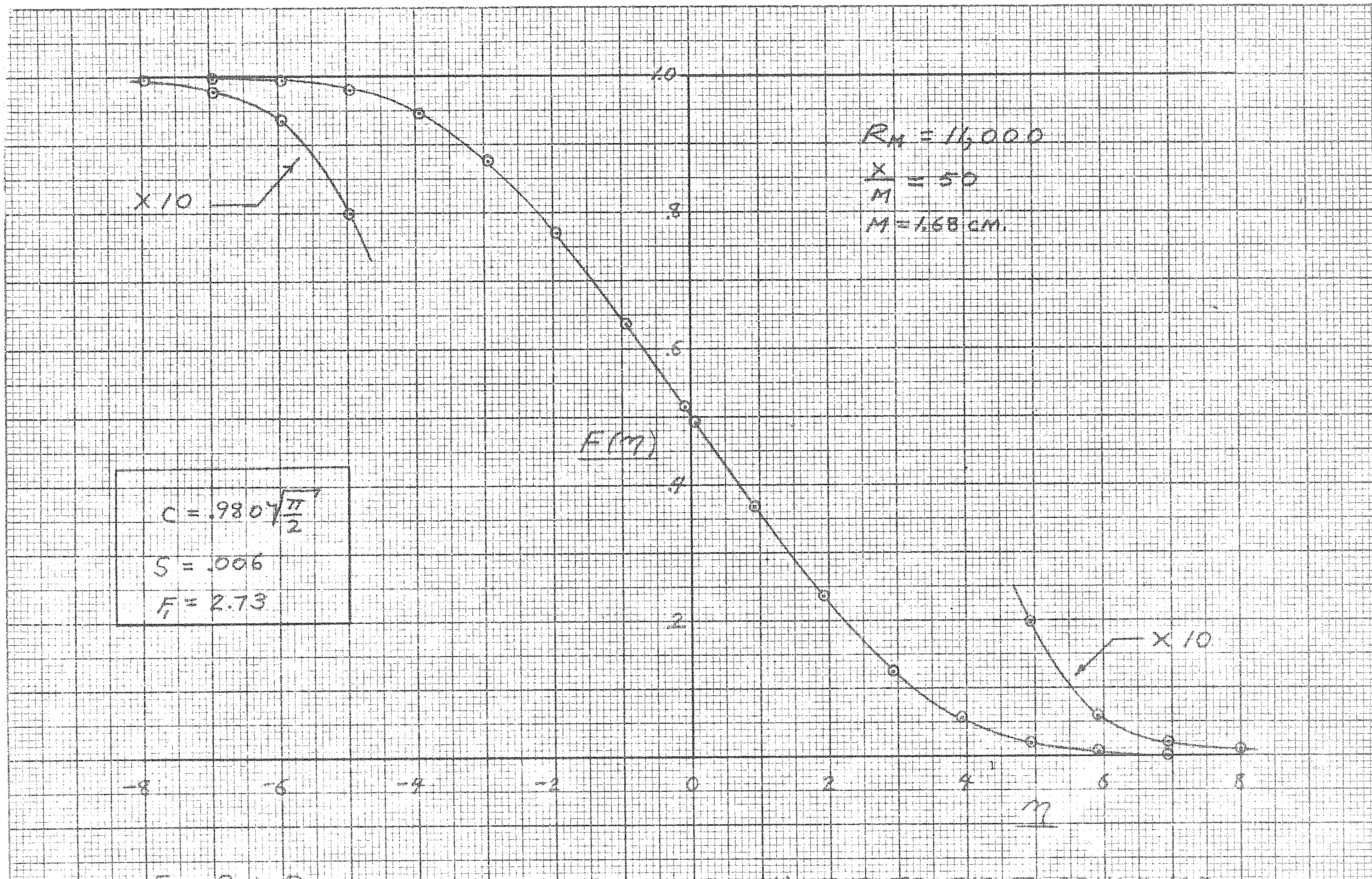


FIG. 9 : DISTRIBUTION FUNCTION FOR u(t) IN ISOTROPIC TURBULENCE

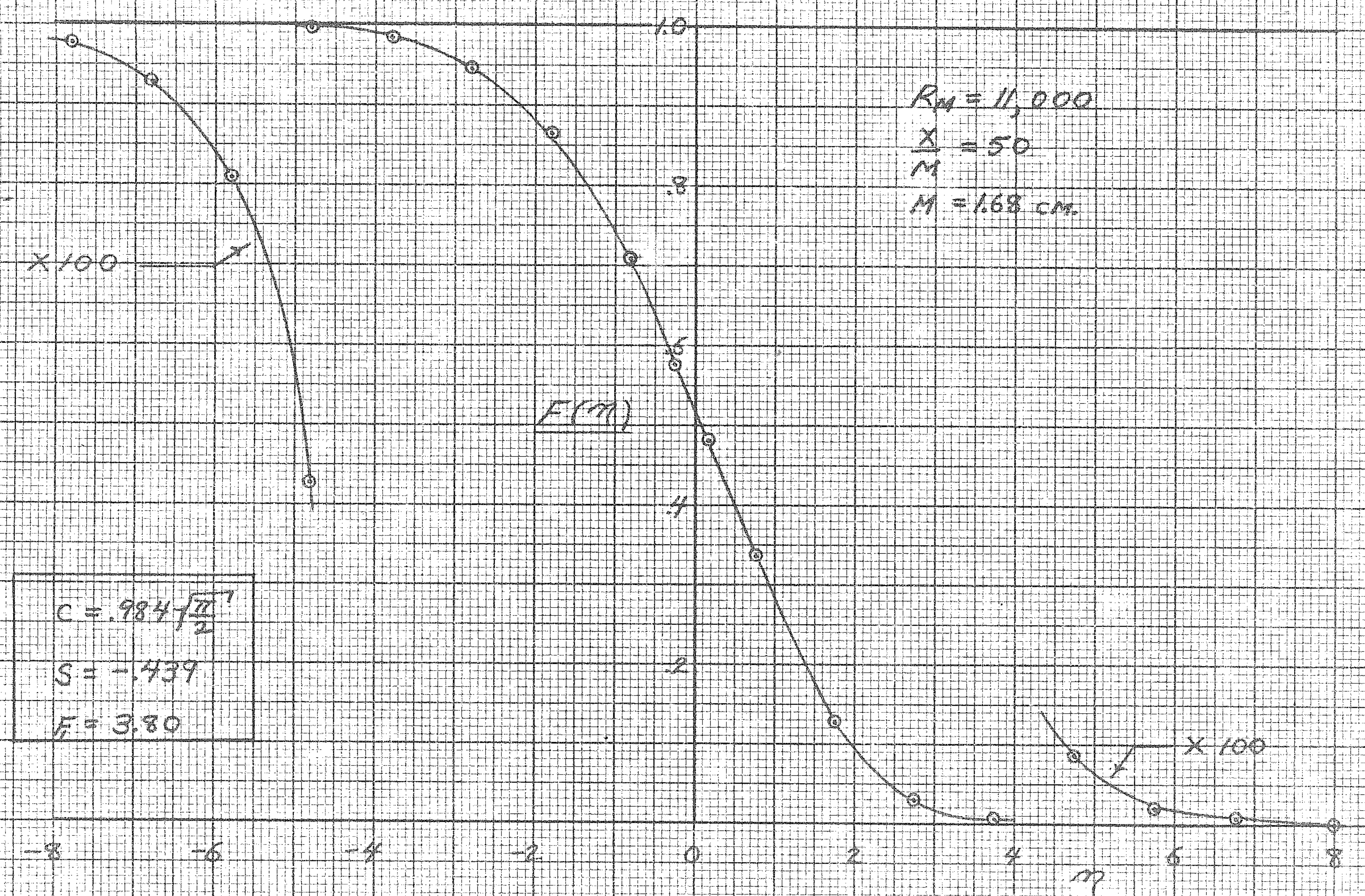


FIG. 10: DISTRIBUTION FUNCTION FOR $\frac{du}{dx}$ IN ISOTROPIC TURBULENCE

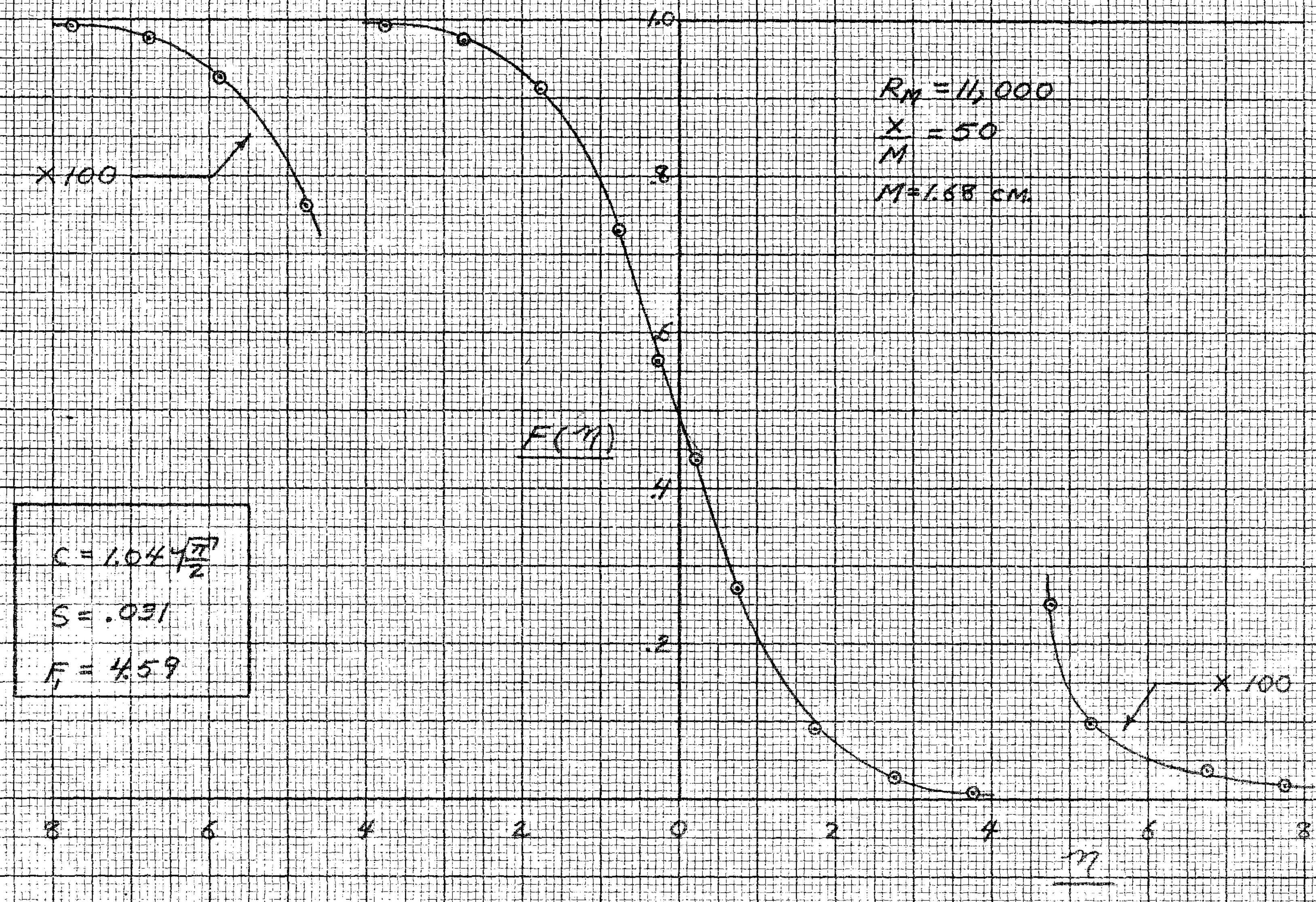


FIG. 11: DISTRIBUTION FUNCTION FOR $\frac{d^2u}{dx^2}$ IN ISOTROPIC TURBULENCE

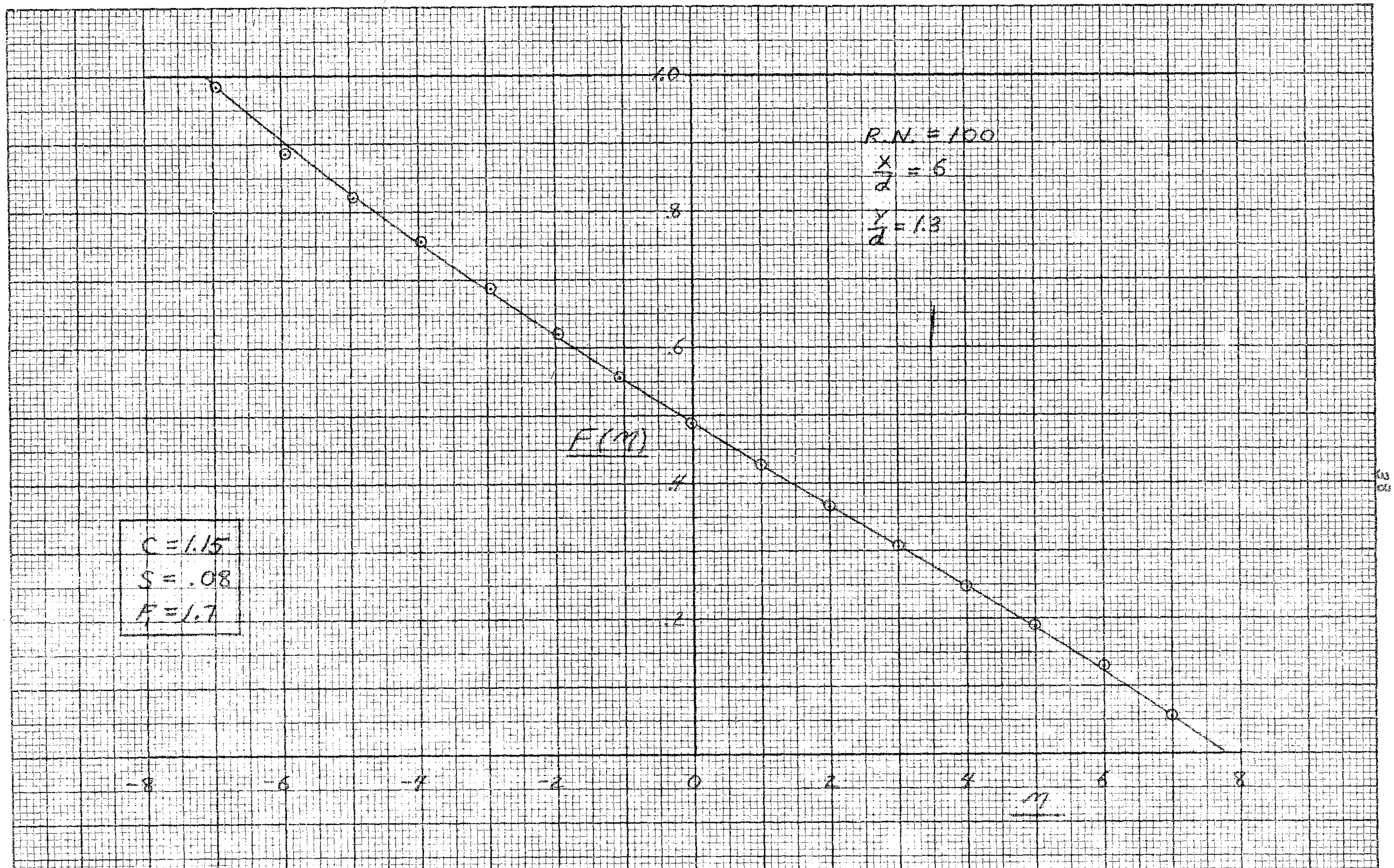
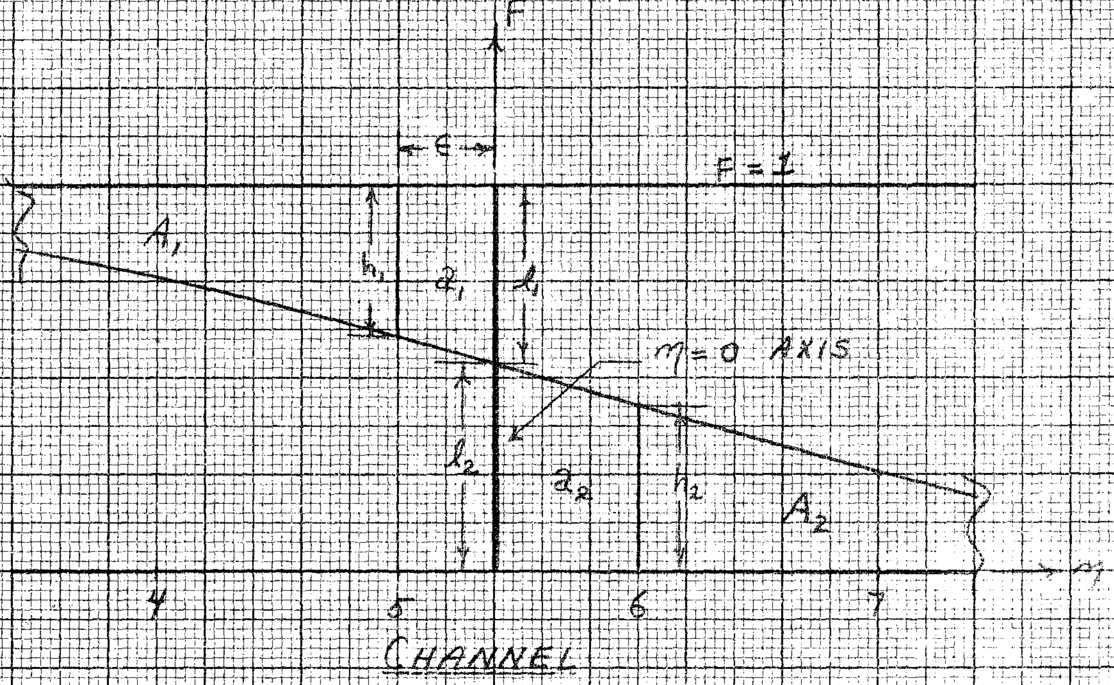


FIG. 12. DISTRIBUTION FUNCTION FOR u(t) IN VORTEX STREET



A_1, A_2, a_1, a_2 ~ AREAS

$l_1, l_2, h_1, h_2, \epsilon$ ~ LENGTHS

FIG. 13: CALCULATION OF $\eta = 0$ AXIS

TABLE OF NOTATION

I	Physical variable
t	Time
T	Time
n	An integer
η	Amplitude
ρ	Probability density
F	Probability distribution function
Δt	Time interval
t_p	Defined on page 8
V	Electric voltage input signal to modulator
ρ'	Defined on page 16
F'	Defined on page 16
N_o	Defined on page 16
c	Defined on page 17
S	Skewness factor
F_1	Flatness factor
u	Air velocity
A_1	Defined in Fig. 13
A_2	Defined in Fig. 13
a_1	Defined in Fig. 13
a_2	Defined in Fig. 13
ϵ	Defined in Fig. 13
$\overline{(\ )}$	Time average

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