STRESSES AND STRAINS IN CYLINDRICAL SHELLS

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1956

ACKNOWLEDGMENTS

The author wishes to express his appreciation to Professor George W. Housner for suggesting this problem, and for his guidance throughout its solution.

The author also wishes to express his gratitude to the Shell Fellowship Committee and to the AiResearch Manufacturing Company for the generous fellowships awarded the author during his residence at the California Institute of Technology.

ABSTRACT

As a consequence of assuming that the circumferential normal strain and in-plane shear strain of a cylindrical shell vanish, a relatively simple differential equation is derived which can be readily solved. An investigation of the frequencies of harmonic vibration of each Fourier component of the shell leads to a criterion determining a range of wave numbers within which the approximate method leads to satisfactory results. Experimental measurements show that solutions of the approximate equation are accurate except in the vicinity of a sharp variation of the load intensity, and so long as the length to diameter ratio of the shell is sufficiently large.

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Chapter I

INTRODUCTION AND HISTORICAL BACKGROUND

A common structural element is the thin-walled cylindrical shell which may be a simple load carrying member such as a monocoque air-frame or a roof panel; or, it may be part of a system of members, as in the case of petroleum production apparatus.

The general problem of the behavior of such a shell under the action of a system of radial and tangential loads has been formulated by many. Notably Love (1), Flugge (2), and Timoshenko (3) have, by the use of the thin shell approximation and linearization techniques, produced the so-called "exact" equations. These equations consist of a system of three partial differential equations in the shell displacements u, v, w, and are of the second, third and fourth orders respectively.

Recently Epstein (4) and Kennard (5) have undertaken a more fundamental formulation of these general equations. They are concerned with the higher order terms which are usually neglected, and they show that these terms have appreciable significance in obtaining the higher vibrational frequencies, especially of torsional modes. Both these papers and the work of Chien (6) take, as a starting point, the basic equations of three-dimensional elasticity, and obtain the equations of the thin shell by a limiting process.

Thus far, only two methods of solution are available for these equations, and these have serious disadvantages. Timoshenko (3) shows that a system of Double Fourier Series will solve the equations in general. However, from the difficulties associated with computing double series, only solutions for the very slowly varying loading functions can be obtained. On the other hand, Yuan (7) solved the Flugge equations for a concentrated force at an arbitrary point by transform methods. However, this solution is not amenable to any generalization to other loading conditions because of its size and complexity.

Other solutions for the general loading condition are available, but are approximations to the "exact" formulation of Love, Flugge and Timoshenko. These solutions resolve themselves into three classes, namely:

1. The Inextensional Theory
2. The Finsterwalder Approximation
3. Donnell's Equations

The theory of inextensional deformations of shells was derived by Lord Rayleigh(8, 9), and appears in his Theory of Sound (10). Love (1, 11) devotes his chapter 13 (1) to a formulation of the general inextensional theory; while Timoshenko (3) considers this theory as a method of solving the problem of concentrated radial loads acting on diametrically opposed points of the surface. Yuan (12) also considered this problem, but his method belongs in the third category. However, Hermes (13) has shown that, for the case of symmetrical line loads placed diametrically opposed, the inextensional theory results in unrealistic implications that fail to give a physically satisfactory representation of the response of the system.

The second class of solutions represents the work of German and Scandanavian civil engineers. In an attempt to solve the problem of the response of long shell-type roof structures to distributed loads, they generally assumed that the effects of the axial bending moment (M_x) and torque (M_{xy}) were unimportant. The first solution in this manner was proposed by Finsterwalder (14), who derived a single partial differential equation in the circumferential bending moment (M_y). This approach was further simplified by Schorer (15) in 1935, when he reduced the differential equation to two terms, again in the variable M_y, by restricting the solution to long spans only. However, even with the simplifications introduced in this paper, any application requires considerable computation. A paper by Odqvist (16) extends Schorer's solution, and presents results from which stresses can be obtained without extensive calculations.

The applicability of the above method to cases in which line or concentrated loads are applied to the structure, raised some doubts in the minds of Hoff (17) and Yuan (12). They replaced the differential equations arising from the condition of equilibrium as proposed by Love (1), Flugge (2) and Timoshenko (3) by a simpler statement proposed by Donnell (18). The so-called Donnell's Equation is obtained from the "exact" equations of Love, etc. by considering certain terms as having less significance than the leading terms. Hoff and associates are able to obtain a solution for the displacements, moments and membrane stresses arising from a sinusoidally distributed line load applied along a generator. Yuan solved the problem of a concentrated radial load on the surface of a finite cylinder using the method of images on a solution derived for a cylinder of infinite length. All the above solutions are applicable to the general problem by the method of superposition, if one assumes the accuracy of the differential equation to be sufficient, and thus represent the solution to the general problem for small displacements.

Therefore, given a distribution of loading on a thin cylindrical shell, one finds that applicable solutions are available only insofar as Hoff's solution and Yuan's solution can be applied by the method of superposition. The inextensional theory has been shown to be inapplicable on the basis of Hermes' work; and the method of Finsterwalder has been shown not to apply to shells in which there is considerable bending. However, the solutions proposed by Hoff and Yuan suffer from the same disadvantage as Yuan's more exact solution, in that they are of considerable complexity, restricting their usefulness to the simplest of loading conditions. In addition, it is doubtful if Donnell's equation is applicable in the vicinity of a discontinuity in loading.

It is the purpose of this paper to describe and develop a method of analysis that leads to solutions of an approximate differential equation. These solutions apply everywhere but in the immediate neighborhood of a discontinuity of loading, and give results in this neighborhood that will have an accuracy equal to or greater than that obtained with the classical equations of Donnell. These solutions have the further advantage of being much simpler in form than the solutions of Hoff and Yuan, and thus have a greater applicability to the general problem of radial loading.

This work is an extension of a previous paper by Hayashi (19) in which the "simple beam" concept of Chapter II, Section 2 first appears.

Chapter II

DESCRIPTION, ASSUMPTIONS AND DERIVATIONS

1. Basic Relations: Consider the following representation of the circular cylindrical shell shown in figure 1.

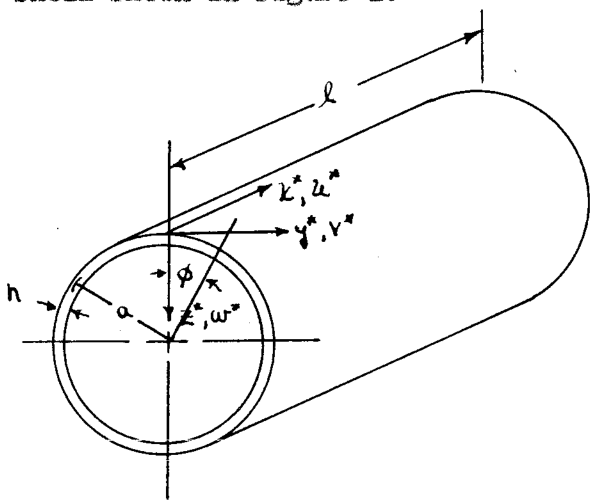


Figure 1.

The displacements of the shell are given as u^*, v^*, w^*, in the x^*, y^*, z^* directions respectively. The non-dimensional distances and displacements are defined by the equations

$$x = x^*/a \qquad \phi = y^*/a \qquad z = z^*/a$$

$$u = u^*/a \qquad v = v^*/a \qquad w = w^*/a \qquad \text{(II-1.1)}$$

where u, v, w are considered functions of (x, ϕ) only.

Considering an element of the cylindrical shell bounded by surfaces x^* = const., ϕ = const. shown in figure 2, the shear forces per unit length (Q) and bending moments per unit length (M) are indicated in their positive sense, along with the proper convention for normal and shear stresses.

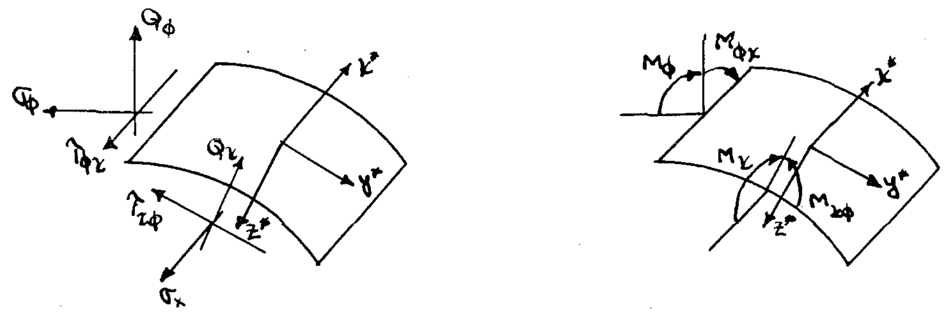


Figure 2.

If it is assumed that the shell is thin, that is, $h/a \ll 1$, one can make the conventional assumption that the shell is in the condition of plane stress, or that

$$\sigma_z \ , \ \gamma_{xz} \ , \ \gamma_{\phi z} \ = 0 \qquad \text{(II-1.2)}$$

so that,

$$\sigma_x = \frac{E}{1-\nu^2} \cdot (\epsilon_x + \nu \epsilon_\phi)$$

$$\sigma_\phi = \frac{E}{1-\nu^2} \cdot (\epsilon_\phi + \nu \epsilon_x)$$

$$\tau_{x\phi} = G \gamma_{x\phi} \qquad G = E/2(1+\nu) \qquad \text{(II-1.3)}$$

If one assumes that the membrane strains have negligible effect on the curvatures, then the strains (ϵ_x , ϵ_ϕ , $\gamma_{x\phi}$) are defined in terms of the membrane strains (ϵ_1 , ϵ_2 , γ) and the local curvatures (χ_x , χ_y , χ_{xy}) as follows:

$$\epsilon_x = \epsilon_1 - z^* \chi_x$$

$$\epsilon_\phi = \epsilon_2 - z^* \chi_y \qquad \text{(II-1.4)}$$

$$\gamma_{x\phi} = \gamma - 2z^* \chi_{xy}$$

where the curvatures are

$$k_x = w^*_{,x^*x^*}$$

$$k_y = \frac{1}{a^2}\left(w^* + w^*_{,\phi\phi}\right) \qquad \text{(II-1.5)}$$

$$k_{xy} = \frac{1}{a}\left(w^*_{,\phi x^*} + v^*_{,x^*}\right)$$

and the membrane strains are

$$\epsilon_1 = u^*_{,x^*}$$

$$\epsilon_2 = \frac{1}{a}\left(v^*_{,\phi} - w^*\right) \qquad \text{(II-1.6)}$$

$$\gamma = \frac{1}{a} u^*_{,\phi} + v^*_{,x^*}$$

The subscripts following a comma indicate differentiation. The bending moments per unit length are defined as

$$M_x = -D\left(k_x + \nu k_y\right)$$

$$M_\phi = -D\left(k_y + \nu k_x\right) \qquad \text{(II-1.7)}$$

$$M_{x\phi} = -M_{\phi x} = D(1-\nu)k_{xy} \qquad D = \frac{E h^3}{12(1-\nu^2)}$$

The shear forces per unit length are given by

$$Q_x = M_{x,x^*} + \frac{1}{a} \cdot M_{\phi x,\phi}$$

$$Q_\phi = \frac{1}{a} M_{\phi,\phi} - M_{x\phi,x^*} \qquad \text{(II-1.8)}$$

In agreement with Kirchhoff's suggestion, the total effective shear force per unit length is defined as

$$Q_{x\,eff} = Q_x - \frac{1}{a} M_{x\phi,\phi}$$
$$Q_{\phi\,eff} = Q_\phi + M_{\phi x,x^*} \qquad \text{(II-1.9)}$$

2. Simple Beam Concept: Consider the circular cylindrical shell shown in figure 1 to be the limiting case of an n-sided polygonal shell shown in figure 3; that is, the limiting case of a structure composed of elements of length ℓ and width $a\Delta\phi$, where $\Delta\phi = 2\pi/n$, as n becomes very large.

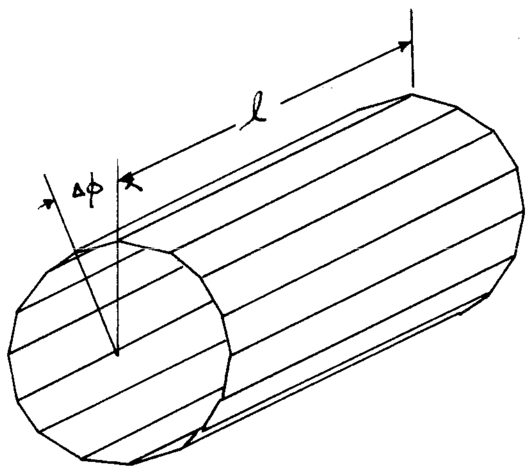


Figure 3.

Consider one of these component elements shown in figure 4. In general, the element will be acted upon by radial loads in the z^* direction, and stresses distributed along the sides $y^* = \pm a\Delta\phi/2$, $x^* = 0, \ell$. The stresses shown in figure 4a, b are those acting on the surface $z^* = 0$, or the membrane stresses.

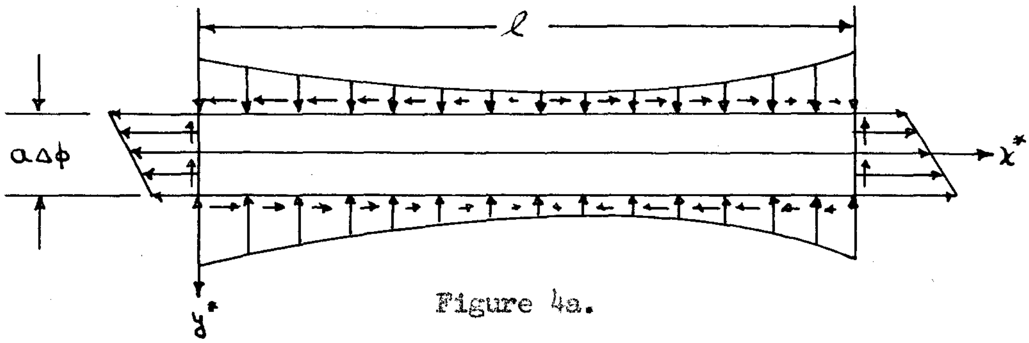


Figure 4a.

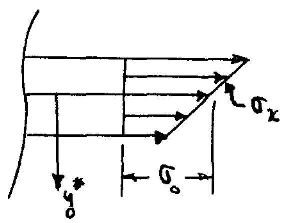


Figure 4b.

Figure 4b illustrates that for small $\Delta\phi$ one can take the axial membrane stress σ_x as being made up of a constant part σ_o plus a bending stress σ_b which varies linearly over the depth $a\Delta\phi$. The simplest method of computing these membrane stresses is to assume that the stresses in the elemental beam follow the simple beam theory. This theory, when applied with the assumption of plane stress, requires that

$$\sigma_b = \sigma_x - \sigma_o = -\frac{E}{1-\nu^2} y^* \cdot v^*_{,x^*x^*}$$

as a result of assuming

$$\epsilon_1 = y^*/r \qquad \text{and} \qquad \sigma_x = \frac{E}{1-\nu^2} \cdot \epsilon_1$$

that is, $\qquad \epsilon_2 = 0$

where r is the local radius of curvature ($\approx 1/v^*_{,x^*x^*}$) of the x^*-axis in the x^*,y^*-plane. Furthermore, if one differentiates σ_b and σ_x with respect to y^* and uses the identity for ϵ_1, it can be shown that this requires

$$\gamma = 0$$

in accordance with simple beam theory.

Thus, applying the concepts of simple beam theory to cylindrical shells conceived as being made up of beam elements of infinitesimal depth, one obtains two relations between the three displacements, that is,

$$v^*_{,\phi} = w^* \qquad \text{II-2.1}$$

$$v^*_{,x} + \frac{1}{a} u^*_{,\phi} = 0 \qquad \text{II-2.2}$$

which state respectively that $\epsilon_x = 0$ and $\gamma = 0$.

3. <u>Derivation of the Approximate Differential Equations</u>: In order to study the behavior of a shell under the restrictions $\epsilon_x, \gamma = 0$, the differential equations governing the behavior of $u(x,\phi)$, $v(x,\phi)$, $w(x,\phi)$ must be formulated and then solved for the functions themselves. The usual manner of formulating these differential equations is to express the three equations of equilibrium in terms of the three displacement functions. However, the restrictions $\epsilon_x, \gamma = 0$ preclude this, for the forces can no longer be expressed in terms of the displacements.

The method that is applicable in this case is that of the Calculus of Variations with a Subsidiary Condition. This method leads to differential equations describing a shell acted upon by a general system of forces in which the deformation of the shell will only be an approximation to the exact solution, but the total energy of the system will be a minimum consistent with $\epsilon_x, \gamma = 0$.

The total energy of the system U is defined as the sum of the strain energy of the shell V and the potential energy of any external forces. This potential energy is equal to the negative of the work (T) done by these external forces acting on the shell as it deforms. Thus,

$$U = V - T \tag{II-3.1}$$

The strain energy per unit volume V_o, as given by Timoshenko (20), when the plane stress approximation (eq. II-1.2) is applied, reduces to

$$V_o = \frac{1}{2E}\cdot(\sigma_x^2 + \sigma_y^2) - \frac{\nu}{E}\cdot\sigma_x\sigma_y + \frac{1}{2G}\gamma_{xy}^2$$

or alternatively, using equation II-1.3, as

$$V_o = \frac{E}{2(1-\nu^2)}\left[\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y + \frac{1-\nu}{2}\gamma_{xy}^2\right] \tag{II-3.2}$$

The total strain energy V is given by the integral of the above expression throughout the volume of the shell, that is,

$$V = \iiint V_o \, a\, dx^* d\phi\, dz^* \tag{II-3.3}$$

where it is assumed that $h/a << 1$ so that the volume element of integration is given by

$$d(\text{VOLUME}) = a\left(1 - \frac{z^*}{a}\right) dx^* d\phi\, dz^* \doteq a\, dx^* d\phi\, dz^*$$

Assuming that the only forces acting on the shell are radial loads over the surface of the shell given by the distribution $q(x, \phi)$, the external work becomes

$$T = \iint q(x^*, \phi)\, w^*\, a\, dx^*\, d\phi \tag{II-3.4}$$

Axial and tangential components of load could be included in the same fashion.

If equations II-1.4, II-1.5, II-1.6 are substituted into equation II-3.2, taking into account equations II-2.1, II-2.2, then V_o becomes a function of u^*, w^* only. Furthermore, if equation II-3.2 is substituted into equation II-3.3, the integration over z^* can be carried out directly, as u^*, w^* are functions of x^*, ϕ only. Finally, substituting the expression for V, T into equation II-3.1, the total energy of the system U becomes

$$U = \iint \left\{ \frac{Eh}{2(1-\nu^2)} \left[u^{*2}_{,x^*} + \frac{h^2}{12} w^{*2}_{,x^*x^*} + \frac{h^2}{12a^4}\left(w^* + w^*_{,\phi\phi}\right)^2 + \frac{(1-\nu)h^2}{6a^2}\cdot\left(w^*_{,x^*\phi} - \frac{1}{a} u^*_{,\phi}\right)^2 + \frac{\nu h^2}{6a^2}\cdot w^*_{,x^*x^*}\left(w^* + w^*_{,\phi\phi}\right) \right] - w^* q \right\} a\, dx^*\, d\phi \tag{II-3.5}$$

where u^*, w^* are related by the subsidiary condition that

$$G = w^*_{,x^*} + \frac{1}{a} u^*_{,\phi\phi} = 0 \tag{II-3.6}$$

which is obtained by eliminating v^* from equations II-2.1, II-2.2.

In order to obtain the differential equations of equilibrium from this expression for total energy U, it is necessary that the variation of U vanish subject to the subsidiary condition of equation II-3.6. This is accomplished, according to the method outlined in Courant and Hilbert (21), by introducing the subsidiary condition into the total energy expression, thus forming a new function U^*. This new function U^* is equal to U plus the result of first multiplying the subsidiary condition G by an arbitrary function Λ and then integrating this product over the surface of the shell, that is

$$U^* = U + \iint \Lambda(x^*, \phi)\, G \cdot dx^* a\, d\phi$$

The function U^* is then required to be a minimum for a suitable value of the parameter Λ. Thus it follows that

$$\delta U^* = 0$$

or that,

$$\delta \iint \Bigg\{ \frac{Eh}{2(1-\nu^2)} \Bigg[u_{,x^*}^{*2} + \frac{h^2}{12} \cdot w_{,x^*x^*}^{*2} + \frac{h^2}{12a^4} \cdot \left(w^* + w_{,\phi\phi}^*\right)^2 + \frac{\nu h^2}{6a^2} w_{,x^*x^*}^* \left(w^* + w_{,\phi\phi}^*\right) + \frac{(1-\nu)h^2}{6a^2} \cdot \left(w_{,x^*\phi}^* - \frac{1}{a} u_{,\phi}^*\right)^2 \Bigg] - w^* q + \Lambda\left(w_{,x^*}^* + \frac{1}{a} u_{,\phi\phi}^*\right) \Bigg\} dx^* a\, d\phi = 0 \qquad \text{(II-3.7)}$$

This operation has been carried out for two cases: 1) the complete cylinder, for which the limits of integration are $x_1^* \leq x^* \leq x_2^*$, $\phi_1 \leq \phi \leq \phi_2$ where $\phi_2 - \phi_1 = 2\pi$; and 2) the curved panel, for which the limits of integration are $x_1^* \leq x^* \leq x_2^*$, $0 \leq \phi \leq \phi_0$ where in both cases $x_2^* - x_1^* = \ell$.

Due to the commutation between the variation and integration processes, and between the differentiation and variation processes, as shown in Hildebrand (22), the requirement that U^* be stationary reduces to:

$$
\begin{aligned}
& 2\cdot\frac{1-\nu}{a}\cdot D\Bigg[\left|\left(w^*_{,r^*\phi}-\frac{1}{a}\cdot u^*_{,\phi}\right)\delta w^*\right|_{\substack{\phi=\phi_2\\ r^*=r^*_2}}-\left|\left(w^*_{,r^*\phi}-\frac{1}{a}\cdot u^*_{,\phi}\right)\delta w^*\right|_{\substack{\phi=\phi_2\\ r^*=r^*_1}} \\
&\qquad +\left|\left(w^*_{,r^*\phi}-\frac{1}{a}\cdot u^*_{,\phi}\right)\delta w^*\right|_{\substack{\phi=\phi_1\\ r^*=r^*_1}}-\left|\left(w^*_{,r^*\phi}-\frac{1}{a}\cdot u^*_{,\phi}\right)\delta w^*\right|_{\substack{\phi=\phi_1\\ r^*=r^*_2}}\Bigg] \\
&+\int_{r^*_1}^{r^*_2}\left|\frac{D}{a}\left[\frac{1}{a^2}\left(w^*+w^*_{,\phi\phi}\right)+\nu\, w^*_{,r^*r^*}\right]\delta\left(w^*_{,\phi}\right)\right|_{\phi_1}^{\phi_2}dr^* \\
&+\int_{r^*_1}^{r^*_2}\left|-\frac{D}{a}\left[\frac{1}{a^2}\cdot w^*_{,\phi}+\frac{1}{a^2}\,w^*_{,\phi\phi\phi}+(2-\nu)\,w^*_{,\phi r^*r^*}-2\cdot\frac{1-\nu}{a}\cdot u^*_{,r^*\phi}\right]\delta w^*\right|_{\phi_1}^{\phi_2}dr^* \\
&+\int_{r^*_1}^{r^*_2}\left|\left[2\cdot\frac{1-\nu}{a^2}\cdot D\left(\frac{1}{a}\cdot u^*_{,\phi}-w^*_{,r^*\phi}\right)-\Lambda_{,\phi}\right]\delta u^*\right|_{\phi_1}^{\phi_2}dr^* \\
&+\int_{r^*_1}^{r^*_2}\left|\Lambda\ \delta\left(u^*_{,\phi}\right)\right|_{\phi_1}^{\phi_2}dr^* \quad+\int_{\phi_1}^{\phi_2}\frac{Eh}{1-\nu^2}\left|u^*_{,r^*}\ \delta u^*\right|_{r^*_1}^{r^*_2}a\,d\phi \\
&+\int_{\phi_1}^{\phi_2}\left|\left[\Lambda-D\left(w^*_{,r^*r^*r^*}+\frac{\nu}{a^2}\cdot w^*_{,r^*}-2\cdot\frac{1-\nu}{a^3}\cdot u^*_{,\phi\phi}+\frac{2-\nu}{a^2}\cdot w^*_{,r^*\phi\phi}\right)\right]\delta w^*\right|_{r^*_1}^{r^*_2}a\,d\phi \\
&+\int_{\phi_1}^{\phi_2}\left|D\left[w^*_{,r^*r^*}+\frac{\nu}{a^2}\left(w^*+w^*_{,\phi\phi}\right)\right]\delta\left(w^*_{,r^*}\right)\right|_{r^*_1}^{r^*_2}a\,d\phi \\
&+\int_{r^*_1}^{r^*_2}\int_{\phi_1}^{\phi_2}\Bigg[\frac{Eh}{1-\nu^2}\left(-u^*_{,r^*r^*}+h^2\cdot\frac{1-\nu}{6a^3}\cdot w^*_{,r^*\phi\phi}-h^2\cdot\frac{1-\nu}{6a^4}\cdot u^*_{,\phi\phi}\right) \\
&\qquad\qquad +\frac{1}{a}\,\Lambda_{,\phi\phi}\Bigg]\delta u^*\,dr^*\,a\,d\phi
\end{aligned}
$$

$$+\int_{x_1^*}^{x_2^*}\int_{\phi_1}^{\phi_2}\Big[D\Big(w^*_{,x^*x^*x^*x^*} + \frac{2}{a^2}\cdot w^*_{,x^*x^*\phi\phi} + \frac{1}{a^4}\cdot w^*_{,\phi\phi\phi\phi} + \frac{1}{a^4}\cdot w^*$$

$$+ \frac{2}{a^4}\cdot w^*_{,\phi\phi} + \frac{2\nu}{a^2}\cdot w^*_{,x^*x^*} - 2\cdot\frac{1-\nu}{a^3}\cdot u^*_{,x^*\phi\phi}\Big)$$

$$-\left(q + \Lambda_{,x^*}\right)\Big]\,\delta w^* dx^* a\, d\phi \quad = \quad 0$$

where all expressions apply to the case of the curved panel, and only the latter five integrals apply to the case of the complete cylinder. The first four terms and the line integrals give rise to the boundary conditions; while the surface integrals produce the required differential equations. Thus, the "natural" boundary conditions for a curved panel are, on the corners of the panel

$$D\cdot\frac{1-\nu}{a}\cdot\left(w^*_{,x^*\phi} - \frac{1}{a}\cdot u^*_{,\phi}\right) = 0 \qquad \text{(II-3.8)}$$

$$\text{or,} \qquad w^* = 0$$

on the curved boundary $(x^* = x_1^*, x_2^*)$,

$$u^*_{,x^*} = 0 \qquad \text{or,} \qquad u^* = 0 \qquad \text{(II-3.9)}$$

$$\Lambda - D\left(w^*_{,x^*x^*x^*} + \frac{\nu}{a^2}\cdot w^*_{,x^*} - 2\cdot\frac{1-\nu}{a^3}\cdot u^*_{,\phi\phi} + \frac{2-\nu}{a^2}\cdot w^*_{,x^*\phi\phi}\right) = 0$$

$$\text{or,} \qquad w^* = 0 \qquad \text{(II-3.10)}$$

$$D\left[w^*_{,r^*r^*} + \frac{\nu}{a^2}\left(w^* + w^*_{,\phi\phi}\right)\right] = 0$$

$$\text{or,} \qquad w^*_{,r^*} = 0 \qquad \text{(II-3.11)}$$

on the straight boundary $\phi = \phi_1 ; \phi_2$,

$$\frac{1-\nu}{a}\cdot D\left(w^*_{,r^*\phi} - \frac{1}{a}\cdot u^*_{,\phi}\right) + \frac{a}{\lambda}\Lambda_{,\phi} = 0$$

$$\text{or,} \qquad u^* = 0 \qquad \text{(II-3.12)}$$

$$-\frac{D}{a}\left[\frac{1}{a^2}w^*_{,\phi} + \frac{1}{a^2}\cdot w^*_{,\phi\phi\phi} + (2-\nu)\,w^*_{,\phi r^* r^*} - 2\cdot\frac{1-\nu}{a}\cdot u^*_{,r^*\phi}\right] = 0$$

$$\text{or,} \qquad w^* = 0 \qquad \text{(II-3.13)}$$

$$D\left[\frac{1}{a^2}\left(w^* + w^*_{,\phi\phi}\right) + \nu\, w^*_{,r^*r^*}\right] = 0$$

$$\text{or,} \qquad w^*_{,\phi} = 0 \qquad \text{(II-3.14)}$$

$$\Lambda = 0 \qquad \text{or,} \qquad u^*_{,\phi} = 0 \qquad \text{(II-3.15)}$$

The "natural" boundary conditions for a complete cylinder are equations II-3.9, II-3.10, II-3.11 as the other integrals vanish from the periodicity condition of the functions u^*, w^*.

It must be noted that, unlike the boundary conditions for a holonomic system, the above conditions are not independent but are related through the subsidiary condition $C = 0$. Thus, for example, we are free to choose the boundary conditions on w^* and its derivatives, but as a result, the boundary conditions on u^* and its derivatives are fixed. This will be illustrated in later examples.

Making use of the definitions given in Chapter II-1 and the identities in equations II-2.1, II-2.2, the above boundary conditions expressed in physical quantities are:

on the corners of the panel,

$$M_{x\phi} = 0 \quad \text{or,} \quad w^* = 0$$

on the curved boundary ($x^* = x_1^*$, x_2^*),

$$N_x = \int \sigma_x \, dz^* = 0$$

$$\text{or,} \quad u^* = 0$$

$$\Lambda + Q_{x_{eff}} = 0 \quad \text{or,} \quad w^* = 0$$

$$M_x = 0 \quad \text{or,} \quad w^*_{,x^*} = 0$$

on the straight boundary ($\phi = \phi_1 , \phi_2$),

$$M_{\phi x} - \frac{a}{2} \cdot \Lambda_{,\phi} = 0$$

$$\text{or,} \quad u^* = 0$$

$$Q_{\phi_{eff}} = 0 \quad \text{or,} \quad w^* = 0$$

$$M_\phi = 0 \quad \text{or,} \quad w^*_{,\phi} = 0$$

$$\Lambda = 0 \quad \text{or,} \quad u^*_{,\phi} = 0$$

where the value of $M_{x\phi}$ at the corners of the panel measures the value of the concentrated reactions at the corners.

Considering the surface integrals that result from the variational procedure, it is noted that the coefficient of the integrand of the first integral is δw^*; while the coefficient of the integrand of the second integral is δu^*. Again, these variations are not independent, and an argument different from that used for holonomic systems must be used to determine the differential equations. If Λ is chosen so that the coefficient of δu^* vanishes, then, since the variation δw^* can be arbitrarily assigned inside $x_1^* \leq x^* \leq x_2^*$; $\phi_1 \leq \phi \leq \phi_2$, the coefficient of δw^* must vanish. Thus, the following differential equations are obtained:

$$-\frac{Eh}{1-\nu^2}\left[u^*_{,x^*x^*} - \frac{1-\nu}{6a^3}\cdot h^2\left(w^*_{,x^*\phi\phi} - \frac{1}{a}\cdot u_{,\phi\phi}\right)\right] + \frac{1}{a}\cdot\Lambda_{,\phi\phi} = 0 \qquad \text{(II-3.16)}$$

$$D\left[\nabla_*^4 w^* + \frac{1}{a^4}\cdot w^* + \frac{2}{a^4}\cdot w^*_{,\phi\phi} + \frac{2\nu}{a^2}\cdot w^*_{,x^*x^*} - 2\cdot\frac{1-\nu}{a^3}\cdot u^*_{,x^*\phi\phi}\right] - q - \Lambda_{,x^*} = 0 \qquad \text{(II-3.17)}$$

in addition to the subsidiary condition (eq. II-3.6), making three differential equations in the three dependent variables w^*, u^*, Λ . Let the following notation be defined

$$\nabla_*^4 w^* = \nabla_*^2\nabla_*^2 w^* = w^*_{,x^*x^*x^*x^*} + \frac{2}{a^2}\cdot w^*_{,x^*x^*\phi\phi} + \frac{1}{a^4}\cdot w^*_{,\phi\phi\phi\phi}$$

If equation II-3.16 is operated on by $a\frac{\partial}{\partial x^*}$ and equation II-3.17 is operated on by $\frac{\partial^2}{\partial\phi^2}$ and the resulting equations added, one obtains an equation independent of Λ , that is,

$$D\left[\nabla_*^4 w^* + \frac{1}{a^4}\cdot w^* + \frac{2}{a^4}\, w^*_{,\phi\phi} + \frac{2\nu}{a^2}\cdot w^*_{,x^*x^*} - 2\cdot\frac{1-\nu}{a^3}\cdot u^*_{,x^*\phi\phi} - q/D\right]_{,\phi\phi}$$

$$- \frac{Eah}{1-\nu^2}\left[u^*_{,x^*x^*} - \frac{1-\nu}{6a^3}\cdot h^2\cdot\left(w^*_{,x^*\phi\phi} - \frac{1}{a}\, u^*_{,\phi\phi}\right)\right]_{,x^*} = 0 \qquad \text{(II-3.18)}$$

This equation, plus the subsidiary condition (eq. II-3.6)

$$w^*_{,x^*} + \frac{1}{a}\cdot u^*_{,\phi\phi} = 0$$

give two equations in the two dependent variables u^*, w^*. If equation II-3.18 is operated on by $\frac{\partial^2}{\partial\phi^2}$ and equation II-3.6 is operated on by $\frac{\partial^3}{\partial x^{*3}}$ and the resulting equations added, one obtains

$$\left[\nabla_*^4 w^* + \frac{2}{a^2}\cdot\nabla_*^2 + \frac{1}{a^4}\cdot w^* - q/D\right]_{,\phi\phi\phi\phi}$$

$$+ 12\left(\frac{a}{h}\right)^2 w^*_{,x^*x^*x^*x^*} + 2\cdot\frac{1-\nu}{a^2}\cdot\left(w^* + w^*_{,\phi\phi}\right)_{,x^*x^*\phi\phi} = 0 \qquad \text{(II-3.19)}$$

Introducing the dimensionless variables (eq. II-1.1), along with

$$p = q a^3/D \qquad\qquad \eta = h/a$$

$$\nabla^4 w = \nabla^2\nabla^2 w = w_{,xxxx} + 2 w_{,xx\phi\phi} + w_{,\phi\phi\phi\phi} \qquad \text{(II-3.20)}$$

one obtains equation II-3.19 in the form

$$\left[\nabla^4 w + 2\nabla^2 w + w - p\right]_{,\phi\phi\phi\phi} + 12/\eta^2\cdot w_{,xxxx} + 2(1-\nu)\left(w + w_{,\phi\phi}\right)_{,xx\phi\phi} = 0 \qquad \text{(II-3.21)}$$

This equation plus the constraints that

$$v_{,\phi} = w \qquad\qquad v_{,x} + u_{,\phi} = 0$$

completely determine the displacements once the loading function and boundary conditions are prescribed.

Equation II-3.21 can be compared with the "exact" equations of Timoshenko (3) which are, in the present notation,

$$u_{,xx} + \frac{1-\nu}{2} \cdot u_{,\phi\phi} + \frac{1+\nu}{2} \cdot v_{,x\phi} - \nu w_{,x} = 0$$

$$v_{,\phi\phi} + \frac{1-\nu}{2} \cdot v_{,xx} + \frac{1+\nu}{2} \cdot u_{,x\phi} - w_{\phi}$$
$$+ \eta^2/12 \; \nabla^2 w_{,\phi} + \eta^2/12 \left(\nabla^2 v - \nu v_{,xx} \right) = 0$$

$$\nabla^4 w - 12/\eta^2 \left(\nu u_{,x} + v_{,\phi} - w \right) + \left[\nabla^2 v + (1-\nu) v_{,xx} \right]_{,\phi} = p$$

Donnell's Equations, simplified by Hoff and Yuan, are

$$\nabla^8 w + \frac{1-\nu^2}{\eta^2} \cdot w_{,xxxx} - \nabla^4 p = 0$$

$$\nabla^4 u = \nu w_{,xxx} - w_{,x\phi\phi}$$

$$\nabla^4 v = (2+\nu) w_{,\phi xx} + w_{,\phi\phi\phi}$$

4. <u>Reduction to a Total Differential Equation:</u> In this section and in the following sections, we shall not consider the case of the complete

cylinder or the curved panel in which one or more of the sides are free. It is only in these cases that the function Λ must be computed. Instead, attention will be restricted to the cases in which a complete cylinder or curved panel is supported such that the deflection w vanishes at the boundaries.

a) <u>The Complete Cylinder:</u> Consider the case of the complete cylinder shown in figure 5.

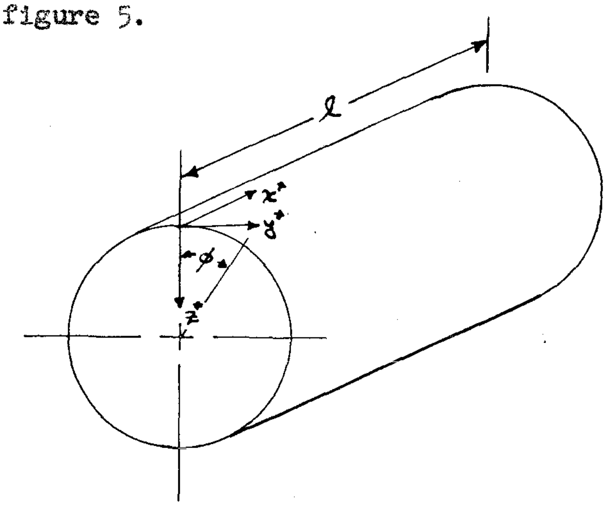


Figure 5.

In the interval $-\pi \leq \phi \leq \pi$, the functions $w(x,\phi)$, $p(x,\phi)$ can be expressed as

$$w(x,\phi) = \sum_{n=0}^{\infty} W(x,n) \cos n\phi \qquad \text{(II-4.1)}$$

$$p(x,\phi) = \sum_{n=0}^{\infty} \bar{p}(x,n) \cos n\phi \qquad \text{(II-4.2)}$$

where $w(x,\phi)$ must satisfy equation II-3.21, and $\bar{p}(x,n)$ is given by

$$\bar{p}(x,n) = \frac{1}{\pi}\int_{-\pi}^{\pi} p(x,\phi) \cos n\phi \, d\phi \qquad \text{(II-4.3)}$$

If equations II-4.1, II-4.2 are substituted into equation II-3.21, one obtains

$$\sum_{n=0}^{\infty} \cos n\phi \left[(n^4 + 12/\eta^2) W_{,xxxx} - 2n^2 W_{,xx} (n^4 - \overline{2-\nu}\, n^2 + 1-\nu) + n^4 W (n^2-1)^2 - n^4 \bar{p} \right] = 0$$

so that,

$$(n^4 + 12/\eta^2) W_{,xxxx} - 2n^2 W_{,xx} (n^4 - \overline{2-\nu}\, n^2 + 1-\nu) + n^4 W (n^2-1)^2 = \bar{p}\, n^4 \qquad \text{(II-4.4)}$$

Thus, the result of assuming $\epsilon_z, \gamma = 0$ is the reduction of the classical 8th order equation as obtained by Flugge (2) to a 4th order equation.

Substituting equation II-4.1 into equations II-3.10, II-3.11 with the definitions contained in equation II-1.1, one obtains the following boundary conditions on x = 0, ℓ/a : either,

$$\left. \begin{aligned} W &= 0 \\ W_{,x} &= 0 \end{aligned} \right\} \text{"Fixed End"} \qquad \text{(II-4.5)}$$

or

$$\left. \begin{aligned} W &= 0 \\ W_{,xx} &= 0 \end{aligned} \right\} \text{"Pin-Ended"} \qquad \text{(II-4.6)}$$

It is apparent with these boundary conditions that for n = 0, W(x,0) = 0, so that equation II-4.1 reduces to:

$$w(x,\phi) = \sum_{n=1}^{\infty} W(x,n) \cos n\phi \qquad \text{(II-4.7)}$$

Introducing the dimensionless variables of equation II-1.1 into equation II-3.6, the subsidiary condition becomes

$$w_{,x} + u_{,\phi\phi} = 0$$

and integrates to

$$u(x,\phi) = \sum_{n=1}^{\infty} \frac{1}{n^2} W_{,x} \cos n\phi + \phi f(x) + g(x) \qquad \text{(II-4.8)}$$

From the requirement of periodicity of $u(x,\phi)$, that is,

$$u(x,\phi) = u(x,\phi + 2\pi)$$

it follows that $f = 0$. If equation II-4.8 is substituted into equation II-3.18, one finds that g must satisfy

$$g_{,xxx} = 0$$

or,

$$g(x) = C_1 x^2 + C_2 x + C_3$$

As u = const. corresponds to translation of the cylinder as a rigid body, which produces no strain, C_3 may be set equal to zero without loss of generality. Furthermore, since radial loads are the only external forces being considered, C_1, C_2 can be taken as equal to zero, as these constants correspond to a shell acting as a column supporting its own weight and to a shell in uniform axial compression respectively. Thus u becomes

$$u(x,\phi) = \sum_{n=1}^{\infty} \frac{1}{n^2} W_{,x} \cos n\phi \qquad \text{(II-4.9)}$$

With equation II-4.9 it is evident that the boundary conditions on u cannot be prescribed independently of the boundary conditions on w. From equation II-4.9, it follows that

$$W_{,x} = \frac{n^2}{\pi} \int_{-\pi}^{\pi} u(x,\phi) \cos n\phi \, d\phi$$

Therefore, requiring u to vanish on the boundary implies that $W_{,x}$ vanishes; or, requiring $u_{,x}$ to vanish implies that $W_{,xx}$ vanishes.

Thus, for the case of the complete cylinder, the deflections w, u are given by equations II-4.7, II-4.9 respectively, and must satisfy the differential equation II-4.4 with the boundary conditions arbitrarily chosen between

$$\left.\begin{aligned} w &= 0 \\ w_{,x} &= 0 \\ u &= 0 \end{aligned}\right\} \text{"Fixed End"}$$

and,

$$\left.\begin{aligned} w &= 0 \\ M_x &= 0 \\ u_{,x} &= 0 \end{aligned}\right\} \text{"Pin-Ended"}$$

b) <u>The Curved Panel:</u> Consider the Case of the curved panel shown in figure 6.

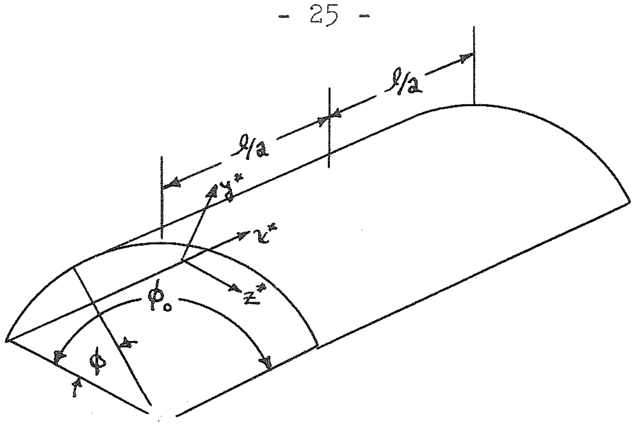


Figure 6

In the interval $0 \leq \phi \leq \phi_0$, the functions $w(x,\phi)$, $p(x,\phi)$ can be expressed as

$$w(x,\phi) = \sum_{n=1}^{\infty} W(x,n) \sin n\pi\phi/\phi_0 \tag{II-4.10}$$

$$p(x,\phi) = \frac{2}{\phi_0} \sum_{n=1}^{\infty} \bar{p}(x,n) \sin n\pi\phi/\phi_0 \tag{II-4.11}$$

where $w(x,\phi)$ must satisfy equation II-3.21, and $\bar{p}(x,n)$ is given by

$$\bar{p}(x,n) = \int_0^{\phi_0} p(x,\phi) \cdot \sin n\pi\phi/\phi_0 \cdot d\phi \tag{II-4.12}$$

However, as will be shown, only certain boundary conditions on $\phi = 0, \phi_0$ can be satisfied.

If equations II-4.10, II-4.11 are substituted into equation II-3.21, one obtains

$$\sum_{n=1}^{\infty} \sin n\pi\phi/\phi_0 \Big[(\bar{n}^4 + 12/\eta^2) W_{,xxxx} - 2\bar{n}^2 W_{,xx} (\bar{n}^4 - \overline{2-\nu}\, n^2 + 1 - \nu)$$

$$+ \bar{n}^4 (\bar{n}^2 - 1)^2 W - 2\bar{n}^4 \bar{p}/\phi_0 \Big] = 0$$

so that,

$$\left(\bar{n}^4 + 12/\eta^2\right)W_{,xxxx} - 2\bar{n}^2 W_{,xx}\cdot\left(\bar{n}^4 - \overline{2-\nu}\,\bar{n}^2 + 1-\nu\right) + \bar{n}^4\left(\bar{n}^2-1\right)^2 W = 2\bar{n}^4 \bar{p}/\phi_o \qquad \text{(II-4.13)}$$

where $\bar{n} = \dfrac{n\pi}{\phi_o}$. Substituting equation II-4.10 into equations II-3.10, II-3.11, the same boundary conditions result as for the complete cylinder, that is, either

$$\left.\begin{matrix} W = 0 \\ W_{,x} = 0 \end{matrix}\right\} \text{"Fixed-End"}$$

or,

$$\left.\begin{matrix} W = 0 \\ W_{,xx} = 0 \end{matrix}\right\} \text{"Pin-Ended"}$$

on $x = \pm\frac{1}{2}\cdot \ell/a$. However, it follows that by substituting equation II-4.10 into equations II-3.13, II-31.4, the boundary conditions $w, M_\phi = 0$ are automatically satisfied on $\phi = 0, \phi_o$. Thus, the use of equation II-4.10 as a substitution is limited to the "Pin-Ended" condition and certain modifications, and as Brown (23) points out, is not applicable to the "Fixed-End" condition. For the "Fixed-End" condition, w,p would have to be expanded in a series of orthogonal functions that automatically satisfy the conditions $w, w_{,\phi} = 0$ on $\phi = 0, \phi_o$.

One finds on substituting equation II-4.10 into the subsidiary condition, that u becomes

$$u(x,\phi) = \sum_{n=1}^{\infty} \frac{1}{\bar{n}^2}\cdot W_{,x} \sin n\pi\phi/\phi_o + \phi f(x) + g(x) \qquad \text{(II-4.14)}$$

Similar to the above result that the condition $w, w_{,\phi} = 0$ cannot be satisfied on $\phi = 0, \phi_0$ by equation II-4.10, it follows that this substitution cannot give $u_{,\phi} = 0$ on $\phi = 0, \phi_0$; but, with $f, g = 0$, it automatically satisfies the condition $u = 0$. Therefore u becomes

$$u(x,\phi) = \sum_{m=1}^{\infty} (\phi_0/m\pi)^2 W_{,x} \sin m\pi\phi/\phi_0 \qquad \text{(II-4.15)}$$

Thus, for the case of the curved panel, the deflections w, u are given by equations II-4.10, II-4.15 respectively, which must satisfy the differential equation II-4.13 subject to the boundary conditions arbitrarily chosen between

$$\left.\begin{aligned} w &= 0 \\ w_{,x} &= 0 \\ u &= 0 \end{aligned}\right\} \text{"Fixed-End"} \qquad \text{(II-4.16)}$$

and,

$$\left.\begin{aligned} w &= 0 \\ M_x &= 0 \\ u_{,x} &= 0 \end{aligned}\right\} \text{"Pin-Ended"} \qquad \text{(II-4.17)}$$

on $x = \pm \ell/2a$; but limited to the conditions that

$$\begin{aligned} w &= 0 \\ M_\phi &= 0 \\ u &= 0 \end{aligned} \qquad \text{(II-4.18)}$$

on $\phi = 0, \phi_0$.

Chapter III

PARTICULAR INTEGRALS

The analysis of Chapter II shows that the assumptions $\epsilon_x, \vartheta = 0$ lead to the differential equations

$$\left[\nabla^4 w + 2\nabla^2 w + w - p\right]_{,\phi\phi\phi\phi} + 12/\eta^2\, w_{,xxxx} + 2(1-\nu)(w + w_{,\phi\phi})_{,xx\phi\phi} = 0$$

$$v_{,\phi} = w \qquad\qquad v_{,x} + u_{,\phi} = 0$$

For the case of the complete cylinder (fig. 5) acted upon by a distribution of forces such that

$$\bar{p}(x,n) = \frac{1}{\pi}\int_{-\pi}^{\pi} p(x,\phi)\cos n\phi \, d\phi$$

the deflections may be expressed as

$$w(x,\phi) = \sum_{n=1}^{\infty} W(x,n)\cos n\phi$$

$$u(x,\phi) = \sum_{n=1}^{\infty} \frac{1}{n^2} W_{,x}\cos n\phi$$

where $W(x,n)$ must satisfy

$$(n^4 + 12/\eta^2)W_{,xxxx} - 2n^2 W_{,xx}(n^4 - \overline{2-\nu}\, n^2 + 1-\nu) + n^4 W (n^2-1)^2 = \bar{p}\, n^4$$

subject to the boundary conditions that either

$$\left.\begin{aligned} w &= 0 \\ w_{,x} &= 0 \\ u &= 0 \end{aligned}\right\} \text{"Fixed-End"}$$

or,

$$\left.\begin{aligned} w &= 0 \\ M_x &= 0 \\ u_{,x} &= 0 \end{aligned}\right\} \text{"Pin-Ended"}$$

on x = 0, ℓ/a.

For the case of the curved panel (fig. 6) acted upon by a distribution of forces such that

$$\bar{p}(x,n) = \int_0^{\phi_0} p(x,\phi)\,\sin n\pi\phi/\phi_0\,d\phi$$

the deflections are

$$w(x,\phi) = \sum_{n=1}^{\infty} W(x,n)\,\sin n\pi\phi/\phi_0$$

$$u(x,\phi) = \sum_{n=1}^{\infty} \left(\phi_0/n\pi\right)^2 W_{,x}\,\sin n\pi\phi/\phi_0$$

where W(x,n) must satisfy

$$(\bar{m}^4 + 12/\eta^2)\,W_{,xxxx} - 2\bar{m}^2 W_{,xx}\left(\bar{m}^4 - \overline{2-\nu}\,\bar{m}^2 + 1-\nu\right) + \bar{m}^4(\bar{m}^2-1)^2 W = 2\bar{m}^4\bar{p}/\phi_0$$

subject to the boundary conditions

$$\left.\begin{aligned} w &= 0 \\ w_{,x} &= 0 \\ u &= 0 \end{aligned}\right\} \text{"Fixed-End"}$$

or,

$$\left.\begin{aligned} w &= 0 \\ M_x &= 0 \\ u_{,x} &= 0 \end{aligned}\right\} \text{"Pin-Ended"}$$

on $x = \pm \ell/2a$; but limited to the conditions that

$$\begin{aligned} w &= 0 \\ M_\phi &= 0 \\ u &= 0 \end{aligned}$$

on $\phi = 0, \phi_o$.

In the following sections, some solutions of the above equations will be presented.

1. The Fundamental Solution for the Pin-Ended Boundary Condition:

The Fundamental Solution, that is, the solution for a unit concentrated force located at an arbitrary point, is of interest since the solution for any distribution of loading can be found from this solution by the method of superposition.

Consider then the case of the complete cylinder (fig. 7), satisfying the "Pin-Ended" boundary conditions (eq. II-4.6), where the loading function represents a concentrated unit radial force acting at ($x^* = x_o^*$, $\phi = 0$).

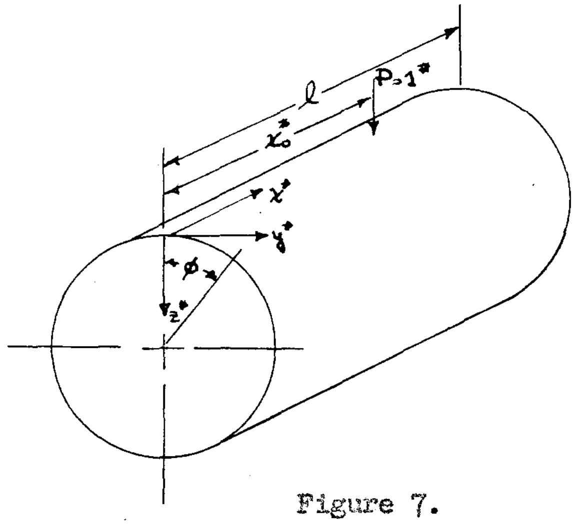


Figure 7.

The concentrated force is expressed by

$$q(x^*, \phi) = \frac{1}{a} \cdot \delta(x^* - x_o^*) \cdot \delta(\phi)$$

where $\delta(x)$ is the Dirac Delta Function. Introducing the dimensionless quantities as defined by equations II-1.1, II-3.20, and using the relation that

$$\delta(ax) = \frac{1}{a} \cdot \delta(x)$$

one finds

$$q(x, \phi) = \frac{1}{a^2} \cdot \delta(x - x_o) \cdot \delta(\phi)$$

so that

$$p(x, \phi) = \frac{a}{D} \cdot \delta(x - x_o) \cdot \delta(\phi)$$

Substituting this dimensionless loading function into equation II-4.3 it follows that

$$\bar{p}(x, n) = \frac{a}{\pi D} \cdot \delta(x - x_o) \qquad \text{(III-1.1)}$$

With $\bar{p}$ (x, n) so defined, equation II-4.4 becomes

$$(n^4 + 12/\eta^2) W_{,xxxx} - 2 n^2 W_{,xx} (n^4 - \overline{2-\nu}\, n^2 + 1-\nu) + n^4 W (n^2-1)^2$$

$$= a n^4/\pi D \cdot \delta(x - x_0)$$

This equation is equivalent to requiring that, for all points save x = x_o

$$(n^4 + 12/\eta^2) W_{,xxxx} - 2 n^2 W_{,xx} (n^4 - \overline{2-\nu}\, n^2 + 1-\nu) + n^4 W (n^2-1)^2 = 0 \qquad \text{(III-1.2)}$$

subject to the condition of continuity of the function W and its first two derivatives at x = x_o , plus the "Jump Condition" given by

$$\left[W_{,xxx} \right]_{x = x_o} = K_n \qquad \text{(III-1.3)}$$

where

$$K_n = \frac{a n^4/\pi D}{n^4 + 12/\eta^2}$$

In addition, the boundary conditions given by equation II-4.6 must be satisfied on x = 0, ℓ/a, that is,

$$W = W_{,xx} = 0$$

For n = 1, equation III-1.2 reduces to

$$W_{,xxxx} = 0$$

The solution of this equation which satisfies the continuity and "Jump" conditions" at $x = x_o$, and the boundary conditions at $x = 0$, ℓ/a is given by

$$W(x,1) = K_1 \left[\frac{x^3}{6}\left(\frac{a x_0}{\ell} - 1\right) + \frac{x_0 \ell}{3a}\cdot\left(1 + \frac{a^2 x_0^2}{2\ell^2} - \frac{3a x_0}{2\ell}\right) x \right]$$
$$(0 \leq x \leq x_0)$$

$$W(x,1) = K_1 \left[\frac{a x_0}{6\ell}\cdot x^3 - \frac{x_0 x^2}{2} + \frac{x_0 \ell}{3a}\cdot\left(1 + \frac{a^2 x_0^2}{2\ell^2}\right) x - \frac{x_0^3}{6} \right] \qquad \text{(III-1.4)}$$
$$(x_0 \leq x \leq \ell/a)$$

For $n > 1$, it is assumed that the solution (W) is proportional to $\exp(mx)$. Making this substitution into equation III-1.2, it is found that m must satisfy

$$(n^4 + 12/\eta^2) m^4 - 2n^2(n^4 - \overline{2-\nu}\, n^2 + 1 - \nu) m^2 + n^4 (n^2 - 1)^2 = 0 \qquad \text{(III-1.5)}$$

If the roots of this equation are

$$m_n = \pm \alpha \pm i\beta \qquad \text{(III-1.6)}$$

the solution takes the form

$$W(x,n) = \cos\beta x \,(A \cosh \alpha x + B \sinh \alpha x)$$
$$+ \sin\beta x \,(C \sinh \alpha x + D' \cosh \alpha x)$$

This solution for the above boundary and continuity conditions becomes

$$W(x,n) = B_1 \cos\beta x \sinh \alpha x + D_1 \sin\beta x \cosh \alpha x \qquad \text{(III-1.7)}$$
$$(0 \leq x \leq x_0)$$

$$W(r, n) = A_2 \cos\beta r \cosh\alpha r + B_2 \cos\beta r \sinh\alpha r$$

$$+ C_2 \sin\beta r \sinh\alpha r + D_2 \sin\beta r \cosh\alpha r \qquad \text{(III-1.8)}$$

$$(r_0 \leq r \leq l/a)$$

where the coefficients are

$$A_2 = -\frac{K_n}{2\alpha\beta(\alpha^2+\beta^2)} \cdot \left(\alpha \sin\beta r_0 \cosh\alpha r_0 - \beta \cos\beta r_0 \sinh\alpha r_0\right)$$

$$B_2 = \frac{1}{\sin^2\beta l/a + \sinh^2\alpha l/a} \cdot \left(- A_2 \cosh\alpha l/a \sinh\alpha l/a + C_2 \sin\beta l/a \cos\beta l/a\right)$$

$$C_2 = -\frac{K_n}{2\alpha\beta(\alpha^2+\beta^2)} \cdot \left(\alpha \cos\beta r_0 \sinh\alpha r_0 + \beta \sin\beta r_0 \cosh\alpha r_0\right)$$

$$D_2 = \frac{1}{\sin^2\beta l/a + \sinh^2\alpha l/a} \cdot \left(- A_2 \sin\beta l/a \cos\beta l/a - C_2 \sinh\alpha l/a \cosh\alpha l/a\right)$$

$$B_1 = \frac{1}{\cos\beta r_0 \sinh\alpha r_0}\Bigg[- A_2 \left(\frac{\sin^2\beta r_0 \cos\beta r_0 \cosh\alpha r_0}{\sin^2\beta r_0 + \sinh^2\alpha r_0} + \right.$$

$$\left. \frac{\cos\beta r_0 \sinh\alpha r_0 \cosh\alpha l/a \sinh\alpha l/a}{\sin^2\beta l/a + \sinh^2\alpha l/a} - \cos\beta r_0 \cosh\alpha r_0 \right)$$

$$- C_2 \left(\frac{\sin\beta r_0 \sinh\alpha r_0 \cosh^2\alpha r_0}{\sin^2\beta r_0 + \sinh^2\alpha r_0} - \right.$$

$$\left. \frac{\cos\beta r_0 \sinh\alpha r_0 \sin\beta l/a \cos\beta l/a}{\sin^2\beta l/a + \sinh^2\alpha l/a} - \sin\beta r_0 \sinh\alpha r_0 \right)\Bigg]$$

$$D_1 = \frac{1}{\sin^2\beta\ell/a + \sinh^2\alpha\ell/a}\Bigg[-A_2\Bigg(\sin\beta\ell/a\,\cos\beta\ell/a -$$

$$\sin\beta x_o\,\cos\beta x_o \times \frac{\sin^2\beta\ell/a + \sinh^2\alpha\ell/a}{\sin^2\beta x_o + \sinh^2\alpha x_o}\Bigg)$$

$$-C_2\Bigg(\sinh\alpha\ell/a\,\cosh\alpha\ell/a -$$

$$\sinh\alpha x_o\,\cosh\alpha x_o \times \frac{\sin^2\beta\ell/a + \sinh^2\alpha\ell/a}{\sin^2\beta x_o + \sinh^2\alpha x_o}\Bigg)\Bigg] \qquad \text{(III-1.9)}$$

2. A Concentrated Force at the Mid-Span Point: Consider the case of the complete cylinder (fig. 8), satisfying the "Pin-Ended" boundary conditions (eq. II-4.6), where the loading function represents a concentrated unit radial force acting at $x = 0$, $\phi = 0$. This solution is identical to the previous solution if $x_o = \ell/2a$. However, rather than to make this substitution, it is more convenient for subsequent computation to take the point of application of the load as a new origin of coordinates (fig. 8).

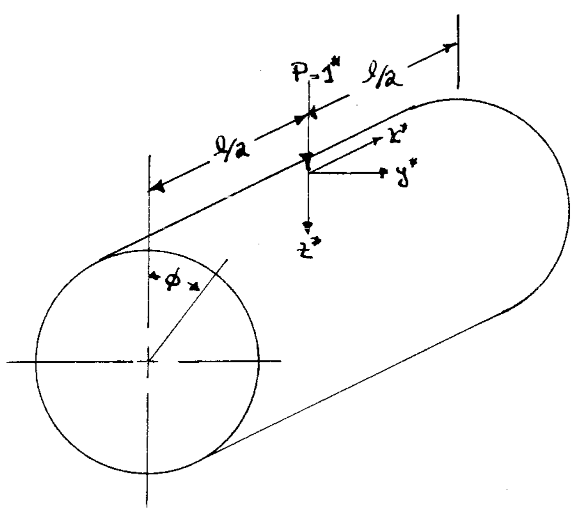


Figure 8

In this case the loading function becomes,

$$q(x^*, \phi) = \frac{1}{a} \cdot \delta(x^*) \cdot \delta(\phi)$$

so that
$$q(x, \phi) = \frac{1}{a^2} \cdot \delta(x) \cdot \delta(\phi)$$

and
$$p(x, \phi) = \frac{a}{D} \cdot \delta(x) \cdot \delta(\phi)$$

with
$$\bar{p}(x, n) = \frac{a}{\pi D} \cdot \delta(x) \tag{III-2.1}$$

With $\bar{p}(x, n)$ so defined, equation II-4.4 reduces to

$$(n^4 + 12/\eta^2) W_{,xxxx} - 2n^2 W_{,xx} (n^4 - \overline{2-\nu}\, n^2 + 1 - \nu) + n^4 (n^2 - 1)^2 W = 0$$

with the requirement that W be an even function of x, and possess a "Jump" in the third derivative given by

$$\Big[W_{,xxx} \Big]_{x=0} = K_n = \frac{a\, n^4 / \pi D}{n^4 + 12/\eta^2}$$

With α, β defined by equations III-1.5, III-1.6, the solution satisfying the boundary, "Jump" and continuity conditions is

$$W(x, 1) = K_1 (\ell/a)^3/48 \left[4\left(\frac{ax}{\ell}\right)^3 - 6\left(\frac{ax}{\ell}\right)^2 + 1 \right]$$
$$(0 \leq x \leq \ell/2a) \tag{III-2.2}$$

and

$$W(x, n) = A \cos\beta x \cosh\alpha x + B \cos\beta x \sinh\alpha x$$
$$+ C \sin\beta x \sinh\alpha x + D' \cosh\alpha x \sin\beta x$$
$$(0 \leq x \leq \ell/2a) \tag{III-2.3}$$

where the coefficients are

$$A = -\frac{B}{\cos^2 \beta l/2a + \sinh^2 \alpha l/2a}\left[\sinh \alpha l/2a \cosh \alpha l/2a - \frac{\alpha}{\beta}\sin \beta l/2a \cos \beta l/2a\right]$$

$$B = -\frac{K_m}{4\alpha(\alpha^2+\beta^2)}$$

$$C = \frac{B}{\cos^2 \beta l/2a + \sinh^2 \alpha l/2a}\left[\sin \beta l/2a \cos \beta l/2a + \frac{\alpha}{\beta}\sinh \alpha l/2a \cosh \alpha l/2a\right]$$

$$D' = -(\alpha/\beta)B \qquad \text{(III-2.4)}$$

3. A Concentrated Force at the Mid-Span Point with a Stiffening Ring: Consider the case of a "Pin-Ended" cylindrical shell with a stiffening ring (fig. 9), acted upon by a concentrated load applied at the mid-span point. The ring is idealized to have infinitesimal width, but finite properties otherwise.

In order to study this shell-ring combination, let us separate the system into three parts: 1) a ring acted upon by an arbitrarily distributed radial load; 2) a shell acted upon by the reaction to the distributed load acting on the ring; and 3) a shell acted upon by a concentrated radial force P. The total deflection of the shell is the sum of that produced by the distributed load and that produced by the concentrated force. The magnitude of the distributed load is obtained by requiring compatibility of the deflections of the ring-shell combination. This method could be applied to any number of stiffening rings, and also extended to include the effect of longeron stiffeners.

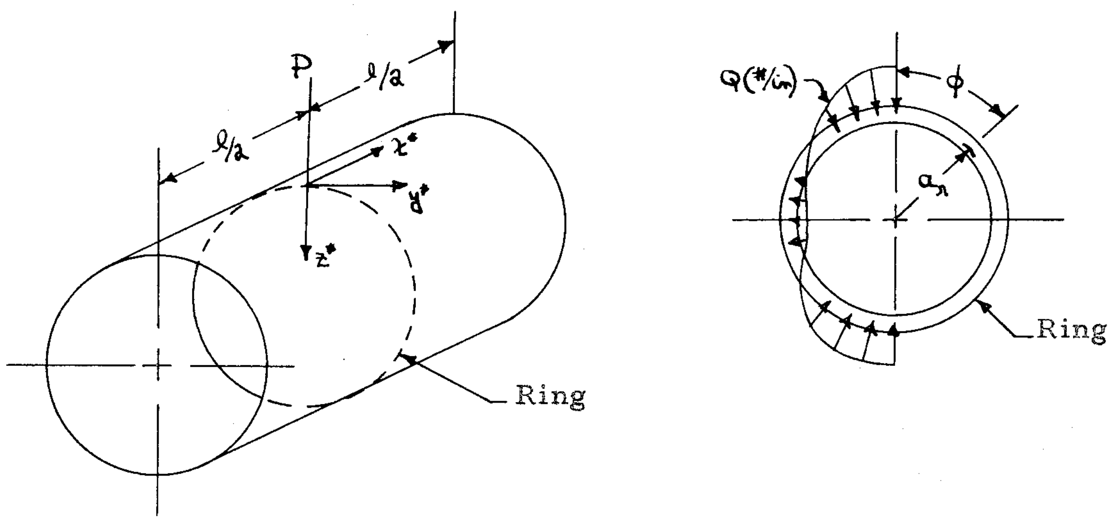


Figure 9.

Consider a thin ring (fig. 9) of radius a_r, moment of inertia I and modulus E_r, acted upon by a distributed loading function $Q(\phi)$. If the depth of the ring is small compared to its radius, it can be shown that only strain energy of bending is important in the expression for the total strain energy V, that is,

$$V = \frac{E_r I}{2 a_r^4} \int_{-\pi}^{\pi} (w^* + w^*_{,\phi\phi})^2 a_r \, d\phi$$

The external work T done on the ring is equal to

$$T = \int_{-\pi}^{\pi} Q(\phi) \, w^* a_r \, d\phi$$

Thus, the total energy of the ring U becomes

$$U = \int_{-\pi}^{\pi} \left[\frac{E_r I}{2 a_r^4} (w^* + w^*_{,\phi\phi})^2 - Q w^* \right] a_r \, d\phi$$

Requiring that $w^*(\phi)$ be such that the total energy of the system is a minimum, that is

$$\delta U = 0$$

it follows that w^* must satisfy

$$w^*_{,\phi\phi\phi\phi} + 2\, w^*_{,\phi\phi} + w^* = Q a_r^4 / E_r I \qquad \text{(III-3.1)}$$

Introducing the dimensionless coordinates

$$\bar{w} = w^*/a_r \qquad \bar{Q} = Q a_r^3 / E_r I \qquad \text{(III-3.2)}$$

equation III-3.1 becomes

$$\bar{w}_{,\phi\phi\phi\phi} + 2\,\bar{w}_{,\phi\phi} + \bar{w} = \bar{Q} \qquad \text{(III-3.1 a)}$$

The substitution

$$\bar{w} = \sum_{n=0}^{\infty} b_n \cos n\phi$$

$$\bar{Q} = \sum_{n=0}^{\infty} c_n \cos n\phi \qquad \text{(III-3.3)}$$

reduces equation III-3.1 a to

$$\sum_{n=0}^{\infty} \cos n\phi \left[b_n (n^2-1)^2 - c_n \right] = 0$$

which requires $\quad b_n = c_n / (n^2-1)^2 \qquad$ (III-3.4)

The loading function (eq. III-3.3) is completely arbitrary, however the condition of equilibrium of the ring as a rigid body requires that the total force on the ring must vanish, that is

$$\int_{-\pi}^{\pi} Q(\phi) \cos\phi \, a_n d\phi = 0$$

$$\int_{-\pi}^{\pi} Q(\phi) \sin\phi \, a_n d\phi = 0$$

The first equation requires $C_1 = 0$, while the second equation is identically satisfied, as Q is an even function of ϕ .

Thus, the deflection of the ring under the arbitrary loading becomes:

$$w^*(\phi) = a_n \sum_{m=0}^{\infty} \frac{C_m}{(m^2-1)^2} \cos m\phi \qquad (C_1 = 0) \qquad \text{(III-3.5)}$$

Consider the deflection of the shell under the reaction to the distributed load acting on the ring, that is

$$q(x^*, \phi) = -\frac{a_n}{a} \cdot \delta(x^*) \, Q(\phi)$$

where it is assumed that the effect of the ring on the shell can be satisfactorily approximated by a distribution of load over the line x = 0. It then follows that

$$q(x, \phi) = -\frac{a_n}{a^2} \cdot \delta(x) \cdot Q(\phi)$$

so that $$p(x, \phi) = -\frac{a\, a_n}{D} \cdot \delta(x) \cdot Q(\phi)$$

and

$$\bar{p}(x, n) = -\frac{E_n a I}{D a_n^2} C_n \cdot \delta(x) \qquad \text{(III-3.6)}$$

With $\bar{p}(x, n)$ so defined, equation II-4.4 becomes

$$(n^4 + 12/\eta^2) W_{,xxxx} - 2n^2 W_{,xx} (n^4 - \overline{2-\nu}\, n^2 + 1 - \nu) + n^4 (n^2 - 1)^2 W = 0$$

subject to the condition of continuity of the function W and its first two derivatives at x = 0, and the "Jump Condition" given by

$$\left[W_{,xxx} \right]_{x=0} = K_n = -\frac{E_n a I}{D a_n^2} \cdot \frac{C_n}{1 + 12/n^4 \eta^2} \qquad \text{(III-3.7)}$$

For n = 0, W_o vanishes as $K_o = 0$. For n = 1, W_1 vanishes as $C_1 = 0$. For n > 1, the solution satisfying the boundary, continuity and "Jump" conditions is

$$W(x, n) = \cos\beta x \,(A \cosh\alpha x + B \sinh\alpha x)$$

$$+ \sin\beta x \,(C \sinh\alpha x + D' \cosh\alpha x)$$

$$(0 \leq x \leq \ell/2a) \qquad \text{(III-3.8)}$$

where

$$A/B = \bar{A} = -\frac{1}{\cos^2\frac{\beta\ell}{2a} + \sinh^2\frac{\alpha\ell}{2a}} \left[\sinh\frac{\alpha\ell}{2a} \cosh\frac{\alpha\ell}{2a} - \frac{\alpha}{\beta} \sin\frac{\beta\ell}{2a} \cos\frac{\beta\ell}{2a} \right]$$

$$B = \frac{1}{4\alpha(\alpha^2 + \beta^2)} \cdot \frac{E_n a I}{D a_n^2} \cdot \frac{C_n}{1 + 12/\eta^2 n^4}$$

$$C/B = \bar{C} = \frac{1}{\cos^2\frac{\beta\ell}{2a} + \sinh^2\frac{\alpha\ell}{2a}} \cdot \left[\sin\frac{\beta\ell}{2a} \cos\frac{\beta\ell}{2a} + \frac{\alpha}{\beta} \cdot \sinh\frac{\alpha\ell}{2a} \cosh\frac{\alpha\ell}{2a} \right]$$

$$D'/B = \bar{D} = -(\alpha/\beta) \tag{III-3.9}$$

Using the solution for the concentrated unit load given by equations III-2.2, III-2.3 and III-2.4, the total deflection is obtained by superposing on the above solution the result of multiplying the solution for the concentrated unit force by P, that is

$$\begin{aligned} w(x,\phi) = {} & \frac{P l^3/48\pi D a^2}{1 + 12/\eta^2}\cdot\left[4\left(\frac{ax}{l}\right)^3 - 6\left(\frac{ax}{l}\right)^2 + 1\right]\cos\phi \\ & + \sum_{n=2}^{\infty} - \frac{a/\pi D}{4\alpha(\alpha^2+\beta^2)}\cdot\frac{\cos n\phi}{1 + 12/n^4\eta^2}\cdot\left(P - \frac{\pi E_r I}{a_r^2}\cdot C_n\right)\times \\ & \left[\cos\beta x\left(\bar{A}\cosh\alpha x + \sinh\alpha x\right) + \sin\beta x\left(\bar{C}\sinh\alpha x + \bar{D}\cosh\alpha x\right)\right] \end{aligned} \tag{III-3.10}$$

Thus, the deflection of the shell is made up of a mode which represents the shell remaining circular (n = 1), plus higher modes which describe the local deviation from roundness.

When applying the compatibility condition to the ring-shell combination, only the higher (n > 2) deflection modes must be equated, as the ring is free to move as a rigid body. Equating the deflection of the shell at x = 0 to the deflection of the ring given by equation III-3.5 leads to

$$\sum_{n=0}^{\infty}\frac{C_n a_r}{(n^2-1)^2}\cdot\cos n\phi = \sum_{n=2}^{\infty} - \frac{\bar{A}a^2/\pi D}{4\alpha(\alpha^2+\beta^2)}\cdot\frac{\cos n\phi}{1 + 12/n^4\eta^2}\cdot\left(P - \frac{\pi E_r I}{a_r^2}\cdot C_n\right)$$

so that, as the coefficient of each term must vanish identically,

$$C_o = 0 \qquad C_1 = 0$$

$$C_m = \frac{P a_r^2 / \pi E_r I}{1 - \dfrac{4\alpha(\alpha^2+\beta^2)}{\bar{A}\,(m^2-1)^2} \cdot \dfrac{D a_r^3}{E_r I a^2} \cdot \left(1 + 12/m^4\eta^2\right)} \qquad \text{(III-3.11)}$$

Finally, the deflected surface of the shell becomes

$$\omega^*(x,\phi) = \frac{P\ell^3/48\pi D a}{1 + 12/\eta^2} \cdot \left[4\left(\frac{ax}{\ell}\right)^3 - 6\left(\frac{ax}{\ell}\right)^2 + 1\right] \cos\phi$$

$$+ \sum_{m=2}^{\infty} \left[\cos\beta x\left(\bar{A}\cosh\alpha x + \sinh\alpha x\right) + \sin\beta x\left(\bar{C}\sinh\alpha x + \bar{D}\cosh\alpha x\right)\right] \times$$

$$\left\{ \frac{\dfrac{P a_r^3 \cos m\phi}{\pi \bar{A}\,(m^2-1)^2 E_r I}}{1 - \dfrac{4\alpha(\alpha^2+\beta^2)}{\bar{A}\,(m^2-1)^2} \cdot \dfrac{D a_r^3}{E_r I a^2}\left(1 + 12/m^4\eta^2\right)} \right\} \qquad \text{(III-3.12)}$$

In order to evaluate the effect of the stiffening ring on the radial deflections, equation III-3.12 can be compared with equation III-2.3 which represents the unstiffened shell. Introducing the notation

$$B_p = - \frac{1}{4\alpha(\alpha^2+\beta^2)} \cdot \frac{Pa/\pi D}{1 + 12/\eta^2 m^4}$$

the deflection of the unstiffened shell becomes

$$w^*(r,\phi) = \frac{P\ell^3/48\pi D a}{1+12/\eta^2}\cdot\left[4\left(\frac{a r}{\ell}\right)^3 - 6\left(\frac{a r}{\ell}\right)^2 + 1\right]\cdot \cos\phi$$

$$+ \sum_{n=2}^{\infty}\left[\cos\beta r\left(\bar{A}\cosh\alpha r + \sinh\alpha r\right) + \sin\beta r\left(\bar{C}\sinh\alpha r + \bar{D}\cosh\alpha r\right)\right]$$

$$\times\ a\ B_p\ \cos n\phi$$

while, for the stiffened shell,

$$w^*(r,\phi) = \frac{P\ell^3/48\pi D a}{1+12/\eta^2}\cdot\left[4\left(\frac{a r}{\ell}\right)^3 - 6\left(\frac{a r}{\ell}\right)^2 + 1\right]\cos\phi$$

$$+ \sum_{n=2}^{\infty}\left[\cos\beta r\left(\bar{A}\cosh\alpha r + \sinh\alpha r\right) + \sin\beta r\left(\bar{C}\sinh\alpha r + \bar{D}\cosh\alpha r\right)\right]$$

$$\times\ a\ B_p\cdot\frac{1}{1+\dfrac{\pi\bar{A}\,(n^2-1)^2 E_n I a B_p}{P a_n^3}}\cdot\cos n\phi$$

As n becomes large, using the results of Appendix I for the behavior of A/B, B as n becomes large, the final factor can be written

$$1\Big/\left(1+\frac{E_n I a^2 n}{4 D a_n^3}\right)$$

Thus, the effect of the ring on the shell is embodied in this latter factor, which reduces the contribution of each of the higher modes to the total deflection. Physically, this means that the effect of the ring is to give the deflected surface a smaller curvature, as a greater proportion of the load is carried in the lower modes.

4. A Concentrated Force on a Curved Panel: The results of Chapter II-4b can be applied to a curved panel (fig. 10), simply supported on all sides, with a concentrated force P applied at the mid-point.

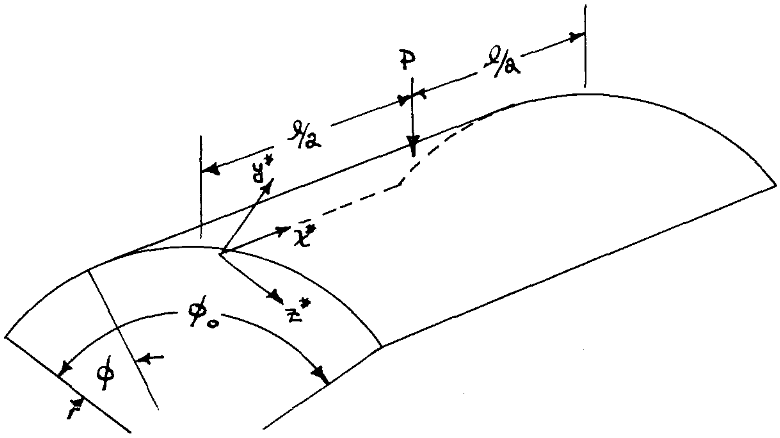


Figure 10.

The loading function q is given by

$$q(x^*,\phi) = \frac{P}{a}\cdot\delta(x^*)\cdot\delta(\phi - \phi_0/2)$$

Also,

$$p(x,\phi) = \frac{Pa}{D}\cdot\delta(x)\cdot\delta(\phi-\phi_0/2)$$

so that,

$$\bar{p}(x,n) = \frac{Pa}{D}\cdot\delta(x)\ \sin n\pi/2 \qquad \text{(III-4.1)}$$

With $\bar{p}(x,n)$ so defined, equation II-4.13 reduces to

$$(\bar{m}^4 + 12/\eta^2)W_{,xxxx} - 2\bar{m}^2 W_{,xx}(\bar{m}^4 - 2\cdot\nu\,\bar{m}^2 + 1-\nu) + \bar{m}^4(\bar{m}^2-1)^2 W = 0 \qquad \text{(III-4.2)}$$

subject to the boundary conditions

$$W = W_{,xx} = 0$$

on $x = \pm \ell/2a$, and the requirement that W be an even function of x and possess a "Jump" in the third derivative given by

$$\left[W_{,xxx}\right]_{x=0} = K_n = \frac{2Pa}{D\,\phi_0} \cdot \frac{\sin n\pi/2}{1 + 12/\eta^2 \bar{n}^4} \qquad \text{(III-4.3)}$$

Since, in most practical cases $\phi_0 < \pi$, the case $\bar{n} = 1$ will not occur, W(x,n) can be taken proportional to exp (mx). With this substitution, m must satisfy

$$(\bar{n}^4 + 12/\eta^2)\, m^4 - 2\bar{n}^2 (\bar{n}^4 - \overline{2-\nu}\,\bar{n}^2 + 1-\nu)\, m^2 + \bar{n}^4(\bar{n}^2 - 1)^2 = 0 \qquad \text{(III-4.4)}$$

If the roots of this equation are

$$m_n = \pm\alpha \pm i\,\beta$$

the assumed solution can be put in the form

$$\begin{aligned} W(x,n) = {} & \cos\beta x\,(A \cosh\alpha x + B \sinh\alpha x) \\ & + \sin\beta x\,(C \sinh\alpha x + D' \cosh\alpha x) \\ & (0 \leq x \leq \ell/2a) \end{aligned} \qquad \text{(III-4.5)}$$

The coefficients which satisfy the above boundary, continuity and "Jump" conditions are

$$\bar{A} = A/B = -\frac{1}{\cos^2\frac{\beta l}{2a} + \sinh^2\frac{\alpha l}{2a}} \cdot \left[\sinh\frac{\alpha l}{2a} \cosh\frac{\alpha l}{2a} - \frac{\alpha}{\beta} \sin\frac{\beta l}{2a} \cos\frac{\beta l}{2a} \right]$$

$$B = -\frac{1}{4\alpha(\alpha^2+\beta^2)} \cdot \frac{K_m}{\bar{m}^4 + \alpha/\eta^2}$$

$$\bar{C} = C/B = \frac{1}{\cos^2\frac{\beta l}{2a} + \sinh^2\frac{\alpha l}{2a}} \cdot \left[\sin\frac{\beta l}{2a} \cos\frac{\beta l}{2a} + \frac{\alpha}{\beta} \sinh\frac{\alpha l}{2a} \cosh\frac{\alpha l}{2a} \right]$$

$$\bar{D} = D'/B = -(\alpha/\beta)$$

(III-4.6)

5. <u>A Uniform Load on a Curved Panel:</u> The results of Chapter II-4.b can also be applied to a curved panel acted upon by a uniform radial load of intensity q_o. This could be an approximation to the case of a panel acted upon by its own weight. In this latter case, the loading function is

$$q = q_o \cos(\phi - \phi_o/2) = q_o - q_o/2! \cdot (\phi - \phi_o/2)^2 + \cdots$$

where there will be a corresponding tangentially distributed load given by

$$q_o \sin(\phi_o/2 - \phi) = q_o(\phi_o/2 - \phi) - q_o/3! \cdot (\phi_o/2 - \phi)^3 + \cdots$$

Thus the assumptions that q = constant (q_o) and the effect of the tangential load is negligible, will be a good approximation to the gravity load problem providing ϕ_o is small.

For the uniform load problem, equation II-4.12 becomes

$$\bar{p}(x,n) = 2 q_0 \phi_0 a^3 / n\pi D \qquad (n = 1, 3, \cdots) \qquad \text{(III-5.1)}$$

so that, for simply supported sides, the assumption that

$$w(x,\phi) = \sum_{n=1}^{\infty} W(x,n) \sin n\pi\phi/\phi_0$$

leads to the reduced equation

$$(\bar{n}^4 + 12/\eta^2) W_{,xxxx} - 2\bar{n}^2 W_{,xx} \cdot (\bar{n}^4 - \overline{2-\nu}\,\bar{n}^2 + 1-\nu) + \bar{n}^4 W (\bar{n}^2 - 1)^2 = 2\bar{n}^4 \bar{p}/\phi_0$$

$$(\bar{n} = n\pi/\phi_0) \qquad \text{(III-5.2)}$$

Taking W(x,n) to be proportional to exp (mx), it is found that m must satisfy the equation

$$(\bar{n}^4 + 12/\eta^2) m^4 - 2\bar{n}^2 (\bar{n}^4 - \overline{2-\nu}\,\bar{n}^2 + 1-\nu) m^2 + \bar{n}^4 (\bar{n}^2 - 1)^2 = 0$$

If the roots of this equation are

$$m_n = \pm \alpha \pm i \beta$$

the solution can be put in the form

$$W(x,n) = W_p + A \cos\beta x \cosh\alpha x + B \sin\beta x \sinh\alpha x$$

$$(0 \leq x \leq l/2a)$$

Imposing the boundary conditions on x = $\pm l/2a$ that

$$W = W_{,xx} = 0$$

the coefficients become

$$W_p = 4 q_o a^3 / n \pi D (n^2-1)^2$$

$$A = -\frac{W_p}{a}\left[(\alpha^2-\beta^2)\sin\frac{\beta l}{2a}\sinh\frac{\alpha l}{2a} + 2\alpha\beta\cos\frac{\beta l}{2a}\cosh\frac{\alpha l}{2a}\right] \Big/$$

$$\left[(\alpha^2-\beta^2)\sin\frac{\beta l}{2a}\cos\frac{\beta l}{2a}\sinh\frac{\alpha l}{2a}\cosh\frac{\alpha l}{2a} + \alpha\beta\left(\cos^2\frac{\beta l}{2a}\cosh^2\frac{\alpha l}{2a} - \sin^2\frac{\beta l}{2a}\sinh^2\frac{\alpha l}{2a}\right)\right]$$

$$B = -\frac{W_p}{a}\left[(\alpha^2-\beta^2)\cos\frac{\beta l}{2a}\cosh\frac{\alpha l}{2a} - 2\alpha\beta\sin\frac{\beta l}{2a}\sinh\frac{\alpha l}{2a}\right] \Big/$$

$$\left[(\alpha^2-\beta^2)\sin\frac{\beta l}{2a}\cos\frac{\beta l}{2a}\sinh\frac{\alpha l}{2a}\cosh\frac{\alpha l}{2a} + \alpha\beta\left(\cos^2\frac{\beta l}{2a}\cosh^2\frac{\alpha l}{2a} - \sin^2\frac{\beta l}{2a}\sinh^2\frac{\alpha l}{2a}\right)\right]$$

$$(n = 1, 3, 5, \cdots)$$

Chapter IV

APPLICABILITY OF THE APPROXIMATION

To establish some criterion for the range of approximate validity of the differential equation that has been derived, an investigation of the frequencies of harmonic vibration of each Fourier component of the deformed surface was made. By comparison with more accurately determined values, one can obtain a range of parameters within which the assumptions that $\epsilon_z, \gamma = 0$ give accurate results. This same criterion can be carried over to the static loading case.

In the following section, the frequencies of harmonic vibration have been computed including the effect of the longitudinal inertia terms, that is, body forces in the x, y-directions. These frequencies are only approximately equal to those obtained from equation II-3.21, as the derivation of this equation did not provide for the inclusion of body forces in the x, y-directions. The body force in the z-direction is obtained by replacing $p(x^*, \phi)$ by $-\rho h w^*_{,\tau\tau}$ where ρ is the mass of the shell per unit volume. However, this effect was shown by Reissner (24) to be unimportant, that is, the frequencies of transverse vibration were relatively unaffected by the presence of the longitudinal inertia terms. Thus, the conclusions drawn from the results of the general frequency equation apply equally as well to equation II-3.21.

1. <u>Derivation of Frequency Factor:</u> Consider the harmonic vibration of a complete cylindrical shell, the ends of which are simply supported according to equation II-4.17. In accordance with the boundary conditions, take the deflections to be of the following form:

$$u^*(x^*,\phi) = \sum_m \sum_n \frac{c_1}{n}\left(\frac{\lambda}{m}\right) \cos\frac{m\pi x^*}{\ell} \cos n\phi \cos\omega\tau$$

$$v^*(x^*,\phi) = \sum_m \sum_n \frac{c_2}{n} \sin\frac{m\pi x^*}{\ell} \sin n\phi \cos\omega\tau$$

$$w^*(x^*,\phi) = \sum_m \sum_n c_3 \sin\frac{m\pi x^*}{\ell} \cos n\phi \cos\omega\tau \qquad \text{(IV-1.1)}$$

where $\lambda = m\pi a/\ell$ and m,n are wave numbers representing the number of axial half-waves and circumferential full waves respectively. Substituting equation IV-1.1 into equation II-3.3 and using the relations given by equation II-3.2 and equation II-1.4, one obtains the following expression for the total strain energy of the shell

$$V(\tau) = \sum_m \sum_n \frac{E h \ell \pi}{4a(1-\nu^2)} \left\{ \lambda^4\left[\left(\frac{c_1}{n^2}\right)^2 + \frac{c_3^2\eta^2}{12}\right] + (c_2 - c_3)^2 \right.$$

$$+ \frac{c_3^2\eta^2}{12}(n^2-1)^2 + 2\nu\left[c_1\left(\frac{\lambda}{n}\right)^2(c_3 - c_2) + c_3^2\,\frac{\lambda^2\eta^2}{12}(n^2-1)\right]$$

$$\left. + \frac{1-\nu}{2}\left[\left(\frac{\lambda}{n}\right)^2(c_1 - c_2)^2 + \frac{\lambda^2\eta^2}{3}\left(c_3^2 n^2 - 2c_2c_3 + c_2^2/n^2\right)\right]\right\} \cos^2\omega\tau$$

(IV-1.2)

Corresponding to the assumptions that ϵ_2, γ vanish, the above expression reduces to

$$V(\tau) = \sum_m \sum_n \frac{E h \ell \pi}{4a(1-\nu^2)} \cdot c^2 \left[\lambda^4\left(\frac{1}{n^4} + \frac{\eta^2}{12}\right) + \eta^2/_{12}\,(n^2-1)^2 \right.$$

$$\left. + \frac{\lambda^2\eta^2\nu}{6}\cdot(n^2-1) + \frac{1-\nu}{6}\cdot\lambda^2\eta^2\left(\frac{n^2-1}{n}\right)^2\right] \cos^2\omega\tau$$

(IV-1.3)

as this assumption requires

$$C_1 = C_2 = C_3 = C$$

If our attention is restricted to frequencies below which the effect of rotary inertia is important, the kinetic energy per unit volume N_o reduces to:

$$N_o = \tfrac{1}{2}\rho \left(u^{*2}_{,\tau} + v^{*2}_{,\tau} + w^{*2}_{,\tau} \right) \qquad \text{(IV-1.4)}$$

Thus, the total kinetic energy of the shell N becomes

$$N(\tau) = \iiint N_o \, a \, dr^* d\phi \, dz^* \qquad \text{(IV-1.5)}$$

Substituting equation IV-1.4 into equation IV-1.5 and using equation IV-1.1, the total kinetic energy becomes

$$N(\tau) = \pi \ell h \omega^2 \rho a/4 \cdot \left(C_3^2 + \tfrac{1}{m^2} \cdot C_2^2 + \tfrac{\lambda^2}{m^4} \cdot C_1^2 \right) \sin^2 \omega\tau \qquad \text{(IV-1.6)}$$

Corresponding to the assumptions that ϵ_{2}, γ vanish, this expression reduces to

$$N(\tau) = \pi \ell h \omega^2 \rho a/4 \cdot C^2 \left(1 + \tfrac{1}{m^2} + \tfrac{\lambda^2}{m^4} \right) \sin^2 \omega\tau \qquad \text{(IV-1.7)}$$

The frequencies of vibration, as defined by a frequency factor ∇ , are obtained by equating the maximum kinetic energy to the maximum strain energy. Hence

$$\nabla = R / S \qquad \text{(IV-1.8)}$$

where
$$\nabla = (1-\nu^2)\rho a^2 \omega^2 / E$$

$$R = \lambda^4\left[\left(\frac{C_1}{m^2}\right)^2 + \frac{C_3^2\eta^2}{12}\right] + (C_2 - C_3)^2 + C_3^2\eta^2/12 \cdot (m^2-1)^2$$

$$+ 2\nu\left[C_1\left(\frac{\lambda}{m}\right)^2(C_3 - C_2) + C_3^2\,\lambda^2\eta^2/12 \cdot (m^2-1)\right]$$

$$+ \frac{1-\nu}{2}\left[\left(\frac{\lambda}{m}\right)^2(C_1 - C_2)^2 + \lambda^2\eta^2/3 \cdot \left(C_3^2 m^2 - 2C_2C_3 + \frac{1}{m^2}C_2^2\right)\right]$$

$$S = C_3^2 + \frac{1}{m^2}\cdot C_2^2 + \left(\frac{\lambda}{m^2}\right)^2 C_1^2 \qquad \text{(IV-1.9)}$$

When $\epsilon_2, \gamma = 0$, the equations in IV-1.9 reduce to

$$\nabla = \frac{\lambda^4\left(\frac{1}{m^4} + \frac{\eta^2}{12}\right) + \frac{\eta^2}{12}(m^2-1)^2 + \frac{\lambda^2\eta^2\nu}{6}(m^2-1) + \frac{1-\nu}{6}\lambda^2\eta^2\left(\frac{m^2-1}{m}\right)^2}{1 + \frac{1}{m^2} + \frac{\lambda^2}{m^4}}$$

(IV-1.10)

2. <u>Analysis and Comparison of Results:</u> As a result of the simplifying assumptions that ϵ_2, γ vanish, a single linear equation for the frequency factor was obtained. The exact method, as followed by Arnold

and Warburton (25), involves substituting equation IV-1.1 into the three "exact" differential equations, thus obtaining three simultaneous, homogeneous, linear equations in the variables C_1 , C_2 , C_3 . The requirement that the determinant of the coefficients must vanish gives a cubic equation in the frequency factor. The ease with which equation IV-1.10 is solved in comparison with the classical cubic equation indicates its possible shortcomings.

A comparison of the results obtained by Arnold and Warburton, and those obtained from equation IV-1.10 is presented in figures 11, 12 for two values of h/a. It can be seen that for the lower circumferential modes (n-small), equation IV-1.10 represents the actual physical system only in the neighborhood of lower axial modes (λ-small). However, as the number of circumferential waves increases, equation IV-1.10 is satisfactory up to an increasing number of axial waves.

This can be interpreted on the basis of the underlying assumptions derived in Chapter II-2. Consider an elemental panel (fig. 13) vibrating in such a way as to satisfy the boundary conditions of a simply supported beam, that is, vanishing of the deflection v^* and the bending moment $v_{,x^*x^*}$ at the ends, and of the shear strain on the upper and lower surfaces.

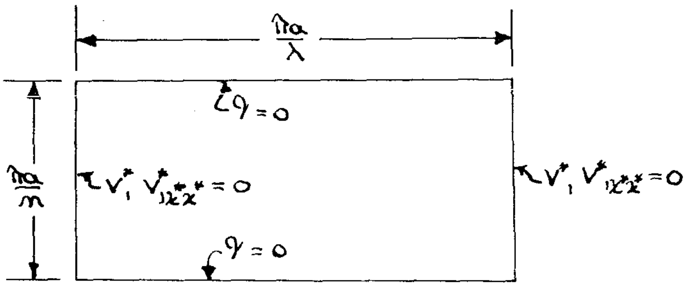


Figure 13.

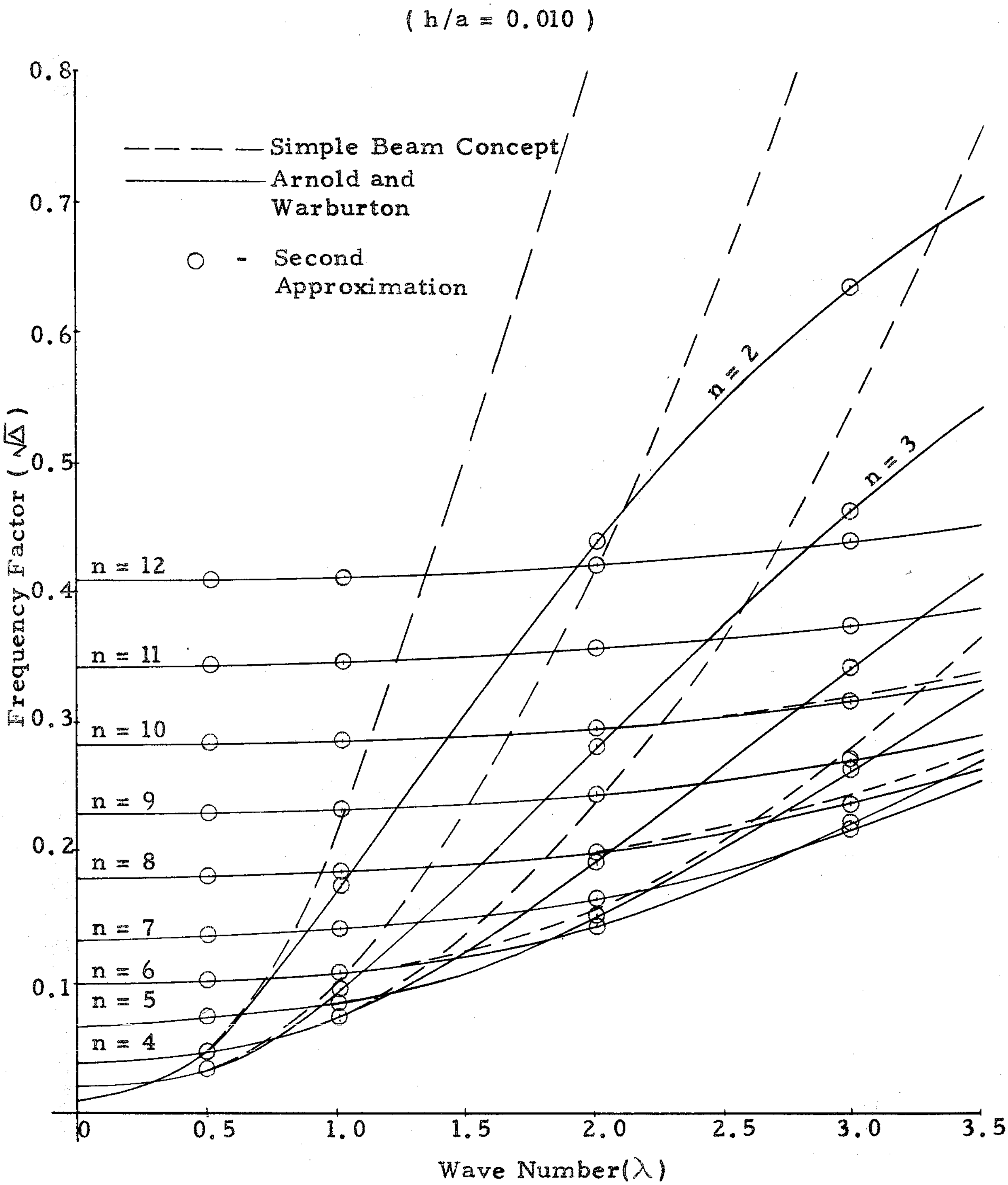


Figure 11 - Frequency Factor

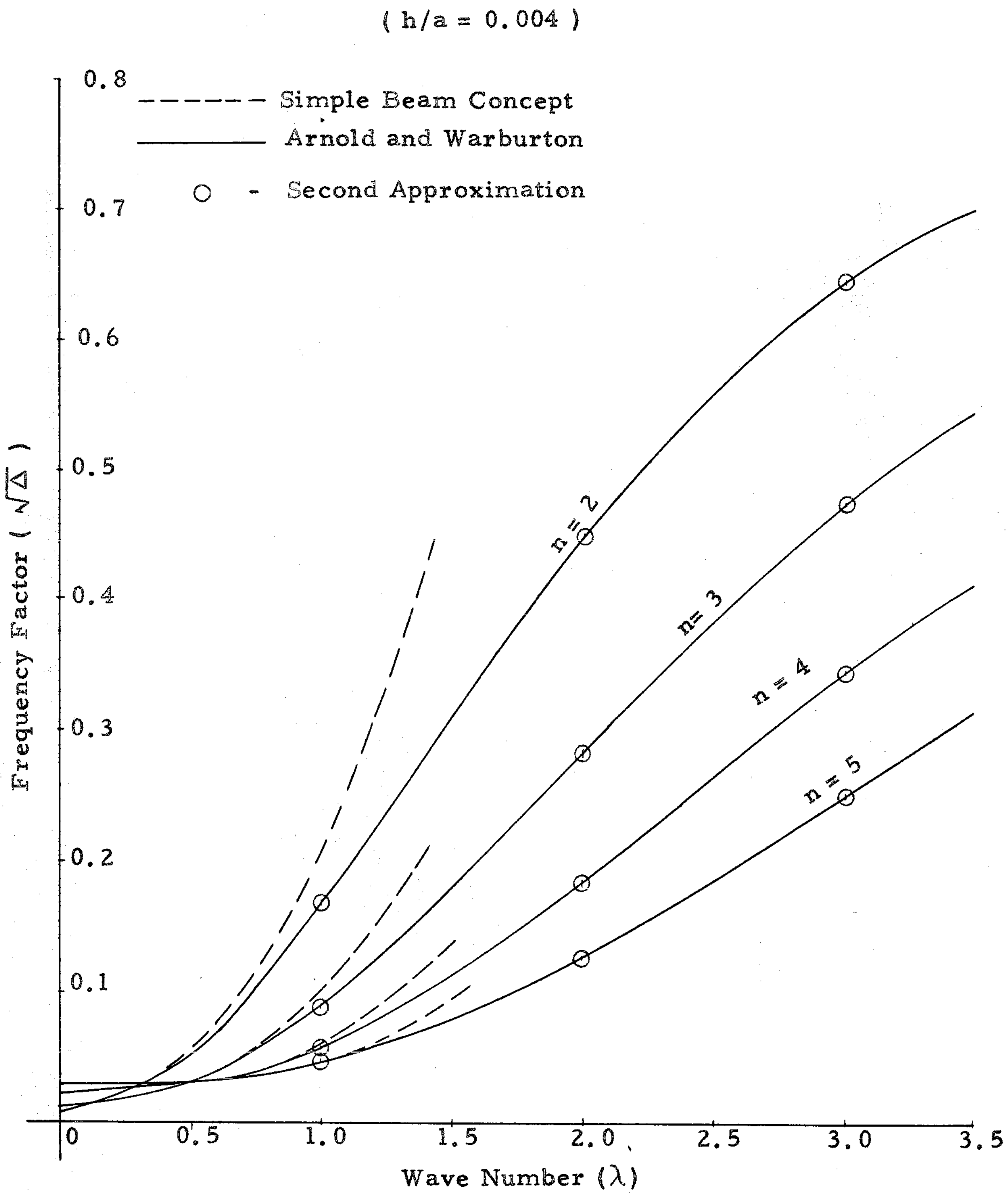


Figure 12 - Frequency Factor

From the equations IV-1.1, one finds that the dimensions are such that the span-to-depth or slenderness ratio is n/λ . The more exact theory of beams, as given by Timoshenko (20), shows that neglecting ϵ_a, γ is justified only if the beam is slender, that is, n/λ of the order of ten. Thus, by analogy, one should not expect agreement with "exact" theories when the model corresponds to a short-deep beam. Actually, the agreement extends satisfactorily to beams whose slenderness ratio is practically three. This is evident in the accompanying figures.

These results can be carried over to the static loading case. By analogy, if the loading function produces a deflected surface whose description by a Double Fourier Series depends strongly on those modes which lie in the range $n/\lambda < 4$, one cannot expect satisfactory agreement. Thus, the deflection due to a strongly varying loading function cannot be expected to be accurate. However, from Saint-Venant's Principle, this discrepancy will be localized in the vicinity of the strong variation. As will be shown in a later section, even for the ultimate case of a concentrated load, the deflected surface can be described reasonably well even near the load.

3. <u>A Second Approximation:</u> Certainly, in the exact treatment of deformation, the strains ϵ_a, γ do not vanish, though their effect may not be large. However, for computing natural frequencies, the restriction of movement caused by the vanishing of ϵ_a, γ gives rise to increasing error as n/λ becomes small, as shown in figures 11, 12. In order to obtain this frequency range more accurately and yet retain some ease of calculation, an iterative scheme will be devised based on the results of neglecting ϵ_a, γ .

As shown in Appendix II, one can apply to thin shells the classical two-dimensional equations of equilibrium. This means that the effect of bending on the in-plane equilibrium of thin shells is negligible. Thus, restricting attention to the net stress acting over the thickness of the shell, that is, the membrane stress, it follows that

$$\sigma_{x,x^*} + \tau_{xy,y^*} = \rho u^*_{,\tau\tau} \qquad \text{(IV-3.1)}$$

$$\sigma_{y,y^*} + \tau_{xy,x^*} = \rho v^*_{,\tau\tau} \qquad \text{(IV-3.2)}$$

where

$$\sigma_x = \frac{E}{1-\nu^2}\left(\epsilon_1 + \nu\epsilon_2\right) \qquad \text{(IV-3.3)}$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\epsilon_2 + \nu\epsilon_1\right) \qquad \text{(IV-3.4)}$$

$$\tau_{xy} = G\gamma \qquad \text{(IV-3.5)}$$

Assuming, for a first approximation, that ϵ_2, γ vanish, equations IV-1.1 become

$$u^*_o(x^*,\phi) = \sum_m \sum_n \frac{C}{m}\left(\frac{\lambda}{m}\right) \cos\frac{m\pi x^*}{\ell} \cos n\phi \cos\omega\tau \qquad \text{(IV-3.6)}$$

$$v^*_o(x^*,\phi) = \sum_m \sum_n \frac{C}{n} \sin\frac{m\pi x^*}{\ell} \sin n\phi \cos\omega\tau \qquad \text{(IV-3.7)}$$

$$w^*_o(x^*,\phi) = \sum_m \sum_n C \sin\frac{m\pi x^*}{\ell} \cos n\phi \cos\omega\tau \qquad \text{(IV-3.8)}$$

Furthermore, u_o^* may be considered the exact deflection, as equation IV-3.6 is perfectly general. Hence v_o^*, w_o^* are derived from u_o^* as a result of the assumption that ϵ_2, γ vanish.

With this first approximation the steps to be followed are

a) Compute ϵ_1 from equation II-1.6.

b) Compute ϵ_2 from equation II-1.6 and substitute ϵ_1, ϵ_2 into IV-3.3 in order to evaluate σ_x.

c) With σ_x, compute τ_{xy}, γ from equations IV-3.1, IV-3.5, and similarly,

d) Compute σ_y from equation IV-3.2.

e) With σ_y, ϵ_1 determine a new value for ϵ_2 from equation IV-3.3.

f) Finally, with u_o^* determine a corrected $v^* = v_1^*$ from equation II-1.6 using γ determined in c); and a new $w^* = w_1^*$ from equation II-1.6 using ϵ_2 computed in e) and $v^* = v_1^*$.

Following the above steps, one finds

a) $$\epsilon_1 = \sum_m \sum_n -\frac{c}{a}\left(\frac{\lambda}{n}\right)^2 \sin\frac{m\pi x^*}{l} \cos n\phi \cos\omega\tau$$

b) $$\sigma_x = \sum_m \sum_n -\frac{E}{1-\nu^2} \cdot \frac{c}{a} \cdot \left(\frac{\lambda}{n}\right)^2 \sin\frac{m\pi x^*}{l} \cos n\phi \cos\omega\tau$$

c) $$\gamma = \sum_m \sum_n \frac{c}{a}\left(\frac{\lambda}{n}\right)\left[\frac{E/(1-\nu^2)}{G}\left(\frac{\lambda}{n}\right)^2 - \rho/G\left(\frac{a\omega}{n}\right)^2\right] \cos\frac{m\pi x^*}{l} \sin n\phi \cos\omega\tau$$

from $$\tau_{xy} = a\int\left(\rho u^*_{,\tau\tau} - \sigma_{x,x^*}\right)d\phi$$

d) $$\sigma_\phi = \sum_m \sum_n \frac{c}{a} \left\{ \rho\left(\frac{a\omega}{n}\right)^2 - \left(\frac{\lambda}{n}\right)^2 \left[\frac{E}{1-\nu^2}\cdot\left(\frac{\lambda}{n}\right)^2 - \rho\left(\frac{a\omega}{n}\right)^2 \right] \right\} \sin\frac{m\pi\nu^*}{\ell} \cos n\phi \cos\omega\tau$$

from $$\sigma_y = a\int\left(\rho V^*_{,\tau\tau} - \tau_{xy,\nu^*}\right) d\phi$$

e) $$\epsilon_\phi = \sum_m \sum_n \frac{c}{a} \left\{ \frac{1-\nu^2}{E}\rho\left(\frac{a\omega}{n}\right)^2 \left[1 + \left(\frac{\lambda}{n}\right)^2\right] + \nu\left(\frac{\lambda}{n}\right)^2 - \left(\frac{\lambda}{n}\right)^4 \right\} \sin\frac{m\pi\nu^*}{\ell} \cos n\phi \cos\omega\tau$$

from $$\epsilon_2 = \frac{1-\nu^2}{2}\sigma_\phi - \nu\,\epsilon_1$$

f) $$V_1^* = \sum_m \sum_n \frac{c}{n} \left[1 + \frac{E/(1-\nu^2)}{G}\left(\frac{\lambda}{n}\right)^2 - \rho/_G\left(\frac{a\omega}{n}\right)^2 \right] \sin\frac{m\pi\nu^*}{\ell} \sin n\phi \cos\omega\tau$$

$$\omega_1^* = \sum_m \sum_n C \left[1 + \frac{E/(1-\nu^2)}{G}\left(\frac{\lambda}{n}\right)^2 - \rho/_G\left(\frac{a\omega}{n}\right)^2 - \frac{1-\nu^2}{E}\rho\left(\frac{a\omega}{n}\right)^2\left(1+\lambda^2/_{n^2}\right) - \nu\left(\frac{\lambda}{n}\right)^2 + \left(\frac{\lambda}{n}\right)^4 \right] \sin\frac{m\pi\nu^*}{\ell} \cos n\phi \cos\omega\tau$$

from $$V_1^* = \int\left(\gamma - \frac{1}{a}u^*_{,\phi}\right) d\nu \qquad \text{and} \qquad \omega^* = V^*_{,\phi} - a\,\epsilon_2$$

This iterative process could be repeated indefinitely, but on comparing the above results with equation IV-1.1, the displacement coefficients are seen to correspond to

$$\begin{aligned} C_1 &= C \\ C_2 &= \left[1 + \frac{2}{1-\nu}\cdot\frac{1}{n^2}\cdot\left(\lambda^2 - \nabla\right) \right] C \\ C_3 &= C_2 - \left[\left(\frac{\lambda}{n}\right)^2\left(\nu + \frac{\nabla}{n^2}\right) + \frac{\nabla}{n^2} - \left(\frac{\lambda}{n}\right)^4 \right] C \end{aligned} \qquad \text{(IV-3.9)}$$

It has been found that these mode shapes give sufficiently accurate frequencies so that further iterations are not warranted. Thus, defining

$$A = C_2 / C_1 \qquad B = C_3 / C_1 \tag{IV-3.10}$$

That is, from equation IV-3.9,

$$A = 1 + \frac{2}{1-\nu} \cdot \frac{1}{m^2} (\lambda^2 - \nabla) \tag{IV-3.11}$$

$$B = A - \left(\frac{\lambda}{m}\right)^2 \left(\nu + \frac{\nabla}{m^2}\right) - \frac{\nabla}{m^2} + \left(\frac{\lambda}{m}\right)^4 \tag{IV-3.12}$$

the frequency factor expression (eq. IV-1.8) becomes

$$\nabla = R' / S' \tag{IV-1.13}$$

where $$R' = \left(\frac{\lambda}{m}\right)^4 + \eta^2/_{12} \left[\lambda^4 + (m^2-1)^2 + 2\nu\lambda^2(m^2-1) + 2(1-\nu)\lambda^2 m^2\right] B^2$$

$$+ (A-B)^2 + 2\nu\left(\frac{\lambda}{m}\right)^2 (B-A) + \frac{1-\nu}{2}\left(\frac{\lambda}{m}\right)^2 (A-1)^2$$

$$- \lambda^2\eta^2/_3 \cdot (1-\nu) A B + (1-\nu)\eta^2/_6 \cdot \left(\frac{\lambda}{m}\right)^2 A^2$$

$$S' = B^2 + \frac{1}{m^2} \cdot A^2 + \left(\frac{\lambda}{m^2}\right)^2$$

It has been found convenient, in calculating frequencies with equation IV-1.13, to use the following procedure:

a) Compute ∇ from equation IV-1.10.

b) Evaluate A,B.

c) Compute ∇ from equation IV-1.13.

d) Repeat from b) until the process converges.

This method has been followed to obtain the frequencies shown in figures 11, 12. Within the range of parameters presented, the frequency curves determined from equation IV-1.13 are not distinguishable from those of Arnold and Warburton.

4. Application to Static Loads: In particular cases where the distribution of radial load is not a strongly varying function, it may be convenient to use a double series solution rather than the methods of Chapter II. This method might be more applicable to automatic computing methods.

Consider the following system (fig. 14) with an arbitrary distribution of radial load q.

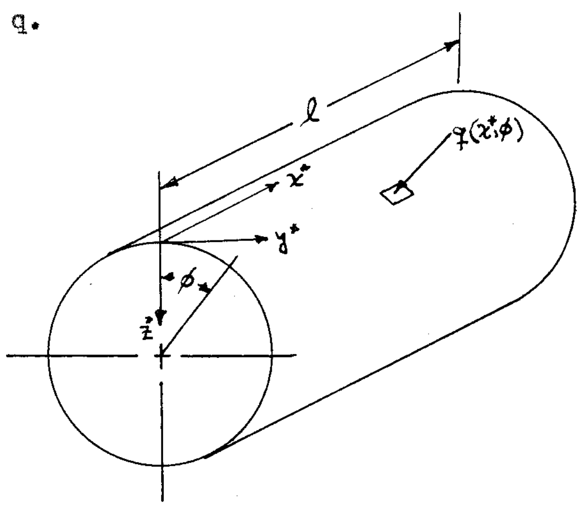


Figure 14.

The displacements may be written

$$u^*(x^*,\phi) = \sum_m \sum_n C_1 \frac{1}{n}\left(\frac{\lambda}{n}\right) \cos\frac{m\pi x^*}{\ell} \cos n\phi$$

$$v^*(x^*,\phi) = \sum_m \sum_n C_2 \cdot \frac{1}{n} \cdot \sin\frac{m\pi x}{\ell} \sin n\phi$$

$$w^*(\nu^*, \phi) = \sum_m \sum_n C_3 \sin \frac{m\pi \nu^*}{l} \cos n\phi \qquad \text{(IV-4.1)}$$

Corresponding to equation IV-1.2, the total energy V of the shell becomes

$$V = \sum_m \sum_n \frac{E h l \pi}{4a(1-\nu^2)} \cdot C_3^2 \, f(n, \lambda) \qquad \text{(IV-4.2)}$$

where

$$f(n,\lambda) = \lambda^4 \left[\left(\frac{1}{B n^2}\right)^2 + \frac{\eta^2}{12} \right] + \left(\frac{A-B}{B}\right)^2 + \eta^2/_{12} \cdot (n^2-1)^2$$

$$+ 2\nu \left[\frac{B-A}{B^2} \left(\frac{\lambda}{n}\right)^2 + \lambda^2 \eta^2/_{12} \cdot (n^2-1) \right]$$

$$+ \frac{1-\nu}{2} \left[\left(\frac{\lambda}{n}\right)^2 \left(\frac{A-1}{B}\right)^2 + \lambda^2 \eta^2/_3 \cdot \left(n^2 - 2 \cdot \frac{A}{B} + \frac{A^2}{n^2 B^2}\right) \right]$$

(IV-4.3)

with A,B defined in equations IV-3.10, IV-3.11 and IV-3.12.

The total external work T corresponding to equation II-3.4 is given by

$$T = \int\int q(\nu^*, \phi) \sum_m \sum_n C_3 \sin \frac{m\pi \nu^*}{l} \cos n\phi \, a \, d\nu^* d\phi \qquad \text{(IV-4.4)}$$

If one makes the approximation that ϵ_a, γ vanish, the Principle of

Least Energy requires that

$$\delta(V - T) = 0 \tag{IV-4.5}$$

or

$$C_3 = \frac{2a^2(1-\nu^2)}{E\,h\,\ell\,\pi\,f(m,\lambda)_{\text{approx.}}}\int \cos m\phi \, d\phi \int q(u,\phi)\,\sin\frac{m\pi u^*}{\ell}\,du^* \tag{IV-4.6}$$

where

$$f(m,\lambda)_{\text{approx}} = \lambda^4\left(\frac{1}{m^4}+\frac{\eta^2}{12}\right)+\frac{\eta^2}{12}(m^2-1)^2+\frac{\nu\lambda^2\eta^2}{6}(m^2-1)+\frac{1-\nu}{6}\lambda^2\eta^2\left(\frac{m^2-1}{m}\right)^2 \tag{IV-4.7}$$

This solution is essentially the one studied by Hayashi (19).

In order to obtain a more accurate solution for a given loading condition, one can apply the results of Chapter IV-3. Equations IV-3.11 and IV-3.12 give C_3 as a definite function of C_1, C_2. If equations IV-4.2 and IV-4.4 are substituted into equation IV-4.5 and the variational procedure carried out with C_3 as the dependent variable, it follows that

$$C_3 = \frac{2a^2(1-\nu^2)}{E\,h\,\ell\,\pi\,f(m,\lambda)}\int \cos m\phi \, d\phi \int q(u,\phi)\,\sin\frac{m\pi u^*}{\ell}\,du^* \tag{IV-4.8}$$

For purposes of computation, it has been found that one can satisfactorily approximate the mode shapes by neglecting the inertial effect ∇, that is, by taking

$$A = 1 + \frac{2}{1-\nu}\left(\frac{\lambda}{m}\right)^2$$

$$B = A - \nu\left(\frac{\lambda}{m}\right)^2 + \left(\frac{\lambda}{m}\right)^4 \tag{IV-4.9}$$

This is shown in figure 15.

The preceding analysis consists of an expansion of the solution in terms of approximate mode shapes. It was seen in Chapter IV-3 that these modes led to frequencies that were indistinguishable from those calculated by more exact methods. However, as the frequencies are obtained from an expression involving the integral of the modes, one must expect the accuracy of the mode shapes to be less than that of the frequencies. This is shown in figure 15. The "exact" mode shapes were obtained from Baron and Bleich (26). The mode shapes resulting from the assumptions that $\epsilon_{\lambda}, \gamma$ vanish lie on the axis, as A, B are identically equal to one. Again, it is evident that this assumption is reasonable only in the vicinity of $\lambda/n < 1$, that is, a model of the shell consisting of long slender elemental beams. The second approximation given by equations IV-3.11 and IV-3.12 is applicable over a much wider range. In fact, it is fortuitous that the error in approximating the mode shapes by equation IV-4.9 seems to be in the direction of closer correspondence with the results of Baron and Bleich. Presumably, if the iterative procedure described in Chapter IV-3 were carried an additional step, it would result in mode shapes that would approximate those of Baron and Bleich more closely.

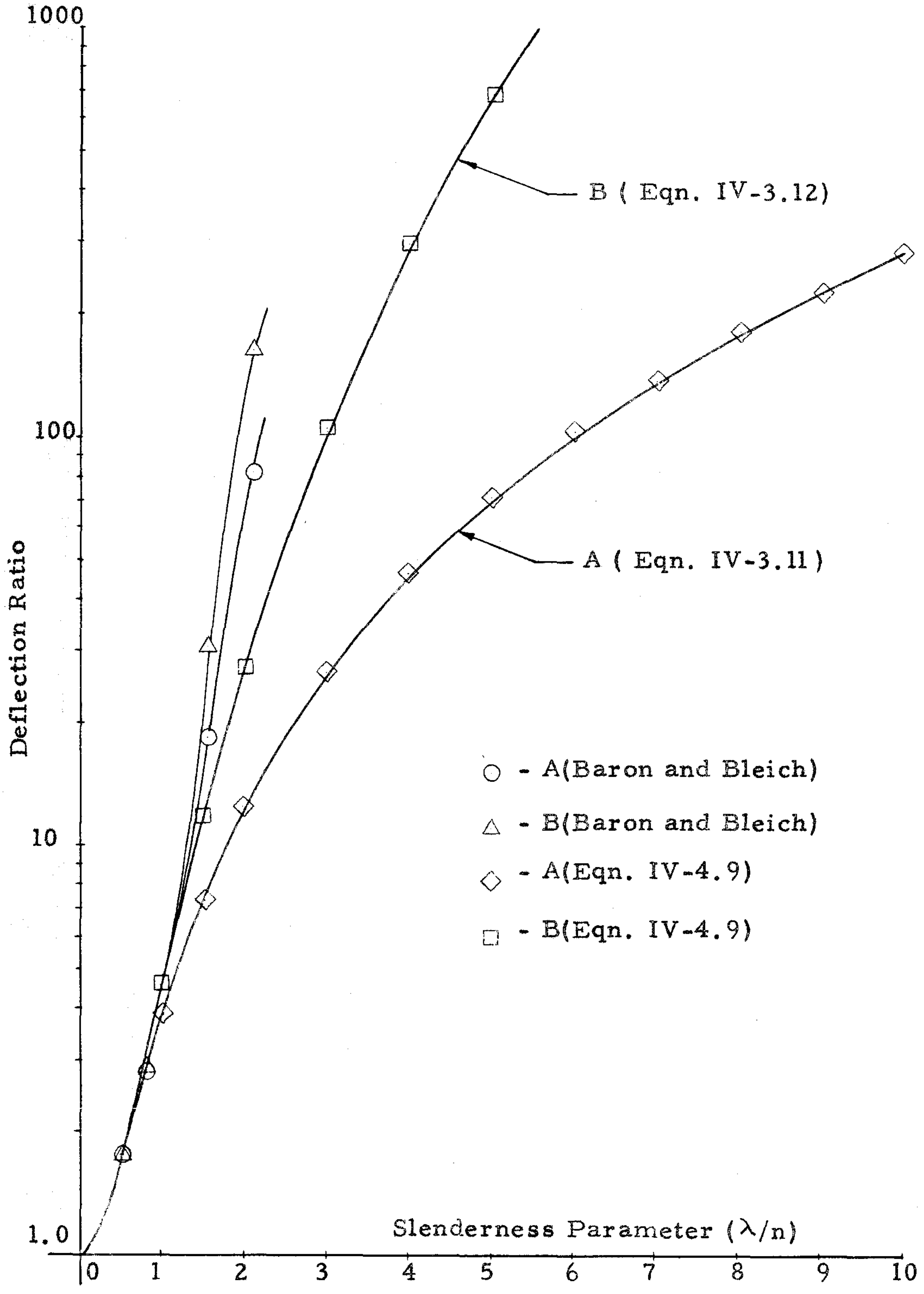


Figure 15 - Deflection Ratio

Chapter V

EXPERIMENTAL PROCEDURE AND RESULTS

It is the purpose of this chapter to examine the behavior of the shell surface in the neighborhood of a concentrated load, which is the ultimate in discontinuous loading functions, in order to determine both how large is the region of discrepancy between the approximate theory and experiment, and to obtain a measure of this discrepancy in both deflection and strain. By extrapolating the results for a concentrated load to the more general loading functions, one can obtain a measure of the accuracy of the approximate method in the general case.

1. Experimental Apparatus: The cylindrical shell used in the experimental investigation was constructed by usual sheet metal technique with a welded longitudinal seam. The material was a low carbon steel that had been cold-rolled. The dimensions are those given in Appendix I. Two rings soldered into the ends of the shell to retain the shape afforded a means of fastening the deflection-measuring fixture.

The load-applying device, shown in figure 16, consists of a rod on which sandbags of any desired weight can be placed, and which is guided by a wooden frame. The load is actually applied to the shell through contact with the end of the rod on which a hemisphere of radius 1/4 inch has been ground.

2. Radial Deflection: The apparatus used in the measurement of radial deflection was the same as that used by Hayashi (19). This fixture, shown in figure 17, consists of a long bar which rotates freely in the shell and on which is mounted a slider holding a dial gage. It was found convenient to use a 1-inch Ames Dial Gage with a least reading of 0.001 in.

Figure 16 - Cylindrical Shell with Load Applying Apparatus

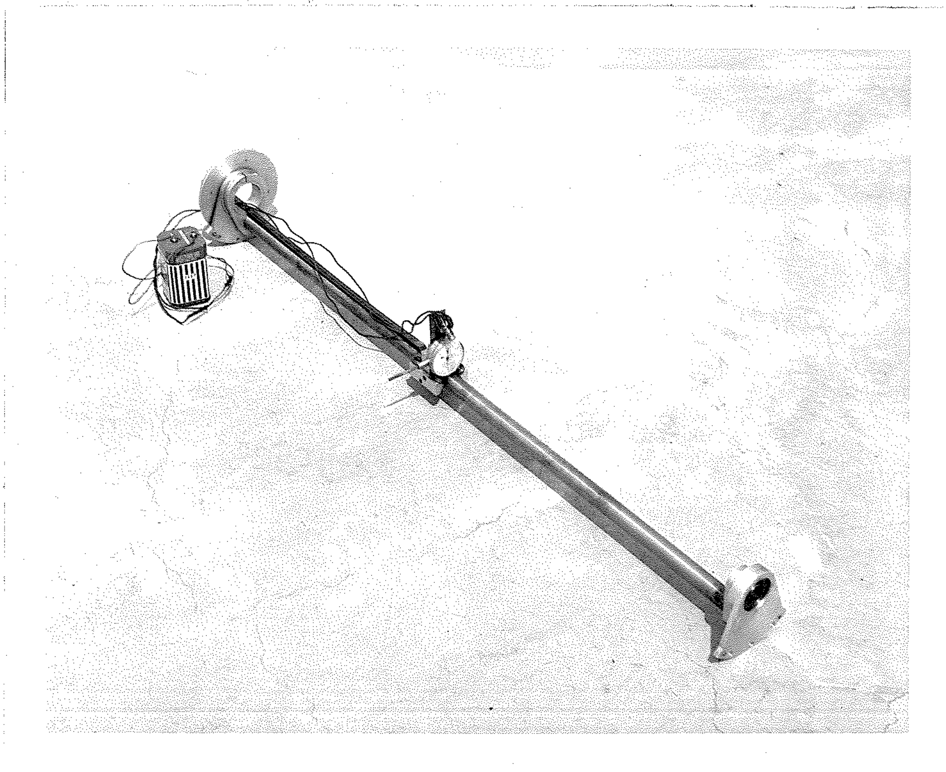


Figure 17 - Radial Deflection Measuring Fixture

The axial position of the slider could be measured by an attached steel tape (least reading 1/16 in.); while the angular orientation could be measured by a protractor fastened to the shell (least reading 1°).

Since the shell was constructed by the usual sheet metal technique, the cross-sections were not perfectly round, although the generators were reasonably straight. The initial surface of the mid-span station is shown in figure 18. The scatter in the observed points is due to the effect of different orientations of the shell, that is, the deflection due to the weight of the shell itself.

Deflection data was taken at several points around the mid-span station with a concentrated load applied at this station, and is presented in figure 19. Each set of symbols applies to a different angular orientation of the point of application of the load which are given in figure 18. It is to be noted that the initial curvature has a marked effect on the final deflected surface. So-called high spots or peaks, as at $\phi = 90^{\circ}$ (fig. 18), give deflections which are low; while at low spots or valleys, as at $\phi = 150^{\circ}$ (fig. 18), the deflections are high. This difference is essentially the result of applying the load alternatively along a major or a minor axis of a locally elliptic surface. The results obtained in the relatively constant-curvature regions, that is $20^{\circ} \leqslant \phi \leqslant 90^{\circ}$ and $200^{\circ} \leqslant \phi \leqslant 270^{\circ}$ (fig. 18), were consistent in the peak readings. However, it was generally found that the surface imperfections had a greater effect on the resulting deflected surface in regions where the shape of the deflected surface was slowly varying, as at a comparatively large distance from the point of application of the load.

A representative set of data is shown in figures 20 and 21, where it is compared with the computed curves of Appendix I. For the circumferential

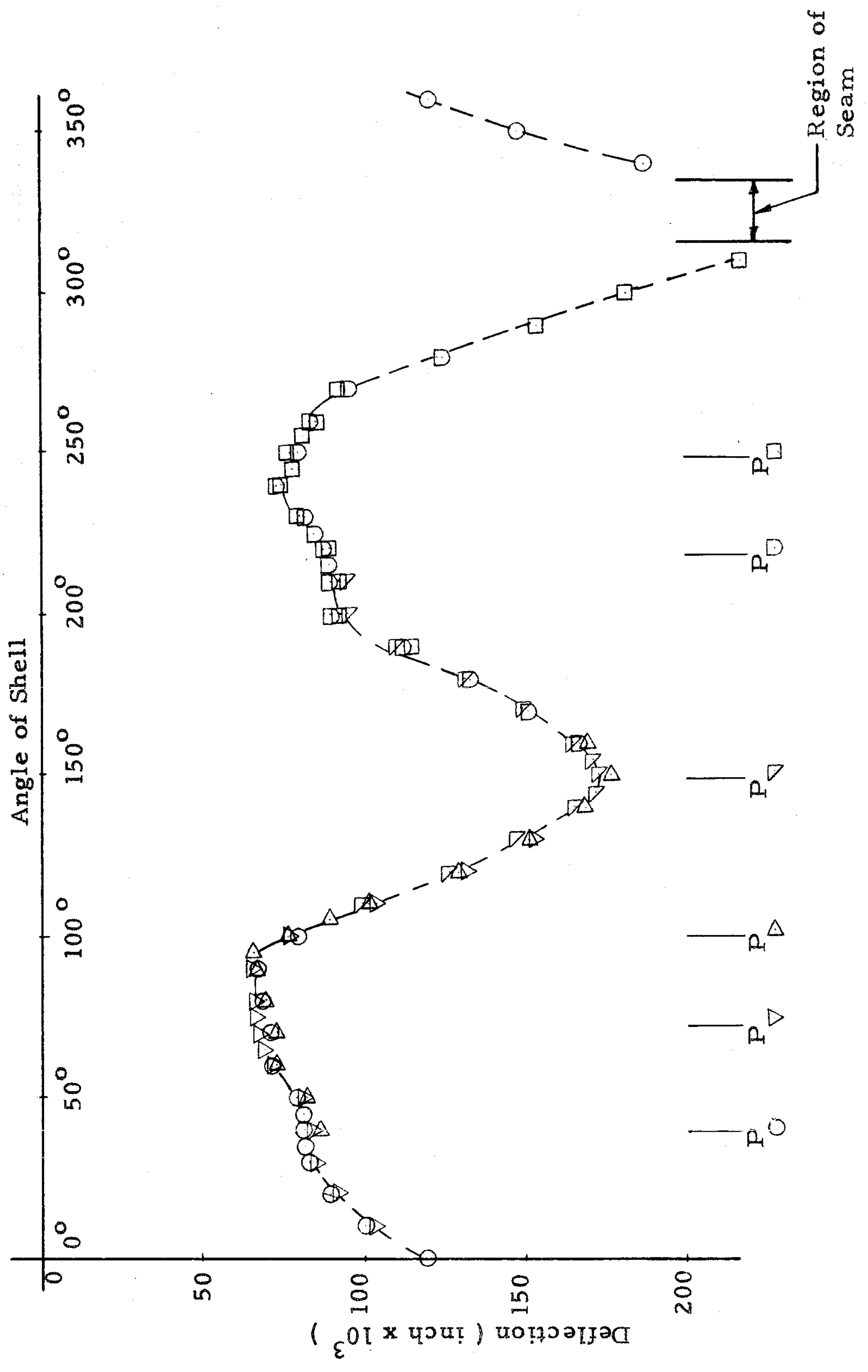


Figure 18 - Unloaded surface; Deflection vs. Angle

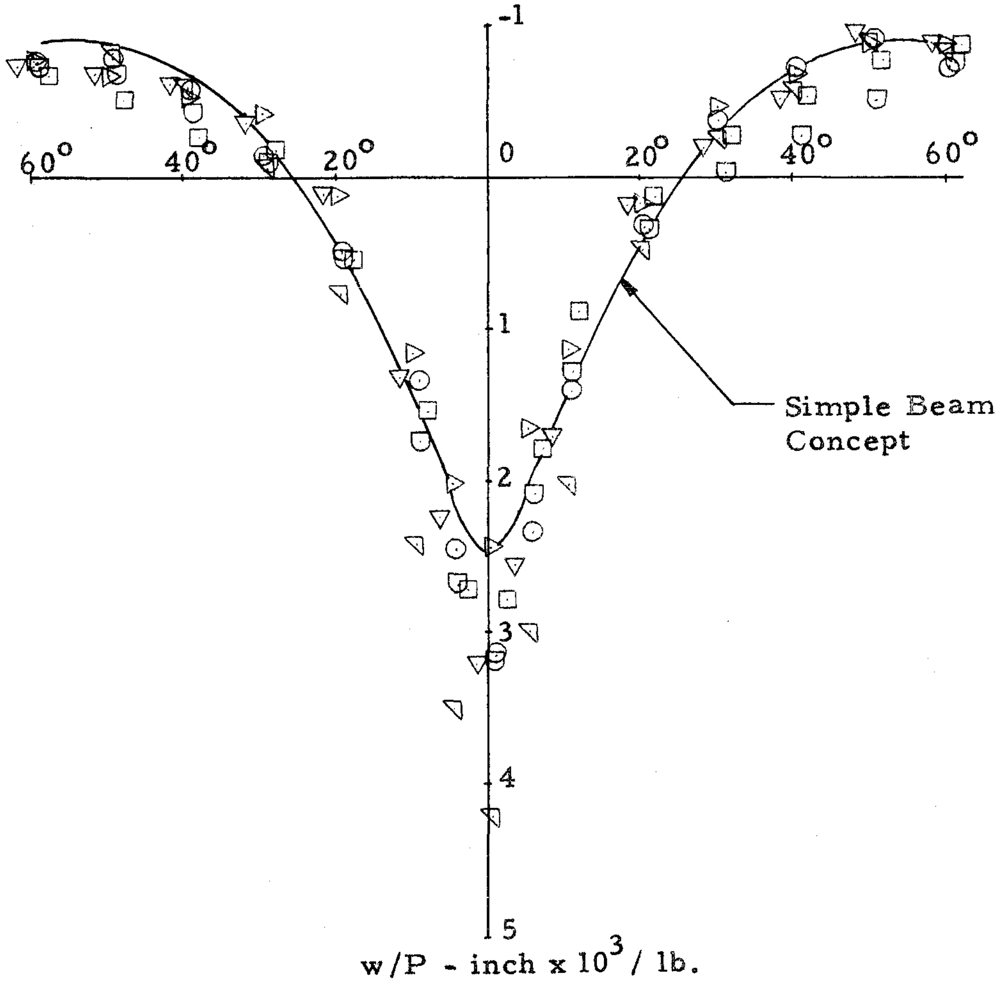


Figure 19 - Radial Deflection vs. Angle - Sta. 0
Comparison With Experiment

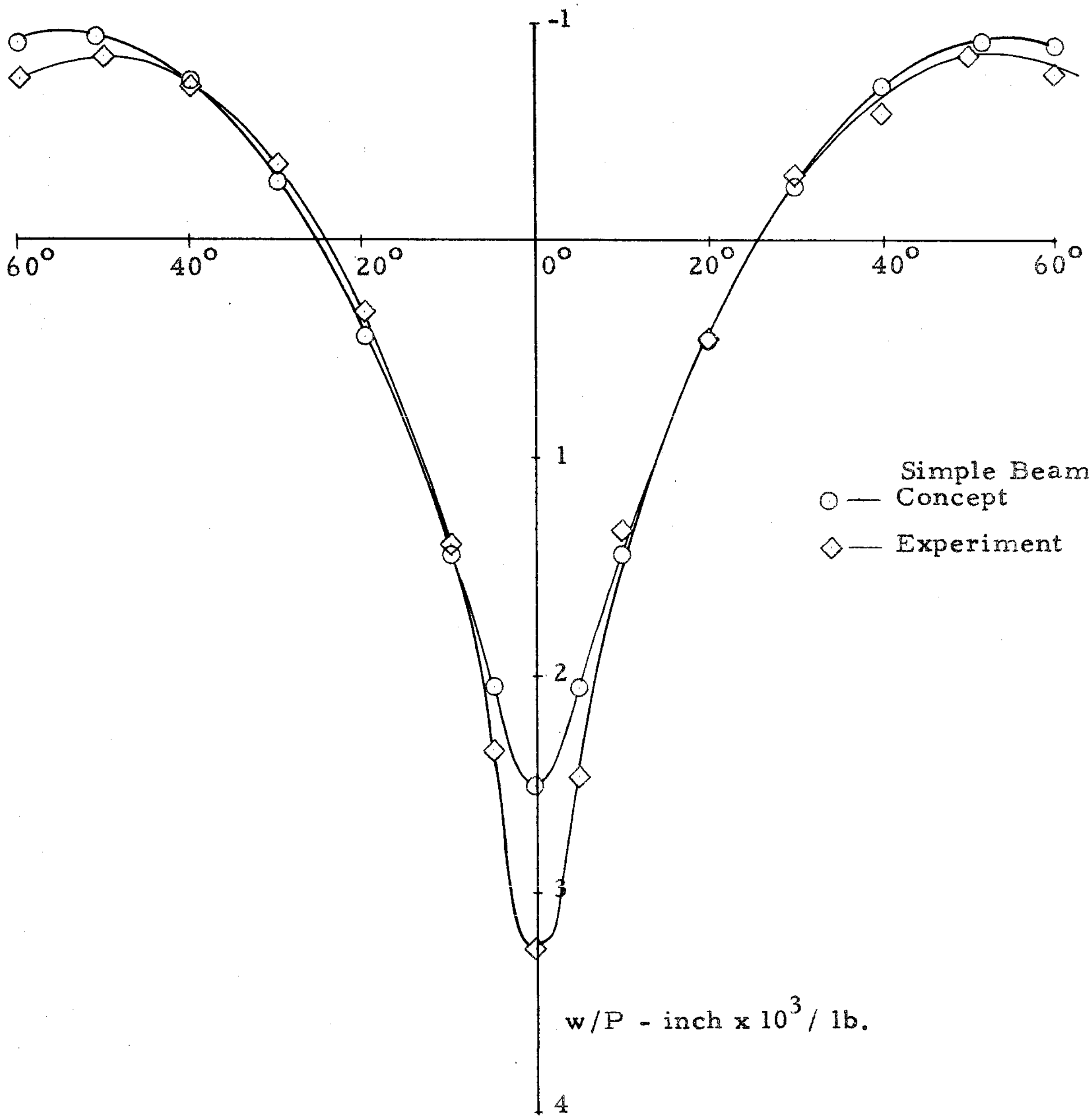


Figure 20 - Radial Deflection vs. Angle - Sta. 0
Comparison With Experiment

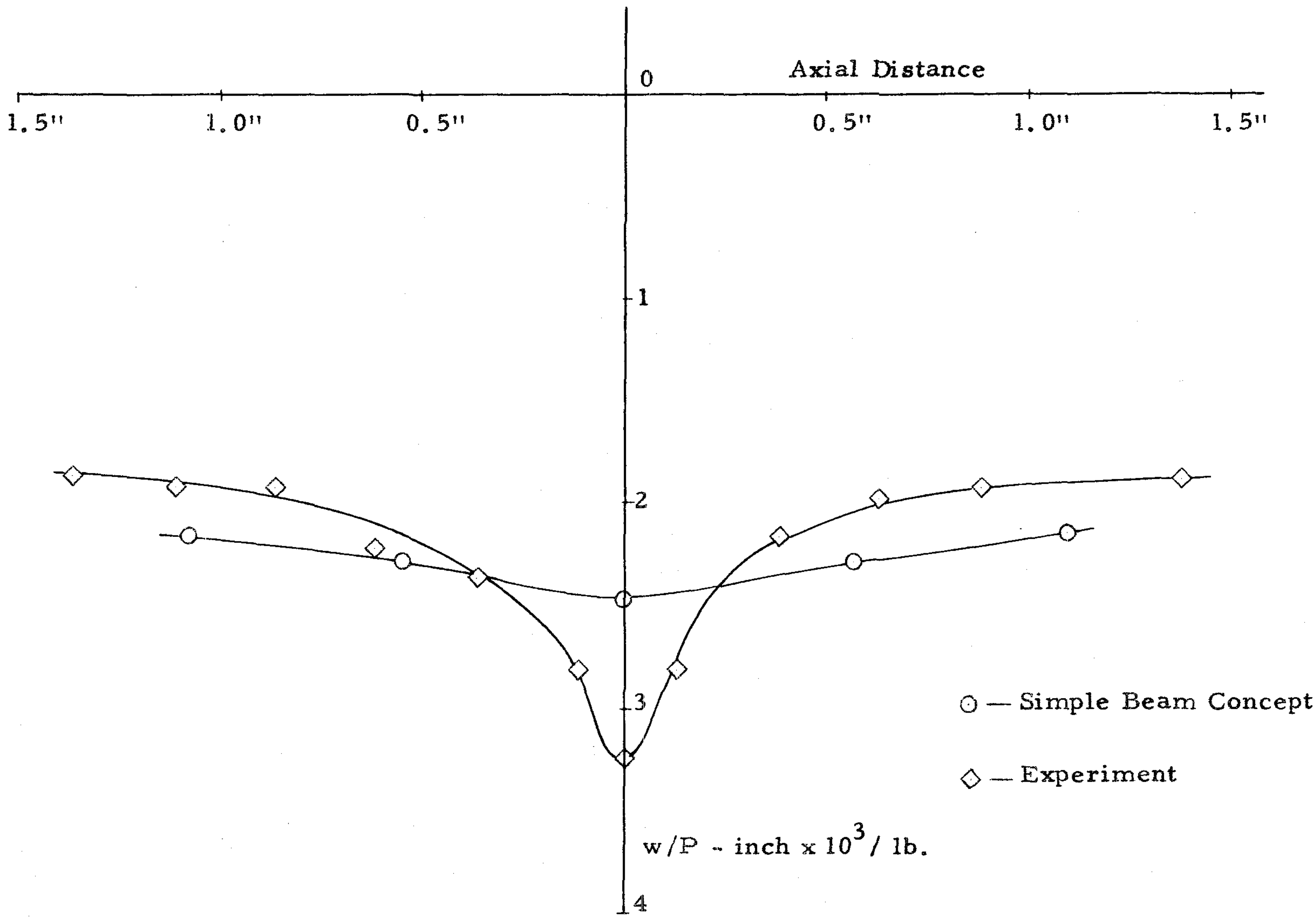


Figure 21 - Radial Deflection vs. Axial Distance on a Generator Thru The Load

variation (fig. 20), very good agreement is found in the region away from the load; while, in the region under the load, the curvatures that were calculated are found to be much smaller than those observed. For the axial variation (fig. 21), the discrepancy again is in an underestimation of the curvature in the neighborhood of the load, but there the effect extends to larger distances from the load.

From these results, it can be concluded that the approximate method underestimates the curvatures in the neighborhood of the load, but gives a good representation elsewhere. This discrepancy is more pronounced in the axial direction where the rate of change of curvature is not as large as in the circumferential direction. Thus, since the strain depends essentially on the local curvature, and the discrepancy in the curvature extends from the load to a greater degree in the axial direction than in the circumferential direction, one can expect the discrepancy in the observed strain to be in the region under the load, but to extend in the axial direction to a greater degree than in the circumferential direction.

3. <u>Axial Membrane Strain</u>: In order to measure the axial strain, electric resistance strain gages (Baldwin, Type A-7) with a gage length of 1/4 inch (width 1/8 inch) were attached to the surface of the shell. Two gages were located at each axial station, one on the inner wall and one on the outer wall. Measurements were taken at axial stations 9/16 inch and 1-3/32 inch from the mid-span station at which the load was applied. The strain was measured by means of a Baldwin Strain Indicator. The apparatus is shown in figure 22.

As the wall thickness of the shell was a nominal 0.015 inch, and the total thickness of the gages (paper + grid + paper) was about 0.005 inch, it was suspected that strain measurements in bending might be greatly

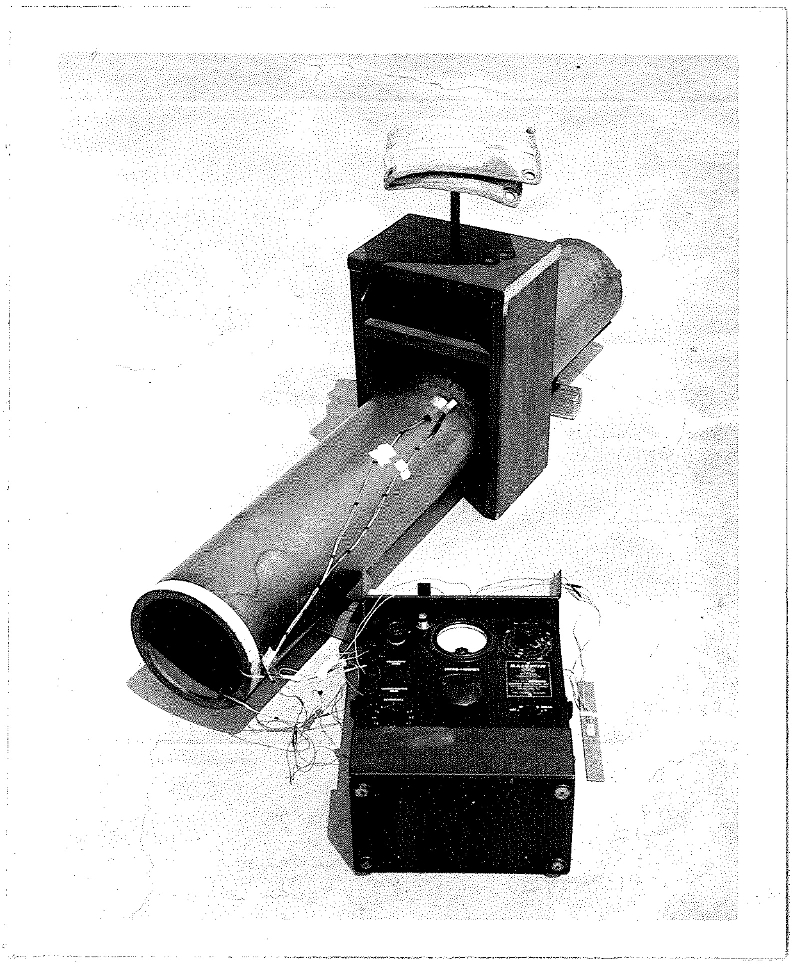
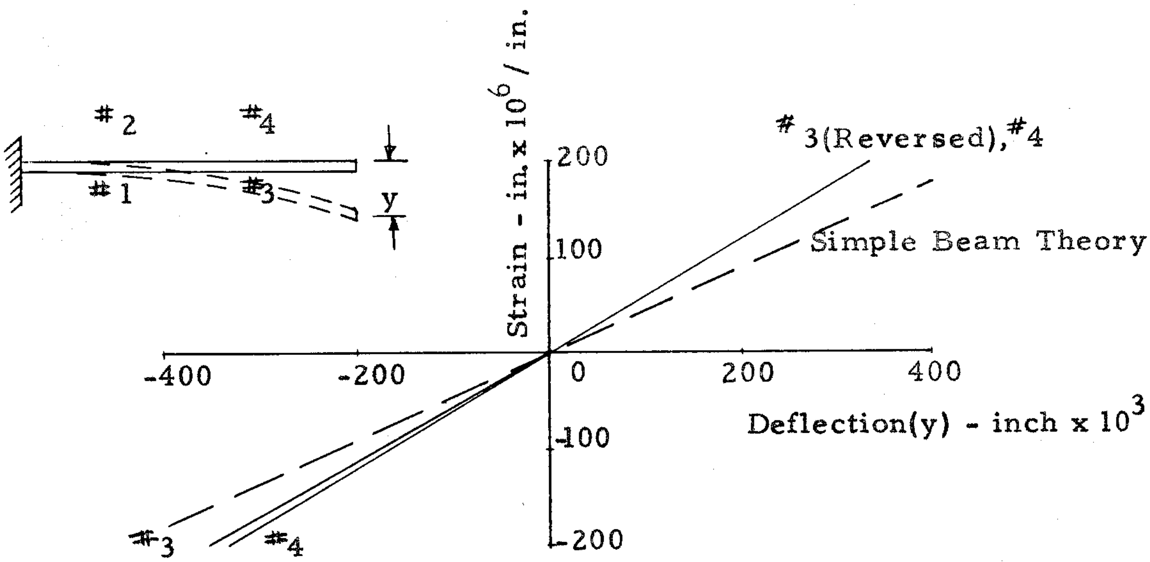


Figure 22 - Cylindrical Shell with Strain Measuring Apparatus

affected by this apparent increase in wall thickness due to the gage. In order to measure this effect, a strip of the same material was tested as a cantilever beam. Four gages of the same type were located on the beam in the same fashion as they were located on the shell. From the results shown in figure 23, it is evident that the reading of the gage can be as much as 20-25 % higher than that expected on the basis of simple beam theory. However, the most significant result is that, due to small differences, as in the amount of glue applied to the gage, it was found that gages at the same station gave results that might differ as much as 5-10 %. Thus, as the membrane strain is taken to be half the sum of the readings of the two gages at a given point, it is possible that the resulting membrane strain can be in error by an amount equal to approximately 2-5 % of the bending strain at that point. Particularly in the neighborhood of the load this can amount to as much as 2-10 % of the membrane strain.

The results of the membrane strain measurements are given in figures 24, 25 corresponding to the axial stations given above. It can be observed that within the limits of the error in the gages mentioned above, the strain results are as predicted from the analysis of the deflection measurements; that is, for any given axial station, the agreement is good away from the generator $\phi = 0$, but near this generator, the lack of curvature given by the approximate theory results in strains that are not in good agreement with those observed.

The strain in the immediate vicinity of the point of application of the load was evidently above the yield point of the material, as plastic flow was visible at the points where the load was applied. Several attempts were made to measure strain in this region, but due to the highly localized behavior of the strain and its magnitude, the gages failed or



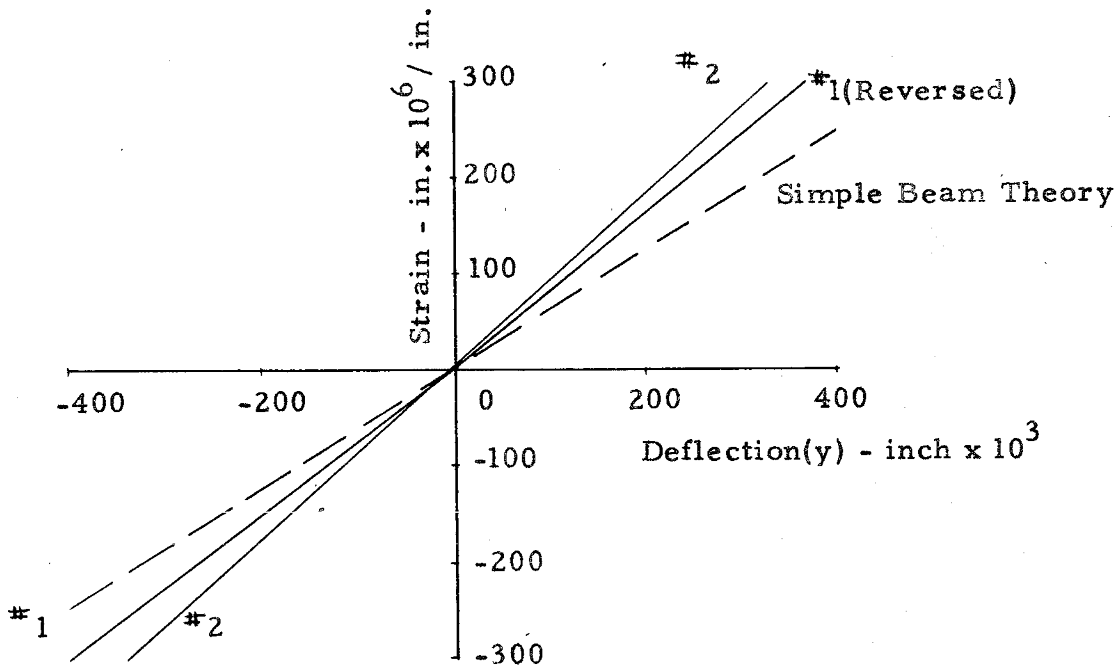


Figure 23 - Strain Gage Calibration

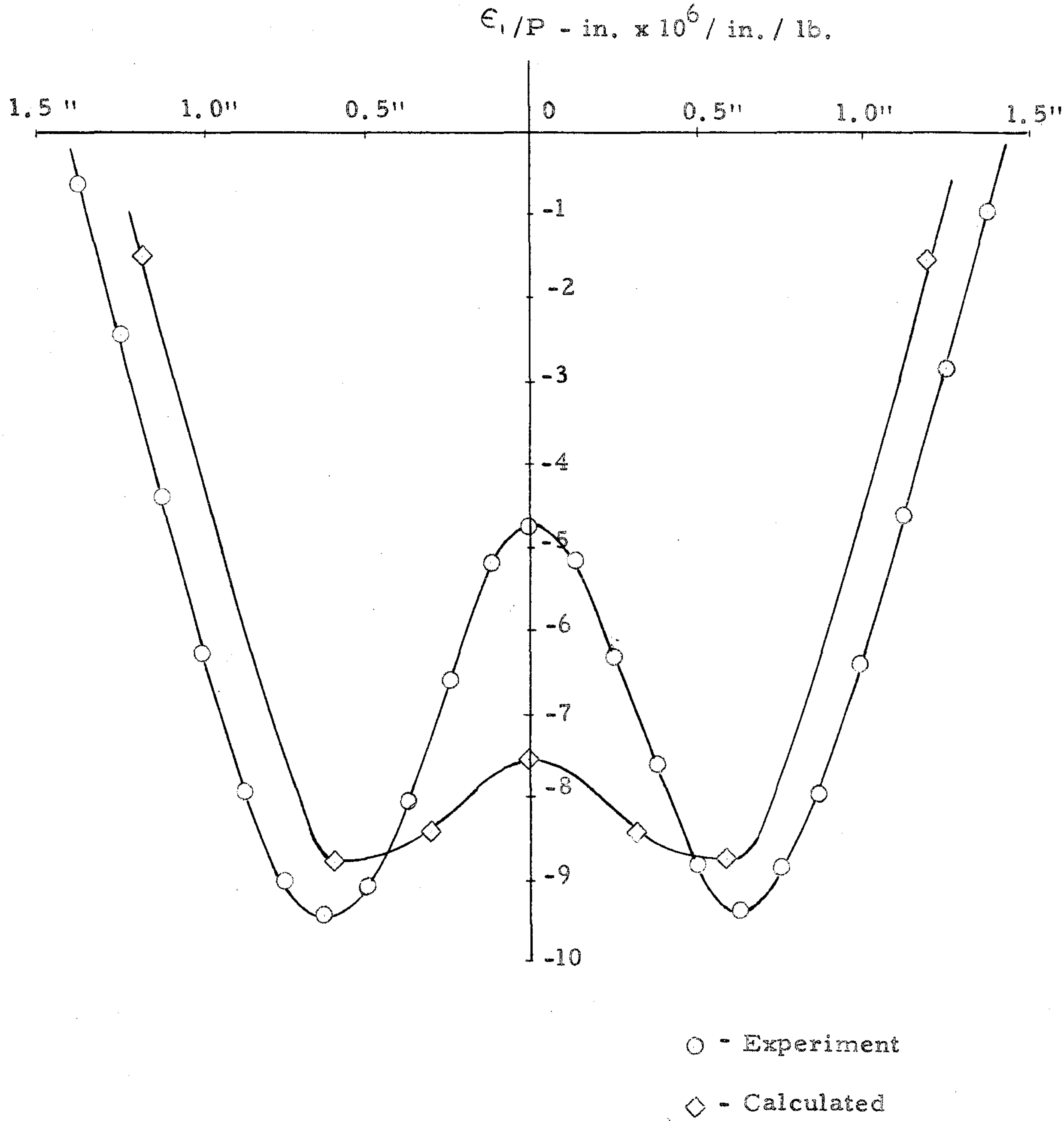


Figure 24 - Axial Membrane Strain vs. Arc Length at Sta. 1-3/32; Comparison With Experiment

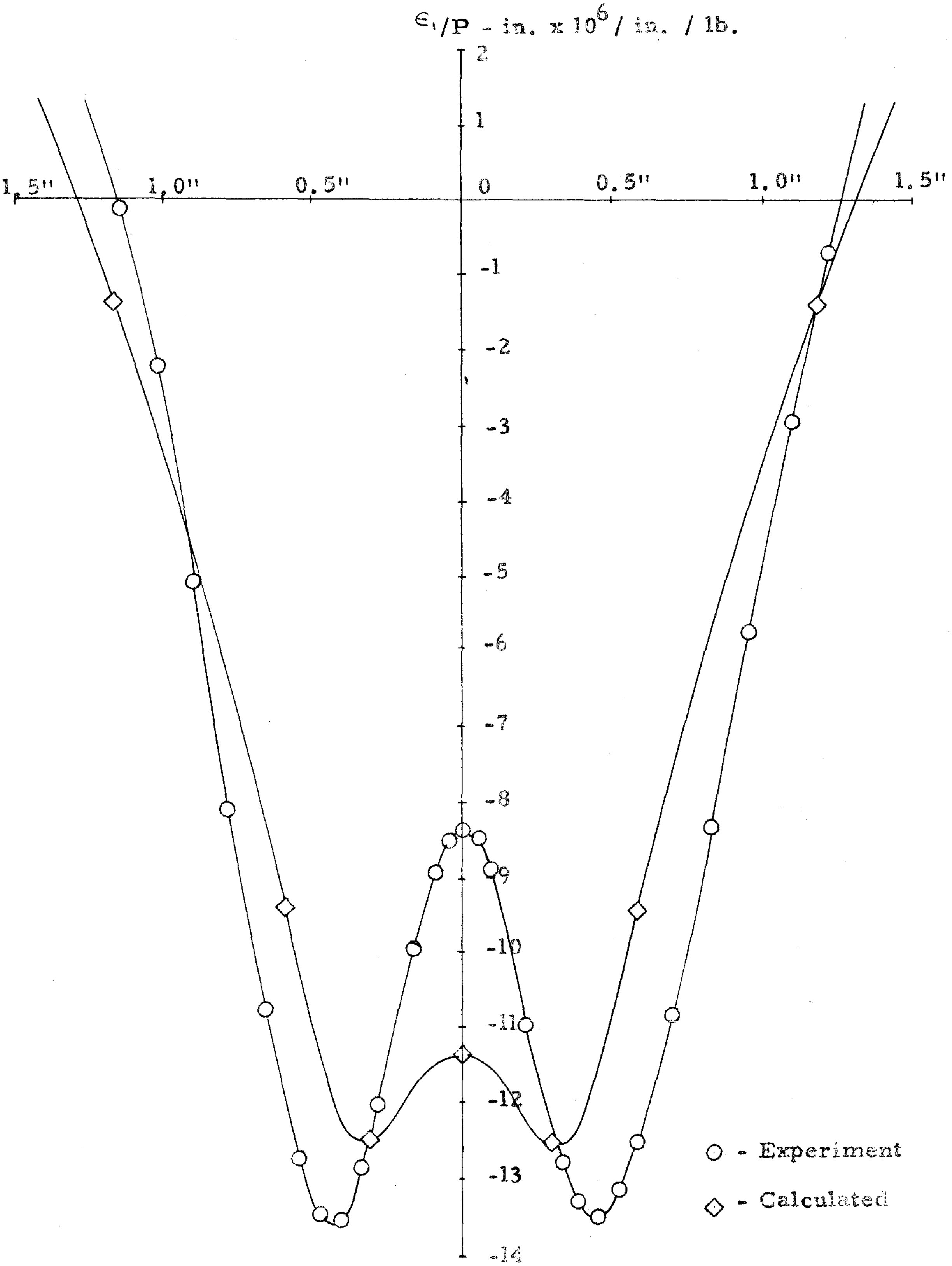


Figure 25 - Axial Membrane Strain vs. Arc Length at Sta. 9/16
Comparison With Experiment

indicated that possibly the adhesive had broken by exhibiting a changing null point.

Thus the strain in the vicinity of the point of application of the load is divided into three regions. The first, closest to the load and extending a distance comparable to several shell thicknesses, is probably describable only in terms of a more complete three dimensional analysis and exhibits a plastic deformation. The second, extending about an inch from the load, is a region in which the strains ϵ_2, γ are important. The third, comprising the remainder of the shell, is a region in which the strains ϵ_2, γ are not significant and the approximate method is sufficient.

SUMMARY

As a result of considering a circular shell to be the limiting case of a polygonal shell, a method of analysis was developed that neglected the contribution of the in-plane shear and circumferential normal strains to the deformation. The classical system of three partial differential equations given by Timoshenko (3) has been reduced by this simplified approach to a less complex set, namely

$$\left(\nabla^4 w + 2\nabla^2 w + w - p\right)_{,\phi\phi\phi\phi} + 12/\eta^2\, w_{,xxxx} + 2(1-\nu)\left(w + w_{,\phi\phi}\right)_{,xx\phi\phi} = 0$$

$$v_{,\phi} = 0$$

$$u_{,\phi} + v_{,x} = 0$$

For the case of a complete shell, the substitution

$$w\,(x,\phi) = \sum_{n=1}^{\infty} W(x,n) \cos n\phi$$

reduces the equation for w to a total differential equation of the fourth order given by

$$\left(n^4 + 12/\eta^2\right) W_{,xxxx} - 2n^2 W_{,xx}\left(n^4 - \overline{2-\nu}\, n^2 + 1-\nu\right) + n^4\left(n^2-1\right)^2 W = 0$$

This is to be compared with Donnell's equations which are

$$\nabla^8 w + \frac{1-\nu^2}{\eta^2} w_{,xxxx} = \nabla^4 p$$

$$\nabla^4 u = \nu w_{,xxx} - w_{,x\phi\phi}$$

$$\nabla^4 v = (2+\nu) w_{,\phi xx} + w_{,\phi\phi\phi}$$

With the same substitution for w, Donnell's equation for w reduces to a total differential equation of the eighth order.

An analysis was made which establishes the conditions of loading under which the approximate solution gives satisfactory values for stresses throughout the shell. It was found that if the loading function could be expanded in a Double Fourier Series such that the principal coefficients corresponded to wave numbers having circumferential wave numbers n greater than or equal to, about three times the axial wave number λ , the approximate solution was in good agreement with the solutions obtained using a more exact approach. This is equivalent to requiring that the principal deflection modes of the shell give rise to elemental panels with a high slenderness ratio (length/depth).

Thus, if one has a strongly varying loading function, the approximate method will give more accurate results if a/ℓ is small. For, with a/ℓ small, as λ is proportional to a/ℓ , then for a given number of axial waves, a greater range of n is within the desired range. When a/ℓ is large, good representation of a strongly varying loading function is not possible, as the principal modes lie outside the desired range.

A computed solution for a concentrated load was compared with experimental measurements of radial deflection and axial membrane strain. It was found that in the immediate neighborhood of the concentrated load, the approximate solution was in error. The observed deflection under the load was about 20% larger than was calculated. No strain data could be obtained in this region. However, outside the region immediately under the load, good correspondence was found between the observed and calculated values of both deflection and strain.

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Appendix I

CALCULATIONS FOR A CONCENTRATED LOAD APPLIED AT THE MID-SPAN POINT OF A COMPLETE CYLINDER

In order to carry out the indicated summation for the case presented in Chapter III-2, the roots α, β of the following equation must be obtained.

$$m^4 - \bar{A}\, m^2 + \bar{B} = 0$$

where

$$\bar{A} = 2m^2 \cdot \frac{m^4 - \overline{2-\nu}\, m^2 + 1-\nu}{m^4 + 12/\eta^2} > 0$$

$$\bar{B} = m^4 \frac{(m^2-1)^2}{m^4 + 12/\eta^2} > 0$$

It follows directly that

$$m^2 = \bar{A}/2 \pm i\sqrt{\bar{B} - (\bar{A}/2)^2}$$

In polar notation, m^2 can be written as

$$m^2 = r\, e^{i(\pm\theta)}$$

where

$$\theta = \tan^{-1}\left[\frac{2}{\bar{A}} \cdot \sqrt{\bar{B} - (\bar{A}/2)^2}\right]$$

$$r = \sqrt{\bar{B}}$$

This is shown graphically in figure A-1. Therefore, the roots m are given by

$$\sqrt{r}\; e^{i(\pm \theta/2)} \qquad\qquad \sqrt{r}\; e^{i(\pm \theta/2 + \pi)}$$

so that

$$\alpha = \sqrt{\tfrac{1}{2}\left(\sqrt{\bar{B}} + \bar{A}/2\right)}$$

$$\beta = \sqrt{\tfrac{1}{2}\left(\sqrt{\bar{B}} - \bar{A}/2\right)}$$

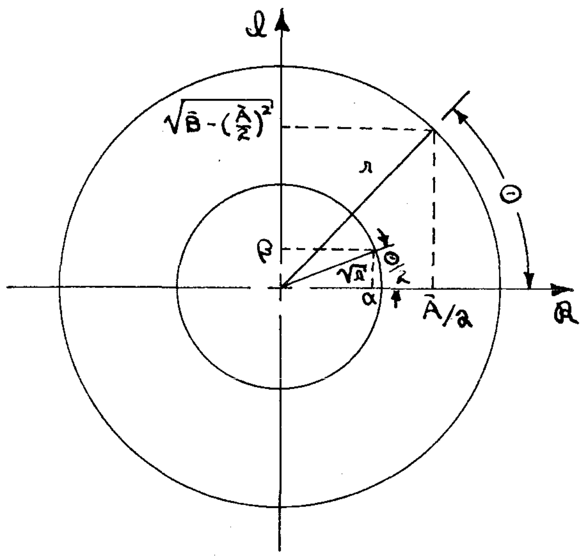

Figure A-1.

For $n \geq 10$, n^2 is negligible in comparison with n^4, so that the following simplifications can be made:

$$\bar{A}/2 \;\approx\; n^6 / (n^4 + 12/\eta^2)$$

$$\bar{B} \;\approx\; n^8 / (n^4 + 12/\eta^2)$$

If N is defined as

$$N = 1/\sqrt{1 + 12/\eta^2 n^4}$$

then

$$\alpha^2 = \frac{m^2}{2} \cdot N(1+N)$$

$$\beta^2 = \frac{m^2}{2} \cdot N(1-N)$$

$$\alpha^2 + \beta^2 = m^2 N \qquad (m \geq 10)$$

$$\alpha^2 - \beta^2 = m^2 N^2$$

Using the following data,

E	= 30 x 10 psi	a	= 3.367 inch
h	= 0.015 inch	ℓ	= 45 inch
ν	= 0.29		

values of α, β , A/B, B, C/B, D'/B have been computed and are presented in Table I for the first thirty-five terms. It is to be noted that all the above items save B are general properties of the differential equation applied to a particular cylinder geometry for arbitrary loading function at the point x = 0 and "Pin-Ended" boundary conditions. Only B is characteristic of the loading function.

For the case of the concentrated force of magnitude P located at the mid-span point, the deflection w and the axial membrane strain ϵ_1 are given by

$$w(x,\phi) = \sum_{m=1}^{\infty} W(x,m) \cos m\phi + W(x,1) \cos \phi$$

$$\epsilon_1(x,\phi) = \sum_{m=1}^{\infty} \frac{1}{m^2} W_{,xx} \cos m\phi + W_{,xx}(x,1) \cos \phi$$

where

$$W(x,m) = B\Big[\cos \beta x \left(\sinh \alpha x + A/B \cosh \alpha x\right)$$

$$+ \sin \beta x \left(C/B \sinh \alpha x - \alpha/\beta \cosh \alpha x\right)\Big]$$

and

$$W_{,xx}(x,n) = B\left\{\cos\beta x\left[-(\alpha^2+\beta^2)\sinh\alpha x + \left(\alpha^2\frac{A}{B} + 2\alpha\beta\frac{C}{B} - \beta^2\frac{A}{B}\right)\cosh\alpha x\right]\right.$$

$$\left. + \sin\beta x\left[\left(\alpha^2\frac{C}{B} - 2\alpha\beta\frac{A}{B} - \beta^2\frac{C}{B}\right)\sinh\alpha x - \frac{\alpha}{\beta}(\alpha^2+\beta^2)\cosh\alpha x\right]\right\}$$

For purposes of carrying out the calculations, the following functions will be introduced:

$$\mathcal{F}(n,x) = \cos\beta x\left(\sinh\alpha x + \frac{A}{B}\cosh\alpha x\right)$$

$$+ \sin\beta x\left(\frac{C}{B}\sinh\alpha x - \frac{\alpha}{\beta}\cosh\alpha x\right)$$

$$\mathcal{S}(n,x) = \frac{1}{n^2}\left\{\cos\beta x\left[-(\alpha^2+\beta^2)\sinh\alpha x + \left(\alpha^2\frac{A}{B} + 2\alpha\beta\frac{C}{B} - \beta^2\frac{A}{B}\right)\cosh\alpha x\right]\right.$$

$$\left. + \sin\beta x\left[\left(\alpha^2\frac{C}{B} - 2\alpha\beta\frac{A}{B} - \beta^2\frac{C}{B}\right)\sinh\alpha x - \frac{\alpha}{\beta}(\alpha^2+\beta^2)\cosh\alpha x\right]\right\} \tag{A-I.1}$$

Thus, the deflection and axial membrane strain relations are given by

$$\omega(x,\phi) = \sum_{n=1}^{\infty} B\cos n\phi \times \mathcal{F}(n,x) + W(x,1)\cos\phi$$

$$\epsilon_1(x,\phi) = \sum_{n=1}^{\infty} B\cos n\phi \times \mathcal{S}(n,x) + W_{,xx}(x,1)\cos\phi \tag{A-I.2}$$

The functions $\mathcal{F}, \mathcal{S}$ can be greatly simplified if the behavior of α, β is determined for n-large. For n such that

$$12/n^4\eta^2 < 1$$

N can be expanded as

$$N = 1/\sqrt{1+12/m^4\eta^2} = 1 - \frac{1}{2}\left(\frac{12}{m^4\eta^2}\right) + \frac{3}{8}\left(\frac{12}{m^4\eta^2}\right)^2 - \cdots$$

Therefore, α^2, β^2 can be expressed as

$$\alpha^2 = m^2\left[1 - \frac{3}{4}\left(\frac{12}{m^4\eta^2}\right) + \frac{11}{16}\left(\frac{12}{m^4\eta^2}\right)^2 - \cdots\right]$$

$$\beta^2 = \frac{3}{m^2\eta^2}\left[1 - \frac{5}{4}\left(\frac{12}{m^4\eta^2}\right) + \frac{11}{8}\left(\frac{12}{m^4\eta^2}\right)^2 - \cdots\right]$$

so that $$\alpha/\beta = \frac{m^2}{\sqrt{3}}\eta\left[1 + \frac{1}{2}\left(\frac{12}{m^4\eta^2}\right) - \frac{1}{16}\left(\frac{12}{m^4\eta^2}\right)^2 - \cdots\right]^{1/2}$$

Thus, as

$$\lim_{m\to\infty}\left\{\frac{\alpha/\beta}{\operatorname{Sinh}\frac{\alpha l}{2a}\operatorname{Cosh}\frac{\alpha l}{2a}}\right\} = 0$$

it is justifiable after a few terms to take

$$\frac{A}{B} = -1.0 \qquad\qquad \frac{C}{B} = \alpha/\beta$$

When this simplification is introduced, the functions $\mathcal{F}, \mathcal{S}$ become

$$\mathcal{F}(m,y) = -e^{-\alpha y}\left(\operatorname{Cos}\beta y + \frac{\alpha}{\beta}\operatorname{Sin}\beta y\right)$$

$$\mathcal{S}(m,y) = \frac{\alpha^2+\beta^2}{m^2}\cdot e^{-\alpha y}\left(\operatorname{Cos}\beta y - \frac{\alpha}{\beta}\operatorname{Sin}\beta y\right) \qquad (m \geq m^{\dagger}) \qquad \text{(A-I.3)}$$

For the values of ℓ, a given previously, this simplification can be applied for $n^{\dagger} = 7$. Defining the angle ψ as

$$\sin\psi = \beta/\sqrt{\alpha^2+\beta^2} \qquad \cos\psi = \alpha/\sqrt{\alpha^2+\beta^2}$$

it follows that

$$\mathcal{F}(x,n) = -\frac{\sqrt{\alpha^2+\beta^2}}{\beta}\cdot e^{-\alpha x}\sin(\psi+\beta x)$$

$$\mathcal{A}(x,n) = \frac{\sqrt{\alpha^2+\beta^2}}{\beta}\cdot e^{-\alpha x}\frac{\alpha^2+\beta^2}{n^2}\sin(\psi-\beta x) \qquad (n \geq n^{\dagger})$$

so, for n-large,

$$\lim_{n\to\infty}|\mathcal{F}| = 0 \qquad \lim_{n\to\infty}|\mathcal{A}| = 0$$

and, for x = 0,

$$\lim_{\substack{x\to 0\\ n>n^{\dagger}}}(\mathcal{F}) = -1.0 \qquad \lim_{\substack{x\to 0\\ n>n^{\dagger}}}(\mathcal{A}) = N$$

The procedure in calculating w, ϵ_1 is therefore to carry out explicitly the summation indicated by equation A-I.2 using the relations of equation A-I.1 for $n < n^{\dagger}$ and the relations of equation A-I.3 for $n \geq n^{\dagger}$. After a number n^* of terms, it becomes reasonable to make the approximation that

$$B/P = -\frac{a}{4\pi D}\cdot\frac{1}{n^3} \qquad (n > n^*)$$

For an error in B/P less than 10%, n^* can be taken equal to thirty-five. Thus the remainder of the series can be taken to be of the form

$$R_m(r,\phi) = -\frac{a}{4\pi D}\sum_{m=36}^{\infty}\mathcal{F}(r,m)\cdot\frac{\cos m\phi}{m^3}$$

$$S_m(r,\phi) = -\frac{a}{4\pi D}\sum_{m=36}^{\infty}\mathcal{S}(r,m)\cdot\frac{\cos m\phi}{m^3}$$

where

$$w(r,\phi) = \sum_{m=1}^{36} B\cos m\phi \times \mathcal{F}(r,m) + R_m(r,\phi) + W(r,1)\cos\phi$$

$$\epsilon_1(r,\phi) = \sum_{m=1}^{36} B\cos m\phi \times \mathcal{S}(r,m) + S_m(r,\phi) + W_{,rr}(r,1)\cos\phi$$

From the behavior of $\mathcal{F}, \mathcal{S}$ described above, it follows that

$$\left|R_m(r,\phi)\right| \leq \frac{a}{4\pi D}\left|\sum_{m=36}^{\infty}\frac{\cos m\phi}{m^3}\right| \times \mathcal{F}(r,35)$$

$$\left|S_m(r,\phi)\right| \leq \frac{a}{4\pi D}\left|\sum_{m=36}^{\infty}\frac{\cos m\phi}{m^3}\right| \times \mathcal{S}(r,35) \qquad \text{(A-I.4)}$$

while

$$R_m(0,\phi) = \frac{a}{4\pi D}\sum_{m=36}^{\infty}\frac{\cos m\phi}{m^3}$$

$$S_m(0,\phi) = -\frac{a}{4\pi D}\sum_{m=36}^{\infty}\frac{\cos m\phi}{m^3}$$

These latter functions can be determined explicitly from a consideration of the following function

$$\Xi(\phi) = \sum_{m=1}^{\infty}\frac{1}{m^3}\cdot\cos m\phi \qquad \text{(A-I.5)}$$

In Jolley (29) and Bromwich (28), the following function is defined

$$\sum_{m=1}^{\infty} \frac{\cos m\phi}{m} = -\ln\left(2 \sin \phi/2\right) \qquad (0 < \phi < 2\pi)$$

Therefore, from the uniformity of a Fourier Series, it follows that

$$\Xi_{,\phi} = -\sum_{m=1}^{\infty} \frac{1}{m^2} \sin m\phi \tag{A-I.6}$$

and
$$\Xi_{,\phi\phi} = \ln\left(2 \sin \phi/2\right) \tag{A-I.7}$$

Integrating once with respect to ϕ results in

$$\Xi_{,\phi} = \phi \ln 2 + 2\int \ln(\sin \psi)\, d\psi + C_1$$

This latter integral can be expressed in terms of a series given in Dwight (29), so that $\Xi_{,\phi}$ becomes

$$\Xi_{,\phi} = \phi \ln 2 + C_1 + 2\left[\frac{\phi}{2}\ln\frac{\phi}{2} - \frac{\phi}{2} - \frac{1}{18}\left(\frac{\phi}{2}\right)^3 - \frac{1}{900}\left(\frac{\phi}{2}\right)^5 - \frac{1}{19{,}845}\left(\frac{\phi}{2}\right)^7 - \cdots\cdots - \frac{2^{2n-1} B_n (\phi/2)^{2n+1}}{n(2n+1)!}\right]$$

where the B_n are Bernoulli Numbers. From equation A-I.6 it follows that

$$\Xi_{,\phi}(0) = 0$$

so that
$$C_1 = -\lim_{\phi \to 0} (\phi \ln \phi) = 0$$

Integrating once more, the function $\Xi(\phi)$ becomes

$$\Xi(\phi) = \left(\frac{\phi}{2}\right)^2(\ln 4 - 3) + \left(\frac{\phi}{2}\right)^2 \ln\left(\frac{\phi}{2}\right)^2 - \frac{1}{18}\left(\frac{\phi}{2}\right)^4\Bigg[1 + \frac{1}{75}\cdot\left(\frac{\phi}{2}\right)^2$$

$$+ \frac{1}{2205}\left(\frac{\phi}{2}\right)^4 + \cdots + \frac{2^{2m-1}B_m}{m(2m+1)!}\cdot\frac{36}{m+1}\cdot\left(\frac{\phi}{2}\right)^{2m-2}\Bigg] + C_2$$

From equation A-I.5 and the fact that

$$\lim_{\phi\to 0}\left(\phi^2 \ln \phi^2\right) = 0$$

the constant C_2 becomes

$$C_2 = \Xi(0) = \zeta(3)$$

where $\zeta(3)$ is the Riemann Zeta Function of argument three. Finally, the function $\Xi(\phi)$ can be evaluated in the following form:

$$\Xi(\phi) = \zeta(3) + \left(\frac{\phi}{2}\right)^2(\ln 4 - 3) + \left(\frac{\phi}{2}\right)^2 \ln\left(\frac{\phi}{2}\right)^2 - \frac{1}{18}\left(\frac{\phi}{2}\right)^4\Bigg[1 +$$

$$\frac{1}{75}\cdot\left(\frac{\phi}{2}\right)^2 + \frac{1}{2205}\left(\frac{\phi}{2}\right)^4 + \cdots + \frac{2^{2m-1}B_m}{m(2m+1)!}\cdot\frac{36}{m+1}\cdot\left(\frac{\phi}{2}\right)^{2m-2}\Bigg] \tag{A-I.7}$$

The remainders R_n, S_n are then calculated by evaluating equation A-I.7 for a particular value of ϕ and subtracting from this the sum of the first thirty-five terms. The results of this calculation are given in Table II for several values of ϕ .

The calculations indicated above have been carried out for several axial stations for both axial membrane strain and deflection. The results are presented in figures A-I.2, A-I.3, A-I.4, A-I.5, A-I.6. The remainder terms for $x \neq 0$, using equation A-I.4, are found to be negligible compared to the contribution of the leading terms.

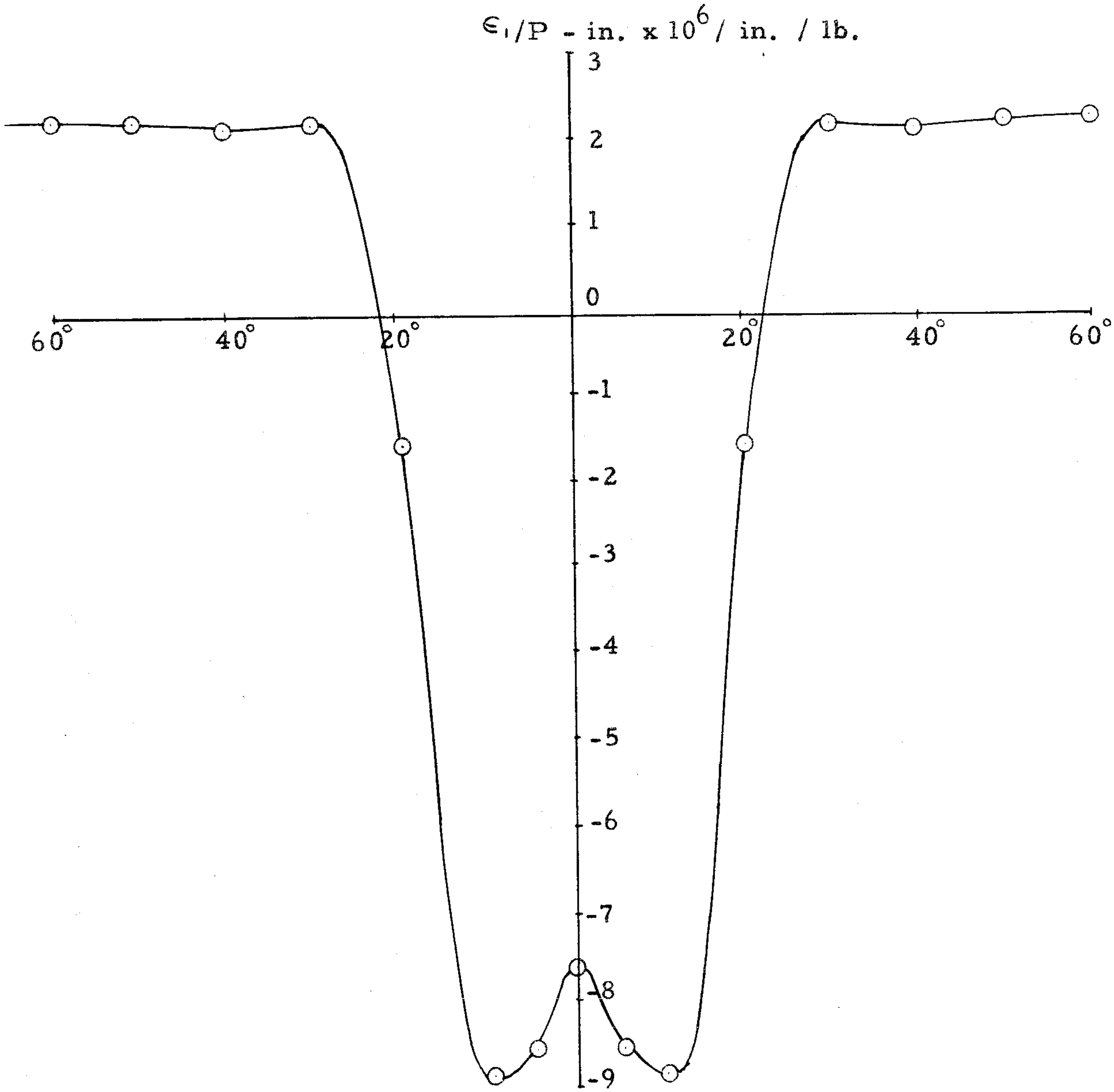

Figure A-1.2 - Axial Membrane Strain vs. Angle at Sta. 1-3/32
Simple Beam Concept

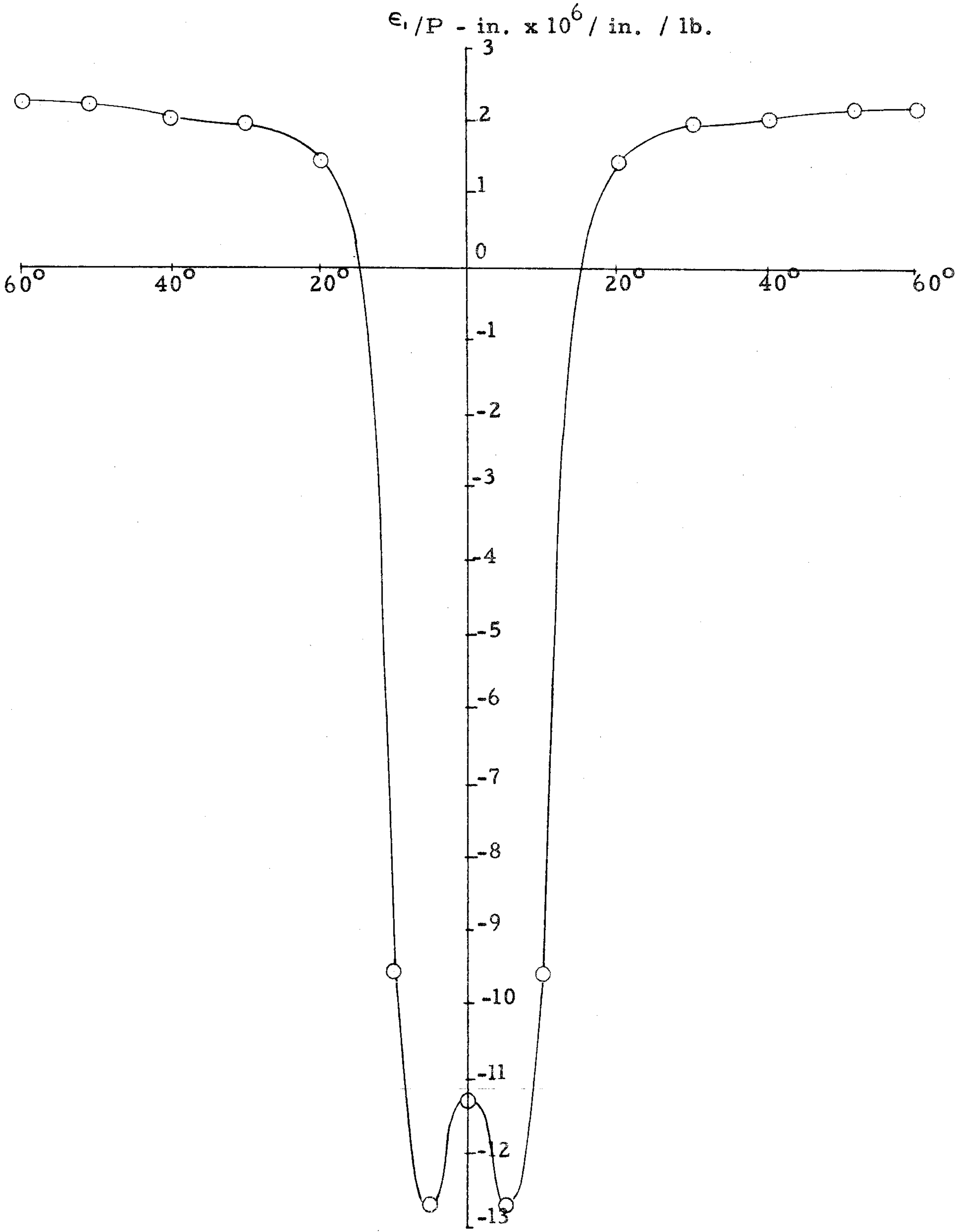

Figure A-1.3 - Axial Membrane Strain vs. Angle at Sta. 9/16
Simple Beam Concept

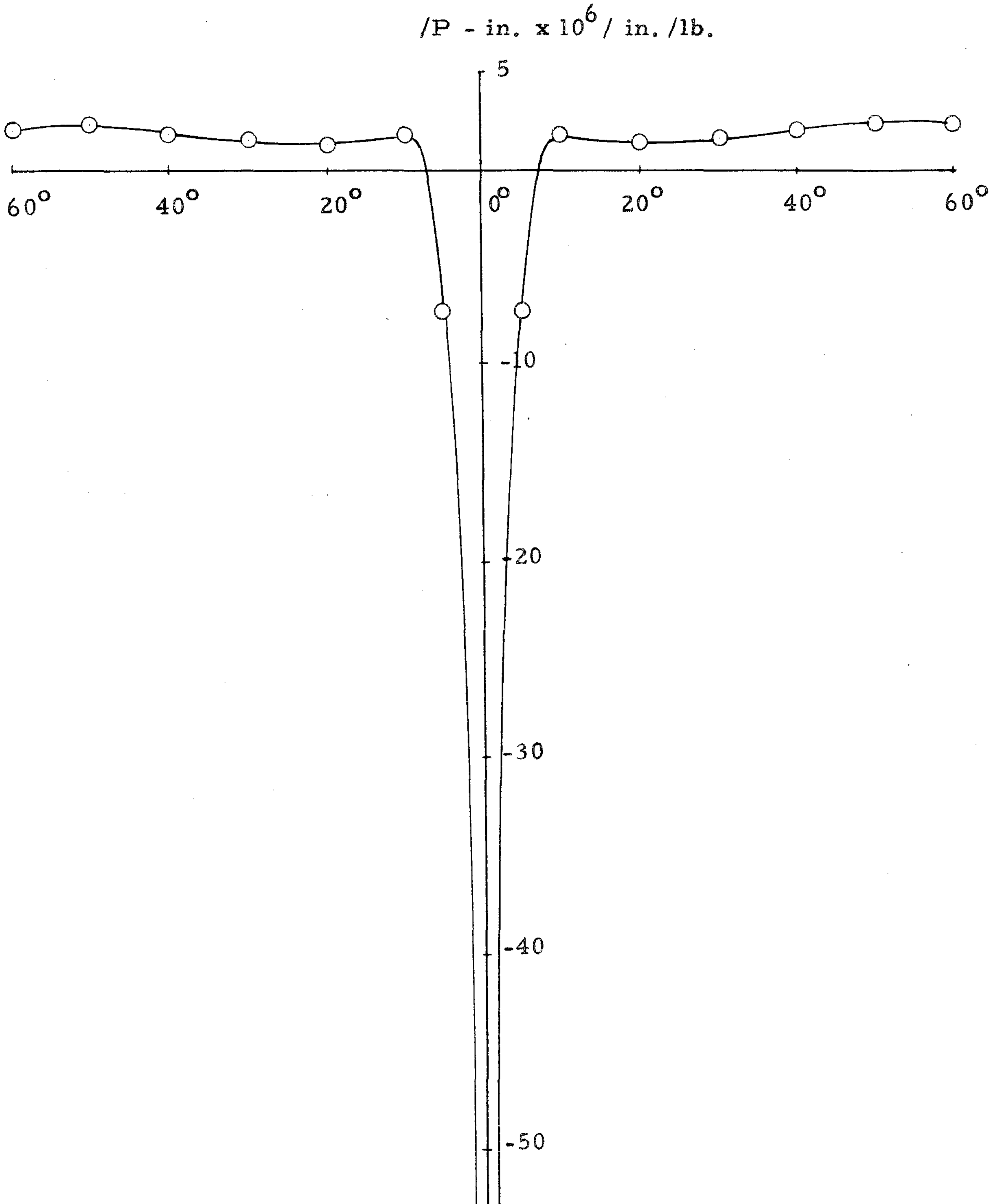

Figure A-1.4 - Axial Membrane Strain vs. Angle at Sta. 0
Simple Beam Concept

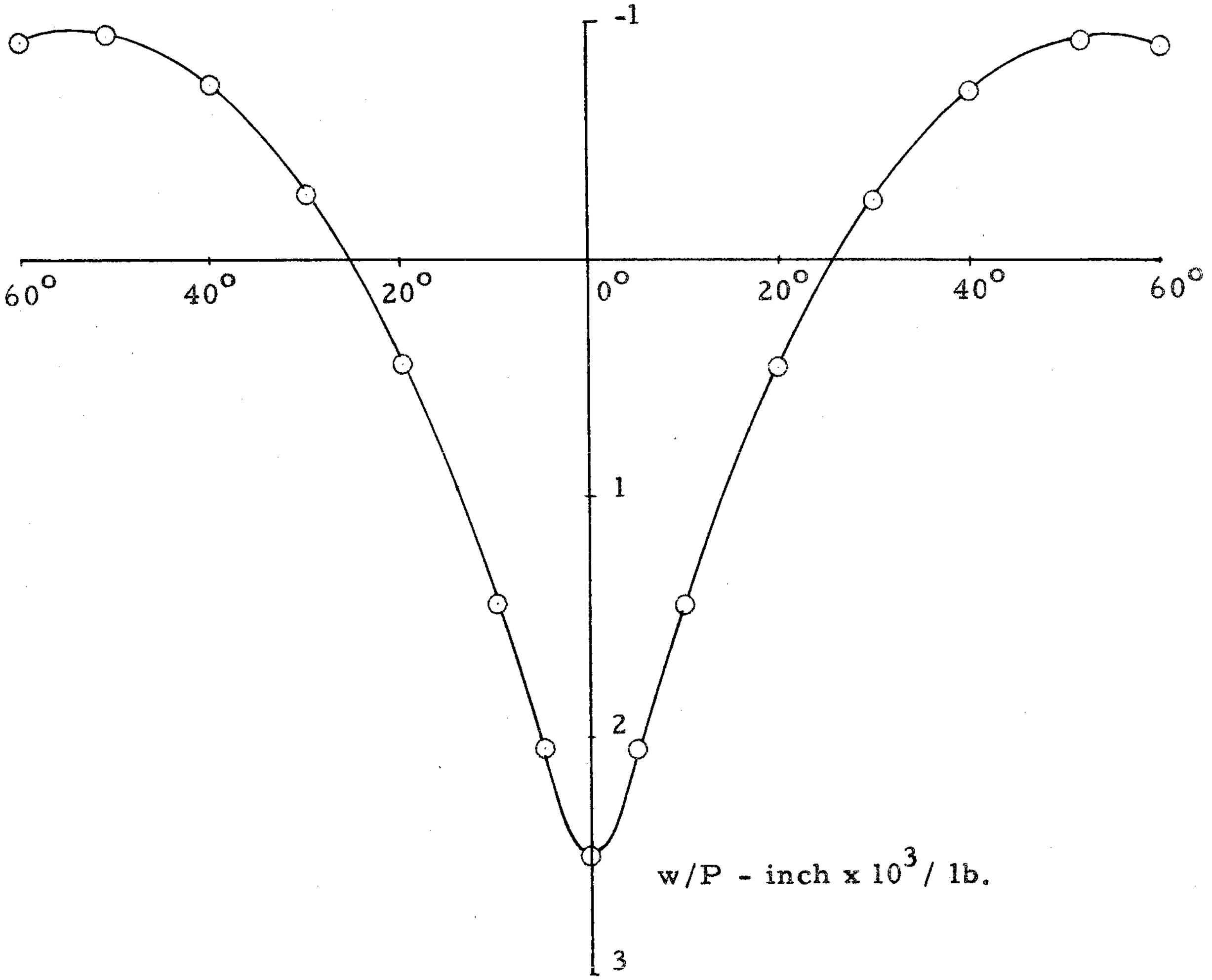

Figure A-1.5 - Radial Deflection vs. Angle at Sta. 0
Simple Beam Concept

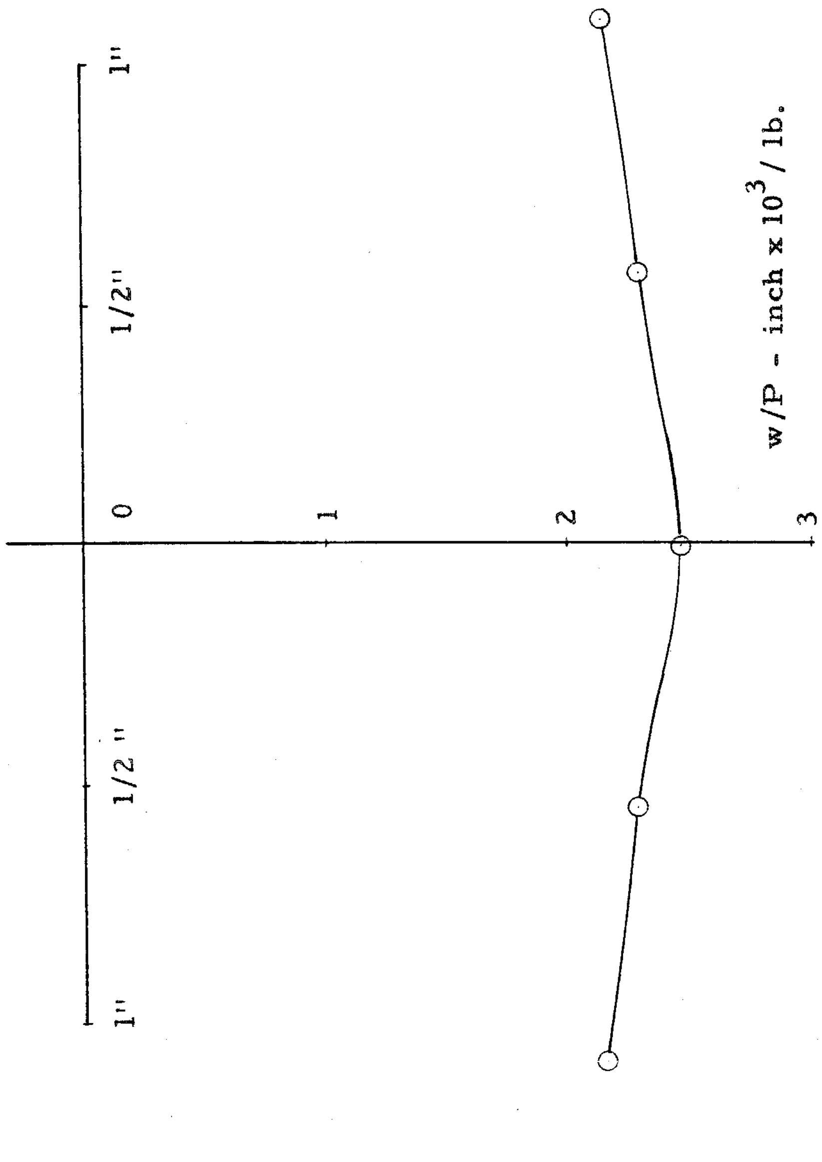

Figure A-1.6 - Radial Deflection vs. Axial Distance On a Generator Thru the Load; Simple Beam Concept

Appendix II

APPROXIMATE EQUILIBRIUM EQUATIONS

From Timoshenko (3), one obtains the following linearized differential equation for the static case:

$$u^*_{,x^*x^*} + \frac{1-\nu}{2a^3} \cdot u^*_{,\phi\phi} + \frac{1+\nu}{2a} \cdot v^*_{,x\phi} = \frac{\nu}{a} w^*_{,x^*}$$

$$\frac{1+\nu}{a} \cdot u^*_{,\phi x^*} + a \cdot \frac{1-\nu}{2} \cdot v^*_{,x^*x^*} + \frac{1}{a} v^*_{,\phi\phi} - \frac{1}{a} w^*_{,\phi}$$

$$+ \frac{h^2}{12a}\left(w^*_{,x^*x^*\phi} + \frac{1}{a^2} w^*_{,\phi\phi\phi}\right) + \frac{h^2}{12a}\left[(1-\nu) v^*_{,x^*x^*} - \frac{1}{a^2} v^*_{,\phi\phi}\right] = 0$$

If the dimensionless ratios given in equation II-1.1 and equation II-3.20 are introduced, the above equations become

$$u_{,xx} + \frac{1-\nu}{2} \cdot u_{,\phi\phi} + \frac{1+\nu}{2} \cdot v_{,x\phi} = \nu w_{,x}$$

$$v_{,\phi\phi} + \frac{1-\nu}{2} \cdot v_{,xx} + \frac{1+\nu}{2} \cdot u_{,x\phi} = w_{,\phi}$$

$$- \eta^2/12 \left(w_{,\phi xx} + w_{,\phi\phi\phi} + \overline{1-\nu}\, v_{,xx} + v_{,\phi\phi}\right) \qquad \text{(A-II.1)}$$

If $\eta^2 << 1$, the above equations can be approximated as

$$u_{,xx} + \frac{1-\nu}{2} \cdot u_{,\phi\phi} + \frac{1+\nu}{2} \cdot v_{,x\phi} = \nu w_{,x}$$

(A-II.2)

$$v_{,\phi\phi} + \frac{1-\nu}{2} \cdot v_{,xx} + \frac{1+\nu}{2} \cdot u_{,x\phi} = w_{,\phi}$$

or equivalently,

$$\left[u_{,r} + \nu\left(v_{,\phi} - w\right)\right]_{,r} + \frac{1-\nu}{2}\cdot\left(u_{,\phi} + v_{,r}\right)_{,\phi} = 0$$

$$\left[v_{,\phi} - w + \nu u_{,r}\right]_{,\phi} + \frac{1-\nu}{2}\cdot\left(u_{,\phi} + v_{,r}\right)_{,r} = 0 \qquad \text{(A-II.3)}$$

Except for a factor of proportionality, these equations are just the differential equations of equilibrium for the static two-dimensional theory of elasticity. If the variables u, v in equation A-II.2 are separated, the equations become

$$\nabla^4 u = \nu w_{,rrr} - w_{,r\phi\phi}$$

$$\nabla^4 v = (2+\nu)\, w_{,\phi rr} + w_{,\phi\phi\phi} \qquad \text{(A-II.4)}$$

It should be noted that equations A-II.4 are well suited for approximate calculations. As the dependence on w can be placed on the right hand side of the equations, w can be treated as a forcing function. If one can determine an approximate form for w by some means, it can be substituted in equation A-II.4 in order to compute u, v. Thus, instead of the methods of Chapter II-4, one might calculate u, v from equation A-II.4 using as an approximate form for w the solution given by equation II-3.21. This method should lead to more accurate strains than the method of Chapter II-4, as, in this case, one obtains the exact u, v corresponding to an approximate w, whereas in the method of Chapter II-4, one obtains only approximate u, v corresponding to an approximate w.

TABLE I

n	α	ρ	A/B	B/P x 10^6	C/B	D'/B
2	0.08802	0.08765	-0.2505	-566.5	1.106	-1.004
3	0.2163	0.2140	-1.077	-194.5	1.159	-1.011
4	0.3967	0.3890	-1.004	-100.4	1.007	-1.020
5	0.6305	0.6111	-1.001	-61.74	1.033	-1.032
6	0.9195	0.8789	-1.000	-41.76	1.046	-1.046
7	1.266	1.190	-1.000	-30.07	1.064	-1.064
8	1.670	1.540	"	-22.64	1.084	-1.084
9	2.138	1.928	"	-17.61	1.109	-1.109
10	2.680	2.357	"	-14.08	1.137	-1.137
11	3.276	2.805	"	-11.28	1.168	-1.168
12	3.937	3.274	"	-9.340	1.203	-1.203
13	4.664	3.759	"	-7.837	1.241	-1.241
14	5.459	4.254	"	-6.642	1.283	-1.283
15	6.321	4.751	"	-5.682	1.330	-1.330
16	7.247	5.244	"	-4.903	1.382	-1.382
17	8.237	5.727	"	-4.251	1.438	-1.438
18	9.288	6.193	"	-3.716	1.500	-1.500
19	10.39	6.633	"	-3.266	1.566	-1.566
20	11.55	7.046	"	-2.878	1.639	-1.639
21	12.74	7.424	"	-2.553	1.716	-1.716
22	13.98	7.765	"	-2.271	1.800	-1.800
23	15.25	8.068	"	-2.027	1.890	-1.890
24	16.54	8.331	"	-1.817	1.985	-1.985
25	17.84	8.551	"	-1.634	2.086	-2.086
26	19.16	8.733	"	-1.473	2.194	-2.194
27	20.49	8.877	"	-1.331	2.308	-2.308
28	21.81	8.985	"	-1.207	2.427	-2.427
29	23.14	9.059	"	-1.097	2.554	-2.554
30	24.45	9.103	"	-0.9998	2.686	-2.686
31	25.76	9.120	"	-0.9131	2.827	-2.827
32	27.06	9.113	"	-0.8358	2.969	-2.969
33	28.35	9.084	"	-0.7665	3.121	-3.121
34	29.62	9.039	"	-0.7043	3.277	-3.277
35	30.88	8.976	"	-0.6491	3.440	-3.440

TABLE II

ϕ	$\sum_{m=1}^{35} \frac{1}{m} \cos m\phi$	$\Xi = \sum_{m=1}^{\infty} \frac{1}{m} \cos m\phi$	Difference
0	1.201660	1.202057	.000397
5°	1.187174	1.187059	-.000116
10°	1.152567	1.152620	.000052
20°	1.046473	1.046499	.000025
30°	0.907466	0.907486	.000020
40°	0.748104	0.748119	.000015
51.428°	0.552023	0.552015	-.000008
60°	0.400672	0.400686	.000103