A Comparison of p-adic Motivic Cohomology and Rigid Cohomology

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ABSTRACT

We study two conjectures introduced by Flach and Morin in [FM18] for schemes over a perfect field of characteristic $p > 0$. The first conjecture relates a p-adic extension of the étale motivic cohomology with compact support on general schemes introduced by Geisser in [Gei06] to rigid cohomology with compact support, and is proved here. The second, relates a p-adic Borel-Moore motivic homology with the dual of rigid cohomology with compact support, and is proved in the smooth case. For this, we also prove a generalization of the comparison theorem from rigid cohomology to overconvergent de Rham-Witt cohomology in [DLZ11].

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Chapter 1

INTRODUCTION

1.1 Notation

k will denote a perfect field of characteristic $p > 0$. $W(k)$ will be its Witt ring, and $K := W(k) \otimes \mathbb{Q}$ will be the field of fractions. We will use $\mathcal{W} = W(k)$ when talking of the formal scheme Spf $\mathcal{W}$ rather than the scheme Spec $W(k)$.

Let Sch^d/k denote the category of separated and finite type schemes over k of dimension $\leq d$. In the case where $d = \infty$ we just use Sch/k. Let $\mathrm{FSch}/\mathcal{W}$ be separated and finite type formal schemes over Spf $\mathcal{W}$. In this thesis, we will consider all schemes and formal schemes to be separated and finite type.

In the derived category $D(\mathcal{A})$ for some abelian category $\mathcal{A}$, and a map $f : A \to B$, let

$$\left[A \xrightarrow{f} B\right] := \mathrm{Cone}(A \xrightarrow{f} B)[-1].$$

For a complex C of abelian groups, let

$$C_{\mathbb{Q}} := C \otimes_{\mathbb{Z}} \mathbb{Q}.$$

1.2 Motivation

For a variety X over a perfect field k of characteristic $p > 0$, there exist various constructions of cohomology theories with coefficients in $\mathbb{Z}_p$ or $\mathbb{Q}_p$, and with a suitable X satisfy the properties of Weil cohomologies (in the sense of [Kle68, 1.2]). For X smooth and proper, crystalline cohomology is a good cohomology theory (see [Ber74] and [BO78]), and can be computed as the hypercohomology of the de Rham-Witt complex $W\Omega^{\bullet}_{X/k}$ by [Ill79, Proposition 2.1]. This endows it with a Frobenius action

$$\phi : \sigma^* R\Gamma(X/W(k)) \to R\Gamma(X/W(k))$$

and a slope filtration on $(H^*(X/W(k)) \otimes K)$.

For a general variety X over k, Berthelot defined rigid cohomology with coefficients in the fraction field K of $W(k)$, by calculating a cohomology of a suitable subcomplex of the de Rham complex in a rigid analytic space over K

related to X. This also has a version with compact support, and has various nice properties such as existence of a Frobenius ([Ber96]), finite dimensionality of cohomology groups ([Ber97b]), and in the case of smooth schemes a Poncairé duality and Künneth formula ([Ber97a]).

On the other hand, we can consider construction on the motivic side: motivic cohomology, motivic cohomology with compact support, Borel-Moore motivic homology, and motivic homology. We can also consider the étale versions of these theories. These are well behaved on smooth quasi-projective schemes. In order to extend this to general varieties over k, [Gei06] used an analog method to Voevodsky's use of cdh topology in order to add abstract blowups to the Nisnevich topology, and considers an eh topology where he adds abstract blowups to the étale topology. Under strong resolution of singularities, this allows to extend the étale motivic cohomology theories to general schemes.

We may consider a p-adic completion of the above étale motivic cohomology and Borel-Moore homology theories. One place where these theories arise is in the study of vanishing order for zeta functions at integers n on proper regular arithmetic schemes as explained in [FM18, Chapter 5].

Based on results on proper smooth schemes over k, we expect certain relations between the p-adic completion of the étale motivic cohomology with compact supports (resp. p-adic completion of the étale Borel-Moore homology) with rigid cohomology with compact support (resp. dual of rigid cohomology with compact support), as stated below in Conjecture 1.3.1 (resp. Conjecture 1.3.2). These relations hold in the case of proper-smooth schemes as shown in [FM18, Proposition 7.21].

1.3 Main Results

Let $\mathbb{Z}(n)$ be the complex of étale sheaves on Sch/k defined in [SV00], and let $\mathbb{Z}^c(n) := z^n(-, 2n-*)$ denote the complex of étale sheaves from Bloch's higher Chow complex defined in [Blo86].

For a scheme X in Sch^d/k, under strong resolution of singularities $R(d)$ (see Definition 3.1.4), let

$$R\Gamma_c(X_{eh}, \mathbb{Q}_p(n)) := \left(R \varprojlim_r R\Gamma_c(X_{eh}, \mathbb{Z}(n)/p^r) \right)_{\mathbb{Q}}$$

as in Definition 3.3.2. Then, we expect the following relation with rigid cohomology with compact support:

Conjecture 1.3.1 (Conjecture A)**.** *Under $R(d)$, for a separated, finite type k-scheme X, and $n \in \mathbb{Z}$, there exists an isomorphism*

$$R\Gamma_c(X_{eh}, \mathbb{Q}_p(n)) \xrightarrow{\sim} \left[R\Gamma_{rig,c}(X/K) \overset{p^n-\phi}{\to} R\Gamma_{rig,c}(X/K) \right]$$

We also define a p-adic Borel-Moore homology theory:

$$R\Gamma(X, \mathbb{Q}_p^c(n)) := \left(R \varprojlim_r R\Gamma_c(X_{et}, \mathbb{Z}^c(n)/p^r) \right)_{\mathbb{Q}} .$$

We expect the following relationship with the dual of rigid cohomology with compact support:

Conjecture 1.3.2 (Conjecture B)**.** *For a separated, finite type k-scheme X of dimension d, and $n \in \mathbb{Z}$, there exists an isomorphism*

$$R\Gamma(X_{et}, \mathbb{Q}_p^c(n)) \xrightarrow{\sim} \left[R\Gamma_{rig,c}(X/K)^* \overset{p^{n-d}-\phi}{\to} R\Gamma_{rig,c}(X/K)^* \right]$$

We prove Conjecture A in Theorem 4.0.1, and we prove Conjecture B in the case where X is smooth over k in Theorem 6.1.1.

The proof of Conjecture B in the smooth case uses a generalization of one of the main results in [DLZ11]:

Theorem 1.3.3. *[DLZ11, Theorem 4.40] Let X be a smooth quasi-projective scheme over k. Then we have a natural quasi-isomorphism*

$$R\Gamma_{rig}(X/K) \xrightarrow{\sim} R\Gamma(X, W^\dagger\Omega^\bullet_{X/k}) \otimes \mathbb{Q}.$$

We generalize this result in order to drop the quasi-projectiveness condition in Theorem 5.5.5 by use of simplicial and cohomological descent methods.

1.4 Outline

In Chapter 2, we introduce the necessary background. Mainly the cohomological descent and simplicial techniques from [Con03], and different p-adic cohomologies and their relations. In particular, we summarize some of the notation and main results from the rigid cohomology version of [CT03], which will allow us to use simplicial methods on rigid cohomology.

In Chapter 3, we explain the construction of the eh-site and extension of étale motivic cohomology to singular varieties done in [Gei06].

In Chapter 4, we prove Conjecture A. In order to do so, for a given scheme X, we first form (under assumption of strong resolution of singularities) a hypercovering in the eh site by smooth schemes, which is also a proper hypercovering. This will allow cohomological descent on the motivic side, and on the rigid side. Doing this functorially, and showing independence of choices will allow to prove Conjecture A.

In Chapter 5, we prove the generalization of [DLZ11, Theorem 4.40] to smooth schemes. In order to transfer their machinery, we find a hypercovering of a given smooth scheme by affine standard smooth schemes, and use some of their results and a vanishing result to generalize the methods.

In Chapter 6, we use the result from Chapter 5 and Poincaré duality on rigid cohomology to prove Conjecture B for smooth schemes.

Chapter 2

BACKGROUND

2.1 Cohomological Descent

Simplicial Objects

We summarize some of the results and notation from [Con03]. Let $\mathcal{C}$ be a category admitting finite inverse limits.

- We denote by Δ^+ the category of objects $[n] = \{0, ..., n\}$ for $n \geq -1$, with morphisms given by non-decreasing maps of ordered sets $[n] \to [m]$.
- We denote by Δ the full subcategory of objects $[n]$ with $n \geq 0$.
- We denote by $\Delta^+_{\leq N}$ the full subcategory of Δ^+ of objects $[n]$ with $-1 \leq n \leq N$.
- We denote by $\Delta_{\leq N}$ the full subcategory of Δ of objects $[n]$ with $0 \leq n \leq N$.

Then, we consider the following:

- $\mathrm{Simp}(\mathcal{C})$ is the category of *simplicial objects in* $\mathcal{C}$. That is, contravariant functors $X_\bullet : \Delta \to \mathcal{C}$, where $X_n = X_\bullet([n])$.
- $\mathrm{Simp}^+(\mathcal{C})$ is the category of *augmented simplicial objects in* $\mathcal{C}$. That is, contravariant functors $X_\bullet/S : \Delta \to \mathcal{C}$, where S denotes the image of $[-1]$.
- $\mathrm{Simp}_N(\mathcal{C})$ is the category of *N-truncated simplicial objects in* $\mathcal{C}$. That is, contravariant functors $X_{\bullet \leq N} : \Delta_{\leq N} \to \mathcal{C}$.
- $\mathrm{Simp}^+_N(\mathcal{C})$ is the category of *N-truncated augmented simplicial objects in* $\mathcal{C}$. That is, contravariant functors $X_{\bullet \leq N}/S : \Delta^+_{\leq N} \to \mathcal{C}$.

Let $\mathrm{sk}_N : \mathrm{Simp}(\mathcal{C}) \to \mathrm{Simp}_N(\mathcal{C})$ and $\mathrm{sk}^+_N : \mathrm{Simp}^+(\mathcal{C}) \to \mathrm{Simp}^+_N(\mathcal{C})$ denote the N-skeleton functor

$$\mathrm{sk}_N(X_\bullet) = X_{\bullet \leq N}.$$

Since $\mathcal{C}$ is taken to have finite inverse limits, we have the following:

Theorem 2.1.1. *[Con03, Theorem 3.9] For any $N \geq 0$, sk_N admits a right adjoint $cosk_N : Simp_N(\mathcal{C}) \to Simp(\mathcal{C})$. Similarly for augmented objects and $N \geq -1$.*

All of the above may be generalized to multisimplicial objects (see [Con03, Definition 3.13]).

Hypercovers

Definition 2.1.2. Let $\mathbf{P}$ be a class of morphisms in $\mathcal{C}$ which is stable under base change, preserved under composition and containing all isomorphisms. A simplicial object $X_\bullet$ in $\mathcal{C}$ is said to be a $\mathbf{P}$-*hypercovering* if, for all $n \geq 0$, the natural adjunction map

$$X_\bullet \to \mathrm{cosk}_n(\mathrm{sk}_n(X_\bullet))$$

induces a map

$$X_{n+1} \to \mathrm{cosk}_n(\mathrm{sk}_n(X_\bullet))_{n+1}$$

in degree $n+1$ which is in $\mathbf{P}$.

Two common examples will be when $\mathcal{C}$ is some category of spaces (*e.g.* schemes), when $\mathbf{P}$ is the class of proper surjective maps, in which case we will talk of *proper hypercoverings*; and when $\mathbf{P}$ is the class of étale surjective maps, in which case we will call them *étale hypercoverings.*

In order to construct hypercoverings, we introduce the notion of split simplicial objects:

Definition 2.1.3. We say that a simplicial object $X_\bullet$ is *split* if there exist subobjects NX_j in each X_j such that the natural map

$$\coprod_{\phi:[n]\twoheadrightarrow[m]} NX_\phi \to X_n$$

is an isomorphism for every $n \geq 0$, where $NX_\phi := NX_m$ for a surjection $\phi : [n] \twoheadrightarrow [m]$, and the natural maps are given by the composition

$$NX_\phi \subset X_m \overset{X_\bullet(\phi)}{\to} X_n.$$

We define truncated and augmented cases similarly.

We denote by $NX_{m,\phi}$ the image of $NX_\phi \subset X_m$ under this isomorphism. Note that for any epimorphism $\phi : [n] \twoheadrightarrow [m]$ we have a commutative map

$$\begin{tikzcd}
 & NX_{m,\mathrm{id}_{[m]}} \arrow[d, "\sim"] & \subset & X_m \arrow[d, "X_\bullet(\phi)"] \\
NX_m \arrow[ur, "\sim"] \arrow[r, "\sim"'] & NX_{n,\phi} & \subset & X_n
\end{tikzcd}.$$

By [Con03, Theorem 4.12], given any split n-truncated augmented simplicial scheme $X_{\bullet\leq n}/S$ with the splitting given by $\{NX_k\}_{0\leq k\leq n}$, in order to extend it to a split $(n+1)$-truncated scheme $X_{\bullet\leq n+1}/S$ it suffices to give an object NX_{n+1} and a morphism

$$\beta : NX_{n+1} \to \mathrm{cosk}_n^S(X_{\bullet\leq n})_{n+1}.$$

Following the notation from [CT03, Section 11.2], we denote the corresponding $n+1$ augmented simplicial object above by

$$\Omega^S_{n+1}(X_{\bullet\leq n}, NX_0, ..., NX_{n+1}) \in \mathrm{Simp}^+_{\leq n+1}(\mathcal{C}).$$

This construction can be done similarly for the non-augmented case.

Remark. Note that if we construct stepwise a split object $X_\bullet$ using the above, by choosing β to be in $\mathbf{P}$ for every n we can form a $\mathbf{P}$-hypercovering.

Cohomological Descent

Consider $\mathcal{C}$ to be some category of spaces, and $X_\bullet$ a simplicial object in $\mathcal{C}$.

Definition 2.1.4. A $\mathscr{F}_\bullet$ *sheaf of sets (resp. groups, resp. rings) on* $X_\bullet$ consists of a collection $\{\mathscr{F}_n\}$ where $\mathscr{F}_n$ is a sheaf of sets (resp. groups, resp. rings) on X_n satisfying some compatibility conditions. More explicitly, given $\phi : [n] \to [m]$, we have a map of sheaves

$$[\phi] : X(\phi)^*(\mathscr{F}_n) \to \mathscr{F}_m$$

satisfying

$$[\phi] \circ X(\phi)^*[\psi] = [\phi \circ \psi]$$

for composable ϕ, ψ.

Given an augmented simplicial object $X_\bullet/S$, let $w_\bullet : X_\bullet \to S$ denote the augmented structure. Then, we have a map of topoi

$$w = (w_\bullet^*, w_{\bullet *}) : \tilde{X}_\bullet \to \tilde{S}$$

where for a sheaf $\mathscr{G}$ on S,

$$(w_\bullet^*(\mathscr{G}))_n := w_n^* \mathscr{G}$$

and for a sheaf $\mathscr{F}_\bullet$ on $X_\bullet$,

$$(w_{\bullet *}(\mathscr{F}_\bullet)) = \ker(w_{0*}\mathscr{F}_0 \to w_{1*}\mathscr{F}_1).$$

Similarly we can do this for abelian sheaves, rings and modules over some ring. We can demonstrate that there are enough injectives and thus we obtain functors

$$w^* : D_+(S) \to D_+(X_\bullet),\ Rw_* : D_+(X_\bullet) \to D_+(S)$$

on the abelian level.

Definition 2.1.5.

- We say that $w : X_\bullet \to S$ is a *morphism of cohomological descent* if the natural transformation
$$\mathrm{id} \to Rw_* \circ w^*$$
on $D_+(S)$ is an isomorphism.
- $w : X_\bullet \to S$ is said to be *universally of cohomological descent* if for every base change $S' \to S$, the augmentation $w' : X_\bullet \times_S S' \to S'$ is of cohomological descent.

2.2 Rigid Cohomology

We use [CT03] definition of rigid cohomology in order to work without assumptions of closed embeddings into a smooth formal scheme. We summarize their main notation and results below.

Definition 2.2.1.

- A *pair of schemes* $(X, \overline{X})$ consists of an open immersion $X \hookrightarrow \overline{X}$ over k.

- A *triple of schemes* $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ consists of a pair $(X, \overline{X})$, and a closed immersion $\overline{X} \hookrightarrow \mathcal{X} \times_{\mathcal{W}} k$ for a formal $\mathcal{W}$-scheme $\mathcal{X}$ of finite type over $\mathcal{W}$. We will denote triples by their corresponding fraktur letter.

- Given a pair $(X, \overline{X})$, a $(X, \overline{X})$-*triple* $\mathfrak{Y} = (Y, \overline{Y}, \mathcal{Y})$ is given by a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \overline{Y} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[dd] \\ X \arrow[r] \arrow[d] & \overline{X} \arrow[d] & \\ k \arrow[r, equal] & k \arrow[r] & \mathcal{W} \end{tikzcd}$$

Morphisms are just pairs (and triples) of morphisms $w = (\mathring{w}, \overline{w}) : (X, \overline{X}) \to (Y, \overline{Y})$ (and $w = (\mathring{w}, \overline{w}, \hat{w}) : \mathfrak{X} \to \mathfrak{Y}$) over (k, k) (and $(k, k, \mathcal{W})$) fitting into the commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathring{w}"] & \overline{X} \arrow[d, "\overline{w}"] \\ Y \arrow[r] & \overline{Y} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d, "\mathring{w}"] & \overline{X} \arrow[r] \arrow[d, "\overline{w}"] & \mathcal{X} \arrow[d, "\hat{w}"] \\ Y \arrow[r] & \overline{Y} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

Definition 2.2.2.

- A morphism of pairs $w : (Y, \overline{Y}) \to (X, \overline{X})$ is *strict* if $Y = \overline{w}^{-1}(X)$.

- A morphism of triples $w = (\mathring{w}, \overline{w}, \hat{w}) : \mathfrak{Y} \to \mathfrak{X}$ is *strict* if $\overline{Y} = \hat{w}^{-1}(\overline{X})$ and $Y = \overline{w}^{-1}(X)$.

Definition 2.2.3. Let $\hat{w} : \mathcal{Y} \to \mathcal{X}$ be a morphism of formal schemes, and let Y be a subset of $\mathcal{Y}$. Then, $\hat{w}$ is *smooth around* Y if there exists an open formal subscheme $\mathcal{U}$ of $\mathcal{Y}$ such that $Y \subset \mathcal{U}$ and $\hat{w}_{|\mathcal{U}} : \mathcal{U} \to \mathcal{X}$ is smooth.

Definition 2.2.4. For a formal scheme $\mathcal{P}$ over $\mathcal{W}$, we have an associated rigid analytic space $\mathcal{P}_K$ over $\mathrm{Spm} K$ in the sense of Raynaud [Ray74], and a specialization morphism

$$\mathrm{sp} : \mathcal{P}_K \to \mathcal{P}.$$

Given a k-subscheme X in the special fiber $\mathcal{P}_0 := \mathcal{P} \times_{\mathcal{W}} k$, we let

$$]X[_{\mathcal{P}} := \mathrm{sp}^{-1}(X)$$

with the induced Grothendieck topology from $\mathcal{P}_K$, and call it the *tube of* X *in* $\mathcal{P}_K$.

Given a morphism of triples $w : \mathfrak{Y} \to \mathfrak{X}$, we naturally get a morphism of rigid analytic spaces

$$\tilde{w} :]\overline{Y}[_{\mathcal{Y}} \to]\overline{X}[_{\mathcal{X}}.$$

Hypercoverings

Definition 2.2.5.

(1) For a simplicial pair $(Y_\bullet, \overline{Y}_\bullet) \to (X, \overline{X})$:

- $(Y_\bullet, \overline{Y}_\bullet) \to (X, \overline{X})$ is an étale-proper hypercovering if $Y_\bullet \to X$ is an étale-hypercovering and $\overline{Y}_\bullet \to \overline{X}$ is proper (i.e. for all n, $\overline{Y}_{n+1} \to \operatorname{cosk}_n^{\overline{X}}(\overline{Y}_{\bullet \leq n})_{n+1}$ is proper, possibly non-surjective).
- $(Y_\bullet, \overline{Y}_\bullet) \to (X, \overline{X})$ is an étale-étale hypercovering if both $Y_\bullet \to X$ and $\overline{Y}_\bullet \to \overline{X}$ are étale hypercoverings, and $(Y_n, \overline{Y}_n) \to (X, \overline{X})$ is strict for all n.
- $(Y_\bullet, \overline{Y}_\bullet) \to (X, \overline{X})$ is an proper-proper hypercovering if $Y_\bullet \to X$ is a proper-hypercovering, $\overline{Y}_\bullet \to \overline{X}$ is proper, and $(Y_n, \overline{Y}_n) \to (X, \overline{X})$ is strict for all n.

(2) A simplicial triple $\mathfrak{Y}_\bullet \to \mathfrak{X}$ is a étale-proper (resp. étale-étale, resp. proper-proper) hypercovering if:

i) $(Y_\bullet, \overline{Y}_\bullet) \to (X, \overline{X})$ is an étale-proper (resp. étale-étale, resp. proper-proper) hypercovering of pairs.

ii) $\operatorname{cosk}_n^{\mathcal{X}}(\mathcal{Y}_{\bullet \leq n})_l \to \operatorname{cosk}_{n-1}^{\mathcal{X}}(\mathcal{Y}_{\bullet \leq n-1})_l$ is smooth around $\operatorname{cosk}_n^{X}(Y_{\bullet \leq n})_l$ for any n and l.

- A simplicial $(X, \overline{X})$-triple $\mathfrak{Y}_\bullet$ is a étale-proper (resp. étale-étale, resp. proper-proper) hypercovering if:

  i) $(Y_\bullet, \overline{Y}_\bullet) \to (X, \overline{X})$ is an étale-proper (resp. étale-étale, resp. proper-proper) hypercovering of pairs.

  ii) $\operatorname{cosk}_n^{\mathcal{W}}(\mathcal{Y}_{\bullet \leq n})_l \to \operatorname{cosk}_{n-1}^{\mathcal{W}}(\mathcal{Y}_{\bullet \leq n-1})_l$ is smooth around $\operatorname{cosk}_n^{X}(Y_{\bullet \leq n})_l$ for any n and l.

- We define truncated versions similarly.

Lemma 2.2.6. *For an n-truncated étale-proper (resp. étale-étale, resp. proper-proper) hypercovering $\mathfrak{Y}_{\bullet\leq n} \to \mathfrak{X}$, we have that*

$$cosk_n^{\mathfrak{X}}(\mathfrak{Y}_{\bullet\leq n}) = (cosk_n^{X}(Y_{\bullet\leq n}), cosk_n^{\overline{X}}(\overline{Y}_{\bullet\leq n}), cosk_n^{\mathcal{X}}(\mathcal{Y}_{\bullet\leq n})) \to \mathcal{X}$$

is an étale-proper (resp. étale-étale, resp. proper-proper) hypercovering.

Proof. This follows from the fact that for $0 \leq n \geq m$, by [Con03, Corollary 3.11] we have a natural isomorphism

$$\operatorname{cosk}_m(\operatorname{sk}_m(-)) \xrightarrow{\sim} \operatorname{cosk}_n(\operatorname{sk}_n(\operatorname{cosk}_m(\operatorname{sk}_m(-)))).$$

□

Overconvergence

We introduce strict neighborhoods, to deal with overconvergence in the case of non-proper schemes:

Definition 2.2.7. [Ber96, Def.1.2.1] Given a triple $(X, \overline{X}, \mathcal{X})$, a subset V of $]\overline{X}[_{\mathcal{X}}$ is called a *strict neighborhood of* $]X[_{\mathcal{X}}$ *in* $]\overline{X}[_{\mathcal{X}}$ if $\{V,]\overline{X} \setminus X[_{\mathcal{X}}\}$ is an admissible covering of $]\overline{X}[_{\mathcal{X}}$. We will simply call V a strict neighborhood if there is no possibility of confusion about $]X[_{\mathcal{X}}$ and $]\overline{X}[_{\mathcal{X}}$.

For admissible open subsets $V \subset U$ of $]\overline{X}[_{\mathcal{X}}$, denote by $j_V^U : V \to U$ the inclusion. In the case $U =]\overline{X}[_{\mathcal{X}}$, simply set $j_V := j_V^{]\overline{X}[_{\mathcal{X}}}$.

By [Ber96, Prop. 1.2.10.(i)], intersections of strict neighborhoods are still strict neighborhoods, so these form a filtered category. Therefore, given a sheaf of abelian groups $\mathscr{F}$ on a strict neighborhood U, we define *the sheaf of overconvergent sections of* $\mathscr{F}$ *on* $]\overline{X}[_{\mathcal{X}}$ *along* $]\overline{X} \setminus X[_{\mathcal{X}}$ as

$$j_U^\dagger \mathscr{F} := \varinjlim_{V \subset U} j_{V*}(j_V^U)^{-1}\mathscr{F}$$

where V runs through strict neighborhoods contained in U. We denote $j^\dagger := j^\dagger_{]\overline{X}[_{\mathcal{X}}}$ for when $U =]\overline{X}[_{\mathcal{X}}$.

If $\mathscr{F}$ is a sheaf of rings on U (resp. $\mathscr{O}$-module for $\mathscr{O}$ a sheaf of rings on U), then $j_U^\dagger \mathscr{F}$ is a sheaf of rings on $]\overline{X}[_{\mathcal{X}}$ (resp. a $j_U^\dagger \mathscr{O}$-module).

Given a morphism of triples $w : \mathfrak{Y} \to \mathfrak{X}$, consider the natural map

$$\tilde{w}^{-1}(j^\dagger \mathscr{O}_{]\overline{X}[_{\mathcal{X}}}) \to j^\dagger \mathscr{O}_{]\overline{Y}[_{\mathcal{Y}}}.$$

For a sheaf E of coherent $j^\dagger\mathscr{O}_{]\overline{X}[_\mathcal{X}}$-modules, we can define

$$w^\dagger E := \tilde{w}^{-1}E \otimes_{\tilde{w}^{-1}(j^\dagger\mathscr{O}_{]\overline{X}[_\mathcal{X}})} j^\dagger\mathscr{O}_{]\overline{Y}[_\mathcal{Y}}$$

which by [CT03, Prop. 2.10.1] gives a functor

$$w^\dagger : \mathrm{Coh}(j^\dagger\mathscr{O}_{]\overline{X}[_\mathcal{X}}) \to \mathrm{Coh}(j^\dagger\mathscr{O}_{]\overline{Y}[_\mathcal{Y}}).$$

Given a simplicial triple $\mathfrak{X}_\bullet$, we get a simplicial objects of rigid spaces

$$]\overline{X}_\bullet[_{\mathcal{X}_\bullet}$$

and we may consider sheaves of rings $\mathscr{O}_{]\overline{X}_\bullet[_{\mathcal{X}_\bullet}}$ as in Definition 2.1.4. We may further apply the $j^\dagger$ at every n to consider sheaves of rings $j^\dagger\mathscr{O}_{]\overline{X}_\bullet[_{\mathcal{X}_\bullet}}$ and sheaves of $j^\dagger\mathscr{O}_{]\overline{X}_\bullet[_{\mathcal{X}_\bullet}}$-modules. We may generalize as follows:

Definition 2.2.8. We say a sheaf $E_\bullet$ of $j^\dagger\mathscr{O}_{]\overline{X}_\bullet[_{\mathcal{X}_\bullet}}$-modules is *coherent* if

- E_n is a sheaf of coherent $j^\dagger\mathscr{O}_{]\overline{X}_n[_{\mathcal{X}_n}}$-modules for all n.
- For any $\phi : [n] \to [m]$, the map

  $$j^\dagger\mathscr{O}_{]\overline{X}_m[_{\mathcal{X}_m}} \otimes_{\tilde{\phi}^{-1}j^\dagger\mathscr{O}_{]\overline{X}_n[_{\mathcal{X}_n}}} \tilde{\phi}^{-1}E_n \to E_m$$

  is an isomorphism, where $\tilde{\phi} :]\overline{X}_m[_{\mathcal{X}_m} \to]\overline{X}_n[_{\mathcal{X}_n}$ is the map induced by ϕ.

Given an augmented simplicial triple

$$w_\bullet : \mathfrak{Y}_\bullet \to \mathfrak{X}$$

and a complex of sheaves $\mathscr{F}_\bullet^\bullet$ of $\tilde{w}_\bullet^{-1}(j^\dagger\mathscr{O}_{]\overline{X}[_\mathcal{X}})-$modules, let $\mathcal{I}_\bullet^\bullet$ be an injective resolution of $\mathscr{F}_\bullet^\bullet$ in $\tilde{w}_\bullet^{-1}(j^\dagger\mathscr{O}_{]\overline{X}[_\mathcal{X}})-$mod. Then, define $Rw_{\bullet *}\mathscr{F}_\bullet^\bullet$ (denoted by $R\mathcal{C}^\dagger(\mathfrak{X}, \mathfrak{Y}_\bullet; \mathscr{F})$ in [CT03]) to be the total complex associated to

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots \\
0 \arrow[r] & \tilde{w}_{0*}\mathcal{I}_0^1 \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & \tilde{w}_{1*}\mathcal{I}_1^1 \arrow[u] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \\
0 \arrow[r] & \tilde{w}_{0*}\mathcal{I}_0^0 \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & \tilde{w}_{1*}\mathcal{I}_1^0 \arrow[u] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & \cdots
\end{tikzcd} \tag{2.1}$$

where the vertical maps come from maps in $\mathcal{I}_p^\bullet$, and the horizontal come from the simplicial structure. Note that these are complexes of abelian sheaves on $]\overline{X}[_{\mathcal{X}}$.

The n-truncated version $w_{\bullet\leq n} : \mathfrak{Y}_{\bullet\leq n} \to \mathfrak{X}$ is defined similarly, taking the total complexes of 2.1 and setting the columns larger than n to 0.

We will be particularly interested in the case when $\mathfrak{X} = (\mathrm{Spec}\ k, \mathrm{Spec}\ k, \mathrm{Spf}\ \mathcal{W})$, in which case we will denote

$$R\Gamma(]\overline{Y}_\bullet[_{\mathcal{Y}_\bullet}, \mathscr{F}_\bullet^\bullet) = Rw_{\bullet*}\mathscr{F}_\bullet^\bullet.$$

Definition 2.2.9. With the same $w_\bullet : \mathfrak{Y}_\bullet \to \mathfrak{X}$, suppose that $\mathcal{Y}_\bullet$ is smooth over $\mathcal{X}$ around $Y_\bullet$. We say that $w_\bullet$ is *de Rham descendable* if, for any sheaf E of coherent $j^\dagger\mathscr{O}_{]\overline{X}[_{\mathcal{X}}}$-modules, the canonical homomorphism

$$E \to Rw_{\bullet*}\left(\tilde{w}_\bullet^\dagger E \otimes_{j^\dagger\mathscr{O}_{]\overline{Y}_\bullet[_{\mathcal{Y}_\bullet}}} j^\dagger\Omega^\bullet_{]\overline{Y}_\bullet[_{\mathcal{Y}_\bullet}/]\overline{X}[_{\mathcal{X}}}\right)$$

is an isomorphism in $D_+(\mathbb{Z}_{]\overline{X}[_{\mathcal{X}}})$.
We say $w_\bullet$ is universally de Rham descendable if, for every morhpism $\mathfrak{Z} \to \mathfrak{Y}$ of triples, the base change

$$\mathfrak{Y}_\bullet \times_{\mathfrak{X}} \mathfrak{Z} \to \mathfrak{Z}$$

is de Rham descendable.

2.3 Definition of rigid cohomology

Definition 2.3.1. Let $\mathfrak{Y}_\bullet$ be a simplicial $(X, \overline{X})$ triple, such that $\mathcal{Y}_n \to \mathcal{W}$ is smooth around Y_n for all n. We say $\mathfrak{Y}_\bullet$ is a *universally de Rham descendable hypercovering of* $(X, \overline{X})$ if for any $(X, \overline{X})$-triple $\mathfrak{Z}$, the base change

$$\mathfrak{Y}_\bullet \times_{(X,\overline{X},\mathcal{W})} \mathfrak{Z} \to \mathfrak{Z}$$

is de Rham descendable.

Proposition 2.3.2. *Given $(X, \overline{X})$, there always exists a universally de Rham descendable hypercovering $\mathfrak{Y}_\bullet$ of $(X, \overline{X})$. Furthermore, if $\mathfrak{Y}_\bullet$ a $(X, \overline{X})$-triple is an étale-proper (resp. étale-étale, resp. proper-proper) hypercovering, then, $\mathfrak{Y}_\bullet$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$*

Proof. The first part is [CT03, Corollary 10.1.5]. For the second part, see [CT03, Example 10.1.6.] for étale-étale and étale-proper cases, and [Tsu03, Proposition 2.2.2.] for the proper-proper case. □

Definition 2.3.3. Given a k-scheme X, consider an open immersion into a proper k-scheme $\overline{X}$; this gives a pair $(X, \overline{X})$. Let $\mathfrak{Y}_\bullet$ be any universally de Rham descendable hypercovering of $(X, \overline{X})$, then set

$$R\Gamma_{\mathrm{rig}}(X/K) := R\Gamma(]\overline{Y}_\bullet[_{\mathcal{Y}_\bullet}, j^\dagger\Omega^\bullet_{]\overline{Y}_\bullet[_{\mathcal{Y}_\bullet}}).$$

Such a $\overline{X}$ always exists by Nagata, and by [CT03], a universally de Rham descendable hypercovering always exists (Corollary 10.1.5), this definition is independent of the choice of universally de Rham descendable hypercovering $\mathfrak{Y}_\bullet$ (Proposition 10.4.3.), compactification $\overline{X}$ (Corollary 10.5.4.) and agrees with Berthelot's original definition of rigid cohomology (Theorem 10.6.1).

Note in the case that X is quasi-projective, we may find some triple $\mathfrak{X} = (X, \overline{X}, \mathcal{X})$ with $\overline{X}$ proper (in fact projective) and $\mathcal{X}$ a smooth formal $\mathcal{W}$-scheme. Then, we may take $\mathfrak{Y}_\bullet$ to be the constant triple over $\mathfrak{X}$ (that is, $\mathfrak{Y}_n = \mathfrak{X}$), and

$$R\Gamma_{\mathrm{rig}}(X/K) \cong R\Gamma(]\overline{X}[_{\mathcal{X}}, j^\dagger\Omega^\bullet_{]\overline{X}[_{\mathcal{X}}}).$$

One important result that we will use later on, is the vanishing of rigid cohomology:

Theorem 2.3.4. *[Tsu04, Theorem 6.4.1] Given a scheme X over k, there exists an integer c such that for $i > c$, $H^i_{rig}(X/K) = 0$.*

2.4 The Tsuzuki Functor

When constructing some simplicial triple $\mathfrak{Y}_\bullet$ to compute rigid cohomology, we may keep control of $(Y_\bullet, \overline{Y}_\bullet)$ using a split construction (see Definition 2.1.2). However, it proves hard to embed into some simplicial formal scheme $\mathcal{Y}_\bullet$, smooth over $\mathcal{W}$, or even to construct it one step at a time.

Noting that for an N-truncated étale-étale (resp. étale - proper, resp. proper-proper) hypercovering $\mathfrak{Y}_{\bullet \leq N}$ of $(X, \overline{X})$, that

$$\operatorname{cosk}_N^{(X,\overline{X},\mathcal{W})}(\mathfrak{Y}_{\bullet\leq N})$$

is also a étale-étale (resp. étale - proper, resp. proper-proper) hypercovering of $(X, \overline{X})$, then we see that we just need to do our construction at the N-truncated level. In fact, the construction below shows that all we need, is a closed immersion of $\overline{Y}_N$ into some smooth formal $\mathcal{W}$-scheme $\mathcal{Y}$. Doing this, we will lose control above N, but for vanishing reasons, this will not affect computations for large enough N.

We use the Tsuzuki functor introduced in [CT03, Section 11.2]. Given a category $\mathcal{C}$ with finite inverse limits, a non-negative integer N, and an object X, we construct a N-truncated simplicial object $\Gamma_N^{\mathcal{C}}(X)$ in $\mathrm{Simp}_{\leq N}(\mathcal{C})$ as follows:

Definition 2.4.1. Set

$$\Gamma_N^{\mathcal{C}}(X)_m := \prod_{\phi:[N]\to[m]} X_\phi$$

where $X_\phi = X$. To define the simplicial maps, given $\alpha : [m'] \to [m]$, we define $\Gamma_\alpha : \Gamma_N^{\mathcal{C}}(X)_m \to \Gamma_N^{\mathcal{C}}(X)_{m'}$ by

$$(c_\phi)_{\phi:[N]\to[m]} \mapsto (d_\psi)_{\psi:[N]\to[m']}$$

where $d_\psi := c_{\alpha\circ\psi}$, and the product is in $\mathcal{C}$.

Note that this is just a product of copies of the given object X. In the case of augmented simplicial objects over some S, we take the product over S.

Given any $Y_{\bullet\leq N}$ in $\mathrm{Simp}_{\leq N}(\mathcal{C})$, and a morphism $f : Y_N \to X$ in $\mathcal{C}$, we construct a morphism

$$Y_{\bullet\leq N} \to \Gamma_N^{\mathcal{C}}(X)$$

by the commutative diagram

$$\begin{tikzcd} Y_m \arrow[r] \arrow[d, "Y(\phi)"'] & \Gamma_N^{\mathcal{C}}(X)_m = \prod_{\psi:[N]\to[m]} X_\psi \arrow[d, "p_\phi"] \\ Y_N \arrow[r, "f"] & X = X_\phi \end{tikzcd}$$

for any m and $\phi : [N] \to [m]$, where p_ϕ is just the projection onto the $\phi : [N] \to [m]$ factor.

Letting $\mathcal{C}$ be the category of formal schemes over $\mathrm{Spf}(\mathcal{W})$ or of schemes over $\mathrm{Spec}(W(k))$, we have the following:

Lemma 2.4.2. *Let $\mathcal{C}$ be as above. If $f : Y_N \to X$ is a (closed) immersion, and $Y_{\bullet\leq N}$ and X are separated, then the induced morphism*

$$Y_{\bullet\leq N} \to \Gamma_N^{\mathcal{C}}(X)_{\bullet\leq N}$$

is a (closed) immersion.

Proof. We will do the case where $\mathcal{C}$ are schemes over $W(k)$, but the case of formal schemes follows identically.

For any $0 \leq m \leq N$, consider any face morphism $\delta : Y_N \to Y_m$ (with $\delta = id_{Y_N}$ if $m = N$), and a corresponding degeneracy map $\sigma : Y_m \to Y_N$ which is a section to δ. Then, we have

$$\begin{tikzcd} Y_N \arrow[r, "\delta"] \arrow[dr] & Y_m \arrow[d] \\ & W(k) \end{tikzcd}$$

where the vertical and diagonal maps are separated. This shows that δ is also separated. Then, by the commutative diagram

$$\begin{tikzcd} Y_m \arrow[r, "\sigma"] \arrow[dr, equal] & Y_N \arrow[d, "\delta"] \\ & Y_m \end{tikzcd}$$

we see that σ is a closed immersion. Finally, by the definition of the map $g_m : Y_m \to \Gamma_N^{W(k)}(X)_m$, we have a commutative diagram

$$\begin{tikzcd} Y_m \arrow[r, "g_m"] \arrow[d, "\sigma"] & \Gamma_N^{W(k)}(X)_m \arrow[r, equal] \arrow[d, "pr_\sigma"] & \prod\limits_{\phi:[N]\to[m]} X \\ Y_N \arrow[r, "f"] & X \end{tikzcd}$$

which shows that $f \circ \sigma$, and thus g_m is a (closed) immersion. $\square$

This will be useful by the following result:

Proposition 2.4.3. *Suppose $(Y_{\bullet\leq N}, \overline{Y}_{\bullet\leq N}) \to (X, \overline{X})$ is an N-truncated étale-proper (resp. étale-étale, resp. proper-proper) hypercovering. Suppose further that there exists a closed immersion $\overline{Y}_N \hookrightarrow \mathcal{Y} \times_k \mathcal{W}$ for some smooth formal $\mathcal{W}$-scheme $\mathcal{Y}$. Then,*

$$cosk_N^{(X,\overline{X},\mathcal{W})}(Y_{\bullet\leq N}, \overline{Y}_{\bullet\leq N}, \Gamma_N^{\mathcal{W}}(\mathcal{Y})) = (cosk_N^X(Y_{\bullet\leq N}), cosk_N^{\overline{X}}(\overline{Y}_{\bullet\leq N}), cosk_N^{\mathcal{W}}(\Gamma_N^{\mathcal{W}}(\mathcal{Y}))$$

is an étale-proper (resp. étale-étale, resp. proper-proper) hypercovering of $(X, \overline{X})$. In particular, $cosk_N^{(X,\overline{X},\mathcal{W})}(Y_{\bullet\leq N}, \overline{Y}_{\bullet\leq N}, \Gamma_N^{\mathcal{W}}(\mathcal{Y}))$ is a universally de Rham descendable hypercovering of $(X, \overline{X})$.

Proof. The first part follows from the proof of [CT03, Prop. 11.4.1.] (note that using $\prod\limits_{0\leq m\leq N} \Gamma^{\mathcal{W}}(\mathcal{Y}_m)$ there instead of just $\Gamma^{\mathcal{W}}(\mathcal{Y})$ does not seem necessary). The second part follows by Proposition 2.3.2. $\square$

2.5 Crystalline Cohomology

The main reference is [BO78]. Given a scheme X over k, we may consider the crystalline site $\mathrm{Cris}(X/W_n)$ with objects given by PD-thickenings $(U \hookrightarrow T, \delta)$ over W_n for Zariski opens U of X. Let $(X/W_n)_{\mathrm{cris}}$ denote its topos. The morphism and topology is explained in §5 *loc. cit.* One can also take the direct limit of the sites $(X/W_n)_{crys}$ (see §7 *loc. cit.*), and obtain a site $\mathrm{Cris}(X/\mathcal{W})$, with a corresponding topos $(X/\mathcal{W})_{\mathrm{cris}}$.

Let $u_{X/W_n} : (X/W_n)_{\mathrm{cris}} \to X_{\mathrm{Zar}}$ and $u_{X/\mathcal{W}} : (X/\mathcal{W})_{\mathrm{cris}} \to X_{\mathrm{Zar}}$ denote the morphism of topoi. Then:

Theorem 2.5.1. *[BO78, Proposition 7.22]*

$$R\Gamma(X/\mathcal{W}, \mathscr{O}_{X/\mathcal{W}}) \cong R\varprojlim_n R\Gamma(X/W_n, O_{X/W_n})$$

$$Ru_{X/\mathcal{W}*}\mathscr{O}_{X/\mathcal{W}} \cong R\varprojlim_n Ru_{X/W_n*}\mathscr{O}_{X/W_n}.$$

2.6 Witt de-Rham Cohomology

The main reference is [Ill79]. For a given smooth scheme X, we may consider the complex of étale sheaves $W_n\Omega^\bullet_{X/k}$ on X. We can consider the pro-complex $W_\bullet\Omega^\bullet_{X/k}$ as a DGA with additional maps

$$F : W_n\Omega^i_{X/k} \to W_{n-1}\Omega^i_{X/k},$$

$$V : W_n\Omega^i_{X/k} \to W_{n+1}\Omega^i_{X/k}$$

satisfying certain compatibility conditions.

We may also consider the limit

$$W\Omega^\bullet_{X/k} := \varprojlim_n W_n\Omega^\bullet_{X/k}$$

and endow it with a Frobenius endomorphism ϕ defined by $\phi = p^iF$ on $W\Omega^i_{X/k}$.

By the proof of [Ill79, Proposition 2.1], we have that the canonical map

$$W\Omega^i_{X/k} \to R\varprojlim_n W_n\Omega^i_{X/k}$$

is a quasi-isomorphism (even though the actual statement of the proposition also assumes properness, we do not need it for this result).

For X proper and smooth, we can then consider the hypercohomology

$$R\Gamma(X, W\Omega^\bullet_{X/k})$$

which is a perfect complex of $W(k)$-modules by [Ill79, Theorem 2.7].

2.7 Comparisons

Theorem 2.7.1. *If X is smooth, and there exists a closed embedding into a smooth formal scheme $\mathcal{X}$ over $\mathcal{W}$, then there exists a natural quasi-isomorphism*

$$Rsp_*\Omega^\bullet_{]X[_\mathcal{X}} \cong Ru_{X/W(k)}\mathscr{O}_{X/W(k),\mathbb{Q}} \cong W\Omega^\bullet_{X,\mathbb{Q}}$$

on X_{Zar}. Therefore, if X is smooth and projective, this induces natural quasi-isomorphisms

$$R\Gamma_{rig}(X/K) \cong R\Gamma(X/W(k))_\mathbb{Q} \cong R\Gamma(X, W\Omega^\bullet_X)_\mathbb{Q}.$$

Proof. From the proof of [Ber97b, Proposition 1.9] we have that

$$R\mathrm{sp}_*\Omega^\bullet_{]X[_\mathcal{X}} \xleftarrow{\sim} \mathrm{sp}_*\Omega^\bullet_{]X[_\mathcal{X}} \cong (\mathrm{sp}_*\mathscr{O}_{]X[_\mathcal{X}}) \otimes \Omega^\bullet_\mathcal{X}.$$

For the first isomorphism, given any open affine formal scheme $U = \mathrm{Spf}A \subset \mathcal{X}$, we see that $\mathrm{sp}^{-1}(U) \cong \mathrm{Spm}(A \otimes K)$ is affinoid and thus quasi-Stein, thus its closed subspace $\mathrm{sp}^{-1}(U)\cap]X[_\mathcal{X}$ is also quasi-Stein, and thus satisfies Kiehl's Theorem B. Therefore, $H^i(\mathrm{sp}^{-1}(U)\cap]X[_\mathcal{X}, \Omega^k_{]X[_\mathcal{X}}) = 0$ for all k and $i > 0$. Since $R^i\mathrm{sp}_*\Omega^k_{]X[_\mathcal{X}}$ is the sheaf associated to the presheaf

$$U \cap X \mapsto H^i(\mathrm{sp}^{-1}(U)\cap]X[_\mathcal{X}, \Omega^k_{]X[_\mathcal{X}})$$

this proves the vanishing of the higher cohomologies.

Then, there exists a natural morphism

$$(\mathrm{sp}_*\mathscr{O}_{]X[_\mathcal{X}}) \otimes \Omega^\bullet_\mathcal{X} \to \hat{\mathcal{P}}(\mathcal{I}) \otimes \Omega^\bullet_{\mathcal{X},\mathbb{Q}}$$

where $\mathcal{I}$ is the ideal of X in $\mathcal{X}$, $\mathcal{P}(\mathcal{I})$ is the PD-envelope of X in $\mathcal{X}$, and $\hat{\mathcal{P}}(\mathcal{I})$ its p-adic completion. This is a quasi-isomorphism when X is smooth. Note that even though properness is assumed in the statement of Berthelot's proposition, we do not require it for this quasi-isomorphism.

Next, by [BO78], we have natural quasi-isomorphisms

$$R\varprojlim_n Ru_{X/W_n(k)}\mathscr{O}_{X/W_n(k)} \overset{[\text{BO78, Th.7.22.2}]}{\cong} Ru_{X/\mathcal{W}*}\mathscr{O}_{X/\mathcal{W}} \overset{[\text{BO78, Th.7.23}]}{\cong} \hat{\mathcal{P}}(\mathcal{I})\otimes\Omega^\bullet_\mathcal{X}.$$

Finally, by [Ill79], we have natural quasi-isomorphisms

$$R\varprojlim_n Ru_{X/W_n(k)}\mathscr{O}_{X/W_n(k)} \overset{[\text{Ill79, II.Th.1.4.}]}{\cong} R\varprojlim_n W_n\Omega^\bullet_X \overset{[\text{Ill79, II.Pr.2.1}]}{\cong} W\Omega^\bullet_X$$

where we again note that even though the Proposition for the last quasi-isomorphism assumes properness, the quasi-isomorphism holds without properness.

Tensoring with $\mathbb{Q}$ these last quasi-isomorphisms we complete the proof. $\square$

Chapter 3

P-ADIC MOTIVIC COHOMOLOGY ON SINGULAR VARIETIES

We recall notation and results from [Gei06].

3.1 The eh-topology

Fix a perfect field k. For $d \in \mathbb{N} \cup \infty$, let Sch^d/k be the category of separated schemes of finite type over k of dimension $\leq d$ (and drop the d in the case $d = \infty$), and Sm^d/k the full subcategory of smooth schemes over k.

Definition 3.1.1. The *étale h-topology* (abbreviated *eh-topology*) on Sch/k, is the Grothendieck topology generated by the following coverings:

1) Étale coverings.

2) Abstract blowups $\{Z \to X, X' \to X\}$ coming from a cartesian square

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"'] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where f is a proper morphism, i a closed embedding, and f induces an isomorphism $X' - Z' \xrightarrow{\sim} X - Z$.

We state the following result (cf. [SV00, Lemma 5.8]:

Lemma 3.1.2. *[Gei06, Lemma 2.2.a] Every proper morphism $p : X' \to X$, such that for every point $x \in X$ there is a point $x \in X$ with $p(x') = x$ which induces an isomorphism on the residue fields, is an eh-covering.*

Definition 3.1.3. We call a covering as in Lemma 3.1.2 a *proper eh-covering.*

Remark. These are called proper cdh-coverings in [SV00].

Definition 3.1.4. For $d \in \mathbb{N} \cup \infty$, we say the *strong form of resolution of singularities holds* for varieties up to dimension d if the following hold:

- For every integral separated scheme $X \in \mathrm{Sch}^d/k$, there is a proper, birational map $f : Y \to X$ with $Y \in \mathrm{Sm}/k$.
- For every smooth scheme $X \in \mathrm{Sm}^d/k$ and every proper birational map $f : Y \to X$, there is a sequence of blow-ups along smooth centers $X_n \to X_{n-1} \to ... \to X_1 \to X$ such that $X_n \to X$ factors through f.

If this holds, we denote it by $R(d)$.

Remark. Some known cases of resolution of singularities:

- $R(\infty)$ when $\mathrm{char}(k) = 0$, by [Hir64].
- $R(2)$ in general, and $R(3)$ for k algebraically closed of $\mathrm{char}(k) = p > 5$, by [Abh56].

Note that $R(d)$ makes all schemes in Sch^d/k locally smooth in the eh-topology (see Lemma 4.1.3).

The inclusion of smooth schemes then induces a morphism of sites

$$\rho_d : (\mathrm{Sch}^d/k)_{eh} \to (\mathrm{Sm}^d/k)_{et}.$$

This in turn induces a morphism of topoi under $R(d)$:

Lemma 3.1.5. *[Gei06, Lemma 2.5.a] Assume $R(d)$ holds. Then the functor ρ_d induces a morphism of topoi*

$$\rho_d : (Sch^d/k)^{\sim}_{eh} \to (Sm^d/k)^{\sim}_{et}.$$

3.2 Cohomology for the eh-topology

By usual arguments, $(\mathrm{Sch}^d/k)^{\sim}_{eh}$ has enough injectives, which allows to define cohomology groups $R\Gamma(X_{eh}, \mathscr{F})$ as the right derived functor for the global section functor $\Gamma(X_{eh}, -)$. One of the advantages of eh-topology is that we can define the cohomology with compact support as follows:

Definition 3.2.1. Let $X \in \mathrm{Sch}^d/k$. Take an open embedding $j : X \to \overline{X}$ with dense image into a proper scheme, and let $i : Z \to X$ be the closed complement with reduce subscheme structure. For $\mathscr{F} \in (\mathrm{Sch}^d/k)^{\sim}_{eh}$, take an injective resolution $\mathscr{F} \to \mathscr{I}^\bullet$ in $(\mathrm{Sch}^d/k)^{\sim}_{eh}$, and define

$$R\Gamma_c(X_{eh}, \mathscr{F}) := \left[\mathscr{I}^\bullet(\overline{X}) \to \mathscr{I}^\bullet(Z)\right].$$

This is independent of the choice of $\overline{X}$ by [Gei06, Lemma 3.4], and has the expected properties, such as contravariance for proper maps, covariance for open embeddings, and long exact sequences for open-closed decompositions.

3.3 Motivic cohomology for singular varieties

Using $R(d)$, we define motivic cohomology on any scheme $X \in \mathrm{Sch}^d/k$ by considering $R\Gamma(X_{eh}, \rho_d^*\mathbb{Z}(n))$, where $\mathbb{Z}(n)$ is Suslin-Voevodsky's motivic complex [SV00, Definition 3.1] for smooth schemes. However, since we will be interested in a p-adic completion of this cohomology, we will use the identification

$$\mathbb{Z}/p^r(n) \cong W_r\Omega^n_{log}[-n]$$

on Sm/k from [GL00], Theorem 8.5, where $W_r\Omega^n_{log}$ (denoted ν_r^n there) is the subsheaf of $W_r\Omega^n$ étale locally generated by sections of the forms $\mathrm{dlog}f_1 ... \mathrm{dlog}f_n$, defined in [Ill79] II.5.7.

Under $R(d)$, by the same proof as [Gei06] Theorem 4.3 mod p^r we get that this motivic cohomology on the eh-site coincides with the usual motivic cohomology in Sm^d/k.

Theorem 3.3.1. *Assuming $R(d)$, for any $n \in \mathbb{N}$ and $r \geq 0$ we have*

$$\mathbb{Z}/p^r(n) \xrightarrow{\sim} R\rho_{d*}\rho_d^*\mathbb{Z}/p^r(n) \text{ on } Sm^d/k.$$

In particular, for any $X \in Sm^d/k$,

$$R\Gamma(X_{et}, \mathbb{Z}/p^r(n)) \cong R\Gamma(X_{eh}, \rho_d^*\mathbb{Z}/p^r(n)).$$

We then consider the p-adic completion of this cohomology:

Definition 3.3.2. Assume $R(d)$. For $X \in \mathrm{Sch}^d/k$ and $n \in \mathbb{Z}$ set

$$R\Gamma(X_{eh}, \mathbb{Z}_p(n)) := R\varprojlim_r R\Gamma(X_{eh}, \rho_d^*\mathbb{Z}(n)/p^r)$$

and

$$R\Gamma(X_{eh}, \mathbb{Q}_p(n)) := R\Gamma(X_{eh}, \mathbb{Z}_p(n))_{\mathbb{Q}},$$

and the cohomology with compact support

$$R\Gamma_c(X_{eh}, \mathbb{Z}_p(n)) := R\varprojlim_r R\Gamma_c(X_{eh}, \rho_d^*\mathbb{Z}(n)/p^r)$$

and

$$R\Gamma_c(X_{eh}, \mathbb{Q}_p(n)) := R\Gamma_c(X_{eh}, \mathbb{Z}_p(n))_{\mathbb{Q}},$$

Chapter 4

CONJECTURE A

In this chapter, we prove Conjecture A:

Theorem 4.0.1 (Conjecture A)**.** *Assume $R(d)$. Let X be in Sch/k with $\dim X \leq d$. Then, for any $n \in \mathbb{N}$,*

$$R\Gamma_c(X_{eh}, \mathbb{Q}_p(n)) \xrightarrow{\sim} \left[R\Gamma_{rig,c}(X/K) \overset{p^n-\phi}{\rightarrow} R\Gamma_{rig,c}(X/K) \right]$$

functorially in X.

In Section 4.1, we construct proper-eh hypercoverings with suitable properties for the proof of Theorem 4.0.1 in Section 4.2.

4.1 Proper eh-hypercoverings

We generalize the notion of a proper hypercovering:

Definition 4.1.1. For an augmented simplicial scheme $a : X_\bullet \to Y$, we say $X_\bullet$ is a *proper eh-hypercovering* if the natural maps

$$f_{n+1} : X_{n+1} \to (\mathrm{cosk}_n sk_n(X_\bullet))_{n+1}$$

are proper-eh coverings for all $n \geq -1$.

We prove some properties of proper eh-coverings which will be useful in the construction of proper-eh hypercoverings:

Lemma 4.1.2. *Proper eh-coverings are stable under base change, preserved under composition and contain all isomorphisms.*

Proof. The only thing that needs proving is the stability under base change. Given a proper eh-cover $p : X \to Z$, and a morphism $f : Y \to Z$, consider $X \times_Z Y$ with morphism p' and f' to X and Y respectively. Then, p' is proper. Also, given any point $y \in Y$, consider a point $x \in X$ lift $f(y) =: z \in Z$ with the same residue field. Then, since $k(x) \otimes_{k(z)} k(y) \cong k(x)$, the point $k(x)$ factors through $X \times_Z Y$. □

Lemma 4.1.3. *Assuming R(d), for every scheme Y in Sch/k of dimension $\leq d$ there exists a proper eh-covering $X \to Y$ with $X \in Sm/k$.*

Proof. Firstly, we can assume that X is integral since the reduced subscheme and disjoint union of irreducible components are both eh-coverings. We proceed by induction on d. The base case $d = 0$ is thus trivial. So we assume it holds true for dimension $< d$. Then, by R(d) we can find a proper birational map

$$Y' \to X$$

with Y' smooth, which is an isomorphism away from some proper closed subscheme Z of X, and thus $Y' \sqcup Z \to X$ is an eh-covering. But by inductive hypothesis, there is a proper eh-covering $Z' \to Z$ with Z' in Sm/k, and thus $Y := Y' \sqcup Z' \to X$ is a proper eh-covering. □

Definition 4.1.4. We say $X_\bullet \to X$ is a *peh-resolution* if it is a split proper-eh hypercovering, and for all n, X_n is smooth over k. We define the truncated version similarly.

Proposition 4.1.5. *Assuming R(d), then for every scheme X in Sch^d/k there exists a split proper eh-hypercovering $X_\bullet \to X$ with*
$X_m \in Sm/k$ for all m.

Proof. Assuming R(d), the proof is identical to the construction of proper hypercoverings (for example in [Con03, Theorem 4.13]), replacing proper surjective maps with proper-eh coverings at every step

$$NX_{n+1} \to \mathrm{cosk}_n^X(X_{\bullet\leq n})_{n+1}$$

using Lemma 4.1.3. □

Constructing peh-resolutions in this manner, we show we can always refine two given ones, and that we can construct them functorially.

Proposition 4.1.6. *Assume R(d). Then,*

i) Given two split proper eh-hypercoverings $U_\bullet, V_\bullet/X$, there exists a peh-resolution $W_\bullet/X$ and morphisms $f_\bullet : W_\bullet \to U_\bullet$, $g_\bullet : W_\bullet \to V_\bullet$ over X.

ii) Given a morphism $f : X \to Y$, and a peh-resolution $b_\bullet : V_\bullet/Y$, then there exists a peh-resolution $a : U_\bullet/X$ with a morphism $f_\bullet : U_\bullet \to V_\bullet$ making

$$\begin{array}{ccc} U_\bullet & \xrightarrow{f_\bullet} & V_\bullet \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

commute.

Proof. For i), we again proceed by proceed by constructing the n-truncation of $W_\bullet$ one step at a time. For $n = 0$, taking a proper eh-covering $NW_0 = W_0 \to U_0 \times_X V_0$ with NW_0 smooth, then this satisfies the lemma at 0-truncation.

Assume we have constructed $W_{\bullet \leq n}$ satisfying the hypothesis. Then, let $(-)'$ denote $\mathrm{cosk}_n^X \mathrm{sk}_n^X(-)$ (e.g. $U'_\bullet = \mathrm{cosk}_n^X \mathrm{sk}_n^X(U_\bullet)$), let NU_k's give the splitting, and denote the proper eh-coverings used to construct the $n+1$ step by β_- (e.g. $\beta_U : NU_{n+1} \to U'_{n+1}$).

Take a proper eh-covering

$$\beta_W : NW_{n+1} \to (W'_{n+1} \times_{U'_{n+1}} NU_{n+1}) \times_{V'_{n+1}} NV_{n+1},$$

where the morphisms W'_{n+1} to U'_{n+1}, V'_{n+1} are defined by functoriality of the cosk_n map and NW_{n+1} is some smooth scheme. Then, looking at the composition

$$NW_{n+1} \overset{p_W}{\to} (W'_{n+1} \times_{U'_{n+1}} NU_{n+1}) \times_{V'_{n+1}} NV_{n+1} \to W'_{n+1} \times_{U'_{n+1}} NU_{n+1} \to W'_{n+1}$$

we see that the two last maps are base changes by proper eh-coverings. So by Lemma 4.1.2 all three are proper eh-coverings, and thus so is the composition $\beta : NW_{n+1} \to W'_{n+1}$ which we can use in the construction of W_{n+1} by the same method as above. This then comes with obvious maps
$NW_{n+1} \to NU_{n+1}, NV_{n+1}$, compatible with the maps on components on lower skeleta, inducing maps $W_{n+1} \to U_{n+1}, V_{n+1}$.

Part ii) follows from the proof of i) by taking a refinement of the peh-resolution $W_\bullet/X$ and $(V_\bullet \times_Y X)/X$. $\square$

Finally, since working on affine schemes will be easier later on, we introduce the following:

Definition 4.1.7. Given an augmented simplicial scheme $X_\bullet/X$, we say an augmented simplicial scheme $X'_\bullet/X$ is a *simplicial affine covering of* $X_\bullet/X$ if there is a morphism $f_\bullet : X'_\bullet \to X_\bullet$ over X such that for all n, $X'_n = \sqcup_{\alpha\in I_n} X_{n,\alpha}$ for a finite open covering by affines $X_n \cong \cup_{\alpha\in I_n} X_{n,\alpha}$ such that the image of each $X_{n,\alpha}$ in X is contained in some affine open subscheme of X.

These will always exist under nice enough conditions, and by a proof similar to 4.1.6 we have:

Lemma 4.1.8. *Let $X_\bullet/X$ be a split proper hypercovering. Then:*

i) [Nak12, Lemma 9.6] There exists some simplicial affine covering $X'_\bullet/X$ of $X_\bullet/X$.

ii) [Nak12, Proposition 6.3.1] Given another split proper hypercovering $Y_\bullet/Y$, and a commutative diagram

$$\begin{tikzcd} X_\bullet \arrow[r, "g_\bullet"] \arrow[d, "a"] & Y_\bullet \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and any simplicial affine covering $Y'_\bullet/Y$ of $Y_\bullet/Y$, then there exists a simplicial affine covering $X'_\bullet/X$ of $X_\bullet/X$ and a morphism $g'_\bullet : X'_\bullet \to Y'_\bullet$ fitting into the commutative diagram

$$\begin{tikzcd} X'_\bullet \arrow[r, "g'_\bullet"] \arrow[d, "a'_\bullet"] & Y'_\bullet \arrow[d, "b'_\bullet"] \\ X_\bullet \arrow[r, "g_\bullet"] \arrow[d, "a"] & Y_\bullet \arrow[d, "b"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

iii) [Nak12, Proposition 6.3.2] Given a two simplicial affine covering $X'_\bullet, X''_\bullet/X$ of $X_\bullet/X$, there exists a third simplicial affine covering $X'''_\bullet/X$ of $X_\bullet/X$ refining $X'_\bullet, X''_\bullet$.

4.2 Proof of Conjecture A

The main ingredient for the proof is the following result:

Theorem 4.2.1. *[Nak12, Theorem 11.6.3] Suppose X is a proper scheme over k. Then there exists a quasi-isomorphism*

$$R\Gamma_{rig}(X/K) \xrightarrow{\sim} R\Gamma(X_\bullet, W\Omega^\bullet_{X_\bullet})_{\mathbb{Q}},$$

functorial in split smooth proper hypercoverings $X_\bullet \to X$.

Proof. We fix some h and some $N > (h+1)(h+2)/2$. Take some simplicial affine covering $X'_\bullet$ of $X_\bullet$, which is possible by Lemma 4.1.8. Let $X_{\bullet,\bullet}$ be the Čech diagram of $X_{\bullet,0}$ over $X_\bullet$, with $X_{lm} := \operatorname{cosk}_0^{X_l}(X'_l)_m = X'_l \times_{X_l} \ldots \times_{X_l} X'_l$.

Take an affine open covering $X = \cup X_\alpha$, with closed embeddings $X_\alpha \hookrightarrow \mathcal{X}_\alpha$ into smooth formal schemes, and let $Z = \sqcup X_\alpha$, $\mathcal{Z} = \sqcup \mathcal{X}_\alpha$. Let

$$(Z_\bullet, \mathcal{Z}_\bullet) = (\operatorname{cosk}_0^X(Z), \operatorname{cosk}_0^{\mathcal{V}}(\mathcal{Z})))$$

be its Čech hypercovering. Then, we set

$$X_{lmn} := \operatorname{cosk}_0^{X_{lm}}(X_{lm} \times_X Z)_n = (X_{lm} \times_X Z) \times_{X_{lm}} \ldots \times_{X_{lm}} (X_{lm} \times_X Z) \cong$$

$$\cong X_{lm} \times_X (Z \times_X \ldots \times_X Z) = X_{lm} \times_X Z_n.$$

Since the X_l are smooth, so are the X_{lm} and X_{lmn}. We then construct a closed immersion $X_{\bullet \leq N,\bullet,\bullet} \hookrightarrow \mathcal{R}_{\bullet \leq N,\bullet,\bullet}$ where $\mathcal{R}_{\bullet \leq N,\bullet,\bullet}$ is a smooth (N,∞,∞)-truncated trisimplicial $\mathcal{W}$-scheme. To do so, since $X_{N0} = X'_N$ is a disjoint union of affine open subschemes of the smooth scheme X_N, we can pick a smooth lift $\mathcal{X}_{N0}$ over $\operatorname{Spf}(\mathcal{W})$. Then, by Lemma 2.4.2 we have a closed immersion

$$X_{\bullet \leq N0} \hookrightarrow \Gamma_N^{\mathcal{W}}(\mathcal{X}_{N0})_{\bullet \leq N}.$$

This in turn gives a closed immersion

$$X_{\bullet \leq N,\bullet} \hookrightarrow \mathcal{R}_{\bullet \leq N,\bullet} := \operatorname{cosk}_0^{\mathcal{W}}(\Gamma_N^{\mathcal{W}}(\mathcal{X}_{N0})_{\bullet \leq N})$$

and

$$X_{\bullet \leq N,\bullet,\bullet} \hookrightarrow \mathcal{R}_{\bullet \leq N,\bullet,\bullet} := \operatorname{cosk}_0^{\mathcal{W}}(\mathcal{R}_{\bullet \leq N,\bullet} \hat{\times}_{\mathcal{W}} \mathcal{Z}))$$

given respectively by

$$X_{lm} = X'_l \times_{X_l} \ldots \times_{X_l} X'_l \hookrightarrow \mathcal{R}_{lm} = \Gamma_N^{\mathcal{W}}(\mathcal{X}_{N0})_l \hat{\times}_{\mathcal{W}} \ldots \hat{\times}_{\mathcal{W}} \Gamma_N^{\mathcal{W}}(\mathcal{X}_{N0})_l$$

and

$$\begin{array}{ccc} X_{lmn} & & \mathcal{R}_{lmn} \\ \| & & \| \\ (X_{lm} \times_X Z) \times_{X_{lm}} ... \times_{X_{lm}} (X_{lm} \times_X Z) & \hookrightarrow & (\mathcal{R}_{lm} \hat{\times}_{\mathcal{W}} \mathcal{Z}) \hat{\times}_{\mathcal{W}} ... \hat{\times}_{\mathcal{W}} (\mathcal{R}_{lm} \hat{\times}_{\mathcal{W}} \mathcal{Z}). \end{array}$$

The following diagram summarizes the morphisms above:

$$\begin{tikzcd} X_{\bullet \leq N, \bullet, \bullet} \arrow[r] \arrow[d] & X_{\bullet \leq N, \bullet} \arrow[r] & X_{\bullet \leq N} \arrow[d] \\ Z_\bullet \arrow[rr] & & X \end{tikzcd}$$

where $X_{\bullet \leq N, \bullet, \bullet}$, $X_{\bullet \leq N, \bullet}$ and $Z_\bullet$ have compatible closed immersions into formally smooth simplicial schemes (note that we do not require any such embeddings for X and $X_{\bullet \leq N}$).

Then, $(X_{\bullet \leq N, \bullet, \bullet}, \mathcal{R}_{\bullet \leq N, \bullet, \bullet}) \to (Z_\bullet, \mathcal{Z}_\bullet)$ induces a map

$$\begin{tikzcd} \tau_{\leq h} R\Gamma_{rig}(X/K) \arrow[r, equal] & \tau_{\leq h} R\Gamma(]Z_\bullet[_{\mathcal{Z}_\bullet}, \Omega^\bullet_{]Z_\bullet[_{\mathcal{Z}_\bullet}}) \arrow[d] \\ & \tau_{\leq h} R\Gamma(]X_{\bullet \leq N, \bullet, \bullet}[_{\mathcal{R}_{\bullet \leq N, \bullet, \bullet}}, \Omega^\bullet_{]X_{\bullet \leq N, \bullet, \bullet}[_{\mathcal{R}_{\bullet \leq N, \bullet, \bullet}}}) \end{tikzcd} \tag{4.1}$$

where $\tau_{\leq h}$ is the canonical truncation, and we show in Lemma 4.2.3 that it is a quasiisomorphism.

Now, by Theorem 2.7.1, we have a natural quasiisomorphism

$$Rsp_* \Omega^\bullet_{]X_{\bullet \leq N, \bullet, \bullet}[_{\mathcal{R}_{\bullet \leq N, \bullet, \bullet}}} \cong W\Omega^\bullet_{X_{\bullet \leq N, \bullet, \bullet}, \mathbb{Q}},$$

which we use to get

$$R\Gamma(]X_{\bullet \leq N, \bullet, \bullet}[_{\mathcal{R}_{\bullet \leq N, \bullet, \bullet}}, \Omega^\bullet_{]X_{\bullet \leq N, \bullet, \bullet}[_{\mathcal{R}_{\bullet \leq N, \bullet, \bullet}}}) \cong \tag{4.2}$$

$$R\Gamma(X_{\bullet \leq N, \bullet, \bullet}, W\Omega^\bullet_{X_{\bullet \leq N, \bullet, \bullet}, \mathbb{Q}}) \cong \tag{4.3}$$

$$R\Gamma(X_{\bullet \leq N, \bullet}, W\Omega^\bullet_{X_{\bullet \leq N}, \bullet, \mathbb{Q}}) \cong \tag{4.4}$$

$$R\Gamma(X_{\bullet \leq N}, W\Omega^\bullet_{X_{\bullet \leq N}, \mathbb{Q}}) \cong R\Gamma(X_{\bullet \leq N, et}, W\Omega^\bullet_{X_{\bullet \leq N}, \mathbb{Q}}) \tag{4.5}$$

where we have used that $X_{\bullet \leq N, \bullet, \bullet} \to X_{\bullet \leq N, \bullet}$ and $X_{\bullet \leq N, \bullet} \to X_{\bullet \leq N}$ are Zariski hypercoverings (and thus satisfy cohomological descent), and that Zariski and étale hypercohomology agrees since the the $W\Omega^i$ are quasi-coherent sheaves (see [Mil80, Remark 3.8]).

This gives

$$\tau_{\leq h} R\Gamma_{rig}(X/K) \cong \tau_{\leq h} R\Gamma(X_{\bullet \leq N, et}, W\Omega^{\bullet}_{X_{\bullet \leq N}, \mathbb{Q}}) \cong \tau_{\leq h} R\Gamma(X_{\bullet, et}, W\Omega^{\bullet}_{X_{\bullet}, \mathbb{Q}}) \tag{4.6}$$

since by the spectral sequence

$$E_1^{p,q} = H^q(X_p, W\Omega^{\bullet}_{X_p}) \Rightarrow H^{p+q}(X_{\bullet}, W\Omega^{\bullet}_{X_{\bullet}})$$

the X_n with $n > N$ don't contribute to $H^i(X_{\bullet}, W\Omega^{\bullet}_{X_{\bullet}})$ for $i \leq h$.

Next, by Theorem 2.3.4 there exists an integer c such that $H^q_{rig}(X/K) = 0$ for $q > c$. Thus, since (4.6) holds for any h, we see that
$H^q(X_{\bullet, et}, W\Omega^{\bullet}_{\bullet, \mathbb{Q}}) = 0$ for $q > c$ also. Taking $h = c$, we can drop the truncation terms and get

$$R\Gamma_{rig}(X/K) \cong R\Gamma(X_{\bullet, et}, W\Omega^{\bullet}_{X_{\bullet}, \mathbb{Q}}) \tag{4.7}$$

Finally, it remains to show independence of all the choices, and prove functoriality, which is Lemma 4.2.2 below. □

Lemma 4.2.2. *With the same assumptions as Theorem 4.2.1:*

i) The isomorphism (4.7) in $D^+(K)$ is independent of choices.

ii) The isomorphism (4.7) in $D^+(K)$ is functorial in split smooth proper hypercoverings $X_{\bullet} \to X$.

Proof.
i) Independence of choices:
We need to show independence of the choices of closed embeddings into smooth formal schemes $X_{N0} \hookrightarrow \mathcal{X}_{N0}$ and $Z \hookrightarrow \mathcal{Z}$, independence of choice of affine Zariski coverings $Z \to X$ and $X_{\bullet 0} \to X_{\bullet}$ and independence of choice of N satisfying $N > (c+1)(c+2)/2$ with c as in the proof. Given two choices $T^i := (X^i_{\bullet 0} \to X_{\bullet}, X^i_{N0} \hookrightarrow \mathcal{X}^i_{N0}, Z^i \to X, Z^i \hookrightarrow \mathcal{Z}^i)$ for $i = 1, 2$, we will set

$$R\Gamma^i_{\bullet,\bullet} := R\Gamma(]X^i_{\bullet \leq N^i, \bullet}[_{\mathcal{R}^i_{\bullet \leq N^i, \bullet}}, \Omega^{\bullet}_{]X^i_{\bullet \leq N^i, \bullet}[_{\mathcal{R}^i_{\bullet \leq N^i, \bullet}}}),$$

$$R\Gamma^i_{\bullet,\bullet,\bullet} := R\Gamma(]X^i_{\bullet \leq N^i, \bullet, \bullet}[_{\mathcal{R}^i_{\bullet \leq N^i, \bullet, \bullet}}, \Omega^{\bullet}_{]X^i_{\bullet \leq N^i, \bullet, \bullet}[_{\mathcal{R}^i_{\bullet \leq N^i, \bullet, \bullet}}})$$

to be the complex formed as in the proof, where if not explicitly defined, we will drop the superscript i (*e.g.* for independence of choice of Z, $T^i = (X_{\bullet 0} \to X_\bullet, X_{N0} \hookrightarrow \mathcal{X}_{N0}, Z^i \to X, Z^i \hookrightarrow \mathcal{Z}^i)$ with only $Z^i, \mathcal{Z}^i$ and $Z^i \hookrightarrow \mathcal{Z}^i$ varying).

By the description of $\mathrm{D}^+(K)$ in terms of right roofs, to show that the two maps

$$\begin{tikzcd} & R\Gamma^i_{\bullet,\bullet,\bullet} & \\ R\Gamma_{rig}(X/K) \arrow[ur] & & R\Gamma(X_\bullet, W\Omega^\bullet_{X_\bullet})_{\mathbb{Q}} \arrow[ul, "\sim"'] \end{tikzcd} \qquad i = 1,2$$

are equivalent, it suffices to find some $R\Gamma^{12}_{\bullet,\bullet,\bullet}$ fitting into the commutative diagram

$$\begin{tikzcd} & R\Gamma^1_{\bullet,\bullet,\bullet} \arrow[d] & \\ R\Gamma_{rig}(X/K) \arrow[ur] \arrow[dr] & R\Gamma^{12}_{\bullet,\bullet,\bullet} & R\Gamma(X_\bullet, W\Omega^\bullet_{X_\bullet})_{\mathbb{Q}} \arrow[ul] \arrow[dl] \\ & R\Gamma^{12}_{\bullet,\bullet,\bullet} \arrow[u] & \end{tikzcd} \tag{4.8}$$

where all maps are quasi-isomorphisms.

- *Independence of choice of* $\mathcal{X}_{N0}$:

  Suppose we choose two different closed embeddings into smooth formal schemes $X_{N0} \hookrightarrow \mathcal{X}^1_{N0}, \mathcal{X}^2_{N0}$. Take the closed embedding into a smooth formal scheme
  $$X_{N0} \hookrightarrow \mathcal{X}^1_{N0} \hat{\times}_{\mathcal{W}} \mathcal{X}^2_{N0} =: \mathcal{X}^{12}_{N0}.$$
  Letting $\Gamma^i := \Gamma^{\mathcal{W}}_N(\mathcal{X}^i_{N0})_{\bullet \leq N}$ for $i = 1, 2, 12$, we then have a commutative diagram of N-truncated simplicial complexes

  $$\begin{tikzcd} & & \Gamma^1 \\ X_{\bullet \leq N, 0} \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & \Gamma^{12} \arrow[ur] \arrow[dr] & \\ & & \Gamma^2 \end{tikzcd}$$

which in turns gives rise to two diagrams

$$\begin{tikzcd}
 & & \mathcal{R}^1_{\bullet\le N,\bullet} \\
X_{\bullet\le N,\bullet} \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & \mathcal{R}^{12}_{\bullet\le N,\bullet} \arrow[ur, "p^1"] \arrow[dr, "p^2"'] & \\
 & & \mathcal{R}^2_{\bullet\le N,\bullet}
\end{tikzcd}$$

of (N,∞), and similarly for (N,∞,∞)-truncated simplicial complexes, where we construct $\mathcal{R}^i_{\bullet\le N,\bullet}$ and $\mathcal{R}^i_{\bullet\le N,\bullet,\bullet}$ as above for $i=1,2,12$. Then, by Berthelot's independence of the choice of a closed immersion into a smooth formal scheme [Ber97b, Théorème 1.4] we have quasi-isomorphisms

$$\Omega^\bullet_{]X_{\bullet\le N,\bullet,\bullet}[_{\mathcal{R}^i_{\bullet\le N,\bullet,\bullet}}} \xrightarrow{\sim} Rp^i_*\Omega^\bullet_{]X_{\bullet\le N,\bullet,\bullet}[_{\mathcal{R}^{12}_{\bullet\le N,\bullet,\bullet}}}$$

for $i=1,2$. This in turn gives rise to a desired diagram such as (4.8).

- *Independence of choice of affine covering* $X_{\bullet 0}$: Given two simplicial affine coverings $X^1_{\bullet 0}, X^2_{\bullet 0} \to X_\bullet$, by Lemma 4.1.8.iii) we can choose a common simplicial affine covering $X^{12}_{\bullet 0}$ fitting into

$$\begin{tikzcd}
 & X^1_{\bullet 0} \arrow[dr] & \\
X^{12}_{\bullet 0} \arrow[ur] \arrow[rr] \arrow[dr] & & X_\bullet \\
 & X^2_{\bullet 0} \arrow[ur] &
\end{tikzcd} ,$$

where X^{12}_{n0} is the disjoint product of some affine coverings of X_n for every n.

Then, we can choose closed embeddings $X^i_{N0} \hookrightarrow \mathcal{X}^i_{N0}$ into smooth formal schemes, compatible with the above maps. This will give maps

$$\begin{tikzcd}
 & (X^1_{\bullet\le N,\bullet} \hookrightarrow \mathcal{R}^1_{\bullet\le N,\bullet}) \\
(X^{12}_{\bullet\le N,\bullet} \hookrightarrow \mathcal{R}^{12}_{\bullet\le N,\bullet}) \arrow[ur] \arrow[dr] & \\
 & (X^2_{\bullet\le N,\bullet} \hookrightarrow \mathcal{R}^2_{\bullet\le N,\bullet})
\end{tikzcd}$$

and similar for its trisimplicial counterpart. We claim that all the vertical arrows in the commutative diagram

$$\begin{tikzcd} R\Gamma^1_{\bullet,\bullet,\bullet} & R\Gamma^1_{\bullet,\bullet} \arrow[l, hook', "\sim"'] \arrow[d] \\ R\Gamma^{12}_{\bullet,\bullet,\bullet} & R\Gamma^{12}_{\bullet,\bullet} \arrow[l, hook', "\sim"'] \\ R\Gamma^2_{\bullet,\bullet,\bullet} \arrow[u] & R\Gamma^2_{\bullet,\bullet} \arrow[l, hook', "\sim"'] \arrow[u] \end{tikzcd} \tag{4.9}$$

with $R\Gamma^1_{\bullet,\bullet,\bullet} \to R\Gamma^{12}_{\bullet,\bullet,\bullet}$ also a vertical arrow, are quasi-isomorphisms, from which independence will follow just as before. It will suffice to show this for the right vertical arrows.

The fact that they are quasi-isomorphisms follows since for a fixed $p \leq N$, and $i = 1, 2, 12$, as X^i_{p0} is a Zariski covering of X_p, $(X^i_{p\bullet}, \mathcal{R}^i_{l\bullet})$ is a universally de Rham descendable hypercovering of X_p by [CT03, Proposition 10.1.4]. Thus, by the independence of choice of universally de Rham descendable hypercovering ([CT03, Lemma 10.4.1]), we have quasi-isomorphisms

$$\begin{array}{c} R\Gamma(]X^1_{p,\bullet}[_{\mathcal{R}^1_{p,\bullet}}, \Omega^\bullet_{]X^1_{p,\bullet}[_{\mathcal{R}^1_{p,\bullet}}}) \\ \downarrow\wr \\ R\Gamma(]X^{12}_{p,\bullet}[_{\mathcal{R}^{12}_{p,\bullet}}, \Omega^\bullet_{]X^{12}_{p,\bullet}[_{\mathcal{R}^{12}_{p,\bullet}}}) \\ \wr\uparrow \\ R\Gamma(]X^2_{p,\bullet}[_{\mathcal{R}^2_{p,\bullet}}, \Omega^\bullet_{]X^2_{p,\bullet}[_{\mathcal{R}^2_{p,\bullet}}}) \end{array}$$

for all $p \leq N$. Together with the spectral sequence

$$E_1^{p,q} = H^q(R\Gamma(]X^i_{p,\bullet}[_{\mathcal{R}^i_{p,\bullet}}, \Omega^\bullet_{]X^i_{p,\bullet}[_{\mathcal{R}^i_{p,\bullet}}}) \Rightarrow H^{p+q}(]X^i_{\bullet\leq N,\bullet}[_{\mathcal{R}^i_{\bullet\leq N,\bullet}}, \Omega^\bullet_{]X^i_{\bullet\leq N,\bullet}[_{\mathcal{R}^i_{\bullet\leq N,\bullet}}})$$

we see that the right vertical arrows in (4.9) are quasi-isomorphisms.

- *Independence of choice of affine covering Z of X and closed immersion $Z \hookrightarrow \mathcal{Z}$*: We already know that the definition of rigid cohomology does not depend on choices of Z and $\mathcal{Z}$ by [CT03, Proposition 10.4.3], so we show that the comparison morphism is also independent.

  Given two affine coverings $Z^i \to X$ $(i = 1, 2)$, with closed immersions $Z^i \hookrightarrow \mathcal{Z}^i$ into smooth formal schemes. we can pick a simplicial affine covering $Z^{12} \to X$ with a compatible closed immersion $Z^{12} \hookrightarrow \mathcal{Z}^{12}$ for some smooth formal scheme. This will give compatible maps $\mathcal{Z}^1_\bullet \leftarrow \mathcal{Z}^{12}_\bullet \to \mathcal{Z}^2_\bullet$ and $\mathcal{R}^1_{\bullet\leq N,\bullet,\bullet} \leftarrow \mathcal{R}^{12}_{\bullet\leq N,\bullet,\bullet} \to \mathcal{R}^2_{\bullet\leq N,\bullet,\bullet}$ giving the commutative diagram

$$\begin{tikzcd} R\Gamma(]Z^1_\bullet[_{\mathcal{Z}^1_\bullet}, \Omega^\bullet_{]Z^1_\bullet[_{\mathcal{Z}^1_\bullet}}) \arrow[r, "\sim"] \arrow[d] & R\Gamma^1_{\bullet,\bullet,\bullet} \arrow[d] \\ R\Gamma(]Z^{12}_\bullet[_{\mathcal{Z}^{12}_\bullet}, \Omega^\bullet_{]Z^{12}_\bullet[_{\mathcal{Z}^{12}_\bullet}}) \arrow[r, "\sim"] & R\Gamma^{12}_{\bullet,\bullet,\bullet} \\ R\Gamma(]Z^2_\bullet[_{\mathcal{Z}^2_\bullet}, \Omega^\bullet_{]Z^2_\bullet[_{\mathcal{Z}^2_\bullet}}) \arrow[r, "\sim"] \arrow[u] & R\Gamma^2_{\bullet,\bullet,\bullet} \arrow[u] \end{tikzcd} \tag{4.10}$$

By [CT03, Lemma 10.4.1], the left vertical maps in (4.10) are quasi-isomorphisms, making all the vertical maps quasi-isomorphisms.

- *Independence of choice of N*: Take $N^2 \geq N^1$ satisfying $N^1 > (h+1)(h+2)/2$ for $h = c$ as above, and choose embeddings $X_{N^i0} \hookrightarrow \mathcal{X}^i_{N^i0}$ to construct $(X_{\bullet\leq N^i,\bullet,\bullet}, \mathcal{R}^i_{\bullet\leq N^i,\bullet,\bullet})$ for $i = 1, 2$. Firstly, by the independence of the embedding, we can replace $\mathcal{R}^1_{\bullet\leq N^1,\bullet,\bullet}$ with the (N^1, ∞, ∞)-truncation $\mathcal{R}^2_{\bullet\leq N^1,\bullet,\bullet}$. Then, consider the natural map

$$\begin{array}{c} \tau_{\leq c} R\Gamma(]X_{\bullet\leq N^1,\bullet,\bullet}[_{\mathcal{R}^2_{\bullet\leq N^1,\bullet,\bullet}}, \Omega^\bullet_{]X_{\bullet\leq N^1,\bullet,\bullet}[_{\mathcal{R}^2_{\bullet\leq N^1,\bullet,\bullet}}}) \\ \downarrow \\ \tau_{\leq c} R\Gamma(]X_{\bullet\leq N^2,\bullet,\bullet}[_{\mathcal{R}^2_{\bullet\leq N^2,\bullet,\bullet}}, \Omega^\bullet_{]X_{\bullet\leq N^2,\bullet,\bullet}[_{\mathcal{R}^2_{\bullet\leq N^2,\bullet,\bullet}}}) \end{array}$$

coming from truncation. By choice of N^1 and N^2, the X_{lmn} and $\mathcal{R}^2_{lmn}$ with $l > N^1$ do not contribute to the cohomology of the bottom complex, so the above is a natural quasi-isomorphism. Similarly, we have a natural quasi-isomorphisms

$$\begin{array}{c} \tau_{\leq c} R\Gamma(X_{\bullet\leq N^1}, W\Omega^\bullet_{X_{\bullet\leq N^1}})_{\mathbb{Q}} \\ \downarrow \\ \tau_{\leq c} R\Gamma(X_{\bullet\leq N^2}, W\Omega^\bullet_{X_{\bullet\leq N^2}})_{\mathbb{Q}} \\ \downarrow \\ \tau_{\leq c} R\Gamma(X_\bullet, W\Omega^\bullet_{X_\bullet})_{\mathbb{Q}} \end{array}$$

compatible with the above. This shows independence of the choice of N.

Functoriality:

Given a diagram of good split proper hypercoverings

$$\begin{tikzcd} X_\bullet \arrow[r] \arrow[d] & Y_\bullet \arrow[d] \\ X \arrow[r] & Y \end{tikzcd},$$

and having chosen a disjoint union W of an open affine covering of Y, and a closed immersion $W \hookrightarrow \mathcal{W}$ into a smooth formal scheme, then we can pick Z to be a disjoint union of an affine open covering of X refining W, and a closed immersion $Z \hookrightarrow \mathcal{Z}$ fitting into the commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\ W \arrow[r] & \mathcal{W}. \end{tikzcd}$$

Similarly, having chosen a simplicial affine covering $Y_{\bullet 0} \to Y_\bullet$, then by Lemma 4.1.8.ii) we can choose some simplicial affine covering $X_{\bullet 0} \to X_\bullet$ fitting into the commutative diagram

$$\begin{tikzcd} X_{\bullet 0} \arrow[r] \arrow[d] & X_\bullet \arrow[d] \\ Y_{\bullet 0} \arrow[r] & Y_\bullet. \end{tikzcd}$$

Then, we can pick $\mathcal{X}_{N0}$ and $\mathcal{Y}_{N0}$ fitting into the commutative diagram

$$\begin{tikzcd} X_{N0} \arrow[r] \arrow[d] & \mathcal{X}_{N0} \arrow[d] \\ Y_{N0} \arrow[r] & \mathcal{Y}_{N0} \end{tikzcd}.$$

This in turn will give morphisms of pairs and triples

$$(X_{\bullet \leq N, \bullet}, \mathcal{R}_{\bullet \leq N, \bullet}) \to (Y_{\bullet \leq N, \bullet}, \mathcal{S}_{\bullet \leq N, \bullet}),$$

$$(X_{\bullet \leq N, \bullet, \bullet}, \mathcal{R}_{\bullet \leq N, \bullet, \bullet}) \to (Y_{\bullet \leq N, \bullet, \bullet}, \mathcal{S}_{\bullet \leq N, \bullet, \bullet}),$$

where $\mathcal{S}_{\bullet \leq N, \bullet}$ and $\mathcal{S}_{\bullet \leq N, \bullet, \bullet}$ are constructed for Y the same as the $\mathcal{R}$ equivalents are for X.

All these maps provide compatible maps at each step of the construction of the comparison morphism. $\square$

Lemma 4.2.3. *The map (4.1) is a quasi-isomorphism.*

Proof. We give an outline of the proof, by drawing directly from the proof of [Nak12, Theorem 11.6], which provides a detailed explanation. We introduce some intermediate multisimplicial pairs to prove the quasi-isomorphism. Consider the proper hypercovering $(X_\bullet \times_X Z)/Z$, and take a projective refinement $V_\bullet/Z$ using [Tsu03, Lemma 4.2.2.(1)]. That is, $V_\bullet/Z$ is a proper hypercovering, and we have a Z-morphism $V_\bullet \to X_\bullet \times_X Z$ with the natural morphisms

$$V_n \to \operatorname{cosk}^Z_{n-1}(\operatorname{sk}_{n-1}(V_\bullet))_n \times_{\operatorname{cosk}^Z_{n-1}(\operatorname{sk}_{n-1}(X_\bullet \times_X Z))_n} (X_\bullet \times_X Z)n$$

are proper surjective for any n, and each V_n is projective over Z. Note that we do not require any smoothness conditions on $V_\bullet$, and that we get an X-morphism $V_\bullet \to X_\bullet$.

Thus, we can choose a closed embedding into a smooth formal scheme $V_N \hookrightarrow \mathcal{P}_N$, and setting $\mathcal{Q}_\bullet := \Gamma^{\mathcal{W}}_N(\mathcal{P}_N)$ we get a closed embedding

$$V_{\bullet \leq N} \hookrightarrow \mathcal{Q}_{\bullet \leq N}.$$

We can then form a (N, ∞)-truncated bisimplicial complex $(V_{\bullet \leq N, \square, \bullet}), \mathcal{Q}_{\bullet \leq N, \square, \bullet})$, where $\square$ stands for an empty spot, by setting

$$V_{l \square n} = \operatorname{cosk}^{X_l}_0(V_l)_n, \quad \mathcal{Q}_{l \square n} = \operatorname{cosk}^{\mathcal{W}}_0(\mathcal{Q}_l \hat{\times}_{\mathcal{W}} \mathcal{Z})_n, \quad 0 \leq l \leq N, n \in \mathbb{N}.$$

This gives a diagram

$$\begin{tikzcd} V_{\bullet \leq N, \square, \bullet} \arrow[r] \arrow[d] & X_{\bullet \leq N} \arrow[d] \\ Z \arrow[r] & X. \end{tikzcd}$$

Also set $X_{l \square n} := \operatorname{cosk}^{X_l}_0(X_l \times_X Z)_n \cong X_l \times_X Z_n$ to define $X_{\bullet \leq N, \square, \bullet}$.

Thus, for any $n \in \mathbb{N}$,

$$(V_{\bullet \leq N, \square, n}, \mathcal{Q}_{\bullet \leq N, \square, n}) \to (Z_n, \mathcal{Z}_n)$$

is a N-truncated proper hypercovering.

We consider the induced morphism

$$R\Gamma(]Z_\bullet[_{\mathcal{Z}_\bullet}, \Omega^\bullet_{]Z_\bullet[_{\mathcal{Z}_\bullet}}) \to R\Gamma(]V_{\bullet \leq N, \square, \bullet}[_{\mathcal{Q}_{\bullet \leq N, \square, \bullet}}, \Omega^\bullet_{]V_{\bullet \leq N, \square, \bullet}[_{\mathcal{Q}_{\bullet \leq N, \square, \bullet}}}).$$

We claim that applying $\tau_{\leq h}(-)$ to this morphism gives a quasi-isomorphism. To see this, consider the spectral sequences

$$E_1^{pq} = H^q(]Z_p[_{\mathcal{Z}_p}, \Omega^\bullet_{]Z_p[_{\mathcal{Z}_p}}) \Rightarrow H^{p+q}(]Z_\bullet[_{\mathcal{Z}_\bullet}, \Omega^\bullet_{]Z_\bullet[_{\mathcal{Z}_\bullet}}),$$

$$E_1^{pq} = H^q(]V_{p,\square,\bullet}[_{\mathcal{Q}_{p,\square,\bullet}}, \Omega^\bullet_{]V_{p,\square,\bullet}[_{\mathcal{Q}_{p,\square,\bullet}}}) \Rightarrow H^{p+q}(]V_{\bullet_{\leq N},\square,\bullet}[_{\mathcal{Q}_{\bullet_{\leq N},\square,\bullet}}, \Omega^\bullet_{]V_{\bullet_{\leq N},\square,\bullet}[_{\mathcal{Q}_{\bullet_{\leq N},\square,\bullet}}}).$$

Since the maps for each p are proper hypercoverings, using [Tsu03, Theorem 2.1.3] we get that the maps on E_1^{pq} for $0 \leq p \leq N$ are isomorphisms, and by our choice of N relative to h, we see that we get the desired isomorphism

$$H^i(]Z_\bullet[_{\mathcal{Z}_\bullet}, \Omega^\bullet_{]Z_\bullet[_{\mathcal{Z}_\bullet}}) \to H^i(]V_{\bullet_{\leq N},\square,\bullet}[_{\mathcal{Q}_{\bullet_{\leq N},\square,\bullet}}, \Omega^\bullet_{]V_{\bullet_{\leq N},\square,\bullet}[_{\mathcal{Q}_{\bullet_{\leq N},\square,\bullet}}})$$

is an isomorphism for $i \leq h$.

Define

$$V_{\bullet_{\leq N},\bullet,\bullet} = \operatorname{cosk}_0^{V_{\bullet_{\leq N},\square,\bullet}}(V_{\bullet_{\leq N},\square,\bullet} \times_{X_{\bullet_{\leq N}}} X_{\bullet_{\leq N},0})$$

so

$$V_{lmn} \cong V_{l\square n} \times_{X_l} X_{lm} \cong V_{l\square n} \times_{X_l \times_X Z_n} (X_{lm} \times_X Z_n) \cong V_{l\square n} \times_{X_{l\square n}} X_{lmn}$$

giving a cartesian diagram

$$\begin{tikzcd} V_{\bullet_{\leq N},\bullet,\bullet} \arrow[r] \arrow[d] & V_{\bullet_{\leq N},\square,\bullet} \arrow[d] \\ X_{\bullet_{\leq N},\bullet,\bullet} \arrow[r] & X_{\bullet_{\leq N},\square,\bullet}. \end{tikzcd}$$

We can then form a closed embedding $V_{\bullet_{\leq N},\bullet,\bullet} \hookrightarrow \mathcal{S}_{\bullet_{\leq N},\bullet,\bullet}$ into a smooth formal scheme, with a pair of morphisms into $X_{\bullet_{\leq N},\bullet,\bullet} \hookrightarrow \mathcal{R}_{\bullet_{\leq N},\bullet,\bullet}$. We claim that the induced morphism

$$\begin{tikzcd} \tau_{\leq h} R\Gamma(]X_{\bullet_{\leq N},\bullet,\bullet}[_{\mathcal{R}_{\bullet_{\leq N},\bullet,\bullet}}, \Omega^\bullet_{]X_{\bullet_{\leq N},\bullet,\bullet}[_{\mathcal{R}_{\bullet_{\leq N},\bullet,\bullet}}}) \arrow[d] \\ \tau_{\leq h} R\Gamma(]V_{\bullet_{\leq N},\bullet,\bullet}[_{\mathcal{S}_{\bullet_{\leq N},\bullet,\bullet}}, \Omega^\bullet_{]V_{\bullet_{\leq N},\bullet,\bullet}[_{\mathcal{S}_{\bullet_{\leq N},\bullet,\bullet}}}) \end{tikzcd}$$

is a quasi-isomorphism. Similar to above, it suffices to check that the induced morphism

$$H^q(]X_{p,\bullet,\bullet}[_{\mathcal{R}_{p,\bullet,\bullet}}, \Omega^\bullet_{]X_{p,\bullet,\bullet}[_{\mathcal{R}_{p,\bullet,\bullet}}}) \to H^q(]V_{p,\bullet,\bullet}[_{\mathcal{S}_{p,\bullet,\bullet}}, \Omega^\bullet_{]V_{p,\bullet,\bullet}[_{\mathcal{S}_{p,\bullet,\bullet}}})$$

is an isomorphism for $p \leq N$. This in turn follows from the commutative diagram

$$\begin{tikzcd} R\Gamma_{rig}(X_p/K) \arrow[r, equal] \arrow[d, equal] & R\Gamma(]X_{p\bullet}[_{\mathcal{R}_{p\bullet}}, \Omega^\bullet_{]X_{p\bullet}[_{\mathcal{R}_{p\bullet}}}) \arrow[r, "\sim"] & R\Gamma(]X_{p\bullet\bullet}[_{\mathcal{R}_{p\bullet\bullet}}, \Omega^\bullet_{]X_{p\bullet\bullet}[_{\mathcal{R}_{p\bullet\bullet}}}) \arrow[d] \\ R\Gamma_{rig}(X_p/K) \arrow[r, "\sim"] & R\Gamma(]V_{p\Box\bullet}[_{\mathcal{S}_{p\Box\bullet}}, \Omega^\bullet_{]V_{p\Box\bullet}[_{\mathcal{S}_{p\Box\bullet}}}) \arrow[r, "\sim"] & R\Gamma(]V_{p\bullet\bullet}[_{\mathcal{S}_{p\bullet\bullet}}, \Omega^\bullet_{]V_{p\bullet\bullet}[_{\mathcal{S}_{p\bullet\bullet}}}) \end{tikzcd}$$

where the horizontal maps are quasi-isomorphisms since they come from Zariski hypercoverings.

Then, with some additional checking and using the spectral sequences

$$E_1^{pq} = H^q(]Z_p[_{\mathcal{Z}_p}, \Omega^\bullet_{]Z_p[_{\mathcal{Z}_p}}) \Rightarrow H^{p+q}(]Z_\bullet[_{\mathcal{Z}_\bullet}, \Omega^\bullet_{]Z_\bullet[_{\mathcal{Z}_\bullet}}) = H^{p+q}_{rig}(X/K),$$

$$E_1^{pq} = H^q(]X_{p,\bullet,\bullet}[_{\mathcal{R}_{p,\bullet,\bullet}}, \Omega^\bullet_{]X_{p,\bullet,\bullet}[_{\mathcal{R}_{p,\bullet,\bullet}}}) \Rightarrow H^{p+q}(]X_{\bullet\leq N,\bullet,\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet,\bullet}}, \Omega^\bullet_{]X_{\bullet\leq N,\bullet,\bullet}[_{\mathcal{R}_{\bullet\leq N,\bullet,\bullet}}})$$

the above isomorphism proves the lemma. $\square$

This allows us to prove conjecture A under $R(d)$:

Proof of Theorem 4.0.1. For general X, take a compactification $X \hookrightarrow \overline{X}$ where $\overline{X}$ is proper over k and has X as a dense open subscheme. Let $Z \subset \overline{X}$ be the complement of X with reduced closed subscheme structure. Since in $D_+(K)$,

$$R\Gamma_c(X_{eh}, \mathbb{Q}_p(n)) = \left[R\Gamma(\overline{X}_{eh}, \mathbb{Q}_p(n)) \to R\Gamma(Z_{eh}, \mathbb{Q}_p(n))\right]$$

and

$$R\Gamma_{rig,c}(X/K) = \left[R\Gamma_{rig}(\overline{X}/K) \to R\Gamma_{rig}(Z/K)\right]$$

it suffices to prove the theorem for X proper, along with functoriality.

Take a peh-resolution $X_\bullet \to X$ using Proposition 4.1.6. Since this is a hypercovering in eh-topology (i.e. every map $X_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n(X_\bullet))_{n+1}$ is a covering in the eh-topology as they are proper eh-coverings), we get by cohomological descent that

$$\begin{aligned} R\Gamma_c(X_{eh}, \mathbb{Q}_p(n)) &= R\varprojlim_r R\Gamma(X_{eh}, \rho^* W_r\Omega^n_{X,log})_{\mathbb{Q}}[-n] \\ &\cong R\varprojlim_r R\Gamma(X_{\bullet,eh}, \rho^* W_r\Omega^n_{X_\bullet,log})_{\mathbb{Q}}[-n] \end{aligned}$$

Now, by Theorem 3.3.1, we have that $\mathbb{Z}(n)/p^r \xrightarrow{\sim} R\rho_*\rho^*\mathbb{Z}(n)/p^r$ on $(\mathrm{Sm}/k)_{et}$, so (using the identification $\mathbb{Z}(n)/p^r \cong W_r\Omega^n_{log}[-n]$ on Sm/k), we have that

$$R\Gamma_c(X_{eh}, \mathbb{Q}_p(n)) \cong R\varprojlim_r R\Gamma(X_{\bullet,et}, W_r\Omega^n_{X_\bullet,log})_{\mathbb{Q}}[-n].$$

Then, by [Ill79],

$$\begin{aligned}
R\Gamma_c(X_{eh}, \mathbb{Q}_p(n)) &\cong R\varprojlim_r R\Gamma(X_{\bullet,et}, W_r\Omega^n_{X_\bullet,log})_{\mathbb{Q}}[-n] \\
&\overset{[\text{Ill79, I.Th.5.7.2}]}{\cong} \left[R\varprojlim_r R\Gamma(X_{\bullet,et}, W_r\Omega^n_{X_\bullet})_{\mathbb{Q}} \overset{1-F}{\to} R\varprojlim_r R\Gamma(X_{\bullet,et}, W_r\Omega^n_{X_\bullet})_{\mathbb{Q}}\right][-n] \\
&\overset{[\text{Ill79, II.Prop.2.1.(a)}]}{\cong} \left[R\Gamma(X_{\bullet,et}, W\Omega^n_{X_\bullet})_{\mathbb{Q}} \overset{1-F}{\to} R\Gamma(X_{\bullet,et}, W\Omega^n_{X_\bullet})_{\mathbb{Q}}\right][-n] \\
&\overset{[\text{Ill79, II.Cor.3.5.]}}{\cong} \left[R\Gamma(X_{\bullet,et}, W\Omega^\bullet_{X_\bullet})_{\mathbb{Q}} \overset{p^n-\phi}{\to} R\Gamma(X_{\bullet,et}, W\Omega^\bullet_{X_\bullet})_{\mathbb{Q}}\right] \\
&\overset{\text{Th.4.2.1}}{\cong} \left[R\Gamma_{rig,c}(X/K) \overset{p^n-\phi}{\to} R\Gamma_{rig,c}(X/K)\right]
\end{aligned}$$

where we have used that $R\Gamma(X_\bullet, W\Omega^\bullet)_{[i,i+1)} = R\Gamma(X_\bullet, W\Omega^i)[-i]$ from the slope decomposition for the second to last equality.

This quasiisomorphism is independent of the choice of peh-resolutions, as by Proposition 4.1.6,i), for any two such peh-resolutions, we can find a peh-resolution which is a common refinement of the two.

As for functoriality, given $f : X \to Y$ of proper schemes over k, we may pick any peh-resolution $Y_\bullet \to Y$, and by Proposition 4.1.6.ii) we may choose another peh-resolution $X_\bullet \to X$ such that we get a commutative diagram

$$\begin{tikzcd}
X_\bullet \arrow[r] \arrow[d] & Y_\bullet \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

so using the functoriality of Theorem 4.2.1 we see that every step is functorial.

□

Chapter 5

COMPARISON OF OVERCONVERGENT WITT DE-RHAM COHOMOLOGY AND RIGID COHOMOLOGY

5.1 Introduction

Let X be a smooth scheme over a perfect field k of characteristic $p > 0$, and consider its overconvergent de Rham-Witt complex of étale sheaves $W^\dagger\Omega^\bullet_{X/k}$, which is defined in [DLZ11] (see Definition 1.1 and Theorem 1.8). One of the main results of *loc. cit.* is that if X is also quasi-projective, then there exists a natural quasi-isomorphism

$$R\Gamma_{\mathrm{rig}}(X/K) \cong R\Gamma(X, W^\dagger\Omega^\bullet_{X/k})_{\mathbb{Q}},$$

where $K = W(k) \otimes \mathbb{Q}$.

The main result of this chapter is Theorem 5.5.5, where we drop the quasi-projectivity condition in the comparison. We outline the approach in [DLZ11] and the one used here.

If $X = \mathrm{Spec}\ A$ is an affine smooth k-scheme, [DLZ11] consider pairs (X, F) given by closed immersions

$$X = \mathrm{Spec}\ A \hookrightarrow F = \mathrm{Spec}\ \tilde{A}$$

into $W(k)$-schemes, called special frames. To this, the authors associate dagger spaces (in the sense of [Gro00]) $]X[^\dagger_{\mathscr{F}}$ functorially in (X, F), which calculate $R\Gamma_{\mathrm{rig}}(X/K)$:

$$R\Gamma_{\mathrm{rig}}(X/K) \xrightarrow{\sim} R\Gamma(]X[^\dagger_{\mathscr{F}}, \Omega^\bullet_{]X[^\dagger_{\mathscr{F}}}) \tag{5.1}$$

(here $\mathscr{F}$ denotes the p-adic completion of F).
So using the specialization maps

$$sp^\dagger_* :]X[^\dagger_{\mathscr{F}} \to X$$

we have that $\mathrm{R}\Gamma_{rig}(X/K) \cong R\Gamma(X, \mathrm{Rsp}^\dagger_*\Omega^\bullet_{]X[^\dagger_{\mathscr{F}}})$.

They also form a quasi-isomorphism of Zariski sheaves on X,

$$\mathrm{sp}_*\Omega^\bullet_{]X[^\dagger_{\hat{F}}} \to W^\dagger\Omega^\bullet_{X/k} \otimes \mathbb{Q}, \tag{5.2}$$

functorial in (X, F).

This all gives a map

$$R\Gamma(\mathrm{sp}_*^\dagger \Omega^\bullet_{]X[^\dagger_{\mathscr{F}}}) \to R\Gamma(X, W^\dagger\Omega^\bullet_{X/k})_{\mathbb{Q}} \tag{5.3}$$

so it suffices to show that the natural map

$$\mathrm{sp}_*^\dagger \Omega^\bullet_{]X[^\dagger_{\mathscr{F}}} \to R\mathrm{sp}_*^\dagger \Omega^\bullet_{]X[^\dagger_{\mathscr{F}}} \tag{5.4}$$

is a quasi-isomorphism.

While vanishing of higher cohomologies in (5.4) is not known in general, we can show it in some instances. In [DLZ11], to globalize the above construction for a smooth quasi-projective X (though possibly not affine), the authors consider an open covering by a particular type of affine smooth schemes, standard smooth schemes, which may be lifted nicely over $W(k)$, which are all coming from localizations in a common projective space provided by the quasi-projectivity of X. This gives a nice description of the intersections of such opens in $\mathrm{cosk}_0^X(X_0)_\bullet$, which allows them to prove the vanishing of higher cohomologies in (5.4), and then complete the proof by means of cohomological descent.

For our case, when X is not quasi-projective, we do not have a common projective space in which all our open affines are open. So instead of working with the 0-coskeleton, we refine it at each level, getting an étale hypercovering $X_\bullet/X$ so that at each level, X_n is a disjoint union of affine standard smooth schemes, which we call a special hypercovering. This is done in Section 5.3.

Considering any compactification $X \hookrightarrow \overline{X}$ to a proper k-scheme $\overline{X}$, we use the Tsuzuki functors $\Gamma_N^{W(k)}(-)$ and $\Gamma_N^{\mathcal{W}}(-)$ introduced in Definition 2.4.1 to construct an N-truncated special frame $(X_{\bullet\leq N}, F_{\bullet\leq N})$ and a N-truncated simplicial version of (5.3):

$$R\Gamma(X_{\bullet\leq N}, W^\dagger\Omega^\bullet_{X_{\bullet\leq N}/k})_{\mathbb{Q}} \to R\Gamma(\mathrm{sp}_*^\dagger \Omega^\bullet_{]X_{\bullet\leq N}[^\dagger_{\mathscr{F}_{\bullet\leq N}}}). \tag{5.5}$$

Then, we show the following:

- Vanishing of $\mathrm{Rsp}_*^i \Omega^\bullet_{]X_n[^\dagger_{\mathscr{F}_n}}$ for $0 \leq n \leq N$ and $i > 0$: we use techniques from the proof of [DLZ11, Proposition 4.35], such as being able to replace the F_n by some F'_n étale over F_n or equal to $F_n \times_{W(k)} \mathbb{A}^r_{W(k)}$ for some r fitting into a special frame (X_n, F'_n).

- Prove independence of choices and functoriality.
- For large enough N, (5.5) gives a map $R\Gamma(X, W^{\dagger}\Omega^{\bullet}_{X/k})_{\mathbb{Q}} \to R\Gamma_{\mathrm{rig}}(X/K)$. This is motivated by [Nak12] and relies on the machinery of [CT03], such as vanishing of higher enough rigid cohomology groups of X, independence of the choices of rigid frames and cohomological descent methods.

5.2 Background

Special Frames and Dagger Spaces

The following is a summary of Section 4 of [DLZ11].

Definition 5.2.1. A *special frame* is a pair (X, F) with a closed embedding $X \hookrightarrow F$, where X and F are smooth affine schemes over k and $W(k)$ respectively.

Given a special frame (X, F), we can choose an embedding $F \hookrightarrow \mathbb{A}^n_{W(k)}$ for some n, and in turn we have an open embedding $E := \mathbb{A}^n_W \hookrightarrow \mathbb{P}^n_{W(k)} =: P$. Let $Q = \overline{F}$ and $\overline{X}$ be the closures of F and X respectively in P, and let $\mathscr{F}$ and $\mathcal{Q}$ be the p-adic completions of F and Q respectively. Then,

$$X \hookrightarrow \overline{X} \hookrightarrow \mathcal{Q}$$

is a frame for rigid cohomology in the sense of Berthelot (i.e. we have an open immersion of X into a proper scheme $\overline{X}$ over k, and a closed immersion $\overline{X} \hookrightarrow \mathcal{Q}$ where $\mathcal{Q}$ is smooth around X). So we may define the rigid cohomology of X as

$$R\Gamma_{\mathrm{rig}}(X/K) = R\Gamma(]\overline{X}[_{\mathcal{Q}}, j^{\dagger}\Omega^{\bullet}_{]\overline{X}[_{\mathcal{Q}}}),$$

where j is the inclusion $]X[_{\mathcal{Q}} \hookrightarrow]\overline{X}[_{\mathcal{Q}}$. Note also that $]X[_{\mathcal{Q}} =]X[_{\mathscr{F}}$.

The authors then give an explicit description of a fundamental system of strict neighborhoods of $]X[_{\mathscr{F}}$ in $]\overline{X}[_{\mathcal{Q}}$, which they use to give a dagger structure (in the sense of [Gro00]) on $]X[_{\mathscr{F}}$, denoted by $]X[^{\dagger}_{\mathscr{F}}$, along with a morphism

$$sp^{\dagger}_{*} :]X[^{\dagger}_{\mathscr{F}} \to X$$

which is independent of the choice of embedding of F into affine and projective spaces. Thus, we have a functorial association

$$\begin{aligned} \text{Special Frames} &\longrightarrow \text{Dagger Spaces} \\ (X, F) &\longmapsto]X[^{\dagger}_{\mathscr{F}}. \end{aligned} \tag{5.6}$$

By [Gro00, Theorem 5.1], this gives quasi-isomorphisms

$$R\Gamma_{\mathrm{rig}}(X/K) = R\Gamma(]\overline{X}[_{\mathcal{Q}}, j^\dagger\Omega^\bullet_{]\overline{X}[_{\mathcal{Q}}}) \xrightarrow{\sim} R\Gamma(]X[^\dagger_{\mathscr{F}}, \Omega^\bullet_{]X[^\dagger_{\mathscr{F}}}). \tag{5.7}$$

To such a frame (X, F), they also form in [DLZ11, (4.32)] a map

$$\mathrm{sp}_*\Omega^\bullet_{]X[^\dagger_{\hat{F}}} \to W^\dagger\Omega^\bullet_{X/k} \otimes \mathbb{Q}, \tag{5.8}$$

which is a quasi-isomorphism of Zariski sheaves and functorial in (X, F).

Standard Smooth Schemes

Definition 5.2.2. We call a ring A a *standard smooth algebra* (over k) if A can be represented in the form

$$A = k[X_1, ..., X_n]/(f_1, ..., f_m),$$

where $m \leq n$ and the determinant

$$\det\left(\frac{\partial f_i}{\partial X_j}\right),\ 1 \leq i, j \leq m$$

is a unit in A. The scheme Spec A is then called a *standard smooth scheme.*

Such schemes are convenient to work with, since for a standard smooth algebra represented as $k[T_1, ..., T_n]/(f_1, ..., f_r)$, we may choose liftings $\tilde{f}_1, ..., \tilde{f}_r$ to $W[T_1, ..., T_n]$, and let $\tilde{A}$ be the localization of $W[T_1, ..., T_n]/(\tilde{f}_1, ..., \tilde{f}_r)$ with respect to det $\left(\frac{\partial \tilde{f}_i}{\partial T_j}\right)$. Then, $\tilde{A}$ is a standard smooth algebra which lifts A over W, which gives a special frame (Spec A, Spec $\tilde{A}$). We note that this may be done functorially in A; that is, given a homomorphism of standard smooth algebras

$$\varphi : A \to B$$

with presentations

$$A \cong k[T_1, .., T_n]/(f_1, .., f_r),\ B \cong k[S_1, ..., S_m]/(g_1, ..., g_s),$$

after choosing liftings $\tilde{f}_i$ to define $\tilde{A}$, we may chose the representation

$$B \cong k[S_1, ..., S_m, T_1, ..., T_n]/(g_1, ..., g_s, f_1, ..., f_r, T_1 - \alpha(T_1), ..., T_r - \alpha(T_r))$$

and then take liftings $\tilde{g}_j, \tilde{\alpha}_i$ over g_j and $\alpha(T_i)$ respectively to form $\tilde{B}$.

Note also that for any such standard smooth scheme $F = \mathrm{Spec}\ \tilde{A}$, we have an étale map

$$F \to \mathbb{A}^n_{W(k)}$$

for some n.

5.3 The hypercovering

Proposition 5.3.1. *Given any étale hypercovering $Z_\bullet \to X$, with Z_n being smooth schemes over k, there exists an étale hypercovering $Y_\bullet \to X$ refining $Z_\bullet \to X$ such that for any n, Y_n is the disjoint union of affine standard smooth schemes giving a finite open covering of Z_n.*

Proof. The proof is nearly identical to [CT03, Proposition 11.3.2], with the only difference being that when we form a finite affine Zariski covering of the smooth scheme

$$\operatorname{cosk}_n^X(Y_{\bullet \leq n})_{n+1} \times_{\operatorname{cosk}_n^X(Z_{\bullet \leq n})_{n+1}} Z_{n+1},$$

we require the covering to be by affine standard smooth schemes also. □

Definition 5.3.2. We say $Y_\bullet \to X$ is a *special hypercovering* if $Y_\bullet$ is a split étale hypercovering of X, and each Y_n is a disjoint union of affine standard smooth schemes which give an open covering of X.

We prove the existence and some functorial property of such hypercoverings, which will be useful to work on the comparison locally.

Proposition 5.3.3. *Given a smooth scheme X:*

i) There exists a special hypercovering $Y_\bullet \to X$.

ii) Given two special hypercoverings $Y_\bullet, Y'_\bullet/X$, there a third special hypercovering $Y''_\bullet/X$ refining them.

iii) Given a morphism $X \to X'$ of smooth schemes, there exist special hypercoverings $Y_\bullet \to X$ and $Y'_\bullet \to X'$ fitting in a commutative diagram

$$\begin{array}{ccc} Y_\bullet & \longrightarrow & Y'_\bullet \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}.$$

Proof. Part i) follows immediately from Proposition 5.3.1 by taking the constant simplicial scheme $Z_\bullet = \operatorname{cosk}_{-1}^X(X)$ (so $Z_n = X$ for all n). For part ii), we just apply Proposition 5.3.1 with

$$Z_\bullet := Y_\bullet \times_X Y'_\bullet,$$

and for part iii) we find some special hypercovering $Y'_\bullet \to X'$, and then again use Proposition 5.3.1 with

$$Z_\bullet := Y'_\bullet \times_{X'} X.$$

$\square$

We will use the following version of Chow's lemma:

Lemma 5.3.4. *Consider a compactification over a scheme S*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "g"] \\ & S \end{tikzcd}$$

where j is an open immersion and g is proper. Suppose that f is quasi-projective. Then, there exists a blowup $Y \to \overline{X}$ at some closed subscheme $Z \subset X$ disjoint from $j(X)$ such that Y is projective over S.

Proof. This follows immediately from [Ray74, Corollaire 5.7.14] since it gives us such a blowup $Y \to \overline{X}$ with Y quasi-projective over S with the only requirements that $\overline{X}$ be quasi-separated and finite type over S. In this case, $Y \to \overline{X} \to S$ is the composition of a blowup and a proper map, so it is proper. Since Y is quasi-projective and proper over S, it is also projective. $\square$

Definition 5.3.5. For a pair $(X, \overline{X})$, we say a simplicial pair $(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X})$ is a *special hypercovering of pairs* if:

(a) Both $X_\bullet$ and $\overline{X}_\bullet$ are split.

(b) $X_\bullet \to X$ is a special hypercovering.

(c) $\overline{X}_\bullet \to \overline{X}$ is proper. In particular, $(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X})$ is an étale-proper hypercovering.

(d) For all $n \geq 0$, $\overline{X}_n$ is projective over k.

Proposition 5.3.6. *Given a smooth scheme X and a compactification $X \hookrightarrow \overline{X}$ over k:*

i) There exists a special hypercovering of pairs $(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X})$.

ii) Let $(X,\overline{X}) \to (Y,\overline{Y})$ be a morphism of compactifications. Then, we may construct special hypercoverings of pairs $(X_\bullet,\overline{X}_\bullet)$ and $(Y_\bullet,\overline{Y}_\bullet)$ fitting into a commutative diagram

$$\begin{tikzcd}
(X_\bullet,\overline{X}_\bullet) \arrow[r] \arrow[d] & (Y_\bullet,\overline{Y}_\bullet) \arrow[d] \\
(X,\overline{X}) \arrow[r] & (Y,\overline{Y})
\end{tikzcd}$$

iii) Let $(X_\bullet,\overline{X}_\bullet),(X'_\bullet,\overline{X}'_\bullet)$ be two special hypercoverings of $(X,\overline{X})$. Then, there exists a third special hypercovering of pairs $(X''_\bullet,\overline{X}''_\bullet)$ fitting into a diagram

$$\begin{tikzcd}
& (X''_\bullet,\overline{X}''_\bullet) \arrow[dl] \arrow[dr] & \\
(X_\bullet,\overline{X}_\bullet) \arrow[dr] & & (X'_\bullet,\overline{X}'_\bullet) \arrow[dl] \\
& (X,\overline{X}) &
\end{tikzcd}$$

Proof. For i), first fix a special hypercovering $X_\bullet \to X$ by Proposition 5.3.3. Let $\{NX_k\}_{k\geq 0}$ denote the splitting of $X_\bullet$.

We construct $\overline{X}_{n+1}$ at step $n+1 \geq 0$, assuming we have constructed $\overline{X}_n$ with a splitting $\{N\overline{X}_k\}_{0\leq k\leq n}$ (the $n=-1$ case is vacuous). By Nagata's compactification theorem we may take a compactification $NX_{n+1} \hookrightarrow N\overline{X}_{n+1}$ fitting into a diagram

$$\begin{tikzcd}
NX_{n+1} \arrow[r] \arrow[d] & N\overline{X}_{n+1} \arrow[d] \\
\mathrm{cosk}_n^X(X_{\bullet\leq n})_{n+1} \arrow[r] & \mathrm{cosk}_n^{\overline{X}}(\overline{X}_{\bullet\leq n})_{n+1}
\end{tikzcd} \tag{5.9}$$

where the vertical arrows are proper maps. For the $n+1=0$, note that $\mathrm{cosk}_{-1}^X(-)_i = X$ for all i (and similar for $\overline{X}$), so the bottom row is just $X \hookrightarrow \overline{X}$.

Since NX_{n+1} is a disjoint union of affine standard smooth schemes by construction, itself is also affine standard smooth. In particular, NX_{n+1} is quasi-projective, and we may use Lemma 5.3.4 to make $N\overline{X}_{n+1}$ be projective over k in (5.9). This allows us to construct $\overline{X}_{\bullet\leq n+1} = \Omega_{n+1}^{\overline{X}}(\overline{X}_{\bullet\leq n}, N\overline{X}_0,\ldots,N\overline{X}_{n+1})$. In particular,

$$\overline{X}_{n+1} \cong \coprod_{\phi:[n+1]\twoheadrightarrow[k]} N\overline{X}_\phi$$

where each $N\overline{X}_\phi$ is projective over k. Since the disjoint union of projective schemes are still projective, it follows that $\overline{X}_{n+1}$ is projective over k. This completes i).

For ii), we first construct compatible $X_\bullet \to Y_\bullet$ over $X \to Y$ using Proposition 5.3.3, and we construct $\overline{Y}_\bullet$ as in i). Then, we build $\overline{X}_\bullet$ similarly, except that at each n, we take a compactification $N\overline{X}_{n+1}$ of NX_{n+1} over

$$\operatorname{cosk}_n^X(X_{\bullet\leq n})_{n+1} \times_{\operatorname{cosk}_n^Y(Y_{\bullet\leq n})_{n+1}} NY_{n+1} \hookrightarrow \operatorname{cosk}_n^{\overline{X}}(\overline{X}_{\bullet\leq n})_{n+1} \times_{\operatorname{cosk}_n^{\overline{Y}}(\overline{Y}_{\bullet\leq n})_{n+1}} N\overline{Y}_{n+1},$$

and by the same argument as i), such that $N\overline{X}_{n+1}$ is projective over k.

This all fits into a commutative diagram

$$\begin{tikzcd}
NX_{n+1} \arrow[r] \arrow[d] & N\overline{X}_{n+1} \arrow[d] \\
\operatorname{cosk}_n^X(X_{\bullet\leq n})_{n+1} \times_{\operatorname{cosk}_n^Y(Y_{\bullet\leq n})_{n+1}} NY_{n+1} \arrow[r] \arrow[d] & \operatorname{cosk}_n^{\overline{X}}(\overline{X}_{\bullet\leq n})_{n+1} \times_{\operatorname{cosk}_n^{\overline{Y}}(\overline{Y}_{\bullet\leq n})_{n+1}} N\overline{Y}_{n+1} \arrow[d] \\
\operatorname{cosk}_n^X(X_{\bullet\leq n}))_{n+1} \arrow[r] & \operatorname{cosk}_n^{\overline{X}}(\overline{X}_{\bullet\leq n})_{n+1}
\end{tikzcd}$$

where all horizontal morphisms are open immersions, and the vertical morphisms on the right are all proper. This gives us the desired functoriallity.

For iii), given $(X_\bullet, \overline{X}_\bullet)$ and $(X'_\bullet, \overline{X}'_\bullet)$, we may construct a special hypercovering $X''_\bullet$ refining $X_\bullet$ and $X'_\bullet$ using Proposition 5.3.3. Then, at each step, we construct $N\overline{X}''_{n+1}$ by taking a compactification of NX''_{n+1} over

$$\begin{tikzcd}
\left(\operatorname{cosk}_n^X(X''_{\bullet\leq n}) \times_{\operatorname{cosk}_n^X(X_{\bullet\leq n})} NX_{n+1}\right) \times_{\operatorname{cosk}_n^X(X'_{\bullet\leq n})} NX'_{n+1} \arrow[d] \\
\left(\operatorname{cosk}_n^{\overline{X}}(\overline{X}''_{\bullet\leq n}) \times_{\operatorname{cosk}_n^{\overline{X}}(\overline{X}_{\bullet\leq n})} N\overline{X}_{n+1}\right) \times_{\operatorname{cosk}_n^{\overline{X}}(\overline{X}'_{\bullet\leq n})} N\overline{X}'_{n+1}
\end{tikzcd}$$

and using the above argument so that $N\overline{X}''_{n+1}$ is projective over k. □

Definition 5.3.7. We call a simplicial pair $(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X})$ as in Proposition 5.3.6 a *special hypercovering* of pairs.

5.4 The simplicial special frame

Ideally, we would like to get a simplicial special frame $(X_\bullet, F_\bullet)$, with an embedding $F_\bullet \hookrightarrow P_\bullet$ for P_n being projective over W, and such that $\overline{X}_\bullet \hookrightarrow P_\bullet$

is the closure of $X_\bullet$. Then, we could do a simplicial version of the comparison in [DLZ11] directly. This problem seems difficult to do simultaneously. However, using the Tsuzuki functor below, it will suffice to do this only for one X_N, rather than all. This will capture all the simplicial structure below N, and for N large enough we will be able to ignore the terms above, as they won't contribute to the cohomology.

For a smooth scheme X and a compactification $X \hookrightarrow \overline{X}$, construct a special hypercovering $(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X})$ as in Proposition 5.3.6. Fix some $N \geq 0$. Then, since $\overline{X}_N$ is projective over k, we may find a closed immersion

$$\overline{X}_N \hookrightarrow \mathbb{P}^{r_N}_k \hookrightarrow \mathbb{P}^{r_N}_{W(k)} =: P$$

for some $r_N \geq 0$.

This gives us an immersion $X_N \hookrightarrow P$, and thus a presentation of X_N (which is affine). We may use this to lift X_N to some standard smooth affine scheme F over $W(k)$. Using the Tsuzuki functor $\Gamma_N(-)$ introduced in Definition 2.4.1, we construct the following:

- $F_\bullet := \Gamma_N^{W(k)}(F)$.
- $P_\bullet := \Gamma_N^{W(k)}(P)$.
- $Y_\bullet$ is the closure of $X_\bullet$ in $P_\bullet$.
- $Q_\bullet$ is the closure of $F_\bullet$.
- $\mathcal{Q}_\bullet$ is the p-adic completion of $Q_\bullet$.

By Lemma 2.4.2, since $\overline{X}_N \hookrightarrow P$ is a closed immersion, we have that $\overline{X}_{\bullet \leq N} \hookrightarrow P_{\bullet \leq N}$ is a closed immersion. Therefore, $X_{\bullet \leq N} \to P_{\bullet \leq N}$ factors through both $\overline{X}_{\bullet \leq N}$ and $Q_{\bullet \leq N}$, giving natural maps from $Y_{\bullet \leq N}$ to both by universal property of the closure.

This all fits into a diagram

$$\begin{tikzcd}
 & \overline{X}_{\bullet \leq N} \arrow[dr] & \\
X_{\bullet \leq N} \arrow[ur] \arrow[r] \arrow[d] & Y_{\bullet \leq N} \arrow[r] \arrow[u] \arrow[d] & P_{\bullet \leq N} \\
F_{\bullet \leq N} \arrow[r] & Q_{\bullet \leq N} \arrow[ur] &
\end{tikzcd} \tag{5.10}$$

Putting all this together:

Definition 5.4.1. Consider a special compactification

$$\begin{tikzcd} X_\bullet \arrow[r] \arrow[d] & \overline{X}_\bullet \arrow[d] \\ X \arrow[r] & \overline{X} \end{tikzcd}$$

with X (resp. $\overline{X}$) being smooth (resp. proper) over k. For a given $N \geq 0$, the information

$$\{F_\bullet, Q_\bullet, \mathcal{Q}_\bullet, Y_\bullet, P_\bullet\}$$

and the diagram from (5.10) is a *N-rigid special frame* of $(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X})$.

Lemma 5.4.2. *Given a commutative diagram of special hypercoverings*

$$\begin{tikzcd} (X_\bullet, \overline{X}_\bullet) \arrow[r] \arrow[d] & (X'_\bullet, \overline{X}'_\bullet) \arrow[d] \\ (X, \overline{X}) \arrow[r] & (X', \overline{X}') \end{tikzcd}$$

and a given $N \geq 0$, we may find N-rigid special frames of $(X_\bullet, \overline{X}_\bullet)$ and $(X'_\bullet, \overline{X}'_\bullet)$ with maps

$$\{F_\bullet, Q_\bullet, \mathcal{Q}_\bullet, Y_\bullet, P_\bullet\} \to \{F'_\bullet, Q'_\bullet, \mathcal{Q}'_\bullet, Y'_\bullet, P'_\bullet\}$$

compatible with the given diagram.

Proof. Consider a morphism $(X_\bullet, \overline{X}_\bullet) \to (X'_\bullet, \overline{X'}_\bullet)$ over $(X, \overline{X}) \to (X', \overline{X}')$. We may construct $\{F'_\bullet, Q'_\bullet, \mathcal{Q}'_\bullet, Y'_\bullet, P'_\bullet\}$ as explained above. Then, since $\overline{X}_N$ and $\overline{X}'_N$ are projective over k, the map between them is projective, and we may find some closed immersion of $\overline{X}_N$ into $P = \mathbb{P}^r_W$ fitting into the commutative diagram

$$\begin{tikzcd} \overline{X}_N \arrow[r] \arrow[d] & P \arrow[d] \\ \overline{X}'_N \arrow[r] & P'. \end{tikzcd}$$

This gives representations of X_N and $\overline{X}'_N$ in compatible affine spaces, and we thus may lift them to standard smooth affine schemes over $W(k)$ as explained in Section 5.2 in a compatible way, giving a commutative diagram

$$\begin{tikzcd} X_N \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & P \arrow[d] \\ X'_N \arrow[r] & F' \arrow[r] & P'. \end{tikzcd}$$

Since the functor $\Gamma_N^{\mathcal{W}}$, p-adic completion and closure are functorial, all the remaining functoriality will follow from that of P and F.

□

This gives an N-truncated special frame

$$(X_{\bullet \leq N}, F_{\bullet \leq N}), \tag{5.11}$$

and an N-truncated étale-proper hypercovering of $(X, \overline{X})$

$$(X_{\bullet \leq N}, Y_{\bullet \leq N}, \mathcal{Q}_{\bullet \leq N}). \tag{5.12}$$

5.5 The comparison theorem

In the course of the proof, we will need some tools from [DLZ11] in order to compare special frames. In *loc. cit.*, Proposition 4.35 proves the first result, and the second result is Proposition 4.37.

Proposition 5.5.1.

i) Given a map of special frames

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & F' \arrow[d] \\ X \arrow[r] & F \end{tikzcd}$$

with the right vertical map being étale, then we get a natural isomorphism of dagger spaces

$$]X[^{\dagger}_{\mathscr{F}'} \cong]X[^{\dagger}_{\mathscr{F}}.$$

ii) For some n, let $(X, F \times \mathbb{A}^n_{W(k)})$ be a special frame such that the map $X \to \mathbb{A}^n_{W(k)}$ factors through the origin. Then, the induced map gives a quasi-isomorphism

$$Rsp_* \Omega^\bullet_{]X[^{\dagger}_{\mathscr{F}}} \xrightarrow{\sim} Rsp_* \Omega^\bullet_{]X[^{\dagger}_{\mathscr{F} \times \mathbb{A}^n_{\mathcal{W}}}}.$$

When proving the comparison, we will need to show vanishing of the higher cohomologies of $\mathrm{Rsp}_*\Omega^\bullet_{]Y_m[^\dagger_{\mathscr{F}_m}}$ for $0 \leq m \leq N$, where (Y_m, F_m) are the special frames constructed in sections 5.3 and 5.4. The above proposition will allow us to reduce it to the following theorem, which follows from the proof of [Ber97b, Theorem 1.10]:

Proposition 5.5.2. *Let (X, F) be a special frame, where F is a lifting of X over $W(k)$. Then,*

$$R^i sp_* \Omega^\bullet_{]X[^\dagger_{\mathscr{F}}} = 0 \text{ for } i>0.$$

We now prove a key ingredient of the comparison theorem:

Proposition 5.5.3. *Let $(X_{\bullet \leq N}, F_{\bullet \leq N})$ be an N-truncated simplicial frame as in (5.11). Then, for $0 \leq m \leq N$ and $i > 0$,*

$$R^i sp_* \Omega^\bullet_{]X_m[^\dagger_{\mathscr{F}_m}} = 0.$$

Proof. Pick any $0 \leq m \leq N$. By splitness of $X_\bullet$, we may write

$$X_m = \bigsqcup_{\phi:[N]\twoheadrightarrow[m]} NX_{m,\phi}.$$

Fix some degeneracy map $\sigma : [N] \twoheadrightarrow [m]$. Then, by construction of $\Gamma_N^{W(k)}(-)$, we have a commutative diagram

$$\begin{tikzcd} X_m \arrow[r] \arrow[d, "X_\bullet(\sigma)"'] & F_m = \prod_{\alpha:[N]\to[m]} F_\alpha \arrow[d, "p_\sigma"] \\ Y_N \arrow[r] & F = F_\sigma \end{tikzcd}$$

where $F_\phi = F$ for any $\phi : [N] \twoheadrightarrow [m]$ was defined in section 5.4 as just a lift of X_N in P_N, and p_σ is the projection, and both horizontal maps and the left vertical map are closed immersions. This gives us a closed immersion

$$X_m \hookrightarrow F_\sigma.$$

Let

$$F'_m := \prod_{\alpha:[N]\to[m],\alpha\neq\sigma} F_\alpha,$$

so $F_m = F'_m \times F_\sigma$. Then, since each of the F_α are standard smooth schemes over $W(k)$, so is their product, and we may get an étale morphism

$$F'_m \to \mathbb{A}^n_{W(k)}$$

for some n. Thus, considering the commutative diagram

$$\begin{tikzcd} X_m \arrow[r] \arrow[d, equal] & F'_m \times F_\sigma \arrow[r, equal] \arrow[d] & F_m \\ X_m \arrow[r] & \mathbb{A}^n_{W(k)} \times F_\sigma & \end{tikzcd}$$

where the right vertical morphism is étale, using Proposition 5.5.1.i) we may reduce to the case of the special frame $(X_m, \mathbb{A}^n_{W(k)} \times F_\sigma)$. Furthermore, we may assume that the map $X_m \to \mathbb{A}^n_{W(k)}$ factors through the origin. To see this, write $X_m = \text{Spec }(A)$ and $F_\sigma = \text{Spec } B$, so $\mathbb{A}^n_{W(k)} \times F_\sigma = \text{Spec } B[T_1, ..., T_n]$. Then, since $B \twoheadrightarrow A$ is surjective (as $X_m \to F_\sigma$ is a closed immersion), we may pick $b_1, ..., b_n \in B$ which map to the images of $T_1, ..., T_n$ respectively in A, and replace T_i by $T'_i := T_i - b_i$, giving a special frame

$$(X_m, \text{Spec } B[T'_1, ..., T'_n]) = (X_m, \mathbb{A}^n_{W(k)} \times F_\sigma)$$

factoring through the origin. Thus, by Proposition 5.5.1.ii), we reduce the proof to the special frame (X_m, F_σ).

Now, since

$$]X_m[^\dagger_{\mathcal{F}_\sigma} =]\coprod_{\phi:[m]\twoheadrightarrow[k]} NX_{m,\phi}[^\dagger_{\mathcal{F}_\sigma} \cong \coprod_{\phi:[m]\twoheadrightarrow[k]}]NX_{m,\phi}[^\dagger_{\mathcal{F}_\sigma}$$

we may reduce to studying the special frames $(NX_{m,\phi}, F_\sigma)$ for any $\phi : [m] \twoheadrightarrow [k]$ and $0 \leq k \leq m$. But notice that by the construction of the frame, for any $\phi : [m] \twoheadrightarrow [k]$, we have a commutative diagram

$$\begin{tikzcd} NX_{m,\phi} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\wr\parallel"] & X_m \arrow[d, hook, "X_\bullet(\sigma)"] & & & \\ NX_{N,\phi\circ\sigma} \arrow[r, phantom, "\subset"] & X_N \arrow[r, hook] & F_\sigma \arrow[r, equal] & F_N \arrow[r, equal] & \coprod_{\psi:[N]\twoheadrightarrow[k']} NF_{N,\psi} \end{tikzcd}$$

where ψ vary over all morphisms $\psi : [N] \twoheadrightarrow [k']$ with $0 \leq k' \leq N$, and the composite map $NX_{m,\phi} \to F_N$ is the map giving the special frame. Thus, $NX_{m,\phi}$ is isomorphic to $NF_{N,\phi\circ\sigma} \subset F_N$, and therefore

$$\text{sp}^{-1}(NX_{m,\phi}) =]NX_{m,\phi}[^\dagger_{\mathcal{F}_\sigma} =]NX_{m,\phi}[^\dagger_{N\mathcal{F}_{N,\phi\circ\sigma}},$$

which reduces the proof to the case of the special frame $(NX_{m,\phi}, NF_{N,\phi\circ\sigma})$.

But by construction, $NF_{N,\phi\circ\sigma}$ is a smooth lift of $NX_{N,\phi\circ\sigma} \cong NX_{m,\phi}$ over $W(k)$, and thus we can apply Proposition 5.5.2 to complete the proof. $\square$

We will need the following to deal with N-truncations, which basically says that for some large enough N, we only need the N-skeleton in the calculations of cohomologies on simplicial objects (such as for rigid cohomology and overconvergent Witt de-Rham). For a complex $A^\bullet$ of K vector spaces, and any h, consider the h-truncated complex

$$\tau_{\leq h}(A^\bullet)^i = \begin{cases} A^i \text{ if } i < h \\ \ker(A^h \to A^{h+1}) \text{ if } i > h \\ 0 \text{ else.} \end{cases}$$

For a double complex $A^{\bullet\bullet}$, let $\tau^{(1)}_{\leq h}(A^{\bullet q}) := \tau_{\leq h}(A^{\bullet q})$, and let $s : C(K) \to K$ be the total complex map.

Lemma 5.5.4. *[Nak12, Lemma 2.2] Consider a double complex $A^{\bullet,\bullet}$ of K vector spaces such that $A^{p,q} = 0$ for $p < 0$ or $q < 0$. Given any*

$$N > \max\{i + (h - i + 1)(h - i + 2)/2 \mid 0 \leq i \leq h\} = (h + 1)(h + 2)/2, \tag{5.13}$$

the natural maps $s(\tau^{(1)}_{\leq N}(A^{\bullet\bullet}) \to s(A^{\bullet\bullet})$ induce a quasi-isomorphism

$$\tau_{\leq h}(s(\tau^{(1)}_{\leq N}(A^{\bullet\bullet})) \xrightarrow{\sim} \tau_{\leq h}(s(A^{\bullet\bullet})).$$

From this, and the formation of the spectral sequence for cohomology on simplicial objects, it follows for example that for some simplicial rigid frame $(Z_\bullet, \overline{Z}_\bullet, \mathcal{Z}_\bullet)$, and h and N as in (5.13), we get natural quasi-isomorphisms

$$\tau_{\leq h} R\Gamma(]\overline{Z}_{\bullet \leq N}[_{\mathcal{Z}_{\bullet \leq N}}, j^\dagger \Omega^\bullet_{]\overline{Z}_{\bullet \leq N}[_{\mathcal{Z}_{\bullet \leq N}}}) \xrightarrow{\sim} \tau_{\leq h} R\Gamma(]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet}, j^\dagger \Omega^\bullet_{]\overline{Z}_\bullet[_{\mathcal{Z}_\bullet}}) \tag{5.14}$$

and that for a smooth simplicial scheme $X_\bullet$,

$$\tau_{\leq h} R\Gamma(X_\bullet, W^\dagger \Omega^\bullet_{X_{\bullet/k}}) \xrightarrow{\sim} \tau_{\leq h} R\Gamma(X_{\bullet \leq N}, W^\dagger \Omega^\bullet_{X_{\bullet \leq N/k}}). \tag{5.15}$$

This is useful because of the vanishing of rigid cohomology from Theorem 2.3.4, which tells us that there exists a c such that

$$\tau_{\bullet \leq c} R\Gamma_{\text{rig}}(X/K) \xrightarrow{\sim} R\Gamma_{\text{rig}}(X/K)$$

so letting $h \geq c$ and N as in (5.13), we may compute rigid chomology with an N-truncated de-Rham descendable hypercovering by (5.14).

We can now prove the main comparison theorem:

Theorem 5.5.5. *Given a smooth scheme X over k, there exists a functorial quasi-isomorphism*

$$R\Gamma_{rig}(X/K) \xrightarrow{\sim} R\Gamma(X, W^\dagger\Omega^\bullet_{X/k}) \otimes \mathbb{Q}.$$

Proof. Choose a compactification $\overline{X}$ of X and construct $\{X_\bullet, \overline{X}_\bullet, F_\bullet, Q_\bullet, \mathcal{Q}_\bullet, Y_\bullet, P_\bullet\}$ as in the previous section.

Then, $(X_{\bullet\leq N}, F_{\bullet\leq N})$ is an N-truncated special frame, so by functoriality of the construction in [DLZ11]

$$\text{Special Frames} \to \text{Dagger Spaces}$$

$$(X, F) \mapsto]X[^\dagger_F$$

we get an N-truncated special frame $]X_{\bullet\leq N}[^\dagger_{\mathscr{F}_{\bullet\leq N}}$.

Furthermore, from (5.8), we get

$$\mathrm{sp}_*\Omega^\bullet_{]X_{\bullet\leq N}[^\dagger_{\mathscr{F}_{\bullet\leq N}}} \to W^\dagger\Omega^\bullet_{X_{\bullet\leq N}/k} \otimes \mathbb{Q} \tag{5.16}$$

to give us a quasi-isomorphism

$$R\Gamma(X_{\bullet\leq N}, \mathrm{sp}_*\Omega^\bullet_{]X_{\bullet\leq N}[^\dagger_{\mathscr{F}_{\bullet\leq N}}}) \xrightarrow{\sim} R\Gamma(X_{\bullet\leq N}, W^\dagger\Omega^\bullet_{X_{\bullet\leq N}/k}) \otimes \mathbb{Q}. \tag{5.17}$$

Next, by Proposition 5.5.3, we have that

$$\mathrm{sp}_*\Omega^\bullet_{]X_{\bullet\leq N}[^\dagger_{\mathscr{F}_{\bullet\leq N}}} \xrightarrow{\sim} \mathrm{Rsp}_*\Omega^\bullet_{]X_{\bullet\leq N}[^\dagger_{\mathscr{F}_{\bullet\leq N}}} \tag{5.18}$$

so that

$$R\Gamma(]X_{\bullet\leq N}[^\dagger_{\mathscr{F}_{\bullet\leq N}}, \Omega^\bullet_{]X_{\bullet\leq N}[^\dagger_{\mathscr{F}_{\bullet\leq N}}}) \cong R\Gamma(X_{\bullet\leq N}, W^\dagger\Omega^\bullet_{X_{\bullet\leq N},k}) \otimes \mathbb{Q} \tag{5.19}$$

Next, note that

$$(X'_\bullet, Y'_\bullet, \mathcal{Q}'_\bullet) := \mathrm{cosk}_N(X_{\bullet\leq N}, Y_{\bullet\leq N}, \mathcal{Q}_{\bullet\leq N})$$

is an étale-proper hypercovering of $(X, \overline{X})$ by Lemma 2.2.6, and thus a de Rham desecendable hypercovering of $(X, \overline{X})$ by Proposition 2.3.2, so

$$R\Gamma_{\mathrm{rig}}(X/K) = R\Gamma(]Y'_\bullet[_{\mathcal{Q}'_\bullet}, j^\dagger\Omega^\bullet_{]Y'_\bullet[_{\mathcal{Q}'_\bullet}}) \tag{5.20}$$

computes the rigid cohomology.

Furthermore, since for any $n \leq N$, Y_n is the closure of X_n in P_n and $\mathcal{Q}_n$ is the p-adic completion of the closure of F_n in P_n, we get an N-truncated simplicial version of (5.7):

$$R\Gamma(]X_{\bullet\leq N}[^{\dagger}_{\mathscr{F}_{\bullet\leq N}}, \Omega^{\bullet}_{]X_{\bullet\leq N}[^{\dagger}_{\mathscr{F}_{\bullet\leq N}}}) \cong R\Gamma(]Y_{\bullet\leq N}[_{\mathcal{Q}_{\bullet\leq N}}, j^{\dagger}\Omega^{\bullet}_{]Y_{\bullet\leq N}[_{\mathcal{Q}_{\bullet\leq N}}}) \tag{5.21}$$

Putting all this together, we will get

$$R\Gamma(]Y_{\bullet\leq N}[_{\mathcal{Q}_{\bullet\leq N}}, j^{\dagger}\Omega^{\bullet}_{]Y_{\bullet\leq N}[_{\mathcal{Q}_{\bullet\leq N}}}) \cong R\Gamma(X_{\bullet\leq N}, W^{\dagger}\Omega^{\bullet}_{X_{\bullet\leq N}/k}) \otimes \mathbb{Q} \tag{5.22}$$

for any N.

We now show that the left-hand side computes $R\Gamma_{\mathrm{rig}}(X/K)$ (compare with (5.20)) and that the right hand side computes $R\Gamma(X, W^{\dagger}\Omega^{\bullet}_{X/k}) \otimes \mathbb{Q}$. Using Proposition 2.3.4, pick some c such that for any $h \geq c$,

$$\tau_{\leq h} R\Gamma_{\mathrm{rig}}(X/K) \xrightarrow{\sim} R\Gamma_{\mathrm{rig}}(X/K),$$

and pick $N = N(h)$ large enough to satisfy (5.13). Then,

$$\begin{aligned} R\Gamma_{\mathrm{rig}}(X/K) \cong & \tau_{\leq h} R\Gamma_{\mathrm{rig}}(X/K) \cong \tau_{\leq h} R\Gamma(]Y'_{\bullet}[_{\mathcal{Q}'_{\bullet}}, j^{\dagger}\Omega^{\bullet}_{]Y'_{\bullet}[_{\mathcal{Q}'_{\bullet}}}) \\ & \cong \tau_{\leq h} R\Gamma(]Y_{\bullet\leq N}[_{\mathcal{Q}_{\bullet\leq N}}, j^{\dagger}\Omega^{\bullet}_{]Y_{\bullet\leq N}[_{\mathcal{Q}_{\bullet\leq N}}}) \end{aligned} \tag{5.23}$$

so the left hand side of (5.22) will compute rigid cohomology.

On the other hand, since $X_{\bullet} \to X$ is an étale hypercovering, and $W^{\dagger}\Omega^{\bullet}_{X/k}$ is an étale sheaf, we have that

$$R\Gamma(X, W^{\dagger}\Omega^{\bullet}_{X/k}) \xrightarrow{\sim} R\Gamma(X_{\bullet}, W^{\dagger}\Omega^{\bullet}_{X_{\bullet}/k})$$

To compare to the truncated version in (5.22), note that for any h and N satisfying (5.13), we have by Lemma 5.5.3 and (5.22) that

$$\begin{aligned} \tau_{\leq h} R\Gamma_{\mathrm{rig}}(X/K) & \cong \tau_{\leq h} R\Gamma(X_{\bullet\leq N}, W^{\dagger}\Omega^{\bullet}_{X_{\bullet\leq N}/k}) \otimes \mathbb{Q} \\ & \cong \tau_{\leq h} R\Gamma(X_{\bullet}, W^{\dagger}\Omega^{\bullet}_{X_{\bullet}/k}) \otimes \mathbb{Q} \cong \tau_{\leq h} R\Gamma(X, W^{\dagger}\Omega^{\bullet}_{X/k}) \end{aligned} \tag{5.24}$$

Varying h (and N), we see that cohomology vanishes in $R\Gamma(X, W^{\dagger}\Omega^{\bullet}_{X/k})$ above c, and thus letting $h \geq c$ we may drop it from (5.24) to obtain

$$R\Gamma_{\mathrm{rig}}(X/K) \cong R\Gamma(X, W^{\dagger}\Omega^{\bullet}_{X/k}) \otimes \mathbb{Q} \tag{5.25}$$

It remains to show independence and functoriality.

Independence:

We must show independence of the choices of $\overline{X}$, $X_\bullet$, $\overline{X}_\bullet$, $F_\bullet$, $Q_\bullet$, $P_\bullet$, and N.

The independence of $\overline{X}$ follows from independence of $\overline{X}$ in computation of rigid cohomology ([CT03, Corollary 10.5.4]).

To show independence of $\{X_\bullet, \overline{X}_\bullet, F_\bullet, Q_\bullet, \mathcal{Q}_\bullet, Y_\bullet, P_\bullet\}$, suppose for a given pair $(X, \overline{X})$, we have made two choices $\{X^i_\bullet, \overline{X}^i_\bullet, F^i_\bullet, Q^i_\bullet, \mathcal{Q}^i_\bullet, Y^i_\bullet, P^i_\bullet\}$ for $i = 1, 2$. Then, by Proposition 5.3.6 it follows that we may find $(X^3_\bullet, \overline{X}^3_\bullet)$ refining them. Furthermore, since $\overline{X}^3_N$ is projective over $\overline{X}^1_N$ and $\overline{X}^2_N$, we can find some P^3 fitting into the diagram

$$\begin{tikzcd}
 & (X^3_N, \overline{X}^3_N, P^3) \arrow[dl] \arrow[dr] & \\
(X^1_N, \overline{X}^1_N, P^1) \arrow[dr] & & (X^2_N, \overline{X}^2_N, P^2) \arrow[dl] \\
 & (X, \overline{X}, W(k)) &
\end{tikzcd}$$

Furthermore, we may take the standard smooth lift F^3 of X^3_N over $W(k)$ to be compatible with the standard smooth lifts F^i of X^i_N over $W(k)$ for $i = 1, 2$.

All this compatibility carries over when applying $\Gamma^{W(k)}_N$, taking closures and completions, and applying $\operatorname{cosk}^X_n, \operatorname{cosk}^{\overline{X}}_N, \operatorname{cosk}^{\mathcal{W}}_N$, which gives a diagrams of N-truncated simplicial special frames

$$\begin{tikzcd}
 & (X^3_{\bullet\leq N}, F^3_{\bullet\leq N}) \arrow[dl] \arrow[dr] & \\
(X^3_{\bullet\leq N}, F^3_{\bullet\leq N}) & & (X^3_{\bullet\leq N}, F^3_{\bullet\leq N})
\end{tikzcd}$$

and of universally de Rham descendable hypercoverings of $(X, \overline{X})$

$$\begin{tikzcd}
 & (X'^3_\bullet, Y'^3_\bullet, \mathcal{Q}'^3_\bullet) \arrow[dl] \arrow[dr] & \\
(X'^1_\bullet, Y'^1_\bullet, \mathcal{Q}'^1_\bullet) & & (X'^2_\bullet, Y'^2_\bullet, \mathcal{Q}'^2_\bullet)
\end{tikzcd}$$

where we use the notation from above. This gives a factorization of all the maps used that shows independence in $D_+(K)$.

For independence of N, we argue similarly to the proof of Lemma 4.2.2. Suppose we fix $(X_\bullet, \overline{X}_\bullet)$. Given to choices N^1 and N^2 satisfying (5.13) for $h = c$, suppose $N^2 \geq N^1$, and construct $F^i_\bullet, Q^i_\bullet, \mathcal{Q}^i_\bullet, Y^i_\bullet, P^i_\bullet\}$ for $i = 1, 2$. We then have a natural map

$$\begin{array}{c} (X'^2_\bullet, Y'^2_\bullet, \mathcal{Q}'^2_\bullet) := \operatorname{cosk}^{(X,\overline{X},\mathcal{W})}_{N^2}(X_{\bullet\leq N^2}, Y^2_{\bullet\leq N^2}, \mathcal{Q}^2_{\bullet\leq N^2}) \\ \Big\downarrow \\ \operatorname{cosk}^{(X,\overline{X},\mathcal{W})}_{N^1}(X_{\bullet\leq N^1}, Y^2_{\bullet\leq N^1}, \mathcal{Q}^2_{\bullet\leq N^1}) \end{array}$$

induced by the maps $\operatorname{cosk}_{N^2} \to \operatorname{cosk}_{N^1} \circ \operatorname{sk}_{N^1} \circ \operatorname{cosk}_{N^2} \cong \operatorname{cosk}_{N^1} \circ \operatorname{sk}_{N^1}$. This all induces a commutative diagram in the diagrams (5.22) for N^2 and N^1. This is compatible with the maps in (5.24). Thus, we may replace N^2 with N^1 (as long as they are both large enough), and then independence above (for N^1 fixed) shows the independence of choices.

Functoriality: Given a map $X^1 \to X^2$, we may choose compatible $\overline{X}^1$ and $\overline{X}^2$, and then pick compatible $(X^i_\bullet, \overline{X}^i_\bullet)$ by Proposition 5.3.6, and by Lemma 5.4.2 we may choose compatible N-rigid special frames.

□

Chapter 6

CONJECTURE B

6.1 Conjecture B

We study the following conjecture:

Conjecture 6.1.1 (Conjecture B)**.** *For a separated, finite type k-scheme X of dimension d, and $n \in \mathbb{Z}$, there exists a quasi-isomorphism*

$$R\Gamma(X_{et}, \mathbb{Q}_p^c(n)) \xrightarrow{\sim} \left[R\Gamma_{rig,c}(X/K)^* \overset{\phi - p^{n-d}}{\to} R\Gamma_{rig,c}(X/K)^* \right] [-2d].$$

Here, $R\Gamma_{rig,c}(X/K)^* := R\mathrm{Hom}(R\Gamma_{rig,c}(X/K), K)$.

We prove this conjecture for X smooth:

Theorem 6.1.2. *If X is smooth, then Conjecture 6.1.1 holds.*

Proof. As before, since X is smooth and we have

$$\mathbb{Z}^c(n)/p^r \cong W_r\Omega^n_{log}[-n],$$

we can identify

$$R\Gamma_{rig}(X, \mathbb{Q}_p^c(n)) \cong R \varprojlim_r R\Gamma(X_{et}, W_r\Omega^n_{X,\log})_{\mathbb{Q}}[-n],$$

and as in the proof of Theorem 4.2.1 we have a short exact sequence

$$0 \to W\Omega^n_{X,\log} \to W\Omega^n_X \overset{1-F}{\to} W\Omega^n_X \to 0$$

in X_{et}, and $W\Omega^n_X \cong R\varprojlim_r W_r\Omega^n_X$, so we have

$$R\Gamma_{rig}(X, \mathbb{Q}_p^c(n)) \cong \left[R\Gamma(X_{et}, W\Omega^n_X) \overset{1-F}{\to} R\Gamma(X_{et}, W\Omega^n_X) \right]_{\mathbb{Q}} [-n].$$

Next, by [Ert14, Corollary 2.4.12], we have that all logarithmic Witt de-Rham sections are overconvergent, and that $1 - F$ is still surjective when restricted to the overconvergent part; so we have a commutative diagram in X_{et}:

$$\begin{tikzcd}
0 \arrow[r] & W\Omega^n_{X,\log} \arrow[r] \arrow[d, equal] & W^\dagger\Omega^n_X \arrow[r, "1-F"] \arrow[d, hook] & W^\dagger\Omega^n_X \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & W\Omega^n_{X,\log} \arrow[r] & W\Omega^n_X \arrow[r, "1-F"] & W^\dagger\Omega^n_X \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are given by inclusion, and both rows are short exact sequences. Thus, we get a natural quasi-isomorphism

$$\left[R\Gamma(X_{et}, W\Omega^n_X) \overset{1-F}{\to} R\Gamma(X_{et}, W\Omega^n_X)\right]_{\mathbb{Q}}[-n]$$
$$\wr|$$
$$\left[R\Gamma(X_{et}, W^\dagger\Omega^n_X) \overset{1-F}{\to} R\Gamma(X_{et}, W^\dagger\Omega^n_X)\right]_{\mathbb{Q}}[-n].$$

We consider the Frobenius on $W^\dagger\Omega^\bullet_X$ by restricting that on $W\Omega^\bullet_X$. So as before, the part with slope p^n must be coming from $W^\dagger\Omega^n$, thus giving

$$\left[R\Gamma(X_{et}, W^\dagger\Omega^n_X) \overset{1-F}{\to} R\Gamma(X_{et}, W^\dagger\Omega^n_X)\right]_{\mathbb{Q}}[-n] \cong \left[R\Gamma(X_{et}, W^\dagger\Omega^\bullet_X) \overset{p^n-\phi}{\to} R\Gamma(X_{et}, W^\dagger\Omega^\bullet_X)\right]_{\mathbb{Q}}.$$

Then, we use the comparison from overconvergent Witt de-Rham cohomology to rigid cohomology for smooth schemes given by Theorem 5.5.5 to get that

$$R\Gamma_{rig}(X/K) \overset{\sim}{\to} R\Gamma(X, W^\dagger\Omega^\bullet_{X/k})_{\mathbb{Q}}$$

and thus

$$R\Gamma_{rig}(X, \mathbb{Q}^c_p(n)) \cong \left[R\Gamma_{rig}(X/K) \overset{p^n-\phi}{\to} R\Gamma_{rig}(X/K)\right].$$

Finally, from [Ber97a, Théorème 2.4] we can use Poincaré duality for rigid cohomology to get non-degenerate pairings

$$H^i_{rig}(X/K) \times H^{2d-i}_{rig,c}(X/K) \to H^{2d}_{rig,c}(X/K) \overset{\sim}{\to} K(-d)$$

compatible as F-crystals, where $K(-d)$ is K with a Frobenius action given by multiplication by p^d. Thus, we have a natural quasi-isomorphism

$$R\Gamma_{rig}(X/K) \overset{\sim}{\to} R\mathrm{Hom}(R\Gamma_{rig,c}(X/K)^*[-2d] := R\mathrm{Hom}(R\Gamma_{rig,c}(X/K), K)[-2d]$$

and therefore,

$$\begin{aligned} R\Gamma_{rig}(X, \mathbb{Q}_p(n)) &\cong \left[R\Gamma_{rig}(X/K_0) \overset{p^n-\phi}{\to} R\Gamma_{rig}(X/K_0)\right] \\ &\cong \left[R\Gamma_{rig,c}(X/K)^* \overset{p^{d-n}-\phi}{\to} R\Gamma_{rig,c}(X/K)^*\right][-2d]. \end{aligned}$$

□

Remark. In order to prove Conjecture B for the general case, one should try and get a map

$$R\Gamma(X, \mathbb{Q}_p^c(n)) \to \left[R\Gamma_{rig,c}(X/K)^* \stackrel{\phi - p^{n-d}}{\to} R\Gamma_{rig,c}(X/K)^* \right] [-2d]$$

for the general case compatible with that used to prove the isomorphism in the smooth case. Then, using localization triangles and an induction on dimension one could prove that it is an isomorphism.

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