IMPERFECT INFORMATION AND OLIGOPOLY
WITH ENDOGENOUS MARKET POWER

Thesis by

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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

1983

(Submitted January 20, 1983)

ACKNOWLEDGEMENTS

I would like to sincerely thank Professor Edward Green, my principal advisor for his enormous help, support, and advice which made this thesis possible. He has been very patient with me and extremely generous with his time. His keen insight, enthusiasm and willingness to share his understanding of economic theory with me has made this research very pleasurable. Not only has he moulded my approach to the subject, but he has taught me that relentless dedication to work and adherence to truth is the hallmark of a scientist.

I would also like to thank the rest of my thesis committee members, Professors Kim Border, Richard McKelvey and Jennifer Reinganum for their thoughtful comments on this thesis. I also thank Professor Border for kindly consenting to be the chairman of my thesis committee and for his extensive comments that have significantly improved the exposition of this thesis.

I gratefully acknowledge the graduate research and teaching assistantships from the Division of Humanities and Social Sciences of the California Institute of Technology which have provided me with financial support during my years there.

I am thankful to Barbara Calli and Jeeva Jonahs for the excellent job of typing this thesis and to Ms. Christina Smith for coordinating the submission of the thesis in Pasadena while I was in Vancouver.

Finally I would like to thank my parents for everything they have been through in order to provide me with the education I have received; and my wife Asha who made it all worthwhile.

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ABSTRACT

This thesis consists of three essays. The first essay describes a model in which a dominant player can be endogenously determined. The model is developed in the context of Cournot and Stackelberg equilibria. Cournot equilibria are obtained in games where players move simultaneously (or sequentially but unobservably), and the extensive form strategy spaces of these players are isomorphic to each other. Stackelberg equilibria, on the other hand, are obtained as the perfect equilibria of perfect information games in which the players move sequentially, with the dominant player or the leader firm moving first and the other player moving second. Thus, the question of how to model an industry--Cournot or Stackelberg--is answered by examining timing and information conditions both of which are presumed exogenous. Firm sizes and technologies and demand characteristics are, in this context irrelevant. What we do instead, is to note that if demand is resolved over time,then firms may face a trade-off between making decisions before the uncertainty in demand is revealed and thereby establishing a "leadership" position, or waiting until after resolution of demand in order to avoid production mistakes. The sequentially rational Nash equilibrium of the resulting game is examined. It is shown that in a market with one large firm (i.e., a firm whose output affects price) and a nonatomic continuum of small firms (i.e., firms whose individual outputs do not affect price), the only equilibrium of the game described above, with nontrivial but small uncertainty, is a Stackelberg equilibrium with the large firm as the endogenously determined dominant

player. The difference between a large and a small firm is also embodied in their respective cost functions.

The second essay answers the question of whether markets with one large firm and several small but atomic firms can be approximated by or can approximate a Stackelberg equilibrium. This is answered by establishing that the equilibrium correspondence of a family of games, each of which has one large firm and several small firms, and the number of small firms increases to infinity, is continuous.

The third essay adapts the model developed in the first essay to a model of noncooperative general exchange in which the traders are in the same strategic position with respect to each other. Thus a noncooperative game is defined in an exchange economy such that a price-setting monopolist is determined endogenously in equilibrium, and this is the unique sequentially rational Nash equilibrium.

A NONCOOPERATIVE EQUILIBRIUM WITH AN ENDOGENOUSLY DETERMINED DOMINANT PLAYER: THE CASE OF COURNOT VERSUS STACKELBERG

1. INTRODUCTION

There have been two classical ways of modeling the behavior of firms in oligopolies. The models differ in their assumptions about firm behaviour and result in different equilibrium outcomes. In one set of models, it is assumed that the firms in a market play a Cournot game with each other. A Cournot game is a noncooperative game in extensive form in which the players are in the same strategic position with respect to each other. That is, the players move simultaneously (or sequentially but unobservably) and their strategy spaces are isomorphic to each other. An example of the Nash equilibrium of such a game when firms choose quantities of production, is the Cournot equilibrium. A more detailed description of a cournot equilibrium follows in section 4.

In the other type of models, it is assumed that the firms play a noncooperative game in which some of the players are in a dominant strategic position with respect to some others. Such a game is called a Stackelberg game. Here, the dominant players move first and have strategy spaces that are not isomorphic to those of the other players. Moreover, these are games of perfect information and the payoffs to a player, among other things, also depend upon when a player moves. An example of the Nash equilibrium of such a game when the dominant firm chooses output quantities and the other firms choose their output quantities as functions of the dominant firm's output is the

Stackelberg equilibrium.

Thus in order to be able to know how to model an industry--Cournot game or Stackelberg game--it would be sufficient to examine timing and information conditions both of which are presumed exogenous. The sizes or technologies of firms, or the characteristics of demand, are in this context, irrelevant. On the other hand there is a "Folk Theorem" that outcomes in oligopolies are best modeled by Cournot equilibria if the firms are of equal size, but by Stackelberg equilibria if they are not. This suggests the possibility that timing and information conditions could be endogenously determined using among other things, firm sizes or technologies as exogenous.

However, unless one is able to obtain a systematic relation between these exogenous characteristics and the choice between a Cournot game or a Stackelberg game to describe an industry, one has to make an ad hoc assumption about the firms' conduct in the industry. Such would be the case as long as one is unable to discern firm behavior in a systematic way using observable data-like firm sizes or demand characteristics. Sometimes this assumption is crucial to policy decisions. Consider for example, a regulator trying to decide whether or not to regulate a duopoly. Let the regulator's objective function be consumer's surplus. Also, let the firms have zero marginal costs. Let the firms' output quantities be denoted by x1 and x2 respectively. Assume that demand is linear and is given by price = y - x1 - x2. Let K be the cost of regulating this industry. Under the assumption that the firms are playing a Cournot game, the consumer surplus is $2y^2/9$ and under the assumption that the firms are playing a Stackelberg game, the consumer surplus is $9y^2/32$. If $2y^2/9 < R < 9y^2/32$ then, while it may be worthwhile regulating the industry under the assumption that it is a Cournot duopoly, it is unprofitable to regulate the same industry if it is assumed that it is a Stackelberg duopoly. Note that in general, using output and demand data, one would not be able to infer the type of equilibrium--Cournot or Stackelberg--without complete information on the cost functions of the firms involved.

The basic purpose of this chapter is to make timing and information conditions endogenous using data on the sizes or technologies of firms and certain characteristics of demand. Thus, we wish to make the choice between a Cournot game and a Stackelberg game endogenous. Since in our model, sizes and technologies are exogenous, we will be able to obtain a rigorous formulation and verification of the "Folk Theorems."

One may try to make timing and information conditions endogenous (i.e., endogenize the choice between a Cournot game and a Stackelberg game) by simply developing a framework in which the firms are allowed to decide which game they want to play. This will not work because in general, in a Stackelberg equilibrium, the leader (dominant) firms are better off than the follower firms and in general, could be better off than in a Cournot equilibrium. Hence all the firms might want to play the Stackelberg game expecting to be the leader. In other words, in order to obtain a Stackelberg equilibrium in which there is a leader and a follower, one would be forced to exogenously assign the dominant player. This assignment would be quite as ad hoc as making the assumption that the firms are playing a Cournot or Stackelberg game. What we do therefore, is to describe a game of imperfect information in which ex ante, the players are in the same strategic position with respect to each other. However, when the sequentially rational Nash equilibrium (see [5]) strategies are being played, it would appear as if the firms are playing the equilibrium strategies of a Stackelberg game or a Cournot game.

In a quantity-setting example of the Cournot game, the strategies of all the firms are output levels. In a quantity-setting example of the Stackelberg game, the dominant firms' strategies are output levels while the other firms' strategies are reaction functions (i.e., output levels that are functions of the dominant firms' output level). On the other hand, if demand uncertainty is resolved over time, then firms may face a trade-off between making quantity decisions early so as to establish a "leadership" position, or waiting until the demand uncertainty has been resolved so as to avoid production decision mistakes. Thus, a larger game is constructed in which there are two logical time periods. There is uncertainty in demand which is revealed between the two time periods. We assume that the firms behave in a sequentially rational way given the information they have in each time period. The firms move simultaneously before the beginning of the first period. The behavior strategy for each firm in the beginning of the game consists of a probability that it would enter in period 1 and the quantity it would produce if it were to enter in period 1. If both firms end up entering in the same period, the sequentially rational Nash equilibrium is Cournot-like, whereas if they end up entering in different periods, it is Stackelberg-like. This will now provide a framework in which we can ask how firms' sizes and technologies and the nature of demand can determine whether an industry is best modelled as Cournot or Stackelberg. An example of the larger game we are alluding to, with two firms and two levels of production for each firm is shown in Figure 6. We will describe this in greater detail in section 2. The basic results of this paper are about the nature of the sequentially rational Nash equilibrium of this larger game and are the following:

1. With the symmetric firms, there is a symmetric equilibrium which is the appropriate generalization of a Cournot equilibrium. (See Section 3 below.) This result assumes a particular technology with linear demand and quadratic costs.

2. In a market with one large firm and a continuum of small firms, the <u>only</u> equilibrium is a Stackelberg Equilibrium.

3. Under some conditions set forth below, firms in this equilibrium will not randomize their "times of entry."[1] Every temporally nonrandomized equilibrium corresponds to either a Cournot or a Stackelberg equilibrium in the strong sense that the quantities produced are exactly those predicted by the respective extensive form concepts.

4. With two symmetric firms, under certain conditions, specified below symmetric equilibrium must be temporally randomized. This is proved in the paper using the technology of result 2.

We now develop some notation in Section 2. In Section 3, the Cournot game, the Stackelberg game and the larger game of our concern are defined. Section 4 examines some properties of the equilibrium of the larger game and Section 5 concludes the paper.

2. DEFINITIONS AND NOTATIONS

We recall the definition of a game in its extensive form as in Kuhn [3]. However, since only a particular game of the form shown in Figure 6 is analyzed, in order to minimize notation, we develop the definitions only with respect to the game depicted in Figure 5. We will refer to this game ad the "larger game."

The game is represented by a tree. The edges that come out of each node represent the alternatives at that node. Nodes which possess alternatives are called moves and those that do not possess alternatives are called terminal nodes. The rank of a node is the number of moves that are on the path from the initial node to itself. The set of moves of a given rank represents a turn for some player. The turns of player 1, player 2 and nature are denoted by (1), (2), and (N) respectively. There is a path or branch running from the initial node to each terminal node. Each such branch is associated with certain payoffs to the players. Often, in a particular players's turn, the player does not have information about which alternative the previous player has chosen. In such a case, the set of nodes at the end points of all those edges is called an information set. For example, the nodes J_{11}, J_{12}, J_{13} form an information set for player 2.

In any game represented in its extensive form by a tree, we could consider the set of nodes in any information set as the set of initial nodes of another game. This is called the subgame of the original game, and the tree that follows this information set is called the subtree of the original tree.[2]

$\hat{N}$ is the number of players. In Figure 5, $\hat{N} = 2$.

The set of vertices that are not terminal vertices are partitioned according to the moves that represent each player's turn. This is the player partition $\{P_1, P_2, \ldots P_N\}$. In Figure 5, the player partition is $\{P_1, P_2\}$ where $P_1 = \{I_1 I_2, \ldots I_7\}$ and $P_2 = \{J_{11}, J_{12}, J_{13}, J_2, \ldots J_{61}, J_{62}, J_{71}, J_{72}\}$.

B is the set of branches (a branch is denoted by b). Since randomized strategies may be played by all players including nature, a probability is assigned to each branch. It is with respect to these probabilities that players make their expected payoff calculations. $G_b(B)$ is the set of probability measures on B, and g_b is an element of G_b.

The information partition is a refinement of the player partition into information sets U_i for each player i. Again in Figure 5,

$$U_1 = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7\}$$

$U_2 = \{\{J_{11},J_{12},J_{13}\},J_2,J_3,J_4,J_5,\{J_{61},J_{62}\}, \{J_{71},J_{72}\}\}.$

A^i_n is the set of edges that come out of node n. Alternatively, one could think of A^i_n as the set of nodes at the end of these edges.

Next, M^i_n is the set of probability measures on A^i_n and $m^i_n \varepsilon M^i_n$. M_i is the product $\prod_{n\varepsilon P_i} M^i_n$.

A <u>behavior strategy</u> for each player is a strategy that consists of randomizing over the alternatives at each move of that player. Further, since in an information set a player cannot distinguish between the nodes, the randomization over the alternatives at each node in the information set should be the same.

Thus for a player i, a behavior strategy at a node n is a probability measure s^i_n on A^i_n such that for every information set u, and all n,n′ ε u, $s^i_n = s^i_{n'}$. For this reason we may ignore the subscript n on s^i_n. Let $S_i \subseteq M_i$ be the set of all behavior strategies of player i, each behavior strategy denoted by $s_i = \prod_{n\varepsilon P_i} s^i_n$. Let the payoff function be given by $\hat{\Pi} : B \to R^N$ and $\hat{\Pi}_i(b)$ be the i^{th} component of $\hat{\Pi}(b)$; i.e., the i^{th} player's payoff.

Next, let

$$S = \prod_{i\varepsilon N} S_i.$$

There is a mapping

$$\mu : S \to G_b(B)$$

induced by the probability measures on the set of branches, due to behavior strategy N-tuples. The measure μ is derived inductively in the following way. Consider a game in its extensive form that has $(\bar{n} + 1)$ turns numbered from 0 to $\bar{n}$. There may be more than one node in each turn, and the particular information set which is of concern to the player when his turn arrives will depend upon the alternative that was chosen in the previous move. Denote the ith node in the jth turn by $\bar{n}_{ji}$, and the set of nodes in that turn by $\bar{n}_j$. Then define

$$\mu_1 = s_{\bar{n}_o}$$

$$\mu_{k+1}(A^*) = \int_{\bar{n}_k} s_{\bar{n}_{kj}} (A_{\bar{n}_{kj}} \cap A^*) d\mu_k .$$

for every μ_{k+1} -- measurable subset A* of $\bar{n}_{k+1}$. Since each terminal node is associated with a unique branch in the game tree, $\mu_{\bar{n}+1}$ defines the function μ mentioned above. The symbol E_μ denotes the expectation with respect to μ and $E_{\mu|(\cdot)}$ is the expectation with respect to μ conditioned on ($\cdot$).

3. An Example with an Endogenously Determined Dominant Player

In this section, results (1) and (2) in the Introduction are proved. To recapitulate, they are stated below.

The equilibrium of the larger game has two forms, one of which is symmetric and the other asymmetric. It will be shown that the symmetric form depends on the nature of the uncertainty. Thus, at the extremes of risk (i.e., zero variance of the demand distribution or a "too diffuse" distribution), it is temporally nonrandomized and corresponds to a Cournot equilibrium, and at intermediate values of the variance it is temporally randomized. A continuous function contained in the equilibrium correspondence links these Cournot end points and all symmetric equilibria lying on this path. It is in this sense remarked to be an appropriate extension of a Cournot equilibrium. This is result (1) of the Introduction. The asymmetric form of the equilibrium will be discussed in Section 4.

Next, we will study how these equilibria depend on relative firm sizes. It will be demonstrated that in a market with one large firm and a continuum of small firms, the only equilibrium is a Stackelberg equilibrium where the large firm moves before the demand is revealed as a leader, and the small firms enter the market as followers after the demand is revealed. This is result (2) of the Introduction.

A very complicated model would be needed to derive these results in complete generality. However, since what is important is the nature of the game tree, rather than the technology of the individual players, the results are proved in the context of a particular technology where the demand is linear and firms have quadratic costs.

The following notation is used in this section:

γ - is the variance of nature's distribution function.

x_i - is a production level of firm i.

x_{iB} – is a value of the production level when firm i decides to enter "Before."

x_{iA} – is a value of the production level when firm i decides to enter "After."

$b|x_{iB},y$ – are the paths that result when x_{iB} and y, the outcome of the random variable are fixed, and x_{jB} or x_{jA} are allowed to vary, for all j, $j \neq i$.

Let the demand be given by $p_r = y - \sum_i x_i$, where p_r is the price and y is the random shift parameter. Let N_f be the set of firms that want to enter the market. At first we shall consider the case of duopoly.

The market demand is given by $p_r = y - x_1 - x_2$. The cost function for both firms is $C(x_i) = (x_i^2)$ for an output level x_i.

Denote by $\hat{\Pi}(b|x_{iB},y)$, the set of values that $\hat{\Pi}$ takes for the different branches represented by $b|x_{iB},y$. Similarly, $\hat{\Pi}(b|x_{i(\cdot)}, x_{j(\cdot)},y)$ is the payoff associated with the particular branch containing $x_{i(\cdot)}$, $x_{j(\cdot)}$ and y. From the tree in Figure 6, it is easy to see that there is always at most one such branch.

The Cournot equilibrium points are then easily seen to be

$$x_{1B} = x_{2B} = \frac{E_y(y)}{5} \qquad (1)$$

and

$$x_{1A} = x_{2A} = \frac{y}{5} \tag{2}$$

While the Stackelberg equilibrium points are:

$$x_{1B} = \frac{3E_y(y)}{14} \tag{3}$$

$$x_{2A} = \frac{y - \frac{3E_y(y)}{14}}{4} \tag{4}$$

The equilibrium of the larger game is obtained as follows.

Denote $E_y(y)$ by E, $E_y(y^2)$ by $\hat{E}$

and $(E_y(y))^2$ by E^2.

Further, let firm i's probability of entering in period B be ν_i and the quantity it decides to produce when entering in period B be x_{iB}.

In the larger game, in the first information set of every player, the player has to decide on a probability of entering in period B and the quantity it would produce if it were to enter in period B. In order to decide on the quantity it would produce if it were to enter in period B, the firm has to choose a quantity so that its ante expected payoff of entering in period B is maximized given certain beliefs about nature's actions and the other player's actions. Thus player i will maximize over his choice variable χ_{iB},

$E_y(\nu_j \cdot \pi_i(\chi_{iB},\chi_{jB},y) + (1 - \nu_j)\pi_i(\chi_{iB}, \chi_{jA},y))\ i \neq j$

noting that χ_{jA} is the best response function given χ_{iB} and y. Thus it obtains an optimal $\hat{\chi}_{iB}$ as a function of χ_{jB}, ν_2 and E. On the other hand, the probability that the i^{th} player will enter in period B is calculated in the following way: Given that it is going to randomize between entering in period A and entering in period B with the optimal amount $\hat{\chi}^*_{iB}$, it must be indifferent (since we are considering only sequentially rational Nash equilibria) between entering in period A and entering in period B.

Thus

$$x_{1B} = \underset{x}{\text{argmax}}\ E_y\{\nu_2 \quad \hat{\Pi}(b|x,x_{2B},y) + (1 - \nu_2) \quad \hat{\Pi}(b|x,x_{2A},y)\}$$

noting that x_{2A} is a measurable function of x and y. It is thus a random variable.

Similarly, we get an expression for x_{2B}. In equilibrium

$\hat{x}_{iB}$ and $\hat{\nu}_i$ satisfy $(i = 1,2)$

(a) $\hat{x}_{iB} = \hat{x}_{jB} = x_B$ and $\hat{\nu}_i = \hat{\nu}_j = \nu$, $i \neq j$

and, given that we are now interested in temporally randomized equilibria,

(b) $$E_y\{\nu \cdot \hat{\Pi}(b|x_B, y) + (1 - \nu) \cdot \hat{\Pi}(b|x_B, x_A, y)\} \tag{5}$$

$$= E_y\{\nu \cdot \hat{\Pi}(b|x_A, x_B, y) + (1 - \nu) \cdot \hat{\Pi}(b|x_A, y)\}$$

It is easy to verify that the values of x_B and of ν thus obtained are the equilibrium values. Solving the maximization problem using (5a), we find

$$x_B = \frac{3E + E\nu}{14 + 6\nu} \tag{6}$$

Finally, substituting for x_B in (5b) above, we obtain

$$\nu^3(650E^2 - 648\hat{E}) + \nu^2(4200E^2 - 4176\hat{E}_2)$$
$$+ \nu(8850E^2 - 8904\hat{E}) + (6300E^2 - 6272\hat{E}) = 0. \tag{7}$$

A solution to the above equation assuming y is normally distributed with mean 1 and variance γ, yields ν as a function of γ, the variance of y. We deduce from equations (5a), (6) and (7) that the equilibrium is symmetric across players.

Also from equation (7), $\nu(0) = 1$, and with $E = 1$, $\nu(.0044) = 0$. Thus when the uncertainty is nontrivial ($\gamma \neq 0$), but sufficiently small ($\gamma < 0.0044$), temporal randomization occurs i.e., $\nu \neq 0,1$. Also, for $\gamma > .0044$, any nonzero ν is not a Nash equilibrium.

Hence there are two equilibria: an asymmetric one corresponding to a Stackelberg equilibrium; and a symmetric one, with the probability of entering "Before" given by the solution to equation (7) and the quantity to be produced a function of that probability as given in (6).

Further, note that in expression (6) $\nu = 1$ yields $x_B = \frac{E}{5}$ which is indeed the Cournot level of production (see equation (1)) should both firms decide to go "Before."

We are now in a position to prove the following theorem.

<u>Theorem 1</u>: In the above duopoly game parameterized by variance as a risk parameter, the equilibrium corresponding to Cournot equilibria (which occur at both extremes of risk) are connected by a continuous path in the graph of the equilibrium correspondence, and the equilibria along this path are symmetric.

<u>Proof</u>: Clearly if $\gamma = 0$, then $\nu = 1$ for both players, and an ex ante Cournot equilibrium results. Further for $\gamma = 0.0044$, (if $E = 1$) $\nu = 0$, and an ex post Cournot equilibrium results. Consider the correspondence $\phi_i : \{\gamma\} \rightrightarrows [0,1]$, with $\phi_i(\gamma) = \nu_i$ obtained from equation (7).

Let us first look at the nature of the correspondence knowing the following facts.

(a) From Proposition 2, (see section 4 below) $\phi_i(\gamma) \neq 0$ or 1 when $\gamma \neq 0$ or $<$ .0044.

(b) The cubic (7) can be written in terms of γ, the variance, i.e.,

$$f(\nu,\gamma) = \nu^3(648(.003086 - \gamma)) + \nu^2(4176(.005 - \gamma))$$
$$+ \nu(8904(-.005 - \gamma)) + (62722(.004 - 1)) = 0.$$

(c) At $\gamma = 0$, by Descartes' Rule of Signs there are 2 or 0 positive roots, 1 negative root and if the discriminant is 0 then there are two identical roots.

(d) At $\gamma = 0.0044$, there are 2 or 0 positive roots, no negative roots, no identical roots, or 1 real root and 2 imaginary roots, and the discriminant is negative.

(e) At $\gamma = 0.003086$, there are 2 or 0 positive roots, no negative roots, and the discriminant is positive; i.e., there are two distinct roots. This is because the polynomial becomes a quadratic at this point.

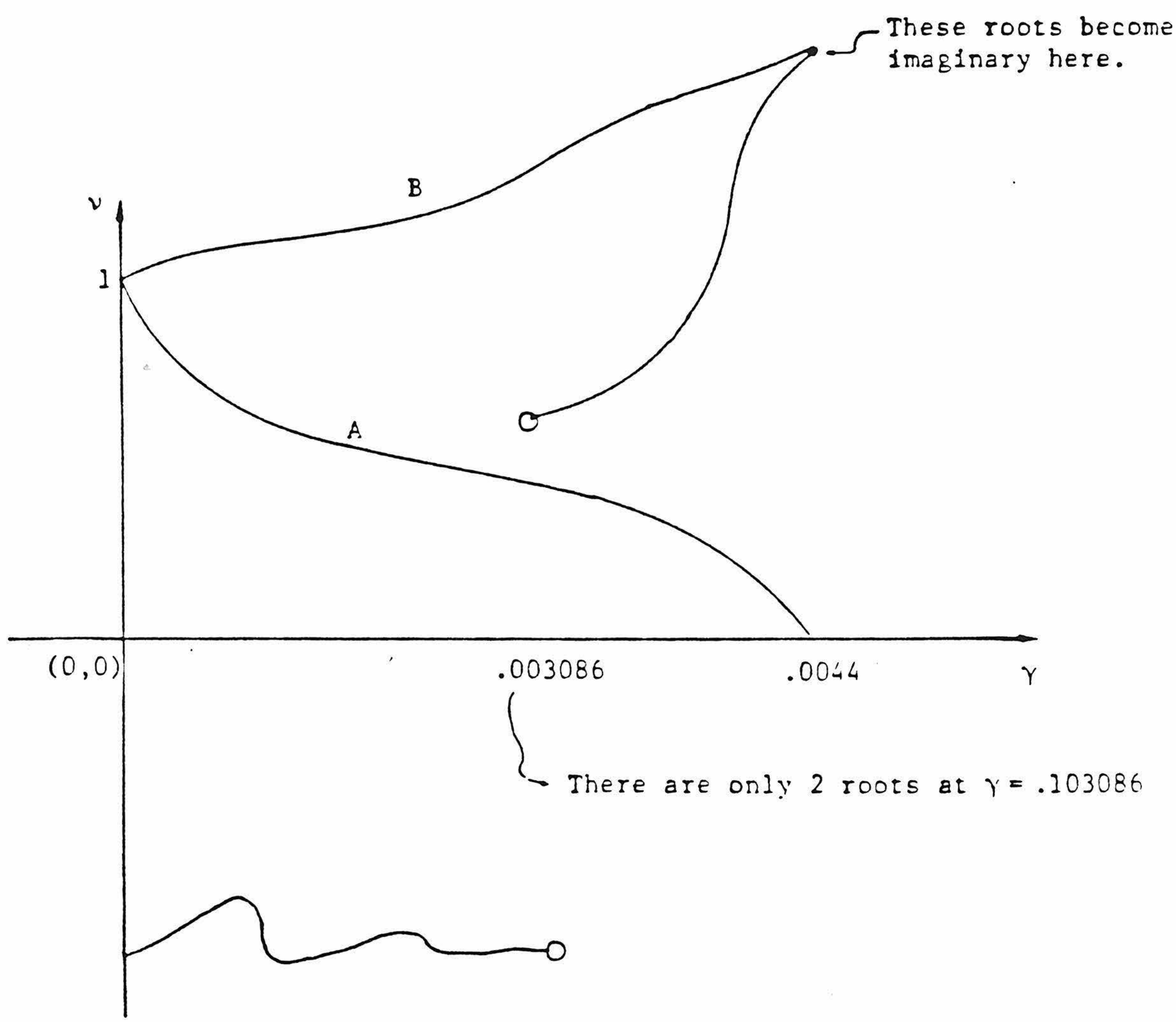
(f) The partial derivative of the polynomial $\frac{\partial f(\nu,\gamma)}{\partial \gamma}$ is never 0 for $\nu \in [0,1]$ and $\frac{\partial f(\nu,\gamma)}{\partial \nu} \neq 0$ for all γ. Thus, the Jacobian of the polynomial is never 0 in the range of concern.

The graph of the correspondence then would look like Figure 1.

[Figure 1 Here]

Now, it is sufficient to show that there is at least one continuous path from the point (0,1) to the point (0.0044, 0).

FIGURE 1


These roots become imaginary here.
ν
B
1
A
(0,0)
.003086
.0044
γ
There are only 2 roots at γ = .103086

Let $f(\nu,\gamma) = 0$ be the polynomial equation under consideration. We know that $\forall(\gamma,\nu) \varepsilon [0, 0.0044] \times [0,1]$, the Jacobian of partial derivatives of first order, $J(f(\nu,\gamma))$ is nonzero.

Then consider any point (γ_0,ν_0), with $\gamma_0 \varepsilon (0, 0.0044]$, and $\nu_0 < 1$ such that $f(\nu_0,\gamma_0) = 0$. Then by the implicit function theorem, there exists a smooth function g and a neighborhood $N(\gamma_0)$ such that for all $\gamma \varepsilon N(\gamma_0)$, $J(f(\nu,\gamma)) \neq 0$.

$$g(\gamma_0) = \nu_0$$

and

$$\forall\gamma \varepsilon N(\gamma_0), \quad f(g(\gamma),\gamma) = 0$$

From observation (e) above, the neighborhood $N(\gamma_0) = [0, 0.0044]$. Thus, the continuous path that is required is the graph of g.

Q.E.D.

Two simple, but interesting, corollaries follow from theorem 2 above. The first corollary states that the continuous path connecting the Cournot extremes is monotone decreasing in the graph of the correspondence ϕ described above. This means that for both firms, in the symmetric equilibrium, the probability of entering "Before" keeps getting smaller as the variance of nature's distribution increases, i.e., as the uncertainty in the demand increases they are less likely to enter the market before the demand information is revealed.

Corollary 1: $\forall\gamma \geq 0,\ \forall\nu \in [0,1],\ \frac{d\nu}{d\gamma} < 0.$

Proof: Writing $\frac{d\nu}{d\gamma}$ as $\frac{-\frac{\partial f}{\partial\gamma}}{\frac{\partial f}{\partial\nu}}$ using the implicit function theorem, the proof is obvious.

The second corollary states that given one firm is more likely to go "After" as demand uncertainty increases, the other firm will want to produce more in the period "Before." Further, this desire to produce more is continuous in the probability of entry "Before," until the other firm will want to produce the Stackelberg leader's quantity when the first firm wants to enter "After" for certain.

Corollary 2: $\forall\nu \in [0,1],\ \frac{dx_B}{d\gamma} > 0.$

Proof: Obvious from expression (6) and corollary 1 above.

In the context of result (2) of the Introduction, we will see that if a mixed set of firms (one large atomic firm and a nonatomic continuum of firms), is contemplating entry into a market with uncertain demand, it is a Nash Equilibrium for the atomic firm to go "Before" as a Stackelberg leader, and for the nonatomic firms to go "After" as followers. The intuitive reason is that each nonatomic firm is so small that it can have no incentive effect on the atomic firm or other nonatomic firms. Furthermore, we will show that since moving "After" is the dominant strategy for the nonatomic firms, and therefore, this is the unique Nash equilibrium.

Denote the large firm's production level by x_1 and the small firms' production level by x_2.

Let the set of "small" firms be indexed by the unit interval $I = [0,1]$, endowed with Lebesgue measure σ. Thus for $S \subseteq I$, $\sigma(s)$, is the proportion of firms belonging to the subset s. Let x(i) denote the amount produced by each firm $i \in I$. The profit associated with x(i) is denoted by $\Pi^i(x(i))$. Let the "large" atomic firm be referred to as the firm of type 1 with a cost function $C_1(x)$. We let the cost function for the atomic firm be $F_1 + C_1x_1^2$ and for the nonatomic firm be $F_2 + C_2x_2^2$. $F_i, C_i \in R$, $i = 1,2$. We are now in a position to state and prove result (2) of the Introduction.

<u>Theorem 2</u>: With one large firm and a continuum of nonatomic firms, with the technology given above, (if there is nontrivial uncertainty), the small firms will enter in period A in the equilibrium of the larger game. If this uncertainty is sufficiently small, then the only equilibrium corresponds to a Stackelberg equilibrium.

<u>Proof</u>: We will show that in the type of markets described above, a Stackelberg equilibrium with the atomic firm entering in period B as leader, is the only equilibrium of the larger game provided there is sufficiently small uncertainty in the demand parameter.

Should it decide to enter early as a Stackelberg leader, firm 1 decides on its production level, as follows:

$$X_1 = \text{argmax } E_y(y - (x_1 + \int_0^1 x(\gamma, x_1)d\mu))x_1 - C_1x_1^2 - F_1),$$

where $x(\gamma, x_1)$ is the follower firm's reaction function.

For a follower, nonatomic firm, x_2 maximizes ex post profits and is the solution to

$$\max_{x_2} (y - x_1 - \int_0^1 x(\gamma, x_1)d\mu)x_2 - C_2x_2^2 - F_2$$

yielding $x_2 = \dfrac{y - x_1 - x(x_1)}{2C_2}$ where $x(x_1) = \int_0^1 x(\gamma, x_1)d\mu$.

Then the profits of a follower are

$$\frac{(y - x_1)^2(2C_2 - 1)}{(2C_2 + 1)^2} - F_2 \tag{8}$$

Since $\int_0^1 x_2 d\mu = x(x_1)$, we have

$$x(x_1) = \int_0^1 \frac{y - x_1 - x(x_1)}{2C_2} d\mu$$

therefore

$$x(x_1) = \frac{y - x_1}{2C_2 + 1} = x_2 \tag{9}$$

Substituting this into the first order condition for firm 1 yields

$$x_1 = \frac{C_2 E}{C_1 + 2C_1C_2 + C_2}. \tag{10}$$

It is easy to show that, when the uncertainty is not too large, the large firm's profits are lower if it decides to enter in Period A. On the other hand, the firm contemplating moving "Before" (i.e., the deviant firm), decides on its production level by maximizing its ex ante profits. Thus,

$$x_2^{de} = \frac{E - x_1 - Ey(x(x_1))}{2C_2}$$

$$= \frac{E - x_1}{2C_2 + 1} \tag{11}$$

so that its ex ante profits are,

$$\frac{(E - x_1)^2(2C_2 - 1)}{(2C_2 + 1)^2} - F_2 \tag{12}$$

Therefore, ex ante, if the deviant firm wants to compare profits, it sees that

$$E_y \left\{ \frac{(y - x_1)^2(2C_2 - 1)}{(2C_2 + 1)^2} \right\} \geq \frac{(E - x_1)^2(2C_2 - 1)}{(2C_2 + 1)^2}$$

since $E_2 - E^2 = \text{var} \geq 0$.

So for the nonatomic firm it is dominant to be a follower and enter "After" for all γ. It can be shown further, that if the amount of uncertainty as measured by γ is larger than a certain value, depending upon the cost characteristics of the large firm, all firms will enter a period A. The proof of this in the case of linear demand--quadratic cost is easy to see. Another example of the argument is used in Chapter 3.

Thus we observe that in the case of two identical firms a symmetric equilibrium results. While in the case of one atomic firm of measure one and a nonatomic continuum of firms, with non-trivial but small uncertainty, the only equilibrium is a Stackelberg equilibrium. A natural question would be: Is it true that, as we increase the cardinality of one set of firms while decreasing the measure of every firm in it, the resulting respective equilibria converge to the case of an asymmetric equilibrium of a mixed market? This will be examined in the next chapter.

4. A More General Model

In this section, a more general from of the Cournot game, Stackelberg game and the larger game are defined, and results (3) and (4) of the introduction are derived.

First, Cournot and Stackelberg equilibria are defined in their general extensive forms. Uncertainty in the market demand is next embedded into the above definitions with players assumed to be Bayesian decision makers whose alternatives at each turn are quantities of production.

In the Stackelberg game, the natural time for the demand uncertainty to be resolved is between the "entry time" of the leader and that of the follower. In Cournot equilibrium, it is possible that uncertainty might be resolved either before or after the time at which firms simultaneously make their quantity decisions.

In the extensive form of the larger game each firm is free to make its quantity decisions either before or after the demand is known. The sequentially rational Nash equilibrium of the subgame that results when both firms decide at the same time (either before or after) is the Cournot equilibrium and the equilibrium of the subgame that results when one firm makes quantity decisions before the information is revealed and the other makes it after, is the Stackelberg equilibrium.

Thus, we want to describe an extensive form game whose Nash equilibria, under certain conditions, correspond to a Cournot or a Stackelberg equilibrium. In order to do this, we construct a game by combining subgames in which the Nash equilibria are precisely the Cournot or the Stackelberg equilibrium. Therefore, we first define these in their extensive forms, and then embed these trees in an extensive form game whose Nash equilibrium we examine.

Cournot Equilibrium in its Extensive Form

Consider the following game, in which there are two players 1 and 2 and two production levels, high (H) and low (L).

[Figure 2 here]

FIGURE 2

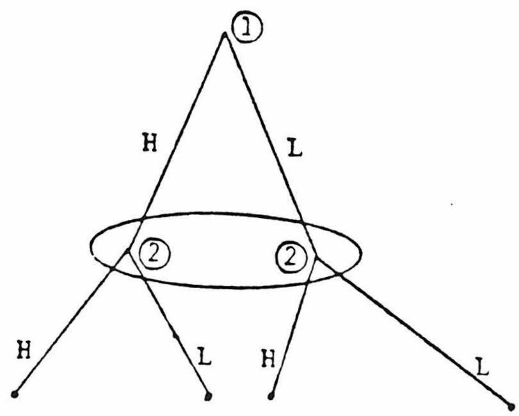
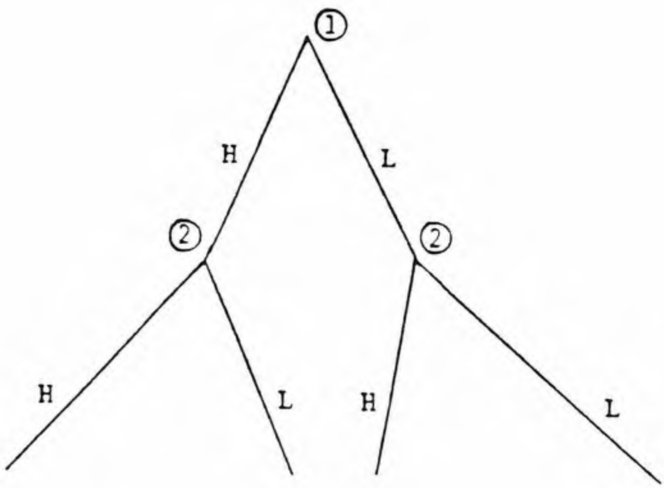


FIGURE 3



In the general multiplayer game with continuously variable production, let $s_i \varepsilon M_i$ be a behavior strategy of player i. A Cournot equilibrium is a vector $s \varepsilon S$ such that $\forall i \varepsilon N \exists$, $s_{i'} \varepsilon S_i$ with

$$E_{\mu(s_1,s_2,\ldots,s_i,\ldots)} \hat{\Pi}_i(b)_{b\varepsilon B} < E_{\mu(s_1,s_2,\ldots,s_{i'},\ldots)} \hat{\Pi}_i(b)_{b\varepsilon B} \quad (13)$$

where E_μ is the expectation over μ.

Stackelberg Equilibrium in its Extensive Form

Consider a game in extensive form whose representative tree for two players, and two production levels is:

[Figure 3 here]

Stackelberg equilibrium is the sequentially rational Nash equilibrium of this game (see [5]) and it is a dominant player equilibrium.

In the general multiplayer case, let $D \varepsilon N$ be the dominant player, who moves first. Again, let $\hat{\Pi} : B \rightarrow R^N$ be the payoff function, and $\mu : S \rightarrow G_b(B)$ be the induced probability measure on the branches.

Then a sequentially rational Nash equilibrium of such a game is a vector $s \varepsilon S$ such that,

(a) $\nexists s_D' \varepsilon M_D$,

$$E_{\mu|s_D} \hat{\Pi}_D(b)_{b\varepsilon B} < E_{\mu|s_D'} \hat{\Pi}_D(b)_{b\varepsilon B}$$

(b) $$\forall s_D' \varepsilon M_D, \forall i \varepsilon N, i \neq D, \nexists s_i' \varepsilon M_i ,$$

$$E_{\mu|s_i,s_D'} \hat{\Pi}_i(b)_{b\varepsilon B} < E_{\mu|s_i',s_D'} \hat{\Pi}_i(b)_{b\varepsilon B} . \qquad (14)$$

Condition (b) ensures that the equilibrium is sequentially rational.

Next, uncertainty in the market demand is embedded into the above definitions. Let there be two "time periods." Assume that demand is revealed between these periods. The periods are referred to as "Before" (B) and "After" (A). Thus, nature is conceived of as having a distribution over a demand shift parameter y. Recall that the reason for introducing demand uncertainty is to describe the strategic relation between the players, allowing them not only to choose production quantities, but "entry times" as well. In a Cournot equilibrium it is possible that uncertainty might be resolved either before or after the time at which firms simultaneously make their quantity decisions. The players are assumed to be Bayesian decision makers.

For instance, if y had two possible values high (h) and low (l), then a representative tree might look like Figure 4a or Figure 4b.

[Figures 4a and 4b Here]

FIGURE 4a

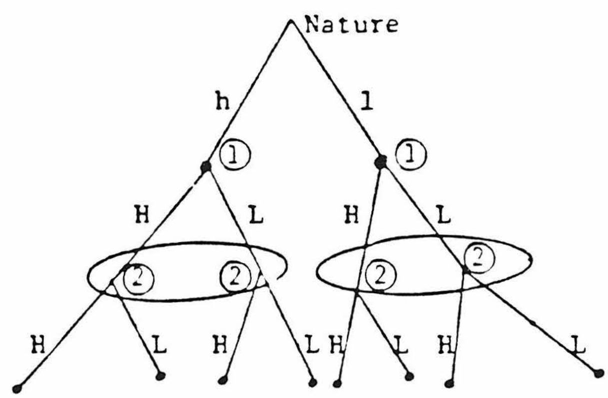
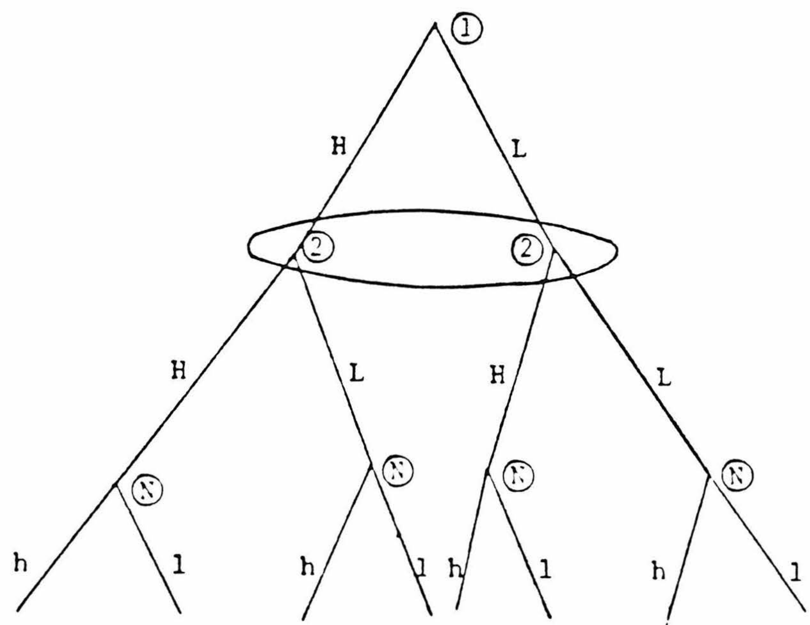


FIGURE 4b



In this figure (N) denotes nature's move.

In general, let nature's strategy be a particular probability measure $g_y(Y)$ on Y which consists of the possible actions nature can take, denoted by y. Y is a subset of R. Let $g_y(Y)$ have a variance γ. The probability measure g_y in an element of $G_y(Y)$, the set of all probability measures on Y. Then if,

$$S = \prod_{i \varepsilon N} M_i \times G_y(Y)$$

and $\mu : S \rightarrow G_b(E)$ is the induced probability measure on the branches, then a Nash equilibrium is a vector $s \varepsilon S$ such that,

$$\forall i \varepsilon N, \nexists s_i' \varepsilon M_i, E_{\mu|s_i,g_y} \hat{\Pi}_i(b)_{b\varepsilon B} < E_{\mu|s_i',g_y} \hat{\Pi}_i(b)_{b\varepsilon B} \quad (15)$$

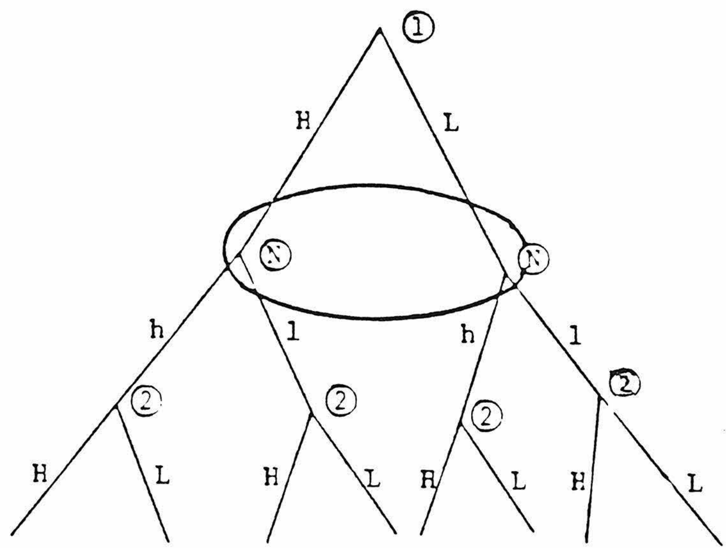
In the Stackelberg game, on the other hand, the natural time for the uncertainty to be resolved is between the "entry time" of the leader and that of the follower. Again, a simple example is given in Figure 5. It is reasonable to assume here that firm 1, which enters before nature's play, is the dominant player.

[Figure 5 Here]

In general, using the notation of the earlier discussion of dominant player equilibrium, and letting

$$S = \prod_{i \varepsilon N} M_i \times G_y(Y),$$

FIGURE 5



In this figure (N) denotes nature's moves.

a dominant player equilibrium is the sequentially rational Nash equilibrium of this type of game. The equilibrium is a vector $s \varepsilon S$, with,

(a) $\nexists s_D' \varepsilon M_D$ such that

$$E_{\mu|s_D} \hat{\Pi}_D(b)_{b \varepsilon B} < E_{\mu|s_D'} \hat{\Pi}_D(b)_{b \varepsilon B}$$

(b) $\forall s_D' \varepsilon M_D, \forall i \varepsilon N, i \neq D, \nexists s_i' \varepsilon M_i$ such that

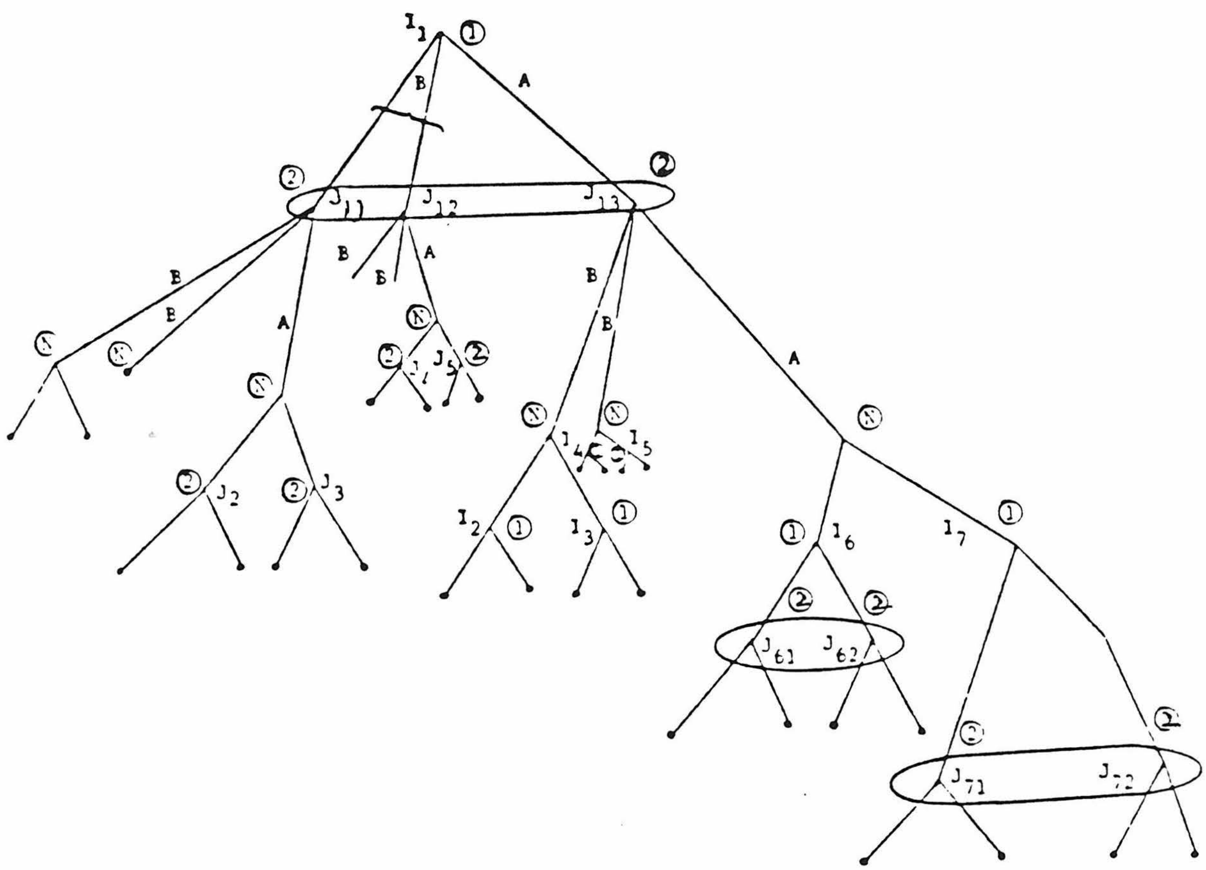
$$E_{\mu|s_i, s_D', y^*} \hat{\Pi}_i(b) < E_{\mu|s_i', s_D', y^*} \hat{\Pi}_i(b). \quad (16)$$

[3]

The equilibrium which endogenizes the Cournot-Stackelberg choice is now defined as the sequentially rational Nash equilibrium of the extensive form game in which each firm is free to make its quantity decisions either before or after the demand is known. The Nash equilibrium of the subgame that results when both firms decide at the same time (either before or after) is the Cournot equilibrium and the equilibrium of the subgame that results when one firm makes quantity decisions before the information is revealed and the other makes it after, is the Stackelberg equilibrium.

A typical tree when there are two players, two levels of production for each player and two values that y can take is shown in Figure 6.

FIGURE 6



(Similar looking paths have not all been completely drawn)

[Figure 6 Here]

To indicate the Nash equilibrium for the general case becomes very complicated and so we will do so for the case of a duopoly. We do this to show the explicit relationship between this equilibrium and the Cournot equilibrium and the Stackelberg equilibrium. Refer to Figure 6.

The vector $(s_1, s_2) \in S_1 \times S_2$ is a sequentially rational Nash equilibrium if,

$$\forall i \neq j,\ \forall s_j' \in M_j,\ \nexists s_i' \in M_i\,,$$

$$E_{\mu | s_i, s_j', g_y} \hat{\Pi}_i(b) < E_{\mu | s_i', s_j', g_y} \hat{\Pi}_i(b) \qquad (17)$$

We are now in a position to compare these equilibria by stating and proving the following proposition in two parts. The second part of the proposition is proved in the body of the proof of the first part.

<u>Proposition 1(a)</u>: For two symmetric firms, every temporally nonrandomized equilibrium of the larger game corresponds to either a Cournot or a Stackelberg equilibrium.

<u>Proof</u>: If an equilibrium of the larger game is temporally

nonrandomized, then at each node n, for every player i, s_n^i is such that the probability of an edge which is an action about only when to enter is 0 or 1. Of course if a strategy is such that the probability of an edge is 0, then $g_b(b) = 0$ for every path that contains that edge.

Thus if $s_{I_1}^1$ is such that the probability of A = 0, and $s_{J_{11}}^2$ is such that the probability of A = 0, then the equilibrium of the larger game is such that from equation (17), the set of paths with nonzero probabilities is the same as the ex ante Cournot game in Figure 4b.

Similarly if $s_{I_1}^1$ is such that the probability of A = 1, and $s_{J_{13}}^2$ is such that the probability of A = 1, then the equilibrium of the larger game corresponds to an ex post Cournot equilibrium. Notice however that this is not a Nash equilibrium of the larger game tree because if temporal randomization is allowed, then at J_{11} (say) player two can find a strategy which will yield him at least as good a payoff as an ex ante Cournot, viz, $s_{J_{11}}^2$ which is such that the probability of A = 1.

Next if $s_{I_1}^1$ is such that the probability of A = 0, and $s_{J_{11}}^2$ is such that the probability of A = 1, then in the equilibrium of the larger game, for player 1:

$$\nexists s_1' \in M_1 \ni$$

$$E_{\mu|s_1,g_y} \hat{\Pi}_1(b) < E_{\mu|s_1',g_y} \hat{\Pi}_1(b)$$

Similarly this condition can be reinterpreted for player 2, and the set of paths of nonzero probabilities is the same as that of the Stackelberg game in Figure 5.

Notice here that this is an asymmetric equilibrium of the larger game even when we allow temporal randomization. For person 2 can do no better against person 1's equilibrium strategy of $s^1_{I_1}$ for which the probability of A = 0.

Of course another equilibrium strategy would be with person 2 having a strategy with $s^2_{J_{11}}$ resulting in the probability of A = 0 and $s^1_{I_1}$ is such that the probability of A = 1.

Also, it is easy to see that a symmetric equilibrium with the probability of A = 1 in both s_1 and s_2 is not an equilibrium of the larger game if we assume that $\gamma \varepsilon (0,\gamma')$, where γ' is some finite value of the variance of nature's distribution such that the gain in being a Stackelberg leader is greater than playing an ex post Cournot game.

Thus for every $\gamma \varepsilon (0,\gamma')$, there can be no symmetric nonrandomized equilibrium of the larger game.

Q.E.D.

<u>Proposition 1(b)</u>: With two symmetric firms there is an equilibrium of the larger game corresponding to a Stackelberg equilibrium. There is

also a symmetric (in both timing and information contingent output) equilibrium. If the uncertainty is nontrivial but sufficiently small so that being a Stackelberg leader is more profitable than being an ex post Cournot firm, then the symmetric equilibrium must be temporally randomized.

5. CONCLUSION

In this chapter, we set out to answer the question: under what circumstances might noncooperative equilibrium take a Cournot form and when might it take a Stackelberg form? In a Cournot game, the players are in the same strategic position with respect to each other and they are assumed to be moving simultaneously (or sequentially but unobservably). In a Stackelberg game, there are some players who are dominant, who move first and who are in a different strategic position with respect to the other players. Thus, a classical way to try and answer the question was to examine timing and information conditions, both of which were presumed exogenous. If these were unobservable, then one was guided by the "Folk Theorem" that outcomes in oligopolies were best modeled by a Cournot equilibrium if the firms were of equal size and by a Stackelberg equilibrium, if they were not. The technologies of the firms and demand characteristics were irrelevant.

On the other hand, we answered the question by defining a game in which ex ante all the players were in the same strategic position with respect to each other, while demand characteristics and sizes and firm technologies were exogenous. The basic idea was that if demand uncertainty was resolved over time, then firms may face a trade-off between making quantity decisions early so as to establish a "leadership" position, or waiting until the demand uncertainty is resolved so as to avoid production mistakes. A sequentially rational Nash equilibrium of the resulting game was Cournot-like if all firms produced at the same time, whereas it was Stackelberg-like if some produced before, and others after, the demand uncertainty was resolved. Equilibrium with respect to this game was studied and it was shown that there are two classes of equilibria, one of which directly corresponded to a Stackelberg equilibrium and the other represented a natural generalization of Cournot equilibrium. We also showed that in a market with one "large" firm and a continuum of "small" firms facing a set of passive consumers, the <u>only</u> equilibrium was the Stackelberg equilibrium with the "large" firm as the leader. There were also some comparative static results on the symmetric form of the equilibrium and how it changed with the amount of uncertainty in demand. This confirmed one part of the Folk Theorem: namely, that when there are firms of different sizes in an industry, it is best modeled by a Stackelberg equilibrium.

On the other hand, we showed through results 1 and 2, that even when an industry has identical firms, a Stackelberg equilibrium is an endogenously determined Nash equilibrium. This refutes the other part of the Folk Theorem: namely, that when an industry has firms of identical sizes, it is best modelled as a Cournot equilibrium.

Further research could adapt the model developed in this paper to the framework of a model of noncooperative exchange where all agents are treated symmetrically, i.e., they are in the same strategic position (such as the noncooperative general exchange model of Shapley). This way one would be able to obtain an endogenously determined price-setting monopolist as an equilibrium of a noncooperative game. Finally, this model can be used to examine advertising and timing of technological innovations as strategic market activities.

FOOTNOTES

1. This result closely resembles an observation made by Guasch and Weiss [4].

2. The game tree could be uncountably infinite--i.e., there could be a continuum of alternatives at some or all of the moves--but of finite play length. Clearly, this might lead to some measurability problems as discussed in Aumann [1]. However, in our game tree the respective spaces are standard measurable spaces as required by Aumann, and therefore these problems do not confront us.

3. We could have more than 1 dominant player. In general let $D \subset N$ be the set of dominant players: the dominant players move together but before the other players. Then, the dominant players equilibrium would be a vector $v \varepsilon V$ with:

(a) $d \varepsilon D$, $s_d' \varepsilon M_d$,

$$E_{\mu | s_d, s_j j \varepsilon D} \Pi_d(b) < E_{\mu | s_d', s_j j \varepsilon D} \Pi_d(b)$$

(b) for every $d \varepsilon D$, $s_d' \varepsilon M_d$, $i \varepsilon N$, $i \varepsilon D$, $s_i' \varepsilon M_i$,

$$E_{\mu | s_i, s_j' j \varepsilon D, y^*} \Pi_i(b) < E_{\mu | s_i', s_j' j \varepsilon D, y^*} \Pi_i(b) \qquad (4')$$

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EQUILIBRIUM WITH ENDOGENOUSLY DETERMINED DOMINANT PLAYERS: CONTINUITY PROPERTIES OF THE EQUILIBRIUM CORRESPONDENCE

1. Introduction

In the last chapter, a noncooperative game was described in which a dominant player was endogenously determined in equilibrium. Using firm sizes and demand characteristics as exogenous, it was shown that in a market with one large firm and a nonatomic continuum of small firms, with nontrivial but small uncertainty in demand, the game had a unique sequentially rational Nash equilibrium in which the large firm was the endogenously determined dominant player. It was also shown, that in a market with two identical firms, there were three Nash equilibrium points. Two of these were asymmetric dominant player (Stackelberg type) equilibria and one was a symmetric (Cournot type) equilibrium. The question this paper addresses is the following: Would it be true that if we considered a sequence of games in which in each game the large firm played against a set of firms, and the individual firm size in these sets converged to zero, the resulting respective equilibria converged in some sense to the unique asymmetric equilibrium of the large firm versus the continuum case noted above? The point is that a continuum set-up is of interest only in so far as it is a model for the behaviour of large but finite markets. Thus, for example, if the equilibrium correspondence of the above sequence of games was upper hemicontinuous, then the description of the equilibrium of a market with one large firm and several small but atomic firms could be approximated by a Stackelberg equilibrium. Hence, the folklore that large firms are leaders and small firms are followers, would formally be an approximation. We will show that this equilibrium correspondence is indeed upper hemicontinuous. What the upper hemicontinuity tells us is that in the game described in the last chapter, the equilibrium, when there are several (but finite) small firms can be approximated by a Stackelberg equilibrium. However, since for non-trivial but small uncertainty, the Stackelberg equilibrium is the unique equilibrium in the limit when there are non-atomic small firms, the correspondence is also lower hemicontinuous. This means that the Stackelberg equilibrium approximates the equilibrium in the game when there is a finite number of small firms.

Section 2 of this chapter develops the basic notation. Section 3 provides an overview of the proof and Section 4 contains the proof.

2. Definitions and Notations

Let N^+ denote the set of natural numbers, and $T = \{t^+, t^-\}$ denote the space of player types. The type t^+ is the type of the large firm and t^- is the type of the small firms.

Let $G = N^+ \cup \{\infty\}$ be the space of games, topologized as the one-point compactification of N^+. Here $g \in G$ represents a game with one large firm and g+1 small firms of size 1/g+1 each. If the large firm is defined to be of size 1, and it has a cost function $C_1(q)$ for an output level q, then a firm of size α has a cost function $C_\alpha(q)$ such that $C_\alpha(\alpha \cdot q) = \alpha \cdot C_1(q)$. Thus, it is sufficient to know C_1 and the game g in order to be able to specify the cost function of the small firms in that game. Cost functions will therefore be parametrized by g and the cost of producing q, for a small firm in game g, will be denoted by C(g,q).

Let M be the monopoly output for the large firm.

As we have noted in the introduction, given the assumption of sequential rationality, the strategy for each player consists of choosing a probability that it will enter in period B and the quantity it will produce if it were to enter in period B. Thus, the strategy space is denoted by $S = [0, M] \times [0, 1]$. This is the same for all players in all games $g \in G$.

A <u>strategic environment</u> assigns a strategy for each player type. Thus, the space of strategic environments is $V = S^2$, and is endowed with the product topology. This means, only strategic environments in which the small firms act symmetrically will be considered. We will show in section 4 that this will be sufficient for the purpose of the proof. We will also discuss this further in section 3. So, for a firm of type t^+, v denotes its own strategy in conjunction with everyone else's. But for a firm of type t^-, v denotes everyone else's strategy. We will in general denote by $\pi_{(\cdot)}$, the projection onto the $(\cdot)$ axis. Thus, $\pi_{t+}(v)$ will be denoted by S_+ and $\pi_{t^-}(v)$ will be denoted by S_-.

A player's <u>decision</u> in a strategic environment is described by identifying the player type and the strategy which he plays. In particular, the ordered pair $(t, \pi_t(v))$ is the decision of a player of type t in the strategic environment v. Thus, every decision is an element of $T \times S$ and this will be denoted by D.

The circumstance of a player in a strategic environment is described by the player's strategy and the environment itself. That is, the ordered pair $(\pi_t(v), v)$ is the circumstance of player t in a strategic environment v. Every circumstance c, is an element of the product set S x V, which will be denoted by C. Players' preferences will compare circumstances rather than strategic environments.

Define r (g,t,x,y,a,b) to be the sequentially rational best response in market g, of a firm of type t, if it were to enter in period A (i.e. wait until after the resolution of uncertainty to decide on a production level). Here x is the aggregate production that is committed to by the firms that have entered in period B, y is the realized value of the demand uncertainty, (Y denotes the space of all y) a is a dummy variable indicating whether the large firm has precommitted or not (a = 1 if it has, a = 0 if not), and b is the proportion of small firms which are entering in period A.

We assume that r is well-defined, single-valued, and uniformly continuous. We will now work out as an example the algebraic form of r in the case of linear demand - quadratic costs for finite g with a = 1, to show that this assumption is consistent.

Thus, let the inverse demand function be given by price = y - (total quantity of production) and for a small firm in game g, the costs of producing q is $\frac{1}{g+1}$. τ. q^2 where τ is some constant. Then, the best response for the small firm in period A is to choose q, its production level, so as to

$$\max_{q}.\ (y-x-bq).\ q - \frac{\tau q^2}{g+1}$$

Therefore, $q = \dfrac{(y-x)}{2\ (b+\frac{\tau}{g+1})}$ is the best response and we see that it is consistent in this case with our assumptions about r.

A correspondence $H:X \to Y$ is a mapping of the topological space X to subsets of the space Y. H is open or closed if its graph $\{(x,y)|y \in H(x)\}$ is open or closed, respectively. H is lower hemicontinuous (l.h.c.) if $\{x|H(x) \cap U \neq \emptyset)$ is open in X for every open set U in Y. H is upper hemicontinuous (u.h.c.) if $\{x|H(x) \subseteq U\}$ is open in X for every open set U in Y and H(x) is nonempty for every x in X.

Each strategic environment results from a combination of strategy choices. These are specified by a correspondence $J: GxV \to D$. The interpretation of $(t,s) \in J\ (g,v)$ means that in a game g, with a strategic environment v, at least one player of type t has chosen strategy s.

The correspondence $F: G \to V$ is the feasible strategy correspondence. $v \in f(g)$ if v is a strategic environment in which every player plays a feasible strategy. Note that for all games $g \in G$, $F\ (g) = ([0, M]\ x\ [0, 1])^2$. The correspondence F is therefore closed, u.h.c. and compact valued.

A correspondence $\overline{A}: G \times T \times C \to C$ is called the alternative correspondence, and $(s', v') \varepsilon \bar{A}(g, t, s, v)$ means that in a game g, a player of type t has changed his strategy from s to s' when the strategic environment was v. The strategic environment now is v'.

There is a <u>preference</u> relation $P \subseteq G \times T \times (S \times V)^2$. Let $\eta: G \times T \times S \times V \times Y \to R$ be the expected profit function of a firm (expectation given the randomization by the other firms), expressed conditionally on the value of y, the realized value of the demand uncertainity, given a strategic environment.

We define

$$P = \{(g,t,s,v,s',v')/E_y\eta\ (g,t,s,v,y) > E_y\eta\ (g,t,s',v',y)\}$$

where E_y denotes the expectation with respect to the random variable y.

Let an individual firm's strategy s be denoted by the ordered pair (q, δ). Here, q is the quantity that the firm will produce if it entered in period B and δ is the probability that it will enter in period B. The strategic environment $v = (q^+, \delta^+, q^-, \delta^-)$, where q^+ and q^- denote the quantity a firm of type t^+ or t^- would respectively produce if it entered in period B, and δ^+ and δ^- are the respective probabilities that those firms will enter in period B.

Deviation from a strategy vector by a single firm is considered via the alternative correspondence and the preference relation. Let (t,s,v) be the normal - form strategy vector in which one firm of type t plays strategy s while all other firms play the strategies prescribed by the strategic environment v. Under this interpretation,

$\overline{A}$ $(g,t^-,s,v) = S \times \{v\}$. The preference relation then, would mean the following:

(g,t,s',v',s,v)   P if a firm of type t gets higher expected profits in game g from the normal - form strategy vector (t,s',v') (with itself being the firm to play s') than from (t,s,v).

An <u>inadmissible-decision</u> correspondence $I:G \times V \to D$ is defined by

$(t,s) \in I(g,v) \Leftrightarrow$ c $[c \in \overline{A}(g,t,[s,v])$ and $(g,t,c,[s,v] \in P]$. The <u>equilibrium</u> correspondence $E:G \to V$ is defined by

$$v \in E(g) \Leftrightarrow [v \in F(g) \text{ and } I(g,v) \cap J(g,v) = \emptyset].$$

A <u>topological family</u> of these games is specified by a relation P and correspondences F, $\overline{A}$ and J which satisfy

P is open,

F is closed,

$\overline{A}$ is lower hemicontinuous, and

J is lower hemicontinuous.

Now, in order to show that the equilibrium correspondence of the family of games we are considering is u.h.c., we will use Theorem 1 and the Corollary of [2]. We will now state that theorem.

<u>Theorem (Green):</u> The equilibrium correspondence of a topological family of games is closed. Furthermore, if E is nonempty-valued and F is u.h.c. and compact-valued, then E is u.h.c.

Thus, we wish to show that E is u.h.c. We are now in a position to give an overview of the proof.

3. An Overview of the Proof:

In order to show that E is u.h.c., we will first show that G is a topological family of games. That is, we will prove that F is closed, $\overline{A}$ is l.h.c., J is l.h.c. and P is open. We will also show that F is compact valued and u.h.c. This is sufficient to establish the upper hemicontinuity of E. The openness of P is proved by computing the profits η and then by showing that $E_y \eta$ is jointly continuous in its arguments. The key to computing profits will be to describe the distribution of the aggregate production (which will be shown to be a random variable), net of the production of the firm whose strategy is under consideration, conditional on that firm's decision and on the value of y. Using this, we will show that for any finite g, $E_y \eta$ is sequentially continuous in its arguments. In order to show the continuity of E_y at $g = \infty$, we will first show that as g approaches ∞, these random variables (the net aggregate expected production) converge in distribution to random variables in which the contribution of the small firms is a deterministic function of the decision of the large firm. Thus, we will know the form of these random variables at $g = \infty$, and using that we will show continuity of $E_y \eta$ at $g = \infty$.

Having done this, we recall that we considered only those strategic environments in which the small firms acted symmetrically. Thus, we will next show that for any given amount of uncertainity in y, there is a finite game g, such that an equilibrium for this finite g and any game larger than g, corresponds to a symmetric (i.e., among small firms) normal - form equilibrium. For the proof that the equilibrium of the game $g = \infty$ corresponds to a symmetric equilibrium, the reader is referred to Lemma 1 in chapter 1. For games less than this finite g, equilibria in which the small firms do not play symmetric strategies do not matter because any correspondence which maps the interval [1,g] to any description of equilibrium strategies of the players will be u.h.c. since [1,g] has the discrete topology. It is due to this that we are able to define the range of E to be strategic environments. We will now prove that E is u.h.c.

4. Proof that the Equilibrium Correspondence is u.h.c.

We will first show through a series of lemmas, that the family of games we are considering is a topological family.

To begin with, since $F(g) = ([0,M] \times [0,1])$, for all $g \in G$, it is easy to see that F is closed, u.h.c. and compact-valued. We will next show that $\bar{A}$ is l.h.c. We will do this in two parts. First it will be shown that $\bar{A}(,t^+ , ,)$ is l.h.c. and then that $\bar{A}(,t^-, ,)$ is l.h.c. Hence, if $C_+ = \{(s,v)/\pi_{t^+}(v) = s\}$, define $A_{gt^+} : S \times V \to S \times V$ such that

$$A_{gt}+ (s,v) = \begin{cases} \{(s',s',\pi_t-(v))/s' \in S\} \\ \text{if } (s,v) \in C_+ \\ S \times V \text{ otherwise.} \end{cases}$$

Also, define $A_{gt}- : S \times V \to S \times V$ such that $\overline{A}_{gt}- (s,v) = A(g,t^-,s,v)$ we will show that $A_{gt}+$ and $A_{gt}-$ are l.h.c. Then, using lemmas 3 and 4, we will show the $\overline{A}$ is l.h.c.

__Lemma 1:__ $A_{gt}+$ is l.h.c.

__Proof:__ Consider a set W open in S x V. Then $W = \hat{W} \cup \overline{W}$ where $\hat{W} \subseteq C_+$ and $\overline{W} \subseteq C \backslash C_+$. The preimage of W under $A_{gt}+$ is $A_{gt}+ (\hat{W}) \cup (C \backslash \hat{C}_+)$. Call this Z.

We will show $(S \times V) \backslash Z$ is closed. $(S \times V) \backslash Z = C_+ \backslash A_{gt}+ (\hat{W})$. Now, $\pi_t- (W)$ is open in S because of openness of projection maps.

$\therefore A_{gt}+ (W) = \{(s', s', \pi_t- (w))/w \in W, s' \in S\}$ is open in $\{(s', s', r)/r \in s_+, s' \in S\} = C_+$.

$\therefore C_+ \backslash A_{gt}+ (\hat{W})$ is closed.

So $(S \times V) \backslash Z$ is closed and this means Z is open in S X V.

Q.E.D.

Lemma 2: A_{gt}^{-} is l.h.c.

Proof: Consider U open in S X V. Then $\{(s,v) | A_{gt}^{-} (s,v) \cap U \quad \emptyset\}$ is equal to $= S \times \{\pi_v (u)$ which is open in S x V. Hence A_{gt}^{-} is l.h.c.

Q.E.D.

Lemma 3: Let L and X be topological spaces and L have the discrete topology. Let $\ell \epsilon L$ index a set of l.h.c. correspondences $H_\ell : X \quad X$. Also, define the correspondence $H : L \times X \quad X$ such that $H(\ell,x) = H_\ell (x)$. Then, H is l.h.c.

Proof: Consider a set U open in X.

Then, consider $\{(\ell,x) | H(\ell,x) \cap U \neq \emptyset\} = \{(\ell,x) | H_\ell(x) \cap U \neq \emptyset\}$.

Now, we know that for all ℓ, $\{x | H_\ell (x) \cap U \neq \emptyset\}$ is open in L x X.

Since arbitrary union of open sets is open,

$\{(\ell,x) | H (\ell,x) \cap U \neq \emptyset\}$ is open in L x X.

Q.E.D.

By letting L in lemma 3 be the space T and let X in lemma 3 be the space SxV then, the correspondence $A_g : T \times S \times V \rightarrow S \times V$ is l.h.c. for every $g \epsilon G$. We will now show that $\overline{A} : G \times T \times S \times V \rightarrow S \times V$ is l.h.c., where $\overline{A} (g, \ , \ , \ ) = A_g (\ , \ , \ )$. This is primarily because A_g is the same correspondence for all $g \epsilon G$.

Lemma 4:

Let K, X, Y be topological spaces. Let k index a set of l.h.c. correspondence, $H_k: X \to Y$. Then, define $H : K \times X \to Y$. Such that $H(k,x) = H_k(x)$. Suppose further that $H_k = H_{k'} = \hat{H}$ for all k', $k \in K$. Then, H is l.h.c.

Proof: Let U be open in Y. Consider

$$\{(k,x) \mid H(k,x) \cap U \neq \emptyset\}$$

$$= \{(k,x) \mid \hat{H}_k(x) \cap U \neq \emptyset\}$$

$$= \{(k, x) \mid \hat{H}(x) \cap U \neq \emptyset\}.$$ This is open in K x X.

Q.E.D.

Hence, let k in lemma 4 be G and X in lemma 4 be T x S x V, and Y be S x V, we then have that $\overline{A}$ is l.h.c. We will now show that the correspondence J is l.h.c. Let $J_g : V \to D$ such that $J_g(v) = J(g,v)$. We will now use lemma 5 to prove that J_g is l.h.c.

Lemma 5: Let X and Y be topological spaces. Let X be finite and have the discrete topology. Further, let Y^X and X x Y have the product topology. Then $H : Y^X \to X \times Y$ such that, $H(\alpha) = \{(x,y) \mid \pi_x(\alpha) = y\}$ is l.h.c. For every α, the correspondence H gives the graph of α.

Proof: Let u be open in X x Y. Then, since X is discrete, u can be written as $\bigcup_{x \in \pi_X(u)} \{x\} \times w_x$ where w_x is open in Y.

Then, consider $\{\alpha \in Y^X | H(\alpha) \cap U \neq \emptyset\}$. This set is equal to $\{(\prod_{x \in \pi_x(u)} w_x)(\prod_{x \in X \setminus \pi_x(u)} Y)\}$ which is open in Y since X is finite and discrete.

Q.E.D.

If in the above lemma, we let X be T and Y be S, J_g is l.h.c.

Furthermore, since J_g is the same for all $g \in G$, by lemma 4, J is l.h.c.

Next, we wish to show that P is open. That p is open is established if we show that $E_y \eta$ is continuous in its arguments. In order to show that $E_y \eta$ is continuous, it is sufficient to prove that it is sequentially continuous. This is because its domain G x T x S x V x Y is a metric space being the product of metric spaces. (G is homeomorphic to the subspace $\{1/n | n = 1,2...\} \cup \{0\}$ of R. See [5], exercise 19 B.2). As noted in section 3 above, we will now write down the aggregate production as a random variable and then compute the expected profits.

Let X_B^- denote the total production net of a firm of type t^-, assuming it precommits.

X_A^- denote the total production net of a firm of type t^-, assuming it enters in period A.

X_B^+ denote the total production net of the firm of type t^+, assuming it enters in period B.

X_A^+ denote the total production net of the firm of type t^+, assuming it enters in period A.

Then, given a normal-form strategy vector,

$(t^-, (q,\delta), (q^+, \varepsilon^+, q^-, \varepsilon^-)$, X^- and X_A^- are random variables and are described as follows: X_B^- takes the value,

$$q(a) + \frac{kq^-}{g+1} + (1-a)\cdot r\left(g,t^+, \frac{q}{g+1} + q(a) + \frac{kq^-}{g+1}, y, a, \frac{g-k}{g+1}\right)$$

$$+ \frac{(g-k)}{g+1} r\left(g,t^-, \frac{q}{g+1} + q(a) + \frac{kq^-}{g+1}, y, a, \frac{g-k}{g+1}\right)$$

with probability

$$\delta(a) \binom{g}{k} (\varepsilon^-)^k (1-\varepsilon^-)^{g-k}$$

Where $k = 0, 1, \ldots g$

$\varepsilon(1) = \varepsilon^+$, $\varepsilon(0) = 1 - \varepsilon^+$

$q(1) = q^+$, $q(0) = 0$

Note that X_B^- is a function of the independent random variables $\frac{k}{g+1}$ in which k is Binomial (ε^-, g) and a which is Binomial $(\varepsilon^+, 1)$. Similarly, X_A^- takes the value,

$$q(a) + \frac{kq^-}{g+1} + (1-a)\cdot r\left(g,t^+, q(a) + \frac{kq^-}{g+1}, y,a, \frac{g+1-k}{g+1}\right)$$

$$+ \frac{(g-k)}{g+1}\cdot r\left(g, t^-, q(a) + \frac{kq^-}{g+1}, y,a,\frac{g+1-k}{g+1}\right)$$

with probability

$$\delta(a) \binom{g}{k} (\delta^-)^k (1-\delta^-)^{g-k}$$

where k = 0, 1 ... g and δ (a) and q (a) are as described above.

Similarly, X_A^- is a function of the independent random variables $\frac{k}{g+1}$ and a.

Again, X_B^+ takes the value

$$\frac{kq^-}{g+1} + \frac{(g-k+1)}{g+1} \cdot r\left(g, t^-, q + \frac{kq^-}{g+1}, y, 1, \frac{g+1-k}{g+1}\right)$$

with probability, $\binom{g+1}{k} (\delta^-)^k (1-\delta^-)^{g-k+1}$. Where k = 0, 1, ... g+1.

while, X_A^+ takes the value

$$\frac{kq^-}{g+1} + \frac{(g-k+1)}{g+1} \cdot r\left(g, t^-, \frac{kq^-}{g+1}, y, 0, \frac{g+1-k}{g+1}\right)$$

with probability $\binom{g+1}{k} (\delta^-)^k (1-\delta^-)^{g-k+1}$

where k = 0, 1, ... g+1. In this case, X_A^+, X_B^+ are functions of the random variables $\frac{k}{g+1}$ in which k is Binomial (δ^-, g+1).

We can now write the profit functions for each firm type and then show that the expected value of profit (expectation over y) for each firm type is continuous in its arguments. This continuity implies that P is open.

Consider a firm of type t^- with the normal-form strategy vector $(t^-, s, v) = (t^-, (q,\delta), (q^+, \delta^+, q^-, \delta^-))$. Its profit function conditional on the value of y is written as follows:

$\eta\ (g, t^-, s, v)$

$$= \int [\ \delta \cdot [D(X_B^-\ (\tfrac{k}{g+1}, a) + q) \quad q - \bar{C}(g, t^-, q)]]\ d\mu\ (\tfrac{k}{g+1}, a)$$

$$+ \int [(1-\delta) \cdot [D(X_A^-\ (\tfrac{k}{g+1}, a) + r) \cdot r\ (g, t^-, X_B^-, y, a, b\ (\tfrac{k}{g+1}))$$

$$- \bar{C}(g, t^-, r)]]\ d\mu\ (\tfrac{k}{g+1}, a),$$

where D is the inverse demand function, $\bar{C}$ is the cost function, μ denotes the joint distribution of $\frac{k}{g+1}$ and a. Since a takes two values, +1 with probability δ^+, and 0 with probability $(1-\delta^+)$, we can rewrite η as follows:

$$\eta = \delta\ [\int \delta^+ \{D(X_B^-\ (\tfrac{k}{g+1}, 1) + q) \cdot q - \bar{C}(g,t^-,q)\}\ d\mu'\ (\tfrac{k}{g+1})$$

$$+ \int (1-\delta^+)\ \{D(X_B^-\ (\tfrac{k}{g+1}, 0) + q) \cdot q - \bar{C}(g,t^-,q)\}\ d\mu'\ (\tfrac{k}{g+1})]$$

$$+(1-\delta)\ [\int \delta^+ \{D(X_A^-\ (\tfrac{k}{g+1}, 1) + r) \cdot r - \bar{C}(g,t^-,q)\}\ d\mu'\ (\tfrac{k}{g+1})$$

$$+(1-\delta^+)\ \{D(X_A^-\ (\tfrac{k}{g+1}, 0) + r) \cdot r\ (g,t^-,X_B^-\ (\tfrac{k}{g+1}, 0), y, 0, b(\tfrac{k}{g+1}))$$

$$-\bar{C}\ (g,t^-,q)\}\ d\mu'(\ \tfrac{k}{g+1}\ )] \qquad \ldots\ldots(0)$$

The profit function for the firm of type t^+ is written as follows: Let the firm have a normal-form strategy vector $(t^+, (q^+, \delta^+), (q^+, \delta^+, q^-, \delta^-))$. Then,

$\eta(g, t^+, s, v)$

$$= \delta^+ \int \{D(X_B^- (\tfrac{k}{g+1}) + q^+)q^+ - \bar{C}(g, t^-, q^+)\} \, d\mu' (\tfrac{k}{g+1})$$

$$+(1-\delta^+)\int \{D(X_A^+ (\tfrac{k}{g+1}) + r) \cdot r - \bar{C}(g, t^+, r)\} \, d\mu' (\tfrac{k}{g+1})$$

Where r denotes $r(g, t^+, X_B^+ (\tfrac{k}{g+1}), y, 0, b (\tfrac{k}{g+1}))$.

In all, we now have six random variables $X_B^- (\cdot, 1)$, $X_A^- (\cdot, 1)$, $X_B^- (\cdot, 0)$, $X_A^- (\cdot, 0)$, X_B^+ and X_A^+. In these expressions, μ' is the probability measure that induces the distribution of $\frac{k}{g+1}$. These are measures on R and given a game g, the support of a measure is the set of values taken by $\frac{k}{g+1}$. The space of these measures is endowed with the topology of weak convergence. In this topology, a sequence of measures $\{\mu_n'\}$ converges to μ_o' iff for all bounded, uniformly continuous real valued functions

$g : R \to R$, $\int_R g d\mu_n' \to \int_R g d\mu_o'$. Assume D and $\bar{C}$ are bounded, uniformly continuous functions. Having defined these random variables, we first note that as g approaches ∞, these random variables converge in distribution to random variables where the contribution of the small firms (i.e. firms of type t^-) is a deterministic function of the decision of the large firm (i.e. the firm of type t^+). In order to prove this, we observe that the randomness created by the small firms in the above expressions is due to $\frac{k}{g+1}$, in which k has the binomial distribution $\binom{g}{k}(\delta^-)^k\,(1-\delta^-)^{g-k}$.

Lemma 6

As $g \to \infty$, the random variable $\frac{k}{g+1}$ converges in distribution to the constant δ^-.

Proof

We know from elementary probability theory (c.f. [6]) that a sequence of random variables converges in distribution to a limiting distribution iff their respective moment generating functions converge to that of the limiting distribution (wherever they all exist).

Now, let $M_{\frac{k}{g+1}}(t)$ denote the moment generating function of the random variable $\frac{k}{g+1}$. Then, $M_{\frac{k}{g+1}}(t) = [\delta^-\ e^{\frac{t}{g+1}} + (1-\delta^-)]^g$. We will now take the limit as $g \to \infty$.

$$\lim_{g\to\infty} [\delta^- e^{\frac{t}{g+1}} + (1-\delta^-)]^g$$

$$= \text{antilog} \lim_{g\to\infty} g\, \ln(\delta^- e^{\frac{t}{g+1}} + (1-\delta^-))$$

$$= \text{antilog} \lim_{g\to\infty} \frac{\ln (\delta^- e^{\frac{t}{g+1}} + (1-\delta^-))}{\frac{1}{g}}.$$

using L' Hospital's rule this expression

$$= \text{antilog} \lim_{g\to\infty} \frac{g^2}{(g+1)^2} \frac{t\delta^- \cdot e^{\frac{t}{g+1}}}{e^{\frac{t}{g+1}} \delta^- + (1-\delta^-)}$$

$$= \text{antilog}\ t\delta^-$$

$$= e^{t\delta^-}$$

Q.E.D.

Thus, as $g \to \infty$, $X_B^- (\cdot, 1)$ converges in distribution to the constant $q^+ + \delta^- q^- + (1-\delta^-) \cdot r (\infty, t^-, q^+ + \delta^- q^-, y, 1, 1-\delta^-)$. The random variables $X_B^- (\cdot, 0)$, $X_A^- (\cdot, 1)$, $X_A^- (\cdot, 0)$, $X_A^+ (\cdot, 0)$, X_A^+, X_B^+ converge to similar constants.

The profit function η for the firm of type t^- has four terms. The profit function for the firm of type t^+ has two terms. We will show that the expected value (with respect to y) of the first term of the profit for the firm of type t^- is sequentially continuous in (g, t^-, S, V). The proof that the expected value (with respect to y) of the other terms is continuous is similar. Further, we know that the finite sum of continuous functions is continuous. Thus $E_y(\eta)$ for the firm of type t^- would be continuous. The proof for t^+ is similar. Denote this first term by $E_y(\eta 1)$.

Lemma 7:

Let $\{(g_n, \bar{t}, s_n, v_n, y)\} \to (g_0, E, s_0, v_0, y)$ then,

$$\{E_y(\eta\, 1(g_n, \bar{t}, s_n, v_n, y))\} \to E_y(\eta 1(g_0, \bar{t}, s_0, v_0, y)).$$

Proof: We will consider two cases: when g_0 is $< \infty$ and when $g_0 = \infty$.

Case 1: Let $g_0 < \infty$. Let $\varepsilon > 0$ be given. Then, because the relative topology of N^+ as a subspace of G is discrete, there exists $N^* \varepsilon N^+$, such that for $n > N^*$, $g_n = g_0$. Next, $E_y(\eta 1(g, \bar{t}, s, v, y))$ can be written down as,

$$\sum_{k=0}^{g+1} E_y \{\delta^+ \cdot D(q^+ + \frac{k\bar{q}}{g+1} + \frac{g-k}{g+1} \cdot r(g, \bar{t}, \frac{q}{g+1} + q^+ + \frac{k\bar{q}}{g+1}, y, 1, \frac{g-k}{g+1}) + q) \cdot q - \bar{C}(g, \bar{t}, q)\} \cdot \{\binom{g}{k} (\delta^-)^k (1-\delta^-)^{g-k}\}.$$

It is now easy to see that each of the (g+1) terms is continuous in (g, t^-, s, v, y).

Thus, given the $\varepsilon > 0$, there exists an $N_i \in N^+$, such that for $n > N_i$, the ith term is less than $\frac{\varepsilon}{g+1}$. Now, consider the expression for $E_y(\eta l\ (g_0, t^-\ s, v, y))$. Then, for $\underline{N} = \max \{\{N_i\}_{i=1}^{g_0+1}, N^*\}$, for every $n > \underline{N}$,

$$\left|E_y(\eta l\ (g_n, t^-, s_n, v_n, y)) - E_y(\eta l\ (g_0, t^-, s_0, v_0, y))\right| < \varepsilon.$$

Hence, $E_y(\eta l)$ is sequentially continuous in (g, t^-, s, v, y).

Case 2: Let $g_0 = \infty$.

Again, note that because of the assumptions on D and $\overline{C}$, the integrand of $E_y(\eta l)$ is a bounded, uniformly continuous function see [4]exercise 4.53). Now, consider a sequence $\{(g_n, t^-, s_n, v_n)\} \to (\infty, t^-, s_0, v_0)$. Let $\varepsilon > 0$ be given. We wish to show that there exists Q such that for $n > Q$,

$$\left|E_y(\eta l\ (g_n, t^-, s_n, v_n, y)) - E_y(\eta l\ (\infty, t^-, s_0, v_0, y))\right| < \varepsilon.$$

Denote the integrand in $E_y(\eta 1)$ by $I(g, \bar{t}\ s, v, y)$ and recall from equation (0) that the measure depends upon g and v. If we examine the integral, we see that for each (g,t,s,v,y), the measure in that equation can be immediately determined. The integral takes on different values due to the different values of $\frac{k}{g+1}$. Hence, we wish to show that, for n large enough

$$\left| E_y \left(\int I (g_n, \bar{t}, s_n, v_n, y)\right) d\, \mu' (g_n, v_n)\right)$$
$$- E_y \left(\int I (\infty, \bar{t}, s_0, v_0, y)\, d\mu' (\infty, v_0)\right) \Big| < \varepsilon$$

Consider the L.H.S. of the above inequality. It can be written as

$$\Big| E_y \left(\int I (g_n, \bar{t}, s_n, v_n, y)\, d\, \mu' (g_n, v_n)\right)$$

$$- E_y \left(\int I (\infty, \bar{t}, s_0, v_0, y)\, d\mu' (g_n, v_n)\right) \Big|$$

$$+ \Big| E_y \left(\int I (\infty, \bar{t}, s_0, v_0, y)\right) d\mu' (g_n, v_n))$$

$$- E_y \left(\int I (\infty, \bar{t}, s_0, v_0, y)\, d\mu' (\infty, v_0)\right) \Big| \qquad -(1)$$

Now, it is easy to show that $I(g_n, \bar{t}, s_n, v_n, y)$ converges uniformly to $I(\infty, \bar{t}, s_0, v_0, y)$. Furthermore, the sequence is uniformly bounded. Thus, for any $\varepsilon > 0$, there exists Q_1, such that for all $n > Q_1$, $|I(g_n, t^-, s_n, v_n, y)) - I(\infty, t^-, s_0, v_0, y)| < \varepsilon/2$. Now, the first term above can be rewritten as

$$\left| E_y \left(\int [I(g_n, t^-, s_n, v_n, y)) - I(\infty, t^-, s_0, v_0, y)] \, d\mu' (g_n, v_n) \right| \right.$$

Hence (by monotonicity and linearity of the integral), for $n > Q_1$, this is $\leq E_y (\int \frac{\varepsilon}{2} \, d\mu' (g_n, v_n))$. But since μ' is a probability measure, the above expression $\leq \varepsilon/2$.

Again, by lemma 4 and by the definition of weak convergence of measures, since I is a bounded, uniformly continuous function, there exists a Q_2 such that for $n > Q_2$, the second term in expression 1 can be made less than $\varepsilon/2$. Hence we have the required inequality.

Q.E.D.

Thus, for each firm of type t, $E_y (\eta(g, t, s, v, y))$ is continuous in (g, t, s, v, y). Hence P is open. We have also shown that F is closed and that $\overline{A}$ and J are l.h.c. Hence G is a topological family of games, also F is u.h.c. and compact valued. Thus, by the theorem of Green (see section 2), its equilibrium correspondence is u.h.c.

Since we had considered only those strategic environments in which the small firms chose symmetric strategies, it now remains to be shown that this is sufficient for the purpose of our proof. This does not mean that in every equilibrium in every game, the small firms choose symmetric strategies. For any given amount of uncertainty (variance of y), there will be a finite number of games in which the small firms will not play symmetric strategies. But, we will show that for each amount of uncertainty, there is a finite game g such that for any game greater than or equal to g, the small firms will play symmetric strategies. The reason why this is sufficient for our purposes is because, $G\setminus\{\infty\}$ has the discrete topology and so the equilibrium correspondence for $g \in G\setminus\{\infty\}$ is trivially u.h.c. Thus, what we are claiming is that for any given amount of uncertainty, the small firms will not play symmetric strategies in every equilibrium up to some finite g. But since $G\setminus\{\infty\}$ has the discrete topology, E is u.h.c. However, for any game greater than or equal to g+1, it will be shown that the small firms will play symmetric strategies and in that case the family of games has been proved to be a topological family and hence the equilibrium correspondence for that part will be u.h.c. Furthermore, since [1, g] and [g+1, ∞] are open in G for any finite g, this part of the equilibrium correspondence is u.h.c. by lemma 8.

<u>**Lemma 8**</u>: Let $H_1 : X_1 \to Y$ and $H_2 : X_2 \to Y$ be u.h.c. correspondences. If X_1 and X_2 are open subsets of X with $X_1 \cap X_2 = \emptyset$, then,

$$H : X_1 \cup X_2 \to Y \text{ such that } H(x) = \begin{cases} H_1(x) & \text{if } x \in X_1 \\ H_2(x) & \text{if } x \in X_2 \end{cases}$$

is u.h.c.

Proof: Let W be open in Y. Then, consider $\{x | H(x) \subseteq W\}$. This set equals

$$\{x \in X_1 | H_1(x) \subseteq W\} \quad \cup \{x \in X_2 | H_2(x) \subseteq W\}.$$

The first set is open in X_1 by upper hemicontinuity of H_1. Furthermore, since X_1 is open in X, it is open in X. Similarly, the second set is open in X. Hence, $\{x | H(x) \subseteq W\}$ is open in X.

Q.E.D.

We will now consider the technology with zero costs and linear demand and show that for any amount of uncertainty, there is a game g* such that for g > g*, all the small firms will play symmetric equilibrium strategies. Let α be the proportion of small firms except for one small firm under consideration that in equilibrium decides to enter in period B with probability 1. The proportion 1-α enters in period A. Consider now the single small firm that is entering in period B with probability 1. We will show that for any given amount of uncertainty, for all α, there is a finite g^*_α such that for $g > g^*_\alpha$ this small firm's ex ante expected profits of entering with probability 1 in period B is less than the ex ante expected profits of entering in period A. Once this one firm deviates and decides to enter in period A, α is changed. But since this result holds for all α > 0, and there are only a finite number of small firms for any g < ∞, there will be only symmetric strategies played amongst the small firms for g > max $\{g^*_\alpha\}$, which is finite. The proportion α is computed as follows. Suppose $g = 10$, and there are 3 small firms that enter in period B. Then, $\alpha = 2/10$ and $1-\alpha = 8/10$.

Thus, in a game g, the figure below gives the proportion of small firms that enter in period B or A given that the small firm under consideration enters in period B or A.

Small firm under consideration ↓	Proportion of small firms entering in → period	B	A
B		$\alpha + \frac{1}{g+1}$	$1-\alpha - \frac{1}{g+1}$
A		α	$1-\alpha$

Now, the large firm can either enter with probability 1 in period B or with the same probability in period A. We will only consider the case where the large firm enters with probability 1 in period B. The proof with the large firm entering in period A with probability 1 is similar.

Recall that the game we are considering has two classes of equilibria. One is a generalization of the Cournot equilibrium (see chapter one). Under this equilibrium, we showed that whenever the amount of uncertainty is nontrivial but small, the firms will randomize their times of entry. Whenever there is temporal randomization, identical firms will have identical strategies. Furthermore, for trivial or too large amounts of uncertainty, there is no temporal randomization, and in the resulting Cournot equilibrium, identical firms will have identical equilibrium strategies. Lastly, in the second class of equilibria, where there is no temporal randomization for any amount of uncertainty, firms entering in the same period will play a Cournot game, and hence here too identical firms will have identical strategies. The basic claim that these three statements make is the following: In a Cournot game, in a symmetric equilibrium, identical firms will have identical equilibrium strategies. For a proof of this, see lemma 9.

Since, for symmetric strategies amongst the small firms, the equilibrium correspondence is u.h.c., the part of the equilibrium correspondence for the generalized Cournot equilibrium case is u.h.c. We will next be proving that for the second class of equilibria, the equilibrium correspondence is again u.h.c, by showing (considering a particular technology) this in this equilibrium the small firms will play symmetric strategies in all games g that are larger than some finite game. Then, by proposition A. III. 1, of [7], E is u.h.c.

We will now proceed to prove that for any given amount of uncertainity, for every $\alpha > 0$, there is some g^* such that for all $g \geq g^*$, a small firm under consideration that is entering in period B will find that its ex ante expected profits of entering with probability 1 in period B is less than that of entering in period A.

First, we will compute the profits to the small firm if it were to enter with probability one in period B.

The large firm's problem would be

$$\max_{q_B^+} \quad E_y \left\{q_B^+ \left(y-q_B^+ - (\alpha + \frac{1}{g+1}) q_B - (1-\alpha- \frac{1}{g+1}) q_B - (1-\alpha- \frac{1}{g+1}) r_B\right)\right\}$$

where r_B is the best response of each small firm that is entering in period A, and q_B is the output of each small firm entering in period B.

A typical small firm i, that is entering in period B solves:

$$\max_{q_{iB}} \quad E_y \left\{q_{iB} \left(y-q_B^{+} - \sum_{\substack{j \varepsilon S_{BB} \\ J \neq i}} \frac{q_{jB}}{g+1} - \frac{q_{iB}}{g+1} - (1-\alpha- \frac{1}{g+1})\, r_B\right)\right\}$$

where S_{BB} is the set of all the small firms entering in period B when the firm under consideration is entering in period B. Similarly, we denote by S_{AB}, the set of small firms entering in period A when the firm under consideration is entering in period B. We also have S_{AA} and S_{BA} defined in the similar way.

A typical small firm entering in period A will solve:

$$\max_{r_{iB}} \quad \left(r_{iB} \left(y-q_B^{+} - (\alpha + \frac{1}{g+1})q_B - \sum_{\substack{j \varepsilon S_{AB} \\ J \neq i}} \frac{r_{jB}}{g+1} - \frac{r_{iB}}{g+1}\right)\right).$$

Solving these, we obtain the optimal outputs of these firms in these three situations. We have (the * represents optimal values and the subscript B refers to the case where the firm under consideration enters in period B),

$$q_B^{+*} = \frac{K_5 - K_6\, K_3}{1 - K_6\, K_4}$$

$$q_B^* = \frac{K_3 - K_4 K_5}{1 - K_6 K_4}$$

$$r_B^* = K_1 - K_2\left(\frac{K_5 - K_6 K_3}{1 - K_6 K_4}\right) + \left(\alpha + \frac{1}{g+1}\right)\left(\frac{K_3 - K_4 K_5}{1 - K_6 K_4}\right)$$

where

$$K_1 = \frac{y}{1-\alpha}$$

$$K_2 = \frac{1}{1-\alpha}$$

$$K_3 = \frac{E_y y - E_{k_1} k_1 \left(1-\alpha- \frac{1}{g+1}\right)}{\left(\alpha+ \frac{2}{g+1}\right)\left(1-k_2 \left(1-\alpha- \frac{1}{g+1}\right)\right)}$$

$$K_4 = \frac{1}{\left(\alpha+ \frac{2}{g+1}\right)}$$

$$K_5 = \frac{K_3}{2}\left(\alpha + \frac{2}{g+1}\right)$$

$$K_6 = \frac{\left(\alpha + \frac{1}{g+1}\right)^2}{2\left(1-K_2 \left(1-\alpha- \frac{1}{g+1}\right)\right)} .$$

Thus, expected profit for the small firm under consideration if it were to enter in period B is:

$$q_B^* E_y y - q_B E_{k_1} k_1 (1-\alpha- \frac{1}{g+1})$$

$$-q_B q_B^{+*} (1-(1-\alpha- \frac{1}{g+1}) K_2)$$

$$-q_B^{*2} (\alpha + \frac{1}{g+1}) (1-(1-\alpha- \frac{1}{g+1}) K_2) \qquad \ldots (X1)$$

On the other hand, with the same α, if the firm under consideration were to enter with probability 1 in period A, then, the large firm's optimization problems would be the following:

$$\max_{q_A^+} \quad E_y \{q_A^+ \; (y-q_A^+ - \alpha \cdot q_A - (1-\alpha) \; r_A)\}$$

The typical small firm i, which is entering in period B solves,

$$\max_{q_{iA}} \quad E_y \{q_{iA} (y-q_A^+ - \sum_{\substack{j \varepsilon S_{BA} \\ J \neq i}} \frac{q_{jA}}{g+1} - \frac{q_{iA}}{g+1} - (1-\alpha) \; r_A)\}$$

and, the typical small firm which is entering in period A solves,

$$\max_{r_{iA}} \quad E_y \{r_{iA} \; (y-q_A^+ - \alpha \cdot q_A - \sum_{\substack{j \varepsilon S_{AA} \\ J \neq i}} \frac{r_{jA}}{g+1} - \frac{r_{iA}}{g+1})\}.$$

The respective optimized quantities now are

$$q_A^{+*} = \frac{K_e - K_f K_e}{1 - K_f K_d}$$

$$q_A^{+*} = \frac{K_e - K_r K_e}{1 - K_f K_d}$$

$$r_A^* = K_a - K_b\left(\frac{K_e - K_f K_c}{1 - K_f K_d} + \alpha\cdot \left(\frac{K_c - K_d K_e}{1 - K_f K_d}\right)\right)$$

where,

$$K_a = \frac{y}{1-(1-\alpha+ \frac{1}{g+1})}$$

$$K_b = \frac{1}{1-\alpha- \frac{1}{g+1}}$$

$$K_c = \frac{E_y y - E_{k_a} k_a (1-\alpha)}{(\alpha+ \frac{1}{g+1}) (1-k_b (1-\alpha))}$$

$$K_d = \frac{1}{(\alpha+ \frac{1}{g+1})}$$

$$K_e = \frac{K_c}{2} (\alpha+ \frac{1}{g+1})$$

$$K_f = \frac{\alpha^2}{2(1-K_b (1-\alpha))}$$

Profits for the small firm, given α, if it were to enter in period A with probability 1 is,

$$K_a y - K_a \overset{+}{q}{}_A^{*} - K_a \alpha q_A^{*} - K_a^2 (1-\alpha)$$

$$+2K_a (1-\alpha) K_b \overset{+}{q}{}_A^{*} + 2K_a (1-\alpha) K_b \alpha q_A^{*}$$

$$-K_b \overset{+}{q}{}_A^{*} y + K_b \overset{+}{q}{}_A^{*2} + K_b \overset{+}{q}{}_A^{*} \alpha q_A^{*}$$

$$-K_b^2 \overset{+}{q}{}_A^{*2} (1-\alpha) - K_b^2 \overset{+}{q}{}_A^{*} (1-\alpha) \alpha q_A^{*}$$

$$-K_b \alpha q_A^{*} y + K_b \alpha q_A^{*} \overset{+}{q}{}_A^{*} + K_b \alpha^2 q_A^{*2}$$

$$-K_b^2 \alpha q_A^{*} (1-\alpha) \overset{+}{q}{}_A^{*} - K_b^2 \alpha^2 q_A^{*} (1-\alpha) \qquad \text{- (Y1)}$$

It is now a trivial algebraic exercise to show that expression (X1) - expected value with respect to y of expression (Y1) can be made negative for some finite g.

All that now remains to be shown is the following:

Lemma 9: Whenever there is no temporal randomization, identical small firms entering in the same period will choose identical optimal quantities of production. When there is temporal randomization, then the small firms will again have the same equilibrium strategies.

Proof: The proof is rather simple and two cases will be considered separately (with and without randomization).

When there is no temporal randomization, then, in a game g, denote the set of small firms entering in period B by S_B. When we consider the small firms that are entering in period A, the proof is similar. Let us also assume, without loss of generality, that the large firm is entering in period A. Thus, the small firm i solves the optimization problem,

$$\max_{q_i} E_y \{D ((q_B^+ + \sum_{\substack{j \in S_B \\ j \neq i}} (\frac{q_j}{g+1}) + \frac{q_i}{g+1} + L), y) \cdot q_i - \bar{C} (q_i, g)\},$$

where L is the aggregate production in period A. Let q_1 be the first argument of D, and q_0 be the aggregate precommitted production.

The first order condition for firm i is

$$\frac{q_i}{g+1} \cdot \frac{\partial E_y D}{\partial q_1} \cdot (1 + \frac{\partial L}{\partial q_0}) - \bar{C}' (q_i, g) + D = 0.$$

Similarly, the first order condition for firm j is

$$\frac{q_j}{g+1} \cdot \frac{\partial E_y D}{\partial q_1} \cdot (1 + \frac{\partial L}{\partial q_0}) - \bar{C}' (q_j, g) + D = 0.$$

Define

$$h(\hat{q}) = \frac{\hat{q}_i}{g+1} \cdot \frac{\partial E_y D}{\partial q_1} \cdot (1 + \frac{\partial L}{\partial q_0}) - \bar{C}' (\hat{q}_i, g) - \frac{\hat{q}}{g+1} \cdot \frac{\partial E_y D}{\partial q_1} \cdot (1 + \frac{\partial L}{\partial q_0}) + \bar{C}' (\hat{q}, g).$$

Now,

$$h' = \overline{C}'' - \frac{1}{g+1} \cdot \frac{\partial E_y D}{\partial q_1} \cdot \left(1 + \frac{\partial L}{\partial q_0}\right) .$$

Thus, $h' > 0$ if $\overline{C}'' > 0$, $D' < 0$, and $L' > -1$. Furthermore, $h(\hat{q}_i) = 0$ and $h(\hat{q}_j) = 0$. Hence, for any i, j, $\hat{q}_i = \hat{q}_j$.

The assumption of $\overline{C}'' > 0$ is consistent with, for example, the quadratic costs case we considered. The functional form of the reaction of individual firms in that case is also consistent with the assumption that $L' > -1$. In fact, in that case

$$L' = - \frac{\text{number of firms entering in period A}}{2 \cdot (\text{number of firms entering in period A} + 1)} .$$

On the other hand, when there is temporal randomization, then let $q(a) = q^+$ if $a = 1$, and $q(a) = 0$ if $a = 0$, and let $\delta(a) = \delta^+$ if $a = 1$ and let $\delta(a) = (1-\delta^+)$ if $a = 0$. Then, a small firm i's optimization. problem is

$$\max_{q_i} \sum_{S \varepsilon P_i} \{q_i \cdot E_y D(\{q(a) + \sum_{k \varepsilon S} q_k + q_i + L(q_0)\}, y)$$

$$- \overline{C}\,(q_i,\, g)\} \cdot \prod_{k \varepsilon S} \delta_k \cdot \prod_{k \varepsilon S} (1-\delta_k) \cdot \delta(a),$$

where P_i is the power set of the set of all firms excluding firm i. Similarly, a small firm j's optimization problem can be written, and by an argument similar to the one above, $(q_i, \delta_i) = (q_j, \delta_j)$ for any i,j.

Q.E.D.

Thus, we have shown that the family of games we considered has an upper hemicontinuous equilibrium correspondence. This means that markets with one large firm and several small, but atomic firms can be approximated by a Stackelberg equilibrium. However, since for non-trivial but small uncertainty, the Stackelberg equilibrium is the unique equilibrium in the limit when there are non-atomic small firms, the correspondence is also lower hemicontinuous. This means that the Stackelberg equilibrium also approximates the equilibrium in the game when there is a finite number of small firms.

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ENDOGENOUSLY DETERMINED PRICE-SETTING MONOPOLY IN AN EXCHANGE ECONOMY

1. Introduction

The behaviour of a perfectly-discriminating monopolist was first explained by the classical theory of monopoly. Later, using Edgeworth's theory of a pure exchange economy and the core as an equilibrium concept of a cooperative game, the perfectly-discriminating monopolist was obtained as the cooperative equilibrium of an exchange economy for particular characteristics of the agents' relative sizes. It was shown that given positive initial allocations, with two agent types one of which was an atomless collection of several identical agents and the other type was a single atom, the final equilibrium allocation was in the Core. Moreover, with those allocations, every nonatomic trader was indifferent between trade and no-trade (see [3]).

On the other hand, it could be shown using offer-curves, that a price-setting monopolist in a pure exchange economy would cause final allocations to lie outside the core. Thus in order to be able to obtain a price-setting monopolist as the equilibrium of some game in an exchange economy, the natural direction to proceed is to consider the pure exchange economy as some form of a noncooperative game and use the noncooperative Nash equilibrium concept. This was made possible with the development of noncooperative exchange models (for example [2]).

However, it was claimed (see [1]) that the examination of market structures similar to the one that had earlier yielded the perfectly-discriminating monopolist, would now only result in trivial no-trade equilibria. This claim was driven by the ad hoc assumption that firms in such a market play a Cournot game with each other. A Cournot game is a noncooperative game in extensive form in which the players are in the same strategic position with respect to each other. That is, the players move simultaneously (or sequentially but unobservably) and their strategy spaces are isomorphic to each other. An example of the Nash equilibrium of such a game is the Cournot equilibrium.

The purpose of this paper is to describe an "intuitively plausible" noncooperative game in an exchange economy, such that a price-setting monopoly is a sequentially rational Nash equilibrium of that game and the price-setting monopolist is determined endogenously.

We recall that another classical way to characterize the quantity-setting equilibrium in an oligopoly, is by the use of a noncooperative game in which some of the players are in a dominant strategic position with respect to some others. Such a game is called a Stackelberg game. Here, the dominant players move first and have strategy spaces that are not isomorphic to those of the other players. Moreover, the payoffs to a firm also depend upon when it moves. An example of the Nash equilibrium of such a game is the Stackelberg equilibrium. It would appear that if we were to define such a game in an exchange economy, that it may be possible to obtain an equilibrium in which the dominant player is the price-setting monopolist. However, such an approach may not be intuitively appealing because there is no a priori reason why a particular firm should be exogeneously assigned to be the dominant player. This assignment would be quite as ad hoc as the assumption that all firms play a Cournot game.

What we do therefore, is to describe a game of imperfect information in which ex ante, the players are in the same strategic position with respect to each other. Next, we will obtain sufficient conditions for such a game to have a unique equilibrium, such that when the equilibrium strategies are being played, it appears as if the traders are playing a Stackelberg game. The sufficient conditions would then also determine when a price-setting monopolist would exist. The main point is, we do not arbitrarily assign who should be the dominant player. Instead we assume that the firms behave rationally in a larger game in which all firms are strategically equivalent in some sense and then examine sufficient conditions on the market structure that would yield a unique equilibrium of this game which results in a price-setting monopolist. The model we will use will be directly adapted from Chapter 1.

We will first describe the basic notation and model developed in [1]. This will be done in Section 2. In Section 3 we will describe our model and derive the main result. Section 4 concludes this Chapter.

2. Noncooperative Exchange With a Syndicate

The unit interval $I = [0, 1]$ represents traders and is endowed with Lebesgue measure μ. I is partitioned into T^0 and T'. The members of T^0 are unorganized traders. The members of T' are organized or syndicated traders or a combination of these with unorganized traders. The members of a syndicate act in concert and decide on a common trade they will all make while those that are unorganized will act individually.

There are two commodities represented by m and y. Both enter the utility functions of the traders.

The utility function for a trader i is represented by U_i which is the same for all members of a particular syndicate. The initial endowments of the trader are represented by w_i^m of good m and w_i^y of good y.

Trade takes place through a market in which the two goods are exchanged against one another. Each trader sends some amount of one or the other commodity into the market. The price ratio is the ratio of the aggregate supplies. We will denote as the price ratio p, the aggregate supply of good y divided by the aggregate supply of good m. For each trader, the amount of the good supplied by him will purchase at the effective price a proportional quantity of the other good. An unorganized trader faces a budget constraint that coincides with that of the usual Walrasian price-taker. However, organized traders acting in concert exert a non-negligible impact on prices. The agents recognize this phenomenon.

Specifically, if each trader were allowed to supply both goods to the market, the amount of good m and good y each trader supplies is called a supply profile and is represented by

$(m_i, y_i) \in \mathbf{R}_+^2$, with $m_i \leq w_i^m$, $y_i \leq w_i^y$

For such a supply profile, the unorganized trader i will receive in return

$$m_i \cdot \frac{\int_I y_\mu \, d\mu}{\int_I m_\mu \, d\mu} \qquad \text{of good y}$$

and

$$y_i \cdot \frac{\int_I m_\mu d\mu}{\int_I y_\mu d\mu} \qquad \text{of good m}$$

The organized trader who belongs to a syndicate $S \subseteq T'$ of measure μ^S will receive in return for a supply of (m_i, y_i),

$$m_i \cdot \frac{\int_{I\ S} y_\mu d\mu \ + \ y_i \mu^S}{\int_{I\ S} m_\mu d\mu \ + \ m_i \mu^S} \qquad \text{of good y}$$

and a similar amount of good m.

Let $M \in (\mathbf{R}_+^2)^I$ be the market supply profile. The space $(\mathbf{R}_+^2)^I$ is endowed with the product topology.

A net trade is an integrable function $X : (\mathbf{R}_+^2)^I \times I \to \mathbf{R}^2$ such that $\int_I X(M,i)d\mu \leq 0$, and it represents the initial endowment vector of each trader net of the amounts supplied to the market and the amounts obtained in exchange from the market.

Thus, for an unorganized trader with the supply profile (m_i, y_i), the net trade X(M,i) is equal to the ordered pair

$$\left((w_i^m + y_i \frac{\int_I m_\mu d_\mu}{\int_I y_\mu d_\mu} - m_i\right), \left((w_i^y + m_i \frac{\int_I y_\mu d_\mu}{\int_I m_\mu d_\mu} - y_i\right).$$

A similar expression obtains for the organized trader belonging to a syndicate S of measure μ^S. Points in the domain of individual utility functions are X(M,i). A trader's utility is higher for higher values of net trade in either commodity. Thus the individual firms in I are playing a game of imperfect information. Each player has only one information set and it represents the information available to him when it is his turn to play. The various alternatives that a player has during his turn are the quantities of good m and good y that he decides to send to the exchange market. If he is a member of T^0, then he does not know what any other player has decided to send to the market and if he is a member of T', then he only knows what the members of his syndicate have decided. The information set of trader i ε T^0 for instance, is the space $(\mathbf{R}_+^2)^{I \setminus \{i\}}$. Thus every trader i's pure strategies are points $(m_i, y_i) \in \mathbf{R}_+^2$.

A strategy vector is an $(\mathbf{R}^2)^I$ - vector of individual strategies (m_i, y_i) for every i ε I. This is a market profile.

The equilibrium concept we will use is that of a sequentially rational Nash equilibrium in pure strategies. At this point we must note that there are examples where such an equilibrium will fail to exist. The reader is referred to Professor Rosenthal's comments following [2] for an example of this, when there are consumption externalities. However, we will assume, as is done in the literature, that such an equilibrium exists.

A sequentially rational Nash equilibrium is a market profile $M \varepsilon (\mathbf{R}_+^{2})^I$ such that for each unorganized trader

$$u_i(X(M,i)) \geq u_i(X(M',i))$$

where $M'(j) = M(j) \forall j \neq i$ and $M'(i) \leq (w_i^m, w_i^y)$ and for each $i \varepsilon S \subseteq T', \nexists M' \varepsilon (\mathbf{R}_+^{2})^I$ with $M'(i) \leq (w_i^m, w_i^y)$, $M'(i) = M(i)$ $\forall i \varepsilon I \backslash S$ and for $i,j \varepsilon S$, $M'(i) = M'(j)$ such that

$$U_i(X(M', i)) > U_i(X(M,i)).$$

It is assumed in this model that for each trader $i \varepsilon I$, $m_i \cdot y_i = 0$ but $m_i + y_i > 0$. This assumption reduces the size

is because if (m,y) are the aggregate amounts of good m and good y in the market and if for a trader $i \varepsilon S \subseteq T'$, (m_i, y_i) is a feasible supply profile then so is $(m_i + \tau, y_i + \tau \frac{y}{m})$ for some $\tau > 0$; and the net trades for that trader are the same in both cases. Furthermore, it is part of the definition of T^0 that $\forall$ i, j ε T^0, $m_i \cdot y_j = 0$.

Given the set-up above it can be argued that in a market with one large trader without a competitive fringe, i.e., all of T' consisting of only one trader, and an unorganized sector T^0, the only equilibrium as defined above, is trivial.

This is because the large trader can make the relative price of the commodity he wants arbitrarily small. Each small trader realizes this together with the fact that he cannot affect the price and that this will result in the commodity he wants being arbitrarily expensive. No trade will therefore be the only Nash equilibrium.

<u>Proposition 1</u>: (Okuno, Postlewaite, Roberts)

Let all of T' be a syndicate with $\mu(T') \neq 0$. Let $\mu(T^0) \neq 0$. Then in an exchange economy with the technology given above, $\vec{0} \in (\mathbf{R}_+^2)^I$ is the only Nash equilibrium.

<u>Proof</u>: We will first show that any $M \neq \vec{0} \in (\mathbf{R}_+^2)^I$ is not a Nash equilibrium. It is then trivial to see that ((0,0), (0,0)) is a Nash equilibrium.

Without loss of generality let the syndicate's supply profile be represented by the 2-Vector (m_ℓ, o) such that m_ℓ is > 0. For the case $m_\ell = o$ it will be seen later, that for every i ε T^0, a supply profile (o,o) is the "best response". Next let firm i ε T^0 have a supply profile (o, y), y > o.

i i

Then the price $p = \frac{\int y_i \, d\mu}{m_\ell}$

with $\frac{\partial p}{\partial m_\ell} = -\frac{\int y_i \, d\mu}{m_\ell^2}$ and

$$\frac{\partial p}{\partial y_i} = \frac{\mu(i)}{m_\ell} = 0.$$

The syndicate's utility u_ℓ is equal to

$$u_\ell\left((w_\ell^m - m_\ell), (w_\ell^y + m_\ell \cdot p)\right)$$

$$\frac{\partial u_\ell}{\partial m_\ell} = -\frac{\partial u_\ell}{\partial 1} + \frac{\partial u_\ell}{\partial 2}\left(m_\ell \cdot \frac{\partial p}{\partial m_\ell} + p\right)$$

Where $\frac{\partial u_\ell}{\partial 1}$ and $\frac{\partial u_\ell}{\partial 2}$ represent the partial derivative of utility with respect to its first and second arguments respectively. Therefore,

$$\frac{\partial u_\ell}{\partial m_\ell} = -\frac{\partial u_\ell}{\partial 1} + \frac{\partial u_\ell}{\partial 2}\left(-\frac{\int y_i d\mu}{m_\ell} + \frac{\int y_i d\mu}{m_\ell}\right)$$

$$= -\frac{\partial u_\ell}{\partial 1}$$

Hence for any $m_\ell > o$, its utility can be increased by decreasing m_ℓ.

The unorganized trader's utility level given any $m_\ell \geq o$ is

$$u_i\left(\left(w_i^m + y_i \cdot \frac{m_i}{\int y_i d\mu}\right), (w_i^m - y_i)\right).$$

$$\frac{\partial u_i}{\partial y_i} = \frac{\partial u_i}{\partial 1}\left(y_i \cdot \frac{\partial p}{\partial y_i} + \frac{1}{p}\right) - \frac{\partial u_i}{\partial 2}$$

$$= \frac{\partial u_i}{\partial 1} \cdot \frac{1}{p} - \frac{\partial u_i}{\partial 2}$$

Since the unorganized trader knows the sign of the partial derivative of the syndicated trader, he realizes that $\frac{1}{p}$ will be very close to zero and hence

$$\frac{\partial u_i}{\partial y_i} \approx - \frac{\partial u_i}{\partial 1} < 0$$

Thus for any $y_i > 0$, the unorganized trader can increase its utility by decreasing y_i. Any $m \neq \vec{0}$ is therefore not a Nash equilibrium. It is trivial to see that ((0,0), (0,0)) is a Nash equilibrium.

Q.E.D.

On the other hand, by incorporating the basic model of Chapter 1 into a general noncooperative exchange context we will find that under any nontrivial uncertainty, the large trader will be a Stackelberg leader, the small traders will be Stackelberg followers and trade will take place in equilibrium. Even under complete certainty the traders are indifferent between playing a Cournot game and playing a Stackelberg game. Hence, although the equilibrium resulting from the Cournot game will be trivial as noted earlier, trade will take place in the equilibrium of the Stackelberg game. Note that the game involves quantity-setting strategies. However, when there are many small traders in the market, those traders can exert only negligible influence on their terms of trade. That is, they are virtually price-takers. Thus, the remaining large trader has latitude to set the price. This situation holds exactly in the limit and is approximated asymptotically as the unorganized traders are made small and numerous. With only a small number of unorganized traders, price-setting equilibrium is not presumed to occur. We will now present our model, which is similar to the one described in the introduction.

3. A Model of Non-cooperative General Exchange with an Endogenously Determined Price-Setting Monopolist

There two time periods denoted by B for period "Before" and A for period "After". There is uncertainty in price and this is revealed between the two periods. We will shortly explain how this uncertainity could arise. The initial endowments of the large trader are (w_ℓ^m, w_ℓ^y) and of a small trader i are (w_i^m, w_i^y).

If a trader decides to "enter" in period B then it has to set aside a supply profile, say a designated fraction α of its initial endowment of good m. Thus $\alpha \cdot w_i^m$ is set aside to be sent to the market. However, the actual amount that the market receives is uncertain and is denoted by $\alpha \cdot w_i^m \cdot k_i$ where k_i is a random variable for the ith trader and whose range is say [0,2]. For example, the traders could be thought of being endowed with seeds, but trade is done in the crops obtained. If a trader decides to "enter" in period A, however, he knows the realized value of the random variable. The crops enter the utility functions of the traders. The random variable could have different independent distributions for different traders.

The extensive form of the game that the traders are assumed to be playing is exactly the one in figure 1. Thus revised initial endowments in period B for a firm i are $(w_i^m \cdot k_i, w_i^y \cdot k_i)$. As before, utility functions are defined on net trades. In the first information set of every trader, its behaviour strategies consist of announcing a probability of entering in period B and the proportion α_i of its initial endowment that it will send to the market if it enters in period B.

We will now consider the situation with the large trader "entering" in period B as the Stackelberg leader and the small traders "entering" in period A as followers. The large traders and the small traders trade in different commodities. Thus for the small traders, the optimal value of the choice variable α_i^* will be a function of the price p and the value of k_i . The aggregate supply of the good (say m) by the small firms will be

$$\int_{T^0} (\alpha_i^* (p, k_i) \cdot w_i^m \cdot k_i) \, d\mu$$

where k_i is the realized value of the random variable.

Thus a trader $i \varepsilon T^0$ will maximize

$$U_i((w_i^m \cdot k_i - m_i), (w_i^y \cdot k_i + m_i p)) \qquad -1-$$

$$\text{s.t.} : 0 \leq w_i^m \cdot k_i$$

The large trader on the other hand realizes that p is a function of α_ℓ^* and for each small trader, therefore, α_i^* is a composite function of α_ℓ^*. He will maximize his expected utility in order to obtain the optimal α_ℓ.

We will show through a lemma that for any non-trivial uncertainty,this situation is a Nash equilibrium of the game described in figure 5 ofChapter 1. Furthermore, it is shown that in this kind of Stackelberg equilibrium, "entering" in period B is a dominant strategy of the small firms.

Lemma 1:

Let $P \subseteq \mathbf{R}$ be the range of a random variable with a distribution function g, $X \subseteq \mathbf{R}$ and let $f: X \times P \to \mathbf{R}$ be a measurable function such that $\operatorname{argmax}_{x \varepsilon X} f(x,p)$ exists for each $p \varepsilon P$. Since f is a random variable, $E(f(x,p))$ is well defined for each $x \in X$. Then if x^{*1} denotes $\underset{x \in X}{\text{argmax}}\, E(f(x,p))$ and $x^*(p)$ denotes $\underset{x \in X}{\text{argmax}}\, f(x,p)$ then

$$E(f(x^*(p),p)) \geq E(f(x^{*1}, p)).$$

Proof:

$$E(f(x^*(p),p)) = \int_{-\infty}^{\infty} f(x^*(p),p)\, g(p)\, dp$$

and

$$E(f(x^{*1}, p)) = \int_{-\infty}^{\infty} f(x^{*1}, p)\, g(p)\, dp$$

Since $x^*(p)$ maximizes $f(x,p)$, $f(x^*(p), p) \geq f(x^{*1}, p)$, and the proof is complete.

Q.E.D.

Thus appealing to the above lemma if we let f be the utility function of the small trader, and p be the price and if we consider the Stackelberg situation discussed above, we see that any small trader that is considering deviating and entering in period B will observe that ex ante its expected utility of entering in period B is never larger than the expected utility of entering in period A. The respective expected utilities can at most be equal when there is no uncertainty.

Note that if the large trader decides to enter in period A, then in equilibrium there is no trade as shown in Proposition 1. It is easy to see that under some non-trivial uncertainty, the expected payoff to the large trader of entering in period B is at least as large as that of entering in period A given that the small traders enter in period A. However, it is important to question whether or not this equilibrium is sensitive to the amount of uncertainty in the random variables. That is, will there be some large enough uncertainty such that ex ante the large trader finds it more profitable to enter in period A? One has good reasons to suspect this since even for risk-neutral firms it is possible for the ex ante expected payoff of making ex post choices to be dependent on the amount of uncertainty. We will now show why this may be the case. Let k be a random variable and suppose a firm has an objective function $f(x,k)$ where x is its choice variable and f is a measurable function of k. Assume f is concave in x. Since expectation is a linear operator, the expected value of f denoted by $E_k(f)$ is also concave in x. Assume further that the firm is risk-neutral. We will show that the result we seek holds even when $\frac{\partial^2 f}{\partial k^2} = 0$. The point is, we are trying to show that even $E_k(f)$ does not depend upon the variance of k, and the firms are risk-neutral, it is possible for the ex ante expected payoff of making ex post choices to depend upon the amount of uncertainty in the random variable. Let us also assume that the random variable k is revealed at some time and the firm has to decide ex ante, whether to choose the optimal amount of x "Before" k is revealed or "After" the revelation. Consider each case separately.

Before:

The firm maximizes $E_x(f(x,k))$ over its choice variable x setting $\frac{\partial E_k f}{\partial x} = 0$.

Thus the optimal $x = x_B^*$ could, in general, a function of $E(k)$. If it is, then the optimized value of the objective function

$f = f_B^*(x_B^*(E(k)),k)$.

Again $\frac{\partial^2 f_B^*}{\partial k^2} = 0$ and the ex ante expected optimized payoff = $E(f^*)$ does not depend upon the amount of uncertainty measured by variance.

After:

The firm now maximizes f(x,k) over its choice variable x setting $\frac{\partial f}{\partial x} = 0$, obtaining an optimal $x = x_A^*$ as a function of k. Thus the optimized value of the objective function $f = f_A^*(x_A^*(k), k)$ and

$$\frac{\partial^2 f_A^*}{\partial k^2} = \frac{\partial f_A^*}{\partial x_A^*} \cdot \frac{\partial^2 x_A^*}{\partial k^2} + \frac{\partial^2 f_A^*}{\partial x_A^{*2}} \cdot \left(\frac{\partial x_A^*}{\partial k}\right)^2 + \frac{\partial^2 f}{\partial k^2} = \frac{\partial^2 f_A^*}{\partial x_A^{*2}} \cdot \left(\frac{\partial x_A^*}{\partial k}\right)^2$$

which need not be equal to zero.

If it does not, then the ex ante expected payoff of deciding after revelation of the random variable is $E(f_A^*)$ and this is a function of the variance of k. Thus ex ante, the expected payoff of deciding "After" changes with the variance of the random variable. In [3] it was shown in fact that as the variance increased, the ex ante expected payoff of deciding to enter "After" increased. This means that although for some amounts of uncertainty the firm found it more profitable to decide to enter "Before" the random variable was revealed, as the amount of uncertainty got sufficiently large, the ex ante expected payoff of deciding to enter "After" got sufficiently large to exceed the constant (relative to amount of uncertainty) ex ante payoff of deciding to enter "Before".

It would therefore be reasonable to expect such a result in the present case. Hence, we examine utility functions which are linear in the random variable but concave in the choice variable α_i. The agents are expected-utility maximizers.

However, we know from Proposition 1 that if the large trader were to enter in period A, then no-trade is the only equilibrium. In other words, x_A^* is a constant (equal to zero) and so

$$\frac{\partial^2 f_A^*}{\partial k^2} = \frac{\partial^2 f_A^*}{\partial x_A^{*2}} \cdot \left(\frac{\partial x_A^*}{\partial k}\right) = 0.$$

Therefore, the ex ante expected payoff of entering in period A does not change with the amount of uncertainty in the random variable. Hence the large trader will always (for any amount of uncertainty) enter in period B. We will now demonstrate this through a simple example.

Example: Let $U = X^\gamma Y^{1-\gamma}$, $0 < \gamma < 1$,

where X is the net trade in good m and Y is the net trade in good y. Assume the constraint in equation 1 is not binding. Also assume for simplicity, that the random variable k is the same for all traders. The small trader's optimal choice α_i is given by

$$\alpha_i = 1 - \gamma - \frac{w_i^y \gamma}{w_i^m p} \tag{2}$$

where $p = \dfrac{w_\ell^y \cdot k \cdot \alpha_\ell}{\int \alpha_i(\alpha_\ell) \cdot w_i^m \cdot k \, d\mu}$ is the market price. Let M be the aggregate initial endowment of good m and G of good y. Also denote the denominator in the price formula by k.I.

Then from equation 2,

$$I = M - M\gamma - \frac{G\gamma}{p} \tag{3}$$

But from the price formula, $I = \dfrac{\alpha_\ell \; w_\ell^y}{p}$.

Therefore

$$p = \frac{\alpha_\ell \; w_\ell^y + G^\gamma}{M - M\gamma} \; . \tag{4}$$

Substituting this expression for p into equation 3 above, we get

$$I = M(1 - \gamma) \frac{\alpha_\ell \; w_\ell^y}{\alpha_\ell \; w_\ell^y + G\gamma} \tag{5}$$

Therefore

$$\frac{\partial I}{\partial \alpha_\ell} = M(1 - \gamma) \frac{w_\ell^y \; G\gamma}{(\alpha_\ell \; w_\ell^y + G\gamma)^2} \; . \tag{6}$$

Large trader: For the large trader, the ex ante expected payoff of entering in period A would be

$$E_x\left(U_\ell((w_\ell^m \cdot k), (w_\ell^y \cdot k))\right) = E(k) \cdot (w_\ell^m)^\gamma \cdot (w_\ell^y)^{1-\gamma} \quad \ldots (7)$$

which does not depend upon the variance of k.

Its ex ante expected payoff of entering in period B is

$$E_k\left((w_\ell^m k + k.I)^\gamma (w_\ell^y \cdot k - \alpha_\ell \cdot w_\ell^y k)\right)^{1-\gamma}$$

$$= E(\kappa) \left(w_\ell^m + I\right)^\gamma (w_\ell^y - \alpha_\ell w_\ell^y)^{1-\gamma}.$$

Thus α_ℓ^* which maximizes this expected payoff, is a function of E(k).

The optimized value of the payoff is therefore

$$E(\kappa) \left(w_\ell^m + I\right)^\gamma \cdot (1 - \alpha_\ell^*(E(k)))^{1-\gamma} \cdot (w_\ell^y)^{1-\gamma}$$

and again this does not depend upon the variance of k.

We now have the following theorem.

Theorem:

In an exchange economy with one large trader and a continuum of small traders given the technology above, there will be non-trivial equilibrium with trade in which the large trader in the Stackelberg leader in the sense discussed above and the small traders will be followers in the same sense. The large trader is the endogenously determined price-setting monopolist. Furthermore, with any non-trivial uncertainty, this is the unique Nash equilibrium.

4. Conclusion

We have been able to define a noncooperative game in an exchange economy such that a price-setting monopolist is endogenously determined and is the only Nash equilibrium of this game. it was pointed out that although such a monopolist could be obtained as the Nash equilibrium of a Stackelberg type game in the exchange economy, such a derivation was not 'sensible' because of the ad hoc assumption that the monopolist was the dominant player in the Stackelberg game. Instead, we defined another game in which the traders were in the same strategic position with respect to each other. The Nash equilibria of such a game depended upon the structure of the market, among other things. It was then shown that in a market with one 'large' trader (i.e., a trader whose output affects price) and several nonatomic traders, the only Nash equilibrium of this game resulted in the large trader being endogenously determined as a price-setting monopolist.

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