COMPLEX BIFURCATION

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Two roads diverged in a wood, and I —
I took the one less traveled by,
And that has made all the difference.

— Robert Frost, *The Road Not Taken*

Abstract

Real equations of the form $g(x, \lambda) = 0$ are shown to have a complex extension $G(u, \lambda) = 0$, defined on the complex Banach space $\mathbb{B} \oplus i\mathbb{B}$. At a singular point of the real equation this extension has solution branches corresponding to both the real and imaginary roots of the Algebraic Bifurcation Equations (ABE's).

We solve the ABE's at simple quadratic folds, quadratic bifurcation points, and cubic bifurcation points, and show that these are complex bifurcation points. We also show that at a Hopf bifurcation point of the real equation there are two families of complex periodic orbits, parametrized by three real parameters.

By taking sections of solutions of complex equations with two real parameters, we show that complex branches may connect disjoint solution branches of the real equation. These complex branches provide a simple and practical means of locating disjoint branches of real solutions.

Finally, we show how algorithms for computing real solutions may be modified to compute complex solutions. We use such an algorithm to find solutions of several example problems, and locate two sets of disjoint real branches.

Table of Contents

I. Introduction

Let (x_0, λ_0) be a singular point of a real, nonlinear functional equation

$$g(x, \lambda) = 0,$$

where

$$x \in \mathbb{B}, \text{ a real Banach space}$$

$$\lambda \in \mathbb{R}$$

$$g : \mathbb{B} \times \mathbb{R} \to \mathbb{B}$$

and $\quad g$ has $k \geq 2$ continuous derivatives in $\mathbb{B} \times \mathbb{R}$.

That is, a solution $g(x_0, \lambda_0) = 0$ at which the Fréchet derivative $g_x^0 \equiv g_x(x_0, \lambda_0)$ is singular, and has finite dimensional nullspaces. The number of solutions in a small neighborhood of the singular point is governed by the number of real roots of the Algebraic Bifurcation Equations (the ABE's). For each isolated real solution there is a distinct half branch of solutions $(x(s), \lambda(s))$, $s > 0$, that passes through the singular point. See, for example, Decker and Keller (2).

If the definition of g is extended to u in the complex Banach space $\mathbb{B} \oplus i\mathbb{B}$, the nonreal roots of the ABE's must be considered as well. For each isolated nonreal root there is a distinct half branch of nonreal solutions. We call this Complex Bifurcation. Figure 1 shows the complex branchs for three types of singular points. Notice that the simple quadratic fold, which is not a bifurcation point, is a *complex* bifurcation point.

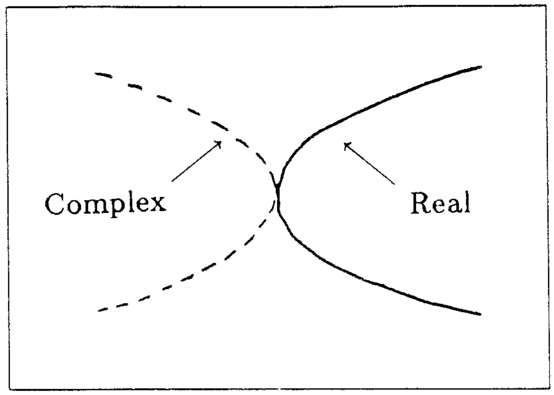
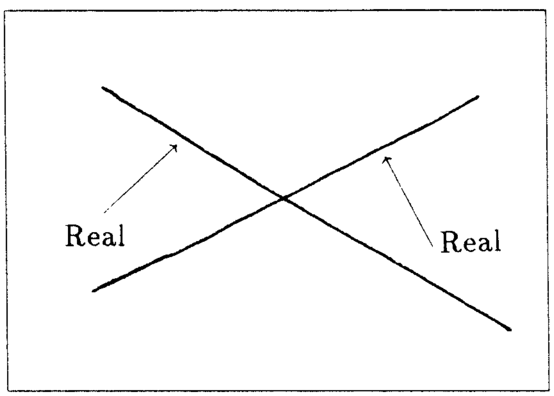


Figure 1a. Complex Branches at a Simple Quadratic Fold



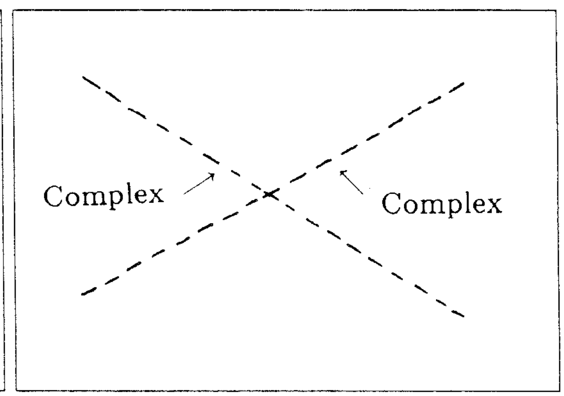


Figure 1b. Complex Branches at a Simple Quadratic Bifurcation

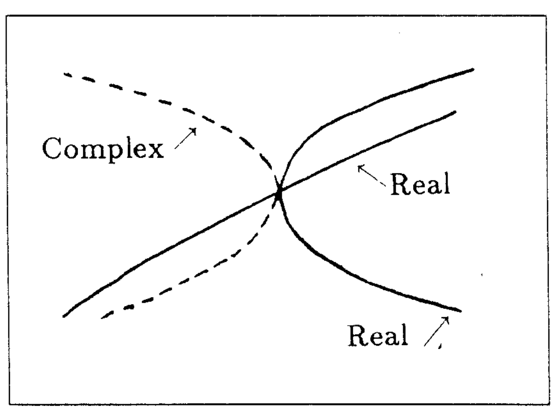


Figure 1c. Complex Branches at a Simple Cubic Bifurcation

In chapter II we rigorously define the complex extension $G(u, \lambda)$, of a real equation. Using an Implicit Function Theorem for mappings of a complex Banach space, we then discuss the behavior of solutions near a regular solution. We are also able to prove a global result, that if G is analytic, and is real for real u, the complex solutions must occur in conjugate pairs, by using the Schwartz Reflection Principle.

In chapter III we prove the existence of complex bifurcation at several simple singular points, using a technique based on a Lyapunov-Schmidt decomposition and the Implicit Function Theorem. We present results for the three bifurcation points shown in Figure 1.

In the first section of chapter IV, we show that it is possible for complex branches to connect two disjoint solution branches of the real equation. We do this by considering sections of solutions of equations with two real parameters. These complex connections provide a practical means of locating disjoint real branches. No knowledge of the existence of the branches is required, and no random searching is necessary.

In the second section of chapter IV, we show that complex bifurcation occurs at Hopf bifurcation points on a real branch of solutions. Complex periodic orbits exist for parameter values above and below the bifurcation point, and two three-parameter families of complex orbits bifurcate.

Chapter IV discusses how to compute complex solution branches numerically. We have modified Keller's pseudo arc-length continuation algorithm (10), and used it to compute the solutions of the examples in chapter VI. We also discuss how to write the complex equation as a pair of real equations, so that existing algorithms can be used.

Finally, in chapter VI, we present several examples that exhibit the various properties of complex bifurcation. We include a problem from Fluid Mechanics, the flow between rotating coaxial disks, that has several real branches of solutions that are connected by complex isolas. We also compute the complex surfaces associated with the Cusp, Swallowtail, and Butterfly catastrophes.

II. Complex Bifurcation

We consider equations of the general form:

$$g(x, \lambda) = 0, \tag{1}$$

where x is an element of a real Banach space $\mathbb{B}$, λ is a real parameter and g is a C^∞ mapping of $\mathbb{B} \times \mathbb{R} \to \mathbb{B}$. These equations have two general types of solutions: regular solutions, at which the Fréchet derivative g_x is nonsingular, and singular solutions, where g_x is singular.

At a regular solution the Implicit Function Theorem can be used to show that a unique smooth arc of solutions of the form $(x(\lambda), \lambda)$ must pass through the regular solution. At a singular solution several solution arcs may touch*. The number of arcs, and the type of contact, is determined by a set of algebraic equations called the Algebraic Bifurcation Equations, or the ABE's. Each isolated real root of the ABE's determines a smooth arc of solutions that passes through the singular point. Since the ABE's are algebraic equations, they may also have isolated solutions which are complex. If the definition of equation (1) is extended to include points in the complex Banach space $\mathbb{B} \oplus i\mathbb{B}$, the complex roots determine complex arcs of solutions through the singular point. When complex solution arcs exist, we call the singular point a Complex Bifurcation point.

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* Other types of solutions, such as surfaces of solutions, may also exist at singular points.

Instead of solving equation (1), we propose to solve its complex extension, $G(u, \lambda) = 0$. In addition to the solutions of (1), the extension has complex solution branches. We have found that these complex solutions can be used to locate solution branches of (1) that are otherwise difficult to find.

In the following section we define this complex extension G, and summarize some of the properties of $\mathbb{B} \oplus i\mathbb{B}$. We then prove a useful global result, that complex solutions of the extension occur in conjugate pairs. The final section of this chapter shows that regular solutions of $G = 0$ lie on smooth arcs, and that if the regular solution is real, the arc it lies on must be real. We also show that a single real solution that lies on a complex arc cannot be a regular solution. It must be a singular solution.

The Complex Extension of a Real Equation

We begin by defining the complex Banach space $\mathbb{B} \oplus i\mathbb{B}$ as

$$\mathbb{B} \oplus i\mathbb{B} \equiv \{u \mid u = x + iy, \quad x, y \in \mathbb{B}\}.$$

If $\mathbb{B}$ has the norm $\| \cdot \|_B$, $\mathbb{B} \oplus i\mathbb{B}$ can be given the norm

$$\| u \|_{B+iB}^2 \equiv \| x \|_B^2 + \| y \|_B^2.$$

Each element of $\mathbb{B} \oplus i\mathbb{B}$ has a complex conjugate $\overline{u}$, defined as

$$\overline{u} \equiv x - iy.$$

It also has an adjoint, or dual u^*. The adjoint satisfies $\Re e(u^* u) = \| u \|^2$, so u^* must be

$$u^* \equiv x^* - iy^*.$$

A linear mapping A, from $\mathbb{B} \oplus i\mathbb{B}$ into itself, can be written

$$Au = (a + ib)(x + iy),$$

where a and b are linear mappings from $\mathbb{B}$ into $\mathbb{B}$. Using the vector notation for elements of $\mathbb{B} \oplus i\mathbb{B}$, we will sometimes write A as a matrix,

$$Au = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The adjoint of A is the linear mapping A^* such that for every $u, v \in \mathbb{B} \oplus i\mathbb{B}$

$$(v^* A)u = (A^* v^*)u.$$

In terms of the adjoints of a and b, A^* is

$$A^* \equiv a^* + ib^*.$$

A nonlinear mapping $G(u, \lambda)$ from $(\mathbb{B} \oplus i\mathbb{B}) \times \mathbb{R}$ into $\mathbb{B} \oplus i\mathbb{B}$ can be written

$$G(u, \lambda) = f(x, y, \lambda) + ih(x, y, \lambda),$$

where f and h are nonlinear mappings of $(\mathbb{B} \times \mathbb{B}) \times \mathbb{R}$ into $\mathbb{B}$. The Fréchet derivative, $G_u(u, \lambda)$, is the bounded linear mapping, if one exists, for which

$$\lim_{\|v\| \to 0} \left\{ \frac{\| G(u+v, \lambda) - G(u, \lambda) + G_u(u, \lambda)v \|}{\| v \|} \right\} = 0.$$

As a consequence og the fact that this limit is independant of the way $\| v \| \to 0$, the analog of the Cauchy-Riemann equations,

$$f_x = h_y \quad \text{and} \quad f_y = -h_x,$$

must hold. By analogy with complex analysis, we say that $G(u, \lambda)$ is analytic if and only if it has a Fréchet derivative at each point in $(\mathbb{B} \oplus i\mathbb{B}) \times \mathbb{R}$.

We can now define the complex extension of a real mapping. Let $g(x, \lambda)$ be a C^∞ mapping of $\mathbb{B} \times \mathbb{R}$ into $\mathbb{B}$. The complex extension of g is the analytic mapping $G(u, \lambda)$ such that

$$G(x + i0, \lambda) = g(x, \lambda) + i0.$$

The mappings f and h must therefore satisfy

$$f(x, 0, \lambda) = g(x, \lambda) \tag{2a}$$

$$\text{and} \quad h(x, 0, \lambda) = 0. \tag{2b}$$

This complex extension is not a typical analytic mapping. Since it is real for real values of u, it satisfies the analog of the Schwartz reflection principle for functions of a complex variable. This can be shown by considering the mapping

$\overline{G}(\overline{u}, \lambda)$, which must also be analytic. Since $\overline{G}(\overline{x + i0}, \lambda)$ and $G(x + i0, \lambda)$ are both equal to $g(x, \lambda) + i0$ at points $u = x + i0$, the identity principle, which Henrici (6) proves for mappings of complex Banach spaces, says that for every u in $\mathbb{B} \oplus i\mathbb{B}$

$$\overline{G}(\overline{u}, \lambda) = G(u, \lambda).$$

The identity principle is proved using the difference of $G(u, \lambda)$ and $\overline{G}(\overline{u}, \lambda)$, which is also an analytic mapping. At points $u_0 = x_0 + i0$ the Taylor expansion of the difference must be zero at all points $u = x_0 + \epsilon x + i0$, for ϵ small. This means that all terms in the expansion must be zero, and so the difference is zero in a neighborhood of u_0. By repeating this argument, using points in the neighborhood, it can be shown that the difference must be zero everywhere.

If we now consider a solution $G(u_0, \lambda_0) = 0$, we have

$$\overline{G}(\overline{u}, \lambda) = f(x_0, -y_0, \lambda_0) - ih(x_0, -y_0, \lambda_0) = 0.$$

If we take the conjugate, then

$$G(\overline{u}, \lambda) = f(x_0, -y_0, \lambda_0) + ih(x_0, -y_0, \lambda_0) = 0.$$

The conjugate of a solution of the complex extension of a real equation is therefore also a solution.

Regular Solutions of the Complex Extension

In this section we define a regular solution of the complex extension of a real equation, and determine the properties of the solution arc passing through it.

Definition: A regular solution of the complex extension $G(u,\lambda) = 0$, *is a point* $(u_0, \lambda_0) \in (\mathbb{B} \oplus i\mathbb{B}) \times \mathbb{R}$ *at which* $G(u_0, \lambda_0) = 0$ *and the Fréchet derivative* $G_u(u_0, \lambda_0)$ *is nonsingular.* ∎

Let (x_0, λ_0) be a regular solution of the real equation $g(x, \lambda) = 0$. The Fréchet derivative of the complex extension at $(x_0 + i0, \lambda_0)$ is

$$\begin{aligned} G_u(x_0 + i0, \lambda_0) &= f_x(x_0, 0, \lambda_0) + i0 \\ &= g_x(x_0, \lambda_0). \end{aligned}$$

Therefore, a regular solution of $g(x, \lambda) = 0$ is also a regular solution of $G(u, \lambda) = 0$.

A regular solution was defined so that we can use the Implicit Function Theorem. The Implicit Function Theorem says that if the Fréchet derivative is nonsingular, $G(u, \lambda) = 0$ implicitly defines u as a function of λ. This version is stated and proved in Nirenberg (14):

The Implicit Function Theorem: Let P *be a Banach space and* $F(p, q)$ *be a mapping of an open set* $U \subseteq P \times \mathbb{R}$ *into* P. *Assume that* F *is* $k \geq 1$ *times continuously differentiable in* U. *Suppose* $(p_0, q_0) \in U$ *and* $F(p_0, q_0) = 0$. *Then if* $F_p(p_0, q_0)$ *is an isomorphism of* P *onto* R*: for some sufficiently small* $r > 0$ *there exists a ball* $B_r(q_0) \equiv \{q \mid \| q - q_0 \| < r\}$ *and a unique* k*-times continuously differentiable mapping* $\omega : B_r(q_0) \rightarrow P$ *such that* $\omega(q_0) = p_0$ *and* $F(\omega(q), q) = 0$.

∎

Since $\mathbb{B} \oplus i\mathbb{B}$ is a Banach space, the Implicit Function Theorem can be applied to regular solutions of complex mappings. Therefore we have the result that at a regular solution of the complex extension, there is a unique mapping $u(\lambda) : \mathbb{R} \rightarrow \mathbb{B} \oplus i\mathbb{B}$ that defines a smooth arc of solutions $(u(\lambda), \lambda)$ on which the regular solution lies.

If G is the complex extension of a real mapping the mapping $u(\lambda)$ is real if the regular point is real. The mapping $u(\lambda)$ is the limit of the iteration

$$G_u(u_0, \lambda_0)u^{n+1}(\lambda) = G_u(u_0, \lambda_0)\left[u^n(\lambda) - G(u^n(\lambda), \lambda)\right],$$

where $u^0(\lambda) = u_0$. If $u_0 = x_0 + i0$, the Fréchet derivative of $G(u, \lambda)$ is

$$G_u(u_0, \lambda_0) = f_x(x_0, 0, \lambda_0) + ih_x(x_0, 0, \lambda_0).$$

Since $h(x_0, 0, \lambda)$ is identically zero, $G_u(u_0, \lambda_0)$ must be real. By induction, if $u^n(\lambda)$ is real, $G(u^n(\lambda), \lambda)$ is real, therefore $u^{n+1}(\lambda)$ must be real. This means that the limit $u(\lambda)$ must be real.

Suppose now that $(u(\lambda), \lambda)$ is an arc of solutions containing a single real point, (u_0, λ_0). If

$$u(\lambda) = x(\lambda) + iy(\lambda),$$

and $y(\lambda) \neq 0$ unless $\lambda = \lambda_0$, then (u_0, λ_0) cannot be a regular solution. If it is regular, it must lie on a real branch. In addition, $(\overline{u}(\lambda), \lambda)$ must also be an arc of solutions. There must therefore be at least two branches of solutions passing through (u_0, λ_0), so it must be a bifurcation point.

We have shown that regular solutions of the complex extension behave in the same way that regular solutions of the real equation do. There can be no complex bifurcation at a regular solution of either the real equation or its complex extension. In the next chapter, we show that complex bifurcation does occur at several types of singular solutions.

III. Complex Bifurcation at Simple Singular Solutions

In this chapter we prove that complex bifurcation occurs at a number of simple singular solutions of the complex extension of a real equation. We do this by a Lyapunov-Schmidt decomposition. This reduces the equation to a single complex scalar equation in one complex variable and the parameter λ. We solve this by making a change of variables and using the Implicit Function Theorem.

For the real equation a simple singular solution is a solution where the Fréchet derivative has one-dimensional null spaces. In the following section we define a simple singular solution of the complex extension in a similar way, and we describe the Lyapunov-Schmidt decomposition. We also show that a simple singular solution of the complex extension is in some ways like a singular solution of the real equation where the Fréchet derivative has two-dimensional null spaces. In addition we prove a Lemma about the singularity of a certain linear operator that will appear later. The final section of this chapter present bifurcation theorems for three types of simple singular solutions of the compelx extension. We prove each for a general simple singular solution, then for the special case of a simple singular solution of the real equation.

Simple Singular Solutions of the Complex Extension of a Real Equation

Definition: A Simple Singular Solution of a complex equation $G(u, \lambda) = 0$ *is a solution* (u_0, λ_0) *at which the Fréchet derivative* $G_u^0 \equiv G_u(u_0, \lambda_0)$ *is singular with*

$$\dim(\mathcal{N}(G_u^0)) = \dim(\mathcal{N}(G_u^{0*})) = 1$$

$$\textit{and} \qquad \text{Range}(G_u^0) \textit{ closed.} \ \blacksquare$$

The bifurcation theorems that we present in the final sections of this chapter use the Lyapunov-Schmidt decomposition, which we describe here. There are two ways of using it. We first consider doing a decomposition of the complex extension.

Let (u_0, λ_0) be a solution of the complex extension $G(u, \lambda) = 0$ of the real equation $g(x, \lambda) = 0$. Suppose that the Fréchet derivative $G_u^0 \equiv G_u(u_0, \lambda_0)$ is singular, and is a Fredholm operator of index 0. That is

$$\dim(\mathcal{N}(G_u^0)) = \dim(\mathcal{N}(G_u^{0*})) = d < \infty$$

$$\text{and} \qquad \text{Range}(G_u^0) \text{ closed.}$$

Then $\mathbb{B} \oplus i\mathbb{B}$ can be split into

$$\mathbb{B} \oplus i\mathbb{B} = \mathcal{N}(G_u^0) \oplus \text{Range}(G_u^{0*})$$

$$\text{or} \qquad \mathbb{B} \oplus i\mathbb{B} = \mathcal{N}(G_u^{0*}) \oplus \text{Range}(G_u^0).$$

Furthermore, the mapping $G_u^0/\text{Range}(G_u^{0*})$ is nonsingular. Let $\{\phi_j\}_1^d$ be a basis for $\mathcal{N}(G_u^0)$, and $\{\psi_j^*\}_1^d$ be a basis for $\mathcal{N}(G_u^{0*})$. By the Fredholm Alternative Theorem

$$\text{Range}(G_u^0) = \{u \in \mathbb{B} \oplus i\mathbb{B} \mid \psi_j^* u = 0, \quad 1 \le j \le d\}$$

$$\text{and} \qquad \text{Range}(G_u^{0*}) = \{u \in \mathbb{B} \oplus i\mathbb{B} \mid u^* \phi_j = 0, \quad 1 \le j \le d\}.$$

The Lyapuñov-Schmidt decomposition begins by splitting the domain of G into $\mathcal{N}(G_u) \oplus \text{Range}(G_u^*)$. Let $\{\xi_j\}_1^d$ be complex scalars, and $\eta \in \text{Range}(G_u^{0*})$, then every element u in $\mathbb{B} \oplus i\mathbb{B}$ can be written as $u = u_0 + \sum_1^d \xi_j \phi_j + \eta$, for some $\{\xi_j\}_1^d$. If we also split the Range of G into $\mathcal{N}(G_u^*) \oplus \text{Range}(G_u)$ we have that $G(u, \lambda) = 0$ if and only if

$$\phi_j \psi_j^* G(u_0 + \sum_1^d \xi_j \phi_j + \eta, \lambda) = 0 \qquad 1 \leq j \leq d, \tag{3a}$$

$$(I - \sum_1^d \phi_j \psi_j^*) G(u_0 + \sum_1^d \xi_j \phi_j + \eta, \lambda) = 0. \tag{3b}$$

The Fréchet derivative of (3b) with respect to η, at $\{\xi_j\} = 0$, $\eta = 0$ is precisely $G_u^0/\text{Range}(G_u^{0*})$, so it is nonsingular. By the Implicit Function Theorem there is a mapping $\eta(\underline{\xi}, \lambda) : \mathbb{C}^d \times \mathbb{R} \to \text{Range}(G_u^{0*})$ such that (3b) is satisfied. The remaining equations (3a) are

$$\psi_j^* G(u_0 + \sum_1^d \xi_j \phi_j + \eta(\underline{\xi}, \lambda), \lambda) = 0 1 \leq j \leq d,$$

and are called the bifurcation equations.

The second way of using the Lyapunov-Schmidt decomposition is to write an equivalent real system for the complex extension and do a decomposition of the real system. Recall that we can write $G(u, \lambda)$ as

$$G(u, \lambda) = f(x, y, \lambda) + ih(x, y, \lambda),$$

where f and $h : (\mathbb{B} \times \mathbb{B}) \times \mathbb{R} \to \mathbb{B}$.

If we replace $G(u,\lambda)=0$ by the equivalent real system

$$\begin{pmatrix} f(x,y,\lambda)=0 \\ h(x,y,\lambda)=0 \end{pmatrix} = 0,$$

the Fréchet derivative $G_u(u_0,\lambda_0)$ is

$$G_u(u_0,\lambda_0) = \begin{pmatrix} f_x(x_0,y_0,\lambda_0) & f_y(x_0,y_0,\lambda_0) \\ h_x(x_0,y_0,\lambda_0) & h_y(x_0,y_0,\lambda_0) \end{pmatrix}. \tag{5}$$

If the bases of the null spaces of G_u^0 and G_u^{0*} are written in terms of their real and imaginary parts

$$\phi_j = \phi_j^r + i\phi_j^i, \qquad \phi_j^r, \phi_j^r \in \mathbb{B}$$

and

$$\psi_j = \psi_j^r + i\psi_j^i, \qquad \psi_j^r, \psi_j^r \in \mathbb{B}, \qquad 1 \le j \le d,$$

then for every j,

$$f_x^0\phi_j^r + f_y^0\phi_j^i = 0$$

and

$$h_x^0\phi_j^r + h_y^0\phi_j^i = 0.$$

The Cauchy-Riemann equations imply that

$$h_y^0\phi_j^r + h_x^0\phi_j^i = 0$$

and

$$h_x^0\phi_j^r + h_y^0\phi_j^i = 0.$$

The null vectors of G_u^* satisfy similar equations. The Fréchet derivative of (4) and its adjoint therefore have $2d$-dimensional null spaces with the bases

$$\left\{\begin{pmatrix} \phi_j^r \\ \phi_j^i \end{pmatrix}, \begin{pmatrix} -\phi_j^i \\ \phi_j^r \end{pmatrix}\right\} \quad \text{and} \quad \left\{\begin{pmatrix} \psi_j^{r*} \\ \psi_j^{i*} \end{pmatrix}, \begin{pmatrix} -\psi_j^{i*} \\ \psi_j^{r*} \end{pmatrix}\right\} \qquad 1 \le j \le d.$$

In this sense, at a simple singular point, the Fréchet derivative of the complex extension can be thought of as having two-dimensional null spaces. This is significant because if the rest of the decomposition is done for the real system (4), the bifurcation equations correspond to those of a multiple bifurcation point instead of a simple bifurcation point.

Complex bifurcation theory can therefore be done in two ways. If the complex extension is considered as a complex equation, and the Lyanpunov-Schmidt decomposition is done using the complex null vectors $\{\phi_j\}$ and $\{\psi_j^*\}$, the complex theory is exactly the same as the real theory, except that complex roots of the ABE's must be allowed.

If, on the other hand, the complex extension is treated as the equivalent pair of real equations (4), a simple bifurcation point of the real equation becomes a multiple bifurcation point of the complex extension.

We prefer the first approach. In the final sections of this chapter we illustrate how the second approach is used in proving a bifurcation theorem for a simple quadratic fold of the real equation. However, we use the first approach for the rest of the proofs.

There remains one more thing to do before we proceed to the bifurcation theorems. When we use the Implicit Function Theorem to solve the bifurcation equations, we must verify that a certain type of linear operator is nonsingular. We therefore present the following Lemma.

Lemma (1): If A is a linear operator of the form

$$A\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} aq + br \\ \bar{c}q + c\bar{q} + 2dr \end{pmatrix}$$

$$A : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$$

where $\quad a, b, c, \text{ and } q \in \mathbb{C}$

and, $\quad d, r \in \mathbb{R},$

then A is nonsingular if and only if

$$d|a|^2 - \Re e(\bar{b}ac) \neq 0.$$

Proof-

We rewrite A as a 3×3 matrix. Let

$$a = a_R + ia_I$$

$$b = b_R + ib_I$$

$$c = c_R + ic_I$$

and $\quad q = q_R + iq_I.$

Then, $A\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if and only if

$$a_R q_R - a_I q_I + b_R r = 0$$

$$a_R q_I + a_I q_R + b_I r = 0$$

and $\quad 2(c_R q_R + c_I q_I) + 2dr = 0.$

This is a matrix equation

$$\begin{pmatrix} a_R & -a_I & b_R \\ a_I & a_R & b_I \\ 2c_R & 2c_I & 2d \end{pmatrix} \begin{pmatrix} q_R \\ q_I \\ r \end{pmatrix} = 0.$$

A is nonsingular if and only if the determinant of this system is zero. That is if

$$2(d|a|^2 - \Re e(\bar{b}ac)) \neq 0. \blacksquare$$

We can now proceed to show that complex bifurcation occurs at three types of simple singular solutions: simple quadratic folds; simple quadratic bifurcation points; and simple cubic bifurcation points.

As a matter of notation, in the following section, the null spaces of G_u^0 and G_u^{0*} at a simple singular solution of a complex equation will be

$$\mathcal{N}(G_u^0) = \text{span}(\phi)$$

$$\text{and} \qquad \mathcal{N}(G_u^{0*}) = \text{span}(\psi^*),$$

where $\phi \in \mathbb{B} \oplus i\mathbb{B}$, and $\psi^* \in \mathbb{B}^* \oplus i\mathbb{B}^*$.

Simple Quadratic Folds

For the real equation a simple quadratic fold, or limit point, is not a bifurcation point. In this section we show that simple quadratic folds are *complex* bifurcation points. We first define a simple quadratic fold of the complex extension, and show that there are two branches of solutiosn that touch at the fold. We then show that a complex branch of solutions exists at a simple quadratic fold of the real equation.

Definition: A Simple Quadratic Fold is a simple singular solution, (u_0, λ_0)*, of a complex equation* $G(u, \lambda) = 0$*, at which*

$$\psi^* G^0_\lambda \neq 0$$

and, $$\psi^* G^0_{uu} \phi\phi \neq 0. \blacksquare$$

Theorem - Bifurcation at a Simple Quadratic Fold: Let (u_0, λ_0) *be a simple quadratic fold of the complex equation* $G(u, \lambda) = 0$*. In a small neighborhood of* (u_0, λ_0)*, there are exactly two branches of solutions. These solution branches have the local expansions*

$$u_1(s) \sim u_0 + se^{-i\alpha/2}\phi + O(s^2)$$
$$\lambda_1(s) \sim \lambda_0 - \frac{1}{2}rs^2 + O(s^3),$$

and

$$u_2(s) \sim u_0 + ise^{-i\alpha/2}\phi + O(s^2)$$
$$\lambda_2(s) \sim \lambda_0 + \frac{1}{2}rs^2 + O(s^3),$$

where

$$re^{i\alpha} = \frac{\psi^* G^0_{uu}\phi\phi}{2\psi^* G^0_\lambda}.$$

Proof -

The proof proceeds by doing a Lyapunov-Schmidt decomposition on the *complex* equation (this is the first approach). The Implicit Function Theorem is then used to solve the bifurcation equations.

Let

$$u = u_0 + \xi\phi + \eta$$

where $\eta \in \text{Range}(G_u^{0*})$

and $\xi \in \mathbb{C}$.

Using the Lyapunov-Schmidt decomposition, we have that $G = 0$ if and only if

(6*a*) $$\phi\psi^* G(u_0 + \xi\phi + \eta, \lambda) = 0$$

(6*b*) and $$(I - \phi\psi^*)G(u_0 + \xi\phi + \eta, \lambda) = 0.$$

The Fréchet derivative of (6*b*) with respect to η is $G^0_u/\text{Range}(G_u^{0*})$ and is nonsingular. So, by the Implicit Function Theorem there is a unique mapping $\eta(\xi, \lambda)$: $\mathbb{C} \times \mathbb{R} \to \text{Range}(G_u^{0*})$ such that $\eta(0, \lambda_0) = 0$ and equation (6*b*) is satisfied. Using this mapping η, equation (6*a*) is satisfied if and only if

$$\psi^* G(u_0 + \xi\phi + \eta(\xi, \lambda), \lambda) = 0.$$

This is a single complex scalar equation in the complex variable ξ and the real parameter λ, and is called the bifurcation equation.

The Fréchet derivative of the bifurcation equation with respect to ξ is identically zero, so the Implicit Function Theorem cannot be used directly. Instead, we introduce a new parameter $\epsilon \in \mathbb{R}$, and look for mappings of the form $\xi(\epsilon)$ and $\lambda(\epsilon)$ that satisfy the bifurcation equations. Taking the derivative of the bifurcation equation, we see that

$$\psi^* G^0_\lambda \dot{\lambda}(0) = 0.$$

Since $\psi^* G^0_\lambda \neq 0$ by assumption, $\dot{\lambda}$ must be zero. We therefore let

$$u = u_0 + \epsilon\xi\phi + \eta$$

and $$\lambda = \lambda_0 + \frac{1}{2}\epsilon^2\varsigma,$$

and scale G by $1/\epsilon^2$ to eliminate the zero Fréchet derivative.

Let

$$F(\xi,\varsigma;\epsilon) \equiv \begin{pmatrix} \frac{2}{\epsilon^2}\psi^* G(u_0 + \epsilon\xi\phi + \eta(\epsilon\xi\phi, \lambda_0 + \frac{1}{2}\epsilon^2\varsigma), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma) & \epsilon \neq 0 \\ \psi^* G^0_{uu}\phi\phi\xi^2 + 2\psi^* G^0_\lambda \varsigma & \epsilon = 0 \\ |\xi|^2 - 1 & \end{pmatrix}.$$

We have chosen $F(\xi,\varsigma;0)$ so that F is continuous at $\epsilon = 0$. The Fréchet deriative of F with respect to ξ and ς is now not identically zero at $\epsilon = 0$. If we can find ξ and ς so that $F(\xi,\varsigma;0) = 0$ and the Fréchet derivative $F_{(\xi,\varsigma)}(\xi,\varsigma;0)$ is nonsingular, we can use the Implicit Function Theorem to find $\xi(\epsilon)$ and $\varsigma(\epsilon)$ that satusfy the bifurcation equation.

We first solve $F(\xi,\varsigma;0) = 0$, the Limit Point Algebraic Bifurcation Equations (LPABE's), and verify that these solutions have nonsingular Fréchet derivatives. The LPABE's are

$$\psi^* G^0_{uu}\phi\phi\xi^2 + 2\psi^* G^0_\lambda \varsigma = 0$$

and $$|\xi|^2 = 1.$$

Keeping in mind that $\xi \in \mathbb{C}$, $\varsigma \in \mathbb{R}$, and that $re^{i\alpha} = \psi^* G^0_{uu}\phi\phi / \psi^* G^0_\lambda$, the only solutions of the LPABE's are:

$$\xi_1 = e^{-\frac{1}{2}i\alpha}, \quad \varsigma_1 = -r;$$

$$\xi_2 = ie^{-\frac{1}{2}i\alpha}, \quad \varsigma_2 = r;$$

$$\xi_3 = -\xi_1, \quad \varsigma_3 = \varsigma_1;$$

and $$\xi_4 = -\xi_2, \quad \varsigma_4 = \varsigma_2.$$

The Fréchet derivative of F at $\epsilon = 0$, $F_{(\xi,\eta)}(\xi,\eta;0)$, satisfies

$$F_{(\xi,\varsigma)}(\xi,\varsigma;0)(\xi',\varsigma') = \begin{pmatrix} 2\psi^* G^0_{uu}\phi\phi\xi\xi' + 2\psi^* G^0_\lambda \varsigma' \\ \overline{\xi}\xi' + \overline{\xi'}\xi. \end{pmatrix}$$

By Lemma (1) the Fréchet derivative is therefore nonsingular if and only if $\Re e\left(\overline{(\psi^* G^0_\lambda)}(\psi^* G\right.$ 0. Since $|\xi| = 1$ at each root of the LPABE's, and $\psi^* G^0_\lambda$ and $\psi^* G^0_{uu}\phi\phi$ are nonzero by assumption, the Fréchet derivatives at all of the roots are nonsingular. The Implicit Function Theorem then guarantees the existance of two distinct pairs of mappings,

$$(\xi_1(\epsilon), \varsigma_1(\epsilon))$$

and $$(\xi_2(\epsilon), \varsigma_2(\epsilon)),$$

that satisfy $F(\xi,\varsigma;\epsilon) = 0$. Since F has the symmetry

$$F(\xi,\varsigma;\epsilon) = F(-\xi,\varsigma;-\epsilon),$$

roots 3 and 4 of the LPABE's are not distinct from roots 1 and 2.

There are therefore exactly two solutions of $G = 0$ for small ϵ:

$$u_1(\epsilon) = u_0 + \epsilon\xi_1(\epsilon)\phi + \eta(\epsilon\xi_1(\epsilon), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_1(\epsilon))$$
$$\lambda_1(\epsilon) = \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_1(\epsilon)$$

and,

$$u_2(\epsilon) = u_0 + \epsilon\xi_2(\epsilon)\phi + \eta(\epsilon\xi_2(\epsilon), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_2(\epsilon))$$
$$\lambda_2(\epsilon) = \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_2(\epsilon). \blacksquare$$

Figure 2 illustrates these solution branches near a simple quadratic fold.

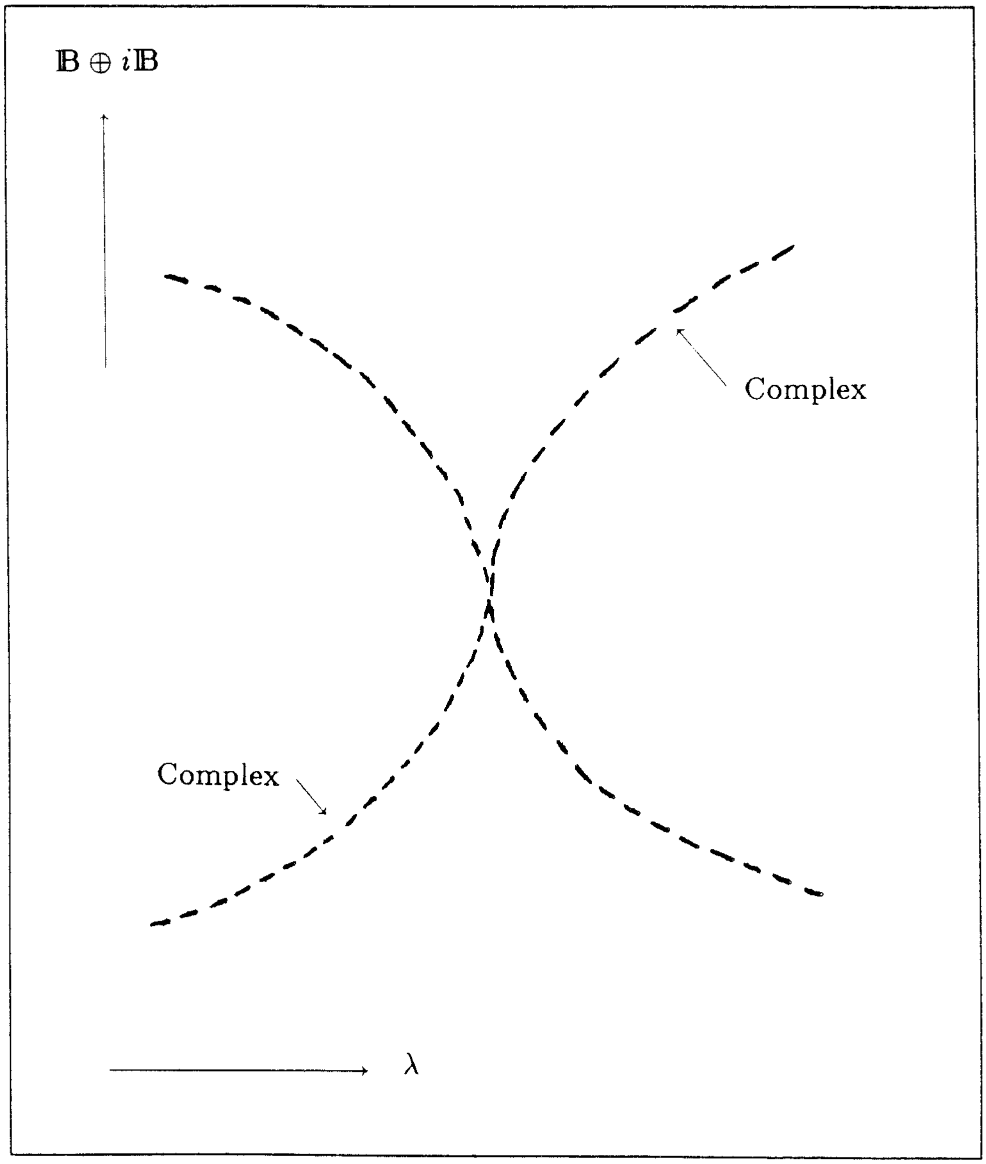

Figure 2. Complex Bifurcation at a Simple Quadratic Fold

We now turn to the special case of a simple quadratic fold (x_0, λ_0) of the real equation $g(x, \lambda) = 0$. We take this opportunity to illustrate how complex bifurcation theory is done when the complex extension is replaced by an equivalent pair of real equations (this is what we called the second approach). For these singular solutions the null vector ϕ and ψ^* are both real, as are the derivatives of G.

Theorem - Bifurcation at a Simple Quadratic Fold of the Real Equation: Let (x_0, λ_0), be a simple quadratic fold of the real equation $g(x, \lambda) = 0$, whose complex extension is the equation $G(u, \lambda) = 0$. The point $(u_0, \lambda) = (x_0 + i0, \lambda_0)$ is a simple quadratic fold of the complex extension. In a small neighborhood of (u_0, λ_0), there are exactly two branches of solutions, one real, and one complex, of $G = 0$. The real branch has the local expansion

$$u_1(s) \sim u_0 + s\phi + O(s^2)$$
$$\lambda_1(s) \sim \lambda_0 - \frac{1}{2} rs^2 + O(s^3)$$

and the complex branch has the local expansion

$$u_2(s) \sim u_0 + is\phi + O(s^2)$$
$$\lambda_2(s) \sim \lambda_0 + \frac{1}{2} rs^2 + O(s^3)$$

where,

$$r = \left| \frac{\psi^* G^0_{uu}\phi\phi}{\psi^* G^0_\lambda} \right| = \left| \frac{\psi^* g^0_{xx}\phi\phi}{\psi^* G^0_\lambda} \right| \in \mathbb{R}.$$

Proof -

For this proof we first replace $G(u, \lambda) = 0$ by a pair of real equations. Recall that

$$G(x + iy, \lambda) = f((x, y), \lambda) + ih((x, y), \lambda)$$

where f and $g : (\mathbb{B} \times \mathbb{B}) \times \mathbb{R} \to \mathbb{B}$. We consider the real system

$$k(x, y, \lambda) = \begin{pmatrix} f((x,y),\lambda) \\ h((x,y),\lambda) \end{pmatrix} = 0,$$

where $k : (\mathbb{B} \times \mathbb{B}) \times \mathbb{R} \to (\mathbb{B} \times \mathbb{B})$. Then

$$k((x_0, 0), \lambda_0) = 0,$$

and $k^0_{(x,y)}$ has two-dimensional null spaces. These null spaces have the bases

$$\{\begin{pmatrix} \phi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi \end{pmatrix}\} \quad \text{and} \{\begin{pmatrix} \psi^* \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \psi^* \end{pmatrix}\}.$$

Let,

$$x = x_0 + \alpha + \xi\phi$$

and $$y = \beta + \eta\phi,$$

where

$$(\alpha, \beta) \in \text{Range}(k^{0*}_{(x,y)})$$

and $$\xi, \eta \in \mathbb{R}.$$

Using the Lyapunov-Schmidt decomposition we split $\mathbb{B} \times \mathbb{B}$. Therefore $k = 0$ if and only if

$$\text{(7a)} \qquad \begin{pmatrix} \phi \\ 0 \end{pmatrix} (\psi^*, 0) k((x_0 + \xi\phi + \alpha, \eta\phi + \beta), \lambda) = 0$$

$$\text{(7b)} \qquad \begin{pmatrix} 0 \\ \phi \end{pmatrix} (0, \psi^*) k((x_0 + \xi\phi + \alpha, \eta\phi + \beta), \lambda) = 0$$

$$\text{(7c)} \qquad (I - \begin{pmatrix} \phi \\ 0 \end{pmatrix} (\psi^*, 0) - \begin{pmatrix} 0 \\ \phi \end{pmatrix} (0, \psi^*)) k((x_0 + \xi\phi + \alpha, \eta\phi + \beta), \lambda) = 0.$$

The Fréchet derivative of (7c) with respect to (α, β) is $k^0_{(x,y)}/\text{Range}(k^{0*}_{(x,y)})$ and is nonsingular. So, by the Implicit Function Theorem there is a unique pair of mappings $\alpha(\xi, \eta, \lambda)$ and $\beta(\xi, \eta, \lambda)$ which map $\mathbb{C}^2 \times \mathbb{R} \to \text{Range}(k^{0*}_{(x,y)})$ such that

$\alpha(0,0,\lambda) = \beta(0,0,\lambda) = 0$ and equation $(7c)$ is satisfied. Using these mappings, equation $(7a)$ and $(7b)$ are

$$\psi^* f((x_0 + \xi\phi + \alpha(\xi,\eta,\lambda), \eta\phi + \beta(\xi,\eta,\lambda)), \lambda) = 0$$

and
$$\psi^* h((x_0 + \xi\phi + \alpha(\xi,\eta,\lambda), \eta\phi + \beta(\xi,\eta,\lambda)), \lambda) = 0.$$

These are a pair of complex scalar equations in the complex variables ξ and η, and the real parameter λ. They are exactly the same as the bifurcation equations obtained in the previous theorem, except that the complex equation has been replaced by a pair of real equations.

The Fréchet derivative of these equations with respect to the variables ξ and η is identically zero, so we cannot use the Implicit Function Theorem directly. We introduce the parameter $\epsilon \in \mathbb{R}$, and look for mappings $\xi(\epsilon)$, $\eta(\epsilon)$, and $\lambda(\epsilon)$ that satisfy the bifurcation equations. The derivatives of these mappings must satisfy

$$\psi^* f^0_\lambda \dot\lambda = 0$$

and
$$\psi^* h^0_\lambda \dot\lambda = 0.$$

Recalling equations $(2ab)$ from chapter II, we see that $\psi^* h^0_\lambda = 0$ and that $\psi^* G^0_\lambda = \psi^* f^0_\lambda \neq 0$. Therefore, $\dot\lambda$ must be zero. We let

$$x = x_0 + \epsilon\xi\phi + \alpha(\epsilon\xi, \epsilon\eta, \lambda_0 + \frac{1}{2}\epsilon^2\varsigma)$$

$$y = \epsilon\eta\phi + \beta(\epsilon\xi, \epsilon\eta, \lambda_0 + \frac{1}{2}\epsilon^2\varsigma)$$

and
$$\lambda = \lambda_0 + \frac{1}{2}\epsilon^2\varsigma,$$

and scale the bifurcation equations by $1/\epsilon^2$ to eliminate the zero derivative.

Let

$$F(\xi,\eta,\varsigma;\epsilon) \equiv \begin{cases} \begin{pmatrix} \frac{2}{\epsilon^2}\psi^* f((x_0 + \epsilon\xi\phi + \alpha, \epsilon\eta\phi + \beta), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma) \\ \frac{2}{\epsilon^2}\psi^* h((x_0 + \epsilon\xi\phi + \alpha, \epsilon\eta\phi + \beta), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma) \end{pmatrix} & \epsilon \neq 0 \\ \begin{pmatrix} \psi^* f^0_{xx}\phi\phi\xi^2 - \psi^* f^0_{xx}\phi\phi\eta^2 + 2\psi^* f^0_\lambda\varsigma \\ 2\psi^* h^0_{xx}\phi\phi\xi\eta \end{pmatrix} & \epsilon = 0 \\ |\xi|^2 + |\eta|^2 - 1 \end{cases}$$

We have defined $F(\xi,\eta,\varsigma;0)$ so F is continuous at $\epsilon = 0$. Notice that equations (2) imply that $f^0_{xx} = h^0_{xx} = g^0_{xx}$ and that $f^0_\lambda = g^0_\lambda$.

The equations $F = 0$ are the Algebraic Bifurcation Equations:

$$\psi^* g^0_{xx}\phi\phi(\xi^2 - \eta^2) + 2\psi^* g^0_\lambda \varsigma = 0$$

$$2\psi^* g^0_{xx}\phi\phi\xi\eta = 0$$

$$|\xi|^2 + |\eta|^2 = 1$$

There are four solutions

$$\xi_1 = 1 \qquad \eta_1 = 0 \qquad \varsigma_1 = -\frac{\psi^* g^0_{xx}\phi\phi}{2\psi^* g^0_\lambda}$$

$$\xi_2 = 0 \qquad \eta_2 = 1 \qquad \varsigma_2 = \frac{\psi^* g^0_{xx}\phi\phi}{2\psi^* g^0_\lambda}$$

$$\xi_3 = 0 \qquad \eta_3 = -\eta_1 \qquad \varsigma_3 = \varsigma_1$$

and,

$$\xi_4 = 0 \qquad \eta_4 = -\eta_2 \qquad \varsigma_4 = \varsigma_2.$$

The Fréchet derivative of F is

$$F_{(\xi,\eta,\varsigma)}(\xi,\eta,\varsigma) = \begin{pmatrix} 2\psi^* g^0_{xx}\phi\phi\xi & -2\psi^* g^0_{xx}\phi\phi\eta & 2\psi^* g^0_\lambda \\ 2\psi^* g^0_{xx}\phi\phi\eta & 2\psi^* g^0_{xx}\phi\phi\xi & 0 \\ 2\xi^* & 2\eta^* & 0 \end{pmatrix}.$$

This is a real matrix, and is singular if and only if

$$-(2\psi^* g^0_\lambda)(4\psi^* g^0_{xx}\phi\phi)(|\xi|^2 - |\eta|^2) = 0$$

Since $\psi^* g_\lambda^0 = \psi^* G_\lambda^0 \neq 0$, and $\psi^* g_{xx}^0 \phi\phi = \psi^* G_{uu}^0 \phi\phi \neq 0$ by assumption, the Fréchet derivative at each of these roots is nonsingular. The Implicit Function Theorem therefore guarantees the existance of two distinct triples of mappings

$$(\xi_1(\epsilon), \eta_1(\epsilon), \lambda_1(\epsilon))$$

and,

$$(\xi_2(\epsilon), \eta_2(\epsilon), \lambda_2(\epsilon))$$

that satisfy the bifurcation equations. Since F has the symmetry

$$F(\xi, \eta, \varsigma) = F(-\xi, -\eta, \varsigma)$$

roots 3 and 4 are not distinct from roots 1 and 2. We have therefore shown that there are exactly two distinct solutions of $k = 0$ for small ϵ:

$$x_1(\epsilon) = x_0 + \epsilon\xi_1(\epsilon) + \alpha(\epsilon\xi_1(\epsilon), \epsilon\eta_1(\epsilon), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_1(\epsilon))$$

$$y_1(\epsilon) = \epsilon\eta_1(\epsilon) + \beta(\epsilon\xi_1(\epsilon), \epsilon\eta_1(\epsilon), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_1(\epsilon))$$

$$\lambda_1(\epsilon) = \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_1(\epsilon))$$

and,

$$x_2(\epsilon) = x_0 + \epsilon\xi_2(\epsilon) + \alpha(\epsilon\xi_2(\epsilon), \epsilon\eta_2(\epsilon), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_2(\epsilon))$$

$$y_2(\epsilon) = \epsilon\eta_2(\epsilon) + \beta(\epsilon\xi_2(\epsilon), \epsilon\eta_2(\epsilon), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_2(\epsilon))$$

$$\lambda_2(\epsilon) = \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_2(\epsilon)).$$

Since $u = x + iy$, this means that there are exactly two solutions

$$u_1(\epsilon) = x_1(\epsilon) + iy_1(\epsilon)$$

and,

$$u_2(\epsilon) = x_2(\epsilon) + iy_2(\epsilon)$$

of $G = 0$.

To show that the solution branch (u_1, λ_1) is real requires writing the contracting map of the Implicit Function Theorem , as we did in chapter II when we showed that a real regular solution lies on a real solution branch. It is not difficult to show that $y_1(\epsilon) = 0$. ∎

The fact that a complex branch bifurcates at a simple quadratic fold of the real equation proves to be very useful. We will show in chapter V that in many cases a second fold is present, and that this complex branch connects the two folds.

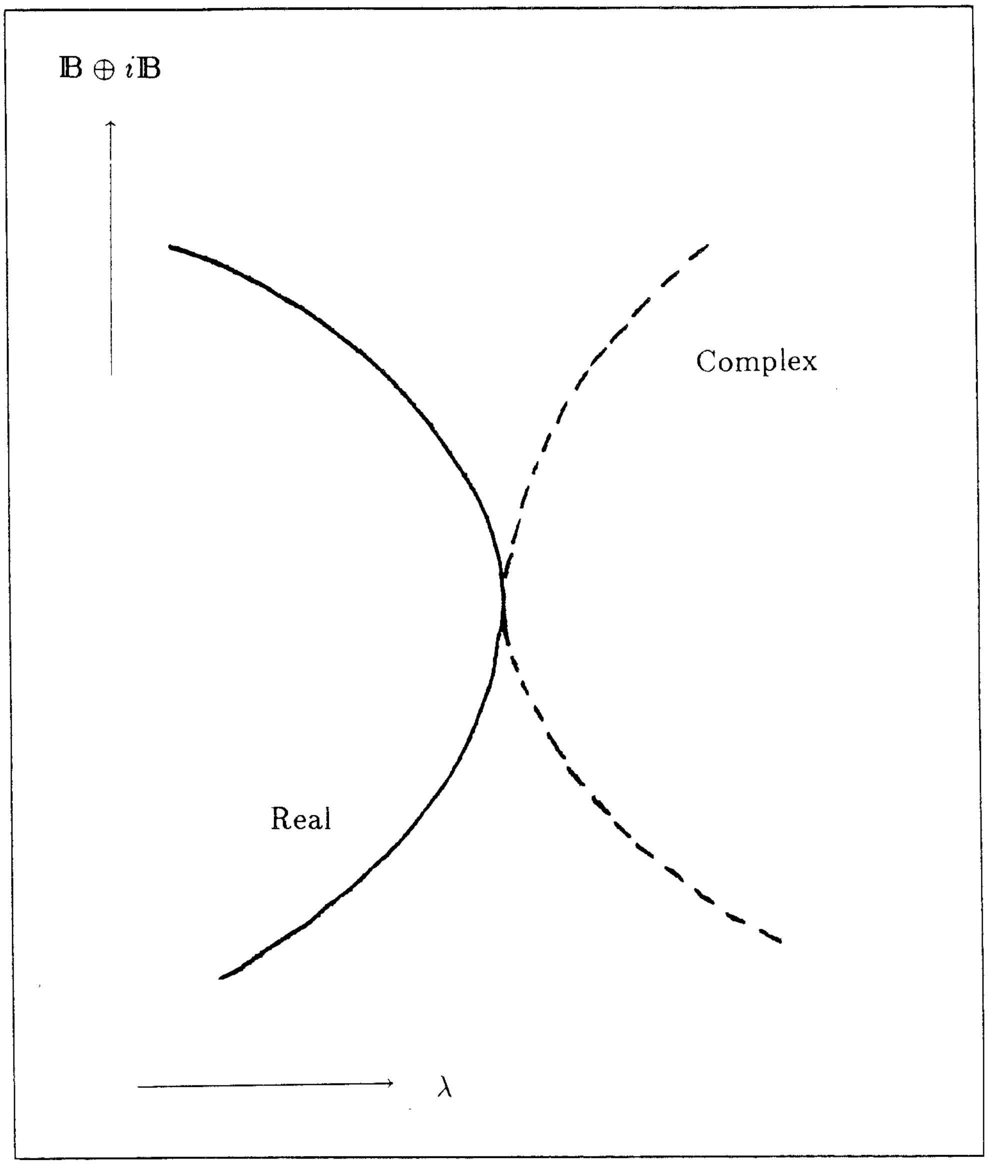

Figure 3. Complex Bifurcation at a Real Simple Quadratic Fold

Simple Quadratic Bifurcation

The next type of simple singular solution that we consider is the simple quadratic bifurcation point. The quantity $\psi^* G^0_\lambda$ is zero, and the tangent $(\dot{u}, \dot{\lambda})$ is no longer vertical. For the real equation there are two possible behaviors, either two real branches intersect at the bifurcation point, or there are no real solutions in the neighborhood of the bifurcation point. We show that in the second case two complex branches of the complex extension intersect at a real solution.

Definition - Simple Quadratic Bifurcation Point: A Simple Quadratic Bifurcation Point is a simple singular point, (u_0, λ_0), of the complex extension $G(u, \lambda) = 0$ of a real equation $g(x, \lambda) = 0$, at which

$$\psi^* G^0_\lambda = 0$$

$$A \equiv \psi^* G^0_{uu} \phi\phi \neq 0$$

and,

$$D^2 \equiv B^2 - AC \neq 0.$$

Where,

$$B \equiv \psi^* G^0_{uu} \phi \eta^0_\lambda + \psi^* G^0_{u\lambda} \phi$$

$$C \equiv \psi^* G^0_{uu} \eta^0_\lambda \eta^0_\lambda + 2\psi^* G^0_{u\lambda} \eta^0_\lambda + \psi^* G^0_{\lambda\lambda}$$

and,

$$(I - \phi\psi^*) G^0_u \eta^0_\lambda + (I - \phi\psi^*) G^0_\lambda = 0. \blacksquare$$

Theorem - Bifurcation at a Simple Quadratic Bifurcation Point: Let (u_0, λ_0) *be a simple quadratic bifurcation point of the complex extension* $G(u, \lambda) = 0$*. In a small neighborhood of* (u_0, λ_0)*, there are exactly two branches of solutions of* $G = 0$*. These two branches have local expansions*

$$u_1(s) \sim u_0 + s\frac{\delta_+}{\sqrt{1 + |\delta_+|^2}}\phi + O(s^2)$$

$$\lambda_1(s) \sim \lambda_0 + \frac{s}{\sqrt{1 + |\delta_+|^2}} + O(s^2)$$

and,

$$u_2(s) \sim u_0 + s\frac{\delta_-}{\sqrt{1 + |\delta_-|^2}}\phi + O(s^2)$$

$$\lambda_2(s) \sim \lambda_0 + \frac{s}{\sqrt{1 + |\delta_-|^2}} + O(s^2).$$

Where,

$$\delta_+ \equiv \frac{-B + D}{A}$$

$$\delta_- \equiv \frac{-B - D}{A}.$$

Proof -

We use a Lyapunov-Schmidt decomposition of the complex extension. Let

$$u = u_0 + \xi\phi + \eta$$

where $\quad \eta \in \text{Range}(G_u^{0*})$

and $\quad \xi \in \mathbb{C}.$

Accordingly, $G = 0$ if and only if

$$(7a) \qquad \phi\psi^* G(u_0 + \xi\phi + \eta, \lambda) = 0$$

$$(7b) \qquad \text{and} \qquad (I - \phi\psi^*)G(u_0 + \xi\phi + \eta, \lambda) = 0.$$

The Fréchet derivative of equation $(7b)$ with respect to η is $G_u^0/\text{Range}(G_u^{0*})$, and so is nonsingular. By the Implicit Function Theorem there is a unique, k-times continuously differentiable mapping $\eta(\xi, \lambda)$ such that $\eta(0, \lambda_0) = 0$, and equation $(7b)$ is satisfied.

Using this mapping η, (7a) is

$$\psi^* G(u_0 + \xi\phi + \eta(\xi, \lambda), \lambda) = 0$$

and is called the bifurcation equation. In order to use the Implicit Function Theorem , we introduce the parameter ϵ, and look for mappings of the form $\xi(\epsilon)$ and $\lambda(\epsilon)$ that satisfy the bifurcation equation. Taking the derivative of the bifurcation equation, we see that the derivatives of both ξ and λ with respect to ϵ may be non zero. We therefore let

$$u = u_0 + \epsilon\xi + \eta$$

$$\text{and} \qquad \lambda = \lambda_0 + \epsilon\varsigma,$$

and scale g by $1/\epsilon^3$ to eliminate the zero derivatives. Let

$$F(\xi, \varsigma; \epsilon) \equiv \begin{cases} \frac{2}{\epsilon^2}\psi^* G(u_0 + \epsilon\xi\phi + \eta(\epsilon\xi\phi, \lambda_0 + \epsilon\varsigma), \lambda_0 + \epsilon\varsigma) & \epsilon \neq 0 \\ \psi^* G_{uu}^0(\xi\phi + \eta_\lambda^0\varsigma)(\xi\phi + \eta_\lambda^0\varsigma) + 2\psi^* G_\lambda^0(\xi\phi + \eta_\lambda^0\varsigma)\varsigma + \psi^* G_{\lambda\lambda}^0\varsigma\varsigma & \epsilon = 0 \\ |\xi|^2 + |\varsigma| - 1 \end{cases}$$

We have defined $F(\xi,\varsigma;0)$ so that F is continuous at $\epsilon = 0$. We first solve $F(\xi,\eta;\epsilon) = 0$, the Quadratic Algebraic Bifurcation Equations (QABE's), then verify that these solutions have nonsingular Fréchet derivative.The QABE's are

$$A\xi^2 + 2B\xi\varsigma + C\varsigma^2 = 0$$

$$|\xi|^2 + |\varsigma|^2 = 1.$$

Keeping in mind that $\xi \in \mathbb{C}$ and $\varsigma \in \mathbb{R}$, the only solutions of the LPABE's are

$$\xi_1 = \frac{\frac{-B+D}{A}}{\sqrt{1+|\frac{-B+D}{A}|^2}}, \quad \varsigma_1 = \frac{1}{\sqrt{1+|\frac{-B+D}{A}|^2}}$$

$$\xi_2 = \frac{\frac{-B-D}{A}}{\sqrt{1+|\frac{-B-D}{A}|^2}}, \quad \varsigma_2 = \frac{1}{\sqrt{1+|\frac{-B-D}{A}|^2}}$$

$$\xi_3 = -\xi_1, \quad \varsigma_3 = -\varsigma_1$$

and $$\xi_4 = -\xi_2, \quad \varsigma_4 = -\varsigma_2.$$

The Fréchet derivative of F at $\epsilon = 0$ is

$$F_{\xi,\varsigma}(\xi,\varsigma;0)(\xi',\varsigma') = \begin{pmatrix} 2A\xi\xi' + 2B\xi'\varsigma + 2B\xi\varsigma' + 2C\varsigma\varsigma' \\ \overline{\xi}\xi' + \overline{\xi'}\xi + 2\varsigma\varsigma' \end{pmatrix}$$

which, by Lemma (1), is nonsingular if and only if

$$2\varsigma|2A\xi + 2B\varsigma|^2 - \Re e(\overline{(2B\xi + 2C\varsigma)}(2A\xi + 2B\varsigma)\xi) \neq 0.$$

If ξ and ς are roots of the QABE's, this is equivalent to

$$\varsigma|D|^2 \neq 0.$$

So, since ς is nonzero at every root, and D is nonzero by assumption, the Fréchet derivative at each of the roots is nonsingular. The Implicit Function Theorem guarantees the existence of two distinct pairs of mappings

$$(\xi_1(\epsilon), \varsigma_1(\epsilon))$$

and

$$(\xi_2(\epsilon), \varsigma_2(\epsilon))$$

that satisfy $F(\xi, \varsigma; \epsilon) = 0$. Since F has the symmetry

$$F(\xi, \varsigma; \epsilon) = F(-\xi, -\varsigma; -\epsilon)$$

roots 3 and 4 of the LPABE's are not distinct from roots 1 and 2.

We have therefore shown that there are exactly two solutions of $g = 0$ for small ϵ:

$$u_1(\epsilon) = u_0 + \epsilon\xi_1(\epsilon)\phi + \eta(\epsilon\xi_1(\epsilon), \lambda_0 + \epsilon\varsigma_1(\epsilon))$$

$$\lambda_1(\epsilon) = \lambda_0 + \epsilon\varsigma_1(\epsilon)$$

and,

$$u_2(\epsilon) = u_0 + \epsilon\xi_2(\epsilon)\phi + \eta(\epsilon\xi_2(\epsilon), \lambda_0 + \epsilon\varsigma_2(\epsilon))$$

$$\lambda_2(\epsilon) = \lambda_0 + \epsilon\varsigma_2(\epsilon).$$

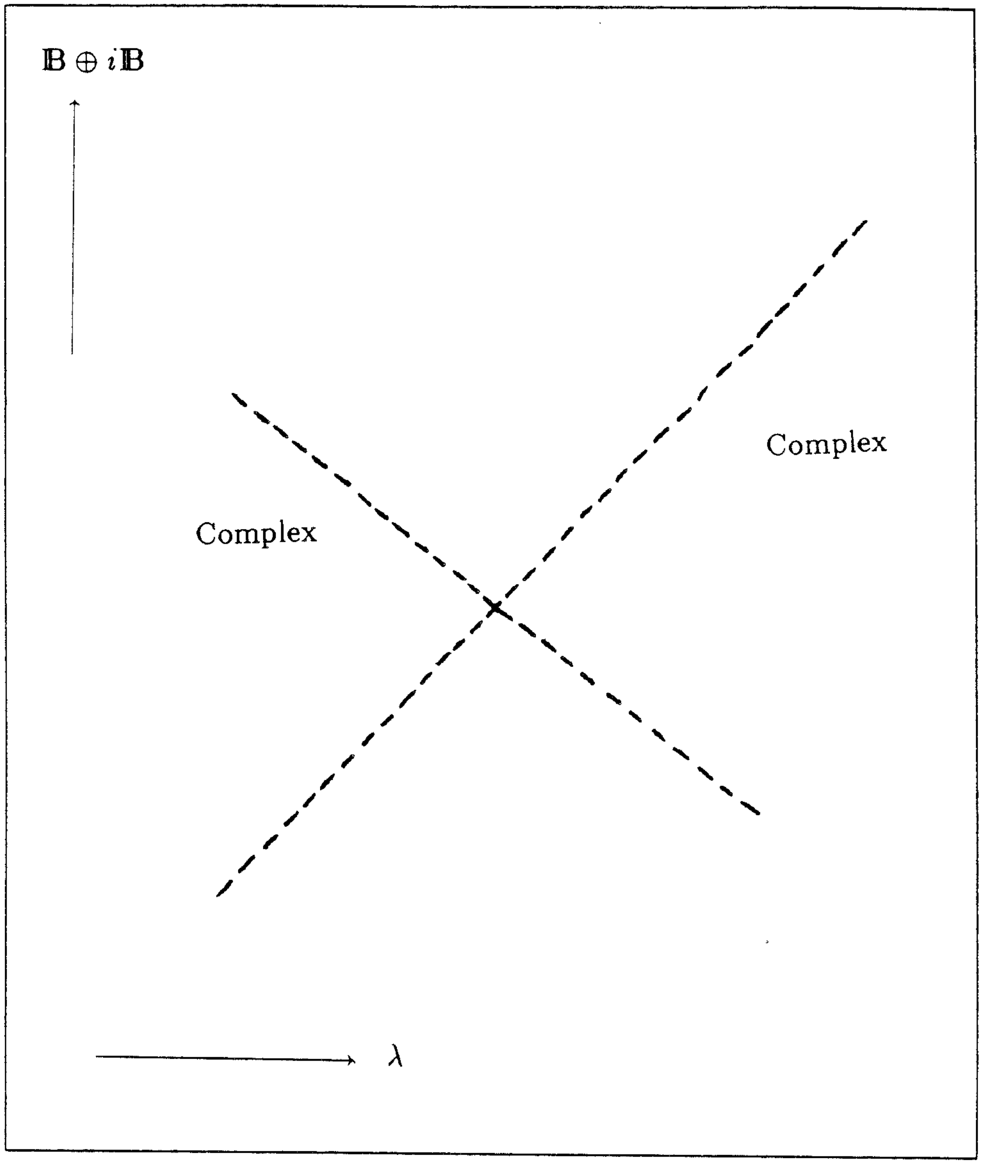

Figure 4. Complex Bifurcation at a Simple Quadratic Bifurcation Point

At a simple quadratic bifurcation point, or transerse bifurcation point, of the real equation, the null vectors ϕ and ψ^* are real, as are the derivatives of G. If the quantity $B^2 - AC$ is positive, there are two real branches of solutions that pass through the bifurcation point. If $B^2 - AC < 0$, the real equation has only a single real solution, an isola center, with no other solutions nearby. We show that when $B^2 - AC < 0$, the complex extension has two complex solution branches that pass through the real bifurcation point.

Theorem - Bifurcation at a Simple Quadratic Bifurcation Point of a Real Equation: Let $x_0, \lambda_0)$, *be a simple quadratic bifurcation point of a real equation* $g(x, \lambda) = 0$. *In a small neighborhood of* $(x_0 + i0, \lambda_0)$, *there are exactly two branches of solutions* $(u_1(s), \lambda_1(s))$, *and* $(u_2(s), \lambda_2(s))$ *of* $G(u, \lambda) = 0$, *the complex extension of* g. *If* $B^2 - AC > 0$ *both of these branches are real, if* $B^2 - AC < 0$, *both are complex. The branches have local expansions*

$$u_1(s) \sim u_0 + s \frac{\delta_+}{\sqrt{1 + |\delta_+|^2}} \phi + O(s^2)$$

$$\lambda_1(s) \sim \lambda_0 + \frac{s}{\sqrt{1 + |\delta_+|^2}} + O(s^2)$$

and,

$$u_2(s) \sim u_0 + s \frac{\delta_-}{\sqrt{1 + |\delta_-|^2}} \phi + O(s^2)$$

$$\lambda_2(s) \sim \lambda_0 + \frac{s}{\sqrt{1 + |\delta_-|^2}} + O(s^2).$$

Where,

$$\delta_+ \equiv \frac{-B + D}{A}$$

$$\delta_- \equiv \frac{-B - D}{A}.$$

Proof -

The proof is almost identical to the proof of the previous theorem. To show that both solution branches are real when $B^2 - AC > 0$, we note that the mapping $\eta(\xi, \lambda)$ can be shown to be real when ξ is real by writing the contracting mapping of the Implicit Function Theorem . The scaled bifurcation equatons are therefore real when ξ is real, and the same argument shows that when the roots (ξ_1, ς_1) and (ξ_2, ς_2) of the ABE's are real, the mappings $(u_1(s), \lambda_1(s))$, and $(u_2(s), \lambda_2(s))$ must also be real. ∎

Figure 5 illustrates the two case $D^2 > 0$ and $D^2 < 0$. The complex branches in the second case make it feasible to locate isola centers of the real equation without resorting to solving the large systems of equations that define the center.

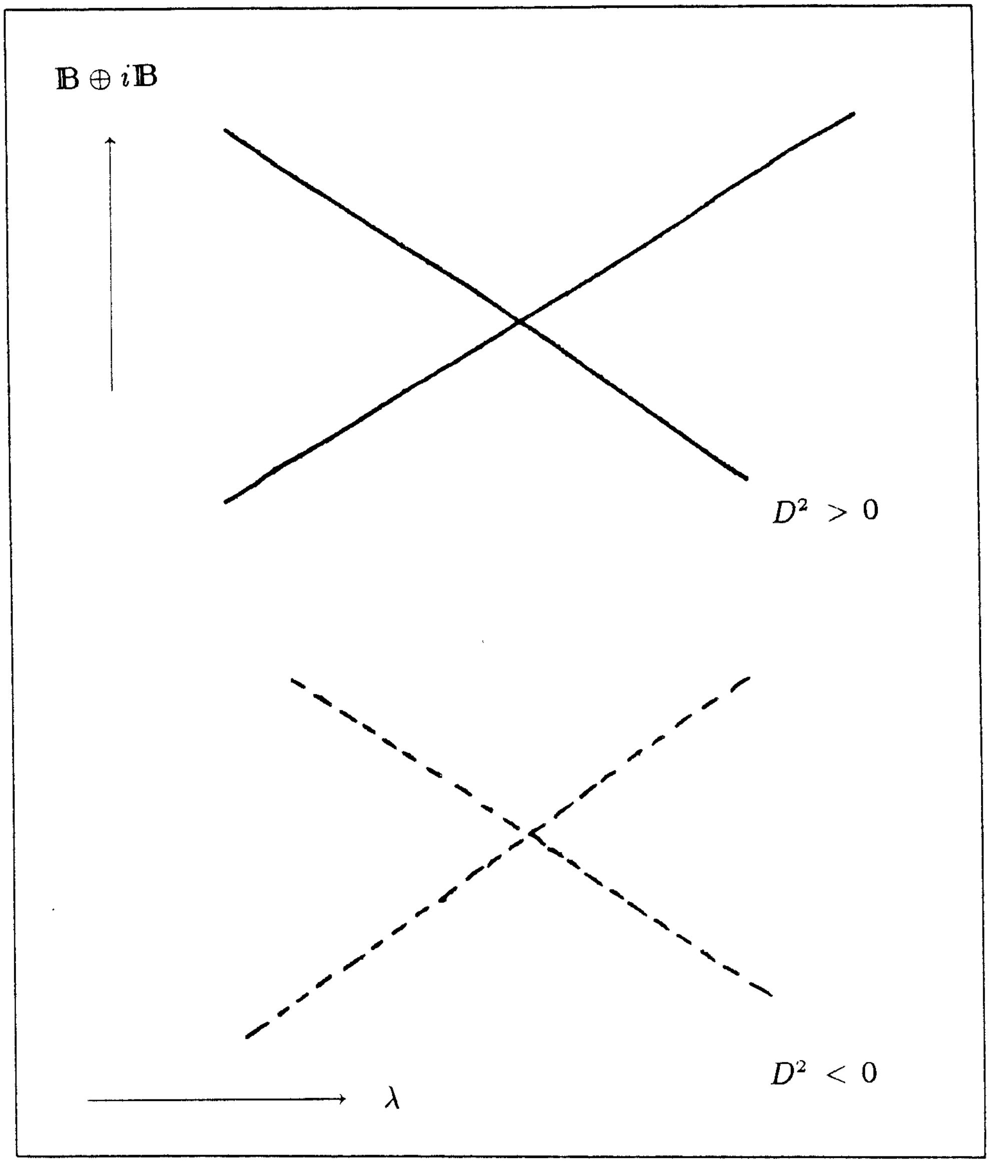

Figure 5. Complex Bifurcation at a Real Simple Quadratic Bifurcation Point

Simple Cubic Bifurcation

The last type of simple singular solution that we consider is the simple cubic bifurcation point, sometimes called a pitchfork, or transcritical bifurcation point. The proof proceeds exactly as for simple quadratic bifurcation, except that one of the roots of the QABE's is singular. We avoid this difficulty by rescaling the bifurcation equations.

Definition - Simple Cubic Bifurcation Point: A Simple Cubic Bifurcation Point is a simple singular solution (u_0, λ_0), *of the complex extension* $G(u, \lambda) = 0$ *of a real equation* $g(x, \lambda) = 0$ *at which*

$$\psi^* G^0_\lambda = 0$$

$$A \equiv \psi^* G^0_{uu} \phi\phi = 0$$

$$D^2 \equiv B^2 - AC \neq 0$$

$$b \equiv \psi^* G^0_{uuu} \phi\phi\phi + 3\psi^* G^0_{uu} \phi \eta^0_{\xi\xi} \neq 0$$

and

$$c \equiv 6\psi^* G^0_{u\lambda} \phi + 6\psi^* G^0_{uu} \phi \eta^0_\lambda \neq 0.$$

Where,

$$B \equiv \psi^* G^0_{uu} \phi \eta^0_\lambda + \psi^* G^0_{u\lambda} \phi$$

$$C \equiv \psi^* G^0_{uu} \eta^0_\lambda \eta^0_\lambda + 2\psi^* G^0_{u\lambda} \eta^0_\lambda + \psi^* G^0_{\lambda\lambda}$$

$$(I - \phi\psi^*) G^0_u \eta^0_\lambda + (I - \phi\psi^*) G^0_\lambda = 0$$

and

$$(I - \phi\psi^*) G^0_u \eta^0_{\xi\xi} + (I - \phi\psi^*) G^0_{uu} \phi\phi = 0. \blacksquare$$

Theorem - Bifurcation at a Simple Cubic Bifurcation Point: Let (u_0, λ_0) be a simple cubic bifurcation point of the complex equation $G(u, \lambda) = 0$. In a small neighborhood of (u_0, λ_0) there are exactly three branches of solutions of $G(u, \lambda) = 0$. These branches have the local expansions

$$u_1(s) \sim u_0 - s\frac{\frac{C}{2B}}{\sqrt{1 + |\frac{C}{2B}|^2}} + O(s^2)$$

$$\lambda_1(s) \sim \lambda_0 + s\frac{1}{\sqrt{1 + |\frac{C}{2B}|^2}} + O(s^2)$$

$$u_2(s) \sim u_0 + se^{-i\alpha/2}\phi + O(s^2)$$

$$\lambda_2(s) \sim \lambda_0 - \frac{1}{2}rs^2 + O(s^3)$$

and,

$$u_3(s) \sim u_0 + ise^{-i\alpha/2}\phi + O(s^2)$$

$$\lambda_3(s) \sim \lambda_0 + \frac{1}{2}rs^2 + O(s^3).$$

Where,

$$re^{i\alpha} = \frac{b}{c}.$$

Proof -

We begin just as in the simple quadratic bifurcation theorem. A Lyapunov-Schmidt decomposition is done for the complex equation, and the bifurcation equations

$$\psi^* G(u_0 + \xi\phi + \eta(\xi, \lambda), \lambda) = 0$$

are obtained. We scale G, u and λ in the same way as for quadratic bifurcation,

$$u = u_0 + \epsilon\xi\phi + \eta(\epsilon\xi, \lambda_0 + \epsilon\varsigma)$$

$$\lambda = \lambda_0 + \epsilon\varsigma,$$

and,

$$F(\xi,\varsigma;\epsilon) \equiv \begin{cases} \frac{2}{\epsilon^2}\psi^* G(u_0 + \epsilon\xi\phi + \eta(\epsilon\xi\phi, \lambda_0 + \epsilon\varsigma), \lambda_0 + \epsilon\varsigma) & \epsilon \neq 0 \\ \psi^* G^0_{uu}(\xi\phi + \eta^0_\lambda\varsigma)(\xi\phi + \eta^0_\lambda\varsigma) + 2\psi^* G^0_\lambda(\xi\phi + \eta^0_\lambda\varsigma)\varsigma + \psi^* G^0_{\lambda\lambda}\varsigma\varsigma & \epsilon = 0 \\ |\xi|^2 + |\varsigma| - 1 & \end{cases}.$$

F has been defined so that it is continuous at $\epsilon = 0$. Letting $\epsilon \to 0$, we obtain the QABE'S

$$A\xi^2 + B\xi\varsigma + C\varsigma^2 = 0$$

$$|\xi|^2 + |\varsigma|^2 = 1.$$

Keeping in mind that $\xi \in \mathbb{C}$ and $\varsigma \in \mathbb{R}$, and that we have assumed that $A = 0$,the only solutions of the QABE's are

$$\xi_1 = -\frac{\frac{C}{2B}}{\sqrt{1 + |\frac{C}{2B}|^2}} \qquad \varsigma_1 = \frac{1}{\sqrt{1 + |\frac{C}{2B}|^2}}$$

$$\xi_2 = e^{i\beta}, \qquad \varsigma_2 = 0$$

and $$\xi_3 = -\xi_1 \qquad \varsigma_3 = -\varsigma_1.$$

Where,

$$0 \le \beta < 2\pi.$$

The Fréchet derivative of F at $\epsilon = 0$ is

$$F_{(\xi,\varsigma)}(\xi,\varsigma;0)(\xi',\varsigma') = \begin{pmatrix} 2B\xi'\varsigma + 2B\xi\varsigma' + 2C\varsigma\varsigma' \\ \overline{\xi}\xi' + \overline{\xi'}\xi + 2\varsigma\varsigma' \end{pmatrix}$$

which, by Lemma (1), is nonsingular if and only if

$$2\varsigma|2B\varsigma|^2 - \Re e(\overline{(2B\xi + 2C\varsigma)}2B\varsigma\xi) \neq 0.$$

If ξ and ς are roots of the QABE's, this is equivalent to

$$\varsigma|D|^2 \neq 0.$$

So, since ς is nonzero for roots 1 and 3, and zero for root 2, and D is nonzero by assumption, roots 1 and 3 are isolated and root 2 is not. The Implicit Function Theorem guarantees the existence of one pair of mappings

$$(\xi_1(\epsilon), \varsigma_1(\epsilon))$$

that satisfiess $F(\xi, \varsigma; \epsilon) = 0$. Because F has the symmetry

$$F(\xi, \varsigma; \epsilon) = F(-\xi, -\varsigma; -\epsilon)$$

root 3 is not distinct from root 1.

The nonisolated root (root 2), corresponds to a pair of branches that are tangent at (u_0, λ_0), and have $\dot{\lambda} = 0$. To find these branches, we rescale the bifurction equations. Let

$$F'(\xi, \varsigma; \epsilon) \equiv \begin{pmatrix} \frac{3!}{\epsilon^3}\psi^* G(u_0 + \epsilon\xi\phi + \eta(\epsilon\xi\phi, \lambda_0 + \frac{1}{2}\epsilon^2\varsigma), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma) & \epsilon \neq 0 \\ b\xi^3 + c\xi\varsigma & \epsilon = 0 \\ |\xi|^2 - 1 & \end{pmatrix}.$$

We have defined $F'(\xi, \varsigma; 0)$ so that F' is continuous at $\epsilon = 0$.

We first solve $F'(\xi, \eta; \epsilon) = 0$, the Cubic Algebraic Bifurcation Equations (CABE's), then check that these solutions have nonsingular Fréchet derivative.

The CABE's are

$$b\xi^3 + c\xi\varsigma = 0$$

$$|\xi|^2 + |\varsigma|^2 = 1.$$

Keeping in mind that $\xi \in \mathbb{C}$ and $\varsigma \in \mathbb{R}$, the only solutions of the QABE's are

$$\xi_1 = e^{-i\frac{\alpha}{2}}, \qquad \varsigma_1 = -r$$

$$\xi_2 = ie^{-i\frac{\alpha}{2}}, \qquad \varsigma_2 = r$$

$$\xi_3 = -\xi_1, \qquad \varsigma_3 = \varsigma_1$$

$$\xi_4 = -\xi_2, \qquad \varsigma_4 = \varsigma_2$$

$$\xi_5 = 0, \qquad \varsigma_5 = 1$$

and

$$\xi_6 = \xi_5, \qquad \varsigma_6 = -\varsigma_5.$$

The Fréchet derivative of F' at $\epsilon = 0$ is

$$F'_{(\xi,\varsigma)}(\xi,\varsigma;0)(\xi',\varsigma') = \begin{pmatrix} 3b\xi^2\xi' + c\xi'\varsigma + c\xi\varsigma' \\ \bar{\xi}\xi' + \bar{\xi}'\xi \end{pmatrix}$$

which, by Lemma (1), is nonsingular if and only if

$$\Re e(\overline{(c\xi)}(3b\xi^2 + c\varsigma)\xi) \neq 0.$$

If ξ and ς are roots of the QABE's, this is equivalent to

$$|\xi|^2 \bar{c} b \xi^2 \neq 0.$$

So, since ξ is nonzero for roots 1 to 4, and zero for roots 5 and 6, and b and c are nonzero by assumption, roots 1 to 4 are isolated and roots 5 and 6 are not. The Implicit Function Theorem guarantees the existence of two pairs of mappings

$$(\xi_2(\epsilon), \varsigma_2(\epsilon))$$

and,

$$(\xi_3(\epsilon), \varsigma_3(\epsilon))$$

that satisfy $F'(\xi, \varsigma; \epsilon) = 0$. Since F' has the symmetry

$$F'(\xi, \varsigma; \epsilon) = F'(-\xi, \varsigma; -\epsilon),$$

roots 3 and 4 are not distinct from roots 1 and 2.

The nonisolated roots (roots 5 and 6) of the QABE's correspond to the branches we found using the mapping F,but are parametrized by ϵ^2 instead of ϵ. We have therefore shown that there are exactly three solutions for small ϵ:

$$u_1(\epsilon) = u_0 + \epsilon\xi_1(\epsilon)\phi + \eta(\epsilon\xi_1(\epsilon), \lambda_0 + \epsilon\varsigma_1(\epsilon))$$

$$\lambda_1(\epsilon) = \lambda_0 + \epsilon\varsigma_1(\epsilon)$$

$$u_2(\epsilon) = u_0 + \epsilon\xi_2(\epsilon)\phi + \eta(\epsilon\xi_2(\epsilon), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_2(\epsilon))$$

$$\lambda_2(\epsilon) = \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_2(\epsilon)$$

and,

$$u_3(\epsilon) = u_0 + \epsilon\xi_3(\epsilon)\phi + \eta(\epsilon\xi_3(\epsilon), \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_3(\epsilon))$$

$$\lambda_3(\epsilon) = \lambda_0 + \frac{1}{2}\epsilon^2\varsigma_3(\epsilon).$$

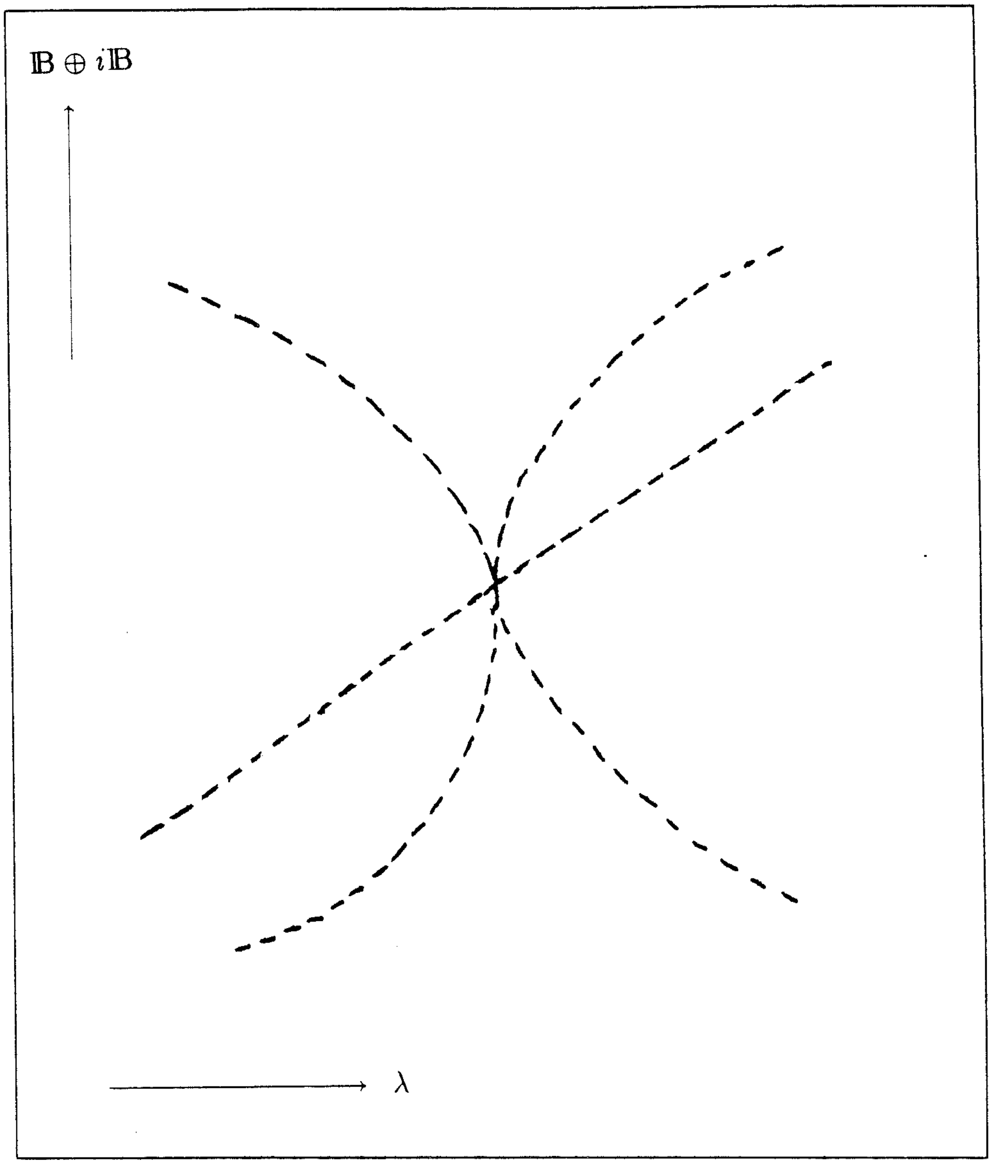

Figure 6. Complex Bifurcation at a Simple Cubic Bifurcation Point

At a simple cubic bifurcation point of a real equation $g(x, \lambda) = 0$ the complex extension has a bifurcation point that is very similar to the superposition of a simple quadratic fold and one branch of a simple quadratic bifurcation. Once again, the null vectors are real, as are the derivatives of G.

Theorem - Bifurcation at a Real Simple Cubic Bifurcation Point: Let (x_0, λ_0) be a simple cubic bifurcation point of a real equation $g(x, \lambda) = 0$. In a small neighborhood of $(x_0 + i0, \lambda_0)$, there are exactly two real branches of solutions, and one complex branch, of the complex extension $G(u, \lambda) = 0$. The real branches have the local expansions

$$u_1(s) \sim u_0 - s\frac{\frac{C}{2B}}{\sqrt{1 + |\frac{C}{2B}|^2}} + O(s^2)$$

$$\lambda_1(s) \sim \lambda_0 + s\frac{1}{\sqrt{1 + |\frac{C}{2B}|^2}} + O(s^2)$$

$$u_2(s) \sim u_0 + se^{-i\alpha/2}\phi + O(s^2)$$

$$\lambda_2(s) \sim \lambda_0 - \frac{1}{2}rs^2 + O(s^3).$$

The complex branch has the expansion

$$u_3(s) \sim u_0 + ise^{-i\alpha/2}\phi + O(s^2)$$

$$\lambda_3(s) \sim \lambda_0 + \frac{1}{2}rs^2 + O(s^3)$$

where,

$$|\frac{b}{c}| = r.$$

Proof -

The proof is the same as the proof of the previous Theorem, except that when the Implicit Function Theorem is used for root 1 of the QABE's, and root 1 of the CABE's, the contracting mapping used in the Implicit Function Theorem is used to show that $u_1(s)$ and $u_2(s)$ are real. ■

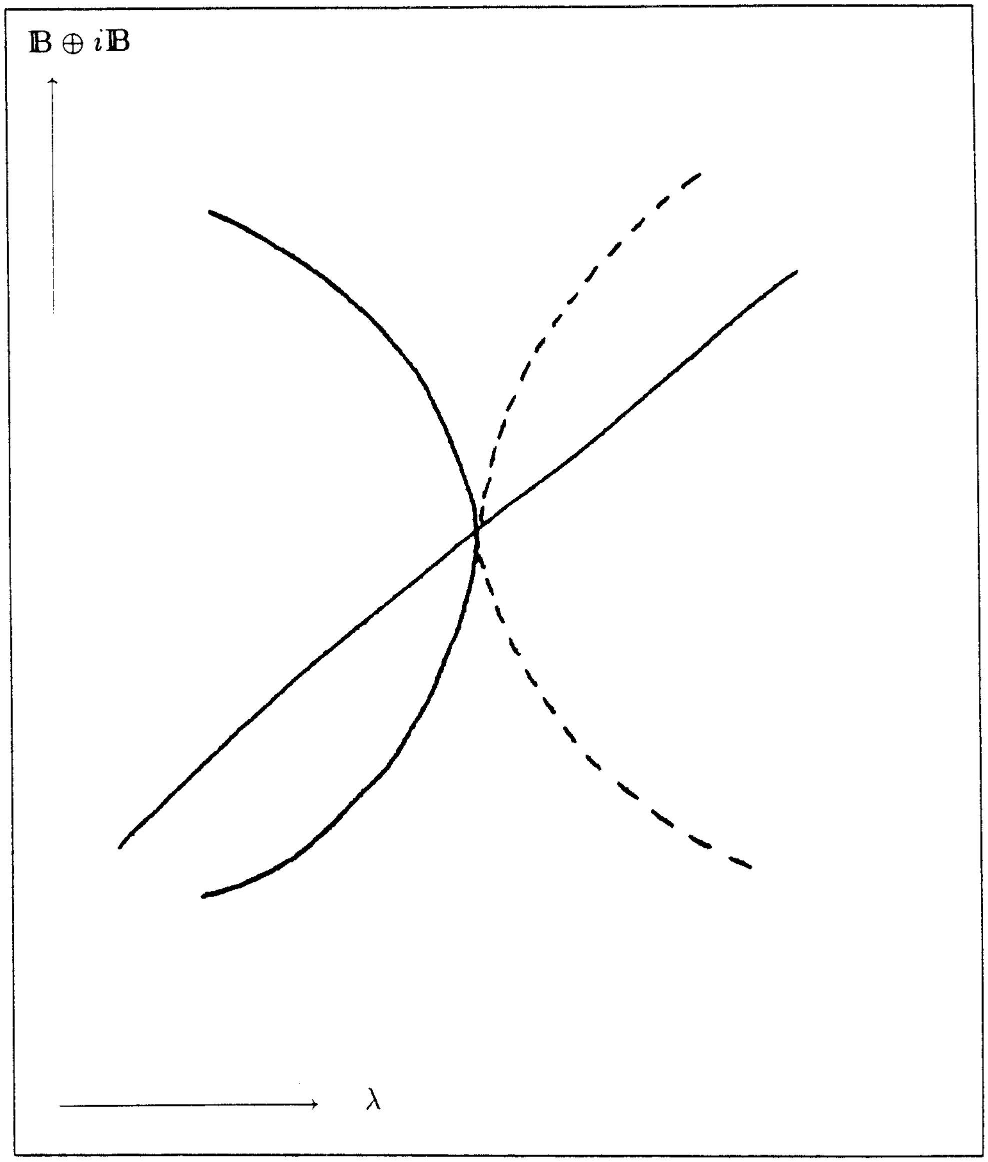

Figure 7. Complex Bifurcation at a Real Simple Cubic Bifurcation Point

In the next chapter we show that pairs of real quadaratic folds can exist that are connected by a complex branch of solutions. This makes complex bifurcation useful when solving real equations numerically, since it reduces the number of initial solutions that are required. We also discuss complex Hopf bifurcation. We show that two three-parameter families of complex periodic solutions are present at a Hopf bifurcation point.

IV. Extensions

In the first half of this chapter we present the result that makes complex bifurcation useful for numerical calculation, i.e. , that complex branches can connect real solution branches that are otherwise disjoint. We begin by considering equations with two real parameters, and use an analysis like that of Jepson and Spence (8) to show that sections of the solutions of these equations can have pairs of simple quadratic folds that are connected by complex solution branches. Most physical problems have two or more parameters, and we have reason to believe that such pairs of folds are common in these problems.

In the second half of this chapter we briefly discuss complex Hopf bifurcation, or the bifuration of periodic orbits from a branch of steady states. We show that two three-parameter families of complex periodic orbits can exist, whereas only a single one-parameter family of real periodic orbits exists.

Equations with Two Real Parameters

In this section we consider complex equations, like those discussed in chapters II and III, but which depend on two real parameters, λ and τ. If the Fréchet derivative $G_u(u_0, \lambda_0, \tau_0)$ at a solution $G(u_0, \lambda_0, \tau_0) = 0$ is nonsingular, the Implicit

Function Theorem can be used to find a solution surface $(u(\lambda,\tau),\lambda,\tau)$. If G is the complex extension of a real equation, and the solution (u_0,λ_0,τ_0) is real, the surface must be real. This can be shown quite easily by writing the contracting map used in the Implicit Function Theorem .

If (u_0,λ_0,τ_0) is not a regular point, various methods have been developed to find solutions. Computationally, the most attractive is to compute paths of singular points. This avoids the problems that arise in computing a surface, yet yields enough information to determine the surface's topology. Jepson and Spence (8) suggest using the extended system

$$F(v,\tau) \equiv \begin{pmatrix} G(u,(\lambda,\tau)) \\ G_u(u,(\lambda,\tau))\phi \\ l^*\phi - 1 \end{pmatrix} = 0, \tag{8}$$

where $v \equiv (u,\phi,\lambda)$, and l^* is an approximation to the null vector of $G_u^*(u,(\lambda,\tau))$.

Suppose that (u_0,λ_0,τ_0) is a simple quadratic fold of G for fixed τ, that is

$$\psi_0^* G_\lambda^0 \neq 0 \tag{9a}$$

$$\psi_0^* G_{uu}^0 \phi_0 \phi_0 \neq 0, \tag{9b}$$

$$\mathcal{N}(G_u^0) = \mathrm{span}(\phi_0) \tag{9c}$$

$$\mathcal{N}(G_u^{0*}) = \mathrm{span}(\psi_0^*). \tag{9d}$$

Jepson and Spence show that the Fréchet derivative of equation (8),

$$F_v((u_0,\phi_0,\lambda_0),\tau_0) \equiv \begin{pmatrix} G_u^0 & 0 & G_\lambda^0 \\ G_{uu}^0\phi_0 & G_u & G_{u\lambda}^0\phi_0 \\ 0 & l^* & 0 \end{pmatrix},$$

is nonsingular if and only if conditions (9) hold. This means that a regular solution of the extended system is a simple quadratic fold of G when τ is fixed. The Implicit

Function Theorem guarantees that a simple quadratic fold lies on a smooth path of quadratic folds.

Jepson and Spence go on to show that a simple quadratic bifurcation point, at which

$$\psi^* G_{uu}\phi\phi \neq 0$$

$$\psi^* G_\lambda = 0$$

$$\text{and,} \qquad \psi^* G_\tau \neq 0,$$

is a simple quadratic fold on the path of folds. Figures 8 and 9 show the two types of these folds, elliptic and hyperbolic. At an elliptic fold, the quantity D, which determines whether the quadratic bifurcation has real or complex branches, is negative. At a hyperbolic fold D is positive.

We can use these folds to show that disjoint real branches can be connected by complex branches. We fix τ, and consider how solutions of $G(u, \lambda, \tau_0)$ depend on τ_0. If (u_0, λ_0, τ_0) is a hyperbolic fold of the real equation, then the solutions at τ_0 consist of two intersecting real branches. As τ is increased, these branches break apart into two disjoint branches with no singular solutions. See Figure 10.

If τ is decreased past τ_0, the intersecting branches break apart the other way, and form a pair of simple quadratic folds, connected by a complex isola. See Figure 11. This complex isola provides a means of computing both disjoint real solution branches numerically, without knowing that a second branch exists. Most algorithms for computing solution branches numerically require an initial solution on each disconnected component of solutions. Finding these initial solutions often involves searching large spaces. By modifying the algorithm to also compute the complex solution branches, the number of real solution branches in each component increases. This reduces the time spent searching for initial solutions. The complex isola is not just a phenomena of perturbed bifurcation, it must exist for τ up to the next singular point on the path of folds. We have found disjoint real branches

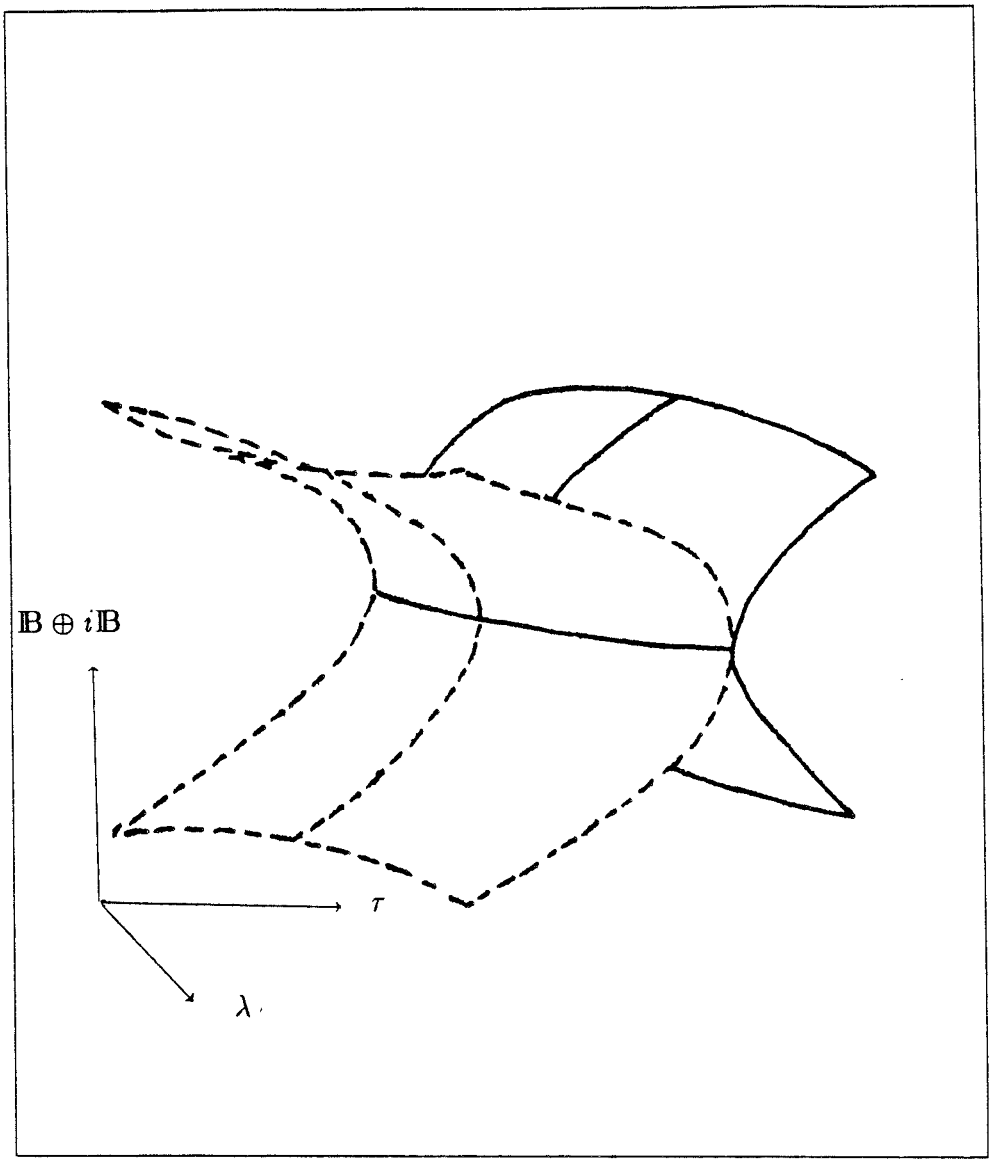

Figure 8. An Elliptic Fold of a Two Parameter Equation

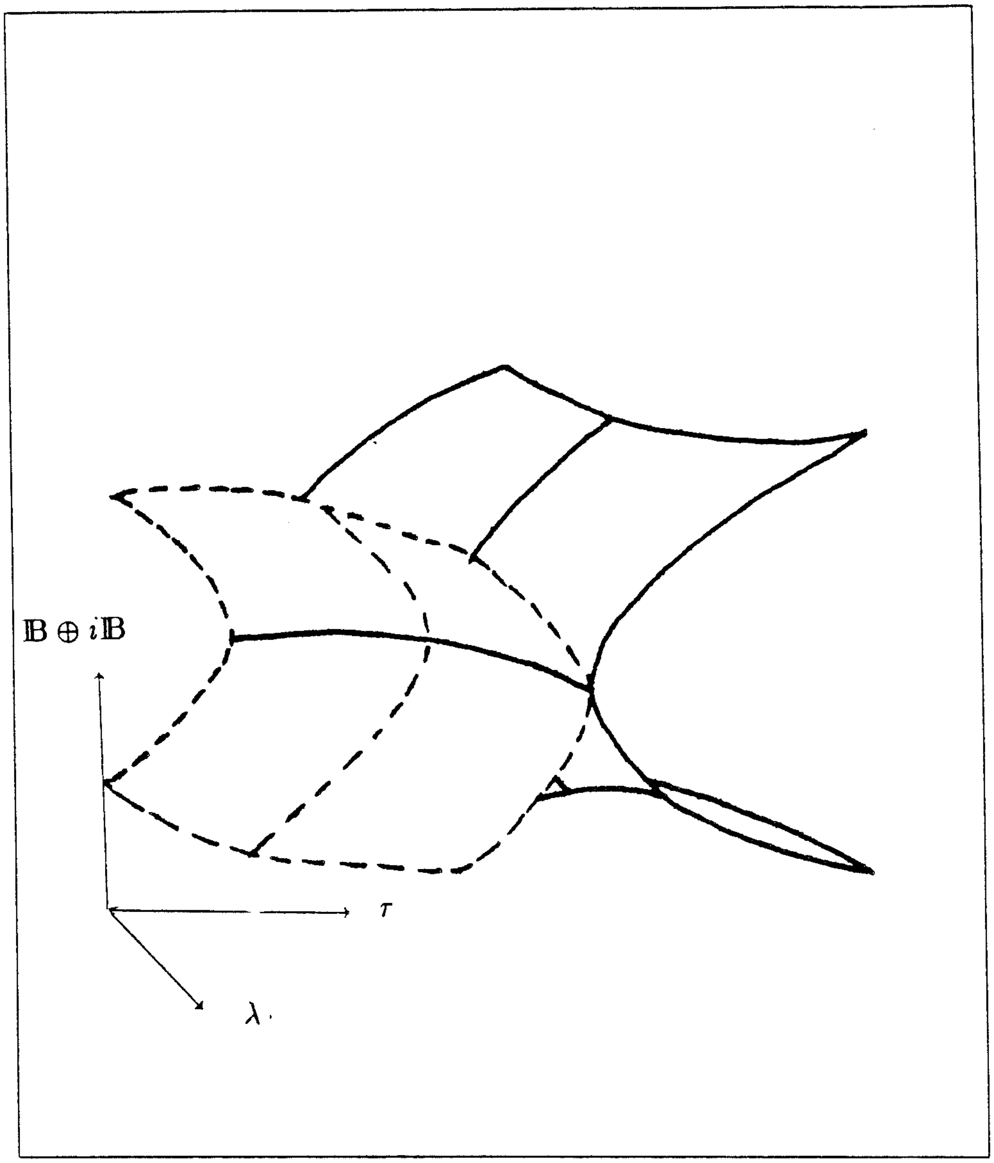

Figure 9. A Hyperbolic Fold of a Two Parameter Equation

that are connected by a complex branch that are quite far apart. See the examples in chapter VI.

We have shown that complex bifurcation occurs at simple quadratic folds, so there should be a complex branch of folds near an elliptic or hyperbolic fold. The tangent vector at such a fold was shown to be $\pm i$ times the null vector of the Fréchet derivative. At a simple quadratic bifurcation point in λ, the Fréchet derivative of F has the null vector

$$\begin{pmatrix} -\frac{B}{A}\phi_0 + \phi_1 \\ -\frac{l^*\phi_2}{l^*\phi_0}\phi_0 + \phi_2 \\ 1 \end{pmatrix},$$

where

$$A = \psi_0^* G_{uu}^0 \phi_0 \phi_0$$

$$B = \psi_0^* G_{uu}^0 \phi_0 \phi_0 + \psi_0^* G_{u\lambda}^0 \phi_0$$

$$G_u^0 \phi_1 = -G_\lambda^0$$

and $$G_u^0 \phi_2 = -(G_{uu}^0 \phi_0 \phi_1 + G_{u\lambda}^0 \phi_0 - \frac{B}{A} G_{uu}^0 \phi_0 \phi_0).$$

Since the λ component is nonzero, the complex path of folds must have complex parameter λ. In many problems there is no distinction between the parameters τ and λ, so that it is rather artificial for λ to be complex while τ is not. Complex paths of folds may however be a useful technique for locating disjoint paths of real folds.

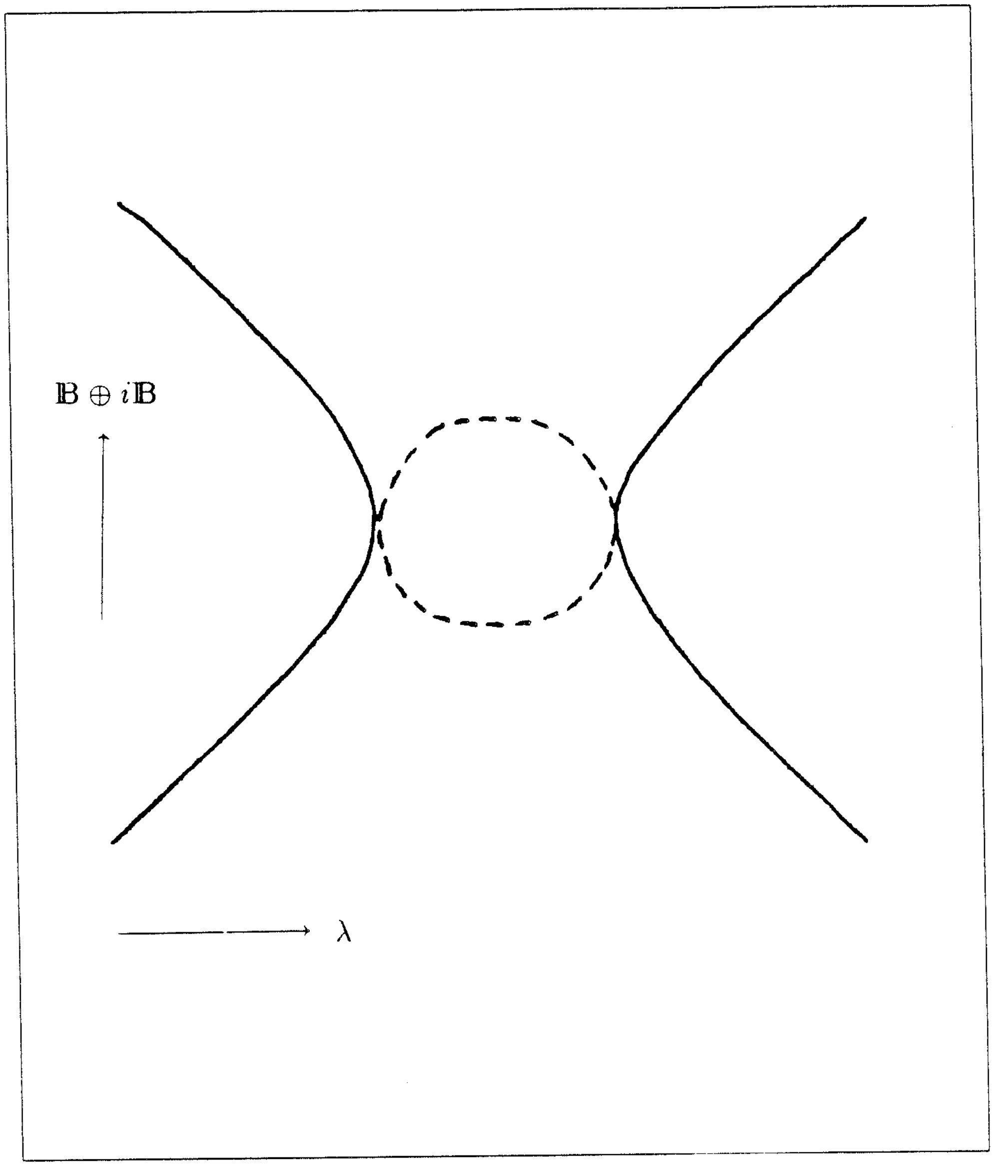

Figure 10. A Section of a Hyperbolic Fold for $\tau < \tau_0$

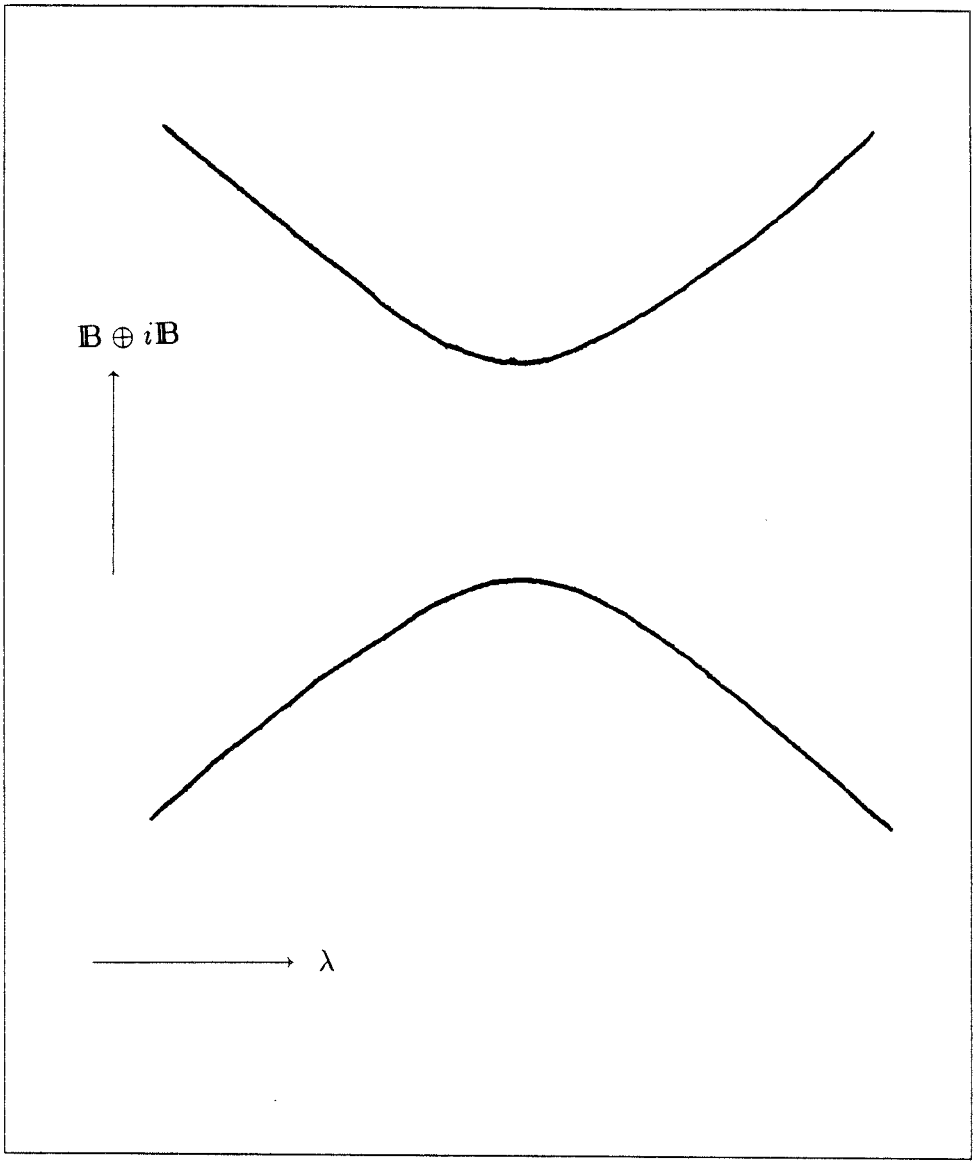

Figure 11. A Section of a Hyperbolic Fold for $\tau > \tau_0$

Complex Hopf Bifurcation

The following example from Jepson (7) demonstrates most of the features of complex Hopf bifurcation:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t - \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u^3 + vu^2 \\ u^2 v + v^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This has the steady state solution $u(\lambda) = 0$. At $\lambda = 0$ the Fréchet derivative has the pair of eigenvalues $\pm i$, and $(0,0)$ is a Hopf bifurcation point. In this example we are able to find exact periodic solutions. They are

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sqrt{-\lambda} \begin{pmatrix} cosh(\theta) sin(t+\tau) + i sinh(\theta) cos(t+\tau) \\ cosh(\theta) cos(t+\tau) - i sinh(\theta) sin(t+\tau) \end{pmatrix}$$

where τ determines the point on the solution at $t = 0$, and θ is an undetermined real constant. For real Hopf bifurcation, θ must be zero.

Figure 12 shows what these solutions look like. There is a steady state branch of solutions, $u(\lambda) = 0$; a set of real periodic orbits, parametrized by λ, which exist for $\lambda > 0$, and have period 2π; and two sheets of complex periodic orbits, $\theta \neq 0$, which are parametrized by λ and θ. These complex orbits exist for all λ, and also have period 2π.

In order to show how these complex orbits arise, we briefly sketch the proof of Hopf bifurcation, and show how the complex orbits arise.

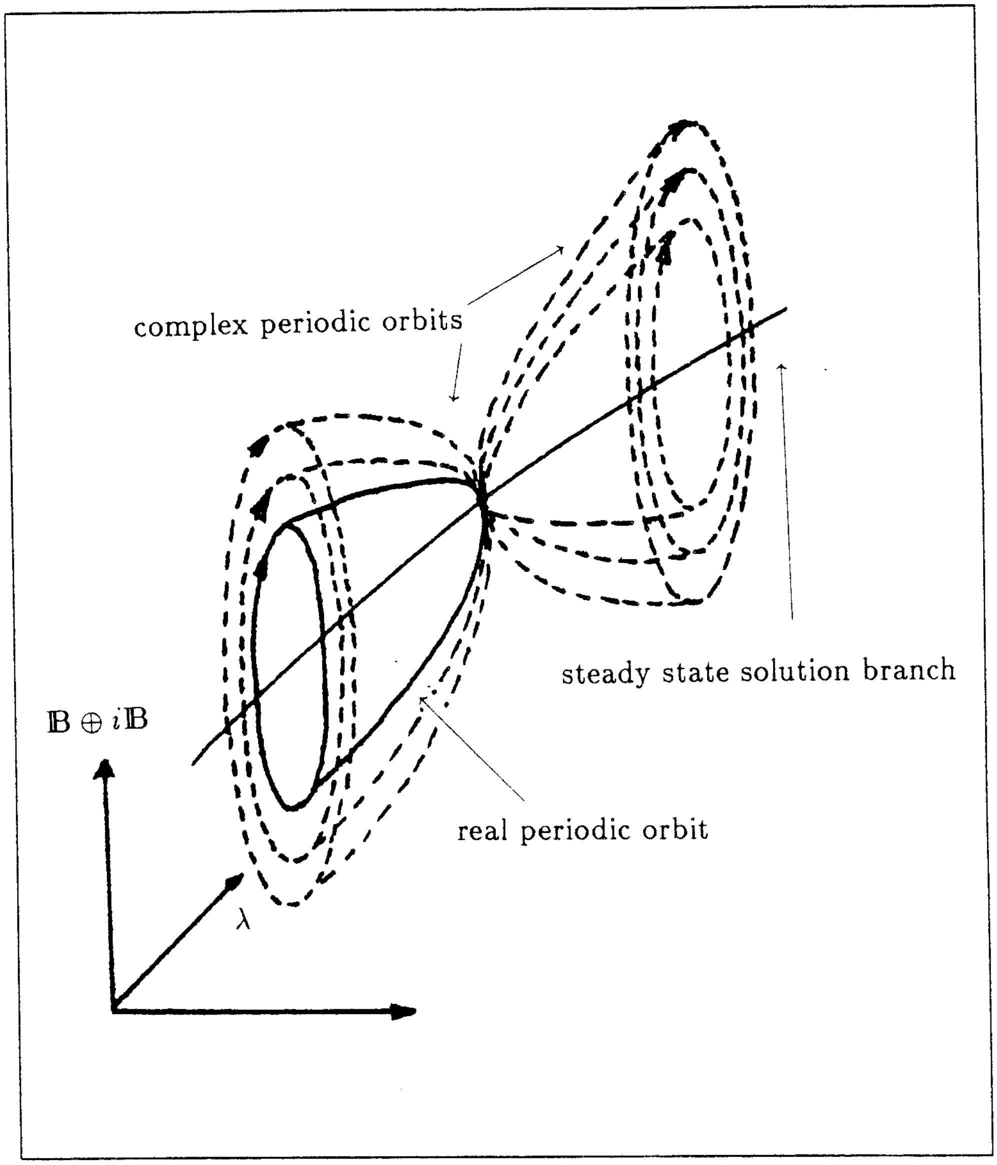

Figure 12. Complex Hopf Bifurcation

Hopf bifurcation concerns solutions of the time dependent problem

$$(15)\qquad \begin{aligned} &u_t - G(u,\lambda) = 0,\\ &u \text{ periodic.}\end{aligned}$$

Suppose that this equation has a steady state solution branch of real solutions $(u^I(\lambda),\lambda)$, and that on this branch, the Fréchet derivative of G, G_u, has a pair of complex eigenvalues $(\alpha \pm i\beta)$. A Hopf bifurcation point (u_0,λ_0) is a point on the steady state branch at which the real part of such a pair goes through zero with nonzero imaginary part. In order to study the bifurcation, a new parameter ϵ is introduced. The solution u is scaled by ϵ and the equation by $1/\epsilon$. Let

$$F(v,\lambda,\epsilon) \equiv \begin{cases} \frac{1}{\epsilon}G(u^I(\lambda)+\epsilon v,\lambda) & \epsilon \neq 0\\ G_u(u^I(\lambda),\lambda) & \epsilon = 0.\end{cases}$$

Then (15) is equivalent to

$$\begin{aligned} &v_t - F(v,\lambda,\epsilon) = 0,\\ &v \text{ periodic,}\end{aligned}$$

or, at $\epsilon = 0$,

$$\begin{aligned} &v_t - G_u(u^I(\lambda),\lambda)v = 0,\\ &v \text{ periodic.}\end{aligned}$$

If the period of u is p, we define the scaled time variable $\tau \equiv pt$, and so have the equation

$$(16)\qquad \begin{aligned} v_\tau - pG_u(u^I(\lambda),\lambda)v &= 0\\ v(1) - v(0) &= 0.\end{aligned}$$

Let

$$A(\lambda) \equiv G_u(u^I(\lambda),\lambda),$$

and $\xi(\lambda)$ and $\eta(\lambda)$ be the eigenfunctions of $A(\lambda)$ and $A^*(\lambda)$ corresponding to the eigenvalues $\alpha(\lambda) + i\beta(\lambda)$. That is

$$A(\lambda)\xi(\lambda) = (\alpha(\lambda) + i\beta(\lambda))\xi(\lambda),$$

$$A^*(\lambda)\eta^*(\lambda) = (\alpha(\lambda) + i\beta(\lambda))\eta^*(\lambda),$$

$$A(\lambda)\overline{\xi}(\lambda) = (\alpha(\lambda) - i\beta(\lambda))\overline{\xi}(\lambda),$$

and $$A^*(\lambda)\overline{\eta}^*(\lambda) = (\alpha(\lambda) - i\beta(\lambda))\overline{\eta}^*(\lambda).$$

The null vectors ξ and η can be normalized so that

$$\eta^*\xi = 1$$

and $$\overline{\eta}^*\overline{\xi} = 1.$$

It can also be easily shown that

$$\overline{\eta}^*\xi = 0$$

and $$\eta^*\overline{\xi} = 0.$$

Equation (16) has the solution

$$\lambda = \lambda_0,$$

$$p = 2\pi/\beta(\lambda_0),$$

$$v(\tau) = e^{p_0 A_0 \tau} v(0),$$

and $$v(0) = a\xi + b\overline{\xi},$$

for some $$a, b \in \mathbb{C}.$$

The constants a and b are determined by introducing the conditions

$$\psi^* v(0) = 1$$

and $$\psi^* A(\lambda_0) v(0) = 0,$$

where ψ^* is an element of the nullspace of $A^*(\lambda_0)$. Therefore

$$\psi = \theta_1 \eta + \theta_2 \overline{\eta},$$

for some complex scalars θ_1 and θ_2. In real Hopf bifurcation, ψ^* must be real, so $\theta_1 = \overline{\theta}_2$. We only require that $|\theta_1|^2 + |\theta_2|^2 = 1$, and that $\theta_1\theta_2 \neq 0$. This means that there are three degrees of freedom in choosing ψ for complex Hopf bifurcation, but only one for the real case. The constants a and b are determined by

$$\begin{pmatrix} \theta_1 & \theta_2 \\ i\beta(\lambda_0)\theta_1 & -i\beta(\lambda_0)\theta_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

If $\theta_1\theta_2 \neq 0$, this system is nonsingular, and has solutions $a = 1/2\theta_1$ and $b = 1/2\theta_2$.

The Fréchet derivative of equation (13) at this solution is

$$\begin{bmatrix} [e^{p_0 A_0} - I] & A_0 v(0) & v_\lambda(1) \\ \psi^* & 0 & 0 \\ \psi^* A^0 & 0 & 0 \end{bmatrix},$$

where $v_\lambda(\tau)$ satisfies

$$v'_\lambda - p_0 A_0 v_\lambda = p_0 A_{0\lambda} v$$

$$\text{and} \qquad v_\lambda = 0.$$

Therefore,

$$v_\lambda(\tau) = p_0 e^{p_0 A^0 \tau} \int_0^\tau \{e^{-p_0 A^0 s} A_\lambda^0 e^{p_0 A^0 s} v(0)\} ds,$$

and so,

$$v_\lambda(1) = pa(\alpha_\lambda + i\beta_\lambda)\xi + pb(\alpha_\lambda - i\beta_\lambda)\overline{\xi} - a(e^{p_0 A^0} - I)\xi_\lambda - b(e^{p_0 A^0} - I)\overline{\xi}_\lambda.$$

The Fréchet derivative can be shown to be nonsingular at this solution, by using Keller's Basic Lemma (10).

Lemma: Let X be a Banach space, and $\mathcal{A}$ a linear operator mapping $X\times\mathbb{R}^2 \to X \times \mathbb{R}^2$, of the form

$$\mathcal{A} = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix},$$

where

$$A : X \to X, \qquad B : \mathbb{R}^2 \to X,$$

$$C^* : X \to \mathbb{R}^2, \qquad D : \mathbb{R}^2 \to \mathbb{R}^2.$$

(i) If A is nonsingular then $\mathcal{A}$ is nonsingular if and only if $(D - C^*A^{-1}B)$ is nonsingular.

(ii) If A is singular and

$$\dim\mathcal{N}(A) = \text{codim Range}(A) = 2,$$

then $\mathcal{A}$ is nonsingular if and only if

(c_0) dim Range*(B)*$= 2$, *(c_1)* Range(B)$\cap$Range(A)$= 0$,

(c_2) dim Range(C^*) $= 2$, *(c_3)* N(A)$\cap$N(C^*)$= 0$.

(iii) If A is singular and

$$\dim\mathcal{N}(A) = \text{codim Range}(A) > 2$$

then $\mathcal{A}$ is singular. ■

Since $\dim(\mathcal{N}(e^{p_0 A^0} - I)) = 2$, case (ii) of this Lemma applies. We first show that (c_2) holds, i.e. , that $\dim(\text{Range}((\{\psi^*, \psi^* A^0\})) = 2$.

$$\begin{aligned}\text{Range}(\begin{pmatrix} \psi^* \\ \psi^* A^0 \end{pmatrix}) &= \text{Range} \begin{pmatrix} \theta_1\eta^* + \theta_2\overline{\eta}^* \\ i\beta(\lambda_0)\theta_1\eta^* - i\beta(\lambda_0)\theta_2\overline{\eta}^* \end{pmatrix} \\ &= \text{Range} \begin{pmatrix} \theta_1 & \theta_2 \\ i\beta(\lambda_0)\theta_1 & i\beta(\lambda_0)\theta_2 \end{pmatrix} \begin{pmatrix} \eta^* \\ \overline{\eta}^* \end{pmatrix} \\ &= \text{Range}((\, \eta^* \quad \overline{\eta}^* \,)).\end{aligned}$$

Since η and $\overline{\eta}$ are independent, $\dim(\text{Range}((\{\psi^*, \psi^* A^0\})) = 2$.

Now, to show that (c_2) holds, let u be an element of $\mathcal{N}(C^*)$. Then

$$\theta_1 \eta^* u + \theta_2 \overline{\eta}^* u = 0$$

$$\text{and} \qquad i\beta(\theta_1 \eta^* u - \theta_2 \overline{\eta}^* u) = 0.$$

If neither θ_1 or θ_2 is zero, this is equivalent to

$$\eta^* u = 0 \qquad \text{and} \qquad \overline{\eta}^* u = 0.$$

Now, $\mathcal{N}(A) = \text{span}(\xi, \overline{\xi})$, so if $u \in \mathcal{N}(A)$,

$$u = \gamma_1 \xi + \gamma_2 \overline{\xi}.$$

If u is also in $\mathcal{N}(C^*)$,

$$\eta^* u = \gamma_1 \eta^* \xi + \gamma_2 \eta^* \overline{\xi} = \gamma_2 \eta^* \xi = 0$$

$$\overline{\eta}^* u = \gamma_1 \overline{\eta}^* \xi + \gamma_2 \overline{\eta}^* \overline{\xi} = \gamma_1 \overline{\eta}^* \overline{\xi} = 0.$$

therefore, $\mathcal{N}(A) \cap \mathcal{N}(C^*) = \emptyset$.

Condition (c_0) is that $\dim(\text{Range}(A_0 v(0), v_\lambda(1))) = 2$. Recall that

$$A_0 v(0) = i\beta_0(a\xi - b\overline{\xi}),$$

$$\text{and} \qquad v_\lambda(1) = p_0 a \kappa_\lambda \xi + p_0 b \overline{\kappa}_\lambda \overline{\xi} - aC_0 \xi_\lambda - bC_0 \overline{\xi}_\lambda,$$

$$\text{where} \qquad \kappa_\lambda = \alpha_\lambda + i\beta(\lambda),$$

$$\text{and} \qquad C_0 = e^{p_0 A_0} - I.$$

Condition (c_0) holds if and only if $A_0 v(0)$ and $v_\lambda(1)$ are independant. Let

$$u = \gamma_1 A_0 v(0) + \gamma_2 v_\lambda(1).$$

Then we show that $u = 0$ implies that $\gamma_1 = \gamma_2 = 0$. The equation $u = 0$ is

$$a(i\beta_0\gamma_1 + p_0\gamma_2\kappa_\lambda)\xi + b(-i\beta_0\gamma_1 + p_0\gamma_2\overline{\kappa}_\lambda)\xi - \gamma_2 C_0(a\xi_\lambda + b\overline{\xi}_\lambda) = 0.$$

We multiply on the left by η^* and $\overline{\eta}^*$, recalling that $\eta^* C_0 = 0$ and $\overline{\eta}^* C_0 = 0$. We obtain the two conditions

$$a(i\beta_0\gamma_1 + p_0\gamma_2\kappa_\lambda) = 0,$$

and

$$b(-i\beta_0\gamma_1 + p_0\gamma_2\overline{\kappa}_\lambda) = 0.$$

Nonzero γ_1 and γ_2 exist if the determinant

$$i\beta_0 abp_0(\kappa_\lambda + \overline{\kappa}_\lambda) \neq 0.$$

Now $ab = 1/4\theta_1\theta_2 \neq 0$, so if $\alpha_\lambda \neq 0$, the determinant is nonzero, and so both γ_1 and γ_2 must be zero. $A_0 v(0)$ and $v_\lambda(1)$ are therefore independant, so (c_0) holds.

Condition (c_1) is shown to hold in a similar way. The condition is that

$$\text{Range}(A_0 v(0), v_\lambda(1)) \cap \text{Range}(e^{p_0 A_0} - I) = \emptyset.$$

For an element u to be in the range of $e^{p_0 A_0} - I$, it must satisfy

$$\eta^* u = 0,$$

and

$$\overline{\eta}^* u = 0.$$

A general element of Range$(A_0 v(0), v_\lambda(1))$ can be written as

$$u = \gamma_1 A_0 v(0) + \gamma_2 v_\lambda(1),$$

so for u to be in both sets, we get a pair of conditions exactly like those obtained above i.e.

$$a(i\beta_0\gamma_1 + p_0\gamma_2\kappa_\lambda) = 0,$$

and

$$b(-i\beta_0\gamma_1 + p_0\gamma_2\overline{\kappa}_\lambda) = 0.$$

Both γ_1 and γ_2 must therefore be zero, and so (c_1) holds.

We have shown that the Fréchet derivative of F is nonsingular. Therefore, by the Implicit Function Theorem , a set of smooth mappings of the form $(v(\epsilon), p(\epsilon), \lambda(\epsilon))$ exist that solve $F = 0$. Such a set exists for each choice of θ_1 and θ_2 for which $\theta_1\theta_2 \neq 0$.

In this chapter we have shown that pairs of simple quadratic folds can be joined by complex branches of solutions. The existence of these complex connections improves the preformance of numerical algorithms for computing solutions of real equations. Fewer initial solutions are required to compute a given set of solutions. In the next chapter we present one numerical algorithm, and suggest ways that it can be modified to compute complex solutions.

V. Computing Complex Solutions Numerically

We have shown that complex bifurcation occurs at several types of simple singular solutions, and that complex branches can connect pairs of real simple quadratic folds. In this chapter we present two ways of using existing algorithms to compute complex solutions numerically.

Several effiecient algorithms exist for computing solutions of real equation, for example Euler-Newton continuation, and pseudo arc-length continuation. There are two main ways to use these real algorithms. We can use complex arithmatic, and modify the algorithm to solve the complex equation, or we can replace the complex equation by an equivalent pair of real equations. Both approaches have their drawbacks. Using an equivalent system of real equations requires no changes to the basic algorithm, but all bifurcation points become multiple bifurcation points, so some care must be used when switching branches. Computing solutions of the complex equation requires changing the basic algorithm, but it makes switching branches simpler, and makes the coding of the Fréchet derivatives trivial. Which method is best depends strongly on how easily changes can be made to a given code.

The first section of this chapter describes pseudo arc-length continuation, which is presented in Keller (10). In the following sections we describe how this algorithm can be used to compute complex solutions. Using Euler-Newton continuation is simpler, and the necessary changes should be obvious from our discussion of pseudo arc-length continuation.

Pseudo Arc-Length Continuation

Let $g(x, \lambda) = 0$ be a real equation. Pseudo arc-length continuation computes solutions of this equation of the form $(x(s), \lambda(s))$, where s is an approximate arc-length. If a solution $(x(s_0), \lambda(s_0))$, and the tangent to the solution branch $(\dot{x}(s_0), \dot{\lambda}(s_0))$ are known, all of the solutions that are connected to (x_0, λ_0) can be computed.

If s were true arc-length, a point on the solution curve would satisfy

$$g(x(s), \lambda(s)) = 0$$

and

$$\| \dot{x}(s) \|^2 + |\dot{\lambda}(s)|^2 = 1.$$

The pseudo arc-length approximation replaces one tangent by the tangent at s_0, and the other by the secant between s and s_0. A solution branch parametrized by pseudo-arc-length satisfies

$$g(x(s), \lambda(s)) = 0 \tag{12a}$$

and

$$N(x(s), \lambda(s)) = 0, \tag{12b}$$

where,

$$N(x(s), \lambda(s)) \equiv \dot{x}_0^T(x(s) - x(s_0)) + \dot{\lambda}_0(\lambda(s) - \lambda(s_0)) - (s - s_0).$$

This has a simple geometric interpretation, shown in Figure 12. The solution at s is required to lie on the intersection of the solution curve and a hyperplane

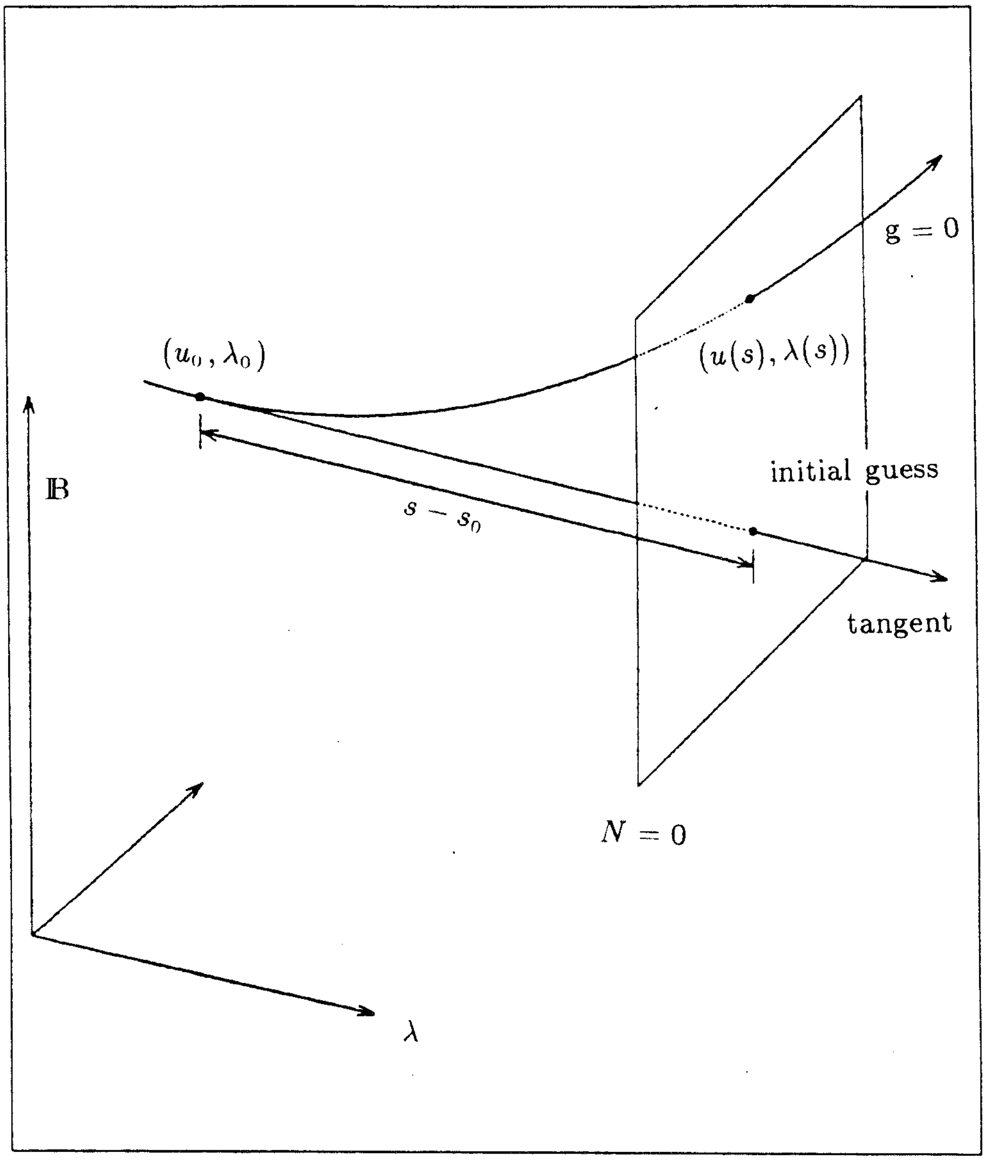

Figure 12. Pseudo Arc-Length Continuation

perpendicular to the tangent at s_0, and at a distance $s-s_0$ from (x_0, λ_0). Using this parametrization, the steepness of the solution curve does not effect the convergence of the algorithm.

For a given stepsize $s - s_0$, solutions of equation (12) are calculated by Newton's method, which is

$$x_{n+1}(s) = x_n(s) + \Delta x_n \tag{13a}$$

$$\lambda_{n+1}(s) = \lambda_n(s) + \Delta\lambda_n \tag{13b}$$

$$\begin{pmatrix} g_x & g_\lambda \\ N_x & N_\lambda \end{pmatrix}_n \begin{pmatrix} \Delta x \\ \Delta\lambda \end{pmatrix}_n = -\begin{pmatrix} g \\ N \end{pmatrix}_n . \tag{13c}$$

The stepsize $s - s_0$ is usually chosen so that it is as large as possible, while still producing an initial guess that is close enough that Newton's method converges. Perozzi (15) discusses several methods for choosing an optimal step size.

Equation (13c) may be solved by a variety of techniques, but in many problems g_x has some structure that can be exploited to speed the solution of linear systems involving g_x. The elements N_x and g_λ destroy this structure. Keller therefore suggests that a block elimination algorithm be used. This requires that a system with g_x be solved for two right hand sides.

The block elimination proceeds as follows. If g_x is nonsingular, let

$$g_x v = g_\lambda \tag{9a}$$

and

$$g_x z = -g. \tag{9b}$$

Then we must have that

$$\Delta x = z - \Delta\lambda \cdot v,$$

and therefore,

$$\Delta\lambda = \frac{(-N_x z - N)}{(N_\lambda - N_x v)}.$$

Keller has shown in (3) that near singular points of g this Newton iteration still converges. If $(s - s_0)$ is large enough that a singular point lies between s_0 and s, the iteration simply converges to a solution on the other side of the singular point. This means, however, that some means of detecting singular points must be included in the algorithm. Some measure of the singularity of g_x is usually monitored as a function of s. Some possiblities are the sign of the determinant of g_x, and its condition number. On most branches through simple singular points the determinant changes sign at the singular point, and the condition number has an extrema.

Once the solution at s is found to a given accuracy, the tangent must be calculated. If s were true arc-length, the tangent would satisfy

$$g_x \dot{x} + g_\lambda \dot{\lambda} = 0$$

and

$$\| \dot{x} \|^2 + |\dot{\lambda}|^2 = 1.$$

The pseudo arc-length approximation is

$$g_x \dot{x} + g_\lambda \dot{\lambda} = 0 \tag{15a}$$

and

$$(x(s) - x(s_0))^T \dot{x}(s) + (\lambda(s) - \lambda(s_0))\dot{\lambda}(s)) = s - s_0. \tag{15b}$$

These equations have the same block structure as (13c). In fact, if the last factorization of g_x is saved, finding the tangent only requires two backsolves.

At a bifurcation point equations (15) are singular, and so the tangents must be found by some other means. Keller (10) suggests several methods, including searching parallel to the known solution branch, and calculating and solving the algebraic bifurcation equations. Once the solutions of the ABE's are known, local expansions, like those in chapter III, can be used to find an initial guess for Newton's method.

The overall organization of the algorithm is fairly simple. Given an initial solution, tangent, and a step size, new points on the solution branch are calculated using Newton's method. After each step the step size is adjusted so that it remains nearly optimal. When a singular point is detected, it is located accurately by bisection, or some other search strategy. The tangents at the singular point are then estimated, and together with the singular point, are used as initial solutions for the algorithm.

Using the Real Algorithm to Compute Complex Solutions

Let $G(u, \lambda) = 0$ be a complex equation. Recall that G can be written in the form

$$G(x + iy, \lambda) \equiv f(x, y, \lambda) + ih(x, y, \lambda),$$

where f and h are mappings of $(\mathbb{B} \times \mathbb{B}) \times \mathbb{R} \to \mathbb{B}$. The complex equation is equivalent to the real system

$$f(x, y, \lambda) = 0$$

$$h(x, y, \lambda) = 0.$$

Using pseudo arc-length continuation, Newton's method is

$$x_{n+1}(s) = x_n(s) + \Delta x_n \tag{16a}$$

$$y_{n+1}(s) = y_n(s) + \Delta y_n \tag{16b}$$

$$\lambda_{n+1}(s) = \lambda_n(s) + \Delta\lambda_n \tag{16c}$$

$$\begin{pmatrix} f_x & f_y & f_\lambda \\ h_x & h_y & h_\lambda \\ N_x^r & N_y^r & N_\lambda^r \end{pmatrix}_n \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta\lambda \end{pmatrix}_n = -\begin{pmatrix} f \\ h \\ N^r \end{pmatrix}_n , \tag{16d}$$

where

$$N^r(x(s), y(s), \lambda(s)) =$$
$$\dot{x}_0^T(x(s) - x(s_0)) + \dot{y}_0^T(y(s) - y(s_0)) + \dot{\lambda}_0(\lambda(s) - \lambda(s_0)) - (s - s_0).$$

If the block elimination algorithm is used to solve (16*d*), the linear system that must be solved is twice as large as for the real equation. Furthermore, unless the variables x and y are interlaced, or the block elimination algorithm is modified, the structure of G_u is destroyed.

At a simple singular solution, where G_u has a one-dimensional null spaces, the real system has two-dimensional null spaces, as we showed in chapter III. This means that simple bifurcation points of the complex equation are multiple bifurcation points of the real system. Most codes do not include routines to handle multiple bifurcation point, as the possibility exists for the bifurcation of surfaces of solutions. Some special purpose code must therefore be written to switch branches at bifurcation points, or it must be done by hand.

A Complex Algorithm

The alternative to solving the equivalent real system is to modify the algorithm to solve the complex equation directly. Most of the required changes involve using complex arithematic to calculate G and it's derivatives. The pseudo arc-length conditions and the block elimination algorithm, however, require other modifications.

The pseudo arc-length algorithm is based on essentially geometrical arguments, and the geometry of the complex solution branches has not changed. We have shown that they are still smooth arcs, and that the arcs are unique near regular points. In order to use the pseudo arc-length condition, it is necessary only to replace the real inner products by the appropriate complex inner product. The resulting pseudo arc-length constraint is

$$N^c(u(s),\lambda(s)) \equiv \Re e[\dot{u}_0^*(u(s)-u(s_0)) + \dot{\lambda}_0(\lambda(s)-\lambda(s_0)) - (s-s_0)] = 0.$$

The block elimination algorithm, which assumes that the linear system is a matrix equation, must also be modified. We again let

$$(17a) \qquad G_u v = G_\lambda$$

$$(17b) \qquad \text{and} \qquad G_u z = -G.$$

Which implies that

$$\Delta u = z - \Delta\lambda \cdot v,$$

and therefore $\qquad \Re e(N_\lambda^c \cdot \Delta\lambda - N_u^c v \cdot \Delta\lambda - N_u^c z - N^c) = 0.$

And so, since $\Delta\lambda$ is real,

$$(18a) \qquad \Delta\lambda = \frac{\Re e(-N_u^c z - N)}{\Re e(N_\lambda^c - N_u^c v)}$$

$$(18b) \qquad \text{and} \qquad \Delta u = z - \Delta\lambda \cdot v.$$

Notice that the linear systems in equation (17) still retain any structure of G_u. If the linear equation solver is modified to use complex arithematic, the operation count remains the same as for the real equation, except that complex multiplications are counted.

In order to detect singular points, it was suggested that either the condition number, or the sign of the determinant of G_u be monitored as a function of s. For a complex branch the condition number still has an extrema at a singular point, but if the sign of the determinant is used some further modifications are necessary. The determinant on a complex branch is a complex quantity, and if the sign is interpreted as the unit in the direction of the determinant, a change of sign is equivalent to the angle between succesive vectors $(\Re e(det), \Im m(det))$ being greater than 180°.

The overall strategy of the algorithm is unchanged. Solution branches are computed until a singular point is detected. There, each tangent is used as an initial solution, and all of the bifurcating branches are computed. However, the number of bifurcating branches is always greater than in the real case.

A well designed code would use the real algorithm, and real arithematic when computing real branches of the complex equation, since we have shown that they remain real. When a complex bifurcation point is identified, the code would switch to the complex algorithm to compute the complex branch.

We have used this algorithm to compute the solutions of the examples in the next chapter. The extra work required to compute the complex solutions is far outweighed by the ability to locate disjoint solutions of the real equation without random searching.

VI. Examples

In this final chapter we apply the numerical techniques of the previous chapter to several example problems. The first is from Keller (10), and was used to demonstrate the pseudo arc-length algorithm. It has a trivial branch of real solutions, with alternate simple quadratic and simple cubic bifurcation points. Its solutions also include several simple quadratic folds.

The second set of examples are the elementary catastrophes from Catastrophe Theory. These are topological models for the solutions of the bifurcation equations, and they may be complex. By taking sections of these catastrophes, we can very simply determine the topology of the complex solution branches at the simple singular points in chapter III. The catastrophes are useful because we can also determine which of the higher order simple singular points are complex bifurcation points.

Our final example shows how complex solutions can be used to locate real solution branches. We consider the axially symmetric flow between a pair of infinite rotating disks. By calculating complex flows, we have been able to find two real branches that are disjoint from our initial solution.

A Nonlinear Two Point Boundary Value Problem

The following example was presented in (10) as a demonstration of the pseudo arc-length continuation algorithm. Its solutions have all of the singular points discussed in chapter III. The problem is to solve the two point boundary value problem

$$G \equiv \begin{cases} u_{xx} + f(x, u; \lambda) = 0 \\ u(0) = 0 \\ u(1) \doteq 0 \end{cases} ,$$

where

$$f(x, u; \lambda) \equiv 2q(\lambda) + \lambda p(u - q(\lambda)x(1-x))$$

$$q(\lambda) \equiv \lambda^2 e^{-\frac{1}{2}\lambda}$$

and

$$p(z) \equiv z - \sum_{i=1}^{8} z^i - 2z^9.$$

Note that the polynomial $p(z)$ given in (10) does not match the figure in (10). This is the correct polynomial for that figure.

Figure 13 shows the integral of the solution as a function of λ. There are bifurcation points on the solution branch $\Gamma_1(\lambda) = q(\lambda) \cdot x(1-x)$ at $\lambda_n = n^2\pi^2$. The first is a simple quadratic bifurcation, the second a simple cubic bifurcation, and so on. The quadratic bifurcation points all have $D^2 > 0$, so there is no complex bifurcation at these points. The cubic bifurcation points and the folds, however, are complex bifurcation points. The branches that bifurcate from the cubic bifurcation point are symmetric, and have the same integral, so appear as a single branch.

In addition to the complex branches which are associated with bifurcation points on real branches, we have been able to find a pair of complex branches Γ_2 and Γ_3, that are disjoint from the real solutions. When the differential equation is approximated by finite differences, an algebraic system is obtained. Bézout's Theorem, from Algebraic Geometry, says that if there is no common factor in this set of equations, the number of complex solutions is 9^n, where n is the number of intervals on $[0, 1]$. We have used 20 intervals to compute the solutions in Figure 13, so there should be 9^{20} complex solutions. Some of these are from bifurcation points outside the range of λ that we considered. We suspect, however, that most of them are like Γ_2 and Γ_3, disjoint from the real solution branches. If a real initial guess is used, these solutions will not be found. We found Γ_2 and Γ_3 by random searching with a complex initial guess.

Notice that the number of solutions is constant away from $\lambda = 0$. This is due the algebraic nature of the difference equations. Even for non-algebraic problems, however, the bifurcation equations are often locally algebraic. So, if the solutions stay bounded, and the only types of singular points have locally algebraic bifurcation equations, the number of the solutions will be independant of λ, since the number of solution branches at a bifurcation point is determined by algebraic equations.

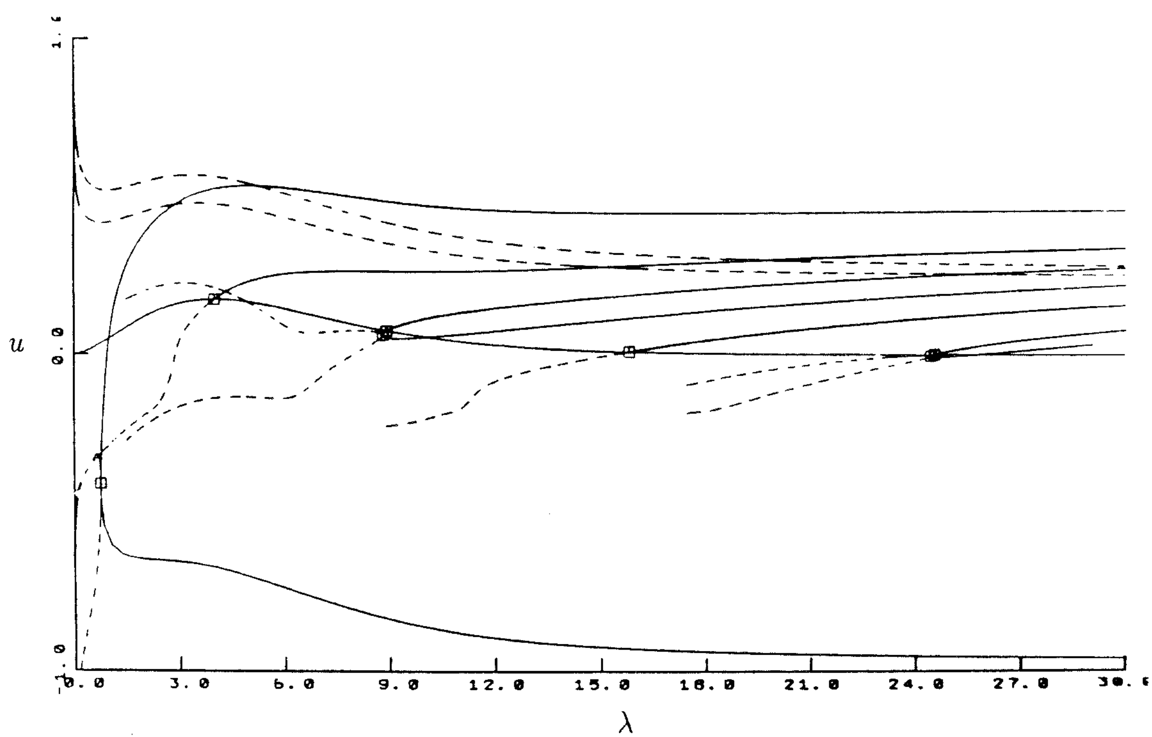

Figure 13. Solutions of a Nonlinear Two Point Boundary Value Problem

The Elementary Catastrophes

Many books have been written recently about Catastrophe Theory. See for example (17), (1), and (16). A catastrophe is essentially a topological model for solutions of the bifurcation equations. At a simple singular solution, if the ABE's are of order less than five, and if fewer than five parameters are involved, the solutions of the bifurcation equations are topologically equivalent to the solutions of one of the following equations

Fold Catastrophe	$u^2 + \lambda_1 = 0$
Cusp Catastrophe	$u^3 + \lambda_1 u + \lambda_2 = 0$
Swallowtail Catastrophe	$u^4 + \lambda_1 u^2 + \lambda_2 u + \lambda_3 = 0$
Butterfly Catastrophe	$u^5 + \lambda_1 u^3 + \lambda_2 u^2 + \lambda_3 u + \lambda_4 = 0.$

The simple quadratic fold, for example, is equivalent to the Fold catastrophe. Recall that the Limit Point Algebraic Bifurcation Equations are $A\xi^2 + D\varsigma = 0$. If ς is scaled by D/A, this becomes the Fold Catastrophe.

A simple quadratic fold is also equivalent to the Fold Catastrophe. Recall that the Quadratic Algebraic Bifurcation Equations are $A\xi^2 + 2B\xi\varsigma + C\varsigma^2 = 0$, which are equivalent to

$$(A\xi + B\varsigma)^2 - D^2\varsigma^2 = 0,$$

where $D = B^2 - AC$. By letting $u = A\xi + B\varsigma$, and $\lambda_1 = -D^2\varsigma$, this becomes the Fold Catastrophe.

The simple cubic bifurcation is the section $\lambda_2 = 0$ of the Cusp Catastrophe. Recall that the Cubic Algebraic Bifurcation Equations are $b\xi^3 + c\xi\varsigma = 0$. A simple scaling transforms this to the Cusp Catastrophe.

Figures 14-17 show the real and complex surfaces of the four catastrophes mentioned above. For the Swallowtail and the Butterfly we have fixed one or more of the parameters in order to remain in three dimensions. The vertical axis in all of these figures is $\Re e(u) + \Im m(u)$.

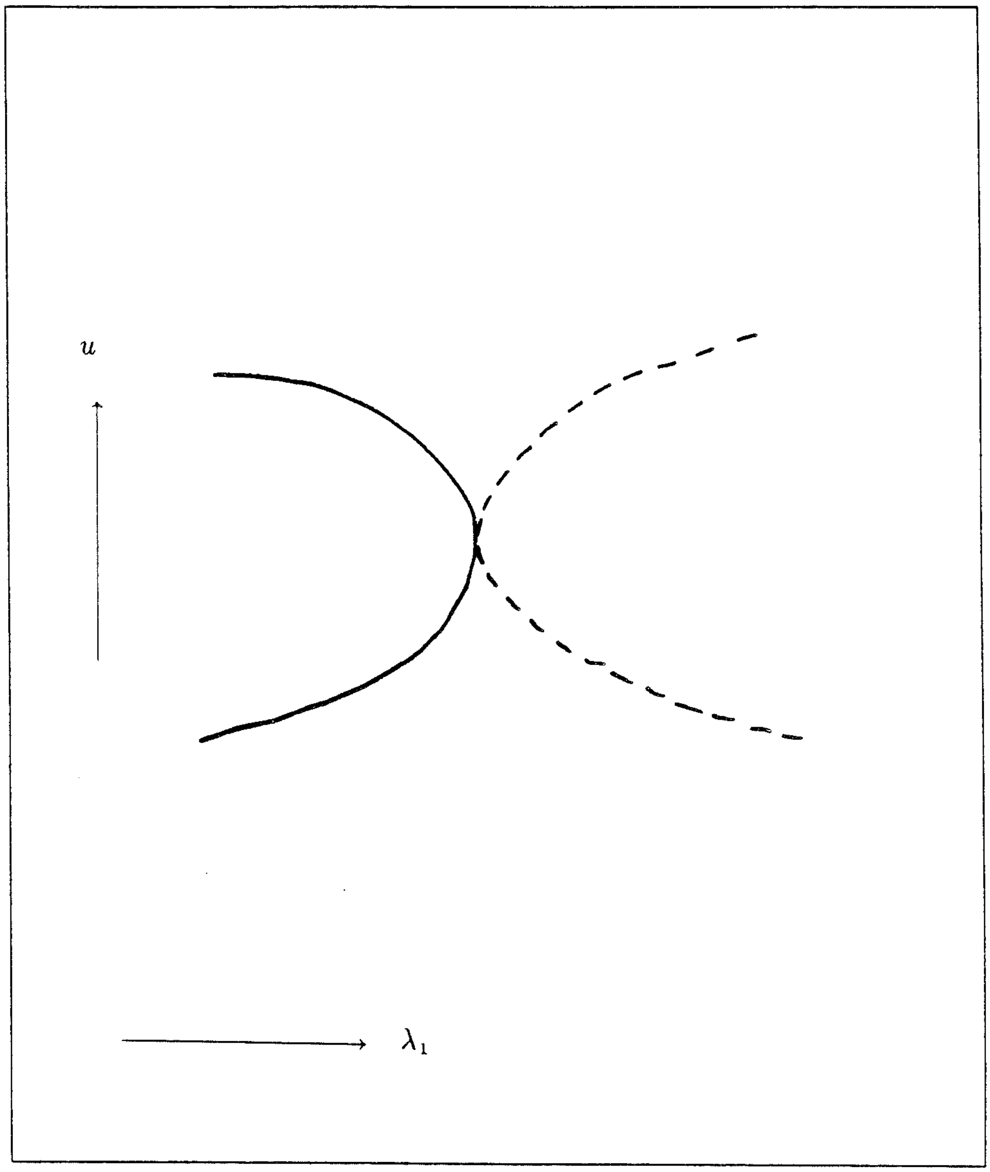

Figure 14. The Fold Catastrophe

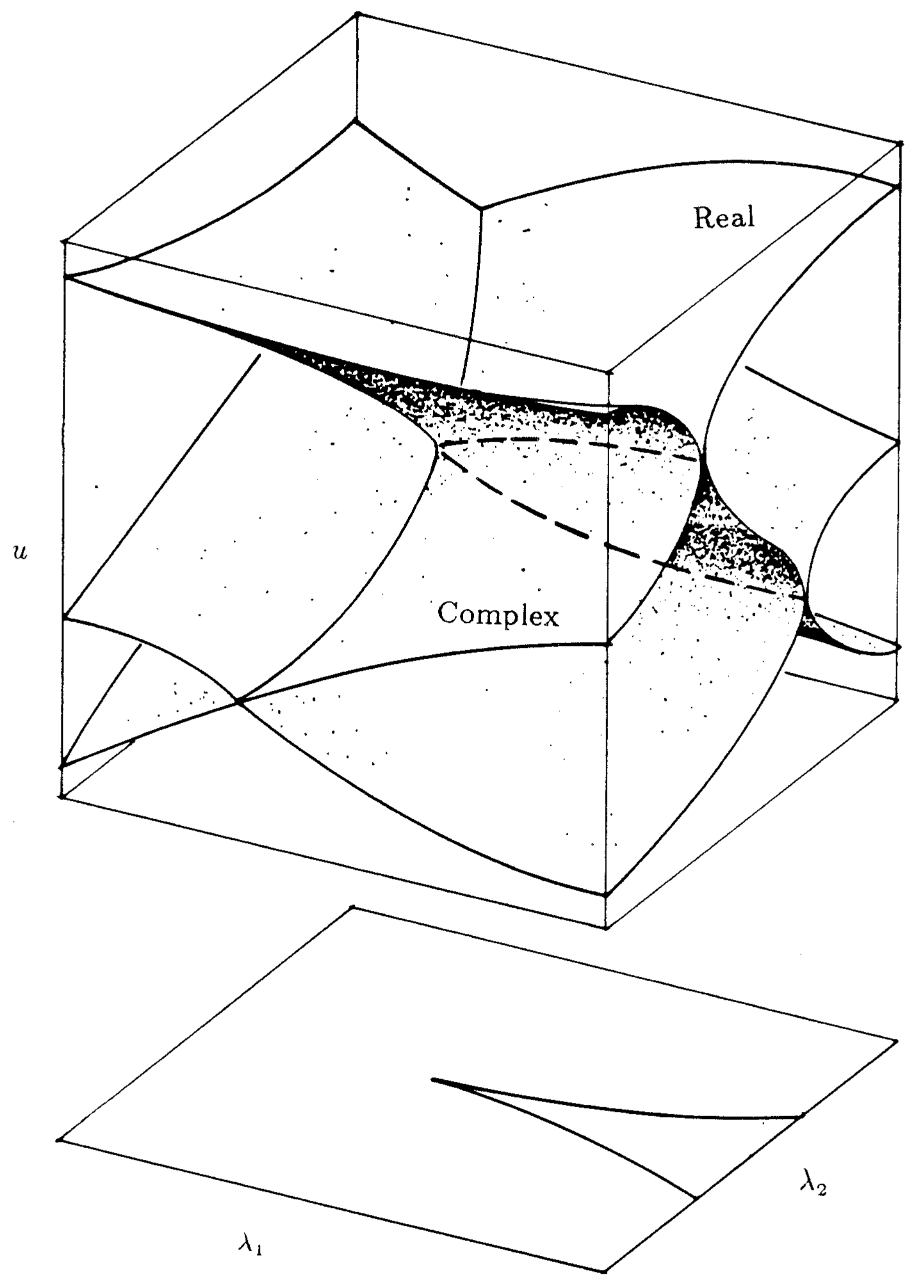

Figure 15. The Cusp Catastrophe

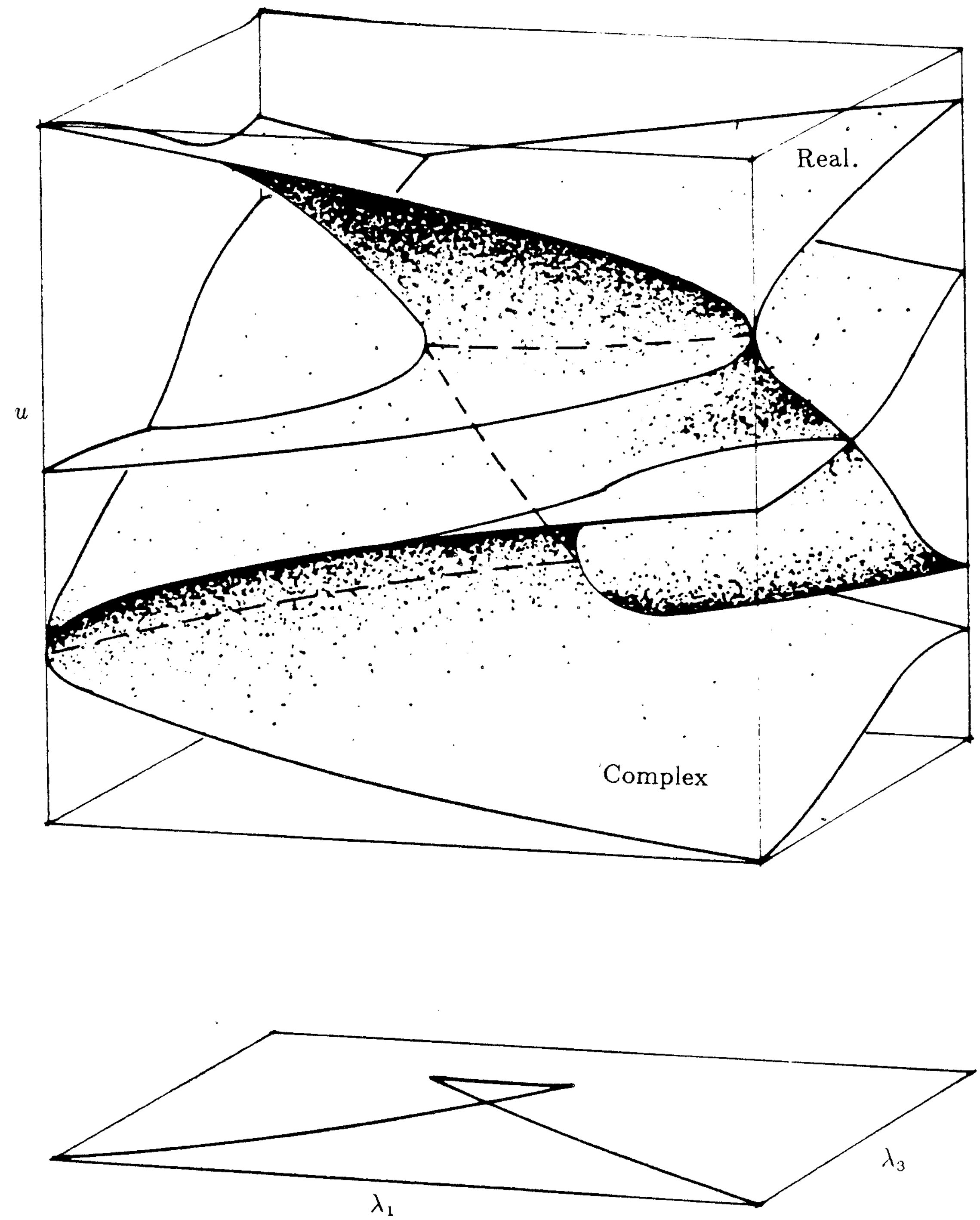

Figure 16. The Swallowtail Catastrophe, $\lambda_2 = 1$

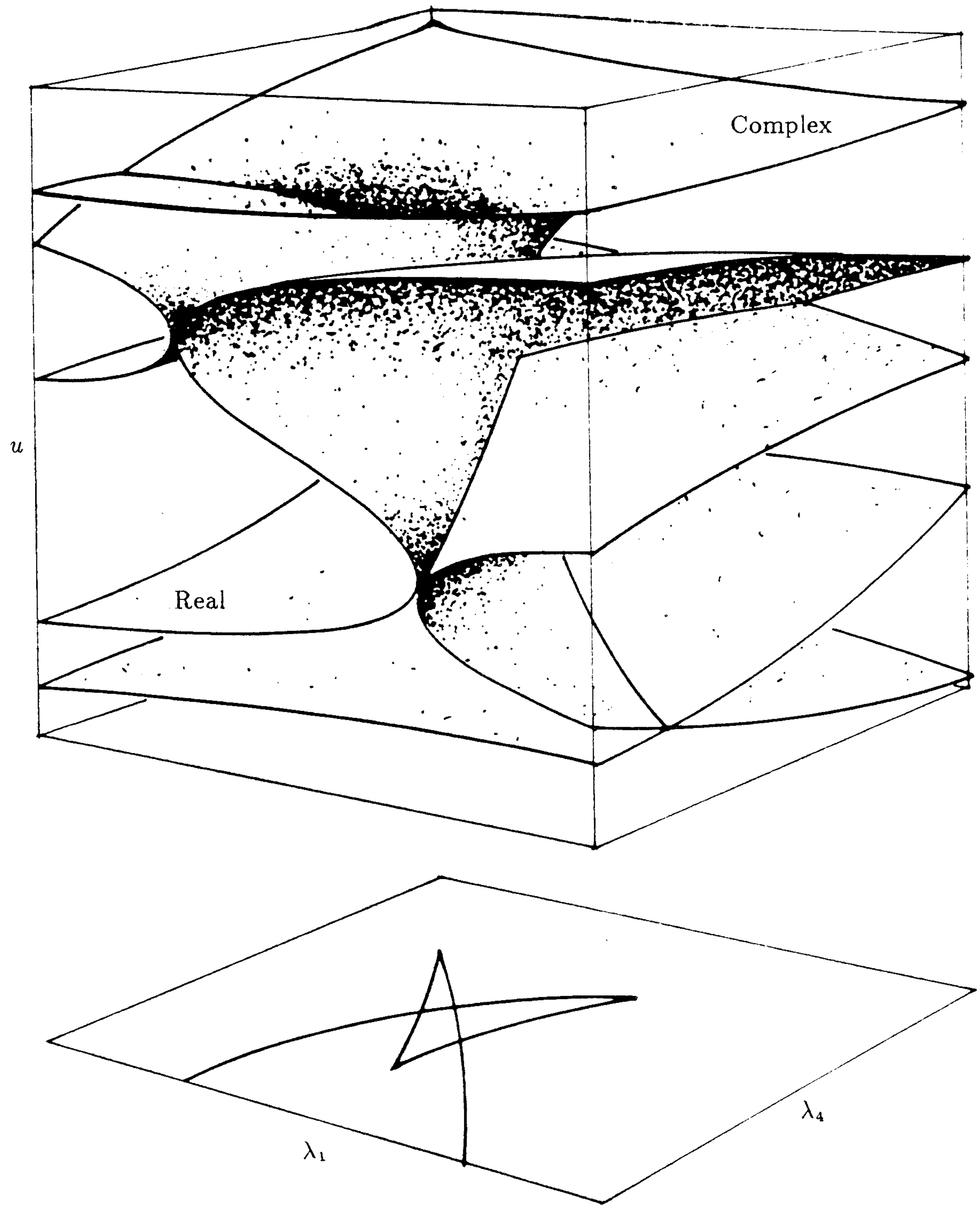

Figure 17. The Butterfly Catastrophe, $\lambda_2 = 1$, $\lambda_3 = 1$

Flow Between Rotating Coaxial Disks

Our final example is the axially symmetric flow of a viscous fluid between two infinite rotating disks. This flow has a similarity solution, which reduces the problem from a three dimensional flow to finding two scalar functions of the axial coordinate. There are two parameters involved, a Reynolds number $R \equiv \Omega_0 d^2/\nu$, and the ratio of the angular speeds of the disks $\gamma \equiv \Omega_1/\Omega_0$, (See Figure 18 for definitions of d, and Ω_0 and Ω_1). Keller and Szeto (12) and Fier (5) have used continuation methods to find paths of simple quadratic folds of solutions. Figure 19 shows the projection of these folds on the (R, γ) plane. Notice that there are two butterfly catastrophes. The similarity solution is

$$u_z = \frac{1}{\Omega_0 d} f(\frac{z}{d})$$

$$u_r = -\frac{1}{2\Omega_0 d} r f'(\frac{z}{d})$$

$$u_\theta = \frac{1}{\Omega_0 d} r g(\frac{z}{d}),$$

where

$$f'''' = R(ff''' + 4gg')$$

$$g'' = R(fg' - gf')$$

$$f(0) = f'(0) = 0 \qquad g(0) = 1$$

and

$$f(1) = f'(1) = 0 \qquad g(1) = \gamma.$$

For our example we have fixed the geometry ($R = 780$) and have investigated how the flow depends on the speed of the disks γ. The branch on which we had an initial guess (obtained from J. Fier) is detailed in Figure 20. There is some reason to suspect that this branch is spurious. It exists for 30 intervals in the axial direction, and the solutions seem smooth for most of the parameter range, but Fier has been unable to locate it for meshes of 200 intervals.

We chose this branch because of the pair of elliptic and hyperbolic folds shown in Figure 20. We showed in chapter IV that sections of a hyperbolic fold have complex isola that connect real branches. This is indeed the case, as shown in Figure 21. This shows how the quantity $\Re e(\kappa) + \Im m(\kappa)$ varies as a function of γ, where $\kappa \equiv \int_0^1 (f + f'' + g) dz$. Notice that the elliptic fold causes the real branches to close and form a real isola, and that another complex isola also exists. Figures 22-34 show solutions along these branches. We used just one starting solution, at $\gamma = 1$, and by following complex branches, found two real branches that cannot be found by ordinary continuation. There are other ways to find these branches, such as continuation in Reynolds number, and computing paths of folds. Complex bifurcation, however, has the advantage that the disjoint branch can be found without resorting to special techniques, and without prior knowledge of its existance.

We have shown that real equations can be extended to equations with complex solutions and real parameters. At bifurcation points, the complex equation has solutions branches corresponding to both the real and complex roots of the Algebraic Bifurcation Equations. These complex branches offer a practical means of locating disjoint real solutions, since they can connect otherwise disjoint branches of real solutions.

We have shown that existing algorithms can be modified very simply to calculate complex solutions, and have used such an algorithm to compute solutions of a non-trivial problem from Fluid Mechanics.

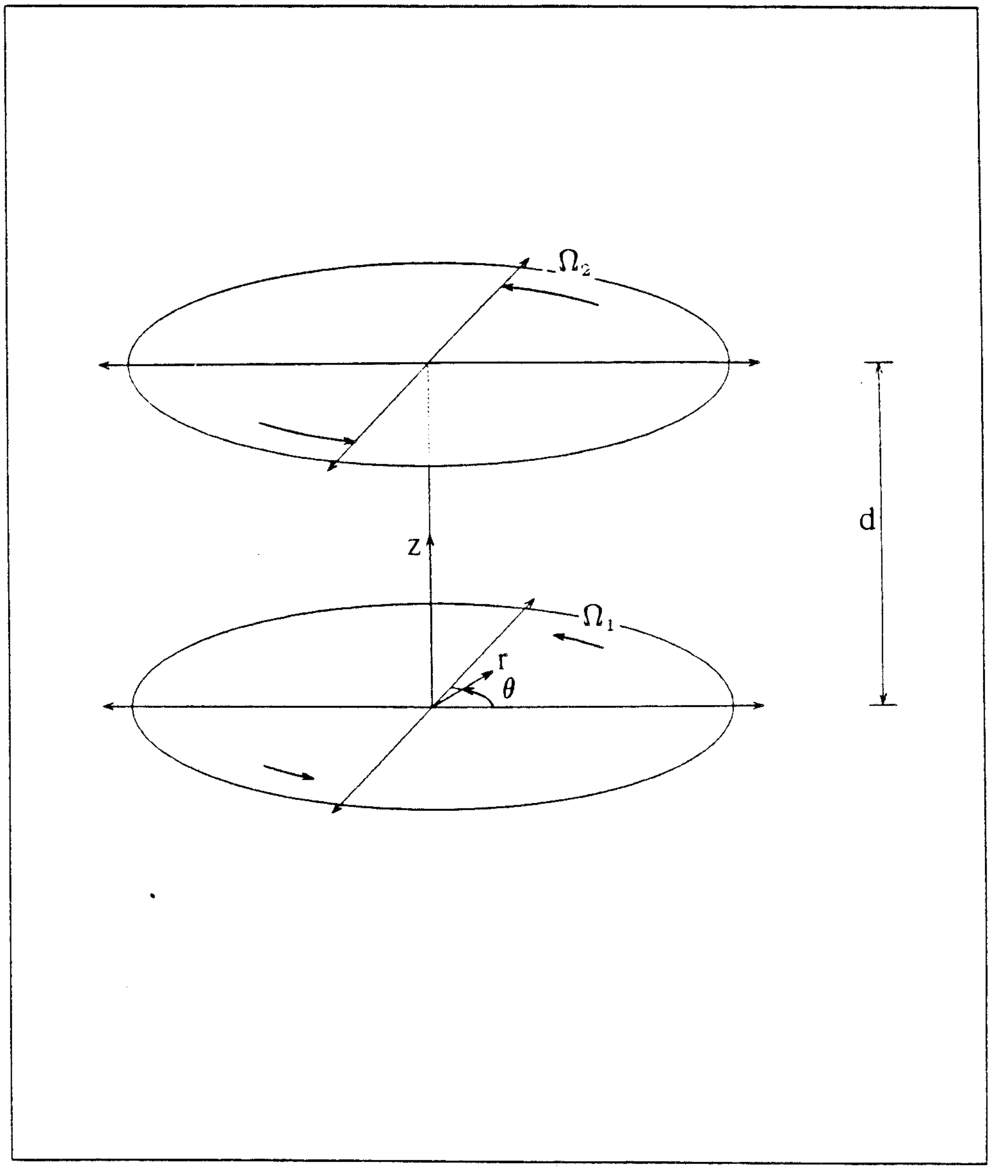

Figure 18. Flow between Rotating Disks

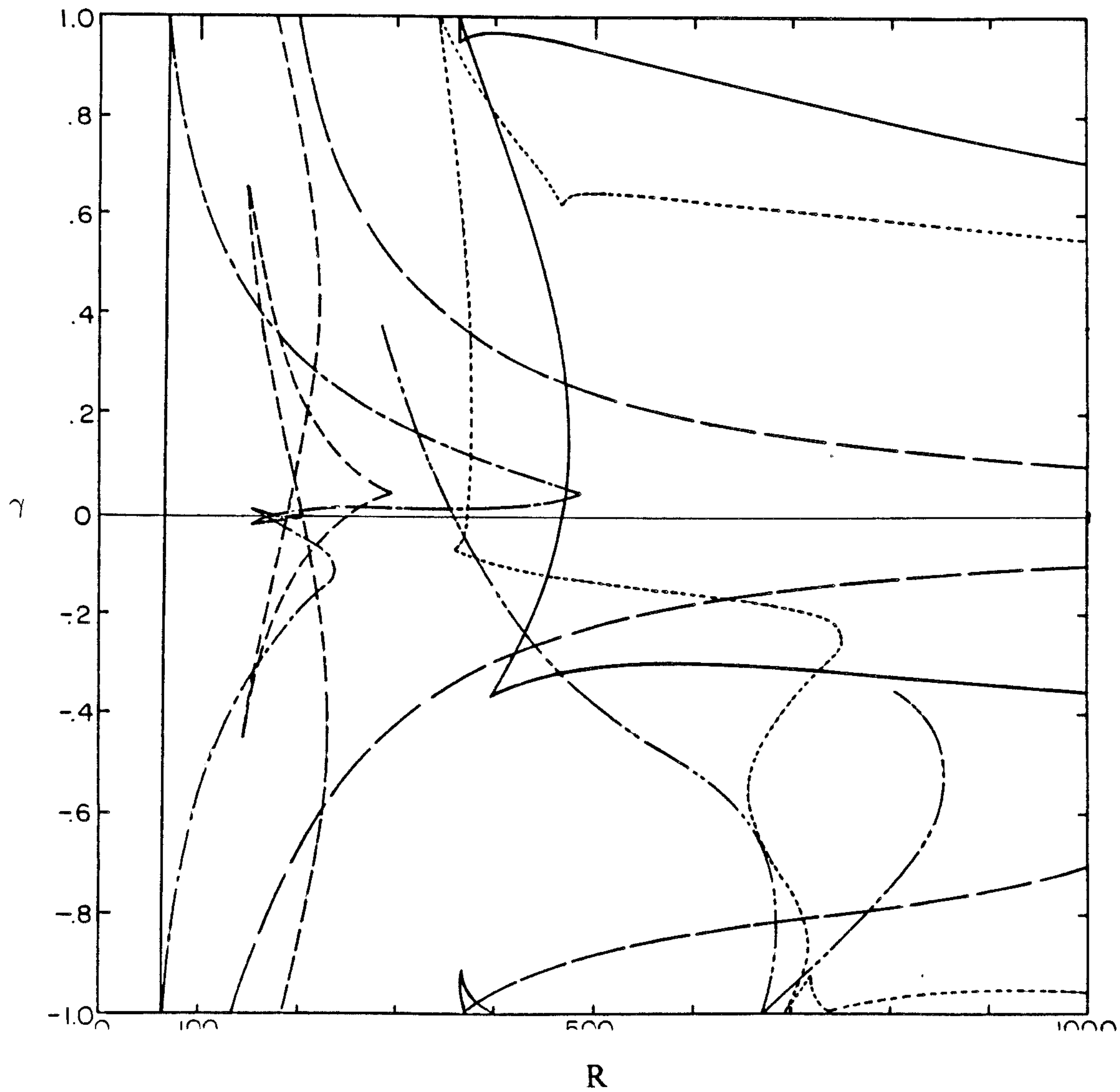

Figure 19. Paths of Folds in the Flow between Rotating Disks

(from Fier (5))

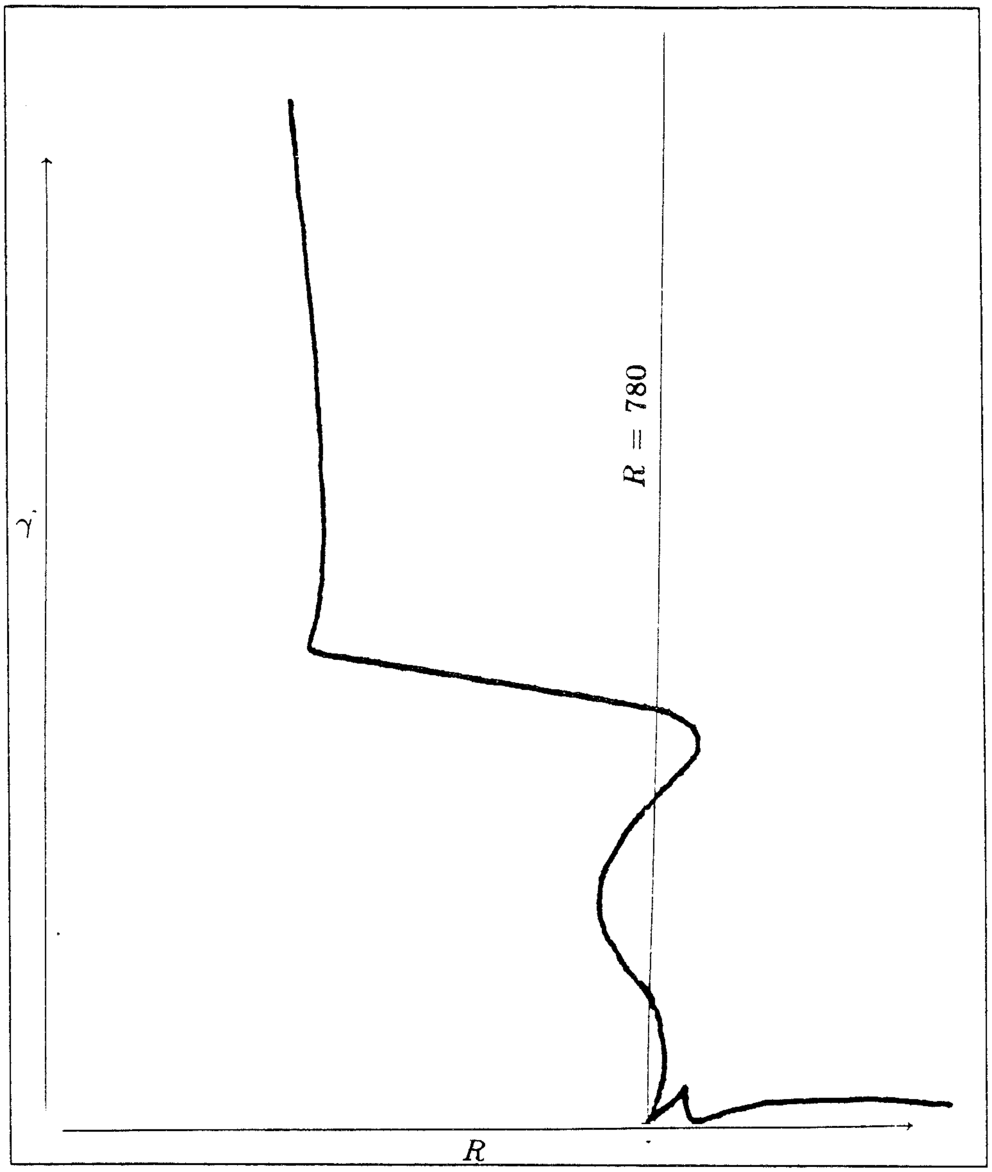

Figure 20. A Specific Path of Folds

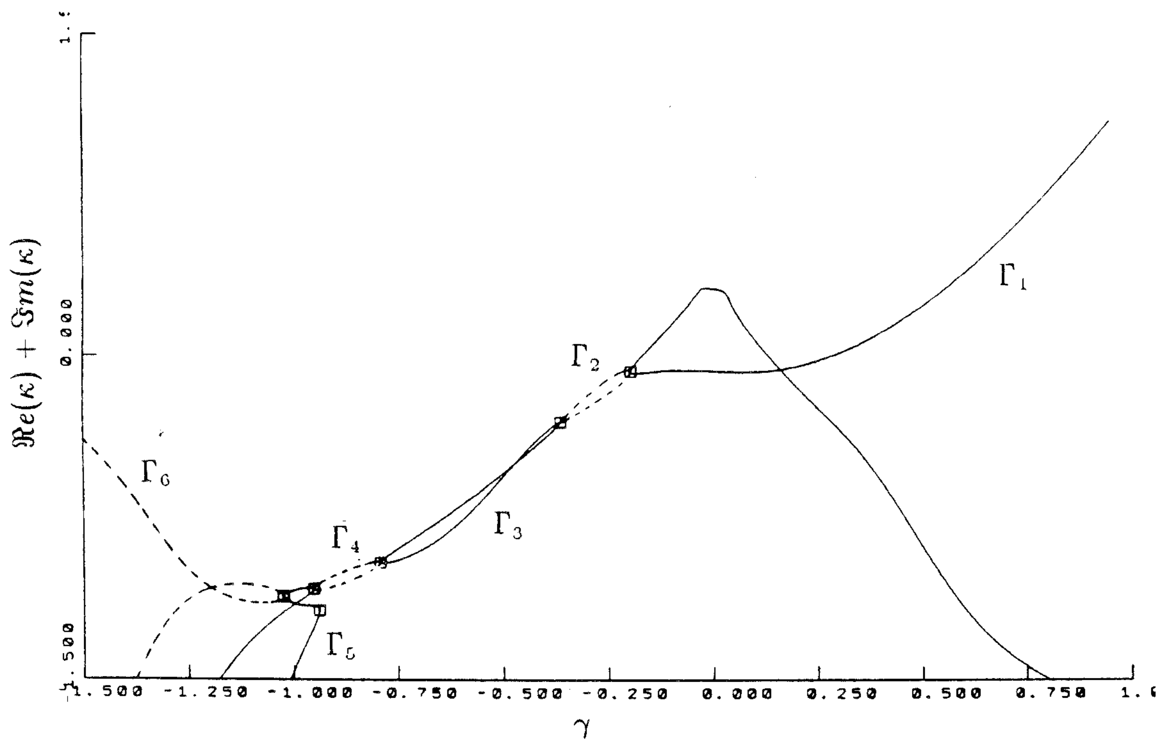

Figure 21. Solutions at $R = 780$

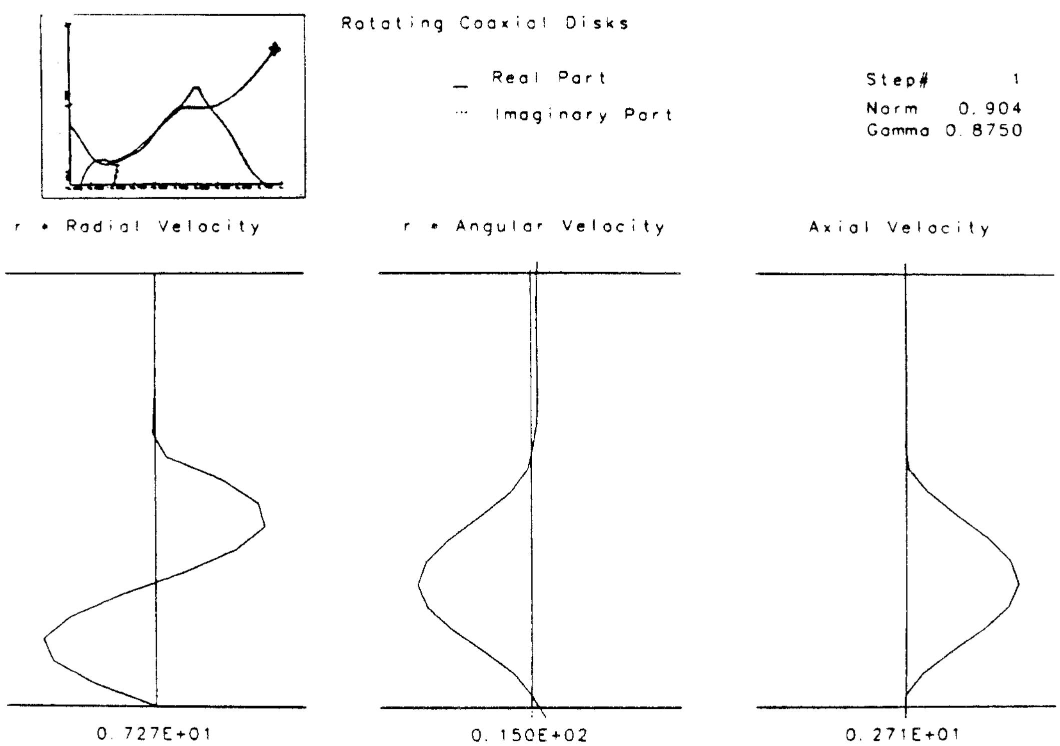

Figure 22. Solution on Γ_1

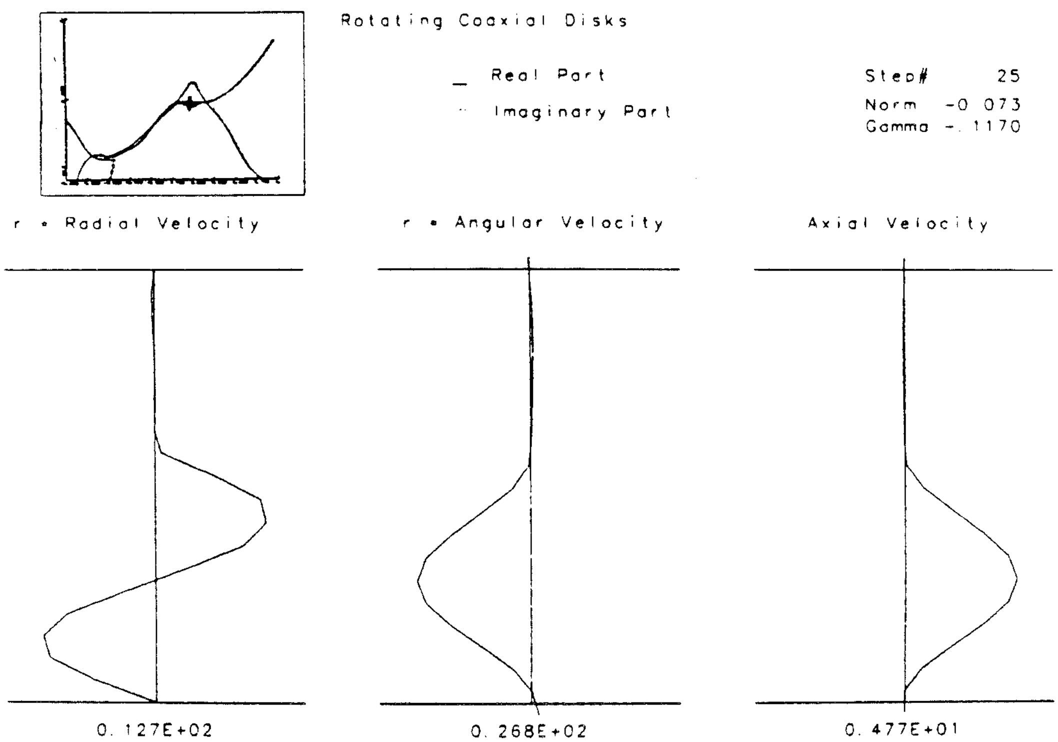

Figure 23. Solution on Γ_1

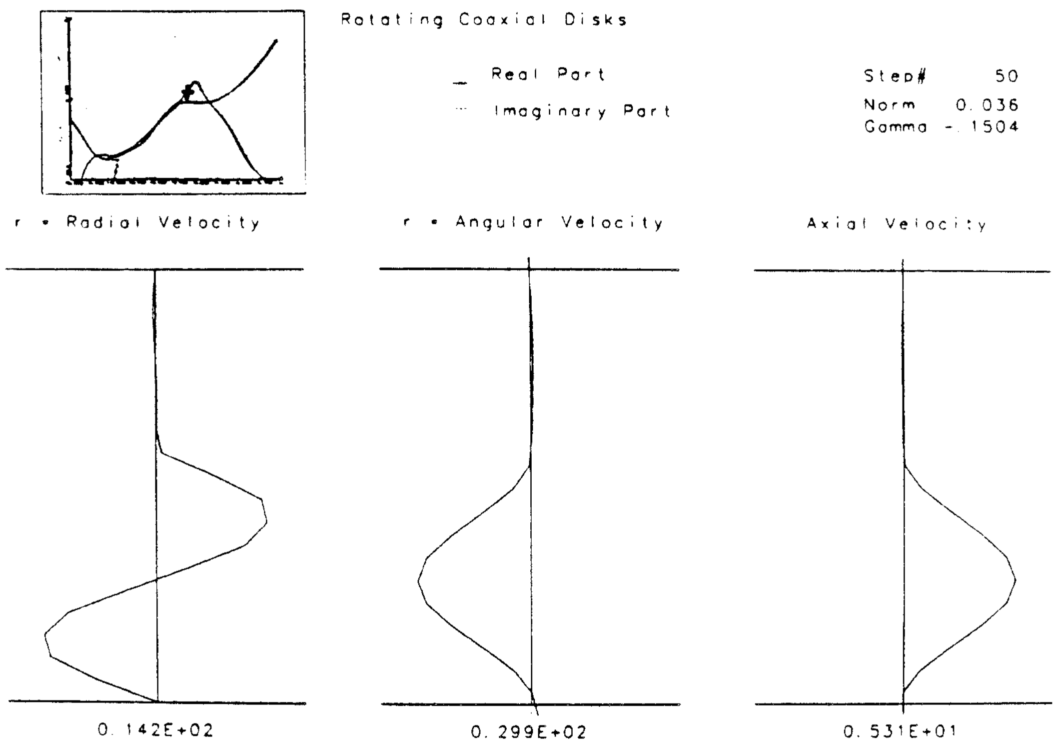

Figure 24. Solution on Γ_1

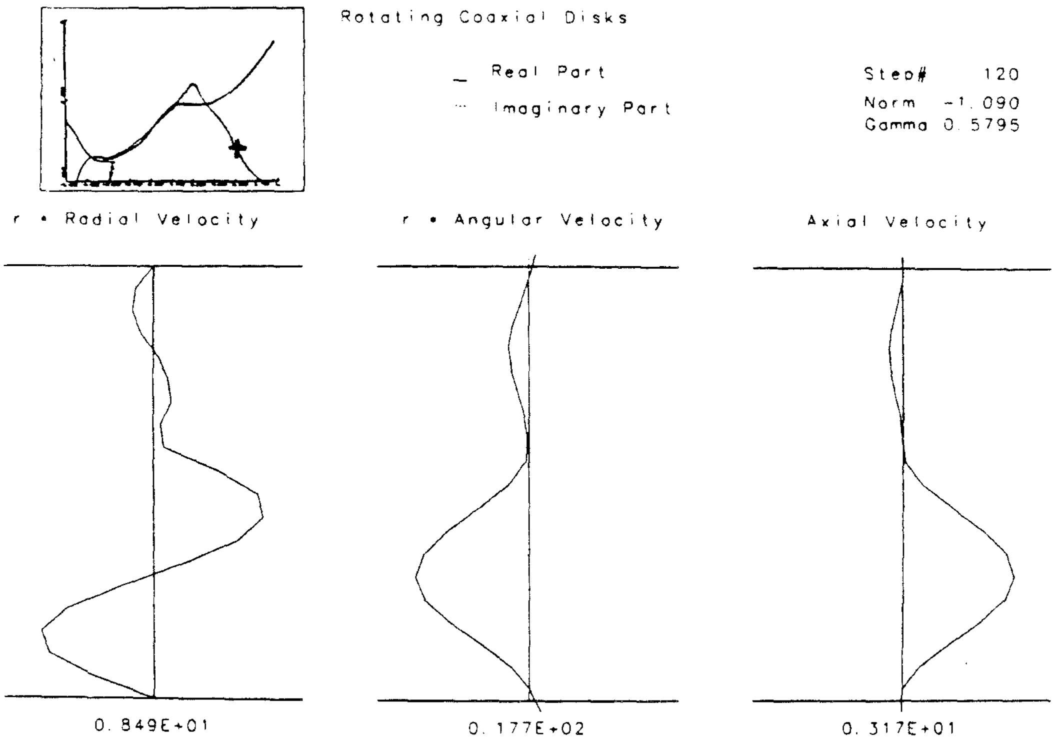

Figure 25. Solution on Γ_1

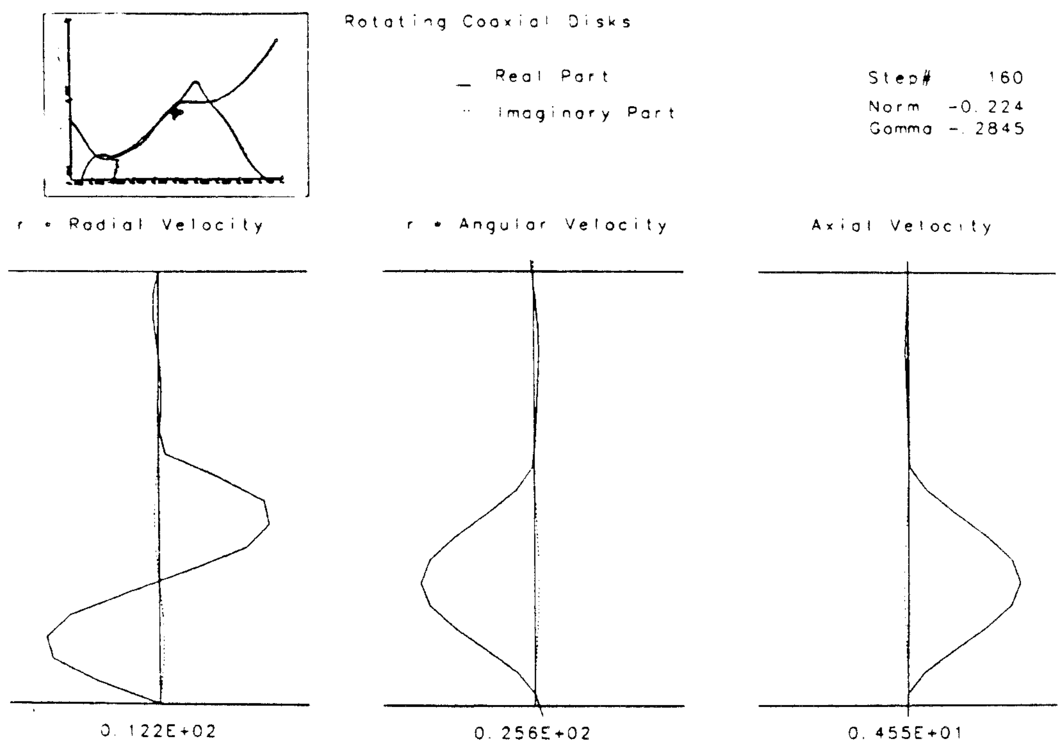

Figure 26. Solution on Γ_2

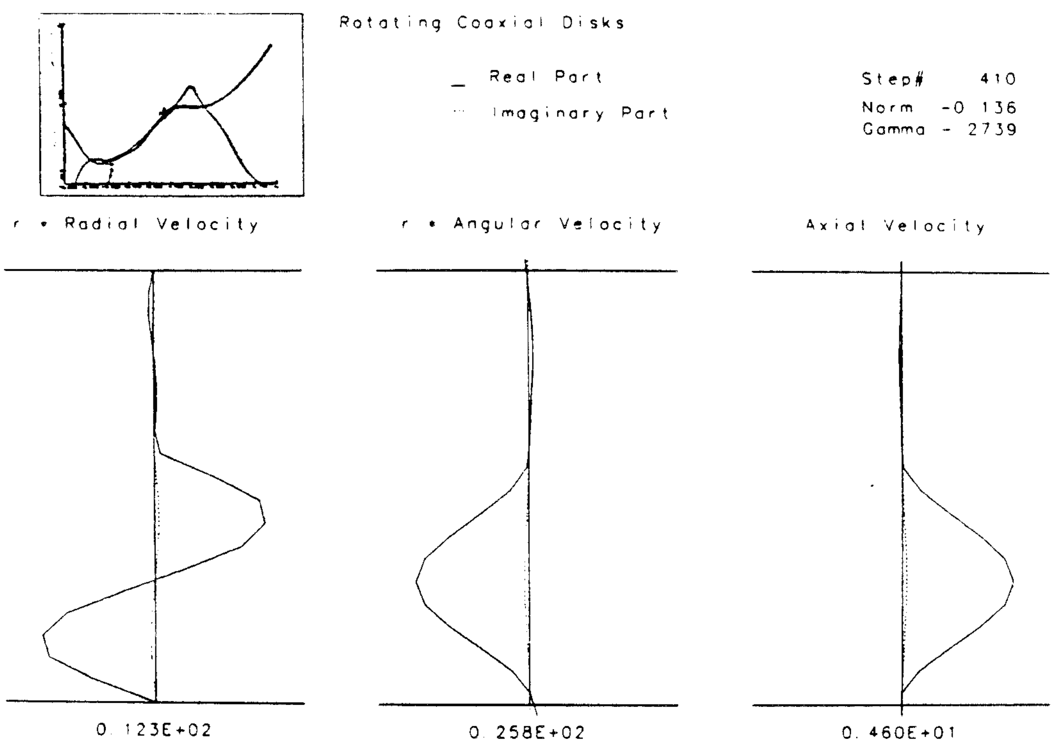

Figure 27. Solution on Γ_2

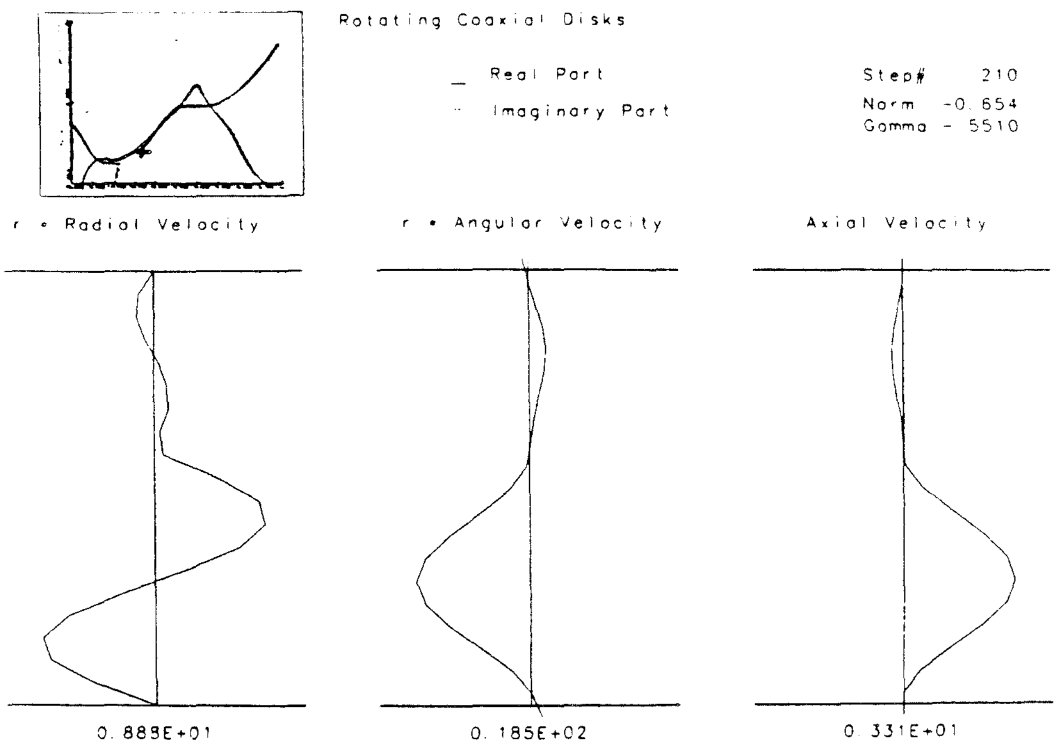

Figure 28. Solution on Γ_3

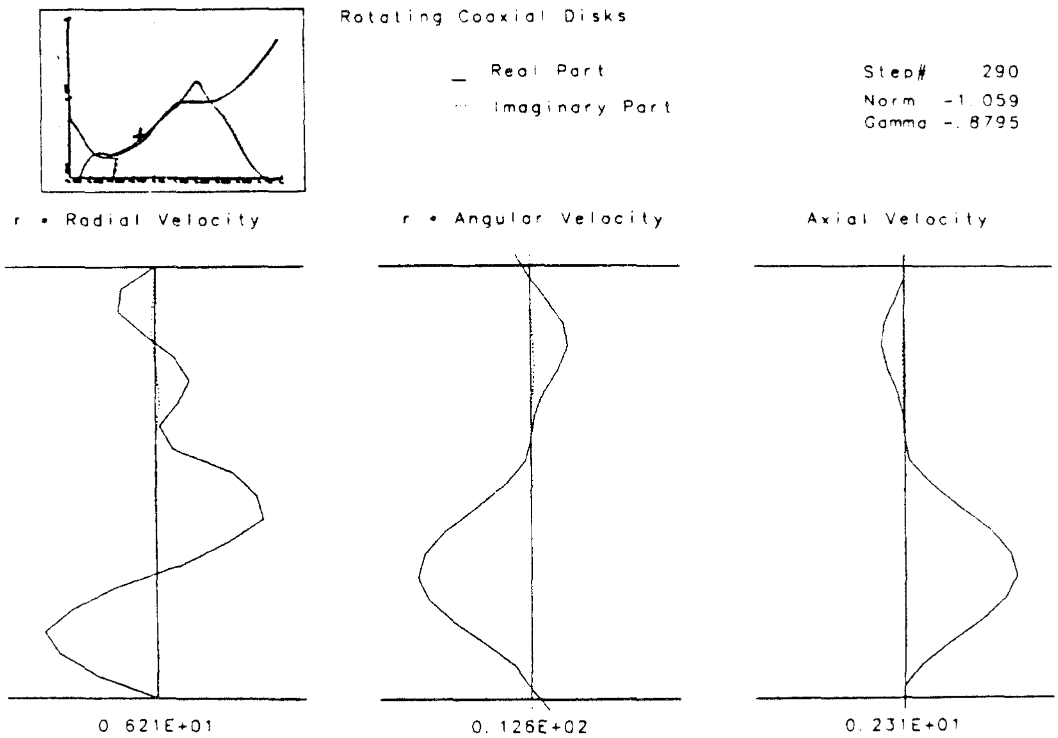

Figure 29. Solution on Γ_3

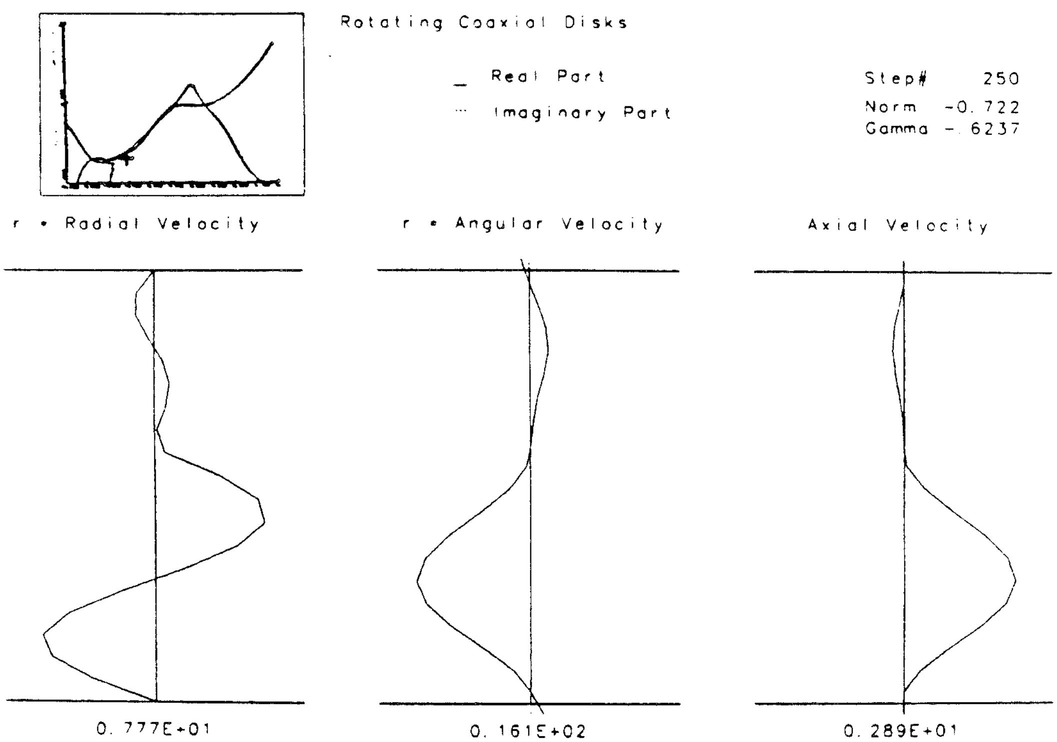

Figure 30. Solution on Γ_4

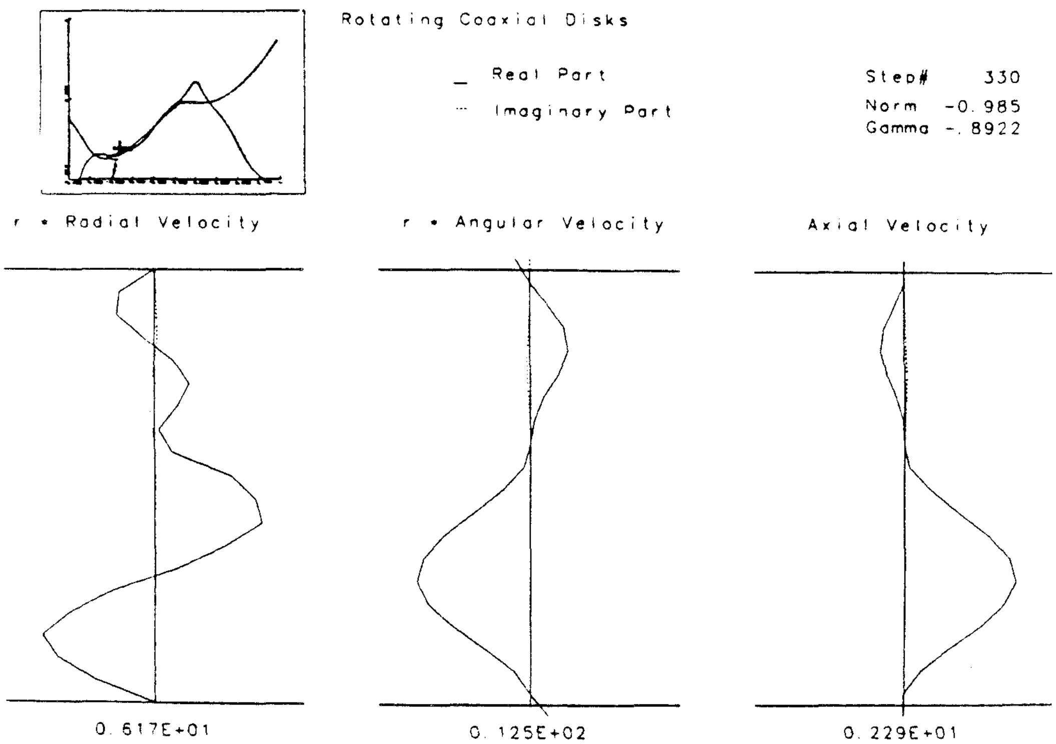

Figure 31. Solution on Γ_4

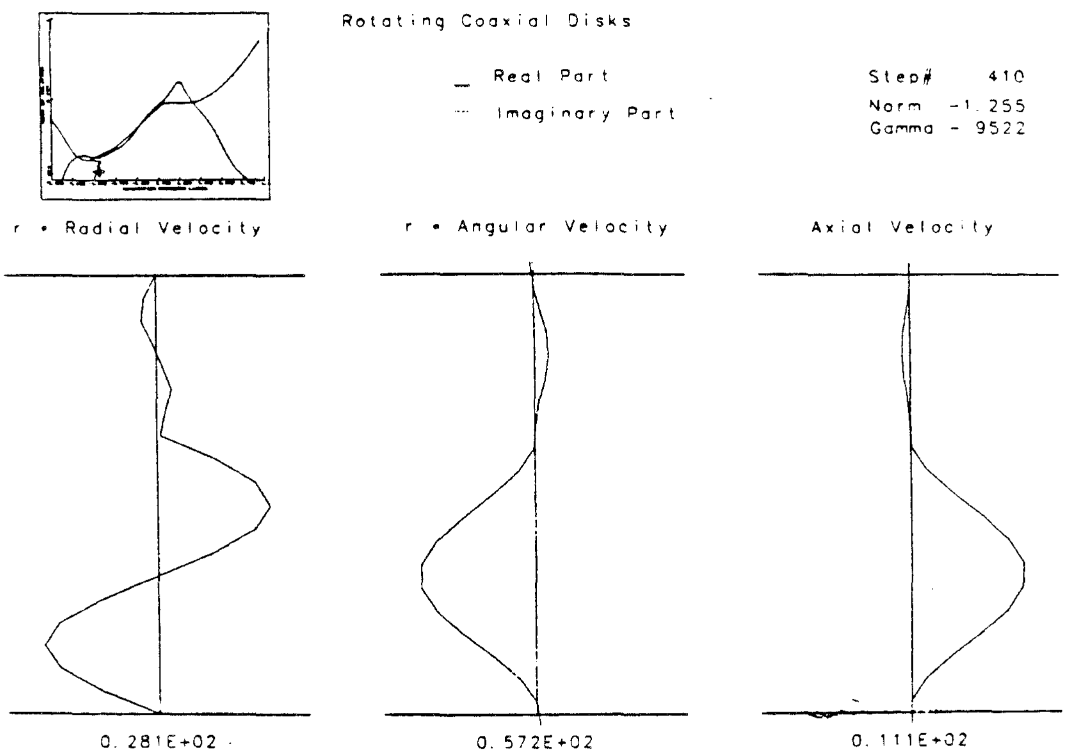

Figure 32. Solution on Γ_5

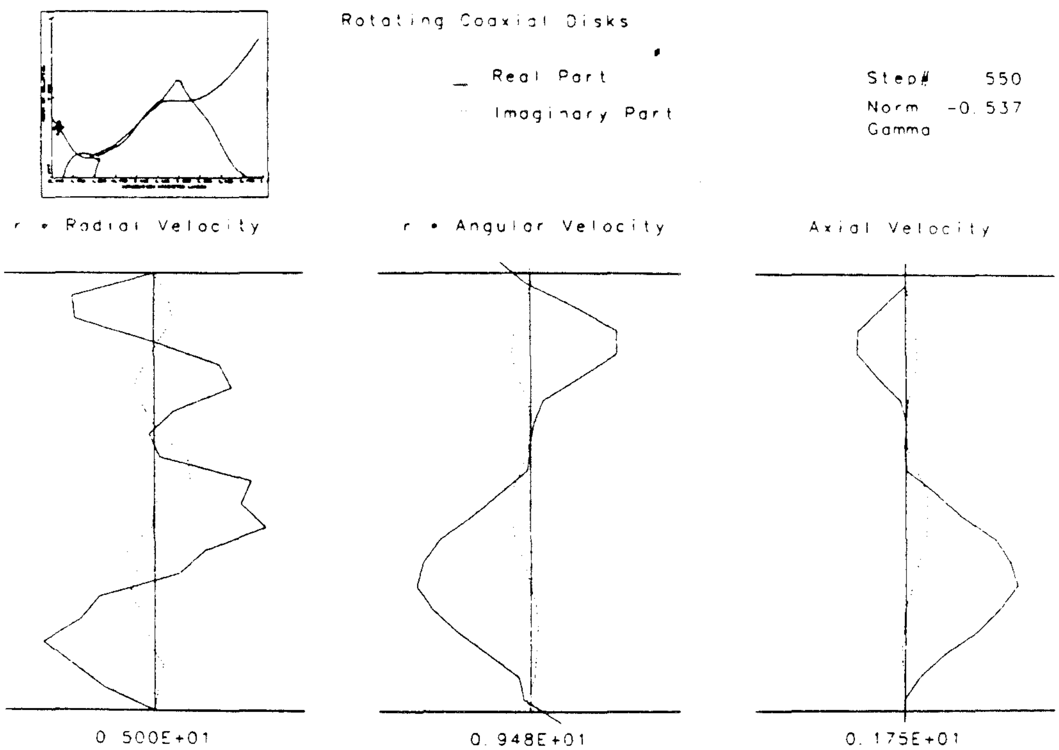

Figure 33. Solution on Γ_6

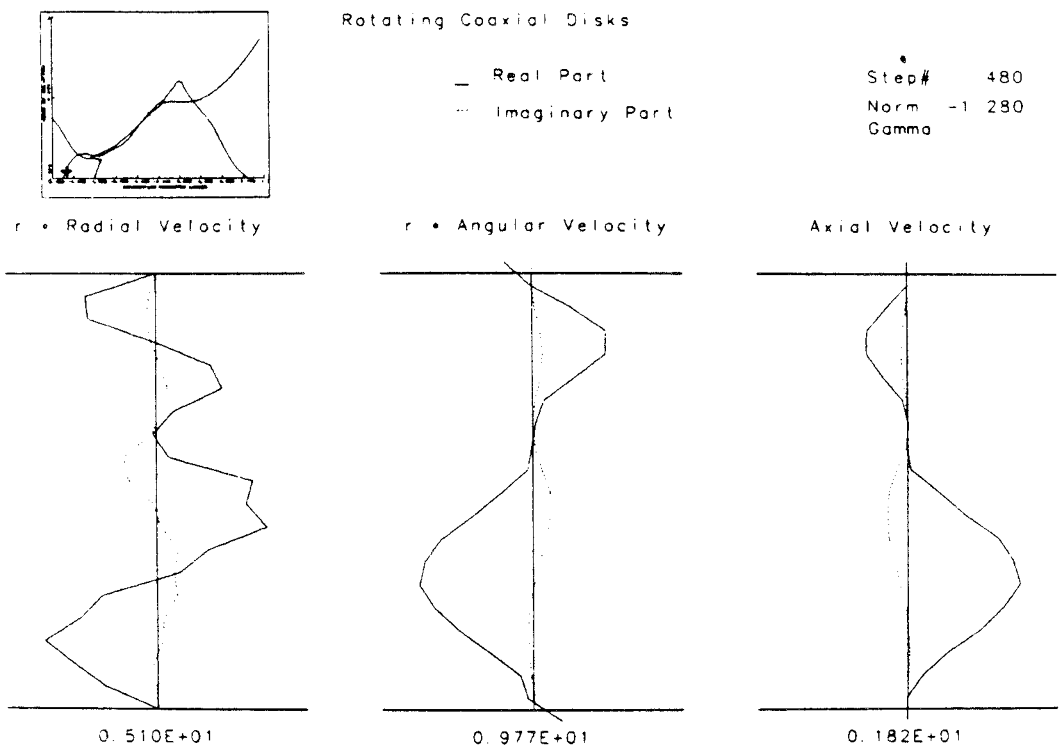

Figure 34. Solution on Γ_6

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