CONTRIBUTIONS TO THE KINETIC THEORY OF TRAFFIC FLOW WITH QUEUING

Thesis by

Charles T. Molloy

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

1985

(Submitted February 14, 1985)

ACKNOWLEDGEMENTS

I wish to thank my advisor, Professor G.B. Whitham, for introducing me to the topic of this thesis and for his many helpful discussions and suggestions. Working with him has been a pleasure and an enlightening experience. I am especially and deeply grateful to him for his efforts on my behalf beyond the purview of this thesis.

I would also like to express my appreciation to the faculty and fellow students of the Applied Mathematics Department for their friendship and their help.

I thank Caltech and the Office of Naval Research, U.S. Navy, for the financial support they have given me.

I am very grateful: to Caroline Sweezey for her inspiration, to Susan Oldham for her encouragement, and to Eileen Williams for her patience.

Lastly, I wish to thank my parents, my ex-wife, and my children for all they have done to help.

ABSTRACT

This thesis contains investigations of the effects of a probability distribution of the desired speeds of the drivers and of the effects of overtaking waiting time. It deals only with traffic for which the density is less than the critical density. Part I concerns simple approaches for assessing the effects for steady-state flow. Part II is a detailed formulation of integro-differential equations for the velocity distribution functions. We prove that solutions to these equations exist, are unique, are nonnegative, and are continuous along characteristics. We make use of the simplifying assumption that, in lighter traffic, a car that has been slowed by one car is unlikely to be slowed still further before passing. We examine the possibility of constant speed, constant shape solutions, and we investigate some special solutions as time approaches infinity. Delta function solutions are found. For one case, we look at the difference between the velocity distribution functions for models with continuous vs. discrete spatial distributions. We compare the steady-state case for the model of Part II with that of Part I. Preliminary comparison with observations is good.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

1. INTRODUCTION

In this thesis we consider several models of traffic flow. We consider the effects of a probability distribution for the desired speeds of the drivers and of the interference effects in overtaking. Each driver travels at his desired speed unless delayed by other drivers; we specify a delay time during which the driver travels at the reduced speed of the car ahead, after which he is free to pass and resume his desired speed. Our models deal with only one lane of traffic at a time, with the presence or absence of other lanes being accounted for in the waiting time.

In a rough way the formulation is analogous to the kinetic theory of gases for fluid flow. We are interested in the velocity distribution of the individual cars within a stream and its effect on the overall flow characteristics. The "collision process" is the delay experienced in overtaking when a faster car arrives behind a slower one. An important difference from kinetic theory is that a probability distribution of the desired speeds of the individual drivers is posed as a given part of the formulation of the problem and this is basic to the whole investigation. On the other hand, in kinetic theory the molecules remain at their new velocities between collisions with no tendency to return to some previous or desired speed.

The thesis is in two main parts. The first part is limited to the steady-state case in which the overall quantities, such as total density and total flow rate, are uniform along the road and independent of time. Individual drivers have a distribution of speeds and experience delays, but the integrated quantities are constant. In this first part, a relatively simple approach is developed by considering the experience of a single driver moving relative to an average background provided by the rest. Then, taking his experience as typical, we derive equations for the average behavior of the flow.

We compare the results of this investigation with actual measurements of traffic flow. For one particular expressway, we find excellent agreement for the left lane of a road with three lanes in one direction. The agreement is poor for the right lane of this road, but this might be expected because of the influence of drivers entering or leaving the road.

The second part is a more detailed study of the mutual interaction between the cars in the stream, and integro-differential equations are obtained for the velocity distribution functions. (This is analogous to the Boltzman equation in kinetic theory, but the form is considerably different because of the different interaction process and the underlying probability distribution of desired speeds.) In this part, the basic formulation is for general time and space dependence of the overall density, flow rate, etc. Various general properties of the integro-differential equations, such as the existence of solutions and the tendency towards equilibrium over large times, are established. This more detailed theory is also applied to the case of uniform flow conditions and compared with the approach in Part I. For an appropriate range of variables, we find close agreement.

2. SOME CHARACTERISTICS OF FREEWAY TRAFFIC FLOW

2.1 Introduction

This section is included with the idea that in order to have an accurate sophisticated mathematical model of traffic flow, it is wise first to have a correct qualitative understanding of the mechanisms involved.

Many researchers have studied traffic flow, continually bringing about a better understanding of the subject. Unfortunately, some of the ideas of some of the researchers neglect some important points. In order to clarify these points for myself, I made a variety of measurements of traffic flow, along with observing traffic without measurements and using the data of other researchers.

This section describes, sometimes quantitatively and sometimes qualitatively, my conclusions and the reasons for them. Some of the measurements permitted solid conclusions, while others pointed in a direction but would need verification at other locations and under varied conditions. The samples frequently consisted of several hundred vehicles so as to give a good measurement at that particular location and amount of traffic.

2.2 Overview

There are two different types of traffic flow, namely, forced flow and subcritical flow.

Forced flow occurs where the capacity at some point in the road is exceeded so that traffic backs up. The flow throughout the section is simply the capacity of the bottleneck. Examples are: a point of reduction in the number of lanes, a point at which an accident or breakdown has occurred (on or off the travelled lanes), an on-ramp adding to the demand so that the capacity is exceeded, and an off-ramp for which the demand exceeds the capacity so that traffic backs up onto the freeway. For forced flow, speeds are usually less than 35 mph.

Subcritical flow occurs on a section of road where the demand does not exceed the capacity anywhere on the section and where no backing up into the section has occurred from a downstream bottleneck. For subcritical flow, speeds are usually greater than 35 mph.

A major point is that some phenomena are important in only one of the two types of flow. Some authors try to use a phenomenon characteristic of one type of flow to explain the other type of flow.

An example is the use of the mechanism of passing in subcritical flow to explain forced flow. Of course, a phenomenon such as the flow-density curve is valid for both types of flow.

2.3 Details of Forced Flow

2.3.1 Passing and Lane Changing: In forced flow, in the densest traffic, I have found that I could change lanes frequently in less than 10 sec and usually in less than 20 sec from the time I formed the desire to change lanes. I was highly motivated to change lanes since my exit was coming up shortly. The only time I could not change lanes quickly was when vehicles in the adjacent lane were at a complete standstill. In measurements I made I found that, on the average, lane changing from a forced flow lane occurred at rates varying from less than .00055 changes/veh-sec to .0043 changes/veh-sec. For a single vehicle this would be from more than 30 min between changes to about 4 min between changes.

Thus, the time necessary to change lanes is much smaller than the average time between lane changes.

Lane changing is not the same as passing. In fact, in the measurements I made, lane 1, the median lane, had a slightly lower average speed than lane 2, yet far more vehicles changed from lane 2 to lane 1 than vice versa. These measurements were taken on the eastbound 134 Freeway slightly before lanes 3 and 4 join the 5 Freeway. Lanes 1 and 2 were in forced flow, while lanes 3 and 4 were in subcritical flow. In 24 min of my observing forced flow, 32 vehicles changed from lane 3 to lane 2, clearly in order to stay on the 134 Freeway. In the same time 24 vehicles changed from lane 2 to lane 1. This was not for the purpose of passing since lane 1 had a lower average speed. Shortly beyond this location the 134 Freeway widens again, adding lanes on the left, so this lane changing may have been to avoid being in the right-hand lane.

In the same 24 min only 3 vehicles changed from lane 1 to lane 2. These could be for the purpose of passing, but the average rate is equivalent to more than 30 min between changes for a single vehicle and so is only a minor phenomenon. For subcritical flow an average single vehicle would change lanes every 1 1/2 to 2 min.

Finally, in the same 24 min only 5 vehicles changed from lane 2 to lane 3. Once again, this is equivalent to a very long time between lane changes for a single vehicle and so is of minor importance whatever the reason for the lane changes.

I believe that, in forced flow, lane changing for the purpose of passing is only a minor phenomenon. If one lane has a higher average speed than the others, a driver can move to that lane, but then he is stuck. Other reasons for lane changing may be to pick a lane the driver is comfortable with when entering the freeway or to get to his exit when leaving the freeway.

2.3.2 Continuity of Speed: Nonmeasured observations indicate that, at any one instant, speed is a smooth function of distance for forced flow as opposed to there being a probability distribution of speeds for subcritical flow. An analogue to forced flow that comes to mind is that for each lane the traffic is like a giant 'Slinky'' spring moving along the road with an occasional wave propagating along it. This indicates that the probability distribution formulation is not appropriate for forced flow in one lane.

Several models which contain a probability distribution of speeds have faster vehicles catching up with slower vehicles. This does not apply in forced flow since, essentially, each vehicle is already caught up with the vehicle ahead. The *Highway capacity manual* (Fig. 9.1, p. 264) gives a flow-velocity curve which is extremely well fitted by assuming an effective car length of 36 ft and 0.92 sec gap from the rear of the lead vehicle to the front of the following vehicle. A vehicle that is 0.92 sec behind another vehicle does not have any catching up to do.

2.3.3 Causes of Speed Variation:

2.3.3.1 Stop and go traffic Stop and go movement in traffic is caused by a lead vehicle slowing slightly (from random acceleration or a car pulling in front of it) and the following vehicle first getting too close then reducing speed below that of the lead vehicle so as to achieve a more satisfactory headway. If enough vehicles do this, the traffic will reach a point where some vehicles are completely stopped. On the other hand, some drivers leave more headway which they use to delay the time at which they must slow down and they do not reduce their speed below that of the lead vehicle. This kind of driver tends to dissipate propagating disturbances. A disturbance will dissipate, persist, or grow to a complete stop, depending on how many of each type of driver there are. A driver can be of one type when the difference between his speed and that of the lead car is small and of another type when the speed difference is large. One simple model for these drivers indicates that a disturbance will dissipate in the presence of drivers with a reaction time that is less than the time gap to the vehicle ahead. If the reaction time is greater than the time gap, the disturbance will grow.

I have seen speed variations of from 0 to 44 mph. Also, I have seen speeds double (or halve) in about one minute, with essentially all vehicles staying in the same lane.

2.3.3.2 Random fluctuations Drivers do not maintain exactly constant speeds. There is a random fluctuation of approximately 2 to 5 mph.

2.3.3.3 Lane changing Cars entering or leaving a lane cause the following drivers to open or close the resulting changed gap. Sometimes they do this immediately and sometimes they do not. A simple model indicates the speed change could be 5 to 10 mph. My own driving behavior is in this range.

In a special case, I have seen a delayed reaction to vehicles entering a lane result in very high flow rates, on a short section of the lane immediately following an on-ramp, for short periods of time. Typical observed flows have been 1.00 veh/sec for 22 sec and .81 veh/sec for 90 sec. The *Highway capacity manual* gives .55 veh/sec as the maximum long-term flow averaged over all lanes and .67 veh/sec as the maximum 5 minute flow for one lane. The phenomenon that occurred was that the on-ramp traffic merged with the shoulder lane traffic and the drivers accepted excessively close spacing for a short time, then changed lanes to allow normal headways.

2.3.3.4 Different lane conditions A condition affecting one lane but not others can cause very large differences in the speeds of the different lanes. One example is the lane going to the Hill Street off-ramp of the southbound Pasadena Freeway moving at 45 mph while the other lanes are stopped during the morning rush hours.

2.3.4 A Simple Model of Forced Flow: One possible model for forced flow is given by:

$x_{n-1}(t)$ = position of a car at time t,

$x_n(t)$ = position of the following car,

T_{Rn} = response time of driver n,

T_{hn} = desired time gap from the front of car n to the rear of car n-1,

L = effective length of a car,

a_n, b_n = magnitude of the acceleration response to velocity differences and space headways, respectively.

T_{Rn}, T_{hn}, a_n, b_n are random functions of n.

$$x_n''(t + T_{Rn}) = a_n(x_{n-1}'(t) - x_n'(t))$$
$$+ b_n(x_{n-1}(t) - x_n(t) - L - T_{hn}x_n'(t)) .$$

Note that for equilibrium $x_n'' = x_{n-1}' = x_n' = 0$, thus $x_{n-1} - x_n - L - T_{hn}x_n' = 0$. (This equation very closely fits the curve in the *Highway capacity manual* (Fig. 9.1, p. 264) for forced flow from jammed conditions almost to the critical density (density at maximum flow) ($L = 36$ ft, $T_{hn} = 0.92$sec)). Several researchers have equations similar to this but that do not have anything to force a reasonable equilibrium solution.

A model for forced flow which is derived from detailed mechanisms ought to include parameters and concepts associated with the above.

A continuous version of the above is given by

$$x_{tt}(y,t + T_R) = a\,h(y,0)\,x_{yt}(y,t)$$
$$+\ b(h(y,0)\,x_y(y,t) - L - T_h\,x_t(y,t))$$

where

$x(y,t)$ = the position of car y at time t,

$x(y,0)$ = 0,

$h(y,t)$ = the space headway of car y at time t.

This is one possible model that could be used in place of a model with a probability distribution of speeds. This continuous model does not have different characteristics for the different drivers.

2.3.5 Desired Speed Desired open road speeds play no part in forced flow since no one ever attains his desired open-road speed. To a first approximation, a driver's speed is the same as that of the driver ahead of him.

As traffic goes from light to moderate to capacity, the desired speeds of the drivers may drop on the order of 10 mph. In particular the slowest drivers' desired speeds go from about 45 mph in light traffic to about 35 mph at capacity. My casual experience has been that I have seen drivers at about 45 mph in light traffic but not at lower speeds. One way to envision the effect of slow drivers is to note that at lighter than capacity flow the slowest drivers can always go at their desired speeds. As traffic goes from light to capacity, the ability of faster drivers to pass and to go at their desired speeds diminishes, until at capacity there is no more room left for increased speeds, so everyone goes at the speed of the slowest drivers.

The desired speeds of drivers do not drop below this. As mentioned before, one lane of the Pasadena Freeway was going 45 mph while the other lanes were stopped. This indicates that even in the heaviest congestion a driver's desired speed (the speed he will go if not blocked) will drop to maybe 45 mph for an average driver and 35 mph for the slowest drivers.

Two sources *(Highway capacity manual*, Figs. 3.26-3.28, and Edie et al. 1980, Fig. 2) indicate that there is a very large decrease in the spread of speeds as traffic goes from light to capacity. This tends to confirm the above.

2.4 Details of Subcritical Flow

2.4.1 Distribution of Speeds: One obvious characterstic of traffic is that vehicles move at different speeds. The standard deviation of velocity varies from 9 mph at low density to 0.5 mph at capacity *(Highway capacity manual*, Figs. 3.26-3.28). Edie et al. (1980, Fig. 2) indicate the standard deviation varies from 7 mph to 2 mph.

The distribution of velocities is close to normal.

A useful calculation for determining the time scale involved with one vehicle catching up with another is the following. For two vehicles, picked at random from a normal distribution with a standard deviation of 5 mph, the speeds being v_1 and v_2, we have the expected value of $|v_1 - v_2| = \frac{2}{\sqrt{\pi}} 5$ mph $\doteq$ 6 mph.

2.4.2 Passing and Lane Changing Another obvious characteristic of subcritical flow is that frequently one vehicle will come up behind another and pass, with or without being delayed. To pass it is necessary, but not sufficient, to change lanes (except for motorcyclists, but they are few and have only a minor impact on traffic). In measurements I made, I got the following rates of lane changing expressed in time between lane changes for an average driver.

Flow veh/sec-lane	.37	.43	.38	.45
Time between changes sec	131	115	94*	1200*

* Freeway interchange probably had a large effect.

My time from forming a desire to change lanes to making the change is usually less than 10 sec and frequently as low as 3 sec in light traffic. This is much less than the average time between changes given above. If a driver is coming up behind a slower driver and he anticipates wanting to change lanes to pass, he can accomplish the lane change with no loss of time during his approach.

One author has equated the time to make a lane change with the waiting time before it is possible to pass another vehicle. This is not justified. The waiting time to pass depends primarily on two things: (1) the time before a driver decides he wants to pass, (2) the presence or absence of vehicles blocking the passing maneuver and the length of time of the blocking effect. I have measured the former to be from 0 sec to several minutes even in light traffic. Some people just like to follow others, or possibly each driver has a range of acceptable speeds. In light or moderate traffic, the first factor will overshadow the second factor.

Incidentally, the average time during a lane change from the time the tires on one side of the vehicle touch the lane line to the time the tires on the other side touch it is 1.8 sec.

PART I

EQUILIBRIUM

Traffic flow can be divided into two regions based on density. Let k_c be the critical density at which the maximum flow occurs. Subcritical flow occurs for $k < k_c$ and is characterized by there being enough room on the road for passing to occur and for some drivers to be able to go at their desired speeds. The distribution of desired and actual speeds, and the waiting time to pass are important phenomena. Forced flow occurs for $k > k_c$. Forced flow usually consists of traffic that has backed up due to the demand exceeding the capacity at some point downstream. Two geometrical configurations that can lead to this are: a reduction in the number of lanes and an on-ramp. Forced flow is characterized by there being no room on the road for a driver to go at his desired speed, by each car going at a speed nearly equal to the speed of the car ahead, by each driver choosing his average headway (distance from one car's front bumper to the front bumper of the car ahead), and by the nature of the response of each driver to acceleration of the car ahead. Essentially no passing occurs in forced flow. A driver can get into the lane that is moving fastest, but then he is stuck. Instability occurs in forced flow, but not in subcritical flow, because in subcritical flow there are only a limited number of cars following one another closely before there is a large gap. A disturbance can start and grow as it moves back, but when it reaches the large gap, it will die. In forced flow there are essentially no large gaps, and disturbances can grow and persist.

Some authors in their models of forced flow have included passing, which is not significant, and have failed to include the reaction of each driver to the acceleration of the car ahead, which is very important since the instability it can cause is one of the major phenomena of forced flow. Of course, there are some phenomena, such as the flow-density relation, which apply to both subcritical and forced flow.

In our model we deal with a uniform stream of cars traveling along one lane of a road. In subcritical flow each driver has his own desired speed at which he goes unless blocked by other cars. In forced flow he has his desired headway, his speed being the same as the car ahead. Our model focuses on subcritical flow and not on the details of forced flow. The desired speed is a function of the average density k of cars on the road since a driver will reduce even his desired speed as critical density is approached. That is, the desired speed refers to the driver's realistic assessment of how fast to go in the particular traffic situation. However, there will be a probability distribution of desired speeds. When a driver, going at his desired speed, comes up behind a slower car, he reduces his speed to that of the slower car. After waiting a specified time W, he passes the slower car and resumes his desired speed. The times taken to decelerate and accelerate can be incorporated, to some extent, into the choice of W. We note that, among other things, W will depend upon the density k, on which lane is under consideration, and upon the number of other lanes.

An individual driver will go at a velocity u which has a dependence on k of the type shown in Figure 1.

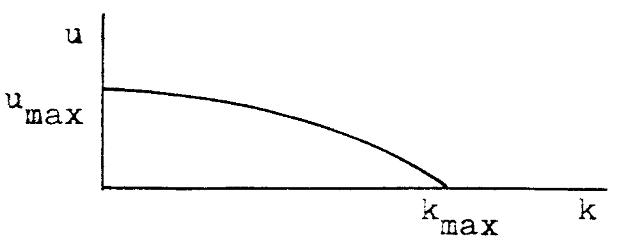


Figure 1. Velocity-density curve.

When the density is low, he will choose a maximum speed u_{max} controlled by his own view of safety or by the speed limit. As k increases the velocity will decrease monotonically to zero as $k \to k_{max}$, the jammed density for the road. Another plot of this would result from considering u as a function of the headway $h = 1/k$.

This plot would take the form shown in Figure 2 where L is the average distance between cars (front bumper to front bumper) in the jammed condition.

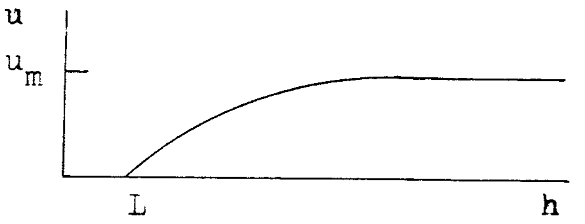


Figure 2. Velocity-headway curve.

In a simple reaction time model, the safe headway is given by

$$h - L = uT$$

where T is the reaction time of driver and car. If this is combined with a cutoff at a maximum speed u_m, we have

$$u = \begin{cases} \frac{h-L}{T} & h \le h_o = u_m T + L \\ u_m & h \ge h_o \,. \end{cases} \tag{1}$$

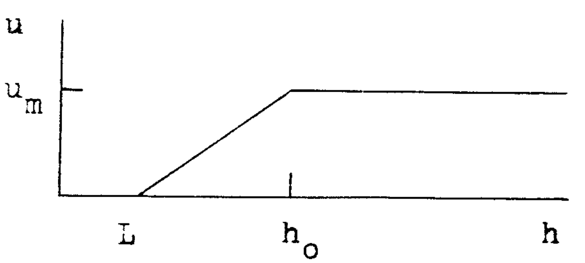


Figure 3. Velocity-headway curve.

If all drivers had the same curve for u(k), then the total flow would be given by

$$q = ku(k) \,.$$

The q-k curve corresponding to Figure 1 is shown in Figure 4.

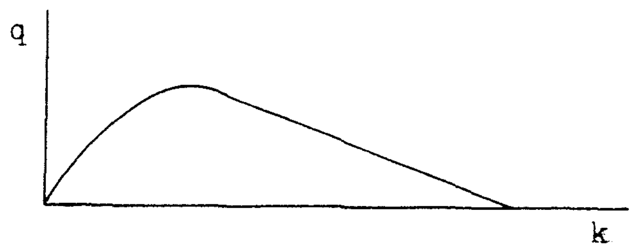


Figure 4. Flow-density curve.

For the reaction time model in Figure 3,

$$q = \begin{cases} (1 - kL)/T & 1/L \geq k \geq 1/h_o \\ ku_m & 0 \leq k \leq 1/h_o \,. \end{cases} \qquad (2)$$

as shown in Figure 5.

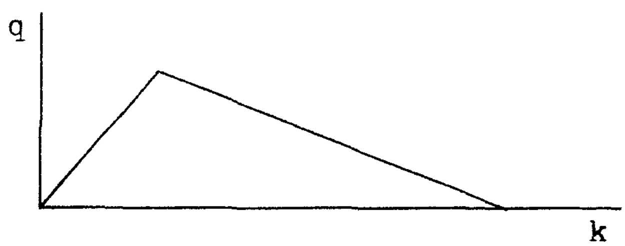


Figure 5. Flow-density curve.

3. THE EFFECT OF RANDOM DESIRED SPEEDS WITH FREE PASSING

In this model we allow each driver to have his own desired velocity-headway curve, thus adding an element of reality that the first model lacked. Suppose the velocity-headway curves are parameterized by the parameter b which represents the type of driver, so that $u = u(h,b)$. Assume that each driver is affected by the same average headway as any other driver. Also assume that drivers are able to pass freely, so that a driver's speed is determined solely by his velocity-headway curve and by the average headway. Let $p(b)$ be the probability density function for b, and let $v(h)$ be the ensemble average of the desired speeds, so that we have the relations

$$v(h) = \int u(h,b)p(b)db \,, \tag{3}$$

$$q \quad = kv(h) \,. \tag{4}$$

To give an example, let $0 \leq b \leq 1$ and let

$$u(h,b) = \begin{cases} 0 & h \leq L \\ (h-L)/T & L \leq h \leq (1-b)h_1 + b\,h_2 \\ \dfrac{((1-b)h_1 + bh_2 - L)}{T} & (1-b)h_1 + b\,h_2 \leq h \end{cases} \tag{5}$$

$$p(b) = 1 \,. \tag{6}$$

This is shown in Figure 6. For this example there is a uniform distribution of desired top speeds. If we let δ be the Dirac delta function and H be the Heaviside step function and $u = (h-L)/T$, we can write the probability density of desired speeds as

$$p_u(s) = \begin{cases} \delta(s-u) & L \le h \le h_1 \\ \dfrac{u_2-u}{u_2-u_1}\delta(s-u) + \dfrac{1}{u_2-u_1}H(s-u_1)\,H(u-s) & h_1 \le h \le h_2 \\ \dfrac{1}{u_2-u_1}H(s-u_1)\,H(u_2-s) & h_2 \le h\,. \end{cases} \qquad (7)$$

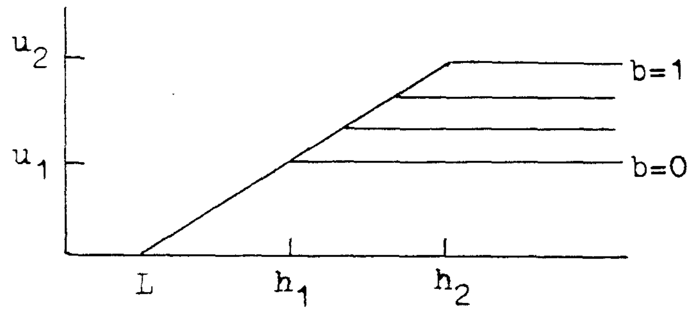


Figure 6. Desired velocity-headway curves.

Upon substituting (5) and (6) into (3) and (4), we get the average velocity-headway relation and the flow-density relation.

$$v(h) = \begin{cases} \dfrac{h-L}{T} & L \le h \le h_1 \\ \dfrac{1}{2(u_2-u_1)}\left(\dfrac{2u_2(h-L)}{T} - \dfrac{(h-L)^2}{T^2} - u_1^2\right) & h_1 \le h \le h_2 \\ \dfrac{u_1+u_2}{2} & h_2 \le h \end{cases} \qquad (8)$$

$$q(k) = \begin{cases} \dfrac{1-kL}{T} & \dfrac{1}{u_1T+L} \le k \\ \dfrac{k}{2(u_2-u_1)}\left(\dfrac{2u_2(1-kL)}{kT} - \dfrac{(1-kL)^2}{k^2T^2} - u_1^2\right) & \dfrac{1}{u_2T+L} \le k \le \dfrac{1}{u_1T+L} \\ \dfrac{k(u_1+u_2)}{2} & k \le \dfrac{1}{u_2T+L}\,. \end{cases} \qquad (9)$$

In Figure 7 the dashed line is the simple flow-density curve, Equation 2, with top desired speed $(u_1+u_2)/2$. So the effect of allowing various maximum desired speeds with free passing is simply to round off the peak of the flow-density curve leaving the remainder unchanged. Moreover, the change is limited to a small range for realistic values of T, L, u_1, and u_2. It is interesting that there is so little

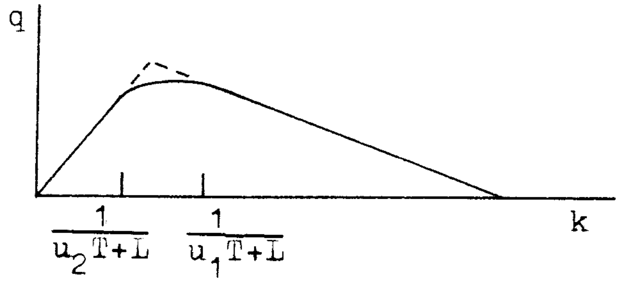


Figure 7. Flow-density curve for free passing.

change that it might well be neglected. However, we get a significant difference when we also incorporate interference with overtaking and a waiting time to pass.

4. RANDOM DESIRED SPEEDS WITH A WAITING TIME TO PASS

We now add to the previous considerations the condition that, before one driver may pass another, he must wait a fixed time W before traffic conditions permit passing. He then instantaneously increases his velocity to his desired speed and maintains this velocity until he again encounters a slower car.

First we consider a stream of cars all with speed v and density k, and one driver who desires to go faster than v, say at speed u. The fast car goes at speed u until blocked by a car in the stream. It then goes at speed v for time W after which it passes the slow car and resumes velocity u. This process is repeated indefinitely. We now consider the average number of delays and determine the average velocity of the fast car. If s is the average speed of the fast car, we argue that in unit time the fast car goes a distance (s-v) farther than the other cars, passing (s-v)k of them. Since it spends time W at speed v behind each of them, the total time at speed v is (s-v)kW. The remainder of the time, 1-(s-v)kW, is spent at speed u. Therefore, the average speed of the fast car is

$$s = (s - v)kWv + (1 - (s - v)kW)u\ .$$

Solving for s we find

$$s = \frac{u + kW(u - v)v}{1 + kW(u - v)}\ . \tag{10}$$

Next we use (10) in a situation where each driver has his own desired speed. If a driver's desired speed is u, we would expect him to be delayed only by drivers with desired speeds less than u. We would further expect his average speed to be given approximately by (10), where v is the average speed of all drivers with desired speed less than u, and k is the density of cars with desired speed less than u.

Equation 10 is exact if all the slow cars have speed v. However, when v is the average speed of the slower cars, (10) is only approximate because the fast car passes *more* cars with speed less than v than cars with speed greater than v. This decreases its average velocity from that given by (10). Improvements on (10) lead to complicated equations. We use (10) with the thought that it is a first approximation and that modification of (10) is not justified unless we make improvements in some of our other assumptions as well.

We now define

$P(u)$ = the fraction of drivers with desired speeds $\leq u$

$p(u) = \frac{dP(u)}{du}$

k_o = the total number of cars per unit length of road.

P(u) and p(u) are the probability distribution and density functions of the desired speeds. Clearly we have

$$k(u) = k_o P(u) \tag{11}$$

for the average density of cars whose drivers have desired speed less than u. Substituting (11) in (10), we have

$$s(u) = \frac{u + Wk_o P(u) v(u) (u - v(u))}{1 + Wk_o P(u) (u - v(u))} \tag{12}$$

where v(u) is the average velocity of all cars whose drivers have desired speed less than u. If we average over the speeds s(u) of individual drivers, we get

$$v(u) = \frac{1}{P(u)} \int_o^u s(z) p(z)\, dz \,. \tag{13}$$

Using (12) in (13), we derive the implicit relation for v(u),

$$v(u) = \frac{1}{P(u)} \int_0^u \frac{z + Wk_o P(z) v(z)\,(z - v(z))}{1 + Wk_o P(z)\,(z - v(z))}\, p(z) dz\,. \tag{14}$$

On differentiation this gives

$$\frac{dv(u)}{du} = \frac{p(u)}{P(u)} \cdot \frac{u - v(u)}{1 + Wk_o P(u)\,(u - v(u))}\,. \tag{15}$$

Thus we have a first order ordinary differential equation for $v(u)$. We may get an appropriate initial condition for this equation by considering the minimum desired speed u_3 of all drivers. Since there are no slower drivers to delay those with desired speed u_3, these drivers go at their desired speed. That is,

$$v(u_3) = u_3\,. \tag{16}$$

Now we wish to solve (15) with initial condition (16). There appear to be no closed form solutions, so we assume $p(u)$ to be some particular function and solve (15) numerically. We assume

$$p(u) = \begin{cases} \dfrac{1}{u_4 - u_3} & u_3 \le u \le u_4 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

which implies

$$P(u) = \begin{cases} 0 & u \le u_3 \\ \dfrac{u - u_3}{u_4 - u_3} & u_3 \le u \le u_4 \\ 1 & u_4 \le u\,. \end{cases} \tag{18}$$

We chose this form of $p(u)$ for its simplicity. It may be considered as a first approximation to the true probability density. At this stage we are interested in the general behavior of our model and not in the many refinements we could make.

Substituting (17) and (18) in (15) we get

$$\frac{dv(u)}{du} = \frac{1}{u-u_3} \cdot \frac{u-v(u)}{1+Wk_o\left(\frac{u-u_3}{u_4-u_3}\right)(u-v(u))} . \tag{19}$$

We eliminate the parameters in (19) by introducing new variables r and $v^*(r)$ as follows:

$$u = \frac{u_4-u_3}{(Wk_o(u_4-u_3))^{1/2}} r + u_3 , \tag{20}$$

$$v(u) = \frac{u_4-u_3}{(Wk_o(u_4-u_3))^{1/2}} v^*(r) + u_3 . \tag{21}$$

Equations 16 and 19 become

$$v^*(0) = 0 , \tag{22}$$

$$\frac{dv^*(r)}{dr} = \frac{1}{r} \cdot \frac{r-v^*(r)}{1+r(r-v^*(r))} . \tag{23}$$

Equations 22 and 23 were solved numerically, the solution being shown in Graph 2.

An approximate solution (within 5%) to (23) is

$$v^*(r) \doteq \frac{1}{2^{1/2}} \tan^{-1}\left(\frac{r}{2^{1/2}}\right) . \tag{24}$$

Equation 24 comes from making a guess for the function $v^*(r)$, substituting this in the right hand side of (23), and then integrating. The guess was $v^*(r) = r/2$ which matches the value and slope of v^* at $r = 0$.

Formally, a power series solution of (23) for r near 0 starts with

$$v^*(r) = \frac{r}{2} - \frac{r^3}{16} + \frac{r^5}{96} + \frac{195}{98304} r^7 + \cdots . \tag{25}$$

Substituting a power series in $\frac{1}{r}$ into (23) for large r gives

$$v^*(r) = 1.16 - \frac{1}{r} + \frac{1}{3r^3} + \frac{.289}{r^4} + \cdots . \tag{26}$$

Thus we have a model which accounts for the effects of waiting time to pass and various desired speeds. From (20) and (21) we see that the solution to our equation is largely controlled by the dimensionless quantity $(Wk_o(u_4 - u_3))^{1/2}$. For the sake of convenience we define

$$a = (Wk_o(u_4 - u_3))^{1/2}.$$

Next, we note that a controls the range of r. Since the range of u is $u_3 \leq u \leq u_4$, (20) implies that the range of r is

$$0 \leq r \leq a.$$

The parameter a contains the combined effects of waiting time to pass, density, and spread of desired speeds. It is interesting to see how these phenomena may work in different ways to produce the same effect. For example, if a is small, then r is small and from (25) we have

$$v^*(r) \doteq \frac{r}{2}.$$

Using (20) and (21) then yields

$$v(u) \doteq \frac{u + u_3}{2}.$$

That is, the average speed achieved by drivers with desired speed less than u is approximately the same as the average of their desired speeds. This can only happen if the average speed achieved by each driver is close to his desired speed.

If a is small due to short waiting time to pass, then each driver, although he may be blocked frequently, spends little time at reduced speed. This causes his average speed to be close to his desired speed.

If a is small due to low density, then each driver is only rarely delayed, so that he spends most of his time at his desired speed. Once again, this causes his average and desired speeds to be approximately equal.

If a is small due to a small spread of desired speeds, then any given driver does not catch up with the driver ahead very quickly because his speed relative to the driver ahead is small. So he is blocked infrequently, and his average and desired speeds are about the same.

If Wk_o is large, then a is large and (26) governs the behavior of the faster drivers. Using only the first term in the series yields

$$v(u) \doteq u_3 + \frac{1.16(u_4 - u_3)}{a},$$

from which we see that the average speed of each driver is close to the minimum desired speed u_3. This may be explained in terms of a long waiting time or a high density.

A feature of this model is that it does not require a knowledge of the fine details of passing, since these are lumped into the single waiting time W. For example, when one car is delaying two others, there are several ways to assign the waiting times and to specify how many cars the rear car passes. Such details do not affect the form of our model, and we expect our model to be approximately true (with appropriate choice of W).

5. APPLICATION TO THE FLOW-DENSITY CURVE

We are interested in the average speed of all cars, and this will be given by $v(u_4)$. Upon using (20) and (21) with $u = u_4$, we get

$$v(u_4) = \left(\frac{u_4 - u_3}{Wk_o}\right)^{1/2} v^*((Wk_o(u_4 - u_3))^{1/2}) + u_3 , \tag{27}$$

and the total flow will be

$$q = k_o v(u_4) = \left(\frac{k_o(u_4 - u_3)}{W}\right)^{1/2} v^*((Wk_o(u_4 - u_3))^{1/2}) + k_o u_3 . \tag{28}$$

Equation 28 gives the flow when the desired speeds of the drivers are uniformly distributed over the range u_3 to u_4. Of course, u_3 and u_4 depend on the headway. Thus (28) becomes

$$q(h) = \left[\frac{k_o(u_4(h) - u_3(h))}{W}\right]^{1/2} v^*([Wk_o(u_4(h) - u_3(h))]^{1/2}) + k_o u_3(h) . \tag{29}$$

In order to apply (29), we must determine $u_3(h)$ and $u_4(h)$ from some other source.

We now apply the waiting time model to the velocity headway curves of Figure 6. For $h \leq h_1$, all drivers have the same speed, so there is no passing. For $h \geq h_2$, the desired speeds are uniformly distributed between u_1 and u_2, so that (29) applies with $u_3 = u_1$ and $u_4 = u_2$.

For $h_1 \leq h \leq h_2$, there will be some drivers with desired speed $u = (h - L)/T$, while the other desired speeds are uniformly distributed between u_1 and u. We average the speeds of these cars as follows: First use (27) to determine the average speed w_1 of cars with uniformly distributed speeds.

$$w_1 = \left[\frac{u - u_1}{Wk_o\left(\frac{u - u_1}{u_2 - u_1}\right)}\right]^{1/2} v^*\left(\left[Wk_o\left(\frac{u - u_1}{u_2 - u_1}\right)(u - u_1)\right]^{1/2}\right) + u_1$$

where $k_o(\frac{u-u_1}{u_2-u_1})$ is the density of the uniformly distributed cars. Next we assume that each driver with desired speed u has only to contend with cars going at speed w_1. Then (10) gives the average speed w_2 of these drivers, the result being

$$w_2 = \frac{u + Wk_o\frac{u-u_1}{u_2-u_1}w_1(u-w_1)}{1 + Wk_o\frac{u-u_1}{u_2-u_1}(u-w_1)}.$$

The average speed v of all the cars is

$$v = \frac{u-u_1}{u_2-u_1}w_1 + \frac{u_2-u}{u_2-u_1}w_2 . \tag{30}$$

Combining the above gives the following formula for flow:

$$q = \begin{cases} \frac{k_o}{T}(h-L) & h \leq h_1 \\ \frac{k_o}{u_2-u_1}((u-u_1)w_1+(u_2-u)w_2) & h_1 \leq h \leq h_2 \\ (\frac{k_o(u_2-u_1)}{W})^{1/2}v^*([Wk_o(u_2-u_1)]^{1/2})+k_ou_1 & h_2 \leq h \end{cases} \tag{31}$$

$$k_o = \frac{1}{h}.$$

6. COMPARISON WITH EMPIRICAL RESULTS

In this section we compare (31) with the measurements from a study by May and Wagner (1960). They investigated the Ford Expressway, near Detroit, which has three lanes in one direction. They present a flow-density curve for each lane. Some of the parameters we use are determined from the measurements of ' ıy and Wagner, while the others are picked to be reasonable and to provide a good fit. First we choose the parameters for the left lane:

At very low density the average speed of all cars was 80.7 ft/sec (55 mph). So we require

$$\frac{u_1 + u_2}{2} = 80.7 .$$

To make a good fit with the empirical curve, we choose

$$u_1 = 66 \text{ ft/sec (45 mph)},$$

$$u_2 = 95.3 \text{ ft/sec (65 mph)}.$$

The waiting time to pass W increases with increasing density. We choose the linear relationship

$$W = 2 \times 10^4 k_o .$$

This is a waiting time of 200 sec at a density of .01 cars/ft (52.8 cars/mile). It is around this density that a substantial reduction of speed starts to occur which one would expect to be accompanied by a relatively long waiting time to pass.

According to May and Wagner (1960), the maximum density is between 200 and 225 vehicles per mile per lane. This is equivalent to 26.4 to 23.5 ft/vehicle. So we choose

$$L = 25 \text{ ft} .$$

A reaction time

$$T = 1.2 \text{ sec} .$$

was chosen to match the curve.

We use the same parameters for the right lane, except that we choose

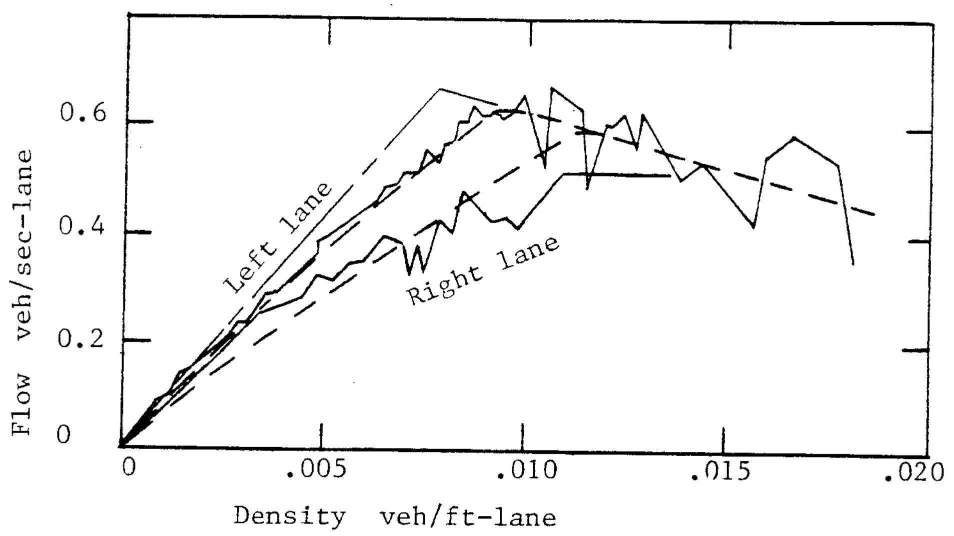
$$u_1 = 50.0 \text{ ft/sec (34 mph)},$$

$$u_2 = 79.2 \text{ ft/sec (54 mph)}.$$

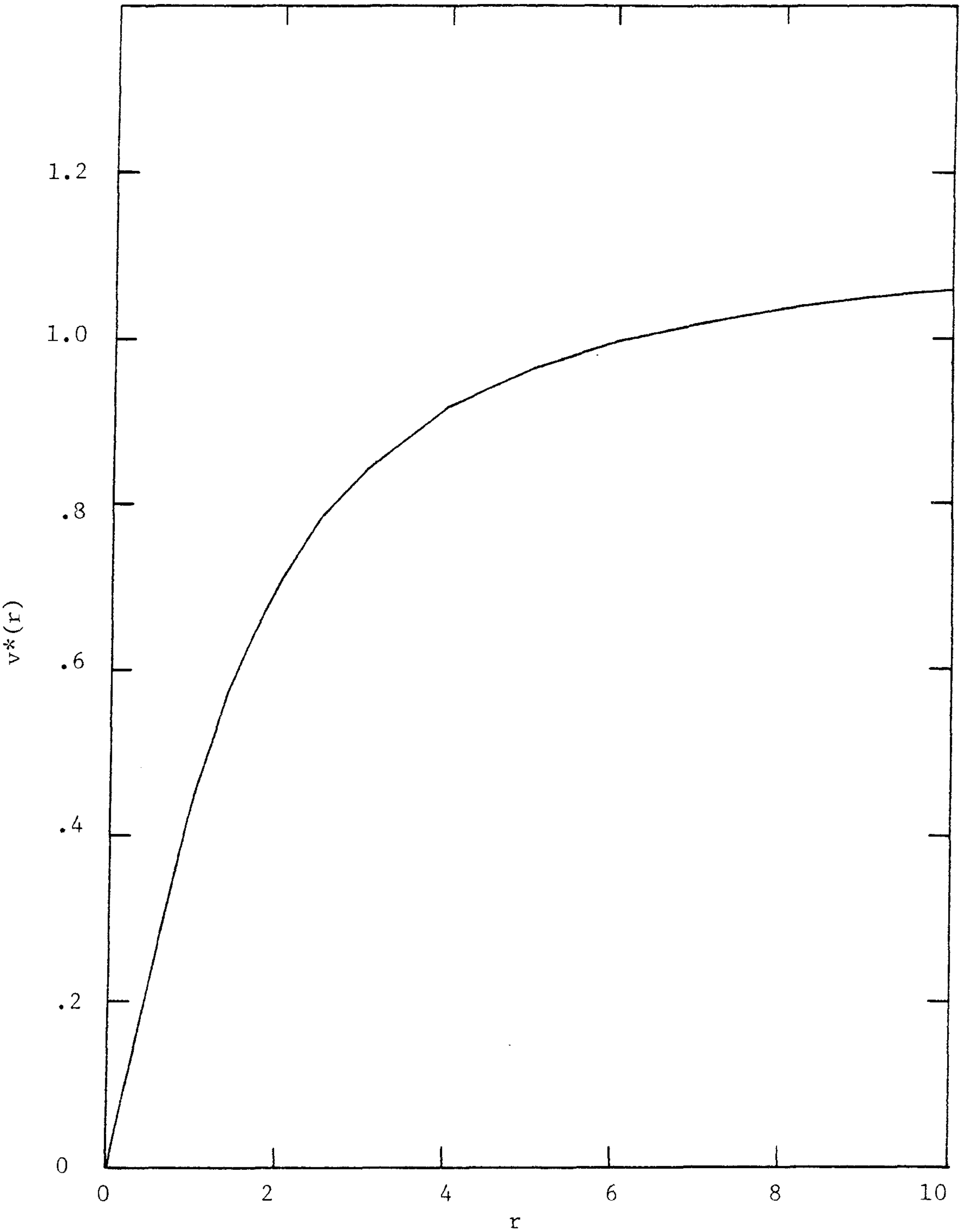
The curves are plotted in Graph 1. The solid lines represent the measurements of May and Wagner (1960). The dashed lines are derived from our waiting time model, while the broken line assumes neither a waiting time nor a range of desired speeds.

For the left lane, Graph 1 shows reasonable agreement between the empirical curve and our model without a range of desired speeds. The use of desired speeds and a waiting time causes the agreement to be excellent.

For the right lane, the agreement is not very good. This could be due to other factors becoming important. One such factor might be the action of vehicles entering or leaving the road which could cause deviations from the velocity-headway curves of Figure 6. Another factor might be that drivers in the slow lane tend not to pass; so that their average speed would be closer to the speed of the slowest driver.



Graph 1. Flow-density curves on the Ford Expressway. Solid lines redrawn from May and Wagner (1960).



Graph 2. Normalized average speed v* as a function of normalized desired speed r.

PART II

THE DETAILED VELOCITY DISTRIBUTION FUNCTIONS

7. ASSUMPTIONS AND EQUATIONS

We now consider a waiting time model with considerably more detail than in Part I. We allow for the study of transient effects and give a very much more detailed description of the velocity distributions of the cars. For this model we assume a continuous distribution of cars along the road (as opposed to a discrete distribution) and include a definite mechanism for passing. We make some approximations to enable us to solve the equations and compare the results with the previous waiting time model.

As before, we consider in this model a stream of cars on one lane of a multilane road where each driver has his own desired speed. Each driver goes at his desired speed until he meets a slower car, at which time he decelerates instantaneously to the lower velocity. If no further delays occur, he accelerates instantaneously to his desired speed after a time W. If, on the other hand, both cars meet a still slower vehicle, they both decelerate instantaneously to the lower velocity. Once again, if no further delays occur, both drivers resume their respective desired speeds at a time W after the last blocking. If the rear driver has the greatest desired speed, he passes both cars ahead of him at the same time in order to achieve his desired speed. This process may continue for any number of delays. The above description of the flow of traffic, together with initial conditions, is sufficiently detailed to uniquely define the position of each car as a function of time.

To keep the model simple, we assume that the cars have zero length. A consequence of this assumption is that each car in a multiple holdup can be viewed as being blocked only by the lead driver, who is going at his desired speed.

To describe the state of traffic, it is convenient to use two velocity distribution functions. The first one g describes the distribution for drivers who are at their desired speeds; the second one f refers to drivers travelling at less than their desired speeds. We let

x = position on the road,

t = time,

s = actual speed of a car,

u = desired speed of a driver,

T = time since the last delay.

The number of drivers with position between x and x + dx, actual speed between s and s + ds, desired speed between u and u + du, and time since the last delay between T and T + dT is given by f(x,t,s,u,T) dx ds du dT. Similarly, the number of drivers going at their desired speed in the box (x,dx) × (u,du) in (x,u) space is g(x,t,u) dx du.

Equations for f and g are obtained by writing equations for the conservation of cars. We consider the conservation for a small box

$$D = dx\,ds\,du\,dT$$

in (x,s,u,T) space. Then, using subscripts to denote partial differentiation, $f_t D$ is the rate of increase in the number of cars in this D box. This increase will be due to three things: streaming in x, streaming in T, and s cars meeting slower cars.

The increase due to streaming is

$$-(sf_x + f_T)\,D\ . \qquad (32)$$

To get the increase (decrease) due to delays, we take expected values from the following probabilistic model. We first consider two cars: one with speed in (s,ds), the other with speed in (z,dz), and both in (x, dx) × (u,du) × (T,dT). For the s car to be delayed, we must have $z < s$. In time dt the s car moves a distance

$(s-z)\,dt$ relative to the z car. We now assume that the positions of the s and z cars are uniform random variables. That is, the probability that the s car is in $(\xi, d\xi)$ is $d\xi(dx)^{-1}$, independent of ξ. The probability that the s car will meet the z car is then $(s-z)dt(dx)^{-1}$. There are $f(x,t,s,u,T)D$, s type cars and $g(x,t,z)\,dx\,dz$, z type cars which can effect a delay. The probability that one of the s cars will meet any of the z cars is

$$(s-z)dt(dx)^{-1}g(x,t,z)\,dx\,dz\ .$$

The expected number of s cars meeting z cars is

$$(s-z)dt(dx)^{-1}g(x,t,z)\,dx\,dz\,f(x,t,s,u,T)\,D\ ,$$

and the rate of such meetings is

$$(s-z)\,g(x,t,z)\,f(x,t,s,u,T)\,D\,dz\ .$$

We integrate to get the rate of s cars slowing to any speed (not just to z), and the result is

$$f(x,t,s,u,T)\int_0^s (s-z)\,g(x,t,z)\,dz\,D\ . \tag{33}$$

On combining (32) and (33), we get

$$f_t(x,t,s,u,T) = -sf_x - f_T - f\int_0^s (s-z)\,g(x,t,z)\,dz\ . \tag{34}$$

At $T = 0$, the rate of change of f is due to four things: streaming in x, cars slowing to speed s, s cars slowing down, and cars passing out of the range $(0, dT)$ by waiting at reduced speed long enough for T to become greater than dT.

The rate of change of f due to streaming in x is

$$-sf_xD\ . \tag{35}$$

The rate of change due to T becoming greater than dT is

$$-f(x,t,s,u,dT)\,dx\,ds\,du\ . \tag{36}$$

The change due to cars slowing to speed s comes from cars at their desired speed u slowing to s and from cars at a speed z ($u > z > s$) slowing to s. As before, the probability of one u car being slowed by a single s car is $(u-s)\,dt\,(dx)^{-1}$. The expected number of u cars slowing to s is

$$(u-s)\,dt\,(dx)^{-1}\,g(x,t,u)\,dx\,du\,g(x,t,s)\,dx\,ds$$

with the rate

$$(u-s)\,g(x,t,u)\,g(x,t,s)\,dx\,ds\,du\ . \tag{37}$$

The rate of z cars slowing to s is

$$(z-s)\,(dx)^{-1}\int_0^W f(x,t,z,u,T)dT\ \ dxdzdu\,g(x,t,s)\,dxds\ .$$

The total rate of z cars ($u > z > s$) being delayed by s cars is

$$g(x,t,s)\int_s^u\int_0^W (z-s)\,f(x,t,z,u,T)\,dT\,dz\ \ dx\,ds\,du\ . \tag{38}$$

As in (33) the rate at which s cars are delayed is

$$f(x,t,s,u,0)\int_0^s (s-z)\,g(x,t,z)\,dz\,D\ . \tag{39}$$

The change in f at $T = 0$ is

$$f_t(x,t,s,u,0)\,D\ , \tag{40}$$

Since D is of smaller order than dx ds du, we can neglect (35), (39), and (40). The conservation of cars at $T = 0$ is expressed by combining (36), (37), and (38) to get

$$f(x,t,s,u,0) = (u-s)\ g(x,t,u)\ g(x,t,s)\ +$$
$$g(x,t,s)\int_s^u\int_0^W (z-s)f(x,t,z,u,T)dTdz\ . \tag{41}$$

The rate of change of g comes from: streaming in x, drivers at their desired speeds being delayed, drivers returning to their desired speeds. Arguments similar to the above give

$$g_t(x,t,u) = -ug_x + \int_0^u f(x,t,z,u,W) - (u-z)g(x,t,u)g(x,t,z)dz \ . \tag{42}$$

Equations (34), (41) and (42), together with initial conditions, determine f and g uniquely. Appropriate initial conditions are

$$f(x,0,s,u,T) = f^*(x,s,u,T) \tag{43}$$

$$g(x,0,u) = g^*(x,u) \ . \tag{44}$$

For consistency we require that $f^*(x,s,u,0)$ satisfy (41). For actual traffic it is certainly reasonable to assume that f^* and g^* are nonnegative and bounded. So we require

$$0 \le f^* \le A \quad \text{and} \tag{45}$$

$$0 \le g^* \le B \tag{46}$$

where A and B are constants. (We also assume that f^* and g^* are Riemann integrable.)

We now discuss the reasons for choosing these particular variables. We could represent the drivers who are at their desired speeds by $f(x,t,u,u,T)$ instead of $g(x,t,u)$. But this presents two difficulties. First, the variable T is not applicable to drivers at their desired speeds. Second, the behavior of $f(x,t,s,u,T)$ for s near u would be that of the Dirac delta function $\delta(u-s)$. In any event, detailed analysis of the equations would require separate consideration of these two difficulties, so there is no increase in complexity due to the use of both f and g.

We make f a function of s, and g a function of u in order to write the delay terms of our equations.

We require f to depend on u to write the f term in (42). This term represents drivers returning to their desired speeds, and, unless f depends on u, we do not know to what speed to return the drivers.

The variable T is used to tell when a driver should return to his desired speed. Another mechanism for the return to desired speed is to have the return occur at a rate proportional to f. The proportionality constant could depend on several variables. The variable T could be included in, or eliminated from, this mechanism.

We can eliminate f from (42) if we assume there are no multiple delays. A multiple delay occurs when a driver, having once reduced his speed, must reduce it again before passing. When there are no multiple delays, a driver will be slowed for exactly a time W, and the rate of drivers returning to their desired speeds at time t is equal to the rate of delays at time t-W. That is, the f term in (42) becomes $\int_o^u f(x,t,z,u,W)dz = \int_o^u (u-z)g(x-zW,t-W,u)g(x-zW,t-W,z)dz$. As we show later, when there are no multiple delays, we can find an explicit formula for f in terms of g, thereby entirely eliminating f from the system of equations.

Next, we integrate our equations along characteristics to get a set of non-linear integro-difference equations to solve. For easy reference we rewrite the integro-differential equations here.

$$f_t(x,t,s,u,T) = -sf_x - f_T - f \int_o^s (s-z)g(x,t,z)dz \,, \tag{34}$$

$$f(x,t,s,u,o) = (u-s)g(x,t,u)g(x,t,s) + g(x,t,s) \int_s^u \int_o^W (z-s)f(x,t,z,u,T)dTdz \,, \tag{41}$$

$$g_t(x,t,u) = -ug_x + \int_o^u f(x,t,z,u,W) - (u-z)g(x,t,u)g(x,t,z)dz \,, \tag{42}$$

$$f(x,o,s,u,T) = f^*(x,s,u,T) \,, \tag{43}$$

$$g(x,0,u) = g^{*}(x,u) \; . \tag{44}$$

$$0 \leq f^{*} \leq A \, , \tag{45}$$

$$0 \leq g^{*} \leq B \, . \tag{46}$$

We first notice that (34) is a linear partial differential equation in f. The characteristics, with parameter y, are given by the system of equations

$$\frac{dt}{dy} = 1, \quad \frac{dx}{dy} = s, \quad \frac{dT}{dy} = 1 \; . \tag{47}$$

Along one of these characteristics we have

$$\frac{df}{dy} = -f \int_o^s (s-z)\, g(x,t,z)dz \; . \tag{48}$$

Upon letting

$$G(x,t,s) = \int_o^s (s-z) g(x,t,z)dz \; , \tag{49}$$

equation (48) becomes

$$\frac{df}{dy} = -fG \; . \tag{50}$$

Suppose that, for y = 0, we have $x = x_o$, $t = t_o$, $T = T_o$. The characteristics are then

$$t = t_o + y \; , \quad x = x_o + sy \; , \quad T = T_o + y \; .$$

We now distinguish between two types of characteristics:

1. those with $t \leq T$ and

2. those with $t > T$.

For the former we set $t_o = 0$ while for the latter we set $T_o = 0$. Figure 8 displays the characteristics. The solution to (50) is

$$f = f\,|_{y=0}\, e^{-\int_o^y G dy} \; . \tag{51}$$

For type 1 characteristics (51) becomes

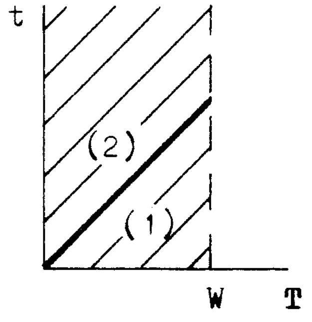


Figure 8. Characteristics for the time variables of Equation 34.

$$f(x_o + sy, y, s, u, T_o + y) = f^*(x_o, s, u, T_o) e^{-\int_o^y G(x_o + s\xi, \xi, s) d\xi} .$$

For type 2 characteristics (51) becomes

$$f(x_o + sy, t_o + y, s, u, y) = f(x_o, t_o, s, u, o) e^{-\int_o^y G(x_o + s\xi, t_o + \xi, s) d\xi} .$$

Upon changing variables, we get

$$f(x,t,s,u,T) = \begin{cases} f^*(x-st,s,u,T-t) e^{-\int_o^t G(x-st+s\xi,\xi,s)d\xi} & \text{for } t \leq T \qquad (52) \\ f(x-sT,t-T,s,u,o) e^{-\int_o^T G(x-sT+s\xi,t-T+\xi,s)d\xi} & \text{for } t > T . \qquad (53) \end{cases}$$

By integrating along characteristics, we may put (42) into the form of an integro-difference equation. The characteristics are

$$x = x_o + uy \qquad t = y, \tag{54}$$

and (42) becomes

$$\frac{dg}{dy} = \int_o^u \{ f(x,t,z,u,W) - (u-z) g(x,t,u) g(x,t,z) \} dz . \tag{55}$$

Therefore,

$$g(x_o + uy, y, u) = g^*(x_o, u) + \int_o^y \int_o^u \{ f(x_o + u\xi, \xi, z, u, W)$$

$$- (u-z) g(x_o + u\xi, \xi, u) g(x_o + u\xi, \xi, z) \} dz d\xi ,$$

and upon changing variables we get

$$g(x,t,u) = g^*(x-ut,u) + \int_0^t\int_0^u \{f(x-ut+u\xi,\xi,z,u,W)$$

$$-(u-z)g(x-ut+u\xi,\xi,u)g(x-ut+u\xi,\xi,\ z)\}\,dz d\xi. \qquad (56)$$

Next we substitute (52) and/or (53) into (56). If $t \leq W$, then in (56) $\xi \leq W$ so we may use (52) alone. If $t \geq W$, we split the integral in (56) into two integrals: one from 0 to W, the other from W to t. Equation 52 is substituted in the first integral, while (53) is used in the second. With the use of (49), we get

$$g(x,t,u) = g^*(x-ut,u) + \int_0^t\int_0^u f^*(x-ut+u\xi-z\xi,z,u,W-\xi)$$

$$e^{-\int_0^\xi\int_0^z (z-\zeta)g(x-ut+u\xi+z\eta-z\xi,\eta,\zeta)d\zeta d\eta}\, dz d\xi$$

$$-\int_0^t\int_0^u (u-z)g(x-ut+u\xi,\xi,u)g(x-ut+u\xi,\xi,z)dz d\xi \quad \text{for } t \leq W, \qquad (57)$$

$$g(x,t,u) = g^*(x-ut,u)$$

$$+\int_0^W\int_0^u f^*(x-ut+u\xi-z\xi,z,u,W-\xi)e^{-\int_0^\xi\int_0^z (z-\zeta)g(x-ut+u\xi-z\xi+z\eta,\eta,\zeta)d\zeta d\eta}\,dz d\xi$$

$$+\int_W^t\int_0^u f(x-ut+u\xi-zW,\xi-W,z,u,o)e^{-\int_0^W\int_0^z (z-\zeta)g(x-ut+u\xi+z\eta-zW,\xi-W+\eta,\zeta)d\zeta d\eta}\,dz d\xi$$

$$-\int_0^t\int_0^u (u-z)g(x-ut+u\xi,\xi,u)g(x-ut+u\xi,\xi,z)dz d\xi \quad \text{for } t > W. \qquad (58)$$

Similarly, when we substitute (49), (52,) and (53) into (41), we get

$$f(x,t,s,u,o) = (u-s)g(x,t,u)g(x,t,s)$$

$$+g(x,t,s)\int_s^u\int_0^t (z-s)f(x-zT,t-T,z,u,o)e^{-\int_0^T\int_0^z (z-\eta)g(x+z\xi-zT,t-T+\xi,\eta)d\eta d\xi}\,dT dz$$

$$+g(x,t,s)\int_s^u\int_t^W (z-s)f^*(x-zt,z,u,T-t)e^{-\int_0^t\int_0^z (z-\eta)g(x+z\xi-zT,\xi,\eta)d\eta d\xi}\,dT dz$$

$$\text{for } t \leq W, \qquad (59)$$

$$f(x,t,s,u,o) = (u-s)g(x,t,u)g(x,t,s)$$

$$+\,g(x,t,s)\int_s^u\int_o^W (z-s)f(x-zT,t-T,z,u,o)e^{-\int_o^T\int_o^z (z-\eta)g(x+z\xi-zT,t-T+\xi,\eta)d\eta d\xi}dTdz$$

$$\text{for } t > W. \qquad (60)$$

We notice that (57) is an integro-difference equation in g alone, and the t argument of g, wherever it appears, is limited to the range $0 \leq t \leq W$. Thus (57) is uncoupled from (58)-(60), so we may work with it independently of the others. The same remarks apply to (59) once the solution to (57) is known. Similarly, we may use (58) and (60) alternately to extend the range of t in steps of W.

8. EXISTENCE AND ELEMENTARY PROPERTIES OF SOLUTIONS

In this section we prove that there exists a solution to the set of integro-difference equations, (52), (53), (57)-(60), which is bounded, nonnegative, and continuous along characteristics. We also prove that there is only one solution in the set of all bounded, integrable functions which satisfy the equations. Note that any solution to these equations also satisfies (56).

First we consider solutions which are integrable and for which a bound $M(t_1,u_1)$ exists; that is,

$$|g(x,t,u)| < M(t_1,u_1) , \tag{61}$$

$$|f(x,t,s,u,T)| < M(t_1,u_1) . \tag{62}$$

for

$$-\infty < x < \infty , \ 0 \le t \le t_1 , \ 0 \le s \le u , \ 0 \le u \le u_1 , \ 0 \le T \le W .$$

Inequalities 61 and 62 hold for all u_1 and t_1, and nowhere is $M(t_1,u_1)$ infinite.

We now show that g is continuous along the characteristic

$$x - ut = c = \text{constant} .$$

Along this characteristic, (56) becomes

$$g(c+ut,t,u) = g^*(c,u) + \int_0^t \int_0^u \{f(c+u\xi,\xi,z,u,W)$$

$$-(u-z)g(c+u\xi,\xi,u)g(c+u\xi,\xi,z)\}dzd\xi . \tag{63}$$

We must show that g(c+ut,t,u) is a continuous function of t. Restricting u to be less than u_1 and t to be less than t_1 yields, by (61) and (62), that f and g are bounded. Since f and g are bounded and integrable, the integrand in (63) is bounded and integrable, and hence the integral in (63) is continuous. $g^*(c,u)$ is a constant function of t. Hence, g(c+ut,u,t) is continuous for $u < u_1$ and $t < t_1$. Since u_1 and t_1 are arbitrary, g is continuous along characteristics for all u and t.

In a similar manner we may use (52) and (53) to show that f is continuous along the characteristics

$$\begin{aligned} x - st &= c = \text{constant}, \\ T - t &= d = \text{constant}. \end{aligned}$$

Next we use a proof by contradiction to show that g is nonnegative for $t \leq W$. The proof that g is nonnegative for $t > W$ and that f is nonnegative follow later, since the nature of the equations forces us to consider first g for $t \leq W$, then f for $t \leq W$, then g for $W < t \leq 2W$, etc. Suppose that for some (x,t_2,u) we have

$$g(x,t_2,u) < 0 . \tag{64}$$

We consider the quantity $g(c+ut,t,u)$ for $0 \leq t \leq t_2$ where $c = x - ut_2$. This gives the values of g on the characteristic through (x,t_2,u). At one end of this characteristic we have $t=0$ and $g = g^*(c,u) \geq 0$, while at the other end $t = t_2$ and $g = g(x,t_2,u) < 0$. Since we have shown that g is continuous along characteristics, there must be some t between 0 and t_2 for which $g = 0$. In fact, there is a largest t, say t_1, such that $g(c+ut_1,t_1,u) = 0$. This t_1 is defined by

$$t_1 = \sup\{t \mid 0 \leq t < t_2 , \quad g(c+ut,t,u) = 0\} .$$

Continuity of g implies that $t_1 < t_2$ and

$$g(c+ut_1,t_1,u) = 0 . \tag{65}$$

Thus we have

$$g(c+ut,t,u) < 0 \qquad \text{for } t_1 < t \leq t_2 . \tag{66}$$

If we evaluate (57) along the characteristic and split the integrals, we get an equation of the form

$$g(c+ut,t,u) = g^*(c,u) + \int_0^{t_1} + \int_{t_1}^{t} . \tag{67}$$

But by (65) we have

$$g^{*}(c,u) + \int_{0}^{t_1} = g(c+ut_1,t_1,u) = 0 .$$

Since the initial function f^{*} is assumed to be nonnegative, the term in (57) containing f^{*} is nonnegative. Upon combining the above, (67) becomes

$$g(c+ut,t,u) \geq -\int_{t_1}^{t}\int_{0}^{u}(u-z)g(c+u\xi,\xi,u)g(c+u\xi,\xi,z)dzd\xi$$

$$\text{for } t_1 \leq t \leq t_2 . \tag{68}$$

Using inequality (61) allows us to write

$$\left|\int_{0}^{u}(u-z)g(c+u\xi,\xi,z)dz\right| < \frac{1}{2}u^2M(t_2,u) .$$

Upon letting $N = \frac{1}{2}u^2M(t_2,u)$ and keeping (66) in mind, (68) becomes

$$g(c+ut,t,u) \geq N\int_{t_1}^{t} g(c+u\xi,\xi,u)d\xi . \tag{69}$$

Repeatedly substituting inequality (69) into itself yields

$$\begin{aligned} g(c+ut,t,u) &\geq N^n\int_{t_1}^{t}\int_{t_1}^{\xi_1}\int_{t_1}^{\xi_2}\cdots\int_{t_1}^{\xi_{n-1}} g(c+u\xi_n,\xi_n,u)d\xi_n \cdots d\xi_3\,d\xi_2\,d\xi_1 \\ &\geq N^n\int_{t_1}^{t}\cdots\int_{t_1}^{\xi_{n-1}} -M(t_2,u)d\xi_n \cdots d\xi_1 \\ &\geq -M(t_2,u)N^n\frac{(t-t_1)^n}{n!} . \end{aligned}$$

Upon letting n approach infinity and $t = t_2$, we find

$$g(x,t_2,u) = g(c+ut_2,t_2,u) \geq 0 ,$$

contrary to assumption (64). Thus we have shown

$$g(x,t,u) \geq 0 \qquad \text{for } 0 \leq t \leq W . \tag{70}$$

In the following proofs of existence, nonnegativity, and uniqueness of solutions, we restrict the range of u to $0 \le u \le u_o$, where u_o is arbitrary. After the main body of the proofs, we remove this restriction.

To show the existence of a solution to (57) for $0 \le t \le W$, suppose we have a nonnegative solution for $0 \le t \le t_1$ (when $t_1 = 0$ the solution simply consists of the initial conditions). We show that we can extend this to a range $0 \le t \le t_1 + dt$, where dt is independent of t_1. Thus the solution can be extended to the range $0 \le t \le W$. Let

$$h_1(x,t,u) = g^*(x-ut,u) + \int_0^{t_1}\int_0^{u} f^*(x-ut+u\xi-z\xi,z,u,W-\xi)\cdot e^{-\int_0^{\xi}\int_0^{z}(z-\zeta)g(x-ut+u\xi+z\eta-z\xi,\eta,\zeta)d\zeta d\eta}\, dzd\xi\ ,$$

$$h_2(x,t,z,u,\xi) = f^*(x-ut+u\xi-z\xi,z,u,W-\xi)\cdot e^{-\int_0^{t_1}\int_0^{z}(z-\zeta)g(x-ut+u\xi+z\eta-z\xi,\eta,\zeta)d\zeta d\eta}\ ,$$

$$h_3(x,t,u) = \int_0^{t_1}\int_0^{u}(u-z)g(x-ut+u\xi,\xi,u)g(x-ut+u\xi,\xi,z)dzd\xi\ ,$$

$$h_4(x,t,u) = h_1(x,t,u) - h_3(x,t,u)\ ,$$

$$C = B + AWu_o\ .$$

Equation 57 implies that

$$g(x,t,u) \le B + \int_0^{t}\int_0^{u} A\,dzd\xi - \int_0^{t}\int_0^{u}(u-z)g\ g\,dzd\xi$$

$$\le B + AWu_o = C \qquad \text{for } 0 \le t \le t_1\ . \tag{71}$$

Since g(x,t,u) is known for $0 \le t \le t_1$, h_1, h_2, and h_3 are known for $0 \le t < \infty$. If we define

$$H_1 = B + AWu_o\ , \tag{72}$$

$$H_3 = W u_o^2 C^2 , \qquad (73)$$

$$H_4 = H_1 + H_3 , \qquad (74)$$

we have the following bounds on h_1, h_2, h_3, and h_4.

$$0 \leq h_1 \leq H_1 ,$$
$$0 \leq h_2 \leq A ,$$
$$0 \leq h_3 \leq H_3 ,$$
$$|h_4| \leq H_4 .$$

Let Q_2 be the operator defined by

$$Q_2(g) = \int_{t_1}^{t} \int_{o}^{u} h_2(x,t,z,u,\xi)\, e^{-\int_{t_1}^{\xi} \int_{o}^{z} (z-\zeta) g(x-ut+u\xi+z\eta-z\xi,\eta,\zeta)\, d\zeta d\eta}\, dz d\xi$$

$$-\int_{t_1}^{t} \int_{o}^{u} (u-z) g(x-ut+u\xi,\xi,u) g(x-ut+u\xi,\xi,z)\, dz d\xi . \qquad (75)$$

Thus

$$g(x,t,u) = h_4(x,t,u) + Q_2(g) .$$

Pick C_1 such that

$$C_1 > H_4 . \qquad (76)$$

Let

$$D_4 = e^{W u_o^2 C_1} , \qquad (77)$$

$$D_5 = A D_4 u_o + u_o^2 H_4^2 , \qquad (78)$$

$$D_6 = A D_4 u_o^3 W + 2 C_1 u_o^2 . \qquad (79)$$

Pick $dt > 0$ such that

$$\frac{D_5}{D_6} [e^{D_6 dt} - 1] < C_1 - H_4 .$$

For $t_1 \leq t \leq t_1 + dt$ define

$$g_o(x,t,u) = h_4(x,t,u) , \tag{80}$$

$$g_{n+1}(x,t,u) = h_4(x,t,u) + Q_2(g_n) , \tag{81}$$

$$k_n(x,t,u) = | g_n(x,t,u) - g_{n-1}(x,t,u) | . \tag{82}$$

We use induction to show that

$$| g_n | \leq C_1 \tag{83}$$

and

$$k_n \leq \frac{D_5}{D_6} D_6^n \frac{(t-t_1)^n}{n!} . \tag{84}$$

Clearly,

$$| g_o | = | h_4 | \leq H_4 < C_1 ,$$

$$k_1 \leq \int_{t_1}^{t} \int_{o}^{u} AD_4 + (u-z)H_4^2 \, dz d\xi$$

$$\leq \int_{t_1}^{t} \int_{o}^{u_o} AD_4 + u_o H_4^2 \, dz d\xi = D_5(t - t_1) .$$

Therefore,

$$|g_1| \leq |g_o| + |g_1 - g_o| \leq H_4 + D_5(t-t_1) \leq H_4 + \frac{D_5}{D_6}[e^{D_6(t-t_1)} - 1] < C_1 .$$

Thus (83) and (84) hold for $n = 1$. Assume they hold for $n-1$; we must show they hold for n. We need a bound on the difference of exponential functions of the type occurring in (75). If $y_1 \leq Y$ and $y_2 \leq Y$, then $|e^{y_1} - e^{y_2}| \leq e^Y |y_1 - y_2|$. Also

$$| -\int_o^{\xi} \int_o^{z} (z-\zeta) g_m(x - ut + u\xi + z\eta - z\xi, \eta, \zeta) d\zeta d\eta | \leq \int_o^{W} \int_o^{u_o} u_o C_1 d\zeta d\eta = W u_o^2 C_1$$

$$\text{for } 0 \leq \xi \leq W , \quad 0 \leq z \leq u_o , \quad 0 \leq m \leq n-1 .$$

Hence,

$$\left| e^{-\int_0^{\xi}\int_0^{z}(z-\zeta)g_{n-1}(x-ut+u\xi+z\eta-z\xi,\eta,\zeta)d\zeta d\eta} - e^{-\int_0^{\xi}\int_0^{z}(z-\zeta)g_{n-2}(x-ut+u\xi+z\eta-z\xi,\eta,\zeta)d\zeta d\eta} \right|$$

$$\leq e^{Wu_o^2C_1}\int_0^{\xi}\int_0^{z}(z-\zeta)k_{n-1}(x-ut+u\xi+z\eta-z\xi,\eta,\zeta)d\zeta d\eta . \tag{85}$$

Upon combining (82), (81), (75), (85), (77), and (73), we get

$$k_n(x,t,u) \leq \int_{t_1}^{t}\int_0^{u} AD_4\int_{t_1}^{\xi}\int_0^{z}(z-\zeta)k_{n-1}(x-ut+u\xi+z\eta-z\xi,\eta,\zeta)d\zeta d\eta\, dz d\xi$$
$$+\int_{t_1}^{t}\int_0^{u}(u-z)C_1[\,k_{n-1}(x-ut+u\xi,\xi,u)+k_{n-1}(x-ut+u\xi,\xi,z)\,]dz d\xi . \tag{86}$$

Inequality (84) then yields

$$k_n(x,t,u) \leq \int_{t_1}^{t}\int_0^{u_o} AD_4\int_{t_1}^{\xi}\int_0^{u_o} u_o \frac{D_5}{D_6}\frac{D_6^{n-1}(\eta-t_1)^{n-1}}{(n-1)!}d\zeta d\eta\, dz d\xi$$
$$+\int_{t_1}^{t}\int_0^{u_o} 2u_oC_1\frac{D_5}{D_6}\frac{D_6^{n-1}(\xi-t_1)^{n-1}}{(n-1)!}dz d\xi$$
$$\leq \int_{t_1}^{t}\int_0^{u_o} AD_4\int_0^{W} u_o^2\frac{D_5}{D_6}\frac{D_6^{n-1}(\xi-t_1)^{n-1}}{(n-1)!}d\eta\, dz d\xi + 2C_1u_o^2\frac{D_5}{D_6}\frac{D_6^{n-1}(t-t_1)^{n}}{n!}$$
$$\leq AD_4u_o^3W\frac{D_5}{D_6}\frac{D_6^{n-1}(t-t_1)^{n}}{n!} + 2C_1u_o^2\frac{D_5}{D_6}\frac{D_6^{n-1}(t-t_1)^{n}}{n!}$$
$$\leq \frac{D_5}{D_6}\frac{D_6^{n}(t-t_1)^{n}}{n!} . \tag{87}$$

Also,

$$|g_n| \leq |g_o| + \sum_{i=1}^{n}|g_i-g_{i-1}| = |g_o| + \sum_{i=1}^{n}k_i \leq H_4 + \sum_{i=1}^{n}\frac{D_5}{D_6}\frac{D_6^{i}(t-t_1)^{i}}{i!} \tag{88}$$
$$\leq H_4 + \frac{D_5}{D_6}[e^{D_6(t-t_1)}-1] < C_1 .$$

So (83) and (84) hold for n = n, thus completing the induction. A slight variation of (88) shows that $\{g_n\}$ is a uniform Cauchy sequence. Let $g = \lim_{n\to\infty} g_n$ and note that $g_n \to g$ uniformly. Since $|g_n| \le C_1$, we have $|g| \le C_1$. The uniform convergence implies that g satisfies (57). Combine this g (with range $t_1 \le t \le t_1 + dt$) with the original g (with range $0 \le t \le t_1$) to get a solution to (57) for $0 \le t \le t_1 + dt$. Note that in (80) and (81), $h_4(x,t_1,u) = g(x,t_1,u)$ and $Q_2 = 0$ for $t = t_1$. Hence $g_n(x,t_1,u) = g(x,t_1,u)$. This means that the original and the extended functions g are the same at t_1. Inequality 70 implies g is nonnegative.

Thus there exists a solution to (57) for $0 \le t \le W$. Each of the finite number of extension steps produces a g with $|g| \le C_1$. Hence g $(0 \le t \le W)$ is bounded and nonnegative.

To show that there is only one nonnegative solution to (57), suppose $\tilde{g}$ is another such solution for $0 \le t \le W$. Redefine D_4, D_5, and D_6 by

$$D_4 = e^{Wu_o^2 C},$$

$$D_5 = 2C,$$

$$D_6 = AD_4 u_o^3 W + 2Cu_o^2.$$

As in (71), $g(x,t,u) \le C$ and $\tilde{g}(x,t,u) \le C$. Let

$$k(x,t,u) = |g(x,t,u) - \tilde{g}(x,t,u)| \le 2C = D_5 .$$

By proceeding in a manner similar to the derivation of (86) and by using (57), we get

$$k(x,t,u) \le \int_0^t \int_0^u AD_4 \int_0^\xi \int_0^z (z-\zeta) k(x - ut + u\xi + z\eta - z\xi, \eta, \zeta)\, d\zeta\, d\eta\, dz\, d\xi$$
$$+ \int_0^t \int_0^u (u-z) C [\, k(x - ut + u\xi, \xi, u) + k(x - ut + u\xi, \xi, z)\,]\, dz\, d\xi .$$

In a manner similar to the derivation of (87), we can show that $k \le D_5 t$ and by induction that

$$k \le D_5 \frac{D_6^n t^n}{n!} .$$

Upon letting n approach ∞, we find k=0, hence $g=\tilde{g}$. That is, there is only one nonnegative solution to (57).

Now that we have proven the existence, uniqueness, nonnegativity and boundedness of g for $0\le t\le W$, we do the same for f for $0\le t\le W$. We start with the proof of the existence of f. Let

$$h_5(x,t,s,u) = (u-s)g(x,t,u)g(x,t,s)$$

$$+\, g(x,t,s)\int_s^u\int_t^W (z-s)f^*(x-zt,z,u,T-t)e^{-\int_o^t\int_o^z (z-\eta)g(x+z\xi-zt,\xi,\eta)d\eta d\xi}\, dTdz ,$$

$$h_6(x,t,s,z,T) = (z-s)g(x,t,s)e^{-\int_o^T\int_o^z (z-\eta)g(x+z\xi-zT,t-T+\xi,\eta)d\eta d\xi}$$

for $s\le z\le u$,

$$H_5 = u_o C^2 + u_o^2 CAW ,$$

$$H_6 = u_o C .$$

Then

$$0\le h_5 \le H_5 , \quad 0\le h_6\le H_6 , \tag{89}$$

and (59) becomes

$$f(x,t,s,u,o) = h_5(x,t,s,u) + \int_s^u\int_o^t h_6(x,t,s,z,T)f(x-zT,t-T,z,u,0)\, dTdz .$$

Let

$$f_o(x,t,s,u,o) = h_5(x,t,s,u) , \tag{90}$$

$$f_{n+1}(x,t,s,u,o) = h_5(x,t,s,u) + \int_s^u\int_o^t h_6(x,t,s,z,T)f_n(x-zT,t-T,z,u,0)\, dTdz . \tag{91}$$

$$j_n(x,t,s,u) = |f_n(x,t,s,u,0) - f_{n-1}(x,t,s,u,o)| . \tag{92}$$

Then we have, for $n\ge 2$,

$$j_n(x,t,s,u) = \left|\int_s^u \int_o^t h_6(x,t,s,z,T) j_{n-1}(x-zT,t-T,z,u,)\,dT\,dz\right| .$$

Equations (89), (90), (91), and (92) yield

$$j_1 = \int_s^u \int_o^t h_6 f_o \, dT\,dz \le H_5 H_6 \int_s^u \int_o^W dT\,dz = H_5 H_6 W(u-s) ,$$

and

$$j_n \le H_5 (H_6 W)^n \frac{(u-s)^n}{n!}$$

follows easily by induction. Note that

$$f_n = f_o + \sum_{i=1}^{n} (f_i - f_{i-1}) ,$$

$$|f_i - f_{i-1}| = j_i \le H_5 \frac{(H_6 W u_o)^i}{i!} . \tag{93}$$

Let $f = \lim_{n\to\infty} f_n = f_o + \sum_{i=1}^{\infty}(f_i - f_{i-1})$. Inequality (93) shows that $\sum_{i=1}^{\infty}(f_i - f_{i-1})$ converges absolutely and uniformly, so f is well-defined and bounded by

$$f \le H_5 e^{H_6 W u_o} .$$

The uniform convergence implies that f satisfies (59). The boundedness implies that f is continuous along characteristics. The nonnegativity of f_o, h_5, and h_6 imply each $f_n \ge 0$; hence $f \ge 0$.

To show that in the set of all bounded integrable functions f is unique, suppose that $\tilde{f}$ is another such solution to (59). Pick $\tilde{A}$ such that $|f - \tilde{f}| \le \tilde{A}$. Let

$$\tilde{j}(x,t,s,u) = |f - \tilde{f}| = \left|\int_s^u \int_o^t h_6(x,t,s,z,T) j(x-zT,t-T,z,u)\,dT\,dz\right| .$$

Algebraic manipulations similar to the above show

$$\tilde{j} \le \tilde{A} H_6^n W^n (u-s)^n / n! .$$

Since n can be arbitrarily large, $\tilde{j} = 0$, hence $\tilde{f} = f$. That is, f is unique.

Next we prove existence, uniqueness, boundedness, and nonnegativity for f and g for $t \geq W$. We use induction and suppose f has these properties for $0 \leq t \leq mW$, while g has these properties for $0 \leq t \leq t_1$, where $mW \leq t_1 < (m+1)W$. First we extend the proof of these properties of g to the interval $t_1 \leq t \leq t_1 + dt$, where dt will be defined shortly. Let

$$h_7(x,t,u) = g^*(x-ut,u)$$

$$+\int_0^W \int_0^u f^*(x-ut+u\xi-z\xi,z,u,W-\xi)e^{-\int_0^{\xi}\int_0^{z}(z-\zeta)g(x-ut+u\xi-z\xi+z\eta,\eta,\zeta)d\zeta d\eta} dzd\xi$$

$$+\int_W^{t_1} \int_0^u f(x-ut+u\xi-zW,\xi-W,z,u,o)e^{-\int_0^{W}\int_0^{z}(z-\zeta)g(x-ut+u\xi+z\eta-zW,\xi-W+\eta,\zeta)d\zeta d\eta} dzd\xi$$

$$-\int_0^{t_1} \int_0^u (u-z)g(x-ut+u\xi,\xi,u)g(x-ut+u\xi,\xi,z)\, dzd\xi\ .$$

Equation 58 implies

$$h_7(x,t,u) = g(x-ut+ut_1,t_1,u) \geq 0\ .$$

By the inductive assumption that f is bounded, suppose

$$f(x,t,s,u,0) \leq A_1 \quad \text{for} \quad 0 \leq t \leq mW\ .$$

Let

$$H_7 = B + Wu_oA + mWu_oA_1$$

then

$$0 \leq h_7 \leq H_7\ .$$

Let

$$h_8(x,t,u,\xi,z) = f(x-ut+u\xi-zW,\xi-W,z,u,0)\ .$$

$$e^{-\int_0^{t_1+W-\xi}\int_0^{z}(z-\zeta)g(x-ut+u\xi+z\eta-zW,\xi-W+\eta,\zeta)d\zeta d\eta}$$

$$\text{for} \quad t_1 \leq \xi \leq t \leq (m+1)W\ , \quad 0 \leq z \leq u\ ,$$

and note that

$$h_8 \leq A_1 .$$

Let Q_3 be the operator defined by

$$Q_3(g) = \int_{t_1}^{t} \int_{o}^{u} h_8 e^{-\int_{t_1}^{\xi} \int_{o}^{z} (z-\zeta) g(x-ut+u\xi+z\eta-z\xi,\eta,\zeta) d\zeta d\eta} dz d\xi$$

$$-\int_{t_1}^{t} \int_{o}^{u} (u-z) g(x-ut+u\xi,\xi,u) g(x-ut+u\xi,\xi,z) \, dz d\xi .$$

Then (58) becomes

$$g(x,t,u) = h_7 + Q_3(g) .$$

Pick C_2 such that $H_7 < C_2$ and define

$$D_7 = e^{W u_o^2 C_2} ,$$

$$D_8 = A_1 u_o + C_2^2 u_o^2 ,$$

$$D_9 = A_1 D_7 u_o^3 W + 2 C_2 u_o^2 .$$

Restrict t to the range $t_1 \leq t \leq t_1 + dt$ where $dt > 0$ and satisfies

$$\frac{D_8}{D_9}(e^{D_9 dt} - 1) \leq C_2 - H_7 .$$

Define

$$g_o = h_7 ,$$

$$g_{n+1} = h_7 + Q_3(g) ,$$

$$k_n = |g_n - g_{n-1}| .$$

It is easily seen that

$$k_1 \leq \int_{t_1}^{t} A_1 u_o + C_2^2 u_o^2 d\xi = (A_1 u_o + C_2^2 u_o^2)(t - t_1) = D_8 (t - t_1) ,$$

$$|g_o| \leq H_7 .$$

To show that

$$k_n(x,t,u) \le \frac{D_8}{D_9}\frac{D_9^n(t-t_1)^n}{n!}, \tag{94}$$

$$|g_n| \le C_2\,, \tag{95}$$

we note that these are true for n=1, and we then use induction. Assuming they hold for n-1, we have

$$k_n = |Q_3(g_{n-1}) - Q_3(g_{n-2})|\,,$$

$$k_n \le \int_{t_1}^{t}\int_{o}^{u_o} A_1D_7\int_{t_1}^{\xi}\int_{o}^{u_o}\frac{u_oD_8}{D_9}\frac{D_9^{n-1}(\eta-t_1)^{n-1}}{(n-1)!}d\zeta d\eta dz d\xi$$

$$+\int_{t_1}^{t}\int_{o}^{u_o} u_o 2C_2\frac{D_8}{D_9}\frac{D_9^{n-1}(\xi-t_1)^{n-1}}{(n-1)!}dz d\xi$$

$$\le \int_{t_1}^{t} A_1D_7u_o^3W\frac{D_8}{D_9}\frac{D_9^{n-1}(\xi-t_1)^{n-1}}{(n-1)!}d\xi + 2C_2u_o^2\frac{D_8}{D_9}\frac{D_9^{n-1}(t-t_1)^n}{n!}$$

$$\le \frac{D_8}{D_9}\frac{D_9^n(t-t_1)^n}{n!}$$

$$|g_n| \le |g_o| + \sum_{i=1}^{n}|g_i - g_{i-1}| = |g_o| + \sum_{i=1}^{n}k_i \le H_7 + \frac{D_8}{D_9}\sum_{i=1}^{n}\frac{D_9^i(t-t_1)^i}{i!} \tag{96}$$

$$\le H_7 + \frac{D_8}{D_9}(e^{D_9dt}-1) \le C_2\,.$$

This completes the induction on n, thus proving (94) and (95). A slight variation of (96) shows that $\{g_n\}$ is a uniform Cauchy sequence. Let $g = \lim_{n\to\infty} g_n$. Since $g_n \to g$ uniformly, g satisfies (58) for $t_1 \le t \le t_1 + dt$. When we note that (68) follows from (58), we see that a proof similar to the proof of nonnegativity of g for $0 \le t \le W$ holds for $t_1 \le t \le t_1 + dt$. Equation (58) and the nonnegativity of g yield

$$0 \le g \le H_7\,.$$

The proof that there is only one nonnegative solution to (58) is analogous to the uniqueness proof for $0\le t\le W$.

Thus we have extended the range of the existence, etc., proofs from $0\le t\le t_1$ to $0\le t\le t_1+dt$. Since dt is independent of t_1, we can, in a finite number of steps of size dt, extend the range to $0\le t\le (m+1)W$.

To extend the range of proofs for f from $0\le t\le mW$ to $0\le t\le (m+1)W$, we note that, with h_5 and h_6 redefined, (60) has the form

$$f(x,t,s,u,o) = h_5(x,t,s,u) + \int_s^u\int_o^W h_6(x,t,s,z,T)f(x-zT,t-T,z,u,o)dTdz \qquad (97)$$

The functions h_5 and h_6 are bounded,

$0\le h_5\le H_5 \quad = \text{constant},$

$0\le h_6\le H_6 \quad = \text{constant}.$

Equation 97 has the same form as (59) except $\int_o^t$ is replaced by $\int_o^W$. In the existence, etc., proofs for f for $0\le t\le W$, if we replace $\int_o^t$ by $\int_o^W$ we find: There exists a solution f for $0\le t\le (m+1)W$, $f\ge 0$ since each $f_n\ge 0$, $0\le f\le |f_o|+\sum|f_i-f_{i-1}| \le H_5e^{H_6W(u-s)}$, and there is only one bounded solution to (60).

In all the above proofs, u was restricted to $u\le u_o$. We now remove that restriction. Since u_o is arbitrary, we could simply take u_o to be the maximum desired speed of any driver so the proofs hold for the range of interest. But, for mathematical convenience, one might want to allow u to be arbitrarily large as, for example, by setting the distribution of desired speeds to be proportional to $\exp(-(u-\bar{u})^2/2\sigma^2)$ for $0\le u<\infty$. In this case note that f and g, for $u=\tilde{u}$, depend only on f,g,f^*,g^* for $u\le \tilde{u}$. So we may pick a sequence

$u_o < u_1 < \cdots < u_n < \cdots$ such that $u_n \to \infty$, and let f_n, g_n be the solution for $u \leq u_n$. For $m < n$ the restriction of f_n, g_n to $0 \leq u \leq u_m$ is a solution to (57)-(60). But bounded solutions are unique; so define

$$f(x,t,s,uT) = f_n(x,t,s,u,T) = f_{n+1}(x,t,s,u,T) = \cdots ,$$

$$g(x,t,u) = g_n(x,t,u) = g_{n+1}(x,t,u) = \cdots ,$$

where n satisfies $u_n > u$. Then f and g satisfy (57)-(60), are nonnegative, are unique, are bounded within any finite range of t and u, and are continuous along characteristics.

9. ASSOCIATED EQUATIONS

We now consider several equations that may be derived from (34), (41) and (42). First, since these equations are based on the conservation of cars, we expect that the usual conservation equation,

$$k_t + q_x = 0 \tag{98}$$

in terms of the density k and the flow q, may be obtained from them. The variables k and q are functions of position and time only, so

$$k(x,t) = \int_o^{\infty} g(x,t,u)du + \int_o^{W} \int_o^{\infty} \int_o^{u} f(x,t,s,uT)dsdudT\,, \tag{99}$$

$$q(x,t) = \int_o^{\infty} ug(x,t,u)du + \int_o^{W} \int_o^{\infty} \int_o^{u} sf(x,t,s,u,T)dsdudT\,. \tag{100}$$

Hence,

$$k_t + q_x = \int_o^{\infty} g_t + ug_x du + \int_o^{W} \int_o^{\infty} \int_o^{u} f_t + sf_x dsdudT\,. \tag{101}$$

Upon substituting (34) and (42) in (101), we get

$$\begin{aligned} k_t + q_x = &\int_o^{\infty} \int_o^{u} [\, f(x,t,z,u,W) - (u-z)g(x,t,u)g(x,t,z)\,]\, dzdu \\ &+ \int_o^{\infty} \int_o^{u} \int_o^{W} [-f_T(x,t,s,u,T) \\ &- f(x,t,s,u,T) \int_o^{s} (s-z)g(x,t,z)dz\,]\, dTdsdu\,, \end{aligned} \tag{102}$$

and using (41) yields

$$\begin{aligned} k_t + q_x = &\int_o^{\infty} \int_o^{u} [\, f(x,t,z,u,W) - f(x,t,z,u,o) \\ &+ g(x,t,z) \int_z^{u} \int_o^{W} (\eta - z) f(x,t,\eta,u,T)dTd\eta\,]\, dzdu \\ &- \int_o^{\infty} \int_o^{u} [\, f(x,t,s,u,W) - f(x,t,s,u,o)\,]dsdu \\ &- \int_o^{\infty} \int_o^{u} \int_o^{W} \int_o^{s} f(x,t,s,u \;\text{[illegible]}\; s-z)g(x,t,z)dzdTdsdu\,, \end{aligned}$$

which after a little manipulating becomes (98).

Thus we have a check on the reasonableness of our equations.

A feature of our model is that if no one has a desired speed less than some speed u_1, then for $t \geq W$, no one has an actual speed less than u_1. To see this, suppose for $s \leq u < u_1$ we have

$$g(x,t,u) = 0 ,$$

$$f(x,t,s,u,T) = 0 .$$

Then (59) and (60) show that for $s < u_1 \leq u$

$$f(x,t,s,u,0) = 0 ,$$

and (53) shows that

$$f(x,t,s,u,T) = 0 .$$

Thus, after a time W after the initial distribution, no one has speed less than u_1.

We can use this fact to translate the variables u and s by u_1 and the variable x by $u_1 t$. This will show that u and s operate as "small" quantities in our equations when they are near u_1. While we are transforming variables, we observe that time can be scaled to eliminate the waiting time W as a variable in the transformed equations. The combined transformation is

$$\bar{x} = x - u_1 t ,$$

$$\bar{t} = t/W ,$$

$$\bar{T} = T/W ,$$

$$\bar{s} = W(s - u_1) , \qquad (103)$$

$$\bar{u} = W(u - u_1) ,$$

$$\bar{g}(\bar{x},\bar{t},\bar{u}) = \frac{1}{W} g(x,t,u) ,$$

$$\bar{f}(\bar{x},\bar{t},\bar{s},\bar{u},\bar{T}) = \frac{1}{W} f(x,t,s,u,T) .$$

The resulting equations are identical to (34), (41) and (42), with barred variables replacing unbarred variables and with W set equal to 1.

Some of the terms in (34) and (41) represent multiple delays; that is, while a driver is at less than his desired speed, he is forced to reduce his speed still further. These terms are the ones which contain the product of an f and a g. To simplify our equations, we delete these terms. The solution to the simplified equations should approximate the solution to the original equations. We expect the approximation to be particularly good for light traffic or when the waiting time to pass, W, is small. To be more specific, let k_o be a typical density of cars, and let u_o be a typical difference in speed between two cars. Then $(u_o k_o)^{-1}$ is a typical time for one car to catch up with another. If $W << (u_o k_o)^{-1}$, a second delay is unlikely, and the terms representing multiple delays will be small. We expect that some general properties possessed by the simplified system are shared by the original system. One such property, which we investigate later, is that of the solution approaching equilibrium as time approaches infinity.

The equations without multiple delays are

$$f_t(x,t,s,u,T) = -sf_x - f_T , \qquad (104)$$

$$f(x,t,s,u,o) = (u-s)g(x,t,u)g(x,t,s) , \qquad (105)$$

$$g_t(x,t,u) = -ug_x + \int_o^u f(x,t,z,u,W) - (u-z)g(x,t,u)g(x,t,z)dz , \qquad (106)$$

$$f(x,o,s,u,T) = f^*(x,s,u,T) , \qquad (107)$$

$$g(x,o,u) = g^*(x,u) . \qquad (108)$$

We now proceed to partially integrate the system of equations, (104)-(108), obtaining a pair of integro-difference equations in g.

Equation 104 is a first order, linear, partial differential equation in f. The characteristics, with parameter y, are given by the system of equations

$$\frac{dt}{dy} = 1 , \qquad \frac{dx}{dy} = s , \qquad \frac{dT}{dy} = 1 .$$

Along these characteristics we have

$$\frac{df}{dy} = 0 .$$

Referring to Figure 8 we see

$$f(x,t,s,u,T) = \begin{cases} f^*(x-st,s,u,T-t) & 0 \le t \le T \\ f(x-sT,t-T,s,u,o) & t \ge T . \end{cases} \tag{109}$$

Equations 105 and 109 combine to give

$$f(x,t,s,u,T) = \begin{cases} f^*(x-st,s,u,T-t) & 0 \le t \le T \\ (u-s)g(x-sT,t-T,u)g(x-sT,t-T,s) & t \ge T . \end{cases} \tag{110}$$

Substituting (110) in (106) yields

$$g_t(x,t,u) + ug_x + \int_o^u (u-z)g(x,t,u)g(x,t,z)dz$$

$$= \begin{cases} \int_o^u f^*(x-zt,z,u,W-t)dz & 0 \le t < W \\ \int_o^u (u-z)g(x-zW,t-W,u)g(x-zW,t-W,z)dz & t > W . \end{cases} \tag{111}$$

Along the characteristics

$$t = y , \qquad x = x_o + uy ,$$

and with G defined by (49), (111) becomes

$$\frac{dg}{dy} + gG = \begin{cases} \int_o^u f^*(x-zt,z,u,W-t)dz & 0 \le t < W \\ \int_o^u (u-z)g(x-zW,t-W,u)g(x-zW,t-W,z)dz & t > W . \end{cases}$$

Hence,

$$g(x_o+uy,y,u) = e^{-\int_o^y G(x_o+u\eta,\eta,u)d\eta}\cdot\{g^*(x_o,u)+\int_o^y e^{\int_o^\xi G(x_o+u\eta,\eta,u)d\eta}\int_o^u f^*(x_o+u\xi-z\xi,z,u,W-\xi)dzd\xi\}$$

$$0\le y\le W,$$

$$g(x_o+uy,y,u) = e^{-\int_o^y G(x_o+u\eta,\eta,u)d\eta}\{g^*(x_o,u)+$$

$$\int_o^W e^{\int_o^\xi G(x_o+u\eta,\eta,u)d\eta}\int_o^u f^*(x_o+u\xi-z\xi,z,u,W-\xi)dzd\xi+$$

$$\int_W^y e^{\int_o^\xi G(x_o+u\eta,\eta,u)d\eta}\int_o^u (u-z)g(x_o+u\xi-zW,\xi-W,u)\cdot$$

$$g(x_o+u\xi-zW,\xi-W,z)dzd\xi\}\quad y\ge W.$$

Upon changing variables and using (49), we get

$$g(x,t,u) = e^{-\int_o^t\int_o^u(u-\zeta)g(x-ut+u\eta,\eta,\zeta)d\zeta d\eta}g^*(x-ut,u)+$$

$$\int_o^t\int_o^u e^{-\int_\xi^t\int_o^u(u-\zeta)g(x-ut+u\eta,\eta,\zeta)d\zeta d\eta}f^*(x-ut+u\xi-z\xi,z,u,W-\xi)dzd\xi$$

$$0\le t\le W, \tag{112}$$

$$g(x,t,u) = e^{-\int_o^t\int_o^u(u-\zeta)g(x-ut+u\eta,\eta,\zeta)d\zeta d\eta}g^*(x-ut,u)+$$

$$\int_o^W\int_o^u e^{-\int_\xi^t\int_o^u(u-\zeta)g(x-ut+u\eta,\eta,\zeta)d\zeta d\eta}f^*(x-ut+u\xi-z\xi,z,u,W-\xi)dzd\xi+$$

$$\int_W^t\int_o^u e^{-\int_\xi^t\int_o^u(u-\zeta)g(x-ut+u\eta,\eta,\zeta)d\zeta d\eta}\cdot$$

$$(u-z)g(x-ut+u\xi-zW,\xi-W,u)g(x-ut+u\xi-zW,\xi-W,z)dzd\xi$$

$$t\ge W. \tag{113}$$

The proof that there exists a nonnegative solution to (112) and (113) and that this solution is unique, bounded for finite t and u, and continuous along characteristics is accomplished easily by taking t in steps of size W and using the

Picard iteration technique. Once g is known, f follows from (110). The above properties hold for f and, in addition, f is constant along characteristics.

10. CONSTANT-SPEED, CONSTANT-SHAPE SOLUTIONS

Up to this point we have required the densities f and g to be bounded ordinary functions. If we allow δ functions, we find there are some solutions with a property which is not shared by ordinary function solutions. In particular, we looked for solutions to the equations without multiple delays which travelled at constant speed with constant shape. When we let u_1 be the minimum desired speed, we prove that there is a range of speeds u_1 to u_1^* for which the only constant-speed, constant-shape, bounded, ordinary function solutions are also independent of x. Then we exhibit a constant-speed, constant-shape solution which involves δ functions.

To show the lack of constant-speed c, constant-shape, ordinary function solutions, we substitute

$$g(x,t,u) = h(x-ct,u) \tag{114}$$

in (111) for $t \geq W$ to get

$$(u-c)h_x(x-ct,u) + \int_{u_1}^{u} (u-z)h(x-ct,u)h(x-ct,z)dz$$
$$= \int_{u_1}^{u} (u-z)h(x-ct-zW+cW,u)h(x-ct-zW+cW,z)dz$$

where the lower limit of integration is u_1 since, for $z < u_1$, $h(,z) = g(, ,z) = 0$. Without loss of generality, we may set $t = 0$ (it is equivalent to renaming the quantity $x - ct$) which yields

$$(u-c)h_x(x,u) = \int_{u_1}^{u} (u-z)\left[-h(x,u)h(x,z) + h(x-zW+cW,u)h(x-zW+cW,z) \right] dz . \tag{115}$$

Now define

$$h^*(u) = h(o,u) , \tag{116}$$

$$k(x,u) = | h(x,u) - h^*(u) | . \tag{117}$$

Note that $h(x,u) = h^*(u)$ is a solution to (115). At this time we limit the range of u to

$$u_1 \le u \le u_1^* \tag{118}$$

where u_1^* will be chosen later. However, we can say now that if $u_1 < c$, then $u_1^* < c$. We can integrate (115) and combine with (117) to get

$$k(x,u) \le \frac{1}{|u-c|} \left| \int_0^x \int_{u_1}^u (u-z) [h(\xi,z)k(\xi,u) + h^*(u)k(\xi,z) \right.$$

$$\left. + h(\xi - zW + cW, z)k(\xi - zW + cW, u) + h^*(u)k(\xi - zW + cW, z)] dz d\xi \right| .$$

Since we are limiting our considerations to functions h which are bounded,

$$h(x,u) < D = \text{constant.} \tag{119}$$

Hence

$$k(x,u) < 2D \tag{120}$$

and

$$k(x,u) \le \frac{D}{|u-c|} \left| \int_0^x \int_{u_1}^u (u-z) [k(\xi,u) + k(\xi,z) + k(\xi - zW + cW, u) \right.$$

$$\left. + k(\xi - zW + cW, z)] dz d\xi \right| . \tag{121}$$

We now show by induction that when $u_1 \ne c$,

$$k(x,u) \le 2D \left(\frac{4D(u_1^* - u_1)^2}{C_1} \right)^n \frac{[|x| + (n-1)C_2 W]^n}{n!} \quad \text{for } u_1 \le u \le u_1^* \tag{122}$$

where

$$C_1 = \begin{cases} u_1 - c & \text{if } c < u_1 \\ c - u_1^* & \text{if } c > u_1 \end{cases}$$

$$C_2 = \begin{cases} u_1^* - c & \text{if } c < u_1 \\ c - u_1 & \text{if } c > u_1. \end{cases}$$

By (120), (122) holds for n=0. Assume (122) holds for n=n, and substitute (122) in (121) to get

$$k(x,u) \le \frac{D}{C_1} \left| \int_o^x \int_{u_1}^{u_1^*} (u_1^* - u_1) 2D \left(\frac{4D(u_1^* - u_1)^2}{C_1} \right)^n \cdot \right.$$

$$\left\{ \frac{(|\xi| + (n-1)C_2W)^n}{n!} + \frac{(|\xi| + (n-1)C_2W)^n}{n!} + \right.$$

$$\left. \left. \frac{(|\xi - zW + cW| + (n-1)C_2W)^n}{n!} + \frac{(|\xi - zW + cW| + (n-1)C_2W)^n}{n!} \right\} dz d\xi \right|$$

$$\le \frac{2D^{n+2}4^n(u_1^* - u_1)^{2n+1}}{C_1^{n+1} n!} \left| \int_o^x \int_{u_1}^{u_1^*} 4(|\xi| + nC_2W)^n dz d\xi \right|$$

$$k(x,u) \le 2D \left(\frac{4D(u_1^* - u_1)^2}{C_1} \right)^{n+1} \frac{1}{n!} \int_o^{|x|} (\xi + nC_2W)^n d\xi$$

$$\le 2D \left(\frac{4D(u_1^* - u_1)^2}{C_1} \right)^{n+1} \frac{(|x| + nC_2W)^{n+1})}{(n+1)!}.$$

This completes the proof by induction of (122) for $u_1 \ne c$. When $u_1 = c$, the factor $(u-z)/|u-c|$ in (121) is less than 1 and so may be deleted without invalidating the inequality. The rest of the proof still holds with $C_1 = 1$, $C_2 = u_1^* - c$ and the factor $(u_1^* - u_1)^2$ replaced by $(u_1^* - u_1)^1$. From Sterling's approximation for the factorial function, we have

$$\frac{(|x| + (n-1)C_2W)^n}{n!} \sim \frac{n^n (C_2W)^n \left[1 + \frac{1}{n} \left(\frac{|x|}{C_2W} - 1 \right) \right]^n}{(2\pi)^{1/2} n^{n+1/2} e^{-n}} \quad \text{as } n \to \infty$$

$$\sim \frac{(C_2We)^n e^{\frac{|x|}{C_2W} - 1}}{(2\pi n)^{1/2}} \quad \text{as } n \to \infty .$$

For n large enough we have

$$\frac{(|x| + (n-1)C_2W)^n}{n!} < \frac{2(C_2We)^n e^{\frac{|x|}{C_2W} - 1}}{(2\pi n)^{1/2}} .$$

Hence (122) becomes

$$k(x,u) \leq \left(\frac{8}{\pi n}\right)^{1/2} De^{\frac{|x|}{C_2W} - 1} E^n \quad \begin{array}{l} \text{for } u_1 \leq u \leq u_1^* \text{ when } u_1 \neq c \\ \text{for } u_1 < u \leq u_1^* \text{ when } u_1 = c \end{array}$$

where

$$E = \begin{cases} \dfrac{4D(u_1^* - u_1)^2 C_2We}{C_1} & u_1 \neq c \\ 4D(u_1^* - u_1) C_2We & u_1 = c . \end{cases}$$

As $u_1^* \to u_1 +$, C_1 and C_2 are bounded above, and C_1 is bounded away from 0, so we may pick $u_1^* > u_1$ such that $E < 1$. Since n can be arbitrarily large, this implies $k(x,u) = 0$ for $u_1 < u \leq u_1^*$. Thus

$$g(x,t,u) = h(x - ct,u) = h^*(u) \quad \text{for } u_1 < u \leq u_1^* ,$$

and (110) for $t \geq T$ yields

$$f(x,t,s,u,T) = (u - s)h^*(u)h^*(s) \quad \text{for } u_1 < s \leq u \leq u_1^* .$$

When $u = u_1 = c$, any function h satisfies (115), but since $u = u_1$ is a set of measure zero, it represents no cars. This concludes the proof that any constant-speed, constant-shape, bounded, ordinary function solution to (110) and (111) has a range of speeds, $u_1 < s \leq u \leq u_1^*$, for which the solution is independent of x and t.

11. DELTA FUNCTION SOLUTIONS

In this section we find some delta function solutions which exhibit some features of the model. Solutions which are discrete in the velocities are of interest for the mathematical simplification they can bring and the fact that a numerical solution will have discrete velocities. Also, in real traffic some drivers will come up behind a slower driver and not pass for several or many minutes. The use of discrete velocities is a step in the direction of representing this phenomenon.

We start with the equations without multiple delays, (109), (110), and (111), and we let t have the range $-\infty < t < \infty$. We then use only the second half of each of these equations which will hold in the range $-\infty < t < \infty$. We integrate (111) along the characteristics

$$t = y, \qquad x = x_o + uy$$

getting

$$\begin{aligned} g(x_o + uy, y, u) = g(x_o, 0, u) \\ &- \int_o^y \int_o^{u-} (u-z) g(x_o + ur, r, u) g(x_o + ur, r, z) dz dr \\ &+ \int_o^y \int_o^{u-} (u-z) g(x_o + ur - zW, r - W, u) \cdot g(x_o + ur - zW, r - W, z) dz dr, \end{aligned}$$

$$\begin{aligned} g(x,t,u) = g(x-ut, 0, u) \\ &- \int_o^t \int_o^{u-} (u-z) g(x-ut+ur, r, u) g(x-ut+ur, r, z) dz dr \\ &+ \int_o^t \int_o^{u-} (u-z) g(x-ut+ur-zW, r-W, u) g(x-ut+ur-zW, r-W, z) dz dr. \end{aligned} \tag{123}$$

The upper limit of the integration with respect to z has been changed from u to u- to reflect the fact that u cars are not delayed by z cars unless $z < u$. Of course, this only makes a difference if we have an integrand of the type $\delta(z-u)$. We now try a solution to (123) of the form

$$g(x,t,u) = g_{10}\delta(x-u_1t)\delta(u-u_1) + g_2(x,t)\delta(u-u_2) \tag{124}$$

where $u_1 < u_2$ and g_{10} is a constant. When we substitute (124) in (123), the terms of the form

$$\int_0^{u_1-} \delta(z-u_1)dz\,,\ \int_0^{u_1-}\delta(z-u_2)dz\,,\ \text{and} \int_0^{u_2-}\delta(z-u_2)dz$$

vanish leaving

$$g_{10}\delta(x-u_1t)\delta(u-u_1) + g_2(x,t)\delta(u-u_2) =$$

$$g_{10}\delta(x-ut)\delta(u-u_1) + g_2(x-ut,0)\delta(u-u_2)$$
$$-\int_0^t\int_0^{u-}(u-z)g_2(x-ut+ur,r)\delta(u-u_2)g_{10}\delta(x-ut+ur-u_1r)\delta(z-u_1)dzdr$$
$$+\int_0^t\int_0^{u-}(u-z)g_2(x-ut+ur-zW,r-W)\delta(u-u_2)\cdot$$
$$g_{10}\delta(x-ut+ur-zW-u_1(r-W))\,\delta(z-u_1)dzdr\,.$$

Separate this into two equations, valid near $u=u_1$ and $u=u_2$ respectively, to get

$$g_{10}\delta(x-u_1t) = g_{10}\delta(x-u_1t)\,, \tag{125}$$

$$g_2(x,t) = g_2(x-u_2t,0) + \begin{cases} -\dfrac{(u_2-u_1)}{(u_2-u_1)}g_2(x-u_2t+u_2(\dfrac{u_2t-x}{u_2-u_1}),\ \dfrac{u_2t-x}{u_2-u_1})g_{10} \\ +\dfrac{(u_2-u_1)}{(u_2-u_1)}g_2(x-u_2t+u_2(\dfrac{u_2t-x}{u_2-u_1})-u_1W,\ \dfrac{u_2t-x}{u_2-u_1}-W)g_{10} \\ \quad \text{for } x-u_2t < 0 < x-u_1t \\ 0 \quad \text{otherwise}\,. \end{cases} \tag{126}$$

Equation 125 is satisfied for all values of g_{10}, while (126) simplifies to

$$g_2(x,t) = g_2(x-u_2t,0) + \begin{cases} -g_{10}g_2(u_1(\dfrac{u_2t-x}{u_2-u_1}),\ \dfrac{u_2t-x}{u_2-u_1}) \\ +g_{10}g_2(u_1(\dfrac{u_2t-x}{u_2-u_1}-W),\ \dfrac{u_2t-x}{u_2-u_1}-W) \\ \quad \text{for } u_1t < x < u_2t \\ 0 \quad \text{otherwise}\,. \end{cases} \tag{127}$$

Figure 9 shows the paths of cars in (127). The number 1 indicates cars represented by the first term on the right-hand side of (127), etc. The second

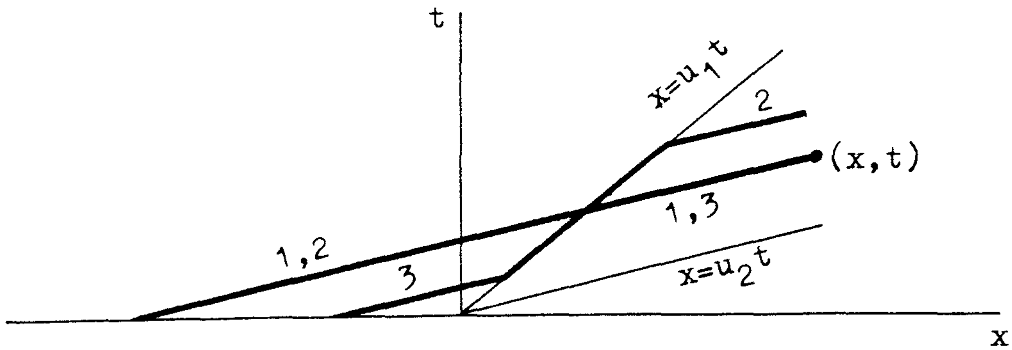


Figure 9. Paths of cars for $g = g_{10}\delta(x - u_1 t)\delta(u - u_1) + g_2(x,t)\delta(u - u_2)$.

term represents cars being delayed.

For simplicity in the following, we talk about "cars" when it would be more accurate to use the phrase "expected value of the density of cars." If $g_{10} = 1$ (representing one u_1 car), then every u_2 car which meets a u_1 car gets delayed. This is exactly what happens in the discrete model from which we derived our continuous model.

If, however, $g_{10} < 1$, then only the fraction g_{10} of the u_2 cars are delayed, while the fraction $1 - g_{10}$ pass freely. If farther down the road is the rest of the u_1 car, then the fraction $1 - g_{10}$ of the delayed u_2 cars will be delayed again, and the same fraction of undelayed u_2 cars will be delayed. The sum of the delayed u_2 cars will be $g_2 dx$, but this will be achieved by some u_2 cars not being delayed, some being delayed once, and some being delayed twice. If we approximate a continuous distribution of u_1 cars by a sum of a large number of delta functions, we see that some u_2 cars (or fractions of u_2 cars) will not be delayed at all, while others will be delayed once, twice, thrice, etc. Thus the passing process has a definite dispersive effect on the u_2 cars.

If $g_{10} > 1$, the second term of (127) represents more than 100 % of the u_2 cars being delayed; in fact, it is possible for g_2 to become negative. This difficulty can easily be avoided by not putting more than one u_1 car at a point on the road.

If we set $g_2(x,t)$ equal to an arbitrary constant g_{20}, we find that (127) is satisfied. Thus a constant speed, constant shape solution to our equations without multiple delays is

$$g(x,t,u) = g_{10}\delta(x-u_1t)\delta(u-u_1) + g_{20}\delta(u-u_2)\,.$$

A more general solution is

$$g(x,t,u) = \sum_{k=1}^{n} g_{10k}\delta(x-x_k-u_1t)\delta(u-u_1) + g_{20}\delta(u-u_2)\,.$$

Next we consider what happens if we do not allow passing. This restriction, when coupled with discrete speeds, allows us to see the delay process more clearly in mathematical terms. It is also of interest on one-lane roads where passing is not possible. The only changes in our equations (with multiple interactions) are that $W=\infty$ and (42) becomes

$$g_t(x,t,u) + ug_x = -g(x,t,u)\int_0^u (u-z)g(x,t,z)dz\,. \tag{128}$$

Along characteristics this becomes

$$\frac{dg}{dy}(x_o+uy,t_o+y,u) = -g(x_o+uy,t_o+y,u)\int_0^{u-}(u-z)g(x_o+uy,t_o+y,z)dz\,. \tag{129}$$

We restrict the speeds to the discrete set $u_1 < u_2 < \cdots < u_n$, so g has the form

$$g(x,t,u) = \sum_{i=1}^{n} g_i(x,t)\delta(u-u_i)\,. \tag{130}$$

In (129) we discard terms of the form $\delta(u-u_i)\delta(z-u_j)$, where $i \le j$, since slow cars are not delayed by cars with equal or greater speed. Thus,

$$\sum_{i=1}^{n} \frac{dg_i(x_o+uy,t_o+y)}{dy}\delta(u-u_i) =$$

$$-\int_0^{u-}\sum_{i=2}^{n}\sum_{j=1}^{i-1}(u-z)g_i(x_o+uy,t_o+y)\delta(u-u_i)g_j(x_o+uy,t_o+y)\cdot\delta(z-u_j)dz\,. \tag{131}$$

Separating (131) into a separate equation for each u_i and performing the integration with respect to z gives

$$\left.\begin{aligned} \frac{dg_1(x_o+u_1y,t_o+y)}{dy} &= 0 \\ \frac{dg_i(x_o+u_iy,t_o+y)}{dy} &= -\sum_{j=1}^{i-1}(u_i-u_j)g_i(x_o+u_iy,t_o+y)\cdot g_j(x_o+u_iy,t_o+y) \\ & \qquad i=2,\cdots n. \end{aligned}\right\} \tag{132}$$

For each g_i this is a first-order, linear, ordinary differential equation. For $t_o=0$ the solution is

$$g_1(x,t)=g_1(x-u_1t,0)$$

$$g_i(x,t)=g_i(x-u_it,0)e^{-\int_0^t\sum_{j=1}^{i-1}(u_i-u_j)g_j(x-u_it+u_iy,y)dy} \tag{133}$$

$$i=2,\cdots,n.$$

Thus no u_1 cars are delayed, and the spatial distribution of the u_1 cars does not distort; it moves with velocity u_1. The density of the u_i cars depends only on the initial conditions and the densities of slower cars. Also, g_i experiences a complex exponential decay being more affected by cars with a greater difference in speed than with a lesser difference. If typical values of g_j for $j<i$ exist, an approximate decay "constant" for g_i could be determined from (133). A simple solution is obtained by considering only two speeds with the initial conditions

$$g_1(x,0)=\begin{cases}0 & x<0\\ g_{10} & x>0\end{cases}$$

$$g_2(x,0)=\begin{cases}g_{20} & x<0\\ 0 & x>0.\end{cases}$$

The solution is

$$g_1(x,t) = \begin{cases} 0 & x-u_1t < 0 \\ g_{10} & x-u_1t > 0 . \end{cases}$$

$$g_2(x,t) = \begin{cases} 0 & u_2t < x \\ g_{20} & x < u_1t \\ g_{20}\, e^{-(x-u_1t)g_{10}} & u_1t < x < u_2t . \end{cases}$$

This is just the simple attenuation that occurs in many physical processes when energy is absorbed as it travels through a medium. Examples are: ultraviolet light travelling through water, γ-rays travelling through a solid material, or sound waves travelling through a porous material. The attenuation is adjusted for the fact that the medium is moving. Equation 133 represents the same type of attenuation occurring in many media simultaneously, each with its own velocity.

We can check the accuracy of the model, focusing on the delay terms, by comparing the solution with the following discrete model. The discrete model has cars entering the road at $x=0$ with a time interval h between successive cars. The speed of the cars is chosen randomly from an approximately normal discrete distribution. Let

$$P(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(u-\mu)^2}{2\sigma^2}}\, du ,$$

$$2Du = u_{j+1} - u_j ,$$

p_j = probability that an entering car has speed u_j, $j = 1, \cdots ,n,$

$$p_j = \frac{P(u_j + Du) - P(u_j - Du)}{P(u_n + Du) - P(u_1 - Du)} .$$

Suppose that car number 0 leaves at time 0 with speed u, while the i^{th} preceding car left at time -hi with speed U_i. The variables U_i are random, while u is fixed. Car 0 does not catch up with car i by the time car 0 reaches position x if and only if

$$\frac{x}{U_i} < \frac{x}{u} + hi \quad \text{or equivalently } U_i > \frac{1}{\frac{1}{u} + \frac{hi}{x}} .$$

Let

$$R_i(x) = \text{Prob}\left\{ U_i > \frac{1}{\frac{1}{u} + \frac{hi}{x}} \right\},$$

$$\begin{aligned} R(x) &= \text{Prob}\,\{\text{car 0 does not catch up with any car} \\ &\qquad \text{by the time it reaches position } x\,\} \\ &= \prod_{i=1}^{m} R_i(x), \end{aligned}$$

where

$$m = [\,(\frac{1}{u_1} - \frac{1}{u})\,\frac{x}{h}]_> - 1\,,$$

$[y]_>$ = smallest integer that is greater than y.

For $i \geq m+1$, we have $\frac{1}{\frac{1}{u} + \frac{hi}{x}} < u_1$, so that $R_i = 1$. Letting $\tilde{g}_i(x)$ be the density of u_i cars in this model gives

$$\tilde{g}_i(x) = g_{io} R(x)\,. \tag{134}$$

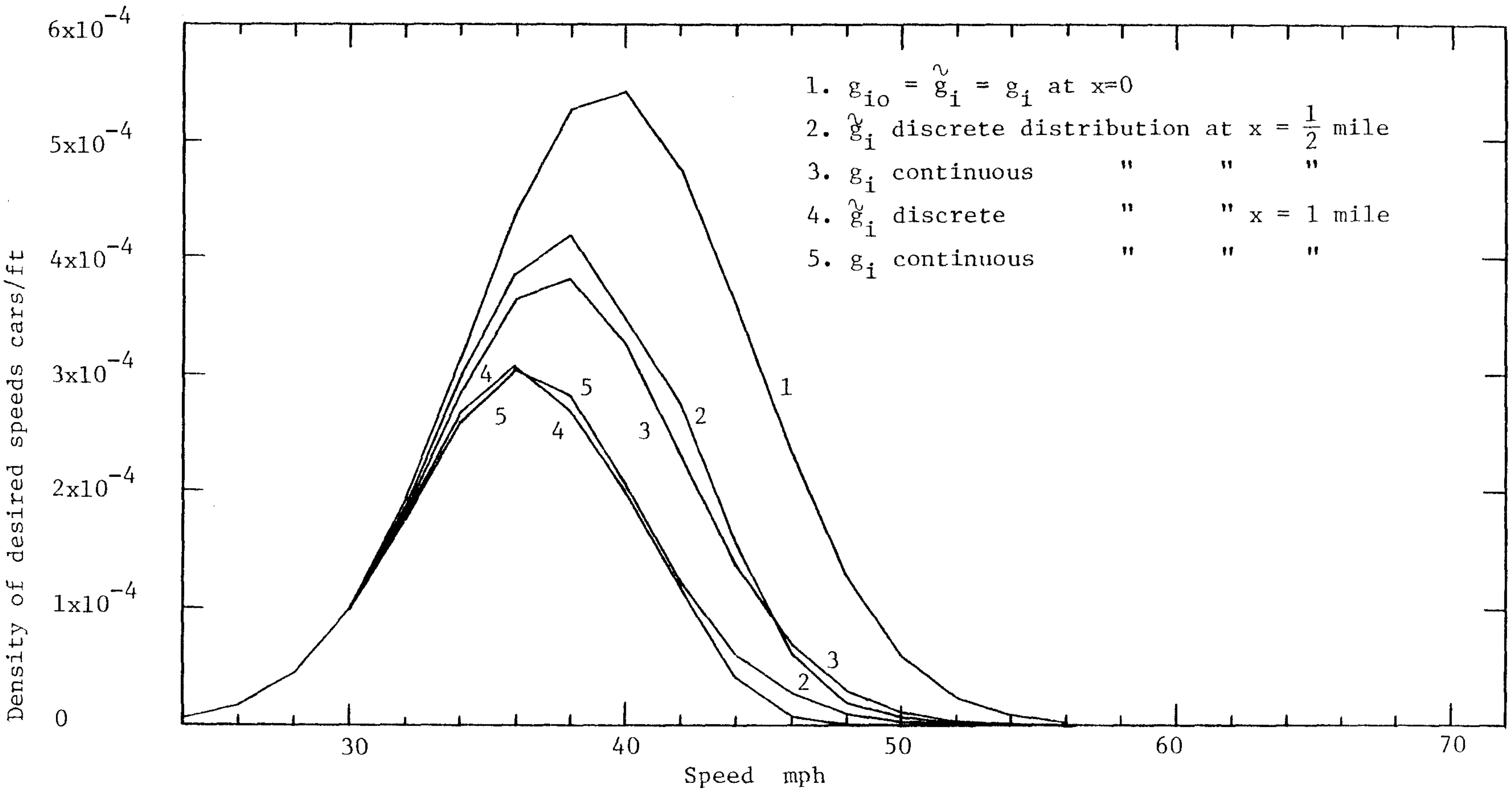
Solving (132) with the boundary conditions $g_i(o,t) = g_{io}$ yields densities which are independent of t,

$$g_1(x) = g_{10}\,,$$

$$g_i(x) = g_{io}\, e^{-\sum_{j=1}^{i-1} (1 - \frac{u_j}{u_i}) \int_o^x g_j(y)dy} . \tag{135}$$

These equations may be solved recursively.

Numerical solutions were found for $g_i(x)$ and $\tilde{g}_i(x)$, using parameters that would fit a city street: $h = 5\,\text{sec}$, $\mu = 40\,\text{mph}$, $\sigma = 5\,\text{mph}$, $u_1 = 24\,\text{mph}$, $u_{17} = 56\,\text{mph}$, $u_{i+1} - u_i = 2\,\text{mph}$, $x = \frac{1}{2}\,\text{mile}$, and $x = 1\,\text{mile}$. The results are shown in Graph 3. Note that curve 1 is not the normal distribution with mean 40 mph since curve 1 is the space density, while it is the input flow that has normally distributed speeds. In fact, equating two expressions for the input flow gives $\frac{1}{h} p_j = u_j g_{jo}$. Hence, $g_{jo} = \frac{1}{h} \frac{p_j}{u_j}$ and the denominator distorts the normal distribution. Graph 3 shows that there are differences between models which are continuous in space and models which are discrete in space. These differences would be large or small depending on the application.



Graph 3. Speed distributions - discrete speeds, continuous vs. discrete spatial distributions.

12. SPECIAL SOLUTIONS AS $t \to \infty$

We now consider what the solutions to our equations are like when t is large. To keep things simple, we consider the equations without the multiple delay terms, that is, equations 110 and 111. To further simplify the problem we let

$$g^*(x,u) = h(u),$$

$$f^*(x,s,u,T) = 0. \tag{136}$$

Since the initial conditions are independent of x, the solution will not depend on x (the solution is unique and may be found by the Picard iteration process which starts and continues with functions that are independent of x). Thus we have

$$g = g(t,u),$$

$$f = f(t,s,u,T). \tag{137}$$

We show that, for u small enough, f approaches a function of s, u, and T only, while g approaches a function of u only.

With the use of (136) and (137), equation 111 becomes

$$g_t(t,u) + \int_0^u (u-z)g(t,u)g(t,z)dz$$

$$= \begin{cases} 0 & 0 \leq t < W \\ \int_0^u (u-z)g(t-W,u)g(t-W,z)dz & t > W. \end{cases} \tag{138}$$

Integrating from 0 to W and from W to t gives

$$g(W,u) + \int_0^W \int_0^u (u-z)g(r,u)g(r,z)dzdr = h(u),$$

$$g(t,u) + \int_{t-W}^t \int_0^u (u-z)g(r,u)g(r,z)dzdr$$

$$= g(W,u) + \int_0^W \int_0^u (u-z)g(r,u)g(r,z)dzdr = h(u). \tag{139}$$

Since g is non-negative, this gives us a bound on g, namely,

$$0 \leq g(t,u) \leq h(u)\ . \qquad (140)$$

We now assume that h(u) is bounded by some constant C so that (140) becomes

$$0 \leq g(t,u) \leq C\ . \qquad (141)$$

We define m by

$$m(t,u) = |g(t,u) - g(t-W,u)| \quad \text{for } t \geq W\ ,$$

and show that m approaches zero as t approaches infinity. A little juggling with (139) yields

$$m(t,u) \leq C \int_{t-W}^{t} \int_{o}^{u} (u-z)[\, m(r,z) + m(r,u)\,]\, dz dr\ . \qquad (142)$$

Inequality (141) implies

$$m(t,u) \leq C \qquad \text{for } t \geq W\ . \qquad (143)$$

Limiting u to the range

$$0 \leq u \leq u_1$$

with

$$C u_1^2 W < 1$$

and using (143) and (142) repeatedly gives

$$m(t,u) \leq C(Cu_1^2W)^n \qquad \text{for } t \geq (n+1)W\ . \qquad (144)$$

Since $Cu_1^2W < 1$, the sequence $\{g(t+nW,u)\}$ $n = 0,1,2,\cdots$ is a Cauchy sequence and hence converges. We must show that any two such sequences, say $\{g(t_1+nW,u)\}$ and $\{g(t_2+nW,u)\}$, with limits B_1 and B_2 actually have the same limit. To do this we use (144) in (138) with the result

$$|g_t(t,u)| \leq C^2u_1^2(Cu_1^2W)^n \quad \text{for } t \geq (n+1)W\ .$$

Hence,

$$|g(t_2+nW,u)-g(t_1+nW,u)| \leq C^2u_1^2(Cu_1^2W)^n\,|t_2-t_1|\,.$$

Therefore,

$$|B_2-B_1| \leq |B_2-g(t_2+nW,u)| + |g(t_2+nW,u)-g(t_1+nW,u)|$$

$$+ |g(t_1+nW,u)-B_1|\,. \qquad (145)$$

As n approaches infinity, the right hand side of (145) approaches zero so that we have

$$B_1 = B_2\,.$$

Thus, as t approaches infinity with $u \leq u_1$, $g(t,u)$ approaches a function of u alone, $\bar{g}(u)$, and by virtue of (110), $f(t,s,u,T)$ approaches $(u-s)\bar{g}(u)\bar{g}(s)$.

13. EQUILIBRIUM

A solution of (34), (41), and (42) which is of particular interest, if it exists, is one which is independent of both position and time. Such a solution could be compared with the previous model.

Upon deleting x and t from f and g, (34), (41), and (42) become

$$f_T(s,u,T)+f(s,u,T)\int_0^s (s-z)g(z)dz=0\,, \tag{146}$$

$$f(s,u,0) = (u-s)g(u)g(s)+g(s)\int_s^u\int_0^W (z-s)f(z,u,T)dTdz\,, \tag{147}$$

$$\int_0^u [\, f(z,u,W)-(u-z)g(u)g(z)]dz=0\,. \tag{148}$$

If we integrate (146), with respect to s from 0 to u, and with respect to T from 0 to W, then add the result to the integral of (147) with respect to s from 0 to u, we get (148). Thus (148) is redundant. This is the same phenomenon that occurred in (98)-(102) where conservation of cars forced cancellation of source and sink terms.

Next we note that (146) is a first order, linear differential equation in f with independent variable T. Solving for f gives

$$f(s,u,T) = f(s,u,0)e^{-T\int_0^s(s-z)g(z)dz}\,. \tag{149}$$

Upon substituting in (147), we find

$$f(s,u,0)=(u-s)g(u)g(s)$$

$$+g(s)\int_0^u\int_0^W(z-s)f(z,u,0)\,e^{-T\int_0^z(z-v)g(v)dv}dTdz\,, \tag{150}$$

or,

$$f(s,u,0) = (u-s)g(u)g(s) + g(s)\int_s^u (z-s)f(z,u,0)\,\frac{1-e^{-W\int_0^z (z-v)g(v)dv}}{\int_0^z (z-v)g(v)dv}\,dz. \tag{151}$$

So we are left with only (151), an equation in f(s,u,0) and g(u), which possesses a multiplicity of solutions. That this is so may be seen by picking g, bounded and integrable but otherwise arbitrary, and noting that for each fixed u (151) is a linear Volterra integral equation for which a solution f exists. What we are missing is the total number, or rather the density, since the range of x is infinite, of drivers with desired speed u. So we define p(u)du to be the fraction of all drivers with desired speed between u and u+du. Clearly,

$$p(u) = \frac{g(u) + \int_0^W \int_0^u f(s,u,T)\,ds\,dT}{k_o} \tag{152}$$

where k_o is the number of cars per unit length of road.

We expect that, when p(u) is given, (149) (151), and (152) define the functions f and g uniquely.

We can eliminate the parameter W by means of a transformation. This transformation is

$$\begin{aligned} f(s,u,T) &= W^{-1/2}\bar{f}(\bar{s},\bar{u},\bar{T})\,, \\ g(u) &= \bar{g}(\bar{u})\,, \\ s &= W^{-1/2}\bar{s}\,, \\ u &= W^{-1/2}\bar{u}\,, \\ k_o &= W^{-1/2}\bar{k}_o\,, \\ T &= W\bar{T}\,, \\ p(u) &= W^{1/2}\bar{p}(\bar{u})\,. \end{aligned} \tag{153}$$

The functions $\bar{f}$ and $\bar{g}$ satisfy (151) and (152) with $W=1$. Note that we achieve the same result with transformation (103) if we include $\overline{k}_o = k_o / W$.

If there is some speed u_1 such that every driver's desired speed is greater than u_1, we have

$$p(u) = 0 \quad \text{for} \ u < u_1 .$$

In this case, since f and g are required to be non-negative, (152) implies that

$$g(u) = 0 \quad \text{for} \ u < u_1 .$$

The previous argument for the time-dependent case gives

$$f(s,u,T) = 0 \quad \text{for} \ s < u_1 .$$

Thus, as before, we have the result that if no one wants to go slower than u_1, then no one is forced to go slower than this speed. This is true as long as traffic is in subcritical flow. As traffic gets heavier, the desired speeds of drivers decrease somewhat. My guess is that the lowest desired speed of the slowest driver on a freeway is around 35 mph at capacity flow since this is about the speed of traffic at capacity flow. A situation I have seen which I think is illuminating is where all lanes except one are at a complete standstill, while the one lane leads to an off-ramp with cars travelling at 45 mph. This indicates to me that a driver's desired speed does not decrease enough to explain the low speeds of forced flow. The only reason an average driver will not go 45 mph or more is that there is a slower car ahead of him. Slower speeds do occur, but these are due to demand exceeding capacity. In this case a driver can choose his headway, but his speed will be approximately that of the driver ahead.

If we translate the speeds in our equations by the transformation

$$f(s,u,T) = \overline{f}(\overline{s},\overline{u},T) ,$$

$$g(u) = \overline{g}(\overline{u}) ,$$

$$p(u) = \bar{p}(\bar{u}) , \tag{154}$$

$$u = \bar{u} + u_1 ,$$

$$s = \bar{s} + u_1 ,$$

then $\bar{f}$ and $\bar{g}$ satisfy (151) and (152), and $\bar{p}(\bar{u})$ is non-zero near $\bar{u} = 0$.

Another transformation of our equations (useful for general information) occurs for the case in which p(u) is constant over some range and zero elsewhere. (Actual traffic has p(u) much more nearly Gaussian.) Thus,

$$p(u) = \begin{cases} u_m^{-1} & \text{for } u_1 \leq u \leq u_1 + u_m \\ 0 & \text{otherwise .} \end{cases} \tag{155}$$

If we transform f and g by

$$f(s,u,T) = W^{-\frac{1}{2}}\left(\frac{k_o}{u_m}\right)^{\frac{3}{2}} \bar{f}(\bar{s},\bar{u},\bar{T}) ,$$

$$g(u) = \frac{k_o}{u_m}\bar{g}(\bar{u}), \tag{156}$$

$$s = u_1 + \left(\frac{u_m}{Wk_o}\right)^{\frac{1}{2}}\bar{s} ,$$

$$u = u_1 + \left(\frac{u_m}{Wk_o}\right)^{\frac{1}{2}}\bar{u} ,$$

$$T = W\bar{T} ,$$

then $\bar{f}$ and $\bar{g}$ satisfy (149), (151), and (152) with the substitutions

$$W = 1 , \quad k_o = 1 , \quad p(u) = 1 .$$

That is, by using (149) and (152) our system becomes

$$\bar{f}(\bar{s},\bar{u},0) = (\bar{u} - \bar{s})\bar{g}(\bar{u})\bar{g}(\bar{s}) + \bar{g}(\bar{s})\int_{\bar{s}}^{\bar{u}} (\bar{z} - \bar{s})\bar{f}(\bar{z},\bar{u},0)\, \frac{1 - e^{-\int_0^{\bar{z}} (\bar{z}-\bar{v})\bar{g}(\bar{v})d\bar{v}}}{\int_0^{\bar{z}} (\bar{z}-\bar{v})\bar{g}(\bar{v})d\bar{v}} d\bar{z} , \tag{157}$$

$$\bar{g}(\bar{u}) = 1 - \int_0^{\bar{u}} \bar{f}(\bar{z},\bar{u},0)\, \frac{1 - e^{-\int_0^{\bar{z}} (\bar{z}-\bar{v})\bar{g}(\bar{v})d\bar{v}}}{\int_0^{\bar{z}} (\bar{z}-\bar{v})\bar{g}(\bar{v})d\bar{v}}\, d\bar{z}\,. \tag{158}$$

The domain of the equations is

$$0 \le \bar{s} \le \bar{u} \le (Wk_o u_m)^{½}\,, \quad 0 \le \bar{T} \le 1\,. \tag{159}$$

13.1 Without Multiple Delays

The final terms in (146) and (147) represent multiple delays. As we discussed previously, it is useful to investigate the equations with these terms deleted. It is also useful to assume that p(u) has the form of (155). With these assumptions, with the transformation (156), and with the deletion of the bar from barred variables, (146), (147), and (152) become

$$f_T(s,u,T) = 0\,, \tag{160}$$

$$f(s,u,0) = (u-s)g(u)g(s)\,, \tag{161}$$

$$g(u) + \int_0^1 \int_0^u f(z,u,T)dzdT = 1. \tag{162}$$

Equations (160) and (161) imply

$$f(s,u,T) = (u-s)g(u)g(s)\,. \tag{163}$$

Substituting (163) in (162) yields

$$g(u) + g(u)\int_0^u (u-z)g(z)dz = 1\,. \tag{164}$$

We now define

$$G(u) = \int_0^u (u-z)g(z)dz$$

and note the relation

$$G''(u) = g(u)\,. \tag{165}$$

Thus (164) becomes

$$G''(u) + G''(u)\,G(u) = 1 \,, \tag{166}$$

$$G(0) = 0 \,,$$

$$G'(0) = 0 \,.$$

This is solved as follows:

$$G'(u)G''(u) = \frac{G'(u)}{1 + G(u)} \,,$$

$$G'(u) = (2\ln(G+1))^{1/2} \,,$$

$$u = \int_0^G (2\ln(r+1))^{-1/2} dr.$$

Letting

$$z^2 = \ln(r+1)$$

gives the form

$$u = 2^{1/2} \int_0^{(\ln(G+1))^{1/2}} e^{z^2} dz \,. \tag{167}$$

Equations 165 and 166 show

$$g(u) = (G(u) + 1)^{-1} \,,$$

and, upon using barred variables again, (167) becomes

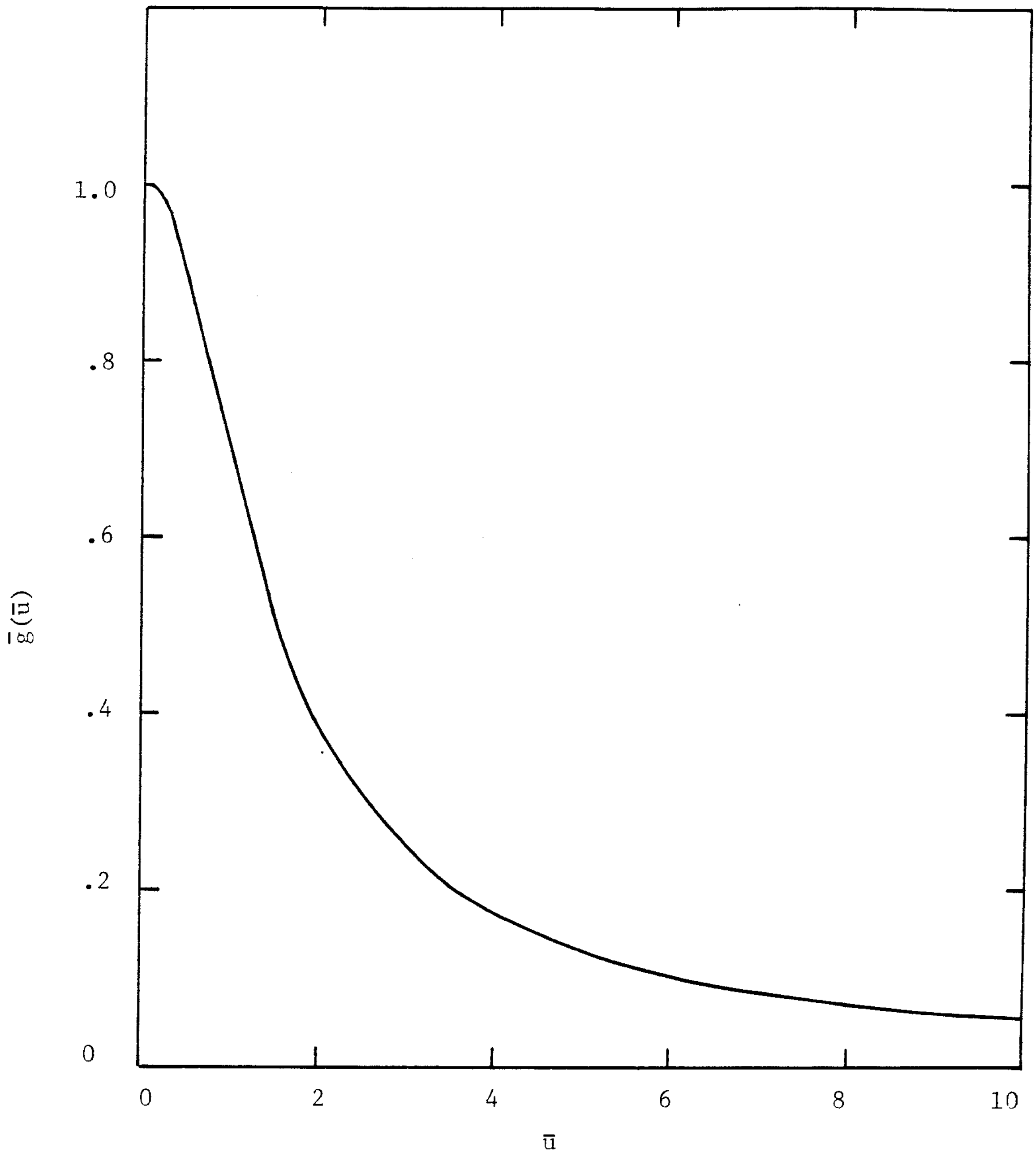
$$\bar{u} = 2^{1/2} \int_0^{(-\ln \bar{g}(\bar{u}))^{1/2}} e^{z^2} dz \,. \tag{168}$$

Applying transformation (156) to get to unbarred variables yields

$$\left(\frac{Wk_o}{2u_m}\right)^{1/2} (u - u_1) = \int_0^{(\ln \frac{k_o}{u_m} - \ln g(u))^{1/2}} e^{z^2} dz \,. \tag{169}$$

The function $\bar{g}(\bar{u})$ is plotted in Graph 4.

The physical interpretation of $\bar{g}(\bar{u})$ is that it represents the fraction of drivers with desired speed u who are at their desired speed. To show this, we note that



Graph 4. Fraction of cars $\bar{g}$ at their normalized speed $\bar{u}$, at equilibrium and without multiple delays.

the number of drivers, per unit length of road, at their desired speed, between u and u+du, is g(u)du. The total number of drivers with these desired speeds is $k_o p(u)du$. Thus the fraction who are at their desired speed is

$$\frac{g(u)du}{k_o p(u)du} = \frac{\frac{k_o}{u_m}\bar{g}(\bar{u})}{k_o \cdot \frac{1}{u_m}} = \bar{g}(\bar{u}) .$$

This quantity also represents the fraction of his total time on the road that a driver spends at his desired speed.

We may see from the graph that as the desired speed of drivers increases, the time spent at the desired speed decreases, and that only when the desired speed corresponds to $\bar{u}=0$, that is $u=u_1$, does the driver continuously travel at his desired speed.

To get an idea of how $\bar{g}(\bar{u})$ corresponds to actual traffic, we take some typical values of W, k_o, and u_m. For example,

$$W = 5 \text{ sec} ,$$
$$k_o = .002 \text{ cars/ft} ,$$
$$u_m = 50 \text{ ft/sec}$$

corresponds to very light traffic. Using (159) we find the range of $\bar{u}$ to be $0 \leq \bar{u} \leq .7$. For this case drivers spend more than 80% of their time at their desired speeds.

The values

$$W = 300 \text{ sec} ,$$
$$k_o = .01 \text{ cars/ft} ,$$
$$u_m = 30 \text{ ft/sec}$$

might be found near capacity flow with $\bar{u}$ in the range $0 \leq \bar{u} \leq 9$. For this case most drivers spend most of their time at less than their desired speed with the fastest drivers going at their desired speed only 6% of the time.

13.2 Comparison with Part I

It is convenient to have an explicit solution to (157) and (158). To get one, we substitute for $\bar{f}$ a power series in $\bar{s}$ and $\bar{u}$ and for $\bar{g}$ a power series in $\bar{u}$. It is easy to show that the coefficients in these power series are all well-defined, so that we have at least a formal expansion of the solution. The first few terms of these power series are

$$\bar{f}(\bar{s},\bar{u},0) = \bar{u} - \bar{s} - \frac{1}{3}(\bar{u}^3 - \bar{s}^3) + \cdots , \tag{170}$$

$$\bar{g}(\bar{u}) = 1 - \frac{1}{2}\bar{u}^2 + \frac{13}{48}\bar{u}^4 + \cdots . \tag{171}$$

It is of interest to compare the present model of traffic flow with the model of Part I. Although the models are similar, there are differences, so we do not expect identical behavior. The quantity we compare is $\bar{v}(u)$, the average velocity of all drivers with desired speed less than u.

By scaling out the various parameters, by using (20) and (21) and by using (25), we have

$$v_1^*(\bar{u}) = \frac{\bar{u}}{2} - \frac{\bar{u}^3}{16} + \frac{\bar{u}^5}{96} + \cdots \tag{172}$$

where the subscript 1 on v_1^* indicates the model of Part I. The corresponding quantity for the present model is defined by

$$\frac{\int_{u_1}^{u} [\, g(z) + \int_0^W \int_z^u f(z,y,T)dydT\,]\, zdz}{k_o \dfrac{u-u_1}{u_m}} ,$$

and with k_o, W and u_m scaled out it is

$$v_2^*(\bar{u}) = \frac{1}{\bar{u}} \int_0^{\bar{u}} [\bar{g}(z) + \int_0^1 \int_z^{\bar{u}} \bar{f}(z,y,T)dydT\,]\, zdz .$$

Upon using (149), (170), and (171) this becomes

$$v_2^*(\bar{u}) = \frac{\bar{u}}{2} - \frac{\bar{u}^3}{12} + \frac{19}{720}\bar{u}^5 + \cdots . \qquad (173)$$

Comparing (172) with (173), we notice that the first terms are identical and that the same powers of $\bar{u}$ are present in each. Some disagreement between (172) and (173) was expected, and in particular we notice that $v_1^*(\bar{u}) > v_2^*(\bar{u})$ when $\bar{u}$ is small enough. This is due at least partially to the following phenomenon which affects v_1^* but not v_2^*. In the model of Part I, a driver with desired speed u is considered to be delayed by cars with speed $\bar{v}(u)$. A more accurate description, as in the present model, considers the car to be delayed by cars with various speeds. In the model of Part I, the use of $\bar{v}(u)$ is equivalent to assigning equal weights to the delaying effects of the various cars. But the u car catches up with slow cars more frequently than with fast cars, so that the delaying effect of a slow car should be given more weight than that of a faster car. This makes v_1^* larger than it would be if these considerations were taken into account.

We are content with the agreement between (172) and (173).

13.3 Small Waiting Time to Pass

In this section we explore the consequences of assuming the waiting time to pass, W, is small. As a preliminary, substitute (149) in (152)

$$k_o p(u) = g(u) + \int_o^u f(z,u,0)\,\frac{1 - e^{-W\int_o^z (z-v)g(v)dv}}{\int_o^z (z-v)g(v)dv}\,dz . \qquad (174)$$

Let

$$H(g,z) = \int_o^z (z-v)g(v)dv . \qquad (175)$$

Easily,

$$\frac{1-e^{-WH}}{H} = W - \frac{H}{2!}W^2 + \frac{H^2}{3!}W^3 - \cdots . \tag{176}$$

Let

$$f(s,u,0) = f_o(s,u) + Wf_1(s,u) + W^2 f_2(s,u) + \cdots . \tag{177}$$

Equation 151 becomes

$$f_o(s,u) + Wf_1(s,u) + W^2 f_2(s,u) = (u-s)g(u)g(s)$$

$$+ g(s)\int_s^u (z-s)\,[\,f_o(z,u) + Wf_1(z,u) + W^2 f_2(z,u,)\,]\,[W - \frac{H}{2!}W^2 + \frac{H^2}{3!}W^3 - \cdots\,]dz . \tag{178}$$

$$f_o(s,u) = (u-s)g(u)g(s) ,$$

$$f_1(s,u) = g(s)\int_s^u (z-s) f_o(z,u)\, dz \tag{179}$$

$$= g(u)g(s)\int_s^u (z-s)(u-z)g(z)dz ,$$

$$f_2(s,u) = g(s)\int_s^u (z-s)\,[f_1(z,u) - f_o(z,u)\frac{H}{2!}]\, dz .$$

This determines f when g is known. To solve (151) and (174) for f and g simultaneously, let

$$g(u) = g_o(u) + Wg_1(u) + W^2 g_2(u) + \cdots . \tag{180}$$

Substituting (177) and (180) in (151) and (174) yields

$$g_o(u) = k_o p(u) ,$$

$$f_o(s,u) = k_o^2 (u-s)p(u)p(s) ,$$

$$g_1(u) = -k_o^2 p(u)\int_o^u (u-z)p(z)dz ,$$

$$f_1(s,u) = k_o^3 p(u)p(s)\{\int_s^u (z-s)(u-z)p(z)dz - \int_o^s (s-z)p(z)dz ,$$

$$-\int_o^u (u-z)p(z)dz\} ,$$

etc.

If $p(u)$ were approximated by a polynomial, the above formulas would be especially simple.

APPENDIX

COMPARISON OF THIS INVESTIGATOR'S WORK WITH THAT OF OTHERS

Our work uses the following variables to describe traffic flow on a freeway.

x = position,

t = time,

s = actual speed of a car,

u = desired speed of a driver,

T = time since a driver last was forced to reduce his speed (applies only to drivers who are at less than their desired speeds),

W = waiting time before a delayed vehicle passes the blocking vehicle,

f(x,t,s,u,T)dx ds du dT = the number of drivers who are at less than their desired speeds with position in (x,x+dx), with actual speed in (s,s+ds), with desired speed in (u,u+du), and with time since last delay in (T,T+dT),

g(x,t,u)dx du = the number or drivers who are at their desired speeds with position in (x,x+dx) and with desired speed in (u,u+du).

With only one exception of which I am aware, all other authors use only one function, f(x,t,s), to describe all cars (not just those at less than their desired speeds). All the authors have terms of the form

$f_t + sf_x = D + R$

where D is a term describing faster cars catching up with slower cars and where

R describes the return to a higher speed of a delayed car.

Prigogine and his fellow investigators in many papers have used a D of the form $\int (s-z)f(s)f(z)dz$. Simply stated, (s-z) is the rate at which an s car catches up with a z car, and the number of delays is proportional to both f(s) and f(z). This leads to the difficulty, admitted by Prigogine, of too many cars being delayed. Lampis (1978) and I (independently) use a D of the form $\int (s-z)f(s)g(z)dz$ since the blocking effect of a queue of cars is the same as the blocking effect of one car, and we each assign the entire effect to the lead car. Lampis reports that for the equilibrium case this solves the above difficulty.

All the other researchers use a return to speed term of the form $R = \frac{1}{\tau}(f_p - f)$ where τ is a relaxation time and f_p is a given speed distribution to which the actual speed distribution tends to return. My model has cars returning to their individual desired speeds after being delayed a fixed time W. Since this model keeps track of desired speeds, the different effects of a situation on drivers with different desired speeds can be examined. Also, this model allows drivers with different desired speeds to flow as they will according to the rules of delay and passing, rather than being forced into an equilibrium with respect to desired speed. This model can be modified easily to allow the waiting time to pass to be an arbitrary probability distribution. When this distribution is a negative exponential, one gets $R = \text{constant} \times f$, which is the same as part of the return to speed term of the other investigators.

This work is the only one of which I am aware with existence and uniqueness and non-negativity proofs (Prigogine's solutions sometimes went negative), with a comparison of the delay phenomenon for the continuous model and a discrete model, with a proof of the lack of constant-speed, constant-shape, ordinary-function solutions (for low desired speeds), and with a simple constant-speed, constant-shape, δ function solution.

We now take a look at a paper by Phillips (1977) in some depth because some other investigators have made some of the same assumptions and because Phillips has written another paper which is largely based on this model. These works have some items in common with our work.

Phillips' basic equation is

$$\frac{\partial(Kf)}{\partial t} + u\,\frac{\partial(Kf)}{\partial x} = \frac{K^2}{n}(1 - P(K))(\overline{u} - u)f + \frac{K}{\tau(K)}(f_p(K,u) - f) \tag{1}$$

where

x = position on road,

t = time,

u = speed,

$\overline{u}$ = average speed = $\int_0^\infty uf(u,x,t)du$,

n = number of lanes,

f(u,x,t)du = Prob(a car at x,t has velocity in (u,u+du)),

K(x,t)dx = number of cars in (x,x+dx),

P(K) = probability that an overtaking car will be able

to change lanes without delay,

$f_p(K,u)du$ = fraction of drivers who return to a speed

in (u,u+du) after a pass,

$\tau(K)$ = a relaxation time for he returning to

speed process.

Generally, Phillips makes many false assumptions about traffic and makes logical and mathematical errors as well. Some of Phillips' graphs contradict his text.

Also, Phillips' results are poor. For subcritical flow, as density increases from zero, the flow-density curve bends slightly due to the phenomenon of cars being delayed by other cars and then passing. Without this phenomenon, $Q = u_o K$, where Q is flow, u_o is average low-density speed and K is density. One typical set of data presented by Phillips (p.48) is fitted very well by the equation

$$Q = u_o K - aK^2 \quad 0 \leq K \leq .00819 \text{ veh/ft}$$

where

u_o = 87.3 ft/sec,

a = 2600 ft^2/veh-sec,

units of Q are veh/sec-lane,

units of K are veh/ft-lane.

Thus it takes only a simple equation with one free parameter, a, to accurately represent the delay-passing phenomenon in subcritical flow. Phillips' equations do not accurately give an equivalent of the aK^2 term, the errors being on the order of the aK^2 term. In all five data sets, Phillips' curves fail to bend at low enough K. This effect does not stand out strongly on a flow-density curve because it is masked by the large value of $u_o K$. However, this effect is the whole reason for including delays and passing in the model. It should be represented accurately.

For the forced flow portion of the flow-density curve, in four out of five of the data sets, the data decrease less rapidly than Phillips' curve. This is probably due in large measure to the fact that Phillips erroneously uses 120 veh/mile-lane as the jam density, whereas May and Wagner give 200 to 225 veh/mile-lane, and my measurements (on a city street) indicate about 225 veh/mile-lane.

Phillips derives the function P(K) based on whether the adjacent lane is occupied next to a driver who wants to pass. That is, P is based on the ability of a driver to change lanes. It should be based on whether he actually changes lanes and whether this results in passing. My observations of traffic indicate driver psychology is equally as important as the existence of a car in an adjacent lane. Some drivers will anticipate the need to pass and will not pass only if the road is blocked in a much larger area than within an effective car length of the slow car. Other drivers will come up behind a slow car and not pass even though there are no cars to block the pass. Some of these drivers wait about 10 to 30 seconds and then pass, while others stay behind the slow car for several minutes even in very light traffic. Phillips' formula for P(K) is derived from

$$P_o(K) = 1 - KL_o/n = 1 - k$$

where

$P_o(K) = P(K)$ if there is only one adjacent lane,

$L_o = n/K_o$ = (jam density for one lane)$^{-1}$,

$k = K/K_o$ = normalized density $0 \leq k \leq 1$.

His argument is, "... the probability of such a car being able to change lanes is just the fraction of the adjacent lane which is not blocked with cars or $P_o = 1 - KL_o/n = 1 - K/K_o = 1 - k$." The quantity L_o should not be (jam density)$^{-1}$ but rather an effective length of the blocking car, considering the speed at which it is going. For example, an open space of 25 ft at 60 mph is a gap of 0.28 sec which will not be acceptable to most drivers (2 to 3 sec gap is the minimum requirement of many drivers). For the sake of completeness, we include Phillips' expression for P(K) based on the number of lanes and the probabilistic argument that the probability of two adjacent lanes being blocked is $(1 - P_o)^2$.

$$P(k) = 1 - \left(\frac{2}{n}k + \frac{n-2}{n}k^2\right) .$$

The term $\frac{K^2}{n}(1 - P(K))(\bar{u} - u)f(u,x,t)$ in (1) accounts for:

i. fast cars catching up with a car at speed u, and either passing immediately or slowing to speed u and

ii. cars at speed u being slowed by slower cars.

This makes sense in subcritical flow but not in forced flow where all drivers are already caught up with the car ahead. Thus, this term should be deleted for forced flow.

The final term in (1) is based on the idea that drivers who have been slowed will, after a time, return to a higher speed (although not necessarily the desired speed). My observations and measurements indicate that essentially no passing occurs in forced flow, so that this term also should be deleted in forced flow.

An expression for the relaxation time τ for drivers returning to a higher speed is derived from the idea that a blockage will disappear in time

$$\tau = \frac{L_0}{2\bar{u}_r}$$

where $\bar{u}_r$ is the average difference of speeds of cars. As in the probability of passing without delay, the driver's psychology plays a major role. Many drivers will not pass even when not blocked. Thus, τ cannot be determined from positions of vehicles alone. Phillips also errs in his determination of $\bar{u}_r$,

$$\bar{u}_r = \int_0^\infty \int_0^\infty |u' - u| f_\infty(u') f_\infty(u) du' du$$

by using a distribution f_∞ which is supposed to hold only for forced flow (and is very badly in error in subcritical flow). He errs in his evaluation of the double integral, and he makes an arithmetic error. His result, with the last two errors corrected, is

$$\tau = 1.71\ kL_0 / (1 - k)u_1$$

where

$u_1 = 38$ ft/sec and comes from a reaction time argument for the spacing of vehicles in forced flow,

gap size, S = time gap x speed,

time gap = L_0/u_1 = (44 ft)/(38 ft/sec) = 1.16 sec.

Phillips intends $f_p(u,K)$ to be an improvement over slowed drivers returning to their desired speeds since they may pass one slow car only to be delayed still by a car that is not as slow as the original blocking car. To do this Phillips uses two speed distributions: f_d, the distribution of desired speeds, and f_m, which is approximately the Weibull distribution

$$f_m \doteq \begin{cases} \dfrac{\pi(2u-\bar{u}_\infty)}{\bar{u}_\infty^2} \exp\left(-\dfrac{\pi}{4}\left(\dfrac{2u-\bar{u}_\infty}{\bar{u}_\infty}\right)^2\right) & u \geq \bar{u}_\infty/2 \\ 0 & u \leq \bar{u}_\infty/2, \end{cases}$$

$$\bar{u}_\infty = u_1(1-k)/k \doteq \text{mean of } f_m \text{ distribution}, \tag{2}$$

$$f_p(u) = \int_u^\infty f_m(u')f_d(u) + f_m(u)f_d(u')du'. \tag{3}$$

The distribution of maximum speeds that the traffic situation will allow a driver to return to is represented by f_m. A driver will return to the minimum of his desired speed and this maximum. The distribution of this minimum is $f_p(u)$. For this formula to be based on sound probabilistic principles, it should include a modification, since the fact that, given that a driver starts at some speed v and given that he is passing, will affect the distributions $f_d(u)$ and $f_m(u')$ (we must have $v \leq u$ and $v \leq u'$). After making these modifications for fixed v, he should average over all v. Equation 2 is at the heart of what makes Phillips model follow the true flow-density curve somewhat. This equation is derived directly from the car-following model where the time gap T between successive cars is constant. Thus the head-to-head distance between cars is $\frac{1}{K} = \bar{u}_\infty T + L_0$.

Hence,

$$\bar{u}_\infty = (\frac{1}{K} - L_o)/T = (\frac{1}{K} - \frac{1}{K_o})/T = (1 - \frac{K}{K_o})/(\frac{K}{K_o})K_oT .$$

This is identical to (2). The result of this is that when K is large f_m has a mean that is correct for forced flow, and (3) gives $f_p \doteq f_m$. When K is small, $\bar{u}_\infty$ becomes larger than the mean of f_d, so (3) gives $f_p \doteq f_d$. Thus (3) provides a smooth transition from the distribution f_d to the distribution f_m. For subcritical flow f_m is a fictitious distribution since it has a mean higher than even the desired speeds. There are sometimes limits on the speed to which a car can return in subcritical flow, but they are not given by f_m.

Phillips gives a formula for the equilibrium distribution derived from (1), $f_o = f_p/(1+\gamma(u-\bar{u}))$, where $\gamma = \tau K(1-P)/n$. Taking $u-\bar{u}$ to be two standard deviations gives $|\gamma(u-\bar{u})| \leq .005$ throughout the range of subcritical flow. This term contains delay and passing effects but is entirely negligible in subcritical flow which is the only region in which it is valid.

Phillips presents graphs for speed distributions with the theoretical distribution superimposed on the experimental distribution. In many cases he would have done much better to assume the distribution is normal and to try to predict μ and σ.

Conclusion:

Phillips' work is supposed to predict the effects of delay and passing but fails to do this. His derivations are incomplete or faulty and do not provide insight except possibly through focusing on what he is trying to achieve. He does not recognize that passing is an insignificant phenomenon in forced flow, and he does not take proper account of driver psychology.

REFERENCES

1. Andrews, F.C. 1970a. "A Statistical Theory of Traffic Flow on Highways - I. Steady-State Flow in Low-Density Limit," *Transpn. Res.*, 4, pp. 359-366.

2. Andrews, F.C. 1970b. "A Statistical Theory of Traffic Flow on Highways - II. Three-Car Interactions and the Onset of Queuing," *Transpn. Res.*, 4, pp. 367-377.

3. Edie, L.C., R. Herman, and T.N. Lam, 1980. "Observed Multilane Speed Distribution and the Kinetic Theory of Vehicular Traffic," *Transpn. Sci.*, 14, pp. 55-76.

4. Gafarian, A.V., P.K. Munjal, and J. Pahl, 1971. "An Experimental Validation of Two Boltzmann-type Statistical Models for Multi-Lane Traffic Flow," *Transpn. Res.*, 5, pp. 211-224.

5. Herman, R. and T. Lam, 1971a. "On the Mean Speed in the 'Boltzmann-like' Traffic Theory: Analytical Derivation," *Transpn. Sci.*, 5, pp. 314-327.

6. Herman, R. and T. Lam, 1971b. "On the Mean Speed in the 'Boltzmann-like' Traffic Theory: A Numerical Method," *Transpn. Sci.*, 5, pp. 418-429.

7. Herman, R., T. Lam, and I. Prigogine, 1972. 'Kinetic Theory of Vehicular Traffic: Comparison with Data," *Transpn. Sci.*, 6, pp. 440-452.

8. Herman, R., T. Lam, and I. Prigogine, 1973. 'Multilane Vehicular Traffic and Adaptive Human Behavior," *Science*, 179, pp. 918-920.

9. *Highway Capacity Manual 1965.* Highway Research Board, Special Report 87, Washington, D.C.

10. Lampis, M. 1978. "On the Kinetic Theory of Traffic Flow in the Case of a Non-negligible Number of Queuing Vehicles," *Transpn. Sci.*, 12, pp. 16-28.

11. Lighthill, M.J. and G.B. Whitham 1955. "On Kinematic Waves II. A Theory of Traffic Flow on Long Crowded Roads," *Proc. Roy. Soc.*, A 229, pp. 317-345.

12. May, A.D., Jr. and F.A. Wagner, Jr. 1960. 'Headway Characteristics and Interrelationships of Fundamental Characteristics of Traffic Flow," *Proc. HRB*, 39, pp. 524-547.

13. O'Mathuna, D. 1978. 'Highway Traffic Kinematics and the Characteristic Relation," *Transpn. Plan. Tech.*, 4, pp. 89-96.

14. Phillips, W.F. 1977. *Kinetic Model for Traffic Flow.* National Technical Information Service PB 280 258, Springfield, VA.

15. Phillips, W.F. 1979. "A Kinetic Model for Traffic Flow with Continuum Implications," *Transpn. Plan. Tech.*, 5, pp. 131-138.

16. Phillips, W.F. 1981. *A New Continuum Model for Traffic Flow.* National Technical Information Service PB 81 249 948, Springfield, VA.

17. Prigogine, I. and F.C. Andrews 1960. "A Boltzmann-like Approach for Traffic Flow," *Opns. Res.*, 8, pp. 789-797.

18. Prigogine, I. and R. Herman 1971. *Kinetic Theory of Vehicular Traffic.* American Elsevier, New York.

19. Whitham, G.B. 1974. *Linear and Nonlinear Waves.* Wiley-Interscience, New York, Chichester, Brisbane, Toronto, Singapore.