SCATTERING OF A RAYLEIGH WAVE

BY THE EDGE OF A THIN SURFACE LAYER

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1975

(Submitted July 3, 1974)

Acknowledgments

Abstract

This investigation treats the problem of the scattering of a Rayleigh wave by the edge of a thin layer which covers half the surface of an elastic half-space. The interaction between the layer and the half-space is described approximately by means of a model in which the effect of the layer is represented by a pair of boundary conditions at the surface of the half-space. Two parameters — one representing mass and the other, stiffness — are found to characterize the layer. The incident Rayleigh wave impinges normally upon the plated region from the unplated side.

In the case where the mass of the layer vanishes, the problem is solved exactly using Fourier transforms and the Wiener-Hopf technique, and numerical results are obtained for the amplitudes of the reflected and transmitted surface waves. In the more general case of a layer possessing both mass and stiffness, a perturbation procedure leads to a sequence of problems, each of which may be solved using Fourier transforms. The zeroth- and first-order problems are solved and the resulting approximate reflection and transmission coefficients are evaluated numerically for various ratios of layer mass to stiffness.

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Introduction

The present work is concerned with the scattering of an elastic surface wave by the edge of a thin surface layer.

The principal motivation for studying the relation between elastic surface waves and thin surface films comes from the field of microwave signal processing. Electromagnetic waves may be converted to elastic waves in solids, and vice versa, by means of piezoelectric transducers. Operations such as delaying and filtering of signals may be performed on the elastic waves by suitably designed devices [1][1]. The advantage of carrying out such operations in the elastic — rather than the electromagnetic — medium comes about primarily from the fact that the phase velocities of elastic waves are typically five orders of magnitude smaller than those of electromagnetic waves [2]. Thus, for example, the length of delay line required for a given delay is smaller, by this ratio of phase velocities, for an elastic wave than for an electromagnetic wave.

Elastic surface waves prove to be more advantageous than body waves because of their accessibility. Thus, for example, surface wave delay lines may be built with numerous taps along the way, providing incrementally varying delays.

The surface of an elastic solid acts in a sense as a waveguide, i.e., energy once transferred into surface waves tends to remain in fields concentrated near the surface. For purposes of microwave

[1]Reference [1] provides a general survey of the design and function of various surface wave signal processing devices.

signal processing it is also desirable to confine — or guide — elastic fields laterally. The basic premise in designing "surface waveguides" is that a strip of the surface on which the characteristic surface wave velocity is lower than that of the surroundings will tend to guide surface waves along its length. This is sometimes implemented by depositing a thin layer of a heavy, soft material on the surface in the form of a strip. The thin layer "loads" the substrate, creating a zone in which the surface wave velocity is lower than that in the unplated region. Conversely, on a surface fully plated with a light, stiff material except for a strip that is left unplated, the strip again acts as a guide since the surface wave velocity is higher in the coated region [2][1].

Straight surface waveguides of the type described above have been analyzed by Tiersten [3], who represents the effect of the surface layer by a boundary condition at the surface of an isotropic substrate. He derives from his model an equation for the velocity of the "straight-crested" (i. e., plane strain) surface waves characteristic of the plated region. He then appeals to a concept introduced by Knowles [4] to demonstrate the existence — in the same plated region — of a more general class of surface waves referred to as "variable-crested" surface waves. The full field in the substrate is synthesized approximately from the variable-crested surface waves characteristic of the plated and unplated regions, by applying an approximate matching condition at the boundary between the regions,

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[1]Reference [2] provides a fuller discussion of these, as well as other techniques for guiding surface waves.

and by requiring the surface waves outside the strip to decay exponentially along the surface away from the strip. This leads to an approximate dispersion relation for the guide, i.e., an equation relating the frequency and velocity of disturbances propagating along the guide. Tiersten and Davis [5] have extended this analysis to the case of a curved surface waveguide.

Freund [6] has treated a structure similar to that treated in [3] using a more exact approach. The surface of the guiding strip considered in [6] is assumed to be traction-free, while the surface outside the guide is assumed to be free of shear stress and to suffer no vertical displacement. The scattering of a single straight-crested surface wave incident on the edge of a semi-infinite guide at arbitrary angle is solved exactly. An approximation to the dispersion relation for a guide of finite width is obtained by superposing incident and reflected straight-crested surface waves and neglecting at one edge of the guide the body waves generated at the other edge. Fossum and Freund [7] have obtained and solved the dispersion equation for the guide by an independent method based on an integral equation, and thereby have validated Freund's original approach.

Apart from the problem solved by Freund in [6] as an intermediate step in his analysis, there are no known exact solutions for scattering of an elastic surface wave by the edge of a surface layer. Approximate solutions employing juxtaposition of straight-crested surface wave modes have been obtained by Li et al [8] and McGarr and Alsop [9].

In contrast, exact solutions do exist for the scattering of certain types of electromagnetic surface waves. Of particular interest is the work of Kay [10], who considers the scattering of an electromagnetic surface wave by a discontinuity in a property of the surface known as normal impedance. An exact solution is obtained by using the Wiener-Hopf technique. The problem is closely analogous to the present one, and in fact has served as a "proving ground" for some of the techniques employed here.

To complete this brief summary of related literature, note should be taken of the study of Koiter [11] of the static problem of load transfer from a stringer into a plate. The problem is one of generalized plane stress, but is essentially identical to that of a half-space in plane strain, half of whose surface is plated with a thin layer, with a static line load acting on the edge of the layer and parallel to the surface. The effect on the substrate of the stiffness of the layer is modeled in precisely the same manner as in the present analysis.

Although the present consideration of the scattering of a surface wave by a thin layer was motivated by the signal-processing applications discussed above, the results of the analysis may also be of interest in connection with seismology.

This study begins in Section 1 with a review of pure surface waves in half-spaces with the surface either fully traction-free or fully covered with a thin layer. In Section 2 the central problem is formulated — that for the scattering of a Rayleigh wave normally

incident on the edge of a semi-infinite layer. In Section 3 the problem is solved exactly for a thin massless layer, and in Section 4 it is solved by a perturbation process for a very thin layer possessing both mass and stiffness.

1. Pure Surface Waves

For the purposes of this study, a pure surface wave is a solution of the elastodynamic equations of motion in a half-space which propagates in a direction parallel to the surface and decays exponentially with depth. Such waves can exist in a half space with the surface either fully traction-free or fully covered by a layer of a different elastic material. In this section the properties of pure surface waves are reviewed, with emphasis upon a method of approximating the effect of a surface layer in the case where the layer is thin. It is necessary to understand these waves before the problem corresponding to the scattering of a surface wave can be formulated.

Consider a half-space occupied by an elastic, isotropic, homogeneous solid with Lamé parameters λ and μ and density ρ, covered by a layer of thickness $2h'$ of a different material with Lamé parameters λ' and μ' and density ρ'. A coordinate system is established as shown in Fig. 1, with a y-axis running out of the plane of the paper. The displacements in the x-, y-, and z-directions in the half-space are denoted by U, V, and W respectively, and in their fullest generality are functions of x, y, z, and time t. The case of plane strain is assumed, so that all displacements are independent of y, and $V \equiv 0$. Further, the time dependence is assumed to be harmonic with

$$\left. \begin{array}{l} U(x, z, t) = u(x, z)e^{-i\omega t} , \\[2mm] W(x, z, t) = w(x, z)e^{-i\omega t} , \quad \omega > 0 . \end{array} \right\} \quad (1.1)$$
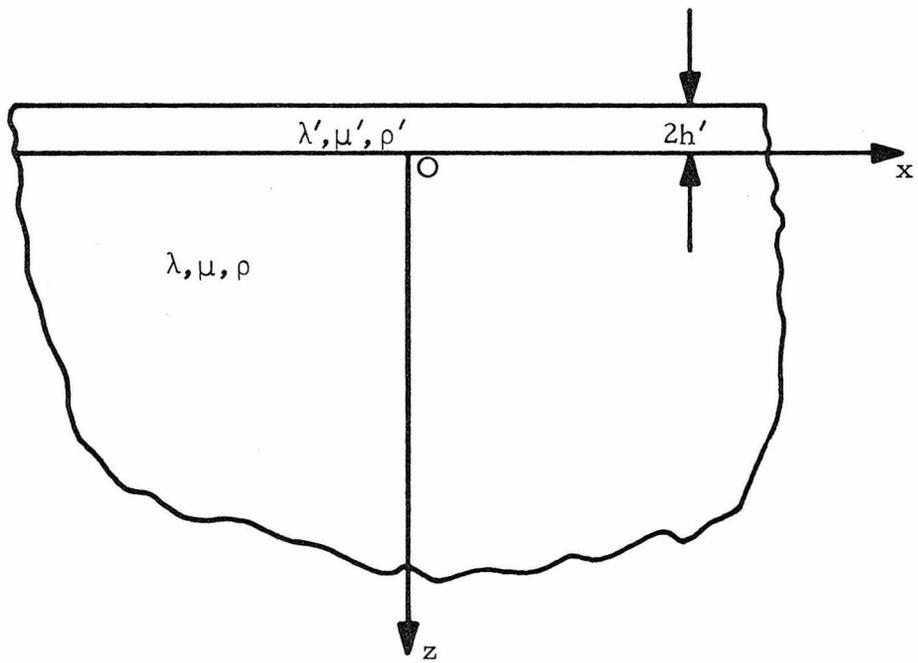
Figure 1. Coordinates for a Half-Space
Covered by a Thin Layer.

For convenience, the amplitudes u, w will henceforth be referred to as displacements. Since actual, physical displacements must be real, those corresponding to the present problem are either the real or imaginary parts of the quantities appearing in (1.1).

Under the foregoing assumptions the elastodynamic equations of motion will be satisfied if the displacements are related to the two-dimensional Lamé potentials $\Phi(x, z)$, $\Psi(x, z)$ by[1]

$$u(x, z) = \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial z} \; , \left.\begin{array}{c}\\\\\end{array}\right\}$$
$$w(x, z) = \frac{\partial \Phi}{\partial z} + \frac{\partial \Psi}{\partial x} \; ,$$

(1.2)

and if the potentials satisfy the reduced wave equations

where

$$\left.\begin{array}{c} \nabla^2 \Phi + k_d^2 \Phi = 0 \; , \\[2mm] \nabla^2 \Psi + k_s^2 \Psi = 0 \; , \quad -\infty < x < \infty, \; z \geq 0, \\[2mm] \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \; , \\[2mm] k_d = \omega / c_d, \; k_s = \omega / c_s \; , \\[2mm] c_d = \sqrt{(\lambda + 2\mu)/\rho} \; , \; c_s = \sqrt{\mu/\rho} \; . \end{array}\right\}$$

(1.3)

The potentials Φ and Ψ are referred to respectively as the dilatation and shear potentials; the speeds c_d and c_s are respectively the dilatation- and shear-wave speeds.

The stresses[2] are related to the potentials by the formulas

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[1]See [12]

[2]For convenience, the quantities appearing in (1.4) are referred to as stresses; the actual, physical stresses are either the real or imaginary parts of these quantities multiplied by $\exp\{-i\omega t\}$.

$$\sigma_{xx} = \mu[(2k_d^2 - k_s^2 + 2\frac{\partial^2}{\partial x^2})\Phi - 2\frac{\partial^2 \Psi}{\partial x \partial z}],$$

$$\sigma_{xz} = \mu[2\frac{\partial^2 \Phi}{\partial x \partial z} - (k_s^2 + 2\frac{\partial^2}{\partial z^2})\Psi],$$

$$\sigma_{zz} = \mu[(2k_d^2 - k_s^2 + 2\frac{\partial^2}{\partial z^2})\Phi + 2\frac{\partial^2 \Psi}{\partial x \partial z}].$$

(1.4)

The pure surface waves of interest here are solutions of (1.3) which decay exponentially with z and satisfy appropriate conditions at $z = 0$. Such solutions take the form

$$\Phi(x, z) = A e^{-n_d(k)z \pm ikx},$$

$$\Psi(x, z) = A\beta e^{-n_s(k)z \pm ikx},$$

where

$$n_p(k) = \sqrt{k^2 - k_p^2}, \quad k > k_p, \quad p = d, s,$$

(1.5)

and where A, β, and the wave number k are constants yet to be determined.

Now assume that the layer covering the half-space is thin enough that the displacements within the layer can be taken approximately as independent of z. By virtue of the requirement that displacements be continous across the interface between the layer and the half-space, the displacements in the layer must then coincide with those in the half-space at $z = 0$.

Consider the forces acting on a differential element of the layer of unit width in the y-direction and of length dx, as shown in Fig. 2. Upon neglecting all internal stress resultants in the layer except those due to simple stretching, the balance of momentum in the x- and z-directions requires
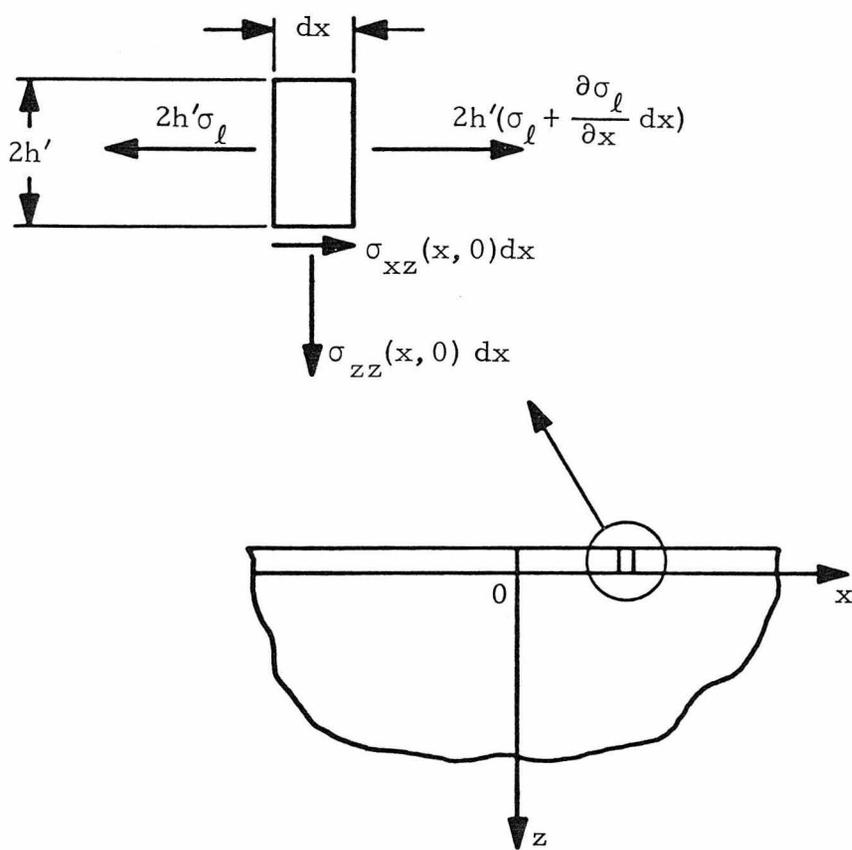
Figure 2. Differential Element of The Surface Layer .

$$2h' \frac{\partial \sigma'_\ell}{\partial x} + \sigma_{xz}(x, 0) + 2h'\rho'\dot{w}^2 u(x, 0) = 0 ,$$

$$\left. \sigma_{zz}(x, 0) + 2h'\rho'\dot{w}^2 w(x, 0) = 0 , \right\} \qquad (1.6)$$

where σ'_ℓ is the axial stress in the layer. The stresses $\sigma_{xz}(x, 0)$ and $\sigma_{zz}(x, 0)$ are those exerted by the substrate on the layer. The axial stress σ'_ℓ is related to the displacement $u(x, 0)$ by

$$\sigma'_\ell(x) = 4\mu' \left(\frac{\lambda' + \mu'}{\lambda' + 2\mu'} \right) \frac{\partial u(x, 0)}{\partial x} . \qquad (1.7)[1]$$

Substitution of (1.7) into (1.6) yields

$$\sigma_{xz}(x, 0) + \mu N \frac{\partial^2 u(x, 0)}{\partial x^2} + \mu M u(x, 0) = 0 ,$$

$$\left. \sigma_{zz}(x, 0) + \mu M w(x, 0) = 0 , \right\} \qquad (1.8)$$

where

$$N = \frac{8h'\mu'}{\mu} \left(\frac{\lambda' + \mu'}{\lambda' + 2\mu'} \right), \quad M = \frac{2h'\rho'\dot{w}^2}{\mu} .$$

Note that N and M are measures of the layer's stiffness and mass, respectively.

Equations (1.8) may be thought of as boundary conditions to be satisfied by the fields in the substrate. Note that in the absence of a covering layer, $M = N = 0$ and (1.8) reduce to the boundary conditions appropriate to a traction-free surface. Tiersten [3] deduced (1.8) from a more general development in which Mindlin's plate equations [13] were taken as a starting point[2]. He argued that for thin plates

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[1]This relationship is derived from the three-dimensional stress-strain law under the assumption of plane strain in the y-direction and vanishing of normal stress in the z-direction.

[2]Achenbach and Keshava [14] used a similar approach which included the effects of axial stretching, bending, shear, and rotatory inertia in the layer.

all terms except those involving axial stretching and inertial reaction are negligible.

Substitution of the potentials (1.5) into (1.2) and (1.4), and thence into the boundary conditions (1.8), yields a pair of homogeneous, linear, algebraic equations in A and Aβ. The condition for the existence of a non-trivial solution is that the determinant of the coefficient matrix vanish. This is

$$R(k) - Nk^2 k_s^2 n_s(k) + M\{k_s^2 [n_s(k) + n_d(k)]$$

$$+ Nk^2 [k^2 - n_s(k) n_d(k)]\} + M^2 \{n_s(k) n_d(k) - k^2\} = 0 \qquad (1.9)$$

where

$$R(k) = (2k^2 - k_s^2)^2 - 4k^2 n_s(k) n_d(k) . \qquad (1.10)$$

Each value of k satisfying (1.9) determines an elastic surface wave through (1.5), provided $k > k_s$ (note that $k_s > k_d$, provided $\lambda > 0$ and $\mu > 0$ — see (1.3)) and provided β has the now common value required by either of the boundary conditions (1.8). If the time dependence is restored in the potentials (1.5), their x- and t-dependence, and hence that of all field quantities, is of the form

$$e^{\pm ikx - i\omega t} = e^{\pm ik(x \mp ct)} \qquad (1.11)$$

where $c = \omega / k$. Thus the surface wave (1.5), while decaying exponentially with z, propagates unchanged in form with speed c in the positive or negative x-direction according to whether the positive or negative sign is used in (1.11).

If the surface layer thickness 2h′ vanishes, M=N=0, the surface becomes traction-free, and (1.9) reduces to

$$R(k) = 0 \ . \tag{1.12}$$

Equation (1.12) determines the propagation speed of Rayleigh waves, an extensive discussion of which may be found on pp. 307 - 309 of Love's text [15]. Achenbach, on p. 190 of [16], proves that there is only one root of (1.12) greater than k_s. This root is denoted by k_R and referred to as the Rayleigh root or the Rayleigh wavenumber. If (1.12) is multiplied by ω^{-4} the resulting equation for $c_R = \omega/k_R$, and hence c_R itself, is independent of ω. The Rayleigh wave is thus said to be non-dispersive, i.e., its propagation speed c_R is independent of frequency ω.

In the more general case Eq. (1.9) may have one or two roots [3]. The smaller one, which will be denoted k_T and referred to as the extended Rayleigh root, reduces continuously to k_R as M and N go to zero, while the other root goes to infinity in this limit. If $N \neq 0$ or $M \neq 0$, the speed corresponding to the extended Rayleigh root is frequency-dependent, so the presence of the layer results in surface waves which are dispersive.

A special case of interest is that of a massless layer, for which, by (1.8), $M = 0$, and (1.9) reduces to

$$R^{*}(k) = R(k) - Nk^{2}k_{s}^{2}n_{s}(k) = 0 \ . \tag{1.13}$$

It is shown in Appendix A that this equation also has but one admissible root. This root is less than k_R, reduces continuously to k_R as $N \to 0$, and corresponds to a dispersive surface wave which propagates faster than a Rayleigh wave but slower than a shear wave. This is

in keeping with the intuitive expectation that a stiffening layer would increase the characteristic surface wave speed.

It must be noted that if the sole objective of the present paper were a study of pure surface waves, it would not be necessary to make any simplifying, approximative assumptions regarding the displacements in the layer. Rather, the layer could be treated as a second elastic continuum, and exact solutions to the field equations in both the layer and the substrate could be obtained, subject to the appropriate continuity conditions at the interface, boundary conditions at the traction-free surface, and decay conditions in the substrate. Auld summarizes this topic on pp. 97 - 103 of [17]; Ewing, Jardetzky, and Press discuss it on pp. 189 - 223 of [18]. Tiersten [3] compares the solutions of the exact dispersion equation with those of (1.9), concluding that the results agree extremely well for layers less than about one sixth of a wavelength thick, and reasonably well up to about half a wavelength. The reason the exact analysis is not used here is that it does not allow the effect of the layer to be fully specified by a simple pair of boundary conditions such as (1.8). The treatment of the effects of the layer through simple boundary conditions is essential to the tractability of the scattering problem which is the main objective of this work.

2. Formulation of the Problem

An elastic, isotropic, homogeneous solid occupies the half-space $-\infty < x, y < \infty$, $z \geq 0$. A thin layer covers and is bonded to the surface $z = 0$, $-\infty < y < \infty$, $x > 0$, as shown in Fig. 3. Notations for coordinates, material properties, and field quantities are exactly as in the previous section. The assumptions of plane strain and harmonic time dependence of the form $e^{-i\omega t}$ are maintained, so that the problem may be formulated in terms of the time-independent Lamé potentials $\Phi(x, z)$, $\Psi(x, z)$. The reduced wave equations governing these potentials are (1.3), the displacement-potential relations are (1.2), and the stress-potential relations are (1.4). The boundary conditions for the unplated surface are

$$\left.\begin{aligned}
\sigma_{zz}(x, 0) &= 0 , \\
\sigma_{xz}(x, 0) &= 0 , \quad -\infty < x < 0 ,
\end{aligned}\right\} \tag{2.1}$$

and for the plated surface, in accordance with the assumptions and results of the previous section,

$$\left.\begin{aligned}
\sigma_{zz}(x, 0) + \mu M w(x, 0) &= 0 , \\
\sigma_{xz}(x, 0) + \mu N \frac{\partial^2 u(x, 0)}{\partial x^2} + \mu M u(x, 0) &= 0 , \quad 0 < x < \infty .
\end{aligned}\right\} \tag{2.2}$$

The potentials Φ, Ψ are required to be twice-continuously differentiable in the interior of the half-space. The quantities appearing in the boundary conditions (2.1), (2.2) are required to be continuous up to the respectively appropriate segment of the boundary, excepting the origin $x = z = 0$. Certain further conditions, which will be explicitly specified in the next section, are required to be satisfied by
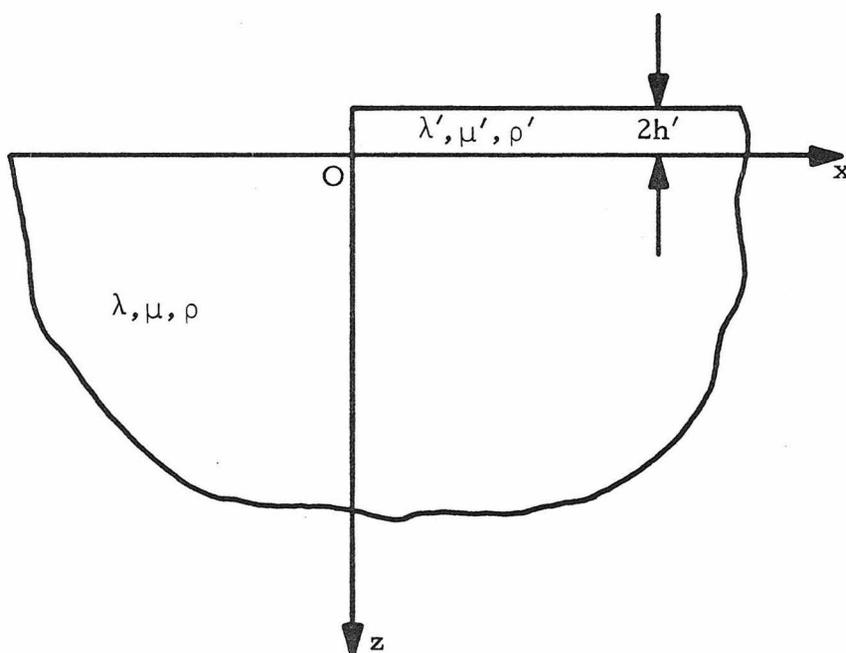
Figure 3. Coordinates for a Half-Space Covered
by a Semi-Infinite Thin Layer.

various field quantities as the origin is approached, these conditions having the effect of ruling out point loads and higher-order singularities at the origin.

The potentials Φ, Ψ are required to be of the form

$$\left.\begin{aligned}
\Phi(x,z) &= A_0 \varphi_{R+}(x,z)H(-x) + A_r \varphi_{R-}(x,z)H(-x) \\
&\quad + A_t \varphi_{T+}(x,z)H(x) + \varphi(x,z) , \\
\Psi(x,z) &= A_0 \psi_{R+}(x,z)H(-x) + A_r \psi_{R-}(x,z)H(-x) \\
&\quad + A_t \psi_{T+}(x,z)H(x) + \psi(x,z) .
\end{aligned}\right\} \quad (2.3)$$

In Eq. (2.3), $\varphi_{R\pm}$, $\psi_{R\pm}$, φ_{T+}, and ψ_{T+} are normalized potentials representing positive- and negative-traveling Rayleigh waves and the positive-traveling extended Rayleigh wave corresponding to the layer, and are given by

$$\left.\begin{aligned}
\varphi_{R\pm} &= e^{-n_d(k_R)z \pm 2k_R x}, \quad \psi_{R\pm} = \pm\beta_R e^{-n_s(k_R)z \pm ik_R x}, \\
\varphi_{T+} &= e^{-n_d(k_T)z + ik_T x}, \quad \psi_{T+} = \beta_T e^{-n_s(k_T)z + ik_T x}, \\
\end{aligned}\right.$$

where

$$\left.\begin{aligned}
\beta_R &= \frac{2k_R^2 - k_s^2}{2ik_R n_s(k_R)} , \\
\beta_T &= \frac{2k_T^2 - k_s^2 - \mu M n_d(k_T)}{2ik_T n_s(k_T) - i\mu M k_T} .
\end{aligned}\right\} \quad (2.4)$$

The constant A_0 is given, the constants A_r and A_t are unknown, $H(x)$ is the unit step function, and φ and ψ are required to satisfy the radiation conditions

$$\left.\begin{aligned}
\varphi(x,z) &\sim F(\theta)\frac{e^{ik_d r}}{\sqrt{r}} , \\
\psi(x,z) &\sim G(\theta)\frac{e^{ik_s r}}{\sqrt{r}} ,
\end{aligned}\right\} \quad (2.5)$$

as $r \to \infty$, uniformly in θ for $0 \le \theta \le \pi$. Here $r = \sqrt{x^2 + z^2}$, $\theta = \mathrm{Tan}^{-1}(z/x)$, $F(\theta)$ and $G(\theta)$ are unknown functions of θ, and the symbol "$\sim$" denotes asymptotic equality in the usual sense.

The motivation behind (2.4) will be discussed presently, but first a matrix notation will be introduced which not only simplifies some of the subsequent manipulations, but consolidates the fundamental concepts of the analysis. Twice-underlined letters will denote 2×2 matrices of partial differential operators, and underlined letters will denote column vectors with two components, the first of which is a dilatational potential, and the second, the corresponding shear potential. Thus the reduced wave equations (1.3) become

$$\underline{\underline{L}}\, \underline{\Phi} = \underline{0}$$

where

$$\underline{\underline{L}} = \begin{pmatrix} \nabla^2 + k_d^2 & 0 \\ 0 & \nabla^2 + k_s^2 \end{pmatrix}, \quad \underline{\Phi} = \begin{Bmatrix} \Phi(x,z) \\ \Psi(x,z) \end{Bmatrix}. \left.\right\} \quad (2.6)$$

The boundary conditions (2.1) and (2.2) become

$$\underline{\underline{R}}\, \underline{\Phi} = \underline{0}, \quad z = 0, \quad -\infty < x < 0,$$

$$(\underline{\underline{R}} + N\underline{\underline{S}} + M\underline{\underline{T}})\underline{\Phi} = 0, \quad z = 0, \quad 0 < x < \infty,$$

where

$$\underline{\underline{R}} = \begin{pmatrix} 2k_d^2 - k_s^2 + 2\dfrac{\partial^2}{\partial z^2} & 2\dfrac{\partial^2}{\partial x \partial z} \\ 2\dfrac{\partial^2}{\partial x \partial z} & -k_s^2 - 2\dfrac{\partial^2}{\partial z^2} \end{pmatrix},$$

$$\underline{\underline{S}} = \begin{pmatrix} 0 & 0 \\ \dfrac{\partial^3}{\partial x^3} & -\dfrac{\partial^3}{\partial x^2 \partial z} \end{pmatrix}, \quad \underline{\underline{T}} = \begin{pmatrix} \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \end{pmatrix}. \left.\right\} \quad (2.7)$$

The required form of the solution (2.3) becomes

$$\underline{\Phi} = A_0 \underline{\varphi}_{R+} H(-x) + A_r \underline{\varphi}_{R-} H(-x) + A_t \underline{\varphi}_{T+} H(x) + \underline{\varphi}$$

where

$$\underline{\varphi}_{R\pm} = \begin{Bmatrix} e^{-n_d(k_R)z} \\ \pm\beta_R e^{-n_s(k_R)z} \end{Bmatrix} e^{\pm ik_R x} ,$$

$$\underline{\varphi}_{T+} = \begin{Bmatrix} e^{-n_d(k_T)z} \\ \beta_T e^{-n_s(k_T)z} \end{Bmatrix} e^{ik_T x} , \quad \underline{\varphi} = \begin{Bmatrix} \varphi(x, z) \\ \psi(x, z) \end{Bmatrix} ,$$

$$(2.8)$$

and where $\varphi(x, z)$ and $\psi(x, z)$ must satisfy the radiation conditions (2.5).

Now it can be seen that in the assumed form (2.8) each of the first three terms represents a surface wave propagating in either the positive or negative x-direction and existing only for either positive or negative x. The first term is the incident wave and has amplitude A_0. The remaining terms taken together constitute the scattered wave and propagate outward from the origin. These terms are referred to respectively as the reflected wave, the transmitted wave, and the radiated wave; the ratios A_r/A_0 and A_t/A_0 are referred to respectively as the reflection and transmission coefficients.

Note that the radiated wave must contain discontinuities across x = 0 which cancel those due to the surface wave terms of (2.7), so as to render the full field $\underline{\Phi}$ twice-continuously differentiable. Morgan and Karp [19] have solved an electromagnetic problem involving surface waves by using a similar representation, in which the full field

is separated into various parts, some representing surface waves,
one representing a radiated wave, and all possessing discontinuities.

3. Solution for a Massless Layer

It is now assumed that the layer's density ρ' vanishes. According to (1.8), this causes M to vanish. Under this condition the governing differential equations (2.6) remain the same, but the boundary conditions (2.7) reduce to

$$\left.\begin{array}{l} \underline{\underline{R}}\,\underline{\Phi} = \underline{0}\,, \quad z = 0\,, \quad -\infty < x < 0\,, \\[2mm] (\underline{\underline{R}} + N\underline{\underline{S}})\underline{\Phi} = \underline{0}\,, \quad z = 0\,, \quad 0 < x < \infty\,. \end{array}\right\} \tag{3.1}$$

The assumed form (2.8) and the radiation conditions (2.5) remain, but the wave number k_T appearing in (2.8) is now a root of the simplified form (1.13) of the more general dispersion relation (1.9).

The order conditions at the origin are taken as

$$\left.\begin{array}{c} \sigma_{xz}(x,0) = O(1)\,, \\[4mm] \dfrac{\partial \sigma_{xz}(x,0)}{\partial x} = O(x^{-p_1}) \text{ as } x \to 0^+\,, \quad p_1 < 1\,, \\[4mm] \sigma_{xz}(x,0) + \mu N \dfrac{\partial^2 u(x,0)}{\partial x^2} = O(|x|^{-p_2}) \text{ as } x \to 0^-\,, \quad p_2 < 1\,. \end{array}\right\} \tag{3.2}$$

The solution to the problem specified by (2.6), (3.1), (2.8), (2.5) and (3.2) will be obtained in closed form in this section.

It is convenient to define a new function $\hat{\underline{\varphi}}(x,z)$ by

$$\underline{\Phi}(x,z) = A_0 \underline{\varphi}_{R+}(x,z) + \hat{\underline{\varphi}}(x,z)\,. \tag{3.3}$$

Recall that A_0 is the amplitude of the incident Rayleigh wave. Thus for $x < 0$, $\hat{\underline{\varphi}}$ comprises the scattered wave, while for $x > 0$ it consists of the scattered wave minus a positive-traveling Rayleigh wave of amplitude A_0. In either case, $\hat{\underline{\varphi}}$ consists only of outwardly-propagating waves.

The differential equations governing $\hat{\underline{\varphi}}$ are obtained by substituting (3.3) into (2.6), and by noting that, according to (1.3) and (2.6), $\underline{\underline{L}} \, \underline{\varphi}_{R+} = \underline{0}$. This yields

$$\underline{\underline{L}} \, \hat{\underline{\varphi}} = \underline{0} \, , \quad -\infty < x < \infty \, , \quad z \geq 0 \, . \tag{3.4}$$

The boundary conditions for $\hat{\underline{\varphi}}$ are obtained by substituting (3.3) into (3.1), and by noting that, according to (1.8), (1.2), (1.4) and (2.7), $\underline{\underline{R}} \, \underline{\varphi}_{R+} = \underline{0}$. This yields

$$\left. \begin{array}{l} \underline{\underline{R}} \, \hat{\underline{\varphi}} = \underline{0} \, , \quad z = 0 \, , \quad -\infty < x < \infty \, , \\[2mm] (\underline{\underline{R}} + N\underline{\underline{S}})\hat{\underline{\varphi}} = \dfrac{A_0 Nik_s^2 k_R}{2} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} e^{ik_R x} \, , \quad z = 0 \, , \quad 0 < x < \infty \, , \end{array} \right\} \tag{3.5}$$

where use has been made of the definition of $\underline{\underline{S}}$ in (2.7) and that of β_R in (2.4).

This problem will be solved with the aid of a Fourier transform on x. Let f(x) be a complex-valued, absolutely integrable function of the real variable x which satisfies

$$|f(x)| < K e^{-\delta |x|} \, , \quad |x| > X_0 \, , \tag{3.6}$$

for some positive constants K, δ and X_0. The exponential Fourier transform of f(x) is defined by

$$\tilde{f}(\varkappa) = \int_{-\infty}^{\infty} f(x) e^{-i\varkappa x} dx \, , \quad -\delta < \text{Im}\{\varkappa\} < \delta \, , \tag{3.7}$$

and the inversion formula is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\varkappa) e^{i\varkappa x} d\varkappa \, , \quad -\infty < x < \infty \, . \tag{3.8}$$

When considered as a function of the complex variable $\varkappa$, the

transform $\tilde{f}(\varkappa)$ is analytic in the domain $-\delta<\text{Im}\{\varkappa\}<\delta$ (see [20], p. 338).

It is clear from (2.8) that for fixed z the potentials Φ, Ψ will not satisfy the decay condition (3.6) for any $\delta>0$, as long as the wave numbers k_R, k_T, k_s and k_d are real. However, application of the Wiener-Hopf technique (which will be explained presently) requires that the transforms be analytic in strips of the complex $\varkappa$-plane such as that defined in (3.7). The difficulty is resolved by temporarily assuming that the frequency ω has a small, positive, imaginary part, while all propagation speeds remain real. This is a standard artifice; it is discussed, for example, on p. 28 of [21]. After the solution is obtained, the imaginary part of ω will be taken to vanish.

Now since $\hat{\varphi}$ is required to contain only outwardly propagating waves, the assumption that $\text{Im}\{\omega\}>0$ (and consequently that $\text{Im}\{k_\alpha\}>0$, $\alpha=d,s,T,R$) renders $\hat{\varphi}$ exponentially decaying in $|x|$ as required by (3.6), with decay constant δ satisfying $0<\delta<\text{Im}\{k_d\}$ (cf. (3.3), (2.8), (2.5)).

Formal application of the transform (3.7) to the differential equations (2.6) yields

$$
\left.
\begin{aligned}
&\frac{\partial^2\tilde{\hat{\varphi}}(\varkappa,z)}{\partial z^2} - n_d^2(\varkappa)\tilde{\hat{\varphi}}(\varkappa,z) = 0, \\[2em]
&\frac{\partial^2\tilde{\hat{\psi}}(\varkappa,z)}{\partial z^2} - n_s^2(\varkappa)\tilde{\hat{\psi}}(\varkappa,z) = 0, \quad -\delta<\text{Im}\{\varkappa\}<\delta, \\[2em]
&\text{where} \quad n_d = \sqrt{\varkappa^2 - k_d^2}, \quad n_s = \sqrt{\varkappa^2 - k_s^2}.
\end{aligned}
\right\} \quad (3.10)
$$

The functions $n_s(\varkappa)$ and $n_d(\varkappa)$ are rendered single-valued in the entire $\varkappa$-plane and analytic in the strip $-\delta < \mathrm{Im}\{\varkappa\} < \delta$ by taking branch cuts as shown in Fig. 4, and by requiring $n_s(\varkappa)$ and $n_d(\varkappa)$ to approach $+\infty$ as $\varkappa \to \infty$ along the positive real $\varkappa$-axis. The solutions of (3.10) are then

$$\left. \begin{aligned} \widetilde{\widetilde{\varphi}}(\varkappa, z) &= A(\varkappa)e^{-n_d(\varkappa)z} + A'(\varkappa)e^{n_d(\varkappa)z} , \\[2mm] \widetilde{\widetilde{\psi}}(\varkappa, z) &= B(\varkappa)e^{-n_s(\varkappa)z} + B'(\varkappa)e^{n_s(\varkappa)z} , \quad -\delta < \mathrm{Im}\{\varkappa\} < \delta , \end{aligned} \right\} \tag{3.11}$$

where $A(\varkappa)$, $A'(\varkappa)$, $B(\varkappa)$, and $B'(\varkappa)$ are unknown functions of $\varkappa$. However, the coefficients $A'(\varkappa)$ and $B'(\varkappa)$ must vanish; otherwise the transforms would grow exponentially with $|\varkappa|$ for $z > 0$, and the inversion integrals (3.8) would not converge. This leaves

$$\left. \begin{aligned} \widetilde{\widetilde{\varphi}}(\varkappa, z) &= A(\varkappa)e^{-n_d(\varkappa)z} , \\[2mm] \widetilde{\widetilde{\psi}}(\varkappa, z) &= B(\varkappa)e^{-n_s(\varkappa)z} , \quad -\delta < \mathrm{Im}\{\varkappa\} < \delta . \end{aligned} \right\} \tag{3.12}$$

The boundary condition (3.5), when written in component form, becomes

$$\left(2k_d^2 - k_s^2 + 2\frac{\partial^2}{\partial z^2}\right)\hat{\varphi} + 2\frac{\partial^2 \hat{\psi}}{\partial x \partial z} = 0 , \quad z = 0 , \quad -\infty < x < \infty , \tag{3.13a}$$

$$2\frac{\partial^2 \hat{\varphi}}{\partial x \partial z} - \left(k_s^2 + 2\frac{\partial^2}{\partial z^2}\right)\hat{\psi} = 0 , \quad z = 0 , \quad -\infty < x < 0 , \tag{3.13b}$$

$$\left(2\frac{\partial^2}{\partial x \partial z} + N\frac{\partial^3}{\partial x^3}\right)\hat{\varphi} - \left(k_s^2 + 2\frac{\partial^2}{\partial z^2} + N\frac{\partial^3}{\partial x^2 \partial z}\right)\hat{\psi}$$

$$= \frac{1}{2}A_0 N i k_s^2 k_R e^{ik_R x} , \quad z = 0 , \quad 0 < x < \infty . \tag{3.13c}$$

Note that (3.13a), which refers to the normal stress at the surface, holds <u>for</u> <u>all</u> <u>values</u> <u>of</u> x <u>at</u> z = 0. This circumstance arises from the
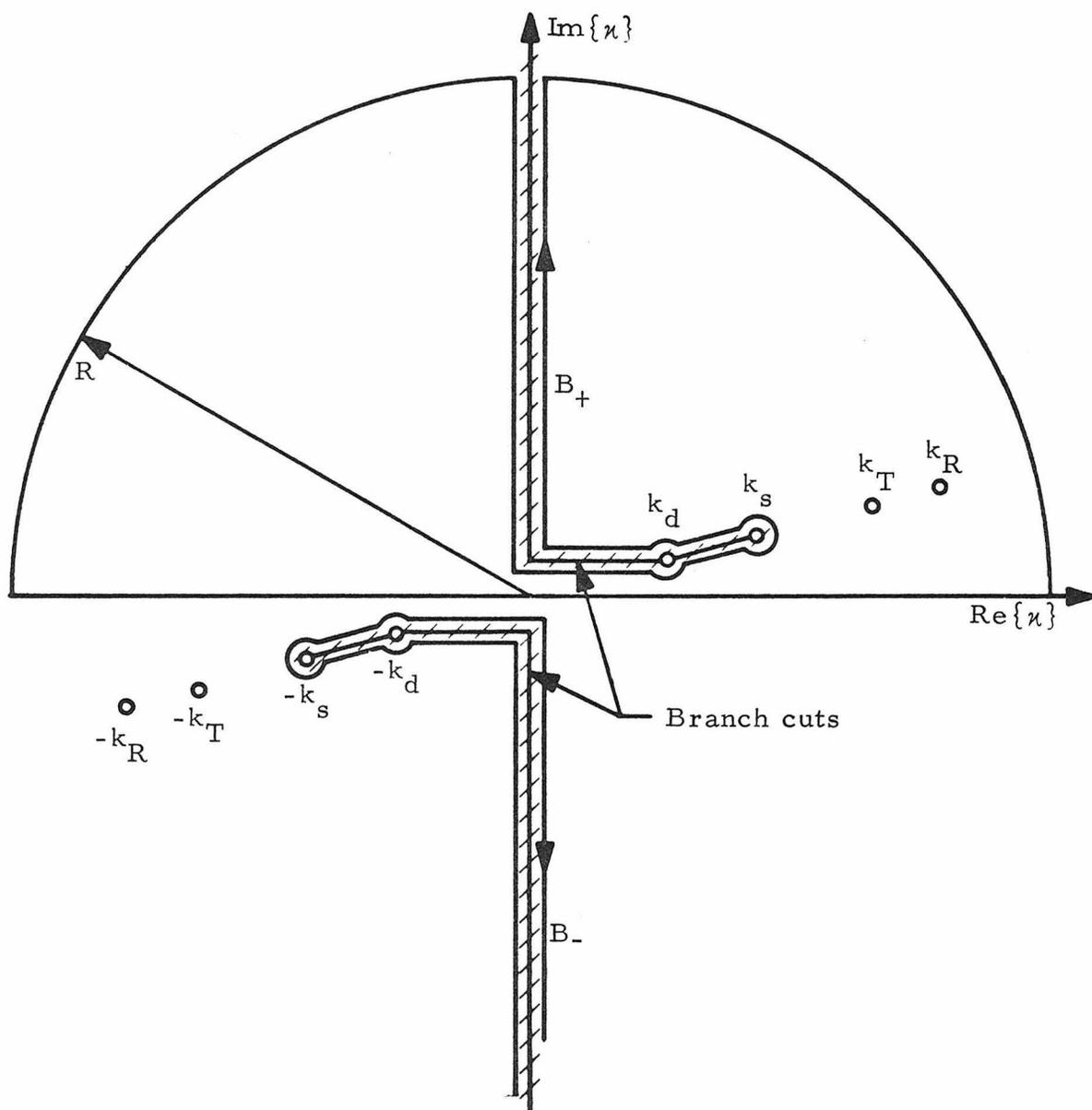
Figure 4. Complex $\varkappa$-Plane.

fact that the mass of the layer is neglected at present, so that $M = 0$ in (2.2). In contrast, the conditions (3.13b, c), which relate to the normal tractions at $z = 0$, take different forms according as $x < 0$ or $x > 0$. The fact that only <u>one</u> of the boundary conditions changes form at $x = 0$ makes possible the application of the Wiener-Hopf procedure in its simplest form.

It is now convenient to define functions $f(x)$ and $g(x)$ on $(0, \infty)$ and $(-\infty, 0)$, respectively, by setting

$$\left. \begin{array}{l} f(x) = 2 \dfrac{\partial^2 \hat{\varphi}}{\partial x \partial z} - \left(k_s^2 + 2 \dfrac{\partial^2}{\partial z^2} \right) \hat{\psi} , \quad z = 0 , \quad 0 < x < \infty , \\[4mm] g(x) = \left(2 \dfrac{\partial^2}{\partial x \partial z} + N \dfrac{\partial^3}{\partial x^3} \right) \hat{\varphi} - \left(k_s^2 + 2 \dfrac{\partial^2}{\partial z^2} + N \dfrac{\partial^3}{\partial x^2 \partial z} \right) \hat{\psi} , \quad z = 0 , \quad -\infty < x < 0 . \end{array} \right\} \quad (3.14)$$

Apart from a factor μ, $f(x)$ and $g(x)$ are the unknown values of σ_{xz} and $\sigma_{xz} + \mu N \partial^2 u / \partial x^2$ under the layer $z = 0$, $0 < x < \infty$, and on the free surface $-\infty < x < 0$, $z = 0$, respectively.

Formal application of the transform (3.7) to (3.13) and (3.14), followed by use of (3.12), yields

$$\left. \begin{array}{l} (2\varkappa^2 - k_s^2) A(\varkappa) - 2i\varkappa n_s(\varkappa) B(\varkappa) = 0 , \\[3mm] -2i\varkappa n_s(\varkappa) A(\varkappa) - (2\varkappa^2 - k_s^2) B(\varkappa) = F^-(\varkappa) , \\[3mm] [-2i\varkappa n_s(\varkappa) - iN\varkappa^3] A(\varkappa) - [2\varkappa^2 - k_s^2 + N\varkappa^2 n_s(\varkappa)] B(\varkappa) \\[3mm] \qquad = \dfrac{A_0 N k_s^2 k_R}{2(\varkappa - k_R)} + G^+(\varkappa) , \quad -\delta < \mathrm{Im}\{\varkappa\} < \delta , \end{array} \right\} \quad (3.15)$$

where

$$F^-(\varkappa) = \int_0^\infty f(x)e^{-i\varkappa x}dx \, , \left.\begin{matrix}\\\\\\\\\\\\\end{matrix}\right\}$$

$$G^+(\varkappa) = \int_{-\infty}^0 g(x)e^{-i\varkappa x}dx \, .$$ (3.16)

The functions f(x) and g(x) as defined by (3.14) are exponentially decaying for large positive and negative x, repsectively, with decay constant δ. Moreover, since $F^-(\varkappa)$ and $G^+(\varkappa)$ are half-range transforms of f(x) and g(x), $F^-(\varkappa)$ and $G^+(\varkappa)$ are analytic functions of $\varkappa$ on the half-planes $\text{Im}\{\varkappa\} < \delta$ and $\text{Im}\{\varkappa\} > -\delta$, respectively (see p. 12 of [21]). The superscripts on $F^-(\varkappa)$ and $G^+(\varkappa)$ are thus seen to be suggestive of the half-planes on which the functions are respectively analytic.

After A($\varkappa$) and B($\varkappa$) are eliminated from (3.15) there remains the single equation

$$F^-(\varkappa) = \frac{R(\varkappa)}{R^*(\varkappa)}\left[G^+(\varkappa) + \frac{A_0 N k_s^2 k_R}{2(\varkappa - k_R)}\right], \quad -\delta < \text{Im}\{\varkappa\} < \delta \, ,$$ (3.17a)

where

$$R(\varkappa) = (2\varkappa^2 - k_s^2)^2 - 4\varkappa^2 n_s(\varkappa)n_d(\varkappa) \, , \left.\begin{matrix}\\\\\\\\\end{matrix}\right\}$$

$$R^*(\varkappa) = R(\varkappa) - N k_s^2 \varkappa^2 n_s(\varkappa) \, .$$ (3.17b)

It is important to note that the domain of validity of (3.17a) is the strip $-\delta < \text{Im}\{\varkappa\} < \delta$.

Now define D($\varkappa$) by

$$D(\varkappa) = \frac{2(k_s^2 - k_d^2)R^*(\varkappa)}{N k_s^2 n_s(\varkappa)R(\varkappa)} \, , \quad -\delta < \text{Im}\{\varkappa\} < \delta \, .$$ (3.18)

It is shown in Appendix B that $D(\varkappa)$ admits the representation

$$D(\varkappa) = \frac{D^-(\varkappa)}{D^+(\varkappa)}$$

where

$$D^\pm(\varkappa) = \exp\left\{-\frac{1}{2\pi i}\int_{-\infty\mp i\delta}^{\infty\mp i\delta}\frac{\text{Log}[D(\alpha)]}{\alpha-\varkappa}d\alpha\right\}, \quad -\delta<\text{Im}\{\varkappa\}<\delta \tag{3.19}$$

and that $D^+(\varkappa)$ and $D^-(\varkappa)$, by trivial analytic continuations, have the same domains of analyticity as $G^+(\varkappa)$ and $F^-(\varkappa)$, respectively. In view of (3.18) and (3.19), Eq. (3.17a) may be rewritten as

$$\frac{Nk_s^2 n_s^-(\varkappa)D^-(\varkappa)F^-(\varkappa)}{2(k_s^2-k_d^2)} - \frac{A_0 Nk_s^2 k_R D^+(k_R)}{2(\varkappa-k_R)n_s^+(k_R)}$$

$$= \frac{D^+(\varkappa)G^+(\varkappa)}{n_s^+(\varkappa)} + \frac{A_0 Nk_s^2 k_R}{2(\varkappa-k_R)}\left[\frac{D^+(\varkappa)}{n_s^+(\varkappa)} - \frac{D^+(k_R)}{n_s^+(k_R)}\right], \quad -\delta<\text{Im}\{\varkappa\}<\delta , \tag{3.20}$$

where $n_s^\pm(\varkappa) = \sqrt{\varkappa\pm k_R}$. The functions $n_s^\pm(\varkappa)$ are rendered single-valued in the entire complex $\varkappa$-plane and analytic in the strip $-\delta<\text{Im}\{\varkappa\}<\delta$ by taking branch cuts as shown in Fig. 4, and by requiring both $n_s^+(\varkappa)$ and $n_s^-(\varkappa)$ to approach $+\infty$ as $\varkappa\to\infty$ along the positive real axis. Note that the superscripts have the same meanings as above.

Equation (3.20) is said to be of Wiener-Hopf type. The method used to solve it is referred to as the Wiener-Hopf technique, and is outlined on p. 37 of [21]. Define functions $H^\pm(\varkappa)$ by

$$H^-(\varkappa) = \frac{Nk_s^2 n_s^-(\varkappa)D^-(\varkappa)F^-(\varkappa)}{2(k_s^2 - k_d^2)} - \frac{A_0 Nk_s^2 k_R D^+(k_R)}{2(\varkappa - k_R)n_s^+(k_R)} \, , \; \text{Im}\{\varkappa\} < \delta \, ,$$

$$H^+(\varkappa) = \frac{D^+(\varkappa)G^+(\varkappa)}{n_s^+(\varkappa)} + \frac{A_0 Nk_s^2 k_R}{2(\varkappa - k_R)}\left[\frac{D^+(\varkappa)}{n_s^+(\varkappa)} - \frac{D^+(k_R)}{n_s^+(k_R)}\right], \; \text{Im}\{\varkappa\} > -\delta \, .$$

(3.21)

The function $H^+(\varkappa)$, by a trivial analytic continuation, is analytic for $\text{Im}\{\varkappa\} > -\delta$, while $H^-(\varkappa)$ is similarly analytic for $\text{Im}\{\varkappa\} < \delta$. According to (3.20), there is a common strip of analyticity in which

$$H^+(\varkappa) = H^-(\varkappa) \, , \; -\delta < \text{Im}\{\varkappa\} < \delta \, . \tag{3.22}$$

It follows that there is an $\underline{entire}$ function $H(\varkappa)$ such that

$$H(\varkappa) = \begin{cases} H^-(\varkappa) \, , \; \text{Im}\{\varkappa\} < \delta \, , \\ H^+(\varkappa) \, , \; \text{Im}\{\varkappa\} > -\delta \, . \end{cases} \tag{3.23}$$

Now it will be proved that both $H^-(\varkappa)$ and $H^+(\varkappa)$ tend to zero as $|\varkappa| \to \infty$ in the appropriate half-plane. The order conditions (3.2) imply the existence of positive constants L and ϵ such that

$$\begin{aligned} &|\sigma_{xz}(x,0)| < L \quad\quad\quad , \; 0 < x \le \epsilon \, , \\[2mm] &|\frac{\partial\sigma_{xz}(x,0)}{\partial x}| < Lx^{-p_1} , \; 0 < x \le \epsilon \, , \\[2mm] &|\sigma_{xz}(x,0) + \mu N\frac{\partial^2 u(x,0)}{\partial x^2}| < L|x|^{-p_2} , \; -\epsilon \le x < 0 \, . \end{aligned}$$

(3.24)

The exponential decay resulting from the complex frequency, together with the assumed continuity of the solution, imply the existence of a constant $K > 0$ such that

$$\left. \begin{array}{l} |\sigma_{xz}(x,0)| < Ke^{-\delta x}, \quad \epsilon \le x < \infty, \\[12pt] \left|\dfrac{\partial\sigma_{xz}(x,0)}{\partial x}\right| < Ke^{-\delta x}, \quad \epsilon \le x < \infty, \\[12pt] \left|\sigma_{xz}(x,0) + \mu N\dfrac{\partial^2 u(x,0)}{\partial x^2}\right| < Ke^{\delta x}, \quad -\infty < x \le -\epsilon. \end{array} \right\} \tag{3.25}$$

Substitution of (1.2) and (1.4) into (3.14), thence the result into (3.16), yields

$$\mu F^-(\varkappa) = \int_0^\infty \sigma_{xz}(x,0)e^{-i\varkappa x}dx, \tag{3.26a}$$

$$\mu G^+(\varkappa) = \int_{-\infty}^0 \left[\sigma_{xz}(x,0) + \mu N\frac{\partial^2 u(x,0)}{\partial x^2}\right]e^{-i\varkappa x}dx. \tag{3.26b}$$

One integration of (3.26a) by parts, followed by use of (3.24) and (3.25), leads to the following asymptotic estimates for $F^-(\varkappa)$ and $G^+(\varkappa)$:

$$\left. \begin{array}{l} |\mu F^-(\varkappa)| = \left|\dfrac{\sigma_{xz}(x,0)}{-i\varkappa}e^{i\varkappa x}\Big|_{0^+}^\infty + \dfrac{1}{i\varkappa}\int_0^\infty \dfrac{\partial\sigma_{xz}(x,0)}{\partial x}e^{-i\varkappa x}dx\right| \\[16pt] \qquad \le \dfrac{L}{|\varkappa|} + \dfrac{1}{|\varkappa|}\left[\int_0^\epsilon Lx^{-p_1}dx + \int_\epsilon^\infty Ke^{-\delta x}dx\right] \\[16pt] \qquad = O(|\varkappa|^{-1}) \quad \text{as} \quad |\varkappa| \to \infty, \quad \text{Im}\{\varkappa\} < \delta, \\[16pt] |\mu G^+(\varkappa)| \le \int_{-\epsilon}^0 L|x|^{-p_2}dx + \int_{-\infty}^{-\epsilon} Ke^{\delta x}dx \\[16pt] \qquad = O(1) \quad \text{as} \quad |\varkappa| \to \infty, \quad \text{Im}\{\varkappa\} > -\delta. \end{array} \right\} \tag{3.27}$$

It is proved in Appendix B that

$$\left.\begin{aligned}|D^+(\varkappa)| &\to 1 \text{ as } |\varkappa| \to \infty, \text{ Im}\{\varkappa\} > -\delta, \\ |D^-(\varkappa)| &\to 1 \text{ as } |\varkappa| \to \infty, \text{ Im}\{\varkappa\} < \delta.\end{aligned}\right\} \tag{3.28}$$

Use of (3.27) and (3.28) in (3.21) yields

$$H^-(\varkappa) \to 0 \text{ as } |\varkappa| \to \infty, \text{ Im}\{\varkappa\} < \delta,$$

$$H^+(\varkappa) \to 0 \text{ as } |\varkappa| \to \infty, \text{ Im}\{\varkappa\} > -\delta.$$

Therefore (3.23) and a trivial extension of Liouville's theorem (p. 125 of [22]) imply that

$$H(\varkappa) \equiv 0. \tag{3.29}$$

In view of (3.29), (3.23), and (3.21), $F^-(\varkappa)$ and $G^+(\varkappa)$ are given by

$$\left.\begin{aligned}F^-(\varkappa) &= \frac{A_0 k_R (k_s^2 - k_d^2) D^+(k_R)}{n_s^-(\varkappa) D^-(\varkappa)(\varkappa - k_R) n_s^+(k_R)}, \text{ Im}\{\varkappa\} < \delta, \\ G^+(\varkappa) &= -\frac{A_0 N k_s^2 k_R}{2(\varkappa - \varkappa_R)}\left[1 - \frac{n_s^+(\varkappa) D^+(k_R)}{D^+(\varkappa) n_s^+(k_R)}\right], \text{ Im}\{\varkappa\} > -\delta.\end{aligned}\right\} \tag{3.30}$$

Substitution from (3.30) into (3.15) yields three equations in $A(\varkappa)$ and $B(\varkappa)$. Any one of these may be considered as redundant and the remaining two solved for $A(\varkappa)$ and $B(\varkappa)$ to give

$$A(\varkappa) = \frac{A_0 N k_R k_s^2 D^+(k_R)\varkappa\, n_s(\varkappa) n_s^+(\varkappa)}{in_s^+(k_R)(\varkappa - k_R) R^*(\varkappa) D^+(\varkappa)} \tag{3.31a}$$

$$= \frac{2A_0 k_R D^+(k_R)(k_s^2 - k_d^2)\varkappa\, n_s(\varkappa)}{in_s^+(k_R) n_s^-(\varkappa)(\varkappa - k_R) R(\varkappa) D^-(\varkappa)}, \tag{3.31b}$$

$$B(\varkappa) = -\frac{A_0 N k_R k_s^2 D^+(k_R)(2\varkappa^2 - k_s^2)}{2n_s^+(k_R)(\varkappa - k_R)R^*(\varkappa)D^+(\varkappa)} \tag{3.31c}$$

$$= -\frac{A_0 k_R D^+(k_R)(k_s^2 - k_d^2)(2\varkappa^2 - k_s^2)}{n_s^+(k_R)n_s^-(\varkappa)(\varkappa - k_R)R(\varkappa)D^-(\varkappa)}, \quad -\delta < \text{Im}\{\varkappa\} < \delta. \tag{3.31d}$$

The alternate forms are obtained by using (3.18) and (3.19).

Application of the Fourier inversion formula (3.8) to (3.12) yields

$$\left.\begin{aligned}
\hat{\varphi}(x, z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\varkappa) e^{-n_d(\varkappa)z + i\varkappa x} d\varkappa, \\[2mm]
\hat{\psi}(x, z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} B(\varkappa) e^{-n_s(\varkappa)z + i\varkappa x} d\varkappa
\end{aligned}\right\} \tag{3.32}$$

as formal candidates for a solution to the original problem for $\hat{\varphi}$, $\hat{\psi}$.

Several steps are now necessary to prove that the functions $\hat{\varphi}$, $\hat{\psi}$ appearing in (3.32) are properly defined and actually satisfy (3.4) and (3.5). These steps are left to Appendix C. It is also proved there that the asymptotic forms of $F^-(\varkappa)$ and $G^+(\varkappa)$ as $|\varkappa| \to \infty$ are in fact as specified in (3.27).

In order to demonstrate that (3.32) contain the surface waves required by (2.8), the integrals in (3.32) must be regarded as complex contour integrals along the real $\varkappa$-axis in the complex $\varkappa$-plane (see Fig. 4). The singularities in the integrands in (3.32) are due to the various singularities of $D^+(\varkappa)$ and $D^-(\varkappa)$, the branch points of $n_s(\varkappa)$ and $n_d(\varkappa)$, and the simple zeros of the denominators associated with $A(\varkappa)$ and $B(\varkappa)$. It is shown in Appendix B that $D^+(\varkappa)$ has branch

points at $-k_s$ and $-k_d$, no zeros, and a simple pole at $-k_T$, while $D^-(\varkappa)$ has branch points at k_s and k_d, no zeros, and a simple pole at k_R. By inspection of (3.17b) it is seen that $R(\varkappa)$ and $R^*(\varkappa)$ have branch points at $\pm k_s$ and $\pm k_d$, and it is proved in Appendix A that their only zeros are located at $\pm k_R$ and $\pm k_T$, respectively. Hence the singularities of the integrands in (3.32) are branch points at $\pm k_s$ and $\pm k_d$, and simple poles at $\pm k_R$ and $\pm k_T$. These are shown in Fig. 4. That they are not on the real axis is due to the assumed small, positive, imaginary part of ω. The branch cuts are taken as shown in Fig. 4. The segments of these cuts between the branch points and the imaginary axis will lie along the real axis in the limit as $\operatorname{Im}\{\omega\} \to 0$.

Now consider an integration around the closed contour formed by the quarter-circular arcs of radius R and the appropriate segments of the branch-line contour and the real axis, as shown in Fig. 4. If the integrals over the quarter-circular arcs vanish in the limit as $R \to \infty$, then by Cauchy's theorem the integral over the real axis may be replaced by that over the branch-line contour plus the contribution from the residues at the poles k_T and k_R. It can easily be shown that this is in fact the case when $x > 0$. The residues are evaluated using (3.31a, c) and (3.32). With the aid of (3.17b) and (3.3) it is seen that the contribution due to the pole at k_R is equal to the incident wave $A_0 \varphi_{R+}(x, z)$ but for its sign. Thus substitution of the result of the contour shift into (3.3) yields

$$\Phi(x, z) = A_t \varphi_{T+}(x, z) + \varphi(x, z) \, , \left.\begin{array}{l}\\ \\ \\ \end{array}\right\}$$

$$\Psi(x, z) = A_t \psi_{T+}(x, z) + \psi(x, z) \, , \ x>0 \, ,$$

(3.33)

where

$$A_t = \frac{A_0 N k_T k_R k_s^2 D^+(k_R) n_s(k_T) n_s^+(k_T)}{(k_T - k_R) n_s^+(k_R) R^{*\,'}(k_T) D^+(k_T)} \, , \tag{3.34}$$

$$R^{*\,'}(k_T) = \left.\frac{dR^*}{d\varkappa}\right|_{\varkappa = k_T} \, ,$$

$$\varphi(x, z) = \frac{1}{2\pi} \int_{B_+} A(\varkappa) e^{-n_d(\varkappa)z + i\varkappa x} d\varkappa \, , \left.\begin{array}{l}\\ \\ \\ \\ \end{array}\right\}$$

$$\psi(x, z) = \frac{1}{2\pi} \int_{B_+} B(\varkappa) e^{-n_s(\varkappa)z + i\varkappa x} d\varkappa \, , \ x>0 \, ,$$

(3.35)

and where B_+ denotes the branch-line contour in the upper half-plane
(see Fig. 4). The full field is thus seen to have the required form
(2.8) for $x>0$, with A_t as the amplitude of the transmitted wave, pro-
vided φ and ψ satisfy the radiation conditions (2.5). That they do may
be proved by analyzing the asymptotic behavior for large $\sqrt{x^2+z^2}$ of
the integrals in (3.35). When $z = 0$, such an analysis may be carried
out with the aid of Watson's lemma, and when $z \neq 0$, by the saddle
point method.

When $x<0$, a similar procedure applied in the lower half of the
$\varkappa$-plane leads to the result

$$\Phi(x, z) = A_0 \varphi_{R+}(x, z) + A_r \varphi_{R-}(x, z) + \varphi(x, z) \, , \left.\begin{array}{l}\\ \\ \\ \end{array}\right\}$$

$$\Psi(x, z) = A_0 \psi_{R+}(x, z) + A_r \psi_{R-}(x, z) + \psi(x, z) \, , \ x<0 \, ,$$

(3.36)

where

$$A_r = - \frac{A_0 k_R (k_s^2 - k_d^2) D^+(k_R) n_s^+(-k_R)}{n_s^+(k_R) D^-(-k_R) R'(-k_R)} , \qquad (3.37)$$

$$R'(-k_R) = \left. \frac{dR}{d\varkappa} \right|_{\varkappa = -k_R} ,$$

$$\left. \begin{aligned} \varphi(x,z) &= \frac{1}{2\pi} \int_{B_-} A(\varkappa) e^{-n_d(\varkappa)z + i\varkappa x} d\varkappa , \\[2mm] \psi(x,z) &= \frac{1}{2\pi} \int_{B_-} B(\varkappa) e^{-n_s(\varkappa)z + i\varkappa x} d\varkappa , \quad x < 0 , \end{aligned} \right\} \qquad (3.38)$$

and where B_- denotes the branch-line contour in the lower half-plane. Thus the full field has the required form for $x < 0$, with A_r as the amplitude of the reflected wave.

Note that the functions $\varphi(x,z)$, $\psi(x,z)$ as defined by (3.35) and (3.38) are discontinuous across $x = 0$, $z > 0$, but the full field $\underline{\Phi}$ is twice-continuously differentiable in the interior of the half-space.

Also note that had (3.32) been derived formally without recourse to the assumption of complex frequency, the inversion contour would appear to pass directly through the poles and branch points (since they would now lie on the real $\varkappa$-axis), and it would be necessary to appeal to some other argument to determine whether this contour should be considered as passing above, below, or through these singularities. The assumption of complex frequency, in addition to providing the overlapping domains of analyticity required for the Wiener-Hopf analysis, serves in effect to resolve this dilemma. Having served its purpose, it may now be discarded.

It is now desired to obtain numerical values for the reflection and transmission coefficients A_r/A_0, A_t/A_0. All quantities in (3.34) and (3.37) can be evaluated straightforwardly except for the functions D^+ and D^-. It is proved in Appendix B that

$$D^-(-\varkappa) = \frac{1}{D^+(\varkappa)} , \quad \text{Im}\{\varkappa\} > -\delta , \qquad (3.39)$$

so for the purpose of determining A_r/A_0 and A_t/A_0 from (3.34) and (3.37), it is only necessary to compute $D^+(k_R)$ and $D^+(k_T)$. This is done by a numerical integration, the details of which are explained in Appendix B. After rewriting (3.34) and (3.37) in dimensionless form it is seen that the coefficients A_r/A_0, A_t/A_0 are functions only of the dimensionless layer parameter Nk_s and Poisson's ratio ν. The dependence on Nk_s of the transmission and reflection coefficients is shown for $\nu = 1/4$ in Fig. 5. The coefficients A_r/A_0 and A_t/A_0 are complex, but for the range of Nk_s shown the imaginary parts never exceed 0.4% of the corresponding real part. A complex coefficient simply signifies a wave whose phase is shifted with respect to the incident wave.

A check of these numerical results may be obtained by computing, from (3.34) and (3.37), the asymptotic forms of A_r/A_0 and A_t/A_0 as $N \to 0$. The wave number k_T is a function of N defined implicitly by (1.13):

$$R[k_T(N)] - Nk_s^2 k_T^2(N) n_s[k_T(N)] = 0 .$$

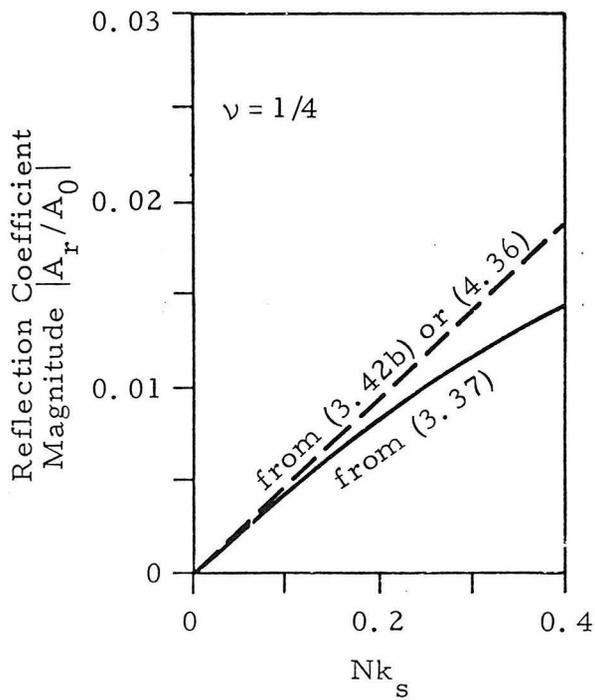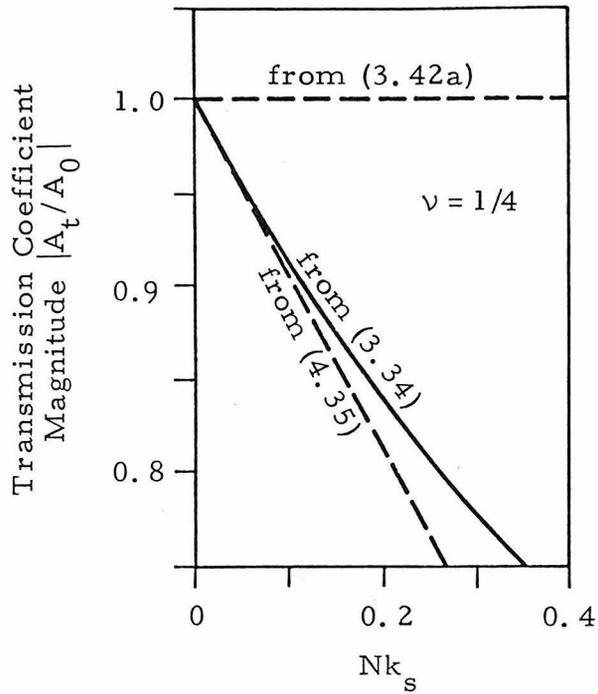This equation may be differentiated with respect to N, yielding

Figure 5. Transmission and Reflection Coefficients
For the Case of a Massless Layer.

$$R'[k_T(N)]k'_T(N) - k_s^2 k_T^2(N) n_s[k_T(N)] - N\frac{d}{dN}\{k_s^2 k_T^2(N) n_s[k_T(N)]\} = 0$$

Upon setting $N = 0$, and noting from (1.13) and (1.12) that $k_T(0) = k_R$, there results

$$k'_T(0) = \frac{k_s^2 k_R^2 n_s(k_R)}{R'(k_R)} .$$

Thus

$$k_T = k_R + \frac{Nk_s^2 k_R^2 n_s(k_R)}{R'(k_R)} + O(N^2) \text{ as } N \to 0 . \qquad (3.40)$$

It is proved in Appendix B that for $k > k_s$

$$D^+(k) = \sqrt{\frac{Nk_s^2(k_s+k)}{2(k_s^2 - k_d^2)}} + o(\sqrt{N}) \text{ as } N \to 0 . \qquad (3.41)$$

Application of (3.39)-(3.41) to (3.34) and (3.37) yields

$$A_t = A_0 + o(1) , \qquad (3.42a)$$

$$A_r = -\frac{Nk_R k_s^2 n_s(k_R)}{2R'(-k_R)} + o(N) \text{ as } N \to 0 . \qquad (3.42b)$$

The first terms of (3.42a) and (3.42b) are plotted in Fig. 5 and it is clear that they do agree with the numerical computation with increasing accuracy as $N \to 0$. Equation (3.42a) expresses nothing more than the fact that the amplitude of the transmitted wave approaches that of the incident wave in the limit of vanishing layer thickness.

4. Approximate Solution for a Very Thin Layer

It is now assumed that the layer is thin enough, when compared with the wavelength of any of the characteristic surface or body waves, that the layer parameters M and N, defined by (1.8), may be taken as perturbation parameters in a sense to be described presently.

Let a constant α be defined by

$$\alpha = \frac{\rho' \omega^2}{4\mu'} \left(\frac{\lambda' + 2\mu'}{\lambda' + \mu'} \right), \tag{4.1}$$

so that

$$M = \alpha N. \tag{4.2}$$

The perturbation process consists of allowing the layer thickness $2h'$ to grow from zero to some small, non-zero value while holding α fixed. Thus, in view of (1.8), (4.1), and (4.2), the problem may be expressed in terms of the single perturbation parameter N.

The differential equation (2.6) for $\underline{\Phi}(x, z)$ remains. By using (4.2), the boundary conditions (2.7) may be rewritten as

$$\left. \begin{array}{l} \underline{\underline{R}}\, \underline{\Phi} = \underline{0}, \ z = 0, \ -\infty < x < 0, \\[2mm] \{\underline{\underline{R}} + N(\underline{\underline{S}} + \alpha \underline{\underline{T}})\}\, \underline{\Phi} = \underline{0}, \ z = 0, \ 0 < x < \infty. \end{array} \right\} \tag{4.3}$$

The assumed form (2.8) also remains appropriate, with the wave number k_T a root of (1.9), but now it is assumed that the amplitudes A_r, A_t appearing in (2.8) may be represented by

$$\left. \begin{array}{l} A_r = A_r^{(0)} + N A_r^{(1)} + o(N), \\[2mm] A_t = A_t^{(0)} + N A_t^{(1)} + o(N) \ \text{as} \ N \to 0, \end{array} \right\} \tag{4.4}$$

and that the radiated wave $\underline{\varphi}$ may be expressed as

$$\underline{\varphi}(x, z) = \underline{\varphi}^{(0)}(x, z) + N\underline{\varphi}^{(1)}(x, z) + o(N) \text{ as } N \to 0 . \qquad (4.5)$$

Each term on the right of (4.5) is assumed separately to satisfy the radiation condition (2.5).

The remainder of this section is concerned with the determination of $A_r^{(m)}$, $A_t^{(m)}$, $m = 0, 1$. The first step is to derive the equations governing $\underline{\varphi}^{(0)}$ and $\underline{\varphi}^{(1)}$. By substitution of (2.8) into (2.6) and (4.3), there results the following problem for $\underline{\varphi}$:

$$\left.\begin{array}{l} \underline{\underline{L}} \, \underline{\varphi} = \underline{0}, \ z > 0, \ -\infty < x < \infty, \ x \neq 0, \\[2mm] \underline{\underline{R}} \, \underline{\varphi} = 0, \ z = 0, \ -\infty < x < 0, \\[2mm] \{\underline{\underline{R}} + N(\underline{\underline{S}} + \alpha\,\underline{\underline{T}})\}\underline{\varphi} = 0, \ z = 0, \ 0 < x < \infty . \end{array}\right\} \qquad (4.6)$$

The requirement that $\underline{\Phi}$ be twice-continuously differentiable, when applied to (2.8), leads to the jump conditions

$$\left.\begin{array}{l} [\underline{\varphi}] - A_0\underline{\varphi}_{R+}(0, z) - A_r\underline{\varphi}_{R-}(0, z) + A_t\underline{\varphi}_{T+}(0, z) = 0, \\[2mm] [\underline{\varphi},_x] - ik_R A_0\underline{\varphi}_{R+}(0, z) + ik_R A_r\underline{\varphi}_{R-}(0, z) \\[2mm] \qquad + ik_T A_t\underline{\varphi}_{T+}(0, z) = \underline{0}, \ 0 < z < \infty , \end{array}\right\} (4.7)$$

where

$$[\underline{\varphi}] = \underline{\varphi}(0^+, z) - \underline{\varphi}(0^-, z) .$$

When (4.2) is substituted into (1.9), there results an equation which defines k_T implicitly as a function of N. By differentiating this equation with respect to N and setting $N = 0$, it follows that

$$k_T^{(1)} \equiv \left.\frac{dk_T}{dN}\right|_{N=0} = \frac{1}{R'(k_R)}\{k_R^2 k_s^2 n_s(k_R) - \alpha k_s^2[n_s(k_R) + n_d(k_R)]\} . (4.8)$$

Equation (4.8), and the fact that $k_T = k_R$ when $N = 0$, lead to

$$k_T = k_R + N k_T^{(1)} + o(N) \text{ as } N \to 0 . \qquad (4.9)$$

The quantity $\underline{\varphi}_{T+}(0, z)$, defined by (2.8), may similarly be expanded in a power series in N, resulting in

$$\varphi_{T+}(0, z) = \underline{\varphi}_{R+}(0, z) + N \underline{\varphi}_{T+}^{(1)}(0, z) + o(N) \text{ as } N \to 0 , \qquad (4.10)$$

where

$$\underline{\varphi}_{T+}^{(1)}(0, z) = \left\{ \begin{array}{l} -n_d^{(1)} z e^{-n_d(k_R)z} \\[2ex] (\beta_T^{(1)} - \beta_R n_s^{(1)} z) e^{-n_s(k_R)z} \end{array} \right\} ,$$

$$\beta_T^{(1)} \equiv \left. \frac{d\beta_T(N)}{dN} \right|_{N=0} , \quad n_s^{(1)} \equiv \left. \frac{d\{n_s[k_T(N)]\}}{dN} \right|_{N=0}$$

Now (4.4), (4.5), (4.9), and (4.10) are substituted into (4.6) and (4.7), and the coefficients of the zeroth and first powers of N in each equation are separately set equal to zero, to yield

$$\left. \begin{array}{l} \underline{\underline{L}} \, \underline{\varphi}^{(0)} = \underline{0} , \ z > 0 , \ -\infty < x < \infty , \ x \neq 0 ; \\[2ex] \underline{\underline{R}} \, \underline{\varphi}^{(0)} = \underline{0} , \ z = 0 , \ -\infty < x < \infty , \ x \neq 0 ; \\[2ex] [\underline{\varphi}^{(0)}] - A_0 \underline{\varphi}_{R+}(0, z) - A_r^{(0)} \underline{\varphi}_{R-}(0, z) + A_t^{(0)} \underline{\varphi}_{R+}(0, z) = \underline{0} , \\[2ex] [\underline{\varphi}_{,x}^{(0)}] - i k_R A_0 \underline{\varphi}_{R+}(0, z) + i k_R A_r^{(0)} \underline{\varphi}_{R-}(0, z) \\[2ex] \qquad + i k_R A_t^{(0)} \underline{\varphi}_{R+}(0, z) = \underline{0} , \ z > 0 ; \end{array} \right\} \quad (4.11)$$

$$\underline{\underline{L}}\,\underline{\varphi}^{(1)} = \underline{0}\,, \quad z>0\,, \quad -\infty<x<\infty\,, \quad x\neq 0\,;$$

$$\underline{\underline{R}}\,\underline{\varphi}^{(1)} = \underline{0}\,, \quad z=0\,, \quad -\infty<x<0\,;$$

$$\underline{\underline{R}}\,\underline{\varphi}^{(1)} = -(\underline{\underline{S}}+\alpha\,\underline{\underline{T}})\,\underline{\varphi}^{(0)}\,, \quad z=0\,, \quad 0<x<\infty\,;$$

$$[\underline{\varphi}^{(1)}] - A_r^{(1)}\underline{\varphi}_{R-}(0,z) + A_t^{(1)}\underline{\varphi}_{R+}(0,z) + A_t^{(0)}\underline{\varphi}_{T+}^{(1)}(0,z) = \underline{0}\,,$$

$$[\underline{\varphi}_{,x}^{(1)}] + ik_R A_r^{(1)}\underline{\varphi}_{R-}(0,z) + ik_R A_t^{(1)}\underline{\varphi}_{R+}(0,z)$$

$$+ ik_R A_t^{(0)}\underline{\varphi}_{T+}^{(1)}(0,z) + ik_T^{(1)}A_t^{(0)}\underline{\varphi}_{R+}(0,z) = \underline{0}\,, \quad z>0\,. \tag{4.12}$$

In order to solve (4.11), it is convenient to define an auxiliary function $\underline{\hat{\varphi}}^{(0)}$ by

$$\underline{\hat{\varphi}}^{(0)} = \underline{\varphi}^{(0)} - A_0\underline{\varphi}_{R+}H(x) + A_r^{(0)}\underline{\varphi}_{R-}H(-x) + A_t^{(0)}\underline{\varphi}_{R+}H(x)\,. \tag{4.13}$$

When (4.13) is substituted into (4.11), the equations governing $\underline{\hat{\varphi}}^{(0)}$ are found to be

$$\underline{\underline{L}}\,\underline{\hat{\varphi}}^{(0)} = \underline{0}\,, \quad z>0\,, \quad -\infty<x<\infty\,,$$

$$\underline{\underline{R}}\,\underline{\hat{\varphi}}^{(0)} = \underline{0}\,, \quad z=0\,, \quad -\infty<x<\infty\,,$$

$$[\underline{\hat{\varphi}}^{(0)}] = [\underline{\hat{\varphi}}_{,x}^{(0)}] = \underline{0}\,, \quad z>0\,. \tag{4.14}$$

In particular, note that the differential equation for $\underline{\hat{\varphi}}^{(0)}$ holds <u>throughout</u> the interior of the half-space. From (4.13) it is seen that $\underline{\hat{\varphi}}^{(0)}$ is everywhere outwardly propagating. This fact together with (4.14) implies that $\underline{\hat{\varphi}}^{(0)} \equiv \underline{0}$. Thus, from (4.13),

$$\underline{\varphi}^{(0)} = (A_0 - A_t^{(0)})\,\underline{\varphi}_{R+}H(x) - A_r^{(0)}\underline{\varphi}_{R-}H(-x)\,. \tag{4.15}$$

But since $\underline{\varphi}^{(0)}$ is a <u>radiated</u> wave, it cannot contain any surface waves, so from (4.15),

$$A_t^{(0)} = A_0 \,, \quad A_r^{(0)} = 0 \,, \\ \underline{\varphi}^{(0)}(x, z) \equiv 0 \,. \Bigg\} \tag{4.16}$$

Equations (4.16) simply express the fact that in the limit as the layer vanishes, the full field $\underline{\Phi}$ consists only of a positive-traveling Rayleigh wave of amplitude A_0.

In view of (4.16), Eqs. (4.12) become

$$\underline{\underline{L}} \, \underline{\varphi}^{(1)} = \underline{0} \,, \quad z > 0 \,, \quad -\infty < x < \infty \,, \quad x \neq 0 \,; \\[6pt]
\underline{\underline{R}} \, \underline{\varphi}^{(1)} = \underline{0} \,, \quad z = 0 \,, \quad -\infty < x < \infty \,, \quad x \neq 0 \,; \\[6pt]
[\underline{\varphi}^{(1)}] - A_r^{(1)} \underline{\varphi}_{R-}(0, z) + A_t^{(1)} \underline{\varphi}_{R+}(0, z) + A_0 \underline{\varphi}_{T+}^{(1)}(0, z) = \underline{0} \,, \\[6pt]
[\underline{\varphi}_{,x}^{(1)}] + ik_R A_r^{(1)} \underline{\varphi}_{R-}(0, z) + ik_R A_t^{(1)} \underline{\varphi}_{R+}(0, z) \\[6pt]
\qquad + ik_R A_t^{(0)} \underline{\varphi}_{T+}^{(1)}(0, z) + ik_T^{(1)} A_t^{(0)} \underline{\varphi}_{R+}(0, z) = \underline{0} \,, \quad z > 0
\Bigg\} \tag{4.17}$$

The solution of (4.17) is facilitated by the introduction of an auxiliary function $\hat{\underline{\varphi}}^{(1)}$ defined by

$$\hat{\underline{\varphi}}^{(1)} = \underline{\varphi}^{(1)} + A_r^{(1)} \underline{\varphi}_{R-} H(-x) + A_t^{(1)} \underline{\varphi}_{R+} H(x) + A_0 \underline{\varphi}_{T+}^{(1)}(0, z) e^{ik_R x} H(x) \\[6pt]
\qquad + \frac{A_0 k_T^{(1)}}{2k_R} \left[\underline{\varphi}_{R+} H(x) + \underline{\varphi}_{R+}(0, z) e^{-ik_R x} H(-x) \right]. \tag{4.18}$$

When (4.18) is substituted into (4.17), the equations for $\hat{\underline{\varphi}}^{(1)}$ are found to be

$$\underline{\underline{L}} \, \hat{\underline{\varphi}}^{(1)} = A_0 H(x) \underline{\underline{L}} \, \{ \underline{\varphi}_{T+}^{(1)}(0, z) e^{ik_R x} \} \,, \quad z > 0 \,, \quad -\infty < x < \infty \,, \tag{4.19a}$$

$$\underline{\underline{R}} \, \hat{\underline{\varphi}}^{(1)} = \frac{A_0 k_T^{(1)}}{2k_R} \, \underline{\underline{R}} \left\{ \underline{\varphi}_{R+}(0, z) e^{-ik_R x} \right\} H(-x) \\[6pt]
\qquad + A_0 \underline{\underline{R}} \left\{ \underline{\varphi}_{T+}^{(1)}(0, z) e^{ik_R x} \right\} H(x), \quad z = 0 \,, \quad -\infty < x < \infty \,, \tag{4.19b}$$

$$[\hat{\underline{\varphi}}^{(1)}] = [\hat{\underline{\varphi}}_{,x}^{(1)}] = \underline{0} , \quad z > 0 . \qquad (4.19c)$$

As in the case of $\hat{\underline{\varphi}}^{(0)}$, the function $\hat{\underline{\varphi}}^{(1)}$ is smooth enough that the differential equation (4.19a) holds <u>throughout</u> the interior of the half-space.

The right-hand sides of (4.19a, b) may be simplified somewhat. Note that

$$\underline{\varphi}_{T+}(x, z) = \underline{\varphi}_{T+}(0, z)e^{ik_T x}$$

$$= \left[\underline{\varphi}_{R+}(0, z) + N\underline{\varphi}_{T+}^{(1)}(0, z) + O(N^2)\right]e^{ik_R x}\left[1 + iNk_T^{(1)}x + O(N^2)\right]$$

$$= \underline{\varphi}_{R+} + N\left[\underline{\varphi}_{T+}^{(1)}(0, z)e^{ik_R x} + ik_T^{(1)}x\underline{\varphi}_{R+}\right] + O(N^2) \text{ as } N \to 0 . \quad (4.20)$$

Since $\underline{\underline{L}}\,\underline{\varphi}_{T+}(x, z) \equiv \underline{0}$, each term of the latter form of (4.20), when operated upon by $\underline{\underline{L}}$, must vanish. Thus

$$\underline{\underline{L}}\,\{\underline{\varphi}_{T+}^{(1)}(0, z)e^{ik_R x}\} = -ik_T^{(1)}\underline{\underline{L}}\,\{x\underline{\varphi}_{R+}(x, z)\} . \qquad (4.21)$$

From (1.8), (1.2), (1.4), (2.7), and (4.2) it follows that

$$\{\underline{\underline{R}} + N(\underline{\underline{S}} + \alpha\,\underline{\underline{T}})\}\,\underline{\varphi}_{T+} = \underline{0} , \quad z = 0 . \qquad (4.22)$$

Substitution of (4.20) into (4.22) leads to

$$\underline{\underline{R}}\,\underline{\varphi}_{R+} + N\left[\underline{\underline{R}}\{\underline{\varphi}_{T+}^{(1)}(0, z)e^{ik_R x}\} + ik_T^{(1)}\underline{\underline{R}}\{x\underline{\varphi}_{R+}\} + (\underline{\underline{S}} + \alpha\underline{\underline{T}})\underline{\varphi}_{R+}\right]$$

$$+ O(N^2) = 0 \text{ as } N \to 0 , \quad z = 0 . \qquad (4.23)$$

Each term of (4.23) must likewise vanish separately, so

$$\underline{\underline{R}}\{\underline{\varphi}_{T+}^{(1)}(0, z)e^{ik_R x}\} = -ik_T^{(1)}\underline{\underline{R}}\{x\underline{\varphi}_{R+}\} - (\underline{\underline{S}} + \alpha\underline{\underline{T}})\underline{\varphi}_{R+} , \quad z = 0 . \qquad (4.24)$$

The right-hand sides of (4.21) and (4.24) may be easily computed by using Leibnitz' rule, and by noting that $\underline{\underline{L}}\,\underline{\varphi}_{R+} = \underline{0}$ and that

$\underline{\underline{R}} \ \underline{\varphi}_{R+} = \underline{0}$ at $z = 0$. The first term in the right-hand side of (4.19b) may also be computed directly. These results are then incorporated into (4.19) to yield

$$\underline{\underline{L}} \ \hat{\underline{\varphi}}^{(1)} = 2A_0 k_R k_T^{(1)} \underline{\varphi}_{R+} H(x) , \ z > 0 , \ -\infty < x < \infty , \qquad (4.25a)$$

$$\underline{\underline{R}} \hat{\underline{\varphi}}^{(1)} = \frac{A_0 k_T^{(1)}(2k_R^2 - k_s^2)}{k_R} \underline{\varphi}_{R-} H(-x) + A_0 \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} e^{ik_R x} , \ z = 0 , \ -\infty < x < \infty, (4.25b)$$

where

$$\left. \begin{aligned} a_1 &= \frac{k_T^{(1)}(2k_R^2 - k_s^2)}{k_R} + \alpha \left[n_d(k_R) - \frac{2k_R^2 - k_s^2}{2n_s(k_R)} \right] , \\ a_2 &= i \left[\frac{k_R k_s^2}{2} + 2n_d(k_R) k_T^{(1)} - \frac{\alpha k_s^2}{2k_R} \right] . \end{aligned} \right\} \qquad (4.26)$$

Equations (4.25) may be solved with the aid of the Fourier transform (3.7). It is once again assumed that the frequency ω has a small, positive, imaginary part. The transform of (4.25a) is then found to be

$$\left. \begin{aligned} \frac{\partial^2 \tilde{\hat{\varphi}}^{(1)}}{\partial z^2}(\varkappa, z) - n_s^2(\varkappa) \tilde{\hat{\varphi}}^{(1)}(\varkappa, z) &= \frac{2A_0 k_R k_T^{(1)} e^{-n_d(k_R)z}}{i(\varkappa - k_R)} , \\ \frac{\partial^2 \tilde{\hat{\psi}}^{(1)}}{\partial z^2}(\varkappa, z) - n_s^2(\varkappa) \tilde{\hat{\psi}}^{(1)}(\varkappa, z) &= \frac{2A_0 k_R k_T^{(1)} \beta_R e^{-n_s(k_R)z}}{i(\varkappa - k_R)} , \\ z > 0 , \ -\delta < Im\{\varkappa\} < \delta , \end{aligned} \right\} \qquad (4.27)$$

where δ is a constant such that $0 < \delta < Im\{k_d\}$. The solution of the homogeneous version of (4.27) is given in (3.11), and again $A'(\varkappa)$, $B'(\varkappa)$ must vanish. The general solution of (4.27) is thus

$$
\tilde{\varphi}^{(1)}(\varkappa, z) = A(\varkappa)e^{-n_d(\varkappa)z} - \frac{2A_0 k_R k_T^{(1)} e^{-n_d(k_R)z}}{i(\varkappa-k_R)^2(\varkappa+k_R)} ,
$$

$$
\tilde{\psi}^{(1)}(\varkappa, z) = B(\varkappa)e^{-n_s(\varkappa)z} - \frac{2A_0 k_R k_T^{(1)} \beta_T e^{-n_s(k_R)z}}{i(\varkappa-k_R)^2(\varkappa+k_R)} ,
$$

$$
z > 0, \quad -\delta < \mathrm{Im}\{\varkappa\} < \delta .
$$
\hfill (4. 28)

Application of the transform (3. 7) to the boundary conditions (4. 25b), followed by use of (4. 28), yields

$$
\begin{pmatrix} 2\varkappa^2-k_s^2 & -2i\varkappa n_s(\varkappa) \\ -2i\varkappa n_d(\varkappa) & -(2\varkappa^2-k_s^2) \end{pmatrix} \begin{Bmatrix} A(\varkappa) \\ B(\varkappa) \end{Bmatrix} - \frac{2A_0 k_R k_T^{(1)}}{i(\varkappa-k_R)^2(\varkappa+k_R)} \begin{pmatrix} 2k_R^2-k_s^2 & -2i\varkappa n_s(k_R) \\ -2i\varkappa n_d(k_R) & -(2k_R^2-k_s^2) \end{pmatrix} \begin{Bmatrix} 1 \\ \beta_R \end{Bmatrix}
$$

$$
= -\frac{A_0 k_T^{(1)}(2k_R^2-k_s^2)}{ik_R(\varkappa+k_R)} \begin{Bmatrix} 1 \\ -\beta_R \end{Bmatrix} + \frac{A_0}{i(\varkappa-k_R)} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} , \quad -\delta < \mathrm{Im}\{\varkappa\} < \delta . \hfill (4. 29)
$$

This is a system of two simultaneous, linear equations in the two unknowns $A(\varkappa)$, $B(\varkappa)$. It possesses a unique solution provided the determinant of the coefficient matrix does not vanish. That determinant is found to be $-R(\varkappa)$, where $R(\varkappa)$ is given by (3. 17b) and is known to vanish only at $\varkappa = \pm k_R$. Since these points lie outside the domain of validity of (4. 29), $A(\varkappa)$ and $B(\varkappa)$ may be computed immediately, and are

$$A(\varkappa) = -\frac{2A_0 k_R k_T^{(1)}}{i(\varkappa-k_R)(\varkappa+k_R)R(\varkappa)} \left\{ \frac{(2\varkappa^2-k_s^2)(2k_R^2-k_s^2)}{k_R} + 4\varkappa n_s(\varkappa)n_d(k_R) \right\}$$

$$+ \frac{A_0 k_T^{(1)}(2k_R^2-k_s^2)}{ik_R(\varkappa+k_R)R(\varkappa)} \left\{ -(2\varkappa^2-k_s^2) - (2k_R^2-k_s^2)\frac{\varkappa n_s(\varkappa)}{k_R n_s(k_R)} \right\}$$

$$- \frac{A_0}{i(\varkappa-k_R)R(\varkappa)} \left\{ -a_1(2\varkappa^2-k_s^2) + 2i\varkappa n_s(\varkappa)a_2 \right\}, \quad (4.30a)$$

$$B(\varkappa) = \frac{2A_0 k_R k_T^{(1)}}{i(\varkappa-k_R)(\varkappa+k_R)R(\varkappa)} \left\{ 2in_d(k_R)(2\varkappa^2-k_s^2) + 2in_d(k_R)\frac{\varkappa}{k_R}(2k_R^2-k_s^2) \right\}$$

$$+ \frac{A_0 k_T^{(1)}(2k_R^2-k_s^2)}{ik_R(\varkappa+k_R)R(\varkappa)} \left\{ 2i\varkappa n_d(\varkappa) + \frac{2ik_R n_d(k_R)}{2k_R^2-k_s^2}(2\varkappa^2-k_s^2) \right\}$$

$$- \frac{A_0}{i(\varkappa-k_R)R(\varkappa)} \left\{ 2i\varkappa n_d(\varkappa)a_1 + a_2(2\varkappa^2-k_s^2) \right\}, \quad -\delta < \text{Im}\{\varkappa\} < \delta . \quad (4.30b)$$

Equation (4.30a), for example, may now be used in the inversion
formula

$$\hat{\varphi}^{(1)}(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[A(\varkappa)e^{-n_d(\varkappa)z} - \frac{2A_0 k_R k_T^{(1)} e^{-n_d(k_R)z}}{i(\varkappa-k_R)^2(\varkappa+k_R)} \right] e^{i\varkappa x} d\varkappa, \quad (4.31)$$

which follows from (3.8) and (4.28).

The singularities of the integrand in (4.31) are branch points
at $\pm k_d$, $\pm k_s$, and poles at $\pm k_R$. The contributions due to the residues
at the poles represent, as in the previous section, the surface waves.
For $x > 0$ the contour may be closed in the upper half-plane, as shown
in Fig. 4, so in this case it is the pole at k_R which is of interest.
Four separate terms appear — three due to $A(\varkappa)$ and the fourth

arising from the last term in (4.31). The residue of each term may be computed separately. The first, third and fourth terms have double poles; the second, a simple pole.

The double poles will lead to secular terms, i.e., terms containing x or z as a factor. It will be seen, however, that none of these terms appears in the asymptotic expansion of $\underline{\varphi}^{(1)}(x, z)$ as $x \to \pm\infty$. This is necessary in order that the asymptotic expansion be valid; otherwise the representation of $\underline{\varphi}^{(1)}(x, z)$ would not be uniformly valid in x and the asymptotic expansion could not be justified. In fact, it is this very requirement which necessitated the seemingly roundabout formulation employed in this section.

From (4.18), for $x > 0$,

$$\varphi^{(1)}(x, z) = \hat{\varphi}^{(1)}(x, z) - A_t^{(1)} \varphi_{R+}(x, z)$$

$$- A_0 \varphi_{T+}^{(1)}(0, z) e^{ik_R x} - \frac{A_0 k_T^{(1)}}{2k_R} \varphi_{R+}(x, z), \quad x > 0. \quad (4.32)$$

When the four residue contributions from $\hat{\varphi}^{(1)}$ are combined with the latter three terms of (4.32), there results

$$\varphi^{(1)}(x, z) \sim (A_t^{(1)'} - A_t^{(1)}) \varphi_{R+}(x, z) \quad \text{as} \quad x \to \infty, \quad (4.33)$$

where

$$A_t^{(1)'} = \frac{A_0 R''(k_R) k_s^2}{2[R'(k_R)]^2} \left\{ -k_R^2 n_s(k_R) + \alpha [n_s(k_R) + n_d(k_R)] \right\}$$

$$+ \frac{A_0 k_s^2 (2k_R^2 - k_s^2)}{n_s(k_R) R'(k_R)} \left[k_R - \frac{2\alpha}{k_R} \right]. \quad (4.34)$$

Note that the secular terms due to the double poles have been cancelled by each other and by the third term on the right-hand side of (4.32). Now since $\varphi^{(1)}(x, z)$ must obey the radiation condition (2.5), it cannot contain any surface waves, so from (4.33) it follows that

$$A_t^{(1)} = A_t^{(1)'} . \tag{4.35}$$

The coefficient $A_r^{(1)}$ may be determined by a procedure analogous to that described above, but with the integration contour closed in the <u>lower</u> half $\varkappa$-plane. Because $A(\varkappa)$ has only a simple pole at $\varkappa = -k_R$, the computation is somewhat simpler. The final result is

$$A_r^{(1)} = \frac{A_0}{2k_R R'(-k_R)}\left\{-k_R^2 n_s(k_R)k_s^2 + \alpha k_s^2[n_s(k_R)-n_d(k_R)]\right\} . \tag{4.36}$$

When (4.16) and (4.34) - (4.36) are used in (4.4), the transmission and reflection coefficients may be computed to first order in N. The resulting approximations to the coefficients are shown in Fig. 6 for $\nu = 1/4$.

If the layer is massless, $M = 0$, $\alpha = 0$, and the results may be compared with those of the previous section. This is done in Fig. 5 for $\nu = 1/4$, and there is seen to be good agreement as $N \to 0$.
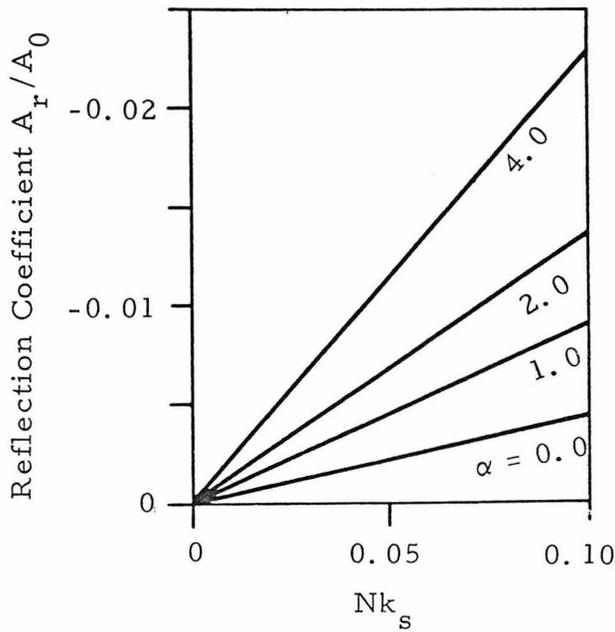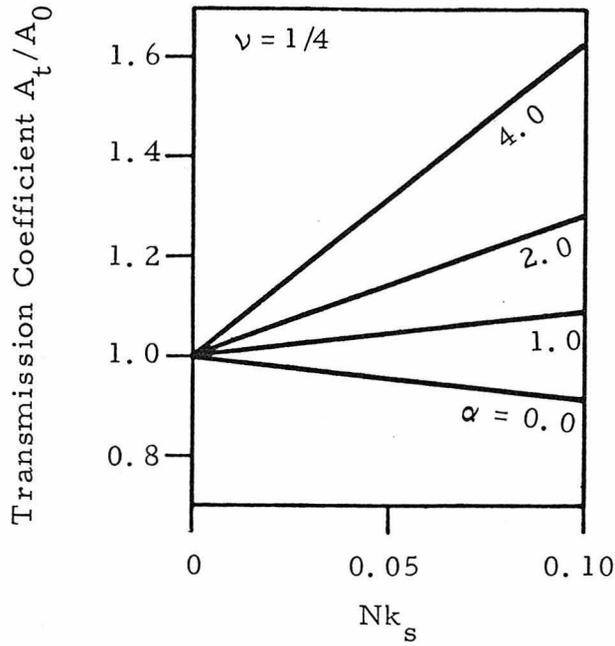
Figure 6.   Approximate Transmission and Reflection
Coefficients for a Very Thin Layer.

5. Conclusions and Comments

The problem of the scattering of a Rayleigh wave by the edge of a thin surface layer is amenable to solution by the Wiener-Hopf technique in the case where the density of the layer vanishes. The amplitudes of the reflected and transmitted surface waves computed by such an analysis are shown in Fig. 5 for $\nu = 1/4$.

A perturbation process leads to an approximate solution for the case of a layer possessing both mass and stiffness. This analysis yields the transmission and reflection coefficients shown in Fig. 6, for $\nu = 1/4$. The results of the two analyses agree closely for very thin, massless layers.

It is reasonable to ask whether the results of Section 3 might be extended to the case of a layer having mass, by treating the layer's mass parameter M as a perturbation parameter, while allowing the stiffness parameter N to remain arbitrary. Such a process leads to a sequence of problems, the zeroth order problem being precisely that solved in Section 3. This procedure has been attempted, but the computations required to obtain the first order solution were found to be of prohibitive complexity. Part of the difficulty arises from the fact that the zeroth-order solution, which is known only as a Fourier inversion integral, appears as a forcing function in the first-order problem. This necessitates the factorization, by the formal procedure used in Section 3, of quantities containing the zeroth order solution.

Appendix A - Roots of $R^*(\varkappa)$

In this section the roots in a suitably cut complex plane of the function

$$R^*(\varkappa) = (2\varkappa^2 - k_s^2)^2 - 4\varkappa^2 n_s(\varkappa) n_d(\varkappa) - N k_s^2 \varkappa^2 n_s(\varkappa) , \quad N > 0 , \quad (A1)$$

are located.

It is assumed that $\omega = 0$, so that k_s and k_d are real. The branch cuts in the complex $\varkappa$-plane are taken as shown in Fig. 7, except that the radii of the cuts in the vicinity of the origin is vanishingly small. The functions $n_s(\varkappa)$ and $n_d(\varkappa)$ are then defined by requiring that they be positive for $\varkappa > k_s$. When $n_s(\varkappa)$ and $n_d(\varkappa)$ are continued analytically throughout the cut $\varkappa$-plane, it can be seen that $R^*(\varkappa)$ is even in $\varkappa$, i.e., $R^*(\varkappa) = R^*(-\varkappa)$. Therefore it is only necessary to consider the right half-plane $\mathrm{Re}\{\varkappa\} \geq 0$.

The number of roots in the right half-plane may be determined by the principle of the argument (see pp. 271-272 of [22]). The principle may be stated as follows: Let a function be analytic inside and on a closed contour C except for at most a finite number of poles inside C. Let f have no zeros on C and at most a finite number of zeros inside C. Then

$$N_0 - N_p = \frac{1}{2\pi} \Delta_C [\arg f(\varkappa)] \qquad (A2)$$

where N_0 is the number of zeros of f inside C, a zero of order m_0 being counted m_0 times, N_p is the number of poles of f inside C, a pole of order m_p being counted m_p times, and $\Delta_C [\arg f(\varkappa)]$ is the change in the argument of $f(\varkappa)$ as C is described in a positive sense.
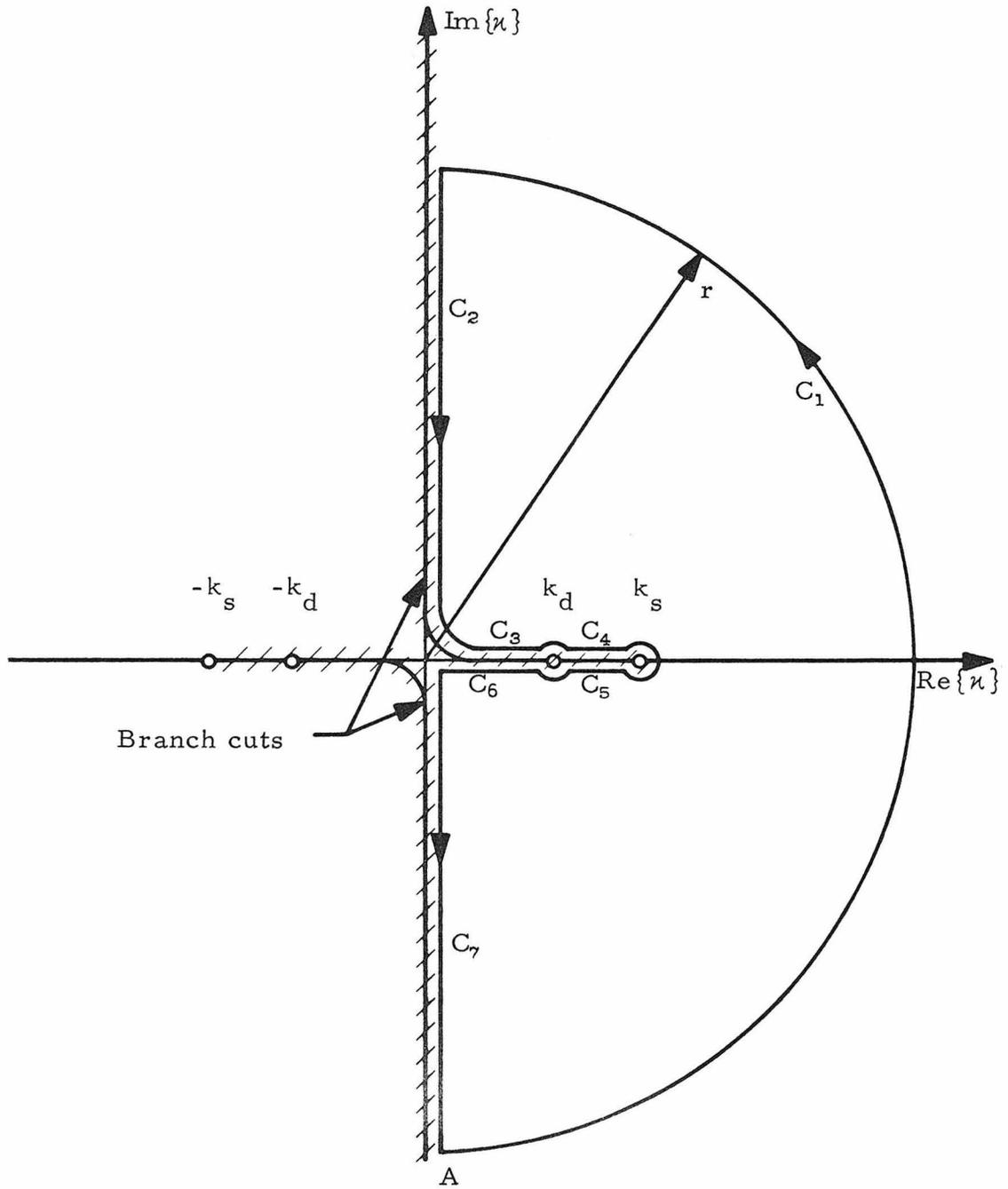
Figure 7.   Complex $\varkappa$-Plane for Determination
of Zeros of $R^*(\varkappa)$.

The curve C is taken as shown in Fig. 7. It is clear by inspection of (A1) that $R^*(\varkappa)$ has no poles inside or on C, so (A2), when applied to $R^*(\varkappa)$, will give the number of zeros of $R^*(\varkappa)$ inside C, provided that $R^*(\varkappa)$ has no zeros on C. To prove that $R^*(\varkappa) \neq 0$ on C, each segment of C must be examined separately. The appropriate values of $n_s(\varkappa)$ and $n_d(\varkappa)$ for each segment of C are determined by analytic continuation from the half-line $Im\{\varkappa\} = 0$, $Re\{\varkappa\} > k_s$. These and the corresponding formulas for $R^*(\varkappa)$ are given in Table 1 in terms of ξ and η, where $\varkappa = \xi + i\eta$. By referring to the table, the following facts are noted: On C_2 and C_7, $Im\{R^*(\varkappa)\} = 0$ only at $\eta = 0$, where $Re\{R^*(\varkappa)\} \neq 0$. On C_3 and C_6, $Im\{R^*(\varkappa)\} = 0$ only at $\xi = 0$, where $Re\{R^*(\varkappa)\} \neq 0$. On C_4 and C_5, $Re\{R^*(\varkappa)\} = 0$ only at $\xi = k_s/\sqrt{2}$, where $Im\{R^*(\varkappa)\} \neq 0$. Hence $R^*(\varkappa) \neq 0$ on C_2 - C_7.

On C_1, $n_s(\varkappa) \sim n_d(\varkappa) \sim re^{i\theta}$ as $r \to \infty$, where $\theta = Tan^{-1}(\eta/\xi)$, so from (A1),

$$R^*(\varkappa) \sim [2(re^{i\theta})^2 - k_s^2]^2 - 4(re^{i\theta})^2(re^{i\theta})^2 - Nk_s^2(re^{i\theta})^2 re^{i\theta}$$

$$\sim -Nk_s^2 r^3 e^{3i\theta} \quad \text{as} \quad r \to \infty \qquad (A3)$$

Thus $R^*(\varkappa) \neq 0$ on C, so the principle of the argument may be applied.

One way to determine $(1/2\pi)\Delta_C[argR^*(\varkappa)]$ is to plot the trajectory of $R^*(\varkappa)$ in the complex R^*-plane as $\varkappa$ describes C in the complex $\varkappa$-plane, and then simply count the net number of times the origin is encircled in a counterclockwise direction. The complex R^*-plane is shown in Fig. 8. Point A, which corresponds to $\varkappa = -ir$, is taken as a starting point. Its image A' in the R^*-plane, by either

Segment of C	Range of $\varkappa$	$n_d(\varkappa)$	$n_s(\varkappa)$	$R^*(\varkappa)$
C_2	$0+i\eta,\ \infty>\eta\geq0$	$i\sqrt{k_d^2+\eta^2}$	$i\sqrt{k_s^2+\eta^2}$	$(2\eta^2+k_s^2)^2 - 4\eta^2\sqrt{k_d^2+\eta^2}\sqrt{k_s^2+\eta^2} + iNk_s^2\eta^2\sqrt{k_s^2+\eta^2}$
C_3	$\xi+i0,\ 0\leq\xi\leq k_d$	$i\sqrt{k_d^2-\xi^2}$	$i\sqrt{k_s^2-\xi^2}$	$(2\xi^2-k_s^2)^2 + 4\xi^2\sqrt{k_d^2-\xi^2}\sqrt{k_s^2-\xi^2} - iNk_s^2\xi^2\sqrt{k_s^2-\xi^2}$
C_4	$\xi+i0,\ k_d\leq\xi\leq k_s$	$\sqrt{\xi^2-k_d^2}$	$i\sqrt{k_s^2-\xi^2}$	$(2\xi^2-k_s^2)^2 - 4i\xi^2\sqrt{\xi^2-k_d^2}\sqrt{k_s^2-\xi^2} - iNk_s^2\xi^2\sqrt{k_s^2-\xi^2}$
C_5	$\xi+i0,\ k_s\geq\xi\geq k_d$	$\sqrt{\xi^2-k_d^2}$	$i\sqrt{k_s^2-\xi^2}$	$(2\xi^2-k_s^2)^2 + 4i\xi^2\sqrt{\xi^2-k_d^2}\sqrt{k_s^2-\xi^2} + iNk_s^2\xi^2\sqrt{k_s^2-\xi^2}$
C_6	$\xi+i0,\ k_d\geq\xi\geq0$	$-i\sqrt{k_d^2-\xi^2}$	$i\sqrt{k_s^2-\xi^2}$	$(2\xi^2-k_s^2)^2 + 4\xi^2\sqrt{k_d^2-\xi^2}\sqrt{k_s^2-\xi^2} + iNk_s^2\xi^2\sqrt{k_s^2-\xi^2}$
C_7	$0+i\eta,\ 0\geq\eta>-\infty$	$-i\sqrt{k_d^2+\eta^2}$	$i\sqrt{k_s^2+\eta^2}$	$(2\eta^2+k_s^2)^2 - 4\eta^2\sqrt{k_d^2+\eta^2}\sqrt{k_s^2+\eta^2} - iNk_s^2\eta^2\sqrt{k_s^2+\eta^2}$

Table 1 – Formulas for $n_s(\varkappa)$, $n_d(\varkappa)$, and $R^*(\varkappa)$.
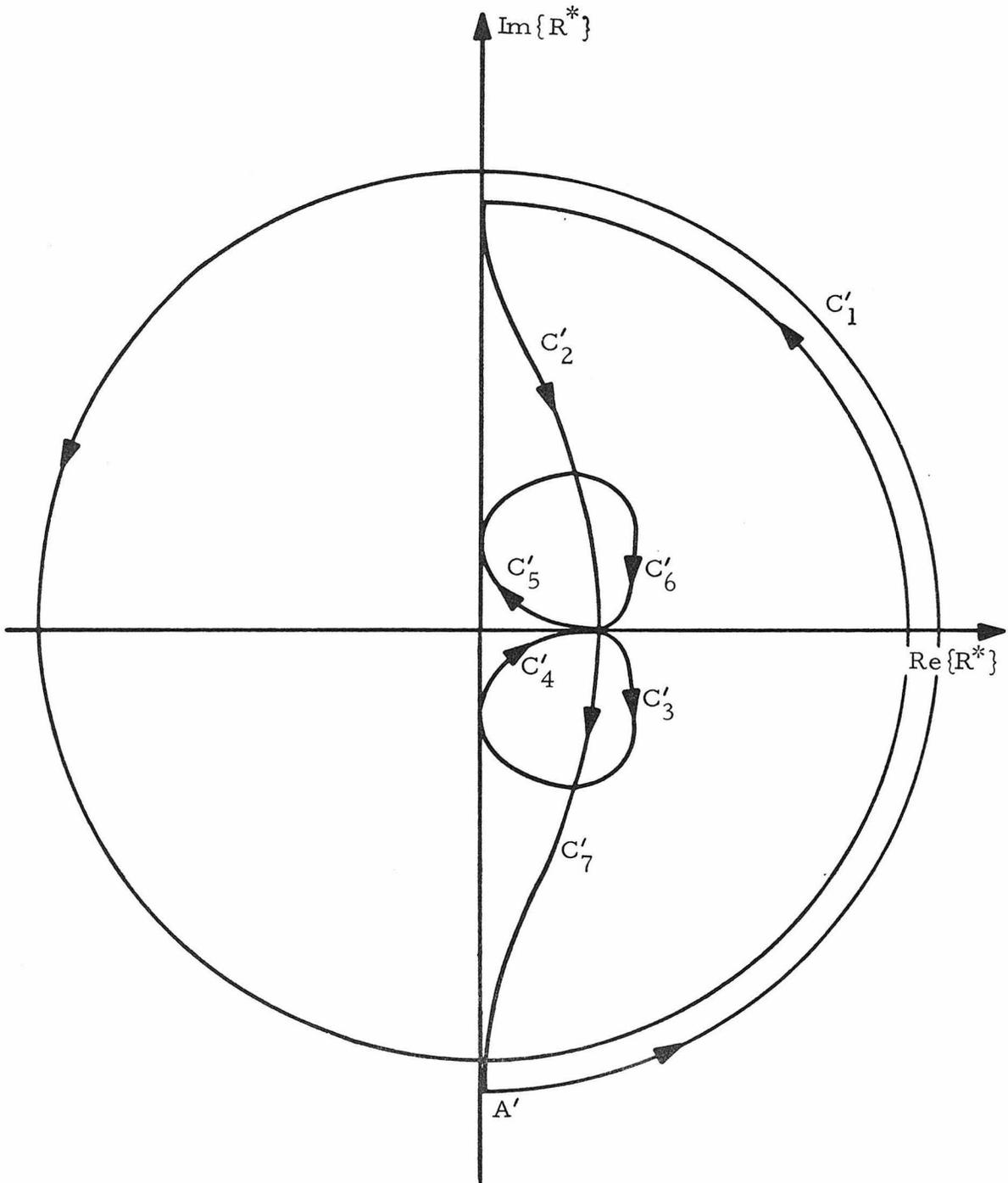
Figure 8. Complex R^*-Plane.

(A3) or Table A1, corresponds to $R^* \sim -iNk_s^2 r^3$. As C_1 is described, θ increases by π while r remains fixed, so by (A3), $\arg(R^*)$ increases by 3π while $|R^*|$ remains constant and large. As C_2 is described, $\mathrm{Re}\{R^*\}$ never vanishes since here $\mathrm{Re}\{R^*(\varkappa)\} = R(\varkappa)$, and $R(\varkappa)$ is known not to vanish on the imaginary axis (see p. 190 of [16]). Thus on C_2 $\mathrm{Re}\{R^*\} > 0$ and $\mathrm{Im}\{R^*\} \geq 0$, equality holding only at $\eta = 0$. On C_3, $\mathrm{Re}\{R^*\} > 0$ and $\mathrm{Im}\{R^*\} \leq 0$, equality holding only at $\xi = 0$. On C_4, $\mathrm{Re}\{R^*\} = 0$ only at $\xi = k_s/\sqrt{2}$, where $\mathrm{Im}\{R^*\} < 0$. Otherwise $\mathrm{Re}\{R^*\} > 0$ and $\mathrm{Im}\{R^*\} \leq 0$, equality holding only at $\xi = k_s$. The images in the R^*-plane of the remaining segments C_5, C_6, and C_7 are the complex conjugates of those of C_4, C_3, and C_2, respectively. Upon returning to the starting point A' in the R^*-plane, the origin has been encircled once in a counterclockwise direction, so by the principle of the argument, there must be precisely one root of $R^*(\varkappa)$ in the right half of the $\varkappa$-plane, and it must be simple.

To locate this root, consider $R^*(\varkappa)$ for $\varkappa = \xi$, $k_s \leq \xi \leq k_R$. Note from (A1) that $R^*(k_s) = k_s^4 > 0$, and $R^*(k_R) = -Nk_s^2 k_r^2 \sqrt{k_R^2 - k_s^2} < 0$ (since $R^*(\varkappa) = R(\varkappa) - Nk_s^2 \varkappa^2 n_s(\varkappa)$ and $R(k_R) = 0$). Hence the root, which is denoted by k_T, must lie in the interval (k_s, k_R). Since $R^*(\varkappa)$ is even in $\varkappa$, it has another root at $-k_T$.

<u>Appendix B - Factorization of</u> $D(\varkappa)$; <u>Properties and Evaluation</u> <u>of</u> $D^+(\varkappa)$ <u>and</u> $D^-(\varkappa)$.

The function $D(\varkappa)$ is defined as

$$D(\varkappa) = \frac{2(k_s^2 - k_d^2)R^*(\varkappa)}{Nk_s^2 n_s(\varkappa)R(\varkappa)}, \quad -\delta \leq \text{Im}\{\varkappa\} \leq \delta, \qquad (B1)$$

where

$$R(\varkappa) = (2\varkappa^2 - k_s^2)^2 - 4\varkappa^2 n_s(\varkappa)n_d(\varkappa),$$

$$R^*(\varkappa) = R(\varkappa) - Nk_s^2 \varkappa^2 n_s(\varkappa),$$

$$n_s(\varkappa) = \sqrt{\varkappa^2 - k_s^2}, \quad n_d(\varkappa) = \sqrt{\varkappa^2 - k_d^2},$$

and where δ is a constant such that $0 < \delta < \text{Im}\{k_d\}$. The functions $n_s(\varkappa)$ and $n_d(\varkappa)$ are rendered single-valued by taking branch cuts as shown in Fig. 4 and by requiring both to approach $+\infty$ as $\varkappa \to \infty$ along the positive real $\varkappa$-axis.

The functions $R(\varkappa)$ and $R^*(\varkappa)$ have the following asymptotic forms:

$$R(\varkappa) = 4\varkappa^4\left[\left(1 - \frac{k_s^2}{2\varkappa^2}\right)^2 - \sqrt{1 - \frac{k_s^2}{\varkappa^2}}\sqrt{1 - \frac{k_d^2}{\varkappa^2}}\right]$$

$$= 4\varkappa^4\left[1 - \frac{k_s^2}{\varkappa^2} - \left(1 - \frac{k_s^2}{2\varkappa^2}\right)\left(1 - \frac{k_d^2}{2\varkappa^2}\right) + O(|\varkappa|^{-4})\right]$$

$$= -2\varkappa^2(k_s^2 - k_d^2) + O(1)$$

$$= -2|\varkappa|^2(k_s^2 - k_d^2) + O(|\varkappa|)$$

$$R^*(\varkappa) = R(\varkappa) - Nk_s^2\varkappa^3\sqrt{1 - \frac{k_s^2}{\varkappa^2}}$$

$$= -Nk_s^2\varkappa^3 + O(|\varkappa|)$$

$$= -Nk_s^2|\varkappa|^3 + O(|\varkappa|^2) \text{ as } |\varkappa| \to \infty, \ -\delta \le \mathrm{Im}\{\varkappa\} \le \delta, \ \mathrm{Re}\{\varkappa\} > 0.$$

$$\tag{B2}$$

In (B2) use was made of the fact $\varkappa = |\varkappa| + O(1)$ as $|\varkappa| \to \infty$, $-\delta \le \mathrm{Im}\{\varkappa\} \le \delta$, $\mathrm{Re}\{\varkappa\} > 0$. Since $R(\varkappa)$ and $R^*(\varkappa)$ are even functions of $\varkappa$, the estimates (B2) also hold as $|\varkappa| \to \infty$, $-\delta \le \mathrm{Im}\{\varkappa\} \le \delta$, $\mathrm{Re}\{\varkappa\} < 0$. Now if (B2) is substituted into (B1) it is seen that

$$D(\varkappa) = 1 + O(|\varkappa|^{-1}) \text{ as } |\varkappa| \to \infty, \ -\delta \le \mathrm{Im}\{\varkappa\} \le \delta. \tag{B3}$$

The functions $D^+(\varkappa)$ and $D^-(\varkappa)$ are formally defined as

$$D^\pm(\varkappa) = \exp\left\{-\frac{1}{2\pi i}\int_{-\infty\mp i\delta}^{\infty\mp i\delta}\frac{\mathrm{Log}\,D(\alpha)}{\alpha - \varkappa}d\alpha\right\}, \ -\delta < \mathrm{Im}\{\varkappa\} < \delta, \tag{B4}$$

where, in view of (B3), $\mathrm{Log}\,D(\alpha)$ is rendered single-valued by requiring it to approach 0 as $|\alpha| \to \infty$, $-\delta \le \mathrm{Im}\{\alpha\} \le \delta$, $\mathrm{Re}\{\alpha\} > 0$. The function $D(\alpha)$ has branch points at $\pm k_s$ and $\pm k_d$, simple poles at $\pm k_R$, and simple zeros at $\pm k_T$ (see (B1)). Since none of these points lies

within the strip $-\delta \leq \text{Im}\{\alpha\} \leq \delta$, the function $\text{Log } D(\alpha)$ is analytic throughout the strip. Since $D(\alpha)$ is an even function of α, it must also be true that $\text{Log } D(\alpha) \to 0$ as $|\alpha| \to \infty$, $-\delta \leq \text{Im}\{\alpha\} \leq \delta$, $\text{Re}\{\alpha\} < 0$. In view of (B3)

$$\text{Log } D(\alpha) = O(|\alpha|^{-1}) \text{ as } |\alpha| \to \infty, \ -\delta \leq \text{Im}\{\alpha\} \leq \delta, \tag{B5}$$

so the integral appearing in (B4) is convergent.

Now by using (B4),

$$\frac{D^-(\varkappa)}{D^+(\varkappa)} = \exp\left\{-\frac{1}{2\pi i} \int_{-\infty+i\delta}^{\infty+i\delta} \frac{\text{Log } D(\alpha)}{\alpha-\varkappa} d\alpha + \frac{1}{2\pi i} \int_{-\infty-i\delta}^{\infty-i\delta} \frac{\text{Log } D(\alpha)}{\alpha-\varkappa} d\alpha\right\}$$

$$= \exp\left\{\frac{1}{2\pi i}\left(\int_{-\infty-i\delta}^{\infty-i\delta} + \int_{\infty+i\delta}^{-\infty+i\delta}\right) \frac{\text{Log } D(\alpha)}{\alpha-\varkappa} d\alpha\right\}. \tag{B6}$$

The only singularity of the integrand in (B6) between the two infinite contours is a simple pole at $\alpha = \varkappa$. The two contours may be joined by two vertical segments of length 2δ at $\text{Re}\{\alpha\} = \pm\infty$, and the residue theorem applied to the resulting closed contour to yield

$$\frac{D^-(\varkappa)}{D^+(\varkappa)} = \exp\{\text{Log } D(\varkappa)\} = D(\varkappa), \ -\delta < \text{Im}\{\varkappa\} < \delta. \tag{B7}$$

The "factorization" procedure exemplified by (B4), (B7) follows the method described by Noble (p. 15 of [21]).

A theorem concerning the domain of analyticity of a function defined by an infinite integral (see p. 110 of [24]), when applied to (B4), leads to the result that $D^+(\varkappa)$ is analytic for $\text{Im}\{\varkappa\} > -\delta$, and $D^-(\varkappa)$ is analytic for $\text{Im}\{\varkappa\} < \delta$.

The singularities of the analytic continuation of $D^+(\varkappa)$ into the lower half-plane are determined by combining (B7) and (B1) to yield

$$D^+(\varkappa) = D^-(\varkappa) \frac{Nk_s^2 n_s(\varkappa)R(\varkappa)}{2(k_s^2 - k_d^2)R^*(\varkappa)} .$$

Thus $D^+(\varkappa)$ has a simple pole at $-k_T$ and branch points at $-k_s$ and $-k_d$. Conversely,

$$D^-(\varkappa) = D^+(\varkappa) \frac{2(k_s^2 - k_d^2)R^*(\varkappa)}{Nk_s^2 n_s(\varkappa)R(\varkappa)} ,$$

so the singularities of the analytic continuation of $D^-(\varkappa)$ into the upper half-plane are a simple pole at k_R and branch points at k_s and k_d.

It is clear from (B4) that $D^+(\varkappa)$ and $D^-(\varkappa)$ can have no zeros in their respective domains of analyticity. It also follows from (B4) and the evenness of $D(\alpha)$ that

$$D^-(-\varkappa) = \exp\left\{-\frac{1}{2\pi i} \int_{-\infty+i\delta}^{\infty+i\delta} \frac{Log\, D(\alpha)}{\alpha+\varkappa} d\alpha\right\} = \exp\left\{\frac{1}{2\pi i} \int_{-\infty-i\delta}^{\infty-i\delta} \frac{Log\, D(-\alpha)}{\varkappa-\alpha} d\alpha\right\}$$

$$= \exp\left\{\frac{1}{2\pi i} \int_{-\infty-i\delta}^{\infty-i\delta} \frac{Log\, D(\alpha)}{\varkappa-\alpha} d\alpha\right\} = D^+(\varkappa) , \quad Im\{\varkappa\} > -\delta .$$

Now it is desired to show that $D^+(\varkappa) \to 1$ as $|\varkappa| \to \infty$, $Im\{\varkappa\} > -\delta$. If the integral appearing in (B4) for $D^+(\varkappa)$ is defined as $I(\varkappa)$, it is sufficient to show that $I(\varkappa) \to 0$ as $|\varkappa| \to \infty$, $Im\{\varkappa\} > -\delta$. In view of (B5) and the continuity of $Log\, D(\alpha)$, there exists a positive constant P such that $|Log\, D(\alpha)| < P|\alpha|^{-1}$, $-\delta \le Im\{\alpha\} \le \delta$. Therefore

$$|I(\varkappa)| = \left| \int\limits_{-\infty-i\delta}^{\infty-i\delta} \frac{\text{Log } D(\alpha)}{\alpha-\varkappa} d\alpha \right| = \left| \int\limits_{-\infty}^{\infty} \frac{\text{Log } D(\alpha-i\delta) d\alpha}{\alpha-i\delta-\varkappa} \right|$$

(B8)

$$\leq P \int\limits_{-\infty}^{\infty} \frac{d\alpha}{|\alpha-i\delta|\,|\alpha-i\delta-\varkappa|} = \frac{P}{|\varkappa|} \int\limits_{-\infty}^{\infty} \frac{d\alpha}{\sqrt{\alpha^2+(\frac{\delta}{|\varkappa|})^2}\sqrt{(\alpha-\cos\theta)^2+(\frac{\delta}{|\varkappa|}+\sin\theta)^2}}$$

where $\theta = \arg(\varkappa)$. It can be shown that the integral appearing in (B8) is $O(\text{Log}|\varkappa|)$ as $|\varkappa| \to \infty$, $\text{Im}\{\varkappa\} > -\delta$, so $I(\varkappa) = O(|\varkappa|^{-1}\text{Log}\,|\varkappa|) \to 0$ and $D^+(\varkappa) \to 1$ as $|\varkappa| \to \infty$, $\text{Im}\{\varkappa\} > -\delta$.

The quantities $D^+(k_R)$ and $D^+(k_T)$ must be evaluated numerically in order to compute the reflection and transmission coefficients. The integral $I(\varkappa)$ appearing in (B4) for $D^+(\varkappa)$ may be evaluated by choosing branch cuts as shown in Fig. 9, closing the contour in the lower half-plane, and allowing $\text{Im}\{w\}$ to vanish. Note that the zeros of $R(\alpha)$ and $R^*(\alpha)$ give rise to logarithmic branch points at $\alpha = \pm k_R$, $\pm k_T$. The integrals along the large quarter-circular arcs vanish as $R \to \infty$, leaving only the integral along the branch-line contour. This contour is subdivided into ten segments numbered sequentially (see Fig. 9). The integral along each segment may be combined conveniently with that from the corresponding segment immediately across the branch cut, yielding

$$I(\varkappa) \equiv \int\limits_{-\infty-i\delta}^{\infty-i\delta} \frac{\text{Log } D(\alpha)}{\alpha-\varkappa} d\alpha = I_{1+10} + I_{2+9} + I_{3+8} + I_{4+7} + I_{5+6}.$$ (B9)

(The contribution from the small semi-circles about the singularities vanishes). By using the appropriate complex values of the square roots and the logarithm, the following formulas may be derived:
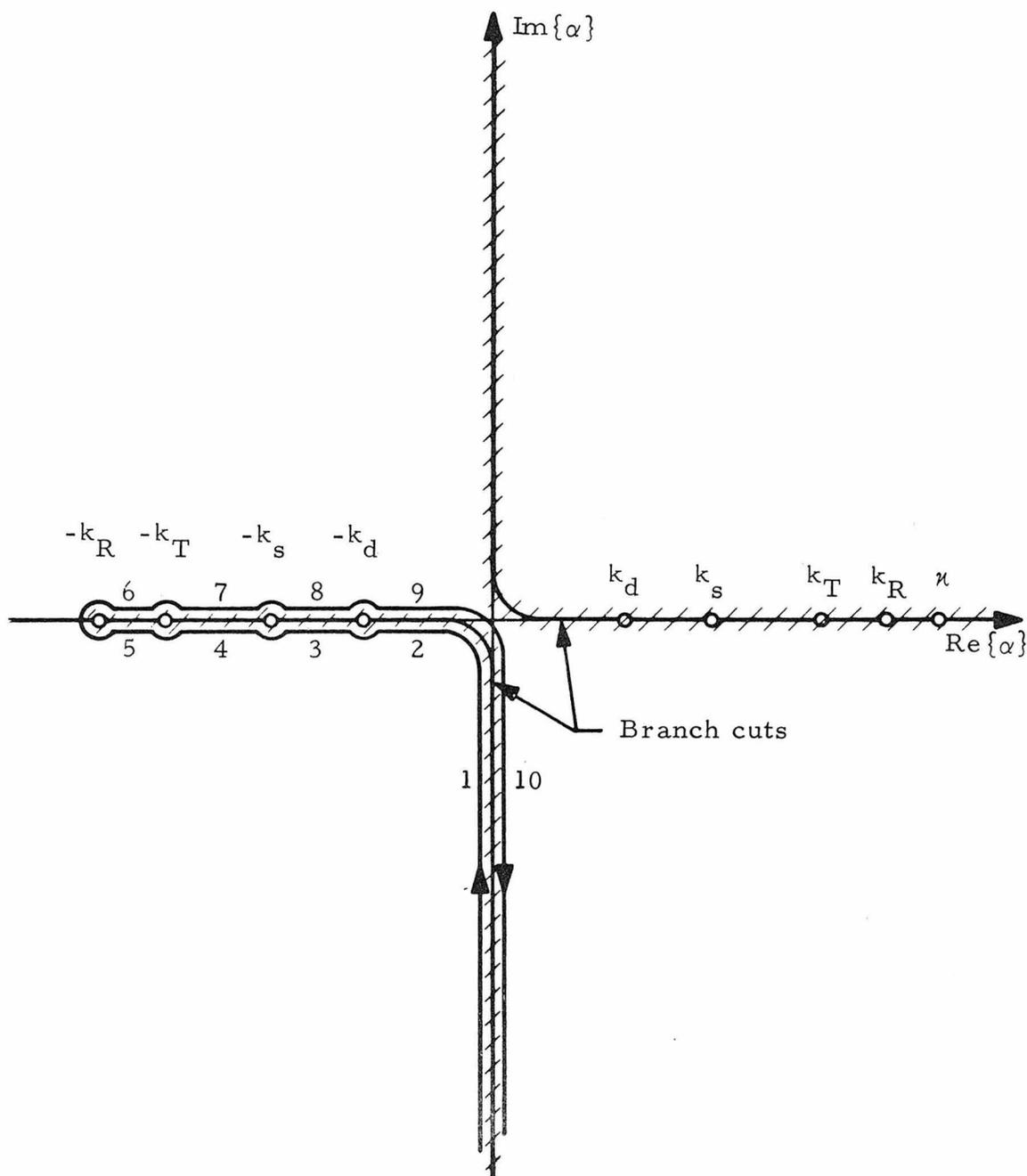
Figure 9.   Complex α-Plane for Evaluation of $D^+(\varkappa)$.

$$I_{1+10} = 2i \int_0^\infty \mathrm{Tan}^{-1}\left[\frac{(2\eta^2+1)^2 - 4\eta^2\sqrt{\eta^2+1}\sqrt{\eta^2+\gamma^2}}{Nk_s\eta^2\sqrt{\eta^2+1}}\right]\frac{d\eta}{\eta-i\varkappa/k_s}, \qquad (B10a)$$

$$I_{2+9} = 2i \int_0^\gamma \left\{\mathrm{Tan}^{-1}\left[\frac{(2\xi^2-1)^2+4\xi^2\sqrt{1-\xi^2}\sqrt{\gamma^2-\xi^2}}{Nk_s\xi^2\sqrt{1-\xi^2}}\right]-\pi\right\}\frac{d\xi}{\xi+\varkappa/k_s}, \qquad (B10b)$$

$$I_{3+8} = 2i \int_\gamma^1 \left\{\mathrm{Tan}^{-1}\left[\frac{\mathrm{Im}\{E(\xi)\}}{\mathrm{Re}\{E(\xi)\}}\right]-\pi\right\}\frac{d\xi}{\xi+\varkappa/k_s} \qquad (B10c)$$

where

$$E(\xi) = \frac{-Nk_s\xi^2\sqrt{1-\xi^2} - 4\xi^2\sqrt{1-\xi^2}\sqrt{\xi^2-\gamma^2} - i(2\xi^2-1)^2}{(2\xi^2-1)^2 - 4i\xi^2\sqrt{1-\xi^2}\sqrt{\xi^2-\gamma^2}},$$

$$I_{4+7} = 0, \qquad (B10d)$$

$$I_{5+6} = -2\pi i\, \mathrm{Log}\!\left(\frac{k_R+\varkappa}{k_T+\varkappa}\right), \qquad (B10e)$$

where $\gamma = k_d/k_s$ and where $-\pi/2 \le \mathrm{Tan}^{-1}(u) \le \pi/2$, $-\infty < u < \infty$. The integrals (B10a - e) have been evaluated numerically with $\varkappa = k_R$ or $\varkappa = k_T$, then the results used in (B9) and (B4) to give the numerical values of $D^+(k_R)$ and $D^+(k_T)$. Note that since γ depends only on Poisson's ratio ν, $D^+(k_R)$ and $D^+(k_T)$ depend only on ν and Nk_s.

The asymptotic form of $D^+(k_R)$ or $D^+(k_T)$ as $N \to 0$ is obtained by asymptotically evaluating (B10a - e) as $N \to 0$. The results of this lengthy analysis are

$$I_{1+10} = \pi i \text{Log} \left[\frac{2(1-\gamma^2)}{N\varkappa} \right] - \frac{\pi^2}{2} + O(N \text{ Log } N),$$

$$I_{2+9} = -\pi i \text{Log} \left[\frac{\gamma + \varkappa/k_s}{\varkappa/k_s} \right] + o(1),$$

$$I_{3+8} = -\pi i \text{Log} \left[\frac{1+\varkappa/k_s}{\gamma + \varkappa/k_s} \right] + o(1),$$

$$I_{5+6} = O(N) \text{ as } N \to 0, \ k_d < \varkappa < \infty .$$

(B11)

When combined, the estimates (B11) yield

$$I(\varkappa) = \pi i \text{Log} \left[\frac{2(1-\gamma^2)}{Nk_s(1+\varkappa/k_s)} \right] - \frac{\pi^2}{2} + o(1) \text{ as } N \to 0 .$$

This is substituted into (B4) to yield

$$D^+(\varkappa) \sim \sqrt{\frac{N(k_s+\varkappa)}{2(1-\gamma^2)}} \, e^{-i\frac{\pi}{4}} \text{ as } N \to 0, \ k_s < \varkappa < \infty .$$

Appendix C - Certain Aspects of the Solution in the

Case of a Massless Layer.

It will now be proved that the formal solution (3.32) satisfies the differential equation (3.4) for $-\infty < x < \infty$, $z > 0$. The steps will be carried out for $\hat{\varphi}$; the procedure for $\hat{\psi}$ is completely analogous.

Appeal is made to the following theorem (see p. 59 of [24]):

Suppose

(a) there exists an $\epsilon > 0$ such that $f(x, \varkappa)$ and $\partial f / \partial x$ are continuous for $x_o - \epsilon < x < x_o + \epsilon$, $-K \leq \varkappa < K$, for all $K > 0$,

(b) $\displaystyle\int_{-\infty}^{\infty} f(x, \varkappa) d\varkappa$ is convergent for $x_o - \epsilon < x < x_o + \epsilon$, and

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} d\varkappa$ is uniformly convergent for $x_o - \epsilon < x < x_o + \epsilon$. Then

$$\frac{d}{dx} \int_{-\infty}^{\infty} f(x, \varkappa) d\varkappa = \int_{-\infty}^{\infty} \frac{\partial f}{\partial x} d\varkappa \quad \text{at } x = x_o.$$

Let $z = z_o > 0$ and hold it fixed. Apply the above theorem to the function

$$f(x, \varkappa) = A(\varkappa) e^{-n_d(\varkappa) z_o + i \varkappa x} \tag{C1}$$

By (3.31), (a) is satisfied for any $\epsilon > 0$. From (3.31) and (C1), $f(x, \varkappa) = O(|\varkappa|^{-\frac{3}{2}} e^{-|\varkappa| z_o})$ uniformly in x, and $f(x, \varkappa)$ is bounded for $-\infty < \varkappa < \infty$, so (b) is satisfied. To verify (c), note that $\partial f / \partial x = O(|\varkappa|^{-\frac{1}{2}} e^{-|\varkappa| z_o})$ uniformly in x and that $\partial f / \partial x$ is bounded for $-\infty < \varkappa < \infty$, so there exists a constant P such that

$$\left| \int_{-\infty}^{\infty} \frac{\partial f}{\partial x} d\varkappa \right| \leq P \int_{-\infty}^{\infty} |\varkappa|^{-\frac{1}{2}} e^{-|\varkappa| z_o} d\varkappa < \infty , \quad x_o - \epsilon < x < x_o + \epsilon . \tag{C2}$$

Thus by the M-test the integral is uniformly convergent and (c) is satisfied. Hence the theorem holds, and the differentiation with respect to x may be carried inside the integral for $-\infty < x < \infty$, $z > 0$. Due to the exponential factor in the integrand, the above process may be repeated for any number of subsequent derivatives with respect to x in the interior of the half-space. A similar procedure may be applied for derivatives with respect to z and for cross-derivatives. Thus derivatives of $\hat{\varphi}(x, z)$ of all orders exist in the interior of the half-space and may be computed by formal differention inside the integrals.

It may now be immediately verified that the differential equation (3.4) is satisfied by taking derivatives inside the integrals in (3.32).

To prove that the derivatives of $\hat{\varphi}(x, z)$ are continuous, use is made of the following theorem (see p. 54 of [25]):

If a function of g(x, z) has uniformly bounded partial derivatives in a region R, then g(x, y) is continuous in R.

To apply this to $\hat{\varphi}(x, z)$, consider $\hat{\varphi}(x, z)$ in any closed, bounded subdomain of the half-space $-\infty < x < \infty$, $z > 0$. The estimate (C2) proves that $\partial\hat{\varphi}/\partial x$ is uniformly bounded, since the constant P must be independent of z_o (see (C1)). Correspondingly, $\partial\hat{\varphi}/\partial z$ is uniformly bounded, so the conditions of the theorem are met on any closed, bounded subdomain of the half-space.

The above argument proves that $\hat{\varphi}(x, z)$ is continuous; it may be repeated to prove that all derivatives of $\hat{\varphi}(x, z)$ are likewise continuous in the interior of the half-space.

The procedure for verifying the boundary condition will be illustrated for the condition

$$\sigma_{zz}(x, 0) = 0 , \quad x < 0 ; \tag{C3}$$

the procedure is analogous for the remaining boundary conditions. In light of the foregoing comments regarding interchange of differentiation and integration in (3.32), the formula for $\sigma_{zz}(x, z)$ in the interior of the half-space may be derived from (1.4), (3.3), (3.32) and (3.31) as

$$\sigma_{zz}(x, z) = \int_{-\infty}^{\infty} E(\varkappa)2i\varkappa n_s(\varkappa)(2\varkappa^2 - k_s^2)(e^{-n_s(\varkappa)z} - e^{-n_d(\varkappa)z})e^{i\varkappa x}d\varkappa, \quad -\infty < x < \infty, \ z > 0 \tag{C4}$$

where

$$E(\varkappa) = \frac{\mu A_0 k_R (k_s^2 - k_d^2) D^+(k_R)}{2\pi n_s^+(k_R) n_s^-(\varkappa)(\varkappa - k_R) R(\varkappa) D^-(\varkappa)} . \tag{C5}$$

It is clear that $\sigma_{zz}(x, z)$ as given in (C4) vanishes <u>at</u> $z = 0$, but it must be proved that $\sigma_{zz}(x, z)$ is <u>continuous up to</u> $z = 0$. Fix $x = x_o < 0$ and close the contour in the lower half $\varkappa$-plane, giving

$$\sigma_{zz}(x_o, z) = \int_{B_-} E(\varkappa)2i\varkappa n_s(\varkappa)(2\varkappa^2 - k_s^2)(e^{-n_s(\varkappa)z} - e^{-n_d(\varkappa)z})e^{i\varkappa x_o}d\varkappa$$

$$+ 2\pi i[\text{Residue at } -k_R], \quad x_o < 0, \ z > 0 , \tag{C6}$$

where B_- is the branch-line contour in the lower half $\varkappa$-plane (see Fig. 4). The residue term will be continuous by inspection so it remains to show that the integral in (C6) is continuous. The contributions to this integral from all segments of B_- except the ones along the negative imaginary axis represent continuous functions of

z since they are integrals along finite segments of a function continuous in both z and $\varkappa$. To prove continuity of the contributions from the infinite segments, first change the variable to $\varkappa = -i\eta$, leaving two integrals in which η runs from 0 to ∞. The integrand in each will be $O(\eta^{\frac{1}{2}}e^{\eta x}o)$ uniformly in z as $\eta \to \infty$, so the integrals can easily be shown by the M-test to be uniformly convergent with respect to z for $0 < z < z_o$, for any $z_o > 0$. Now apply the following theorem (see pp. 25-26 of [24]):

If $f(\eta, z)$ is continuous for $0 \le \eta \le K$, $0 \le z \le z_o$ for all $K > 0$, and the integral

$$F(z) = \int_0^\infty f(\eta, z) d\eta$$

converges uniformly with respect to z for $0 < z < z_o$, then $F(z)$ is a continuous function of z in this interval, the continuity being one-sided at the end-points.

By letting $f(\eta, z)$ equal either of the two integrands, $F(z)$ then represents the contribution to $\sigma_{zz}(x_o, z)$ from the respective segment of B_-. The conditions of the theorem are met so these contributions are continuous up to $z = 0$. Thus $\sigma_{zz}(x, z)$ is continuous up to $z = 0$ for $x < 0$, the limit as $z \to 0$ may be taken inside the integral in (C4), and the boundary condition (C3) is satisfied.

The functions $G^+(\varkappa)$ and $F^-(\varkappa)$ are defined by (3.26) and their asymptotic forms are obtained in (3.27), provided that order conditions (3.2) hold as $x \to 0$ along the surface. It will now be shown that conditions (3.2) hold for the solution given by (3.3) and (3.32).

When (3.3) and (3.32) are substituted into (1.4) and the integration contour closed by large arcs in the upper half plane, there results

$$\sigma_{xz}(x, 0) = 2\pi i [\Sigma \text{Residues}] + \int_{B_+} C(\varkappa) e^{i\varkappa x} d\varkappa,$$

(C7)

$$\frac{\partial \sigma_{xz}}{\partial x}\bigg|_{z=0} = 2\pi i [\Sigma \text{Residues}] + \int_{B_+} i\varkappa C(\varkappa) e^{i\varkappa x} d\varkappa,$$

where $C(\varkappa) \sim P |\varkappa|^{\frac{3}{2}}$ as $|\varkappa| \to \infty$ along B_+, P is a constant, and the integrands are continuous along B_+. The residue terms and the contributions to the integrals from all segments of B_+ except those along the positive imaginary axis represent functions continuous up to $x = 0$. If the change of variable $\varkappa = i\eta$ is made in the portions of the branch line integrals along the imaginary axis, they take the forms

$$\int_{B_+} C(\varkappa) e^{i\varkappa x} d\varkappa = \int_0^{\infty} C^*(\eta) e^{-\eta x} d\eta + O(1) \text{ as } x \to 0, \qquad \text{(C8a)}$$

$$\int_{B_+} i\varkappa C(\varkappa) e^{i\varkappa x} d\varkappa = -\int_0^{\infty} \eta C^*(\eta) e^{-\eta x} d\eta + O(1) \text{ as } x \to 0, \qquad \text{(C8b)}$$

where $C^*(\eta) \sim P^* \eta^{-\frac{3}{2}}$ as $\eta \to \infty$. The integral in (C8a) is clearly bounded as $x \to 0$. To analyze the integral in (C8b), the following extended Abel limit theorem may be used (see pp. 288-289 of [26]):

If $\gamma > -1$ and $f(t)/At^{\gamma} \to 1$ as $t \to \infty$, then

$$\frac{s^{1+\gamma} \int_0^{\infty} f(t) e^{-st} dt}{A\Gamma(\gamma+1)} \to 1 \text{ as } s \to 0^+.$$

When this is applied to (C8b) and the result substituted into (C7), it follows that $\partial\sigma_{xz}/\partial x\big|_{z=0} = O(x^{-\frac{1}{2}})$ as $x \to 0^+$.

An analogous procedure may be used to verify that

$$\sigma_{xz}(x, 0) + \mu N \partial^2 u/\partial x^2 \big|_{z=0} = O(1) \text{ as } x \to 0^-.$$

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