LARGE N GAUGE THEORY AT STRONG COUPLING WITH CHIRAL FERMION

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Abstract

The properties of the $SU(N)$ lattice gauge theory are investigated at strong coupling ($\lambda \simeq \infty$). We use a Euclidean formulation with naive fermions which preserves all the chiral symmetries of the continuum theory, and solve the theory exactly in the limit $N \to \infty$, $\lambda \to \infty$. It is shown how the hopping parameter expansion in the inverse quark mass can be summed to all orders. This method of resummation is first applied to a calculation of the order parameter of chiral symmetry, $\langle\bar{\psi}\psi\rangle$. We compute the first two terms in the strong coupling expansion for this quantity but neglect internal fermion loops, and show that at sufficiently strong coupling, the chiral symmetry spontaneously breaks. After considering several mechanisms, we conclude that chiral symmetries break when the gauge forces are strong enough to make a quark - anti-quark bound state.

Next, we use the resummation to find the spectrum of the $N = \lambda = \infty$ theory as a function of the bare mass of the quarks, and calculate the first correction in λ^{-1} to this spectrum. The mesons are pseudo-Goldstone bosons, and the baryons acquire masses of order N through the spontaneous breakdown of chiral symmetry. These calculations also determine the spectrum of the strongly coupled theory at finite N in the approximation of no internal quark loops. We compare these masses to those from numerical simulations and experiment.

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Chapter 1

INTRODUCTION

Symmetries play an important role in physics, both because they permit an understanding of many aspects of natural phenomena and because they sometimes put very stringent constraints on the dynamics. Hadron physics has many symmetries, exact or approximate, manifest or hidden. These symmetries alone, without any dynamical input, account for most of the qualitative features of the hadron spectrum, and a simple model such as the quark model which incorporates these symmetries can give good agreement with experiment. To describe the dynamics of hadrons, one needs a candidate quantum field theory with the correct symmetries. If in addition, one demands that the interactions between quarks be described by a renormalizable asymptotically free gauge theory, one obtains Quantum Chromodynamics (QCD) as the only viable possibility. However, more than a decade after it was proposed, much of the dynamics of QCD still escapes us. The agreement of renormalization group improved perturbation theory with experiment has given us reason to believe that QCD is indeed the correct theory of the strong interactions. Unfortunately, even the very high energy processes are entangled with non perturbative low energy phenomena. Other features such as confinement or the existence of bound states are beyond the reach of weak coupling perturbation theory. These non perturbative effects come from two sources:

1) QCD has a nontrivial topological structure so there are instanton configurations which contribute to the path integral as $\exp(-g^{-2})$ relative to the classical vacuum configurations.

2) the renormalization group equations show that Greens functions and more generally all dimensionful quantities have contributions that depend

on the coupling constant as $\exp(-g^{-2})$ whether or not there are instantons.†

This makes it clear that before one can quantitatively understand QCD, it is necessary to establish a suitable framework for studying these non perturbative effects. Attempts have been made to understand the contributions of instantons, and this has lead to a good qualitative picture of confinement and of chiral symmetry breaking [1], as well as to a resolution of the $U(1)$ problem [2]. However reliable calculations are extremely difficult because the instanton gas is not dilute. On the other hand, it has been realized that the lattice is a natural way to define QCD so that <u>all</u> the nonperturbative physics is accessible, one is no longer restricted to a semi-classical analysis. In 1974, Wilson [3] showed how the lattice regulator could be made to preserve the gauge invariance of the theory. Since then, much progress has been made, new calculational schemes have been developed, and today we are perhaps not so far from being able to determine the consequences of QCD. It is probable that in a few years, the lattice numerical methods will give us a first principles calculation of the hadron spectrum to a few percent.

Meanwhile, lattices can provide qualitative information about QCD. In the region of small cut-off (strong coupling), confinement and chiral symmetry breaking can be shown by simple and reliable means.* It is necessary however to determine whether these qualitative features are artifacts of the cut-off or whether they are preserved as the cut-off is taken to infinity. That is one asks whether singularities are encountered when taking the continuum limit. QCD is strongly interacting at large distances so the strongly coupled lattice is probably a good first approximation. Usually one relies on the

† These two features are not generally related. For instance the O(N) models in 1+1 are asymptotically free for $N \geq 3$ and thus have a $\exp(-g^{-2})$ behavior, but only the $N=3$ theory has instantons. On the other hand, the O(2) model has vortices but is not asymptotically free.

* Actually, some quantitative features in the strong coupling region such as the masses of the lightest hadrons are often within 30 percent of experiment. It is plausible that these quantities do not depend sensitively on the dynamics, explaining the success of quark models.

numerical methods to show whether the behavior as a function of the cut-off is smooth or not.

The lattice can also be used as a laboratory for testing ideas about the dynamics. For instance, one can hope to learn what is the mechanism responsible for confinement. A striking feature of QCD is the way it realizes its chiral symmetries. Our present understanding of chiral theories is that the chiral symmetries should spontaneously break if the effective coupling constant is sufficiently large. In this dissertation, we shall consider chiral symmetry breaking on the lattice at strong coupling. Since for QCD the effective coupling constant is large at the scale of the hadrons, the strongly coupled lattice theory is perhaps adequate to determine whether or not the chiral symmetry dynamically breaks. After all, one is only asking a qualitative question, not a quantitative one. Unless chiral symmetry breaking depends very much on the details of the dynamics, we should obtain the right answer. The lattice thus enables us to examine in a very concrete way the various mechanisms proposed for breaking chiral symmetries. This is the subject of Chapter 3. The strongly coupled limit of the lattice theory also provides a standard with which to compare the results of the numerical simulations. One can then see which predictions of these calculations are non trivial. This comparison is done in Chapter 4.

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Chapter 2

ASPECTS OF LATTICE GAUGE THEORIES

1. THE LATTICE AS A GOOD REGULATOR

The lattice can be regarded as simply another way to regularize a field theory. However, it has an important feature which other cut-offs do not have: the theory with fixed cut-off is well defined independently of perturbation theory. The Feynman Path Integral defines the cut-off theory unambiguously because the integration is over a discrete space of variables instead of over a space of functions. This means that the lattice is particularly well suited to the study of non perturbative phenomena. Many aspects of field theories such as QCD are intrinsically non-perturbative in nature. For instance, the spectrum of an asymptotically free theory is not analytic in the coupling constant (the masses behave exponentially in $-g^{-2}$), and confinement cannot be seen in perturbation theory. The lattice cut-off enables one to use new methods which are not available with other cut-offs in order to study these non-perturbative phenomena. This has had a significant impact on our understanding of field theories. Let us mention two "applications" of the lattice formulation:

1) It is currently believed that $SU(N)$ pure gauge theories confine static quarks, i.e., the Wilson loop has an area law behavior. Recently, Tomboulis [1] has shown, by using correlation inequalities, that the $SU(2)$ lattice gauge theory confines quarks in the fundamental representation for all values of the coupling constant, thus possibly giving a proof of confinement.

2) One is approaching a rigorous proof that $\lambda\varphi^4$ theory in four dimensions is trivial. In perturbation theory, it is a well behaved interacting theory. However, because it seems to have no non-Gaussian fixed point, it is believed that non-perturbative phenomena drastically change the

properties of this theory and necessarily render it non-interacting in the continuum limit [2].

This shows that the lattice regulator has many merits: it provides us with new tools and it serves as an intermediate step in our understanding of field theories. In fact, it enables one to obtain with little effort qualitative information about the quantum field theory, such as what the symmetries of the vacuum are, and what the nature of the asymptotic states is. In the above illustrations, the main thrust was the use of correlation inequalities. It is also possible to determine quantitatively the properties of the cutoff field theory by approximate (as opposed to rigorous) methods which have their analogues in statistical physics: high temperature expansions and numerical simulations. It is probable that in the not so distant future, the spectrum of QCD will be determined by Monte-Carlo techniques. Preliminary results are encouraging [3]. Less quantitatively, the lattice can be used as a laboratory for testing models of dynamics in quantum field theory. In particular, the lattice may help us understand the mechanisms responsible for confinement and dynamical symmetry breaking.

Unfortunately, the lattice regulator also has its inconveniences. First it destroys Lorentz invariance in a very strong way. A consequence of this is that the "bare" Lagrangian used in the lattice theory is not constrained to be of any particular form by renormalizability arguments. Another problem with the lattice regulator is that it cannot describe non-vectorlike chiral theories without explicitly breaking the chiral symmetry. This is unfortunate since chiral symmetries play a very important role in nature. This also means that lattices cannot incorporate most supersymmetrical theories without explicitly destroying the sypersymmetry. We shall avoid theories which can not be put onto the lattice in a clean way, and thus we shall deal only with vector theories which have enough fermion species. Although Lorentz invariance will be explicitly broken, we do not expect this to affect our qualitative picture of chiral symmetry breaking.

2. RENORMALIZATION ON THE LATTICE

Free field theory on the lattice

Consider a hypercubical lattice of spacing a which discretizes space-time. We shall always work in Euclidean space, and our units are such that $\hbar = c = 1$. The action for a free scalar field obtained by discretizing derivatives in the continuum Lagrangian $L = \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}m_q^2\,\varphi^2$ is simply

$$S[\varphi] = \sum_{x,\mu} \frac{1}{2}(\varphi_{x+\mu}-\varphi_x)^2\,\mathrm{a}^2 + \frac{1}{2}\,m_q^2 \sum_x \varphi_x^2\,\mathrm{a}^4 \qquad \text{(II.2.1)}$$

where x labels the sites of the lattice and μ runs from 1 to 4 in four dimensions. The generating functional for this lattice theory is

$$Z(J) = \int[d\varphi]\, e^{-S[\varphi] + \sum_x J_x\cdot\varphi_x} \quad . \qquad \text{(II.2.2)}$$

This theory describes a free scalar of mass E given by

$$4\ \mathrm{s\,h}^2\,(\frac{E\mathrm{a}}{2}) = \sum_i 4\,\sin^2(\frac{p_i\mathrm{a}}{2}) + m_q^2\mathrm{a}^2 \qquad \text{(II.2.3)}$$

as can be seen from the poles in the propagator. To recover the continuum field theory, simply take $\mathrm{a}\to 0$. No renormalization is necessary for this gaussian theory, and one obtains the usual relativistic dispersion relation $E^2 = p^2 + m_q^2$.

We have dealt with a space-time lattice for simplicity, but a purely spatial lattice is also a possible regulator. In that case, the degrees of freedom on the sites are quantum mechanical in nature. The Hamiltonian of this (continuous time) theory can be obtained from the above theory by taking the time spacing to zero with the use of transfer matrix methods [4].

Gauge fields on the lattice

The lattice regulator is particularly useful for gauge theories because the cutoff can preserve the gauge invariance.* Such a construction is due independently to Polyakov, Wegner and Wilson [5]. In this section, we briefly review how the cutoff preserves the gauge symmetry.

Suppose we wished to obtain a lattice discretization of the *QED* Lagrangian $L = \frac{1}{4}\sum_{\mu,\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$. To see how one should define the lattice degrees of freedom, consider transporting a test charge in a background gauge field A. The field of the charged particle is multiplied by $\exp\{ ig\int_x^y \vec{A}\cdot d\vec{l}\,\}$ under transport from x to y, and the line integral is along the path taken. On the lattice, the paths are sequences of discrete steps, so for $y = x + \mu$, we take the phase to be

$$\mathrm{U}_{x,\mu} = e^{\,i a g A_{x,\mu}} \tag{II.2.4}$$

where g is the charge of the test particle. Transport from $x+\mu$ to x simply gives the inverse factor $\mathrm{U}_{x+\mu,-\mu} = \mathrm{U}_{x,\mu}^{-1}$. Thus we see that one should associate the gauge degrees of freedom with the links connecting the sites of the lattice. On each directed link is a gauge variable $A_{x,\mu}$. A gauge invariant discretization of L is

$$L_{lattice} = \frac{1}{4}\sum_{\mu,\nu} F_{x,\mu,\nu}^2 \quad ; \tag{II.2.5}$$

$$F_{x,\mu,\nu} = (\, A_{x+\mu,\nu} - A_{x,\nu} - A_{x+\nu,\mu} + A_{x,\mu}\,)\ \mathrm{a}^{-1}$$

which is just the discrete curl on the lattice. The set of links $(x, x+\mu), (x+\mu, x+\mu+\nu), (x+\nu+\mu, x+\nu), (x+\nu, x)$ encloses an elementary square called a plaquette.

* In fact, pure gauge theories are very well suited to lattice studies. The only problem seems to be the restoration of Lorentz invariance which may be associated with a mild roughening transition.

The above construction does not lead to a gauge invariant lattice theory for non abelian groups. If instead of using eq. (II.2.5), we work with the phases $\mathbf{U}_{x,\mu}$, (the gauge degrees of freedom are then compact), we can take

$$L_{lattice} = \frac{1}{2a^4g^2}\sum_{\mu,\nu}(1 - \mathbf{U}_{x,\mu}\cdot\mathbf{U}_{x+\mu,\nu}\cdot\mathbf{U}_{x+\mu+\nu,-\mu}\cdot\mathbf{U}_{x+\nu,-\nu}) \ .$$

For *QED* this gives

$$L_{lattice} = \frac{1}{2a^4g^2}\sum_{\mu,\nu}(1 - e^{ia^2gF_{x,\mu\nu}}) = \frac{1}{4}\sum_{\mu,\nu}F^2_{x,\mu,\nu} + \cdots \qquad \text{(II.2.6)}$$

which gives the previous non-compact Lagrangian density plus some terms with higher powers of the lattice spacing. This Lagrangian generalizes to non-abelian theories and maintains the gauge invariance. $\mathbf{U}_{x,\mu}$ is then a matrix which describes the color rotation of a colored matter field under transport from site x to site $x+\mu$. The gauge transformations on the $\mathbf{U}$'s are

$$\mathbf{U}_{x,\mu} \rightarrow \mathbf{V}_x\ \mathbf{U}_{x,\mu}\,\mathbf{V}^{-1}_{x+\mu} \ . \qquad \text{(II.2.7)}$$

The action is up to an overall constant

$$S_{lattice} = \frac{-1}{2g^2}\sum_{x,\mu,\nu} tr(\mathbf{U}_{x,\mu}\cdot\mathbf{U}_{x+\mu,\nu}\cdot\mathbf{U}_{x+\mu+\nu,-\mu}\cdot\mathbf{U}_{x+\nu,-\nu}) \qquad \text{(II.2.8)}$$

where tr stands for the trace in some representation. It is manifestly invariant under gauge transformations because it is made of traces of products of $\mathbf{U}$ matrices along ordered closed paths. The continuum limit is obtained by taking a to zero. Expanding the exponential of eq. (II.2.4) in powers of a,

$$\mathbf{U}_{x,\mu} = 1 + iagA_{x,\mu} - a^2g^2A^2_{x,\mu} \cdots \qquad \text{(II.2.9)}$$

one obtains $L_{lattice} \rightarrow \frac{1}{4}\sum_{\mu,\nu}F^2_{\mu,\nu}$ to leading order in a, with

$$F_{x,\mu,\nu} = (A_{x+\mu,\nu} - A_{x,\nu} - A_{x+\nu,\mu} + A_{x,\mu})\,a^{-1} + i\,g\,[A_\mu, A_\nu] \ .$$

However, when taking the cut-off away, a renormalization of the coupling constant g is necessary. One has to tune the bare parameters (here g), i.e., adjust g as a function of the lattice spacing a so that quantities of physical interest come out finite. For $SU(N)$ gauge theories, g must have the following dependence on the cut-off:

$$g^2 \underset{a\to 0}{\sim} \frac{1}{b_1 \ln(\Lambda^2 a^2) - \frac{b_2}{b_1}\ln(\ln(\Lambda^2 a^2)) + O(g^2)} \qquad \text{(II.2.10)}$$

as given by the renormalization group. b_1 and b_2 are the first two coefficients in the expansion of the beta function in powers of the coupling constant: $\beta(g) = a\frac{dg}{da} = b_1 g^3 + b_2 g^4 + \cdots$. Λ is some physical scale such as Λ_{QCD}. One then has

$$a \underset{g\to 0}{\sim} \Lambda^{-1}\,[\, b_1\, g^2(a)]^{(-b^2/2b_1)} \exp(\frac{-1}{2b_1\, g^2(a)}) \ .$$

Note that again a continuous time version of this lattice theory could have been taken [6].

Renormalization and approach to the continuum limit

In the above theory we saw that one must make the bare parameters depend on the lattice spacing in a well defined way in order to remove the cut-off. This is a general feature which corresponds to renormalizing the coupling constants. It can be interpreted in the language of statistical physics as approaching a second order phase transition. A scale in the lattice theory which is typically the mass gap must become infinite in lattice units at this critical point. The nature of the divergence of this scale is given by the renormalization group: it tells us how to extrapolate the results with a finite but large cut-off to the infinite cutoff limit. However, the absolute magnitude of this scale which is the quantity of interest for the field theory must be determined by other means. We now review the various scenarios for estimating the magnitudes of such scales and the methods developed for extrapolating them to the continuum limit.

The first step is to find the location of the critical points in bare coupling space. This is done by determining the phases of the theory as a function of the parameters. The edge of this phase diagram usually corresponds to trivial or soluble theories and can give information about the phase structure near the boundaries. It may be that there are several critical points, so one must determine which one describes the quantum field theory one is studying. When this is done, one is to approach the chosen critical point from some direction in coupling space. This direction will be specified by the properties of the phases near the critical point. For instance, if the quantum field theory is to have a mass gap, the critical point is to be approached from a disordered phase where the correlation functions fall off exponentially.

Although the physical quantities of the quantum field theory must be obtained by going arbitrarily close to the critical point, qualitative and semi-quantitative information can be obtained further away from the critical point. Thus it is useful to choose a path or trajectory in coupling space which starts far away from the critical point and determine the absolute magnitudes of relevant quantities along such a trajectory. If the path does not cross any phase boundaries, the symmetries of the ground state of the quantum field theory can be read off from any point along this path, in particular in the region of small cut-off which can be analyzed very reliably. For definiteness, consider $SU(3)$ pure gauge theory which has only one coupling constant g. There is a single critical point and it is at $g=0$. At $g \simeq \infty$, perturbation theory in g^{-1} (strong coupling) can be used to determine the scales of observables in lattice units. This kind of perturbation theory is called a high temperature expansion in statistical physics. For $g \simeq \infty$, the lattice spacing a is large (the lattice provides only a very coarse grid), so the quantities of physical interest such as the mass gap are higher than the cut-off. For instance in QCD, the hadron radii will be smaller than the lattice spacing. The next section will discuss these perturbation expansions in greater detail. As g decreases and becomes of order 1, the g^{-1} series becomes misbehaved. This breakdown shows up for instance by a sensitive dependence of the observable on the order of the (truncated) series. The corresponding correlation length will be of order 1. Beyond this region, one resorts to numerical estimates by using Monte-Carlo simulations. These permit one in principle to go as close to the critical point as one wants. However, this requires vast computing power because the number of lattice sites must become very large, and also the dynamics of the simulation shows critical slow-down typical of a second order phase transition. Thus in practice one must extrapolate the results of small correlation lengths to the infinite correlation length limit.* The renormalization group gives the dependence of the masses on g near the critical point $g=0$. One can reliably extrapolate when deviations from "scaling" as given by eq. (II.2.10) are small. This will not always be the case in practice because g will still be of order 1 when the Monte-Carlo becomes unreliable. Luckily for four dimensional gauge theories it seems that scaling according to eq. (II.2.10) does set in rapidly so that correlation lengths of two to three are perhaps sufficient if one is satisfied with 20% accuracy. It is still useful to improve upon the above extrapolation method. Since g is still large, it is better to use in eq. (II.2.10) the beta function to higher orders in g. This is precisely what is effectively done if one calculates only ratios of masses: the masses have corrections to scaling that go as g^2 whereas ratios of masses have corrections in a that go as $(\frac{a}{\xi})^2$, i.e., as $\exp(-b_1^{-1} \cdot g^{-2})$.

3. THE HIGH TEMPERATURE OR STRONG COUPLING EXPANSION

In the regime where g is large, the correlation length is smaller than the lattice spacing, so the fields are dominantly random. To be specific, consider the $SU(N)$ pure gauge theory on a space-time lattice. Its generating function is

$$Z = \int [dU] \, e^{\beta \sum_{\square} \mathrm{Re}\, tr(UUU^\dagger U^\dagger)} \tag{II.3.1}$$

* For four dimensional gauge theories, the Monte-Carlo has been useful in the range $1<\xi<3$ where ξ is the correlation length, whereas for lower dimensional models one can go beyond $\xi=10$.

with $\beta = N \cdot g^{-2}$ and we have normalized our traces so that $tr\,1 = 1$. $\sum_{\blacksquare}$ means that one is to sum over all plaquettes. β can be thought of as the inverse temperature of the statistical mechanics problem, so that the expansion in powers of β gives a high temperature series. For small β, expand the exponential:

$$Z = \int [dU] \sum_k \frac{\beta^k}{k!} \left(\sum_{\blacksquare} \mathrm{Re}\; tr(UUU^\dagger U^\dagger) \right)^k \; . \tag{II.3.2}$$

One can determine the observables of interest by using this β series. To calculate a term of the series, we need the following group integrals:

$$\int [dU]\; U_{i,j}\; U^\dagger_{k,l} \;=\; \frac{\delta_{i,l} \cdot \delta_{j,k}}{N} \tag{II.3.3}$$

$$\int [dU]\; U_{i,j}\; U_{k,l} \;\cdots\; U_{y,z} \;=\; \frac{\varepsilon^{i,k,\ldots y} \;.\; \varepsilon^{j,l\,\cdots z}}{N!} \tag{II.3.4}$$

for a product of N matrix elements, and $\varepsilon^{i,k,\cdots y}$ is totally antisymmetric.

Let us illustrate this expansion by determining the average plaquette in strong coupling. One has

$$< tr(UUU^\dagger U^\dagger) > = \frac{\int [dU]\; tr(UUU^\dagger U^\dagger)\; e^{\beta \sum_{\blacksquare} \mathrm{Re} tr(UUU^\dagger U^\dagger)}}{\int [dU]\; e^{\beta \sum_{\blacksquare} \mathrm{Re}\, tr(UUU^\dagger U^\dagger)}} \tag{II.3.5}$$

The denominator is $1 + O(\beta^2)$ whereas the numerator is of order β :

$$< tr(UUU^\dagger U^\dagger) > = \frac{\beta}{2} \int [dU]\; tr(U_{\blacksquare})\; tr(U_{\blacksquare}^\dagger) = \frac{\beta}{2N} \; . \tag{II.3.6}$$

Each term of the series in eq. (II.3.2) can be interpreted as a graph on the lattice. Contractions in group indices must be such that a singlet of the group can be made at each link, which means that the only graphs that contribute are those which correspond to 'tiling' the observable.

More sophisticated methods of expansion are useful if one wants to go to relatively high orders in this series. Fig. 1 shows the result for the string tension to order β^{12} [7] and fig. 2 shows those for the glueball mass to order β^8 [8]. Such improved methods use the cluster expansions developed in statistical physics [9]. For the case of gauge theories, they make use of character expansions which correspond to Fourier analysis on the group manifold. The expansion parameter will be a function of β which goes like β as $\beta \rightarrow 0$. For instance, the parameter for the Z_2 model is $\tanh(\beta)$. These expansions are also more reliable than the 'naive' expansion because they effectively include higher order graphs so the series can be truncated sooner. Note that these strong coupling series can typically be taken to much higher orders than in weak coupling perturbation theory.

These high temperature series are guaranteed to converge in a finite region about $\beta=0$. The behavior at $\beta=0$ is analytic as can be shown by using bounds on the number of graphs at fixed order in β and the fact that the range of the variables is bounded [10]. This is to be contrasted with the case of weak coupling expansions which are only asymptotic. For some theories, the region of convergence of the high temperature series may reach all the way to the critical point of interest. Thus if one could sum the series to all orders, the theory would be solved. This has been done for the Gross-Neveu model at large N [11]. However resumming such series will not be possible for realistic theories, and at best one can resum a subset of graphs. For most cases though, the series will not have such a large radius of convergence. Another problem is that the sum of the strong coupling series in the region of convergence does not uniquely determine the theory everywhere. There exist cases such as the Eguchi-Kawai and the quenched Eguchi-Kawai models where two theories have identical strong coupling expansions but are different in weak coupling [12]. For pure gauge $SU(N)$ theories, the strong coupling series seems to have a bad behavior at $g^{-2} \simeq .39\, N$ which is close to the estimated roughening transition [13]. Thus a major problem is that of extrapolating results of truncated series beyond their radius of convergence. Pade approximants have been used extensively and very successfully in statistical physics where the correlation length behaves as $\xi \sim | T-T_c |^{\nu}$. These same methods have been used for gauge theories but they do not give as good results, probably because the scaling law at the critical point is very different.

If one is to include fermions into the theory, a great complication arises. Even if the coupling constant is very large, there are very light particles such as the pion if the bare mass of the quarks is small. Since high temperature series depend on a small correlation length, their use here is problematical. However, if the quark part of the theory is taken into account exactly, this problem can be overcome, and we shall see in Chapters 3 and 4 that the first few orders of the expansion are still tractable.

Let us also mention that a g^{-1} expansion can also be used for the Hamiltonian lattice theory. The techniques (Raleigh Schrodinger perturbation theory) are of course very different. The glueball spectrum at strong coupling has been studied within this approach in reference [14].

4. PROPERTIES OF LATTICE QCD

The behavior of lattice QCD is now rather well understood. The pure gauge theory confines quarks at strong enough coupling and there is good numerical evidence that there is no deconfining phase transition as one goes from strong to weak coupling [15]. The string tension follows the scaling law of eq. (II.2.10) quite early, and thus the lattice theory incorporates both confinement and asymptotic freedom. The numerical [16] and strong coupling [7] estimates of the string tension are shown in fig. 1. As mentioned in section 1, there may now be a rigorous proof that pure gauge $SU(2)$ confines. It is expected that this proof can be generalized to all $SU(N)$'s.*

The inclusion of quark fields significantly complicates the dynamics of the theory. Much still remains to be done here. The lattice is not as well suited to this problem mainly because of the difficulties related to chiral symmetries. It is expected that one will have to go to larger correlations lengths than for the pure gauge theory to get reliable quantitative results. So far, almost all

* For $N \geq 4$, pure gauge $SU(N)$ theory on the lattice has a first order phase transition on the Wilson axis, but this is not expected to spoil confinement.

numerical studies have been done in the "quenched" approximation where only valence quarks are included in the calculations. Fig. 3 shows some results for the $SU(3)$ QCD spectrum in this approximation. These results are to be contrasted with those from strong coupling estimates either in the Hamiltonian [17] or Euclidean [18] formalisms. Some of the formalisms used such as the Euclidean lattice theory with Kogut-Susskind fermions have a continuous chiral symmetry. In these cases, the chiral symmetry dynamically breaks and the pion appears as a Goldstone boson. The numerical results give $f_\pi \sim 150$ mev. (experiment gives 93 mev.) in the current algebra relation $m_\pi^2 f_\pi^2 = \langle\bar{\psi}\psi\rangle m_q$, but this quantity is more sensitive to the cut-off than the masses are [19].

Let us also mention some of the properties of lattice QCD at finite temperature. The pure gauge $SU(3)$ theory has a first order deconfining phase transition at finite temperature [20]. Below T_c, the free energy of an isolated quark in the fundamental representation is infinite, whereas above T_c, this free energy becomes finite and quarks are liberated. This first order phase transition also has the effect of restoring the chiral symmetry. However, there is good evidence that confinement of fundamental quarks is not a necessary condition for chiral symmetry breaking to occur [21]. It is possible that dynamical quarks substantially change this picture, especially if they are light [22].

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Figure Captions

[1] String tension times the lattice spacing squared as a function of coupling:

a) Strong coupling results for $SU(2)$ [7]. The solid line is the result of a cluster expansion up to 12'th order. The dots are are from the first reference in [15].

b) Monte Carlo results the coefficient of the area law for various sized loops for $SU(3)$. The string tension is given by the envelope of the data points [16].

[2] The mass gap as a function of coupling for $SU(3)$. Shown are the strong coupling series to 8'th order and Pade approximants [8].

[3] Monte Carlo results for the $SU(3)$ spectrum in the quenched approximation at $\beta=6.0$, from Gupta and Patel, reference 3. The values are plotted as a function of the hopping parameter.

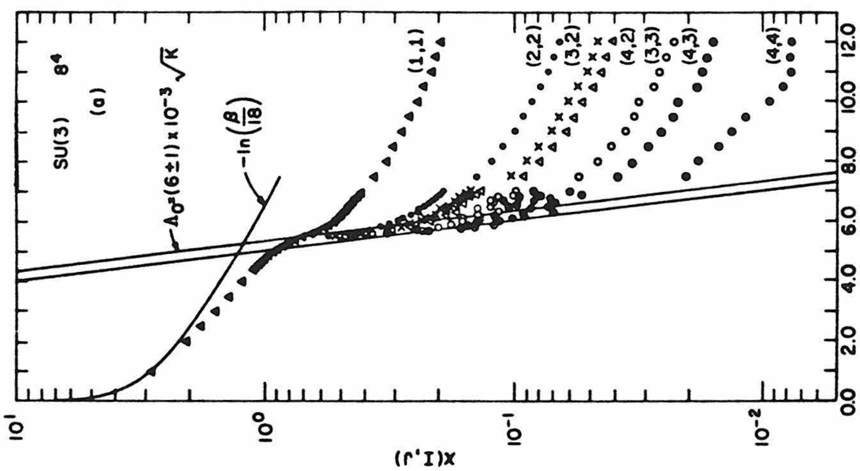
SU(3) 8⁴
(a)
Λ₀=(6±1)×10⁻³ √K
−ln(β/18)
(1,1)
(2,2)
(3,2)
(4,2)
(3,3)
(4,3)
(4,4)
χ(I,J)
β

Fig. 1b

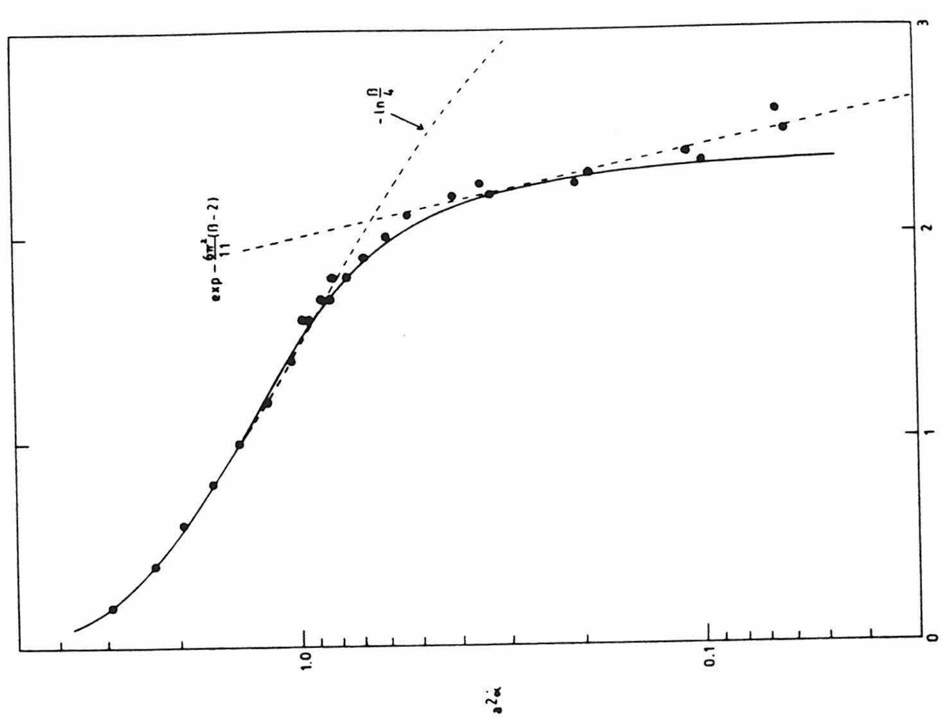
exp − 6π²/11 (β − 2)
− ln β/4
β
a²κ

Fig. 1a

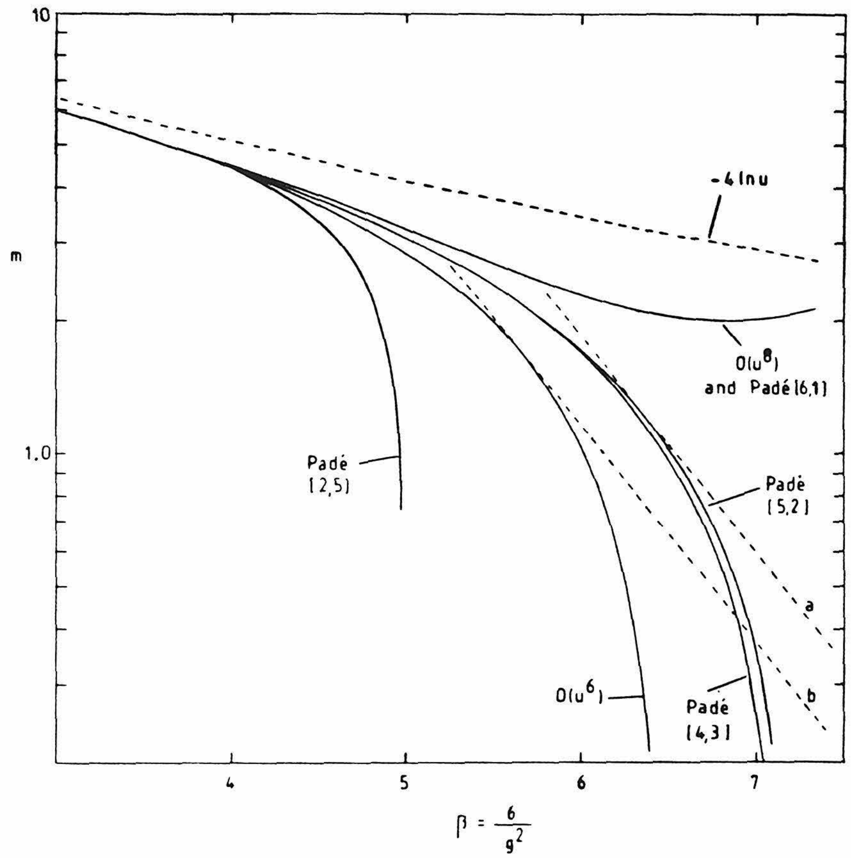
10
1.0
m
-4 ln u
O(u^8)
and Padé [6,1]
Padé
[2,5]
Padé
[5,2]
a
b
O(u^6)
Padé
[4,3]
4
5
6
7
β = 6/g^2

Fig. 2

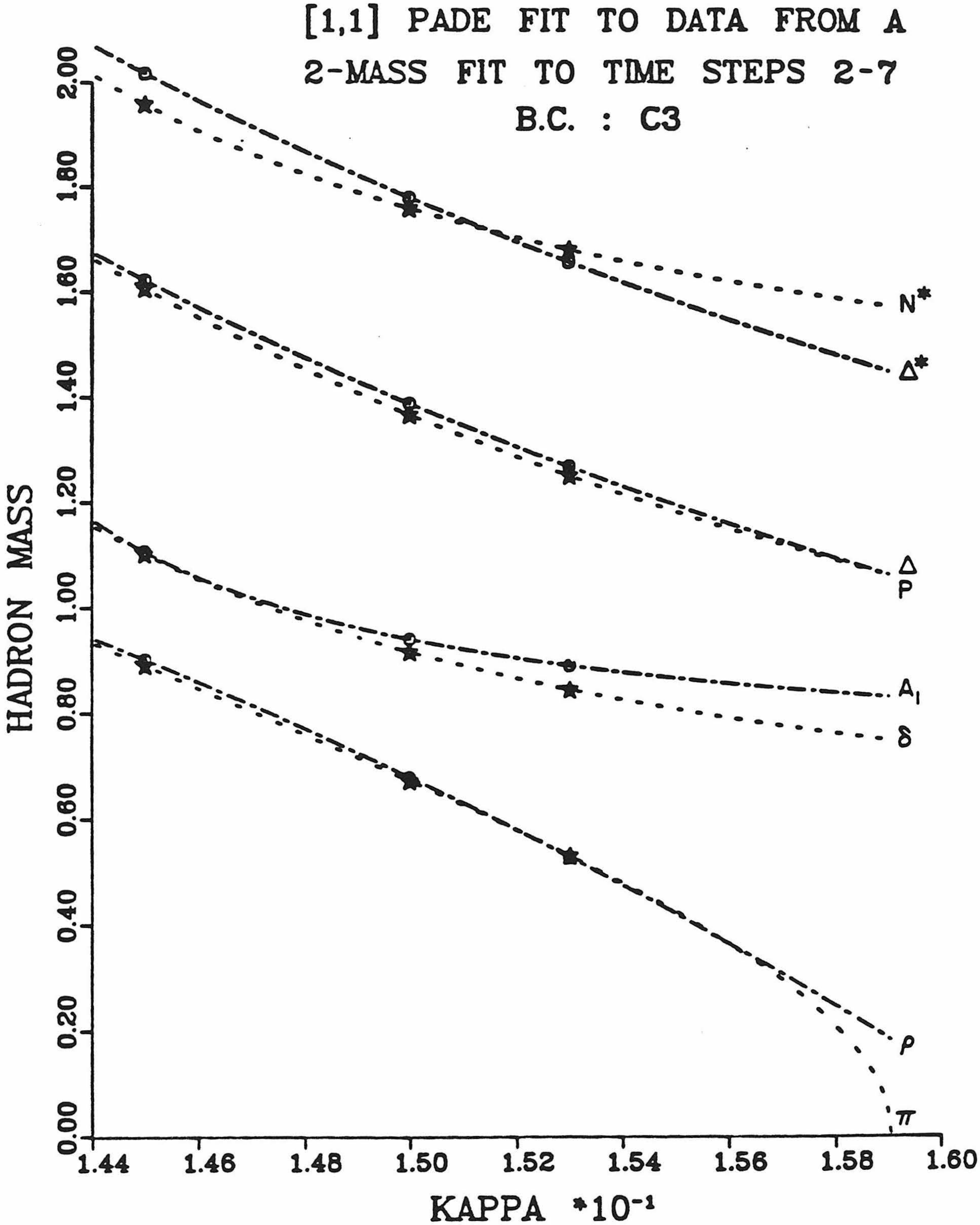
[1,1] PADE FIT TO DATA FROM A
2-MASS FIT TO TIME STEPS 2-7
B.C. : C3
HADRON MASS
0.00
0.20
0.40
0.60
0.80
1.00
1.20
1.40
1.60
1.80
2.00
1.44
1.46
1.48
1.50
1.52
1.54
1.56
1.58
1.60
KAPPA *10⁻¹
N*
Δ*
Δ
P
A1
δ
ρ
π

Fig. 3

Chapter 3

CHIRAL SYMMETRY BREAKING IN LATTICE GAUGE THEORIES

1. INTRODUCTION

It was mentioned earlier that one would like to understand better the reason why QCD generates the pattern of symmetry breaking it does. Since it is only a qualitative feature, we can hope to make some progress by studying models which need not necessarily give a very accurate description of the details of QCD. We will address the following two questions:

1) Is chiral symmetry breaking a generic phenomenon of gauge theories?
2) What is the mechanism responsible for chiral symmetry breaking?

This last question is of great importance for model building because chiral symmetries play a crucial role in dynamical symmetry breaking schemes and in models where quarks and leptons are composite. Such chiral symmetries would protect fermions from acquiring masses and this would build in a scale hierarchy as long as the symmetries do not spontaneously break. However some consistency conditions discovered by 't Hooft show that most gauge theories must see their chiral symmetries spontaneously broken. These are very strong and useful constraints for model building. One would hope that the models that do satisfy 't Hooft's conditions could be further studied on the lattice. Unfortunately, as showed by Nielsen and Ninomiya [1], the lattice regulator cannot preserve non-vector (left-right asymmetric) chiral symmetries without introducing extra unwanted fermion species of the opposite handedness.* Thus theories with unpaired Weyl spinors such as the weak interactions as well as most models of interest cannot be satisfactorily put

* However their theorem can be evaded, for instance by introducing long range forces. This has been used to investigate a class of haplon models in ref [2].

onto the lattice.

In this dissertation, we shall deal only with vector theories, which allows us to completely preserve the chiral symmetry. The price paid is a multiplicity of fermion species. Although this may affect some details of the dynamics, we expect that the mechanism responsible for chiral symmetry breaking will not be very sensitive to this feature. We shall argue that a sufficiently strong force, even if it is short range, is sufficient to break chiral symmetry. It thus seems that chiral symmetry breaking is generic to strong gauge forces. We study this by working at strong coupling for an $SU(N)$ gauge theory in the limit $N \to \infty$. The chiral theory we investigate has been previously considered by other groups: Klumberg-Stern, Morel, Napoly and Petersson [3], as well as by Hoek, Kawamoto and Smit [4]. They showed that at infinite bare coupling, the order parameter $<\bar{\psi}\psi>_{m_q=0}$ is non zero so that the chiral symmetry is spontaneously broken. Our results also determine $<\bar{\psi}\psi>_{m_q}$ for finite N but in the so called quenched approximation where one neglects the contribution of internal quark loops.

Some problems related to fermions on the lattice will first be discussed in section 2. Then we review some of the mechanisms which have been proposed for chiral symmetry breaking. In section 4, we present our calculation of the order parameter $<\bar{\psi}\psi>$ for an $SU(N)$ gauge theory with naive fermions on a hypercubical lattice in d dimensions at large N. The appendix deals with the extension of this calculation to higher orders in the inverse coupling constant.

2. CHIRAL SYMMETRIES

Spontaneously broken symmetries

Let us begin by reviewing some elementary facts about chiral symmetries. Consider first a single free Dirac fermion in continuum 4 dimensions. The Lagrangian is

$$L = \bar{\psi}_L \not{p} \, \psi_L + \bar{\psi}_R \not{p} \, \psi_R + m_q \, (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \qquad \text{(III.2.1)}$$

where we have used the left and right components of the fermion field which are the eigenstates of chirality: $\gamma_5 \psi_L = -\psi_L$ and $\gamma_5 \psi_R = \psi_R$. If $m_q = 0$, this one flavor Lagrangian has a $U_L(1) \times U_R(1)$ symmetry which corresponds to invariance under independent rotation of the left and right components of the field. The symmetries can also be thought of as invariance under rotation of the fields together (this is the vector symmetry $U_V(1)$) and under relative rotation of left and right components (this is the axial symmetry $U_A(1)$). The mass term explicitly breaks the $U_A(1)$ but preserves the $U_V(1)$ symmetry which is associated with fermion number. If one had n identical flavors instead of just one, the $U_L(1) \times U_R(1)$ symmetry would generalize to $U_L(n) \times U_R(n)$ and the mass terms would destroy the $U_A(n)$, leaving only the $U_V(n)$.

Suppose now that we take an $SU(N)$ gauge theory with n flavors of identical Dirac fermions:

$$L = \frac{1}{4} F_{\mu,\nu} F^{\nu,\mu} + \sum_{f=1}^{n} \{ \bar{\psi} \not{D} \, \psi + m_q \, \bar{\psi} \psi \} \ . \qquad \text{(III.2.2)}$$

where $\not{D} = \not{p} - ig \not{A}$ is the covariant derivative and we have suppressed flavor indices. When $m_q = 0$, the flavor and chiral symmetries of this Lagrangian are $U_L(n) \times U_R(n) = U_V(1) \times U_A(1) \times SU_V(n) \times SU_A(n)$. However, the $U_A(1)$ is 'anomalous', i.e., it disappears when the theory is regularized and it turns out not to be a symmetry of the quantum theory. What are the consequences of the remaining symmetries? The $U_V(1)$ corresponds to the conservation of the total fermion number. In QCD ($N=3$), hadrons are either baryons (fermion number 3) or mesons (fermion number 0), so that all physical states belong to representations of $U_V(1)$. What about $SU_V(n) \times SU_A(n)$? The hadrons also fall into multiplets corresponding to representations of $SU_V(n)$, but not so for $SU_A(n)$.* For systems with an infinite number of degrees of

* For simplicity, suppose that the quarks in the real world are massless.

freedom, symmetries can appear in the Nambu-Goldstone mode, and then physical states do not fall into representations of the symmetry group. Such symmetries are called 'hidden' or 'spontaneously broken.' Then Goldstone's theorem asserts that there must be a massless boson for each symmetry generator that is broken. If we take QCD with two flavors (up and down) of massless quarks, one obtains $n^2-1 = 3$ massless Goldstone bosons, π^+, π^0, π^-. $SU_V(2) \times SU_A(2)$ has spontaneously broken down to $SU_V(2)$. Associated with this breaking is a degeneracy of the vacuum: there exists a family of vacua which transform into each other under these $SU(2)$ chiral rotations. Each possible vacuum lives in a distinct Hilbert space which satisfies clustering. In the language of statistical mechanics, one identifies the Hilbert space with a pure phase. To find out whether a pure phase is chirally symmetric or not, it is necessary to introduce a non chirally invariant perturbation and determine the properties of this new theory as the perturbation is taken away. We have chosen this perturbation to be a mass term $m_q \bar{\psi}\psi = m_q(\bar{\psi}_l \psi_r + \bar{\psi}_r \psi_l)$ which rotates left components into right components and vice versa. It serves as an order parameter for the chiral symmetry. One must see how the system behaves as $m_q \to 0$. We will show that the resulting vacuum is not chirally invariant, $\lim_{m_q \to 0} <\bar{\psi}\psi>_{m_q} \neq 0$. Associated with the breaking of this continuous symmetry are Goldstone bosons as shown in Chapter 4. Before giving the details of the graphical method, we first discuss some problems related to putting fermions on the lattice.

Fermions on the lattice

Consider putting the previous theory of a free Dirac fermion on the lattice by discretizing the derivatives:

$$L \to \bar{\psi}_x \gamma^\mu \frac{(\psi_{x+\mu} - \psi_{x-\mu})}{2a} + m_q \bar{\psi}\psi \ . \tag{III.2.3}$$

The inverse propagator of this free theory on the lattice is

$$i \sum_{\mu} \gamma_\mu \sin(p_\mu \mathrm{a}) + m_q \mathrm{a} \quad , \quad -\frac{\pi}{\mathrm{a}} < p_\mu \le \frac{\pi}{\mathrm{a}} \quad . \qquad \text{(III.2.4)}$$

The corresponding dispersion relation is

$$\sum_{\mu} \sin^2 p_\mu \mathrm{a} + m_q^2 \mathrm{a}^2 = 0 \quad , \quad p_4 = iE \quad . \qquad \text{(III.2.5)}$$

At small energies, this gives $E^2\mathrm{a}^2 \simeq \sum_{i=1}^{3} \sin^2(p_i \mathrm{a}) + m_q^2 \mathrm{a}^2$, and this is shown in fig. 1 for two dimensions. This is to be compared with the case of a boson shown in fig. 2: $E^2\mathrm{a}^2 \simeq 4 \sum_{i=1}^{3} \sin^2(\frac{p_i \mathrm{a}}{2}) + m_q^2 \mathrm{a}^2$. For fermions, there exist very low energy states near $p = \frac{\pi}{\mathrm{a}}$, so that in the limit $\mathrm{a} \to 0$, we do not obtain the relativistic free theory we set out to discretize. The lattice has introduced additional degrees of freedom which appear in the low energy spectrum and can be interpreted as additional flavors in the continuum limit. Each corner of the Brillouin zone $p : 0, \frac{\pi}{\mathrm{a}}$ gives rise to one of these, so that in d dimensions, there are 2^d such flavors. This multiplicity of flavors on the lattice has been called the fermion doubling problem. This sickness is intrinsic to lattice fermion theories as was precisely enunciated by Nielsen and Ninomiya [1]. Their theorem can be briefly stated as follows:

"Any lattice theory which preserves the chiral charges of a compact continuous group and has only local interactions has a doubling problem".

Their proof relies on the topological structure of the lattice and has since been restated in a more geometrical form by Rabin [5].

The problem of the extra modes must nevertheless be dealt with. There have been several approaches:

1) Sacrifice the chiral symmetry. One can throw away all the chiral symmetries, e.g., by introducing into the Lagrangian a term which gives to the unwanted species masses comparable to the cutoff [6]. One can also keep only a subset of all the symmetries by thinning out the degrees of freedom of the Dirac spinor [7], and thus reducing the number of extra flavors.

2) Sacrifice the Osterwalder-Schrader inequalities. This can be done by explicitly breaking certain symmetries of the lattice, e.g., by destroying the cubic

symmetry [8] or by destroying all the space-time symmetries such as in a random lattice [9]. Another possibility is to make the interactions infinite range as in the Slac Lagrangian [10].

The problems associated with chiral symmetries should not be considered a disaster for lattice theories. On the contrary, they suggest that some continuum field theories may have problems. This is illustrated by the $U(1)$ anomaly mentioned where the symmetry of the continuum bare Lagrangian is not a symmetry of the theory. A lattice theory for which the $U_A(1)$ is a good symmetry will not have the right number of flavors and thus it will not give rise to a $U(1)$ problem. Another example is the $SU(2)$ chiral theory of Witten. It cannot be put onto the lattice, and in fact, it is ill-defined [11]. However these chiral problems do drastically complicate the program of determining by lattice methods the mechanism responsible for chiral symmetry breaking. One has several options. If one explicitly breaks the chiral symmetry on the lattice, one must study its restoration in the continuum limit where the terms which break the symmetries should be irrelevant. In particular, it seems that the Wilson formalism does lead to the restoration of chiral symmetries since in weak coupling perturbation theory, it correctly reproduces the $U(1)$ anomaly and PCAC [12]. For QCD, the chiral symmetry should reappear in the spontaneously broken form: the π-π scattering amplitude at zero four momentum should vanish and the results from current algebra should become valid. It is difficult to determine the symmetry breaking mechanism in this approach, and in particular a strong coupling analysis will not do. One could also look at some remnants of chiral symmetry such as there exists for Kogut-Susskind fermions in the Lagrangian formalism, and see if these are spontaneously broken. This has the advantage that it can be done at strong coupling. We have chosen to take the case where the symmetry is as difficult as possible to break. Naive fermions have all the chiral symmetries of the continuum (and unfortunately some extra ones). In fact, as explained below they have an $U(4)\times U(4)$ symmetry which gets spontaneously broken down to a $U_\Delta(4)$. Because of the multiplicity of flavors in this model, it is not phenomenologically realistic, but we hope that one can learn something about the mechanism responsible for chiral symmetry breaking independently of the doubling problem.

Consider the naive lattice action for a free Dirac fermion as given by eq. (III.2.3). Let o be an arbitrary origin on the lattice, and $T_{(o)}$ and $S_{(o)}$ be any $U(4)$ transformations in spinor space. For a site r on the lattice, $r = (r_1, r_2, r_3, r_4)$, define $\Delta_{(r)}$ to be

$$\Delta_{(r)} = \gamma_1^{r_1} \gamma_2^{r_2} \gamma_3^{r_3} \gamma_4^{r_4} \ .$$

Define space dependent transformations

$$T_{(r)} = \Delta_{(r)}^{-1} T_{(o)} \Delta_{(r)} \ ,$$

$$S_{(r)} = \Delta_{(r)}^{-1} S_{(o)} \Delta_{(r)} \ .$$

Then the action of eq. (III.2.3) with $m_q = 0$ is invariant under the transformations

$$\psi_{(r)} \to T_{(r)} \psi_{(r)}$$

$$\overline{\psi}_{(r)} \to \overline{\psi}_{(r)} S_{(r)}^{-1}$$

for even sites ($r_1 + r_2 + r_3 + r_4$ even), and

$$\psi_{(r)} \to S_{(r)} \psi_{(r)}$$

$$\overline{\psi}_{(r)} \to \overline{\psi}_{(r)} T_{(r)}^{-1}$$

for odd sites. Thus the global symmetry group of the naive action in the chiral limit is $U(4) \times U(4)$, which is a much larger group than the continuum theory had. When $m_q \neq 0$, we are left with a single $U_\Delta(4)$ symmetry which corresponds to $S_{(r)} = T_{(r)}$. The only smooth transformations which have a continuum interpretation are precisely the $U_V(1) \times U_A(1)$ of the continuum with 1 flavor. To have smoothness, we must have $S_{(r)} = T_{(r)} = T_{(o)}$, which indeed leaves only the usual $U_V(1)\times U_A(1)$. If we had n flavors, we would have instead $U_V(n)\times U_A(n)$. These symmetries are preserved after $SU(N)$ gauge fields are introduced, the $U_A(1)$ not being anomalous in this lattice model. To study chiral symmetry breaking, we will explicitly break the $U_A(1)$ by a small quark mass, and will take the limit $m_q \to 0$. In section 4, we shall show that the resulting theory at strong coupling has a vacuum which is not a singlet under chiral rotations, so that the $U_A(1)$ is spontaneously broken and a massless pion appears. For n flavors, the $U_A(1)\times SU_A(n)$ would spontaneously break. Actually, the mass term explicitly breaks a $U(4)$ symmetry so that one has 16 Goldstone bosons as will be shown in Chapter 4.

3. MECHANISMS FOR BREAKING CHIRAL SYMMETRIES

There have been many suggestions for how chiral symmetries spontaneously break. Historically, the first model where spontaneous chiral symmetry breaking was convincingly shown was the Nambu Jona-Lasinio model [13]. The theory involved no gauge bosons, only fermions with 4 point couplings. For a sufficiently large coupling constant, the Hartree-Fock approximation they used indicated that chiral symmetry should spontaneously break. Spontaneous chiral symmetry breaking can also be seen through perturbation theory about a chirally symmetric vacuum, for instance by the appearance of tachyons or by the breakdown of clustering. One can also show that the Greens functions can satisfy self consistent equations and yet not be chirally invariant [14]. All these models require a coupling constant which gets large. In 1979 , t' Hooft derived the anomaly and decoupling conditions which are a set of consistency equations that must be satisfied by confining theories if their chiral symmetries are <u>not</u> spontaneously broken [15]. These conditions are sufficient to prove that all $SU(N)$ vector theories with even N must spontaneously break their chiral symmetries. However, the previous evidence indicates that chiral symmetries spontaneously breakdown when the gauge force is sufficiently strong, even if there are no anomalies, and confinement is not necessary. This has been substantiated by the extensive numerical work of Kogut et al. [16] who have shown that a purely Coulombic force will break chiral symmetry if it is sufficiently strong. Thus we may expect, for instance, that all confining theories break chiral symmetry, since such theories have strong forces.

Let us now see how a strong gauge force could give rise to spontaneous chiral symmetry breaking. First, we follow the discussion of Casher [17]. Consider a chirally invariant two body force sufficiently strong to make an S wave $q-\bar{q}$ bound state. Within a semi classical picture, the q and $\bar{q}$ world lines must turn around when they reach the 'edge' of the bound state, but this is not possible because chirality is conserved and equal to helicity. In the vacuum of the quantum field theory, we have many virtual $q-\bar{q}$ pairs. To get around the above problem while minimizing the energy of the ground state, these pairs must be able to exchange their constituents. The exchanged particles must be correlated in helicity, with the result that there is a coherence of the vacuum pairs. This means that chiral symmetry breaking has occurred. One must ensure that the $q-\bar{q}$ pairs are long lived for the coherence not to decay. In two dimensions, the quark loops will make the coherence length finite because the chiral symmetry cannot break by Coleman's theorem.

This argument was put into a slightly different form by Banks and Casher [18]. They considered the propagator of a Dirac particle in a background field expressed as a path integral:

$$Q_{oo}^{-1} = \int_0^\infty dT\, e^{-m^2 T} \int_{x(0)=x(T)} dx(\tau)\, \exp\{-\int_0^T d\tau\, \frac{\dot{x}^2}{4}\} \times \exp[i\int dx_\mu A_\mu - \frac{1}{2}\int_0^T d\tau\, \sigma\cdot F] \qquad \text{(III.3.1)}$$

where we have dropped internal fermion loops and T labels the length of the quark path. The condition for chiral symmetry breaking is that $\lim_{m_q\to 0} \langle m_q\, Q_{oo}^{-1}\rangle \neq 0$ averaged over the gauge field distribution. This imposes that the contribution of the large T paths go as $T^{-\frac{1}{2}}$ before the factor $e^{-m^2 T}$ is introduced. Now for free field theory in d dimensions, the contribution goes as $T^{-\frac{d}{2}}$ so that the condition for chiral symmetry breaking is that the paths returning to the origin behave as if they were one dimensional. A lesser constraint which is sufficient is that the one dimensional paths contribute a finite fraction to this path integral. Banks and Casher argued that this should happen if there is a bound state. Although this is a very appealing picture, no one has yet proved its validity. They then studied the strongly coupled limit of the gauge theory on the lattice in the quenched approximation. We shall do this also, but with a different formalism for the fermions. As we shall see, the quark must always screen itself from the gauge fields, and in doing so, the chiral symmetry breaks. The contributions to $<\bar{\psi}\psi>$ can be thought of as quark paths that will necessarily become very long as $m_q \to 0$. From perturbation theory in the mass, any given path of length T gives a chiral rotation angle that goes as $m_q \cdot T$. The paths that contribute to $<\bar{\psi}\psi>$ thus have a length that goes as m_q^{-1}.

What happens as the strength of the interaction decreases? The paths open up and the phases for various paths tend to cancel so $<\bar{\psi}\psi>$ decreases. In free field theory these contributions cancel completely as $m_q \to 0$. If the theory is analytic at zero coupling, we expect the chiral symmetry to be restored at some finite value of the coupling. This is probably what happens for QED as shown numerically by Kogut et al. [16]. They considered non-compact QED on the lattice in the quenched approximation. This lattice theory has a purely Coulombic force and thus is non-confining. At zero coupling, one has a free field theory of massless fermions with no bound states, and the vacuum is chirally symmetric. As the gauge coupling g increases, the fermions interact more strongly via the coulomb force, but they stay massless because they are protected by the chiral symmetry. When the coupling becomes sufficiently strong, a bound state forms. It is made of massless fermions, and is itself massless. As g increases from this critical value g_c, the radius of the bound state decreases from infinity. Above g_c, chiral symmetry is spontaneously broken, the bound state is in fact the Goldstone boson so it stays massless, and the fermions dynamically acquire a mass due to $<\bar{\psi}\psi> \neq 0$. The phase transition at g_c is continuous, that is second order or higher. The inclusion of quark loops may change the order of the transition; it should make it smoother since screening by virtual quarks allows a $q-\bar{q}$ pair to separate even at infinite coupling, and thus it is more difficult to make a bound state. Note that to all orders in weak coupling perturbation theory, the vacuum is chirally symmetric. For a confining theory like QCD, one looses analyticity at zero coupling, and the spontaneously broken phase probably extends all the way to $g = 0$.

4. THE ORDER PARAMETER AT STRONG COUPLING AND LARGE N

Our previous discussion indicates that strong forces are most probably responsible for chiral symmetry breakdown. We would like a model with strong forces where we have a good handle on the dynamics. Since such a regime is non perturbative, we shall study it on the lattice. We shall calculate, for an $SU(N)$ lattice gauge theory, the quantity $<\bar{\psi}\psi>$ at infinite coupling, and some corrections in the inverse coupling constant (see the appendix). We neglect internal quark loops. In two dimensions, these loops in fact restore the chiral invariance of the ground state by Coleman's theorem.

Let us first summarize previous results for studies of chiral symmetry breaking using chirally symmetric lattice fermions. In the Hamiltonian theory, the ground state at infinite coupling is known exactly only for Kogut-Susskind fermions [19]. For naive fermions, several different approximation schemes indicate that the vacuum is not chirally symmetric [20]. Slac fermions give rise to the same qualitative picture [21]. In the action formulation, Blairon et al. [22] have used a mean field technique which can be viewed as a $\frac{1}{d}$ limit (d is the dimension of space-time). It is worthwhile to improve upon this result because symmetries generically break in large dimension. A non graphical approach was later used by Klumberg-Stern et al. [3] and by Kawamoto et al. [4] for all d. They considered the large N theory (where quark loops are suppressed) and determined $<\bar{\psi}\psi>$ at infinite coupling, $\lambda = Ng^2$. They found

$$<\bar{\psi}\psi>_{m_q} = \frac{d\sqrt{2d-1+m_q^2}-(d-1)m_q}{d^2+m_q^2} \quad . \tag{III.4.1}$$

Interestingly, this is a decreasing function of d so that a $\frac{1}{d}$ expansion is perhaps more reliable than one would naturally expect. The methods used were an exact evaluation of the generating function of the lattice gauge theory at infinite coupling with arbitrary sources. A more detailed discussion is given in Chapter 4. Graphical techniques on the other hand have the advantage of making the mechanism of chiral symmetry breaking clearer and the calculation is much more straightforward. It also allows one to consider other types of quarks, e.g., in other representations, and one can perhaps go to higher orders in the strong coupling expansion. For clarity, we first first present the calculation in the $\frac{1}{d}$ expansion, and then show the exact determination of $<\bar{\psi}\psi>$.

The order parameter

The lattice theory is defined on a d dimensional hypercubical lattice of spacing a (which we set to 1 in the following) by the generating functional:

$$Z = \int [d\bar{\psi}][d\psi][dU] \; e^{-S_{Y-M}} \; e^{-S_{matter}} . \tag{III.4.2}$$

The Yang-Mills degrees of freedom are embodied by $SU(N)$ or $U(N)$ matrices on the links $(r, \hat{n})$; we use the standard Wilson action for these fields:

$$S_{Y-M} = -\frac{N}{2 \cdot g^2} \sum_{\blacksquare} tr \; (\; U_{\blacksquare} + h.c. \;) \quad , \quad \lambda = Ng^2 \; , \tag{III.4.3}$$

where $U_{\blacksquare}$ is the product of matrices around a plaquette. The large N limit is obtained by $N \to \infty$, λ fixed. For ease of presentation, we shall consider naive fermions, but the value of $<\bar{\psi}\psi>_{m_q}$ for Kogut-Susskind fermions follows if no internal quark loops are included. If we call $\not{D}$ the discrete lattice derivative, we have

$$S_{matter} = \bar{\psi}(\not{D}+m_q)\psi = \tag{III.4.4}$$

$$\sum_{r,n} \bar{\psi}(r)\frac{\not{n}}{2}U(r,n)\psi(r+n) + \sum_{r} m_q \sum_{r} \bar{\psi}(r)\psi(r) .$$

The matter fields $\bar{\psi},\psi$ live on sites r and have spin and color degrees of freedom. This theory is gauge invariant and has a $U_V(C)\times U_A(C)$ symmetry if $m_q=0$ (C is the number of spin components in d dimensions). We choose the conventions

$$\not{n} = n^{\mu}\gamma_{\mu} \quad ; \quad [\gamma_{\mu},\gamma_{\nu}]_+ = 2\delta_{\mu,\nu} \quad ; \quad tr(1)=1 \quad . \tag{III.4.5}$$

We wish to find $<\bar{\psi}\psi>_{m_q}$ as $m_q\to 0$ since it is an order parameter of the chiral symmetry. To do this, we shall expand Z in powers of m_q^{-1}, and then expand each of those terms in powers of λ^{-1}. It will be necessary to resum the $\frac{1}{m_q}$ series exactly. The vacuum expectation value of $\bar{\psi}_o\psi_o$ is given by

$$<\bar{\psi}_o\psi_o>_{m_q} = \frac{\int[dU]\,\det(\not{D}+m_q)\,(\not{D}+m_q)^{-1}_{o,o}\quad e^{-S_{Y-M}}}{\int[dU]\,\det(\not{D}+m_q)\quad e^{-S_{Y-M}}} \tag{III.4.6}$$

where we have done the integration over the the Grassmann fields $\bar{\psi}$ and ψ, and o is the origin on the lattice. Let us expand $(\not{D}+m_q)$ in powers of m_q^{-1} :

$$(\not{D}+m_q)^{-1} = \frac{1}{m_q}\;(1-\frac{\not{D}}{m_q}+\frac{\not{D}^2}{m_q^2}-\frac{\not{D}^3}{m_q^3}+\;\cdots\;) \;. \tag{III.4.7}$$

Since $\not{D}$ is a bounded operator, this series converges for $m_q > d$. Expand $\det(\not{D}+m_q) = e^{tr\,\log(\not{D}+m_q)}$ in powers of m_q also. These series lead to products of terms like

$$\frac{1}{m_q^k}\,tr(\Pi_C U)\;tr'(\Pi_C \not{n}) \;.$$

We shall call them graphs since they can be represented by a set of closed oriented paths on the lattice (see fig. 3). For each such graph, associate a factor of $\frac{1}{m}$ for each site along the path, and the matrix factors $-\frac{\gamma}{2}$ and $U(r,\hat{n})$ for each link $(r,\hat{n})$. One is then to take the trace over spin and color. To calculate $<\bar{\psi}\psi>_{m_q}$, we are to sum terms of the form

$$\int[dU] tr(\Pi_{C_1}U) tr(\Pi_{C_1}\gamma_1) tr(\Pi_{C_2}U) \cdots tr(\Pi_{C_l}U) tr(\Pi_{C_l}\gamma_l)\ e^{-S_{Y-M}}$$

where the C_i are Wilson loops and the last trace comes from the expansion of $(\not{D}+m_q)^{-1}$ so that the loop C_l is to begin at the origin. We now show how to calculate this at $N=\lambda=\infty$.

Imagine expanding $e^{-S_{Y-M}}$ in powers of $\frac{N}{g^2}$. Each non trivial color trace leads to a $\frac{1}{N^2}$ suppression, so the terms which contain a power of $\frac{1}{g^2}$ will vanish as $\lambda\to\infty$. Thus at $\lambda=\infty$ we can formally set $S_{Y-M}=0$. In this limit, the only nonvanishing contributions to eq. (III.4.6) come from graphs which traverse each link 0 times modulo N. (Different directions are counted with opposite sign.) Consider now Wilson loops that obey this constraint and in addition satisfy $\Pi_C U=1$ for all values of the U's. Such graphs are called "tree" graphs because they enclose zero area. A typical tree graph is drawn in fig. 4. Classify the graphs coming from eq. (III.4.6) into trees and non-trees. In the limit of large N, one has the factorization property

$$\int[dU]\ tr(U_1)\ tr(U_2) = \int[dU]\ tr(U_1)\cdot\int[dU]\ tr(U_2) + O(\frac{1}{N^2}) \quad .$$

The graphs which are not tree graphs lead to contributions to $<\bar{\psi}\psi>_{m_q}$ which are of higher order in $\frac{1}{N}$ than what one gets for tree graphs. This is true for all graphs which involve any non trivial trace over the gauge degrees of freedom. So to this order ($\lambda=N=\infty$), we need only consider tree graphs for $(\not{D}+m_q)^{-1}$ and the numerator of eq. (III.4.6) factors, leaving us with

$$<\bar{\psi}_o\psi_o>_{m_q} = \frac{C}{m_q}\sum_{L=0}^{\infty}(-1)^L\cdot\frac{A(L)}{(2m_q)^{2L}} \quad . \qquad \text{(III.4.8)}$$

The $(-1)^L$ factor comes from the spin traces, C is the number of spin components in d dimensions and $A(L)$ is the number of tree graphs of length $2L$ which have their origin at o. This is the series in m_q^{-1} which we will first sum in the $\frac{1}{d}$ limit and then exactly in the last section.

The large d limit

The above sum can be easily evaluated at large d. This was done using a mean field theory by Blairon et al. [22]. We shall do it differently in order to prepare for the next section. Set up a recurrence which enumerates the paths of length $L+1$ in terms of the number of paths of length L. Consider first the trees of the type in fig. 5 denoted by $A_1(L)$. They correspond to a tree with a single trunk and arbitrary branches so that $A_1(L+1) = 2dA(L)$ or equivalently

$$A_1(L+1) = 2d \sum_{\sum l_i = L} A_1(l_1) \cdots A_1(l_p) \quad , \tag{III.4.9}$$

where $2d$ counts the number of choices of directions for the trunk. There are other graphs as in fig. 6 with t trunks which one must also count. Since they can be considered as t successive graphs of the previous type, one has

$$A_t(L) = \sum_{\sum l_i = L} A_1(l_1) \cdots A_1(l_t) \quad .$$

To solve these recursion relations, it is convenient to use the Laplace transforms:

$$W_A(x) = \sum_{l=0}^{\infty} A(l)\, x^l \quad .$$

Then the above recursion relations become equations relating the various W's:

$$W_{A_1} = 2dx W_A = \frac{2dx}{1-W_{A_1}} \qquad ; \qquad W_{A_t} = W_{A_1}{}^t \quad .$$

We must sum all possible graphs with all values of t:

$$W_A = \sum_{t=0}^{\infty} W_{A_t} = \frac{1}{1-W_{A_1}} \quad .$$

Thus we have a self consistency equation for W_A :

$$W_A = \frac{1}{1-2dx\,W_A} \quad , \tag{III.4.10}$$

so that

$$W_A(x) = \frac{1-\sqrt{1-8dx}}{4dx} \quad ,$$

which gives for the order parameter

$$<\bar{\psi}\psi> = C\cdot\frac{\sqrt{m_q^2+2d}\; - m_q}{d} \quad . \tag{III.4.11}$$

As $m_q \to 0$, $<\bar{\psi}\psi> \to C\sqrt{\frac{2}{d}}$ **and chiral symmetry is spontaneously bro-**
ken.

If d is finite, this estimate is inexact. The problem is that one has to make sure that each graph on the lattice is counted once and only once. The recursion relation in eq. (III.4.9) does not conserve the character of A_1, i.e. the graphs of length $L+1$ constructed from the shorter ones will not necessarily have a single trunk. The branches on the stump in fig. 7 can come back to the origin and add new trunks to the graph. One must take into account this problem when constructing the graphs recursively. The next section shows how this can be done by imposing appropriate constraints on these branches.

Exact treatment in finite d

A single graph of length $2L$ is determined by its origin and a sequence of directions :

$$\hat{n}_1, \hat{n}_2, \hat{n}_3, \cdots \hat{n}_{2L}.$$

It is a tree (zero area) if $\Pi_C U = 1$ for all U. To sum the series (III.4.8), it is useful to introduce the notion of an irreducible tree graph at the origin (ITGo).

A graph is called irreducible at o if the above sequence cannot be truncated at $\hat{n}_k$ and still give a tree graph $\hat{n}_1, \hat{n}_2, \hat{n}_3, \cdots \hat{n}_k$ at o. An example of such a graph is shown in fig. 5. The ITGo are the building blocks for making general tree graphs. Any tree graph can be uniquely specified by its "first" ITGo and by the rest of its sequence which is a general tree graph as illustrated in fig. 6. Then one has the recursion relation

$$A(L) = \sum_{l=1}^{L} I(l)\, A(L-l) \qquad L\geq 1 \qquad A(0)=1 \qquad \text{(III.4.12)}$$

where $I(l)$ is the number of ITGo of length $2l$. An analogous recurrence relation can be derived for $I(l)$ by noticing that each ITGo is specified by its first step $\hat{n}_1$ followed by a sequence of irreducible tree graphs at $o+\hat{n}_1$, and finally by its last step, necessarily $-\hat{n}_1$. This corresponds to building ITGo by grafting irreducible tree graphs onto a stump as was done in fig. 5. In such a description, each of the ITG at $o+\hat{n}_1$ "begins" in a direction different from $-n_1$ because the overall graph is irreducible at o. If we take into account this constraint which was disregarded in the previous section, the labeling is one to one, so one has

$$I(L) = 2d \cdot \sum_{\sum l_i = L-1} I(l_1)\, I(l_2) \,....\, I(l_p) \cdot \frac{(2d-1)^p}{(2d)^p} \qquad \text{(III.4.13)}$$

with $L \geq 2$ and $l_i > 0$. The initial conditions are $I(0)=0$, $I(1)=2d$. To solve this recurrence relation, consider the generating function

$$W_I(x) = \sum_{0}^{\infty} I(l)\ x^l \quad .$$

Eq. (III.4.13) then gives

$$W_I(x) = 2d \cdot x \sum_{k=0}^{\infty} \frac{(2d-1)^k}{(2d)^k} \cdot W_I^k(x)$$

so that

$$W_I(x) = d \cdot \frac{1-\sqrt{1-(2d-1)\cdot 4\cdot x}}{(2d-1)} \quad . \qquad \text{(III.4.14)}$$

This is essentially the generating function of the l'^{th} Catalan number $\frac{1}{2l+1}\frac{(2l)!}{l!l!}$ which often appears in problems dealing with trees [23]. Now we are ready to solve eq. (III.4.8). If W_A is the generating function for $A(l)$, eq.

(III.4.12) leads to

$$W_A(x) = 1 + W_A(x)\, W_I(x) \quad .$$

We could have obtained this result by noticing that all tree graphs are sequences of ITGo :

$$A(L) = \sum_{l_1+l_2 \cdots + l_k = L} I(l_1)\, I(l_2) \ldots .\, I(l_k)$$

so that

$$W_A(x) = \sum_1^{\infty} W_I^k(x) = \frac{1}{1-W_I(x)} \quad . \qquad \text{(III.4.15)}$$

Finally, for $N=\lambda=\infty$,

$$<\bar{\psi}_o \psi_o>_{m_q} = \frac{C}{m_q}\, W_A\left(\frac{-1}{4\, m_q^2}\right) = \qquad \text{(III.4.16)}$$

$$C \cdot \frac{d \cdot \sqrt{2d-1+m_q^2} - (d-1)m_q}{d^2+m_q^2} \quad .$$

As $m_q \to 0$, $<\bar{\psi}\psi>_{m_q} \to \frac{C}{d}\sqrt{2d-1}$, showing that the continuous chiral symmetry has been spontaneously broken.

Some comments are in order.

(i) The analytic extension to masses outside of the region of convergence of the series (III.4.8) is not guaranteed to give us the correct answer. It is possible that one obtains an unphysical value by reaching a metastable state instead of the true equilibrium state.

(ii) It may be interesting to note that the above calculation is unchanged if the theory is defined on a finite lattice with periodic boundary conditions on the Grassmann fields (instead of the antiperiodic as one should) because $A(l)$ remains the same. This occurs because we are in the $N=\infty$ case: it is well known that the finite volume pure gauge model (reduced model) can be identified with the infinite volume model [24].

(iii) Finally, one should show that the $\frac{1}{N}$ corrections are finite. Blairon et al. [22] have shown that this is indeed the case for $d>2$, so the continuous chiral symmetry will break for strong enough coupling λ in $d>2$.

5. Comments and Conclusions

Other Vacua

In the model studied, chiral symmetry was shown to spontaneously break. As a consequence, there should be a family of degenerate vacua which transform into each other under $U_A(1)$ chiral rotations. The vacuum we obtained was selected by the mass perturbation $m_q \bar{\psi}\psi$. To get a rotated vacuum, we can apply the perturbation $m_q \,\bar{\psi}\, e^{i\alpha\gamma_5}\,\psi$. This should simply have the effect of rotating the order parameter. This fact can be explicitly checked with the above graphical expansion.*

Consider the order parameter for the theory with the new perturbation $m_q \sum \bar{\psi}\, e^{i\alpha\gamma_5}\,\psi$:

$$<\bar{\psi}_o \psi_o>_{m_q} = \frac{\int [dU] \det(\not{D}+m_q e^{i\alpha\gamma_5})\, (\not{D}+m_q e^{i\alpha\gamma_5})^{-1}_{o,o} \quad e^{-S_{Y-M}}}{\int [dU] \det(\not{D}+m_q e^{i\alpha\gamma_5}) \quad e^{-S_{Y-M}}} .$$

The inverse propagator can be expanded in m_q^{-1}:

$$(\not{D}+m_q e^{i\alpha\gamma_5})^{-1} = \frac{e^{-i\alpha\gamma_5}}{m_q} \left(1 - \frac{e^{-i\alpha\gamma_5}\not{D}}{m_q} + \frac{(e^{-i\alpha\gamma_5}\not{D})^2}{m_q^2} - \cdots\right) .$$

Using the algebra of the γ matrices, the fact that γ_5 anticommutes with $\not{D}$, and that only even terms will contribute to $<\bar{\psi}\psi>$, this series reduces to

$$(\not{D}+m_q\, e^{i\alpha\gamma_5})^{-1}_{oo} = \frac{e^{-i\alpha\gamma_5}}{m_q} \left(1 - \frac{\not{D}}{m_q} + \frac{\not{D}^2}{m_q^2} - \frac{\not{D}^3}{m_q^3} + \cdots\right)_{oo} .$$

This shows that

* We are grateful to N. Christ for pointing this out.

$$<\bar{\psi}\psi>_\alpha = <\bar{\psi}\, e^{-i\alpha\gamma_5}\,\psi>_{\alpha=0} \quad ,$$

i.e., the order parameter in the vacuum obtained by the perturbation $m_q\, \bar{\psi}\, e^{i\alpha\gamma_5}\, \psi$ is related to the order parameter in the original vacuum by a chiral rotation.

The Quenched Approximation

This chapter has dealt with the large N limit of the lattice gauge theory at strong coupling. However, the result quoted for $<\bar{\psi}\psi>$ is the same as that obtained in the finite N case if we replace the determinant $\det(\not{D} + m_q)$ by 1. This approximation, where internal quark loops are neglected, has recently been called the valence or quenched approximation. For finite N, it probably enhances the chiral symmetry breaking, since screening by internal quark loops tends to randomize the m_q^{-1} series.

The fact that the chiral symmetry breaks without the inclusion of dynamical quarks may seem confusing because one usually thinks of a condensation of $q-\bar{q}$ pairs in the vacuum. This condensation occurs if a bound state becomes overbound, and $<\bar{\psi}\psi>$ then receives contributions from graphs such as given in fig. 8. Can this same mechanism be described within the quenched approximation? Suppose we have a very small mass term m_q. Quark paths of length L contribute to $<\bar{\psi}\psi>$ as $L \cdot m_q$, so that paths of finite length become unimportant as $m_q \to 0$. However, the world line of a single quark will try to imitate the virtual bound states that occur in the previous way of looking at things: the path will become very long, go off to infinity and self screen. In other words, the $q-\bar{q}$ pairs in the 'condensate' can come from a single world line if we follow the line all the way to infinity. Thus internal quark loops are not necessary for chiral symmetry breaking. We then expect that the mechanism of chiral symmetry breaking as proposed in reference [17] and in reference [18] are identical. Note that the background gauge field has no memory of the chiral symmetry breaking direction. That is,

$$\det(\not{D}+m_q e^{i\alpha\gamma_5})$$

is independent of α, and thus, in the chiral limit, we can simply set m_q in this determinant to zero. Only for the propagators in the external gauge field do we need to take the limit $m_q \to 0$ to get the correct results.

Adjoint quarks

So far, we have studied only quarks in the fundamental representation of color. The strong coupling calculation of $<\bar{\psi}\psi>$ can also be done for quarks in other representations such as adjoint quarks which can be screened by the gauge fields, and thus are not confined. The corresponding contribution to $<\bar{\psi}\psi>$ at $\lambda=\infty$ is the same as for fundamental quarks, showing that adjoint quarks also break the chiral symmetry at sufficiently strong coupling. To see the role played by the center of the gauge group, it is necessary to go to much higher orders in λ^{-1}. This is because screening is energetically favored only for large enough loops. Small loops still have an area law dependence so screening cannot affect chiral symmetry breaking at large enough couplings.*

Conclusions

We have seen that spontaneous chiral symmetry breaking occurs at strong coupling for the $N = \infty$ theory with naive fermions. Our results can be extended to Kogut-Susskind fermions by noting that each spin trace leads to a factor of C, and one can set $C = 1$ for these kinds of fermions. This is because naive fermions are simply four copies of Kogut-Susskind fermions. If we do not include internal quark loops, the Greens functions of the two theories are identical up to factors of C.

For the two orders calculated in the strong coupling expansion of $<\bar{\psi}\psi>$, we did not need to explicitly evaluate the determinant $\det(\not{D}+m_q)$. This is

* This same argument was given in [22].

no longer the case if one includes $\frac{1}{N}$ corrections, so that these are more difficult to estimate. In particular, one has graphs of the type shown in fig. 9. It may then be advantageous to take the large d approximation. By comparing the exact result for $<\bar{\psi}\psi>_{\lambda=\infty}$ with that of eq. (III.4.11), we see that the large d approximation is in fact quite accurate, with an error of only 7% for $d = 4$. Thus we expect that one can go to quite high orders in λ^{-1} if a large d expansion is used, and one can also try to improve on the quenched approximation for finite N by doing a loop expansion.

What have we learned about the mechanism of chiral symmetry breaking? Our evidence mainly comes from lattice models, but it is hoped that for theories which are strongly interacting at large distances, the strongly coupled lattice provides an adequate model of the relevant degrees of freedom. We have presented strong evidence that chiral symmetries break when gauge forces are sufficiently strong. In particular, it is plausible that confining theories always break chiral symmetry, but confinement is not a necessary condition. A purely Coulombic force, or even a short range force such as that between adjoint quarks, can break chiral symmetry if it is very strong. It was also argued that dynamical quarks are not necessary for chiral symmetry breaking. In fact, the mechanism responsible for chiral symmetry breaking is best seen in the approximation where internal quark loops are thrown out. $<\bar{\psi}\psi>$ then receives contributions from closed quark lines which self screen. As the mass perturbation goes to zero, the quark paths become very long, but the amplitudes of these paths do not cancel sufficiently to average $<\bar{\psi}\psi>$ to zero if self screening is intense.

Appendix : λ^{-1} corrections to $<\overline{\psi}\psi>$

In this appendix, we calculate the second term in the strong coupling expansion for $<\overline{\psi}\psi>$. The graphs to sum enclose a single plaquette. We show how to sum the corresponding m_q^{-1} series exactly.

The Graphs to Sum

We shall calculate here the lowest order corrections in $\frac{1}{\lambda}$ with $N=\infty$, i.e., we do not include internal quark loops. As previously, divide the graphs coming from $(\not{D}+m_q)^{-1}_{oo}$ into trees at o and the rest, which we denote by "$tree_o$" and "$other_o$". Expand $e^{-S_{Y-M}}$ in powers of $\frac{N}{g^2}$. To the order we are interested in,

$$<\overline{\psi}_o\psi_o>=\frac{\int[dU](tree_o+other_o)\cdot det(\not{D}+m_q)\cdot(1+\frac{N}{2g^2}\sum_r tr\ U_{\blacksquare r}+c.c.)}{\int[dU]det(\not{D}+m_q)\cdot(1+\frac{N}{2g^2}\sum_x tr\ U_{\blacksquare x}+c.c.)}$$

The "$tree_o$" term in the numerator can be factored out since its color factor is trivial, and this gives the result $<\overline{\psi}\psi>_{\lambda=\infty}$ derived in Chapter 3. Now consider the "$other_o$" term. The 1 from the gauge action leads to contributions that are down by N^{-2} and thus can be disregarded. We are left with at least two color traces, one coming from the "$other_o$" graph and one from the plaquette at r. Any additional color trace will lead to terms of order $\frac{1}{N}\cdot\frac{1}{\lambda}$ at least. This means that the graphs coming from the determinant are necessarily trees. Note also that to this order, we can set S_{Y-M} in the denominator to zero. The above equation then reduces to

$$<\overline{\psi}_o\psi_o> = <\overline{\psi}_o\psi_o>_{\lambda=\infty} +$$
$$\frac{1}{m_q}\ \frac{N}{2g^2}\sum_{r,other_o} tr'(other_o)\int[dU]\ tr(other_o)\cdot tr(U_{\blacksquare r}+h.c.)$$

where tr denotes the color trace and tr' the spin trace. The color integral will vanish unless the graph $other_o$ of contour C leads to a color trace over the plaquette located at r, $\Pi_C U=U_{\blacksquare r}$ or $\Pi_C U=U^{\dagger}_{\blacksquare r}$. These graphs are characterized by their origin o and by the fact that their contour C can be obtained by dressing the plaquette at r. This is illustrated in fig. 10. Denote by $D_r(l)$ the number of such graphs of length $2l$. Then

$$<\bar{\psi}_o \psi_o> = <\bar{\psi}_o \psi_o>_{\lambda=\infty} -$$

$$\frac{C}{m_q} \frac{N}{2g^2} \int [dU]\, tr(U_\bullet) \cdot tr(U_\bullet^\dagger) \cdot \sum_r \sum_{l=2}^{\infty} (-1)^l \frac{D_r(l)}{(2m)^{2l}}$$

where the $-C$ factor comes from the spin trace. Now we must determine $D_r(l)$. We need to find a convenient labeling of these graphs. Just as for the calculation of $<\bar{\psi}\psi>_{\lambda=\infty}$, the fundamental building blocks are the irreducible tree graphs. First fix the plaquette (x,μ,ν) which leads to the color trace and its orientation. Since instead of $\Pi_C U=1$, we have for instance $\Pi_C U=U_\bullet$, each graph can be built from this single plaquette by adding arbitrary "branches" at the corners.

Our first approach will be to use translation invariance which shows that $\sum_r D_r(l)$ is also the number of graphs with an arbitrary initial point but which lead to a color trace over a plaquette at the origin in the first quadrant of space-time. This last constraint ensures the uniqueness of the translation which takes a plaquette to the origin. We shall find $D(l)$, the number of paths which enclose the plaquette $(o,x)(o+x,y)(o+x+y,-x)(o+y,-y)$. Paths with different origins are not distinguished. The total number of graphs will then be $\binom{2}{d}\cdot 2\cdot D(l)$, where the prefactor counts the number of ways of choosing a plane in d dimensions and the directions in which the plaquette is enclosed. The sum of all graphs will then follow.

The second approach will be more direct. The origin o will be fixed and we shall perform the sum over l first, then over r.

Labeling of graphs: the open path case.

First consider all the paths which are obtained by dressing an open path as in fig. 11. This will be the set of graphs of interest for the propagator. We can label these graphs as follows. Begin at the first step of the path (call it o) and find the last step such that this partial path is a tree T_o at o. Go to the next step (1) and find again the last step where one has made a tree at 1. Iterate this until the end of the path. The graph has now been decomposed as $T_o, n_1 T_1 n_2 \cdots n_l T_l$. However, there are constraints on the T_i , $i \geq 1$ because of the construction. Clearly, the first step of T_1 cannot be $-n_1$. In fact, we saw that any tree is a sequence of irreducible tree graphs, so that $T_i = I_1 I_2 \cdots I_p$. The I's are constrained to not begin along the direction

$-n_i$. This condition completely satisfies the constraints, and we have a one to one labeling of the graphs. However this construction implicitly supposed that the original path to be dressed was non backtracking, which clearly is a constraint one can impose without loss of generality. We now count the number of possible branches. Denote by $\tilde{A}(l)$ the number of ways to make T_i , $i \geq 1$ of length $2l$. By the decomposition into irreducible graphs, we have

$$\tilde{A}(l) = \sum_{\sum l_j = l} I(l_1)\, I(l_2) \cdots I(l_p)\, \left(\frac{2d-1}{2d}\right)^p$$

where the $(\frac{2d-1}{2d})^p$ factor incorporates the constraints. The generating function for $\tilde{A}$ is then

$$W_{\tilde{A}}(x) = \frac{1}{1 - \frac{(2d-1)}{2d} W_I(x)} \ .$$

The first tree T_0 has no constraint, so the total number of graphs which dress a non backtracking path of length P is

$$W_A \cdot W_{\tilde{A}}^P \quad ; \quad W_A = \frac{1}{1 - W_I} \ .$$

Labeling of graphs: the closed path case.

Now we consider the case of graphs obtained by dressing the plaquette $(o,x)(o+x,y)(o+x+y,-x)(o+y,-y)$. This dressing can be done in two stages. First dress the plaquette with steps that belong to the plaquette as in fig. 12. Then dress the sites of this graph with an arbitrary number of irreducible tree graphs that are constrained to have their first step off of the plaquette. Consider all the steps (o,x) after the first dressing of the plaquette. There may be several such links as in fig. 12. Now find the only one such that if we take it to be the first step of the graph, one never comes back to the origin having a tree. There is one and only one such link. It exists as can be seen by considering all the (o,x) links successively. If all were followed by tree graphs, the over all color trace would be 1, not $tr(U_\blacksquare)$. On the other hand, if there were two such links, the trace would involve higher powers of $U_\blacksquare$. In fact, the number of these links precisely counts the winding number of the graph, which we have imposed to be one. Now that we have picked out a special link of the graph, we can label it the in same way as we did for open paths. The special step will be the first step (o,x). Then we proceed with T_1 which is constrained not to come back along $-x$. Iterate this process to get the number of graphs at the first stage of the dressing. The generating function for these graphs is

$$y^2 \, W^{4}_{\tilde{A},d=1}(y) \ .$$

Now we complete the dressing to the full lattice. This is implemented by substituting

$$y \rightarrow \frac{x}{\left(1-\frac{(2d-2)}{(2d)}\cdot W_I(x)\right)^2}$$

which corresponds to dressing each site of the graph with a sequence of irreducible tree graphs of first step off of the plaquette. This accounts for the factor $(2d-1)/2d$. One now has the generating function for all graphs $D(l)$ which dress our plaquette:

$$W_D(x) = \frac{x^2}{\left(1-\frac{(2d-2)}{(2d)}\cdot W_I(x)\right)^4} \; W^{4}_{\tilde{A},d=1}\,(y(x)) \;=\; x^2 \, W^{4}_{\tilde{A}} \ .$$

This is the answer one would have naturally guessed. The reason we used the intermediate dressing is that there are graphs such as in fig. 13 which have (o, x) links that do not belong to the plaquette in the sense we want. It is easy to see that we can define the 'unique' link in spite of this problem, and then the correct answer is obtained immediately. This result clearly generalizes to arbitrary shapes of contour to dress. The generating function for the closed graphs which lead to the color factor $U_1 \, U_2 \cdots U_P$ is just $x^{P/2} \, W^{P}_{\tilde{A}}$. The factor $W_{\tilde{A}}$ can be thought of as a mass renormalization term.

We have now found the number of graphs $D(l)$ of length $2l$ which dress a given plaquette. To obtain the number of graphs which contribute to $<\bar{\psi}\psi>$, we must account for the multiplicity of the origin of the graph along the path. The number of graphs of length $2l$ is $2\,l\;d(d-1)\;D(l)$, i.e., the generating function we want is $2d(d-1)\,x\,W_D'$, where $'$ denotes differentiation with respect to x. Finally, to get the sum we want, we substitute $x \rightarrow -(4m_q^2)^{-1}$. The result to lowest order in m_q leads to

$$<\bar{\psi}\psi> \;=\; <\bar{\psi}\psi>_{\lambda=\infty} - \frac{C}{\lambda} d(d-1)\{1+m_q \frac{d-1}{\sqrt{2d-1}}\} \ ,$$

where $<\bar{\psi}_o\psi_o>_{\lambda=\infty}$ is as given eq. (III.4.16). We have used $\int [dU] tr(U_\bullet) \cdot tr(U_\bullet^\dagger) = \frac{1}{N^2}$. The corrections are seen to decrease the value of $<\bar{\psi}\psi>$. The sign of the correction of this order is due to the spin trace.

The Direct Method

In this paragraph, we show how to derive the above formula without using translation invariance. This method will be more in line with the techniques used in Chapter 4, where the corrections to the meson masses are calculated. We begin again with the graphs $D_r(l)$. These graphs lead to a color trace $tr(U_\blacksquare)$. This means that the color factor before taking the trace is $U_T\ U_\blacksquare\ U_T{}^\dagger$ as is illustrated in fig. 14. We can for this fixed color factor follow the methods developed in the previous paragraphs. The color path gives us an open bare graph which is to be dressed; if it is of length $2l$, the dressing is described by the generating function $W_A \cdot W_{\tilde{A}}^{2l}$. One must set up a recurrence relation for summing over all possible bare graphs.

We decompose the bare graph into a trunk U_T followed by a plaquette $U_\blacksquare$. For each trunk of non zero length, there are $4(d-1)^2$ choices for the plaquette. The trunks can be constructed recursively. A trunk of length $L+1$ is obtained from a trunk of length L by adding a single step. We have

$$T(0) = 8\binom{2}{d} \quad ; \quad T(1) = 8d(d-1)^2 \quad ; \quad T(L) = (2d-1)\ T(L-1) \quad , \quad L>1$$

The trunks are non backtracking. The generating function for the total number of graphs is

$$\sum_{L=0}^{\infty} T(L)\ x^{L+2}\ W_A(x)\ W_{\tilde{A}}^{2L+4}(x) =$$

$$x^2 W_A\ W_{\tilde{A}}^4\ \{\ 4d(d-1)+x\ W_{\tilde{A}}^2\ (\ \sum_{L=0}^{\infty} x^L\ W_{\tilde{A}}^{2L}\ (2d-1)^L\ )\ T(1)\ \}$$

$$= x^2 W_A\ W_{\tilde{A}}^4\ 4d(d-1)\ \{\ 1 + \frac{x\ W_{\tilde{A}}^2\ 2(d-1)}{1-x\ W_{\tilde{A}}^2\ (2d-1)}\}\ .$$

This expression can be checked to be identical to that derived in the previous paragraph.

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Figure Captions

[1] Dispersion relation for free naive fermions in 1 + 1 dimensions.

[2] Dispersion relation for a free boson in 1 + 1 dimensions.

[3] A closed graph on the lattice.

[4] A tree graph has zero area and no loops.

[5] An irreducible tree graph at o.

[6] A general (reducible) tree graph can be decomposed into a sequence of irreducible tree graphs.

[7] Example of a sub dominant graph at large dimension.

[8] Quark condensate breaking chiral symmetry.

[9] A graph with internal quark loops contributing to the $\frac{1}{N}$ corrections.

[10] A graph enclosing a plaquette contributing to the λ^{-1} corrections.

[11] Dressing (thin lines) of an open path (thick line).

[12] First stage of the dressing of the plaquette: the thin lines are constrained to stay on the plaquette.

[13] A graph that does not appear at the first stage of the dressing of the plaquette.

[14] A bare graph made of a trunk followed by a plaquette.

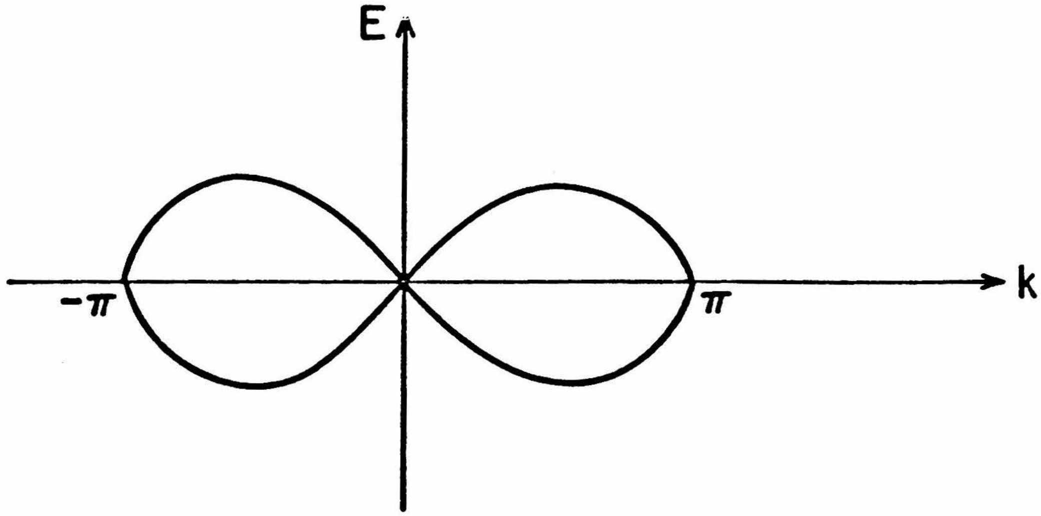

Fig. 1

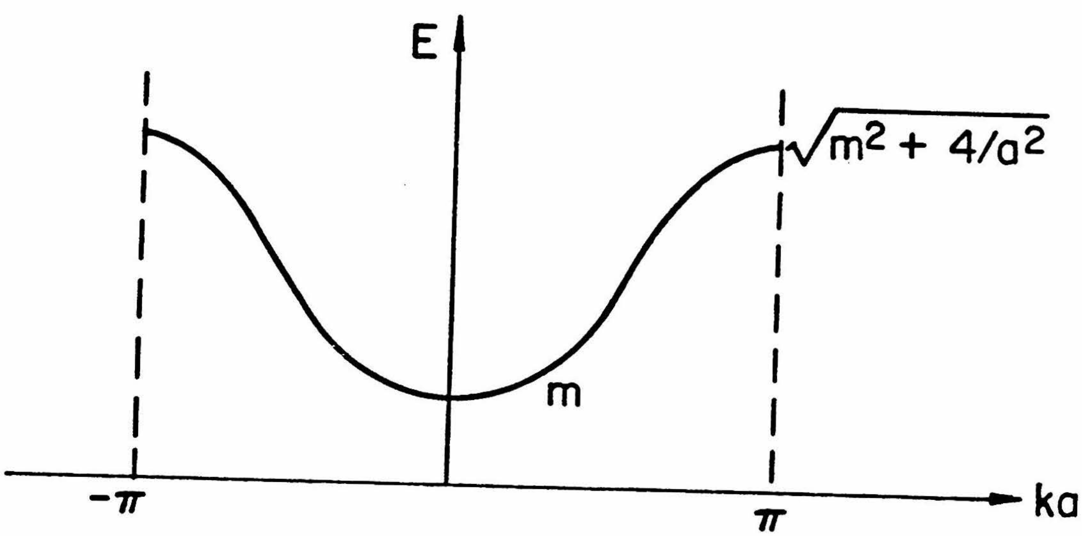

Fig. 2

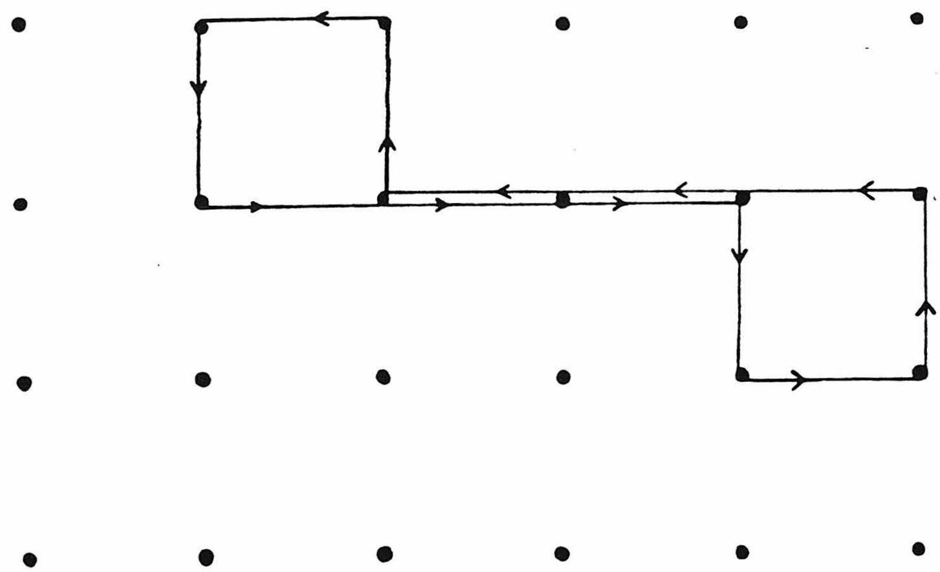

Fig. 3

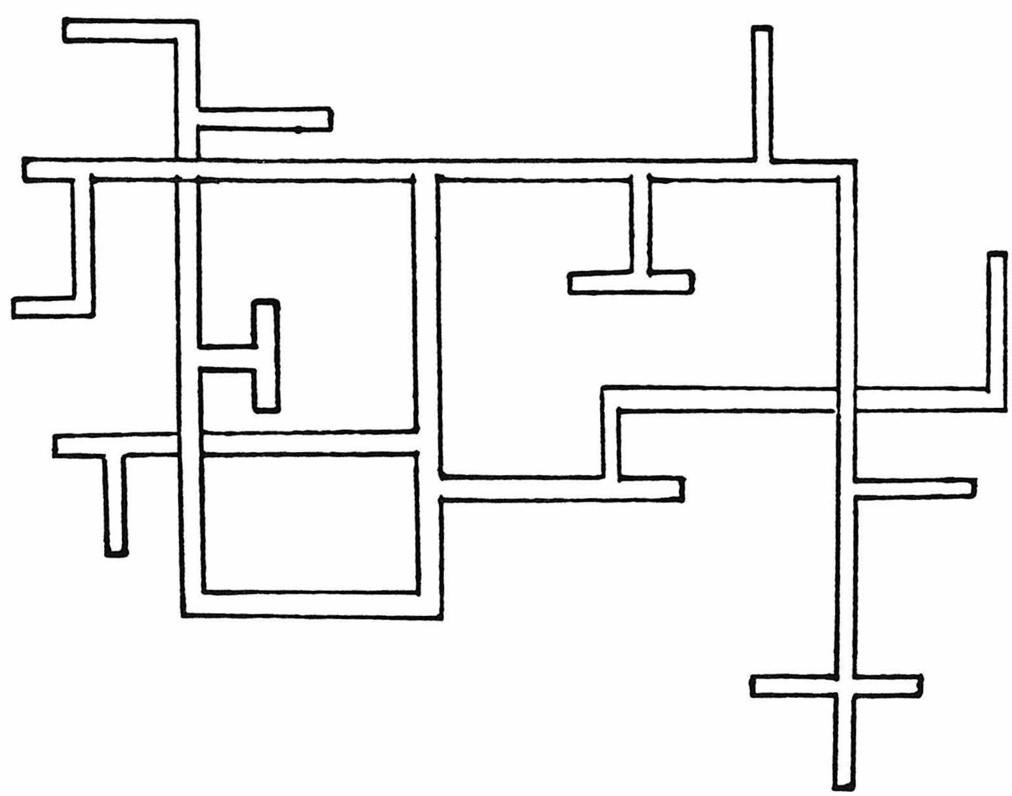

Fig. 4

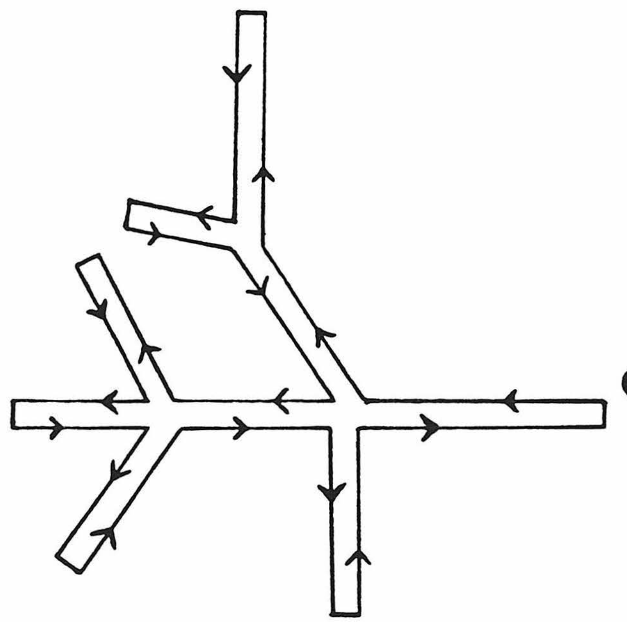

Fig. 5

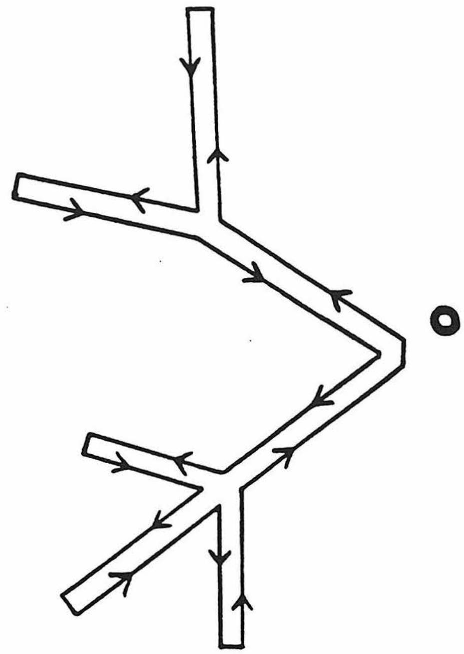

Fig. 6

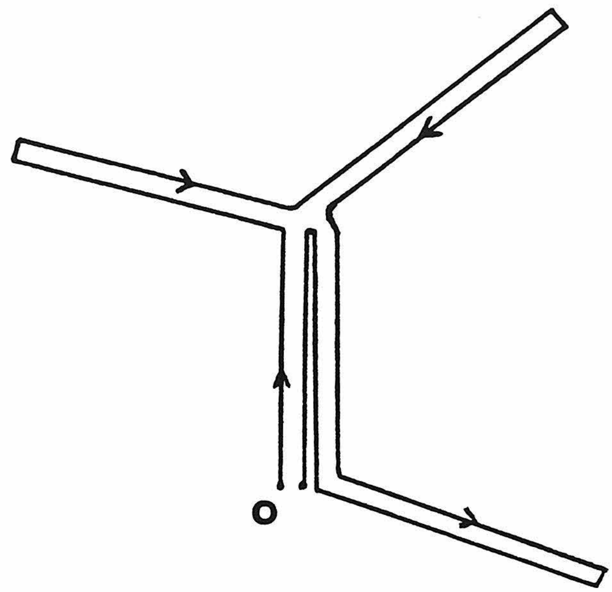

Fig. 7

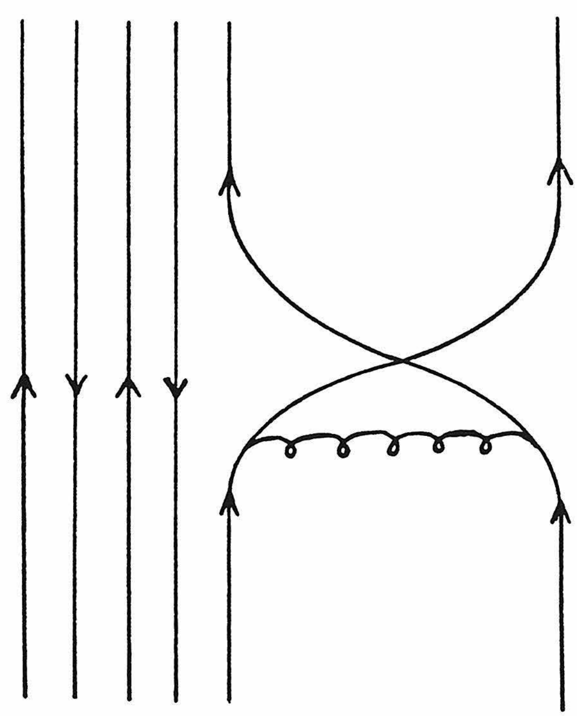

Fig. 8

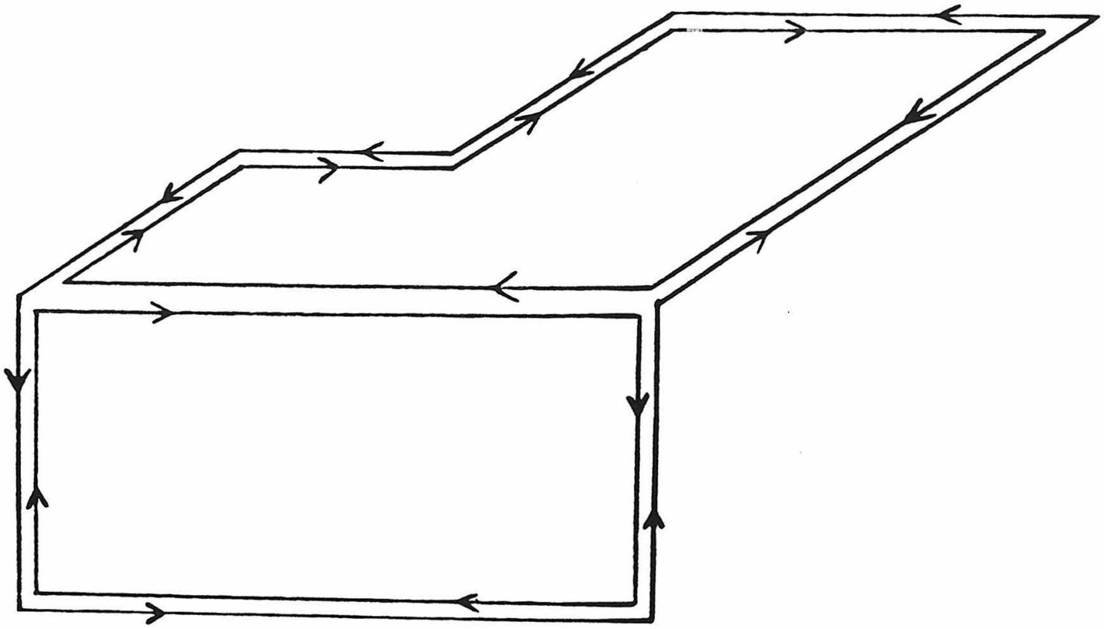

Fig. 9

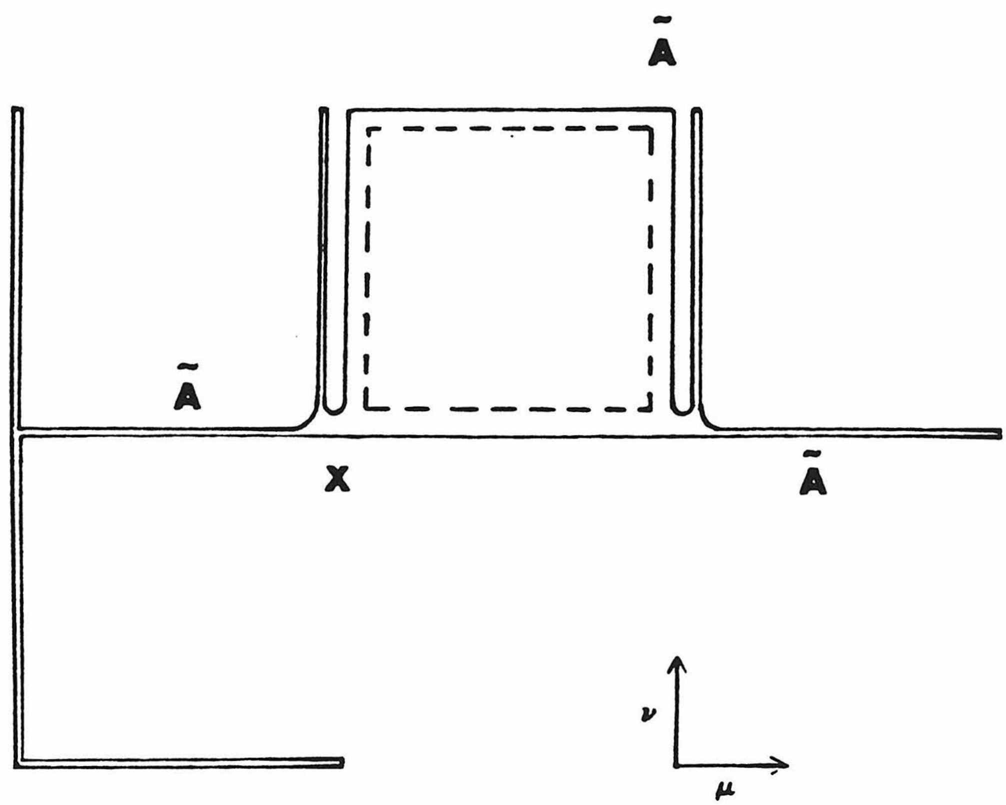

Fig. 10

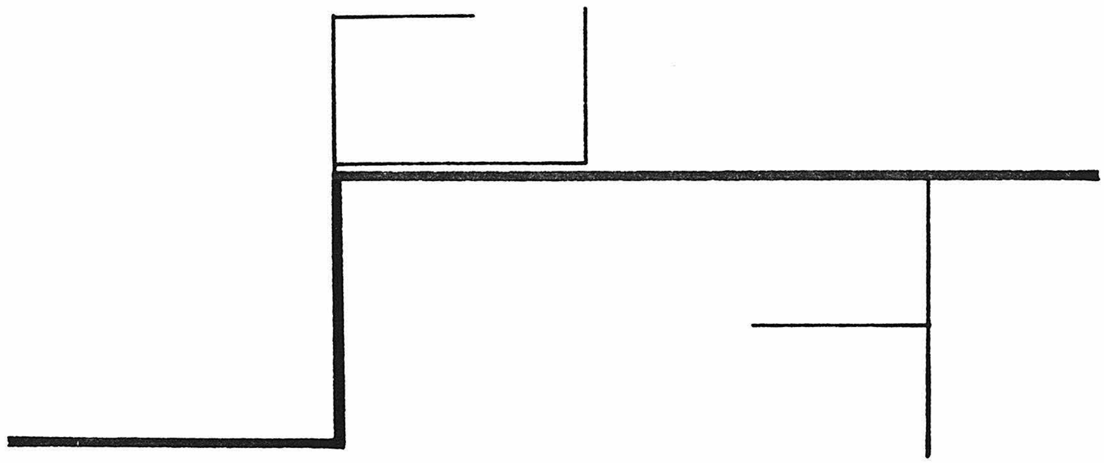

Fig. 11

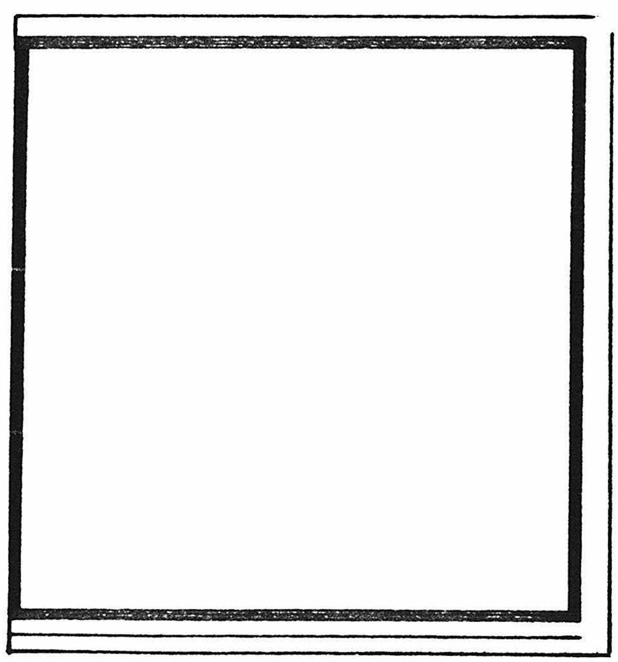

Fig. 12

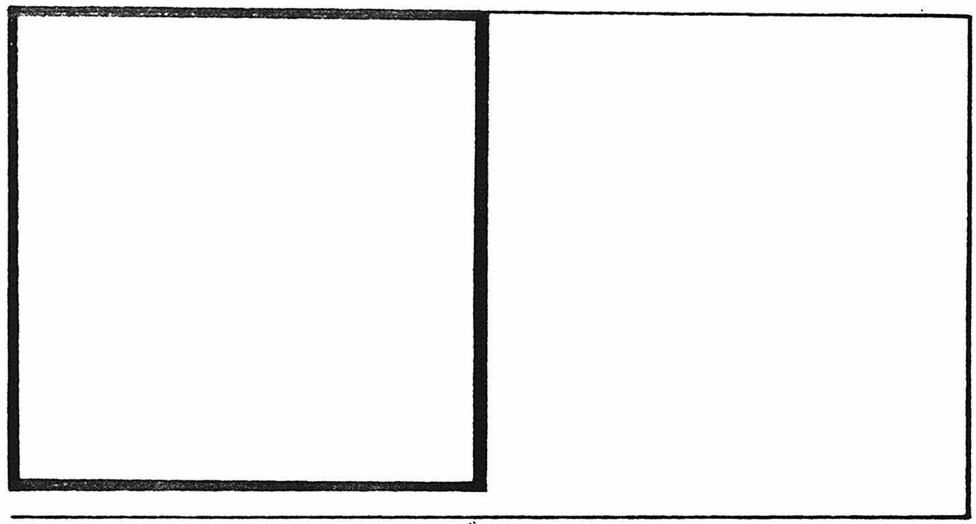

Fig. 13

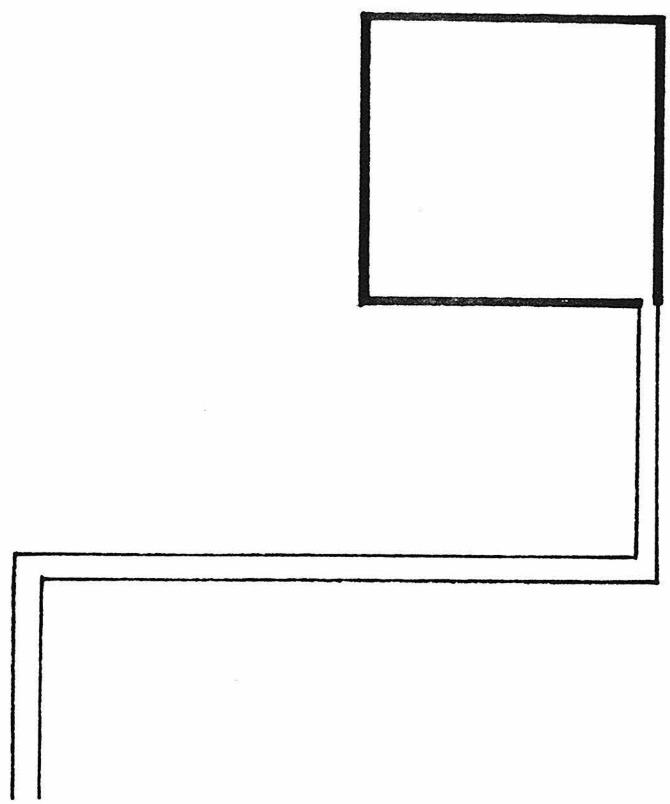

Fig. 14

Chapter 4

THE LARGE N SPECTRUM AT STRONG COUPLING

1. INTRODUCTION

In the previous chapter, we argued that chiral symmetry breaking is most likely a generic phenomenon in gauge theories whenever the effective coupling constant is large. In particular, a confining theory such as *QCD* will dynamically break chiral symmetry, and the Goldstone boson that must appear will be a quark - anti-quark bound state with an interpolating field given by $\pi = \overline{\psi}\gamma_5\psi$, since γ_5 is the generator of chiral transformations. In QCD, this particle is the pion.

In this chapter, we shall study the spectrum of the strongly coupled $SU(N)$ gauge theory on the lattice with naive fermions. The generating functional of this theory is

$$Z=\int[d\overline{\psi}][d\psi][dU]\, e^{-S_{Y-M}}\, e^{-S_{naive}} \quad , \tag{IV.1.1}$$

$$S_{naive} = \overline{\psi}(\not{D}+m)\psi = \sum_{r,\vec{n}} \overline{\psi}(r)\frac{\not{n}}{2}U(r,\vec{n})\psi(r+\vec{n}) + m_q \sum_{r} \overline{\psi}(r)\psi(r) \quad .$$

We saw that when $m_q=0$, this theory has the usual $U_R(1)\times U_L(1)$ symmetry of the continuum theory, and in addition, some symmetries which have no smooth limit as the lattice spacing $a\rightarrow 0$. (See Chapter 3 for details .) The full symmetry of the lattice theory is $U(4)\times U(4)$ which gets spontaneously broken down to $U_\Delta(4)$, and in particular, $U_R(1)\times U_L(1)\rightarrow U_V(1)$. This was explicitly shown in the previous chapter in the quenched approximation for sufficiently strong coupling. Because of Goldstone's theorem, we expect $4^2 = 16$ Goldstone bosons to appear as $m_q\rightarrow 0$, one for each symmetry generator which is broken. One of these Goldstone bosons corresponds to the pion, $\overline{\psi}\gamma_5\psi$. The other 15 have interpolating fields $\overline{\psi}\gamma_5\gamma_\mu\psi$, $\overline{\psi}\gamma_5\gamma_\mu\gamma_\nu\psi$, $\overline{\psi}\gamma_5\gamma_\mu\gamma_\nu\gamma_\delta\psi$, and $\overline{\psi}\gamma_5\gamma_\mu\gamma_\nu\gamma_\delta\gamma_\gamma\psi = \overline{\psi}\psi$. These massless particles are lattice artifacts since they correspond to spurious broken symmetry generators which treat even and odd sites differently. At least one component of their four-momentum will be at $\frac{\pi}{a}$. Now if $m_q \neq 0$, so that there is an explicit but soft breaking of chiral symmetry, the above 16 mesons become pseudo-Goldstone bosons of mass given by the current algebra relation

$$f^2 m^2 = <\bar{\psi}\psi> \cdot m_q$$

as $m_q \to 0$.* f describes the wave function of the pseudo Goldstone boson at the origin.

By using graphical methods similar to those used in the previous Chapter for calculating $<\bar{\psi}\psi>$, we shall determine the masses of mesons and baryons of the above theory at strong coupling in the quenched approximation. Our results give the exact spectrum at strong coupling of the large N limit of the $SU(N)$ theory with naive fermions. This $N=\infty$ spectrum has been previously obtained in the limit of infinite coupling by two groups, Klumberg-Stern, Morel, Napoly, Petersson [2] and Hoek, Kawamoto, Smit [3], by techniques very different from ours. We present our graphical method mainly because of its simplicity. It provides an intuitive picture of the physics, permits the calculation of corrections to the $\lambda=\infty$ limit, and can probably be taken to high order if used in conjunction with a d^{-1} expansion. This Chapter is organized as follows. In section 2, we discuss the large N theory and briefly explain the effective action methods used by the above two groups to derive the spectrum in the infinite coupling limit. Section 3 begins with an exposition of some random walk techniques, after which we determine the two-point functions at $\lambda=\infty$. This calculation is first done in the limit of large d and then for finite d. An illustrative calculation to order λ^{-1} is given in the appendix. In section 4, we compare the strong coupling spectrum in the quenched approximation with the results from numerical simulations and experiment.

* For an illustration of current algebra on the lattice, see ref [1].

2. SPECTRUM OF THE $N=\lambda=\infty$ THEORY

Since our results determine the spectrum of the $N=\lambda=\infty$ theory, it is important that this theory be qualitatively similar to QCD. At the scale of the hadrons, QCD is strongly interacting, so a strongly coupled lattice theory is probably a good effective theory of the low energy physics of QCD. It has also been argued that the $N=\infty$ theory is very similar to the $N=3$ theory. The pure gauge $N=\infty$ theory is expected to confine static quarks and to have a spectrum of massive glueball excitations. If quarks are included, the chiral symmetries spontaneously break according to the pattern we see in the real world, as was shown by Coleman and Witten by using very reasonable hypotheses [4]. This argument is independent of the coupling constant, so that at $\lambda=\infty$, we also expect to obtain the correct pattern. Another feature of the large N limit is the suppression of internal quark loops so that the Okubo-Zweig-Iizuka rule is automatically satisfied. This rule may account for the partial success of the Monte Carlo estimates of the QCD spectrum within the quenched approximation.

The $N=\infty$ theory on the lattice at strong coupling is now well understood. The spectrum of the pure gauge theory can be determined by standard high temperature expansions [5]. The glueball masses are rather insensitive to N for $N\geq 3$. Suppose fermions are now included into the theory. In the Hamiltonian formulation, the vacuum is infinitely degenerate at $\lambda=\infty$. When determining the spectrum by perturbation theory in λ^{-1}, it is thus necessary to make many approximations and these become less reliable as a particle such as the pion becomes massless. The spectrum in this formulation is thus only qualitatively understood, and in fact this is also the case for finite N [6]. In the Lagrangian formulation, the quarks are represented by Grassmann fields, and thus the fermionic part of the action can not be considered as a Boltzmann factor to be used in a high temperature series. Nevertheless, one can expand $e^{-S_{matter}}$ in powers of the fields ψ and $\overline{\psi}$. Good convergence of this series requires that the kinetic piece of the fermion fields be sufficiently small, i.e. that the fermions be essentially static. This will not be the case if there is a massless particle such as the pion, and then one must in fact resum the series exactly. This was first done for Wilson fermions by using graphical methods similar to the ones we shall use [7]. Later, the spectrum for chirally symmetric fermions was found through the use of effective action methods by two groups: Klumberg-Stern, Morel, Napoly, Petersson [2] and Hoek, Kawamoto, Smit [3]. The first group has since estimated the spectrum at $N=3$, $\lambda=\infty$ within a d^{-1} expansion and the results are quantitatively very close to the $N=\infty$ case.* The second group in fact calculated the spectrum of the theory with the action of eq. (IV.3.1) which includes both naive fermions (r = 0) and Wilson fermions (r = 1). We shall now briefly discuss the effective action methods they used.

The lattice theory is defined as in Chapter 3 on a d dimensional hypercubical lattice of spacing a=1. The generating functional is

$$Z=\int[d\overline{\psi}][d\psi][dU]\; e^{-S_{Y-M}}\; e^{-S_{matter}}.$$

The Yang-Mills degrees of freedom are $SU(N)$ or $U(N)$ matrices labeled by links $(r,\hat{n})$. In the large N limit, these two theories become identical. The fermionic part of the action S_{matter} is a bilinear form in the fermion fields, which we denote by $\overline{\psi}Q\psi$. In this section, we shall consider only naive fermions, where S_{naive} is given by eq. (IV.1.1). For the rest of this chapter, we use the conventions

$$\not{n} = n^{\mu}\gamma_{\mu} \;\;;\;\; [\gamma_{\mu},\gamma_{\nu}]_{+} = 2\delta_{\mu,\nu} \;\;;\;\; \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \;\;;\;\; tr(1)=1 \;\; .$$

Now consider the strong coupling limit $\lambda\to\infty$. Just as for $<\overline{\psi}\psi>$, one can formally set $S_{Y-M}=0$. The integration over the gauge fields then factors into independent integrals at each link:

* The hadrons become noninteracting in the limit $d\to\infty$ for any N, and each internal quark loop is suppressed by a factor of d^{-1}.

$$Z = \int [d\bar{\psi}][d\psi]\, e^{-m_q \sum_r \bar{\psi}(r)\psi(r)} \prod_{r,\mu} z(r,\mu)$$

with

$$z(r,\mu) = \int [dU]\, e^{\bar{a}Ub + \bar{b}U^\dagger a}$$

$$a = -\frac{1}{2}\gamma_\mu \psi(r) \qquad \bar{a} = \bar{\psi}(r)$$

$$b = \frac{1}{2}\gamma_\mu \psi(r+\mu) \qquad \bar{b} = \bar{\psi}(r+\mu) \ .$$

The link integral $z(r,\mu)$ resembles very much the one link integral

$$\tilde{z}(r,\mu) = \int [dU]\, e^{UM^\dagger + U^\dagger M} \ .$$

This last integral has been studied extensively [8], and the very same techniques can be used to evaluate z, though some care is necessary because of the Grassmann nature of the sources. Begin with a Schwinger-Dyson equation for z: differentiate z with respect to the sources, and use the unitarity of U to get

$$\sum_{i=1}^{N} \frac{\partial^2 z}{\partial \bar{a}_i^\beta \partial a_i^\alpha} = z \sum_{i=1}^{N} \bar{b}_i^\alpha b_i^\beta \ .$$

It is useful to find the corresponding equation for W where

$$z = e^{NW} \ .$$

Since the lattice theory is gauge invariant, W can be expressed in terms of the gauge invariant quantities

$$A^{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N} \bar{a}_i^\alpha a_i^\beta \qquad B^{\alpha\beta} = \frac{1}{N}\sum_{i=1}^{N} \bar{b}_i^\alpha b_i^\beta \ .$$

There is also an invariance in spin space under $U(4)\times U(4)$, so that W must be a function only of $tr(AB)^q$. Dropping non leading terms in the Schwinger-Dyson equation for W leads to the solution

$$W(AB) = -tr\{\, (1+4AB)^{1/2} - 1 - Log\,(\frac{1+(1+4AB)^{1/2}}{2})\, \} \ .$$

The spectrum of the resulting purely fermionic theory

$$Z = \int [d\bar{\psi}][d\psi]\, e^{-m_q \sum_r \bar{\psi}\psi + NW(AB)}$$

can be obtained in the large N limit by steepest descent. The stationary point of lowest action is not chirally symmetric as $m_q \to 0$. This gives

$$<\bar{\psi}\psi>_{m_q} = \frac{d\sqrt{2d-1+m^2}-(d-1)m}{d^2+m_q^2} ,$$

and one has 16 degenerate pseudo-Goldstone bosons of mass given by

$$ch\, E = 1 + \frac{2m_q \sqrt{1-<\bar{\psi}\psi>^2}}{<\bar{\psi}\psi>} , \qquad \text{(IV.2.1)}$$

where ch is the hyperbolic cosine. This effective action method can be made to include baryons for the $SU(N)$ theory. This requires special care since baryons acquire masses that grow like N and thus decouple in the large N limit. These masses have been determined to leading order in N and $\frac{1}{d}$ in ref [3] :

$$m_B = NLog\left(\frac{2}{<\bar{\psi}\psi>}\right) . \qquad \text{(IV.2.2)}$$

When $m_q=0$, this gives

$$m_B = \frac{N}{2}Log(2d) . \qquad \text{(IV.2.3)}$$

In the next section we will see how these results can be extended by using graphical techniques rather than by calculating the generating functional in the presence of arbitrary sources. This will also enable us to find the baryon masses exactly.

3. GRAPHICAL METHOD FOR OBTAINING THE SPECTRUM

The problem addressed in this section is that of calculating the spectrum of $SU(N)$ gauge theory with naive fermions in the limit $Ng^2=\lambda\to\infty$ and $N\to\infty$ by graphical methods. The corrections in $\frac{1}{\lambda}$ are considered in the appendix.

The expansion parameter

The fermionic part of the action $\overline{\psi}Q\psi$ has a diagonal piece in the space indices and a hopping term from site to site. For example, we have for Wilson fermions

$$\overline{\psi}Q\psi = \sum_{x}\overline{\psi}(x)\psi(x) - \qquad \text{(IV.3.1)}$$

$$\kappa\sum_{x,\mu}\{\overline{\psi}(x)(r-\gamma_\mu)U(x,\mu)\psi(x+\mu) + \overline{\psi}(x+\mu)(r+\gamma_\mu)U^\dagger(x,\mu)\psi(x)\} \ ,$$

whereas for naive fermions we have

$$S_{naive} = \overline{\psi}(\not{D}+m_q)\psi = \qquad \text{(IV.3.2)}$$

$$\sum_{r,\vec{n}}\overline{\psi}(r)\frac{\not{n}}{2}U(r,\vec{n})\psi(r+\vec{n}) + m_q\sum_{r}\overline{\psi}(r)\psi(r) \ .$$

Consider the meson two-point function:

$$<\overline{\psi}_x^\gamma\psi_x^\delta\overline{\psi}_o^\alpha\psi_o^\beta> = \frac{\int[dU][d\overline{\psi}][d\psi]\ \overline{\psi}_x^\gamma\psi_x^\gamma\overline{\psi}_o^\delta\overline{\psi}_o^\alpha\psi_o^\beta\ e^{-S_{Y-M}-S_{matter}}}{\int[dU][d\overline{\psi}][d\psi]\ e^{-S_{Y-M}-S_{matter}}}$$

where the sum over color is implicit. After integrating over the Grassmann variables, this reduces to

$$<\overline{\psi}_x\psi_x\overline{\psi}_o\psi_o> = \frac{\int[dU]\ (\ Q_{o,x}^{-1}\ \ Q_{x,o}^{-1} + Q_{o,o}^{-1}\ \ Q_{x,x}^{-1})\ \det(Q)\ e^{-S_{Y-M}}}{\int[dU]\ \det(Q)\ \ e^{-S_{Y-M}}}$$

Expand the numerator and denominator in the off diagonal part of Q. For Wilson fermions, this means a κ expansion, and for naive fermions, a m_q^{-1} expansion. In both cases, one will want to resum this series to all orders in order to locate the pole in the propagator reliably. However, as stressed by Wilson [7], the series in κ is expected to converge as long as there is not a pole in the propagator, whereas the m_q^{-1} series is divergent before one reaches a pole, so that the result must be analytically continued to find the particle mass. This can be seen for instance for free fields. Thus a truncated κ series may still yield useful information, but a truncated m_q^{-1} series will not. Each term of this expansion has a graphical interpretation as in Chapter 3. The numerator is graphically represented in fig. 1, and the denominator is just a sum over closed paths with appropriate weights. In the large N limit, we can throw away all the graphs which have internal quark loops, so we can effectively replace the determinant by 1. Factorization allows us also to keep only the connected piece of the numerator. Now consider the strong coupling limit, $\lambda \to \infty$. Just as for $\langle\bar{\psi}\psi\rangle$, one can formally set $S_{Y-M}=0$. In this limit, all graphs must have zero area. To find the dependence of the two-point function on the separation x, we need to enumerate all the graphs on the lattice and sum their amplitudes.

Wilson fermions

The case of Wilson fermions is reviewed here to introduce some important techniques which we shall use later, and to show some differences with the case of naive fermions. Reference [7] is a good introduction to the relevant random walk techniques.

The expansion parameter κ is the strength of the kinetic energy of the fermions. If $Q=1-\kappa T$, (T for transport), the series

$$Q^{-1} = 1 + \kappa T + \kappa^2 T^2 \cdots \tag{IV.3.3}$$

generates walks on the lattice. Each term in this series for the propagator can be interpreted as a quark world line from o to x. A considerable simplification occurs if we use the specific form of Q given in eq. (IV.3.1) with $r=1$. For each fermion (respectively anti-fermion) step, we have in the κ expansion a spin factor $\kappa\ (1\pm\gamma_\mu)$ (respectively $\kappa\ (1\mp\gamma_\mu^t)$). Since $\gamma_\mu^2=1$, any quark or anti quark line which has a step which backtracks on itself has a zero weight.* Now in the strong coupling limit, the only paths which survive are those with trivial color factors, i.e., the paths which enclose zero area. Thus, the set of graphs which we need to count are just the non-backtracking

* In the generalization of the action in eq. (IV.3.1) where $1\pm\gamma_\mu$ is replaced by $r\pm\gamma_\mu$, this is no longer true.

random walks from o to x of the $q-\bar{q}$ pair, with appropriate spin factors. There is no self screening. The problem of summing over all quark and anti-quark world lines has been reduced to summing over a single meson world line. In fact, we can sum over all random walks of the meson since those that do backtrack don't contribute to the amplitude if we calculate things correctly. Each step of the random walk contributes a $spin \times \overline{spin}$ factor $(1 \pm \gamma_\mu) \times (1 \mp \gamma_\mu)^t$ to the overall propagator. One can solve for the sum of all random walks analogously to the case where there are no spin degrees of freedom by going to momentum space. This gives the propagator, and the poles are then directly obtained. The matrix which generates the world line of the meson is

$$\varphi^\dagger \widetilde{T} \varphi = \kappa^2 \sum_{r,\mu} \{ \varphi_r^* (1-\gamma_\mu) \times (1+\gamma_\mu)^t \varphi_{r+\mu} + \varphi_{r+\mu}^* (1+\gamma_\mu) \times (1-\gamma_\mu)^t \varphi_r \}$$

where we have introduced for convenience the meson φ_i which has $4 \times \bar{4}$ "spin" indices. The $spin \times \overline{spin}$ tensor products of operators have been denoted by $\times$. The two-point function's expansion is precisely the expansion of $(1-\widetilde{T})^{-1}$ in powers of κ. Thus the meson propagator $G(o,x)_{\alpha\beta}^{\gamma\delta}$ is simply obtained by inverting $1-\widetilde{T}$. This is best done by going to momentum space:

$$G^{-1}(k) = 1- \qquad \text{(IV.3.4)}$$

$$\kappa^2 \sum_\mu \{ 2\cos(k_\mu) + 2i\sin(k_\mu)\gamma_\mu \times 1 - 2i\sin(k_\mu) 1 \times \gamma_\mu^t - 2\cos(k_\mu)\gamma_\mu \times \gamma_\mu^t \} \ .$$

The particle spectrum can be obtained by doing a Fierz transformation on $G^{-1}(p)$. It is useful to use the symmetries of the theory for this. One obtains a partial diagonalization of $G^{-1}(p)$ by using parity and the discrete Euclidean symmetries. In particular, there is a single pseudo-scalar (the pion), and three vectors (the rho). The pion is the lightest particle; its energy E at zero momentum is given by

$$ch\ E = 1 + \frac{(1-16\kappa^2)(1-4\kappa^2)}{8\kappa^2(1-6\kappa^2)} \ . \qquad \text{(IV.3.5)}$$

This mass is seen to vanish at the critical value $\kappa_c = \frac{1}{4}$. Note that the series of eq. (IV.3.3) converges until the pole is reached.

We now proceed to chiral fermions in the limit $d=\infty$, for there a random walk picture still applies. A special treatment will be necessary in finite dimension.

Naive fermions in the large d limit

The chirally symmetric transport matrix $\overline{\psi}\not{D}\psi$ will now be used instead of the Wilson form. The expansion parameter is m_q^{-1} rather than κ. We shall follow closely Blairon et al. [9]. The large d limit considerably simplified the calculation of $<\overline{\psi}\psi>$, and the same is true of the meson propagator. In the case of Wilson fermions, the paths to be summed were meson world lines. The condition of no backtracking prevented the quark from self screening. Naive fermions can self screen, so we no longer have a problem with a single meson line. However, we can construct the quark and anti-quark paths of zero area as follows. Take a $q-\overline{q}$ line (call it a trunk) as in fig. 2, and dress each site with branches, i.e., quark lines of zero area. (See fig. 3.) Each branch is then just a graph that contributes to $<\overline{\psi}\psi>$ averaged over color as was shown in Chapter 3. The contribution of all graphs can be calculated by first summing over the graphs with the same trunk, and then summing over all possible trunks.

For each step of the $q-\overline{q}$ trunk, there is a $spin \times \overline{spin}$ factor $\gamma_\mu \times (-\gamma_\mu)^t$. The amplitude of the sum of all graphs with a fixed trunk, but arbitrary branches, is a product of gamma matrices for the trunk, times the factor $<\overline{\psi}\psi>^2$ raised to the number of links on the trunk coming from the dressing of the trunk. The series for the propagator is then up to an unimportant constant

$$<\overline{\psi}_x \psi_x \overline{\psi}_o \psi_o> = \sum_{trunks\{o\to x\}} (-1)^L \frac{<\overline{\psi}\psi>^{2L}}{2^{2L}} [\gamma \cdots \gamma] \times [\gamma^t \cdots \gamma^t]$$

where we have omitted the indices on the spin matrices. L is the length of the trunk, and the 2^{2L} comes from the factor of 2 in $\not{D}$. The dressing of the trunk has simply the effect of renormalizing the bare mass. The paths from o to x can now be summed analogously to the Wilson case. The corresponding $\widetilde{T}$ matrix is

$$\varphi^\dagger \widetilde{T} \varphi = \frac{<\bar{\psi}\psi>^2}{4} \sum_{r,\mu} \varphi_r^* \gamma_\mu \times (-\gamma_\mu)^t \varphi_{r+\mu} ,$$

so that the propagator in momentum space is

$$G(k) = \frac{1}{1 + \frac{<\bar{\psi}\psi>^2}{4} \sum_\mu 2\cos(k_\mu)\, \gamma_\mu \times (-\gamma_\mu)^t} .$$

Diagonalization of $G(k)$ can be done via a Fierz transformation. There are 16 poles corresponding to the 16 corners of the Brillouin zone. The pion $(\bar{\psi}\gamma_5\psi)$ is the only particle which has low energy at small spatial momentum. Its mass E is given by

$$ch\ E = \frac{2}{<\bar{\psi}\psi>^2} - (d-1) \ . \qquad \text{(IV.3.6)}$$

Using the value of $<\bar{\psi}\psi>$, this gives to order m_q , $E^2 = 2dm_q<\bar{\psi}\psi>$, and in this large d approximation $f_\pi^2 = \frac{2}{d}$. We emphasize that the spectrum is rather trivial: there are no radial or orbital excitations of finite mass since the strength of the gauge force is infinite.

Finally one can calculate the masses of baryons. We suppose that N is odd so that the baryons are fermions. Begin with a trunk of N quark lines, and dress each line independently. As before, this has the effect of renormalizing the mass. The spin contribution of each step of the trunk has the form $\gamma_\mu \times \gamma_\mu \cdots \times \gamma_\mu$, where there are N such terms, so the propagator is

$$\frac{1}{1 - (\frac{<\bar{\psi}\psi>}{2})^N \sum_\mu 2i\sin(k_\mu)\, \gamma_\mu \times \cdots \gamma_\mu} .$$

Since the quarks in the baryon are totally antisymmetric in color, and they are symmetric in space, they must be symmetric in spin (we have a single flavor), so the energy of baryons at zero momentum satisfies

$$sh\ E = \frac{1}{2}\left(\frac{2}{<\bar{\psi}\psi>}\right)^N \ . \qquad \text{(IV.3.7)}$$

As $N \to \infty$, $E \to N \cdot Log\left(\frac{2}{<\bar{\psi}\psi>}\right)$, and we can think of each quark in the baryon as acquiring a mass $E = Log\left(\frac{2}{<\bar{\psi}\psi>}\right)$ due to chiral symmetry breaking.

Naive fermions at finite d

We now extend the previous analysis to finite dimensions. One can calculate the graphs of the two-point function exactly for any d, and determine the spectrum exactly. Why are the graphs incorrectly counted in the above paragraph? It is the assignment of links to a branch that is ambiguous if d is finite. This is illustrated in fig. 4, where the 'dress' can be assigned to the sites 1 or 2. This same graph would be counted twice in the above formalism. As $d \to \infty$, this particular graph becomes unimportant because it does not contribute to the leading behavior in d^{-1}. We must find a way to label the graphs in a unique way. We show here a possible labeling.

First we need to locate a trunk uniquely. Consider for instance the quark line. Starting at o, go along the line until the last occurrence of a passage through o such that this partial path encloses zero area, i.e., is a tree graph. This defines the part of the graph which dresses the quark line at the site o. Proceed to the next site and define analogously the part of the graph which is considered to dress it. Thus the dress in fig. 4 is assigned to the site 1. Iteration of this procedure defines what is called the trunk versus a branch for an arbitrary graph. However, because of the construction, one obtains a trunk which is non backtracking. The dressings also satisfy some constraints. Each dress is a tree graph and thus is a sequence of irreducible tree graphs (ITG). Since the dresses are defined successively as "maximal tree graphs" at the considered site, the ITG are constrained not to begin in the direction of the previous step of the trunk. Now consider the set of all graphs and the set of all labelings, i.e., a specification of the trunk and of the dressings at each site which satisfy the above constraints. It is not difficult to see that the graphs are in one to one correspondence with the labelings: for each graph, there is a single labeling and vice-versa. Thus we avoid the over-counting problem of the previous paragraph at finite d.

We can now compute the propagator. Just as in the infinite d case, we shall first sum over all graphs which have a given trunk. Take the non backtracking trunk to be of length L, and allow for all possible dressings that satisfy the above mentioned constraints. As before, the dressing simply renormalizes the mass. Denote by $W_I(x)$ the generating function of the ITG, and by $D(L)$ the number of ways to dress a given site. The constraint that the ITG in the dressing do not begin in a specific direction gives

$$D(L) = \sum_{\sum l_i = L} I(l_1) I(l_2) \cdots I(l_p) \left(\frac{2d-1}{2d}\right)^p \quad ,$$

from which the generating function of D follows:

$$W_D = \frac{1}{1 - \frac{2d-1}{2d} W_I} = 2\, m_q \cdot \alpha^{-1} \quad ,$$

with $\alpha = m_q + \sqrt{m_q^2 + 2d - 1}$. This renormalizes the mass, so each step of the trunk is associated with a factor $\alpha^{-2}\, \gamma_\mu \times (-\gamma_\mu)^t$. The series for the meson propagator is now

$$<\overline{\psi}_x \psi_x \overline{\psi}_o \psi_o> = \sum_{trunks\{o \to x\}}{}' \, (-1)^L \, \alpha^{-2L} \, [\gamma \cdots \gamma] \times [\gamma^t \cdots \gamma^t]$$

where $\sum'$ denotes summation over non-backtracking walks only. In the large d limit, this constraint becomes unimportant and we find the result of the previous section.

The remaining problem is that of summing over random walks which are non backtracking. This can be reduced to a matrix problem because the walk remembers only a finite number of steps. For the purpose of definiteness, take d = 4. First label the paths by the direction of their last step; they then fall into 8 classes. Let us relate the number of paths of length L+1 to those of length L. Fix the origin of all paths to be at o, and let the space-time label x be free. Each graph of length L gives rise to 7 (2d-1) new paths of length L+1. Let $B_\mu(L)$ be the number of graphs of length L of last step along the direction μ, μ in [1,8], with, e.g., 5 = 1+4 = -1. The recurrence relation is then:

$$B_\mu(L+1) = \sum_{\nu \neq -\mu} B_\nu(L) \; .$$

The propagator in momentum space is

$$<\bar{\psi}_x \psi_x \bar{\psi}_o \psi_o>_{(k)} = \sum_x \sum_\mu B_\mu(L)\, e^{ik\cdot x}\, (-1)^L\, \alpha^{-2L}\, [\gamma \cdots \gamma] \times [\gamma^t \cdots \gamma^t] \; .$$

We thus need the recursion relation for $B_\mu(L)$ in momentum space:

$$\tilde{B}_\mu(L+1) = e^{ik_\mu} \sum_{\nu \neq -\mu} \tilde{B}_\nu(L) \; .$$

This can be written in matrix form as

$$[\tilde{B}]_{L+1} = M(k)\, [\tilde{B}]_L$$

where $M(k)$ is given in table 1. The series for the propagator can now be summed since it is geometric, giving

$$G(k) = [1]^t \; \frac{1}{1+\alpha^{-2} M(k) \times [\gamma] \times [\gamma]^t} \; [1] \; ,$$

where the additional vector [1] is introduced to sum over the μ index, i.e., over all the $B_\mu(L)$. The poles of the propagator give the value of the masses. Evaluate the determinant of $M+\alpha^2 I$ to determine the condition on the momenta k_μ. If the spatial momenta are set to zero, a pole appears when $k_4 = iE$ satisfies

$$ch\; E = \frac{2(2d-2) - (1+\alpha^2)(2d-3-\alpha)}{2(2d-2)\alpha - 2\alpha(2d-3-\alpha)} = \frac{2\alpha+\alpha^2-1+2d(1-\alpha)}{2\alpha} \; .$$

The eigenvector corresponds to the pion $\bar{\psi}\gamma_5\psi$. The above result is valid in any dimension, and of course, gives the values quoted in the previous paragraph when $d \to \infty$. This is the same expression for the mass as that given in reference [3],

$$\cosh(E) = 1 + \frac{(\alpha-1)(\alpha+1-2d)}{2\alpha} \ .$$

When $m_q \to 0$, this gives

$$m_\pi^2 = \frac{4(d-1)}{\sqrt{2d-1}}\, m_q \ ,$$

so that $f_\pi^2 = 7/12\ a^{-2}$ in four dimensions. The other eigenvectors of low mass live on the corners of the Brillouin zone, but have same energy. They are degenerate because the broken symmetry generators are all on an equal footing. More concretely, the propagator of the $\overline{\psi}\gamma_5\gamma_\mu\psi$ meson at four momentum k, for instance, can be seen to be the same as that of $\overline{\psi}\gamma_5\psi$ at $k + \sum_{\nu\neq\mu} \hat{x}_\nu \frac{\pi}{a}$.

It is possible to calculate the mass of baryons also. The corresponding propagator is analogously

$$[1]^t \ \frac{1}{1-\alpha^{-N} M(k) \times [\gamma \times \gamma \cdots \times \gamma]} \ [1] \ .$$

Since we have only one flavor, the masses are given by the equation

$$ch\ E = \frac{\alpha^N}{2} \ .$$

To leading order in N, this gives $m_B = N \cdot Log\,(m_q + \sqrt{2d-1+m_q^2}\,)$. This agrees with the result of reference [3] to leading order in d^{-1}.

We mention briefly the results for the corrections in $\frac{1}{\lambda}$. To order $\frac{1}{\lambda}$, one allows the trunk to create openings of 1 plaquette large, and the same for the dressing at each site. One thus sums over paths of the type given in fig. 5. Again a linear recurrence relation can be derived and the geometrical series summed. The results for the masses to lowest order in m_q in four dimensions are

$$m_\pi^2 = m_q \ (\frac{12}{\sqrt{7}} + \frac{3}{\lambda}\ \frac{1}{4\ 7^{3/2}}) \ .$$

Details are presented in the appendix.

4. Quenched Fermions at Finite N

The large N limit was taken mainly because the graphical method described becomes cumbersome if internal quark loops are included. However, the approximation of neglecting internal quark loops is commonly used and thus it is interesting to compare our results when we set $N=3$ with those, e.g., of the numerical simulations. Let us pause to describe these latter results [11]. The Monte Carlos are run at a finite value of g. The string tension is then finite rather than infinite as it is in our case. The averages over the gauge degrees of freedom are done by a biased random sampling of configurations. The two-point functions in the background field are calculated by relaxation methods, and then averaged over configurations. One thus has an estimate of the hadron masses which converges statistically with the sample size. There are finite volume effects which make the approach to the chiral limit difficult. To compare with experiment, one sets the scale of the lattice spacing by fitting the string tension, and then one uses the pion mass to determine the bare mass to be used in the Lagrangian. In practice, only the lowest states of any quantum number which does not mix with the vacuum can be measured reliably.

To compare the estimates of these simulations with our analytical results, we use the following convention for defining the masses of various states [3]. Consider the large time behavior of the propagator:

$$\int_{-\pi}^{\pi} \frac{dk_4}{2\pi} e^{ik_4 x_4} \frac{1}{\cos k_4 - b} = (-1)^{x_4} e^{-Ex_4} (b^2-1)^{-\frac{1}{2}}$$

with

$$ch\ E(p) = \pm b(p) \quad , \quad \pm b > 0 \ .$$

We define the mass m of the particle to be the energy at zero spatial momentum $E(p=0)$. Note however that m will not always be a local minimum of the energy surface. This definition gives different masses to the particles $\overline{\psi}\gamma_0\psi$ and $\overline{\psi}\gamma_1\psi$, so we work only with those states which have $k_4 \simeq 0$, throwing away those with $k_4 \simeq \frac{\pi}{a}$. For instance, in our model, the δ particle, $\overline{\psi}\psi$, propagates when all four of its momentum components are near $\frac{\pi}{a}$, and thus we consider its appearance a pure lattice artifact. (In the quark model, the δ is an orbital excitation, so its existence in the strong coupling limit is unexpected. Similarly, the A_1 is an excitation, but the field $\overline{\psi}\gamma_0\psi$ has $k_4 \simeq 0$, so we have included it in table 2.)

As mentioned at the end of Chapter 3, naive fermions are simply four copies of Kogut-Susskind fermions. The spectrum of the theory with naive fermions is the same as in the theory with Kogut-Susskind fermions, but with a fourfold multiplicity. We obtain a total of four energy levels:

$$ch\ E = \frac{7 - 6\alpha^2 + \alpha^4}{2\alpha^2} + 2i \quad ; \quad i = 0, \cdots 3 .$$

In table 2, we compare these values to experiment by using the pion and rho masses as input. Also shown are Monte Carlo results for Wilson fermions and for Kogut-Susskind fermions [from the first two references in 11].

The main features of these calculations are qualitative agreement with experiment, too small spin splittings, and baryons which are too heavy. This qualitative agreement is not unexpected. As long as there is chiral symmetry breaking, the pion will be light, and quarks should acquire a constituent mass. The other mesons will thus be approximately twice this mass, and the baryons three times. The smallness of the spin splittings is also understandable. The sign is correctly given by the Monte Carlo, but the magnitude is too small. The lattices used are smaller than the size of hadrons. The quarks cannot explore enough of the confining potential under these conditions. In the strong coupling limit, the quarks are always on top of each other so the spin dependence of the force cannot have its effect.

It would be good to take the strong coupling expansion to higher orders. Kawamoto and Shigemoto [12] have estimated the dependence of the masses on g by taking the expansion to high order and by using various approximations. The masses, as shown in fig. 6, are not sensitive to g (or rather their ratios are not). There does seem to be a dependence in the cross over region though. A similar analysis should be done for chirally symmetric fermions.

5. Conclusions

We have seen how the large N limit of the $SU(N)$ lattice gauge theory can be solved exactly at strong coupling. The corrections to this $\lambda=\infty$ limit are calculable, though tedious. A systematic d^{-1} expansion would be useful to go beyond the two orders we have calculated. The resulting spectrum would probably agree better with the Monte Carlo results which are obtained at finite N in the quenched approximation. The resummation methods we introduced may also be useful to calculate the propagators in background gauge fields instead of using the relaxation methods commonly used in numerical simulations. This could be applied for instance to a study of chiral symmetry breaking at finite temperature. Finally, these graphical methods give an appealing picture of the dynamics of strong gauge forces, and an explicit realization of baryon mass generation via chiral symmetry breaking.

Appendix : λ^{-1} Corrections to the Meson Masses

In this appendix, we calculate the corrections in λ^{-1} to the meson masses. The trunks have openings of one plaquette at a time, and the dressing at each site can enclose at most one plaquette. A consistency check on the calculation is that the pion be a pseudo-Goldstone boson order by order in λ^{-1}.

The Graphs to be Summed

We shall restrict ourselves to the pion, but since the propagators of the other mesons are simply related by shifts in the momentum, there is no loss of generality.

Again we think of the meson propagator as being constructed from a trunk. Each step of the trunk will lead to a $spin \times \overline{spin}$ factor when it is dressed, and we will get a geometric series for the propagator

$$1 + \omega\Gamma + \omega^2\Gamma^2 \cdots$$

To pick out the λ^{-1} correction to the pion mass, we need to include at each step of the trunk only contributions of order λ^{-1}. This means that we can restrict ourselves to bare trunks which have an opening of only one plaquette at a time. The dressing at each site will also enclose at most one plaquette. Fig. 7 shows all the possibilities for steps of the trunk.

Labeling of Graphs

Consider first a bare trunk which has openings of at most one plaquette at a time as illustrated in fig. 8.a. This trunk can be schematically described by a walk on the lattice as in fig. 8.b. The steps join nearest neighbor sites or sites diagonally opposite on a plaquette. Just as for the $\lambda=\infty$ case, the trunk has some constraints to avoid over-counting in finite dimension. A diagonal bond (μ,ν) cannot be followed by either $(-\mu,-\nu)$, $(-\mu)$ or $(-\nu)$, and a (μ) bond cannot be followed by a $(-\mu)$ bond. Now dress this trunk. Take for instance the quark line. The dressing at a site either encloses a plaquette or it doesn't. If it doesn't, we are reduced to the case of $\lambda=\infty$. If it does, it gives rise to a color factor $U_T U_{\blacksquare} U_T^{\dagger}$, and we can calculate the factor it contributes to the propagator by using the technique developed in the appendix of Chapter 3. Let us dress a quark line like that in fig. 9. With the usual notation, the generating function for the dressed graph enclosing a plaquette is

$$\sum_{L=1}^{\infty} T(L)\, x^{L+2}\, W_{\tilde{A}}^{2L+5}(x) =$$

$$x^2\, W_{\tilde{A}}^5\, \Big(\sum_{L=1}^{\infty} x^L\, W_{\tilde{A}}^{2L}\, (2d-1)^L \Big)\, T(1)$$

$$= x^3\, W_{\tilde{A}}^5\, 8\,(d-1)^3\, \frac{x\, W_{\tilde{A}}^2\,(2d-1)}{1-x\, W_{\tilde{A}}^2\,(2d-1)} .$$

We have taken into account the fact that $T(1)$ preserves the non backtracking nature of the quark line:

$$T(1) = 8\,(d-1)^3 .$$

The amplitude associated with each step of the trunk is shown in fig. 10. To implement the constraints the trunk must satisfy, we label the trunks by their last step just as in (IV.3). However, the steps now can be in the (μ,ν) direction, so the matrix $M(k)$ is much larger (32 x 32 in four dimensions). This is shown in table 3. Since we are only interested in the λ^{-1} contributions, the determinant of $M(k)$ can be reduced to 8 x 24 determinants of size 8 x 8. The calculation can be arranged so that in fact most of these vanish. The expression for the determinant is quite complicated, but to order m_q, it gives at $p_i=0$ and in $d=4$:

$$m_\pi^2 = m_q \cdot \Big(\frac{12}{\sqrt{7}} + \frac{3}{\lambda} \frac{1}{4\, 7^{3/2}} \Big) .$$

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Figure Captions

[1] Quark anti-quark lines for the meson propagator: connected and disconnected graphs.

[2] A bare non backtracking trunk on the lattice.

[3] The dressing (thin lines) of a trunk on the lattice.

[4] Example of a sub dominant graph at large dimension.

[5] A graph contributing to the λ^{-1} corrections to the propagator. In this example, both plaquettes are openings in the trunk.

[6] Hadron masses as a function of coupling from reference [12].

[7] All possible steps for a trunk with at most one plaquette openings.

[8] a) A bare trunk.
b) Schematic representation of the above trunk: the steps join nearest neighbor sites or sites on the diagonal of a plaquette.

[9] A dressing which encloses a plaquette.

[10] Amplitudes associated with the trunk steps.

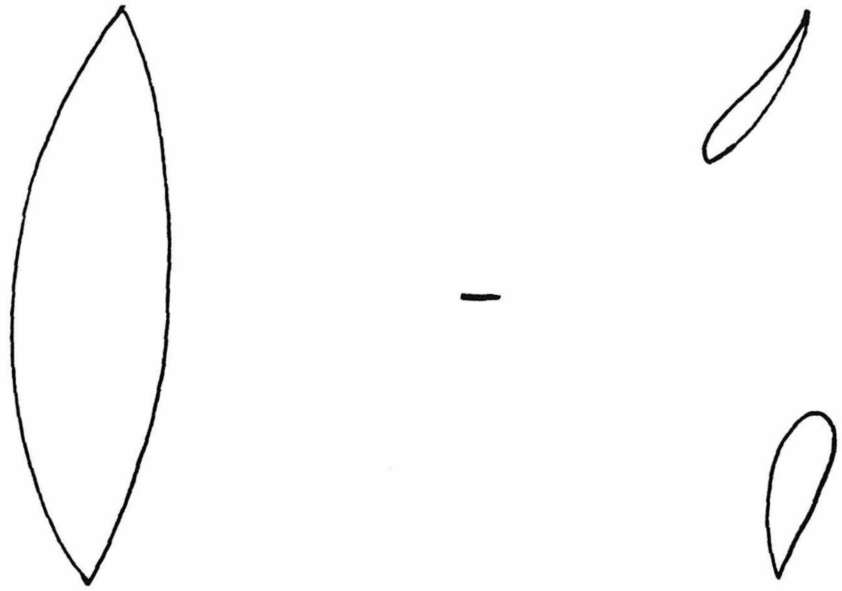


Fig. 1

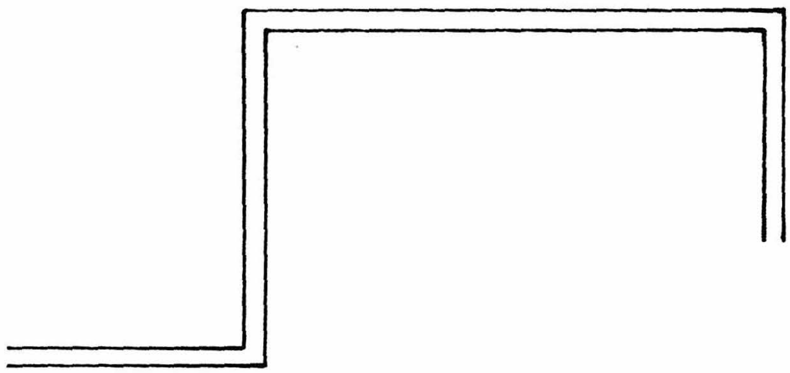


Fig. 2

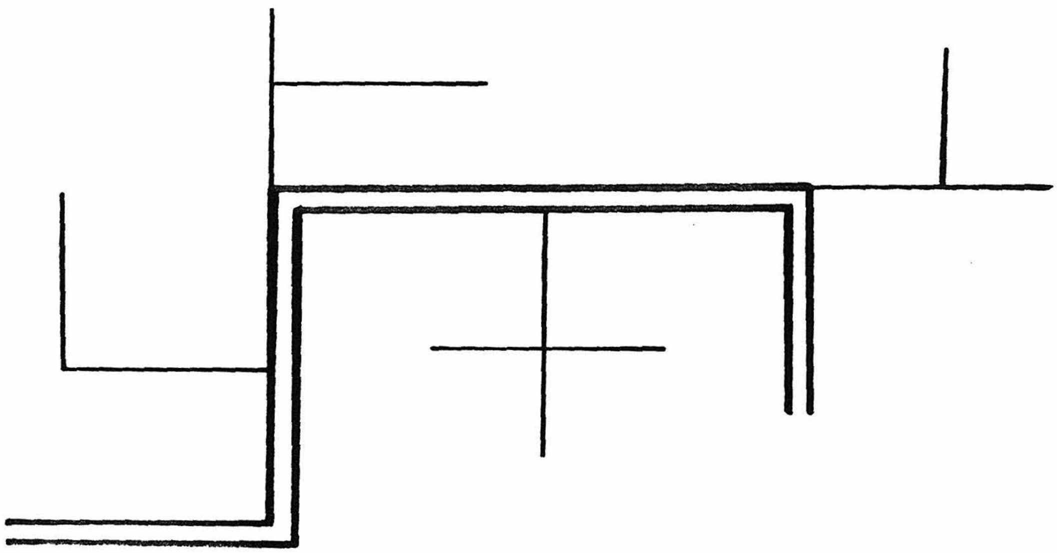


Fig. 3

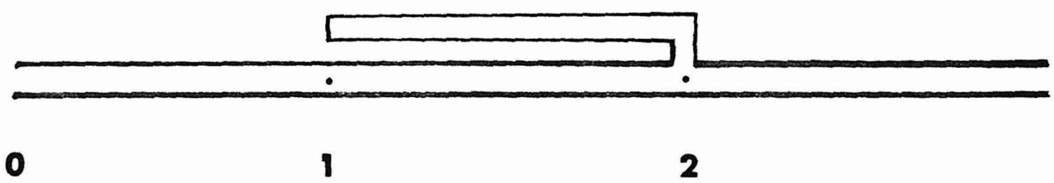


Fig. 4

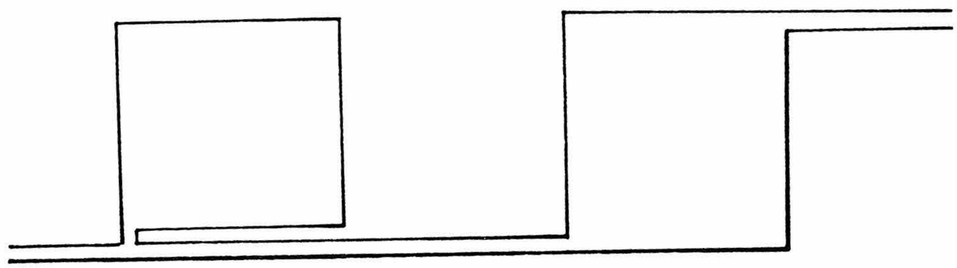


Fig. 5

Predictions of the meson masses in MeV.

	m_{K^*}	m_ϕ	m_D	m_{D^*}	m_F	m_{F^*}	m_{η_c}
strong-coupling limit	868	956	2048	2065	2111	2126	3098
strong-coupling approximation	866	951	2064	2077	2126	2137	3096
mean-field approximation	866	950	2066	2080	2127	2139	3096
experiment	892	1020	1868	2006	2030	2140	2979

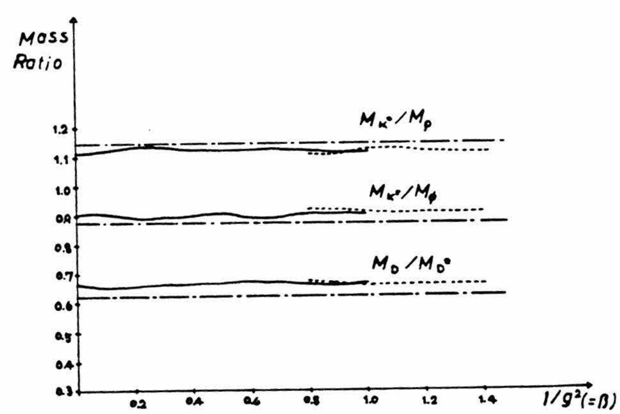


Fig. 6

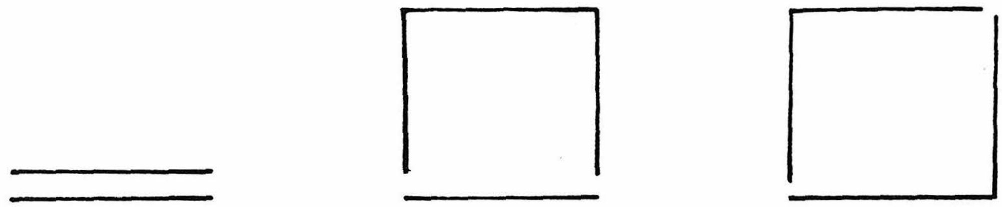


Fig. 7

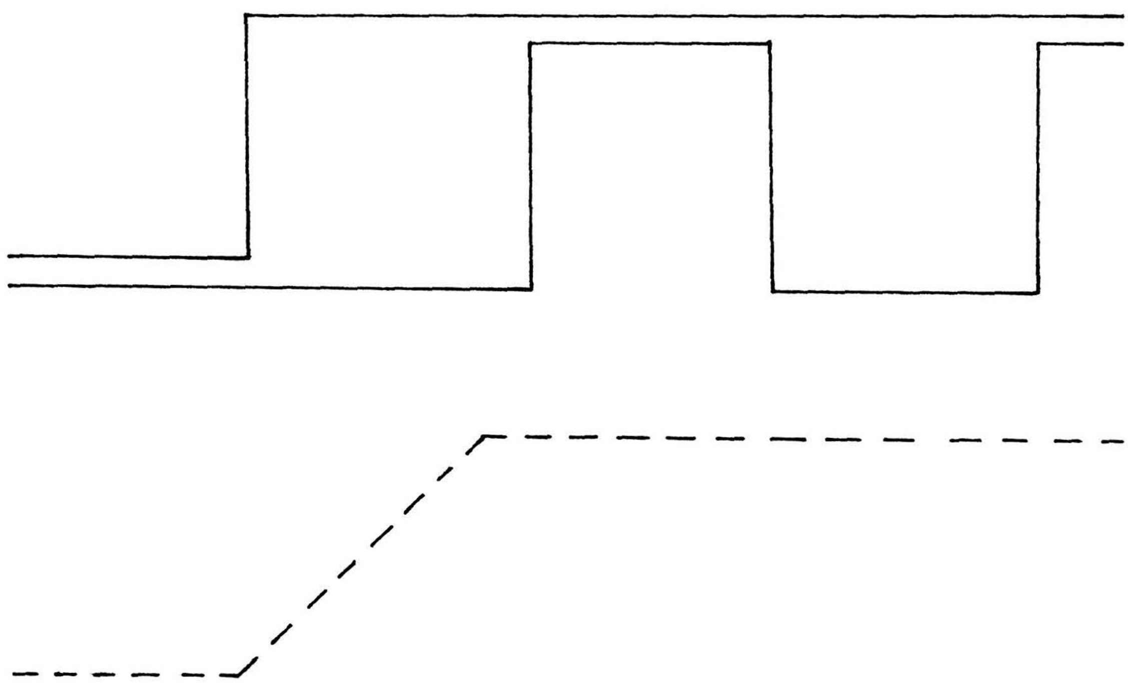


Fig. 8

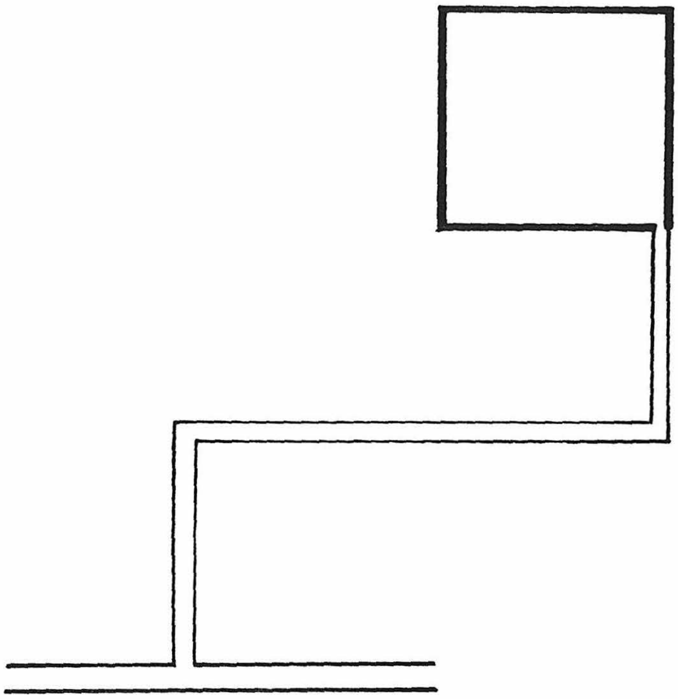
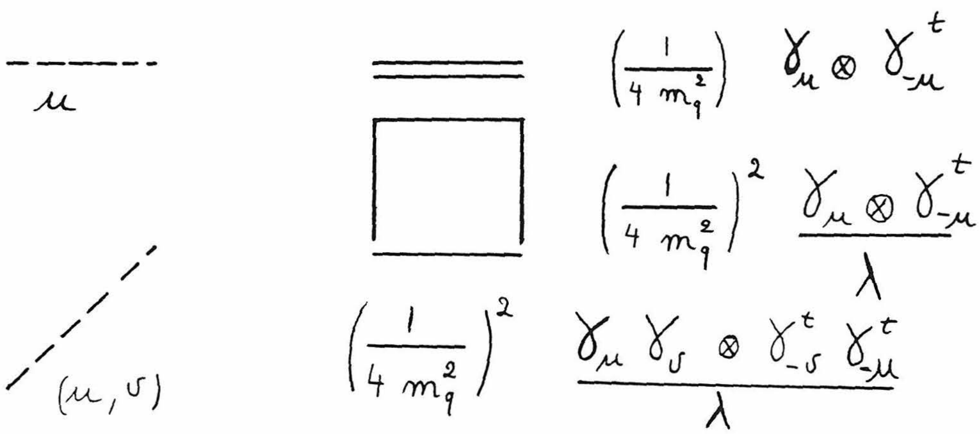


Fig. 9



μ
(μ, ν)
$\left(\frac{1}{4\, m_q^2}\right) \quad \gamma_\mu \otimes \gamma_{-\mu}^t$
$\left(\frac{1}{4\, m_q^2}\right)^2 \quad \frac{\gamma_\mu \otimes \gamma_{-\mu}^t}{\lambda}$
$\left(\frac{1}{4\, m_q^2}\right)^2 \quad \frac{\gamma_\mu \gamma_\nu \otimes \gamma_{-\nu}^t \gamma_{-\mu}^t}{\lambda}$

Fig. 10

Table Captions

Table 1

8x8 matrix determining the recursion relation in momentum space of non backtracking walks on the lattice.

Table 2

Masses of mesons and baryons.

a) naive fermions at infinite bare coupling.

b) experiment.

c) Kogut-Susskind fermions at $g^2 = 0.55$ from the second reference of [11].

d) Wilson fermions at $g^2 = 1$ from the first reference of [11].

Table 3

32x32 matrix determining the recursion relation in momentum space of walks contributing to the propagator to $O(\lambda^{-1})$.

e^{ik_1}	e^{ik_1}	e^{ik_1}	e^{ik_1}	0	e^{ik_1}	e^{ik_1}	e^{ik_1}
e^{ik_2}	e^{ik_2}	e^{ik_2}	e^{ik_2}	e^{ik_2}	0	e^{ik_2}	e^{ik_2}
e^{ik_3}	e^{ik_3}	e^{ik_3}	e^{ik_3}	e^{ik_3}	e^{ik_3}	0	e^{ik_3}
e^{ik_4}	e^{ik_4}	e^{ik_4}	e^{ik_4}	e^{ik_4}	e^{ik_4}	e^{ik_4}	0
0	e^{-ik_1}	e^{-ik_1}	e^{-ik_1}	e^{-ik_1}	e^{-ik_1}	e^{-ik_1}	e^{-ik_1}
e^{-ik_2}	0	e^{-ik_2}	e^{-ik_2}	e^{-ik_2}	e^{-ik_2}	e^{-ik_2}	e^{-ik_2}
e^{-ik_3}	e^{-ik_3}	0	e^{-ik_3}	e^{-ik_3}	e^{-ik_3}	e^{-ik_3}	e^{-ik_3}
e^{-ik_4}	e^{-ik_4}	e^{-ik_4}	0	e^{-ik_4}	e^{-ik_4}	e^{-ik_4}	e^{-ik_4}

Table 1

		Naive fermions	Experiment	Susskind fermions	Wilson fermions
$\bar{\psi}\gamma_5\psi$	m_π	$\sqrt{\frac{12ma}{\sqrt{7}}}\ \frac{1}{a}$	140	$\sqrt{(6.5\pm0.1)ma}\ \frac{1}{a}$	$\sqrt{6ma}\ \frac{1}{a}$
$\bar{\psi}\gamma_\mu\psi$	m_ρ	$\frac{1.76 + .81ma}{a}$	780	800±80	800±100
$\bar{\psi}\psi$	m_δ	—	980	950±100	1000±100
$\bar{\psi}\gamma_5\gamma_\mu\psi$	m_{A_1}	$\frac{2.63 + .32ma}{a} = 1155$	1270	—	1200±100
$\psi\psi\psi$	m_p m_Δ	$\frac{2.91 + 1.14ma}{a}$ $= 1285$	940 1230	— —	950±100 1300±100
INPUT		m_π, m_ρ		m_π string tension	m_π string tension

Table 2

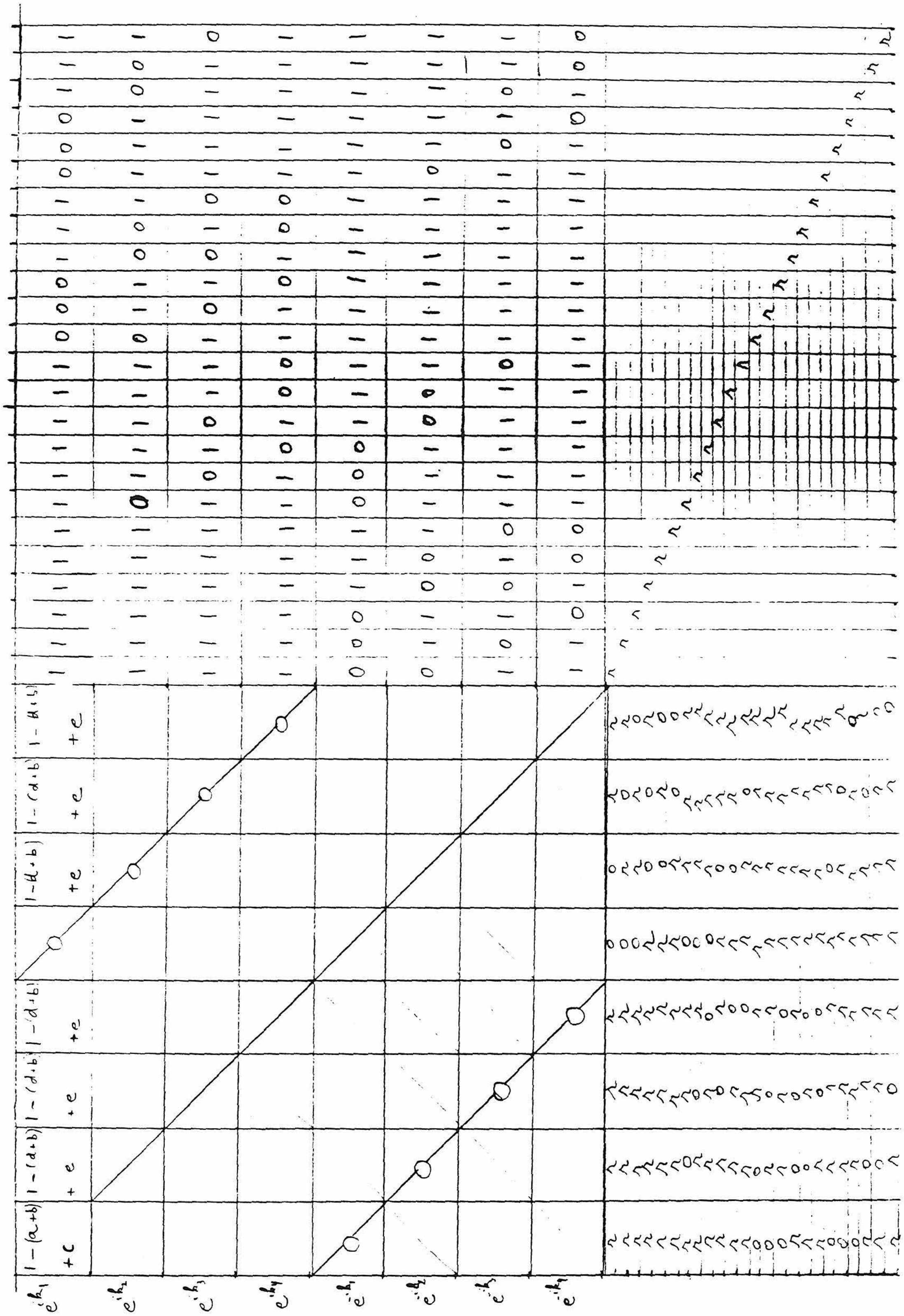


Table 3