Topics in Supersymmetry Theory:

1. A Superspace Action for Ten-Dimensional Supersymmetric Yang-Mills Theory in Terms of Four-Dimensional Superfields;

2. Gauge Groups for Type-I Superstrings

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To my parents,
Giovanni and Iole Sagnotti,
and to my wife
Francesca

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Abstract

This thesis is an account of most of the work that I did in Supersymmetry and Supergravity over the last two years. It deals with two major topics, the construction of a new superspace action for ten-dimensional supersymmetric Yang-Mills theory in terms of four-dimensional superfields, and the classification of the gauge groups allowed at the classical level in the type-I superstring theory. In addition, it contains a discussion of work that I did showing the uniqueness of supergravity in eleven dimensions and the uniqueness of the free Rarita-Schwinger action for massless and massive spin-$\frac{3}{2}$ fields.

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Chapter 1

Introduction to Supersymmetry

1.1. Introduction

Supersymmetry [1] is a symmetry of Relativistic Field Theory relating fields of different spin. It corresponds, from the algebraic point of view, to enlarging the Poincaré algebra by the inclusion of extra spinorial generators whose anticommutator generates the translations †.

Supersymmetry has stimulated much theoretical interest over the last years, mainly because of its undeniable aesthetic appeal. To be honest, however, one has to admit that to date there is no compelling phenomenological evidence that forces supersymmetry into our picture of the fundamental interactions, contrary to what happened previously for the Poincaré symmetry and for the internal symmetries of hadron physics. The main problem in this respect is that, as we will see, particles in the same irreducible representation of the supersymmetric generalization of the Poincaré algebra are sets of bosons and fermions in equal numbers and *all with the same mass*, a circumstance that clearly does not correspond to observation. As a symmetry of nature, supersymmetry must therefore be a broken symmetry. Why bother, then? The motivation for dealing with supersymmetry is actually to be found in the mathematical structure of the theories. A somewhat loose, but very successful analog of such a situation can be found in the electroweak theory, where the SU(2) x U(1) symmetry, hidden as spontaneously broken to $U(1)_{em}$, is mathematically appealing in that it leads to a renormalizable theory ‡. In a similar fashion, introducing supersymmetry, spontaneously or even explicitly

† Corresponding to the number (N) of supersymmetry generators in the algebra, one talks about N-extended supersymmetry.

‡ The analogy is not too strong, because the SU(2) x U(1) multiplets are observed in weak interactions.

broken, in gauge theories, leads to improved ultraviolet behavior, resulting from cancellations between contributions from fermionic and bosonic members of the same supermultiplet. Far more important, in the opinion of the author, is the possibility of constructing supersymmetric theories of gravity [2], where the metric tensor is unified via supersymmetry with matter fields. In this case, the cancellation of ultraviolet divergences is more crucial, as it leads to the first examples of couplings of matter to gravity *which do not introduce any divergences in the S matrix at one loop*.

Continuing in this direction, one also realizes that the supersymmetric theories of gravity, known as supergravity theories, are descendants of multilocal field theories defined in ten-dimensional space time, known as superstring theories [3]. These are the first examples of string theories without tachyons in their spectrum. Moreover, when interpreted in four dimensions via compactification of six of the nine spatial dimensions, they appear as "corrected" supergravity theories, with further *hope* of providing a perturbatively calculable quantum theory of gravity. Again, to be honest, one should stress here that no ultraviolet divergences have been found to date in the S matrix of pure gravity, or in the S matrix of extended supergravity. In some sense, however, there is no reason why the S matrix of dimensionally regularized pure gravity should not contain $\frac{1}{\varepsilon}$ divergences in two loops, and no unquestionable reason why the same should not happen in supergravity for the $\frac{1}{\varepsilon}$ terms in three loops. Furthermore, only one-loop amplitudes have been calculated so far in the superstring theories. Even though string theory amplitudes are less singular than the corresponding amplitudes in supergravity, in that they converge in higher dimensions, where these diverge, nothing is known to date beyond one loop.

If one takes a positive attitude toward supergravity and superstrings, the natural thing to do in this context would then be trying to fit the known particles into the spectrum of the most suitable supergravity theory, or in the massless spectrum of a superstring theory. Again, the impact with this program (a very ambitious one, indeed), is not totally encouraging, as to date there is no widely agreed upon scheme for extracting the low-energy gauge theories from supergravity. Three different mechanisms have been proposed. One has to do with the conjectured formation of bound states providing propagating bosons that would mediate the local SU(8) symmetry of the N=8 supergravity theory, for which the gauge fields are composites of the scalars at the classical level. The second mechanism attempts to relate the gauge symmetries observed at low energy to compactified solutions of higher-dimensional supergravity theories. Finally, the third mechanism has to do with the option of adding dimensionless coupling constants to extended supergravity theories.

All these approaches have problems, which I will now mention very briefly. In the first case, the conjectured formation of bound states is motivated by the situation encountered in two-dimensional nonlinear σ-models, and there is no proof that this would also occur for supergravity. Even leaving this problem aside, the formation of bound states would lead to a large number of unwanted states, and there are at best highly disputable mechanisms invoked to explain why the unwanted states should acquire large masses and be unobservable at low energies. The compactification program is a more straightforward one to carry out at the moment, and it is giving the first results. As anticipated, it has to do with the possibility of writing supergravity theories in higher-dimensional spacetimes, up to and including D=11. To reconcile this with our four-dimensional perception of space time, one interprets (D-4) of the spatial coordinates as parametrizing a compact manifold, and extracts the four-dimensional physics from the study of the lowest modes of oscillation about the compactified solution. Symmetries of the ground state solution then translate into gauge symmetries. Several exact solutions of the field equations of D=11 supergravity are known by now [4]. It is remarkable in this respect that solutions to the field equations with four-dimensional anti-de Sitter symmetry and the seven remaining spatial coordinates parametrizing a manifold with SU(3) x SU(2) x U(1) symmetry have been found [5]. While this program can in principle be carried out up to the end, the choice of one manifold rather than another for compactification appears to be somewhat unmotivated and *ad hoc*. Finally, the last mechanism is the most conventional one. It has to do with the spontaneous breaking of the local symmetries of the gauged extended supergravity theories. In this respect, the main problem was pointed out a long time ago [6]: *even* SO(8) is too small to contain SU(3) x SU(2) x U(1). If, however, one tries to fit only a subgroup of the low-energy group, the explicit study of the extrema of the potentials in extended supergravities acquires some immediate interest [7]. This approach and the compactification approach are expected, although not proved, to be intimately connected, in that extrema of the potentials of four-dimensional supergravity would correspond to compactified solutions of the higher-dimensional field equations and vice versa. One problem common to both these approaches is that one ends up with a large negative cosmological constant.

Supersymmetry in general, and supergravity and superstrings in particular, also present some interesting problems of a more formal and technical nature. In this context, a main puzzle has to do with the off-shell closure of supersymmetry algebras. In general, we are not very skilled at present in writing supersymmetric theories. All we can do in most cases is to consider a set of fields, each of which is associated with one irreducible representation of the Poincaré group, and adjoin them into actions which are made supersymmetric by the choice of a few parameters. The supersymmetry, as we remarked, is related to a suitable choice of parameters, and is therefore not an easily recognizable property. Moreover, the set of fields at our disposal is *not* in general an off-shell representation of supersymmetry, and correspondingly the supersymmetry algebra closes on the fields only when the field equations are used. The recipe for closing the supersymmetry algebra off-shell can be found in the more familiar case of the Lorentz algebra. Just as one, in this case, adds to the transverse propagating components of, say, the electrodynamic potential two more components to complete the four-vector A_μ, so one completes an on-shell supersymmetry multiplet by adding extra gauge and auxiliary degrees of freedom. The problem is that *auxiliary fields are not known for most supersymmetric theories*, and there are formal arguments suggesting that they cannot be found for all cases of interest [8].

A consequent, but equally interesting, problem, is learning how to describe supersymmetric theories in terms of superfields [9]. Superfields are generalizations of ordinary fields defined in superspace, an enlargement of space time by the inclusion of extra spinorial coordinates. On account of the anticommuting nature of the spinorial coordinates, superfields are just a compact-looking regrouping of a finite set of fields, including those of an off-shell supermultiplet. It thus seems that the natural thing to do would be to mimic the steps that one takes in going from the two transverse propagating components of the electrodynamic potential to the four-vector A_μ, completing supersymmetry multiplets by the addition of auxiliary fields, and finally attempting to regroup the resulting set of fields into one (or more) superfields. Even if the auxiliary fields are known, this last step is usually a highly nontrivial one, because there are, in general, several extra gauge degrees of freedom in the superfields, which are not known *a priori*. Moreover, the actions one writes in superspace are usually complicated-looking, nonpolynomial and not manifestly gauge covariant.

The main hopes attached to superspace formulations have to do with the expectation that making the supersymmetries manifest can say something about the unknown quantum behavior of supergravity. Indeed, assuming that a a formulation of N=8 supergravity in terms of N=8 superfields can be found boosts from three to seven loops the first possible occurrence of ultraviolet divergences in the S matrix of the theory.

In this first chapter, we will start by describing the N=1 super-Poincaré algebra (and the corresponding super-de Sitter algebra) in four-dimensional space time and some of their most interesting higher-dimensional analogues. Then we will study the particle representations of the super-Poincaré algebra, and we will describe in some detail the component formulations of a few basic models with N=1 supersymmetry in four dimensions, the Wess-Zumino model, the N=1 supersymmetric Yang-Mills theory and N=1 supergravity, and of supergravity in eleven dimensions. In particular, we will show that, contrary to what happens in four dimensions, supergravity cannot accommodate a cosmological term consistent with local supersymmetry in eleven dimensions. The appendices contain some useful material about supersymmetry algebras, and a proof that the Rarita-Schwinger action is the unique one for a spin-$\frac{3}{2}$ field. Most of this first chapter is a pedagogical introduction to several issues encountered in supersymmetry and supergravity, and consequently most of the material here is well known. We chose to write in this form, as it allows us to collect several otherwise unconnected topics we have touched upon in learning about supersymmetry, thus providing at the same time some background material for the discussion of the new superspace action presented in chapter 2. The contributions of the author are in part in the discussion of the algebras, in the generalization to eleven dimensions of the original study of the particle representations of the Poincaré algebra, in the discussion of the uniqueness of supergravity in eleven dimensions, and in the discussion of the uniqueness of the Rarita-Schwinger equation.

1.2. Supersymmetry algebras

In the previous section we have remarked that, from the algebraic point of view, N-extended supersymmetry corresponds to an enlargement of the Poincaré group by the inclusion of N sets of anticommuting charges, transforming as N spinors under the Lorentz group. The simplest case is the one in which only one spinorial generator is present. This is the N=1 super Poincaré algebra, which we now describe.

Given the Poincaré algebra †

$$[P_\mu, P_\nu] = 0 \ , \tag{1.2.1a}$$

$$[P_\mu, J_{\nu\rho}] = -2i\, \eta_{\mu[\nu} P_{\rho]} \ , \tag{1.2.1b}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = 4i\, J_{[\mu[\rho}\eta_{\nu]\sigma]} \ , \tag{1.2.1c}$$

where [] denotes antisymmetrization with unit strength, we supplement the set of translation generators P_μ and Lorentz generators $J_{\mu\nu}$ by the inclusion of a spinor Q satisfying the Majorana condition

$$Q = C\,\overline{Q}^T \ . \tag{1.2.2}$$

C, known as the charge conjugation matrix, is *antisymmetric* and satisfies‡

$$C^{-1}\gamma_\mu C = -\gamma^T{}_\mu \ . \tag{1.2.3}$$

The extension of (1.2.1) into the super-Poincaré algebra then follows from only two elements, the postulated behavior of Q under Lorentz transformations, and the requirement that, in analogy with eqs. (1.2.1), no dimensionful parameters

† We use the signature (-+..+) throughout.

‡ We denote four-dimensional Dirac matrices by γ_μ, and higher-dimensional Dirac matrices by Γ_μ. Moreover, $\gamma_{\mu_1..\mu_n}$ ($\Gamma_{\mu_1..\mu_n}$) is a product of Dirac matrices antisymmetrized in all the indices with strength one.

enter the algebra.

Indeed, it is simple to show that all the Jacobi identities are satisfied for

$$[P_\mu, P_\nu] = 0 \ , \tag{1.2.4a}$$

$$[P_\mu, J_{\nu\rho}] = -2i\,\eta_{\mu[\nu} P_{\rho]} \ , \tag{1.2.4b}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = 4i\, J_{[\mu[\rho}\eta_{\nu]\sigma]} \ , \tag{1.2.4c}$$

$$[Q_\alpha, P_\mu] = 0 \ , \tag{1.2.4d}$$

$$[Q_\alpha, J_{\mu\nu}] = \frac{i}{2}(\gamma_{\mu\nu})_{\alpha\beta}\, Q_\beta \ , \tag{1.2.4e}$$

$$\{Q_\alpha, Q_\beta\} = -(\gamma^\mu C)_{\alpha\beta} P_\mu \ . \tag{1.2.4f}$$

This is the N=1 super Poincaré algebra. It is the global algebra of theories invariant under simple supersymmetry in four-dimensional space time. The new equations (1.2.4d) and (1.2.4e) simply state that Q is a spinor of four-dimensional space time invariant under translations. Far more interesting is eq. (1.2.4f), which allows one to recover the translation generators from the anticommutator of two spinorial charges. The Q's, therefore, are the more fundamental objects. The consistency of eq. (1.2.4f) rests heavily on eqs. (1.2.2) and (1.2.3). In fact, these imply that $(\gamma^\mu C)$ is a symmetric matrix, a necessary requirement in view of the form of eq. (1.2.4f).

The N-extended super-Poincaré algebra corresponds to enlarging (1.2.4) by endowing the Q's with an SO(N) vector index. The resulting N spinorial charges are still invariant under translations and rotate as independent spinors under the Lorentz group, so that eqs. (1.2.4d) and (1.2.4e) are replaced by

$$[Q^i_\alpha, P_\mu] = 0 \tag{1.2.5a}$$

and

$$[Q^i_\alpha, J_{\mu\nu}] = \frac{i}{2}(\gamma_{\mu\nu})_{\alpha\beta} Q^i_\beta \quad , \tag{1.2.5b}$$

respectively. A more interesting modification can take place for (1.2.4f). The naive modification would be writing

$$\{Q^i_\alpha, Q^j_\beta\} = -\delta^{ij}(\gamma^\mu C)_{\alpha\beta} P_\mu \quad . \tag{1.2.6}$$

Extra terms, however, can be present on the r.h.s of (1.2.6). These terms are usually denoted as "central charges" to indicate that they appear on the r.h.s. of the equations defining the superalgebras, but commute with all generators of the superalgebras. Rather than writing out directly the terms involving central charges, we will now show how their very appearance in the algebras finds a natural explanation in the existence of higher-dimensional analogs of the super-Poincaré algebra (1.2.4). This will be of some relevance to our discussion in chapter 2.

Higher-dimensional analogs of the super-Poincaré algebra are, indeed, very useful. By setting some of the higher-dimensional momenta to zero, they provide very convenient compact rewritings of extended supersymmetry algebras in four dimensions. A higher-dimensional analog of the N=1 super-Poincaré algebra would differ from (1.2.4) only in two respects. First of all, the vector indices would span the D values $(0,..,D-1)$, rather than the four values (0,1,2,3). The second, and more important, modification is that the lowest spinor representations of higher-dimensional Lorentz groups are direct sums of an even number (actually, a power of two) of spinor representations of the four-dimensional Lorentz group. Therefore, the analog of the charge Q^α of (1.2.4) would regroup an even number of four-dimensional charges. As to the signature of the higher-dimensional spacetime, one is forced to keep only one time dimension in order not to generate scalar fields of wrong metric (ghosts) by dimensionally reducing higher-dimensional tensors.

We can now ask ourselves what values of the dimensionality D of space time allow a direct generalization of eqs. (1.2.4). It should be emphasized that the restrictions do not originate from the bosonic part of (1.2.4) which, as is evident, can be written (i.e, the corresponding Jacobi identities close) in any space time dimension, but from the presence of the spinorial charge Q. This charge is required to satisfy a covariant constraint, the Majorana condition (1.2.2), and the question can be rephrased as follows: for what values of D does an antisymmetric C, satisfying (1.2.3), exist? This question can be answered in different ways. For example, the explicit construction of Dirac matrices in $D>4$ will suffice for studying a few cases. This can be done most simply by taking direct products of the 2 x 2 hermitian matrices $(1_2, \sigma_1, \sigma_2, \sigma_3)$, where where 1_2 is the 2 x 2 unit matrix, and σ_i are the Pauli matrices. The result is that an antisymmetric C satisfying (1.2.3) can be found only for (D=2,3,4,10,11,..). In general, following ref. [10], we can note that the very existence of an *antisymmetric* matrix C satisfying (1.2.3) implies that antisymmetric products of Γ matrices

$$\Gamma_{\mu_1 \cdot\cdot \mu_n} = \Gamma_{[\mu_1} \cdot\cdot \Gamma_{\mu_n]}$$

are such that $(\Gamma_{\mu_1 \cdot\cdot \mu_n} C)$ is *symmetric* for $n = (1,2)\ mod 4$, and is antisymmetric for $n = (0,3)\ mod 4$. This gives a total of 2^D matrices, out of which $2^{[D/2]-1}(2^{[D/2]}+1)$ must be symmetric. This also leads in general to a contradiction, as the $\Gamma_{\mu_1 \cdot\cdot \mu_n}$'s can also be counted directly by looking at their index structure. The result is that *an antisymmetric C exists only for D = 2,3,4 (mod 8)*.

Indeed, the discussion following eqs. (1.2.4) indicates that the condition that an antisymmetric C exists is somewhat too strong, and can therefore be relaxed. All we need to write eq. (1.2.4f) is an object $\widehat{C}$ such that $(\Gamma^\mu \widehat{C})$ is a symmetric matrix for all values of μ. This can be certainly realized with an antisymmetric

C satisfying (1.2.3), but another possibility exists, namely a *symmetric* C' satisfying

$$(C')^{-1}\Gamma_{\mu} C' = \Gamma_{\mu}^{T} \ . \tag{1.2.7}$$

Such a C′ can indeed be found in D=0,1,2 (mod 8). This completes our list and shows that the super-Poincaré algebra (1.2.4) exists in spacetimes of dimensionality D=2,3,4,8,9,10,11,..

Going back to central charges, the point is that a higher-dimensional analog of (1.2.4f) would read

$$\{Q_{\alpha}, Q_{\beta}\} = -(\Gamma^{\mu} C)_{\alpha\beta} P_{\mu} - (\Gamma^{I} C)_{\alpha\beta} P_{I} \ , \tag{1.2.8}$$

where P_I are the generators of translations in the extra dimensions, and $(I = 4,..,D-1)$. From the four-dimensional point of view, the P_I annihilate all physical states, and commute with the four-dimensional subset of the Lorentz generators. They are thus central charges [11].

In discussing higher-dimensional analogs of (1.2.4), we have also learned that supersymmetric theories are bound to look more compact, and thus simpler, in spacetimes of higher dimensionality, a very useful supplement to our somewhat limited skills in constructing them. The reason for this is simply that, just as happens with the generators in the algebra, higher-dimensional spinors regroup several four-dimensional spinors, and higher-dimensional tensors regroup several four-dimensional tensors. Indeed, this was the original motivation for considering supersymmetric theories in spacetimes of higher dimensionality, and soon after the construction of ten- dimensional supersymmetric Yang-Mills theory was shown to yield the N=4 supersymmetric Yang-Mills theory in four dimensions [10,12] by dimensional reduction, similar steps were followed leading to the construction of N=8 supergravity [13] by dimensional reduction of supergravity in eleven dimensions [14]. In this respect, it should be noted that an early attempt to construct N=8 supergravity directly in four-dimensions [15] could not be completed because of the sheer complexity of the theory.

As is well known, the Poincaré algebra (1.2.1) can be recovered by taking a singular limit (contraction) of either of the two simple algebras

$$[P_\mu, P_\nu] = \pm \frac{J_{\mu\nu}}{R^2} \ , \tag{1.2.9a}$$

$$[P_\mu, J_{\nu\rho}] = -2i\, \eta_{\mu[\nu} P_{\rho]} \ , \tag{1.2.9b}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = 4i\, J_{[\mu[\rho}\eta_{\nu]\sigma]} \ . \tag{1.2.9c}$$

The limit corresponds to letting $R \to \infty$. The algebra (1.2.9) is denoted as de Sitter algebra if the "plus" sign is chosen, and as anti-de Sitter if the "minus" sign is chosen.

This raises the question of which simple superalgebras give rise to the super-Poincaré algebra (1.2.4) upon contraction. The answer is known and, contrary to eqs. (1.2.9), is unique. The bosonic subalgebra must correspond to choosing the "minus" sign in eq (1.2.9). It is therefore *always an anti-de Sitter algebra*. The other (anti)commutators are:

$$[Q_\alpha, P_\mu] = \frac{1}{R}(\gamma_\mu)_{\alpha\beta}\, Q_\beta \ , \tag{1.2.10a}$$

$$[Q^i_\alpha, J_{\mu\nu}] = \frac{i}{2}(\gamma_{\mu\nu})_{\alpha\beta}\, Q^i_\beta \tag{1.2.10b}$$

$$\{Q_\alpha, Q_\beta\} = -(\gamma^\mu C)_{\alpha\beta} P_\mu - \frac{1}{2R}(\gamma^\mu C)_{\alpha\beta} J_{\mu\nu} \ . \tag{1.2.10c}$$

The appearance of $J_{\mu\nu}$ on the r.h.s eq. (1.2.10b) is not surprising, because now, on account of the bosonic algebra (1.2.9), the P_μ's and $J_{\mu\nu}$'s play a similar role. We note that the closure of the new Jacobi identities introduced in enlarging (1.2.1) and (1.2.4) is an immediately recognizable property. All one needs is that the γ_μ's be a set of (anti)hermitian Dirac matrices out of which matrices $\gamma_{\mu\nu}$ reproducing the algebra of the bosonic operators can be constructed. This also applies to the super-de Sitter algebra (1.2.9)-(1.2.10), and it is what forces the "minus" sign in eq. (1.2.9a). It also fixes all the relative coefficients in (1.2.10). The new feature, as compared to the super-Poincaré algebra (1.2.4), is that there is now an additional nontrivial Jacobi identity, obtained from the (anti)commutator of three Q's. The identity in question is

$$[\{Q_\alpha, Q_\beta\}, Q_\gamma] + [\{Q_\beta, Q_\gamma\}, Q_\alpha] + [\{Q_\gamma, Q_\alpha\}, Q_\beta] = 0 \tag{1.2.11}$$

and, using (1.2.10), can be written in the form

$$\frac{1}{R}[(\gamma^\mu C)_{\alpha\beta}(\gamma_\mu)_{\gamma\delta} + (\gamma^\mu C)_{\beta\gamma}(\gamma_\mu)_{\alpha\delta} + (\gamma^\mu C)_{\gamma\alpha}(\gamma_\mu)_{\beta\delta} -$$

$$\frac{1}{2}(\gamma^{\mu\nu} C)_{\alpha\beta}(\gamma_{\mu\nu})_{\gamma\delta} - \frac{1}{2}(\gamma^{\mu\nu} C)_{\beta\gamma}(\gamma_{\mu\nu})_{\alpha\delta} - \frac{1}{2}(\gamma^{\mu\nu} C)_{\gamma\alpha}(\gamma_{\mu\nu})_{\beta\delta}] = 0 \ . \tag{1.2.12}$$

This condition is trivial in the Poincaré limit ($R \to \infty$), but needs separate and detailed investigation otherwise [16,17]. We will now show how to carry out such an analysis, in a way that directly generalizes to higher dimensions. This is essentially all one needs to study super-de Sitter algebras in arbitrary space-time dimensions. Before doing so, however, we want to remark that, by the same reasoning as for the super-Poincaré case or, equivalently, by inspection of of (1.2.10b), we see that now C must satisfy a stronger condition than it did in the super-Poincaré case. Indeed, now not only $(\Gamma_\mu C)_{\alpha\beta}$, but also $(\Gamma_{\mu\nu} C)_{\alpha\beta}$, must be symmetric matrices. Notice that from (1.2.3) it follows that

$$(\Gamma_\mu C) = (\Gamma_\mu C)^T \tag{1.2.13}$$

and, as anticipated in discussing the super-Poincaré case, the same follows from (1.2.7). We can also see that (1.2.13) implies

$$(\Gamma_\mu \Gamma_\nu C) = -(\Gamma_\nu \Gamma_\mu C)^T \quad , \tag{1.2.14}$$

and therefore not only $(\Gamma_\mu C)$, but also $(\Gamma_{\mu\nu} C)$ are symmetric matrices. However, the choice of a symmetric C′ as in eq. (1.2.7) introduces one extra "minus" sign, and the result is that $(\Gamma_{\mu\nu} C')$ is antisymmetric. Going back again to what said for the super-Poincaré algebra, we see that only the cases of D=2,3,4,10,11 are left as possibilities for writing the N=1 super-de Sitter algebra (1.2.9)-(1.2.10).

To decide which of these cases works, we must solve (1.2.12). This equation is somewhat complicated-looking, but, as we now show, is amenable to a simple treatment using Fierz identities. Consider first the case of four dimensions, and define the two quantities

$$x_1^{\alpha\beta\gamma\delta} = (\gamma^\mu C)_{\alpha\beta} (\gamma_\mu)_{\gamma\delta} + (\gamma^\mu C)_{\beta\gamma} (\gamma_\mu)_{\alpha\delta} + (\gamma^\mu C)_{\gamma\alpha} (\gamma_\mu)_{\beta\delta} \tag{1.2.15a}$$

and

$$x_2^{\alpha\beta\gamma\delta} = \frac{1}{2}[(\gamma^{\mu\nu} C)_{\alpha\beta} (\gamma_{\mu\nu})_{\gamma\delta} + (\gamma^{\mu\nu} C)_{\beta\gamma} (\gamma_{\mu\nu})_{\alpha\delta} + (\gamma^{\mu\nu} C)_{\gamma\alpha} (\gamma_{\mu\nu})_{\beta\delta}] \tag{1.2.15b}$$

The important point to notice here is that, on account of the explicit cyclic symmetrization and of the symmetry of $(\gamma_\mu C)$ and $(\gamma_{\mu\nu} C)$, x_1 and x_2 are totally symmetric in their first three indices. The other possible quantities, obtained from x_1 by replacing γ_μ with 1, $\gamma_{\mu\nu\rho\sigma}$ ($\approx \gamma_5$) and $\gamma_{\mu\nu\rho}$ ($\approx \gamma_\mu \gamma_5$), on the other hand, are all antisymmetric. It follows that Fierz identities must exist connecting just x_1 and x_2. All we need to proceed further is the Fierz identity

$$(C)^{\alpha\beta} \delta^{\gamma\delta} = \frac{1}{4} (\gamma_\mu)^{\alpha\delta} (\gamma_\mu C)^{\beta\gamma} - \frac{1}{8} (\gamma_{\mu\nu})^{\alpha\delta} (\gamma_{\mu\nu} C)^{\beta\gamma} \cdots \quad , \tag{1.2.16}$$

where the other terms can be ignored, as they are not relevant to this analysis. From this it follows that

$$(\gamma^\mu C)^{\alpha\beta} (\gamma_\mu)^{\gamma\delta} = \frac{1}{4} (\gamma_\mu \gamma_\rho \gamma^\mu)^{\alpha\delta} (\gamma^\rho C)^{\beta\gamma}$$

$$- \frac{1}{8} (\gamma_\mu \gamma_{\rho\sigma} \gamma^\mu)^{\alpha\delta} (\gamma^{\rho\sigma} C)^{\beta\gamma} + \cdots \tag{1.2.17a}$$

and

$$(\gamma^{\mu\nu} C)^{\alpha\beta} (\gamma_{\mu\nu})^{\gamma\delta} = \frac{1}{4} (\gamma_{\mu\nu} \gamma_\rho \gamma^{\mu\nu})^{\alpha\delta} (\gamma^\rho C)^{\beta\gamma}$$

$$- \frac{1}{8} (\gamma_{\mu\nu} \gamma_{\rho\sigma} \gamma^{\mu\nu})^{\alpha\delta} (\gamma^{\rho\sigma} C)^{\beta\gamma} + \cdots \quad . \tag{1.2.17b}$$

Using the results given in Appendix A, these relations can be written

$$(\gamma^\mu C)^{\alpha\beta} (\gamma_\mu)^{\gamma\delta} = - \frac{1}{2} (\gamma_\mu \gamma_\rho \gamma^\mu)^{\alpha\delta} + \cdots \tag{1.2.18a}$$

and

$$(\gamma^{\mu\nu} C)^{\alpha\beta} (\gamma_{\mu\nu})^{\gamma\delta} = - \frac{1}{2} (\gamma_{\mu\nu} \gamma_\rho \gamma^{\mu\nu})^{\alpha\delta} + \cdots \quad . \tag{1.2.18b}$$

Adding to these equations the two cyclic permutations in $(\alpha\beta\gamma)$ and using the symmetry of $(\gamma_\mu C)$ and $(\gamma_{\mu\nu} C)$ in their spinor indices yields equations for x_1 and x_2:

$$x_1^{\alpha\beta\gamma\delta} = -\frac{1}{2} x_1^{\alpha\beta\gamma\delta}$$

$$x_2^{\alpha\beta\gamma\delta} = -\frac{1}{2} x_2^{\alpha\beta\gamma\delta} \quad . \tag{1.2.19}$$

It follows that *in four dimensions* all the Jacobi identities are satisfied for the super-de Sitter algebra.

We can now ask ourselves whether the super-de Sitter algebra closes also in the other cases where a C can be defined, and especially in eleven dimensions and in ten dimensions, two cases of particular relevance in supersymmetry theory. Consider first the case of D=11. As always in spacetimes of odd dimensionality, the analogue of the Γ_5 of four dimensions is now one of the Γ matrices carrying a spacetime index. As a consequence, the eleven-dimensional Γ matrices satisfy the algebraic constraint

$$\Gamma^{a_1..a_n} = \frac{(-1)^{\frac{n(n-1)}{2}}}{(11-n)!}\, \varepsilon^{a_1..a_n b_1..b_{11-n}}\, \Gamma_{b_1..b_{11-n}} \ , \qquad (1.2.20)$$

and only Γ_μ, $\Gamma_{\mu\nu}$, $\Gamma_{\mu\nu\rho}$, $\Gamma_{\mu\nu\rho\sigma}$ and $\Gamma_{\mu\nu\rho\sigma\tau}$ are independent. Moreover, it follows from the properties of C in eq. (1.2.3) that $\Gamma_\mu C$, $\Gamma_{\mu\nu} C$ and $\Gamma_{\mu\nu\rho\sigma\tau} C$ are symmetric, whereas $\Gamma_{\mu\nu\rho} C$ and $\Gamma_{\mu\nu\rho\sigma} C$ are antisymmetric. Thus, we can now find at best identities connecting $x_1^{\alpha\beta\gamma\delta}$, $x_2^{\alpha\beta\gamma\delta}$ and $x_5^{\alpha\beta\gamma\delta}$, where x_1 and x_2 are defined in eqs. (1.2.14), and

$$x_5^{\alpha\beta\gamma\delta} = \frac{1}{5!}[(\Gamma^{\mu_1..\mu_5} C)^{\alpha\beta} (\Gamma_{\mu_1..mu_5} C)^{\gamma\delta} + (\Gamma^{\mu_1..\mu_5} C)^{\beta\gamma} (\Gamma_{\mu_1..\mu_5} C)^{\alpha\delta} +$$

$$(\Gamma^{\mu_1..\mu_5} C)^{\gamma\alpha} (\Gamma_{\mu_1..\mu_5} C)^{\beta\delta}] \ . \qquad (1.2.21)$$

Correspondingly, eqs. (1.2.17) are replaced by

$$(\Gamma^\mu C)^{\alpha\beta} (\Gamma_\mu)^{\gamma\delta} = \frac{1}{32} (\Gamma^\mu \Gamma^\rho \Gamma_\mu)^{\alpha\delta} (\Gamma_\rho C)^{\beta\gamma} - \frac{1}{64} (\Gamma^\mu \Gamma^{\rho\sigma} \Gamma_\mu)^{\alpha\delta} (\Gamma_{\rho\sigma} C)^{\beta\gamma} +$$

$$\frac{1}{32 \cdot 5!} (\Gamma^\mu \Gamma^{\rho_1..\rho_5} \Gamma_\mu)^{\alpha\delta} (\Gamma_{\rho_1..\rho_5} C)^{\beta\gamma} + \ \cdots \ , \qquad (1.2.22a)$$

$$(\Gamma^{\mu\nu} C)^{\alpha\beta} (\Gamma_{\mu\nu})^{\gamma\delta} = \frac{1}{32} (\Gamma^{\mu\nu} \Gamma^\rho \Gamma_{\mu\nu})^{\alpha\delta} (\Gamma_\rho C)^{\beta\gamma}$$

$$- \frac{1}{64} (\Gamma^{\mu\nu} \Gamma^{\rho\sigma} \Gamma_{\mu\nu})^{\alpha\delta} (\Gamma_{\rho\sigma} C)^{\beta\gamma} +$$

$$\frac{1}{32 \cdot 5!} (\Gamma^{\mu\nu} \Gamma^{\rho_1..\rho_5} \Gamma_{\mu\nu})^{\alpha\delta} (\Gamma_{\rho_1..\rho_5} C)^{\beta\gamma} + \ \cdots \ , \qquad (1.2.22b)$$

and

$$(\Gamma^{\mu_1..\mu_5} C)^{\alpha\beta} (\Gamma_{\mu_1..\mu_5})^{\gamma\delta} = \frac{1}{32} (\Gamma^{\mu_1..\mu_5} \Gamma^\rho \Gamma_{\mu_1..\mu_5})^{\alpha\delta} (\Gamma_\rho C)^{\beta\gamma}$$

$$- \frac{1}{64} (\Gamma^{\mu_1..\mu_5} \Gamma^{\rho\sigma} \Gamma_{\mu_1..\mu_5})^{\alpha\delta} (\Gamma_{\rho\sigma} C)^{\beta\gamma} +$$

$$\frac{1}{32 \cdot 5!} (\Gamma^{\mu_1..\mu_5} \Gamma^{\rho_1..\rho_5} \Gamma_{\mu_1..\mu_5})^{\alpha\delta} (\Gamma_{\rho_1..\rho_5} C)^{\beta\gamma} + \ \cdots \ . \qquad (1.2.22c)$$

Using the results in Appendix A and adding the two cyclic permutations in $(\alpha\beta\gamma)$, these equations can be converted into as many relations between x_1, x_2 and x_5:

$$4x_1 + 7x_2 + x_5 = 0 \ ,$$

$$35x_1 + 13x_2 - 5x_5 = 0 \ ,$$

$$x_1 - x_2 + x_5 = 0 \ . \tag{1.2.23}$$

The problem is that, in contrast to what happens in four dimensions, these relations are not enough to set x_1 and x_2 simultaneously to zero, because they are linearly dependent. They do provide one relation between x_1 and x_2,

$$5x_1 + x_2 = 0 \ , \tag{1.2.24}$$

but this does not close the super-de Sitter algebra, as eq. (1.2.12) demands that x_1 be equal to x_2. Consequently, the super-de Sitter algebra does not generalize to eleven dimensions.

Next we consider the case of ten dimensions. Now there is no "a priori" algebraic dependence of the Γ matrices, as the space time has even dimensionality. However, in ten dimensions one can simultaneously impose both the Majorana and the Weyl conditions†. It follows that, when working with the Weyl projected Γ matrices

$$\Gamma_\mu = \frac{1}{2}(1 \pm \Gamma_{11})\Gamma_\mu \ , \tag{1.2.25}$$

one finds that, as in D=11, only Γ_μ, $\Gamma_{\mu\nu}$, $\Gamma_{\mu\nu\rho}$, $\Gamma_{\mu\nu\rho\sigma}$ and $\Gamma_{\mu\nu\rho\sigma\tau}$ are independent. The problem, however, is that now the presence of the second term in eq.

† In general [10], this can be done for D=2 (mod 8).

(1.2.10c) is incompatible with the Weyl condition that the Majorana charge satisfies, and correspondingly *the super-de Sitter algebra does not exist in ten dimensions*. The same arguments rule out the super-de Sitter algebra in the case of two dimensions as well, as there also one can define Majorana-Weyl spinors. Finally, in three dimensions the super-de Sitter algebra does exist, as it can be seen directly that x_1 equals x_2, without the need of any Fierz transformation.

Summarizing, we have seen that the N=1 super-Poincaré algebra exists in 2,3, up to 4 dimensions. It should be noted that the cases of D=10 and D=11 are very different. In D=10, as we have seen, the super-Poincaré algebra does not have a simple extension, just because this is incompatible with the Weyl property of the spinorial charge Q. On the other hand, in eleven dimensions a simple extension of the super Poincaré algebra does exist. It is just more complicated than the de Sitter algebra (1.2.10), and is obtained by adjoining to P_μ and $J_{\mu\nu}$ extra bosonic generators grouped into a fifth-rank antisymmetric tensor of SO(1,10), $G_{\alpha_1..\alpha_5}$. The P_μ, $J_{\mu\nu}$ and $G_{\alpha_1..\alpha_5}$ then generate the Sp(32) algebra. We thus obtain the (anti)commutators of the super-de Sitter algebra, with an extra term in the $\{Q,Q\}$ anticommutator, which now reads

$$\{Q_\alpha, Q_\beta\} = -(\Gamma^\mu C)_{\alpha\beta} P_\mu - \frac{i}{2R}(\Gamma^{\mu\nu} C)_{\alpha\beta} J_{\mu\nu}$$

$$- \frac{1}{5!\,R}(\Gamma^{\mu_1..\mu_5} C)_{\alpha\beta} G_{\mu_1..\mu_5} , \tag{1.2.26a}$$

In addition, there are the following extra commutators involving G:

$$[P_\mu, G_{\alpha_1..\alpha_5}] = \frac{-i}{5!\,R}\,\varepsilon_{\mu\alpha_1..\alpha_5}{}^{\beta_1..\beta_5} G_{\beta_1..\beta_5} , \tag{1.2.26b}$$

$$[J_{\mu\nu}, G_{\alpha_1..\alpha_5}] = 10\,i\, G_{[\mu[\alpha_1..\alpha_4}\eta_{\nu]\alpha_5]} , \tag{1.2.26c}$$

$$[G_{\alpha_1..\alpha_5}, G_{\beta_1..\beta_5}] = -i\,R\,\varepsilon_{\alpha_1..\alpha_5\beta_1..\beta_5\mu} P^\mu - i\,5\cdot 5!\,J_{[\alpha_1[\beta_1}\eta_{\alpha_2\beta_2}..\eta_{\alpha_5]\beta_5]}$$

$$+ \frac{5}{3} i \, G_{\gamma_1 \cdot\cdot \gamma_5} \varepsilon^{\gamma_1 \cdot\cdot \gamma_5 [\alpha_1 \alpha_2 \alpha_3 [\beta_1 \beta_2 \beta_3} \eta^{\alpha_4 \beta_4} \eta^{\alpha_5] \beta_5]} \quad . \tag{1.2.26d}$$

This is the Osp(1|32) algebra, and is the minimal simple extension of the super-Poincaré algebra in eleven dimensions. To prove closure, one proceeds as in the super-de Sitter case. The only nontrivial Jacobi identity is the one involving three Q's. One can then use the identities derived above to show that closure follows directly from the third of eqs. (1.2.23).

1.3. Particle representations of the super-Poincaré algebra

The particle representations of the super-Poincaré algebra (1.2.4) can be studied using a remarkably simple method, originally due to Salam and Strathdee [18], and later generalized by Gell-Mann and Ne'eman [19] to the more complicated case of N-extended super-Poincaré algebras. The idea is to reduce the problem to the simpler one of studying the particle representations of the Poincaré algebra, the solution of which has long been known. In this last case we know that it is necessary to distinguish between massless states ($P^2 = 0$) and massive states ($P^2 = m^2 > 0$). The states are classified picking a canonical momentum, say $P^\mu = (1001)$ for the massless case, and $P^\mu = (1000)$ for the massive case, and considering the subgroup of the Lorentz group which leaves the canonical momentum invariant (the little group of the canonical momentum), which is SO(2) in the massless case, and SO(3) in the massive case. Then, given one state, applying to it the operators of the little group has the effect of completing it into an irreducible representation of the little group. This leads to the familiar chains of $(2J + 1)$ states in the massive case and to the familiar sets of one (or two, if CPT is enforced) helicity states in the massless case. The crucial point is that, as in the Poincaré case, the operator $P^\mu P_\mu$ commutes with all the generators in the algebra, and is therefore a Casimir invariant. This is what allows one to study representations of the super-Poincaré algebra by fixing canonical momenta and by looking at representations of the corresponding little groups.

In order to extend the discussion to the case of the super-Poincaré algebra, it is also necessary to distinguish between massive and massless states. Consider first massive states. Then the nontrivial (anti)commutators in the algebra are

$$[Q_\alpha, \vec{J}] = \frac{i}{2} \vec{\sigma}_{\alpha\beta} Q_\beta \qquad (1.3.1a)$$

and

$$\{Q_\alpha, Q_\beta\} = -(\gamma^0 C)_{\alpha\beta} \ . \tag{1.3.1b}$$

In order to proceed further, we need an explicit representation of the γ matrices, and the standard one

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ; \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{1.3.2}$$

will suffice. The charge conjugation matrix is then:

$$C = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \ , \tag{1.3.3}$$

and writing

$$Q = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} \ , \tag{1.3.4}$$

one sees that the Majorana condition $Q = C\,\overline{Q}^T$ results in the two conditions

$$S_3 = -S_2^\dagger \quad ; \qquad S_4 = S_1^\dagger \ . \tag{1.3.5}$$

It follows that (1.3.1b) can be written

$$\{S_\alpha, S^\dagger{}_\beta\} = \delta_{\alpha\beta} \qquad (\alpha,\beta = 1,2) \ , \tag{1.3.6}$$

a very suggestive expression, as it closely resembles the algebra of creation and annihilation operators for a system of fermions with two states available. Together with this relation, we have (1.3.1a), which in terms of the S's is written

$$[S_\alpha, \vec{J}] = \tfrac{1}{4}\vec{\sigma}_{\alpha\beta} S_\beta \ , \tag{1.3.7}$$

with $\vec{\sigma}$ a Pauli matrix, and the angular momentum commutation relations between the J's:

$$[\vec{J},\vec{J}] = i\vec{J} \ . \tag{1.3.8}$$

The algebra (1.3.6)-(1.3.8) is still somewhat complicated-looking, essentially because of (1.3.7). If one could diagonalize it, then its representation could be studied very simply. This can indeed be achieved by abandoning the J's in favor of the operators

$$\vec{j} = \vec{J} - \frac{1}{2} S^\dagger \vec{\sigma} S \tag{1.3.9}$$

which, on account of (1.3.6) and (1.3.8), also satisfy the angular momentum algebra

$$[\vec{j},\vec{j}] = i\vec{j} \ . \tag{1.3.10}$$

Moreover, the relative factor between the two terms in eq. (1.3.9) ensures that the $\vec{j}$'s commute with the S's. It follows that the representations of the algebra (1.3.6), (1.3.7) and (1.3.10) are simply obtained starting from the representations of the SO(3) algebra (1.3.10) and extending them to realize (1.3.6). If we start with a chain of states with $-j \leq j_3 \leq j$ which are a Clifford vacuum, i.e which satisfy $S^{\dagger a}|jj_3\rangle = 0$, we can construct from a state $|jj_3\rangle$ the four states

$$|j\,j_3 n_1 n_2\rangle = S_1^{n_1} S_2^{n_2}\,|j\,j_3\rangle \qquad (n_1,n_2 = 0,1) \ , \tag{1.3.11}$$

which are all eigenstates of j^2 with eigenvalue $j(j+1)$ and of j_3 with eigenvalue j_3. The problem is that $\vec{j}$ is not an easily recognizable object, as it contains the angular momentum generators $\vec{J}$ together with additional fermionic bilinears. The simplest way of disentangling the angular momentum content of the chains is to use the third component of (1.3.9) to extract the eigenvalues of j_3 for the four states (1.3.11). The result is:

$$J_3\,|j\,j_3 0 0\rangle = j_3\,|j\,j_3 0 0\rangle$$

$$J_3\,|j\,j_3\,1\,0> = (j_3 - \tfrac{1}{2})\,|j\,j_3\,1\,0>$$

$$J_3\,|j\,j_3\,0\,1> = (j_3 + \tfrac{1}{2})\,|j\,j_3\,0\,1>$$

$$J_3\,|j\,j_3\,1\,1> = j_3\,|j\,j_3\,1\,0> \quad . \tag{1.3.12}$$

Looking at the highest j_3 value then tells us that we have four chains, two of which have opposite parity (as the product $S_1 S_2$ has negative parity) and angular momentum j. The remaining two chains have angular momentum $(j + \frac{1}{2})$ and $(j - \frac{1}{2})$ respectively. We wish to emphasize that ***all the states in a given supermultiplet have the same mass m.*** Moreover, in a given multiplet there are equal numbers of Bose and Fermi degrees of freedom. The multiplet with lowest spins contains a scalar, a pseudoscalar and a Majorana spinor. The corresponding field theory model is the massive Wess- Zumino model.

We now turn to the case of massless states. This appears to be more interesting, because exact gauge symmetries force particles of spin ≥ 1 to be exactly massless, and with them all their superpartners must also be massless †. In this case the canonical momentum is $P^\mu = (1001)$, and (1.2.4f) becomes

$$\{Q_\alpha, Q_\beta\} = -\,((\gamma^0 - \gamma^3)\,C)_{\alpha\beta} \qquad (\alpha,\beta = 1,..,4) \quad . \tag{1.3.13}$$

Rather than dealing with the four components of Q, as defined in eq. (1.3.4), it is now more convenient to work with

$$s_1 = S_1$$

$$s_2 = S^\dagger{}_1 + S_2 \quad , \tag{1.3.14}$$

as they satisfy the easily recognizable algebra

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†See, however, the discussion in section 2.4.

$$\{s_1,s^\dagger{}_1\} = 1 \ , \tag{1.3.15a}$$

$$\{s_2,s^\dagger{}_2\} = 0 \ , \tag{1.3.15b}$$

$$\{s_1,s_2\} = 0 \ , \tag{1.3.15c}$$

$$\{s_1,s^\dagger{}_2\} = 0 \ . \tag{1.3.15d}$$

Consequently the two-element Clifford algebra that we had in the massive case is now a one-element Clifford algebra, together with a one-element Grassman algebra. Since states of positive norm can only represent trivially the Grassman algebra (1.3.15b), all we are left with is one fermionic creation operator, say s_1, and the corresponding annihilation operator $s^\dagger{}_1$. Given one irreducible representation of the Poincaré algebra, which in the massless case we are now dealing with consists of only one state, we form irreducible representations of the super- Poincaré algebra by adjoining it with another state differing from it in helicity by $\frac{1}{2}$ unit. Consequently *massless irreducible representations contain couples of states of adjacent helicity*. Enforcing CPT then leads to add to a couple of states of helicities $(j, j+\frac{1}{2})$ two more states of helicities $(-j, -j-\frac{1}{2})$. The $(2,\frac{3}{2})$ multiplet is perhaps the most remarkable one, as it leads to the supersymmetric generalization of Einstein's general theory of Relativity, N=1 supergravity [20]. The $(1,\frac{1}{2})$ multiplet leads to the N=1 supersymmetric Yang-Mills theory, where an adjoint set of Majorana (or Weyl) spinors is minimally coupled to a Yang-Mills boson.

The case of extended super-Poincaré algebras was also studied a long time ago [19]. There are now more fermionic charges, and therefore more fermionic creation and annihilation operators. As a consequence, one gets longer multiplets. There are also upper bounds on the number of supersymmetries for a given maximum helicity. In particular, maximum helicity $\frac{1}{2}$ (Wess-Zumino model) is possible only for N=1 and N=2 supersymmetry. Maximum helicity 1 is possible up to N=4 supersymmetry, and maximum helicity 2 is possible up to N=8 supersymmetry. The particle multiplets for supersymmetric Yang-Mills and for supergravity are summarized below:

SupersymmetricYang –Mills

spin	N=0	N=1	N=2	N=3	N=4
1	1	1	1	1	1
$\frac{1}{2}$	-	1	2	4	4
0	-	-	2	6	6

Supergravity

spin	N=0	N=1	N=2	N=3	N=4	N=5	N=6	N=7	N=8
2	1	1	1	1	1	1	1	1	1
$\frac{3}{2}$	-	1	2	3	4	5	6	8	8
1	-	-	1	3	6	10	16	28	28
$\frac{1}{2}$	-	-	-	1	4	11	26	56	56
0	-	-	-	-	2	10	30	70	70

We now want to show how a simple generalization of the arguments presented above allows to study the particle representations of higher-dimensional super-Poincaré algebras. In particular, we will discuss the massless representations of the D=11 super-Poincaré algebra [16], thus showing why D=11 is an upper bound for the construction of supergravity theories. The starting point is the super-Poincaré algebra (1.2.4), rewritten for the case of D=11, so that the vector indices now run over the eleven values (0,..,10), and the spinorial charge Q has 32, rather than 4, components. The next step is then picking a canonical momentum, which we take to be $P^{\mu} = (1001..0)$. The little group of P^{μ} is the SO(9) subgroup of the eleven-dimensional Lorentz group that leaves our P^{μ} invariant. Massless states in D=11 are therefore classified by representations of this transverse SO(9) group, and the relevant part of the super-Poincaré algebra now becomes

$$[Q_{\alpha}, J_{IJ}] = \frac{i}{2} (\Gamma_{IJ})_{\alpha\beta} Q_{\beta} \ , \qquad (1.3.16a)$$

$$[J_{IJ}, J_{KL}] = 4i \, J_{[I[K} \delta_{J]L]} \qquad (I,J \neq 0,3) \ , \qquad (1.3.16b)$$

$$\{Q_{\alpha}, Q_{\beta}\} = -((\Gamma^0 - \Gamma^3) \, C)_{\alpha\beta} \ . \qquad (1.3.16c)$$

As in the four-dimensional case, the next step consists in picking an explicit representation of the Dirac algebra, and subjecting the charge Q to the Majorana condition. We wish to emphasize that now the Γ matrices are rather complicated objects, as they are 32-dimensional. Choosing a convenient representation of the eleven-dimensional Dirac algebra can therefore simplify matters considerably. The main feature a convenient representation must have is to split simply when specializing to the SO(9) subgroup in eq. (1.3.16b). Such a representation is discussed in Appendix B.

Using the explicit form of the Γ matrices in Appendix B, we see that the Majorana condition forces the charge Q to have the form

$$Q = \sqrt{2p} \begin{pmatrix} U \\ D \\ -D^* \\ U^* \end{pmatrix} , \tag{1.3.17}$$

where U and D are eight-component vectors. Eq. (1.3.16c) then implies the following anticommutation relations:

$$\{U_\alpha, U_\beta\} = \delta_{\alpha\beta}$$

$$\{U,U\} = \{U,D\} = \{U,D^*\} = \{D,D\} = \{D,D^*\} = 0 \ . \tag{1.3.18}$$

It follows that the D's satisfy a Grassman algebra, and therefore annihilate all particle states, which have positive norm. Restricting (1.3.16a) to such positive norm states gives

$$<\psi'|[Q_\alpha, J_{IJ}]|\psi> = \frac{i}{2}(\Gamma_{IJ})_{\alpha\beta} \begin{pmatrix} <\psi'|U|\psi> \\ 0 \\ 0 \\ <\psi'|U^\dagger|\psi> \end{pmatrix} . \tag{1.3.19}$$

Defining

$$u_\alpha = \begin{pmatrix} U \\ U^* \end{pmatrix} \tag{1.3.20}$$

and using the results in Appendix B, we can write eqs. (1.3.16) in terms of the 16 x 16 matrices $\widetilde{\Gamma}_{IJ}$:

$$\{U_\alpha, U_\beta^\dagger\} = \delta_{\alpha\beta} \ ; \qquad \{U_\alpha, U_\beta\} = 0 \tag{1.3.21a}$$

$$[u_\alpha, J_{IJ}] = \frac{i}{2}(\widetilde{\Gamma}_{IJ})_{\alpha\beta} u_\beta \ , \tag{1.3.21b}$$

$$[J_{IJ}, J_{KL}] = 4i\, J_{[I[K}\delta_{J]L]} \ . \tag{1.3.21c}$$

In analogy with what was done in the D=4 case, in order to disentangle the algebra (1.3.21) we define the operator

$$\widetilde{J}_{IJ} = J_{IJ} + i\, y\, u^T \widetilde{C}' \widetilde{\Gamma}_{IJ} u \ . \tag{1.3.22}$$

Here $\widetilde{C}'$ is obtained suppressing the central 16 rows and the central 16 columns of C', a symmetric matrix satisfying

$$C' \Gamma_I (C')^{-1} = \Gamma^{\dagger}{}_I \qquad (I \neq 0,3) \ , \tag{1.3.23}$$

which in our representation takes the form

$$C' = i \begin{pmatrix} & & & 1_8 \\ & & -1_8 & \\ & -1_8 & & \\ 1_8 & & & \end{pmatrix} \tag{1.3.24}$$

Thus,

$$\widetilde{C}' = i \begin{pmatrix} 0 & 1_8 \\ 1_8 & 0 \end{pmatrix} \ . \tag{1.3.25}$$

Demanding that (1.3.21c) hold for the $\widetilde{J}_{ij}$ then gives $y = \frac{1}{2}$, and the operators

$$\widetilde{J}_{IJ} = J_{IJ} + \frac{i}{2} u^T (\widetilde{C}') \widetilde{\Gamma}_{IJ} u \quad (I,J \neq 0,3) \tag{1.3.26}$$

generate an SO(9) algebra, which we denote as $\widetilde{SO}(9)$, to emphasize that its generators are the $\widetilde{J}_{IJ}$, not the J_{IJ}. They also commute with the u's and u †'s. The algebra (1.3.21) correspondingly takes the simpler form

$$\{U_\alpha, U^\dagger_\beta\} = \delta_{\alpha\beta} \quad ; \quad \{U_\alpha, U_\beta\} = 0$$

$$[u_\alpha, \widetilde{J}_{IJ}] = 0 \ ; \ [\widetilde{J}_{IJ}, \widetilde{J}_{KL}] = 4i\, \widetilde{J}_{[I[K}\delta_{J]L]} \ , \tag{1.3.27}$$

and its irreducible representations, in analogy with the four-dimensional case, are constructed starting from an irreducible representation of the bosonic algebra, and extending it by applying to its states the U's and the U†'s. To be more precise, one starts from a representation of the bosonic algebra which is a Clifford vacuum, i.e. is such that

$$U_\alpha\,|v> = 0 \qquad (\alpha = 1,..,8) \tag{1.3.28}$$

on all its states $|v>$, and applies to all states all the independent nonvanishing products of the U†'s.

As in the four-dimensional case, the main problem is that the physical meaning of the $\tilde{J}_{IJ}$ is not clear. Again, however, we can classify the states simply by looking at their helicity content, i.e. by looking at their J_{12} eigenvalue. The procedure we have discussed here is exhaustive and, in principle can lead to construct all the irreducible representations of the eleven-dimensional super-Poincaré algebra. There is one case, however, which deserves some attention, and which we will now describe in some detail. We want to study what the smallest irreducible representation is. Since the fermionic operators just enlarge representations of the bosonic algebra by a factor 2^8, it follows that the smallest representation is constructed starting from a state $|v>$ which is an $\tilde{SO}(9)$ singlet, i.e such that

$$\tilde{J}_{IJ}\,|v> = 0 \qquad (I,J \neq 0,3)\ . \tag{1.3.29}$$

Then the helicity content of this state is obtained by simply noting that $\tilde{J}_{12}\,|v> = 0$, and $U_\alpha\,|v> = 0$. Indeed, these two conditions imply

$$J_{12}\,|v> \equiv -\frac{i}{2}u^T(\tilde{C}')\tilde{\Gamma}_{12}u =$$

$$= \frac{1}{4}((U^{*\alpha})(U^\alpha) - (U^\alpha)(U^{*\alpha}))\,|v> = -2\,|v>\ . \tag{1.3.30}$$

The choice $U^{\alpha}|v> = 0$ thus leads to a Clifford vacuum $|v>$ which has helicity -2. The other states of the smallest irreducible representation then follow by applying to $|v>$ the U^{*}'s. This leads to the following set of states:

states	helicity	multiplicity
$\|v>$	-2	1
$U^{*a_1}\|v>$	$-\frac{3}{2}$	8
$U^{*[a_1}U^{*a_2]}\|v>$	-1	28
$U^{*[a_1}U^{*a_2}U^{*a_3]}v>$	$-\frac{1}{2}$	56
$U^{*[a_1}U^{*a_2}U^{*a_3}U^{*a_4]}\|v>$	0	70

together with the CPT conjugates of the states of nonzero helicity. This representation is the only one we can find in eleven dimensions with maximum helicity 2, and has the helicity content of N=8 supergravity. As is well known, the corresponding supergravity theory has actually been constructed by Cremmer, Julia and Scherk [14]. Before concluding this section, we want to show how the group theory analysis presented before leads directly to the set of fields corresponding to a given multiplet. This example is preliminary to our discussion of supergravity in eleven dimensions in the next section, but is also meant to illustrate a general point: the group theory analysis of the multiplets practically amounts to writing down the linearized lagrangian field theory for a given multiplet. We start by noting that it is more natural to describe the states of a multiplet in terms of SO(9) representations, rather than in terms of their helicity content. Consider first the 128 Bose states. The need to accommodate one and only one helicity 2 state leads necessarily to the O(9) representation corresponding to the traceless part of the Young tableau , which corresponds to a two-index symmetric and traceless tensor of O(9) and describes 44 Bose degrees of freedom with helicity content $1\cdot(\lambda=\pm2) + 7\cdot(\lambda=\pm1) + 28\cdot(\lambda=0)$. The remaining $21\cdot(\lambda=\pm1) + 42\cdot(\lambda=0)$ Bose degrees of freedom can then only be described by either one of the two O(9) representations:

 or 

each one of which decomposes under O(2) into: $21\cdot(\lambda=\pm1) + 42\cdot(\lambda=0)$. The tensors having the transformation properties described by the above Young tableaux are the totally antisymmetric $A_{[ijk]}$ and $A_{[ijklmn]}$, respectively.

Thus, as far as the Bose states are concerned, they can only be described by the fields

$$g_{\mu\nu} = g_{\nu\mu} \quad \text{and} \quad A_{[\mu\nu\rho]}$$

or by

$$g_{\mu\nu} = g_{\nu\mu} \quad \text{and} \quad A_{[\mu\nu\rho\sigma\tau]}$$

For the Fermi degrees of freedom the only possible field choice is a Majorana vector-spinor ψ_μ, because the defining ($\underline{9}$) and the spinor ($\underline{16}$) representations of O(9) decompose under O(2) as follows:

$$\underline{9} \sim 1\cdot(\lambda=\pm1) + 7\cdot(\lambda=0)$$

$$\underline{16} \sim 8 \cdot (\lambda = \pm \tfrac{1}{2})$$

so that the product $\underline{9} \times \underline{16}$ contains exactly the helicity states we want, after we gauge away (using the gauge invariance of the 1 + 10-dimensional form of the Rarita-Schwinger Lagrangian) the $8 \cdot (\lambda = \pm \frac{1}{2})$ extra degrees of freedom.

Any other choice of fermion fields like, for instance, the generalization of the field $(\psi_{(\alpha\beta\gamma)}(x), \psi_{(\dot{\alpha}\dot{\beta}\dot{\gamma})}(x))$ of 1 + 3 dimensions, leads to too many helicity -$\frac{3}{2}$ or helicity -$\frac{1}{2}$ states.

1.4. Field theory models with simple supersymmetry

In this section† we want to describe three field theory models possessing invariance under simple supersymmetry in four dimensions, the Wess-Zumino model, the N=1 supersymmetric Yang-Mills theory, and the N=1 supergravity theory. Then we will discuss supergravity in eleven dimensions, showing in particular its uniqueness.

Let us start by considering the simplest supersymmetric model in four dimensions. It is a free action for a scalar, a pseudoscalar and a Majorana spinor, all massless. It was the model that first showed four- dimensional supersymmetry, and is known as the Wess-Zumino model [22].

The action is

$$S = \int d^4x \, \left\{ \, \frac{1}{2} A \Box A + \frac{1}{2} B \Box B - \frac{i}{2} \bar{\lambda} \gamma^{\mu} \partial_{\mu} \lambda \, \right\} \, , \tag{1.4.1}$$

where A is a scalar and B is a pseudoscalar, and is invariant under the global supersymmetry transformations:

$$\begin{aligned} \delta A &= i \, \bar{\varepsilon} \lambda \ , \\ \delta B &= -i \, \bar{\varepsilon} \gamma_5 \lambda \ , \\ \delta \lambda &= \partial\!\!\!/ \, (A + \gamma_5 B) \, \varepsilon \ , \end{aligned} \tag{1.4.2}$$

with ε a constant Majorana spinor. Indeed, varying S in eq. (1.4.1) gives, apart from a total divergence,

$$\delta S = \int d^4x \, \left\{ i \, (\bar{\varepsilon} \lambda) \Box A - (\bar{\varepsilon} \gamma_5 \lambda) \Box B - i \, \bar{\lambda} \Box (A + i \, \gamma_5 B) \, \varepsilon \right\} \ , \tag{1.4.3}$$

which vanishes on account of the symmetries of the spinor matrix elements

† We use the Minkowski metric (-+..+) throughout.

induced by the Majorana property of λ. Before proceeding to introduce supersymmetric interactions in this model, we want to compute the commutator of two supersymmetry transformations on the fields. We start by computing the commutator on A, and write

$$\delta_2 \delta_1 A = i \bar{\varepsilon}_1 \not{\partial} \, (A + i \gamma_5 B) \varepsilon_2 \ . \tag{1.4.4}$$

Subtracting the same expression with ε_1 and ε_2 interchanged then gives

$$[\delta_2, \delta_1] A = 2i \, (\bar{\varepsilon}_1 \gamma^\mu \varepsilon_2) \, \partial_\mu A \ . \tag{1.4.5}$$

Repeating the same exercise for B gives the same answer, i.e.

$$[\delta_2, \delta_1] B = 2i \, (\bar{\varepsilon}_1 \gamma^\mu \varepsilon_2) \, \partial_\mu B \ . \tag{1.4.6}$$

The situation is different, however, when we go on to compute the commutator of two supersymmetries on the spinor λ. Now we get:

$$\delta_2 \delta_1 \lambda = i \not{\partial} \, \left\{ (\bar{\varepsilon}_2 \lambda) - \gamma_5 (\bar{\varepsilon}_2 \gamma_5 \lambda) \right\} \varepsilon_1 \ , \tag{1.4.7}$$

and antisymmetrizing in 1 and 2 gives

$$[\delta_2, \delta_1] \lambda = i \not{\partial} \, \left\{ (\bar{\varepsilon}_2 \lambda) \varepsilon_1 - (\bar{\varepsilon}_1 \lambda) \varepsilon_2 - (\bar{\varepsilon}_2 \gamma_5 \lambda) \gamma_5 \varepsilon_1 + (\bar{\varepsilon}_1 \gamma_5 \lambda) \gamma_5 \varepsilon_2 \right\} \tag{1.4.8}$$

To write this quantity in a form similar to (1.4.5), we must perform a Fierz transformation to bring ε_1 and ε_2 together. To this end, we note that, since (1.4.7) is manifestly antisymmetric, only the γ_μ and $\sigma_{\mu\nu}$ terms will enter the Fierz expansion. The result is

$$[\delta_2, \delta_1] \lambda = 2i \, (\bar{\varepsilon}_1 \gamma^\mu \varepsilon_2) \, \partial_\mu \lambda - i \, (\bar{\varepsilon}_1 \gamma^\mu \varepsilon_2) \gamma_\mu \not{\partial} \, \lambda \ , \tag{1.4.9}$$

and is actually simpler, as it contains only γ_μ terms. The first term is clearly the same as the corresponding terms in eqs. (1.4.4) and (1.4.5). On the other hand, the second term in (1.4.9) has no analogue in the case of the bosons. It is proportional to the equation of motion of the spinor, and therefore disappears altogether for an on-shell spinor. This illustrates a recurrent feature of supersymmetric models, the nonclosure of the supersymmetry algebras on some of the fields off the mass-shell. The discussion of the particle representations of the super-Poincare' algebra given in section 3 also tells us why this happened. Off-shell the Majorana spinor λ describes four degrees of freedom, rather than two, and equality of the numbers of Bose and Fermi degrees of freedom can be achieved only at the cost of introducing extra nonpropagating degrees of freedom (auxiliary fields) that disappear altogether on the mass shell. To attain equality of Bose and Fermi degrees of freedom in our case we need extra nonpropagating degrees of freedom with an excess of two bosons over fermions. To determine the auxiliary fields, we perform one more Fierz transformation on the second term in eq. (1.4.9), and rewrite (1.4.9) in the equivalent form

$$[\delta_2,\delta_1]\lambda = 2i(\bar{\varepsilon}_1\gamma^\mu\varepsilon_2)\partial_\mu\lambda + i\,(((\varepsilon_2)(\bar{\varepsilon}_1) - (\gamma_5\varepsilon_2)(\bar{\varepsilon}_1\gamma_5)) - (1\rightleftarrows 2)) \; . \qquad (1.4.10)$$

The structure of the last two terms then suggests the solution to the problem. We modify the transformation of λ and write:

$$\delta\lambda = F\varepsilon + \gamma_5 G\varepsilon + \not{\partial}\,(A - \gamma_5 B)\varepsilon \; , \qquad (1.4.11a)$$

$$\delta F = \bar{\varepsilon}\not{\partial}\lambda \; , \qquad (1.4.11b)$$

$$\delta G = \bar{\varepsilon}\gamma_5\not{\partial}\lambda \; . \qquad (1.4.11c)$$

The transformations of A and B, on the other hand, remain those in eqs. (1.4.2). Then the new terms we introduce in the commutator on λ cancel the second term in eq. (1.4.10), and lead to

$$[\delta_2,\delta_1]\lambda = 2i\,(\bar{\varepsilon}_2\gamma^\mu\varepsilon_1)\,\partial_\mu\lambda \ , \tag{1.4.12}$$

even off the mass shell. Furthermore, one can show that

$$[\delta_2,\delta_1]\,F = 2i\,(\bar{\varepsilon}_2\gamma^\mu\varepsilon_1)\,\partial_\mu F \ ,$$

$$[\delta_2,\delta_1]\,G = 2i\,(\bar{\varepsilon}_2\gamma^\mu\varepsilon_1)\,\partial_\mu G \ . \tag{1.4.13}$$

The action itself is invariant under the modified transformation laws (1.4.11) provided we add nonpropagating kinetic terms for F and G, and write

$$S = \int d^4x \left\{ \tfrac{1}{2}A\Box A + \tfrac{1}{2}B\Box B - \frac{i}{2}\bar{\lambda}\not{\partial}\,\lambda + \frac{F^2}{2} + \frac{G^2}{2} \right\} . \tag{1.4.14}$$

One can also introduce supersymmetric interactions in this model. This requires adjusting the relative strengths of spinor-scalar couplings and scalar self-couplings, and modifying the on-shell supersymmetry transformations (1.4.2) by the addition of nonlinear terms.

The next model we wish to consider is the N=1 supersymmetric Yang-Mills theory. It describes the interactions of an adjoint multiplet of Majorana (or Weyl) spinors, minimally coupled to Yang-Mills bosons. The action is:

$$S = \int d^4x \left\{ -\frac{1}{4}F^a_{\mu\nu}\,F^{a\mu\nu} - \frac{i}{2}\bar{\lambda}^a\,(\not{\nabla})^{ab}\,\lambda^b \right\} , \tag{1.4.15}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,f^{abc}\,A^b_\mu\,A^c_\nu$$

$$(\nabla_\mu)^{ab} = \delta^{ab}\,\partial_\mu - g\,f^{abc}\,A^c_\mu \ , \tag{1.4.16}$$

with f^{abc} totally antisymmetric structure constants of a semisimple Lie algebra. The action (1.4.15) is invariant under the supersymmetry transformations

$$\delta A^a{}_\mu = -i\,\bar{\varepsilon}\gamma_\mu\lambda^a \ , \tag{1.4.17a}$$

$$\delta\lambda^a = -i\,\sigma^{\mu\nu}F^a_{\mu\nu}\,\varepsilon\ , \tag{1.4.17b}$$

with ε a constant Majorana spinor. Indeed, inserting (1.4.17) into (1.4.15) leads to

$$\delta S = tr\int d^4x\ \Big\{-i\ (\bar\varepsilon\gamma_\nu[\nabla_\mu,F^{\mu\nu}] + \frac{i}{2}\bar\varepsilon\gamma_{\mu\nu}\gamma^\rho[\nabla_\rho,\lambda]F^{\mu\nu}$$

$$+\ \frac{1}{2}\bar\lambda[(\bar\varepsilon\gamma^\mu\lambda)\gamma_\mu,\lambda]\Big\} \tag{1.4.18}$$

or, using the Bianchi identity, to

$$\frac{1}{2}\int d^4x\ \bar\lambda[(\bar\varepsilon\gamma^\mu\lambda)\gamma_\mu,\lambda]\Big\} \tag{1.4.19}$$

All we need now is the Fierz identity for four dimensions which, using the results in Appendix A, is seen to imply

$$f^{abc}\,(\bar\varepsilon\gamma^\mu\lambda^a)(\bar\lambda^b\,\gamma_\mu\lambda^c) = \frac{1}{2}f^{abc}\,(\bar\varepsilon\gamma^\mu\lambda^c)(\bar\lambda^b\,\gamma_\mu\lambda^a)\ . \tag{1.4.20}$$

Relabeling a and c above then shows that (1.4.19) vanishes, and therefore proves that the action (1.4.18) is supersymmetric.

Next we consider the commutator of two supersymmetry transformations. Commuting two supersymmetries on A_μ we get

$$[\delta_2,\delta_1]A^a_\mu = -2i\,(\bar\varepsilon_1\gamma_\nu\varepsilon_2)F^a_{\mu\nu}\ . \tag{1.4.21}$$

This can be written:

$$[\delta_2,\delta_1]A^a_\mu = 2i\,(\bar\varepsilon_1\gamma_\nu\varepsilon_1)\,\partial_\nu A^a_\mu + \delta_{gauge}\,(-2i\,(\bar\varepsilon_1\gamma^\nu\varepsilon_2)A^a_\nu)\ . \tag{1.4.22}$$

This illustrates a new feature, peculiar to supersymmetric theories containing gauge fields: the commutator of two supersymmetries closes on the gauge fields only modulo a gauge transformation. This result, however, should not be very surprising, as it occurs also for the more familiar case of the Lorentz transformations. Alternatively, it could be noted that the r.h.s. of (1.4.21) is gauge covariant. This has an effect on the commutator on the other fields: the spatial derivative in the translation part will be replaced by a covariant derivative.

Next we consider the commutator of two supersymmetry transformations on the spinor λ. Counting degrees of freedom indicates an excess of Fermi degrees of freedom by one unit off-shell. As a consequence, we expect nonclosure of the supersymmetry algebra on λ. This is indeed the case, and the result can be written, after a Fierz rearrangement,

$$[\delta_2,\delta_1]\lambda^a = 2i\,(\bar{\varepsilon}_1\gamma^\mu\varepsilon_2)\,(\nabla_\mu\lambda)^a + i\,((\gamma_5\varepsilon_1)\,(\bar{\varepsilon}_2\gamma_5(\not{\nabla}\lambda)^a)$$

$$- (1\rightleftarrows 2))\ . \tag{1.4.23}$$

Working in analogy with what done for the Wess-Zumino model, we see that we can dispose of the additional terms in (1.4.23) by introducing an adjoint multiplet of pseudoscalar fields D^a such that, under a supersymmetry transformation,

$$\delta D^a = -i\,\bar{\varepsilon}\gamma_5(\not{\nabla}\lambda)^a\ , \tag{1.4.24}$$

and by modifying the supersymmetry transformation of λ into

$$\delta\lambda^a = \gamma_5\varepsilon\, D^a - i\,\sigma^{\mu\nu}F^a_{\mu\nu}\,\varepsilon\ . \tag{1.4.25}$$

Then commuting two supersymmetries on D^a also gives

$$[\delta_2,\delta_1]D^a = 2i\,(\bar{\varepsilon}_1\gamma^\mu\varepsilon_2)\,(\nabla_\mu D)^a\ , \tag{1.4.26}$$

and closure of the algebra off the mass-shell is achieved. The action is then made invariant by adding a suitably normalized nonpropagating kinetic term for D^a. The final result can be written:

$$S = \mathrm{tr}\int d^4x \left\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{i}{2}\bar{\lambda}[\nabla\!\!\!/,\lambda] - \frac{D^2}{2} \right\} \ . \tag{1.4.27}$$

We wish to emphasize that the interacting theory has a field-dependent supersymmetry algebra that coincides with the super-Poincare' algebra only in the limit of vanishing fields. This is a new feature that one encounters when dealing with interacting supersymmetric theories.

Having discussed the Wess-Zumino model and the N=1 supersymmetric Yang-Mills theory, we now turn to consider the simplest supersymmetric generalization of Einstein's general theory of Relativity [20]. This model describes interactions of the gravity field (here described by means of a vierbein $V^m{}_\mu$) and a Majorana spinor-vector ψ_μ, all governed by a single dimensionful coupling constant k. The resulting action is invariant under one *local* supersymmetry transformation.

Describing the supergravity action requires separate discussions of the actions for $V^m{}_\mu$ and ψ_μ. As anticipated, the gravity field in supergravity is described by a vierbein field $V^m{}_\mu$, rather than by the metric tensor $g_{\mu\nu}$. Here curved vector indices will be denoted by small Greek letters, and flat vector indices will be denoted by small Latin letters. Replacing $g_{\mu\nu}$ with $V^m{}_\mu$ provides the "flat to curved" index converter necessary to put spinors in curved space-time, but this apparently increases the number of components of the gravity field from 10 to 16. However, the set of local symmetries of the Einstein action is correspondingly increased by the addition of the local Lorentz symmetry in the tangent space, which effectively gauges away the extra six components.

To write the pure Einstein action in the vierbein formalism, we introduce the spin connection $\omega_\mu{}^{mn}$, which is the gauge field for the local Lorentz transformations in the tangent space, and we define the curvature

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac}\,\omega_{\nu c}{}^{b} - \omega_\nu{}^{ac}\,\omega_{\mu c}{}^{b} \ . \tag{1.4.28}$$

Finally, we postulate the action

$$S = \int d^4x \left\{ \frac{-V}{4k^2} V^{\mu}{}_a V^{\nu}{}_b R_{\mu\nu}{}^{ab}(\omega) \right\} . \tag{1.4.29}$$

This expression does not bear an obvious resemblance to the conventional way of writing the Einstein action in terms of the metric field and of the Christoffel symbols

$$S = \int d^4x \left\{ -\frac{\sqrt{-g}}{4k^2} R^{\mu}{}_{\nu\rho\sigma} \delta^{\rho}{}_{\mu} g^{\nu\sigma} \right\} . \tag{1.4.30}$$

The equivalence between the two formulations is established as follows. First of all one relates the metric tensor to the vierbein by writing

$$g_{\mu\nu} = V^m{}_{\mu} V_{m\nu} , \tag{1.4.31}$$

which already implies that $V = \sqrt{-g}$. Then one eliminates ω in eq. (1.4.32) using its nonpropagating equation of motion, which is

$$\partial_{\mu} V^m{}_{\nu} - \partial_{\nu} V^m{}_{\mu} + \omega_{\mu}{}^{mn} V_{\nu n} - \omega_{\nu}{}^{mn} V_{\mu n} = 0 . \tag{1.4.32}$$

Using the antisymmetry of ω in its two flat indices, this equation can be solved to give

$$\omega_{\mu}{}^{mn} = \frac{1}{2}(V^{n\nu}\partial_{\nu} V^m{}_{\mu} - V^{m\nu}\partial_{\nu} V^m{}_{\mu}) -$$

$$\frac{1}{2}(V^{n}\nu\,\partial_{\mu} V^m{}_{\nu} - V^{m\nu}\partial_{\mu} V^n{}_{\nu}) + \frac{1}{2} V_{b\mu}(V^{m\rho} V^{n\nu} - V^{n\rho} V^{m\nu})\partial_{\nu} V^b{}_{\rho} \tag{1.4.33}$$

Using this expression and the conventional (i.e torsionless) expression of the Christoffel symbols in terms of the metric

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) , \tag{1.4.34}$$

together with the relation (1.4.31) between $V^m{}_{\mu}$ and $g_{\mu\nu}$, one can then show that (1.4.32) and (1.4.33) fit together into the simple condition

$$\partial_\mu V^m{}_\nu + \omega_\mu{}^{mn} V_{n\nu} - \Gamma^\alpha{}_{\mu\nu} V^m{}_\alpha = 0 \quad , \tag{1.4.35}$$

which states that the vierbein is covariantly constant. Finally, using (1.4.35) one can show that

$$R_{\mu\nu}{}^{mn}(\omega) = R^\rho{}_{\sigma\mu\nu}(\Gamma)\, V^m{}_\rho\, V^{n\sigma} \quad , \tag{1.4.36}$$

and the equivalence between (1.4.29) and (1.4.30) is proved. This presentation is,of course, not the only way of looking at the problem, and one could as well turn the argument around and take (1.4.35) as the starting point.

In the presence of spinors, one is forced to work with the vierbein, and the action (1.4.29) is the starting point in the construction of the supergravity action. Our next task will be to describe the Rarita-Schwinger action for the gravitino field ψ_μ. The free action for a massless Majorana gravitino can be written

$$S = \int d^4x \left\{ -\tfrac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma \right\} \quad . \tag{1.4.37}$$

It is invariant under the gauge transformation $\delta\psi_\mu = \partial_\mu \varepsilon$, with ε an anticommuting Majorana spinor. The corresponding free action for a massive gravitino follows by adding to (1.4.37) the mass term

$$S_m = \int d^4x \left\{ \tfrac{1}{2} m\, \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu \right\} \quad . \tag{1.4.38}$$

As (1.4.37) and (1.4.38) are not very familiar objects, we find it convenient to discuss their properties in some detail. In order not to interrupt our description of supergravity, however, we prefer to relegate the discussion of the properties of the Rarita-Schwinger action to Appendix C. There we will show [25] that (1.4.37)-(1.4.38) is the unique free action (up to field redefinitions of the form $\psi_\mu \to \psi_\mu + \lambda\gamma_\mu \gamma\cdot\psi$) which propagates only spin-$\frac{3}{2}$ modes, is linear in the derivative ∂_μ, does not contain nonlocalities and yields identical equations of motion for the fields that enter it.

For our present purpose, what is most important is that the gauge invariance $\delta\psi_\mu = \partial_\mu \varepsilon$ has a parameter which, apart from its being space-time dependent, looks like the supersymmetry parameters encountered for the Wess-Zumino model and for the N=1 supersymmetric Yang-Mills theory, as it is an anticommuting Majorana spinor. The gravitino thus suggests itself as the gauge field of supersymmetry. Constructing the supergravity action amounts to following steps conceptually similar (if technically far more difficult) to those one would follow in constructing the full self-interacting Yang-Mills theory from the corresponding free theory. In this case we must start from a globally supersymmetric model (and here is where gravity already enters) and make the supersymmetry local by the addition of suitable extra couplings.

Our starting point is therefore the Einstein action in the form (1.4.29), together with the Rarita-Schwinger action (1.4.37), covariantized with respect to general coordinate transformations, i.e.

$$S = \int d^4x \left\{ \frac{-V}{4k^2} R(V,\omega) - \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \overline{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma \right\} . \tag{1.4.39}$$

No vierbein determinant appears in front of the second term above, because ε is a tensor density. γ-matrices bearing a curved index are related to ordinary constant γ matrices with flat indices via contraction with the vierbein, so that $\gamma_\mu = V^m{}_\mu \gamma_m$. Finally, the covariant derivative D_ρ can be written omitting the symmetric Christoffel connection altogether, as this would drop out because of the antisymmetrization. Thus

$$D_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{2} \omega_\mu{}^{ab} \sigma_{ab} \psi_\nu . \tag{1.4.40}$$

The next step is writing supersymmetry transformations under which (1.4.42) is invariant. The natural candidate for the gravitino transformation is the generalization of the gauge invariance of eq. (1.4.37) to include covariantization with respect to general coordinate transformations, i.e.

$$\delta \psi_\mu = \frac{1}{k} D_\mu \varepsilon \ . \tag{1.4.41}$$

Together with this transformation, we need the transformation of $V^m{}_\mu$, and the transformation of $\omega_\mu{}^{mn}$. Actually, one can forget about ω altogether, since it is possible to arrange things in order that it does not contribute to the variation, by taking it to be that function of V and ψ that satisfies its algebraic equation of motion. This trick, usually denoted as 1.5 order formalism, follows from the observation that in varying the action one produces

$$\frac{\delta S}{\delta V^m{}_\mu} \delta V^m{}_\mu + \frac{\delta S}{\delta \psi_\mu} \delta \psi_\mu + \frac{\delta S}{\delta \omega_\mu{}^{ab}} \delta \omega_\mu{}^{ab} \ , \tag{1.4.42}$$

and the last term vanishes identically, irrespective of what $\delta\omega_\mu{}^{ab}$ is, if the spin connection satisfies its equation of motion. The important point to notice, however, is that the addition of the $\psi\omega\psi$ coupling in eq. (1.4.39) modifies the equation for ω from its form (1.4.35) to

$$\partial_\mu V^m{}_\nu - \partial_\nu V^m{}_\mu + \omega_\mu{}^{mn} V_{\nu n} - \omega_\nu{}^{mn} V_{\mu n} = - \frac{k^2}{2} (\overline{\psi}_\mu \gamma^m \psi_\nu) \ . \tag{1.4.43}$$

This, in turn, can be solved to give

$$\omega_\mu{}^{mn} = \omega^o{}_\mu{}^{mn}$$
$$+ \frac{k^2}{2} (\overline{\psi}_n \gamma_m \psi_\mu + \overline{\psi}_\mu \gamma_n \psi_m - \overline{\psi}_m \gamma_\mu \psi_n) \ , \tag{1.4.44}$$

where $\omega^o{}_\mu{}^{mn}$ is given in eq. (1.4.33). Eq. (1.4.44) can equivalently be restated in the form of a condition that the vierbein be covariantly constant, as in eq. (1.4.35), but with the Christoffel symbols no more symmetric in their two lower indices, and therefore possessing a torsion part

$$S^a{}_{\mu\nu} = -\frac{k^2}{2}\overline{\psi}_\nu\gamma^a\psi_\mu \ . \tag{1.4.45}$$

Summarizing, the requirement that ω in eq. (1.4.39) satisfy its nonpropagating equation of motion has had the effect of hiding in its definition quadratic terms in the gravitino ψ_μ, which in turn introduce quartic spinor couplings in eq. (1.4.39).

We now vary $V_\mu{}^m$ and ψ_μ in eq. (1.4.39), which gives

$$\delta S = \int d^4x \ \Big\{ \ \frac{V}{4k^2}\delta V_\mu{}^m (R^\mu{}_m - \frac{1}{2}V_m{}^\mu R) - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\delta\overline{\psi}_\mu)\gamma_5\gamma_\nu D_\rho\psi_\sigma - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_5(\delta\gamma_\nu)(D_\rho\psi_\sigma) - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_5\gamma_\nu D_\rho\delta\psi_\sigma \Big\} \ . \tag{1.4.46}$$

Inserting (1.4.41) above and integrating by part the second term then gives:

$$\delta S = \int d^4x \ \Big\{ \ \frac{V}{4k^2}\delta V_\mu{}^m (R^\mu{}_m - \frac{1}{2}V_m{}^\mu R) - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_5\gamma_\nu[D_\rho,D_\sigma]\varepsilon - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\delta V^m{}_\nu)\overline{\psi}_\mu\gamma_5\gamma_m D_\rho\psi_\sigma + \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\overline{\varepsilon}\gamma_5\gamma_m D_\rho\psi_\sigma(D_\mu V^m{}_\nu - D_\nu V^m{}_\mu) \Big\} \ , \tag{1.4.47}$$

where the last term originates from integrating by parts the second term in eq. (1.4.46). We can now rewrite more conveniently (1.4.46) by using (1.4.43) and the relation between the commutator of two covariant derivatives on ε and the curvature tensor in eq. (1.4.28). The result is:

$$\delta S = \int d^4x \ \Big\{ \ \frac{V}{4k^2}(\delta V^m{}_\mu)(R^m{}_\mu - \frac{1}{2}V^m{}_\mu R) - \frac{V}{4k}(R^m{}_\mu - \frac{1}{2}V^m{}_\mu R) - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\delta V^m{}_\nu)\overline{\psi}_\mu\gamma_5\gamma_m D_\rho\psi_\sigma + \frac{k^2}{8}\varepsilon^{\mu\nu\rho\sigma}\overline{\varepsilon}\gamma_5\gamma_m D_\rho\psi_\sigma(\overline{\psi}_\mu\gamma^m\psi_\nu) \tag{1.4.48}$$

where we notice that all terms involving the Riemann tensor in the variation of the Rarita-Schwinger action have reduced to terms involving the combination $(R_\mu{}^m - \frac{1}{2}V^m{}_\mu R)$. Requiring cancellation of these terms then determines the transformation of the vielbein to be

$$\delta\, V^m{}_\mu = k\, \bar{\varepsilon} \gamma^m \psi_\mu \ , \tag{1.4.49}$$

and one is left with

$$\delta S = \int d^4x \,\Big\{ \frac{1}{8} \varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon} \gamma_5 \gamma_a\, D_\rho \psi_\sigma)(\bar{\psi}_\mu \gamma^a \psi_\nu)$$

$$- \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} (\bar{\psi}_\mu \gamma_5 \gamma_a\, D_\rho \psi_\sigma)(\bar{\varepsilon} \gamma^a \psi_\nu) \Big\} \ , \tag{1.4.50}$$

which vanishes upon a Fierz rearrangement. Consequently the action for N=1 supergravity in four dimensions is

$$S = \int d^4x \,\Big\{ - \frac{V}{4k^2} R(V,\omega) - \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma \Big\} \ , \tag{1.4.51}$$

with ω determined by eq. (1.4.42), and is invariant under the local supersymmetry transformations

$$\delta\, V^m{}_\mu = k\, \bar{\varepsilon} \gamma^m\, \psi_\mu \tag{1.4.52a}$$

and

$$\delta \psi_\mu = \frac{1}{k} D_\mu(\omega)\, \varepsilon \ . \tag{1.4.52b}$$

The Einstein action for pure gravity (1.4.29) embodies three main features, the invariance with respect to general coordinate transformations and local Lorentz transformations, and the absence of terms containing more than two derivatives. As is well known, however, these two features are also preserved if we modify (1.4.29) by the addition of a cosmological term, and write

$$S = \int d^4x \,\Big\{ - \frac{V}{4k^2} V_a^\mu\, V_b^\nu R_{\mu\nu}{}^{ab}(\omega) + \lambda\, V \Big\} \ , \tag{1.4.53}$$

where the cosmological constant λ can have any sign. It is then natural to ask whether the action of simple supergravity in eq. (1.4.51) can also accommodate a cosmological term compatibly with local supersymmetry. The answer to this question is contained in the discussion of superalgebras in section 2. There we have seen that the N=1 super-Poincare' algebra in four dimensions can be recovered by contracting a simple algebra, the super-de Sitter algebra. The peculiar feature is that in this case the bosonic algebra is constrained to be anti-de Sitter, rather than de Sitter. This, in turn, fixes the sign of the cosmological term, which is bound to be negative. To construct this theory [23], we start by adding to (1.4.51) a negative cosmological term, and modify action and transformation laws in order to attain local supersymmetry of the resulting action. The first modification one needs is the addition of a mass term for the gravitino which, as shown in Appendix C, is of the form $\overline{\psi}_\mu \sigma^{\mu\nu} \psi_\nu$ once we choose for the kinetic term the form (1.4.37). We are thus led to consider

$$S = \int d^4x \left\{ \frac{-V}{4k^2} R(V,\omega) - \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \overline{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma \right.$$
$$\left. - \frac{\lambda V}{2k} \overline{\psi}_\mu \sigma^{\mu\nu} \psi_\nu + \frac{3V\lambda^2}{4k^4} \right\} . \tag{1.4.54}$$

Then, requiring that the variation of all terms quadratic in ψ cancels leads to modify the supersymmetry transformation of ψ into

$$\delta \psi_\mu = \frac{1}{k} D_\mu \varepsilon - \frac{\lambda}{4k^2} \gamma_\mu \varepsilon \ , \tag{1.4.55}$$

and this fixes the action, apart from quartic spinor terms arising from the variation of the vierbein in the mass and kinetic terms. No additional quartic spinor terms are needed, however, as

$$\delta S = \int d^4x \left\{ \frac{\lambda}{8} \varepsilon^{\mu\nu\rho\sigma} (\overline{\psi}_\mu \gamma_5 \varepsilon)(\overline{\psi}_\rho \gamma_\nu \psi_\sigma) - \frac{\lambda V}{4} (\overline{\varepsilon} \gamma^\rho \psi_\rho)(\overline{\psi}_\mu \sigma^{\mu\nu} \psi_\nu) + \right.$$
$$\left. \frac{\lambda V}{4} (\overline{\psi}_\mu \sigma^{\rho\nu} \psi_\nu)(\overline{\varepsilon} \gamma^\mu \psi_\rho) \right\} , \tag{1.4.56}$$

which can be shown to vanish by a Fierz rearrangement.

The last model with simple supersymmetry we wish to describe here is simple supergravity in eleven dimensions. As we have shown in section 2, group theory directly sorts out D=11 as the largest dimensionality of spacetime where a supersymmetric multiplet containing no states of helicity greater than two exists. We have also remarked that the smallest supermultiplet of the D=11 super-Poincare' algebra contains a set of states with the same helicity content as the 256 states of N=8 supergravity. Indeed, as we have seen, the information provided by the study of the super-Poincare' algebra is almost equivalent to knowing the noninteracting form of the corresponding field theory. The only ambiguity is related to different choices of fields that on-shell describe a given set of states. Supergravity in eleven dimensions provides an example of such an ambiguity. As we have seen in section 3, the ambiguity has to do with the bose fields, since the 128 fermionic degrees of freedom sort out a Majorana gravitino as the only possible field choice. Of the 128 bose degrees of freedom, 44 are accommodated by the elfbein $V^m{}_\mu$, whereas for the remaining 84 there are two possible choices of field, a third-rank antisymmetric tensor gauge field $A_{\mu\nu\rho}$, and a sixth-rank antisymmetric tensor gauge field $A_{\mu_1 .. \mu_6}$. It is then straightforward to write the corresponding free theories and their supersymmetry transformations. The first case gives the free action

$$S = -\int d^{11}x \, \Big\{ \, [\frac{V}{4k^2} R(V,\omega)]_{lin} - \frac{1}{2} \bar{\psi}_\mu \Gamma^{\mu\nu\rho} \partial_\nu \psi_\rho - \frac{V}{2 \cdot 4!} (F_{\mu\nu\rho\sigma})^2 \Big\} \, , \tag{1.4.57}$$

and the corresponding supersymmetry transformations

$$\delta V^m{}_\mu = k \, \bar{\varepsilon} \, \Gamma^m \, \psi_\mu \tag{1.4.58a}$$

$$\delta \psi_\mu = \frac{1}{k} [D_\mu \varepsilon]_{lin} - \frac{1}{144} (\Gamma_\mu{}^{\alpha\beta\gamma\delta} + 8 \, \delta_\mu{}^\alpha \Gamma^{\beta\gamma\delta}) \varepsilon \, F_{\alpha\beta\gamma\delta} \tag{1.4.58b}$$

$$\delta A_{\mu\nu\rho} = -\frac{3}{2}\bar{\varepsilon}\,\Gamma_{[\mu\nu}\psi_{\rho]} \ . \tag{1.4.58c}$$

Here ε is a constant Majorana spinor, [] denotes antisymmetrization with strength one, and the field strength

$$F_{\alpha\beta\gamma\delta} = 4\,\partial_{[\alpha}A_{\beta\gamma\delta]} \tag{1.4.59}$$

is invariant under the gauge transformations

$$\delta A_{\alpha\beta\gamma} = \partial_{[\alpha}A_{\beta\gamma]} \ . \tag{1.4.60}$$

The second case gives the action

$$S = \int d^{11}x \,\Big\{ -[\frac{V}{4k^2}R(V,\omega)]_{lin} - \frac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\partial_\nu\psi_\rho + \frac{V}{2\cdot 7!}(F_{\alpha_1..\alpha_7})^2 \Big\} \ , \tag{1.4.61}$$

and the corresponding supersymmetry transformations

$$\delta V^m{}_\mu = k\,\bar{\varepsilon}\,\Gamma^m\psi_\mu \tag{1.4.62a}$$

$$\delta\psi_\mu = [\frac{1}{k}D_\mu]_{lin} + \frac{1}{6\cdot 7!}(2\,\Gamma_\mu{}^{\alpha_1..\alpha_7} + 7\,\delta_\mu^{\alpha_1}\Gamma^{\alpha_2..\alpha_7})\,\varepsilon\,F_{\alpha_1..\alpha_7} \tag{1.4.62b}$$

$$\delta A_{\alpha_1..\alpha_6} = 3\,\bar{\varepsilon}\,\Gamma_{[\alpha_1..\alpha_5}\psi_{\alpha_6]} \ , \tag{1.4.62c}$$

where ε is again a *constant* Majorana spinor, and the field strength

$$F_{\alpha_1..\alpha_7} = 7\,\partial_{[\alpha_1}A_{\alpha_2..\alpha_7]} \tag{1.4.63}$$

is invariant under the gauge transformations

$$\delta A_{\alpha_1..\alpha_6} = \partial_{[\alpha_1}\Lambda_{\alpha_2..\alpha_6]} \ . \tag{1.4.64}$$

In attempting to promote the supersymmetry to a local symmetry, one follows steps similar to those described for the case of N=1 supergravity in four dimensions. There are two technical complications in this case. First of all, the actions contain one extra field, apart from ψ_μ and $V^m{}_\mu$, and correspondingly more couplings are possible and, in general, needed. Moreover, the eleven-dimensional Dirac algebra is considerably more cumbersome than the four-dimensional one, in that one is dealing with antisymmetrized products of several Γ matrices that must be multiplied and combined together. The main tool one needs is eq. (1.B.3), which provides the decomposition for the product of two antisymmetrized products of Γ matrices.

We consider first (1.4.57). As in the discussion of N=1 supergravity in four dimensions, we use 1.5 order formalism and do not vary ω. As a first step, the full nonlinear vielbein content is introduced, by demanding invariance under general coordinate transformations. Then, in order to make the supersymmetry local, one needs to add the Noether term

$$S_N = \int d^{11}x \left\{ -\frac{k\,V}{96}\bar{\psi}_\mu (\Gamma^{\mu\nu\alpha\beta\gamma\delta} + 12 g^{\mu\alpha} g^{\nu\beta}\Gamma^{\gamma\delta})\psi_\nu F_{\alpha\beta\gamma\delta} \right\} \quad (1.4.65)$$

in order to cancel the residual $\partial\varepsilon$ terms from the variation of the kinetic term of ψ_μ in eq. (1.4.57). The terms of the form $k\varepsilon FF$, generated by varying $\delta\psi \approx \varepsilon F$ in the Noether term, and by varying the vielbein in the kinetic term of $A_{\mu\nu\rho}$, do not completely cancel against one another, but add up to

$$\delta\tilde{S} = \int d^{11}x \left\{ \frac{k}{16\cdot 144}\varepsilon^{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'\mu\nu\rho}(\bar{\varepsilon}\Gamma_{\mu\nu}\psi_\rho) F_{\alpha\beta\gamma\delta}F_{\alpha'\beta'\gamma'\delta'} \right\} , \quad (1.4.66)$$

which is canceled by adding the gauge-invariant term

$$S' = \int d^{11}x \left\{ \frac{2k}{(144)^2}\varepsilon^{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'\mu\nu\rho} A_{\mu\nu\rho} F_{\alpha\beta\gamma\delta}F_{\alpha'\beta'\gamma'\delta'} \right\} . \quad (1.4.67)$$

One then fixes the cubic terms in the transformation law for ψ_μ and the quartic terms in the action by supercovariantization. The final answer is the action of Cremmer, Julia and Scherk [14]:

$$S = \int d^{11}x \left\{ -\frac{V}{4k^2} R(V,\omega) - \frac{V}{2} \bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu(\frac{\omega+\hat{\omega}}{2})\, \psi_\rho - \frac{V}{2\cdot 4!} (F_{\alpha\beta\gamma\delta})^2 \right.$$

$$- \frac{\kappa V}{192} \bar{\psi}_\mu (\Gamma^{\mu\nu\alpha\beta\gamma\delta} + 12 g^{\mu\alpha} g^{\nu\beta} \Gamma^{\gamma\delta}) \psi_\nu (F_{\alpha\beta\gamma\delta} + \hat{F}_{\alpha\beta\gamma\delta})$$

$$\left. + \frac{2\kappa}{(144)^2} \varepsilon^{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'\mu\nu\rho} A_{\mu\nu\rho} F_{\alpha\beta\gamma\delta} F_{\alpha'\beta'\gamma'\delta'} \right\} , \qquad (1.4.68)$$

with

$$\hat{F}_{\mu\nu\rho\sigma} = F_{\mu\nu\rho\sigma} + 3\kappa \bar{\psi}_{[\mu} \Gamma_{\nu\rho} \psi_{\sigma]} \qquad (1.4.69a)$$

and

$$\hat{\omega}_{\mu ab} = \omega_{\mu ab} - \frac{i\kappa^2}{4} \bar{\psi}_\alpha \Gamma_{\mu ab}{}^{\alpha\beta} \psi_\beta \qquad (1.4.69b)$$

the supercovariant extensions of F and ω respectively. Correspondingly, the supersymmetry transformations are given by

$$\delta V_\mu^a = \kappa \bar{\varepsilon} \Gamma^a \psi_\mu, \qquad (1.4.70a)$$

$$\delta A_{\alpha\beta\gamma\delta\varepsilon\zeta} = 3i \bar{\varepsilon} \Gamma_{[\alpha\beta\gamma\delta\varepsilon} \psi_{\zeta]}, \qquad (1.4.70b)$$

$$\delta\psi_\mu = \frac{1}{\kappa} D_\mu(\omega)\varepsilon + \frac{1}{6\cdot 7!} (2\Gamma_\mu^{\alpha\beta\gamma\delta\varepsilon\zeta\eta} + 7\delta_\mu^\alpha \Gamma^{\beta\gamma\delta\varepsilon\zeta\eta}) \varepsilon F_{\alpha\beta\gamma\delta\varepsilon\zeta\eta}. \qquad (1.4.70c)$$

We now discuss what happens when one tries to construct an interacting theory in eleven dimensions starting from the action in eq. (1.4.61) [16,24]. As before, one starts by covariantizing the action with respect to general coordinate transformations. Then, to make the supersymmetry local, the addition of the Noether term

$$S_N = \int d^{11}x \left\{ \frac{k\, V}{4\cdot 7!} \bar{\psi}_\mu (\Gamma^{\mu\nu\alpha_1..\alpha_7} + 42 g^{\mu\alpha_1} g^{\nu\alpha_2} \Gamma^{\alpha_3..\alpha_7}) \psi_\nu F_{\alpha_1..\alpha_7} \right\} \qquad (1.4.71)$$

is necessary and, again, the $k\varepsilon FF$ terms in the variation of the lagrangian do not cancel completely, but lead to

$$\delta S = \int d^{11}x \left\{ \frac{-k}{4 \cdot 4! \cdot 7!} \varepsilon^{\alpha_1..\alpha_7\beta_1..\beta_4} (\bar{\varepsilon}\Gamma^{\mu\nu}\psi^{\rho}) F_{\alpha_1..\alpha_7} F_{\beta_1..\beta_4\mu\nu\rho} \right\} . \quad (1.4.72)$$

The analogue of the term in eq. (1.4.67) is not available now, simply because (1.4.72) does not have the right index structure. The only possibility of canceling it requires adding a cosmological constant but, as we now show, even this attempt fails, and it is not possible to construct an interacting supersymmetric theory with $A_{\alpha_1..\alpha_6}$ [16]. To show that this is the case, we add to the action the term

$$S' = \int d^{11}x \left\{ \frac{-B\,k^3}{8 \cdot 4! \cdot 7!} \varepsilon^{\alpha_1..\alpha_7\beta_1..\beta_4} (\bar{\psi}^{\mu}\Gamma^{\nu}\psi^{\rho}) F_{\alpha_1..\alpha_7} F_{\beta_1..\beta_4\mu\nu\rho} \right\} \quad (1.4.73)$$

and modify the supersymmetry transformation of ψ_μ to include the term

$$\delta\psi_\mu = \frac{1}{B\,k^2}\Gamma_\mu \varepsilon \ , \quad (1.4.74)$$

which is enough to cancel (1.4.72). But the effect of the new transformation for the gravitino on the remaining terms in the action then leads to the addition of a mass term for ψ_μ and of a cosmological constant. We are thus led to consider the action

$$S = -\int d^{11}x \left\{ \frac{V}{4\kappa^2} R(V,\omega) - \frac{V}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu(\omega)\psi_\rho - \frac{9V}{2B\kappa}\bar{\psi}_\mu\Gamma^{\mu\nu}\psi_\nu + \frac{90V}{B^2\kappa^4} + \right.$$

$$+ \frac{V}{2 \cdot 7!}(F_{\alpha\beta\gamma\delta\varepsilon\zeta\eta})^2 + \frac{kV}{4 \cdot 7!}\bar{\psi}_\mu(\Gamma^{\mu\nu\alpha\beta\gamma\delta\varepsilon\zeta\eta} + 42 g^{\mu\alpha}g^{\nu\beta}\Gamma^{\gamma\delta\varepsilon\zeta\eta})\psi_\nu F_{\alpha\beta\gamma\delta\varepsilon\zeta\eta}$$

$$\left. - \frac{B\kappa^3}{8 \cdot 4! \cdot 7!} \varepsilon^{\alpha\beta\gamma\delta\varepsilon\zeta\eta abcd} (\bar{\psi}^\mu \Gamma^\nu \psi^\rho) F_{\alpha\beta\gamma\delta\varepsilon\zeta\eta} F_{abcd\mu\nu\rho} \right\} . \quad (1.4.75)$$

A critical test is now to check whether the terms generated by varying $\delta\psi \approx \frac{\varepsilon}{k^2}$ in the Noether term cancel against the terms generated by varying $\delta\psi \approx \varepsilon F$ in the mass term. This does not happen, but rather we are left with

$$\delta S_m + \delta S_N = \int d^{11}x \left\{ \frac{-V}{5!\,B\,k} (\bar{\psi}^{\alpha_1}\Gamma^{\alpha_2..\alpha_7}\varepsilon) F_{\alpha_1..\alpha_7} \right\} . \quad (1.4.76)$$

In the same way [16], one can show that it is impossible to modify the Cremmer, Julia Scherk theory to include a cosmological term consistent with local supersymmetry. Starting from (1.4.68), one adds a cosmological term and the necessary (by supersymmetry) ψ_μ mass term, and modifies the ψ_μ transformation by the addition of the term (1.4.75). But again one arrives at the same difficulty as the one encountered in the six-index case: the variation $\delta\psi \approx \varepsilon F$ in the mass term does not cancel against the variation $\delta\psi \approx \frac{\varepsilon}{k^2}$ in S_N, but rather they add up to

$$\delta S_m + \delta S_N = \int d^{11}x \left\{ \frac{i\,V}{4B\,k} (\bar{\psi}_\mu \Gamma^{\mu\alpha\beta\gamma\delta} \varepsilon) F_{\alpha\beta\gamma\delta} \right\} , \qquad (1.4.77)$$

which is the analog of (1.4.76) and is impossible to cancel.

The failure of these attempts has an explanation in terms of group theory. It has to do with the fact that, as we have seen, no super-de Sitter algebra exists in eleven dimensions, as such an algebra would be the global algebra of a model with cosmological term and an Abelian gauge symmetry for the antisymmetric tensor $A_{\mu\nu\rho}$ (or for $A_{\alpha_1 .. \alpha_6}$). The conclusion is that, in eleven dimensions, supergravity has the unique form (1.4.68), built out of the set of fields ψ_μ, $V^m{}_\mu$ and $A_{\mu\nu\rho}$ and, differently from the case of four dimensions, it cannot be modified to accommodate a cosmological term consistently with local supersymmetry.

Appendix A

The equations derived in section 2 when discussing the closure of the super-de Sitter algebra in several dimensions involve the evaluation of terms like $(\Gamma^{\mu_1..\mu_m}\Gamma_{\nu_1..\nu_n}\Gamma_{\mu_1..\mu_m})$. This can be done very simply using the following result†:

$$(\Gamma^{\beta_1..\beta_m})_{ab}\,(\Gamma^{\alpha_1..\alpha_n}\Gamma_{\beta_1..\beta_m}\Gamma_{\alpha_1..\alpha_n})_{cd} = (-1)^{\frac{n(n-1)}{2}+n\,m}\,n!$$

$$\times\,(\Gamma_{\beta_1..\beta_m})_{ab}\,(\Gamma_{\beta_1..\beta_m})_{cd}\sum_{k=0}^{min(m,n)}(-1)^k\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}D-m\\n-k\end{bmatrix}\,, \qquad (1.A.1)$$

where $D>m+n$. As usual, $\Gamma_{\alpha_1..\alpha_k}$ denotes a product of k Γ matrices, totally antisymmetrized and normalized so that it equals $\Gamma_{\alpha_1}..\Gamma_{\alpha_k}$ when the indices are all different.

In order to prove (1.A.1), we start by considering the case in which k indices out of the set $(\alpha_1,..,\alpha_m)$, say $(\alpha_1,..,\alpha_k)$, coincide with k indices out of the set $(\beta_1,..,\beta_m)$, say $(\beta_1,..,\beta_k)$. This circumstance corresponds, for example, to the arrangement

$$\Gamma^{\alpha_1..\alpha_k\alpha_{k+1}..\alpha_n}\,\Gamma_{\beta_1..\beta_k\beta_{k+1}..\beta_m}\,\Gamma_{\alpha_1..\alpha_k\alpha_{k+1}..\alpha_n}\ . \qquad (1.A.2)$$

There are several dispositions of the indices which are equivalent to the one shown in eq. (1.A.2), corresponding to different possible choices of the indices $(\beta_1,..,\beta_k)$ and of the indices $(\alpha_1,..,\alpha_k)$, and to different relative permutations of the sets $(\alpha_1,..,\alpha_k)$ and $(\beta_1,..,\beta_k)$. The total number of these cases is:

$$\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}k!\ . \qquad (1.A.3)$$

† I thank J. H. Schwarz for teaching me this.

The remaining indices $(\alpha_{k+1}..\alpha_n)$ in (1.A.2) contract against the indices in the other $\Gamma_{\alpha_1..\alpha_n}$, and this corresponds to

$$(D-m)..(D-m-n+k+1) = \begin{pmatrix} D-m \\ n-k \end{pmatrix} (n-k)! \qquad (1.A.4)$$

possibilities. Multiplying (1.A.3) by (1.A.4) then gives the total factor for this case,

$$\begin{pmatrix} m \\ k \end{pmatrix} \begin{pmatrix} n \\ k \end{pmatrix} k! \qquad (1.A.5)$$

where the overall sign accounts for the exchanges of Γ matrices with respect to the original permutation. If $D<m+n$, some of the m indices $(\beta_1,..,\beta_m)$ must equal some of the n indices $(\alpha_1,..,\alpha_n)$, and the sum in eq. (1.A.1) therefore starts from $\max(D-m-n,0)$, rather than from 0. In general, we thus have:

$$(\Gamma^{\beta_1..\beta_m})_{ab} (\Gamma^{\alpha_1..\alpha_n} \Gamma_{\beta_1..\beta_m} \Gamma_{\alpha_1..\alpha_n})_{cd} = (-1)^{\frac{n(n-1)}{2} + nm} n!$$

$$(\Gamma_{\beta_1..\beta_m})_{ab} (\Gamma_{\beta_1..\beta_m})_{cd} \sum_{max(0,D-m-n)}^{min(m,n)} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} \begin{pmatrix} D-m \\ n-k \end{pmatrix} . \qquad (1.A.6)$$

From this result we obtain the following 3 tables, where we collect some identities for D=4,10,11.

D	$(\Gamma^\mu \Gamma_\alpha \Gamma_\mu)/(\Gamma_\alpha)$	$(\Gamma^\mu \Gamma_{\alpha\beta} \Gamma_\mu)/(\Gamma_{\alpha\beta})$	$(\Gamma^\mu \Gamma_{\alpha_1..\alpha_5} \Gamma_\mu)/(\Gamma_{\alpha_1..\alpha_5})$
4	-2	0	0
10	-8	6	0
11	-9	7	-1

D	$(\Gamma^{\mu\nu}\Gamma_\alpha\Gamma_{\mu\nu})/(\Gamma_\alpha)$	$(\Gamma^{\mu\nu}\Gamma_{\alpha\beta}\Gamma_{\mu\nu})/(\Gamma_{\alpha\beta})$	$(\Gamma^{\mu\nu}\Gamma_{\alpha_1..\alpha_5}\Gamma_{\mu\nu})/(\Gamma_{\alpha_1..\alpha_5})$
4	0	4	0
10	-54	-26	10
11	-70	-38	10

D	$(\Gamma^{\mu_1..\mu_5}\Gamma_\alpha\Gamma_{\mu_1..\mu_5})$	$(\Gamma^{\mu_1..\mu_5}\Gamma_{\alpha\beta}\Gamma_{\mu_1..\mu_5})$	$(\Gamma^{\mu_1..\mu_5}\Gamma_{\alpha_1..\alpha_5}\Gamma_{\mu_1..\mu_5})$
4	0	0	0
10	0	-3360	0
11	-5040	-5040	-1200

Appendix B

In this section we summarize our conventions for the eleven-dimensional Dirac algebra which we use in the discussion of the algebras and in the discussion of supergravity in eleven dimensions. We use the Minkowski metric $\eta_{MN} = diag(-1,+1,..,+1)$, $M,N = 0,1,..,10$ and our Γ_M's satisfy the Dirac algebra $\{\Gamma_M,\Gamma_N\} = 2\eta_{MN}$ and are chosen to obey the condition:

$$\Gamma^0\Gamma^1\Gamma^2..\Gamma^{10} = -1.$$

Antisymmetric products of Γ-matrices are defined by:

$$\Gamma_{M_1M_2..M_n} = \Gamma_{[M_1}\Gamma_{M_2}..\Gamma_{M_n]} \qquad (1.B.1)$$

where the square brackets denote antisymmetrization with unit strength. One can now prove that those antisymmetrized products are related by:

$$\Gamma^{a_1a_2..a_n} = \frac{(-)^{n(n-1)/2}}{(11-n)!}\,\varepsilon^{a_1a_2..a_nb_1b_2..b_{11-n}}\Gamma_{b_1b_2..b_{11-n}} \qquad (1.B.2)$$

and consequently all the products of Γ-matrices can be expressed in terms of 1, Γ_m, Γ_{mn}, Γ_{mnp}, Γ_{mnpq} and Γ_{mnpqr}, which multiplied with factors of i where necessary in order to become Hermitian, form a complete set of 1024 32×32 Hermitian matrices.

Antisymmetric products of Γ-matrices can be combined according to

$$\Gamma_{a_1a_2..a_m}\Gamma^{b_1b_2..b_n} = \sum_{k=0}^{min(m,n)} \frac{m!n!}{(m-k)!(n-k)!k!}\,\Gamma_{[a_1..a_k}{}^{[b_{k+1}..b_n}\eta_{a_m}^{b_1}\eta_{a_{m-k+1}]}^{b_k]} \qquad (1.B.3)$$

The coefficients in the expansion are determined by the condition that all traces be removed from the r.h.s.

Fierz rearrangements are performed using:

$$M^{\alpha\beta}N^{\gamma\delta} = \sum_j \frac{(NO_jM)^{\gamma\beta}}{Tr(O_j^2)}\,(O_j)^{\alpha\delta} \qquad (1.B.4)$$

where M, N are any 32×32 matrices and O_j is a shorthand for any matrix in the set $\{1, \Gamma_m, \Gamma_{mn}, \Gamma_{mnp}, \Gamma_{mnpq}, \Gamma_{mnpqr}\}$.

Explicit evaluation of the traces leads to:

$$M^{\alpha\beta} N^{\gamma\delta} = \frac{1}{32} \{(NM)^{\gamma\beta} \delta^{\alpha\delta} + (N\Gamma^m M)^{\gamma\beta} \Gamma_m^{\alpha\delta} - \frac{1}{2} (N\Gamma^{mn} M)^{\gamma\beta} \Gamma_{mn}{}^{\alpha\delta} -$$

$$- \frac{1}{3!} (N\Gamma^{mnp} M)^{\gamma\beta} \Gamma_{mnp}{}^{\alpha\delta} + \frac{1}{4!} (N\Gamma^{mnpq} M)^{\gamma\beta} \Gamma_{mnpq}{}^{\alpha\delta} +$$

$$+ \frac{1}{5!} (N\Gamma^{mnpqr} M)^{\gamma\beta} \Gamma_{mnpqr}{}^{\alpha\delta}\}. \tag{1.B.5}$$

A convenient choice of the Dirac matrices Γ^M, $M = 0,1,\ldots,10$, in 11-dimensions used in Section I of the text is:

$$\Gamma^\mu = i\,\gamma^\mu \times 1_2 \times 1_4; \quad \mu = 0,1,2,3$$

$$\Gamma^{3+i} = i\,\gamma_5 \times \rho_1 \times \alpha_i; \quad i = 1,2,3$$

$$\Gamma^{6+j} = i\,\gamma_5 \times \rho_3 \times \beta_j; \quad j = 1,2,3$$

$$\Gamma^{10} = \Gamma^0 \Gamma^1 \ldots \Gamma^9 = i\,\gamma_5 \times \rho_2 \times 1_4. \tag{1.B.6}$$

The 4×4 matrices γ^μ are the Dirac matrices in four dimensions, they satisfy: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} = 2\,diag(-+++)$ and are taken in the form (spinorial representation):

$$\gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\,\tau_1 \times 1_2, \quad \gamma^i = \begin{pmatrix} 0 & -i\,\sigma_i \\ i\,\sigma_i & 0 \end{pmatrix} = \tau_2 \times \sigma_i; \quad \gamma_5 \equiv \gamma^0\gamma^1\gamma^2\gamma^3 = -i\tau_3 \times 1_2$$

The 4×4 matrices α_i and β_i, $i = 1,2,3$ are:

$$\alpha_1 = \tau_2 \times \tau_1, \quad \alpha_2 = 1_2 \times \tau_2, \quad \alpha_3 = \tau_2 \times \tau_3 \ ,$$

$$\beta_1 = \tau_1 \times \tau_2, \quad \beta_2 = \tau_2 \times 1_2, \quad \beta_3 = \tau_3 \times \tau_2$$

satisfying

$$\alpha_i^T = -\alpha_i,\ \beta_i^T = -\beta_i,\ \alpha_i^2 = 1 = \beta_j^2,\ i,j = 1,2,3. \tag{1.B.7}$$

In these conventions, one can immediately verify that

$$\{\Gamma_M,\Gamma_N\} = 2\eta_{MN} = 2\, diag(-1,+1,+1,+1,..,+1)\ \ M,N = 0,1,2,..,10$$

$$\Gamma_0^\dagger = \Gamma_0,\ \Gamma_I^\dagger = -\Gamma_I,\ \ I = 1,2,..,10$$

$$\Gamma_0^T = \Gamma_0,\ \Gamma_2^T = \Gamma_2,\ \Gamma_I^T = -\Gamma_I \text{ for } I \neq 0,2. \tag{1.B.8}$$

The charge conjugation matrix which in 1 + 10 dimensions must satisfy

$$C\Gamma^M C^{-1} = -\Gamma^{M^T} \text{ and } C^T = -C \tag{1.B.9}$$

takes, in our conventions, the explicit form

$$C = i\Gamma^0\Gamma^2 = i\gamma^0\gamma^2 \times 1_8 \tag{1.B.10}$$

and also satisfies

$$C^2 = -1_{32}. \tag{1.B.11}$$

Given the matrices Γ_M, $M = 0,1,2,\ldots,10$ which obey the Dirac algebra, one can prove that the matrices:

$$\Sigma_{MN} \equiv \frac{i}{4}\,[\Gamma_M,\Gamma_N] = \frac{i}{2}\,\Gamma_{MN}, \tag{1.B.12}$$

satisfy the SO(1,10) algebra

$$[\Sigma_{MN},\Sigma_{\Lambda P}] = -i\,(\Sigma_{M\Lambda}\eta_{NP}+\Sigma_{NP}\eta_{M\Lambda}-\Sigma_{MP}\eta_{N\Lambda}-\Sigma_{N\Lambda}\eta_{MP})$$

$$\equiv -4i\Sigma_{[M_{[\Lambda}}\eta_{N]_{P]}} \tag{1.B.13}$$

and, in particular, the matrices Σ_{IJ}, $I,J \neq 0,3$ satisfy the O(9) algebra:

$$[\Sigma_{IJ},\Sigma_{KL}] = 4i\Sigma_{[I_{[K}}\delta_{J]_{L]}}. \tag{1.B.14}$$

We have $[\Gamma^0,\Sigma_{IJ}] = 0$, $I,J \neq 0$, which means that the 32×32 matrices Σ_{IJ} with $I,J = 1,2,\ldots,10$ form a reducible representation of $O(10)$. This can be decomposed into its two 16 dimensional irreducible pieces $(\underset{\sim}{16}+\overline{\underset{\sim}{16}})$ according to the eigenvalue of Γ^0 they correspond to, using the projection matrices:

$$\frac{1\pm\Gamma^0}{2}\,. \tag{1.B.15}$$

Under $O(9)$ the two 16-dimensional representations are equivalent and the decomposition is actually

$$\begin{aligned} \underset{\sim}{32}_{\,spinor} \ of \ SO(1,10) &\sim \underset{\sim}{16} + \overline{\underset{\sim}{16}} \ of \ SO(10) \quad \subseteq \quad SO(1,10) \\ &\sim \underset{\sim}{16} + \underset{\sim}{16} \ of \ SO(9) \quad \subseteq \quad SO(1,10). \end{aligned}$$

We now discuss a few properties of the matrices Γ_M valid in the representation (1.B.6), (1.B.7) and which are used in the text.

We first split the indices $I,J,\ldots \neq 0,3$ into two subsets, denoted by:

$$\begin{aligned} i,j,\ldots &= 1,2 \\ \hat{i},\hat{j},\ldots &= 4,5,\ldots,10 \end{aligned}$$

Statement 1. As can be checked using (1.B.6) and (1.B.7) the matrices Γ_{IJ} have the form:

$$\Gamma_{IJ} = -i\begin{pmatrix} A_{IJ} & 0 & K_{IJ} \\ 0 & B_{IJ} & 0 \\ K_{IJ}{}^{\dagger} & 0 & M_{IJ} \end{pmatrix} \quad I,J \neq 0,3 \tag{1.B.16}$$

where A, K, M are 8×8 matrices, while B is 16×16, and $A_{IJ}{}^{\dagger} = A_{IJ}$, $M_{IJ}{}^{\dagger} = M_{IJ}$, $B_{IJ}{}^{\dagger} = B_{IJ}$ are required by the antihermiticity of the Γ_{IJ}'s.

From the form (1.B.16), it follows that the 16×16 matrices:

$$\tilde{\Gamma}_{IJ} \equiv -i\begin{pmatrix} A_{IJ} & K_{IJ} \\ K_{IJ}{}^{\dagger} & M_{IJ} \end{pmatrix}; \quad I,J \neq 0,3 \tag{1.B.17}$$

form the 16-dimensional spinor representation of O(9), since $\tilde{\Sigma}_{IJ} = \frac{i}{2}\tilde{\Gamma}_{IJ}$ satisfy eq. (1.B.14). The same is true for the B_{IJ} matrices.

Statement 2. The matrices $\tilde{\Gamma}_{ij}$ and $\tilde{\Gamma}_{\widehat{ij}}$ have the form

$$\tilde{\Gamma}_{12} = -i\begin{pmatrix} I_8 & 0 \\ 0 & -I_8 \end{pmatrix}$$

$$\tilde{\Gamma}_{\widehat{ij}} = -i\begin{pmatrix} A_{\widehat{ij}} & 0 \\ 0 & A_{\widehat{ij}} \end{pmatrix}$$

with

$$A^{\dagger}_{\widehat{ij}} = A_{\widehat{ij}} \text{ and } A^{T}_{\widehat{ij}} = -A_{\widehat{ij}}. \tag{1.B.18}$$

As for the matrices $\tilde{\Gamma}_{i\hat{j}}$, they all have the form:

$$\tilde{\Gamma}_{1\hat{j}} = -i\begin{pmatrix} 0 & K_{1\hat{j}} \\ K_{1\hat{j}} & 0 \end{pmatrix}, \quad K^{\dagger}_{1\hat{j}} = K_{1\hat{j}}, \quad K^{T}_{1\hat{i}} = -K_{1\hat{i}}$$

$$\tilde{\Gamma}_{2\hat{j}} = -i\begin{pmatrix} 0 & K_{2\hat{j}} \\ -K_{2\hat{j}} & 0 \end{pmatrix}, \quad K^{\dagger}_{2\hat{j}} = -K_{2\hat{j}}, \quad K^{T}_{2\hat{i}} = -K_{2\hat{j}}. \tag{1.B.19}$$

The proof of this statement is based on the observation that

$$[\gamma^1,\gamma^2] = -2i\, 1_2 \times \sigma_3 = -2i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = -2i\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$(\gamma_5)^2 = -1_4 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad [\gamma^1,\gamma^5] = -2i\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$$

and

$$[\gamma^2,\gamma_5] = -2i\begin{pmatrix} & & 0 & -i \\ & & i & 0 \\ 0 & -i & & \\ i & 0 & & \end{pmatrix}.$$

The relative sign of the (11) and (44) or of the (14) and (41) elements of the above matrices determines the relative signs of the elements of the matrices $\tilde{\Gamma}_{ij}$, $\tilde{\Gamma}_{\widehat{\widehat{ij}}}$ or $\tilde{\Gamma}_{1\hat{i}}$, $\tilde{\Gamma}_{2\hat{i}}$, respectively. The rest is determined by the hermiticity of the $\tilde{\Gamma}$'s and from the symmetry properties:

$$\tilde{\Gamma}_{12}^{T} = \tilde{\Gamma}_{12},\ \tilde{\Gamma}_{\widehat{\widehat{ij}}}^{T} = -\tilde{\Gamma}_{\widehat{\widehat{ij}}},\ \tilde{\Gamma}_{1\hat{i}}^{T} = -\tilde{\Gamma}_{1\hat{i}},\ \tilde{\Gamma}_{2\hat{j}}^{T} = \tilde{\Gamma}_{2\hat{j}}$$

which, in turn, follow from (1.B.8) and (1.B.16).

The matrices $\tilde{\Sigma}_{12}$ and $\tilde{\Sigma}_{\widehat{\widehat{ij}}}$ generate the O(2) × O(7) maximal subalgebra of O(9), while the rest, $\tilde{\Sigma}_{\widehat{ij}}$, belong to O(9) - O(2) × O(7).

Because of (A.18) the matrices $\frac{1}{2} A_{\widehat{ij}}$ satisfy the O(7) algebra forming its eight-dimensional spinor representation.

As explained above, using the Dirac matrices in D-dimensions, one can construct the so-called spinor representations of the O(D) groups. For example, we showed that starting with the matrices Γ_M in (1.B.8), the matrices Σ_{MN} in (1.B.12) satisfy the SO(1,10) algebra (1.B.12) and, in particular, using the matrices Γ_I, $I = 1,2,\dots,10$, one can construct Σ_{IJ} which generate a 32-dimensional reducible representation of SO(10) *subset* SO(1,10).

Assume now that we can construct a matrix C or C' such that:

$$C\Gamma_X C^{-1} = -\Gamma_X^T \text{ and } C^T = -C \ (X = 0,1,\dots,D-1)$$

or

$$C'\Gamma_X C'^{-1} = \Gamma_X^T \text{ and } C'^T = C'.$$

Then in either case we will have:

$$\underline{C}\Sigma_{XY}\underline{C}^{-1} = -\Sigma_{XY}^T$$

with

$$\underline{C} = C \text{ or } C'.$$

But then if Q is a spinor under SO(D), i.e., if under an SO(D) rotation with parameters $\omega_{XY} = -\omega_{YX}$

$$Q \overset{\omega}{\rightarrow} S(\omega)Q = e^{\frac{i}{2}\omega_{XY}\Sigma_{XY}} Q$$

the quantities $\underline{C}Q$ transform according to:

$$\underline{C}Q \overset{\omega}{\rightarrow} \underline{C}S(\omega)Q = (e^{-\frac{i}{2}\omega_{XY}\Sigma_{XY}})^T \underline{C}Q = (S(\omega)^{-1})^T \underline{C}Q.$$

Thus using Q and $\underline{C}Q$ we can immediately build a scalar bilinear in Q, namely:

$$(\underline{C}Q)^T Q = Q^T \underline{C}^T Q$$

is a scalar under SO(D).

The definition of Σ_{XY} and the Dirac algebra lead to:

$$S(\omega)\Gamma_X S^{-1}(\omega) = R_{XY}(\omega)\Gamma_Y$$

with $R(\omega)$ the D $\times$ D orthogonal matrix representing the SO(D) rotation with parameters ω_{XY} in the defining D-dimensional representation.

The quantities

$$Q^T \underline{C}^T \Gamma_{X_1}\Gamma_{X_2}\ldots\Gamma_{X_n} Q$$

are then tensors under SO(D) and, in particular:

$$Q^T \underline{C}^T [\Gamma_X, \Gamma_Y] Q$$

is an antisymmetric two-tensor under SO(D).

Restriction of the values of X and Y gives an antisymmetric two-tensor under the corresponding SO(D') (D' < D).

Appendix C

We want to show that the free Rarita-Schwinger action (1.4.40), including the mass term (1.4.41), is the unique free action (apart from linear field redefinitions) [25] that satisfies the following conditions:

(1). It propagates only spin $\frac{3}{2}$ modes ;

(2). It is linear in the derivative operator ;

(3). It contains no nonlocal terms ;

(4). It yields identical field equations for the two fields it appearing in it.

To this end, we start by recalling the discussion given in ref. [2], and we notice that the condition that no spin-$\frac{1}{2}$ modes propagate can be replaced by the requirement that, on shell, the two conditions $\gamma \cdot \psi = 0$ and $\partial \cdot \psi = 0$ hold. To deal with this problem, we find it convenient to introduce spin-projection operators. This just amounts to rewriting the most general kinetic term linear in the derivative operator for ψ_μ,

$$L = \overline{\psi}_\mu (\alpha_1 \gamma_\mu \partial_\nu + \alpha_2 \gamma_\nu \partial_\mu + \alpha_3 \gamma_\mu \not{\partial} \gamma_\nu + \alpha_4 \not{\partial} \eta_{\mu\nu}) \psi_\nu \tag{1.C.1}$$

in terms of an equivalent, but orthonormal, basis. The projectors we need are:

$$\begin{aligned}
(P^{\frac{3}{2}})_{\mu\nu} &= \eta_{\mu\nu} - \omega_\mu \omega_\nu - \frac{1}{3} \hat{\gamma}_\mu \hat{\gamma}_\nu \ ; \\
(P^{\frac{1}{2}}_{11})_{\mu\nu} &= \frac{1}{3} \hat{\gamma}_\mu \hat{\gamma}_\nu \\
(P^{\frac{1}{2}}_{12})_{\mu\nu} &= \frac{1}{\sqrt{3}} \hat{\gamma}_\mu \omega_\nu \\
(P^{\frac{1}{2}}_{21})_{\mu\nu} &= \frac{1}{\sqrt{3}} \omega_\mu \hat{\gamma}_\nu \\
(P^{\frac{1}{2}}_{22})_{\mu\nu} &= \omega_\mu \omega_\nu \ ,
\end{aligned} \tag{1.C.2}$$

where

$$\hat{\gamma}_\mu = \gamma_\mu - \omega_\mu$$

$$\omega_\mu = \frac{\partial_\mu \partial\!\!\!/}{\Box} \ . \tag{1.C.3}$$

The projectors in eq. (1.C.2) satisfy the following properties:

(1). *orthonormality* : $(P^I{}_{ij})_{\mu\nu}(P^J{}_{kl})_{\nu\rho} = \delta^{IJ}\,\delta_{jk}\,(P^I{}_{il})_{\mu\rho}$;

(2). *decomposition of unity* : $1 = P^{\frac{3}{2}} + (P^{\frac{1}{2}})_{11} + (P^{\frac{1}{2}})_{22}$.

Moreover, they are complete in that they span the set of all operators entering (1.C.1). Thus we start by writing a free massive field equation of the form

$$(C_1 P^{\frac{3}{2}} + c_2 P^{\frac{1}{2}}_{11} + c_3 P^{\frac{1}{2}}_{22} + c_4 P^{\frac{1}{2}}_{12} + c_5 P^{\frac{1}{2}}_{21})_{\mu\nu}\partial\!\!\!/\, \psi_\nu = M\psi_\mu \ . \tag{1.C.4}$$

Next, we require that the operator above square to $\Box P^{\frac{3}{2}}$, which is tantamount to requiring that no spin $\frac{1}{2}$ components propagate, because of the orthogonality properties of the projectors, and of the form of the decomposition of unity. The obvious solution consists in starting with $P^{\frac{3}{2}}\partial\!\!\!/$, but it has to be rejected, as it leads to a nonlocal field equation.

Alternatively, one is led to

$$[P^{\frac{3}{2}} + \alpha(P^{\frac{1}{2}}_{11} + P^{\frac{1}{2}}_{22} + \beta P^{\frac{1}{2}}_{12} + \beta^{-1}P^{\frac{1}{2}}_{21})]_{\mu\nu}\partial\!\!\!/\, \psi_\mu \ . \tag{1.C.5}$$

Demanding that (1.C.5) be a local equation then leads to one relation between α and β, namely

$$2(1-\alpha) = -\sqrt{3}\,\alpha\,(\beta - \beta^{-1}) \ . \tag{1.C.6}$$

We have thus found a one-parameter family of suitable field equations. The next problem is writing an action which yields (1.C.5) upon variation with respect to one of the fields in it. The obvious candidate is:

$$L = \overline{\psi}_\mu [P^{\frac{3}{2}} + \alpha(P^{\frac{1}{2}}_{11} + P^{\frac{1}{2}}_{22} + \beta P^{\frac{1}{2}}_{12} + \beta^{-1} P^{\frac{1}{2}}_{21})]_{\mu\nu} \partial\!\!\!/ \, \psi_\nu + M \overline{\psi}^\mu \psi_\mu \tag{1.C.7}$$

but the problem here is that varying with respect to $\overline{\psi}_\mu$ yields a different equation from what one obtains by varying with respect to ψ_ν. To overcome this difficulty, one shifts the field in the field equation before multiplying with $\overline{\psi}_\mu$, which gives the new action

$$L = \overline{\psi}_\mu [P^{\frac{3}{2}} + \alpha(P^{\frac{1}{2}}_{11} + P^{\frac{1}{2}}_{22} + \beta P^{\frac{1}{2}}_{12} + \beta^{-1} P^{\frac{1}{2}}_{21})]_{\mu\nu} \partial\!\!\!/ \, (\psi_\nu + \lambda \gamma_\nu \gamma \cdot \psi) + M \overline{\psi}_\mu (\psi_\mu + \lambda \gamma_\mu \gamma \cdot \psi) \ . \tag{1.C.8}$$

Varying $\overline{\psi}_\mu$ now gives

$$[P^{\frac{3}{2}} + \alpha(P^{\frac{1}{2}}_{11} + P^{\frac{1}{2}}_{22} + \beta P^{\frac{1}{2}}_{12} + \beta^{-1} P^{\frac{1}{2}}_{21})]_{\mu\nu} \partial\!\!\!/ \, (\psi_\nu + \lambda \gamma_\nu \gamma \cdot \psi) = M (\psi_\mu + \lambda \gamma_\mu \gamma \cdot \psi) \ , \tag{1.C.9}$$

which, upon iteration, becomes

$$\Box (P^{\frac{3}{2}})_{\mu\nu} (\psi_\nu + \lambda \gamma_\nu \gamma \cdot \psi) = M^2 (\psi_\mu + \lambda \gamma_\mu \gamma \cdot \psi) \ . \tag{1.C.10}$$

Multiplying both sides of this equation by $P^{\frac{3}{2}}$ then gives:

$$\psi_\mu + \lambda \gamma_\mu \gamma \cdot \psi = P^{\frac{3}{2}} (\psi_\mu + \lambda \gamma_\mu \gamma \cdot \psi) \tag{1.C.11}$$

and, contracting with γ_μ,

$$(1 + 4\lambda) \gamma \cdot \psi = 0 \ . \tag{1.C.12}$$

Consequently, the condition that $\gamma \cdot \psi$ vanish on-shell is preserved by the new choice of action, at least provided $\lambda \neq -\frac{1}{4}$. In terms of the projectors (1.C.2), the l.h.s. of the field equation (1.C.9) reads

$$[P^{\frac{3}{2}} + \alpha([1 + \lambda(3 - \sqrt{3}\beta)] P^{\frac{1}{2}}_{11} + [1 + \lambda(1 - \sqrt{3}\beta^{-1})] P^{\frac{1}{2}}_{22} +$$

$$[\beta + \lambda(\beta - \sqrt{3})]\,P^{\frac{1}{2}}_{12} + [\beta^{-1} + \lambda(3\beta^{-1} - \sqrt{3})]\,P^{\frac{1}{2}}_{21}\,]_{\mu\nu}\,\not\partial\,\psi_\nu \,. \qquad (1.C.13)$$

To derive the corresponding equation that follows from varying with respect to ψ_ν, we first note that $P^{\frac{1}{2}}_{12}$ and $P^{\frac{1}{2}}_{21}$ turn into each other upon Majorana flipping ~~and anticommute with $\not\partial$, whereas the other projectors are~~ left unchanged by Majorana flipping and commute with $\not\partial$. Consequently we obtain:

$$\Big\{ P^{\frac{3}{2}} + \alpha\big([1 + \lambda(3 - \sqrt{3}\beta)]\,P^{\frac{1}{2}}_{11} + [1 + \lambda(1 - \sqrt{3}\beta^{-1})]\,P^{\frac{1}{2}}_{22} -$$

$$[\beta + \lambda(\beta - \sqrt{3})]\,P^{\frac{1}{2}}_{21} - [\beta^{-1} + \lambda(3\beta^{-1} - \sqrt{3})]\,P^{\frac{1}{2}}_{12}\Big\}_{\mu\nu}\,\not\partial\,\psi_\nu \,, \qquad (1.C.14)$$

and comparison with (1.C.13) yields one single condition that determines λ to be

$$\lambda = \frac{\beta + \beta^{-1}}{2\sqrt{3} - \beta - 3\beta^{-1}} \qquad (1.C.15)$$

The final result is:

$$L = \overline{\psi}_\mu \Big\{ P^{\frac{3}{2}} + \frac{2}{(\beta - \sqrt{3}\beta^{-1})^2}\,[P^{\frac{1}{2}}_{22}\beta^{-1} - \beta P^{\frac{1}{2}}_{11} + P^{\frac{1}{2}}_{12} - P^{\frac{1}{2}}_{21}]\Big\}_{\mu\nu}\,\not\partial\,\psi_\nu$$

$$+\, M\,\overline{\psi}_\mu \Big\{ P^{\frac{3}{2}} + \frac{\beta + \beta^{-1}}{(\beta - \sqrt{3}\beta^{-1})^2}\,(P^{\frac{1}{2}}_{12} + P^{\frac{1}{2}}_{21}) + \frac{2\sqrt{3} - 2\beta^{-1}}{(\beta - \sqrt{3}\beta^{-1})^2}$$

$$+ \frac{2\sqrt{3} + 2\beta}{(\beta - \sqrt{3}\beta^{-1})^2}\,P^{\frac{1}{2}}_{11} \qquad (1.C.16)$$

Apart from its unfamiliar-looking form, this expression gives a one-parameter family of actions which can be seen to reduce to the usual form (1.4.40) of the Rarita-Schwinger action in the limit $\beta \to \infty$. These actions in the massless case possess the gauge invariance

$$\delta \psi_\mu = \gamma_\mu \partial\!\!\!/ \, \varepsilon - (\sqrt{3}\beta + 1)\, \partial_\mu \varepsilon \tag{1.C.17}$$

which also reduces to the standard form in the limit $\beta \to \infty$. The one-parameter family of actions in eq. (1.C.16) satisfies all the four conditions listed at the beginning of this appendix, and it may appear puzzling that one still has the freedom of shifting the field *in the action* (not in a field equation, as we did to derive (1.C.16)) according to

$$\psi_\mu \to \psi_\mu + \tau \gamma_\mu \gamma \cdot \psi \tag{1.C.18}$$

as this clearly preserves the symmetry of the differential operator in (1.C.16). We can do this for example by rewriting (1.C.17) in terms of the projectors as

$$\psi_\mu \to [P^{\frac{3}{2}} + (1 + 3\tau)P^{\frac{1}{2}}_{11} + (1 + \tau)P^{\frac{1}{2}}_{22} + \sqrt{3}\tau (P^{\frac{1}{2}}_{12} + P^{\frac{1}{2}}_{21})]_{\mu\nu} \psi_\nu \tag{1.C.19}$$

This essentially replaces some γ-matrix algebra by the multiplication rules for the projection operators.

The uniqueness of the Rarita-Schwinger action follows from one peculiar feature of (1.C.16): if the fields are shifted in the action according to eq. (1.C.19), the *net result* is that β transforms according to

$$\beta \to \frac{-1}{\sqrt{3}} + \frac{(\beta + \frac{1}{\sqrt{3}})(1 + 4\tau)}{1 + \tau + \sqrt{3}\tau\beta} \tag{1.C.20}$$

This in turn implies that every value of β can be reached from any other one via a simple field redefinition, and therefore the uniqueness of the Rarita-Schwinger action.

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Chapter 2

Ten-Dimensional Supersymmetric Yang-Mills Theory in Terms of Four-Dimensional Superfields

2.1. Introduction

The N=4 supersymmetric Yang-Mills theory (SSYM) [1,2] is very interesting for several reasons. It is the maximally supersymmetric interacting theory not containing gravity, and as such is unique. Like N=8 supergravity [3], it is a limit of a ten-dimensional superstring theory [4]. It is invariant under a very large symmetry group, the N=4 superconformal group. Finally, and most importantly, its perturbation expansion appears to be free of ultraviolet divergences [5], even after the addition of suitable mass terms breaking the symmetries of the theory [6]. While finiteness is not presently regarded as a crucial feature for a renormalizable theory, it offers some hope that supersymmetry may eliminate the nonrenormalizable ultraviolet divergences of gravity in the framework of the N=8 supergravity theory, or in the framework of the theory of superstrings [7].

The original component formulation of N=4 SSYM was obtained from the N=1 SSYM theory in ten dimensions [1,2], a theory consisting simply of a Majorana-Weyl spinor in the adjoint representation of a gauge group, minimally coupled to a Yang-Mills boson. The ten-dimensional theory is reduced to four dimensions by assuming independence of the fields on six of the ten space-time coordinates. The resulting component theory consists of a vector, four Weyl spinors and six scalars, all in the adjoint representation of the gauge group. Correspondingly, the simple ten-dimensional supersymmetry breaks into four four-dimensional supersymmetries, and the Lorentz group breaks into the direct product of the four-dimensional Lorentz group and a global internal $SU(4) \approx SO(6)$ group. In the component formulation the $SU(4)$ symmetry is manifest, but the four supersymmetries are not.

As the main problem with making supersymmetries manifest is the existence of auxiliary fields for the off-shell closure of the supersymmetry algebra, one approach is to forego the use of auxiliary fields altogether, eliminate all the gauge and auxiliary degrees of freedom, and formulate the theory in terms of light-cone superfields [5,8]. The Bose and Fermi degrees of freedom then match even off shell, but manifest Lorentz covariance is lost. Each covariant supersymmetry spinor generator splits into two parts, one of which is a manifest symmetry of the resulting action. The action also has a manifest internal SO(4), or even the full SU(4) [8,9] invariance. However, only an E_2 subgroup of the Lorentz group is linearly realized on the fields.

A formulation of N=4 SSYM in terms of N=1 covariant superfields is also known [10,11]. In this case the Lorentz symmetry, one of the supersymmetries and an $SU(3)\otimes U(1)$ subgroup of the SU(4) are manifest, but the extra supersymmetries and the $SU(4)/(SU(3)\otimes U(1))$ symmetries are realized as complicated nonlinear transformations of the superfields. Recent progress in understanding the superspace formulation of N=1 SSYM in six dimensions [12,13] (which yields N=2 SSYM in four dimensions upon dimensional reduction) also allows a formulation in terms of N=2 superfields. The ultimate goal in this kind of approach would be a formulation in terms of N=4 superfields. This would also be of interest for the superspace formulation of N=4 supergravity, where N=4 SSYM would enter as a compensator†. The formulation in terms of N=4 superfields, if it exists, is expected to possess uncommon features, in order to circumvent, for instance, the counting argument of ref. [14], which suggests that a set of auxiliary fields leading to closure of the supersymmetry algebra cannot be found for N=4 SSYM, or for N=1 SSYM in ten dimensions. In this context, a clear sign of trouble would be finding ultraviolet divergences in six dimensions at two loops, as suggested by superstring counting arguments [4],

† See ref. [19] for a discussion of compensators.

but excluded by superspace counting arguments based on the assumption that an N=4 superfield formulation exists [15]. This problem is currently under investigation.

A possible way to attack the auxiliary field problem is to first seek an off-shell formulation of the ten-dimensional theory in terms of covariant superfields, and then derive the four-dimensional N=4 superfield formulation by dimensional reduction in superspace, in analogy with what was originally done in components. For this approach, it is interesting to investigate what our present, incomplete understanding of the four-dimensional problem can teach us about the ten-dimensional theory. This requires undoing a dimensional reduction (an operation that might be called dimensional oxidation!), and is the subject of the present chapter. Here we show that it is possible to extend the known formulation of N=4 SSYM in terms of N=1 four-dimensional superfields (extended to depend on all ten space-time coordinates) to provide an interesting, if somewhat unusual, description of the ten-dimensional theory. While several symmetries of the action are not manifest, this formalism is the only one known in which *all the fields are geometrical objects*. This theory can be dimensionally reduced, in the normal manner, to give a four-dimensional superspace formulation in any $4 \leq D \leq 10$. This is the first instance in which four dimensional superfields have been used to describe a higher-dimensional theory. For the time being, this result may be regarded as an arcane application of four-dimensional unextended superspace. Our hope, however, is that it may also serve as a useful starting point in the search for the complete off-shell ten-dimensional action. In fact, the formalism does suggest one tensor auxiliary field that should be included in the complete off-shell theory.

The plan of this chapter is as follows. We start in Section 2 by describing the component form of ten-dimensional supersymmetric Yang-Mills theory [1,2].

This is then dimensionally reduced in section 3, thus recovering the SU(4) covariant form of N=4 SSYM in four dimensions [2]. Section 4 addresses the question of the existence of the auxiliary fields for this theory. Here we show how considering massive representations of supersymmetry provides enough information to recover the sets of auxiliary fields for some simple multiplets [16]. Whereas the results we present here are very preliminary, and the auxiliary fields are known already for the multiplets we discuss, the method we introduce gives nonetheless a very simple way of deriving them. It also allows us to explain better the problems encountered with N=4 SSYM, and in particular to describe the argument of Siegel and Roček [14] that indicates that no auxiliary fields exist that close the supersymmetry algebra off the mass-shell for N=4 SSYM. The rest of this chapter is devoted to a description of N=4 SSYM in terms of N=1 superfields and of the corresponding new ten-dimensional action. These sections are based on a paper written by the author on the subject, in collaboration with Neil Marcus and Warren Siegel [18]. The appendices contain a few comments about notation and conventions, as well as the derivation of a formula used in proving the gauge invariance of the ten-dimensional action.

2.2. The component form of ten-dimensional SSYM [1,2]

The ten-dimensional supersymmetric Yang-Mills theory describes the interactions of an adjoint multiplet of Majorana-Weyl spinors, minimally coupled to Yang-Mills bosons. The Majorana condition

$$\lambda = C\bar{\lambda}^T \tag{2.2.1}$$

and the Weyl condition

$$\lambda = \Gamma_{11}\lambda \tag{2.2.2}$$

are compatible in ten-dimensional spacetime, and reduce the 32 complex components of λ to 16 real components off the mass-shell, which correspond to 8 independent propagating components. This number equals the number of propagating components of a massless vector in ten dimensions, which suggests that the model should be supersymmetric. We write the ten-dimensional action as

$$S = \mathrm{tr}\int d^{10}x \left\{ -\frac{1}{4} F^{\hat{\mu}\hat{\nu}} F_{\hat{\mu}\hat{\nu}} - \frac{i}{2}\bar{\lambda}[\nabla\!\!\!/ ,\lambda] \right\}, \tag{2.2.3}$$

where

$$A_{\hat{\mu}} = A_{\hat{\mu}}^{\;a}\, T^a \quad , \quad \lambda = \lambda^a\, T^a \quad , \tag{2.2.4}$$

with a an adjoint index and

$$\mathrm{tr}(T^a\, T^b) = \delta^{ab} \quad , \tag{2.2.5}$$

and the covariant derivative is

$$(\nabla_{\hat{\mu}})^{ac} = \delta^{ac}\,\partial_{\hat{\mu}} + g\, f^{abc}\, A^b_{\hat{\mu}} \quad , \tag{2.2.6}$$

with the hatted indices running from zero to nine. The non-Abelian field strength is therefore

$$F^a_{\widehat{\mu\nu}} = \partial_{\hat{\mu}} A^a_{\hat{\nu}} - \partial_{\hat{\mu}} A^a_{\hat{\nu}} + g\, f^{abc} A^b_{\hat{\mu}} A^c_{\hat{\nu}} \quad . \tag{2.2.7}$$

Indeed, as we now show, the supersymmetry transformations are

$$\begin{aligned} \delta A^a_{\hat{\mu}} &= -i\, \bar{\varepsilon} \Gamma_{\hat{\mu}} \lambda^a \\ \delta \lambda^a &= \frac{1}{2} \Gamma_{\widehat{\mu\nu}} F^{a\widehat{\mu\nu}} \varepsilon \quad . \end{aligned} \tag{2.2.8}$$

Varying S and substituting in it the transformations (2.2.8) yields

$$\begin{aligned} \delta S = \operatorname{tr} \int d^{10}x \, \Big\{ &-i (\bar{\varepsilon} \Gamma_{\hat{\nu}} \lambda) [\nabla_{\hat{\mu}}, F^{\widehat{\mu\nu}}] + \frac{i}{2}\, \bar{\varepsilon} \Gamma_{\widehat{\mu\nu}} \Gamma^{\hat{\rho}} [\nabla_{\hat{\rho}}, \lambda] F^{\widehat{\mu\nu}} \\ &+ \frac{1}{2} \bar{\lambda} [(\bar{\varepsilon} \Gamma_{\hat{\mu}} \lambda) \Gamma^{\hat{\mu}}, \lambda] \Big\} \quad . \end{aligned} \tag{2.2.9}$$

Then, composing the Γ matrices according to

$$\Gamma_{\widehat{\mu\nu}} \Gamma^{\hat{\rho}} = \Gamma^{\hat{\rho}}{}_{\widehat{\mu\nu}} + \delta^{\hat{\rho}}_{\hat{\nu}} \Gamma_{\hat{\mu}} - \delta^{\hat{\rho}}_{\hat{\mu}} \Gamma_{\hat{\nu}} \quad , \tag{2.2.10}$$

and integrating by parts the covariant derivative onto F, leads to cancellation of the first term in eq. (2.2.9), and one is left with

$$\delta S = \operatorname{tr} \int d^{10}x \, \Big\{ \frac{i}{2} (\bar{\varepsilon} \Gamma^{\hat{\rho}}{}_{\widehat{\mu\nu}} \lambda) [\nabla_{\hat{\rho}}, F^{\widehat{\mu\nu}}] + \frac{i}{2} \bar{\lambda} [(\bar{\varepsilon} \Gamma_{\mu} \lambda) \Gamma^{\hat{\mu}}, \lambda] \Big\} \quad , \tag{2.2.11}$$

the first term of which vanishes on account of the Bianchi identity

$$[\nabla_{[\hat{\mu}}, F_{\widehat{\nu\rho}]}] = 0 \quad . \tag{2.2.12}$$

Then

$$\delta S = \operatorname{tr} \int d^{10}x \, \Big\{ \frac{i}{2} (\bar{\varepsilon} \Gamma_{\hat{\mu}} \lambda) \bar{\lambda} [\Gamma^{\hat{\mu}}, \lambda] \Big\} \quad . \tag{2.2.13}$$

This term can then be analyzed by means of the Fierz identity which, for ten-dimensional Majorana-Weyl spinors, is written

$$\lambda^b \bar{\lambda}^a = \frac{-1}{16} \Gamma^{\hat{\mu}} (\bar{\lambda}^a \Gamma_{\hat{\mu}} \lambda^b) + \frac{1}{6 \cdot 16} \Gamma^{\widehat{\mu\nu\rho}} (\bar{\lambda}^a \Gamma_{\widehat{\mu\nu\rho}} \lambda^b)$$

$$- \frac{1}{32 \cdot 5!} \Gamma^{\hat{\mu}_1 \cdot\cdot \hat{\mu}_5} (\bar{\lambda}^a \Gamma_{\hat{\mu}_1 \cdot\cdot \hat{\mu}_5} \lambda^b) \ . \tag{2.2.14}$$

More convenient for our purposes is the identity we can derive from (2.2.14) by multiplying from the left by $\Gamma^{\hat{\alpha}}$ and from the right by $\Gamma_{\hat{\alpha}}$, since then

$$\Gamma^{\hat{\alpha}} \Gamma_{\hat{\mu}_1 \cdot\cdot \hat{\mu}_5} \Gamma_{\hat{\alpha}} = 0 \ , \tag{2.2.15}$$

and one is left with

$$\Gamma^{\hat{\alpha}} \lambda^b \bar{\lambda}^a \Gamma_{\hat{\alpha}} = \frac{1}{2} \Gamma^{\hat{\mu}} (\bar{\lambda}^a \Gamma_{\hat{\mu}} \lambda^b) - \frac{1}{24} \Gamma^{\widehat{\mu\nu\rho}} (\bar{\lambda}^a \Gamma_{\widehat{\mu\nu\rho}} \lambda^b) \ . \tag{2.2.16}$$

Then

$$f^{abc} (\bar{\varepsilon} \Gamma_{\hat{\mu}} \lambda^a)(\bar{\lambda}^b \Gamma^{\hat{\mu}} \lambda^c) = \frac{1}{2} f^{abc} (\bar{\varepsilon} \Gamma_{\hat{\mu}} \lambda^c)(\bar{\lambda}^a \Gamma^{\hat{\mu}} \lambda^b)$$

$$- \frac{1}{24} f^{abc} (\bar{\varepsilon} \Gamma^{\widehat{\mu\nu\rho}} \lambda^c)(\bar{\lambda}^a \Gamma_{\widehat{\mu\nu\rho}} \lambda^b) \ , \tag{2.2.17}$$

where the second term vanishes because it is symmetric under the interchange of λ^a and λ^b. Relabeling cyclically the indices in the first term on the r.h.s. of eq. (2.2.17) then leads to

$$f^{abc} (\bar{\varepsilon} \Gamma_{\hat{\mu}} \lambda^a)(\bar{\lambda}^b \Gamma^{\hat{\mu}} \lambda^c) = 0 \ , \tag{2.2.18}$$

which proves the supersymmetry of the theory.

2.3. Dimensional Reduction to Four Dimensions and N=4 SSYM [2]

Dimensionally reducing a higher-dimensional field theory means imposing that its fields depend only on a subset of the total set of coordinates. In particular, to reduce the ten-dimensional theory of section 2 to four dimensions, we assume that both $A^a_{\hat{\mu}}$ and λ^a depend on x^0, x^1, x^2 and x^3 only, thus dropping derivatives with respect to the extra six spatial coordinates. Expanding the kinetic term of the non-Abelian vector field in eq. (2.2.3) gives:

$$-\frac{1}{4}F^a_{\widehat{\mu\nu}}F^{a\widehat{\mu\nu}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}F^a_{\mu I}F^{a\mu I} - \frac{1}{4}F^a_{IJ}F^{aIJ} \quad , \qquad (2.3.1)$$

where μ, ν run from zero to three , I and J are SO(6) vector indices, and

$$\begin{aligned} -\frac{1}{2}F^a_{\mu I}F^{a\mu I} &= -\frac{1}{2}(\partial_\mu A^a_I + g\, f^{abc} A^b_\mu A^c_I)(\partial^\mu A^{aI} + g\, f^{ade} A^{d\mu} A^{eI}) \\ &= -\frac{1}{2}(D_\mu A^a_I)(D^\mu A^{aI}) \qquad (2.3.2) \end{aligned}$$

gives four-dimensional gauge-covariantized kinetic terms for the six scalars A^a_I, once the derivatives ∂_I have been dropped. Moreover,

$$-\frac{1}{4}F^a_{IJ}F^{aIJ} = -\frac{g^2}{4} f^{abc} f^{ade} A^b_I A^c_J A^{dI} A^{eJ} \qquad (2.3.3)$$

gives quartic self-interactions for the six scalars. The dimensional reduction for the bosons thus gives:

$$-\frac{1}{4}F^a_{\mu\nu} - \frac{1}{2}(D_\mu A^a_I)(D^\mu A^{aI}) - \frac{g^2}{4} f^{abc} f^{ade} A^b_I A^c_J A^{dI} A^{eJ} \quad . \qquad (2.3.4)$$

The SO(6) notation for the scalars $A^a{}_I$ can be translated into SU(4) notation by regrouping the six scalars into a self-dual antisymmetric tensor of SU(4) as follows. One defines

$$\varphi^a_{14} = \frac{1}{\sqrt{2}}\,(A^a{}_4 + i\,A^a{}_5)$$

$$\varphi^a_{24} = \frac{1}{\sqrt{2}}\,(A^a{}_6 + i\,A^a{}_7)$$

$$\varphi^a_{34} = \frac{1}{\sqrt{2}}\,(A^a{}_8 + i\,A^a{}_9) \quad . \tag{2.3.5}$$

Then one fixes the remaining components of φ^a_{ij} by antisymmetry and by imposing the duality-reality condition

$$\varphi^{aij} = \frac{1}{2}\,\varepsilon^{ijkl}\,\varphi^a_{kl} = (\varphi^a_{ij})^* \quad . \tag{2.3.6}$$

What is important for our purposes is that the choice of normalization in eq. (2.3.5) gives

$$A^a_I A^{bI} = \varphi^a_{ij}\,\varphi^{aij} \quad . \tag{2.3.7}$$

On account of the self-duality, pairs of SU(4) indices can be freely raised and lowered in the contraction of two φ's. The reduction of the vector kinetic term thus takes the final form

$$-\frac{1}{4}F^a_{\mu\nu}\,F^{a\mu\nu} - \frac{1}{2}(D_\mu\varphi^a_{ij})(D^\mu\varphi^{aij}) - \frac{g^2}{4}f^{abc}f^{ade}\,\varphi^b_{ij}\,\varphi^c_{kl}\,\varphi^{dij}\,\varphi^{ekl} \tag{2.3.8}$$

Next one considers the spinor kinetic term, which is written

$$-\frac{i}{2}\,\bar\lambda^a\,\Gamma^\mu[\nabla_\mu,\lambda]^a - \frac{i\,g}{2}\,f^{abc}\,(\bar\lambda^a\,\Gamma^I\,\lambda^c)\,A^b_I \quad . \tag{2.3.9}$$

In terms of the SU(4) notation for the six-dimensional indices, this becomes

$$-\frac{i}{2}\,\bar\lambda^a\,\Gamma^\mu[\nabla_\mu,\lambda]^a - \frac{i\,g}{2}\,f^{abc}\,(\bar\lambda^a\,\Gamma^{ij}\,\lambda^c)\,\varphi^b_{ij} \quad , \tag{2.3.10}$$

where the definitions of the Γ_{ij} in terms of the Γ_I are the same as those of the φ^a_{ij} in terms of the A^a_I. The ten-dimensional Majorana-Weyl spinor λ^a, however, is not an irreducible representation of the four-dimensional Lorentz group, and to complete the reduction one must split it accordingly. To this end one needs a suitable representation of the ten-dimensional Γ matrices, which we take to be

$$\Gamma_\mu = \gamma_\mu \otimes 1_8 \qquad (\mu = 0,..,3) \quad ,$$

$$\Gamma_4 = -\gamma_5 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \ ,$$

$$\Gamma_5 = \gamma_5 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \ ,$$

$$\Gamma_6 = \gamma_5 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \ ,$$

$$\Gamma_7 = \gamma_5 \otimes \sigma_2 \otimes \sigma_2 \otimes 1_2 \ ,$$

$$\Gamma_8 = -\gamma_5 \otimes \sigma_1 \otimes 1_2 \otimes \sigma_2 \ ,$$

$$\Gamma_9 = \gamma_5 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \ , \tag{2.3.11}$$

where γ_μ is a 4 x 4 representation of the four-dimensional Dirac algebra, σ_i are the Pauli matrices, 1_2 is the 2 x 2 unit matrix and $\otimes$ denotes outer product of two matrices. Correspondingly, the Γ_{ij} take the form

$$\Gamma_{ij} = \frac{1}{\sqrt{2}} \gamma_5 \otimes \begin{bmatrix} 0 & (\rho_{ij}) \\ (\rho'_{ij}) & 0 \end{bmatrix} \ , \tag{2.3.12}$$

where

$$(\rho_{ij})_{kl} = \delta_{ik}\, \delta_{jl} - \delta_{il}\, \delta_{jk}$$

$$(\rho'_{ij})_{kl} = \varepsilon_{ijkl} \ . \tag{2.3.13}$$

The charge conjugation matrix induced by the choice (2.3.11) is then

$$C_{10} = C_4 \otimes \begin{bmatrix} 0 & 1_4 \\ 1_4 & 0 \end{bmatrix} \ , \tag{2.3.14}$$

where 1_4 denotes a 4 x 4 unit matrix, and the ten-dimensional helicity matrix Γ_μ is:

$$\Gamma_{11} = i\, \Gamma_0 .. \Gamma_9 = -\gamma_5 \begin{bmatrix} 1_4 & 0 \\ 0 & -1_4 \end{bmatrix} \ . \tag{2.3.15}$$

The ten-dimensional Majorana-Weyl spinor λ is obtained from a general 32-component spinor by imposing the constraints (2.2.1) and (2.2.2). The result can be expressed in terms of four independent four-dimensional Weyl spinors and their Majorana conjugates, and takes the form

$$\lambda = \begin{bmatrix} L\,\chi^i \\ R\,\tilde{\chi}_i \end{bmatrix} , \tag{2.3.16}$$

where

$$L = \frac{1}{2}(1+\gamma_5) \tag{2.3.17a}$$

and

$$R = \frac{1}{2}(1-\gamma_5) \tag{2.3.17b}$$

are the usual four-dimensional helicity projectors. Corresponding to eq. (2.3.16), we have

$$\bar{\lambda} = (\bar{\chi}_i\,R \;\; ; \;\; \tilde{\bar{\chi}}^i\,L) \ , \tag{2.3.18}$$

and the first term in eq. (2.3.10) translates into

$$-\frac{i}{2}(\bar{\chi}_i\,R\,\gamma^\mu \;\; ; \;\; \tilde{\bar{\chi}}^i\,L\,\gamma^\mu) \begin{bmatrix} [\nabla_\mu, L\chi^i] \\ [\nabla_\mu, R\tilde{\chi}_i] \end{bmatrix} \tag{2.3.19}$$

which, expanding and undoing the Majorana conjugations, can be recast into the familiar form for the kinetic term for four Weyl spinors:

$$-i\;\bar{\chi}_i\,\gamma^\mu\,[\nabla_\mu, L\,\chi^i] \ . \tag{2.3.20}$$

To conclude the construction of the four-dimensional action, one must reduce the second term in eq. (2.3.10), which will give Yukawa couplings between the six scalars and the four Weyl spinors. To this end, we write

$$\bar{\lambda}\Gamma_{ij}\,\lambda = \frac{i}{\sqrt{2}}\,(\bar{\chi}_i\,R\,\gamma^{\mu}\,;\,\tilde{\bar{\chi}}^i\,L\,\gamma^{\mu}) \begin{bmatrix} 0 & (\rho_{ij}) \\ (\rho'_{ij}) & 0 \end{bmatrix} \begin{bmatrix} L\chi^i \\ R\tilde{\chi}_i \end{bmatrix} . \tag{2.3.21}$$

Consequently

$$-\frac{ig}{2} f^{abc}\,\varphi^{bij}\,\bar{\lambda}^a\,\Gamma_{ij}\,\lambda^c = \frac{g}{\sqrt{2}} f^{abc}\,(\varphi^c_{ij}\,\tilde{\bar{\chi}}^{ai}\,L\,\chi^{bj} - \varphi^{cij}\,\bar{\chi}^a_i R\,\tilde{\chi}^b_j) \tag{2.3.22}$$

using the definitions of ρ_{ij} and ρ'_{ij} given in eq. (2.3.13) and the self-duality of φ^c_{ij}.

The final result is therefore

$$S = \int d^4x \left\{ \frac{-1}{4} F^a_{\mu\nu}\,F^{a\mu\nu} - \frac{1}{2}(D_\mu \varphi^a_{ij})(D^\mu \varphi^{aij}) - \right.$$

$$\frac{g^2}{4} f^{abc} f^{ade}\,\varphi^b_{ij}\,\varphi^c_{kl}\,\varphi^{dij}\,\varphi^{ekl} - i\,\bar{\chi}^a{}_i\,\Gamma^\mu D_\mu (L\,\chi^{ai})$$

$$\left. + \frac{g}{\sqrt{2}} f^{abc}\,(\varphi^c_{ij}\,\tilde{\bar{\chi}}^{ai}\,L\,\chi^{bj} - \varphi^{cij}\,\bar{\chi}^{ai}\,R\,\tilde{\chi}^b_j) \right\} , \tag{2.3.23}$$

which is the N=4 SSYM theory in four dimensions, written in manifestly SU(4) covariant form. Working with Majorana spinors rather than with Weyl spinors in four dimensions would lead to an SO(4) invariant form of N=4 SSYM. One should notice that the SU(4) invariance cannot be extended to a $U(4)$ invariance, because of the duality condition on φ^a_{ij}.

To conclude, we notice that the ten-dimensional supersymmetry transformations in eqs. (2.2.8) translate in four dimensions into

$$\delta A^a_\mu = i\,(\bar{\varepsilon}_i\,\gamma_\mu L\,\chi^{ia} - \bar{\chi}^a_i \gamma_\mu L\,\varepsilon^i) ,$$

$$\delta\varphi^a_{ij} = \frac{i}{\sqrt{2}}(\bar{\chi}_i\,R\,\tilde{\chi}_j - \bar{\chi}_j\,R\,\tilde{\chi}_i) - \frac{i}{\sqrt{2}}\,\varepsilon_{ijkl}\,\tilde{\bar{\chi}}^k\,L\,\chi^l ,$$

$$\delta(L\,\chi^a{}_i) = \frac{1}{2}\gamma_{\mu\nu} F^a_{\mu\nu}\,L\,\varepsilon^i - \frac{1}{\sqrt{2}}(D^\mu \varphi^{aij})\,\gamma_\mu R\,\tilde{\varepsilon}_j$$

$$+ J^{abc} \varphi^{bij} \varphi^{c}_{jk} L \varepsilon^{k} \quad , \tag{2.3.24}$$

where the ten-dimensional Majorana-Weyl spinor parameter ε has been split into four four-dimensional Weyl spinor parameters ε^i,in analogy with what was done for λ, i.e.

$$\varepsilon = \begin{pmatrix} L\,\varepsilon^i \\ R\,\tilde{\varepsilon}_i \end{pmatrix} \quad . \tag{2.3.25}$$

2.4. Auxiliary Fields and N=4 SSYM

In chapter 1 we have shown how the sets of auxiliary fields necessary to close the supersymmetry algebra emerge in the two very simple cases of the Wess-Zumino model and of the N=1 SSYM. These two examples have been chosen because of their simplicity, but they illustrate very clearly several features of the general case. First of all, the nonclosure of the supersymmetry algebra always manifests itself as a consequence of Fierz rearrangements one has to perform when computing the commutator on the Fermi fields. Another way of looking at the thing is saying that, in a theory without auxiliary fields, one can have nonclosure only on the fermions, as the supersymmetry algebra produces one spatial derivative, which is enough to generate a fermionic equation of motion, but is not enough to generate a bosonic equation of motion. The second remark is that the search for auxiliary fields is essentially a problem concerning free field theories. The inclusion of interactions is only a technical complication. It can at best select some of the sets of auxiliary fields one determines in the free case as the only consistent ones, but the mechanism responsible for the cancellations is already clearly operating at the level of free theories. From now on we will therefore deal exclusively with free field theories. The next observation we want to make is that there is evidently some sort of rationale that selects one set of auxiliary fields rather than another for a given model. The first rule of thumb we have seen operating is that the auxiliary fields add as many nonpropagating degrees of freedom[†] as are needed to balance the numbers of bose and Fermi degrees of freedom off-shell. There is actually more to it, as in the case of the N=1 SSYM this would just tell us that we need one extra bose degree of freedom whereas, as we have seen, closure can only be

[†] We have only encountered bosonic auxiliary fields, but in more complicated cases also fermionic auxiliary fields are found.

achieved provided one adds a *pseudoscalar* auxiliary field D.

In chapter 1 we have discussed the particle representations of the super-Poincaré algebra for the two cases of massive and massless states. Actually, we have stressed that massless representations lead to the more interesting models, just because gauge invariance demands that the fields in multiplets containing gauge fields be exactly massless. As we will now see, massive representations of supersymmetry serve another, equally interesting purpose, as they *determine the sets of auxiliary fields closing the supersymmetry algebra of the massless models off-shell* [16]†. All we need to recall is that massive representations of simple supersymmetry contain two chains of spin J, one chain of spin $(J+\frac{1}{2})$ and one chain of spin $(J-\frac{1}{2})$. Moreover, the last two chains always have opposite parity. Thus, if we consider the N=1 SSYM, we see directly that the vector A_μ describes a spin-1 chain off-shell (the time component can be gauged away). Moreover, the Majorana spinor λ describes one propagating spin-$\frac{1}{2}$ multiplet and one auxiliary spin-$\frac{1}{2}$ multiplet, and completing a massive multiplet necessarily requires a pseudoscalar field. In the same way, the Wess-Zumino model requires the addition of one scalar and one pseudoscalar auxiliary field to achieve off-shell closure of the algebra, because again the Majorana spinor λ describes one propagating and one auxiliary spin-$\frac{1}{2}$ multiplet, and one scalar and one pseudoscalar is all one needs to complete two massive representations of supersymmetry.

The procedure is completely general. One starts with the on-shell content of the multiplet, and determines what massive representations are described by the given fields off-shell. Then one fits these components in massive

† de Wit and Ferrara [17] and Siegel and Roček [14] have previously remarked that a close connection exists between massive representation of supersymmetry and off-shell massless representations. These authors, however, do not use this as a quantitative tool for predicting the form of off-shell supermultiplets.

representations of supersymmetry, and the missing components, determined in number, spin and parity, are the auxiliary fields that are needed to close the off-shell algebra. These results are very preliminary, and the procedure has been applied so far only to a few relatively simple cases, apart from the two described above, namely N=1 and N=2 supergravity, N=2 SSYM and the N=2 scalar multiplet (in this case limitedly to the simplest solution, involving an off-shell central charge), reproducing *unambiguously* the known sets of auxiliary fields. We found it worthwhile to present this discussion because the result is so simple and elegant, and already adds some clarity to the description of off-shell multiplets. The more complicated cases of the N=2 scalar multiplet with vanishing off-shell central charge, of the N=3 supergravity, and of the N=4 SSYM are under investigation. The crucial step in being able to carry out this procedure up to the end is choosing the form for the nonpropagating kinetic terms of the auxiliary fields, because this determines what spin components they describe. For example

$$A^\mu A_\mu \tag{2.4.1}$$

describes one spin-1 chain and one spin-0 chain, whereas using a gauge-invariant kinetic term (which requires other bosonic auxiliary fields, apart from A_μ) would actually eliminate the spin-0 part.

This discussion also allows us to introduce the counting argument of Siegel and Roček [14] that suggests that no auxiliary fields can be found to close the supersymmetry algebra off-shell for N=4 SSYM. Before proceeding, we note that kinetic terms for auxiliary fermions involve two distinct nonpropagating spinors, just because spinors have half-integer dimensionality, and two identical ones cannot make up four powers of mass. Thus, we will consider fermionic auxiliary fields to have nonpropagating kinetic terms of the form

$$\bar{\chi}\psi \; , \tag{2.4.2}$$

with, say, χ of dimension $\frac{5}{2}$ and ψ of dimension $\frac{3}{2}$. The discussion presented above tells us that an off-shell representation of supersymmetry contains an integer number of massive representations of supersymmetry. The dimensionality of the smallest of such representations can be determined by counting the number of fermionic creation operators one has in the algebra, and is 2^{2N}. Since the bosons and fermions occur in equal numbers in the representation, it follows that the fermions add up to a total of 2^{2N-1}. On the other hand, the form of the kinetic term in eq. (2.4.2) implies that fermionic auxiliary fields always enter the action in pairs, or that the number of auxiliary Fermi components is an *even* multiple of the number of components (4) of a four-dimensional spinor. Moreover, if we restrict our attention to the case of even number (N) of supersymmetries, we have a stronger constraint in that all representations of SU(N) (N even) with an odd number of indices of the fundamental representation have a number of components that is an integer multiple of the number of components of the fundamental representation, and fermionic auxiliary fields always bear an odd number of internal SU(N) indices. Therefore, the total number of components (physical and auxiliary) of the physical Fermi fields must differ from the total Fermi dimensionality of the off-shell representation by an integral multiple of 8N. This gives compatible answers for all multiplets of interest, apart from N=4 SSYM, where 2^{2N-1} equals 128, which is not the total number of components of the four physical Fermi fields (=16) modulo 8N (=32). The same difficulty is encountered when dealing with N=1 SSYM in ten dimensions.

This implies that, if some of the assumptions made above cannot be relaxed, no covariant superfield formulation can be found for N=4 SSYM or for N=1 SSYM in ten dimensions. In the next section we give a temporary (or, possibly, permanent) alternative to a complete covariant superfield formulation of ten-dimensional SSYM in superspace, by showing that it is possible to extend the known formulation of N=4 SSYM in four dimensions in terms of N=1 superfields to provide a description of the ten-dimensional theory.

2.5. The New Superspace Action for Ten-Dimensional Supersymmetric Yang-Mills Theory in Terms of Four-Dimensional Superfields

2.5.a. Geometry and Field Equations

As stated in the introduction, the component form of N=4 SSYM in four dimensions [1,2] consists of a vector, four Weyl spinors and six scalars, all in the adjoint representation of a gauge group, and interacting via a single coupling constant. The corresponding formulation in terms of N=1 superfields [10,11] fits the six scalars and three of the spinors in an SU(3) triplet of chiral superfields, and the remaining spinor and the vector in a real scalar superfield. The action is :

$$S_4 = \mathrm{tr}\int d^4x \left\{ \int d^4\theta\, e^{-gV}\bar{\varphi}^i\, e^{gV}\varphi_i - \frac{1}{g^2}\int d^2\theta\, W^\alpha W_\alpha \right.$$

$$\left. + \frac{ig}{3!}\int d^2\theta\, \varepsilon^{ijk}\varphi_i[\varphi_j,\varphi_k] + \frac{ig}{3!}\int d^2\bar{\theta}\, \varepsilon_{ijk}\bar{\varphi}^i[\bar{\varphi}^j,\bar{\varphi}^k] \right\} \quad , \qquad (2.5.1)$$

where $W_\alpha = \bar{D}^2(e^{-gV}D_\alpha e^{gV})$ is the chiral field strength of the real superfield, and the trace is over the group indices, with $\mathrm{tr}(T^a T^b) = \delta^{ab}$. The action (2.5.1) is invariant under the gauge transformations

$$\delta_\Lambda e^{gV} = i(\bar{\Lambda}e^{gV} - e^{gV}\Lambda) \quad , \qquad (2.5.2a)$$

$$\delta_\Lambda \varphi_i = i[\Lambda, \varphi_i] \quad , \qquad (2.5.2b)$$

with Λ an infinitesimal Lie-algebra valued chiral parameter. In the rest of this section we shall, for simplicity, set $g=1$.

The chiral superfields transform as matter fields under the four-dimensional gauge transformations in eqs. (2.5.2). In the ten-dimensional theory, however, the $\theta = 0$ parts of φ_i are components of the ten-dimensional vector, and are thus gauge fields. This suggests, as a first step, that the gauge transformation of φ_i in eq. (2.5.2b) be modified by the addition of terms involving derivatives of the gauge parameter with respect to the extra dimensions. The chirality of φ_i and SU(3)-covariance then lead uniquely to :

$$\delta_\Lambda e^V = i(\bar{\Lambda} e^V - e^V \Lambda)$$

$$\delta_\Lambda \varphi_i = \partial_i \Lambda + i[\Lambda, \varphi_i] \quad . \tag{2.5.3}$$

Here the derivatives with respect to the extra six spatial coordinates have been grouped into a $\underset{\sim}{3}$ of SU(3), in analogy with the grouping of the six four-dimensional scalars into the $\theta = 0$ parts of the chiral superfields.

With *every* field now being a gauge field, it is of interest to consider the covariant derivatives $\bar{\nabla}_{\dot{\alpha}}$, ∇_α, ∇_i and ∇^i. As usual, $\nabla_{\alpha\dot{\alpha}}$ is defined to be the anticommutator of the spinorial covariant derivatives. Working in the chiral representation, one demands that the covariant derivatives all transform as

$$\nabla_A \to e^{i\Lambda} \nabla_A e^{-i\Lambda} \tag{2.5.4}$$

under a gauge transformation, with Λ a chiral parameter. The covariant derivatives are therefore :

$$\bar{\nabla}_{\dot{\alpha}} \equiv \bar{D}_{\dot{\alpha}} - i\,\bar{\Gamma}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} \quad ,$$

$$\nabla_\alpha \equiv D_\alpha - i\,\Gamma_\alpha = e^{-V} D_\alpha e^V \quad ,$$

$$\nabla_i \equiv \partial_i - i\,\Gamma_i = \partial_i - i\,\varphi_i \quad ,$$

$$\nabla^i \equiv \bar{\partial}^i - i\,\Gamma^i = e^{-V}(\bar{\partial}^i - i\,\bar{\varphi}^i)\,e^V\ . \tag{2.5.5}$$

Taking commutators of covariant derivatives then generates field strengths, which by construction transform covariantly under gauge transformations. The *nonvanishing* field strengths are :

$$W_\alpha \equiv [\bar{\nabla}^{\dot{\alpha}}, \{\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}\}] = \bar{D}^2\,(e^{-V} D_\alpha e^V)\ ,$$

$$W_{\dot{\alpha}} \equiv [\nabla^\alpha, \{\bar{\nabla}_{\dot{\alpha}}, \nabla_\alpha\}] = e^{-V} D^2\,(e^V \bar{D}_{\dot{\alpha}} e^{-V})\,e^V\ ,$$

$$F_{\alpha i} \equiv [\nabla_\alpha, \nabla_i] = -\,iD_\alpha \varphi_i - \partial_i(e^{-V} D_\alpha e^V) - i\,[(e^{-V} D_\alpha e^V), \varphi_i]\ ,$$

$$F_{\dot{\alpha}}{}^{i} \equiv [\bar{\nabla}_{\dot{\alpha}}, \nabla^i] = \bar{D}_{\dot{\alpha}}(e^{-V}\bar{\partial}^i e^V) - i\,\bar{D}_{\dot{\alpha}}(e^{-V}\bar{\varphi}^i e^V)\ ,$$

$$F_{ij} \equiv [\nabla_i, \nabla_j] = -i\,(\partial_i \varphi_j - \partial_j \varphi_i - i\,[\varphi_i, \varphi_j])\ ,$$

$$F^{ij} \equiv [\nabla^i, \nabla^j] = -i\,e^{-V}(\bar{\partial}^i \bar{\varphi}^j - \bar{\partial}^j \bar{\varphi}^i - i\,[\bar{\varphi}^i, \bar{\varphi}^j])\,e^V\ ,$$

$$\begin{aligned} F_i{}^j \equiv [\nabla_i, \nabla^j] &= \partial_i\,(e^{-V}\bar{\partial}^j e^V) - i\,\partial_i(e^{-V}\bar{\varphi}^j e^V) + i\,\bar{\partial}^j \varphi_i \\ &+ i\,[(e^{-V}\bar{\partial}^j e^V), \varphi_i] + [(e^{-V}\bar{\varphi}^j e^V), \varphi_i]\ . \end{aligned} \tag{2.5.6}$$

W_α, $W_{\dot{\alpha}}$, $F_{\alpha i}$ and $F_{\dot{\alpha}}{}^{i}$ have dimension $\frac{3}{2}$, whereas F_{ij} , F^{ij} and $F_i{}^j$ have dimension 2.

It should be noted that the form (2.5.5) of the covariant derivatives follows from the constraints

$$F_{\alpha\beta} \equiv \{\nabla_\alpha, \nabla_\beta\} = 0 \ ,$$

$$F_{\dot\alpha\dot\beta} \equiv \{\overline{\nabla}_{\dot\alpha}, \overline{\nabla}_{\dot\beta}\} = 0 \ ,$$

$$F_{\alpha\dot\alpha} \equiv \{\nabla_\alpha, \overline{\nabla}_{\dot\alpha}\} - \frac{i}{2}\nabla_{\alpha\dot\alpha} = 0 \ , \tag{2.5.7a}$$

$$F_\alpha^{\ i} \equiv [\nabla_\alpha, \nabla^i] = 0 \ ,$$

$$F_{\dot\alpha i} \equiv [\overline{\nabla}_{\dot\alpha}, \nabla_i] = 0 \ , \tag{2.5.7b}$$

where, in the chiral representation, we choose $\overline{\nabla}_{\dot\alpha} = \overline{D}_{\dot\alpha}$. The new constraints are easily understood by examining the field strengths. At $\theta = \overline{\theta} = 0$, F_{ij}, F^{ij} and $F_i^{\ j}$ contain the part of the non-abelian field strength $F_{\underset{\sim}{\mu\nu}}$ corresponding to the extra six dimensions, and W_α, $W_{\dot\alpha}$, $F_{\alpha i}$ and $F_{\dot\alpha}^{\ i}$ contain the components of the ten-dimensional spinor. The new constraints (2.5.7b) simply state that the field strengths $F_\alpha^{\ i}$ and $F_{\dot\alpha i}$ vanish. This is necessary as at $\theta = \overline{\theta} = 0$ they would contain 3 *new* physical spinors, which do not exist in the component theory.

We can now ask ourselves what covariant equations of motion we can write for the ten-dimensional theory using the field strengths in eqs. (2.5.6). Interestingly, the answer is almost uniquely determined by dimensionality and SU(3)-covariance. Indeed, the equation of motion for V can only be

$$2\{\nabla^\alpha, W_\alpha\} + \gamma F_i^{\ i} = 0 \ , \tag{2.5.8}$$

and the equation of motion for φ_i can only be

$$i\{\nabla^\alpha, F_{\alpha i}\} + \delta\, \varepsilon_{ijk} F^{jk} = 0 \ . \tag{2.5.9}$$

The constants γ and δ are then fixed by comparison with the four-dimensional theory to be $\gamma = -1$, $\delta = -i/2$. They could also be determined by demanding the covariance of the field equations under the non-manifest symmetries of the theory.

2.5.b. The Action

The next problem is to construct an action that yields these equations of motion. To this end, we notice that, under arbitrary variations of the fields, the ten-dimensional action must vary as :

$$\Delta S_{10} = \mathrm{tr}\int d^{10}x \ \Big\{ \int d^4\theta \, \Delta V \, (\, 2\{\nabla^\alpha, W_\alpha\} - F_i{}^i)$$

$$+ \int d^2\bar{\theta} \, (e^{-V}\delta\bar{\varphi}^i e^V) (\, i\, \{\nabla^\alpha, F_{\alpha i}\} - \frac{i}{2}\, \varepsilon_{ijk}\, F^{jk})$$

$$+ \int d^2\theta \, \delta\varphi_i \, (\, i\, \{\bar{\nabla}^{\dot{\alpha}}, F_{\dot{\alpha}}{}^i\} - \frac{i}{2}\, \varepsilon^{ijk}\, F_{jk}) \Big\} \ , \qquad (2.5.10)$$

where $\Delta V \equiv e^{-V}\delta e^V$ is the gauge covariant variation of V, and where $e^{-V}\delta\bar{\varphi}^i e^V$ is the chirally covariant variation of $\bar{\varphi}^i$. Reconstructing the action from this variation is not straightforward, as the equations of motion mix the various field strengths and involve the field strengths themselves, not only their derivatives. To proceed further, it is convenient to introduce in eq. (2.5.10) the explicit form of the field strengths in terms of the fields. One obtains :

$$\Delta S_{10} = \mathrm{tr}\int d^{10}x \ \Big\{ \int d^4\theta \ \Delta V \Big\{ \, 2\{e^{-V} D^\alpha e^V, \bar{D}^2(e^{-V} D_\alpha e^V)\}$$

$$- \partial_i \, (\, e^{-V}\bar{\partial}^i e^V) + i \, (\partial_i \, (\, e^{-V}\bar{\varphi}^i e^V) - e^{-V}\bar{\partial}^i \, (e^V\varphi_i) - i\, [\varphi_i \, , e^{-V}\bar{\varphi}^i e^V] \,) \Big\}$$

$$+ \int d^2\theta \ \delta\varphi_i \ \Big\{ i\bar{D}^2 \, (\, e^{-V}\bar{\partial}^i e^V - i\, e^{-V}\bar{\varphi}^i e^V) - \frac{1}{2}\varepsilon^{ijk} \, (\, \partial_j\varphi_k - \partial_k\varphi_j \, - i\, [\varphi_j \, , \varphi_k \,]) \Big\}$$

$$+ \int d^2\bar{\theta} \, \delta\bar{\varphi}^i \ \Big\{ iD^2 (e^V\partial_i e^{-V} - i\, e^V\varphi_i e^{-V})$$

$$- \frac{1}{2}\varepsilon_{ijk} \, (\bar{\partial}^j\bar{\varphi}^k - \bar{\partial}^k\bar{\varphi}^j - i\, [\bar{\varphi}^j \, , \bar{\varphi}^k \,]) \Big\} \Big\} . \qquad (2.5.11)$$

Most of the terms in the action can indeed be guessed, using eq. (2.5.11) and comparing with the four-dimensional action given in eq. (2.5.1). One is thus led to consider

$$S_{10} = \mathrm{tr}\int d^{10}x\, \Big\{\int d^4\theta\, e^{-V}\bar\varphi^i e^V \varphi_i - \int d^2\theta\, W^\alpha W_\alpha + \frac{i}{3!}\int d^2\theta\, \varepsilon^{ijk}\varphi_i[\varphi_j,\varphi_k]$$

$$+ \frac{i}{3!}\int d^2\bar\theta\, \varepsilon_{ijk}\bar\varphi^i[\bar\varphi^j,\bar\varphi^k] - \frac{1}{2}\int d^2\theta\, \varepsilon^{ijk}\varphi_i\partial_j\varphi_k - \frac{1}{2}\int d^2\bar\theta\, \varepsilon_{ijk}\bar\varphi^i\bar\partial^j\bar\varphi^k$$

$$+ i\int d^4\theta (\partial_i e^{-V})\bar\varphi^i e^V - i\int d^4\theta\, e^V\varphi_i(\bar\partial^i e^{-V})$$

$$+ \frac{1}{2}\int d^4\theta (e^{-V}\bar\partial^i e^V)(e^{-V}\partial_i e^V)\Big\} . \qquad (2.5.12)$$

This action in fact yields the correct equations of motion for φ_i and $\bar\varphi^i$. In the vector equation, however, the terms with two ∂_i's are not reproduced correctly, as the last term in eq. (2.5.12) varies as

$$-\frac{1}{2}\int d^{10}x\, d^4\theta\, \Delta V\Big[\partial_i\,(e^{-V}\bar\partial^i e^V) + \bar\partial^i\,(e^{-V}\partial_i e^V)\Big] , \qquad (2.5.13)$$

whereas the corresponding term in eq. (2.5.11) is

$$-\int d^{10}x\, d^4\theta\, \Delta V\,\partial_i\,(e^{-V}\bar\partial^i e^V) . \qquad (2.5.14)$$

One therefore needs to find a new term to be added to the action that varies into

$$-\frac{1}{2}\int d^{10}x\, d^4\theta\, \Delta V\Big[\partial_i\,(e^{-V}\bar\partial^i e^V) - \bar\partial^i\,(e^{-V}\partial_i e^V)\Big] . \qquad (2.5.15)$$

Such a term must be very similar in structure to the last term in eq. (2.5.12), but must be odd under the interchange of ∂_i and $\bar\partial^i$, and *cannot be written in* *terms of the potential* e^V *only, but must also contain the prepotential* V *explicitly.* In order to complete the construction of the action, it is convenient to expand the last term in eq. (2.5.12) in powers of V using

$$e^{-V}\partial_i e^V = \frac{1-e^{-L_V}}{L_V}(\partial_i V) \ , \tag{2.5.16}$$

where $L_V X \equiv [V,X]$. The result is

$$\mathrm{tr}\int d^{10}x\, d^4\theta\, (\bar{\partial}^i V)\frac{\cosh L_V - 1}{L_V^2}(\partial_i V) \quad , \tag{2.5.17}$$

and the missing term is

$$\mathrm{tr}\int d^{10}x\, d^4\theta\, (\bar{\partial}^i V)\frac{\sinh L_V - L_V}{L_V^2}(\partial_i V) \quad , \tag{2.5.18}$$

which contains the odd function of L_V corresponding to (2.5.17), and is thus odd under the interchange of ∂_i and $\bar{\partial}^i$ †. A proof that varying this term yields (2.5.15) can be found in appendix B.

It is interesting to note that the term in eq. (2.5.18) can be also recovered from eq. (2.5.15) using a prescription recently given by Koller [13]. One reconstructs the term from its variation simply by replacing V with tV, ΔV with V, and by integrating over t from zero to one. This trick replaces functional integrations with respect to the fields with integrations over scalar parameters, thus undoing the combinatorics of functional differentiation. In our case, the method works particularly simply if we start from eq. (2.5.14), and write

$$\mathrm{tr}\int d^{10}x\, d^4\theta\, (\bar{\partial}^i V)\int_0^1 dt\ \frac{e^{tL_V}-1}{L_V}(\partial_i V) \ . \tag{2.5.19}$$

Performing the t-integration then clearly leads to the sum of (2.5.17) and

† Note that $\mathrm{tr}(A\, L_V^n B) = (-1)^n\, \mathrm{tr}(B\, L_V^n A)$.

(2.5.18).

We have thus found that

$$S_{10} = \mathrm{tr}\int d^{10}x \,\Big\{\int d^4\theta\, e^{-V}\overline{\varphi}^i e^V\varphi_i - \int d^2\theta\, W^\alpha W_\alpha + \frac{i}{3!}\int d^2\theta\, \varepsilon^{ijk}\varphi_i[\varphi_j,\varphi_k]$$

$$+ \frac{i}{3!}\int d^2\overline{\theta}\, \varepsilon_{ijk}\overline{\varphi}^i[\overline{\varphi}^j,\overline{\varphi}^k] - \frac{1}{2}\int d^2\theta\, \varepsilon^{ijk}\varphi_i\partial_j\varphi_k - \frac{1}{2}\int d^2\overline{\theta}\, \varepsilon_{ijk}\overline{\varphi}^i\overline{\partial}^j\overline{\varphi}^k$$

$$+ i\int d^4\theta\,(\partial_i e^{-V})\overline{\varphi}^i e^V - i\int d^4\theta\, e^V\varphi_i(\overline{\partial}^i e^{-V}) + \frac{1}{2}\int d^4\theta\,(e^{-V}\overline{\partial}^i e^V)(e^{-V}\partial_i e^V)$$

$$+ \int d^4\theta\,(\overline{\partial}^i V)\frac{\sinh L_V - L_V}{L_V^2}(\partial_i V)\Big\} \tag{2.5.20}$$

yields the equations of motion (2.5.8) and (2.5.9) (with $\gamma = -1$ and $\delta = -i/2$) and is invariant under the gauge transformations in eqs. (2.5.3). We wish to emphasize that the action in eq. (2.5.20) yields gauge covariant equations of motion, even though it is not expressible in terms of field strengths and covariant derivatives only. Only the purely chiral (or antichiral) terms are obviously, if not manifestly, gauge invariant, as they have the form of a gauge invariant mass term for a three-dimensional non-abelian gauge theory. The lack of manifest gauge invariance will be a common feature of all extended superspace formulations of supersymmetric theories, as increasing the number of anticommuting coordinates lowers the dimensionality of the volume element, and does not leave room for squares of curvatures, which have at least dimension 3.

Starting from eq. (2.5.20), one can recover the usual component form of the ten-dimensional theory as follows. First, for simplicity, one goes to a Wess-Zumino gauge eliminating the chiral and antichiral parts of V and reducing the action to a polynomial function of V. Then one integrates out the θ's, replacing the integrals by spinorial derivatives and using the definitions of the component fields in terms of the superfields given in appendix A. The following changes of notation are then required. First of all, the spinors are grouped together into a ten- dimensional spinor. Then the four-dimensional spinor indices are eliminated in favor of four-component notation. Finally, the complex SU(3) triplets of spatial derivatives ∂_i and of field components A_i are regrouped according to the conventional SO(6) vector notation. The result of these manipulations is

$$S_{10} = \mathrm{tr}\int d^{10}x \,\Big\{ -\frac{1}{4}F^{\widehat{\mu\nu}}F_{\widehat{\mu\nu}} - \frac{i}{2}\bar{\lambda}\gamma^{\hat{\mu}}D_{\hat{\mu}}\lambda$$

$$+ \| F^i - \frac{i}{2}\varepsilon^{ijk}(\partial_j A_k - \partial_k A_j - i[A_j, A_k]) \|^2$$

$$+ (D - \frac{i}{2}(\bar{\partial}\cdot A - \partial\cdot\bar{A} - i[\bar{A}^i, A_i]))^2 \Big\} , \qquad (2.5.21)$$

which is the usual component form of the ten-dimensional action, together with extra terms that vanish when the equations of motion for the auxiliary fields are used. It is interesting to note that the equations of motion for the auxiliary fields are

$$F_i = \frac{i}{2}\varepsilon_{ijk}(\bar{\partial}^j \bar{A}^k - \bar{\partial}^k \bar{A}^j - i[\bar{A}^j, \bar{A}^k])) \qquad (2.5.22a)$$

$$D = \frac{i}{2}(\bar{\partial}\cdot A - \partial\cdot\bar{A} - i[\bar{A}^i, A_i]) \ . \qquad (2.5.22b)$$

The right-hand sides of eqs. (2.5.22a) and (2.5.22b) are, respectively, the SU(3) singlet and triplet parts of the $\underline{15}$ of SO(6)

$$G_{IJ} = \partial_I A_J - \partial_J A_I - \frac{i}{\sqrt{2}}[A_I, A_J] \ . \qquad (2.5.23)$$

This suggests that the bosonic auxiliary fields F_i, $\overline{F}^i$ and D would appear in the complete ten-dimensional action together with extra auxiliary fields completing a $\underset{\sim}{45}$ of SO(9,1), $G_{\underset{\sim}{\mu}\underset{\sim}{\nu}}$. There should, of course, be other auxiliary fields as the number of bosonic auxiliary fields must exceed the number of fermionic auxiliary fields by only 7, for the off-shell equality of Bose and Fermi degrees of freedom.

2.5.c. Global Symmetries of the Ten-Dimensional Action

The four-dimensional action in eq. (2.5.1), besides being gauge invariant, possesses several global symmetries [19]. It is invariant under the direct product of the four-dimensional Lorentz group and an SO(6) ≈ SU(4) group corresponding to spatial rotations in the extra dimensions. Moreover, it is invariant under four global supersymmetries. Indeed, as stated in the introduction, the four-dimensional action possesses the full N=4 superconformal symmetry. This, however, does not concern us, as we are interested in symmetries that generalize to the ten-dimensional theory.

Consider first the supersymmetry that corresponds to the N=1 superspace coordinates. Its parameter fits into an x-independent real scalar superfield ζ, which also contains, in its nongauge part, the parameters of four-dimensional translations and of the U(1) subgroup of the SU(4) realized as combined chiral rotations of the fermionic superspace coordinates and of the chiral superfields (R-transformations). These transformations correspond to shifts of the superspace coordinates. In four dimensions, by adding a gauge transformation of parameter

$$\Lambda = -\bar{D}^2[(D^\alpha \zeta)(e^{-V} D_\alpha e^V)] \ , \tag{2.5.24}$$

they can be written in the covariant form

$$\Delta V = i\,[(\nabla^\alpha \zeta)\, W_\alpha - (\bar{\nabla}^{\dot{\alpha}} \zeta)\, W_{\dot{\alpha}}] \ ,$$

$$\delta\varphi_i = -i\,\bar{\nabla}^2[\,(\nabla^\alpha \zeta)(\nabla_\alpha \varphi_i) + \tfrac{1}{3}(\nabla^2 \zeta)\,\varphi_i\,] \ . \tag{2.5.25}$$

The modified transformations in eq. (2.5.25), however, are not a symmetry of the ten-dimensional action as they stand. A signal of this is that they are not gauge invariant (not even up to a gauge transformation) in $D>4$, in contradiction with the commutativity of supersymmetry transformations and gauge transformations. Moreover, invariance of the fifth and sixth terms in eq. (2.5.20) demands that the transformations be modified by the addition of orbital pieces. The correct transformations are :

$$\Delta V = i\,[\,(\nabla^\alpha \zeta)\,W_\alpha - (\overline{\nabla}^{\dot\alpha} \zeta)\,W_{\dot\alpha}] + \frac{i}{3}(\overline{\nabla}^2 \nabla^2 \zeta)\,e^{-V}(\overline{x}\cdot\overline{\partial} - x\cdot\partial)\,e^{V} \ ,$$

$$\delta\varphi_i \ = -i\,\overline{\nabla}^2[\,(\nabla^\alpha\zeta)\ i\,F_{\alpha i} + \tfrac{1}{3}(\nabla^2\zeta)\,\varphi_i\,] + \frac{i}{3}(\overline{\nabla}^2\nabla^2\zeta)\,(\overline{x}\cdot\overline{\partial} - x\cdot\partial)\,\varphi_i \quad (2.5.26)$$

Apart from the orbital pieces, the changes amount only to the replacement of the non-covariant quantity $\nabla_\alpha \varphi_i$ with the field strength $F_{\alpha i}$.

Next we consider the three extra four-dimensional supersymmetries. Their parameters, together with the parameters of central charge transformations Z_i and the parameters of SU(4)/(SU(3)⊗U(1)) transformations ω_i, fit into an SU(3) triplet of x-independent chiral superfields χ_i

$$\chi_i \ = Z_i + \theta^\alpha \varepsilon_{\alpha i} + \theta^2 \omega_i \quad . \qquad (2.5.27)$$

The four-dimensional action is indeed invariant under

$$\Delta V = i\,(e^{-V}\chi_i\,\overline{\varphi}^i\,e^{V} - \overline{\chi}^i\,\varphi_i)$$

$$\delta\varphi_i \ = \varepsilon_{ijk}\overline{\nabla}^2(\overline{\chi}^j\,e^{-V}\,\overline{\varphi}^k\,e^{V}) + 2i(\nabla^\alpha\chi_i)\,W_\alpha \ . \qquad (2.5.28)$$

In finding the corresponding transformations for the ten-dimensional action, it is useful to note that the central charge transformations become translations in the extra dimensions. Moreover, the SU(4)/(SU(3)⊗U(1)) transformations are Lorentz transformations in the extra dimensions and, as such, acquire orbital parts. The correct transformations for the ten- dimensional action are :

$$\Delta V = i\,(e^{-V}\chi_i\,\bar\varphi^i\,e^V - \bar\chi^i\,\varphi_i) - e^{-V}(\bar\chi\cdot\partial + \chi\cdot\bar\partial)\,e^V$$

$$- \varepsilon_{ijk}\,(\bar\nabla^2\bar\chi^j)\,x^k\,e^{-V}\bar\partial^i\,e^V - \varepsilon^{ijk}\,(\nabla^2\chi_j)\,\bar x_k\,e^{-V}\,\partial_i\,e^V$$

$$\delta\varphi_i = \varepsilon_{ijk}\,\bar\nabla^2[\,\bar\chi^j\,e^{-V}(\bar\varphi^k + i\,\bar\partial^k)\,e^V] + 2i\,(\nabla^\alpha\chi_i)\,W_\alpha - \chi\cdot\bar\partial\,\varphi_i$$

$$- \varepsilon_{jkl}\,(\bar\nabla^2\bar\chi^j)\,x^k\,\bar\partial^l\,\varphi_i - \varepsilon^{jkl}(\nabla^2\chi_j)\,\bar x_k\,\partial_l\,\varphi_i \quad . \qquad (2.5.29)$$

By adding a gauge transformation of parameter

$$\Lambda = -\,\varepsilon_{ijk}\,\bar\nabla^2[\bar\chi^j\,x^k\,e^{-V}(i\,\bar\partial^i + \bar\varphi^i)\,e^V] - \varepsilon^{ijk}\,(\nabla^2\chi_j)\,\bar x_k\,\varphi_i \quad , \qquad (2.5.30)$$

the χ-transformations can be cast in a more elegant form, involving the field strengths of eqs. (2.5.6) :

$$\Delta V = i\,(e^{-V}\chi\cdot\bar\varphi\,e^V - \bar\chi\cdot\varphi) - e^{-V}(\chi\cdot\bar\partial + \bar\chi\cdot\partial)\,e^V + 2\,\varepsilon_{ijk}\,x^k\,(\bar\nabla^{\dot\alpha}\bar\chi^j)\,F_{\dot\alpha}{}^i$$

$$- 2\varepsilon^{ijk}\,\bar x_k\,(\nabla^\alpha\chi_j)\,F_{\alpha i} + \varepsilon_{ijk}\,\bar\chi^j\,x^k\,\{\bar\nabla^{\dot\alpha}, F_{\dot\alpha}{}^i\} - \varepsilon^{ijk}\,\chi_j\,\bar x_k\,\{\nabla^\alpha, F_{\alpha i}\} \quad ,$$

$$\delta\varphi_i = i\,\varepsilon_{jkl}\,\bar\nabla^2(\bar\chi^j\,F_i{}^l)\,x^k + 2i\,(\nabla^\alpha\chi_i)\,W_\alpha$$

$$+ \varepsilon^{jkl}\,(\nabla^2\chi_j)\,\bar x_k\,F_{li} - \chi\cdot\bar\partial\,\varphi_i \quad . \qquad (2.5.31)$$

This result, unlike the ζ-transformations, contains, as well as covariant terms, non-covariant ones which cannot be eliminated because of the chirality of φ_i . As a consequence, χ-transformations commute with gauge transformations only up to a gauge transformation of parameter $-\chi\cdot\bar\partial\,\Lambda$.

Finally, we consider the remaining Lorentz transformations, corresponding to $SO(9,1)/SU(4)$. Of these, the purely four-dimensional Lorentz transformations are an obvious symmetry of the action, manifest in the way the spinor indices are contracted together, with the superfields V, φ_i and $\overline{\varphi}^i$ transforming as scalars under them. On the other hand, the "off-diagonal" Lorentz transformations which rotate the four-dimensional coordinates into those of the extra six dimensions, and therefore have no analogue in the four-dimensional theory, are not an obvious symmetry and require direct investigation. It is natural to try to fit the corresponding parameters $\lambda_i^{\alpha\dot{\alpha}}$ into an SU(3) triplet of complex x-independent superfields. This, however, does not lead to a symmetry of the action, which is not surprising, as the extra parameters do not correspond to symmetries of the component action. Restricting the complex superfields to be of the form $\lambda_i = \lambda_i^{\alpha\dot{\alpha}}\theta_\alpha\overline{\theta}_{\dot{\alpha}}$, i.e. demanding that they only have a nonvanishing $\theta_\alpha\overline{\theta}_{\dot{\alpha}}$ component, is indeed enough to ensure that the transformations

$$\Delta V = 2\left(e^{-V}\lambda\cdot\overline{\varphi}\, e^{V} + \overline{\lambda}\cdot\varphi\right) + \overline{\lambda}^{j\alpha\dot{\alpha}} e^{-V}\left[(x_{\alpha\dot{\alpha}} - i\theta_\alpha\overline{\theta}_{\dot{\alpha}})\partial_j - \overline{x}_j\,\partial_{\alpha\dot{\alpha}}\right]e^{V}$$

$$+ \lambda_j^{\alpha\dot{\alpha}} e^{-V}\left[(x_{\alpha\dot{\alpha}} + i\theta_\alpha\overline{\theta}_{\dot{\alpha}})\overline{\partial}^j - x^j\,\partial_{\alpha\dot{\alpha}}\right]e^{V}$$

$$\delta\varphi_i = 4\overline{D}^2\left[(D^\alpha\lambda_i)(e^{-V}D_\alpha e^{V})\right] + 2i\,\varepsilon_{ijk}\overline{D}^2\left[\overline{\lambda}^j e^{-V}(\overline{\varphi}^k + i\overline{\partial}^k)e^{V}\right]$$

$$+ \lambda_j^{\alpha\dot{\alpha}}\left[(x_{\alpha\dot{\alpha}} + i\theta_\alpha\overline{\theta}_{\dot{\alpha}})\overline{\partial}^j - x^j\,\partial_{\alpha\dot{\alpha}}\right]\varphi_i$$

$$+ \overline{\lambda}^{j\alpha\dot{\alpha}}\left[(x_{\alpha\dot{\alpha}} + i\theta_\alpha\overline{\theta}_{\dot{\alpha}})\partial_j - \overline{x}_j\,\partial_{\alpha\dot{\alpha}}\right]\varphi_i \qquad (2.5.32)$$

be an invariance of the ten-dimensional action. We note that the chirality of φ_i demands that the four-dimensional coordinates $x_{\alpha\dot{\alpha}}$ appear in the orbital parts of these transformations only in the chiral combination $(x_{\alpha\dot{\alpha}} + i\,\theta_\alpha\overline{\theta}_{\dot{\alpha}})$, and thus the explicit $\theta_\alpha\overline{\theta}_{\dot{\alpha}}\lambda_j^{\alpha\dot{\alpha}}$'s cannot be absorbed in a general superfield λ_j.

2.5.d. Quantization and Feynman Rules

We now turn to the problem of quantization. Following the standard procedure for quantizing gauge theories, we must add a gauge-fixing term and the corresponding Faddeev-Popov ghost action. To this end, we notice that the lagrangian in eq. (2.5.20) contains quadratic terms mixing the vector and chiral multiplets, a situation similar to that of spontaneously broken Yang-Mills theory, where the kinetic terms mix scalar and vector fields. The nonlocal gauge-fixing term

$$S_{\mathrm{GF}} = -\,\mathrm{tr}\int d^{10}x\, d^4\theta (\bar{D}^2 V + i\frac{\bar{D}^2}{\Box_4}\partial\cdot\bar{\varphi})(D^2 V - i\frac{D^2}{\Box_4}\bar{\partial}\cdot\bar{\varphi}) \quad . \tag{2.5.33}$$

generalizes the four-dimensional Feynman-type gauge and diagonalizes the kinetic term in an SU(3) covariant way. This is the gauge-fixing term associated with the gauge

$$D^2\bar{D}^2 V + i\partial\cdot\bar{\varphi} = 0 \quad . \tag{2.5.34}$$

The Faddeev-Popov ghost lagrangian is then determined to be :

$$S_{\mathrm{FP}} = -\mathrm{tr}\int d^{10}x\, d^4\theta \Big\{ (c' + \bar{c}')[L_{\frac{V}{2}}(c + \bar{c}) + L_{\frac{V}{2}}\coth L_{\frac{V}{2}}(c - \bar{c})]$$

$$- c'\frac{\Box_6}{\Box_4}\bar{c} + i(\partial_i c')\frac{1}{\Box_4}[\bar{c},\bar{\varphi}^i] + \bar{c}'\frac{\Box_6}{\Box_4}c - i(\bar{\partial}^i\bar{c}')\frac{1}{\Box_4}[c,\varphi_i]\Big\} \tag{2.5.35}$$

which also contains nonlocal terms. The nonlocalities are only introduced by our gauge choice, and it turns out that rearranging the covariant derivatives according to the standard rules of superfield perturbation theory [10] always cancels the nonlocal terms in Green functions not involving external ghosts.

The propagators are obtained by inverting the quadratic part of the gauge-fixed lagrangian

$$\mathrm{tr} \int d^{10}x \, d^4\theta \, \Big\{ \bar{\varphi}^i \varphi_i - \tfrac{1}{2} V \Box_{10} V - \tfrac{1}{2} \varepsilon^{ijk} \varphi_i \partial_j \frac{D^2}{\Box_4} \varphi_k - \tfrac{1}{2} \varepsilon_{ijk} \bar{\varphi}^i \bar{\partial}^j \frac{\bar{D}^2}{\Box_4} \bar{\varphi}^k$$

$$- (\partial \cdot \bar{\varphi}) \frac{1}{\Box_4} (\bar{\partial} \cdot \varphi) + \bar{c}' c + \bar{c}' \frac{\Box_6}{\Box_4} c - c' \bar{c} - c' \frac{\Box_6}{\Box_4} \bar{c} \Big\} . \qquad (2.5.36)$$

There is now a φ-φ propagator of the form

$$-i \varepsilon_{ijk} \frac{\bar{p}^k}{p_{10}^2} \frac{D^2}{p_4^2} \delta(\theta_1 - \theta_2) \ , \qquad (2.5.37)$$

and a corresponding $\bar{\varphi}$-$\bar{\varphi}$ propagator, resembling those of a massive chiral multiplet. The other propagators differ from those of the four-dimensional theory in the corresponding Feynman-type gauge only by the obvious replacement of $\Box_4$ with $\Box_{10}$. There are also some additional vertices in the theory coupling vectors to a single chiral superfield, additional purely vector vertices and new couplings of chiral fields to ghosts. The propagators and cubic vertices of the theory are shown in figure 1. It should be noted that the new contributions to the purely vector vertex do not involve spinorial derivatives, and therefore usually do not contribute to loop diagrams.

In order to compute quantum corrections, one must, as usual, regularize the theory to localize and control the infinities of the Feynman diagrams. As one wants the regularization scheme to preserve as many of the symmetries of the theory as possible, one can use the only freedom in the theory: the fact that it can be written in any space-time dimension $D \le 10$ by dimensional reduction. We should point out that here we use two different dimensional reductions. The first is a classical procedure. If one wants to work in $D < 10$ dimensions, one must set some of the ∂_i's to 0 † (i.e. the fields are taken to be independent of

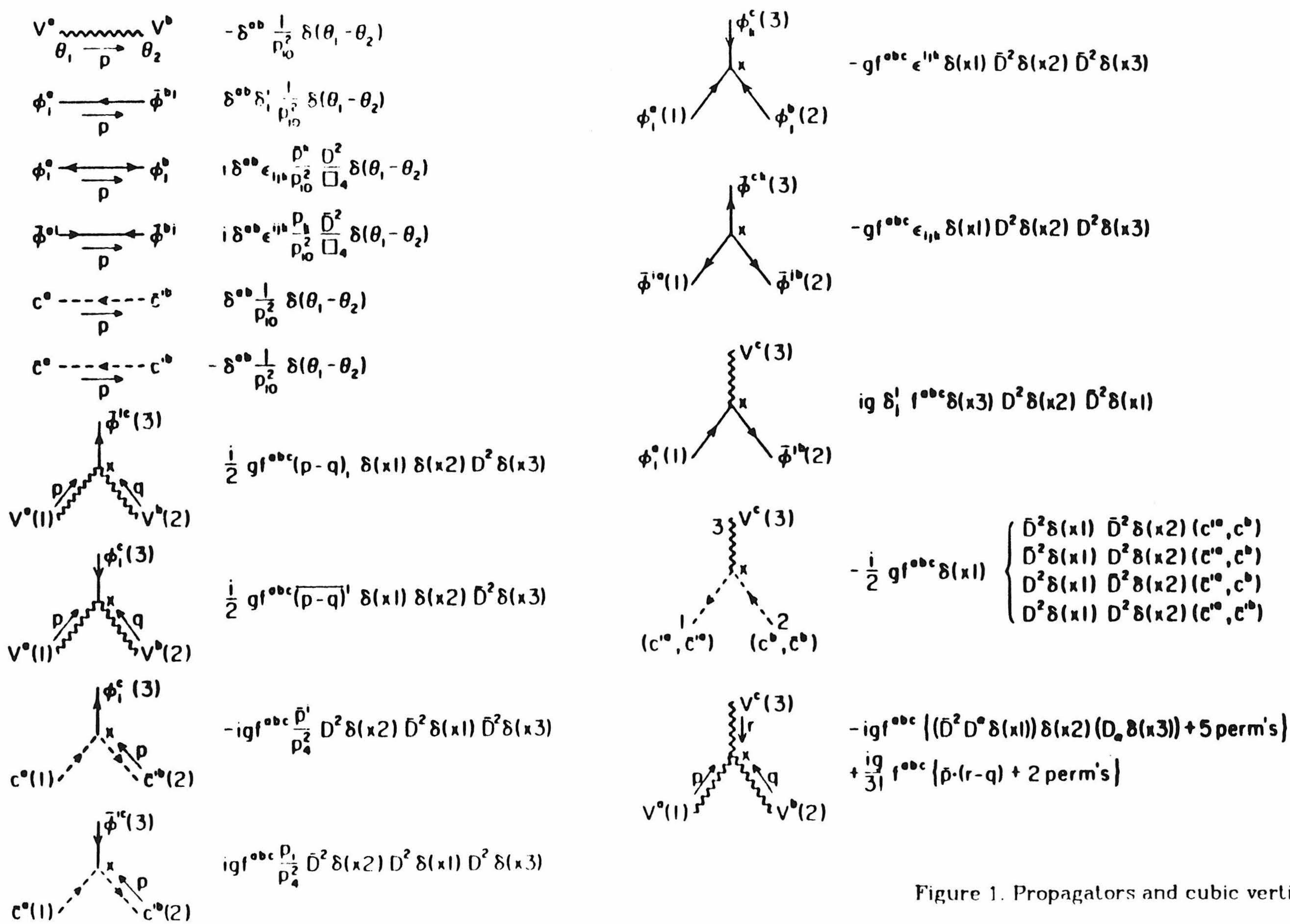

Figure 1. Propagators and cubic vertices

some of the x^i's). Thus, in D=6, for example, one would set $\partial_2 = \partial_3 = 0$. The second is adapted from the dimensional reduction scheme of ref. [20], in which one keeps the indices of the fields and of the D operators fixed, while varying the range of the indices of the momenta. In our case, as we encounter terms where a ∂_i is contracted with an ε tensor, we shall keep the SU(3) indices running over an integral number of values, and let the 4-dimensional momenta become $4 - 2\varepsilon$ dimensional. Thus, in D=6, for example, we would work with ∂_1, $\bar{\partial}^1$ and ∂_μ, with $\mu = 1, \cdots, 4-2\varepsilon$.

Calculations with this model parallel those in four dimensions. They are somewhat more laborious, however, because the addition of the new vertices, and especially the presence of the new chiral propagators, increases considerably the number of diagrams contributing to a given process. As an example, consider the one-loop corrections to the propagators in $D > 4$. The relevant diagrams are shown in figure 2, where we have taken care to distinguish between diagrams contributing in four dimensions and extra diagrams introduced by the new vertices of the ten-dimensional action. In D>4 the diagrams containing vertices of the four-dimensional theory only do not separately add up to zero, because the D-algebra generates terms like $\dfrac{k_4^2}{k_D^2 (k+p)_D^2}$, which only vanish in four dimensions, where they are massless tadpoles. However, when the new diagrams are added, one finds that, as in the corresponding gauge in four-dimensions, all the one-loop propagator corrections *vanish identically* in this theory in the gauge (2.5.34) for any $D \leq 10$.

In four dimensions, all three particle vertices are finite, as suggested by superfield power counting rules and N=4 supersymmetry. As with the propagators, one might hope that this feature would persist in higher dimensions. The

†In odd dimensions, because of our complex SU(3) notation, it is necessary to set one $\partial_i = \bar{\partial}^i$.

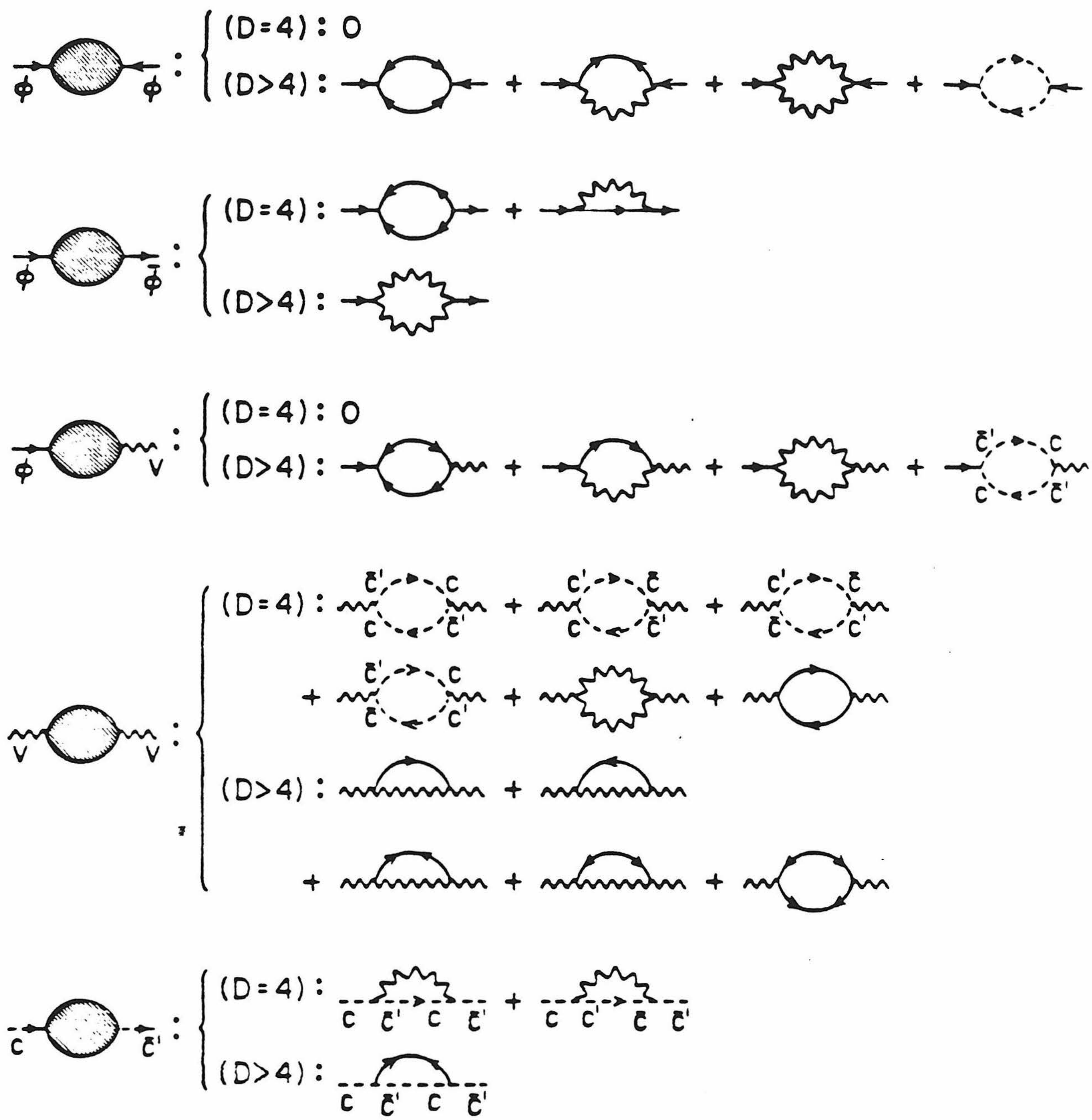

Figure 2. One-loop propagator diagrams

situation for the higher-point Green functions, however, is somewhat more complicated. For example, the triple chiral vertex is corrected by the appropriate diagrams in figure 3 that, in six dimensions, give the divergent contribution

$$\frac{i}{4(4\pi)^3\varepsilon}\int d^6x\, d^4\theta\, \varepsilon^{ijk}\,(D^2\varphi_i)\,[\,\varphi_j\,,\varphi_k\,]\ , \tag{2.5.38}$$

which must be removed by adding a counterterm. It thus appears that the theory is already divergent at this level. As Green functions are gauge dependent, however, we must check whether this divergence is a physical one or not. One's first thought may be that, as in $D>4$ there is no phase space for massless three-particle interactions, divergences in three-point functions are irrelevant. This is clearly not true, however, as they can contribute in higher point non $1PI$ S-matrix amplitudes. It is thus necessary to study whether the counterterm of (2.5.38) contributes as an insertion in S-matrix amplitudes.

A well known example of a "harmless" divergence, familiar from ordinary renormalizable field theories, is that of wavefunction renormalization. In non-renormalizable theories the existence of dimensionful coupling constants allows this concept to be generalized to arbitrary nonlinear field redefinitions. Generically, if we shift a field Ψ by $\Psi \to \Psi + h\Delta\Psi$, the action transforms as

$$S[\Psi] \to S[\Psi] + h\,\frac{\delta S}{\delta\Psi}\,\Delta\Psi\ , \tag{2.5.39}$$

where $\frac{\delta S}{\delta\Psi}$ can be recognized as the equation of motion for Ψ. Divergences proportional to equations of motion can therefore be absorbed at one loop by field redefinitions which, as is well known, do not affect the S matrix [21]. These infinities are familiar from the case of pure gravity at one loop [22], and are the only kind of divergences allowed in non-renormalizable theories.

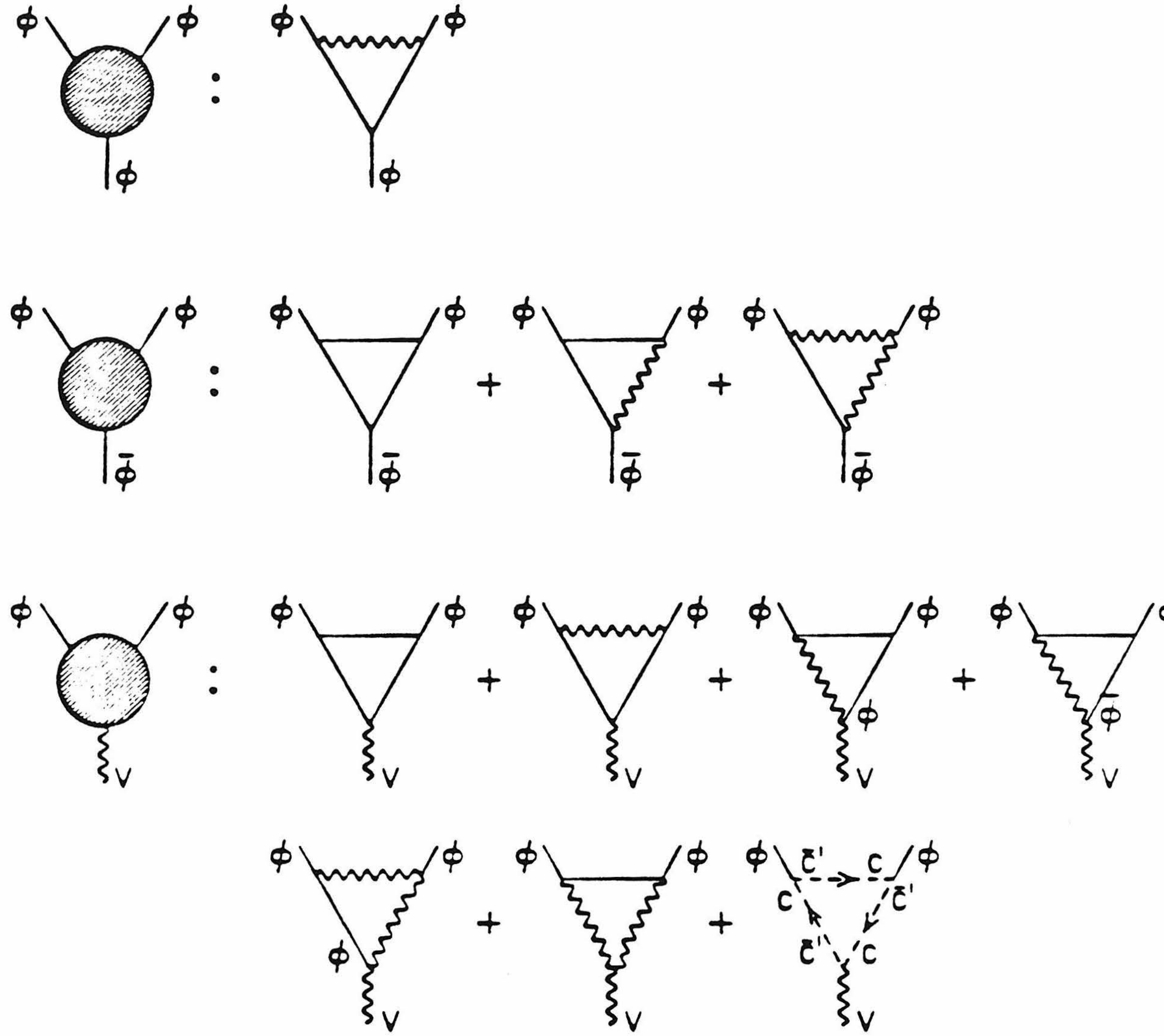

Figure 3 One-loop $\phi\phi\phi$, $\phi\phi\bar{\phi}$ and $\phi\phi V$ diagrams

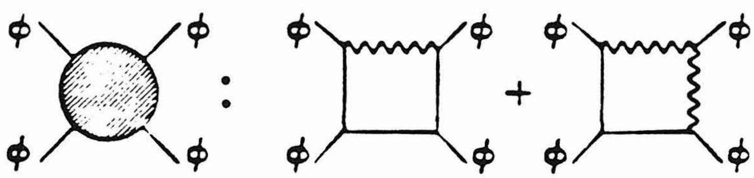

Figure 4. One-loop $\phi\phi\phi\phi$ diagrams

The counterterm of (2.5.38) does not appear to vanish when the equation of motion of φ_i is used, as the linearized equation of motion for φ_i relates it to V and $\overline{\varphi}^m$. However, when the divergent parts of the other three-point functions are added, (2.5.38) mixes with suitable contributions from the $\varphi\varphi\overline{\varphi}$ and $\varphi\varphi V$ vertex corrections in figure 3 to become

$$\frac{i}{4(4\pi)^3\varepsilon}\int d^6x\, d^4\theta\, \varepsilon^{ijk}\,[\varphi_j\,,\varphi_k\,]\,(\,(D^2\varphi_i) - \varepsilon_{ilm}\,\overline{\partial}^l\overline{\varphi}^m \,- iD^2\partial_i\,V) \quad (2.5.40)$$

which is proportional to the linearized equation of motion of $\overline{\varphi}^i$. This can be eliminated by the field redefinition

$$\overline{\varphi}^i \to \overline{\varphi}^i - \frac{i}{4(4\pi)^3\varepsilon}\,\varepsilon^{ijk}\,D^2[\,\varphi_j\,,\varphi_k\,]\ . \quad (2.5.41)$$

Similarly, it can be shown that *all* the one-loop infinities of the three point functions in $D>4$ are field redefinitions. It may be noted that, while field redefinitions do not occur in four dimensions for the three-point functions, they do occur for the four-point functions in superfield SSYM theory [23].

The next step is to consider the $1PI$ four point Green functions, the least divergent of which is the $\varphi\varphi\varphi\varphi$ amplitude. The diagrams contributing to this amplitude are shown in figure 4. Superfield power counting now shows that the amplitude becomes divergent in eight dimensions, and the one-loop $\varphi\varphi\varphi\varphi$ S-matrix is thus finite in six dimensions. In eight dimensions, however, the divergence of this amplitude is not a gauge artifact, and the S matrix itself now diverges. This can be seen as, after the contribution from the $1PI$ amplitude is added to eq. (2.5.40) and the nonlinear field equations are used, the resulting divergent $\varphi\varphi\varphi\varphi$ terms do not vanish. Therefore, we conclude that at one loop the S matrix starts to be ultraviolet divergent in eight dimensions. This result agrees with the superstring calculations of ref. [4].

We conclude by drawing to the attention of the reader the remarkable ghost-free gauge $\varphi_1 = 0$[†], that in six dimensions reduces the action to

$$S_{10} = \mathrm{tr}\int d^{10}x \left\{ \int d^4\theta e^{-V}\bar{\varphi}^i e^V \varphi_i - \int d^2\theta W^\alpha W_\alpha \right.$$

$$+ \int d^4\theta (\bar{\partial} V) \frac{\sinh L_V - L_V}{L_V^2} (\partial V)$$

$$\left. + \frac{1}{2}\int d^4\theta (e^{-V}\bar{\partial}e^V)(e^{-V}\partial e^V) \right\} , \qquad (2.5.42)$$

where i is now an SU(2) index. Many of the interactions involving chiral fields have disappeared, leaving only a minimal coupling of the scalar superfield to the remaining chiral superfields φ_2 and φ_3. The price for this, however, is a complicated vector propagator :

$$\frac{1}{\Box_6} \left\{ 1 + 2\frac{D^\alpha \bar{D}^2 D_\alpha}{\Box_{10}} \right\} . \qquad (2.5.43)$$

This propagator involves four spinorial derivatives, which considerably complicates the evaluation of graphs.

[†]It may appear puzzling that one can set *both* φ_1 and $\bar{\varphi}^1$ to zero. However, a linear combination of $\bar{\varphi}^1$ and φ_1 is transferred to the lower components of V.

Appendix A

We use two-component notation for the four-dimensional indices throughout. Our conventions are those of ref. [24], so that our spinorial covariant derivatives are

$$D_\alpha = \frac{i}{2}(\partial_\alpha + i\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}})$$

$$\bar{D}_{\dot{\alpha}} = -\frac{i}{2}(\bar{\partial}_{\dot{\alpha}} + i\theta^\alpha\partial_{\alpha\dot{\alpha}}) \; , \tag{2.A.1}$$

satisfying

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = \frac{i}{2}\partial_{\alpha\dot{\alpha}} \; . \tag{2.A.2}$$

The derivatives with respect to the extra six coordinates are grouped into the three complex quantities ∂_i and their complex conjugates $\bar{\partial}^i$. For example :

$$\partial_1 = \frac{\partial}{\partial x^4} + i\,\frac{\partial}{\partial x^5} \; ; \tag{2.A.3}$$

It follows that

$$\bar{\partial}^i\,\partial_i = \Box_6 \; , \tag{2.A.4}$$

where $\Box_6$ denotes the part of the D'Alembertian operator corresponding to the extra six coordinates. The definition we use for the components of vectors differs from (2.A.3) by a normalization factor so that, for example

$$A_1 = \frac{1}{\sqrt{2}}(A_4 + i\,A_5) \; , \tag{2.A.5}$$

and one goes from SO(6) vector indices to SU(3) indices according to

$$A^I B_I = \bar{A}^i B_i + A_i \bar{B}^i$$

and

$$A^I \partial_I = \frac{1}{\sqrt{2}}(\bar{A}^i \partial_i + A_i \bar{\partial}^i) \ . \tag{2.A.6}$$

Ten-dimensional vector indices are denoted by hatted Greek letters.

The component fields are defined in terms of the covariant derivatives and field strengths as

$$A_{\alpha\dot{\alpha}} = \sqrt{2}\,\Gamma_{\alpha\dot{\alpha}} \quad ; \quad \lambda_\alpha = \sqrt{2} W_\alpha \quad ; \quad D = \{\nabla^\alpha , W_\alpha\}$$

$$A_i = \Gamma_i \quad ; \quad \lambda_{\alpha i} = i\sqrt{2} F_{\alpha i} \quad ; \quad F_i = i\{\nabla^\alpha , F_{\alpha i}\} \ , \tag{2.A.7}$$

at $\theta = \bar{\theta} = 0$. Here $\Gamma_{\alpha\dot{\alpha}}$ is the connection in the anticommutator of ∇_α and $\bar{\nabla}_{\dot{\alpha}}$ and Γ_i is the connection in ∇_i. In a Wess-Zumino gauge these definitions become

$$A_{\alpha\dot{\alpha}} = \sqrt{2}[\bar{D}_{\dot{\alpha}}, D_\alpha] V \quad ; \quad \lambda_\alpha = \sqrt{2}\bar{D}^2 D_\alpha V \quad ; \quad D = D^\alpha \bar{D}^2 D_\alpha V$$

$$A_i = \varphi_i \quad ; \quad \lambda_{\alpha i} = \sqrt{2} D_\alpha \varphi_i \quad ; \quad F_i = D^2 \varphi_i \ . \tag{2.A.8}$$

Appendix B

We want to show that the variation of the term in eq. (2.5.18) is indeed (2.5.15). To this end, it is convenient to rewrite (2.5.18), using an exponential parametrization, as :

$$\frac{1}{2}\,\mathrm{tr}\int d^{10}x\, d^4\theta \int_0^1 dx \int_0^x dy\, (\bar\partial^i V)(e^{yL_V} - e^{-yL_V})(\partial_i V) \quad . \qquad (2.B.1)$$

Then, in order to perform the variation, all one needs is the following formula for the varying the exponential of a commutator

$$(\delta e^{\sigma L_V})\, A = e^{\sigma L_V}[\, \frac{1 - e^{-\sigma L_V}}{L_V}\,\delta V\,,\, A\,] \quad . \qquad (2.B.2)$$

The variation of (2.B.1) can be written, using another exponential parametrization,

$$\frac{1}{2}\int d^{10}x \int d^4\theta \int_0^1 dx \int_0^x dy \int_0^y dz\;\delta V \,\Big\{\, e^{yL_V}[\,(\partial_i V)\,,e^{-zL_V}(\bar\partial^i V)]$$

$$+\, e^{zL_V}[\,(\partial_i V)\,,e^{-yL_V}(\bar\partial^i V)] + e^{yL_V}[\,e^{-zL_V}(\partial_i V)\,,(\bar\partial^i V)]$$

$$+\, e^{zL_V}[\,e^{-yL_V}(\partial_i V)\,,(\bar\partial^i V)\,]$$

$$+\,[\,e^{-zL_V}(\partial_i V)\,,e^{-yL_V}(\bar\partial^i V)\,] + [\,e^{-yL_V}(\partial_i V)\,,e^{-zL_V}(\bar\partial^i V)\,]\,\Big\} \qquad (2.B.3)$$

or, using the symmetry of the integrand above under the interchange of y and z,

$$\frac{1}{2}\,\mathrm{tr}\int d^{10}x\, d^4\theta \int_0^1 dx \int_0^x dy \int_0^x dz\;\Big\{ e^{yL_V}[\,(\partial_i V)\,,e^{-zL_V}(\bar\partial^i V)\,]$$

$$+\, e^{yL_V}[\,e^{-zL_V}(\partial_i V)\,,(\bar\partial^i V)\,] + [\,e^{-yL_V}(\partial_i V)\,,e^{-zL_V}(\bar\partial^i V)\,]\,\Big\} \quad . \qquad (2.B.4)$$

The answer then follows after performing the y and z integrations and using the identity

$$e^{\sigma L_V}(A\,B) = (e^{\sigma L_V}A)(e^{\sigma L_V}B) \ , \tag{2.B.5}$$

which is a direct consequence of $e^{L_V}A = e^V A\, e^{-V}$, and the identity

$$[\frac{1}{L_V}A, \frac{1}{L_V}B] = \frac{1}{L_V}[A, \frac{1}{L_V}B] + \frac{1}{L_V}[\frac{1}{L_V}A, B] \ , \tag{2.B.6}$$

which is just a convenient rewriting of the Jacobi identity.

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Chapter 3

Gauge Groups for Type-I Superstrings

3.1. Introduction

In Chapter 1 we have given a description of the super-Poincaré algebras. We have also emphasized how the study of their particle representations alone suggests the existence of several local-field theory models possessing invariance under *global* or *local* supersymmetries. As we have seen, the theories in the latter category are rather remarkable, as they are supersymmetric generalizations of Einstein's General Theory of Relativity possessing invariance under a number (≤ 8) of local supersymmetries, and thereby providing unifications of the gravity field with other matter fields of lower spin. In particular, the maximally extended of these supergravity theories is a model of an unprecedented complexity, with an on-shell content consisting of as many as 256 states (128 bose and 128 fermi) connected by a symmetry principle (the N=8 supersymmetry), so that they span a single irreducible representation.

The main attempts to relate supergravity to low energy ($\approx$100 GeV) phenomenology are presently connected with the N=8 model. The motivation of this program has to do with one property of this theory, namely the very tight constraints imposed by supersymmetry, which make its multiplet of states the unique one compatible with N=8 supersymmetry. Supersymmetry itself also stimulates the *hope* that N=8 supergravity will provide a quantum theory of gravitation and matter with an S-matrix free from nonrenormalizable ultraviolet divergences. On the other hand, connecting N=8 supergravity to phenomenology appears not to be a straightforward task, as neither the multiplet of states it describes, nor their interactions, bears an obvious resemblance to the low-energy phenomenological theories based on Yang-Mills type interactions. Moreover, to date the *hope* that N=8 supergravity is free from ultraviolet divergences is *only* sustained by a few simple power-counting arguments that, based on more or less favorable assumptions, can at most stretch the expected onset of ultraviolet divergences up to seven loops. Settling this issue completely by means of formal arguments alone appears to be very difficult, because the non-renormalizable interactions in supergravity can in principle produce ultraviolet divergent expressions which are necessarily of more complicated structure as the order of the expansion is increased. Explicit calculations could settle this issue if divergences are found in corrections to S-matrix amplitudes with a given number (≥ 3) of loops, but such calculations appear to be too difficult to tackle within the presently known formulations of supergravity. It is undeniable, however, that supersymmetry does have a softening effect on the ultraviolet divergences of local field theories, and one may wish to keep it as a feature of a fundamental theory of gravitation, even if one abandons the hope of using N=8 supergravity as such a theory.

It is most remarkable, in this respect, that N=8 supergravity is already known not to be the end of the road. Rather, it is a special case (indeed, a singular limit) of a multilocal field theory defined in ten-dimensional spacetime, known as type II superstring theory [1,2]. This theory combines the multiplet of massless states of N=8 supergravity and an infinite number of massive supersymmetry multiplets into the set of excitations of an extended object (a string moving in ten-dimensional spacetime). The resulting field theory has *local* interactions, corresponding to the joining and splitting of strings. The interactions in this model are all governed by two parameters, one of which characterizes the strength of the gravitational interaction, whereas the other determines the masses of the excited states of the strings. However, only the cubic couplings have been formulated in a field-theoretic language so far, and the construction of the full theory represents one of the main challenges for the near future in this context.

From the point of view of ten-dimensional spacetime, the ground state (i.e., the massless sector) of a superstring of type II is the set of states of N=2 supergravity †. There is also another, equally remarkable, ten-dimensional superstring theory. This is known as superstring theory I, and its ground state includes the set of states of N=1 supersymmetric Yang-Mills theory in ten dimensions, together with the states of N=1 supergravity in ten dimensions. This second model describes Yang-Mills type interactions also, corresponding to a gauge group G.

As is well known, Yang-Mills theories can be constructed for any gauge group G which is a direct product of simple groups and U(1) groups. Our concern in this chapter will be extending this analysis to the theory of type I superstrings, thus providing a classification of all gauge groups allowed in this case (at least at the classical level). In doing this, we shall find it worthwhile to provide some basic results about superstring theory. We will only present the material we need in order to discuss the problem of introducing Yang-Mills gauge groups in the theory of type I superstrings. Further details can be found in ref. [6].

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† To be precise, there are two forms of N=2 supergravity in ten dimensions [3,4,5], only one of which can be obtained by reducing the eleven-dimensional supergravity theory. Correspondingly, there are two forms of type II superstring theory.

3.2. Gauge Groups of Type-I Superstrings

The quantum mechanics of pointlike particles can be described in terms of a sum over paths, with each path weighted by the classical action computed along it. Correspondingly, local field theories describe the amplitudes for scattering of particles in terms of sets of Feynman diagrams that link initial and final states by means of the interactions in the theories. The quantum mechanics of strings is described in terms of a sum over world sheets swept out by the strings in the course of their motion, the weight being the classical action on the surfaces. Again, the field theory of strings involves operators creating and destroying strings, and the scattering of strings can be described in terms of diagrams. However, because of the extended nature of the strings, the diagrams are two-dimensional surfaces. If one borrows the result, familiar from local field theory, that a given amplitude is determined by the complete set of corresponding topologically inequivalent diagrams, one is lead to classifying topologically inequivalent surfaces (open and closed) in order to describe the perturbation expansion of string amplitudes. In particular, tree amplitudes correspond to intermediate states involving only one string, and therefore to surfaces with no holes or handles. The problem is that, just as happens with diagrams in local field theory, surfaces which naively look different may be equivalent upon suitable "stretching". This property alone, without any extra dynamical input, imposes a constraint on the amplitudes. It implies that the amplitudes provided by the theory possess total cyclic symmetry in the external states. This leads, in particular, to the equality of the s-channel and t-channel tree amplitudes in four-particle scattering, which is commonly called "duality". Figure 1 illustrates this property of tree-level string theory diagrams for a four-particle amplitude. The relation

$$A(1,2,3,4) = A(4,1,2,3) \qquad (3.2.1)$$

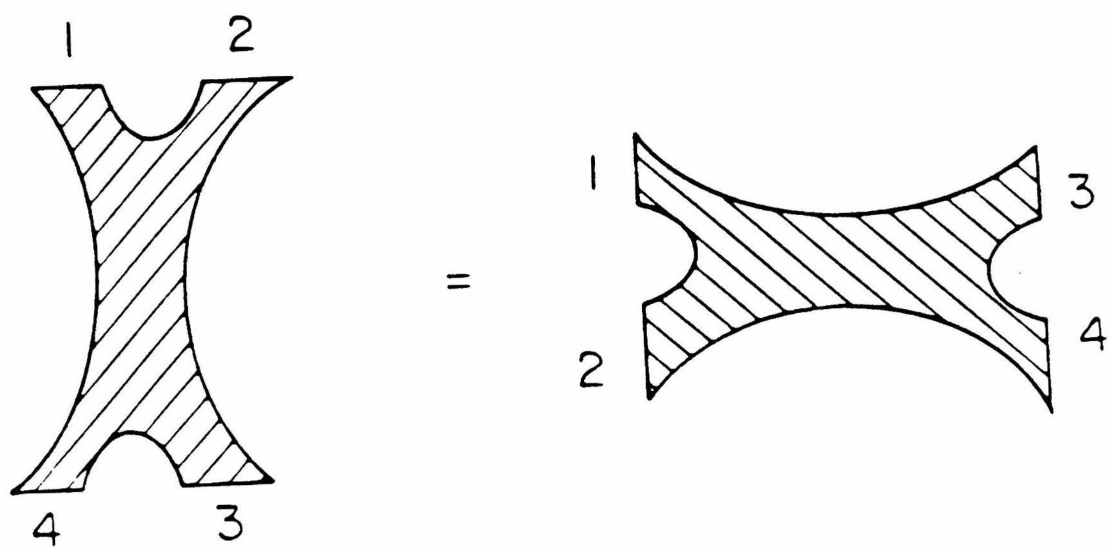

Figure 1. Duality for a four-point amplitude

then evidently holds. This is the standard form of the duality condition. In general, for N-point tree amplitudes, one can deform diagrams in several ways, thus exhibiting different channels that connect two subsets of the total set of external states. The generalization of the condition in eq. (3.2.1) then requires that the tree amplitudes are totally symmetric under the interchange of the labels of the external states. More importantly for our purposes, this picture leads directly to the constraints on the amplitudes due to unitarity. These conditions follow from the structure of the residues of the poles in the various intermediate channels. The poles in the intermediate channels must describe the same particle states as the external states, i.e they must have the right quantum numbers consistent with the known spectrum of the theory. This condition, as we will see, imposes severe restrictions on the gauge groups allowed in type I superstring theory.

Before proceeding, however, we need to describe two essential ingredients. First of all, tree amplitudes possess one more symmetry of a topological nature. This occurs because all the states of the strings are eigenstates of an operator, the twist operator, that exchanges the endpoints of the strings. More quantitatively, the twist operator for on-mass-shell states has the simple form

$$T = (-1)^{m+1} \quad , \tag{3.2.2}$$

where m is an operator which gives the mass squared of an excited state of the string relative to that of the first excited state. States at even mass levels are thus odd under twisting, whereas states at odd mass levels are even under twisting. The corresponding symmetry of the tree amplitudes then follows by twisting all their external legs, and is

$$A(1,2,...,N) = (-1)^{N+\sum_1^N m_i} A(N,...,2,1) \quad , \tag{3.2.3}$$

where $m_1, \cdots, m_N$ are the mass levels of the states $1,...,N$.

The next point we wish to mention is how a Yang-Mills gauge group can be introduced in the theory of type-I superstrings. As we have explained, the tree amplitudes provided by the theory possess cyclic symmetry in their external legs. The cyclic symmetry is clearly preserved if each amplitude is multiplied by a group-theory factor that is also cyclically symmetric. To construct such a factor we consider, to start with, a set of matrices representing the generators of a Lie algebra. This is done because the massless states of the string are the states of N=1 SSYM in ten dimensions which, as we know, form adjoint multiplets. On the other hand, we do not know, *a priori*, which representations correspond to the excited states of the string. Taking the trace of the product of a number of such matrices provides us with a quantity that has just the desired properties, namely it is cyclically symmetric and bears a set of adjoint indices. The tree-approximation S-matrix amplitude for a process with N external adjoint representation states is obtained by adding together products of the A(1,...,N) and of the trace factors $tr(\Lambda^{a_1} \cdots \Lambda^{a_N})$ corresponding to the (N-1)! cyclically inequivalent permutations of the external legs, i.e.

$$S(1,..,N) = \sum_{perms} A(1,..,N) tr(\Lambda^{a_1}..\Lambda^{a_N}) \quad . \tag{3.2.4}$$

The natural condition to impose is that, corresponding to a massless pole in the channel (1,...,r), i.e. corresponding to a set of external momenta such that $(p_1+...+p_r)^2 = 0$, for which the residue of $A(1,...,N)$ is $A(1,...,r)A(r+1,...,N)$ (see fig. 2), the group theory factor can also be split according to

$$tr(\Lambda^{a_1}..\Lambda^{a_n}) = \sum_I tr(\Lambda^{a_1}..\Lambda^{a_r}\Lambda^I) tr(\Lambda^I \Lambda^{a_{r+1}}..\Lambda^{a_n}) \quad . \tag{3.2.5}$$

It is clear that, using a suitably large set of matrices Λ^{a_i}, this condition can always be satisfied. It becomes nontrivial, however, if we require that the Λ^{a_i} be matrices of the right kind to describe massless states, i.e. that they label the adjoint representation of a group. This provides a constraint both on the gauge group and on the representation that is used for the matrices Λ^{a_i}. As in Yang-Mills theory, one deals only with compact Lie algebras, and correspondingly the Λ's can all be taken to be antihermitian. It is therefore clear that the condition (3.2.5) has one obvious solution, the defining representation of a U(N) group, which consists of the complete set of N x N antihermitian matrices. This was recognized a long time ago by Paton and Chan [7], who emphasized the particular cases of U(2) and U(3), then attractive in connection with the attempts to use dual models to describe hadron physics.

This problem has been reconsidered recently by Schwarz [8]. His motivation in doing this was a completely different one. It had to do with attempts at basing a description of fundamental interactions on the theory of superstrings, regarded as a well-behaved generalization of supergravity. The main new input in his discussion has to do with the twist symmetry. This, as we have anticipated, implies the relation (3.2.3) between the amplitudes, and tells us that, when factoring (3.2.4) at a massless pole in the channel $(1,\ldots,r)$, we must consider not only the configuration in fig. 2, but the three more shown in fig. 3. At the massless pole the corresponding amplitudes are all proportional to $A(1,\ldots,r)\,A(r+1,\ldots,n)$, on account of (3.2.3). The corresponding four terms are

$$\begin{aligned}
&tr(\Lambda^{a_1}\cdots\Lambda^{a_r}\Lambda^{a_{r+1}}\cdots\Lambda^{a_n})\,A(1,\ldots,r,I)\,A(I,r+1,\ldots,n)\,+\\
&tr(\Lambda^{a_1}\cdots\Lambda^{a_r}\Lambda^{a_n}\cdots\Lambda^{a_{r+1}})\,A(1,\ldots,r,I)\,A(I,n,\ldots,r+1)\,+\\
&tr(\Lambda^{a_r}\cdots\Lambda^{a_1}\Lambda^{a_{r+1}}\cdots\Lambda^{a_n})\,A(r,\ldots,1,I)\,A(I,r+1,\ldots,n)\,+\\
&tr(\Lambda^{a_r}\cdots\Lambda^{a_1}\Lambda^{a_n}\cdots\Lambda^{a_{r+1}})\,A(r,\ldots,1,I)\,A(I,n,\ldots,r+1)\ .
\end{aligned} \qquad (3.2.6)$$

Using eq. (3.2.3), these terms can be grouped together and written in the form

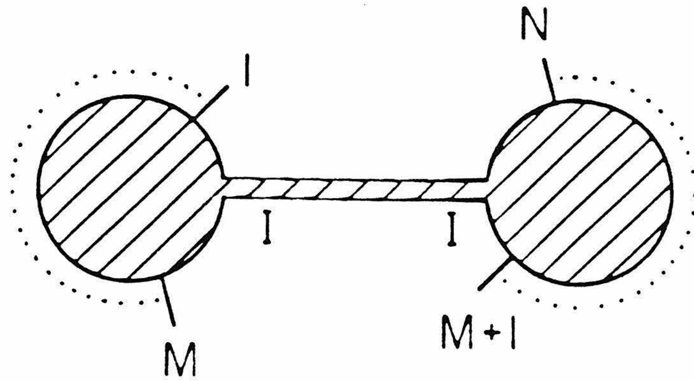

Figure 2. Factorization in the $(1,\ldots,M)$ channel

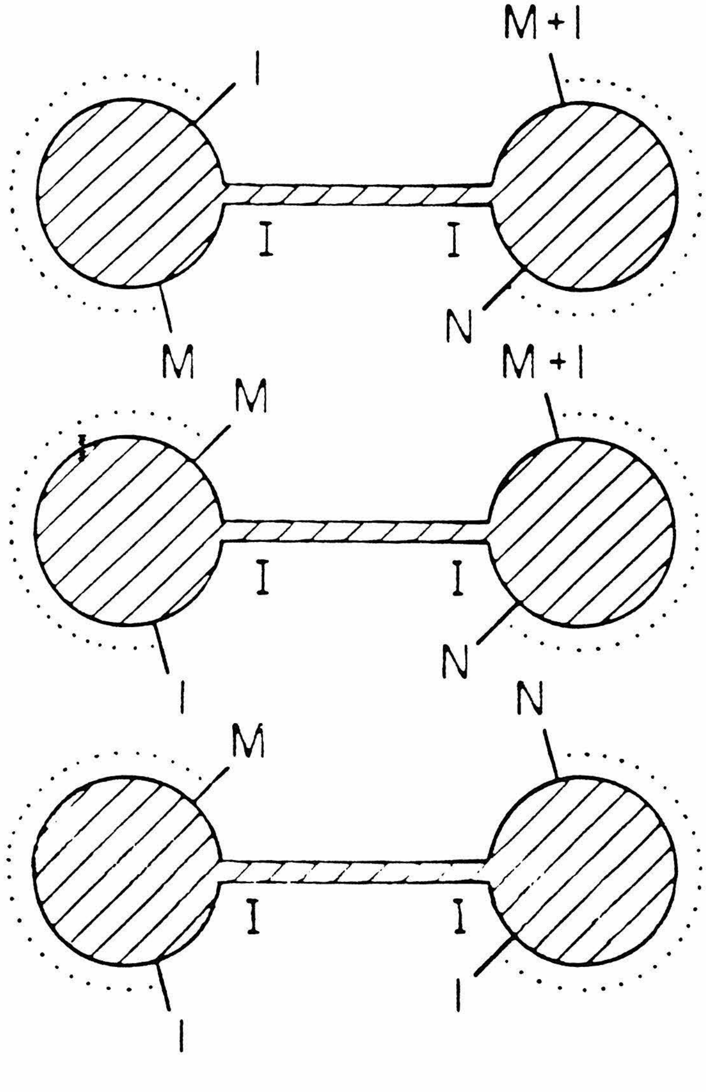

Figure 3 Diagrams related by twisting to the one in fig 2

$$A(1,\ldots,r,I)A(I,r+1,\ldots,n)tr((\Lambda^{a_1}\cdots\Lambda^{a_r}-(-1)^r\Lambda^{a_r}\cdots\Lambda^{a_1})$$

$$\times(\Lambda^{a_{r+1}}\cdots\Lambda^{a_n}-(-1)^{(n-r)}\Lambda^{a_n}\cdots\Lambda^{a_{r+1}})) \quad . \qquad (3.2.7)$$

We must require that (3.2.7) can also be written in the form

$$A(1,\ldots,r,I)\,A(I,r+1,..,n)\,tr((\Lambda^{a_1}..\Lambda^{a_r}-(-1)^r\Lambda^{a_r}..\Lambda^{a_1})\Lambda^I)$$

$$tr(\Lambda^I(\Lambda^{a_{r+1}}..\Lambda^{a_n}-(-1)^{(n-r)}\Lambda^{a_n}..\Lambda^{a_{r+1}})) \quad , \qquad (3.2.8)$$

corresponding to a sum over intermediate states, *with the index* I *being an adjoint representation index.* The factorization leading from (3.2.7) to (3.2.8) can clearly proceed if the matrices multiplying Λ^I are a linear combination of adjoint matrices†, because adjoint matrices satisfy an orthogonality condition of the form

$$tr(\Lambda^a\Lambda^b)=\delta^{ab} \quad . \qquad (3.2.9)$$

This therefore leads to an infinite set of conditions on the Λ's, which we write

$$\Lambda^{a_1}..\Lambda^{a_n}-(-1)^n\Lambda^{a_n}..\Lambda^{a_1} \quad \in \quad \lambda \ , \qquad (3.2.10)$$

to indicate that these particular combinations of the Λ's must be adjoint matrices for any n and for any choice of $a_1,..,a_n$. The n=2 case is trivial, as it just implies that the Λ's represent the generators of a Lie algebra. The conditions for $n>2$, however, are nontrivial, and require separate investigation. Three classes of solutions emerge by inspection of (3.2.10) [8]. They correspond to the antihermitian matrices Λ being the matrices of the defining representations of the classical algebras U(N), SO(N) and USp(2N). For example, to see that the matrices of the defining representation of U(N) are a solution, we construct

† By adjoint matrices we mean a set of matrices representing the generators of a Lie algebra.

$$\Lambda^{a_1}..\Lambda^{a_n} - (-1)^n \Lambda^{a_n}..\Lambda^{a_1} \quad . \tag{3.2.11}$$

Then, taking the hermitian conjugate of this expression and using $(\Lambda^{a_i})^\dagger = -\Lambda^{a_i}$ gives

$$(-1)^n \Lambda^{a_n}..\Lambda^{a_1} - \Lambda^{a_1}..\Lambda^{a_n} \quad , \tag{3.2.12}$$

so that (3.2.11) is also antihermitian, and therefore a matrix of the defining representation of U(N). The case of SO(N) also works in the same way, as antisymmetric matrices are just particular (real) antihermitian matrices. Finally, to discuss the case of USp(2N), it is most convenient to regard it as U(N,Q), the set of unitary matrices over the quaternions. Then the discussion above directly leads to the conclusion that USp(2N) also satisfies the conditions (3.2.10).

This still leaves open the possibility that more solutions exist, corresponding to other representations of the classical algebras, or to exceptional algebras. To study the solutions of (3.2.10) in general [9], a few comments are in order. First of all, the form of the twist operator in eq. (3.2.2) implies that *all states at even levels are odd under twisting* , and *all states at odd levels are even under twisting*. Consequently (3.2.10) would be recovered by studying factorization at any even-level pole, rather than only at a massless one. By a minor modification of the arguments that led to (3.2.10), we can also recognize that factorization at an odd-level pole would require that

$$\Lambda^{a_1}..\Lambda^{a_n} + (-1)^n \Lambda^{a_n}..\Lambda^{a_1} \tag{3.2.13}$$

be a matrix describing one of the odd-level states. The n=2 case here is nontrivial already, and tells us that odd-level states are associated with anticommutators of the matrices describing the even-level states. This shows that both the group and the representation selected for the Λ matrices are important.

In order to proceed further, it is convenient to note that the conditions for the even and odd-level poles can be combined into the equivalent, but simpler, algebraic condition that the Λ's form an algebra over the reals, i.e. that arbitrary linear combinations of the Λ's with *real* coefficients form a vector space closed under multiplication. We stress here that the conditions in (3.2.10) and (3.2.13) are equivalent to stating that the Λ's form an algebra *only* if the linear combinations are restricted to have real coefficients, as only in this case the hermiticity properties of the Λ's and of their anticommutators can be preserved.

Classifying the solutions to eqs. (3.2.10) and (3.2.11) thus amounts to classifying the real algebras (*not* Lie algebras, as we are now dealing with products, rather than with commutators). The problem of classifying algebras, as the corresponding one of classifying Lie algebras, is much simpler if the restriction that the coefficients be real is temporarily removed, thus considering the complex extension of the algebra. Eventually, of course, one must find some way of "taking the real part" of the result. We also notice that, since we are dealing with basis elements in the algebra which are either hermitian or antihermitian, the matrices Λ are either an irreducible set, or are a completely reducible one, which means they can be reduced to block diagonal form. Consequently, without any loss of generality, we can assume that the Λ's are a set of irreducible matrices, which is commonly expressed by saying that the algebra they generate is a *simple algebra*. The simple complex algebras are classified by Wedderburn's theorem [10], which states that the only simple complex algebras are the full algebras of complex matrices, GL(N,C). This result is much simpler than the corresponding one for Lie algebras, where many simple algebras exist. It corresponds to the intuitive idea that any complex N x N matrix can be obtained from arbitrary products of a "nontrivial" set of N x N matrices. It should also be clear why allowing arbitrary complex linear combinations of the Λ's leads to a simple result: the Λ's themselves can contain complex elements, and combining them with real coefficients leads in general to a subset of the GL(N,C) matrices, rather than to the whole set of them. In order to complete the classification, and thus to be able to apply our results to solve (3.2.10), we must learn how to "take the real part" of GL(N,C). That is, we must find all subsets of the n^2 GL(N,C) matrices that are closed under multiplication and, when multiplied by arbitrary complex factors, reproduce the whole set of GL(N,C) matrices. To this end, it is very helpful to note that GL(N,C), being an algebra, is also a Lie algebra, and it is known how to "take the real part" of a Lie algebra, just by looking at the known list of its real forms. It is necessary to distinguish between two cases. If the original algebra contains $\sqrt{-1}$, it coincides with its complexified form GL(N,C). If not, it is one of the real forms of GL(N,C) that, besides being a Lie algebra, is also an algebra. As such, it is to be found among the real forms of GL(N,C) containing the unit matrix $\underline{1}$ and not containing $\sqrt{-1}\,\underline{1}$, which are

$$R \otimes SU(N,C) \qquad\qquad R \otimes SU(p,q,C) \;\; (p+q=N)$$

$$GL(N,R) \qquad\qquad U^{*}(N) \;\; (N \text{ even}) \; . \tag{3.2.14}$$

Here direct product with R is a shorthand for the condition that all products with *real* multiples of the unit matrix are allowed, and $U^{*}(N)$ is the subset of SU(2N) generated by all antihermitian N x N matrices over the quaternions. GL(N,R) is clearly an algebra. Moreover, $U^{*}(N)$ is also an algebra, as it is isomorphic to the general linear algebra over the quaternions GL(N,Q). Finally, R $\otimes$ SU(N) and R $\otimes$ SU(p,q) are algebras only in the two-dimensional cases, where the isomorphisms SU(2) $\approx$ $SU^{*}(2)$ and SU(1,1) $\approx$ Sl(2,R) hold.

The solutions of eqs. (3.2.10) and (3.2.11) are therefore the general linear algebras over the real, complex and quaternionic fields. The matrices corresponding to the even-level states span the corresponding maximal compact subalgebras SO(N), U(N) and USp(2N), while the matrices corresponding to the odd-level states generate the coset spaces GL(N,R)/SO(N), GL(N,C)/U(N) and $U^*(2N)$/USp(2N). The even and odd states transform under the representations of the gauge group collected in the table below:

group	gauge group	even states	odd states
GL(d,R)	SO(d,R)	$\frac{d(d-1)}{2}$	$\frac{d(d+1)}{2} - 1$; 1
GL(d,C)	U(d,C)	d^2	d^2
$U^*(2d)$	USp(2d)	d(2d+1)	d(2d−1) − 1 ; 1

These results can be summarized by saying that the YM gauge groups allowed in superstring theory I are classical compact groups. *Exceptional groups are excluded.* Moreover, the reduced symmetry of the kinematic factors one is dealing with in superstring theory not only restricts the gauge groups, but also requires using the matrices of the defining representations for the groups allowed.

It should be stressed that eq. (3.2.10) was originally derived by considering the factorization of tree-level amplitudes. One may therefore wonder whether considering the factorization of higher-loop amplitudes would place further restrictions on the set of gauge groups allowed in superstring theory I. We will now discuss this for the case of one-loop amplitudes. This is a relatively simple case and still illustrates the general features of the factorization. Moreover, it allows to derive simply from first principles which kinds of diagrams one must consider at one-loop in the various cases, and the relative factors between them.

There are three distinct topologies for one-loop string theory diagrams, which are shown in fig.4. The *planar* diagrams are characterized by having the external particles all on the same boundary, and the two boundaries are distinct. The *nonplanar* diagrams are like the planar diagrams, but with the difference that some of the external particles are emitted from one boundary, and some from the other. Finally, the *nonorientable* diagrams have only one boundary, and look like a Mobius strip. The usual prescription is that the planar diagrams get an extra factor of N, corresponding to tr(1) for the boundary with no external particles, and that the nonorientable diagrams are absent in the case of the U(N) groups, when the ends of the strings can be thought of as carrying different, and inequivalent, quantum numbers (N and $\overline{N}$ respectively). We will now show how factorizing one-loop amplitudes leads directly to these results.

We start by considering the configuration in fig. 5, where the factorization of a one-loop amplitude is shown. As usual, we must consider the amplitude in fig. 5 together with other amplitudes related to it by twisting. For definiteness, we restrict ourselves to the case of external particles corresponding to even levels. We proceed in opposite order from what we did in the discussion of tree-level unitarity, and write the factorized amplitude, to then combine the group theory factors using the properties of the classical algebras summarized in the appendix. This will then give the imaginary part of the corresponding one-loop amplitude by unitarity, and from it we can read off the relative factors with which the three types of diagrams enter in the resulting amplitude.

Consider the factorized amplitude

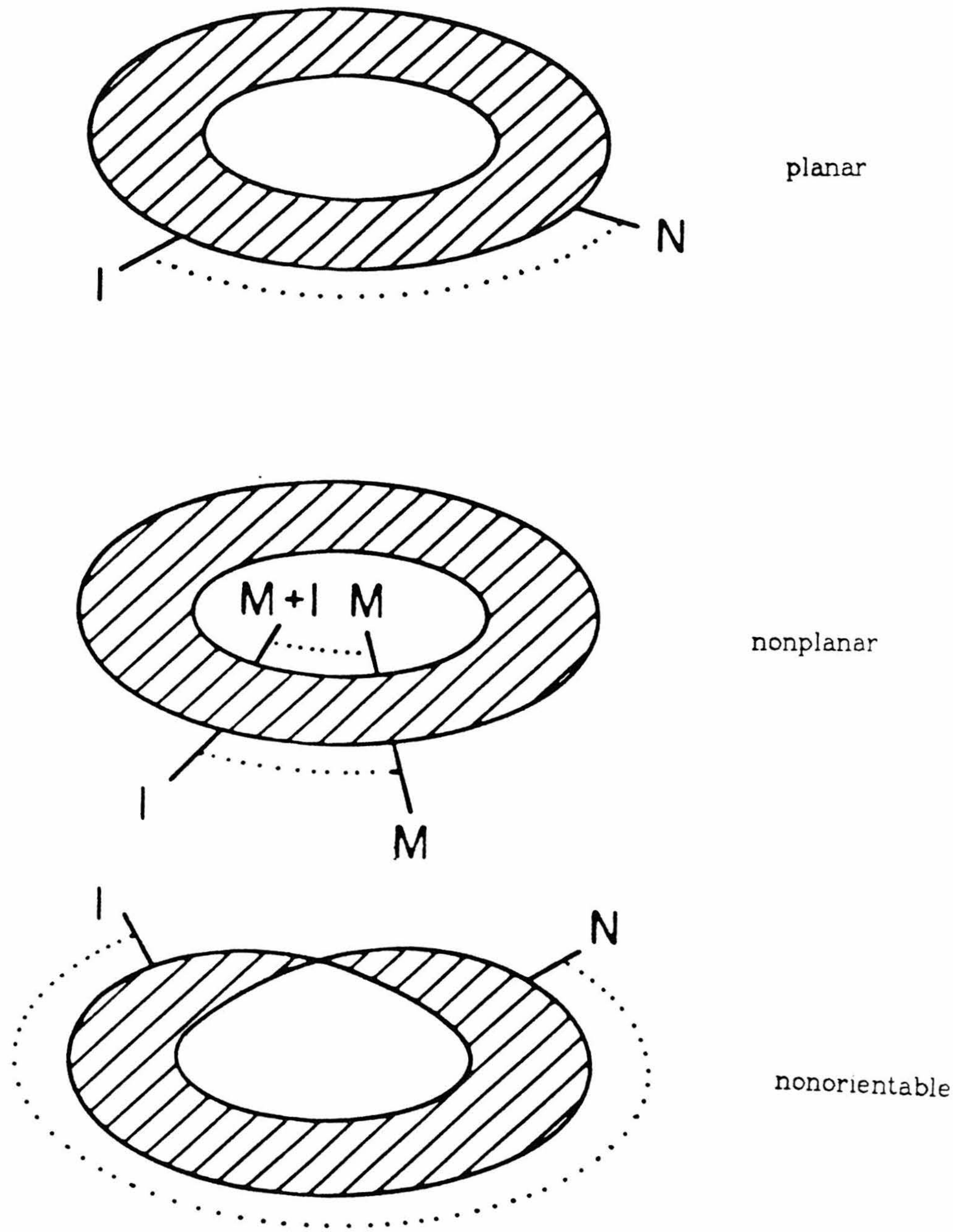

Figure 4. Topologies for one-loop diagrams

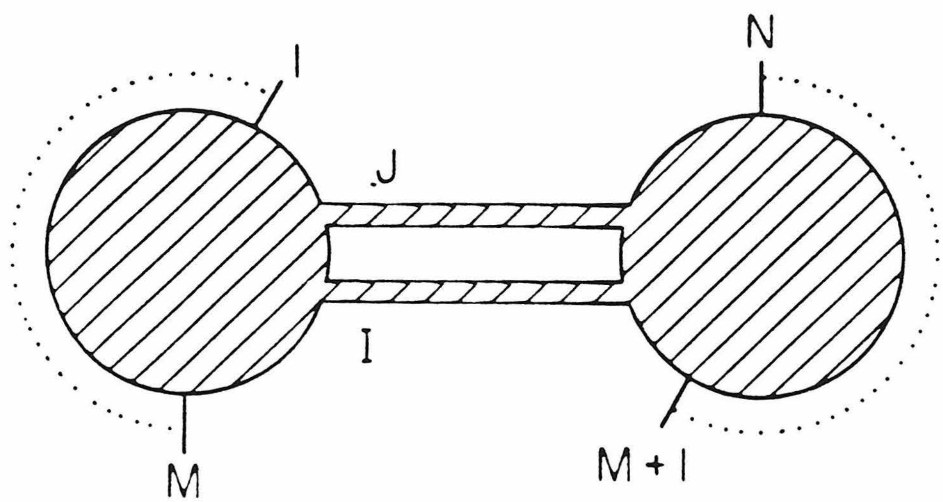

Figure 5. Factorization of a one-loop amplitude

$$(A(M+1,...,N,I,J)tr(\Lambda^{M+1}\cdots\Lambda^N\Lambda^I\Lambda^J)+A(N,...,M+1,I,J)$$

$$\times\, tr(\Lambda^N\cdots\Lambda^{M+1}\Lambda^I\Lambda^J)$$

$$+A(M+1,...,N,J,I)tr(\Lambda^{M+1}\cdots\Lambda^N\Lambda^J\Lambda^I)+A(N,...,M+1,J,I)$$

$$\times\, tr(\Lambda^N\cdots\Lambda^{M+1}\Lambda^J\Lambda^I))$$

$$\times\,(A(1,...,M,I,J)tr(\Lambda^1\cdots\Lambda^M\Lambda^I\Lambda^J)+A(M,...,1,I,J)tr(\Lambda^M\cdots\Lambda^1\Lambda^I\Lambda^J)$$

$$+A(1,...,M,J,I)tr(\Lambda^1\cdots\Lambda^M\Lambda^J\Lambda^I)+A(M,...,1,J,I)$$

$$\times\, tr(\Lambda^M\cdots\Lambda^1\Lambda^J\Lambda^I)) \qquad (3.2.15)$$

Using the twist condition (3.2.3) this can be written

$$(A(1,...,M,I,J)tr((\Lambda^1\cdots\Lambda^M\Lambda^I+(-1)^{M+m_I+m_J}\Lambda^I\Lambda^M\cdots\Lambda^1)\Lambda^J)$$

$$+A(1,...,M,J,I)tr((\Lambda^I\Lambda^1\cdots\Lambda^M+(-1)^{M+m_I+m_J}\Lambda^M\cdots\Lambda^1\Lambda^I)\Lambda^J))$$

$$\times\,(A(M+1,...,N,I,J)tr(\Lambda^J(\Lambda^{M+1}\cdots\Lambda^N\Lambda^I+(-1)^{N-M+m_I+m_J}\Lambda^I\Lambda^N\cdots\Lambda^{M+1}))$$

$$+A(M+1,..,N,J,I)tr(\Lambda^J(\Lambda^I\Lambda^{M+1}..\Lambda^N$$

$$+(-1)^{N-M+m_I+m_J}\Lambda^N..\Lambda^{M+1}\Lambda^I))), \qquad (3.2.16)$$

where m_I and m_J denote the mass levels of the intermediate states. We can now expand this expression and contract together products of traces, thus eliminating the Λ^J's. This requires the use of the conditions (3.2.10) and (3.2.11) on the matrices. The result, apart from an overall factor, is

$$tr((\Lambda^1\cdots\Lambda^M\Lambda^I+(-1)^{M+m_I+m_J}\Lambda^I\Lambda^M\cdots\Lambda^1)$$

$$\times\,(\Lambda^{M+1}\cdots\Lambda^N\Lambda^I+(-1)^{N-M+m_I+m_J}\Lambda^I\Lambda^N\cdots\Lambda^{M+1}))$$

$$\times A(1,...,M,I,J)A(M+1,...,N,I,J)$$

$$+ tr((\Lambda^1 \cdots \Lambda^M\Lambda^I + (-1)^{M+m_I+m_J}\Lambda^I\Lambda^M \cdots \Lambda^1)$$

$$\times (\Lambda^I\Lambda^{M+1} \cdots \Lambda^N + (-1)^{N-M+m_I+m_J}\Lambda^N \cdots \Lambda^{M+1}\Lambda^I))$$

$$\times A(1,...,M,I,J)A(M+1,...,N,J,I)$$

$$+ tr((\Lambda^I\Lambda^1 \cdots \Lambda^M + (-1)^{M+m_I+m_J}\Lambda^M \cdots \Lambda^1\Lambda^I)$$

$$\times (\Lambda^{M+1} \cdots \Lambda^N\Lambda^I + (-1)^{N-M+m_I+m_J}\Lambda^I\Lambda^N \cdots \Lambda^{M+1}))$$

$$\times A(1,...,M,J,I)A(M+1,...,N,I,J)$$

$$+ tr((\Lambda^I\Lambda^1 \cdots \Lambda^M + (-1)^{M+m_I+m_J}\Lambda^M \cdots \Lambda^1\Lambda^I)$$

$$\times (\Lambda^I\Lambda^{M+1} \cdots \Lambda^N + (-1)^{N-M+m_I+m_J}\Lambda^N \cdots \Lambda^{M+1}\Lambda^I))$$

$$\times A(1,...,M,J,I)A(M+1,...,N,J,I) \quad . \tag{3.2.17}$$

We wish to stress that the step that led to eq. (3.2.17), starting from eq. (3.2.16), is the crucial one, as undoing it amounts to factorizing the traces. We have thus shown that the conditions (3.2.10) are also sufficient to achieve factorization at the one-loop level, as they are at the tree level.

The next observation has to do with the nature of the terms in (3.2.17). This equation is clearly more complicated than the corresponding tree-level result, as there is still one dummy index to be eliminated. To proceed further we need identities to simplify terms like

$$\Lambda^I\Lambda^1 \cdots \Lambda^r\Lambda^I \quad . \tag{3.2.18}$$

Such identities are actually group dependent, and are noticeably different in the case of U(N) groups than in the cases of SO(N) and USp(2N) groups. In particular, in the appendix it is shown that the following results hold:

(1) for SO(N) :

$$\Lambda^I \Lambda^I = (1 - N(-1)^{N_I})$$

$$\Lambda^I \Lambda^1 \cdots \Lambda^r \Lambda^I = (-1)^r \Lambda^r \cdots \Lambda^1 - (-1)^{N_I} tr(\Lambda^1 \cdots \Lambda^M) \ , \qquad (3.2.19)$$

where N_I is the mass level to which the matrices Λ^I correspond;

(2) for USp(2N) :

$$\Lambda^I \Lambda^I = (-1 - N(-1)^{N_I})$$

$$\Lambda^I \Lambda^1 \cdots \Lambda^r \Lambda^I = -(-1)^M \Lambda^M \cdots \Lambda^1 - (-1)^{N_I} tr(\Lambda^1 \cdots \Lambda^M) \ , \quad (3.2.20)$$

where N_I is again the mass level to which the matrices Λ^I correspond;

(3) for U(N) :

$$\Lambda^I M \Lambda^I = -2tr(M) \ . \qquad (3.2.21)$$

Eq. (3.2.17) can be written, by relabeling some terms

$$(tr((\Lambda^1 \cdots \Lambda^M \Lambda^I + (-1)^{M+m_I+m_J} \Lambda^I \Lambda^M \cdots \Lambda^1)$$

$$\times (\Lambda^{M+1} \cdots \Lambda^N \Lambda^I + (-1)^{N-M+m_I+m_J} \Lambda^I \Lambda^N \cdots \Lambda^{M+1}))$$

$$+ tr((\Lambda^I \Lambda^1 \cdots \Lambda^M + (-1)^{M+m_I+m_J} \Lambda^M \cdots \Lambda^1 \Lambda^I)$$

$$\times (\Lambda^I \Lambda^{M+1} \cdots \Lambda^N + (-1)^{N-M+m_I+m_J} \Lambda^N \cdots \Lambda^{M+1} \Lambda^I)))$$

$$\times A(1,\ldots,M,I,J) A(M+1,\ldots,N,I,J)$$

$$+ (tr((\Lambda^1 \cdots \Lambda^M \Lambda^I + (-1)^{M+m_I+m_J} \Lambda^I \Lambda^M \cdots \Lambda^1)$$

$$\times (\Lambda^I \Lambda^{M+1} \cdots \Lambda^N + (-1)^{N-M+m_I+m_J} \Lambda^N \cdots \Lambda^{M+1} \Lambda^I))$$

$$+ \, tr((\Lambda^I \Lambda^1 \cdots \Lambda^M + (-1)^{M+m_I+m_J} \Lambda^M \cdots \Lambda^1 \Lambda^I)$$

$$\times \, (\Lambda^{M+1} \cdots \Lambda^N \Lambda^I + (-1)^{N-M+m_I+m_J} \Lambda^I \Lambda^N \cdots \Lambda^{M+1})))$$

$$\times \, A(1,\ldots,M,I,J) A(M+1,\ldots,N,J,I) \quad . \tag{3.2.22}$$

Expanding the traces then gives, apart from an overall factor,

$$A(1,\ldots,M,I,J)A(M+1,\ldots,N,I,J)\{ - (-1)^{m_I} tr(\Lambda^1 \cdots \Lambda^M) tr(\Lambda^{M+1} \cdots \Lambda^N)$$

$$+ [\pm 1 \pm (-1)^{m_I+m_J} - N(-1)^{m_I}](-1)^{M+m_I+m_J} tr(\Lambda^1 \cdots \Lambda^M \Lambda^N \cdots \Lambda^{M+1})) \}$$

$$+ \, A(1,\ldots,M,I,J)A(M+1,\ldots,N,J,I)\{ - (-1)^{M+m_J} tr(\Lambda^1 \cdots \Lambda^M) tr(\Lambda^{M+1} \cdots \Lambda^N)$$

$$+ \, [(\pm 1 \pm (-1)^{m_I+m_J} - N(-1)^{m_I}] tr(\Lambda^1 \cdots \Lambda^N) \} \tag{3.2.23}$$

in the SO(N) and USp(2N) cases, where the upper signs apply to the SO(N) case and the lower signs apply to the USp(2N) case, and

$$A(1,\ldots,M,I,J)A(M+1,\ldots,N,I,J) \, \mathrm{Re}[-(-1)^{M+m_I+m_J} N$$

$$\times \, tr(\Lambda^1 \cdots \Lambda^M \Lambda^N \cdots \Lambda^{M+1}) - (-1)^{m_I} tr(\Lambda^1 \cdots \Lambda^M) tr(\Lambda^{M+1} \cdots \Lambda^N)]$$

$$+ \, A(1,\ldots,M,I,J)A(M+1,\ldots,N,J,I)$$

$$\times \, \mathrm{Re}[-N \, tr(\Lambda^1 \cdots \Lambda^N)$$

$$- \, (-1)^{m_I+m_J} tr(\Lambda^1 \cdots \Lambda^M) tr(\Lambda^{M+1} \cdots \Lambda^N)] \tag{3.2.24}$$

in the U(N) case.

This result illustrates the peculiarity of string theories constructed using U(N) groups that we have anticipated. It has to do with the absence of N-independent terms containing a single trace in eq. (3.2.24), as compared with eq. (3.2.23), which applies to the SO(N) and USp(2N) cases. This result can be interpreted as follows: the planar diagrams get an extra factor of N corresponding to the boundary with no external particles, whereas the nonorientable diagrams do not get this factor, just because they have only one boundary, which contains all the external particles. Finally, the nonplanar diagrams correspond to products of traces, because some of the particles are on one of their boundaries and some are on the other. The absence of N-independent terms containing a single trace in (3.2.24) is just telling us that the perturbation expansion for open strings includes nonorientable diagrams for the cases of SO(N) and USp(2N), but does not include them for the case of U(N).

Appendix A

For SO(N) we can use as basis elements the matrices [11]

$$\eta_{ij}^{(N)} = \frac{1}{\sqrt{2}} (E_{ij}^{(N)} - E_{ji}^{(N)}) \tag{3.A.1}$$

for the even levels, and the matrices

$$\omega_{ij}^{(N)} = \frac{1}{\sqrt{2}} (E_{ij}^{(N)} + E_{ji}^{(N)}) \qquad (i \neq j)$$

$$\omega_{ii}^{(N)} = \sqrt{2}\, E_{ii}^{(N)} \tag{3.A.2}$$

for the odd levels. Here $E_{ij}^{(N)}$ denotes an N x N matrix with (i,j) element equal to one and all the other elements equal to zero. It is a very convenient object, as it multiplies according to

$$E_{ij}^{(N)}\, E_{kl}^{(N)} = \delta_{jk}\, E_{il}^{(N)} \quad , \tag{3.A.3}$$

and its trace is

$$tr\, (E_{ij}^{(N)}) = \delta_{ij} \quad . \tag{3.A.4}$$

As anticipated in the text, the matrices for the even levels are antihermitian, and the matrices for the odd levels are hermitian. Moreover, it can be readily verified that they satisfy the following trace relations:

$$tr\, (\eta_{ij}^{(N)}\, \eta_{kl}^{(N)}) = 2\, (\delta_{jk}\, \delta_{il} - \delta_{ik}\, \delta_{jl}) \quad ,$$

$$tr\, (\eta_{ij}^{(N)}\, \omega_{kl}^{(N)}) = 0 \quad ,$$

$$tr\, (\omega_{ij}^{(N)}\, \omega_{kl}^{(N)}) = 2\, (\delta_{jk}\, \delta_{il} + \delta_{ik}\, \delta_{jl}) \quad , \tag{3.A.5}$$

Then, using the definitions in eqs. (3.A.1) and (3.A.2), one can prove the following relations:

$$\Lambda^I \eta_{ij}^{(N)} \Lambda^I = -\eta_{ij}^{(N)} \ ,$$

$$\Lambda^I \omega_{ij}^{(N)} \Lambda^I = \omega_{ij}^{(N)} \ ,$$

$$\Lambda^I \Lambda^I = 1 - N(-1)^{m_I} \ , \tag{3.A.6}$$

where the Λ^I's are matrices either of the odd levels or of the even levels. Eqs. (3.A.6) then imply

$$\Lambda^I \Lambda^1 \cdots \Lambda^M \Lambda^I = (-1)^M \Lambda^M \cdots \Lambda^1 - (-1)^{m_I} tr(\Lambda^1 \cdots \Lambda^M) \ , \tag{3.A.7}$$

which is obtained decomposing the product $\Lambda^1 \cdots \Lambda^M$ into its irreducible pieces and using eqs. (3.A.6), and where $\Lambda^1, .., \Lambda^M$ are taken to be matrices for the even levels.

The case of USp(2N) can be treated along the same lines [11]. For the even levels one uses antihermitian matrices constructed out of

$$\alpha_{ij}^{(N)} = E_{ij}^{(N)} - sign(i) sign(j) E_{-i-j}^{(N)} \ , \tag{3.A.8}$$

(i,j = -N,..,N) and for the odd levels one uses hermitian matrices constructed out of

$$\beta_{ij}^{(N)} = E_{ij}^{(N)} + sign(i) sign(j) E_{-i-j}^{(N)} \ . \tag{3.A.9}$$

One thus arrives at the following results:

$$\Lambda^I \Lambda^I = -1 - N(-1)^{m_I}$$

$$\Lambda^I \Lambda^1 \cdots \Lambda^M \Lambda^I = -(-1)^M \Lambda^M \cdots \Lambda^1 - (-1)^{m_I} tr(\Lambda^1 \cdots \Lambda^M) \ . \tag{3.A.10}$$

More simply, one can use quaternionic notation and write the generators in the form

$$A \qquad \text{and} \qquad i\,\sigma \otimes B \tag{3.A.11}$$

with $A = -A^T$ and $B = B^T$ for the even levels, and $A = A^T$ and $B = -B^T$ for the odd levels. Then

$$\Lambda^I (X + i\,\sigma\otimes Y)\Lambda^I = A\,(X + i\,\sigma\otimes Y)\,A$$

$$- \sigma\otimes B\,(X + i\,\sigma\otimes Y)\,\sigma\otimes B \qquad (3.A.12)$$

and, using

$$\sum_i \sigma^i\,\sigma^i = 3 \quad \text{and} \quad \sum_i \sigma^j\,\sigma^i\,\sigma^j = -\,\sigma^i \qquad (3.A.13)$$

one obtains directly eqs. (3.A.10).

The case of U(N) is quite different, as now the same matrices (apart from a factor i) are used for both the even and the odd levels. For example, the matrices for the even levels are η_{ij} and $i\,\omega_{ij}$. From eqs. (3.A.6) it then follows that, for U(N), eqs. (3.A.7) and (3.A.8) are replaced by

$$\Lambda^I\,M\,\Lambda^I = -2tr(M) \quad . \qquad (3.A.14)$$

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