Two holomorphic extremal problems in Teichmüller theory

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ABSTRACT

In this thesis, we study the complex geometry of the Teichmüller space of conformal structures on a finite-type Riemann surface. We give partial answers to two structural questions: (1) Which holomorphic disks in Teichmüller space are holomorphic retracts of Teichmüller space? (2) What are the holomorphic and Kobayashi-isometric submersions between Teichmüller spaces? In both cases, the answers have to do with the geometry of the underlying surfaces, while the methods require developing and applying novel analytic tools.

Question (1) is equivalent to asking the following: on which pairs of points in Teichmüller space do the Carathéodory and Teichmüller metrics coincide? Markovic showed that the Carathéodory and Teichmüller metrics on Teichmüller space are not the same. On the other hand, Kra earlier showed that the metrics coincide when restricted to a Teichmüller disk generated by a differential with no odd-order zeros. We conjecture the converse: the Carathéodory and Teichmüller metrics agree on a Teichmüller disk if and only if the Teichmüller disk is generated by a differential with no odd-order zeros. We prove this conjecture for the Teichmüller spaces of the five-times punctured sphere and the twice-punctured torus. As a key analytic step in the proof, we study the family of holomorphic retractions from the polydisk onto its diagonal. In particular, we analyze the asymptotics of the orbit of such a retraction under the conjugation action of a unipotent subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

Question (2) concerns holomorphic and isometric submersions between Teichmüller spaces of finite-type surfaces. We prove that, with potential exceptions coming from low-genus phenomena, any such map is a forgetful map $\mathcal{T}_{g,n} \to \mathcal{T}_{g,m}$ obtained by filling in punctures. This generalizes a classical result of Royden and Earle-Kra asserting that biholomorphisms between finite-type Teichmüller spaces arise from mapping classes. As a key step in the argument, we prove that any $\mathbb{C}$-linear embedding $Q(X) \hookrightarrow Q(Y)$ between spaces of integrable quadratic differentials is, up to scale, pull-back by a holomorphic map. We accomplish this step by adapting methods developed by Markovic to study isometries of infinite-type Teichmüller spaces. The main analytic tool used is a theorem of Rudin on isometries of L^p spaces.

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Chapter 1

INTRODUCTION

Let $S_{g,n}$ denote the surface of genus g with n punctures. The *Teichmüller space* $\mathcal{T}_{g,n}$ of $S_{g,n}$ parametrizes marked complex structures on $S_{g,n}$. The Teichmüller space supports a natural complex structure making it into a complex manifold of dimension $3g - 3 + n$. The complex-analytic properties of Teichmüller space reflect the geometry and topology of the surface $S_{g,n}$. In this thesis, we study two analytic extremal problems on Teichmüller space and establish in certain cases that the solutions have simple descriptions in terms of the geometry of the surfaces involved.

The goals of this thesis are two-fold.

1. Chapters II and III are devoted to the problem of determining the complex dimension one *holomorphic retracts* of Teichmüller space. That is, we aim to classify holomorphic embeddings of the unit disk $\tau : \mathbb{D} \to \mathcal{T}_{g,n}$ which admit a *holomorphic retraction*, i.e., a holomorphic map $F : \mathcal{T}_{g,n} \to \mathbb{D}$ so that $F \circ \tau = \mathrm{id}_{\mathbb{D}}$.

2. In Chapter IV, we pursue the problem of classifying the holomorphic and *isometric submersions* between Teichmüller spaces. That is, we seek to determine the holomorphic maps $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$ so that the pushforward F_* maps the Teichmüller unit ball of each domain tangent space onto to the unit ball of the target tangent space. Here, we use the infinitesimal Teichmüller metric on tangent spaces.

A key player in our analysis is the intrinsically defined Kobayashi pseudometric on a complex manifold X and the corresponding Kobayashi pseudonorm on tangent spaces T_pX. Recall that the Kobayashi pseudometric K_X on X is the largest pseudometric so that all maps out of the disk $\mathbb{D}$ with its Poincaré metric into X are distance non-decreasing. Any holomorphic map of complex manifolds $X \to Y$ is non-expanding with respect to Kobayashi pseudometrics on the source and target. By a classic theorem of Royden [33], the Kobayashi metric on $\mathcal{T}_{g,n}$ coincides with the classical Teichmüller metric defined in terms of quasiconformal maps (see Section 1.2).

The two problems we deal with are related in the sense that they are both extremal problems. Since holomorphic maps decrease Kobayashi distance, both of our goals in this thesis can be described as classification of maps which are *extremal* in a given class. Any holomorphic $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$ sends the unit ball of each tangent space into the unit ball of the target; in Chapter IV, we seek F so that the image is the entire unit ball. By the classical Schwarz lemma, a composition of holomorphic maps $\mathbb{D} \xrightarrow{\tau} \mathcal{T}_{g,n} \xrightarrow{F} \mathbb{D}$ is non-expanding for the hyperbolic distance on the disk; in Chapter III we seek pairs so that $F \circ \tau$ is an isometry. We mention another connection between our analyses of submersions and retractions: a key input to the study of retractions $\mathcal{T}_{g,n} \to \mathbb{D}$ in Chapter III is an analysis of the space of Kobayashi-isometric submersions $\mathbb{D}^k \to \mathbb{D}$ in Chapter II. (In Chapter II, Kobayashi-isometric submersions on the polydisk are referred to as *extremal* maps. These maps were classified by Knese in [23].)

1.1 Outline of results and methods

For further background on Teichmüller theory, see Section 1.2 and the introductions to Chapters II, III, and IV.

Translation flow

The focus of Chapter II is the proof of a purely complex-analytic result used as input in Chapter III. Recall that the upper half-plane $\mathbb{H}$ is biholomorphic to the unit disk $\mathbb{D}$. In Chapter II, we study the space of holomorphic retractions $\mathbb{H}^n \to \mathbb{H}$ of the polydisk onto its diagonal. Let $f : \mathbb{H}^n \to \mathbb{H}$ be a holomorphic retraction onto the diagonal, i.e., a holomorphic map satisfying $f(\lambda, \ldots, \lambda) = \lambda$ for all $\lambda \in \mathbb{H}$. For each $t \in \mathbb{R}$, define $f_t : \mathbb{H}^n \to \mathbb{H}$ by $f_t(\lambda_1, \ldots, \lambda_n) = f(\lambda_1 - t, \ldots, \lambda_n - t) + t$. We call the action $(f, t) \mapsto f_t$ the *translation flow* on the space of holomorphic retractions of $\mathbb{H}^n$ onto its diagonal. The main result of Chapter II asserts roughly that, for most t, the map f_t is close to a convex combination of the coordinate functions. To state this more precisely, let $\alpha_j = \frac{\partial f}{\partial \lambda_j}(i, \ldots, i)$ and define the map $g : \mathbb{H}^n \to \mathbb{H}$ by $g(\lambda_1, \ldots, \lambda_n) = \sum_j \alpha_j \lambda_j$. Then for any neighborhood U of g in the compact-open topology, the set of times $\{t \in \mathbb{R} | f_t \in U\}$ has density 1 in $\mathbb{R}$.

We briefly describe the idea of the proof. First, we reduce to the case that the partial derivatives $\frac{\partial f}{\partial \lambda_j}(i, \ldots, i)$ are all equal to $\frac{1}{n}$, so that $g = \frac{1}{n} \sum_j \lambda_j$. Note that the translation flow preserves the space C of retractions $f : \mathbb{H}^n \to \mathbb{H}$ onto the diagonal with $\frac{\partial f}{\partial \lambda_j}(i, \ldots, i) = \frac{1}{n}$ for $j = 1 \ldots, n$. An argument using the Poisson integral formula shows that g is the unique element of C invariant under the entire flow. We use this to argue that the average $\lim_{s\to\infty}\frac{1}{2s}\int_{-s}^{s} f_t dt$ is well-defined and equal to the linear function g. On the other hand, using the fact that g satisfies equality in the Schwarz lemma for maps $\mathbb{H}^n \to \mathbb{H}$, we show that g is an extreme point of C, meaning that g cannot be written as a non-trivial convex combination of elements of C. Since f_t averages to g to over the orbit, and since g is an extreme point of C, it must be that the orbit f_t is concentrated at g. To make the last step of this argument precise, we use compactness of the space of Borel measures on C.

Holomorphic retractions

The motivating problem of Chapter III is the classification of holomorphic maps $\tau : \mathbb{D} \to \mathcal{T}_{g,n}$ which admit a holomorphic retraction, i.e., a holomorphic map $F : \mathcal{T}_{g,n} \to \mathbb{D}$ so that $F \circ \tau = \mathrm{id}_{\mathbb{D}}$. Because holomorphic maps decrease Kobayashi distance, any holomorphic retract $\tau : \mathbb{D} \to \mathcal{T}_{g,n}$ must be an isometric embedding for the Teichmüller-Kobayashi metric. Holomorphic and isometric embeddings for the Teichmüller metric are called *Teichmüller disks*. There is a bijective correspondence between unit-area quadratic differentials and Teichmüller disks. Each Teichmüller disk is generated through affine deformations of a quadratic differential. Thus, our motivating problem can be rephrased as a question about quadratic differentials: which quadratic differentials generate a Teichmüller disk which is a holomorphic retract of Teichmüller space? Using the period mapping on the Teichmüller space of a closed surface, Kra [24] showed that if all of the zeros of a quadratic differential have even order, then the associated Teichmüller disk is a holomorphic retract of Teichmüller space. On the other hand, Markovic [27] recently showed that not all Teichmüller disks are retracts. It remains to determine precisely which Teichmüller disks are retracts. We conjecture that the converse of Kra's result holds, namely that a quadratic differential generates a retract if and only if all of its zeros are of even order. In Chapter III, we prove this result for the Teichmüller spaces of the five-times punctured sphere and the twice-punctured torus.

As a key step in the proof, we observe that the property of generating a holomorphic retract is invariant under the actions of the mapping class group and the group $\mathrm{SL}_2(\mathbb{R})$ on the space of marked holomorphic quadratic differentials. Moreover, this property is closed. Thus, the property of generating a retract descends to a closed $\mathrm{SL}_2(\mathbb{R})$-invariant property on the moduli space of (unmarked) quadratic differentials. On the other hand, by a theorem of Smillie and Weiss [36], the closure of the $\mathrm{SL}_2(\mathbb{R})$ orbit of any quadratic differential contains a *Jenkin-Strebel* differential. (Recall that a Jenkin-Strebel is, informally, a quadratic differential built out of cylinders.) Thus, it suffices to show that any Jenkins-Strebel differential ϕ with an odd-order zero does not generate a holomorphic retract of Teichmüller space. Corresponding to an n-cylinder Jenkins-Strebel differential ϕ is a holomorphic map out of the polydisk $\mathcal{E}^{\phi} : \mathbb{H}^n \to \mathcal{T}_{g,n}$ obtained by deforming each of the cylinders of the ϕ individually. The diagonal of this map is the Teichmüller disk $\tau^{\phi} : \mathbb{H} \to \mathcal{T}_{g,n}$ generated by ϕ. We prove that if the disk τ^{ϕ} is a holomorphic retract, then there is a retraction $\mathcal{T}_{g,n} \to \mathbb{H}$ which restricts to a linear combination of the coordinate functions on the polydisk $\mathcal{E}^{\phi}(\mathbb{H}^n)$. The key facts used to linearize the retraction are (a) the main result of Chapter II and (b) equidistribution of linear flows in the torus. Now, to establish the classification of retracts for $\mathcal{T}_{0,5}$ we use the dynamical result of Smillie and Weiss to reduce to the case of the L-shaped pillowcase. We assume for the sake of contradiction that the pillowcase generates a retract and linearize the retraction on the associated bidisk. To finish the argument, we apply an argument of Markovic [27] to obtain a contradiction on the differentiability of the retraction near the boundary of the bidisk.

Isometric submersions

By a classic theorem of Royden [33], any biholomorphism of a finite-type Teichmüller space $\mathcal{T}_{g,n}$ is induced a mapping class of the underlying surface $S_{g,n}$ (with some exceptions in low complexity cases). Royden's first step was to establish that the Teichmüller metric is the same as the intrinsically defined Kobayashi metric and is thus invariant under biholomorphisms. Thus, Royden's theorem amounted to a classification of the holomorphic isometries of Teichmüller space. In Chapter IV, we study the more general class of holomorphic, Kobayashi-isometric submersions between finite-type Teichmüller spaces. Recall that an isometric submersion is a C^1 map which sends the unit ball of each tangent space onto the unit ball of the target tangent space. Equivalently, each induced map of cotangent spaces is an isometric embedding. In Chapter IV, we generalize Royden's theorem by showing that (with potential low-genus exceptions) the holomorphic and isometric submersions between Teichmüller spaces are all *forgetful maps* $\mathcal{T}_{g,m} \to \mathcal{T}_{g,n}$, induced by filling $m-n$ punctures of the Riemann surface $S_{g,m}$. We restrict our analysis to the case that the target of the submersion is not biholomorphic to a genus 0 Teichmüller space $\mathcal{T}_{0,n}$.

As in the proof of Royden's theorem, they key is to work first on the infinitesimal level, using the natural identification of the cotangent space to $\mathcal{T}_{g,m}$ at a marked Riemann surface X with the space of L^1 holomorphic quadratic differentials $Q(X)$.

Given two Riemann surfaces X,Y, we prove that any L^1 isometric embedding $Q(X) \to Q(Y)$ is, up to scale, pullback by a holomorphic map. To do this, we adapt a method used by Markovic [26] to prove the infinite dimensional case of Royden's theorem. Specifically, we use a theorem of Rudin [34] on isometries of L^p spaces to relate the bi-canonical embeddings of X and Y. Now, given a holomorphic and isometric submersion $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$, we get for each $Y \in \mathcal{T}_{g,n}$ an isometric embedding $Q(F(Y)) \to Q(Y)$ of cotangent spaces, which is in turn induced by a holomorphic map $h_Y : \widehat{Y} \to \widehat{F(Y)}$, where $\widehat{Y}, \widehat{F(Y)}$ are the compact surfaces obtained by filling in all punctures of of $Y, F(Y)$ respectively. By a dimension count, it is not the case that all genus g surfaces cover a surface of genus $0 < k < g$ and so $k = g$ and the maps $h_Y : \widehat{Y} \to \widehat{F(Y)}$ are biholomorphisms. The fact that h_Y pulls $Q(F(Y))$ to $Q(Y)$ implies h_Y restricts to an inclusion $h_Y : Y \to F(Y)$. Finally, an argument involving the universal curves over $\mathcal{T}_{g,n}$ and $\mathcal{T}_{g,m}$ shows that h_Y varies continuously in the parameter Y and so the map $Y \mapsto F(Y)$ is induced by a fixed mapping class, followed by a filling of punctures.

1.2 Definitions and background

We recall basic definitions and background related to Teichmüller theory. Good references for this material include [19] and [40].

Teichmüller space

Let $\mathcal{T}_{g,n}$ be the Teichmüller space of marked complex structures on a finite-type, orientable surface $S_{g,n}$ of genus g with n punctures. A point of $\mathcal{T}_{g,n}$ is specified by a Riemann surface X and a *marking*, meaning a homeomorphism $f : S_{g,n} \to X$ [1]. Two elements (X_1, f_1) and (X_2, f_2) are considered equivalent if there is a biholomorphism $g : X \to Y$ which preserves the marking, meaning that $g \circ f_1$ is isotopic to f_2. Teichmüller space is the space of equivalence classes $[(X, f)]$. To simplify notation, we will often suppress the marking and refer to an element of Teichmüller space by the underlying Riemann surface X.

Quasiconformal maps and Teichmüller distance

The Teichmüller distance between two marked surfaces X,Y in $\mathcal{T}_{g,n}$ measures the minimal conformal distortion of a marking-preserving map $f : X \to Y$. We make this precise below:

A map of Riemann surfaces $f : X \to Y$ is said to be *quasiconformal* if

[1] We require that a neighborhood of each puncture in X be isomorphic to $\mathbb{D} \setminus \{0\}$.

1. f is a homeomorphism.

2. The distributional first derivatives of f are represented by locally L^2 functions.

3. The *Beltrami coefficient*
$$\mu^f = \frac{f_{\bar{z}}}{f_z}\frac{d\bar{z}}{dz} \tag{1.1}$$
has essential supremum less than 1: $\|\mu^f\|_\infty < 1$.

The third condition expresses the act that f has bounded conformal distortion: Assume f is differentiable at $p \in X$. The image under the derivative df_p of a circle in T_pX is an ellipse, whose eccentricity is $K_p = \frac{1+|\mu^f|(p)}{1-|\mu^f|(p)}$. The condition $\|\mu^f\|_\infty < 1$ implies that the dilatation $\mathrm{dil}_f = \mathrm{essup}|K_p|$ is finite.

Given two surfaces $X, Y \in \mathcal{T}_{g,n}$, the *Teichmüller distance* is defined to be

$$K_{\mathcal{T}}(X,Y) = \frac{1}{2}\inf_f \log\left(\mathrm{dil} f\right),$$

where f ranges over all marking-preserving quasiconformal homeomorphisms $f : X \to Y$.

Infinitesimal structure and the Teichmüller norm

The Teichmüller distance also has a description as the path metric associated to a certain Finsler norm on tangent spaces. We first need a description of the tangent and cotangent spaces to a point $X \in \mathcal{T}_{g,n}$. Let $B(X)$ denote the space of L^∞ (1,-1) tensor fields on a Riemann surface $X \in \mathcal{T}_{g,n}$ and let $B_1(X)$ denote the unit ball. A marking-preserving quasi-conformal map $f : X \to Y$ yields an element $\mu^f \in B_1(X)$ as described in equation (1.1). Conversely, according to the measurable uniformization theorem, given $\mu \in B_1(X)$, there is a Riemann surface Y and a marking-preserving quasiconformal $f : X \to Y$ with $\mu^f = \mu$. Moreover, Y is unique up to Teichmüller equivalence. Thus, we have a natural surjection $\pi : B_1(X) \to \mathcal{T}_{g,n}$. Differentiating π at zero yields an identification of the tangent space $T_X\mathcal{T}_{g,n}$ with the quotient $B(X)/\ker(d\pi_0)$. There is a natural integration pairing between $B(X)$ and the space $Q(X)$ of L^1 holomorphic quadratic differentials on X. [2]. It turns out that $\ker(d\pi_0)$ is precisely the orthocomplement $Q(X)^\perp$ with respect to the pairing. Thus, $Q(X)$ identifies naturally with the cotangent space

[2]A holomorphic quadratic differential is L^1 if and only if it has at worst a first-order pole at each puncture.

$T_X^*\mathcal{T}_{g,n}$. The infinitesimal Teichmüller norm is the Finsler norm on $T_X\mathcal{T}_{g,n}$ dual to the L^1 norm on $Q(X)$, and the global Teichmüller-Kobayashi metric $K_{\mathcal{T}}$ is the corresponding path metric on $\mathcal{T}_{g,n}$.

The complex structure on Teichmüller space

The natural complex structure on $\mathcal{T}_{g,n}$ is the unique one making $\pi : B_1(X) \to \mathcal{T}_{g,n}$ a holomorphic submersion. There are many other routes to define the complex structure on $\mathcal{T}$, but the complex structure is characterized by the fact that holomorphic maps $\mathbb{D} \to \mathcal{T}_{g,n}$ are in bijection with holomorphic submersions $p : F \to \mathbb{D}$ over $\mathbb{D}$ whose fibers $p^{-1}(z)$ are Riemann surface of type g, n.

Quadratic differentials and Teichmüller disks

Given a unit-norm differential $\phi \in Q(X)$, the Beltrami differential $\mu_\phi = \frac{|\phi|}{\phi}$ is the unique unit vector satisfying $(\phi, \mu_\phi) = 1$. The *Teichmüller disk* $\tau^\phi : \mathbb{D} \to \mathcal{T}_{g,n}$ generated by ϕ is defined by $\tau^\phi(z) = \pi(z\mu_\phi)$. The Teichmüller disk τ^ϕ is an isometric embedding with respect to the Poincaré metric of curvature -4 on $\mathbb{D}$ and the Teichmüller metric on $\mathcal{T}$.

Half-translation surfaces and the $\mathrm{GL}_2^+(\mathbb{R})$ action.

The Teichmüller disk τ^ϕ may be described more geometrically in terms of the *half translation structure* associated to ϕ. Away from the zeros and poles of ϕ, X admits charts in which ϕ takes the form $(dz)^2$. The transitions between these *flat charts* are of the form $z \mapsto \pm z + c$ with $c \in \mathbb{C}$. Thus, ϕ induces on the complement of the zeros and poles of ϕ a flat metric and a distinguished foliation by horizontal geodesics. These two pieces of data give X the structure of a *half-translation surface*. Conversely, the data of a half-translation surface determines a Riemann surface with a quadratic differential. The group $\mathrm{GL}_2^+(\mathbb{R})$ of 2-by-2 matrices with positive determinant acts on the space of quadratic differentials (equivalently, half-translation surfaces) by post-composition of flat charts. One checks that (the image of) the Teichmüller disk generated by ϕ is the projection to Teichmüller space of the orbit $\mathrm{GL}_2^+(\mathbb{R})\phi$. In other words, the Teichmüller disk associated to ϕ is the collection of Riemann surfaces obtained via affine deformations of ϕ.

Mapping classes and biholomorphisms

The mapping class group

$$\mathrm{MCG}_{g,n} = \mathrm{Homeo}^+(S_{g,n})/\mathrm{Homeo}_0(S_{g,n})$$

is the group of orientation-preserving homeomorphisms of $S_{g,n}$ modulo the homeomorphisms isotopic to the identity. The mapping class group acts properly discontinuously on $\mathcal{T}_{g,n}$ by precomposition of markings, and The action of each mapping class is a biholomorphism. The quotient $\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\mathrm{MCG}_{g,n}$ is the moduli space of (unmarked) complex structures on $S_{g,n}$. A key type of a mapping class is a *Dehn twist* about an essential simple closed curve $\gamma \in S_{g,n}$. A Dehn twist is obtained by cutting out a normal neighborhood $\mathbb{R}/\mathbb{Z} \times [0,1]$ of γ, twisting by $(s,t) \mapsto (s, s+t)$ and regluing. Dehn twists about core curves of Jenkins-Strebel differentials will be key in Chapters II and III.

Chapter 2

ASYMPTOTICS OF THE TRANSLATION FLOW

2.1 Introduction

Let $\mathbb{H}$ be the upper half-plane $\mathbb{H} = \{\lambda \in \mathbb{C} | \text{Im}(\lambda) > 0\}$. The poly-plane $\mathbb{H}^n = \mathbb{H} \times \cdots \times \mathbb{H}$ is the n-fold product of $\mathbb{H}$ with itself.

Let $\mathcal{D}$ be the family of holomorphic functions $f : \mathbb{H}^n \to \mathbb{H}$ which restrict to the identity on the diagonal, i.e., $f(\lambda, \ldots, \lambda) = \lambda$ for all $\lambda \in \mathbb{H}$. Fix $t \in \mathbb{R}$. If f is in $\mathcal{D}$, then so is the map f_t defined by

$$f_t(z_1, \ldots, z_n) = f(z_1 - t, \ldots, z_n - t) + t. \tag{2.1}$$

The action $(f, t) \mapsto f_t$ is called the *translation flow* on $\mathcal{D}$.

In this paper, we study the asymptotics of the translation flow. Suppose $f \in \mathcal{D}$, and let $\alpha_j = \frac{\partial f}{\partial z_j}(i, \ldots, i)$ for $j = 1, \ldots, n$. Our main result is that for "most" $t \in \mathbb{R}$, f_t is "close" to the translation-invariant function $\mathbf{g}(z_1, \ldots, z_n) = \sum_{j=1}^n \alpha_j z_j$. More precisely, we prove

Theorem 2.4.1 . *Let U be any open neighborhood of* $\mathbf{g}$ *in the compact-open topology. Choose t uniformly at random in* $[-r, r]$. *The probability that f_t is in U tends to 1 as $r \to \infty$.*

The motivation for this work comes from the study of the Kobayashi and Carathéodory metrics on Teichmüller space (see Section 2.1). Let $\mathcal{T}$ denote the Teichmüller space of a finite-type orientable surface. A *Teichmüller disk* $\tau : \mathbb{H} \to \mathcal{T}$ is a complex geodesic for the Kobayashi metric on $\mathcal{T}$. It is an open problem to classify Teichmüller disks on which the Kobayashi and Carathéodory metrics coincide. To say that the metrics agree on $\tau(\mathbb{H})$ means exactly that there is a *holomorphic retraction* onto τ, i.e., a holomorphic $\Psi : \mathcal{T} \to \mathbb{H}$ so that $\Psi \circ \tau = \text{id}_{\mathbb{H}}$.

In recent work with Markovic [15], we classify holomorphic retracts in the Teichmüller space of the five-times punctured sphere. Key to our argument is the observation that certain Teichmüller disks τ factor as

$$\mathbb{H} \xrightarrow{\Delta} \mathbb{H}^n \xrightarrow{\mathcal{E}} \mathcal{T},$$

where Δ is the diagonal mapping and $\mathcal{E}$ is a particular naturally-defined holomorphic embedding. If $\Psi : \mathcal{T} \to \mathbb{H}$ is a holomorphic retraction onto $\tau(\mathbb{H})$, then $f = \Psi \circ \mathcal{E} : \mathbb{H}^n \to \mathbb{H}$ is a holomorphic retraction onto the diagonal, i.e., f is in $\mathcal{D}$. In [15], we use the properties of $\mathcal{D}$ developed in this paper to glean information about holomorphic maps out of Teichmüller space.

The translation flow (2.1) should be viewed in the context of unipotent dynamics. The translation flow on $\mathcal{D}$ extends to an action of $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R})$ (See Section 2.5). Equation (2.1) gives the action of the unipotent subgroup

$$U = \left\{ \left(\begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right) \middle| t \in \mathbb{R} \right\}.$$

Analogously, there is a natural $\mathrm{PSL}_2(\mathbb{R})$ action on the unit cotangent bundle $T_1^*\mathcal{T}$ of Teichmüller space. The restriction of this action to U is called the *horocycle flow*. Our methods in [15] are summarized as follows: First, use results on horocycle flow in $T_1^*\mathcal{T}$ [36] to reduce to an appropriate class of Teichmüller disks. Next, use translation flow in $\mathcal{D}$ and the results of this paper to analyze retractions onto disks in that class.

Generalizing from the case of translations acting on holomorphic maps $\mathbb{H}^n \to \mathbb{H}$, it is natural to ask the following question: given two Hermitian symmetric spaces X_1 and X_2, what can one say about the dynamics of subgroups of $\mathrm{Aut}(X_1) \times \mathrm{Aut}(X_2)$ acting on subsets of the space of holomorphic functions $\mathcal{O}(X_1, X_2)$? To our knowledge, there is no previous work in the literature explicitly addressing this question. There has however been much interest in the dynamics of linear operators acting on holomorphic function spaces (see [7]). In Section 2.1 we use results [9][16] on linear dynamics to study the analogue of translation flow for maps $\mathbb{C}^k \to \mathbb{C}$. In this context, the flow is chaotic and behaves quite differently than the flow on maps $\mathbb{H}^k \to \mathbb{H}$.

The key tool in the proof of our main result is a multivariate version of the Schwarz lemma (see Section 2.1). Our methods are inspired by Knese's work [23] on extremal maps $\mathbb{D}^n \to \mathbb{D}$.

The Carathéodory and Kobayashi metrics on Teichmüller space

The Carathéodory pseudometric d_C on a complex manifold X assigns to two points $p, q \in X$ the distance

$$d_C(p,q) \equiv \sup_f d_{\mathbb{H}}(f(p), f(q)),$$

where the supremum is taken over all holomorphic maps $f : X \to \mathbb{H}$, and $d_{\mathbb{H}}$ is the Poincaré metric. In other words, d_C is the smallest pseudometric on X so that every holomorphic map from X to $\mathbb{H}$ is length-decreasing.

The Kobayashi pseudometric d_K on X is defined in terms of maps $\mathbb{H} \to X$. It is the largest pseudometric on X so that every holomorphic map from $\mathbb{H}$ to X is length-decreasing.

The Kobayashi and Carathéodory metrics on $\mathbb{H}^n$ are both given by

$$d_{\mathbb{H}^n}(z, w) = \max_j d_{\mathbb{H}}(z_j, w_j).$$

In general, the Schwarz lemma implies $d_C \leq d_K$ for any complex manifold. However, it is usually difficult to determine if $d_C = d_K$ for a given complex manifold X.

In [27], Markovic proves that d_C and d_K do not agree on the Teichmüller space of a closed orientable surface of genus ≥ 2. Let $\mathcal{T}$ be the Teichmüller space of a finite-type orientable surface. Given a rational Strebel differential ϕ with characteristic annuli $\Pi_1, \ldots, \Pi_n$, Markovic defines a holomorphic map $\mathcal{E}^{\phi} : \mathbb{H}^n \to \mathcal{T}$. The marked surface $\mathcal{E}^{\phi}(z_1, \ldots z_n)$ is constructed by applying the affine transformation $x + iy \mapsto x + z_j y$ to Π_j. In particular, the restriction of $\mathcal{E}^{\phi}$ to the diagonal is the Teichmüller disk generated by ϕ. Let $\alpha_j = \left(\int_{\Pi_j} |\phi|\right) / \|\phi\|_1$. Markovic proves the following:

Proposition 2.1.1. *If the metrics d_C and d_K agree on the Teichmüller disk generated by ϕ, then there is a holomorphic function $\Psi : \mathcal{T} \to \mathbb{H}$ and a real constant T so that $f = \Psi \circ \mathcal{E}^{\phi}$ satisfies*

$$f(\lambda, \ldots, \lambda) = \lambda \tag{A}$$

$$\frac{\partial f}{\partial z_j}(\lambda, \ldots, \lambda) = \alpha_j \tag{B}$$

$$f(z_1 + T, z_2 + T, \ldots, z_n + T) = f(z_1, z_2, \ldots, z_n) + T \tag{C}$$

for all $\lambda \in \mathbb{H}$, $(z_1, \ldots, z_n) \in \mathbb{H}^n$, $j = 1, \ldots, n$.

Markovic then proves

Proposition 2.1.2. *For $n = 2$, the only holomorphic $f : \mathbb{H}^2 \to \mathbb{H}$ satisfying conditions* (A),(B),(C) *is $f(z_1, z_2) = \alpha_1 z_1 + \alpha_2 z_2$.*

So if ϕ has exactly two characteristic annuli, there is a $\Psi : \mathcal{T} \to \mathbb{H}$ such that $\Psi \circ \mathcal{E}^\phi = \alpha_1 z_1 + \alpha_2 z_2$. This criterion is then used to show that d_C and d_K do not agree on the Teichmüller disk generated by an L-shaped pillowcase with rational edge lengths.

As a corollary of our main result Theorem 2.4.1, we obtain the generalization of Proposition 2.1.2 to arbitrary n:

Corollary 2.4.3. *The only holomorphic* $f : \mathbb{H}^n \to \mathbb{H}$ *satisfying* (A),(B),(C) *is* $f(z_1, \dots, z_n) = \sum_{j=1}^n \alpha_j z_j$.

Taken together, Proposition 2.1.1 and Corollary 2.4.3 yield the following criterion for determining whether d_C and d_K agree on the Teichmüller disk generated by a rational Strebel differential.

Proposition 2.1.3. *Let ϕ be a rational Jenkins-Strebel differential, with characteristic annuli $\Pi_1, \dots, \Pi_n$. Suppose d_C and d_K agree on the Teichmüller disk generated by ϕ. Then there exists a holomorphic map $\Phi : \mathcal{T} \to \mathbb{H}^n$ such that*

$$\Phi \circ \mathcal{E}^\phi(z_1, \dots, z_n) = \alpha_1 z_1 + \cdots + \alpha_n z_n,$$

where $\alpha_j = \left(\int_{\Pi_j} |\phi| \right) / \|\phi\|_1$.

Remark: Markovic showed that there are Teichmüller disks on which $d_C \neq d_K$. On the other hand, Kra [24] proved that $d_C = d_K$ on every Teichmüller disk generated by a holomorphic quadratic differential with no odd-order zeros. This raises a natural question: for which quadratic differentials do the Carathéodory and Kobayashi metrics on the corresponding disk agree? A natural conjecture is that the converse of Kra's result holds: $d_C = d_K$ on a Teichmüller disk if and only if the generating differential has no odd-order zeros. In a recent paper [15] we prove this conjecture in the case of the five-times punctured sphere and twice-punctured torus. Key to the proof is the fact that Proposition 2.1.3 continues to hold without the rationality assumption. This fact in turn hinges on the main result Theorem 2.4.1 of this paper. (The weaker result Corollary 2.4.3 is insufficient to deal with the irrational case.)

The Schwarz lemma and extremal maps

Let $\mathbb{D}$ be the open unit disk in the complex plane. The classical Schwarz lemma states that, if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$(1 - |z|)^2 |f'(z)| \leq 1 - |f(z)|^2, \tag{2.2}$$

for all $z \in \mathbb{D}$. If equality holds in (2.2) for some $z \in \mathbb{D}$, then it holds for all z. In this case, f is a conformal automorphism of $\mathbb{D}$.

The Schwarz lemma has the following generalization for holomorphic maps f from the polydisk $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$ to $\mathbb{D}$ (see page 179 of [35]):

$$\sum_{j=1}^{n} (1 - |z_j|^2) \left| \frac{\partial f}{\partial z_j}(z) \right| \leq 1 - |f(z)|^2 , \tag{2.3}$$

for every $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$. To understand (2.3), we recall the following definitions: A *balanced disk* in $\mathbb{D}^n$ is a copy of $\mathbb{D}$ embedded in $\mathbb{D}^n$ by a map of the form

$$\Phi : z \mapsto (\phi_1(z), \ldots, \phi_n(z)) ,$$

where $\phi_i \in \mathrm{Aut}(\mathbb{D})$. A balanced disk Φ is called *extreme* for f if the restriction $f \circ \Phi$ is in $\mathrm{Aut}(\mathbb{D})$. The content of (2.3) is that the restriction of f to every balanced disk satisfies the classical Schwarz lemma. Equality in (2.3) means that z is contained in some extreme disk for f.

The *extreme set* $X(f)$ is the union of the extreme disks of f. In other words, $X(f)$ is the set of points $z \in \mathbb{D}^n$ for which equality holds in (2.3). In [23], Knese classifies maps $f : \mathbb{D}^n \to \mathbb{D}$ for which $X(f) = \mathbb{D}^n$. Such maps are called *everywhere extremal*, or simply *extremal*. Knese shows that extremal maps $\mathbb{D}^n \to \mathbb{D}$ form a special class of rational functions parameterized by $(n + 1) \times (n + 1)$ symmetric unitary matrices.

The upper half-plane $\mathbb{H}$ is conformally equivalent to $\mathbb{D}$ via the Cayley transform $z \mapsto \frac{i-z}{i+z}$. For holomorphic maps $f : \mathbb{H}^n \to \mathbb{H}$, the generalized Schwarz lemma becomes

$$\sum_{j=1}^{n} \mathrm{Im}(z_j) \left| \frac{\partial f}{\partial z_j}(z) \right| \leq \mathrm{Im} f(z). \tag{2.4}$$

The families $\mathcal{D}, \mathcal{C}$

Consider the family $\mathcal{D}$ of holomorphic maps $f : \mathbb{H}^n \to \mathbb{H}$ which restrict to the identity on the diagonal:

$$f(\lambda, \ldots, \lambda) = \lambda \tag{2.5}$$

for all $\lambda \in \mathbb{H}$. $\mathcal{D}$ is a natural class to consider; it is the collection of maps $\mathbb{H}^n \to \mathbb{H}$ with a distinguished extreme disk. After pre- and post-composing by biholomorphisms, any holomorphic map $\mathbb{H}^n \to \mathbb{H}$ with an extreme disk becomes an element of $\mathcal{D}$.

Differentiating both sides of (2.5) with respect to λ yields

$$\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(\lambda, \dots, \lambda) = 1.$$

But by the generalized Schwarz lemma (2.4),

$$\sum_{j=1}^{n} \left| \frac{\partial f}{\partial z_j}(\lambda, \dots, \lambda) \right| \leq 1.$$

So $\frac{\partial f}{\partial z_j}(\lambda, \dots, \lambda) \geq 0$ for all $\lambda \in \mathbb{H}$ and $j = 1, \dots, n$. By the open mapping theorem, $\lambda \mapsto \frac{\partial f}{\partial z_j}(\lambda, \dots, \lambda)$ is constant. So f satisfies

$$\frac{\partial f}{\partial z_j}(\lambda, \dots, \lambda) = \alpha_j$$

for all $\lambda \in \mathbb{H}$, for some collection of nonnegative constants α_j summing to 1.

In the rest of the paper, we assume without loss of generality that $\alpha_j = \frac{1}{n}$. To reduce the general case to this one, suppose $f \in \mathcal{D}$ and $\frac{\partial f}{\partial z_j}(i, \dots, i) = \alpha_j$. Define $g \in \mathcal{D}$ by

$$g(z) = \sum_{j=1}^{n} \left(\frac{1 - \alpha_j}{n - 1} \right) z_j.$$

Then

$$\tilde{f} = \frac{1}{n} f + \frac{n-1}{n} g$$

is in $\mathcal{D}$ and satisfies $\frac{\partial \tilde{f}}{\partial z_j}(i, \dots, i) = \frac{1}{n}$. Since g is invariant under the translation flow, it suffices to consider the translation orbit of $\tilde{f}$.

With these considerations in mind, we define C to be the family of holomorphic maps $\mathbb{H}^n \to \mathbb{H}$ satisfying

$$f(\lambda, \dots, \lambda) = \lambda, \tag{A}$$

$$\frac{\partial f}{\partial z_j}(\lambda, \dots, \lambda) = \frac{1}{n}, \tag{B}$$

for all $\lambda \in \mathbb{H}$ and $j = 1, \dots, n$.

When convenient, we view C as the family of maps $\mathbb{D}^n \to \mathbb{D}$ satisfying the same conditions. (Conjugation by the Cayley transform $\mathbb{H} \to \mathbb{D}$ preserves (A),(B).)

Remark: Conditions (A) and (B) hold for all $\lambda \in \mathbb{H}$ iff they both hold for some $\lambda \in \mathbb{H}$.

Extremal maps in dimension two

In [23], Knese showed that extremal maps $g : \mathbb{D}^2 \to \mathbb{D}$ satisfying $g(0,0) = 0$ are all of form

$$g(z,w) = \mu \frac{az + bw - zw}{1 - \overline{b}z - \overline{a}w},$$

where $|\mu| = |a| + |b| = 1$. Imposing $f(\lambda,\lambda) = \lambda$ and $\frac{\partial f}{\partial z}(\lambda,\lambda) = \frac{\partial f}{\partial w}(\lambda,\lambda) = \frac{1}{2}$, we find that the extremal elements of C are the functions of form

$$g_\nu(z,w) = \frac{\nu(\frac{z}{2} + \frac{w}{2}) - zw}{\nu - (\frac{z}{2} + \frac{w}{2})}$$

with $\nu \in \partial\mathbb{D}$.

A direct computation shows that, for any $\gamma \in \mathrm{Aut}(\mathbb{D})$,

$$\gamma \cdot g_\nu = g_{\gamma(\nu)},$$

where $(\gamma \cdot g_\nu)(z_1, z_2) = \gamma g_\nu(\gamma^{-1} z_1, \gamma^{-1} z_2)$. Thus, the set of extremals in C is in $\mathrm{Aut}(\mathbb{D})-$equivariant bijection with $\partial\mathbb{D}$.

Remark: The situation for $n > 2$ is more complicated; one can show using Knese's classification of extremals that the extremals in C constitute a manifold of dimension $\frac{n(n-1)}{2}$.

Conjugating by the Cayley transform, we get a description of the extremal maps $\mathbb{H}^2 \to \mathbb{H}$ in C. They are the functions of form

$$h_r(z,w) = \frac{r(\frac{z}{2} + \frac{w}{2}) - zw}{r - (\frac{z}{2} + \frac{w}{2})},$$

with $r \in \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. In particular,

$$h_\infty(z,w) = \frac{z}{2} + \frac{w}{2}.$$

One can check that the extreme disks for h_∞ are precisely those of form $\{(z, az + b) \,|\, z \in \mathbb{H}\}$, where $a > 0$ and $b \in \mathbb{R}$. It follows, more generally, that the extreme disks for h_r are those of form $\{(z, \phi(z)) \,|\, z \in \mathbb{H}\}$, with $\phi \in \mathrm{Stab}(r)$.

Example In [23], Knese constructed a holomorphic map $\mathbb{D}^2 \to \mathbb{D}$ which has two extreme disks, yet is not everywhere extremal. Below, we give an example of a map $\mathbb{H}^2 \to \mathbb{H}$ which is extremal on every disk of the form $\{(z, az) | z \in \mathbb{H}\}$ with $a > 0$, yet is not everywhere extremal.

Given $r, s \in \partial\mathbb{H}$, $\mathrm{Stab}(r) \cap \mathrm{Stab}(s)$ is the set of isometries preserving the hyperbolic geodesic with endpoints r, s. For example, $\mathrm{Stab}(0) \cap \mathrm{Stab}(\infty)$ consists of isometries preserving the positive imaginary axis; these are of form $z \mapsto az$ with $a > 0$. So the disks $D_a = \{(z, az) | z \in \mathbb{H}\}$ are extreme for both $h_\infty(z,w) = \frac{z+w}{2}$ and $h_0(z,w) = \frac{2zw}{z+w}$. In fact, the D_a are extreme for any convex combination

$$f^t = th_\infty + (1-t)h_0,$$

with $t \in (0,1)$. Indeed,

$$f^t(z, az) = \left(t\frac{1+a}{2} + (1-t)\frac{2a^2}{1+a}\right)z.$$

So the extreme set $X(f^t)$ contains a set of real dimension 3. Yet f^t is not everywhere extremal, as $f^t \neq h_r$ for any $r \in \partial\mathbb{H}$.

Translation flow in dimension 2

In dimension 2, C can be parameterized explicitly using Nevanlinna-Pick interpolation on the bidisk. The maps $\mathbb{D}^2 \to \mathbb{D}$ belonging to C are precisely those of form

$$f(z,w) = \frac{1}{2}(z+w) + \frac{1}{4}(z-w)^2 \frac{\Theta(z,w)}{1 - \frac{1}{2}(z+w)\Theta(z,w)}, \tag{2.6}$$

where Θ is any holomorphic map from $\mathbb{D}^2$ to the closed disk $\overline{\mathbb{D}}$. (See page 189 of [1].)

To parameterize maps $\mathbb{H}^2 \to \mathbb{H}$ in C, we conjugate (2.6) by the Cayley transform. We get the same general form, with Θ any holomorphic map from $\mathbb{H}^2$ to the closure $\overline{\mathbb{H}}$ of $\mathbb{H}$ in the Riemann sphere. Substituting $\Theta = -\frac{1}{\Phi}$, (2.6) becomes

$$f(z,w) = \frac{\frac{z+w}{2} \cdot \Phi(z,w) + zw}{\Phi(z,w) + \frac{z+w}{2}}. \tag{2.7}$$

The extremal map h_r corresponds to $\Phi \equiv -r$. In particular, $h_\infty(z,w) = \frac{z+w}{2}$ corresponds to $\Phi \equiv \infty$.

Applying translation flow to (2.7) yields

$$f(z-t, w-t) + t = \frac{\frac{z+w}{2} \cdot [\Phi(z-t, w-t) - t] + zw}{[\Phi(z-t, w-t) - t] + \frac{z+w}{2}}. \tag{2.8}$$

One can show that for randomly chosen real t, $|\Phi(z-t, w-t) - t|$ is very large, so that (2.8) is very close to $\frac{z+w}{2}$. This yields a proof of Theorem 2.4.1 in dimension 2.

Translation flow for maps $\mathbb{C}^k \to \mathbb{C}$

Let $\mathcal{D}'$ denote the space of holomorphic maps $f : \mathbb{C}^k \to \mathbb{C}$ satisfying $f(z, \ldots, z) = z$ for all $z \in \mathbb{C}$. Define translation flow on $\mathcal{D}'$ by the same formula

$$f_t(z_1, \ldots, z_k) = f(z_1 - t, \ldots, z_k - t) + t$$

as the flow on $\mathcal{D}$.

The main results of this paper state that translation flow on $\mathcal{D}$ is "unchaotic." Theorem 2.4.1 asserts that the orbit any $f \in \mathcal{D}$ is concentrated at a single point, while 2.4.3 states that the periodic points lie in a finite-dimensional subspace of $\mathcal{D}$. In stark contrast, the flow on $\mathcal{D}'$ has orbits which equidistribute; moreover, the set of periodic points is dense. This contrast should be viewed in light of the fact that, unlike $\mathcal{D}$, the space $\mathcal{D}'$ is not compact.

Proposition 2.1.4. *There is a probability measure μ on $\mathcal{D}'$ which is ergodic with respect to translation flow and whose support is the entire space $\mathcal{D}'$. In particular, a dense set of $f \in \mathcal{D}'$ have μ-equidistributed orbits under translation flow.*

Remark: Another way of stating the main result Theorem 2.4.1 is that any ergodic probability measure for translation flow on $\mathcal{D}$ is a delta measure supported at a point $g \in \mathcal{D}$ of form $g(\mathbf{z}) = \sum_j \alpha_j z_j$ (see Proposition 2.4.2).

Proposition 2.1.5. *The set of periodic points for the translation flow on $\mathcal{D}'$ is dense.*

Propositions 2.1.4 and 2.1.5 follow easily from the following results on linear dynamics:

Proposition 2.1.6 (Bonilla, Grosse-Erdmann [9]). *Let L be any continuous linear operator on $\mathcal{O}(\mathbb{C}^n)$ which commutes with the differential operators $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$. Then L is ergodic with respect to a full-support probability measure.*

Proposition 2.1.7 (Godefrey, Shapiro [16]). *Under the hypotheses of Proposition 2.1.6, L has a dense set of periodic points.*

Proof of Propositions 2.1.4 and 2.1.5: Let $S : \mathcal{D}' \to \mathcal{D}'$ be the time-one translation $f \mapsto f_1$. It suffices to show that S has a dense set of periodic points and an ergodic probability measure μ with full support. (To obtain the desired flow-invariant measure, average μ over the flow from time 0 to time 1.)

Propositions 2.1.6 and 2.1.7 apply to the operator L on $\mathcal{O}(\mathbb{C}^n)$ defined by

$$L\phi(z_1,\ldots,z_n) = \phi(z_1-1,\ldots,z_n-1).$$

It thus suffices to exhibit a continuous surjection

$$\mathcal{O}(\mathbb{C}^n) \to \mathcal{D}',$$

intertwining the actions of L and S.

To this end, define $g \in \mathcal{O}(\mathbb{C}^k)$ by $g(z_1,\ldots,z_n) = \frac{1}{n}\sum_j z_j$, and let $\mathbf{1} \in \mathbb{C}^n$ denote the vector with all entries equal to 1. The map $F : \mathcal{O}(\mathbb{C}^n) \to \mathcal{D}'$ associating to each $\phi \in \mathcal{O}(\mathbb{C}^n)$ the function

$$f(\mathbf{z}) = \phi(\mathbf{z}) - \phi(g(\mathbf{z})\mathbf{1}) + g(\mathbf{z})$$

is the desired surjection. It is easy to check F intertwines the actions of L and S. Moreover, the map $\mathcal{D}' \to \mathcal{O}(\mathbb{C}^n)$ sending $f \in \mathcal{D}'$ to $\phi(\mathbf{z}) = f(\mathbf{z}) - g(\mathbf{z})$ is a right inverse for F. □

Outline

The rest of the paper will focus on the proof of our main result, Theorem 2.4.1. The key observation is that $\mathbf{g}(z) = \frac{1}{n}\sum_{j=1}^n z_j$ is an everywhere extremal map from $\mathbb{H}^n$ to $\mathbb{H}$.

In Section 2.2, we show that extremals in C are extreme points of C, in the sense of convex analysis. More precisely, we prove

Proposition 2.2.3. *If $g \in C$ is extremal and μ is a Borel probability measure on C such that*

$$\int_C f(z)d\mu(f) = g(z)\ \forall z \in \mathbb{H}^n,$$

then μ is the Dirac measure δ_g concentrated at the point $g \in C$.

Then, in Section 2.3 we show that the average of any $f \in C$ over the translation flow is $\mathbf{g}(z) = \frac{1}{n}\sum_{j=1}^n z_j$. That is, we prove

Proposition 2.3.1. *Let $f \in C$. For each $t \in \mathbb{R}$, define $f_t(z_1,\ldots,z_n) = f(z_1 - t,\ldots,z_n - t) + t$. Then $\frac{1}{2r}\int_{-r}^{r} f_t(z)dt$ converges locally uniformly to $\mathbf{g}(z)$ as $r \to \infty$.*

In Section 2.4, we prove the main result. To apply Proposition 2.2.3, we consider the measure μ_r on C obtained by pushing forward the uniform probability measure on $[-r,r]$ via the map $t \mapsto f_t$. The desired result is that $\mu_r \to \delta_{\mathbf{g}}$ as $r \to \infty$. Propositions 2.2.3, 2.3.1 imply that $\delta_{\mathbf{g}}$ is the only accumulation point of $\{\mu_r\}_{r>0}$. The main result then follows by the Banach-Alaoglu theorem.

In Section 2.5, we rephrase our results in a more invariant form, in terms of the conjugation action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathcal{D}$. In Section 2.6, we establish a rigidity result used in the proof of Proposition 2.3.1, and in the Appendix, we discuss generalizations of the classical polarization principle.

2.2 Convexity and extreme points

Let C be the family of holomorphic maps $\mathbb{H}^n \to \mathbb{H}$ satisfying

$$f(\lambda, \dots, \lambda) = \lambda, \tag{A}$$

$$\frac{\partial f}{\partial z_j}(\lambda, \dots, \lambda) = \frac{1}{n} \tag{B}$$

for all $\lambda \in \mathbb{H}$ and $j = 1, \dots, n$.

Recall that an extremal map $g : \mathbb{H}^n \to \mathbb{H}$ is a holomorphic function satisfying

$$\sum_{j=1}^{n} \mathrm{Im}(z_j) \left| \frac{\partial g}{\partial z_j}(z) \right| = \mathrm{Im}\, g(z)$$

for all $z = (z_1, \dots, z_n) \in \mathbb{H}^n$.

Observe that C is a convex subset of the holomorphic functions on $\mathbb{H}^n$. Our next result is that every extremal in C is an extreme point in the sense of convex analysis.

Proposition 2.2.1. *Suppose $g \in C$ is extremal. If $g = tf_1 + (1-t)f_2$, with $f_i \in C$ and $t \in (0,1)$, then $f_1 = f_2 = g$.*

Proof: We have

$$\begin{aligned} t\,\mathrm{Im}(f_1) + (1-t)\mathrm{Im}(f_2) &= \sum_{j=1}^{n} \mathrm{Im}(z_j) \left| t\frac{\partial f_1}{\partial z_j} + (1-t)\frac{\partial f_2}{\partial z_j} \right| \\ &\leq \sum_{j=1}^{n} \mathrm{Im}(z_j) \left[t\left|\frac{\partial f_1}{\partial z_j}\right| + (1-t)\left|\frac{\partial f_2}{\partial z_j}\right| \right] \\ &\leq t\,\mathrm{Im}(f_1) + (1-t)\mathrm{Im}(f_2), \end{aligned} \tag{2.9}$$

where in the first line we've used that g is extremal, and in the third we've applied (2.4) to f_1, f_2. Thus,

$$\left| t\frac{\partial f_1}{\partial z_j}(z) + (1-t)\frac{\partial f_2}{\partial z_j}(z) \right| = t\left|\frac{\partial f_1}{\partial z_j}(z)\right| + (1-t)\left|\frac{\partial f_2}{\partial z_j}(z)\right|$$

for $j = 1, \ldots, n$ and all $z \in \mathbb{H}^n$.

So

$$\left(\frac{\partial f_1}{\partial z_j}\right)\left(\frac{\partial g}{\partial z_j}\right)^{-1} \geq 0,$$

whenever $\frac{\partial g}{\partial z_j} \neq 0$, and similarly for f_2. Let $U \subset \mathbb{H}^n$ be the complement of the zero set of $\frac{\partial g}{\partial z_j}$. By (B), $\frac{\partial g}{\partial z_j}$ is not identically zero, so U is a dense connected subset of $\mathbb{H}^n$. The open mapping theorem now implies that $\left(\frac{\partial f_1}{\partial z_j}\right)\left(\frac{\partial g}{\partial z_j}\right)^{-1}$ is a nonnegative constant on U. Again by (B),

$$\frac{\partial f_1}{\partial z_j} = \frac{\partial g}{\partial z_j}$$

on U and, thus, on all of $\mathbb{H}^n$. Since the first derivatives of f_1 and g are the same, f_1 and g differ by a constant. By (A), $f_1 = g$. Similarly, $f_2 = g$. □

The last result implies that if a finite convex combination

$$g = \sum_k t_k f_k$$

of elements of C is extremal, then the f_k are all equal to g. We will show, more generally, that if μ is a Borel probability measure on C such that

$$g = \int_C f \, d\mu(f)$$

is extremal, then $\mu = \delta_g$. Before we consider Borel measures on the space C, we need to understand the space's basic topological properties.

Proposition 2.2.2. *The family C is compact and metrizable in the compact-open topology.*

Proof: Metrizability is standard: Choose a compact exhaustion $K_1, K_2, \ldots$ of $\mathbb{H}^n$, and set $d_j(f,g) = \sup_{z \in K_j} |f(z) - g(z)|$. Then the metric

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(f,g)}{1 + d_j(f,g)}$$

induces the compact-open topology.

To prove compactness, we need to show that C is precompact and closed in $\mathcal{O}(\mathbb{H}^n)$. By the definition of the Carathéodory metric, any holomorphic map $\mathbb{H}^n \to \mathbb{H}$ decreases Carathéodory distance. Thus, every $f \in C$ satisfies

$$d_{\mathbb{H}}(f(z_1, \ldots, z_n), i) \leq d_{\mathbb{H}^n}\left((z_1, \ldots, z_n), (i, \ldots, i)\right).$$

The right side of the inequality is continuous in the z_j. So C is locally uniformly bounded and thus precompact. The inequality also implies that any accumulation point of C has image contained in $\mathbb{H}$. Furthermore, (A) and (B) are closed conditions. Thus, C is closed in $\mathcal{O}(\mathbb{H}^n)$. □

Let μ be a Borel probability measure on C. For each $z \in \mathbb{H}^n$, the evaluation map $f \mapsto f(z)$ is a continuous function on the compact space C. So the evaluation map is μ-integrable. We denote its integral by $\int_C f(z)d\mu(f)$.

Proposition 2.2.3. *Suppose $g \in C$ is extremal. Let μ be a Borel probability measure on C. Suppose $\int_C f(z)d\mu(f) = g(z)$ for all $z \in \mathbb{H}^n$. Then μ is δ_g, the Dirac measure concentrated at g.*

Proof: Though this result can be derived as a formal consequence of Proposition 2.2.1, we prefer to give a direct proof.

The proof is similar to that of Proposition 2.2.1. To establish the analog of equality (2.9), we need to differentiate $\int_C f(z)d\mu(f)$ under the integral sign; Proposition 2.2.2 implies that the family $\{\frac{\partial f}{\partial z_j} | f \in C\}$ is locally uniformly bounded, which justifies switching $\int$ and $\frac{\partial}{\partial z_j}$.

Let U be the complement of the zero set of $\frac{\partial g}{\partial z_j}$. Fix $z \in U$. Arguing as before, we get

$$\left(\frac{\partial f}{\partial z_j}(z)\right)\left(\frac{\partial g}{\partial z_j}(z)\right)^{-1} \geq 0, \tag{2.10}$$

for μ-almost-every f. A countable intersection of full-measure subsets of C has full measure. Thus, for μ-a.e. f, (2.10) holds at all $z \in U$ with rational coordinates. By continuity, μ-a.e. f satisfies (2.10) on U. We conclude that μ-a.e. f is equal to g. This means that $\mu = \delta_g$. □

2.3 Averaging over translations

Let $f : \mathbb{H}^n \to \mathbb{H}$ be a holomorphic map. For each $t \in \mathbb{R}$, we define

$$f_t(z_1, \ldots, z_n) = f(z_1 - t, \ldots, z_n - t) + t.$$

The action $(f,t) \mapsto f_t$ is the translation flow on $\mathcal{O}(\mathbb{H}^n)$. The family C is invariant under the translation flow.

For each $f \in C$ and $r > 0$, we define the average $\mathcal{A}_r[f] \in C$ by

$$\mathcal{A}_r[f](z) = \frac{1}{2r}\int_{-r}^{r} f_t(z)dt.$$

One might expect that averaging $f \in C$ over the entire flow yields an invariant element. This is indeed the case:

Proposition 2.3.1. *For each $f \in C$, $\mathcal{A}_r[f]$ converges locally uniformly to $\mathbf{g}(z) = \frac{1}{n}\sum_{j=1}^{n} z_j$ as $r \to \infty$.*

Proof: Fix $z \in \mathbb{H}^n$. By Proposition 2.2.2, there is a $C(z) > 0$ so that

$$|f(z)| < C(z) \tag{2.11}$$

for all $f \in C$.

Fix $s \in \mathbb{R}$. We use (2.11) to compare $\mathcal{A}_r[f]$ and the translate $(\mathcal{A}_r[f])_s$:

$$\begin{aligned} |\mathcal{A}_r[f](z) - (\mathcal{A}_r[f])_s(z)| &= \frac{1}{2r}\left|\int_{-r}^{-r+s} f_t(z)dt - \int_{r}^{r+s} f_t(z)dt\right| \\ &\le \frac{s}{r}C(z). \end{aligned}$$

Thus, any limit point of the family $\{\mathcal{A}_r[f]\}_{r>0}$ along a sequence with $r \to \infty$ is invariant under all translations. But, as we will show in Proposition 2.6.2, the only translation-invariant element of C is $\mathbf{g}$. Since C is sequentially compact, we get the desired result. □

2.4 The main result

We now use Propositions 2.2.3, 2.3.1 and the Banach-Alaoglu theorem to prove the main result.

Theorem 2.4.1. *Suppose $f : \mathbb{H}^n \to \mathbb{H}$ is holomorphic and satisfies $f(\lambda, \dots, \lambda) = \lambda$ for all $\lambda \in \mathbb{H}$. Let $\alpha_j = \frac{\partial f}{\partial z_j}(i, \dots, i)$, and define $\mathbf{g}(z) = \sum_{j=1}^{n} \alpha_j z_j$. Fix $\varepsilon > 0$, and let U be any open neighborhood of $\mathbf{g}$ in the compact-open topology. Then for sufficiently large r, the set $\{t \in [-\frac{r}{2}, \frac{r}{2}] | f_t \in U\}$ has measure at least $(1-\varepsilon)r$.*

Proof: We may assume without loss of generality that $\alpha_j = \frac{1}{n}$ for $j = 1, \dots, n$. So $f \in C$. Let μ_r be the pushforward to C of the uniform probability measure on $[-r, r]$, via the continuous map $t \mapsto f_t$. Then the desired result is equivalent to the assertion that $\mu_r \to \delta_{\mathbf{g}}$ weakly as $r \to \infty$.

By the Banach-Alaoglu theorem, the space of Borel probability measures on the compact metric space C is sequentially compact. It thus suffices to show that any limit point μ of $\{\mu_r\}_{r>0}$ along a sequence with $r \to \infty$ is $\delta_{\mathbf{g}}$. Proposition 2.3.1 says that $\int_C h(z) d\mu_r(h) \to \mathbf{g}(z)$, as $r \to \infty$. So μ satisfies

$$\int_C h(z) d\mu(h) = \mathbf{g}(z)$$

for all $z \in \mathbb{H}^n$. By Proposition 2.2.3, $\mu = \delta_{\mathbf{g}}$. This completes the proof. □

The Birkhoff ergodic theorem yields following restatement of the main result.

Proposition 2.4.2. *The only invariant measure for translation flow on C is the delta measure $\delta_{\mathbf{g}}$.*

Remark: We do not know if $\lim_{t\to\infty} f_t = \mathbf{g}$ for all $f \in C$.

As a corollary to the main result, we obtain the generalization of Proposition 2.1.2 to maps $\mathbb{H}^n \to \mathbb{H}$.

Corollary 2.4.3. *Suppose $f : \mathbb{H}^n \to \mathbb{H}$ is holomorphic and satisfies $f(\lambda, \ldots, \lambda) = \lambda$ for all $\lambda \in \mathbb{H}$. Suppose in addition that $f(z_1 + T, \ldots, z_n + T) = f(z_1, \ldots, z_n) + T$ for some $T > 0$ and all $(z_1, \ldots, z_n) \in \mathbb{H}^n$. Then f is equal to the function $\mathbf{g}(z) = \sum_{j=1}^n \alpha_j z_j$, where $\alpha_j = \frac{\partial f}{\partial z_j}(i, \ldots, i)$.*

Proof: Assume WLOG $\alpha_j = \frac{1}{n}$. The hypothesis on f means that it is a periodic point of the translation flow, with period T. Thus, $\mu_T = \lim_{r\to\infty} \mu_r = \delta_{\mathbf{g}}$. Since $t \mapsto f_t$ is continuous, it follows that $f_t = \mathbf{g}$ for all $t \in [-T, T]$. In particular, $f = \mathbf{g}$, as claimed.

2.5 Unipotent subgroups acting on $\mathcal{D}$

In this section, we restate our results in terms of the action of $\mathrm{Aut}(\mathbb{H})$ on $\mathcal{D}$.

The group $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R})$ acts on $\mathcal{D}$ by conjugation: an element $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ sends $f \in \mathcal{D}$ to the function $\gamma \cdot f$ given by

$$(\gamma \cdot f)(z_1, \ldots, z_n) = \gamma f(\gamma^{-1} z_1, \ldots, \gamma^{-1} z_n).$$

By the chain rule, $\gamma \cdot f$ has the same first partials at $(i, \ldots, i)$ as f. So C is invariant under the action.

An element of $\mathrm{PSL}_2(\mathbb{R})$ is called *unipotent* (or *parabolic*) if it fixes exactly one point in $\partial\mathbb{H}$. A *unipotent subgroup* of $\mathrm{PSL}_2(\mathbb{R})$ is a nontrivial one-parameter subgroup whose non-identity elements are unipotent. Every unipotent subgroup is conjugate to the group of translations $z \mapsto z + t$.

The following generalization of our results is immediate:

Theorem 2.5.1. *Let $\mathcal{D}$ be the family of holomorphic maps $\mathbb{H}^n \to \mathbb{H}$ which restrict to the identity on the diagonal. Let $f \in \mathcal{D}$. For each j, $\lambda \mapsto \frac{\partial f}{\partial z_j}(\lambda, \ldots, \lambda)$ is identically equal to some nonnegative constant α_j.*
Let $\{\gamma_t\} \subset \mathrm{PSL}_2(\mathbb{R})$ be a unipotent subgroup. There is a unique γ_1-invariant holomorphic $\mathbf{g} \in \mathcal{D}$ satisfying $\frac{\partial \mathbf{g}}{\partial z_j}(\lambda, \ldots, \lambda) = \alpha_j$ for all $\lambda \in \mathbb{H}$ and $j = 1, \ldots, n$.
Let μ_r be the pushforward to $\mathcal{D}$ of the uniform measure on $[-r, r]$, by the map $t \mapsto \gamma_t \cdot f$. Then $\mu_r \to \delta_{\mathbf{g}}$ weakly as $r \to \infty$.

Remark: Theorem 2.5.1 holds exactly as stated with $\mathbb{H}$ replaced by $\mathbb{D}$.

2.6 A rigidity result

Below, we establish the rigidity result we used in the proof of Proposition 2.3.1, namely that any $f \in \mathcal{D}$ which is invariant under all translations is a convex combination of the coordinate functions.

First, we need a lemma.

Lemma 2.6.1. *Let $\phi : \mathbb{C} \to \mathbb{R}$ be a harmonic function with $\phi(0) = \frac{\partial \phi}{\partial x}(0) = \frac{\partial \phi}{\partial y}(0) = 0$. Suppose there is a $C > 0$ so that $\phi(z) \geq -C|z|$ for all $z \in \mathbb{C}$. Then ϕ is identically zero.*

Proof: The idea is to use the Poisson integral formula to show that ϕ has sublinear growth.

Write $\phi = \phi_+ - \phi_-$, where $\phi_+(z) = \max\{0, \phi(z)\}$, and $\phi_-(z) = \max\{0, -\phi(z)\}$. Fix $r > 0$, and set

$$A = \int_0^1 \phi_+(re^{2\pi i\theta})d\theta, \ B = \int_0^1 \phi_-(re^{2\pi i\theta})d\theta.$$

By the mean value property, $A - B = \phi(0) = 0$. We compute

$$\begin{aligned}\int_0^1 |\phi(re^{2\pi i\theta})| d\theta &= A + B \\ &= 2B \\ &= 2\int_0^1 \phi_-(re^{2\pi i\theta}) d\theta \\ &\leq 2Cr,\end{aligned}$$

where in the last inequality, we've used $\phi(z) \geq -C|z|$. Now, for any z with $|z| = \frac{r}{2}$, the Poisson integral formula for the ball $B_r(0)$ yields

$$|\phi(z)| = \left| \int_0^1 \frac{r^2 - \left(\frac{r}{2}\right)^2}{r\left|z - re^{2\pi i\theta}\right|} \phi(re^{2\pi i\theta}) d\theta \right| \leq \sup_{\theta\in[0,2\pi]} \left(\frac{3r}{4\left|z - re^{2\pi i\theta}\right|}\right) \cdot \int_0^1 |\phi(re^{2\pi i\theta})| d\theta \leq 3Cr.$$

Since r was arbitrary, we have $|\phi(z)| \leq 6C|z|$ for all z. Since ϕ is harmonic and has sublinear growth, ϕ is affine, that is, $\phi(x + iy) = ax + by + c$ for some $a, b, c \in \mathbb{C}$. (Indeed, the higher derivatives of ϕ at 0 vanish, as we can see by differentiating Poisson's formula on $B_r(0)$ under the integral and letting r tend to infinity.) By assumption, ϕ and its first derivatives vanish at the origin, so ϕ is identically 0.

□

We now prove the main result of this section.

Proposition 2.6.2. *Fix positive constants α_j with $\sum_{j=1}^n \alpha_j = 1$. Let $f : \mathbb{H}^n \to \mathbb{H}$ be a holomorphic function satisfying*

$$f(\lambda, \dots, \lambda) = \lambda, \tag{A}$$

for all $\lambda \in \mathbb{H}$.

$$\frac{\partial f}{\partial z_j}(\lambda, \dots, \lambda) = \alpha_j, \tag{B}$$

for all $\lambda \in \mathbb{H}$ and $j = 1, \dots, n$.

$$f(z_1 + t, \dots, z_n + t) = f(z_1, \dots, z_n) + t, \tag{C}$$

for all $(z_1, \dots, z_n) \in \mathbb{H}^n$ and all $t \in \mathbb{R}$. Then f is the function $f(z_1, \dots, z_n) = \sum_{j=1}^n \alpha_j z_j$.

Proof: As usual, we assume $\alpha_j = \frac{1}{n}$. The idea is to first show that f is of form

$$\frac{1}{n}\sum_{j=1}^{n} z_j + H(z_2 - z_1, z_3 - z_2, \dots, z_n - z_{n-1}),$$

for some holomorphic $H : \mathbb{C}^{n-1} \to \mathbb{C}$. Then we use Lemma 2.6.1 to show that $H \equiv 0$.

Let

$$g(z_1, \dots, z_n) = f(z_1, \dots, z_n) - \frac{1}{n}\sum_{j=1}^{n} z_j.$$

In terms of g, conditions (A), (B), (C) become

$$g(\lambda, \dots, \lambda) = 0. \tag{A'}$$

$$\frac{\partial g}{\partial z_j}(\lambda, \dots, \lambda) = 0. \tag{B'}$$

$$g(z_1 + t, \dots, z_n + t) = g(z_1, \dots, z_n), \text{ for all } t \in \mathbb{R}. \tag{C'}$$

Condition C' implies that

$$g(z_1 + c, \dots, z_n + c) = g(z_1, \dots, z_n), \tag{2.12}$$

for all complex c with $\mathrm{Im}(c) > -\min_j \mathrm{Im}(z_j)$. Indeed, fixing $z_1, \dots, z_n \in \mathbb{H}$, the holomorphic function $c \mapsto g(z_1 + c, \dots, z_n + c) - g(z_1, \dots, z_n)$ vanishes on the real axis and, thus, on the whole domain.

Now, write $g(z_1, \dots, z_n) = h(a, d_1, \dots, d_{n-1})$, where

$$a = \frac{1}{n}\sum_{j=1}^{n} z_j \text{ and } d_j = z_{j+1} - z_j \text{ for } j = 1, \dots, n-1,$$

and h is holomorphic on the image Ω of $\mathbb{H}^n$ under the coordinate change.

For each $a \in \mathbb{H}$, let

$$\Omega(a) = \{(d_1, \dots, d_n) \in \mathbb{C}^{n-1} \mid (a, d_1, \dots, d_n) \in \Omega\}.$$

Define $h^a : \Omega(a) \to \mathbb{C}$ by

$$h^a(d_1, \dots, d_{n-1}) = h(a, d_1, \dots, d_{n-1}).$$

For each $a \in \mathbb{H}$, $\Omega(a)$ is a convex open set containing the origin. Moreover, $\Omega(ta) = t\Omega(a)$ for $t > 0$. It follows that $\Omega(it_1) \subset \Omega(it_2)$ for $0 < t_1 < t_2$, and that $\bigcup_{t>0} \Omega(it) = \mathbb{C}^{n-1}$.

Now, (2.12) implies $h^{it_1}(d_1,\dots,d_{n-1}) = h^{it_2}(d_1,\dots,d_{n-1})$ whenever $(d_1,\dots,d_{n-1}) \in \Omega(it_1)$ and $t_1 < t_2$. Since $\bigcup_{t>0}\Omega(it) = \mathbb{C}^{n-1}$, there is a holomorphic $H : \mathbb{C}^{n-1} \to \mathbb{C}$ so that $h^{it} = H|_{\Omega(it)}$. Again by (2.12), $h^{x+iy} = h^{iy}$ for all $x + iy \in \mathbb{H}$. So

$$h^a = H|_{\Omega(a)},\ \forall a \in \mathbb{H}.$$

It thus suffices to show that H is identically 0.

Recall that

$$f = \frac{1}{n}\sum_{j=1}^{n} z_j + g(z_1,\dots,z_n) = a + h(a,d_1,\dots,d_{n-1}).$$

Since f maps into $\mathbb{H}$, h^a maps $\Omega(a)$ into the strip $\{z \mid \mathrm{Im}(z) > -\mathrm{Im}(a)\}$. Thus, H maps each $\Omega(it)$ to $\{\mathrm{Im}(z) > -t\}$.

Recall that $\Omega(i)$ is open and contains 0. So $\Omega(i)$ contains an open Euclidean ball $B_r(0)$ centered at the origin. Then $B_{rt}(0) \subset \Omega(it)$, so $H(B_{rt}) \subset \{\mathrm{Im}(z) > -t\}$, for all $t > 0$. Thus, if $\left(\sum |d_j|^2\right)^{1/2} = rt$, then

$$\mathrm{Im}\,[H(d_1,\dots,d_{n-1})] \geq -t.$$

In other words, we have

$$\mathrm{Im}\,[H(d_1,\dots,d_{n-1})] \geq -\frac{\left(\sum |d_j|^2\right)^{1/2}}{r}, \tag{2.13}$$

for all $(d_1,\dots d_{n-1}) \in \mathbb{C}^{n-1}$.

Condition (A') implies $h(a,0,\dots,0) = 0$, so

$$H(0,\dots,0) = 0. \tag{2.14}$$

Finally, condition (B') and the chain rule imply that the derivatives $\frac{\partial h}{\partial d_j}(a,0,\dots,0)$ are 0, so that

$$\frac{\partial H}{\partial d_j}(0,\dots,0) = 0\ \forall j. \tag{2.15}$$

We reduce to Lemma 2.6.1. Fix arbitrary $(d_1,\dots,d_{n-1})$ with $\sum_j |d_j|^2 = 1$. By (2.13),(2.14), and (2.15) the harmonic function

$$\phi(z) = \mathrm{Im}\,[H(d_1 z,\dots,d_{n-1}z)]$$

satisfies the conditions of the lemma, with $C = \frac{1}{r}$. We conclude that $\mathrm{Im}(H)$, and thus H, are identically 0. □

2.7 Appendix: Polarization

Markovic's proof in [27] of Proposition 2.1.2 uses the classical polarization principle. The proof generalizes almost verbatim to a proof of the corresponding result for maps $\mathbb{H}^n \to \mathbb{H}$ (Corollary 2.4.3), but the polarization principle must be replaced by the following fact:

Proposition 2.7.1. *Let V be the real vector subspace of $\mathbb{C}^n$ consisting of points the form $(r + t_1 i, \ldots, r + t_n i)$ with r and $t_1, \ldots, t_n$ real and $\sum_{j=1}^n t_j = 0$. Let $U \subset \mathbb{C}^n$ be a domain such that $U \cap V$ is nonempty. If $h : U \to \mathbb{C}$ is holomorphic and vanishes on $U \cap V$, then h is identically 0 on U.*

(The polarization principle is the $n = 2$ case of the above result.) We will prove Proposition 2.7.1 as a corollary of the following well-known proposition.

Proposition 2.7.2. *Let $U \subset \mathbb{C}^n$ be a domain, and let $M \subset U$ be a nonempty smooth submanifold. Suppose for each $p \in M$ that T_pM and $i\left(T_pM\right)$ together span $\mathbb{C}^n$. Let $h : U \to \mathbb{C}$ be a holomorphic function which vanishes on M. Then h is identically 0 on U.*

Proof: Let $p \in M$, and consider the differential $dh_p : \mathbb{C}^n \to \mathbb{C}$. Since f vanishes on M, dh_p vanishes on T_pM. Since dh_p is complex-linear, it vanishes also on $i\left(T_pM\right)$. But since $T_pM + i\left(T_pM\right) = \mathbb{C}^n$, $dh_p = 0$. Since p was arbitrary, we conclude the first partial derivatives $\frac{\partial h}{\partial z_j}$ vanish on M. Applying the same argument to $\frac{\partial h}{\partial z_j}$, we find that the second partials $\frac{\partial^2 h}{\partial z_k \partial z_j}$ also vanish on M. Continuing inductively, we find that all higher derivatives vanish on M. Since h is analytic, it follows that h is identically 0 on U. □

Proof of Proposition 2.7.1: If $p \in U \cap V$, T_pV identifies naturally with V. The vector space V has (real) dimension n, and $V \cap iV = \{0\}$, so $\mathbb{C}^n = V \oplus iV$. So Proposition 2.7.2 applies, with $M = U \cap V$.

□

Chapter 3

CLASSIFYING COMPLEX GEODESICS FOR THE CARATHÉODORY METRIC

3.1 Introduction

Let $\mathcal{T} := \mathcal{T}_{g,n}$ denote the Teichmüller space of a finite-type orientable surface $S_{g,n}$. Let $\mathbb{H}$ denote the upper half-plane, equipped with its Poincaré metric $d_{\mathbb{H}}$. The Carathéodory metric on $\mathcal{T}$ is the smallest metric so that every holomorphic map $\mathcal{T} \to (\mathbb{H}, d_{\mathbb{H}})$ is nonexpanding. On the other hand, the Kobayashi metric on $\mathcal{T}$ is the largest metric so that every map $(\mathbb{H}, d_{\mathbb{H}}) \to \mathcal{T}$ is nonexpanding. Royden [33] proved that the Kobayashi metric is the same as the classical Teichmüller metric. Whether or not the Carathéodory metric is also the same as the Teichmüller metric was a longstanding open problem.

Let $\tau : \mathbb{H} \to \mathcal{T}$ be a Teichmüller disk. Then the Carathéodory and Kobayashi metrics agree on $\tau(\mathbb{H})$ if and only if there is a *holomorphic retraction* onto $\tau(\mathbb{H})$, i.e., a holomorphic map $F : \mathcal{T} \to \mathbb{H}$ so that $F \circ \tau = \mathrm{id}_{\mathbb{H}}$. Thus, the problem of determining whether the Carathéodory and Kobayashi metrics agree reduces to checking whether each Teichmüller disk is a holomorphic retract of Teichmüller space. In 1981, Kra [24] showed that if a holomorphic quadratic differential has no odd-order zeros, then its associated Teichmüller disk is a holomorphic retract. However, it was recently shown [27] that not all Teichmüller disks in $\mathcal{T}_g$ are retracts, and so the Carathéodory and Kobayashi metrics are different.

It remains to classify the Teichmüller disks on which the two metrics agree. In other words, we would like to know which Teichmüller disks are holomorphic retracts of Teichmüller space. Put another way, our aim is to classify the complex geodesics for the Carathéodory metric on Teichmüller space.

We conjecture the converse of Kra's result:

Conjecture 3.1.1. *A Teichmüller disk is a holomorphic retract if and only if it is generated by a quadratic differential with no odd-order zeros.*

In this paper, we suggest a program for proving the conjecture. We carry out the program for the spaces $\mathcal{T}_{0,5}$ and $\mathcal{T}_{1,2}$ of complex dimension two. That is, we prove

Theorem 3.1.2 (Main Result). *A Teichmüller disk in $\mathcal{T}_{0,5}$ or $\mathcal{T}_{1,2}$ is a holomorphic retract if and only if the zeros of ϕ are all even-order.*

Dynamics on the moduli space of quadratic differentials plays a key role in the proof. If the quadratic differential ϕ generates a holomorphic retract, then so does every differential in its $\mathrm{SL}_2(\mathbb{R})$ orbit closure. On the other hand, combined results of Minsky-Smillie [30] and Smillie-Weiss [36] show that the orbit closure contains a Jenkins-Strebel differential in the same stratum as ϕ. To prove Conjecture 3.1.1, it thus suffices to consider Jenkins-Strebel differentials. To this end, we develop a complex-analytic criterion, Theorem 3.4.1, characterizing Jenkins-Strebel differentials which generate retracts. The criterion involves a certain holomorphic embedding $\mathcal{E}^{\phi} : \mathbb{H}^k \to \mathcal{T}$, called a *Teichmüller polyplane*, of the k-fold product of $\mathbb{H}$ into Teichmüller space. To prove Conjecture 3.1.1 for $\mathcal{T}_{0,5}$, we reduce to the case that ϕ is an *L-shaped pillowcase*. Then, following the argument in [27], we use our analytic criterion to show that an L-shaped pillowcase does not generate a retract.

The Carathéodory and Kobayashi pseudometrics

A *Schwarz-Pick system* is a functor assigning to each complex manifold M a pseudometric d_M satisfying the following conditions:

(i) The metric assigned to the upper half-plane $\mathbb{H} = \{\lambda \in \mathbb{C} | \mathrm{Im}\lambda > 0\}$ is the Poincaré metric of curvature -4:

$$d_{\mathbb{H}}(\lambda_1, \lambda_2) = \tanh^{-1} \left| \frac{\lambda_1 - \lambda_2}{\lambda_1 - \overline{\lambda}_2} \right|.$$

(ii) Any holomorphic map $f : M \to N$ between complex manifolds is non-expanding:

$$d_N\left(f(p), f(q)\right) \le d_M(p, q)$$

for all $p, q \in M$.

Distinguished among the Schwarz-Pick systems are the Carathéodory and Kobayashi pseudometrics. The Carathéodory pseudometric C_M on a complex manifold M is the smallest pseudometric so that all holomorphic maps from M to $\mathbb{H}$ are nonexpanding. More explicitly,

$$C_M(p, q) = \sup_{f \in \mathcal{O}(M, \mathbb{H})} d_{\mathbb{H}}\left(f(p), f(q)\right). \tag{3.1}$$

On the other hand, the Kobayashi pseudometric K_M is the largest pseudometric so that all holomorphic maps $\mathbb{H} \to M$ are nonexpanding. Thus, the Kobayashi pseudometric is bounded above by

$$\delta_M(p,q) = \inf_{f \in \mathcal{O}(\mathbb{H},M)} d_{\mathbb{H}} \left(f^{-1}(p), f^{-1}(q) \right),$$

and if δ_M happens to satisfy the triangle inequality, then $K_M = \delta_M$. In general,

$$K_M(p,q) = \inf \sum_{j=1}^{n} \delta_M(p_{j-1}, p_j), \tag{3.2}$$

where the infimum is taken over all sequences $p_0, \ldots, p_n$ with $p_0 = p$ and $p_n = q$. The Schwarz-Pick lemma implies that the assignments $M \mapsto K_M$ and $M \mapsto C_M$ satisfy condition (i), while condition (ii) is a formal consequence of definitions (3.1) and (3.2). In case M is biholomorphic to a bounded domain in $\mathbb{C}^k$, the pseudometrics K_M, C_M are nondegenerate and are thus referred to as the Carathéodory and Kobayashi metrics, respectively.

From the definitions, we see that every Schwarz-Pick system d satisfies

$$C_M \leq d_M \leq K_M$$

for all complex manifolds M. Thus, if $C_M = K_M$, then every Schwarz-Pick system assigns to M the same pseudometric. The problem of determining for which manifolds M the two pseudometrics agree has attracted a great deal of attention.

Complex geodesics and holomorphic retracts

Let d be a Schwarz-Pick system and M a complex manifold. A *complex geodesic* for d_M is a holomorphic and isometric embedding $(\mathbb{H}, d_{\mathbb{H}}) \to (M, d_M)$. A holomorphic map $\tau \in \mathcal{O}(\mathbb{H}, M)$ is said to be a *holomorphic retract* of M if there exists a map $F \in \mathcal{O}(M, \mathbb{H})$ so that $F \circ \tau = \mathrm{id}_{\mathbb{H}}$. We also say that τ *admits a holomorphic retraction*. The main point of the following well-known lemma is that τ is a complex geodesic for the Carathéodory metric if and only if it admits a holomorphic retraction [20].

Lemma 3.1.3. *Let $\tau : \mathbb{H} \to M$ be a holomorphic map into a connected complex manifold. The following are equivalent:*

(a) *There is a pair of distinct $z, w \in \mathbb{H}$ so that $C_M(\tau(z), \tau(w)) = d_{\mathbb{H}}(z, w)$.*

(b) *τ is a holomorphic retract of M.*

(c) *τ is a complex geodesic for C_M.*

(d) *τ is a complex geodesic for K_M and the restrictions of C_M and K_M to $\tau(\mathbb{H})$ coincide.*

Proof:

(a) $\implies$ (b): There is a sequence of holomorphic maps $F_j : M \to \mathbb{H}$ with $d_{\mathbb{H}}(F_j \circ \tau(z), F_j \circ \tau(w))$ converging to $d_{\mathbb{H}}(z,w)$. Postcomposing each F_j by a Möbius transformation, we may assume $F_j \circ \tau$ fixes z and maps w to a point on the geodesic segment connecting z and w. Then the F_j form a normal family. Any subsequential limit F of the F_j satisfies $F \circ \tau(z) = z$ and $F \circ \tau(w) = w$. By the Schwarz-Pick lemma, $F \circ \tau = \mathrm{id}_{\mathbb{H}}$.

(b) $\implies$ (c): Suppose $F : M \to \mathbb{H}$ is holomorphic and satisfies $F \circ \tau = \mathrm{id}_{\mathbb{H}}$. Then for any pair of points z and w in $\mathbb{H}$,

$$d_{\mathbb{H}}(z,w) = d_{\mathbb{H}}(F \circ \tau(z), F \circ \tau(w)) \leq C_M(\tau(z), \tau(w))$$

because F is holomorphic. Also,

$$C_M\,(\tau(z), \tau(w)) \leq d_{\mathbb{H}}(z,w)$$

because τ is holomorphic.

(c) $\implies$ (d): For any z, w in $\mathbb{H}$,

$$d_{\mathbb{H}}(z,w) = C_M\,(\tau(z), \tau(w)) \leq K_M(\tau(z), \tau(w)).$$

Since holomorphic maps decrease Kobayashi distance, the inequality must be an equality.

(d) $\implies$ (a): Obvious.

□

Remark: Let p be a point in a connected complex manifold M. To prove the implication (a) $\implies$ (b), we used the fact that a family $\{F_j\}$ of holomorphic maps $M \to \mathbb{H}$ is precompact in $O(M, \mathbb{H})$ if and only if $\{F_j(p)\}$ is precompact in $\mathbb{H}$. (This is essentially a rephrasing of Montel's theorem.) We will use this fact throughout the paper.

Symmetric spaces vs. Teichmüller space

In case M is a Hermitian symmetric space, the Kobayashi and Carathéodory metrics coincide. Indeed, we have

1. Each pair of points in M is contained in the image of a complex geodesic for K_M.
2. Every complex geodesic for K_M is a holomorphic retract of M.

So from Lemma 3.1.3, we get $K_M = C_M$. In fact, by a theorem of Lempert ([25];[20] Chapter 11) , the Kobayashi and Carathéodory metrics coincide for all bounded convex domains. (Every Hermitian symmetric space is biholomorphic to a bounded convex domain.) Whether the two metrics agree for all bounded $\mathbb{C}$-convex domains is an open question.

Given the many parallels between Teichmüller spaces and symmetric spaces, it is natural to ask whether the Carathéodory metric on Teichmüller space is the same as Teichmüller-Kobayashi metric. As is the case with Hermitian symmetric spaces, any pair of points in $\mathcal{T}$ is contained in a complex geodesic for $K_{\mathcal{T}}$. The problem of determining whether the Kobayashi and Carathéodory metrics agree on Teichmüller space thus reduces to checking whether each complex geodesic for $K_{\mathcal{T}}$ is a holomorphic retract.

Abelian Teichmüller disks

Complex geodesics for the Teichmüller-Kobayashi metric $K_{\mathcal{T}}$ are called *Teichmüller disks*. A Teichmüller disk is determined by the initial data of a point in $\mathcal{T}$ and a unit cotangent vector at that point. In other words, the disk is determined by a unit-norm holomorphic quadratic differential ϕ. We say ϕ *generates* the Teichmüller disk τ^ϕ. (See Section 3.2.) If ϕ is the square of an Abelian differential, τ^ϕ is called an *Abelian Teichmüller disk*.

Kra [24] showed that the Kobayashi and Carathéodory metrics agree on certain subsets of Teichmüller space. Namely, he proved

Theorem 3.1.4. *[24] Let $\mathcal{T}$ be the Teichmüller space of a finite-type orientable surface. If ϕ is a quadratic differential with no odd-order zeros, then the restrictions of the metrics $K_{\mathcal{T}}$ and $C_{\mathcal{T}}$ to $\tau^\phi(\mathbb{H})$ coincide. That is, τ^ϕ is a complex geodesic for $C_{\mathcal{T}}$ and thus a holomorphic retract of $\mathcal{T}$.*

The key tool in the proof is the Torelli map from the Teichmüller space $\mathcal{T}_g$ of a closed surface to the Siegel upper half-space $\mathcal{Z}_g$. Kra showed that the Torelli map sends every Abelian Teichmüller disk in $\mathcal{T}_g$ to a complex geodesic in the symmetric space $\mathcal{Z}_g$. Post-composing by a holomorphic retraction $\mathcal{Z}_g \to \mathbb{H}$ onto this complex geodesic yields a retraction $\mathcal{T}_g \to \mathbb{H}$ onto the Abelian Teichmüller disk. A covering argument then extends the result to all differentials ϕ with no odd-order zeros. (Note that, at a puncture, ϕ can have a simple pole or a zero of any order, and the theorem still holds.)

Carathéodory ≠ Teichmüller

However, in [27] it was shown that the Carathéodory and Kobayashi metrics on Teichmüller space do not coincide:

Theorem 3.1.5. *[27] The Kobayashi and Carathéodory metrics on the Teichmüller space of a closed surface of genus at least two do not coincide; if $g \geq 2$, there is a Teichmüller disk in $\mathcal{T}_g$ which is not a holomorphic retract.*

An elementary covering argument, outlined in the appendix of this paper, extends the result to all Teichmüller spaces $\mathcal{T}_{g,n}$ of complex dimension at least two:

Theorem 3.1.6. *Suppose* $\dim_{\mathbb{C}} \mathcal{T}_{g,n} := 3g - 3 + n \geq 2$. *The Kobayashi and Carathéodory metrics on $\mathcal{T}_{g,n}$ are different.*

Remark: The one-dimensional spaces $\mathcal{T}_{0,4}$ and $\mathcal{T}_{1,1}$ are biholomorphic to $\mathbb{H}$, so the Kobayashi and Carathéodory metrics are equal to the Poincaré metric.

Theorem 3.1.6 has consequences for the global geometry of Teichmüller space. Teichmüller space is biholomorphic via Bers' embedding to a bounded domain in $\mathbb{C}^k$. However, combined with Lempert's theorem, Theorem 3.1.6 implies

Theorem 3.1.7. *The Teichmüller space $\mathcal{T}_{g,n}$ is not biholomorphic to a bounded convex domain in $\mathbb{C}^k$, whenever* $\dim_{\mathbb{C}} \mathcal{T}_{g,n} \geq 2$.

See [17] for related convexity results. See [5] and [6] for other recent results comparing the complex geometry of Teichmüller spaces and symmetric spaces.

Outline

It remains to characterize the quadratic differentials which generate holomorphic retracts. Our Conjecture 3.1.1 is that the converse of Kra's result holds – τ^{ϕ} is a holomorphic retract if and only if ϕ has no odd-order zeros.

In the rest of the paper, we develop some tools towards a proof of Conjecture 3.1.1. The conjecture is obviously true for the Teichmüller spaces $\mathcal{T}_{1,1}$ and $\mathcal{T}_{0,4}$ of complex dimension one. Our **main result**, Theorem 3.1.2, is that the conjecture holds for the spaces $\mathcal{T}_{0,5}$ and $\mathcal{T}_{1,2}$ of complex dimension two.

The idea of the proof is as follows. We first show that the property of generating a holomorphic retract is a closed condition on the bundle $\widetilde{Q}$ of marked quadratic differentials. The condition is also invariant under the actions of $\mathrm{SL}_2(\mathbb{R})$ and the mapping class group. Thus, it descends to a closed, $\mathrm{SL}_2(\mathbb{R})$-invariant condition on the moduli space Q of unmarked quadratic differentials. In other words, if ϕ generates holomorphic retract, then so does every element of its $\mathrm{SL}_2(\mathbb{R})$ orbit closure in Q. On the other hand, the $\mathrm{SL}_2(\mathbb{R})$-orbit closure of any quadratic differential contains a Jenkins-Strebel differential in the same stratum [30][36].

To prove Conjecture 3.1.1, it thus suffices to establish that no Jenkins-Strebel differential with an odd-order zero generates a retract. To this end, we prove an analytic criterion characterizing Jenkins-Strebel differentials which generate retracts. Given a Jenkins-Strebel differential ϕ with k cylinders, we define a holomorphic map $\mathcal{E}^\phi : \mathbb{H}^k \to \mathcal{T}$ called a *Teichmüller polyplane*. The marked surface $\mathcal{E}^\phi(\lambda_1, \ldots, \lambda_k)$ is obtained by applying the map $x + iy \mapsto x + \lambda_j y$ to the jth cylinder (Figure 3.2). The Teichmüller disk τ^ϕ is the diagonal of $\mathcal{E}^\phi$, so if $F : \mathcal{T} \to \mathbb{H}$ is a holomorphic retraction onto $\tau^\phi(\mathbb{H})$, then the composition $f = F \circ \tau : \mathbb{H}^k \to \mathbb{H}$ restricts to the identity on the diagonal:

$$f(\lambda, \ldots, \lambda) = \lambda.$$

Our analytic criterion, Theorem 3.4.1, states that if a unit area Jenkins-Strebel differential ϕ generates a retract, then the retraction F can be chosen so that $f = F \circ \mathcal{E}^\phi$ is a convex combination of the coordinate functions:

$$f(\lambda_1, \ldots, \lambda_k) = \sum_j a_j \lambda_j,$$

where a_j is the area of the jth cylinder. In other words, τ^ϕ is a retract if and only if the linear function $\sum_j a_j \lambda_j$ on the polyplane $\mathcal{E}^\phi(\mathbb{H}^k)$ extends to a holomorphic map $\mathcal{T} \to \mathbb{H}$. As a corollary of this criterion, we observe that if a Jenkins-Strebel differential ϕ generates a retract, then so does any differential obtained by horizontally shearing the cylinders of ϕ.

To prove Conjecture 3.1.1 for $\mathcal{T}_{0,5}$, let $\phi \in Q_{0,5}$ be a Jenkins-Strebel differential with an odd-order zero. So ϕ has a simple zero and five poles. Using the results of [30],[36] and a simple combinatorial argument, we show that ϕ contains in its $\mathrm{SL}_2(\mathbb{R})$ orbit closure a *two-cylinder* Jenkins-Strebel differential ϕ' with a simple zero. Shearing the cylinders of ϕ' yields an *L-shaped pillowcase differential* ψ (Figure 3.6). Now, assume for the sake of contradiction that ϕ generates a retract. Then so does the differential ϕ' in $\overline{SL_2(\mathbb{R})\phi}$, and so does the L-shape ψ obtained by shearing the cylinders of ϕ'. But in [27], Markovic shows that an L-shape does not satisfy our criterion; there is no holomorphic map $F : \mathcal{T}_{0,5} \to \mathbb{H}$ extending $a_1\lambda_1 + a_2\lambda_2$. The idea of Markovic's proof is to assume a holomorphic extension F exists and then, using the Schwarz-Christoffel formula, obtain a contradiction on the smoothness of F at the boundary of the bidisk $\mathcal{E}^{\psi}(\mathbb{H}^2)$.

This proves the Conjecture 3.1.1 for the sphere with five punctures. The isomorphism $\mathcal{T}_{0,5} \cong \mathcal{T}_{1,2}$ yields the corresponding result for the twice-punctured torus.

3.2 Dynamics on moduli space

In this section, we describe the role of dynamics in the classification of complex geodesics for the Carathéodory metric. After recalling some basic definitions, we show how the $\mathrm{GL}_2^+(\mathbb{R})$ action on Q allows us to reduce Conjecture 3.1.1 to the case of Jenkins-Strebel differentials. Using ergodicity of the Teichmüller geodesic flow, we show that most quadratic differentials do not generate holomorphic retracts.

The GL_2^+ action on $\widetilde{Q}$

Let $\mathcal{T} = \mathcal{T}_{g,n}$ be the Teichmüller space of marked complex structures on a finite-type, orientable surface $S_{g,n}$ of genus g with n punctures. Let $\widetilde{Q} = \widetilde{Q}_{g,n}$ denote the bundle of marked, nonzero, integrable, holomorphic quadratic differentials over $\mathcal{T}$. Equivalently, $\widetilde{Q}$ is the bundle of marked half-translation structures on $S_{g,n}$ (Figure 3.1). The group GL_2^+ of orientation-preserving linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ acts on $\widetilde{Q}$ by post-composition of flat charts. In other words, the action is by affine deformations of the polygonal decomposition of a differential (Figure 3.2).

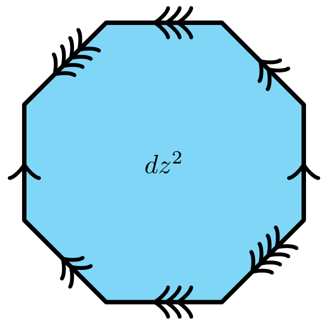


Figure 3.1: A quadratic differential on the surface of genus two. The vertices glue up to a single cone point of angle 6π, corresponding to an order four zero of the differential.

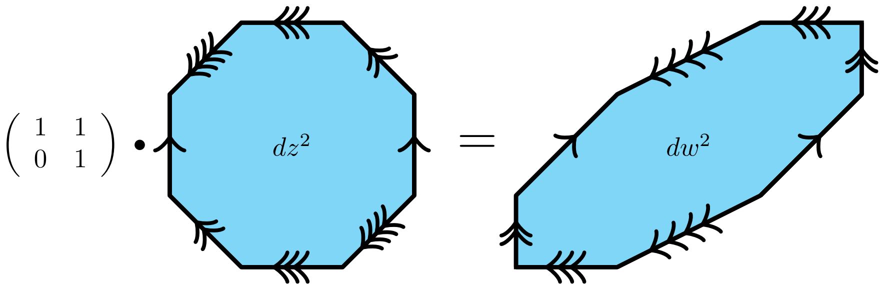


Figure 3.2: The action of a shear on the differential in Figure 3.1. The underlying surface of the resulting quadratic differential is $\tau^\phi(1+i)$.

Teichmüller disks

Let $p : \widetilde{\mathcal{Q}} \to \mathcal{T}$ denote the projection sending a quadratic differential to its underlying Riemann surface. The action of a conformal linear transformation does not change the underlying Riemann surface of a quadratic differential. Therefore,

$$\tau^\phi : g \mapsto p(g \cdot \phi)$$

is a well-defined map from the upper half-plane $\mathbb{H} \cong \mathbb{C}^\times \backslash \mathrm{GL}_2^+$ to Teichmüller space $\mathcal{T}$. To give a more explicit description of the map τ^ϕ, note that $\mathbb{H}$ sits in GL_2^+ as the subgroup generated by vertical stretches and horizontal shears:

$$\left\{ \begin{pmatrix} 1 & \mathrm{Re}(\lambda) \\ 0 & \mathrm{Im}(\lambda) \end{pmatrix} \middle| \lambda \in \mathbb{H} \right\}.$$

The Teichmüller disk generated by ϕ is

$$\lambda \mapsto p(\lambda \cdot \phi). \tag{3.3}$$

Written in complex coordinates, the action of the matrix $\begin{pmatrix} 1 & \mathrm{Re}(\lambda) \\ 0 & \mathrm{Im}(\lambda) \end{pmatrix}$ is

$$x + iy \mapsto x + \lambda y,$$

which has Beltrami coefficient

$$\frac{i-\lambda}{i+\lambda}\frac{\overline{dz}}{dz}.$$

Thus, $\tau^\phi(\lambda)$ is the quasiconformal deformation of $X := p(\phi)$ with Beltrami coefficient

$$\frac{i-\lambda}{i+\lambda}\mu_\phi,$$

with $\mu_\phi := \phi^{-1}\,|\phi|$. The Teichmüller disk τ^ϕ is the unique Kobayashi geodesic with initial data $\tau(i) = X$ and $\tau'(i) = \frac{i}{2}\mu_\phi$.

Orbit closures

The mapping class group MCG of $S_{g,n}$ acts on $\mathcal{T}$ and $\widetilde{Q}$ by changes of marking. The action of each mapping class is a biholomorphism, and by a theorem of Royden [33], every biholomorphism of Teichmüller space arises in this way (as long as $\dim_{\mathbb{C}} \mathcal{T} \geq 2$). The quotient of Teichmüller space by the MCG action is the moduli space of complex structures on $S_{g,n}$. The quotient of $\widetilde{Q}$ by the MCG action is the space Q of (unmarked) half-translation structures on $S_{g,n}$. The GL_2^+ action on $\widetilde{Q}$ descends to an action on Q.

Let $\phi \in \widetilde{Q}$ and $\alpha \in$ MCG. Then the disk τ^ϕ is a holomorphic retract if and only if $\tau^{\alpha(\phi)}$ is. Indeed, if $F : \mathcal{T} \to \mathbb{H}$ is a retraction onto τ^ϕ, then $F \circ \alpha^{-1}$ is a retraction onto $\tau^{\alpha(\phi)} = \alpha \circ \tau^\phi$. Thus, we will say $\phi \in Q$ generates a retract if every element of its preimage in $\widetilde{Q}$ does. The property of generating a retract is also invariant under the GL_2^+ action. Indeed, if

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+,$$

and $m : \mathbb{H} \to \mathbb{H}$ is the associated Möbius transformation

$$m(\lambda) = \frac{d\lambda + b}{c\lambda + a},$$

then $\tau^{g\cdot\phi} = \tau^\phi \circ m$.

The following Proposition is key:

Proposition 3.2.1. *If ϕ generates a holomorphic retract, so does every element in the orbit closure $\overline{\mathrm{GL}_2^+\,\phi} \subset Q$.*

Proof: From the above discussion, we know that every element of the orbit $\mathrm{GL}_2^+\,\phi$ generates a retract. The desired result will follow from the next Lemma.

Lemma 3.2.2. *Suppose the sequence $\phi^1, \phi^2, \ldots$ converges to ϕ in $\widetilde{Q}$. Let τ^N, τ be the Teichmüller disks generated by ϕ^N and ϕ, respectively. If each τ^N is a holomorphic retract, then so is τ.*

Proof: This follows by the continuity of the Kobayashi and Carathéodory metrics on $\mathcal{T}$ (see [12]). However, we prefer to give a direct proof.

For each N, let $F^N : \mathcal{T} \to \mathbb{H}$ be a holomorphic map satisfying $F^N \circ \tau^N = \mathrm{id}_{\mathbb{H}}$. Then $\{F^N\}$ is a normal family. To see this, let X^N and X denote the marked surfaces $p(\phi^N), p(\phi)$. Then

$$d_{\mathbb{H}}\left(F^N(X), i\right) = d_{\mathbb{H}}\left(F^N(X), F^N(X^N)\right) \leq K_{\mathbb{H}}(X, X^N),$$

which is uniformly bounded in N. Thus, passing to a subsequence, we may assume F^N converges locally uniformly to a holomorphic map $F : \mathcal{T} \to \mathbb{H}$. By continuity of the GL_2^+ action on $\widetilde{Q}$, the sequence τ^N converges locally uniformly to τ. Therefore,

$$\Phi \circ \tau^\phi = \lim_{N\to\infty} \Phi^N \circ \tau^N = \mathrm{id}_{\mathbb{H}}.$$

□

Let Q_1 denote the space of unit-area half-translation surfaces. The *Teichmüller geodesic flow* on Q_1 is the action of the subgroup

$$\left\{ \left(\begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right) \middle| \, t \in \mathbb{R} \right\} \subset \mathrm{GL}_2^+.$$

With respect to a suitable probability measure on Q_1, the Teichmüller geodesic flow is ergodic [28], [38]. In particular, for almost all $\phi \in Q_1$, the orbit $\mathrm{GL}_2^+ \phi$ is dense in Q_1. Combined with Proposition 3.2.1 and Theorem 3.1.6, this implies

Theorem 3.2.3. *For almost every quadratic differential $\phi \in Q_1$, the Teichmüller disk $\tau^\phi(\mathbb{H})$ is not a holomorphic retract.*

The horocycle flow and Jenkins-Strebel differentials

Recall that $\phi \in Q$ is said to be *Jenkins-Strebel* if its nonsingular horizontal trajectories are compact.

The *horocycle flow* on Q is the action of the subgroup

$$H = \left\{ \left(\begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right) \middle| \, t \in \mathbb{R} \right\} \subset \mathrm{GL}_2^+.$$

Minsky and Weiss [30] showed that every closed H-invariant set of Q_1 contains a minimal closed H-invariant subset. If Q_1 were compact, this would follow by a standard Zorn's lemma argument. The main point of their proof is a *quantitative nondivergence* result which states that each H-orbit spends a large fraction of its time in the ε-thick part of Q_1.

Subsequently, Smillie and Weiss [36] showed that every minimal closed H-invariant set is the H orbit closure of a Jenkins-Strebel differential. In particular, the orbit closure $\overline{H\phi}$ of any $\phi \in Q$ contains a Jenkins-Strebel differential. As Smillie and Weiss observed, the above results continue to hold for the action of H on each stratum of Q. In particular, this means that if ϕ has an odd-order zero, then $\overline{H\phi}$ contains a Jenkins-Strebel differential with an odd-order zero.

An advantage of working with the horocycle flow, rather than the full GL_2^+ action, is that H preserves horizontal cylinders. That is, if ϕ has a horizontal cylinder, then every element of $\overline{H\phi}$ has a cylinder of the same height and length. Suppose in addition that the cylinder of ϕ is not dense in $S_{g,n}$. Then since $\|h \cdot \phi\| = \|\phi\|$ for all h in H, a Jenkins-Strebel differential in $\overline{H\phi}$ has at least two cylinders.

We summarize the above in the following theorem.

Theorem 3.2.4. *Let $\phi \in Q$ be a quadratic differential.*

(a) *The closure $\overline{H \cdot \phi}$ contains a Jenkins-Strebel differential ψ. If ϕ has an odd-order zero, then ψ can be taken to also have an odd-order zero.*

(b) *If ϕ has a horizontal cylinder which is not dense in $S_{g,n}$ then ψ has at least two cylinders.*

By Proposition 3.2.1 and Theorem 3.2.4(a), our Conjecture 3.1.1 reduces to the case of Jenkins-Strebel differentials. That is, it suffices to show that no Jenkins-Strebel differential with an odd-order zero generates a holomorphic retract.

3.3 Jenkins-Strebel differentials and Teichmüller polyplanes

In this section, we begin our analysis of Teichmüller disks generated by Jenkins-Strebel differentials. The key observation is that such a disk is the diagonal of a certain naturally defined polydisk holomorphically embedded in $\mathcal{T}$.

Teichmüller polyplanes

The core curves of a Jenkins-Strebel differential form a collection of essential simple closed curves, which are pairwise disjoint and non-homotopic (Figure 3.3). We will call such a collection of curves a *disjoint curve system*. Let $C = \{\gamma_1, \ldots, \gamma_k\}$ be a disjoint curve system on $S_{g,n}$. We say a Jenkins-Strebel differential is of type C if the cores of its cylinders are homotopic to $\gamma_1, \ldots, \gamma_k$. We define an action of the k-fold product $\mathbb{H}^k = \mathbb{H} \times \cdots \times \mathbb{H}$ on the differentials of type C. The tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \mathbb{H}^k$ acts on the jth cylinder of ϕ by the affine map $x + iy \mapsto x + \lambda_j y$ (Figure 3.4). Since this map takes horizontal circles isometrically to horizontal circles, the result is a well-defined Jenkins-Strebel differential $\boldsymbol{\lambda} \cdot \phi$. Projecting the orbit of ϕ to the Teichmüller space, we get a map $\mathcal{E}^\phi : \mathbb{H}^k \to \mathcal{T}$ defined by

$$\mathcal{E}^\phi(\boldsymbol{\lambda}) = p(\boldsymbol{\lambda} \cdot \phi).$$

We call $\mathcal{E}^\phi$ the *Teichmüller polyplane* associated to ϕ. Below, we list some properties of Teichmüller polyplanes.

The Teichmüller disk associated to ϕ is the diagonal of the polyplane:

$$\tau^\phi(\lambda) = \mathcal{E}^\phi(\lambda, \ldots, \lambda).$$

The Teichmüller polyplane mapping sends translations to Dehn twists (See Figure 3.5.) To make this precise, let m_j denote the modulus (height divided by length) of the jth cylinder. Let T_j denote the Dehn twist about the core curve γ_j. Then

$$\mathcal{E}^\phi\left(\boldsymbol{\lambda} + (0, \ldots, m_j^{-1}, \ldots, 0)\right) = T_j \circ \mathcal{E}^\phi(\boldsymbol{\lambda}). \tag{3.4}$$

Equation (3.4) is crucial. It will allow us to relate the analysis of holomorphic maps $\mathbb{H}^k \to \mathbb{H}$ to the geometry of Teichmüller space.

The polyplane mapping is a holomorphic embedding. The mapping is holomorphic because the Beltrami coefficient of $x + iy \mapsto x + \lambda_j y$ is holomorphic in λ_j. Holomorphicity implies that $\mathcal{E}^\phi$ is nonexpanding for the Kobayashi metrics on $\mathbb{H}^k$ and $\mathcal{T}$:

$$K_{\mathcal{T}}\left(\mathcal{E}^\phi(\boldsymbol{\lambda}_1), \mathcal{E}^\phi(\boldsymbol{\lambda}_2)\right) \leq K_{\mathbb{H}^k}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \tag{3.5}$$

(Recall that the Kobayashi metric $K_{\mathbb{H}^k}$ is the supremum of the Poincaré metrics on the factors.) We prove that $\mathcal{E}^\phi$ is an embedding in Theorem 3.3.3 below.

However, the mapping is not proper. Indeed, there are no proper holomorphic maps $\mathbb{H}^k \to \mathcal{T}$. (See Page 75, Corollary 1 of [37].) As one of the one of the coordinate functions approaches the real axis and the height of the corresponding cylinder goes to 0, the sequence of image points may converge in $\mathcal{T}$. This lack of properness was critical in the proof [27] that the Kobayashi and Carathéodory metrics on $\mathcal{T}$ are different.

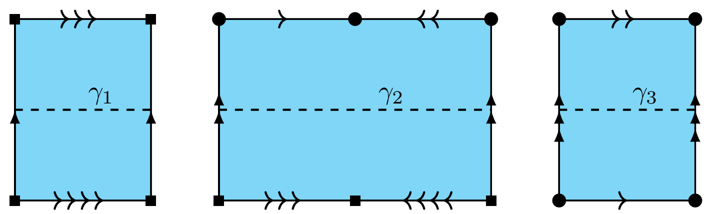


Figure 3.3: A Jenkins-Strebel differential on a genus 2 surface. The differential has two order 2 zeros, indicated by the dot and the square.

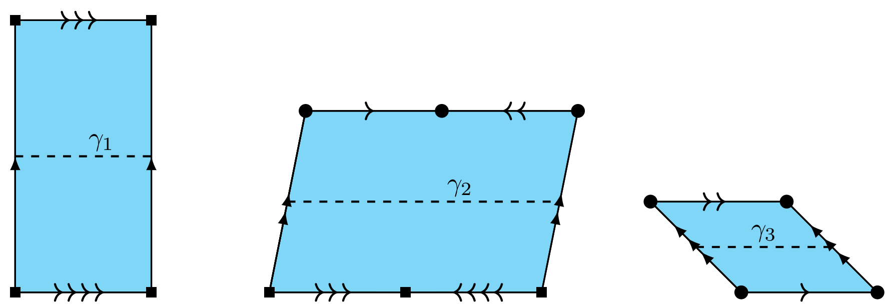


Figure 3.4: The action of an element of $\mathbb{H}^3$ on the differential from Figure 3.3. The resulting Riemann surface is $\mathcal{E}^{\phi}(1.5i, .2 + i, -.5 + .5i)$

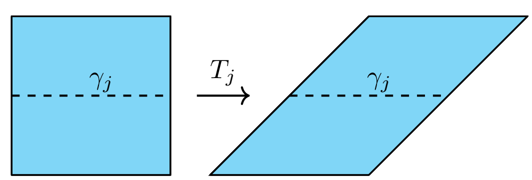


Figure 3.5: Translation by m_j^{-1} in $\mathbb{H}^k$ corresponds to a Dehn twist T_j about γ_j in the Teichmüller space.

Teichmüller polyplanes are embedded

Below, we prove that the Teichmüller polyplane $\mathcal{E}^\phi : \mathbb{H}^k \to \mathcal{T}$ is an embedding. We do not use this fact in the proof of our main result; the reader may choose to skip this section.

Let ϕ be a quadratic differential and γ a closed curve. We denote by $L_\phi(\gamma)$ the ϕ-length of γ − the shortest length of a curve homotopic to γ, measured in the flat metric associated to ϕ. If γ is the core of a cylinder of ϕ, then $L_\phi(\gamma)$ is the circumference of the cylinder. The following result of Jenkins [21] asserts uniqueness of Jenkins-Strebel differential with given length data.

Proposition 3.3.1. *Let $C = \{\gamma_1, \ldots, \gamma_k\}$ be a disjoint curve system, and let $l_1, \ldots, l_k$ be positive numbers. There is at most one Jenkins-Strebel differential ϕ whose core curves are homotopic to a subset of C and which satisfies $L_\phi(\gamma_j) = l_j$ for $j = 1, \ldots, k$.*

Combined with Proposition 3.3.1, the following result implies injectivity of $\mathcal{E}^\phi$.

Lemma 3.3.2. *The action of $\mathbb{H}^k$ on Jenkins-Strebel differentials of given type $\{\gamma_1, \ldots, \gamma_k\}$ is free. In other words, each orbit map $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda} \cdot \phi$ is injective.*

Proof: Suppose $\boldsymbol{\lambda} \cdot \phi = \boldsymbol{\mu} \cdot \phi$. Then the height of the jth cylinder is the same for $\boldsymbol{\lambda} \cdot \phi$ and $\boldsymbol{\mu} \cdot \phi$, so $\mathrm{Im}(\lambda_j) = \mathrm{Im}(\mu_j)$.

We need to show $\mathrm{Re}(\lambda_j) = \mathrm{Re}(\mu_j)$. Suppose not. Then $\boldsymbol{\mu} = \boldsymbol{\lambda} + \mathbf{t}$ for some nonzero vector $\mathbf{t} \in \mathbb{R}^k$. The equation $(\boldsymbol{\lambda} + \mathbf{t}) \cdot \mu = \boldsymbol{\lambda} \cdot \mu$ combined with the fact that $(\boldsymbol{\lambda}, \phi) \mapsto \boldsymbol{\lambda} \cdot \phi$ is a group action implies

$$(\boldsymbol{\lambda} + N\mathbf{t}) \cdot \phi = \boldsymbol{\lambda} \cdot \phi$$

for every positive integer N. Projecting to Teichmüller space, we get

$$\mathcal{E}^\phi(\boldsymbol{\lambda} + N\mathbf{t}) = \mathcal{E}^\phi(\boldsymbol{\lambda}).$$

But this is impossible since for large N, $\mathcal{E}^\phi(\boldsymbol{\lambda} + N\mathbf{t})$ is bounded distance from a translate of $\mathcal{E}^\phi(\boldsymbol{\lambda})$ by a big Dehn multi-twist.

To make the argument precise, let $\mathbf{v}^N$ denote the vector with jth entry

$$\frac{\lfloor N t_j m_j \rfloor}{m_j}.$$

(The vector $\mathbf{v}^N$ is an element of the lattice $\bigoplus m_j^{-1}\mathbb{Z}$ which approximates $N\mathbf{t}$.) Now by (3.4), $\mathcal{E}^\phi(\boldsymbol{\lambda}+\mathbf{v}^N)$ is the marked surface obtained by twisting $\lfloor Nt_j m_j \rfloor$ times about γ_j. By proper discontinuity of the action of the mapping class group on $\mathcal{T}$, the sequence $\mathcal{E}^\phi(\boldsymbol{\lambda}+\mathbf{v}^N)$ leaves every compact set as $N\to\infty$. However, by (3.5),

$$\begin{aligned} K_{\mathcal{T}}\left(\mathcal{E}^\phi(\boldsymbol{\lambda}+\mathbf{v}^N),\mathcal{E}^\phi(\boldsymbol{\lambda})\right) &= K_{\mathcal{T}}\left(\mathcal{E}^\phi(\boldsymbol{\lambda}+\mathbf{v}^N),\mathcal{E}^\phi(\boldsymbol{\lambda}+N\mathbf{t})\right) \\ &\le K_{\mathbb{H}^k}(\boldsymbol{\lambda}+\mathbf{v}^N,\boldsymbol{\lambda}+N\mathbf{t}), \end{aligned}$$

which is bounded by a constant independent of N. This is a contradiction. □

We now come to the main result of this section.

Theorem 3.3.3. *The Teichmüller polyplane $\mathcal{E}^\phi:\mathbb{H}^k\to\mathcal{T}$ is a holomorphic embedding.*

Proof: We have already seen that $\mathcal{E}^\phi$ is holomorphic. To prove injectivity, suppose $\mathcal{E}^\phi(\boldsymbol{\lambda})=\mathcal{E}^\phi(\boldsymbol{\mu})$. Then $\boldsymbol{\lambda}\cdot\phi$ and $\boldsymbol{\mu}\cdot\phi$ are quadratic differentials on the same marked Riemann surface. By construction, the cylinders of these two differentials have the same lengths. So $\boldsymbol{\lambda}\cdot\phi=\boldsymbol{\mu}\cdot\phi$ by Proposition 3.3.1. Now, by Lemma 3.3.2, $\boldsymbol{\lambda}=\boldsymbol{\mu}$.

It remains to show that $\mathcal{E}^\phi$ is a homeomorphism onto its image. To this end, let $\boldsymbol{\lambda}^1,\boldsymbol{\lambda}^2,\ldots$ be a sequence which leaves all compact subsets of $\mathbb{H}^k$. We must verify that $\mathcal{E}^\phi(\boldsymbol{\lambda}^1),\mathcal{E}^\phi(\boldsymbol{\lambda}^2),\ldots$ does not converge to an element of $\mathcal{E}(\mathbb{H}^k)$. It suffices to check the following cases:

(i) The imaginary part of some component of $\boldsymbol{\lambda}^N$ goes to infinity as $N\to\infty$.

Then the modulus of the corresponding cylinder goes to infinity. Therefore, the extremal length (see, e.g., [22]) of the core curve converges to 0. Thus, $\mathcal{E}^\phi(\boldsymbol{\lambda}^N)$ leaves all compact subsets of $\mathcal{T}$.

(ii) The imaginary parts of all components $\boldsymbol{\lambda}^N$ are bounded above but at least one of them converges to zero.

Suppose $\mathcal{E}^\phi(\boldsymbol{\lambda}^N)$ converges to a marked surface X. Since the imaginary parts of $\boldsymbol{\lambda}^N$ stay bounded, the norms $\|\boldsymbol{\lambda}^N\cdot\phi\|$ stay bounded. Passing to a subsequence, we may thus assume that $\boldsymbol{\lambda}^N\cdot\phi$ converges to a Jenkins-Strebel differential ψ on X. By continuity, ψ has the same length data as every element of the orbit $\mathbb{H}^k\cdot\phi$, that is $L_\psi(\gamma_j)=L_\phi(\gamma_j)$ for $j=1,\ldots,k$. However, if the jth coordinate λ_j^N converges to 0, then the jth cylinder of ψ is degenerate. In other words, γ_j is not a core curve of ψ. So ψ is not in $\mathbb{H}^k\cdot\phi$. By Proposition 3.3.1, X is not in $\mathcal{E}^\phi(\mathbb{H}^k)$.

(iii) All of the imaginary parts stay bounded between two positive numbers, but some of the real parts go to infinity.

Arguing as in the proof of Lemma 3.3.2, there is a sequence of multi-twists $\alpha^N : \mathcal{T} \to \mathcal{T}$ so that

$$K_{\mathcal{T}}\left(\mathcal{E}(\boldsymbol{\lambda}^N), \alpha^N \circ \mathcal{E}^{\phi}(i)\right)$$

is bounded uniformly in N. Since $\alpha^N \circ \mathcal{E}^{\phi}(i)$ leaves all compact subsets of $\mathcal{T}$, so does $\mathcal{E}(\boldsymbol{\lambda}^N)$. □

Case (ii) accounts for the fact that $\mathcal{E}^{\phi}$ is not proper.

3.4 The analytic criterion

The goal of this section is to prove the following analytic criterion characterizing Jenkin-Strebel differentials which generate retracts. The criterion generalizes results from [14] [27].

Theorem 3.4.1. *Let $\phi \in \widetilde{Q}$ be a unit-area Jenkins-Strebel differential, and let a_j denote the area of the jth cylinder of ϕ. Let $\mathcal{E}^{\phi} : \mathbb{H}^k \to \mathcal{T}$ be the Teichmüller polyplane associated to ϕ. Then the Teichmüller disk τ^{ϕ} is a holomorphic retract if and only if there exists a holomorphic map $G : \mathcal{T} \to \mathbb{H}$ so that*

$$G \circ \mathcal{E}^{\phi}(\boldsymbol{\lambda}) = \sum_{j=1}^{k} a_j \lambda_j. \tag{3.6}$$

In other words, τ^{ϕ} is a retract if and only if the function $\sum_{j=1}^{k} a_j \lambda_j$ on $\mathcal{E}^{\phi}(\mathbb{H}^k)$ admits a holomorphic extension to the entire Teichmüller space. Heuristically, the more cylinders ϕ has, the stronger the criterion is. If ϕ has one cylinder, the criterion is vacuous. If the core curves of ϕ form a maximal disjoint curve system, then the polyplane is an open submanifold of $\mathcal{T}$, so the criterion says that $\sum_{j=1}^{k} a_j \lambda_j$ has a unique extension to a holomorphic map $\mathcal{T} \to \mathbb{H}$.

Remark 1: Alex Wright has pointed out to us a proof, based on his work in [39], that the orbit closure of a differential ϕ with an odd-order zero contains a Strebel differential with at least two cylinders, except potentially if ϕ is a pillowcase cover. So the criterion in Theorem 3.4.1 almost always gives us at least some nontrivial information.

Remark 2: A potential program to prove Conjecture 3.1.1 is to

(i) Use the criterion to identify a class of Jenkins-Strebel differentials which do not generate retracts.

(ii) Show that the orbit closure of any differential with an odd-order zero contains a Jenkins-Strebel differential of that class.

We will carry out this program for $\mathcal{T}_{0,5}$ by working with the class of differentials with two cylinders.

We return to the proof of Theorem 3.4.1. The "if" direction is easy; if there is a holomorphic map $G : \mathcal{T} \to \mathbb{H}$ satisfying (3.6), then

$$\begin{aligned} G \circ \tau^{\phi}(\lambda) &= G \circ \mathcal{E}^{\phi}(\lambda, \dots, \lambda) \\ &= \sum_{j=1}^{k} a_j \lambda = \lambda. \end{aligned}$$

Thus, G is a holomorphic retraction onto $\tau^{\phi}(\mathbb{H})$.

To prove the other direction, suppose there is a map $F : \mathbb{H} \to \mathcal{T}$ so that $F \circ \tau^{\phi} = \mathrm{id}_{\mathbb{H}}$. The idea of the proof is to approximate the desired G by maps of the form $t + F \circ \alpha$, with $t \in \mathbb{R}$ a translation and $\alpha \in \mathrm{MCG}$ a multi-twist. First, we recall a lemma from [27].

Lemma 3.4.2. *Let $f : \mathbb{H}^k \to \mathbb{H}$ be the composition $F \circ \mathcal{E}^{\phi}$. Then f satisfies*

$$f(\lambda, \dots, \lambda) = \lambda \tag{3.7}$$

and

$$\frac{\partial f}{\partial \lambda_j}(i, \dots, i) = a_j, \tag{3.8}$$

where a_j is the area of the jth cylinder of the unit-area differential ϕ.

Proof: Equation (3.7) is a restatement of $F \circ \tau^{\phi} = \mathrm{id}_{\mathbb{H}}$. To prove (3.8), let $X = \tau^{\phi}(i)$ be the underlying surface of ϕ. We first show that the cotangent vector $dF_X \in T^*_X\mathcal{T}$ is represented by the quadratic differential $-2i\phi$. To this end, let μ_{ϕ} be the Beltrami differential $\phi^{-1}|\phi|$. By definition, $\left(\tau^{\phi}\right)'(i) = \frac{i}{2}\mu_{\phi}$. So by the chain rule,

$$dF_X\left(\frac{i}{2}\mu_{\phi}\right) = 1.$$

But also

$$\int_X (-2i\phi)\left(\frac{i}{2}\mu_{\phi}\right) = \|\phi\| = 1.$$

Since F is holomorphic, dF_X has norm at most 1 with respect to the infinitesimal Kobayashi metrics on $T_X\mathcal{T}$ and $T_i\mathbb{H}$. Since the infinitesimal Kobayashi metric on $T_i\mathbb{H}$ is half the Euclidean metric, the unit norm ball of $\mathrm{Hom}_{\mathbb{C}}(T_X\mathcal{T}, T_i\mathbb{H})$ corresponds to the 2-ball of $Q(X)$. But $-2i\phi$ is the unique differential in the 2-ball of $Q(X)$ which pairs to 1 against $\frac{i}{2}\mu_\phi$. Thus, dF_X is integration against $-2i\phi$, as claimed.

By construction, the tangent vector $\frac{\partial \mathcal{E}^\phi}{\partial \lambda_j}(i,\ldots,i)$ is represented by the Beltrami differential which is equal to $\frac{i}{2}\mu_\phi$ on the jth cylinder Π_j and zero elsewhere. To obtain (3.8), compute

$$\begin{aligned}\frac{\partial f}{\partial \lambda_j}(i,\ldots,i) &= dF_X\left(\frac{\partial \mathcal{E}^\phi}{\partial \lambda_j}(i,\ldots,i)\right)\\ &= \int_{\Pi_j}(-2i\phi)\left(\frac{i}{2}\mu_\phi\right)\\ &= \int_{\Pi_j}|\phi|\\ &= a_j.\end{aligned}$$

□

The key tool in the proof of Theorem 3.4.1 is a complex-analytic result concerning the space $\mathcal{D}$ of holomorphic functions $\mathbb{H}^k \to \mathbb{H}$ which satisfy condition (3.7). Consider the conjugation of action of $\mathbb{R}$ on $\mathcal{D}$:

$$f_t(\lambda_1,\ldots,\lambda_k) = f(\lambda_1 - t,\ldots,\lambda_k - t) + t.$$

We call the map $(t, f) \mapsto f_t$ the *translation flow* on $\mathcal{D}$. The translation flow is well-behaved; for any $f \in \mathcal{D}$, the orbit $\{f_t\}$ spends most of its time close to a linear function. More precisely, we have the following result, proven in [14].

Theorem 3.4.3. *Let $f \in \mathcal{D}$. Define $\mathbf{g} \in \mathcal{D}$ by*

$$\mathbf{g}(\lambda_1,\ldots,\lambda_n) = \sum_{j=1}^{k} a_j\lambda_j, \text{where } a_j = \frac{\partial f}{\partial \lambda_j}(i,\ldots,i).$$

Let U be any neighborhood of $\mathbf{g}$ in the compact-open topology. Then the set

$$S = \{t \in \mathbb{R} | f_t \in U\}$$

has density 1 in $\mathbb{R}$:

$$\lim_{r\to\infty}\frac{m(S\cap[-r,r])}{2r} = 1,$$

where m is the Lebesgue measure.

We now prove the main result of this section.

Proof of Theorem 3.4.1: Let $F : \mathcal{T} \to \mathbb{H}$ be a holomorphic map such that $F \circ \tau^{\phi} = \mathrm{id}_{\mathbb{H}}$. Then $f = F \circ \mathcal{E}^{\phi}$ satisfies equations (3.7) and (3.8). To apply Theorem 3.4.3 in the present context, we need to approximate translations in the polyplane $\mathcal{E}^{\phi}(\mathbb{H}^k)$ by Dehn multi-twists. We will find a sequence $t^1, t^2, \ldots$ of real numbers and a sequence $\alpha^1, \alpha^2, \ldots$ of mapping classes so that $t^N + F \circ \alpha^N$ converges to the desired map G.

Let m_j denote the modulus of the jth cylinder of ϕ. Fix $\varepsilon > 0$. We claim we can choose $t \in \mathbb{R}$ so that

(i) The distance from t to the nearest point in $m_j^{-1}\mathbb{Z}$ is less than ε, for each $j = 1, \ldots, k$.

(ii)
$$d\left(f_t, \mathbf{g}\right) < \varepsilon,$$
where $\mathbf{g}(\boldsymbol{\lambda}) = \sum a_j \lambda_j$ and d is a fixed metric inducing the compact-open topology on $\mathcal{O}(\mathbb{H}^k)$.

To see this, let S_1 be the set of $t \in \mathbb{R}$ satisfying Condition (i), and let S_2 be the set satisfying the Condition (ii). By standard results on equidistribution of linear flows on the k-torus, the set S_1 has positive density in $\mathbb{R}$. By Theorem 3.4.3, the set S_2 has density 1. Therefore, the intersection $S_1 \cap S_2$ is nonempty, which is what we need.

So pick t satisfying the above conditions and find integers N_j so that $\left|t - \frac{N_j}{m_j}\right| < \varepsilon$. Let α be the multi-twist which twists N_j times about the jth cylinder. Now, set

$$G_\varepsilon = t + F \circ \alpha^{-1}.$$

Then
$$d(G_\varepsilon \circ \mathcal{E}^{\phi}, \mathbf{g}) \leq d(G_\varepsilon \circ \mathcal{E}^{\phi}, f_t) + d(f_t, \mathbf{g}).$$

By Condition (ii), the second term is less than ε. By Condition (i), the first term is small; to see this, write

$$\begin{aligned} d_{\mathbb{H}}\left(G_\varepsilon \circ \mathcal{E}^{\phi}(\boldsymbol{\lambda}), f_t(\boldsymbol{\lambda})\right) &= d_{\mathbb{H}}\left(f\left(\lambda_1 - \frac{N_1}{m_1}, \ldots, \lambda_k - \frac{N_k}{m_k}\right), f\left(\lambda_1 - t, \ldots, \lambda_k - t\right)\right) \\ &\leq \max d_{\mathbb{H}}\left(\lambda_j - \frac{N_j}{m_j}, \lambda_j - t\right) \\ &\leq \max d_{\mathbb{H}}\left(\lambda_j, \lambda_j + \varepsilon\right), \end{aligned}$$

where, in the equality, we have used (3.4) and, in the first inequality, we have used the fact that the Kobayashi distance on $\mathbb{H}^k$ is the max of the Poincaré distances on the factors. The last displayed quantity goes to 0 locally uniformly as $\varepsilon \to 0$.

Thus, the sequence

$$G_{\frac{1}{2}}, G_{\frac{1}{3}}, G_{\frac{1}{4}}, \ldots$$

is a normal family and any subsequential limit G satisfies $d\left(G \circ \mathcal{E}^{\phi}, \mathbf{g}\right) = 0$, i.e., $G \circ \mathcal{E}^{\phi} = \mathbf{g}$. □

If $\psi = \boldsymbol{\mu} \cdot \phi$ is another element of the orbit $\mathbb{H}^k\phi$, then $\tau^{\psi}(\lambda)$ is the surface obtained by applying to the jth cylinder of ϕ the linear transformation

$$\begin{pmatrix} 1 & \mathrm{Re}\,(\lambda) \\ 0 & \mathrm{Im}\,(\lambda) \end{pmatrix} \begin{pmatrix} 1 & \mathrm{Re}\left(\mu_j\right) \\ 0 & \mathrm{Im}\left(\mu_j\right) \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{Re}\left[\mathrm{Im}\left(\mu_j\right)\lambda + \mathrm{Re}\left(\mu_j\right)\right] \\ 0 & \mathrm{Im}\left[\mathrm{Im}\left(\mu_j\right)\lambda + \mathrm{Re}\left(\mu_j\right)\right] \end{pmatrix},$$

so

$$G \circ \tau^{\psi}(\lambda) = \sum_j a_j \left[\mathrm{Im}\left(\mu_j\right)\lambda + \mathrm{Re}\left(\mu_j\right)\right] = c\lambda + d,$$

where $c = \sum_j \mathrm{Im}\left(\mu_j\right)$ and $d = \sum_j \mathrm{Re}\left(\mu_j\right)$. So $\frac{G-d}{c}$ is a retraction onto the disk generated by ψ. Thus, we have proved

Corollary 3.4.4. *Let ϕ be a Jenkin-Strebel differential with k cylinders. If ϕ generates a holomorphic retract, then so does every differential in its $\mathbb{H}^k$ orbit.*

3.5 The L-shaped pillowcase

Doubling a right-angled L-shaped hexagon along its boundary yields a Jenkins-Strebel differential on $S_{0,5}$ called an *L-shaped pillowcase* (Figures 3.6, 3.7). In this section, we sketch the proof [27] that an L-shaped pillowcase does not generate a holomorphic retract, and thus that the Carathéodory and Kobayashi metrics on $\mathcal{T}_{0,5}$ do not coincide. We then prove Theorem 3.1.6, which states the two metrics on $\mathcal{T}_{g,n}$ are different whenever $\dim_{\mathbb{C}} \mathcal{T}_{g,n} \geq 2$.

The L-shaped pillowcase has two cylinders, Π_1 and Π_2. We let h_i denote the heights of Π_i. We denote the length of Π_1 by q and normalize so that the length of Π_2 is 1. We call the resulting quadratic differential $\phi(h_1, h_2, q)$ and its underlying marked surface $X(h_1, h_2, q)$.

Theorem 3.5.1. *[27] The Teichmüller disk generated by $\phi(h_1, h_2, q)$ is not a holomorphic retract.*

Sketch: By Corollary 3.4.4, it suffices to show that $\phi_0 := \phi(1,1,q)$ does not generate a retract. Suppose to the contrary that the disk τ^{ϕ_0} is a holomorphic retract of $\mathcal{T}_{0,5}$. Then by Theorem 3.4.1, there is a holomorphic $G : \mathcal{T}_{0,5} \to \mathbb{H}$ so that

$$G \circ \mathcal{E}^{\phi_0} = a_1\lambda_1 + a_2\lambda_2,$$

with $a_1 = \frac{q}{1+q}$ and $a_2 = \frac{1}{1+q}$.

The idea is to reach a contradiction by examining the regularity of G at the boundary of $\mathcal{E}^{\phi_0}(\mathbb{H}^2)$. To this end, note that the differential $\phi(0,1,q)$ obtained by collapsing Π_1 is a well-defined element of $\widetilde{\mathcal{Q}}_{0,5}$ (Figure 3.8a). Now observe that $\gamma(t) = X(0,1,q-t)$ is a smooth path in Teichmüller space (Figure 3.8b). Since G is holomorphic, $G \circ \gamma$ is a smooth path in $\mathbb{H}$.

Now, an argument using the Schwarz-Christoffel mappings shows that $X(0,1,q-t)$ is in $\mathcal{E}^{\phi}(\mathbb{H}^2)$ for each $t \in (0,q)$. In fact, there is a unique pair of positive numbers $h_1(t), h_2(t)$ so that

$$X(0,1,q-t) = X(h_1(t), h_2(t), q).$$

Thus,

$$\begin{aligned} G \circ \gamma(t) &= G \circ E^{\phi}(h_1(t)i, h_2(t)i) \\ &= [a_1h_1(t) + a_2h_2(t)]\, i. \end{aligned}$$

An involved computation with the Schwarz-Christoffel mappings determines the asymptotics of $h_1(t), h_2(t)$ for small positive t. See [27] Sections 8,9 for details. The end result is that there are constants β_1 and $\beta_2 \neq 0$ so that

$$G \circ \gamma(t) - G \circ \gamma(0) = \beta_1(1+o(1))\frac{t}{\log t^{-1}} + \beta_2(1+o(1))\frac{t^2}{\log t^{-1}} + o\left(\frac{t^2}{\log t^{-1}}\right),$$

which is incompatible with the fact that $G \circ \gamma$ is thrice-differentiable. □

Proof of Theorem 3.1.6: We have seen that the Kobayashi and Carathéodory metrics on $\mathcal{T}_{0,5}$ are different. To prove the corresponding fact for $\mathcal{T}_{g,n}$, we note that there is an embedding $\iota : \mathcal{T}_{0,5} \to \mathcal{T}_{g,n}$ which is holomorphic and isometric for the Kobayashi metric. The embedding is constructed using an elementary covering argument; we give the details in the Appendix.

Now, let $\phi \in \widetilde{\mathcal{Q}}_{0,5}$ be an L-shaped pillowcase. Then $\iota \circ \tau^{\phi}$ is a Teichmüller disk in $\mathcal{T}_{g,n}$. A holomorphic retraction $F : \mathcal{T}_{g,n} \to \mathbb{H}$ onto this disk would yield a retraction $F \circ \iota$ onto τ^{ϕ}, contradicting Theorem 3.5.1. So $\iota \circ \tau^{\phi}$ is not a retract and thus the two metrics on $\mathcal{T}_{g,n}$ are different. □

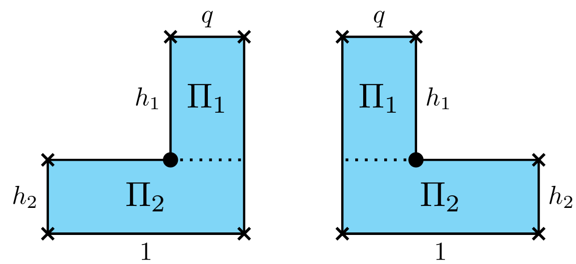


Figure 3.6: Gluing two copies of the L along corresponding edges yields the differential $\phi(h_1, h_2, q)$ on $S_{0,5}$. The crosses indicate poles and the dot indicates a simple zero.

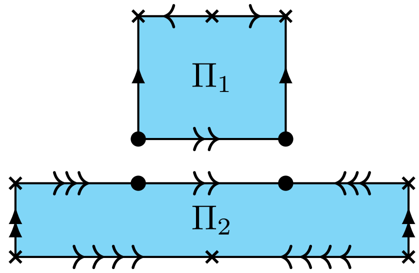
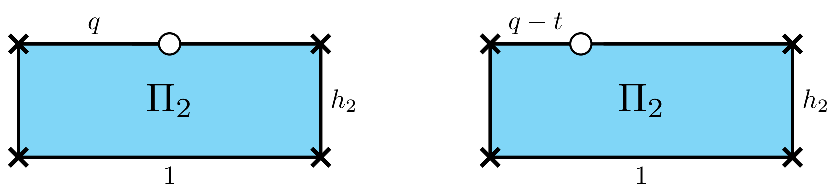


Figure 3.7: The cylinders of the L-shaped pillowcase.



(a) $X(0, h_2, q)$ (b) $\gamma(t) = X(0, h_2, q - t)$

Figure 3.8: Collapsing the top cylinder of an L-shaped pillowcase yields a well-defined quadratic differential. A simple pole and zero "cancel" to give a regular point at a puncture. We get a path in $\mathcal{T}_{0,5}$ by moving this puncture horizontally.

3.6 The five-times punctured sphere and twice-punctured torus

Jenkins-Strebel differentials on $S_{0,5}$

In this section, we classify the Jenkins-Strebel differentials on the five-times punctured sphere $S_{0,5}$. We establish the following:

Proposition 3.6.1.

(a) *Any differential with an odd-order zero has a Jenkins-Strebel differential with two cylinders in its* GL_2^+ *orbit closure.*

(b) *Any differential with two cylinders has an L-shaped pillowcase in its* $\mathbb{H}^2$-

orbit.

Combined with Proposition 3.2.1, Corollary 3.4.4, and Theorem 3.5.1, this proves our main result that Conjecture 3.1.1 holds in the case of $\mathcal{T}_{0,5}$. We summarize the proof of the main result in Section 3.6.

An integrable holomorphic quadratic differential ϕ on $S_{0,5}$ either has five poles and a simple zero or four poles and no zeros. In the second case, τ^{ϕ} is a retract by Theorem 3.1.4. (Alternatively, note that the forgetful map $F : \mathcal{T}_{0,5} \to \mathcal{T}_{0,4}$ which "fills in" the puncture at the regular point restricts to a biholomorphism $F \circ \tau^{\phi} : \mathbb{H} \to \mathcal{T}_{0,4}$.)

Conjecture 3.1.1 for $\mathcal{T}_{0,5}$ reduces to showing that, in case ϕ has a simple zero, τ^{ϕ} is not a retract. As we observed in Section 3.2, it suffices to consider the case that ϕ is Jenkins-Strebel. Let Γ be the critical graph of ϕ, i.e., the union of the horizontal rays emanating from zeros and poles. Since ϕ is Jenkins-Strebel, each horizontal ray emanating from a singularity ends at a singularity. Thus, Γ is a finite graph on S^2 with a valence three vertex at the simple zero and valence one vertices at each of the poles. (In this context, the valence of a vertex is the number of *half-edges* incident on it; a loop counts twice towards the valence of the incident vertex.) Each boundary component of an ε-neighborhood of Γ is a closed horizontal curve. There are two possibilities:

(Case 1) All three of the horizontal rays emanating from the zero terminate at simple poles. (See Figure 3.9a.) In this case, the two remaining poles are joined by a horizontal segment. The ε-neighborhood of Γ has two boundary components. Thus, ϕ is Jenkins-Strebel with one cylinder.

By shearing ϕ appropriately, we may assume that there is a vertical geodesic connecting a pole in one component of Γ to a pole in the other component. Then ϕ has a vertical cylinder which is not dense in $S_{0,5}$ (Figure 3.10). So by Theorem 3.2.4(b), the horocycle orbit closure of the rotated differential

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \phi$$

contains a Jenkins-Strebel differential with two cylinders.

(Case 2) One of the horizontal rays emanating from the zero terminates at the zero. (See Figure 3.9b.) Another ray emanating from the zero terminates at a pole. Let Γ_z denote the component of Γ containing the zero. Let M_z denote the

ε-neighborhood of Γ_z. There are two remaining pairs of poles; each pair is joined by a horizontal segment. Let Γ_1, Γ_2 denote these horizontal segments. Let M_1, M_2 denote their ε-neighborhoods.

The boundary ∂M_z has two components, one longer than the other. Thus, one of the components of ∂M_z, say the shorter one, is homotopic to ∂M_1, and the other is homotopic to ∂M_2. So ϕ is Jenkins-Strebel with two cylinders.

Now, shear a cylinder of ϕ so that there is a vertical segment connecting the zero in Γ_z and a pole in Γ_1. Next, shear the other cylinder so that there is a vertical geodesic connecting the pole of Γ_z to a pole of Γ_2. The resulting differential is an L-shaped pillowcase in the same $\mathbb{H}^2$-orbit as ϕ.

Proof of Proposition 3.6.1: To prove the first part, suppose $\phi \in \mathcal{Q}_{0,5}$ has an odd-order zero. Let $\psi \in \overline{\mathrm{GL}_2^+ \phi}$ be a Jenkins-Strebel differential with a simple zero. If ψ has two cylinders, we are done. If it has only one, then we are in **Case 1**, so ψ has a differential with two cylinders in its orbit closure.

To prove the second part, suppose $\phi \in \widetilde{\mathcal{Q}}_{0,5}$ has two cylinders. Then we are in **Case 2**, so ϕ is in the same $\mathbb{H}^2$ orbit as an L-shaped pillowcase. □

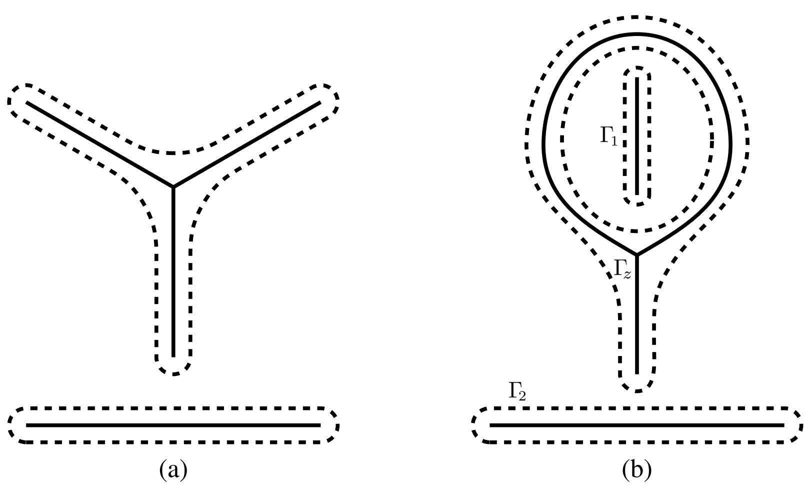


Figure 3.9: If $\phi \in \mathcal{Q}_{0,5}$ has a simple zero, then its critical graph is of one of the two indicated types.

Proof of the main theorem

We collect our results and classify Carathéodory geodesics in $\mathcal{T}_{0,5}$ and $\mathcal{T}_{1,2}$.

Proof of Theorem 3.1.2: We first prove the result for $\mathcal{T}_{0,5}$. The "if" direction follows from Kra's Theorem 3.1.4. For the "only if" direction, let $\phi \in \mathcal{Q}_{0,5}$ be a differential

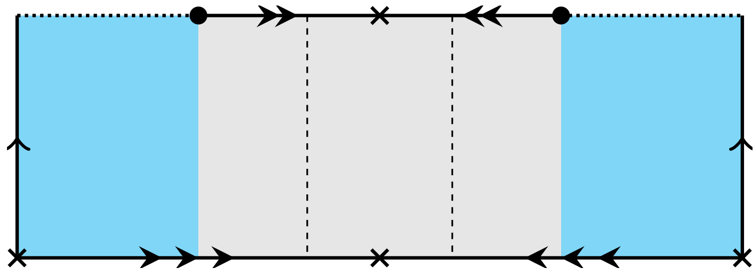


Figure 3.10: A Strebel differential on $S_{0,5}$ with one cylinder and an odd-order zero. Crosses indicate poles, and the large dot indicates a zero. The dashed curve is the core of a closed vertical cylinder.

with an odd-order zero. Suppose for the sake of contradiction that ϕ generates a holomorphic retract. By Theorem 3.2.4 and Proposition 3.6.1(a), the GL_2^+ orbit closure of ϕ contains a Jenkins-Strebel differential ϕ' with two cylinders and an odd-order zero. By Proposition 3.6.1(b), the orbit $\mathbb{H}^2 \cdot \phi'$ contains an L-shaped pillowcase ϕ''. By Proposition 3.2.1 and Corollary 3.4.4, $\tau^{\phi''}$ is a holomorphic retract. This contradicts Theorem 3.5.1.

To prove the result for $\mathcal{T}_{1,2}$, recall that there is an orientation-preserving involution $\alpha \in \mathrm{MCG}_{1,2}$ which fixes every point of $\mathcal{T}_{1,2}$. For each $X \in \mathcal{T}_{1,2}$, the class α is represented by a conformal involution of X which fixes four points and swaps the punctures. The quotient of X by the involution is a surface X' of genus zero with five marked points. There is a unique complex structure on X' making the quotient $f : X \to X'$ a holomorphic double cover branched over four marked points.

The map $\mathcal{T}_{1,2} \to \mathcal{T}_{0,5}$ sending X to X' is a biholomorphism. Let ϕ be a quadratic differential on X', and let $f^*\phi$ be its pullback to X. Then the Teichmüller disk τ^{ϕ} in $\mathcal{T}_{0,5}$ corresponds to the disk $\tau^{f^*\phi}$ in $\mathcal{T}_{1,2}$. To complete the proof, we observe that $f^*\phi$ has an odd-order zero if and only if ϕ does. Indeed, a simple zero of ϕ is necessarily unramified and thus lifts to two simple zeroes of $f^*\phi$. On the other hand, ramified poles and ramified regular points lift to regular points and double zeroes, respectively. So if ϕ has no odd-order zeros, neither does $f^*\phi$. □

3.7 Appendix: $\mathcal{T}_{0,5}$ embeds in $\mathcal{T}_{g,n}$

We used the following Lemma in Section 3.5 to deduce that the Kobayashi and Carathéodory metrics on $\mathcal{T}_{g,n}$ are different.

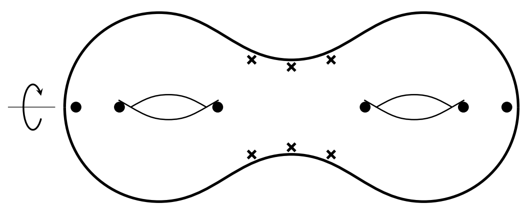
Lemma 3.7.1. *Whenever* $\dim_{\mathbb{C}} \mathcal{T}_{g,n} = 3g - 3 + n \geq 2$, *there is a holomorphic and isometric embedding* $\mathcal{T}_{0,5} \to \mathcal{T}_{g,n}$.

Proof: If $n = 2m$ is even, then $S_{g,n}$ admits an involution α fixing $2g + 2$ points, none of which are marked (Figure 3.11a). If $n = 2m + 1$ is odd, then $S_{g,n}$ admits an involution α fixing $2g + 2$ points, one of which is marked (Figure 3.12b). In either case, the quotient is S_{0,n_1} where $n_1 = m + 2g + 2 \geq 5$. The space $\mathcal{T}_{0,n_1}$ embeds holomorphically and isometrically in $\mathcal{T}_{g,n}$ as the fixed point set of the action of α (see, e.g., [13] p. 370). (In case $g = 1$ and $n = 2$, we obtain the isomorphism $\mathcal{T}_{0,5} \cong \mathcal{T}_{1,2}$ described in the last section. This construction also yields the other two coincidences $\mathcal{T}_{0,4} \cong \mathcal{T}_{1,1}$ and $\mathcal{T}_{0,6} \cong \mathcal{T}_{2,0}$.) If $n_1 = 5$, we are done. Otherwise, by the same construction as above, the surface S_{0,n_1} admits an involution with quotient S_{0,n_2} and $5 \leq n_2 < n_1$ (Figure 3.12). So we have embeddings

$$\mathcal{T}_{0,n_2} \hookrightarrow \mathcal{T}_{0,n_1} \hookrightarrow \mathcal{T}_{g,n}.$$

Continuing inductively, we get an embedding $\mathcal{T}_{0,5} \hookrightarrow \mathcal{T}_{g,n}$. □

(a)



(b)

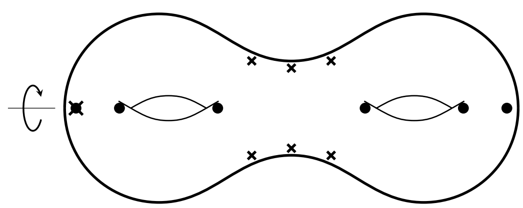


Figure 3.11: The desired involution is 180 degree rotation about the indicated axis. Crosses indicate marked points. Dots indicate fixed points of the involution.

(a) (b)

Figure 3.12: Involutions of spheres.

3.8 Acknowledgements

We would like to thank Alex Wright for helpful discussion.

Chapter 4

ISOMETRIC SUBMERSIONS OF TEICHMÜLLER SPACES

4.1 Introduction

Biholomorphisms of Teichmüller spaces

Let $S_{g,n}$ denote the surface of genus g with n punctures and let $\mathcal{T}_{g,n}$ denote the corresponding Teichmüller space. A central theme in Teichmüller theory is the interplay between the analytic structure of $\mathcal{T}_{g,n}$ and the topology and geometry of the underlying finite-type surface $S_{g,n}$. This theme is exemplified by the result of Royden [33] asserting that every biholomorphism of $\mathcal{T}_g$ with $g \geq 2$ arises from the action of a mapping class of S_g. To prove this, Royden first established that the Teichmüller metric is an invariant of the complex structure on $\mathcal{T}_g$ – it coincides with the intrinsically defined Kobayashi metric. Thus, any biholomorphism of $\mathcal{T}_g$ is an isometry for the Teichmüller metric. Then, by analyzing the infinitesimal properties of the Teichmüller norm, Royden showed that any holomorphic isometry is induced by a mapping class. Earle and Kra [10] later extended Royden's result to the finite-dimensional Teichmüller spaces $\mathcal{T}_{g,n}$. Finally, Markovic [26] generalized to the infinite dimensional case, proving for any Teichmüller space of complex dimension ≥ 2, that the biholomorphisms are induced by quasi-conformal self-maps of the underlying Riemann surface.

Isometric submersions between finite-type Teichmüller spaces

Royden, Earle-Kra, and Markovic characterized holomorphic isometries between Teichmüller spaces – except in a few low-complexity cases, these are induced by identifications of the underlying surfaces. In this paper, we characterize a broader class of maps between finite-type Teichmüller spaces: the holomorphic and isometric submersions. Recall that a C^1 map between Finsler manifolds is an *isometric submersion* if the derivative maps the unit ball of each tangent space onto the unit ball of the target tangent space. Our **main result** is that the holomorphic isometric submersions between Teichmüller spaces are all of geometric origin; with some low genus exceptions, these submersions are precisely the forgetful maps $\mathcal{T}_{g,n} \to \mathcal{T}_{g,m}$.

Theorem 4.1.1 (Main Result)**.** *Let $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$ be a holomorphic map which is an isometric submersion with respect to the Teichmüller metrics on the domain and* *range. Assume* (k,m) *satisfies the following conditions:*

The type (k,m) *is* non-exceptional*:* $2k+m \geq 5$. (4.1)

The genus k *is positive:* $k \geq 1$. (4.2)

Then $g = k$, $n \geq m$, *and up to pre-composition by a mapping class,* $F : \mathcal{T}_{g,n} \to \mathcal{T}_{g,m}$ *is the forgetful map induced by filling in the last* $n-m$ *punctures of* $S_{g,n}$.

Remark 1: Recall that we have isomorphisms $\mathcal{T}_{2,0} \cong \mathcal{T}_{0,6}$ and $\mathcal{T}_{1,2} \cong \mathcal{T}_{0,5}$ induced by hyperelliptic quotients. Thus, our hypothesis on the type (k,m) can be rephrased as follows – $\mathcal{T}_{k,m}$ is of complex dimension at least 2 and is not biholomorphic to a genus zero Teichmüller space $\mathcal{T}_{0,m}$. We expect that it is possible to remove the genus condition:

Conjecture 4.1.2. *Any holomorphic and isometric submersion between finite-dimensional Teichmüller spaces of complex dimension at least 2 is a composition of*

1. *Forgetful maps* $\mathcal{T}_{g,n} \to \mathcal{T}_{g,m}$ *with* $m < n$.
2. *Mapping classes* $\mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$.
3. *The isomorphisms* $\mathcal{T}_{2,0} \cong \mathcal{T}_{0,6}$ *and* $\mathcal{T}_{1,2} \cong \mathcal{T}_{0,5}$.

Remark 2: The complex dimension 1 Teichmüller spaces $\mathcal{T}_{0,4}$, $\mathcal{T}_{1,0}$, and $\mathcal{T}_{1,1}$ are all biholomorphic to the unit disk $\mathbb{D}$. There are many isometric submersions $\mathcal{T}_g \to \mathbb{D}$ – the diagonal entries of the canonical period matrix are examples (see [29] Theorem 5.2).

Infinitesimal geometry of the cotangent space

The proofs of Royden's theorem and its generalizations hinge on the analysis of the infinitesimal geometry of the Teichmüller norm. Fix a marked Riemann surface $X \in \mathcal{T}_{g,n}$. Then the cotangent space $T^*_X \mathcal{T}_{g,n}$ identifies with the space $Q(X)$ of integrable holomorphic quadratic differentials on X. With respect to this identification, the dual Teichmüller norm is the L^1 norm $\|\phi\| = \int_X |\phi|$. Thus, a holomorphic isometry $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$ induces for each $X \in \mathcal{T}_{g,n}$ a bijective, $\mathbb{C}$-linear isometry of quadratic differential spaces $Q(F(X)) \to Q(X)$. The core step in the proof of Royden's theorem is showing that, up to scale by a constant $e^{i\theta}$, any such isometry is pullback by a biholomorphism $X \to F(X)$.

Our study of isometric submersions between Teichmüller spaces follows a similar tack. The key observation is that an isometric submersion induces an isometric embedding of cotangent spaces (see Section 4.2). An important step in our analysis is the following classification result, which is of independent interest.

Theorem 4.1.3. *Let X and Y be finite-type Riemann surfaces. Let $\widehat{X}$ and $\widehat{Y}$ be the compact surfaces obtained by filling the punctures of X and Y. Assume the type (k,m) of X is non-exceptional: $2k+m \geq 5$. Let $T : Q(X) \hookrightarrow Q(Y)$ be a $\mathbb{C}$-linear isometric embedding. Then there is a holomorphic map $h : \widehat{Y} \to \widehat{X}$ and a constant $c \in \mathbb{C}$ of magnitude $deg(h)^{-1}$ so that $T = c \cdot h^*$.*

Remark: Suppose X is of exceptional type (k,m), so $2k+m \leq 4$. Then one of the following holds:

1. $\dim_{\mathbb{C}} Q(X) \leq 1$
2. (k,m) is $(2,0)$ or $(1,2)$, in which case $Q(X)$ identifies naturally with the quadratic differential space of a surface of non-exceptional type $(0,6)$ or $(0,5)$, respectively.

Thus, Theorem 4.1.3 amounts to a complete classification of $\mathbb{C}$-linear isometric embeddings $Q(X) \to Q(Y)$ for X and Y of finite type.

To prove Theorem 4.1.3, we use methods developed by Markovic [26] in his proof of the infinite-dimensional generalization of Royden's theorem. (See also the paper of Earle-Markovic [11] and the thesis of S. Antonakoudis [3].) Recall the *bi-canonical map* $\widehat{X} \to \mathbb{P}Q(X)^*$ sending each $x \in \widehat{X}$ to the hyperplane in $Q(X)$ of quadratic differentials vanishing at x. The idea is to relate the bi-canonical images of X and Y using a result of Rudin [34] on isometries of L^p spaces. The fact that $T : Q(X) \to Q(Y)$ is an isometric embedding implies via Rudin's theorem that $T^* : \mathbb{P}Q(Y)^* \to \mathbb{P}Q(X)^*$ carries the bi-canonical image of $\widehat{Y}$ onto the bi-canonical image of $\widehat{X}$. So, there is a unique $h : \widehat{Y} \to \widehat{X}$ making the following diagram commute:

$$\begin{array}{ccc} \mathbb{P}Q(Y)^* & \xrightarrow{T^*} & \mathbb{P}Q(X)^* \\ \uparrow & & \uparrow \\ \widehat{Y} & \xrightarrow{h} & \widehat{X} \end{array}$$

In fact, Rudin's result gives us more: for any $\phi \in Q(X)$, the map h pushes the $|T\phi|$-measure on $\widehat{Y}$ to the $|\phi|$-measure on $\widehat{X}$. Thus, we obtain the following intermediate result:

Proposition 4.1.4. *Let X and Y be finite-type Riemann surfaces, with X of non-exceptional type. Suppose $T : Q(X) \hookrightarrow Q(Y)$ is a $\mathbb{C}$-linear isometric embedding. Then there is a holomorphic map $h : \widehat{Y} \to \widehat{X}$ with the following property: For any $\phi \in Q(X)$ and any measurable $K \subset \widehat{X}$,*

$$\int_K |\phi| = \int_{h^{-1}(K)} |T\phi| \,.$$

We then use Proposition 4.1.4 to derive the classification result Theorem 4.1.3.

Infinitesimal to global

The last step is to obtain the global Main Result, Theorem 4.1.1, from the infinitesimal Theorem 4.1.3. We are given a holomorphic and isometric submersion $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$, with (k, m) satisfying hypotheses (4.1) and (4.2). Since (k, m) is assumed non-exceptional, Theorem 4.1.3 gives for each $Y \in \mathcal{T}_{g,n}$ a holomorphic branched cover $h_Y : \widehat{Y} \to \widehat{F(Y)}$. By a dimension count, it is not the case that every Riemann surface of genus g is a branched cover of a surface of genus k with $1 \leq k < g$. We then obtain that $g = k$. Finally, an argument involving the universal families over $\mathcal{T}_{g,n}$ and $\mathcal{T}_{g,m}$ shows that the map $h_Y : Y \to F(Y)$ varies continuously in $Y \in \mathcal{T}_{g,n}$. Thus, the topological type of h_Y is constant in Y. We conclude that the map F is induced by a (fixed) mapping class composed with the inclusion map on the underlying surfaces, filling in punctures.

Related work

In this paper, we generalize Royden's theorem on isometries by studying isometric submersions between Teichmüller spaces. Dually, one can attempt to generalize Royden's theorem by classifying of the holomorphic and isometric *embeddings* between Teichmüller spaces. A claimed result of S. Antonakoudis states that the isometric embeddings all arise from covering constructions.

Our result on submersions complements a classic theorem of Hubbard [18] asserting that there are no holomorphic sections of the forgetful map $\mathcal{T}_{g,1} \to \mathcal{T}_g$, except for the six sections in genus 2 obtained by marking fixed points of the hyperelliptic involution. Earle and Kra [10] later extended the result to the setting of forgetful maps between finite-type Teichmüller spaces $\mathcal{T}_{g,n} \to \mathcal{T}_{g,m}$. Combined, our result and the theorem of Hubbard-Earle-Kra have the following interpretation:

1. Holomorphic and isometric submersions between finite-dimensional Teichmüller spaces are of geometric origin. (They are forgetful maps.)
2. These submersions do not admit holomorphic sections, unless there is a geometric reason (fixed points of elliptic involutions in genus 1 and hyperelliptic involutions in genus 2).

We mention also a result of Antonakoudis-Aramayona-Souto [4] stating that any holomorphic map $\mathcal{M}_{g,n} \to \mathcal{M}_{k,m}$ between moduli spaces is forgetful, as long as $g \geq 6$ and $k \leq 2g - 2$. One can see this is as a parallel of our result, with our metric constraint replaced by an equivariance condition.

S. Antonakoudis [3] was the first to study isometric submersions in the context of Teichmüller theory. He proved that there is no holomorphic and Kobayshi-isometric submersion between a finite-dimensional Teichmüller space and a bounded symmetric domain, provided each is of complex dimension at least two.

The classification of holomorphic isometric submersions between bounded symmetric domains is an interesting problem. See the paper of Knese [23] for the classification of holomorphic Kobayashi-isometric submersions from the polydisk $\mathbb{D}^n$ to the disk $\mathbb{D}$. For Teichmüller-theoretic applications of this class of functions on the polydisk, see [14] and [15].

Outline

The rest of the paper is devoted to the proofs of Theorems 4.1.1 and 4.1.3.

Section 4.2 focuses on the infinitesimal geometry of isometric submersions between Teichmüller spaces. In 4.2, we recall basic facts on isometric submersions between Finsler manifolds. In 4.2 we establish that forgetful maps between Teichmüller spaces are holomorphic and isometric submersions. In 4.2, we review a theorem of Rudin concerning isometries between L^p spaces, and in 4.2, we discuss the bi-canonical embedding $X \hookrightarrow \mathbb{P}Q(X)^*$ of a Riemann surface. Then, in 4.2 we follow the argument of [26] to obtain Proposition 4.1.4. Finally, in 4.2, we obtain the classification Theorem 4.1.3 of isometric embeddings between quadratic differential spaces.

Section 4.3 focuses on the global geometry of isometric submersions $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$ and the proof of the main result, Theorem 4.1.1. In 4.3, we use Theorem 4.1.3 to obtain for each $Y \in \mathcal{T}_{g,n}$ a non-constant holomorphic map $h_Y : \widehat{Y} \to \widehat{F(Y)}$. Then we use a dimension-counting argument to show that $g = k$. In 4.3, we use properties of the universal family to show that the collection of maps $h_Y : Y \to X$ varies continuously in the parameter $Y \in \mathcal{T}_{g,n}$. Finally, in 4.3, we finish the proof of the main result.

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4.2 Infinitesimal Geometry

Isometric submersions of Finsler manifolds

We review basic properties of isometric submersions, following [2]. First, we recall the relevant notion from linear algebra. An *isometric submersion* between normed vector spaces V and W is a linear map $V \to W$ so that the image of the closed unit ball in V is the closed unit ball in W. Isometric submersions and isometric embeddings of normed vector spaces are dual in the following sense.

Lemma 4.2.1. *Let $T : V \to W$ be a linear map between normed vector spaces.*

1. *If T is an isometric submersion, then the dual map $T^* : W^* \to V^*$ is an isometric embedding.*

2. *If T is an isometric embedding, then $T^* : W^* \to V^*$ is an isometric submersion.*

The proof of the first assertion of the Lemma is elementary. The second assertion is a restatement of the Hahn-Banach theorem.

An *isometric submersion between Finsler manifolds M, N* is a C^1 submersion $F : M \to N$ such that the derivative $dF_m : T_m M \to T_{F(m)} N$ is an isometric submersion between tangent spaces with respect to the Finsler norms, for each $m \in M$. We will use the characterization of isometric submersions in terms of isometric embeddings of cotangent spaces.

Corollary 4.2.2. *Let $F : M \to N$ be a C^1 map of Finsler manifolds. Then F is an isometric submersion if and only if for each $m \in M$, the coderivative*

$$dF_m^* : T^*_{F(m)}N \to T^*_m M$$

is an isometric embedding of cotangent spaces with respect to the dual Finsler norms.

Forgetful maps between Teichmüller spaces

We recall basic properties of forgetful maps between Teichmüller spaces, and in particular observe that these maps are holomorphic and isometric submersions. Let $F : \mathcal{T}_{g,1} \to \mathcal{T}_g$ be the forgetful map; for each $X \in \mathcal{T}_{g,1}$, $F(X)$ is the marked Riemann surface obtained by filling in the puncture of X. The cotangent space $T^*_X \mathcal{T}_{g,1} = Q(X)$ consists of holomorphic quadratic differentials on X with at worst a simple pole at the puncture, while $T^*_{F(X)} \mathcal{T}_g = Q(F(X))$ consists of those quadratic differentials on X which extend holomorphically over the puncture. The co-derivative dF^*_X is the inclusion $Q(F(X)) \hookrightarrow Q(X)$, which is clearly isometric and complex-differentiable. Thus, F is a holomorphic and isometric submersion. The same reasoning shows that any forgetful map $\mathcal{T}_{g,n} \to \mathcal{T}_{g,m}$ is an isometric submersion.

Rudin's Equimeasurability Theorem

We will need a general result of Rudin concerning isometries between subspaces of L^p spaces. Markovic [26] used this result in the $p = 1$ case to extend Royden's theorem to Teichmüller spaces of infinite dimension, and Earle-Markovic [11] used the result to give a new and illuminating proof of Royden's theorem in the finite-dimensional case.

Proposition 4.2.3 (Rudin [34], Theorem 1)**.** *Let p be a positive real number which is not an even integer. Let X and Y be sets with finite positive measures μ and ν respectively. Let l be a positive integer. Suppose $f_1, \ldots, f_l$ in $L^p(\mu, \mathbb{C})$, and $g_1, \ldots, g_l$ in $L^p(\nu, \mathbb{C})$ satisfy the following condition:*

$$\int_X \left|1 + \sum_{j=1}^{l} \lambda_j f_j\right|^p d\mu = \int_Y \left|1 + \sum_{j=1}^{l} \lambda_j g_j\right|^p d\nu, \text{ for all } (\lambda_1, \ldots, \lambda_l) \in \mathbb{C}^l. \quad (4.3)$$

If $F = (f_1, \ldots, f_l)$ and $G = (g_1, \ldots, g_l)$, then the maps $F : X \to \mathbb{C}^l$ and $G : Y \to \mathbb{C}^l$ satisfy the following equimeasurability condition:

$$\mu(F^{-1}(E)) = \nu(G^{-1}(E)) \text{ for each Borel set } E \subseteq \mathbb{C}^l. \quad (4.4)$$

Equation (4.3) is an assumption on the moments of the $\mathbb{C}^l$-valued random variables F and G. The conclusion (4.4) is that F and G have the same distribution. In other words, the pushforward measures $F_*(\mu)$ and $G_*(\nu)$ on $\mathbb{C}^l$ are equal.

Projective embeddings of Riemann surfaces

In this section, we establish the setting for our application of Rudin's theorem. Let L be a holomorphic line bundle over a compact Riemann surface $\widehat{X}$, and let $\mathcal{O}(L)$ denote the space of holomorphic sections of L. There is a holomorphic map $\widehat{X} \to \mathbb{P}\mathcal{O}(L)^*$ sending $x \in \widehat{X}$ to the hyperplane in $\mathcal{O}(L)$ consisting of sections which vanish at x. An argument using the Riemman-Roch theorem (see [32] p. 55) shows that if the degree of L is at least $2g + 1$, then the map $\widehat{X} \to \mathbb{P}\mathcal{O}(L)^*$ is an embedding.

Now, let X be a Riemann surface of type (g, n). Denote by $\widehat{X}$ the compact, genus g Riemann surface obtained by filling in the punctures of X. The space $Q(X)$ consists of quadratic differentials which are holomorphic on X and have at most simple poles at the punctures $\widehat{X} \setminus X$. Thus, elements of $Q(X)$ correspond to sections of a line bundle on $\widehat{X}$ of degree $4g - 4 + n$. By the preceding discussion, the associated *bi-canonical map* $\widehat{X} \to \mathbb{P}Q(X)^*$ is an embedding provided $4g - 4 + n \geq 2g + 1$, or $2g + n \geq 5$. Thus, the surfaces X of non-exceptional type are precisely those for which $\widehat{X} \to \mathbb{P}Q(X)^*$ is an embedding.

Applying the equimeasurability theorem

In this section, we apply the methods of [26] to prove Proposition 4.1.4. We acknowledge some overlap with [3] Section 5, particularly in the proof of the fact that the surface $\widehat{Y}$ covers the surface $\widehat{X}$ if there is a $\mathbb{C}$-linear isometric embedding $Q(X) \hookrightarrow Q(Y)$.

Proof of Proposition 4.1.4: Let X and Y be Riemann surfaces of finite type. Assume X is of non-exceptional type, and denote by $\Phi : \widehat{X} \hookrightarrow \mathbb{P}Q(X)^*$ the bi-canonical embedding associated to X. Let $T : Q(X) \to Q(Y)$ be a $\mathbb{C}$-linear isometric embedding. Denote by Ψ the composition $\widehat{Y} \to \mathbb{P}Q(Y)^* \to \mathbb{P}Q(X)^*$ of the bi-canonical map of Y with the dual of T. To describe the maps Φ and Ψ more concretely, fix a basis $\phi_0, \ldots, \phi_k$ for $Q(X)$ and let $\psi_i = T\phi_i$ denote the images in $Q(Y)$. In terms of local coordinates z, w for $\widehat{X}$ and $\widehat{Y}$, respectively, the maps $\Phi : \widehat{X} \to \mathbb{P}^l$ and $\Psi : \widehat{Y} \to \mathbb{P}^l$ are given by

$$\Phi(z) = [\phi_0(z) : \ldots : \phi_l(z)], \; \Psi(w) = [\psi_0(w) : \ldots : \psi_l(w)].$$

Now, consider the rational functions $f_i = \frac{\phi_i}{\phi_0}$ on $\widehat{X}$ and $g_i = \frac{\psi_i}{\psi_0}$ on $\widehat{Y}$, with $i = 1, \ldots, l$. Form the $\mathbb{C}^l$-valued maps $F = (f_1, \ldots, f_l)$ and $G = (g_1, \ldots, g_l)$. The maps F and G are just Φ and Ψ viewed as rational maps to $\mathbb{C}^l$.

Let μ denote the $|\phi_0|$-measure on $\widehat{X}$; that is,

$$\mu(K) = \int_K |\phi_0|$$

for any measurable $K \subset \widehat{X}$. Similarly, let ν denote the $|\psi_0|$-measure on $\widehat{Y}$. Then f_i and g_i are L^1 functions with with respect to the measures μ and ν. The assumption that T is isometric and $\mathbb{C}$-linear translates precisely to the hypothesis (4.4) of Rudin's theorem:

$$\begin{aligned}\int_{\widehat{X}} \left| 1 + \sum_{i=1}^{l} \lambda_i f_i \right| d\mu &= \int_{\widehat{X}} \left| \phi_0 + \sum_{i=1}^{l} \lambda_i \phi_i \right| \\ &= \int_{\widehat{Y}} \left| \psi_0 + \sum_{i=1}^{l} \lambda_i \psi_i \right| = \int_{\widehat{Y}} \left| 1 + \sum_{i=1}^{l} \lambda_i g_i \right| d\nu.\end{aligned}$$

Note that we used $\mathbb{C}$-linearity of T in the second equality. We conclude that the measures $F_*(\mu)$ and $G_*(\nu)$ on $\mathbb{C}^l$ are equal. What amounts to the same thing, the measures $\Phi_*(\mu)$ and $\Psi_*(\nu)$ on $\mathbb{P}^k$ are equal.

We now show that Φ and Ψ have the same image. To this end, note that the measure $\Psi_*(\nu) = \Phi_*(\mu)$ has as its support the compact set $\Phi(\widehat{X})$. Since Ψ is continuous and since ν assigns nonzero measure to each open set of $\widehat{Y}$, we conclude $\Psi(\widehat{Y}) \subset \Phi(\widehat{X})$. Thus, there is a unique holomorphic map $h : \widehat{Y} \to \widehat{X}$ so that $\Psi = \Phi \circ h$. Obviously, Ψ is not constant and so neither is h. In particular, h is a branched cover and $\Psi(\widehat{Y}) = \Phi(\widehat{X})$.

In terms of the map h, the equimeasurability condition $\Psi_*(\nu) = \Phi_*(\mu)$ becomes simply $h_*(\nu) = \mu$. Thus, for any measurable $K \subset \widehat{X}$ we have

$$\int_K |\phi_0| = \mu(K) = \nu\left(h^{-1}(K)\right) = \int_{h^{-1}(K)} |T\phi_0| \,.$$

Since ϕ_0 was chosen arbitrarily, we have the desired equality

$$\int_K |\phi| = \int_{h^{-1}(K)} |T\phi|$$

for any $\phi \in Q(X)$ and any measurable $K \subset \widehat{X}$. This completes the proof of Proposition 4.1.4.

Completing the classification of isometric embeddings

Let $\phi \in Q(X)$ and write $\psi = T\phi$. Proposition 4.1.4 says

$$\int_{h^{-1}(K)} |\psi| = \int_K |\phi| \tag{4.5}$$

for any measurable $K \subset \widehat{X}$. To complete the proof of Theorem 4.1.3, we must show that ψ is a scalar multiple of the pullback $h^*\phi$. By working over an appropriate coordinate chart in X, we will reduce the proof to the following elementary lemma.

Lemma 4.2.4. *Let g be a real-valued function defined on a domain in $\mathbb{C}$. If both g and e^g are harmonic, then g is constant.*

Proof: Compute

$$0 = (e^g)_{z\overline{z}} = e^g\,(g_z g_{\overline{z}} + g_{z\overline{z}}) = e^g g_z g_{\overline{z}}.$$

Thus, g is either holomorphic or anti-holomorphic. Since g is real-valued, it follows that it is constant. □

Returning to the proof of Theorem 4.1.3, fix a coordinate chart (U, z) in X on which $\phi = (dz)^2$. (Recall that one achieves this by integrating a local square root of ϕ.) Shrinking U if necessary, assume U is evenly covered by h and that ψ has no zeros or poles in $h^{-1}(U)$. Write $h^{-1}(U)$ as a disjoint union of coordinate charts (U_i, z_i), with coordinate functions chosen so that $h : (U_i, z_i) \to (U, z)$ is the identity function:

$$z(h(y)) = z_i(y).$$

Let $\psi_i(z_i)(dz_i)^2$ denote the local expression for ψ in U_i. Let $K \subset U$ be measurable. Then equation (4.5) yields

$$\int_K \left(\sum_{i=1}^{\deg(h)} |\psi_i(z)| \right) |dz| = \int_K |dz|\,.$$

Since K was arbitrary, we have

$$\sum_{i=1}^{\deg(h)} |\psi_i(z)| = 1,$$

identically on U. Recall that the absolute value of a holomorphic function of one variable is subharmonic. So the function

$$|\psi_1(z)| = 1 - \sum_{i=2}^{\deg(h)} |\psi_i(z)|$$

is both subharmonic and superharmonic. That is, $|\psi_1(z)|$ is harmonic. But, since $\psi_1(z)$ is holomorphic and non-vanishing, $\log|\psi_1(z)|$ is also harmonic. By Lemma 4.2.4, $\psi_1(z)$ is identically equal to some constant c. In other words,

$$\psi = c \cdot h^*\phi$$

on the open set U_1 and thus on all of X. Since $\phi \in Q(X)$ was arbitrary and $T : Q(X) \to Q(Y)$ is linear, we have

$$T\phi = c \cdot h^*\phi$$

for all $\phi \in Q(X)$, with c independent of ϕ. Since T is an isometric embedding, we have

$$|c| = \frac{\|\phi\|}{\|h^*\phi\|} = \deg(h)^{-1}.$$

This completes the proof of Theorem 4.1.3.

4.3 The main result

Set-up

We begin the proof of the main result Theorem 4.1.1. Let $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$ be a holomorphic and isometric submersion of Teichmüller spaces. Assume $2k + m \geq 5$ and $k \geq 1$. By Corollary 4.2.2, we have for each $Y \in \mathcal{T}_{g,n}$ that the induced map of cotangent spaces $Q(F(Y)) \to Q(Y)$ is an isometric embedding. Since $2k + m \geq 5$, Theorem 4.1.3 tell us that the embedding is, up to scale, pull-back by a holomorphic branched cover of compact surfaces

$$h_Y : \widehat{Y} \to \widehat{F(Y)}.$$

We conclude in particular that every Riemann surface of genus g admits a holomorphic branched cover of a surface of genus h. We now use our assumption that $k \geq 1$. The following elementary lemma implies that $g = k$.

Lemma 4.3.1. *Suppose $g \geq 2$. It is not the case that every $X \in \mathcal{T}_g$ admits a holomorphic cover of a surface of genus k with $1 \leq k < g$.*

Proof: The proof is by a dimension comparison. Suppose $1 \leq k < g$ and let $f : S_g \to S_k$ be a degree d branched cover. Recall the Riemann-Hurwitz formula:

$$2 - 2g = d \cdot (2 - 2k) - b,$$

where b is the total branch order of the cover.

We distinguish the cases $k = 1$ and $k \geq 2$. If $k \geq 2$, we have $\dim \mathcal{T}_g = 3g - 3$ and $\dim \mathcal{T}_k = 3k - 3$, so we get

$$\dim \mathcal{T}_g = d \cdot \dim \mathcal{T}_k + \frac{3}{2} b.$$

On the other hand, for a fixed topological type of branched cover, the space of surfaces in $Y \in \mathcal{T}_g$ which admit a holomorphic cover $Y \to X$ of that type has dimension at most

$$\dim \mathcal{T}_k + b,$$

which is less than $\dim \mathcal{T}_g$ since $g > k$ and thus $d > 1$.

If $k = 1$, then $\dim \mathcal{T}_g = \frac{3}{2} b$ and the dimension of the locus of $X \in \mathcal{T}_g$ which admit a holomorphic cover of the given type is at most b. Since $g > k = 1$, the cover must have $b > 0$ and so $b < \frac{3}{2} b = \dim \mathcal{T}_g$.

Thus, the locus of $X \in \mathcal{T}_g$ covering a surface of genus less than g and greater than 0 is a countable union of lower-dimensional subvarieties. The lemma follows. $\square$

Remark: The locus of $X \in \mathcal{T}_g$ which cover the square torus is dense. This follows from the fact that the locus of abelian differentials with rational period coordinates is dense in the Hodge bundle over $\mathcal{T}_g$ [40].

We conclude that $g = k$, so our submersion F maps from $\mathcal{T}_{g,n}$ to $\mathcal{T}_{g,m}$ with $m \leq n$. We are almost done: If $g \geq 2$, the covering maps $h_Y : \widehat{Y} \to \widehat{F(Y)}$ must be biholomorphisms. If $g = 1$, we know a priori only that h_Y are (unbranched) holomorphic covers. Since the pullback h_Y^* sends $Q(F(Y))$ into $Q(Y)$, each preimage of a puncture p in $F(Y)$ must be a puncture of Y. (Otherwise, h_Y pulls a differential with a pole at p back to a differential which is not in $Q(Y)$.) Thus, h_Y restricts to a map between the (potentially punctured) surfaces Y and X. The map $h_Y : Y \to X$ and the markings $S_{g,n} \to Y$, $S_{g,m} \to X$ fit into a diagram

$$\begin{array}{ccc} S_{g,n} & \longrightarrow & S_{g,m} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{h_Y} & X \end{array} .$$

It remains to establish two facts.

1. The maps h_Y are biholomorphisms in the $g = 1$ case.

2. The isotopy class of $S_{g,n} \to S_{g,m}$, is independent of $Y \in \mathcal{T}_{g,n}$.

The key to establishing both is showing that the family $h_Y : \widehat{Y} \to \widehat{F(Y)}$ varies continuously in the variable Y. To make this precise, we observe that the maps $h_Y : \widehat{Y} \to \widehat{F(Y)}$ fit together into a map of universal curves $H : \mathcal{C}_{g,n} \to \mathcal{C}_{g,m}$ covering the map $F : \mathcal{T}_{g,m} \to \mathcal{T}_{g,m}$ of Teichmüller spaces:

$$\begin{array}{ccc} \mathcal{C}_{g,n} & \xrightarrow{H} & \mathcal{C}_{g,m} \\ \downarrow & & \downarrow \\ \mathcal{T}_{g,n} & \xrightarrow{F} & \mathcal{T}_{g,m} \end{array}$$

We will show in the next section that H is continuous. Recall h_Y was constructed using the maps $X \to \mathbb{P}Q(X)^*$ and $Y \to \mathbb{P}Q(Y)^*$. We will leverage properties of the bundle of quadratic differentials over Teichmüller space to prove that H is in fact holomorphic.

The universal curve and the cotangent bundle

We start by recalling the properties of the universal curve $\pi : \mathcal{C}_{g,n} \to \mathcal{T}_{g,n}$. A good reference for this material is [31].

The map $\pi : \mathcal{C}_{g,n} \to \mathcal{T}_{g,n}$ is a holomorphic submersion whose fiber over $X \in \mathcal{T}_{g,n}$ is exactly the compact Riemann surface $\widehat{X}$. The locations of the punctures are encoded by canonical holomorphic sections

$$s_i : \mathcal{T}_{g,n} \to \mathcal{C}_{g,n} \; i = 1, \dots, n.$$

The point $s_i(X) \in \widehat{X}$ is the ith puncture of X. Moreover, there is a canonical topological trivialization

$$\mathcal{F}_{g,n} : \mathcal{T}_{g,n} \times S_{g,n} \to \mathcal{C}_{g,n} \setminus \bigcup_{i=1}^{n} s_i(\mathcal{T}_{g,n}),$$

unique up to fiberwise isotopy, so that the induced marking of each fiber

$$S_{g,n} \to \{X\} \times S_{g,n} \overset{\mathcal{F}_{g,n}}{\to} X$$

agrees with the marking defining X as a point of $\mathcal{T}_{g,n}$. The family $(\pi, \{s_i\}_{i=1}^n, \mathcal{F}_{g,n})$ is universal among n-pointed marked holomorphic families of genus g Riemann surfaces (see [31]).

Now, let $Q_{g,n} \to \mathcal{T}_{g,n}$ denote the bundle of integrable holomorphic quadratic differentials over Teichmüller space. Let $\mathbb{P}Q^*_{g,n} \to \mathcal{T}_{g,n}$ denote the associated holomorphic bundle of projectivized dual spaces. The bi-canonical maps $\widehat{X} \to \mathbb{P}Q(X)^*$ fit into a map

$$\Psi : C_{g,n} \to \mathbb{P}Q^*_{g,n}$$

covering the projections to Teichmüller space. We need to show that this map of bundles is holomorphic.

Proposition 4.3.2. *The fiberwise bi-canonical map $\Psi : C_{g,n} \to \mathbb{P}Q^*_{g,n}$ is holomorphic. If the type (g,n) is non-exceptional, then the map is a biholomorphism onto its image.*

Proof: Since π is a holomorphic submersion, $C_{g,n}$ is covered by product neighborhoods $U \times V$, with U open in $\mathcal{T}_{g,n}$ and V open in $\mathbb{C}$. Each $U \times V$ maps biholomorphically to an open neighborhood of $C_{g,n}$ by a map commuting with the projections:

$$\begin{array}{ccc} U \times V & \longrightarrow & C_{g,n} \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{T}_{g,n} \end{array}$$

Given $X \in U$, the slice $\{X\} \times V$ is a holomorphic coordinate chart for the Riemann surface $\widehat{X}$. For this reason, the product neighborhoods $U \times V$ are called *relative coordinate charts* for the family $C_{g,n}$.

Recall $Q_{g,n} \to \mathcal{T}_{g,n}$, the bundle of integrable holomorphic quadratic differentials over Teichmüller space. A section $q : \mathcal{T}_{g,n} \to Q_{g,n}$ can be thought of as a fiberwise quadratic differential on $C_{g,n}$. In a relative coordinate chart $U \times V$, the differential q takes the form $q(X,z)(dz)^2$. It follows by a result of Bers [8] that a section $q : \mathcal{T}_{g,n} \to Q_{g,n}$ is holomorphic if and only if $(X,z) \mapsto q(X,z)$ is meromorphic in each relative chart $U \times V$.

Now, let $U \times V$ be a relative coordinate chart for $C_{g,n}$ and let $q_0, \ldots, q_k$ be a holomorphic frame for $Q_{g,n} \to \mathcal{T}_{g,n}$ over U. With respect to the choice of coordinates and frame, the fiberwise bi-canonical map $C_{g,n} \to \mathbb{P}Q^*_{g,n}$ is expressed as the map $U \times V \to \mathbb{P}^k$ given by

$$(X,z) \mapsto [q_0(X,z) : q_1(X,z) : \cdots : q_k(X,z)], \tag{4.6}$$

which is holomorphic since the $q_i(X,z)$ are meromorphic.

We conclude that $\Psi : C_{g,n} \to \mathbb{P}Q^*_{g,n}$ is holomorphic, as claimed. If (g,n) is non-exceptional, then Ψ restricts to an embedding on the fibers of $C_{g,n} \to \mathcal{T}_{g,n}$. Since the fibers are compact, Ψ is a biholomorphism onto its image. □

We now prove the main result of this subsection.

Proposition 4.3.3. *The map $H : \mathcal{C}_{g,n} \to \mathcal{C}_{g,m}$ defined in the last section is holomorphic.*

Proof: Consider the following diagram.

$$\begin{array}{ccccccc} \mathcal{C}_{g,n} & \xrightarrow{\Psi} & \mathbb{P}Q^*_{g,n} & \xrightarrow{F_*} & \mathbb{P}Q^*_{g,m} & \xleftarrow{\Phi} & \mathcal{C}_{g,m} \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & \mathcal{T}_{g,n} & \xrightarrow{F} & \mathcal{T}_{g,m} & & \end{array}$$

Here, Ψ and Φ denote the fiberwise bi-canonical maps, which are holomorphic by Proposition 4.3.2. The map F_* can be viewed in two ways.

1. F_* is the projectivization of the derivative of the holomorphic map F.

2. On the fiber over $Y \in \mathcal{T}_{g,n}$, F_* is the dual of the isometric embedding $dF^*_Y : Q\left(F(Y)\right) \hookrightarrow Q(Y)$.

The first interpretation shows that F_* is holomorphic. The second interpretation, combined with the results of Section 4.2, shows that $F_* \circ \Psi$ has the same image as Φ. Moreover, $H : \mathcal{C}_{g,n} \to \mathcal{C}_{g,m}$ is the unique map so that

$$F_* \circ \Psi = \Phi \circ H.$$

But since (g,m) is non-exceptional, Φ is a biholomorphism onto its image. Thus, H can be expressed as the composition of holomorphic maps

$$\mathcal{C}_{g,n} \overset{F_*\circ\Psi}{\to} \Phi\left(\mathcal{C}_{g,m}\right) \overset{\Phi^{-1}}{\to} \mathcal{C}_{g,m}.$$

□

Completing the proof of Theorem 1.1

As discussed at the end of Section 4.3, each map $h_Y : \widehat{Y} \to \widehat{X}$ sends Y to X. Thus, there is a unique map $G : \mathcal{T}_{g,n} \times S_{g,n} \to \mathcal{T}_{g,m} \times S_{g,m}$ fitting into the diagram

$$\begin{array}{ccc} \mathcal{T}_{g,n} \times S_{g,n} & \xrightarrow{G} & \mathcal{T}_{g,m} \times S_{g,m} \\ \downarrow{\scriptstyle \mathcal{F}_{g,n}} & & \downarrow{\scriptstyle \mathcal{F}_{g,m}} \\ \mathcal{C}_{g,n} & \xrightarrow{H} & \mathcal{C}_{g,m}, \end{array}$$

where the vertical maps are the canonical trivializations discussed in the last section. Since H is continuous, the maps $S_{g,n} \to S_{g,m}$ obtained by restricting G to fibers are all isotopic. Restricting the above commutative square to fibers, we conclude that there is a fixed $f : S_{g,n} \to S_{g,m}$ so that

$$\begin{array}{ccc} S_{g,n} & \xrightarrow{f} & S_{g,m} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{h_Y} & F(Y) \end{array} .$$

commutes up to isotopy for all $Y \in \mathcal{T}_{g,n}$. By construction, the vertical arrows are the markings defining Y and $F(Y)$ as points of Teichmüller space. If $g \geq 2$, we already know that $f : S_{g,n} \to S_{g,m}$ is one-to-one. Thus, up to pre-composition by a mapping class, $Y \mapsto F(Y)$ is the forgetful map filling in the last $n - m$ punctures. This completes the proof when $g \geq 2$.

To finish the proof in the case $g = 1$, it suffices to establish that $f : S_{1,n} \to S_{1,m}$ is one-to-one. We prove this by another dimension argument. The point is that, if the degree of f is greater than 1, then not every $X \in \mathcal{T}_{1,n}$ admits a non-constant holomorphic map to a $Y \in \mathcal{T}_{1,m}$.

In more detail: Let d denote the degree of the cover $S_1 \to S_1$ obtained by extending f over the punctures. Then f factors through a degree d (unbranched) cover $S_{1,dm} \to S_{1,m}$.

$$\begin{array}{ccc} S_{1,n} & \xrightarrow{\quad f \quad} & S_{1,m} \\ & \searrow \quad \nearrow & \\ & S_{1,dm} & \end{array}$$

The covering $S_{1,dm} \to S_{1,m}$ induces an isometric embedding of Teichmüller spaces $\mathcal{T}_{1,m} \hookrightarrow \mathcal{T}_{1,dm}$, while the injective map $S_{1,n} \to S_{1,dm}$ induces a forgetful map $\mathcal{T}_{1,n} \twoheadrightarrow \mathcal{T}_{1,dm}$. These fit into the diagram

$$\begin{array}{ccc} \mathcal{T}_{1,n} & \xrightarrow{\quad F \quad} & \mathcal{T}_{1,m} \\ & \searrow\!\!\searrow \quad \swarrow & \\ & \mathcal{T}_{1,dm} & \end{array}$$

Thus, $\mathcal{T}_{1,m} \hookrightarrow \mathcal{T}_{1,dm}$ is surjective, which implies $d = 1$. □

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