A MICHELL OSEEN-FLOW THEORY FOR THIN SHIPS

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ABSTRACT

A linear theory is developed for the steady free surface flow of a viscous fluid past a general system of submerged flow disturbances (a point mass source and three orthogonal point forcelets). The viscous character of the flow is approximated by using the Oseen linearization of the Navier-Stokes equations.

Solution of the fundamental problem (point flow disturbances) using double Fourier transforms furnishes formal representations of all the interesting flow quantities: the wave height, the three components of the perturbation velocity, and the dynamic pressure. Asymptotic expansions are presented for the 'free' or propagating parts of the flow quantities as they would appear far downstream.

Centerplane distributions of the flow disturbance singularities are used to model the flow about a symmetric thin ship. From the application of the momentum theorem, general formulae are derived for the total fluid drag on a ship in a viscous flow. These results are then specialized for use with the Oseen equations. The wave resistance formulae are of particular interest because they contain the strengths of the three forcelet distributions as well as the mass source distribution.

A numerical example of a wave resistance calculation is presented in which the four distribution functions are prescribed. The results are compared to known experimental curves. These indicate that significant features in the character of ship wave resistance can be qualitatively described by including the strengths of local viscous forces acting on the body.

THE CONTENTS

NOTATION*

b, b_1	$= [y^2+(z-\zeta)^2]^{\frac{1}{2}},\ [y^2+(z+\zeta)^2]^{\frac{1}{2}};\ \zeta = -h$ in Chapters (II - IV)
B	= beam of ship hull
$\mathfrak{c}_m(x, z)$	$= M_o(x, z)\big/\frac{1}{2} U$, mass source distribution (nondimensional)
$(\mathfrak{c}_x, \mathfrak{c}_y^D, \mathfrak{c}_z)$	$= (X_o, L^{-1} Y_o^D, Z_o)\big/\frac{1}{2} \rho U^2$, forcelet distributions (nondimensional)
C_m	$= m\big/\frac{1}{2} U\ell^2$
(C_x, C_y, C_z)	$= (X, Y, Z)\big/\frac{1}{2} \rho U^2 \ell^2$
$D_1(k_p)$	$= \left[\frac{\partial \Delta}{\partial k}\right]_{k_p}$
$\vec{F}$	$= (X, Y, Z)$
F_ℓ, F_L	$= U/\sqrt{g\ell},\ U/\sqrt{gL}$ Froude number
g	= acceleration of gravity
h	= depth of submergence of point singularities
$h(x, z)$	= equation of ship hull shape
i^2	$= -1$
k	= transform variable
k_p	= approximate simple root of $\Delta(k, \theta) = 0$
ℓ	= reference length in Chapters (II-IV); $\ell = L/2$ in Chapter VIII.
L	= length of waterline of ship
m	= strength of mass source[length^3 / time]

*This list includes only symbols of general use throughout the thesis, except where noted. The remaining nomenclature is identified as it is introduced.

$M_o(x,z)$	= mass source distribution (length/time)
p	$= p_o + p_1$, hydrodynamic pressure
p_1	$= p_1^{(o)} + p_s$
P	= total pressure $= p - \rho g z$
$\vec{q}$	$= \vec{q}_o + \vec{q}_1$, perturbation velocity
$\vec{q}_1$	$= \vec{q}_1^{(o)} + \vec{q}_s$
r	$= (x^2+y^2)^{\frac{1}{2}}$
R, R_1	$= [(x-\xi)^2 + y^2 + (z-\zeta)^2]^{\frac{1}{2}}, \; [(x-\xi)^2 + y^2 + (z+\zeta)^2]^{\frac{1}{2}}$
R_ℓ, R_L	$= U\ell/\nu$, UL/ν Reynolds number
R_v	= viscous resistance (see Eq. (5.21))
R_{w_t}	= wave resistance (see Eq. (5.20))
S_o	= centerplane area of ship hull
S_2	= downstream control surface (cf. Chapter V)
T	= surface tension constant in Chapters (II-IV); draft of ship in Chapters (V-VIII)
(u,v,w)	= perturbation velocity components
U	= free stream velocity, forward velocity of ship
$\vec{V}$	= total velocity $= U\hat{e}_x + \vec{q}$
(x,y,z)	= Cartesian coordinates, Figs. (2.1) and (6.1)
x_D	= distance to downstream control surface
(X,Y,Z)	= point forcelet strengths (force)
$(X_o, L^{-1}Y_o^D, Z_o)$	= forcelet distributions for symmetrical disturbance [force/length^2]
(α, β)	= transform variables

$\Delta(k,\theta)$	= function appearing in the denominator of the $(\vec{q}_s, p_s)$ solution
ζ	= wave elevation, measured upward from $z=0$
θ	= transform variable
κ	= $U/2\nu$
κ_o	= g/U^2
μ	= dynamic viscosity
ν	= kinematic viscosity
ρ	= fluid density
σ_ℓ, σ_L	= $g\ell/U^2$, gL/U^2 Froude number parameter
ω	= $\tan^{-1}(y/x)$, $x = r\cos\omega$, $y = r\sin\omega$

I. INTRODUCTION

1. Background

Linear theories for the calculation of ship wave resistance have thus far produced results that cannot satisfactorily predict the measured wavemaking properties of realistic ship forms without considerable empirical modification. Most of the existing work on ship flows has been based on the assumption that the wavemaking features of the flow can be adequately represented by potential theory. (e.g., see the general references: Havelock (1963), Japanese Society of Naval Architects (1957), Kostyukov (1968), Lunde (1951), Michell (1898), Newman (1970), and Wehausen and Laitone (1960))*. These potential theories predict qualitatively the presence of humps and hollows in the wave resistance curve as a function of the Froude number, but they generally fail in several important respects. The theoretically calculated wave resistance curves tend to have exaggerated humps and hollows , while for the measured curves, these features are much less pronounced. Also, there is often a large quantitative discrepancy between theoretical results and measured curves. A common feature indicated by the comparison between theory and experiment is a shift in the Froude number at which a local peak or local minimum of the wave drag occurs. This feature is usually more noticeable at low Froude numbers. (see, for example Wigley and Lunde (1948), Lewison (1963), Fig. 5; and Lackenby (1965), Fig. 5). A subtler

*a name followed by a date in parentheses indicates an entry in the list of references.

effect is illustrated in Sharma's (1963, 1969) comparisons between measured and theoretical wave spectra. Especially in the 1969 reference, it is evident that even for very thin ship forms, there are contributions to each of the two components of the free wave spectra that are not well predicted by the thin ship potential theory. Although the resultant amplitude function in Sharma's (1969) work appears to be fairly well predicted, the shifts in phase and magnitude of the component free-wave spectra are disturbing. These effects are apparently not too serious in terms of the net wave resistance computed for thin strut-like bodies such as those tested by Weinblum, Kendrick, and Todd (1952), and by Sharma (1969). But the qualitative deficiency in the thin ship potential theory becomes very important for more realistic ship hull shapes. It is probable that potential flow results can be improved by using higher order ship theories. (see, for example Wehausen (1969) and his cited references). However, it seems that these may never give a complete physical picture.

2. Effects of Viscosity

A likely source of error in unmodified potential flow calculations is the neglect of the viscous effects. Experiments by a number of investigators have indicated that there is a measurable interaction between viscous and wavemaking components of ship drag. See, for example, references by J. Wu and Landweber (1963), Lackenby (1965), Shearer and Cross (1965), Townsin (1967), and Tzou and Landweber (1968). There is a strong motiviation to understand these effects and to study how they can be included in a theoretical analysis of the problem.

There are two main aspects of the viscous character of the flow: (1) the boundary layer at the ship hull and (2) the viscous wake. Vorticity generated at the surface of a body by viscous shear stresses diffuses outward and is convected downstream. The continuous production of vorticity along the length of the solid surface causes this rotational flow regime to grow within a thin layer whose thickness increases along the body. At the rear, the boundary layer flow from all around the body fuses into the wake regime. The wake, though lacking a source of new vorticity, is nevertheless a region of retarded and rotational flow that stretches downstream behind the body. It grows in cross-sectional area as the convective velocity defect gradually decreases in magnitude. When the free surface is included in the problem, the gross flow picture is basically unchanged except for the generation of waves within the Kelvin wedge downstream of the disturbance.

Unfortunately, the detailed flow picture for the ship is complicated by interactions between various features of the actual flow. The ship-generated wave system causes an undulatory pressure variation along the body, which in turn has an influence on the growth of the boundary layer. It also has an undulatory effect on the magnitude of the local shear stress within the boundary layer. Consequently the overall skin friction drag could be expected to display some Froude number dependence. Wave generation, and hence also wave resistance, depends on the effective shape of the body. Variations in the boundary layer displacement thickness change the virtual hull shape slightly. Therefore the wave resistance could be expected to contain some Reynolds number effect.

There are other troublesome real flow effects. Flow separation near the stern of blunt ship bodies is perhaps the most difficult problem of this type. Even disregarding separation, the extra thick boundary layer region far aft on a ship and the presence of the rotational wake can alter and diminish the effectiveness of wave production near the stern. Streamlines of the flow are prevented from closing in at the stern as they would in an idealized flow. The variation of the velocity defect across the wake can cause a complex focusing and scattering of wave systems that further confuse the flow picture (Lau (1968), Savitsky (1970)). Complicated secondary flows are also possible, especially near the stern (Chow (1967), Gadd (1970)). Another important real flow phenomenon is the breaking of the bow wave created by blunt ships. Baba (1969) has studied this effect and has measured the accompanying resistance by means of wake profiles.

There is no question that the real ship flow is exceedingly complicated. A detailed analysis is probably impossible. The issue then becomes how to model the situation approximately in order to obtain both understanding and useful results. A number of attempts have been made to study various aspects of the problem. Some of these are reviewed here.

Concerning the boundary layer influence, the effect of adding a prescribed displacement thickness to a hull shape to model the boundary layer has been studied by Havelock (1948), Laurentieff (1952), Wigley (1963), and T. Wu (1963). From the curves of wave resistance computed for the effective hull forms, the influence of displacement thickness along most of the length of the ship is known to be small. This is because the slope of the virtual hull form is changed so little by the boundary layer. However, extending the displacement thickness beyond the stern of the hull tends to smooth out the pronounced humps and hollows of the calculated wave resistance curves (Havelock (1948)). It also tends to reduce the magnitude of wave resistance somewhat, although this is probably due mostly to the virtual lengthening of the hull.

Some research has been carried out to investigate the modification of local skin friction by the pressure variations due to the surface wave profile. For example, Steele and Pearce (1968) and Shearer and Steele (1970) have presented experimental results indicating that there is a definite undulatory behavior of the local shear stress in the flow along ship hulls. T. Wu's (1963) approximate two-dimensional theory includes some results that show a small Froude number effect on skin friction.

The influence of a wake region on ship flows has also been investigated. Milgram (1969) considered the effect on Michell's integral of an idealized wake geometry. His wake consists of a narrow constant width semi-infinite extension of the body itself. The wake streamlines are joined smoothly onto the hull shape, and the fluid within the wake is supposed to move with the velocity of the ship. The results for wave drag are similar to Havelock's (1948).

Tatinclaux (1970) and Brard (1970) have presented independent investigations in which the wake behind a ship is simulated by a prescribed volume distribution of vorticity. Tatinclaux studied the problem of an infinitely deep rotational wake behind a two-dimensional ogive. He discussed the effect of the rotational flow region on the wave profiles and resulting wave drag. Brard's work also is based on the assumption that the volume distribution of vorticity inside the combined wake and boundary layer region is known. He used source distributions over the surface of a body shape to model the ship hull, and has calculated general formulae for various components of total wave resistance, including terms that depend on wake vorticity.

Beck (1970) has proposed a theory for modelling the wake region behind a ship, using a constant width U-shaped vortex sheet stretching to infinity. The vortex sheet is attached to the hull at some position along the length which can be adjusted to give a total wave resistance which agrees with experimental results. The value of the strength of the vortex sheet is related by a simple formula to the measured viscous drag of the hull at various Froude numbers.

3. Present Study

The present work is an approach from a somewhat different point of view. It is an investigation of a linearized approximation to the complete viscous free-surface flow problem. The theory developed here contains the basic features of viscous flow in the sense of the Oseen linearization. Thus, for large Reynolds numbers, the solution becomes asymptotically valid in the far field. One interesting feature of this approach is that it includes the possibility of relating the properties of the viscous boundary layer and rotational wake to the shape of the ship hull.

A brief consideration is given here to some general characteristics of a three-dimensional turbulent wake behind a body, where from the present point of view, the forces on the body appear merely as concentrated flow disturbances. The Reynolds number UL/ν, based on some body length L, is assumed to be large. When the flow near the disturbance is nearly axisymmetrical, Fig. (I.1) gives a schematic representation of the turbulent wake flow (near wake) and its eventual decay to a laminar wake (or far wake) for the case when the drag force D is large compared with Y, a small yaw force. The decay of the turbulent wake structure occurs as the scale of the turbulent eddies gradually decreases (Townsend (1956)). Using similarity arguments, the half width of the wake $b(x)$ and the mean perturbation velocity components u, v on the centerline can be estimated for the fully turbulent wake region. These are indicated in the sketch. What is important here is that for distances far enough downstream of the disturbance, the flow is well represented by the Oseen equations, regardless of the details closer to the body. It is this fact that justifies the use of the Oseen linearization in the present case, because our interest is also in the far field flow regime.

The idea of studying concentrated or singular flow disturbances is especially useful in linear analysis, because various fundamental

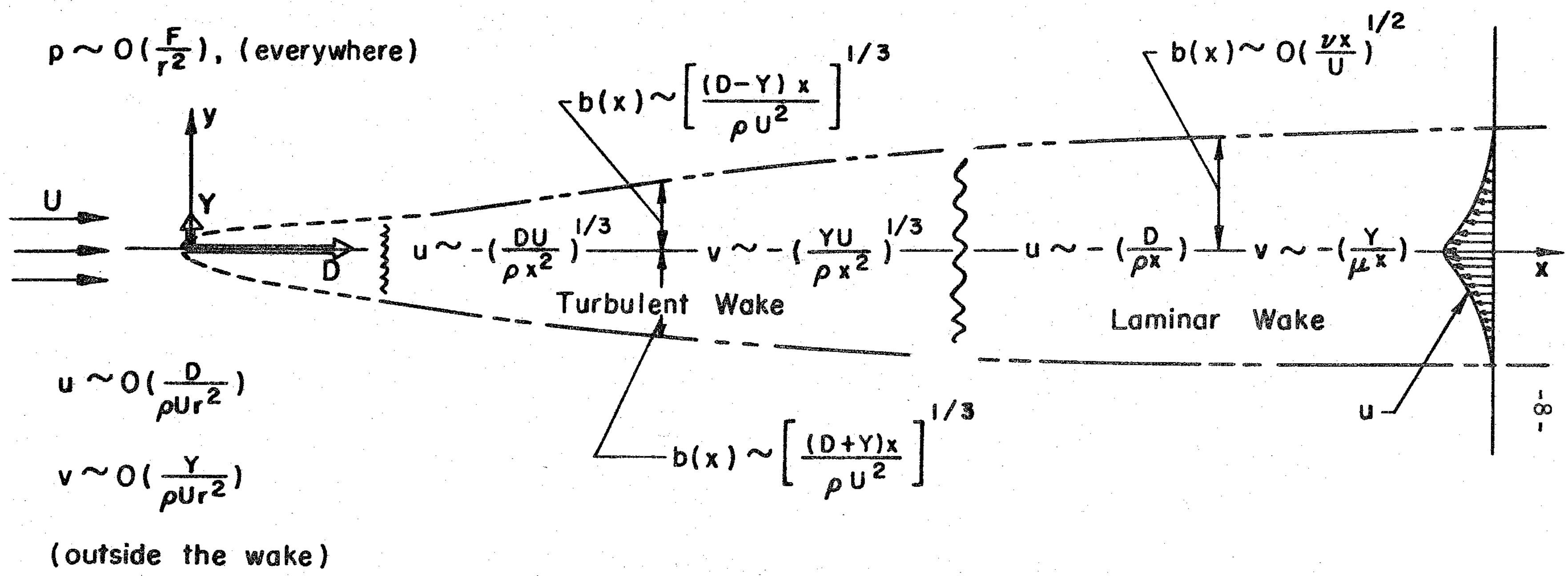

Fig. (1.1) Decay of a fully turbulent nearly-axisymmetric wake to a laminar far wake. Forces on the body F are the drag D and a small yaw force Y. The velocities u and v are perturbation quantities.

solutions can be added together to form a composite flow. In the first part of this thesis (Chapters II - IV), the fundamental solutions are obtained for the free surface Oseen-flow past a set of submerged disturbances represented by a singular mass source m and a singular force $\vec{F} = (X, Y, Z)$. Some preliminary explanation is desirable regarding these two systems of flow disturbances. The ultimate aim of the theory is to use the mass source m and forcelet $\vec{F}$ as distributions to simulate the flow around a ship form. The mass source distribution acts to displace streamlines of the flow around the gross shape of the hull outside the boundary layer. Similarly, one may regard the forcelets (X, Y, Z) as 'sources' of fluid stresses giving rise to the vorticity contained within the thin boundary layer and wake region. Of course the fundamental solutions are determined here without any assumptions about thinness or slenderness of the body to be modeled. It is interesting to note that the wake flow viewed in this way contains automatically a continuous mixture of both the 'Betz wake sources' (Tulin (1951)) and rotational flow properties from the forcelets.

Sretenskii (1957) presented a result for thin ship wave resistance in Oseen-flow, using only a mass source to simulate the flow about the hull. It will be seen here that including singularities to model the fluid shear stresses introduces new and interesting features to the classical thin ship results.

Formal representations of the flow quantities are derived in Chapter III using integral transforms. Then from the integral expressions, asymptotic formulae are obtained for the 'free' or propagating wave flow quantities as they would appear far away from the disturbance. These are discussed in Chapter IV.

In the second part of this work, the fundamental solutions are used to model the thin ship flow problem. Using the results of the momentum theorem discussed in Chapter V, expressions are derived in Chapter VI for various components of ship resistance. The boundary conditions and resulting integral equations for the unknown flow disturbance functions are outlined and solved approximately in Chapter VII. Finally, in Chapter VIII a sample numerical calculation of the wave resistance is presented.

II. GENERAL FORMULATION

This chapter describes a mathematical formulation of the free surface viscous flow problem under consideration. The basic approach is to linearize both the governing equations of fluid motion and the free surface boundary conditions in order to reduce the real problem to one of tractable form.

The fundamental problem treated here is an extension of some previous investigations on waves in a viscous fluid. Discussion and a listing of the early researches on this subject are given by Wehausen and Laitone (1960), §25. Some of the more recent work should also be mentioned here.

The effect of viscosity on two-dimensional free surface waves generated by disturbances was studied extensively by Wu and Messick (1958). Detailed solutions for both the far field and near field were obtained by a Fourier transform technique. The disturbances considered in that work were two concentrated orthogonal stresses applied on the undisturbed free surface.

Cumberbatch (1965) extended the theory to the case of Kelvin ship waves generated by a concentrated normal stress applied on the free surface. He provided asymptotic results for the free surface elevation far downstream of the disturbance when the Reynolds number is large.

The case of a submerged point drag force was considered by Lurye (1968) who solved the problem formally by a double Fourier transform approach, but produced no solutions in terms of the physical variables of the problem.

Allen (1968) also dealt with the problem of Kelvin ship waves in a viscous fluid, using Cumberbatch's work as a starting point. Instead of concentrated stress loads, however, he considered patches of applied normal stress and discussed the relative importance of the effects of viscosity and wave interference on the damping of the resulting wave systems.

In the present work, the problem of the free surface viscous flow caused by submerged singularities is attacked in a manner very similar to Lurye's. In addition to a drag forcelet, singularities representing a lift forcelet, a yaw forcelet, and a point mass source are included.

1. Statement of the Fundamental Problem

Consider a point disturbance moving steadily beneath the free surface of an otherwise undisturbed infinitely deep half space of a viscous fluid. The disturbance moves with a constant horizontal velocity U in the $-x$ direction at a distance h below the free surface. The fluid has a mass density ρ, dynamic viscosity μ, and is assumed to be homogeous and incompressible. The acceleration of the external gravity field is g, acting downward. This problem is equivalent (by a Galilean transformation) to that of a uniform flow past the fixed point disturbance.

We prescribe the disturbance in terms of a submerged mass source of strength m and a submerged point force $\vec{F} = (X, Y, Z)$ represented by delta function singularities in the equations of continuity and momentum, respectively. A Cartesian coordinate system is fixed at the free surface of the fluid, directly above the system of point disturbances, with z pointing up. Figure (2.1) is a sketch of the flow geometry of the fundamental problem.

Perturbation flow quantities (velocity and pressure) are assumed to vanish at infinity. In addition, the perturbation velocities (u, v, w), in comparison with the free stream velocity U, are assumed to be small so that only the linear terms of these quantities are retained. The same assumption applies to the surface wave elevation ζ in terms of its maximum slope. Throughout this chapter and the next, the surface tension effect is included for completeness in the general formulation. This effect will be dropped in the work that follows thereafter.

2. Basic Equations of Motion

We denote the total flow velocity by $\vec{V} = U\hat{e}_x + \vec{q} = (U+u, v, w)$. The perturbation velocity $\vec{q}$ satisfies the continuity equation

$$\operatorname{div} \vec{q} = m\delta(\vec{x}+h\hat{e}_z) \quad , \tag{2.1}$$

where $\delta(\vec{x}+h\hat{e}_z) = \delta(x)\delta(y)\delta(z+h)$, $\delta(x)$ being the Dirac delta function, and m the strength of the mass source. Here, $\hat{e}_x, \hat{e}_y$, and $\hat{e}_z$ denote the unit vector in the x, y, z direction, respectively. The Navier-Stokes equation for steady flow with a submerged point force of strength $\vec{F}$ can be written as

$$(\vec{V}\cdot\nabla)\vec{V} = -\frac{1}{\rho}\nabla(P+\rho gz) + \nu\nabla^2\vec{V} + \frac{\vec{F}}{\rho}\delta(\vec{x}+h\hat{e}_z) \quad , \tag{2.2}$$

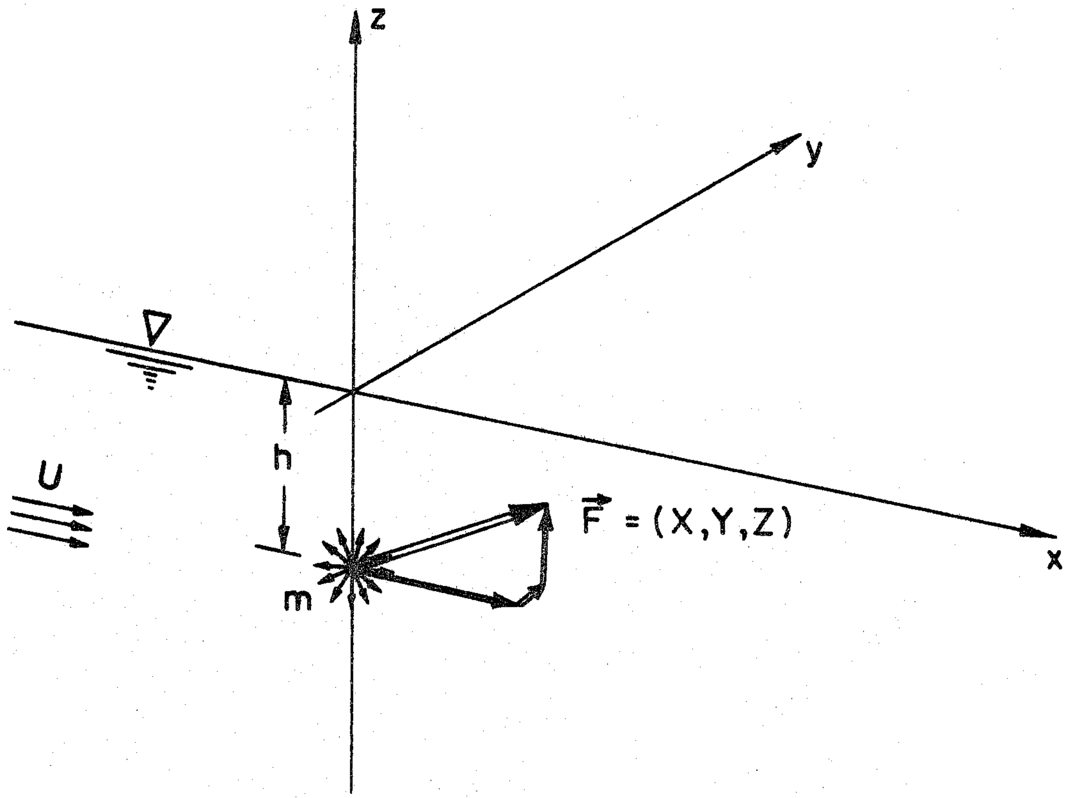

Fig. (2.1) Schematic representation of the flow disturbances for the fundamental problem. The mass source strength m has dimensions $(\text{length})^3/(\text{time})$; the force strength components X, Y, Z have dimensions of (force).

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

By assuming that the perturbation velocity $\vec{q}$ is small compared to U, (2.2) is linearized to give the Oseen equation

$$U \frac{\partial \vec{q}}{\partial x} = - \frac{1}{\rho} \nabla(P+\rho g z) + \nu \nabla^2 \vec{q} + \frac{\vec{F}}{\rho} \delta(\vec{x}+h\hat{e}_z) , \qquad (2.3)$$

where $P = p+p_h$ is the total pressure; $p_h = -\rho g z$ being the hydrostatic pressure and p the hydrodynamic pressure; $\nu = \mu/\rho$ is the kinematic viscosity.

3. Boundary Conditions

There are three boundary conditions involving the flow variables $(\vec{q}, p; \zeta)$, where ζ is the elevation of the free surface, measured upward from $z = 0$. The flow velocity is uniform at infinity and the kinematic and dynamic boundary conditions are to be satisfied on the free surface.

The kinematic boundary condition states that fluid particles on the free surface remain on it. Physically, this condition requires that the flow velocity of a free surface particle be tangential to the surface elevation ζ. Mathematically, this condition for the case of steady flow is

$$\left[(U+u) \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}\right] (z-\zeta) = 0 \quad \text{on} \quad z = \zeta(x,y) \quad . \qquad (2.4)$$

Linearizing this equation, and expanding various quantities about the mean free surface $z = 0$ for small ζ, the approximate kinematic condition becomes

$$w = U \frac{\partial \zeta}{\partial x} \qquad (z = 0) \tag{2.5}$$

Now, the dynamic free surface condition specifies that the tangential fluid stress be continuous across the free surface interface, while the normal stress undergoes a jump equal to the surface tension T times the total (or Gaussian) curvature of the surface. Assuming that the pressure above the free surface is a constant P_a, and neglecting the shear stresses in the medium above the surface, the dynamic free surface condition reads

$$\vec{n} \times [(PI-\tau) - P_a I]\vec{n} = 0 \qquad (z=\zeta) \quad , \tag{2.6}$$

$$[(P\delta_{ik}-\tau_{ik}) - P_a \delta_{ik}] n_i n_k = TK_s \qquad (z=\zeta) \quad , \tag{2.7}$$

where δ_{ik} is the Kronecker delta; I is the identity matrix; τ_{ik} is the component of the viscous stress tensor τ

$$\tau_{ik} = \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \quad , \tag{2.8}$$

n_i is the i-th component of the unit normal to the free surface; T is the coefficient of surface tension; and K_s is the sum of the principal curvatures

$$K_s = \operatorname{div} \vec{n} \quad , \tag{2.9}$$

which, after linearization becomes

$$K_s(x,y) \simeq -(\zeta_{xx}+\zeta_{yy}) \quad . \tag{2.10}$$

The unit normal to the free surface $\vec{n}$ is given by the normalized gradient of the free surface function $F(x,y,z) = z-\zeta(x,y) = 0$.

$$\vec{n} = \frac{\nabla F}{|\nabla F|} = \frac{-\zeta_x \hat{e}_x - \zeta_y \hat{e}_y + \hat{e}_z}{\sqrt{\zeta_x^2 + \zeta_y^2 + 1}} \tag{2.11}$$

so the linearized form is

$$\vec{n} \simeq -\zeta_x \hat{e}_x - \zeta_y \hat{e}_y + \hat{e}_z \quad . \tag{2.12}$$

where the surface elevation ζ and its derivatives are all assumed to be small. The constant pressure P_a is set equal to zero (with no loss of generality). Further, Eqs. (2.6) and (2.7) may be linearized by neglecting the cross products of the small quantities $\zeta_x, \zeta_y, \frac{\partial u_i}{\partial x_k}$. Then the linearized dynamic free surface conditions are

$$\tau_{xz} = 0 = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \qquad (z=0) \quad , \tag{2.13}$$

$$\tau_{yz} = 0 = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \qquad (z=0) \quad , \tag{2.14}$$

$$-P + \tau_{zz} + TK_g = 0 = -p + \rho g \zeta + 2\mu \frac{\partial w}{\partial z} - T\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) \quad (z=0) \quad , \tag{2.15}$$

where in (2.15) we have used the fact that on $z = 0$, the total pressure $P = p(x, y, 0) - \rho g \zeta(x, y)$.

It is convenient to split the flow quantities $(\vec{q}, p)$ formally into two parts

$$\vec{q} = \vec{q}_0 + \vec{q}_1 \quad , \qquad p = p_0 + p_1 \quad , \tag{2.16}$$

where $(\vec{q}_0, p_0)$ represents the unbounded flow due to the mass source m and forcelet $\vec{F}$ in a uniform stream velocity $U\hat{e}_x$ with the singularities located at $z = -h$. The $(\vec{q}_1, p_1)$ is the complementary part, defined only in the flow region. The $(\vec{q}_1, p_1)$ problem is coupled to the first by the free surface conditions. Substituting (2.16) into the continuity equation (2.1) and the Oseen equations (2.3), the unbounded flow problem $(\vec{q}_o, p_o)$ satisfies the equations of motion

$$\mathrm{div}\,\vec{q}_o = m\delta(\vec{x}+h\hat{e}_z) \quad , \tag{2.17}$$

$$\nabla^2\vec{q}_o - 2\kappa \frac{\partial \vec{q}_o}{\partial x} = \frac{1}{\mu}\nabla p_o - \frac{\vec{F}}{\mu}\delta(\vec{x}+h\hat{e}_z) \quad , \tag{2.18a}$$

where

$$\kappa = \frac{U}{2\nu} \quad . \tag{2.18b}$$

The $(\vec{q}_1, p_1)$ problem is governed by the equations

$$\mathrm{div}\,\vec{q}_1 = 0 \tag{2.19}$$

$$\nabla^2\vec{q}_1 - 2\kappa \frac{\partial \vec{q}_1}{\partial x} = \frac{1}{\mu}\nabla p_1 \, . \tag{2.20}$$

For the present discussion, it is sufficient to state that the solution to the unbounded Oseen flow problem $(\vec{q}_o, p_o)$ is known (e.g., Lagerstrom (1964)). Some details of this solution are discussed in Chapter III.

To deal with the $(\vec{q}_1, p_1)$ problem, we first take the divergence of Eq. (2.20) and use $\mathrm{div}\,\vec{q}_1 = 0$ to obtain

$$\mathrm{div}\,\mathrm{grad}\,p_1 = \nabla^2 p_1 = 0 \quad . \tag{2.21}$$

We apply the curl operator twice on Eq. (2.20) and use the identity $\mathrm{curl}\,\mathrm{grad}\,p_1 = 0$ to obtain

$$\left(\nabla^2 - 2\kappa \frac{\partial}{\partial x}\right) \mathrm{curl}\,\mathrm{curl}\,q_1 = 0 \quad . \tag{2.22}$$

Using the vector identity $\mathrm{curl}\,\mathrm{curl}\,\vec{q}_1 = \mathrm{grad}\,\mathrm{div}\,\vec{q}_1 - \nabla^2\vec{q}_1$ and $\mathrm{div}\,\vec{q}_1 = 0$, we may rewrite (2.22) as

$$\left(\nabla^2 - 2\kappa \frac{\partial}{\partial x}\right) \nabla^2 \vec{q}_1 = 0 \quad . \tag{2.23}$$

Thus the equations of motion for $(\vec{q}_1, p_1)$ reduce to the convenient form

$$\nabla^2 p_1 = 0 \quad , \tag{2.24}$$

$$\nabla^2 L_1 \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = 0 \quad , \tag{2.25}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \quad , \tag{2.26}$$

where $L_1 = \nabla^2 - 2\kappa \frac{\partial}{\partial x}$.

The connection between the component $(\vec{q}_1, p_1)$ and the unbounded flow $(\vec{q}_o, p_o)$ arises entirely through the free surface conditions of Eqs. (2.13) - (2.15). In their split form, these boundary conditions are

$$\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x} = -\left(\frac{\partial u_o}{\partial z} + \frac{\partial w_o}{\partial x}\right) \qquad (z=0) \quad , \tag{2.27}$$

$$\frac{\partial v_1}{\partial z} + \frac{\partial w_1}{\partial y} = -\left(\frac{\partial v_o}{\partial z} + \frac{\partial w_o}{\partial y}\right) \qquad (z=0) \quad , \tag{2.28}$$

$$-\frac{p_1}{\rho} + g\zeta + 2\nu \frac{\partial w_1}{\partial z} - T(\zeta_{xx}+\zeta_{yy}) = \frac{p_o}{\rho} - 2\nu \frac{\partial w_o}{\partial z} \qquad (z=0) \ . \tag{2.29}$$

$$w_o + w_1 = U \frac{\partial \zeta}{\partial x} \quad , \qquad (z=0) \quad , \tag{2.30}$$

where $T_1 = T/\rho$. These equations together with (2.24) - (2.26) comprise the mathematical statement for the equations governing the component $(\vec{q}_1, p_1)$. Also, the perturbation velocities and hydrodynamic pressure are required to vanish as $x^2 + y^2 + z^2 \rightarrow \infty$ within the flow region. In contrast to the free surface potential flow problem, nothing further needs to be said about the radiation condition. Inclusion of the viscous effect guarantees that propagating free surface disturbances appear only downstream of the submerged singularities, as will be seen later.

4. Introduction of Integral Transforms

An important formal simplification of the three-dimensional problem is achieved by using the double Fourier transform on the quantities, u_1, v_1, w_1, p_1 and ζ. The transform and its inversion used in this work are defined by

$$\tilde{f}(\alpha, \beta, z) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{-i(\alpha x+\beta y)} f(x, y, z) dx\, dy \quad , \tag{2.31}$$

$$f(x, y, z) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{i(\alpha x+\beta y)} \tilde{f}(\alpha, \beta, z) d\alpha\, d\beta \quad , \tag{2.32}$$

assuming that $|f|$ is integrable with respect to x and y.

The problem may be recast in a nondimensional form using U as the reference velocity, some characteristic length ℓ as the length scale, and ρU^2 as the reference for pressure and stresses. We denote the dimensionless quantities by variables with underlines

$$\begin{aligned} &\vec{x} = \ell \underline{\vec{x}} \;, && p = \rho U^2 \underline{p} \;, \\ &\zeta = \ell \underline{\zeta} \;, && \ell\alpha = \underline{\alpha} \;,\quad \ell\beta = \underline{\beta} \;, \\ &\vec{q} = U\underline{\vec{q}} \;, && \ell k_o = \underline{k}_o \;,\quad \ell k_1 = \underline{k}_1 \;. \end{aligned} \tag{2.33}$$

We define the dimensionless parameters Reynolds number R_ℓ , Froude number F_ℓ, and Weber number W_ℓ by

$$R_\ell = U\ell/\nu = 2\kappa\ell \quad , \quad F_\ell = U/\sqrt{g\ell} \quad , \quad W_\ell = U^2\ell\rho/T = U^2\ell/T_1 \quad . \qquad (2.34)$$

It is more convenient to use the inverse Froude number parameter σ_ℓ defined as

$$\sigma_\ell = g\ell/U^2 = 1/F_\ell^2 \quad . \qquad (2.35)$$

Applying (2.31) to the equations of motion (2.24) - (2.26), and incorporating the dimensionless quantitites (2.33) - (2.35), we obtain

$$\tilde{\nabla}^2 \underline{p}_1 = 0 \qquad (\underline{z} < 0) \quad , \qquad (2.36)$$

$$\tilde{\nabla}^2 \tilde{L}_1 \begin{bmatrix} \tilde{\underline{u}}_1 \\ \tilde{\underline{v}}_1 \\ \tilde{\underline{w}}_1 \end{bmatrix} = 0 \qquad (\underline{z} < 0) \quad , \qquad (2.37)$$

$$i\underline{\alpha}\tilde{\underline{u}}_1 + i\underline{\beta}\tilde{\underline{v}}_1 + \frac{\partial \tilde{\underline{w}}_1}{\partial \underline{z}} = 0 \qquad (\underline{z} < 0) \quad , \qquad (2.38)$$

where

$$\nabla^2 = \frac{d^2}{d\underline{z}^2} - \underline{k}_o^2 \quad , \quad \tilde{L}_1 = \frac{d^2}{d\underline{z}^2} - \underline{k}_1^2 \quad ,$$

and

$$\underline{k}_o^2 = (\underline{\alpha}^2 + \underline{\beta}^2) \quad , \quad \underline{k}_1^2 = (\underline{\alpha}^2 + \underline{\beta}^2 + i R_\ell \underline{\alpha}) \quad .$$

The transformed nondimensional free surface boundary conditions are

$$\frac{d\tilde{\underline{u}}_1}{d\underline{z}} + i\underline{\alpha}\tilde{\underline{w}}_1 = \tilde{M}_o(\underline{\alpha}, \underline{\beta}) \qquad (\underline{z} = 0) \quad , \qquad (2.39)$$

$$\frac{d\tilde{\underline{v}}_1}{d\underline{z}} + i\underline{\beta}\,\tilde{\underline{w}}_1 = \tilde{N}_o(\underline{\alpha},\underline{\beta}) \qquad (\underline{z} = 0) \quad , \qquad (2.40)$$

$$-i\underline{\alpha}\,\tilde{\underline{p}}_1 + \left(\frac{2i\underline{\alpha}}{R_\ell}\right)\frac{d\tilde{\underline{w}}_1}{d\underline{z}} + \left(\sigma_\ell + \frac{\underline{k}_o^2}{W_\ell}\right)\tilde{\underline{w}}_1 = \tilde{P}_o(\underline{\alpha},\underline{\beta}) \quad (\underline{z} = 0) \quad , \qquad (2.41)$$

$$i\underline{\alpha}\,\tilde{\underline{u}}_1 + i\underline{\beta}\,\tilde{\underline{v}}_1 + \frac{d\tilde{\underline{w}}_1}{d\underline{z}} = 0 \qquad (\underline{z} = 0) \quad , \qquad (2.42)$$

where the transformed version of (2.30),

$$\tilde{\underline{\zeta}}(\alpha,\beta) = \frac{1}{i\alpha}\left(\tilde{\underline{w}}_o + \tilde{\underline{w}}_1\right)_{\underline{z}=o} \quad , \qquad (2.43)$$

has been used to eliminate $\tilde{\underline{\zeta}}$ from the boundary conditions. The nondimensional functions $\tilde{M}_o$, $\tilde{N}_o$, and $\tilde{P}_o$ are assumed to be known because they depend entirely upon the transformed $(\vec{q}_o, p_o)$ system.

$$\tilde{M}_o(\underline{\alpha},\underline{\beta}) = -\left[\frac{d\tilde{\underline{u}}_o}{d\underline{z}} + i\underline{\alpha}\,\tilde{\underline{w}}_o\right]_{\underline{z}=o} \quad , \qquad (2.44)$$

$$\tilde{N}_o(\underline{\alpha},\underline{\beta}) = -\left[\frac{d\tilde{\underline{v}}_o}{d\underline{z}} + i\underline{\beta}\,\tilde{\underline{w}}_o\right]_{\underline{z}=o} \quad , \qquad (2.45)$$

$$\tilde{P}_o(\underline{\alpha},\underline{\beta}) = \left[i\underline{\alpha}\,\tilde{\underline{p}}_o - \frac{2i\underline{\alpha}}{R_\ell}\frac{d\tilde{\underline{w}}_o}{d\underline{z}} - \left(\sigma_\ell - \frac{\underline{k}_o^2}{W_\ell}\right)\tilde{\underline{w}}_o\right]_{\underline{z}=o} \quad . \qquad (2.46)$$

This completes the formulation of the fundamental problem, with the mathematical statement of the nondimensional problem embodied in Eqs. (2.36) - (2.42). There are four unknown functions $\tilde{\underline{u}}_1$, $\tilde{\underline{v}}_1$, $\tilde{\underline{w}}_1$, and $\tilde{\underline{p}}_1$ with four free surface boundary conditions. The fifth flow quantity of interest, $\tilde{\underline{\zeta}}$, can be obtained from (2.43).

III. FUNDAMENTAL SOLUTION

The free surface flow problem posed in Chapter II is solved formally in this chapter. Surface tension is included only up to a certain point in order to illustrate how it affects the analytical behavior of the Fourier integral representation. It is then dropped from further consideration after Section 2 of this chapter.

1. Unbounded Flow Solution $(\vec{q}_o, p_o)$

The solution of the unbounded singular flow problem described by Eqs. (2.17) and (2.18) is known from results found in existing literature (e.g., Lagerstrom (1964)). It can be written as

$$\vec{q}_o = \frac{\vec{F}}{4\pi\mu}\left(\frac{e^{\kappa(x-R)}}{R}\right) + \nabla\left\{\frac{-m}{4\pi R} - \frac{1}{4\pi\rho U}\left(1 - e^{\kappa(x-R)}\right)(\vec{F}\cdot\nabla)\ell n(R-x)\right\}, \quad (3.1)$$

$$p_o = \frac{1}{4\pi}\left(\rho m U \frac{\partial}{\partial x} - \vec{F}\cdot\nabla\right)\frac{1}{R} + \mu m \delta(\vec{x} + h\hat{e}_z) \quad , \quad (3.2)$$

where

$$R^2 = x^2 + y^2 + (z+h)^2 \quad , \quad \kappa = U/2\nu \quad .$$

Here the strength of the forcelet, $\vec{F}$, represents a point force acting on the fluid with components (X, Y, Z) in the positive (x, y, z) directions respectively. For instance, a singular drag force would be written as drag = $-X$.

There is an interesting property regarding further splitting of the unbounded flow component $(\vec{q}_o, p_o)$, a property which is common to the component $(\vec{q}_1, p_1)$ and is fundamental to the Oseen equation.

This property can be exploited to good effect when the ship flow is analyzed, so it is worth reviewing here briefly.

Consider the following steady-flow Oseen equations

$$U \frac{\partial \vec{v}}{\partial x} = - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} \quad ,$$
$$\operatorname{div} \vec{v} = 0 \quad , \tag{3.3}$$

where U is the free stream velocity, and the total fluid velocity is $\vec{V} = U\hat{e}_x + \vec{v}$. It has been shown (e.g., Lamb (1945), Lagerstrom (1964)) that any solution $(\vec{v}, p)$ of these equations may be decomposed uniquely into two parts: a longitudinal component $(\vec{v}_L, p)$ and a solenoidal component $(\vec{v}_T, 0)$.

A solution of (3.3) is called a longitudinal component if the velocity is irrotational $\nabla \times \vec{v}_L = 0$, and it satisfies the equations

$$U \frac{\partial \vec{v}_L}{\partial x} = - \frac{1}{\rho} \nabla p \quad ,$$
$$\operatorname{div} \vec{v}_L = 0 \quad , \tag{3.4}$$
$$\operatorname{curl} \vec{v}_L = 0 \quad .$$

The viscous term in (3.4) has dropped out, since $\nabla^2 \vec{v}_L = \operatorname{grad} \operatorname{div} \vec{v}_L - \operatorname{curl} \operatorname{curl} \vec{v}_L = 0$. Hence the longitudinal component carries all the pressure and no vorticity, and can be represented by a potential φ

$$\vec{v}_L = \operatorname{grad} \varphi \quad .$$

Simultaneously, the solenoidal component, defined by $\operatorname{div} \vec{v}_T = 0$, carries all the vorticity but no pressure. It satisfies the equations

$$U \frac{\partial \vec{v}_T}{\partial x} = \nu \nabla^2 \vec{v}_T = -\nu \text{ curl curl } \vec{v}_T \quad , \qquad \text{div } \vec{v}_T = 0 \quad . \tag{3.5}$$

The notation $\vec{v}_T$ from Lagerstrom (1964) is retained, but the name solenoidal replaces 'transversal' to avoid confusion with the term 'transverse free surface wave system' usually associated with ship waves. The sum $\vec{v} = \vec{v}_L + \vec{v}_T$ satisfies the original system (3.3). Uniqueness of this splitting can be proved when the conditions at infinity are specified (e.g., analogous to the present problem, $\vec{v} = 0$ and $p \to p_\infty$ at infinity). For proof and extensive discussion, see Lagerstrom, et al (1949).

It should be noted that the decomposition of the perturbation velocity $\vec{q} = \vec{q}_o + \vec{q}_1$ discussed in Chapter II is not a splitting into longitudinal and solenoidal components. Each of $\vec{q}_o$ and $\vec{q}_1$ are themselves made up of these components.

Referring to Eqs. (3.1) and (3.2), the longitudinal and solenoidal components are easily identified by inspection. The terms of the solenoidal components contain the factor $e^{\kappa(x-R)}$, whereas the longitudinal components do not. There are some interesting features of the longitudinal component flow $\vec{q}_{o_L}$. Carrying out the indicated operations, we have

$$\vec{q}_{o_L} = \frac{1}{4\pi} \text{ grad}\left[\frac{-m}{R}\right] + - \frac{1}{4\pi\rho U} \text{ grad}\left\{- \frac{X}{R} + Y \frac{y}{(y^2 + (z+h)^2)}\left[1 + \frac{x}{R}\right] + Z \frac{(z+h)}{y^2 + (z+h)^2}\left[1 + \frac{x}{R}\right]\right\}. \tag{3.6}$$

The X-force has the same velocity potential part as a point sink, whereas the Y-force and Z-force have the same velocity potential parts as horseshoe vortices with their lifts oriented in the $-\hat{e}_y$ and $-\hat{e}_z$ directions respectively (see Fig. (3.1)).

The wake character of the $\vec{q}_o$ flow is governed by its solenoidal component $\vec{q}_{o_T}$. For large x downstream of the origin, the radius of the wake $r_w = \sqrt{y^2 + (z+h)^2}$ has the simple equation

$$r_w = \sqrt{\frac{2C_w x}{\kappa}} \quad , \tag{3.7}$$

where the constant $C_w > 0$ defines the scale of the effect of the wake. C_w is chosen so the wake flow quantities are diminished by a factor e^{-C_w}. Hence the influence of the solenoidal velocity $\vec{q}_{o_T}$ is confined within the paraboloidal region defined by (3.7), and is exponentially small throughout the remaining flow regime.

2. Transform of the $(\vec{q}_o, p_o)$ System

The transform quantities $\tilde{u}_o, \tilde{v}_o, \tilde{w}_o$, and $\tilde{p}_o$ are required to compute the functions $\tilde{M}_o, \tilde{N}_o$, and $\tilde{P}_o$ of Eqs. (2.44) - (2.46). The nondimensional form of $(\vec{q}_o, p_o)$ given by Eqs. (3.1) and (3.2) is obtained by using the definitions in (2.33) and the nondimensional force and mass source coefficients defined as

$$\vec{C}_F = \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \frac{1}{\frac{1}{2}\rho U^2 \ell^2} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} , \qquad C_m = \frac{m}{\frac{1}{2} U \ell^2} \quad . \tag{3.8}$$

In the subsequent discussions it is convenient to <u>drop the underlines</u>

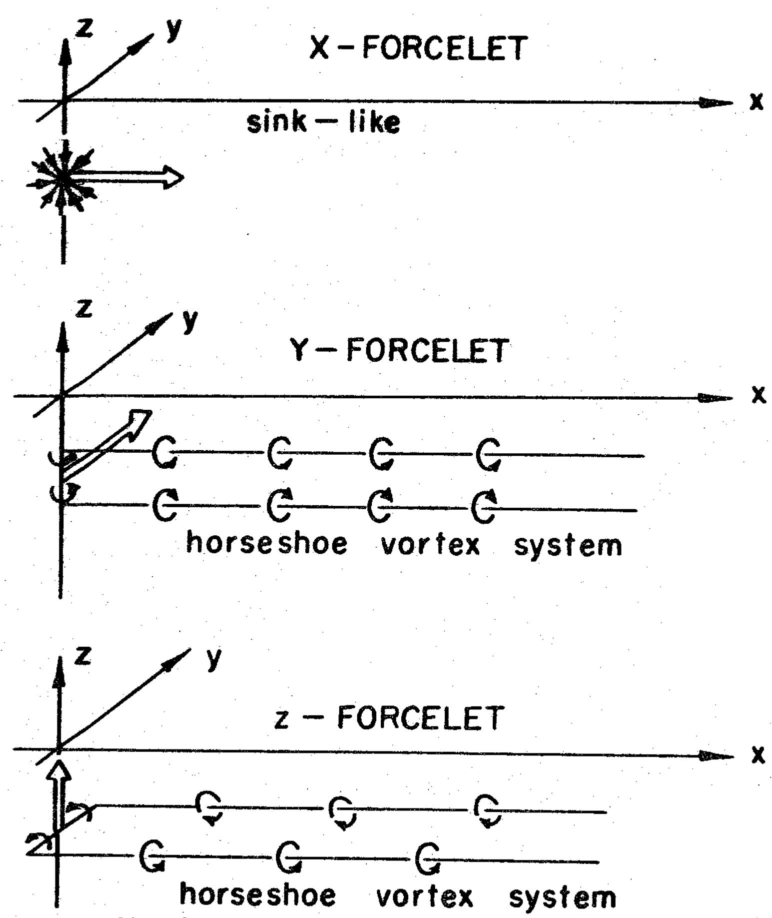

Fig. (3.1) Outside the wake region the flow is dominated by the longitudinal velocity component, as indicated above for the forcelets X, Y, Z. Within the wake, this behavior is cancelled by part of the $\vec{q}_{o_T}$ component.

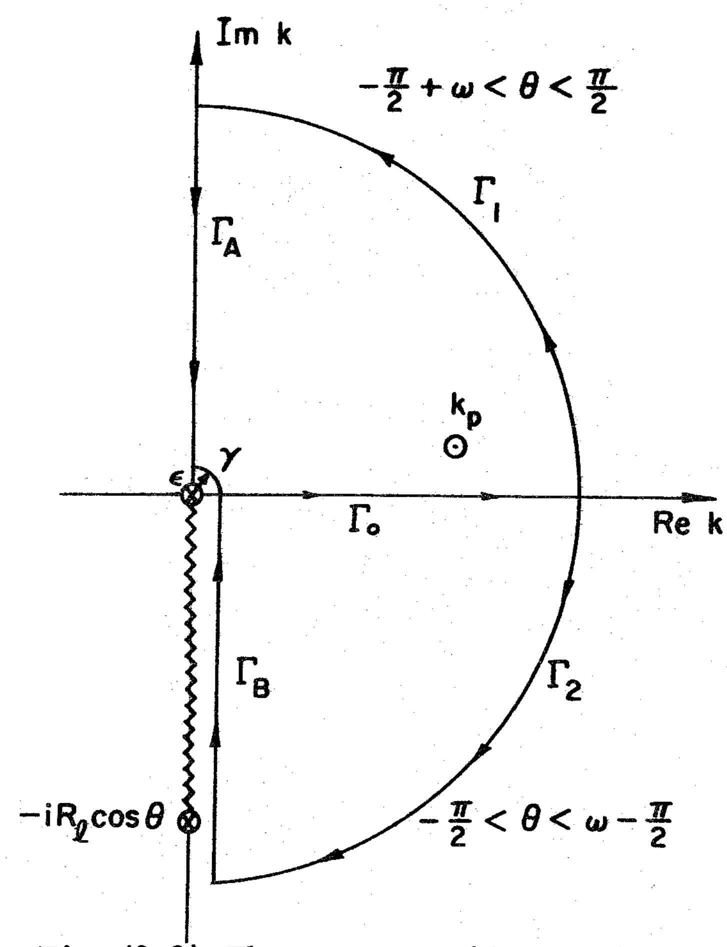

Fig. (3.2) The contours of integration in the complex k-plane.

from the nondimensional quantities. All conversions of equations to dimensional quantities are clearly stated in the text.

The Fourier transform of $(\vec{q}_o, p_o)$ is determined by using the following integral representations:

$$\left.\begin{aligned}
&\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\alpha x+\beta y)}\,\frac{1}{R}\,dx\,dy = \frac{e^{-k_o|z+h|}}{k_o} \quad ,\\
&\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\alpha x+\beta y)}\,\frac{e^{\frac{R_\ell}{2}(x-R)}}{R}\,dx\,dy = \frac{e^{-k_1|z+h|}}{k_1} \quad ,\\
&\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\alpha x+\beta y)}\left(1-e^{\frac{R_\ell}{2}(x-R)}\right)\frac{\partial}{\partial x}\,\ell n(R-x)\,dx\,dy\\
&\qquad = \left[-\frac{e^{-k_o|z+h|}}{k_o} + \frac{e^{-k_1|z+h|}}{k_1}\right] \quad ,\\
&\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\alpha x+\beta y)}\left(1-e^{\frac{R_\ell}{2}(x-R)}\right)\frac{\partial}{\partial y}\,\ell n(R-x)\,dx\,dy\\
&\qquad = \frac{\beta}{\alpha}\left[-\frac{e^{-k_o|z+h|}}{k_o} + \frac{e^{-k_1|z+h|}}{k_1}\right] \quad ,\\
&\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-i(\alpha x+\beta y)}\left(1-e^{\frac{R_\ell}{2}(x-R)}\right)\frac{\partial}{\partial z}\,\ell n(R-x)\,dx\,dy\\
&\qquad = \frac{i}{\alpha}\left[-e^{-k_o|z+h|} + e^{-k_1|z+h|}\right]\operatorname{sgn}(z+h) \quad ,
\end{aligned}\right\}\quad (3.9)$$

where

$$R = \sqrt{x^2 + y^2 + (z+h)^2} \quad ,$$

$$k_o = \sqrt{\alpha^2 + \beta^2} \quad ,$$

$$k_1 = \sqrt{\alpha^2 + \beta^2 + iR_\ell \alpha} \quad .$$

Then, for $z < 0$, the nondimensional $(\tilde{\vec{q}}_o, \tilde{p}_o)$ are given by the following functions of (α, β, z)

$$\begin{bmatrix} \tilde{u}_o \\ \tilde{v}_o \\ \tilde{w}_o \end{bmatrix} = \begin{bmatrix} i\alpha \\ i\beta \\ -k_o \end{bmatrix} G_1 \frac{e^{-k_o|z+h|}}{k_o} + \begin{bmatrix} i\alpha \\ i\beta \\ -k_1 \end{bmatrix} G_2 \frac{e^{-k_1|z+h|}}{k_1} + \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} \frac{R_\ell}{8\pi} \frac{e^{-k_1|z+h|}}{k_1} \quad , \tag{3.10}$$

$$\tilde{p}_o(\alpha, \beta, z) = -i\alpha[G_1]e^{-k_o|z+h|} + \frac{C_m}{4\pi R_\ell}\delta(z+h) \quad , \tag{3.11}$$

where the groups of terms G_1 and G_2 are functions of α, β and the nondimensional singularity strengths C_m, C_x, C_y, C_z,

$$G_1 = \frac{1}{8\pi}\left[-C_m + C_x + \frac{\beta}{\alpha} C_y + \frac{ik_o}{\alpha} C_z \operatorname{sgn}(z+h)\right] \quad , \tag{3.12}$$

$$G_2 = \frac{1}{8\pi}\left[-C_x - \frac{\beta}{\alpha} C_y - \frac{ik_1}{\alpha} C_z \operatorname{sgn}(z+h)\right] \quad . \tag{3.13}$$

Then the functions $\tilde{M}_o, \tilde{N}_o$, and $\tilde{P}_o$ are evaluated at $z = 0$ in terms of G_1 and G_2.

$$\widetilde{M}_o = 2i\alpha G_1 e^{-k_o h} + \left[2i\alpha G_2 + \frac{R_\ell C_x}{8\pi} - \frac{i\alpha}{k_1}\frac{R_\ell C_z}{8\pi}\right] e^{-k_1 h} \quad ,$$

$$\widetilde{N}_o = 2i\beta G_1 e^{-k_o h} + \left[2i\beta G_2 + \frac{R_\ell C_y}{8\pi} - \frac{i\beta}{k_1}\frac{R_\ell C_z}{8\pi}\right] e^{-k_1 h} \qquad (3.14)$$

$$\widetilde{P}_o = \left[\frac{\alpha^2}{k_o} - \frac{2i\alpha k_o}{R_\ell} + \sigma'\right] G_1 e^{-k_o h} +$$

$$+ \left[\left(-\frac{2i\alpha k_1}{R_\ell} + \sigma'\right) G_2 + \left(\frac{2i\alpha k_1}{R_\ell} - \sigma'\right)\frac{R_\ell C_z}{8\pi k_1}\right] e^{-k_1 h} \quad ,$$

where

$$\sigma' = \sigma_\ell + \frac{k_o^2}{W_\ell} \quad , \quad \sigma_\ell = g\ell/U^2 \quad , \quad W_\ell = U^2 \ell \rho / T \quad .$$

3. Solution for $(\vec{q}_1, p_1, \zeta)$

The solutions for the complementary flow quantities, $\widetilde{p}_1, \widetilde{u}_1, \widetilde{v}_1$, $\widetilde{w}_1$ follow directly from the differential equations (2.36) and (2.37)

$$\widetilde{p}_1(\alpha, \beta, z) = -Ae^{k_o z} \qquad (z < 0) \quad , \qquad (3.15)$$

$$\widetilde{u}_1(\alpha, \beta, z) = B_u e^{k_o z} + C_u e^{k_1 z} \qquad (z < 0) \quad , \qquad (3.16)$$

$$\widetilde{v}_1(\alpha, \beta, z) = B_v e^{k_o z} + C_v e^{k_1 z} \qquad (z < 0) \quad , \qquad (3.17)$$

$$\widetilde{w}_1(\alpha, \beta, z) = B_w e^{k_o z} + C_w e^{k_1 z} \qquad (z < 0) \quad , \qquad (3.18)$$

where $A, B_u, C_u, B_v, C_v, B_w, C_w$ are coefficients to be determined below. These solutions vanish at infinity $(z \to -\infty)$ provided the real parts of k_o and k_1 are positive

$$\mathrm{Re}\sqrt{\alpha^2 + \beta^2} > 0 \quad , \quad \mathrm{Re}\sqrt{\alpha^2 + \beta^2 + iR_\ell \alpha} > 0 \quad , \qquad (3.19)$$

The nondimensional form of Eq. (2.19),

$$\left(\nabla^2 - R_\ell \frac{\partial}{\partial x}\right)\vec{q}_1 = R_\ell \nabla p_1 \quad , \qquad (3.20)$$

has the following Fourier transform

$$\tilde{L}_1 \begin{bmatrix} \tilde{u}_1 \\ \tilde{v}_1 \\ \tilde{w}_1 \end{bmatrix} = \begin{bmatrix} i\alpha \\ i\beta \\ \frac{d}{dz} \end{bmatrix} R_\ell \tilde{p}_1 \qquad (z \leqslant 0) \quad , \qquad (3.21)$$

where

$$\tilde{L}_1 = \frac{d^2}{dz^2} - k_1^2 \quad .$$

Substitution of Eqs. (3.15) - (3.18) into (3.21) shows that B_u, B_v, B_w are related to the coefficient A by

$$\begin{bmatrix} B_u \\ B_v \\ B_w \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\beta}{\alpha} \\ \frac{k_o}{i\alpha} \end{bmatrix} A \quad .$$

Next, the four unknown coefficients $A, C_u, C_v,$ and C_w can be reduced to three unknowns by invoking the continuity equation. This gives

$$C_u = \frac{i}{\alpha}\,(i\beta\, C_v + k_1 C_w) \quad . \qquad (3.22)$$

The formal solutions of Eqs. (3.15) - (3.18) thus reduce to

$$\tilde{p}_1(\alpha, \beta, z) = -Ae^{k_o z} \quad , \qquad (3.23)$$

$$\widetilde{u}_1(\alpha,\beta,z) = Ae^{k_o z} + \frac{i}{\alpha}(i\beta C_v + k_1 C_w)e^{k_1 z} \quad , \tag{3.24}$$

$$\widetilde{v}_1(\alpha,\beta,z) = \frac{\beta}{\alpha} A e^{k_o z} + C_v e^{k_1 z} \quad , \tag{3.25}$$

$$\widetilde{w}_1(\alpha,\beta,z) = \frac{k_o}{i\alpha} A e^{k_o z} + C_w e^{k_1 z} \quad , \tag{3.26}$$

for $z \leqslant 0$. Using these relations, the free surface boundary conditions (2.39) - (2.41) provide a system of three linear equations in three unknowns A, C_v, and C_w.

$$\left.\begin{aligned} &\left[i\alpha + \frac{2k_o^2}{R_\ell} - \frac{i\sigma' k_o}{\alpha}\right] A + \left[\frac{2i\alpha k_1}{R_\ell} + \sigma'\right] C_w = \widetilde{P}_o \quad , \\ &2k_o A - \frac{\beta}{\alpha} k_1 C_v + \left[\frac{ik_1^2}{\alpha} + i\alpha\right] C_w = \widetilde{M}_o \quad , \\ &2\,\frac{\beta}{\alpha} k_o A + k_1 C_v + i\beta C_w = \widetilde{N}_o \quad . \end{aligned}\right\} \tag{3.27}$$

Solution of this system along with (3.22) yields the four coefficients A, C_u, C_v, C_w which connect the $(\vec{q}_1, p_1, \zeta)$ flow system to the unbounded flow system via the free surface conditions. The results are

$$A = i\alpha[G_1]\frac{e^{-k_o h}}{k_o} + \frac{1}{\Delta}\Bigg\{\left(\frac{2i\alpha^3}{k_o^2} + \frac{8\alpha^2}{R_\ell} - \frac{8i\alpha k_o^2}{R_\ell^2}\right)[G_1]e^{-k_o h} +$$

$$+ \frac{1}{8\pi}\left[C_x + \frac{\beta}{\alpha} C_y + \frac{ik_o^2}{\alpha k_1} C_z\right]\left(-\frac{4\alpha^2 k_1}{k_o R_\ell} + \frac{8i\alpha k_o k_1}{R_\ell^2}\right)e^{-k_1 h}\Bigg\} \quad , \tag{3.28}$$

$$C_u = i\alpha[G_2]\,\frac{e^{-k_1 h}}{k_1} + R_\ell\,\frac{C_x}{8\pi}\,\frac{e^{-k_1 h}}{k_1} +$$

$$+ \frac{1}{\Delta}\left\{\left(-\frac{4\alpha^2 k_1}{k_o R_\ell} + \frac{8i\alpha k_o k_1}{R_\ell^2}\right)[G_1]\,e^{-k_o h} + \frac{C_x}{8\pi}\left(\frac{8\alpha^2}{R_\ell} - \frac{8i\alpha k_o^2}{R_\ell^2}\right)e^{-k_1 h}\right.$$

$$\left. + \frac{C_y}{8\pi}\left(\frac{\beta}{\alpha}\right)\left(\frac{8\alpha^2}{R_\ell} - \frac{8i\alpha k_o^2}{R_\ell^2}\right)e^{-k_1 h} + \frac{C_z}{8\pi}\left(\frac{8k_o^2 k_1}{R_\ell^2}\right)e^{-k_1 h}\right\}, \qquad (3.29)$$

$$C_v = i\beta[G_2]\,\frac{e^{-k_1 h}}{k_1} + R_\ell\,\frac{C_y e^{-k_1 h}}{8\pi k_1} +$$

$$+ \frac{1}{\Delta}\left\{\frac{\beta}{\alpha}\left(-\frac{4\alpha^2 k_1}{k_o R_\ell} + \frac{8i\alpha k_o k_1}{R_\ell^2}\right)[G_1]\,e^{-k_o h} + \frac{C_x}{8\pi}\left(\frac{\beta}{\alpha}\right)\left(\frac{8\alpha^2}{R_\ell} - \frac{8i\alpha k_o^2}{R_\ell^2}\right)e^{-k_1 h}\right.$$

$$\left. + \frac{C_y}{8\pi}\left(\frac{\beta^2}{\alpha^2}\right)\left(\frac{8\alpha^2}{R_\ell} - \frac{8i\alpha k_o^2}{R_\ell^2}\right)e^{-k_1 h} + \frac{C_z}{8\pi}\left(\frac{\beta}{\alpha}\right)\left(\frac{8k_o^2 k_1}{R_\ell^2}\right)e^{-k_1 h}\right\}, \qquad (3.30)$$

$$C_w = [G_2]\,e^{-k_1 h} - R_\ell\,\frac{C_z}{8\pi}\,\frac{e^{-k_1 h}}{k_1} +$$

$$+ \frac{1}{\Delta}\left\{-\frac{ik_o^2}{\alpha k_1}\left(-\frac{4\alpha^2 k_1}{k_o R_\ell} + \frac{8i\alpha k_o k_1}{R_\ell^2}\right)[G_1]\,e^{-k_o h} + \frac{1}{8\pi}\left[C_x + \frac{\beta}{\alpha}\,C_y + \frac{ik_o^2}{\alpha k_1}\,C_z\right]\right.$$

$$\left. \times\left(-\frac{8k_o^2 k_1}{R_\ell^2}\right)e^{-k_1 h}\right\}. \qquad (3.31)$$

The denominator Δ is

$$\Delta(\alpha,\beta) = \sigma' - \left(\frac{\alpha^2}{k_o} - \frac{4i\alpha k_o}{R_\ell}\right) + \frac{4k_o^3}{R_\ell^2} - \frac{4k_o^2 k_1}{R_\ell^2}, \qquad (3.32)$$

where

$$\sigma' = \sigma_\ell + \frac{k_o^2}{W_\ell}, \quad k_o = \sqrt{\alpha^2 + \beta^2}, \quad k_1 = \sqrt{\alpha^2 + \beta^2 + iR_\ell\alpha} .$$

To complete the formal solution of the free surface problem, the nondimensional transform of the wave elevation $\tilde{\zeta}(\alpha,\beta)$ is determined from Eq. (2.43) to yield

$$\tilde{\zeta}(\alpha,\beta) = \frac{1}{i\alpha}\left[\tilde{w}_o(\alpha,\beta,0) + \frac{k_o}{i\alpha}A + C_w\right] ,$$

which, upon making use of (3.9), (3.28), and (3.31), gives

$$\tilde{\zeta}(\alpha,\beta) = \frac{1}{4\pi}\left(\frac{i\alpha}{k_o} + \frac{2k_o}{R_\ell}\right)\left[C_m - C_x - \frac{\beta}{\alpha}C_y - \frac{ik_o}{\alpha}C_z\right]\frac{e^{-k_o h}}{\Delta}$$
$$+ \frac{1}{4\pi R_\ell}\left[2k_1\left(C_x + \frac{\beta}{\alpha}C_y\right) + \frac{2ik_o^2}{\alpha}C_z\right]\frac{e^{-k_1 h}}{\Delta} \quad . \qquad (3.33)$$

By inserting the expressions for G_1 and G_2 into Eqs. (3.28) - (3.31), the resulting solutions for the $[(\tilde{u}_1, \tilde{v}_1, \tilde{w}_1), \tilde{p}_1]$ system exhibit the following decomposition

$$\tilde{p}_1 = \tilde{p}_1^{(o)} + \tilde{p}_s \quad , \qquad (3.34)$$

$$\begin{bmatrix}\tilde{u}_1\\ \tilde{v}_1\\ \tilde{w}_1\end{bmatrix} = \begin{bmatrix}\tilde{u}_1^{(o)}\\ \tilde{v}_1^{(o)}\\ \tilde{w}_1^{(o)}\end{bmatrix} + \begin{bmatrix}\tilde{u}_s\\ \tilde{v}_s\\ \tilde{w}_s\end{bmatrix} \quad , \qquad (3.35)$$

where $[(\tilde{u}_1^{(o)}, \tilde{v}_1^{(o)}, \tilde{w}_1^{(o)}), \tilde{p}_1^{(o)}]$ is an image flow system (singular at $\vec{x} = +h\hat{e}_z$ above the free surface), and $[(\tilde{u}_s, \tilde{v}_s, \tilde{w}_s), \tilde{p}_s]$ represents the remaining surface effect. Now, the surface flow system can be further decomposed into two parts corresponding to the 'longitudinal' and 'solenoidal' components discussed earlier,

$$[(\tilde{u}_s, \tilde{v}_s, \tilde{w}_s), \tilde{p}_s] = [(\tilde{u}_{s_L}, \tilde{v}_{s_L}, \tilde{w}_{s_L}), \tilde{p}_s] + [(\tilde{u}_{s_T}, \tilde{v}_{s_T}, \tilde{w}_{s_T}), 0] \quad . \qquad (3.36)$$

We have anticipated this decomposition from the form of Eqs. (3.23) - (3.26). The transformed pressure $\tilde{p}_1$ is associated with the factor $e^{k_o z}$, while the velocity transforms $(\tilde{u}_1, \tilde{v}_1, \tilde{w}_1)$ contain the factors $e^{k_o z}$ and $e^{k_1 z}$. We know from the Fourier transforms listed in Eqs. (3.9) that the $e^{k_o z}$ terms are associated with the potential flow parts, and hence identify the longitudinal components. The $e^{k_1 z}$ terms, on the other hand, are associated with solenoidal components containing $e^{\frac{R_\ell}{2}(x-R_o)}$, $R_o = \sqrt{x^2+y^2+z^2}$.

Collecting these results and using the Fourier inversion formula (2.32), the solution of $(\vec{q}_1, p_1; \zeta)$ is now summarized.

$$
\begin{aligned}
p_1 &= p_1^{(o)} + p_s \quad , \\
\vec{q}_1 &= \vec{q}_1^{\,(o)} + \vec{q}_s \quad ,
\end{aligned}
\tag{3.37}
$$

where

$$(\vec{q}_s, p_s) = (\vec{q}_{s_L}, p_s) + (\vec{q}_{s_T}, 0) \quad .$$

The image flow system $(\vec{q}_1^{\,(o)}, p_1^{(o)})$ is

$$p_1^{(o)} = \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{i(\alpha x+\beta y)} \tilde{p}_1 \, d\alpha \, d\beta \quad , \tag{3.38}$$

$$\begin{bmatrix} u_1^{(o)} \\ v_1^{(o)} \\ w_1^{(o)} \end{bmatrix} = \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{i(\alpha x+\beta y)} \begin{bmatrix} \tilde{u}_1^{(o)} \\ \tilde{v}_1^{(o)} \\ \tilde{w}_1^{(o)} \end{bmatrix} d\alpha \, d\beta \quad , \tag{3.39}$$

with

$$\tilde{p}_1^{(o)} = -i\alpha[G] \frac{e^{k_o(z-h)}}{8\pi k_o} \quad , \tag{3.40}$$

$$\begin{bmatrix} \tilde{u}_1^{(o)} \\ \tilde{v}_1^{(o)} \\ \tilde{w}_1^{(o)} \end{bmatrix} = \begin{bmatrix} i\alpha \\ i\beta \\ k_o \end{bmatrix} G \frac{e^{k_o(z-h)}}{8\pi k_o} + \begin{bmatrix} i\alpha \\ i\beta \\ k_1 \end{bmatrix} H \frac{e^{k_1(z-h)}}{8\pi k_1} + \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} \frac{R_\ell e^{k_1(z-h)}}{8\pi k_1} ,$$

(3.41)

and where

$$G = \left[-C_m + C_x + \frac{\beta}{\alpha} C_y + \frac{ik_o}{\alpha} C_z \right] , \qquad (3.42)$$

$$H = \left[-C_x - \frac{\beta}{\alpha} C_y - \frac{ik_1}{\alpha} C_z \right] . \qquad (3.43)$$

The longitudinal component $[\vec{q}_{s_L}(x,y,z), p_s(x,y,z)]$ is

$$p_s(x,y,z) = - \iint_{-\infty}^{\infty} d\alpha \, d\beta \, \frac{e^{i(\alpha x+\beta y)}}{8\pi^2 \Delta} \left\{ \mathscr{A}_{p_s} e^{k_o(z-h)} + \mathscr{B}_{p_s} e^{k_o z - k_1 h} \right\} ,$$

(3.44)

$$\begin{bmatrix} u_{s_L} \\ v_{s_L} \\ w_{s_L} \end{bmatrix} = \iint_{-\infty}^{\infty} d\alpha \, d\beta \, \frac{e^{i(\alpha x+\beta y)}}{8\pi^2 \Delta} \left\{ \begin{bmatrix} \mathscr{A}_{u_s} \\ \mathscr{A}_{v_s} \\ \mathscr{A}_{w_s} \end{bmatrix} e^{k_o(z-h)} + \begin{bmatrix} \mathscr{B}_{u_s} \\ \mathscr{B}_{v_s} \\ \mathscr{B}_{w_s} \end{bmatrix} e^{k_o z - k_1 h} \right\} ,$$

(3.45)

where

$$\mathscr{A}_{p_s} = \left(\frac{i\alpha^3}{k_o^2} + \frac{4\alpha^2}{R_\ell} - \frac{4i\alpha k_o^2}{R_\ell^2} \right) [G \ (\alpha, \beta)] , \qquad (3.46)$$

$$\mathscr{B}_{p_s} = \left(- \frac{2\alpha^2 k_1}{k_o R_\ell} + \frac{4i\alpha k_o k_1}{R_\ell^2} \right) \left[C_x + \frac{\beta}{\alpha} C_y + \frac{ik_o^2}{\alpha k_1} C_z \right] , \qquad (3.47)$$

$$\begin{bmatrix} \mathscr{A}_{u_s} \\ \mathscr{A}_{v_s} \\ \mathscr{A}_{w_s} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\beta}{\alpha} \\ \frac{k_o}{i\alpha} \end{bmatrix} \left(\frac{i\alpha^3}{k_o^2} + \frac{4\alpha^2}{R_\ell} - \frac{4i\alpha k_o^2}{R_\ell^2} \right) [G(\alpha,\beta)] \ , \tag{3.48}$$

$$\begin{bmatrix} \mathscr{B}_{u_s} \\ \mathscr{B}_{v_s} \\ \mathscr{B}_{w_s} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\beta}{\alpha} \\ \frac{k_o}{i\alpha} \end{bmatrix} \left(- \frac{2\alpha^2 k_1}{k_o R_\ell} + \frac{4i\alpha k_o k_1}{R_\ell^2} \right) \left[C_x + \frac{\beta}{\alpha} C_y + \frac{ik_o^2}{\alpha k_1} C_z \right] \ . \tag{3.49}$$

The solenoidal component $[\vec{q}_{s_T}(x,y,z), 0]$ is

$$\begin{bmatrix} u_{s_T} \\ v_{s_T} \\ w_{s_T} \end{bmatrix} = \iint_{-\infty}^{\infty} d\alpha \, d\beta \ \frac{e^{i(\alpha x \beta y)}}{8\pi^2 \Delta} \left\{ \begin{bmatrix} \mathscr{C}_{u_s} \\ \mathscr{C}_{v_s} \\ \mathscr{C}_{w_s} \end{bmatrix} e^{k_1 z - k_o h} + \begin{bmatrix} \mathscr{D}_{u_s} \\ \mathscr{D}_{v_s} \\ \mathscr{D}_{w_s} \end{bmatrix} e^{k_1 (z-h)} \right\} \ , \tag{3.50}$$

where

$$\begin{bmatrix} \mathscr{C}_{u_s} \\ \mathscr{C}_{v_s} \\ \mathscr{C}_{w_s} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\beta}{\alpha} \\ -\frac{ik_o^2}{\alpha k_1} \end{bmatrix} \left(- \frac{2\alpha^2 k_1}{k_o R_\ell} + \frac{4i\alpha k_o k_1}{R_\ell^2} \right) [G(\alpha,\beta)] \ , \tag{3.51}$$

$$\left.\begin{aligned}
\mathscr{D}_{u_s} &= \left(\frac{2\alpha^2}{R_\ell} - \frac{4i\alpha k_o^2}{R_\ell^2}\right)\left[C_x + \frac{\beta}{\alpha} C_y\right] + \frac{4k_o^2 k_1}{R_\ell^2} C_z \\
\mathscr{D}_{v_s} &= \frac{\beta}{\alpha}\left(\frac{2\alpha^2}{R_\ell} - \frac{4i\alpha k_o^2}{R_\ell^2}\right)\left[C_x + \frac{\beta}{\alpha} C_y\right] + \left(\frac{\beta}{\alpha}\right)\frac{4k_o^2 k_1}{R_\ell^2} C_z \\
\mathscr{D}_{w_s} &= \left(-\frac{4k_o^2 k_1}{R_\ell^2}\right)\left[C_x + \frac{\beta}{\alpha} C_y + \frac{ik_o^2}{\alpha k_1} C_z\right] .
\end{aligned}\right\} \qquad (3.52)$$

Finally, the free surface elevation from (3.33) is expressed in abbreviated form

$$\zeta(x,y) = \iint_{-\infty}^{\infty} d\alpha\, d\beta\, \frac{e^{i(\alpha x+\beta y)}}{8\pi^2 \Delta}\left\{A_\zeta e^{-k_o h} + B_\zeta e^{-k_1 h}\right\} , \qquad (3.53)$$

where

$$A_\zeta = \left(\frac{i\alpha}{k_o} + \frac{2k_o}{R_\ell}\right)[-G(\alpha,\beta)] , \qquad (3.54)$$

$$B_\zeta = \frac{2k_1}{R_\ell}\left[C_x + \frac{\beta}{\alpha} C_y + \frac{ik_o^2}{\alpha k_1} C_z\right] . \qquad (3.55)$$

The factor $G(\alpha, \beta) = \left[-C_m + C_x + \frac{\beta}{\alpha} C_y + \frac{ik_o}{\alpha} C_z\right]$ is a grouping of terms which appears often throughout this work.

It may be remarked here that when surface tension effects are omitted $(T = 0)$, and if $C_m = C_x = C_y = 0$, leaving only C_z directed in the $+z$ direction at the free surface $(h = 0)$, we have the special case of a normal surface force acting at the point $x, y, z = 0$. If we also put the reference length $\ell = U^2/g$, the formula for $\zeta(x, y)$ given above reduces to

$$\zeta(x,y) = \frac{(-C_z)}{8\pi^2} \iint_{-\infty}^{\infty} \frac{e^{i(\alpha x+\beta y)}}{\Delta(\alpha,\beta)} \, d\alpha \, d\beta \quad , \qquad (3.56)$$

where $\sigma_\ell = 1$ and $W_\ell \to \infty$ in $\Delta(\alpha,\beta)$ of (3.32). This is a nondimensional version of Cumberbatch's (1965) result.

Certain symmetry properties of the free surface solution are evident in Eqs. (3.38) - (3.55). (a) For symmetrical disturbances (terms involving C_m, C_x, and C_z) $\tilde{u}_1, \tilde{w}_1, \tilde{p}_1$, and $\tilde{\zeta}$ are even in β, and the physical flow quantities u_1, w_1, p_1, and ζ are thus even in y. The quantity $\tilde{v}_1$ is odd in β, and hence v_1 is consequently odd in y. (b) For the antisymmetrical disturbance C_y, by the same reasoning, u_1, w_1, p_1, and ζ are odd in y, and v_1 is now even in y.

4. Partial Inversion of the Formal Solution

The nondimensional image flow system of Eq. (3.38) - (3.41) is inverted immediately by inspection, giving

$$\begin{bmatrix} u_1^{(o)} \\ v_1^{(o)} \\ w_1^{(o)} \end{bmatrix} = \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} \frac{R_\ell}{8\pi} \frac{e^{\left(\frac{R_\ell}{2}\right)(x-R_1)}}{R_1} - \frac{1}{8\pi} \nabla \left\{ \frac{C_m}{R_1} + \left(1 - e^{\left(\frac{R_\ell}{2}\right)(x-R_1)} \right) (\vec{C}_F \cdot \nabla) \ell n (R_1 - x) \right\}$$

(3.57)

$$p_1^{(o)} = \frac{1}{8\pi} \left(C_m \frac{\partial}{\partial x} - \vec{C}_F \cdot \nabla \right) \frac{1}{R_1} \quad , \qquad (3.58)$$

where

$$R_1 = \sqrt{x^2 + y^2 + (z-h)^2} \quad , \qquad \vec{C}_F = (C_x, C_y, C_z) \quad .$$

For the remaining flow quantities $(\vec{q}_s, p_s; \zeta)$, it is convenient to introduce new transform variables (k, θ) defined by

$$\alpha = k \cos\theta \quad , \qquad \beta = k \sin\theta \quad , \tag{3.59}$$

and to use polar coordinates for the space variables (x, y)

$$x = r \cos\omega \quad , \qquad y = r \sin\omega \quad . \tag{3.60}$$

The Fourier integral representation in (2.32) becomes

$$f(r, \omega, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{\infty} \tilde{f}(k, \theta, z) e^{ikr\cos(\theta-\omega)} k\,dk\,d\theta \quad , \tag{3.61}$$

with the notations k_o and k_1 becoming

$$\begin{aligned} k_o^2 &= \alpha^2 + \beta^2 = k^2 \\ k_1^2 &= \alpha^2 + \beta^2 + iR_\ell \alpha = k^2 + iR_\ell k \cos\theta \quad . \end{aligned} \tag{3.62}$$

The factor $\Delta(\alpha, \beta)$ in the denominator of all the integrands then becomes

$$\Delta(k, \theta) = \sigma_\ell + \frac{k^2}{W_\ell} - k\left(\cos^2\theta - \frac{4ik\cos\theta}{R_\ell} - \frac{4k^2}{R_\ell^2}\right) - \frac{4k^2\sqrt{k^2 + iR_\ell k\cos\theta}}{R_\ell^2} \quad . \tag{3.63}$$

Considering k as a complex variable, contributions to the k-integrations depend on the nature and number of the singularities of the function $\Delta(k, \theta)$. There are two branch points at $\sqrt{k^2 + iR_\ell k\cos\theta} = 0$, i.e., at

$$k = 0 \qquad \text{and} \qquad k = -iR_\ell \cos\theta \quad . \tag{3.64}$$

A branch cut along the imaginary k-axis between these two points renders $\Delta(k)$ a single-valued analytic function in the cut k-plane. The branch point $k = -iR_\ell \cos\theta$ will be along the positive or negative imaginary k-axis depending on the values of the polar angle θ. See Fig. (A.1) of Appendix A.

The poles of the integrands are the zeros of the function $\Delta(k,\theta)$. Unfortunately, the roots of the equation $\Delta(k) = 0$ cannot be determined exactly in a closed form. However, they can be found approximately by expanding in a power series in $R_\ell^{-\frac{1}{2}}$, for large R_ℓ. The details of this approximation are discussed in Appendix A. There, it is also shown (using as a guide, the reference by Wu and Messick (1958)) that if surface tension is included there are exactly two simple zeros of $\Delta(k)$ in the cut k-plane. These roots are (for $R_\ell \gg 1$)

$$k_1(\theta) = \lambda_1 + \frac{i\,4\lambda_1^2}{\left(\cos\theta - \frac{2\lambda_1}{W_\ell}\sec\theta\right)R_\ell} + \frac{4e^{i\frac{\pi}{4}}\lambda_1^{5/2}\sqrt{\cos\theta}}{\left(\cos^2\theta - \frac{2\lambda_1}{W_\ell}\right)R_\ell^{3/2}}\,, \qquad (3.65)$$

$$k_2(\theta) = \lambda_2 + \frac{i\,4\lambda_2^2}{\left(\cos\theta - \frac{2\lambda_2}{W_\ell}\sec\theta\right)R_\ell} + \frac{4e^{i\frac{\pi}{4}}\lambda_2^{5/2}\sqrt{\cos\theta}}{\left(\cos^2\theta - \frac{2\lambda_2}{W_\ell}\right)R_\ell^{3/2}}\,, \qquad (3.66)$$

valid in the range $0 < \theta < \theta_o$, where

$\theta_o = \cos^{-1}\left(\frac{4\sigma_\ell}{W_\ell}\right)^{\frac{1}{4}}$. A critical value of U is determined by

$\frac{4\sigma_\ell}{W_\ell} = 1 \Rightarrow U_c = \left(\frac{4gT}{\rho}\right)^{\frac{1}{4}}$. The two inviscid roots λ_1 and λ_2 are

$$\lambda_1(\theta) = \frac{W_\ell}{2}\left[\cos^2\theta - \sqrt{\cos^4\theta - \frac{4\sigma_\ell}{W_\ell}}\,\right], \qquad (3.67)$$

$$\lambda_2(\theta) = \frac{W_\ell}{2}\left[\cos^2\theta + \sqrt{\cos^4\theta - \frac{4\sigma_\ell}{W_\ell}}\;\right]. \qquad (3.68)$$

Note that in the limit as $T \to 0$,

$$\lambda_1 \to \sigma_\ell \sec^2\theta \quad \text{(gravity-dominated term)},$$

$$\lambda_2 \to W_\ell \cos^2\theta \quad \text{(surface-tension dominated)}.$$

The root $k_1(\theta)$ is in the first quadrant of the k-plane, and $k_2(\theta)$ is in the fourth quadrant. Surface tension introduces interesting complications to free surface flows. These have been studied in the plane viscous flow case by Wu and Messick (1958), and by Crapper (1964) for the Kelvin problem in inviscid potential flow. Webster (1966) explored the influence of surface tension for potential flow ship resistance. However, this subject is dropped from further discussion here.

Neglecting surface tension altogether $(T = 0,\ W_\ell = \infty)$, and dropping the terms of $O(R_\ell^{-3/2})$ or smaller, it can be shown (see Appendix A) that there is one simple zero $k_p(\theta)$ of $\Delta(k) = 0$. Hence throughout the remainder of this work, the single zero of the function $\Delta(k)$ is taken as

$$k_p = \sigma_\ell \sec^2\theta + i\,\frac{4\sigma_\ell^2 \sec^5\theta}{R_\ell} + O(R_\ell^{-3/2}) \qquad (3.69)$$

valid for $R_\ell \gg 1$. It is neither necessary nor desirable to carry out any of the expansions in this work to $O(R_\ell^{-3/2})$. The gain in accuracy is not worth the algebraic complexity. One of the effects of viscosity is to move the pole off the Rek axis into the first quadrant of the complex k-plane. Not only does this introduce viscous damping effects, but it also removes the necessity of a radiation condition to insure that waves appear only downstream.

The case of small Reynolds number flow, $R_\ell \ll 1$, is not considered in this work.

Now, neglecting all the terms smaller than $O(R_\ell^{-1})$ in Eqs. (3.46) - (3.52), and using the Fourier integral representation (3.61), the approximate solutions of the $(\vec{q}_s, p_s; \zeta)$ flow quantities are summarized below.

The $(\vec{q}_{s_L}, p_s)$ system is

$$p_s(r,\omega,z) = -\int_{-\pi}^{\pi} d\theta \int_0^{\infty} dk \, \frac{e^{ikr\cos(\theta-\omega)}}{8\pi^2 \Delta(k,\theta)} \left\{ A_{p_s} e^{k(z-h)} + B_{p_s} e^{kz-Kh} \right\}, \tag{3.70}$$

$$\begin{bmatrix} u_{s_L} \\ v_{s_L} \\ w_{s_L} \end{bmatrix} = \int_{-\pi}^{\pi} d\theta \int_0^{\infty} dk \, \frac{e^{ikr\cos(\theta-\omega)}}{8\pi^2 \Delta(k,\theta)} \left\{ \begin{bmatrix} A_{u_s} \\ A_{v_s} \\ A_{w_s} \end{bmatrix} e^{k(z-h)} + \begin{bmatrix} B_{u_s} \\ B_{v_s} \\ B_{w_s} \end{bmatrix} e^{kz-Kh} \right\}. \tag{3.71}$$

The $(\vec{q}_{s_T}, 0)$ system is

$$\begin{bmatrix} u_{s_T} \\ v_{s_T} \\ w_{s_T} \end{bmatrix} = \int_{-\pi}^{\pi} d\theta \int_0^{\infty} dk \, \frac{e^{ikr\cos(\theta-\omega)}}{8\pi^2 \Delta(k,\theta)} \left\{ \begin{bmatrix} C_{u_s} \\ C_{v_s} \\ C_{w_s} \end{bmatrix} e^{Kz-kh} + \begin{bmatrix} D_{u_s} \\ D_{v_s} \\ D_{w_s} \end{bmatrix} e^{K(z-h)} \right\}, \tag{3.72}$$

and the free surface elevation is

$$\zeta(r,\omega) = \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} dk \, \frac{e^{ikr\cos(\theta-\omega)}}{8\pi^2 \Delta(k,\theta)} \left\{ A_\zeta e^{-kh} + B_\zeta \, {}^{-Kh} \right\} , \qquad (3.73)$$

where the functions $A_{js}, B_{js}, C_{js}, D_{js}, A_\zeta, B_\zeta\,(j=p,u,v,w)$ are all functions of (k,θ), and where

$$K = \sqrt{k^2 + iR_\ell k\cos\theta} \quad ,$$

$$\Delta(k,\theta) = \sigma_\ell - k\left(\cos^2\theta - \frac{4ik\cos\theta}{R_\ell}\right) + \frac{4k^3}{R_\ell^2} - \frac{4k^2 K}{R_\ell^2} \quad , \qquad (3.74)$$

$$A_{p_s} \simeq \left(ik^2\cos^3\theta + \frac{4k^3\cos^2\theta}{R_\ell}\right) [G(\theta)] \quad ,$$

$$B_{p_s} \simeq \frac{2k^2 K\cos^2\theta}{R_\ell}[-C_x - \tan\theta\, C_y] - \frac{2ik^3\cos\theta}{R_\ell} C_z \quad , \qquad (3.75)$$

$$\begin{bmatrix} A_{u_s} \\ A_{v_s} \\ A_{w_s} \end{bmatrix} \simeq \begin{bmatrix} 1 \\ \tan\theta \\ -i\sec\theta \end{bmatrix} \left(ik^2\cos^3\theta + \frac{4k^3\cos^2\theta}{R_\ell}\right)[\,G(\theta)] \quad , \qquad (3.76)$$

$$\begin{bmatrix} B_{u_s} \\ B_{v_s} \\ B_{w_s} \end{bmatrix} \simeq \begin{bmatrix} 1 \\ \tan\theta \\ -i\sec\theta \end{bmatrix} \left\{ \frac{2k^2 K\cos\theta}{R_\ell}[-C_x - \tan\theta\, C_y] - \frac{2ik^3\cos\theta}{R_\ell} C_z \right\} , \qquad (3.77)$$

$$\begin{pmatrix} C_{u_s} \\ C_{v_s} \end{pmatrix} \simeq \begin{pmatrix} 1 \\ \tan\theta \end{pmatrix} \frac{2k^2 K \cos^2\theta}{R_\ell} \quad [-G(\theta)] \quad ,$$

$$C_{w_s} \simeq \frac{2ik^3 \cos\theta}{R_\ell} \quad [G(\theta)] \quad , \tag{3.78}$$

$$\begin{pmatrix} D_{u_s} \\ D_{v_s} \end{pmatrix} \simeq \begin{pmatrix} 1 \\ \tan\theta \end{pmatrix} \frac{4k^3 \cos^2\theta}{R_\ell} [\, C_x + \tan\theta\, C_y] \quad ,$$

$$D_{w_s} \simeq 0 \quad , \tag{3.79}$$

$$A_\zeta = \left(ik\cos\theta + \frac{2k^2}{R_\ell} \right) \quad [-G(\theta)] \quad ,$$

$$B_\zeta = \frac{2kK}{R_\ell} [\, C_x + \tan\theta\, C_y] + \frac{2ik^2 \sec\theta}{R_\ell}\, C_z \quad , \tag{3.80}$$

with

$$G(\theta) = [\, -C_m + C_x + \tan\theta\, C_y + i\sec\theta\, C_z] \quad . \tag{3.81}$$

We proceed with a partial inversion of the formal solution by computing the k-integrations indicated in (3.70) - (3.73). Consider the general form of these integrals

$$I(r,\omega,z) = \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} dk \; \frac{e^{ik\tilde{\omega}_o}}{\Delta(k,\theta)} \left\{ A(k,\theta) e^{k(z-h)} + B(k,\theta) e^{kz-Kh} \right\} , \tag{3.82}$$

$$J(r,\omega,z) = \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} dk \; \frac{e^{ik\tilde{\omega}_o}}{\Delta(k,\theta)} \left\{ C(k,\theta) e^{Kz-kh} + D(k,\theta) e^{K(z-h)} \right\} , \tag{3.83}$$

where $ik\tilde{\omega}_o = ik(x\cos\theta + y\sin\theta)$, $K = \sqrt{k^2 + iR_\ell k\cos\theta}$. By suitable changes of variables and combination of terms, I can be rewritten as

$$I = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \; \mathrm{Re}\int_{o}^{\infty} dk \; \frac{e^{ikr\cos(\theta-\omega)}}{\Delta(k,\theta)}\left\{Ae^{k(z-h)} + Be^{kz-Kh}\right\} , \qquad (3.84)$$

where $0 \leq \omega \leq \pi$ includes the entire range of interest in the physical space. The integral J can be simplified in precisely the same fashion, reducing to a form analogous to (3.84).

Now, referring to Fig. (3.2), the appropriate contour for the k-integration consists of the path Γ_o along the positive real axis starting at ϵ and extending out to R_k. When $\cos(\theta-\omega) > 0$, the choice of the contour $\Gamma_o+\Gamma_1+\Gamma_A+\gamma$ guarantees that the contribution from Γ_1 is zero in the limit $R_k \to \infty$. Also, since $0 < \omega < \pi$, the condition $\cos(\theta-\omega) > 0$ is satisfied when $-\frac{\pi}{2} < (\theta-\omega) < \frac{\pi}{2} - \omega$, so that the range of θ is restricted to $-\frac{\pi}{2} + \omega < \theta < \frac{\pi}{2}$. The contour in this case encloses the pole k_p.

When $\cos(\theta-\omega) < 0$, the contour must be deformed to make $\mathrm{Im}\,k < 0$ so that the contribution from Γ_2 goes to zero as $R_k \to \infty$. The conditions $\cos(\theta-\omega) < 0$ and $0 < \omega < \pi$ imply that for the contour $\Gamma_o+\Gamma_2+\Gamma_B$, the range of θ is restricted to $-\frac{\pi}{2} < \theta < -\frac{\pi}{2} + \omega$.

Applying the residue theorem to the closed contours discussed above, we obtain

$$\int_{\Gamma_o} + \int_{\Gamma_1} + \int_{\Gamma_A} + \int_{\gamma} + \int_{\Gamma_1} + \int_{\Gamma_B} g(k,\theta)dk = 2\pi i \sum \text{Residues} \quad , \qquad (3.85)$$

where $g(k,\theta)$ represents the integrand of (3.84). On the arc γ, we put $k = \epsilon^{i\varphi}$, so the integral on k becomes an integral on φ between

$\frac{\pi}{2}$ and 0. On Γ_B, we put $k = \epsilon - i\lambda$, where $0 < \lambda < R_k$. Then in the limit $\epsilon \to 0$, $R_k \to \infty$, the integral on Γ_o is the desired integral I; the contributions from Γ_1 and Γ_2 vanish; and the contribution from γ also goes to zero as $\epsilon \to 0$, because the functions $A(\epsilon e^{i\varphi}, \theta)$ and $B(\epsilon e^{i\varphi}, \theta)$ all vanish in that limit (see Eqs. (3.75) - (3.80)). Rearranging, we have the general result

$$I(r,\omega,z) = 2\mathrm{Re}\int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \, \frac{2\pi i F(k_p,\theta,z)}{D_1(k_p,\theta)} \, e^{ik_p r\cos(\theta-\omega)} +$$

$$+ 2\mathrm{Re}\int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_o^{\infty} e^{-\lambda r\cos(\theta-\omega)} \, \frac{F(i\lambda,\theta,z)}{\Delta(i\lambda,\theta)} \, (id\lambda) +$$

$$- 2\mathrm{Re}\int_{-\frac{\pi}{2}}^{\omega-\frac{\pi}{2}} d\theta \int_o^{\infty} e^{\lambda r\cos(\theta-\omega)} \, \frac{F(-i\lambda,\theta,z)}{\Delta(-i\lambda,\theta)} \, (id\lambda) \quad , \quad (3.86)$$

where

$$F(k,\theta,z) = [\,A(k,\theta)e^{k(z-h)} + B(k,\theta)e^{kz-Kh}] \quad ,$$

$$K = K(k,\theta) = \sqrt{k^2 + iR_\ell k\cos\theta} \quad ,$$

$$D_1(k_p,\theta) = \left[\frac{\partial\Delta}{\partial k}\right]_{k=k_p} \simeq -\cos^2\theta\left(1 - i\,\frac{8\sigma_\ell \sec^3\theta}{R_\ell}\right) + O(R_\ell^{-3/2}) \quad ,$$

$$k_p = \sigma_\ell \sec^2\theta + i\,\frac{4\sigma_\ell^2\sec^5\theta}{R_\ell} \quad .$$

This result for I is split into two parts. The first integral on the right hand side of (3.86) represents a 'free' or propagating disturbance

$I^{(f)}$, while the second and third integrals represent a localized contribution $I^{(\ell)}$ (a flow disturbance that is not swept downstream). We are primarily interested in the propagating disturbance $I^{(f)}$ because it gives the dominant contribution far downstream.

The integral J of Eq. (3.83) can be integrated with respect to k in an analogous fashion, giving a parallel result to (3.86).

It is desirable to simplify all the terms containing the radical $\sqrt{k_p^2+iR_\ell k_p\cos\theta}$. Expanding for large R_ℓ and choosing the proper sign of $\sqrt{i}$, to keep $\mathrm{Re}\sqrt{k_p^2+iR_\ell k_p\cos\theta} > 0$, we keep only the first term in the expansion

$$\sqrt{k_p^2+iR_\ell k_p\cos\theta} \simeq \sqrt{\frac{\sigma_\ell R_\ell}{2}\sec\theta}\ (1+i) + O(R_\ell^{-\frac{1}{2}}) \quad . \tag{3.87}$$

For θ very near $\pm\frac{\pi}{2}$, this expansion is incorrect because then $\left|\sigma_\ell \frac{\sec^3\theta}{R_\ell}\right| \not< 1$. However, as will be seen later, the principal contribution of interest come from values of $|\theta| < \sec^{-1}\left(\frac{R_\ell}{\sigma_\ell}\right)^{\frac{1}{3}}$. It can be shown that any contributions coming from values $\sec^{-1}\left(\frac{R_\ell}{\sigma_\ell}\right)^{\frac{1}{3}} < |\theta| \le \frac{\pi}{2}$ are exponentially small and are neglected.

The results for the 'free disturbances' $(\vec{q}_s^{(f)}, p_s^{(f)}; \zeta^{(f)})$ are now summarized. The superscript f indicates that only the propagating parts of the flow disturbance are included (contributions from the pole k_p).

$$p_s^{(f)}(r,\omega,z) = -\frac{Re}{2\pi}\int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\, \frac{i}{D_1(k_p)}\left\{A_{p_s}(k_p)e^{m_1(k_p)} + B_{p_s}(k_p)e^{m_2(k_p)}\right\} , \tag{3.88}$$

$$\begin{bmatrix} u^{(f)}_{s_L} \\ v^{(f)}_{s_L} \\ w^{(f)}_{s_L} \end{bmatrix} = \frac{\mathrm{Re}}{2\pi} \int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \; \frac{i}{D_1(k_p)} \left\{ \begin{bmatrix} A_{u_s} \\ A_{v_s} \\ A_{w_s} \end{bmatrix}_{k_p} e^{m_1(k_p)} + \begin{bmatrix} B_{u_s} \\ B_{v_s} \\ B_{w_s} \end{bmatrix}_{k_p} e^{m_2(k_p)} \right\} , \tag{3.89}$$

$$\begin{bmatrix} u^{(f)}_{s_T} \\ v^{(f)}_{s_T} \\ w^{(f)}_{s_T} \end{bmatrix} = \frac{\mathrm{Re}}{2\pi} \int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \; \frac{i}{D_1(k_p)} \left\{ \begin{bmatrix} C_{u_s} \\ C_{v_s} \\ C_{w_s} \end{bmatrix}_{k_p} e^{m_3(k_p)} + \begin{bmatrix} D_{u_s} \\ D_{u_s} \\ D_{w_s} \end{bmatrix}_{k_p} e^{m_4(k_p)} \right\} , \tag{3.90}$$

$$\zeta^{(f)}(r,\omega) = \frac{\mathrm{Re}}{2\pi} \int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \; \frac{i}{D_1(k_p)} \left\{ A_\zeta(k_p) e^{m_o(k_p)} + B_\zeta(k_p) e^{n_o(k_p)} \right\} , \tag{3.91}$$

with

$$\left.\begin{aligned}
D_1^{-1}(k_p) &\simeq -\sec^2\theta[1+i(8\sigma_\ell \sec^3\theta)/R_\ell] , \\
m_o &= ik_p r\cos(\theta-\omega) - k_p h , \\
n_o &= ik_p r\cos(\theta-\omega) - h\sqrt{k_p^2 + iR_\ell k_p \cos\theta} , \\
m_1 &= ik_p r\cos(\theta-\omega) + k_p(z-h) , \\
m_2 &= ik_p r\cos(\theta-\omega) + k_p z - h\sqrt{k_p^2 + iR_\ell k_p \cos\theta} , \\
m_3 &= ik_p r\cos(\theta-\omega) + z\sqrt{k_p^2 + iR_\ell k_p \cos\theta} - k_p h , \\
m_4 &= ik_p r\cos(\theta-\omega) + (z-h)\sqrt{k_p^2 + iR_\ell k_p \cos\theta} .
\end{aligned}\right\} \tag{3.92}$$

The functions $A_{js}, B_{js}, C_{js}, D_{js} (j = p, u, v, w)$ are evaluated at $k_p(\theta)$

$$A_{p_s}(k_p) \simeq \left(i\sigma_\ell^2 \sec\theta - \frac{4\sigma_\ell^3 \sec^4\theta}{R_\ell} \right) [G(\theta)] \quad ,$$

$$B_{p_s}(k_p) \simeq 2\sigma_\ell^2 \sqrt{\frac{\sigma_\ell}{2R_\ell}} \sec^{5/2}\theta (1+i)[-H_1(\theta)] - \frac{2i\sigma_\ell^3 \sec^5\theta}{R_\ell} C_z \quad , \tag{3.93}$$

$$\begin{bmatrix} A_{u_s} \\ A_{v_s} \\ A_{w_s} \end{bmatrix} \simeq \begin{bmatrix} 1 \\ \tan\theta \\ -i\sec\theta \end{bmatrix} \left(i\sigma_\ell^2 \sec\theta - \frac{4\sigma_\ell^3 \sec^4\theta}{R_\ell} \right) [G(\theta)] \quad , \tag{3.94a}$$

$$\begin{bmatrix} B_{u_s} \\ B_{v_s} \\ B_{w_s} \end{bmatrix} \simeq \begin{bmatrix} 1 \\ \tan\theta \\ -i\sec\theta \end{bmatrix} \left\{ 2\sigma_\ell^2 \sqrt{\frac{\sigma_\ell}{2R_\ell}} \sec^{5/2}\theta (1+i)[-H_1(\theta)] - \frac{2i\sigma_\ell^3 \sec^5\theta}{R_\ell} C_z \right\} \quad , \tag{3.94b}$$

$$\begin{pmatrix} C_{u_s} \\ C_{v_s} \end{pmatrix} \simeq \begin{pmatrix} 1 \\ \tan\theta \end{pmatrix} 2\sigma_\ell^2 \sqrt{\frac{\sigma_\ell}{2R_\ell}} \sec^{5/2}\theta (1+i)[-G(\theta)] \quad ,$$

$$C_{w_s} \simeq \frac{2i\sigma_\ell^3 \sec^5\theta}{R_\ell} [G(\theta)] \quad , \tag{3.95a}$$

$$\begin{pmatrix} D_{u_s} \\ D_{v_s} \end{pmatrix} \simeq \begin{pmatrix} 1 \\ \tan\theta \end{pmatrix} \frac{4\sigma_\ell^3 \sec^4\theta}{R_\ell} [H_1(\theta)] \quad ,$$

$$D_{w_s} \simeq 0 \quad , \tag{3.95b}$$

$$A_\zeta(k_p) = \left(i\sigma_\ell \sec\theta - \frac{2\sigma_\ell^2 \sec^4\theta}{R_\ell}\right)[-G(\theta)] \quad ,$$

$$B_\zeta(k_p) = 2\sigma_\ell \sqrt{\frac{\sigma_\ell}{2R_\ell}} \sec^{5/2}\theta(1+i)[H_1(\theta)] + \frac{2i\sigma_\ell^2 \sec^5\theta}{R_\ell} C_z \quad , \qquad (3.96)$$

where

$$G(\theta) = [-C_m + C_x + \tan\theta C_y + i\sec\theta C_z] \quad ,$$

$$H_1(\theta) = [C_x + \tan\theta C_y] \quad .$$

IV. ASYMPTOTIC RESULTS FOR THE FUNDAMENTAL SOLUTION

In its present form, the formal solution for $(\vec{q}_s^{(f)}, p_s^{(f)}; \zeta^{(f)})$ given in Eqs. (3.88) - (3.97) does not readily exhibit any physical features of the flow. Unfortunately, the θ-integrals involved in the complete inversion of the solution cannot be performed exactly analytically. However, some useful asymptotic results can be obtained. In this chapter the method of stationary phase is used to derive solutions of the fundamental problem valid in the far field flow regime $r \gg 1$.

Section 1 of this chapter deals in detail with the wave elevation $\zeta^{(f)}$, and Section 2 is a summary of results for the velocity components $\vec{q}_s^{(f)}$.

1. Wave Elevation $\zeta^{(f)}(r,\omega)$

The free wave system $\zeta^{(f)}$ in (3.91) is split into two parts $\zeta^{(f)} = \zeta_1^{(f)} + \zeta_2^{(f)}$, where

$$\zeta_1^{(f)} = \frac{\mathrm{Re}}{2\pi} \int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \, \frac{i}{D_1(k_p)} \, A_\zeta(k_p) e^{m_o(k_p)} \quad , \tag{4.1}$$

$$\zeta_2^{(f)} = \frac{\mathrm{Re}}{2\pi} \int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \, \frac{i}{D_1(k_p)} \, B_\zeta(k_p) e^{n_o(k_p)} \quad , \tag{4.2}$$

and $A_\zeta(k_p), B_\zeta(k_p)$, and $D_1^{-1}(k_p)$ are given in Eqs. (3.96), (3.97). This separation is made because there are two different oscillatory functions found in $e^{m_o(k_p)}$ and $e^{n_o(k_p)}$ respectively. Evaluating these exponentials at the pole k_p, we have

$$e^{m_o(k_p)} = \exp\left[-h\sigma_\ell \sec^2\theta - \frac{4\sigma_\ell^2 \sec^5\theta}{R_\ell} r\cos(\theta-\omega) - i\frac{4\sigma_\ell^2 \sec^5\theta}{R_\ell} h\right] \times e^{i(\sigma_\ell r)\psi_o(\theta,\omega)} , \quad (4.3)$$

$$e^{n_o(k_p)} = \exp\left[-h\sqrt{\frac{\sigma_\ell R_\ell}{2}} \sec\theta - \frac{4\sigma_\ell^2 \sec^5\theta}{R_\ell} r\cos(\theta-\omega)\right] \times e^{i\left[\sigma_\ell r\psi_o(\theta,\omega) - h\sqrt{\frac{\sigma_\ell R_\ell}{2}}\sec\theta\right]} ,$$

$$(4.4)$$

where

$$\psi_o(\theta,\omega) = \sec^2\theta\cos(\theta-\omega) \ . \quad (4.4a)$$

An asymptotic representation is obtained separately for each of the two parts of $\zeta^{(f)}$.

(a) <u>Wave Component $\zeta_1^{(f)}$.</u> We consider the first integral

$$\zeta_1^{(f)} = \frac{\text{Re}}{2\pi}\int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{i}{D_1(k_p)} A_\zeta(k_p)\exp\left[-h\sigma_\ell\sec^2\theta - \frac{4\sigma_\ell^2\sec^5\theta}{R_\ell} r\cos(\theta-\omega) - i\frac{4\sigma_\ell^2\sec^5\theta}{R_\ell} h\right] e^{i(\sigma_\ell r)\psi_o(\theta,\omega)} . \quad (4.5)$$

For $\sigma_\ell r$ large, the dominant oscillatory function is $e^{i(\sigma_\ell r)\psi_o(\theta,\omega)}$, and the stationary phase analysis in this case is based on the phase function $\psi_o(\theta,\omega)$. The other oscillatory part of the integrand is $\exp\left[-i\frac{4\sigma_\ell^2\sec^5\theta}{R_\ell} h\right]$, whose argument for most of the range of θ is very small because $R_\ell \gg 1$, and $h \ll 1$. It should be recalled that h is dimensionless with respect to some length ℓ, with the underline omitted.

Application of the method of stationary phase to an integral of the form of Eq. (4.5) is based on the idea that the main contribution to the value of the integral comes from a small interval centered about the 'critical point' along θ. The critical point occurs where the rate of change of $\psi_o(\theta, \omega)$ with respect to θ vanishes. The rapid oscillations of the function $e^{i\sigma_\ell r \psi_o(\theta,\omega)}$ away from the critical point tends to cancel out the contributions of the remaining values of θ. Points of stationary phase or critical points are determined by solving the equation $\frac{\partial \psi_o}{\partial \theta} = 0$. In the present case, the phase function $\psi_o(\theta, \omega)$ is a very familiar one in the classical theory of water waves. The details concerning its stationary points, the Taylor series expansions about those points, and the general form of the resulting asymptotic solutions are outlined in Appendix B.

It is evident that there are three ranges of the angle $\omega = \tan^{-1}\left(\frac{y}{x}\right)$ which must be studied separately: (1) $0 < \omega < \omega_c$, (2) ω near ω_c, and (3) $\omega_c < \omega \leq \pi$. The Kelvin angle is $\omega_c = \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) = 19°28'$.

(a.1) $0 \leq \omega < \omega_c$. When ω is strictly inside the Kelvin angle, there are two first order stationary phase points θ_1 and θ_2, where from Eq. (B.3)

$$\begin{pmatrix} \tan\theta_1 \\ \tan\theta_2 \end{pmatrix} = \frac{-1 \pm \sqrt{1-8\tan^2\omega}}{4\tan\omega} \tag{4.6}$$

The point θ_1 is a local minimum of ψ_o and has a range of values $\theta_2 < \theta_1 < 0$; θ_2 is a local maximum of ψ_o and has the range of values $-\frac{\pi}{2} < \theta_2 < \theta_c$. Expanding the phase function $\psi_o(\theta,\omega)$ about θ_1 and θ_2 as discussed in Appendix B, and the applying principle of stationary phase*, we obtain the asymptotic formula valid in $0 < \omega < \omega_c$,

$$\begin{aligned}
\zeta_1^{(f)}(r,\omega) \sim \frac{1}{\sqrt{2\pi}}\frac{\sigma_\ell}{\sqrt{\sigma_\ell r}} \sum_{j=1}^{2} \Bigg\{ & \frac{\sec^3\theta_j}{\sqrt{|\psi_{o_{\theta\theta}}(\theta_j)|}} \mathscr{F}(\theta_j) e^{-h\sigma_\ell \sec^2\theta_j} \\
\times \Bigg\langle & \left[(C_m - C_x - \tan\theta_j C_y) + C_z \frac{10\sigma_\ell \sec^4\theta_j}{R_\ell} \right] \cos\Big(\sigma_\ell r \psi_1(\theta_j) + \\
& \qquad + \frac{\pi}{4}\operatorname{sgn}(\psi_{o_{\theta\theta}}(\theta_j)) \Big) + \\
+ & \left[\sec\theta_j C_z - (C_m - C_x - \tan\theta_j C_y) \frac{10\sigma_\ell \sec^3\theta_j}{R_\ell} \right] \sin\Big(\sigma_\ell r \psi_1(\theta_j) + \\
& \qquad + \frac{\pi}{4}\operatorname{sgn}(\psi_{o_{\theta\theta}}(\theta_j)) \Big) \Bigg\rangle \Bigg\} + \\
+ O\left(\frac{1}{\sigma_\ell r}\right) + O\left(\frac{1}{\sqrt{\sigma_\ell r}\, R_\ell^{3/2}}\right) ,
\end{aligned} \tag{4.7}$$

where

$$\psi_1(\theta) = \left[\psi_o(\theta,\omega) - \frac{4\sigma_\ell h}{R_\ell r}\sec^5\theta \right], \qquad \psi_{o_{\theta\theta}}(\theta) = \psi_o(\theta,\omega)[1 - 2\tan^2\theta] \quad ,$$

$$\mathscr{F}(\theta_j) = \exp\left[-\frac{4\sigma_\ell^2 \sec^5\theta_j}{R_\ell} r\cos(\theta_j - \omega) \right] .$$

*For a concise description of the method applied to ship waves in potential flow, see for example Plesset and Wu (1960).

The influence of the oscillatory part $\exp\left[-i\,\frac{4\sigma_\ell^2 h}{R_\ell}\sec^5\theta\right]$ has now been included in the resulting wave-phase function of (4.7), but its contribution is evaluated at the stationary points of the $\psi_o(\theta,\omega)$ function alone. It will be seen that this approximation is a good one everywhere in $0 \leqslant \omega < \omega_c$ except for the diverging wave system near $\omega = 0$.

Two different wave systems are represented in the sum over θ_1 and θ_2 in Eq. (4.7). The terms arising from $\exp[i\sigma_\ell r\psi_o(\theta_1,\omega) + i\,\frac{\pi}{4}]$ are associated with the transverse wave system. The locus of constant phase lines of this system is determined approximately from

$$\sigma_\ell r\psi_o(\theta_1,\omega) - \frac{4\sigma_\ell^2 h}{R_\ell}\sec^5\theta_1 = \text{const} = \sigma_\ell C_t \quad , \tag{4.8}$$

where C_t is a constant. Equation (4.8) reduces to

$$x\cos\theta_1 + y\sin\theta_1 = C_t\cos^2\theta_1 + \frac{4\sigma_\ell h}{R_\ell}\sec^3\theta_1 \quad . \tag{4.9}$$

Using the relationship between ω and θ_1 determined from $\frac{\partial\psi_o}{\partial\theta} = 0$ (Eq. (B.2)), we obtain

$$\frac{y}{x} = \frac{-\tan\theta_1}{(1+2\tan^2\theta_1)} = \frac{-\sin\theta_1\cos\theta_1}{(1+\sin^2\theta_1)} \quad . \tag{4.10}$$

Equation (4.9) and (4.10) can be used to solve parametrically for x and y in terms of the angle θ_1. The loci of crests and troughs of the transverse wave system are

$$\left.\begin{aligned} x &= C_t \cos\theta_1 (1+\sin^2\theta_1) + \frac{4\sigma_\ell h}{R_\ell} \sec^4\theta_1 (1+\sin^2\theta_1) \quad , \\ y &= C_t(-\sin\theta_1)\cos^2\theta_1 + \frac{4\sigma_\ell h}{R_\ell} (-\sin\theta_1)\sec^3\theta_1 \quad , \end{aligned}\right\} \qquad (4.11)$$

for $\theta_c < \theta_1 \leqslant 0$. The effect of viscosity is essentially negligible for this case because $\frac{h}{R_\ell}$ is small. For example, on the x-axis $\omega \to 0$ and $\theta_1 \to 0$, so the crests and troughs are located at

$$x = C_t + \frac{4\sigma_\ell h}{R_\ell} \quad , \qquad (4.12)$$

in which the constant $\left(\frac{4\sigma_\ell h}{R_\ell}\right)$ is actually negligible to the order of accuracy maintained throughout this work.

In determining the constant C_t, we must deal separately with the sine and cosine parts of (4.7). For wave crests, and assuming $C_m > 0$, $C_x < 0$, $C_y > 0$, and $C_z > 0$, C_t changes its value by $\frac{2\pi}{\sigma_\ell}$ for each successive crest according to the relations

Cosine part: $$C_t = \frac{1}{\sigma_\ell}\left(2\pi k - \frac{\pi}{4}\right) \quad k = 1, 2, 3, \ldots , \qquad (4.13a)$$

Sine part: $$C_t = \frac{1}{\sigma_\ell}\left(2\pi k + \frac{\pi}{4}\right) \quad k = 0, 1, 2, \ldots . \qquad (4.13b)$$

Next, we deal with the terms arising from $\exp\left[i\sigma_\ell r\psi_o(\theta_2, \omega) - i\,\frac{\pi}{4}\right]$ which are associated with the diverging wave system. Following the same procedure used with the θ_1-terms, the loci of crests and troughs of the diverging wave system are determined parametrically to be

$$\left.\begin{aligned} x &= C_d \cos\theta_2 (1+\sin^2\theta_2) + \frac{4\sigma_\ell h}{R_\ell} \sec^2\theta_2 (1+\sin^2\theta_2) \\ y &= C_d(-\sin\theta_2)\cos^2\theta_2 + \frac{4\sigma_\ell h}{R_\ell} (-\sin\theta_2)\sec^3\theta_2 \quad , \end{aligned}\right\} \tag{4.14}$$

for $-\frac{\pi}{2} < \theta_2 < \theta_c$, where C_d is the constant for the diverging system analogous to C_t. For wave crests and using the same assumptions as above regarding the signs of C_m, C_x, C_y, and C_z the values of C_d are given by

$$\text{Cosine part:} \qquad C_d = \frac{1}{\sigma_\ell}\left(2\pi k + \frac{\pi}{4}\right) \qquad k = 0, 1, 2, \ldots , \tag{4.15a}$$

$$\text{Sine part:} \qquad C_d = \frac{1}{\sigma_\ell}\left(2\pi k + \frac{3\pi}{4}\right) \qquad K = 0, 1, 2, \ldots . \tag{4.15b}$$

The parametric equations (4.14) are valid for $0 < \omega < \omega_c$. They become meaningless for $\omega \to 0$ (along $y = 0$). The reason that these equations fail for $\omega \to 0$, $\theta_2 \to -\frac{\pi}{2}$ is that when computing stationary points, the error committed by ignoring $\exp\left[-i\,\frac{4\sigma_\ell^2 h}{R_\ell}\sec^5\theta\right]$ is no longer small as $\theta \to -\frac{\pi}{2}$. To determine the loci of constant phase for the diverging wave system, one would have to find the stationary phase points of $\psi_1(\theta,\omega)$ for the complete oscillatory function

$$e^{i\sigma_\ell r\left[\psi_o(\theta,\omega) - \frac{4\sigma_\ell h}{R_\ell r}\sec^5\theta\right]} = e^{i\sigma_\ell r \psi_1(\theta,\omega)} \quad . \tag{4.16}$$

This would be a fruitless exercise because the amplitude of the diverging system clearly becomes exponentially small near $\omega = 0$ due to the terms $\exp\left[-h\sigma_\ell \sec^2\theta_2 - \frac{4\sigma_\ell^2 \sec^5\theta_2}{R_\ell} r\cos(\theta_2 - \omega)\right]$. The region of validity of the parametric equations (4.14) is restricted to values of

$\left(\frac{\sigma_\ell h}{32R_\ell}\right)^{1/5} \lesssim \frac{y}{x} < \frac{1}{2\sqrt{2}}$. This is based on determining when $\frac{\sigma_\ell h}{R_\ell} \sec^5 \theta_2$ is of order unity, with $\sec\theta_2 \sim \frac{1}{2\omega}$ as $\omega \to 0$.

It is interesting to consider the stationary points of the total phase function $\psi_1(\theta, \omega)$ when $y = 0$. From (4.16), with $\omega = 0$,

$$\psi_1(\theta, 0) = \sec\theta - \frac{4\sigma_\ell h}{R_\ell x} \sec^5\theta \quad , \tag{4.17}$$

$$\frac{\partial\psi_1}{\partial\theta}(\theta, 0) = \tan\theta\sec\theta\left[1 - \frac{20\sigma_\ell h}{R_\ell x}\sec^4\theta\right] \quad , \tag{4.18}$$

$$\frac{\partial^2\psi_1}{\partial\theta}(\theta, 0) = \sec^3\theta\left[1 - \frac{20\sigma_\ell h}{R_\ell x}\sec^4\theta\right] + \tan^2\theta\sec\theta\left[1 - \frac{100\sigma_\ell h}{R_\ell x}\sec^4\theta\right] . \tag{4.19}$$

Stationary points occur at $\theta = 0$ and at $\theta = \sec^{-1}\left(\frac{R_\ell x}{20\sigma_\ell h}\right)^{\frac{1}{4}}$. In either case, there exist transverse waves for $x > \frac{20\sigma_\ell h}{R_\ell}$ and diverging waves for $x < \frac{20\sigma_\ell h}{R_\ell}$. Our interest in this work is for x large, certainly larger than $\frac{20\sigma_\ell h}{R_\ell}$, so we omit any further discussion of the details of the locus of constant wave-phase for the diverging wave system when x is small and $y < \left(\frac{\sigma_\ell h}{32R_\ell}\right)^{1/5} x$.

(a. 2) Extent of wave region.

It is expected that viscosity has little effect on the angle limiting the extent of the wave region, which in potential flow analysis is the Kelvin angle ω_c. We can estimate the effect by expanding the value of $\omega = \omega_*(r;h, R_\ell)$ near the boundary $\omega = \omega_c$, and $\theta = \theta_*$ near the value θ_c

$$\omega_*(r;h,R_\ell) = \omega_c + \frac{\omega_1(r,h)}{R_\ell} \quad , \tag{4.20}$$

$$\theta_*(r;h,R) = \theta_c + \frac{\theta_{c_1}(r,h)}{R_\ell} \quad . \tag{4.21}$$

The phase function $\psi_1(\theta,\omega)$ from (4.16) is written as

$$\psi_1(\theta,\omega) = \psi_o(\theta,\omega) + \frac{\varphi_o(\theta,r,h)}{R_\ell} \quad , \tag{4.22}$$

where

$$\varphi_o(\theta,r,h) = -4\sigma_\ell\left(\frac{h}{r}\right)\sec^5\theta \quad ,$$

and the condition of stationary phase is

$$\frac{\partial\psi_1(\theta,\omega)}{\partial\theta} = \frac{\partial\psi_o}{\partial\theta} + \frac{1}{R_\ell}\,\frac{\partial\varphi_o}{\partial\theta} = 0 \quad . \tag{4.23}$$

Substituting (4.20) and (4.21) into (4.23), using the required expansions, we obtain equations for ω_1 by equating like orders of R_ℓ^{-1}.

$$\frac{\partial\psi_o}{\partial\theta}(\theta_c,\omega_c) = 0 \quad , \tag{4.24}$$

$$\theta_{c_1}\frac{\partial^2\psi_o}{\partial\theta^2}(\theta_c,\omega_c) + \omega_1\frac{\partial^2\psi_o}{\partial\omega\partial\theta}(\theta_c,\omega_c) + \frac{\partial\varphi_o}{\partial\theta}(\theta_c,\omega_c) = 0 \quad . \tag{4.25}$$

Equation (4.24) is satisfied for $\theta_c = \tan^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, and $\omega_c = \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$. It happens that θ_c is also the double root of the stationary phase condition when $\omega = \omega_c$, so $\frac{\partial^2\psi_o}{\partial\theta^2}(\theta_c,\omega_c) = 0$. Solving (4.25) for ω_1, we obtain

$$\omega_1(r,h) = -\left[\frac{\frac{\partial\varphi_o}{\partial\theta}(\theta_c)}{\frac{\partial^2\psi}{\partial\omega\partial\theta}(\theta_c,\omega_c)}\right] \quad , \tag{4.26}$$

and evaluating this ratio as indicated, we find

$$\omega_1(r,h) = -\left[\frac{-20\sigma_\ell\left(\frac{h}{r}\right)\sec^3\theta_c\tan\theta_c}{\cos(\theta_c-\omega_c)(1+2\tan\theta_c\tan(\theta_c-\omega_c))}\right] = -\frac{15}{2}\sigma_\ell\left(\frac{h}{r}\right) \quad . \tag{4.27}$$

Thus, the boundary of the wave region for $y > 0$ is

$$\omega_*(r;h,R_\ell) = \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) - \frac{15}{2}\frac{\sigma_\ell}{R_\ell}\left(\frac{h}{r}\right) + O\left(\frac{1}{R_\ell^2}\right) \quad . \tag{4.28}$$

The influence of depth of submergence and viscosity is to narrow slightly the wedge of the wave region. However, the effect is essentially negligible and becomes even smaller as r increases.

(a.3) ω near ω_c.

At $\omega = \omega_c$, there is a double root $\theta_c(= \theta_1 = \theta_2)$ of the stationary phase equation $\frac{\partial\psi_o}{\partial\theta} = 0$. All along this cusp line, the asymptotic formula of Eq. (4.7) gives a solution with infinite amplitude. This is not an acceptable result, and the correct asymptotic solution requires a re-examination of the expansion of the phase function specifically near $\omega = \omega_c$ and $\theta = \theta_c$. This expansion is discussed in Appendix B, along with the general form of the resulting asymptotic formulae. Repeating the type of calculation described in detail by Plesset and Wu (1960), we find the asymptotic solution

$$\zeta_1^{(f)}(r,\omega) \sim \left(\frac{3}{2}\,\frac{\sigma_\ell}{(\sigma_\ell r)^{\frac{1}{3}}}\right) A_i(Z_\omega^\circ) e^{-\frac{3}{2}\sigma_\ell h - \frac{9\sigma_\ell^2 r}{R_\ell}\left[\cos\omega - \frac{\sqrt{2}}{2}\sin\omega\right]}$$

$$\times\left\{\left[\left(C_m - C_x + \frac{\sqrt{2}}{2}C_y\right) + \frac{45\sigma_\ell}{2R_\ell}C_z\right]\cos\left(\sigma_\ell r\,\frac{\sqrt{3}}{2}\left(1-\sqrt{2}\,\bar{\omega} + \frac{11}{2}\bar{\omega}^2\right) - 4\left(\frac{3}{2}\right)^{5/2}\frac{\sigma_\ell^2 h}{R_\ell}\right) + \right.$$

$$\left. + \left[\sqrt{\frac{3}{2}}\,C_z - 10\left(\frac{3}{2}\right)^{3/2}\frac{\sigma_\ell}{R_\ell}\left|C_m - C_x + \frac{\sqrt{2}}{2}C_y\right|\right]\sin\left(\sigma_\ell\sqrt{\frac{3}{2}}\left(1-\sqrt{2}\,\bar{\omega} + \frac{11}{2}\bar{\omega}^2\right) - 4\left(\frac{3}{2}\right)^{5/2}\frac{\sigma_\ell^2 h}{R_\ell}\right)\right\} +$$

$$+ O\left(\frac{1}{\sigma_\ell r}\right)^{\frac{2}{3}} + O\left(\frac{1}{(\sigma_\ell r)^{1/3} R_\ell^{3/2}}\right) , \qquad (4.29)$$

where $A_i(Z_\omega^\circ)$ is the Airy function, with

$$Z_\omega^\circ = \frac{3}{\sqrt{2}}\,(\sigma_\ell r)^{\frac{2}{3}}\,\bar{\omega}\,(1-2\sqrt{2}\,\bar{\omega}) \quad , \quad \bar{\omega} = \omega - \omega_c \quad .$$

This result could be rewritten in terms of transverse and diverging wave systems, using the identity

$$e^{i\Omega} = e^{i\left(\Omega + \frac{\pi}{3}\right)} + e^{i\left(\Omega - \frac{\pi}{3}\right)} , \qquad (4.30)$$

where, for example, $\cos\left(\Omega + \frac{\pi}{3}\right)$ would be associated with the cosine transverse wave and $\cos\left(\Omega - \frac{\pi}{3}\right)$ would represent the cosine diverging wave system. A more careful analysis of the entire range of $0 \leqslant \omega \leqslant \omega_c$ using the method of steepest descent* would show that the

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*see Ursell (1960)

constant angle in the wave-phase is $\pi/4$ on $y = 0$ and increases to $\pi/3$ at $\omega = \omega_c$.

(a.4) Viscous Decay Factor.

An interesting feature of the asymptotic solutions given in (4.7) and (4.29) is the exponential damping factor appearing in the amplitude of the wave formulae. This exponential factor is

$$\mathscr{F}(\theta_j) = e^{-\frac{4\sigma_\ell^2 \sec^5\theta_j}{R_\ell} r\cos(\theta_j - \omega)} , \qquad (4.31)$$

where $\tan\theta_j = \frac{-1 \mp \sqrt{1-8\tan^2\omega}}{4\tan\omega}$, in which $j = 2$ corresponds to the (−) and $j = 1$ corresponds to the (+). The factor $\mathscr{F}$ can be shown to be identical with Cumberbatch's (1965) result for the viscous damping factor obtained by slightly different techniques.

Allen (1968) presents Cumberbatch's result in the form

$$\mathscr{F} = e^{-\frac{r}{F_\ell^4 R_\ell} \cos\omega \, B_o} , \qquad (4.32)$$

where

$$B_o = \frac{4A_o^6}{(2A_o^2 - 1)} ,$$

$$A_o^2 = \frac{1}{2}\left[1 + \frac{1}{\tan\omega}\left(\frac{1}{4\tan\omega} \pm \frac{\sqrt{1-8\tan^2\omega}}{4\tan\omega}\right)\right] ,$$

$$F_\ell = \text{Froude number} = U/\sqrt{g\ell} .$$

Using the expression $\tan\theta_j = \frac{-1 \mp \sqrt{1-8\tan^2\omega}}{4\tan\omega}$, we find that

$$A_o^2 = \sec^2\theta_j . \qquad (4.33)$$

Then (4.32) yields the formula

$$\mathscr{F}(\theta_j) = e^{-\frac{4r}{F_\ell^4 R_\ell}\left(\cos\omega \sec^4\theta_j\right)\frac{(1+\tan^2\theta_j)}{(1+2\tan^2\theta_j)}} \quad . \tag{4.34}$$

It is a simple matter to show that Eqs. (4.31) and (4.34) are indeed identical. We have used the fact that $\sigma_\ell^2 = F_\ell^{-4}$. For further details concerning the influence of this viscous decay factor on the amplitudes of the transverse and diverging wave systems, discussion can be found in Cumberbatch (1965) and Allen (1968).

(a.5) $\omega_c < \omega \leqslant \pi$.

When ω is outside the cusp line $y = \frac{1}{2\sqrt{2}}\, x$, the asymptotic solution can be determined by integration by parts since there are no points where $\frac{\partial\psi_o}{\partial\theta} = 0$. Using the result outlined in Appendix B, we obtain

$$\begin{aligned}\zeta_1^{(f)}(r,\omega) \sim \frac{\csc\omega}{2\pi r}\, e^{-h\sigma_\ell \csc^2\omega} \Bigg\{ & \left[\csc\omega \cdot C_z - \frac{10\sigma_\ell \csc^3\omega}{R_\ell}\left(C_m - C_x - \cot\omega\, C_y\right)\right]\cos\left(\frac{4\sigma_\ell^2 h}{R_\ell}\csc^5\omega\right) + \\ & + \left[\left(C_m - C_x - \cot\omega\, C_y\right) + \frac{10\sigma_\ell \csc^4\omega}{R_\ell}\right] C_z \sin\left(\frac{4\sigma_\ell^2 h}{R_\ell}\csc^5\omega\right)\Bigg\} + \\ & + O\left(\frac{1}{\sigma_\ell r}\right)^2 + O\left(\frac{1}{\sigma_\ell r R_\ell^{3/2}}\right) \quad ,\end{aligned} \tag{4.35}$$

which has an amplitude at most $O\left(\frac{1}{r}\right)$.

<u>(b) Wave Component $\zeta_2^{(f)}$.</u>

We now turn to the integral of Eq. (4.2)

$$\zeta_2^{(f)} = \frac{\text{Re}}{2\pi} \int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \, \frac{i}{D_1(k_p)} \, B_\zeta(k_p)$$

$$\times \exp\left[-h\sqrt{\frac{\sigma_\ell R_\ell}{2}} \sec\theta - \frac{4\sigma_\ell^2 \sec^5\theta}{R_\ell} \, r\cos(\theta-\omega)\right] e^{i\sigma_\ell r \Phi(\theta,\omega;\hat{h})} , \tag{4.36}$$

where

$$\Phi(\theta,\omega;\hat{h}) = \psi_o(\theta,\omega) - \hat{h}\sqrt{\sec\theta} , \tag{4.37}$$

$$\hat{h} = \frac{h}{r}\sqrt{\frac{R_\ell}{2\sigma_\ell}} .$$

The stationary phase analysis in this case is based on the combined phase function $\Phi(\theta,\omega;\hat{h})$. Because of the exponential factor $\exp\left[-h\sqrt{\frac{\sigma_\ell R_\ell}{2}} \sec\theta\right]$, the interesting values of h in (4.36) are those for which $\hat{h} = \frac{h}{r}\sqrt{\frac{R_\ell}{2\sigma_\ell}}$ is less than unity. Appendix C contains an approximate determination of the stationary phase points, the expansions about those points, and the related asymptotic solutions based on the assumption of small $\hat{h}$. The ranges of ω that must be treated separately are the same as before.

(b.1) $0 \leqslant \omega < \omega_c$.

The two first order stationary points t_1 and t_2 for ω inside the Kelvin angle are given approximately in (C.5), (C.6) of Appendix C, accurate up to $O(\hat{h})$, as

$$\begin{pmatrix} \tan t_1 \\ \tan t_2 \end{pmatrix} = \begin{pmatrix} \tan\theta_1 \\ \tan\theta_2 \end{pmatrix} \left[1 + \hat{h} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right] . \tag{4.38}$$

where $\tan\theta_{1,2}$ are the stationary points of $\psi_o(\theta,\omega)$ and

$$\gamma_{1,2} = \frac{1}{2\cos\omega\sqrt{\sec\theta_{1,2}}\left[\pm\sqrt{1-8\tan^2\omega}\right]} \quad , \tag{4.38a}$$

with the subscript 1 corresponding to the (+) and 2 corresponds to the (-).

The asymptotic solution is

$$\zeta_2^{(f)}(r,\omega) \sim \frac{1}{\sqrt{\pi R_\ell}} \frac{\sigma_\ell^{3/2}}{\sqrt{\sigma_\ell r}} \sum_{j=1}^{2} \Bigg\{$$

$$\frac{\sec^{9/2} t_j}{\sqrt{|\Phi_{\theta\theta}(t_j)|}} \mathscr{F}(t_j) e^{-h\sqrt{\frac{\sigma_\ell R_\ell}{2}}\sec t_j}$$

$$\times\Bigg\langle \left[(C_x + \tan t_j C_y) + \sqrt{\frac{2\sigma_\ell}{R_\ell}} \sec^{5/2} t_j C_z \right] \cos\left(\sigma_\ell r\Phi(t_j) + \frac{\pi}{4}\,\mathrm{sgn}(\Phi_{\theta\theta}(t_j))\right) +$$

$$+ [C_x + \tan t_j C_y] \sin\left(\sigma_\ell r\Phi(t_j) + \frac{\pi}{4}\,\mathrm{sgn}(\Phi_{\theta\theta}(t_j))\right) \Bigg\rangle \Bigg\} +$$

$$+ O\left(\frac{1}{\sigma_\ell r\sqrt{R_\ell}}\right) \quad , \tag{4.39}$$

where

$$\Phi_{\theta\theta}(t_j) = \psi_o(t_j,\omega)[1 + 6\tan^2 t_j - 4\tan t_j \tan(t_j - \omega)] - \frac{\hat{h}}{2}\sqrt{\sec t_j}\left(1 + \frac{3}{2}\tan^2 t_j\right) ,$$

$$\Phi_{\theta\theta}(t_1) > 0 \quad , \quad \Phi_{\theta\theta}(t_2) < 0 \quad , \quad \mathscr{F}(t_j) \text{ from Eq. (4.31)} \quad .$$

Analogous to the case of $\zeta_1^{(f)}$, the transverse and diverging wave systems are represented in the sum over t_1 and t_2 in Eq. (4.39). The transverse wave system involves the terms arising from $\exp\left[i\sigma_\ell r\Phi(t_1,\omega;\hat{h}) + i\,\frac{\pi}{4}\right]$, and the lines of constant phase are determined from

$$\sigma_\ell r\psi_o(t_1,\omega) - h\sqrt{\frac{\sigma_\ell R_\ell}{2}}\;\sec t_1 = \text{const} = \sigma_\ell K_t \quad , \tag{4.40}$$

where K_t is a constant (different from C_t) and where t_1 is known approximately in terms of $\theta_1(\omega)$ from Eq. (4.37). An approximate parametric representation of the loci of crests and troughs can be obtained by solving (4.40) together with the formula (4.10) relating ω and θ_1. The result is

$$\left.\begin{aligned} x &= K_t\cos\theta_1(1+\sin^2\theta_1) + h\sqrt{\frac{R_\ell}{2\sigma_\ell}}\;(1+\sin^2\theta_1)\sqrt{\cos\theta_1}\left(1+\frac{\hat{h}}{2}\gamma_1\sin^2\theta_1\right) \\ y &= K_t(-\sin\theta_1)\cos^2\theta_1 + h\sqrt{\frac{R_\ell}{2\sigma_\ell}}\;(-\sin\theta_1)\cos^{3/2}\theta_1\left(1+\frac{\hat{h}}{2}\gamma_1\sin^2\theta_1\right) \quad , \end{aligned}\right\} \tag{4.41}$$

for $\theta_c < \theta_1 < 0$. The quantity K_t is a constant for any given crest or trough, and changes by $\frac{2\pi}{\sigma_\ell}$ from one constant-phase curve to the next.

Since $\theta_1 \to 0$ as $\omega \to 0$, we find from (4.41) that along the x-axis the transverse wave system of $\zeta_2^{(f)}(x,0)$ has crests or troughs located at

$$x = K_t + h\sqrt{\frac{R_\ell}{2\sigma_\ell}} \quad . \tag{4.42}$$

So, the location of the surface wave $\zeta_2^{(f)}$ along x is shifted downstream of the location that would have been observed in potential flow. The shift depends on the depth of submergence, the Reynolds number, and the Froude number; it is caused by the arrival of the wake at the free surface.

It should be noted however that along the x-axis, the distance between crests is still $\frac{2\pi}{g\ell} U^2$, equal to the wavelength of the waves in potential flow.

The diverging wave system arises from terms involving $\exp\left[i\sigma_\ell r\, \Phi(t_2, \omega; \hat{h}) - i\, \frac{\pi}{4}\right]$. Repeating the approximate solution for the parametric equations of the loci of crests and troughs, this time for the t_2-terms, we obtain

$$\left.\begin{aligned} x &= K_d \cos\theta_2 (1+\sin^2\theta_2) + h\sqrt{\frac{R_\ell}{2\sigma_\ell}}\,(1+\sin^2\theta_2)\sqrt{\cos\theta_2}\left(1+\frac{\hat{h}}{2}\gamma_2 \sin^2\theta_2\right), \\ y &= K_d(-\sin\theta_2)\cos^2\theta_2 + h\sqrt{\frac{R_\ell}{2\sigma_\ell}}\,(-\sin\theta_2)\cos^{3/2}\theta_2\left(1+\frac{\hat{h}}{2}\gamma_2\sin^2\theta_2\right), \end{aligned}\right\} \quad (4.43)$$

for $-\frac{\pi}{2} < \theta_2 < \theta_c$, and where K_d is a constant.

(b.2) Extent of wave region.

To determine the effect of viscosity on the extent of the wave region, we proceed as in Eqs. (4.20) - (4.23). In this case, the angles $\omega = \omega_*$ and $\theta = t_*$ are expanded near the boundary $\omega = \omega_c$ and $\theta = \theta_c$ in terms of small $\hat{h}$

$$\omega_*(r; h, R_\ell) = \omega_c + \hat{h}\,\omega_{c_1} \quad (4.44)$$

$$t_* = \theta_c + \hat{h}\,\theta_{c_1} \quad . \quad (4.45)$$

After substitution of these into $\frac{\partial\Phi}{\partial\theta} = 0$ and expanding as necessary, it can be shown that to $O(\hat{h})$

$$\omega_{c_1} = -\frac{1}{6}\left(\frac{2}{3}\right)^{\frac{1}{4}} \quad . \quad (4.46)$$

Hence, the boundary of the wave region for $y > 0$ is

$$\omega_*(r;h,R_\ell) = \omega_c - \frac{\hat{h}}{6}\left(\frac{2}{3}\right)^{\frac{1}{4}} + O(\hat{h}^2) \quad , \tag{4.47}$$

where

$$\hat{h} = \frac{h}{r}\sqrt{\frac{R_\ell}{2\sigma_\ell}} \quad .$$

The wedge of the wave region is thus narrowed by the influence of viscosity and depth of submergence. However, (4.47) is valid only far downstream (r large), and the effect diminishes rapidly as r increases. The near field influence of viscosity on ω_* would require special study and is not discussed here.

(b.3) ω near ω_c.

When $\omega = \omega_c$, the asymptotic result given in (4.39) breaks down. The expansions for $\tan t_{1,2} = \tan\theta_{1,2}[1 + \hat{h}\gamma_{1,2}]$ fail because $\gamma_{1,2}$ becomes infinite along the cusp line. Using the correct approximate expansion for $\Phi(\theta,\omega;\hat{h})$ developed in Appendix C, the stationary phase method for ω near ω_c yields the asymptotic result

$$\begin{aligned}\zeta_2^{(f)} \sim \left(\frac{3}{2}\right)^{7/4} & \sqrt{\frac{2\sigma_\ell}{R_\ell}}\ \frac{\sigma_\ell}{(\sigma_\ell r)^{1/3}}\ A_i(Z_\omega)\mathscr{F}(t_c)e^{-h\sqrt{\frac{\sigma_\ell R_\ell}{2}}\ \sec t_c} \\ & \times\left\{\left[(C_x + \tan t_c C_y) + \sqrt{\frac{2\sigma_\ell}{R_\ell}}\ \sec^{5/2} t_c C_z\right]\cos(\Omega_c r)\right. \\ & \left. \qquad + [C_x + \tan t_c C_y]\sin(\Omega_c r)\right\} + \\ & + O\left(\frac{1}{(\sigma_\ell r)^{2/3}\sqrt{R_\ell}}\right) + O\left(\frac{\hat{h}}{(\sigma_\ell r)^{1/3}\sqrt{R_\ell}}\right) \quad ,\end{aligned} \tag{4.48}$$

where

$A_i(Z_\omega)$ is the Airy function, $\mathscr{F}(t_c)$ is given in Eq. (4.31),

$$Z_\omega = \frac{3}{\sqrt{2}}(\sigma_\ell r)^{\frac{2}{3}}\left\{\bar{\omega}(1-2\sqrt{2}\,\bar{\omega}) + \frac{\hat{h}}{6}\left(\frac{2}{3}\right)^{\frac{1}{4}}\left[1 - \frac{85\sqrt{2}}{48}\,\bar{\omega} + \frac{43\sqrt{2}}{3}\,\bar{\omega}^2\right]\right\} ,$$

$$\Omega_c = \sigma_\ell\,\frac{\sqrt{3}}{2}\left\{\left(1-\sqrt{2}\,\bar{\omega} + \frac{11}{2}\,\bar{\omega}^2\right) - \hat{h}\sqrt{2}\left(\frac{2}{3}\right)^{\frac{1}{4}}\left[1 - \frac{15\sqrt{2}}{16}\,\bar{\omega} + \frac{57}{16}\,\bar{\omega}^2\right]\right\} ,$$

$$\bar{\omega} = \omega - \omega_c ,$$

$$\tan t_c = -\frac{\sqrt{2}}{2}\left[1 - \hat{h}\,\frac{7\sqrt{2}}{16}\left(\frac{2}{3}\right)^{\frac{1}{4}}\right] ,$$

$$\sec t_c = \sqrt{\frac{3}{2}}\left[1 - \hat{h}\,\frac{7\sqrt{2}}{48}\left(\frac{2}{3}\right)^{\frac{1}{4}}\right] .$$

Using the identity (4.30), this wave solution could also be written in a form displaying transverse and diverging wave systems.

(b.4) $\omega_c < \omega \leqslant \pi$.

For ω outside the cusp line $\omega = \omega_c$, the asymptotic formula for $\zeta_2^{(f)}$ would exhibit the same general character as (4.35), having an amplitude in this case no large than $O\left(\frac{1}{\sqrt{R_\ell}\, r}\right)$. This is very small, and the result is omitted here.

2. Perturbation Velocity $\vec{q}_s^{(f)}$

It is useful to deal with the $\vec{q}_s^{(f)}$ velocity in two parts: the longitudinal component $\vec{q}_{s_L}^{(f)}$ and the transverse component $\vec{q}_{s_T}^{(f)}$. The formulas for the free disturbance quantities $\vec{q}_{s_L}^{(f)}$ and $\vec{q}_{s_T}^{(f)}$ of (3.75) and (3.76) have an obvious resemblance to the formula for $\zeta_1^{(f)}$ in (3.77), whose asymptotic results are discussed in Section 1 of this chapter. Therefore the form of the asymptotic solutions given previously can be carried over directly to the solutions for the velocity components. What follows here is a summary of final results.

Evaluating the exponentials $e^{m_1(k_p)}$, $e^{m_2(k_p)}$, $e^{m_3(k_p)}$, and $e^{m_4(k_p)}$ at the pole k_p, we obtain

$$e^{m_1(k_p)} = \exp\left[(z-h)\sigma_\ell \sec^2\theta - \frac{4\sigma_\ell^2 \sec^5\theta}{R_\ell} r\cos(\theta-\omega) + i\frac{4\sigma_\ell^2\sec^5\theta}{R_\ell}(z-h)\right] e^{i\sigma_\ell r\psi_o(\theta,\omega)} , \quad (4.49)$$

$$e^{m_2(k_p)} = \exp\left[z\sigma_\ell \sec^2\theta - h\sqrt{\frac{\sigma_\ell R_\ell}{2}}\sec\theta - \frac{4\sigma_\ell^2 \sec^5\theta}{R_\ell} r\cos(\theta-\omega) + i\frac{4\sigma_\ell^2\sec^5\theta}{R_\ell} z\right] e^{i\sigma_\ell r[\psi_o(\theta,\omega)-\hat{h}\sqrt{\sec\theta}]} , \quad (4.50)$$

$$e^{m_3(k_p)} = \exp\left[z\sqrt{\frac{\sigma_\ell R_\ell}{2}}\sec\theta - h\sigma_\ell\sec^2\theta - \frac{4\sigma_\ell^2 \sec^5\theta}{R_\ell} r\cos(\theta-\omega) - i\frac{4\sigma_\ell^2\sec^5\theta}{R_\ell} h\right] e^{i\sigma_\ell r[\psi_o(\theta,\omega)+\hat{z}\sqrt{\sec\theta}]} , \quad (4.51)$$

$$e^{m_4(k_p)} = \exp\left[(z-h)\sqrt{\frac{\sigma_\ell R_\ell}{2}}\sec\theta - \frac{4\sigma_\ell^2 \sec^5\theta}{R_\ell} r\cos(\theta-\omega)\right] \times e^{i\sigma_\ell r[\psi_o(\theta,\omega)+(\hat{z}-\hat{h})\sqrt{\sec\theta}]} , \quad (4.52)$$

where

$$\hat{z} = \frac{z}{r}\sqrt{\frac{R_\ell}{2\sigma_\ell}} , \qquad \hat{h} = \frac{h}{r}\sqrt{\frac{R_\ell}{2\sigma_\ell}} .$$

All the results discussed in Appendices B and C are applicable to the stationary phase analysis of the phase functions appearing in Eqs. (4.49) - (4.52). We restrict the range of z such that

$|\hat{z}| = \left| \frac{z}{r}\sqrt{\frac{R_\ell}{2\sigma_\ell}} \right| < 1$, analogous to the similar restriction on $\hat{h}$.

(a) $0 \leqslant \omega < \omega_c$. For ω taken strictly inside the Kelvin angle, we obtain the asymptotic solutions valid for $z \leqslant 0$ as sums over the stationary phase points. First, the longitudinal components are

$$\begin{bmatrix} u_{s_L}^{(f)} \\ v_{s_L}^{(f)} \\ w_{s_L}^{(f)} \end{bmatrix} \sim \frac{-1}{\sqrt{2\pi}} \frac{\sigma_\ell^2}{\sqrt{\sigma_\ell r}} \sum_{j=1}^{2} \left\{ \frac{\sec^3\theta_j}{\sqrt{|\psi_{o_{\theta\theta}}(\theta_j)|}} \mathscr{F}(\theta_j) e^{(z-h)\sigma_\ell \sec^2\theta_j} \right.$$

$$\left\langle \begin{bmatrix} K_{u_o}(\theta_j) \\ K_{v_o}(\theta_j) \\ K_{w_o}(\theta_j) \end{bmatrix} \cos\left(\sigma_\ell r \Omega_o(\theta_j) + \frac{\pi}{4}\,\mathrm{sgn}(\psi_{o_{\theta\theta}}(\theta_j))\right) + \right.$$

$$\left. \left. + \begin{bmatrix} S_{u_o}(\theta_j) \\ S_{v_o}(\theta_j) \\ S_{w_o}(\theta_j) \end{bmatrix} \sin\left(\sigma_\ell r \Omega_o(\theta_j) + \frac{\pi}{4}\,\mathrm{sgn}(\psi_{o_{\theta\theta}}(\theta_j))\right) \right\rangle \right\} +$$

$$-\sqrt{\frac{\sigma_\ell}{\pi R_\ell}} \frac{\sigma_\ell^2}{\sqrt{\sigma_\ell r}} \sum_{j=1}^{2} \left\{ \frac{\sec^{9/2} t_j}{\sqrt{|\Phi_{\theta\theta}(t_j)|}} \mathscr{F}(t_j) e^{z\sigma_\ell \sec^2 t_j - h\sqrt{\frac{\sigma_\ell R_\ell}{2}}\sec t_j} \right.$$

$$\left\langle \begin{bmatrix} K_{u_1}(t_j) \\ K_{v_1}(t_j) \\ K_{w_1}(t_j) \end{bmatrix} \cos\left(\sigma_\ell r \Omega_1(t_j) + \frac{\pi}{4}\,\mathrm{sgn}(\Phi_{\theta\theta}(t_j))\right) + \right.$$

(continued)

$$+\begin{bmatrix} S_{u_1}(t_j) \\ S_{v_1}(t_j) \\ S_{w_1}(t_j) \end{bmatrix} \sin\left[\sigma_\ell r\Omega_1(t_j)+\frac{\pi}{4}\,\mathrm{sgn}(\Phi_{\theta\theta}(t_j)\,)\right]\Bigg\rangle\Bigg\}, \tag{4.53}$$

where

$$K_{u_o}(\theta_j) = [\,C_m - C_x - \tan\theta_j C_y] + (12\sigma_\ell \sec^4\theta_j / R_\ell)\,[\,C_z] \quad ,$$

$$K_{v_o}(\theta_j) = [\,\tan\theta_j] K_{u_o}(\theta_j) \quad , \tag{4.54}$$

$$K_{w_o}(\theta_j) = [\,-\sec\theta_j] S_{u_o}(\theta_j) \quad ,$$

$$S_{u_o}(\theta_j) = \sec\theta_j C_z - (12\sec^3\theta_j / R_\ell)\,[\,C_m - C_x - \tan\theta_j C_y]$$

$$S_{v_o}(\theta_j) = [\,\tan\theta_j] S_{u_o}(\theta_j) \quad , \tag{4.55}$$

$$S_{w_o}(\theta_j) = [\,\sec\theta_j] K_{u_o}(\theta_j) \quad ,$$

$$K_{u_1}(t_j) = [\,C_x + \tan t_j C_y] + (2\sigma_\ell / R_\ell)^{1/2} \sec^{5/2} t_j [\,C_z] \quad ,$$

$$K_{v_1}(t_j) = [\,\tan t_j] K_{u_1}(t_j) \quad , \tag{4.56}$$

$$K_{w_1}(t_j) = [\,-\sec t_j] S_{u_1}(t_j) \quad ,$$

$$S_{u_1}(t_j) = [\,C_x + \tan t_j C_y] \quad ,$$

$$S_{v_1}(t_j) = [\,\tan t_j] S_{u_1}(t_j) \quad , \tag{4.57}$$

$$S_{w_1}(t_j) = [\,\sec t_j] K_{u_1}(t_j) \quad ,$$

and

$$\Omega_o(\theta) = \left[\psi_o(\theta,\omega) + \frac{4\sigma_\ell \sec^5\theta (z-h)}{R_\ell\; r}\right] \quad , \tag{4.58}$$

$$\Omega_1(\theta) = \left[\psi_o(\theta,\omega) + \frac{4\sigma_\ell \sec^5\theta}{R_\ell}\left(\frac{z}{r}\right) - \hat{h}\sqrt{\sec\theta}\,\right] \quad . \tag{4.59}$$

The functions $\tan\theta_{1,2}$, $\tan t_{1,2}$, $\psi_o(\theta)$, and $\Phi(\theta)$ are given in Eqs. (4.6), (4.38), (4.40), and (4.37) respectively. Then, the solenoidal components are

$$\begin{bmatrix} u_{s_T}^{(f)} \\ v_{s_T}^{(f)} \\ w_{s_T}^{(f)} \end{bmatrix} \sim \sqrt{\frac{\sigma_\ell}{\pi R_\ell}}\,\frac{\sigma_\ell^2}{\sqrt{\sigma_\ell r}} \sum_{j=1}^{2} \left\{ \frac{\sec^{9/2}\tau_j}{\sqrt{|\Phi_{2_{\theta\theta}}(\tau_j)|}}\,\mathscr{F}(\tau_j)\, e^{-h\sigma_\ell \sec^2\tau_j + z\sqrt{\frac{\sigma_\ell R_\ell}{2}}\sec\tau_j} \right.$$

$$\left\langle \begin{bmatrix} K_{u_2}(\tau_j) \\ K_{v_2}(\tau_j) \\ K_{w_2}(\tau_j) \end{bmatrix} \cos\left(\sigma_\ell r \Omega_2(\tau_j) + \frac{\pi}{4}\,\mathrm{sgn}(\Phi_{2_{\theta\theta}}(\tau_j))\right) \quad + \right.$$

$$\left. \left. + \begin{bmatrix} S_{u_2}(\tau_j) \\ S_{v_2}(\tau_j) \\ S_{w_2}(\tau_j) \end{bmatrix} \sin\left(\sigma_\ell r \Omega_2(\tau_j) + \frac{\pi}{4}\,\mathrm{sgn}(\Phi_{2_{\theta\theta}}(\tau_j))\right) \right\rangle \right\} +$$

$$+ \frac{2\sigma_\ell^3}{R_\ell}\sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{\sigma_\ell r}} \sum_{j=1}^{2} \left\{ \frac{\sec^6 T_j}{\sqrt{|\Phi_{3_{\theta\theta}}(T_j)|}}\,\mathscr{F}(T_j)\, e^{(z-h)\sqrt{\frac{\sigma_\ell R_\ell}{2}}\sec T_j} \right.$$

$$\left. \left\langle \begin{bmatrix} S_{u_3}(T_j) \\ S_{v_3}(T_j) \\ S_{w_3}(T_j) \end{bmatrix} \sin\left(\sigma_\ell r \Omega_3(T_j) + \frac{\pi}{4}\,\mathrm{sgn}(\Phi_3(T_j))\right) \right\rangle \right\} \quad , \tag{4.60}$$

where

$$K_{u_2}(\tau_j) = [\,C_m - C_x - \tan\tau_j C_y - \sec\tau_j C_z\,] \quad ,$$

$$K_{v_2}(\tau_j) = [\,\tan\tau_j\,] K_{u_2}(\tau_j) \quad , \tag{4.61}$$

$$K_{w_2}(\tau_j) = -\,(2\sigma_\ell / R_\ell)^{1/2} \sec^{5/2}\tau_j [\,C_m - C_x - \tan\tau_j C_y\,] \quad ,$$

$$S_{u_2}(\tau_j) = [\,C_m - C_x - \tan\tau_j C_y + \sec\tau_j C_z\,] \quad ,$$

$$S_{v_2}(\tau_j) = [\,\tan\tau_j\,] S_{u_2}(\tau_j) \quad , \tag{4.62}$$

$$S_{w_2}(\tau_j) = -\,(2\sigma_\ell / R_\ell)^{1/2} \sec^{7/2}\tau_j [\,C_z\,] \quad ,$$

$$S_{u_3}(T_j) = [\,C_x + \tan T_j C_y\,] \quad ,$$

$$S_{v_3}(T_j) = [\,\tan T_j\,] S_{u_3} \tag{4.63}$$

$$S_{w_3}(T_j) = 0 \quad ,$$

and

$$\Phi_2(\theta) = [\,\psi_o(\theta,\omega) + \hat{z}\sqrt{\sec\theta}\,] \quad , \tag{4.64}$$

$$\Phi_3(\theta) = [\,\psi_o(\theta,\omega) + (\hat{z}-\hat{h})\sqrt{\sec\theta}\,] \quad , \tag{4.65}$$

$$\Omega_2(\theta) = \left[\psi_o(\theta,\omega) - \frac{4\sigma_\ell \sec^5\theta}{R_\ell}\left(\frac{h}{r}\right) + \hat{z}\sqrt{\sec\theta}\,\right] \quad , \tag{4.66}$$

$$\Omega_3(\theta) = \Phi_3(\theta) \quad , \tag{4.67}$$

$$\tan\tau_{1,2} = \tan\theta_{1,2}\,[\,1 - \hat{z}\gamma_{1,2}\,] \quad , \tag{4.68}$$

$$\tan T_{1,2} = \tan\theta_{1,2}\,[\,1 + (\hat{h}-\hat{z})\gamma_{1,2}\,] \quad . \tag{4.69}$$

<u>(b) ω near ω_c</u>. In the neighborhood of $\omega = \omega_c$, we are

interested only in the longitudinal components. Using results analogous to Eqs. (4.29) and (4.48), we obtain for $z \leqslant 0$,

$$\begin{bmatrix} u^{(f)}_{s_L} \\ v^{(f)}_{s_L} \\ w^{(f)}_{s_L} \end{bmatrix} \sim -\left(\frac{3}{2}\right)\frac{\sigma_\ell^2}{(\sigma_\ell r)^{\frac{1}{3}}} A_i(Z^o_\omega) e^{\frac{3}{2}(z-h)\sigma_\ell - \frac{9\sigma_\ell^2 r}{R_\ell}\left(\cos\omega - \frac{\sqrt{2}}{2}\sin\omega\right)} \Bigg\{$$

$$\left\langle \begin{bmatrix} U_{c_1} \\ V_{c_1} \\ W_{c_1} \end{bmatrix} \cos\left(\sigma_\ell r \frac{\sqrt{3}}{2}\left(1-\sqrt{2}\,\bar{\omega}+\frac{11}{2}\bar{\omega}^2\right)+4\left(\frac{3}{2}\right)^{5/2}\frac{\sigma_\ell^2(z-h)}{R_\ell}\right) + \right.$$

$$\left. + \begin{bmatrix} U_{s_1} \\ V_{s_1} \\ W_{s_1} \end{bmatrix} \sin\left(\sigma_\ell r \frac{\sqrt{3}}{2}\left(1-\sqrt{2}\,\bar{\omega}+\frac{11}{2}\bar{\omega}^2\right)+4\left(\frac{3}{2}\right)^{5/2}\frac{\sigma_\ell^2(z-h)}{R_\ell}\right)\right\rangle\Bigg\} +$$

$$-\left(\frac{3}{2}\right)^{7/4}\sqrt{\frac{2\sigma_\ell}{R_\ell}}\;\frac{\sigma_\ell^2}{(\sigma_\ell r)^{\frac{1}{3}}} A_i(Z_\omega)\mathscr{F}(t_c) e^{z\sigma_\ell \sec^2 t_c - h\sqrt{\frac{\sigma_\ell R_\ell}{2}}\sec t_c} \Bigg\{$$

$$\left\langle \begin{bmatrix} U_{c_2} \\ V_{c_2} \\ W_{c_2} \end{bmatrix} \cos\left(\Omega_c r + \frac{4\sigma_\ell^2 \sec^5 t_c}{R_\ell} z\right) + \right.$$

(continued)

$$+\begin{bmatrix} U_{s_2} \\ V_{s_2} \\ W_{s_2} \end{bmatrix} \sin\left(\Omega_c r + \frac{4\sigma_\ell^2 \sec^5 t_c}{R_\ell} z\right)\Bigg\rangle\Bigg\} , \qquad (4.70)$$

where

$$U_{c_1} = \left[C_m - C_x + \frac{\sqrt{2}}{2} C_y\right] + (27\sigma_\ell / R_\ell)[C_z] ,$$
$$V_{c_1} = [-\sqrt{2}/2] U_{c_1} , \qquad (4.71)$$
$$W_{c_1} = [-\sqrt{3/2}] U_{s_1}$$

$$U_{s_1} = \sqrt{3/2}[C_z] - (12(3/2)^{3/2}\sigma_\ell / R_\ell)\left[C_m - C_x + \frac{\sqrt{2}}{2} C_y\right]$$
$$V_{s_1} = [-\sqrt{2}/2] U_{s_1} , \qquad (4.72)$$
$$W_{s_1} = [\sqrt{3/2}] U_{c_1} ,$$

$$U_{c_2} = [C_x + \tan t_c C_y] + (2\sigma_\ell / R_\ell)^{1/2} \sec^{5/2} t_c [C_z] ,$$
$$V_{c_2} = [\tan t_c] U_{c_2} , \qquad (4.73)$$
$$W_{c_2} = [-\sec t_c] U_{s_2} ,$$

$$U_{s_2} = [C_x + \tan t_c C_y] ,$$
$$V_{s_2} = [\tan t_c] U_{s_2} , \qquad (4.74)$$
$$W_{s_2} = [\sec t_c] U_{c_2} ,$$

and where the notations associated with (4.29) and (4.48) apply here, and are not repeated.

(c) $\underline{\omega_c < \omega \leqslant \pi}$. With no stationary points in the range $\omega_c < \omega \leqslant \pi$ for any of the phase functions of Eqs. (4.49) - (4.52), the integrals of (3.75) and (3.76) can be performed asymptotically by integration by parts. The results would all have amplitudes at most $O\left(\frac{1}{r}\right)$ or $O\left(\frac{1}{\sqrt{R_\ell}\, r}\right)$, and would resemble the solution (4.35). These results are omitted here.

V. MOMENTUM THEOREM

The principles of conservation of mass and conservation of momentum are used to obtain a general formula for the forces acting on a body moving on the free surface of a viscous fluid. For this calculation we require knowledge of certain flow quantities measured at control surfaces far away from the body. This is consistent with the approximations already introduced into the problem. In particular, the Oseen linearization is fully justifiable in the far field for $UL/\nu \gg 1$, where L is the characteristic length of the body.

1. Conservation of Momentum

Consider a uniform stream of velocity U moving past a ship in the free surface of a viscous fluid. We take the coordinates as shown in Fig. (5.1). A control surface S fixed in space with respect to the x, y, z coordinate system encloses a volume $\forall$ of the viscous incompressible fluid. In this view, the fluid flows into the volume through S_1, is disturbed by the ship and then flows out of S carrying the superposed perturbation velocities. The surface S consists of all the boundaries of the fluid within $\forall$. It includes the wetted surface of the body S_B and the fluid free surface S_{FS}.

Conservation of momentum is a statement of Newton's law for the fluid system. It is convenient to use the tensor component form of the momentum equation (a repeated subscript indicates summation over $\alpha = 1,2,3$). In the case of steady flow, we have

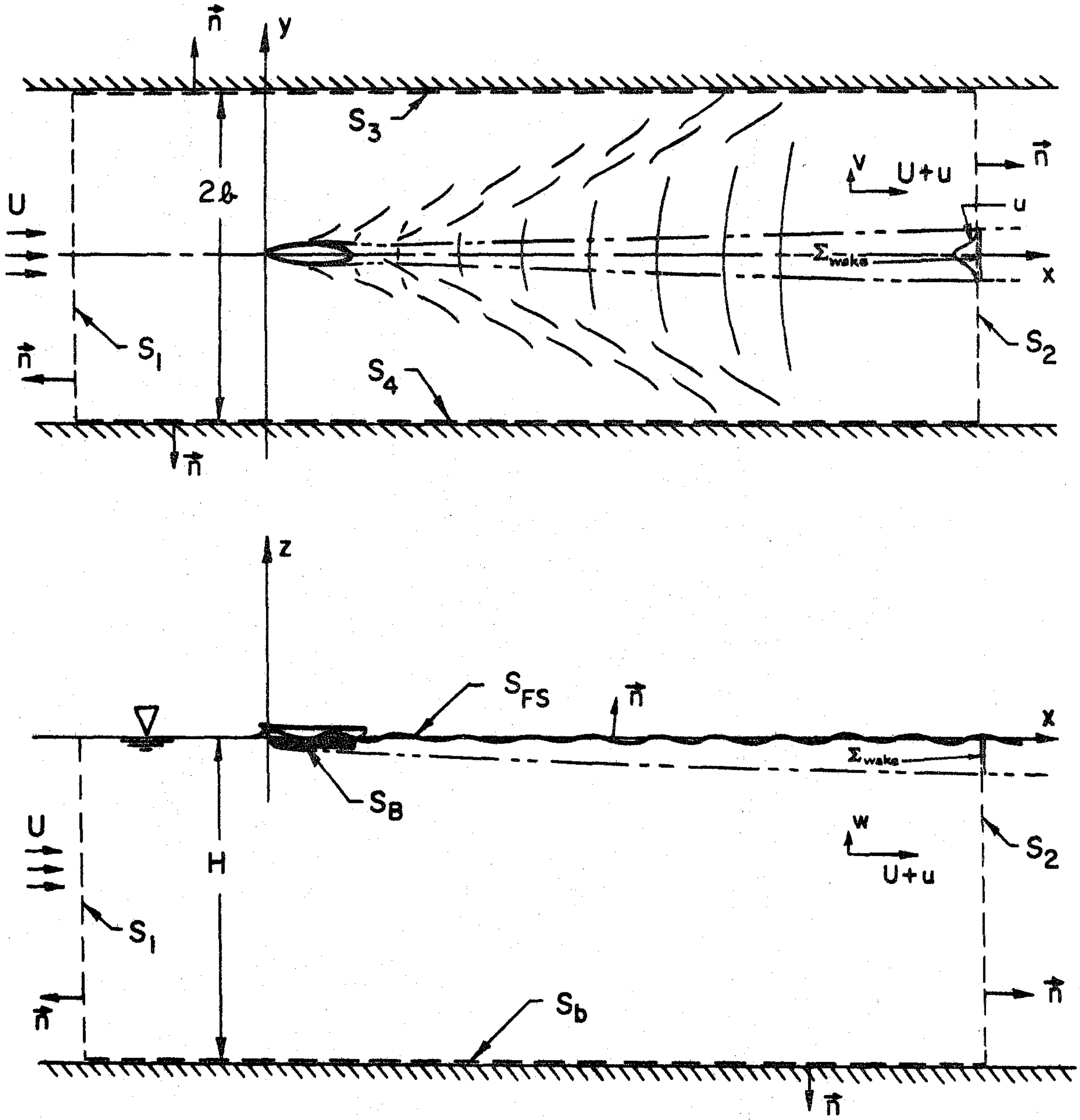


Fig. (5.1) Definition sketch for the control volume ∀ contained by S. For the derivation of the final resistance formula (Eqs. (5.20) and (5.21)), the surfaces S_3, S_4, and S_b are withdrawn to infinity.

$$0 = -\int_S \rho V_i(V_\alpha n_\alpha)dS + \int_S [-P\delta_{i\alpha}+\tau_{i\alpha}]n_\alpha dS + \int_{\forall} \rho F_i^{(b)} d\forall \quad , \quad (5.1)$$

where V_i is the i-th component of the total velocity $(U+u, v, w)$; P is the total pressure; n_α is a component of $\vec{n}$, the outward normal to S; $\tau_{i\alpha}$ is the viscous stress tensor component; $F_i^{(b)}$ is the gravity body force per unit mass of fluid; and $dS, d\forall$ are elemental surface area and volume, respectively.

$$\tau_{i\alpha} = \mu\left(\frac{\partial V_i}{\partial x_\alpha} + \frac{\partial V_\alpha}{\partial x_i}\right) \quad ,$$

$$\int_{\forall} \rho F_i^{(b)} d\forall = -\int_{\forall} \rho g \frac{\partial x_3}{\partial x_i} d\forall \quad , \quad (5.2)$$

$$(x_1, x_2, x_3) \longleftrightarrow (x, y, z)$$

$$S = S_1 + S_2 + S_3 + S_4 + S_b + S_{FS} + S_B \quad .$$

The force exerted on the body by the fluid in the direction $\hat{e}_i$ is denoted by F_i, and can be computed by integrating the pressure and shear stress over the body surface S_B.

$$F_i = \int_{S_B} \left[P\delta_{i\alpha} - \mu\left(\frac{\partial V_i}{\partial x_\alpha} + \frac{\partial V_\alpha}{\partial x_i}\right)\right] n_\alpha dS \quad , \quad (5.3)$$

where n_α is normal to S_B and points into the body. Upon substituting (5.3) into (5.2),

$$F_i = \int_{\forall} \left(-\rho g \frac{\partial x_3}{\partial x_i}\right) d\forall - \int_S \rho V_i(V_\alpha n_\alpha)dS - \int_{\Sigma} \left[P n_i - \mu\left(\frac{\partial V_i}{\partial x_\alpha} + \frac{\partial V_\alpha}{\partial x_i}\right) n_\alpha\right] dS \quad , \quad (5.4)$$

where

$$\Sigma = S - S_B = S_1 + S_2 + S_3 + S_4 + S_{FS} + S_b \quad .$$

2. Conditions on the Boundary Surfaces of S

Let us suppose that the boundary surfaces S_3, S_4, and S_b coincide, respectively, with the two sides and bottom of a towing tank. Later it is shown that if these surfaces are withdrawn to infinity, the resulting resistance formula is the same. For the present, we consider the S_3, S_4, and S_b to be material surfaces located at a distance sufficiently far away from the body. The physical boundary conditions used to simplify (5.4) are outlined below.

(a) On the material surfaces S_3, S_4, and S_b. The no-slip condition must be applied on all material surfaces

$$V_i = 0 \qquad \text{on} \qquad S_3, S_4, S_b \quad . \tag{5.5}$$

This means that the second integral of (5.4) is zero for these portions of S. Next, we anticipate calculating the total resistance on the body, $F_1 = R$, with $i = 1$ in (5.4). We note that the outward normals on S_3, S_4, and S_b are $\hat{e}_2$, $-\hat{e}_2$, and $\hat{e}_3$ respectively. For these surfaces the pressure terms in the third integral on the right in (5.4) are all zero (for $i = 1$)

$$\int_{S_3+S_4+S_b} (P\delta_{1\alpha}) n_\alpha dS = 0 \quad , \tag{5.6}$$

since $n_1 = 0$ on these surfaces.

(b) The no-slip condition is also applied on the body surface,

S_B. This eliminates the contribution from S_B from the second integral in (5.4).

(c) On the free surface S_{FS}. There are two conditions to be satisfied on the free surface, repeated here from Chapter II. The kinematic condition specifies that the fluid particles in the S_{FS} have velocities tangent to the free surface shape

$$V_\alpha n_\alpha = 0 \qquad \text{on} \qquad x_3 = \zeta(x_1, x_2) \quad . \tag{5.7}$$

The dynamic boundary condition states that, in the absence of surface tension, the stress is continuous across S_{FS}, so that with zero pressure above the S_{FS}, and for $i = 1, 2, 3$,

$$Pn_i - \mu\left(\frac{\partial V_i}{\partial x_\alpha} + \frac{\partial V_\alpha}{\partial x_i}\right)n_\alpha = 0 \qquad \text{on} \qquad \zeta(x_1, x_2) \quad . \tag{5.8}$$

Equations (5.7) and (5.8) together eliminate the third integral in (5.4), computed over S_{FS}.

3. Resistance Formula

To calculate the drag, we put $i = 1$, and use the boundary conditions to obtain

$$\begin{aligned} F_1 = R = \int_{\forall}\left(-\rho g \frac{\partial z}{\partial x}\right)d\forall - \int_{S_1} [\rho(U+u)^2 + P - \tau_{11}](-dy\,dz) + \\ - \int_{S_2} [\rho(U+u)^2 + P - \tau_{11}]dy\,dz + \\ + \int_{S_3} \tau_{12}\,dx\,dz + \int_{S_4} \tau_{12}(-dx\,dz) + \int_{S_b} \tau_{13}(-dx\,dy) \quad . \end{aligned} \tag{5.9}$$

The first integral vanishes. All the shear stress terms on S_3, S_4, S_b are negligibly small since these boundary surfaces are outside the wake region. Upstream on S_1, the perturbation velocity u and dynamic pressure p are both small compared to their wave region values, and hence can be ignored. The viscous stress term $\tau_{11} = 2\mu \frac{\partial u}{\partial x}$ is even smaller than u on S_1.

Using the boundary conditions on S_3, S_4, S_b, S_B, and S_{FS}, the continuity equation

$$\int_S \rho \vec{V} \cdot \vec{n}\, dS = 0 \tag{5.10}$$

reduces to

$$\int_{S_2} \rho(U+u) dy\, dz = \int_{S_1} \rho U\, dy\, dz \quad . \tag{5.11}$$

Separating out the hydrostatic pressure $-\rho gz$ from $P = p - \rho gz$, and using the results above, the resistance formula becomes

$$R = \int_{S_1} (-\rho gz) dy\, dz + \int_{S_2} (\rho gz) dy\, dz - \int_{S_2} [p + \rho u(U+u) - \tau_{11}] dy\, dz \quad , \tag{5.12}$$

where

$$\int_{S_1} dy\, dz = \int_{-\ell}^{\ell} dy \int_{-H}^{0} dz \quad ; \quad \int_{S_2} dy\, dz = \int_{-\ell}^{\ell} dy \int_{-H}^{\zeta} dz \quad ,$$

ℓ = half width of tank

H = depth of tank.

The first two integrals combine to give

$$\int_{\substack{-\ell \\ S_2}}^{\ell} dy \int_0^{\zeta} \rho g \, z \, dz = \frac{1}{2} \rho g \int_{\substack{-\ell \\ S_2}}^{\ell} \zeta^2 \, dy \quad . \tag{5.13}$$

We make use of the splitting of the perturbation velocities into longitudinal and solenoidal components, $\vec{q} = \vec{q}_L + \vec{q}_T$. Then the group of terms in the third integral of Eq. (5.12) is written as

$$p + \rho u(U+u) - \tau_{11} = p + \rho u_L U + \rho u_L^2 + [\, \rho u_T(U+u_T+2u_L) - \tau_{11}] \quad . \tag{5.14}$$

Out side the wake, where u_T is exponentially small, we can legitimately apply the Bernoulli equation for the longitudinal components which carry the pressure p. It is known that the components u_L, v_L, w_L contain terms that are $O(R^{-\frac{1}{2}})$ near $z = 0$ because of the free surface effect. This justifies the inclusion of the squares of the velocity components

$$p + \frac{1}{2} \rho [\, (U+u)^2 + v^2 + w^2] = \frac{1}{2} \rho U^2 \quad , \tag{5.15}$$

so that at the downstream station $x = x_D$,

$$p + \rho u_L U + \rho u_L^2 = - \frac{1}{2} \rho (v_L^2 + w_L^2 - u_L^2) \quad . \tag{5.16}$$

The group of terms $[\, \rho u_T(U+u_T+2u_L) - \tau_{11}]$ is negligibly small outside the wake.

Within the wake, the pressure is still carried by the longitudinal components, and the group of terms neglected outside the wake cannot be omitted without more careful study. The resistance formula (5.12) becomes

$$R = \frac{1}{2}\rho g \int\limits_{\substack{-\ell \\ S_2\,(x=x_D)}}^{\ell} \zeta^2 dy + \frac{1}{2}\rho \int\limits_{\substack{-H \\ (S_2\,-\,\Sigma_{wake})}}^{\zeta} \int\limits_{-\ell}^{\ell} (v_L^2 + w_L^2 - u_L^2) dy\, dz +$$

$$-\iint\limits_{\Sigma_{wake}(x=x_D)} \left\{ \rho U u_T + \rho u_T(u_T+u_L) + \rho u_L u_T - \frac{1}{2}\rho(v_L^2 + w_L^2 - u_L^2) - \tau_{11} \right\} dy\, dz \quad , \qquad (5.17)$$

where Σ_{wake} is the area of the wake at station $x = x_D$. We note that the second integral can be extended through the wake region, since the only velocity components which contribute to that integral at the far downstream station x_D are the longitudinal components $\frac{1}{2}\rho(v_L^2 + w_L^2 - u_L^2)$. The integral of $-\rho U u_T$ over the wake area is the familiar viscous dissipation term. The remaining quadratic terms are negligible everywhere in the wake area Σ_{wake}, except possibly near the free surface $z = 0$, in which region the terms $\rho u_L u_T$ or ρu_T^2 can give contributions comparable with u_L^2 because of the surface stress condition. After combining terms as noted above, we arrive at the resistance formula

$$R = \frac{1}{2}\rho g \int\limits_{\substack{-\ell \\ S_2\,(x=x_D)}}^{\ell} \zeta^2\, dy + \frac{1}{2}\rho \int\limits_{\substack{-H \\ S_2(x=x_D)}}^{\zeta} \int\limits_{-\ell}^{\ell} (v_L^2 + w_L^2 - u_L^2) dy\, dz +$$

$$-\iint\limits_{\Sigma_{wake}(x=x_D)} \{\rho U u_T + \rho u_T(u_T + 2u_L) - \tau_{11}\} dy\, dz \quad . \qquad (5.18)$$

The expression (5.18) can be further simplified to a form more appropriate for theoretical calculation of ship resistance. Suppose that the surfaces S_3, S_4, and S_b are withdrawn to infinity. Instead of specifying the conditions pertinent to a material surface as in (5.5), we need only observe that the velocities V_i $\perp$ these surfaces (v and w) are negligibly small in the limit. The same is true for the dynamic pressure p on these surfaces, while the hydrostatic pressure simply cancels out as before except for the contributions from S_1 and S_2. The shear stress terms appearing in (5.9) are vanishingly small in the limit. So the development of the resistance formula proceeds exactly as previously. The downstream control surface S_2 now extends to infinity laterally and down into the fluid, and is located at a large but finite value $x = x_D$. The second integral in (5.18) is approximated as follows

$$\int_{-\infty}^{\infty}\limits_{S_2} dy \int_{-\infty}^{\zeta} dz(v_L^2+w_L^2-u_L^2) \simeq \int_{-\infty}^{\infty}\limits_{S_2} dy \int_{-\infty}^{0} dz(v_L^2+w_L^2-u_L^2) \quad ,$$

because the neglected term would be of order $O(\zeta u_L^2)$.

Since the solenoidal component u_T is exponentially small outside the wake, the integration of $-\rho U u_T$ across the wake in the third integral of (5.18) is extended to infinity. The final result for the unrestricted flow resistance formula is then divided formally into two general components: total wave resistance R_{w_t} and viscous resistance R_v

$$R = R_{w_t} + R_v \quad , \tag{5.19}$$

where

$$R_{w_t} = \frac{1}{2}\rho g \int_{-\infty}^{\infty} \zeta^2(x=x_D)dy + \frac{1}{2}\rho \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dz \left[v_L^2 + w_L^2 - u_L^2\right]_{x=x_D} , \tag{5.20}$$

$$R_v = -\rho U \int_{-\infty}^{\infty} dy \int_{-\infty}^{0} u_T(x=x_D) dz - \iint_{\Sigma_{wake}} \left[\rho u_T(u_T + 2u_L) - \tau_{11} \right]_{x=x_D} dy\, dz. \quad (5.21)$$

It may appear that this splitting confines the influence of viscosity to the viscous resistance component R_v. There are, however, formal interaction effects present in each of these components. This means that there is an influence of viscosity on certain terms of R_{w_t}, and also that undulatory free surface effects appear in R_v. We assert that these interactions exist without further comment here. These effects are dealt with more explicitly in Chapter VI.

VI. DRAG ON THIN SHIPS

The fundamental flow solution and the resistance formulae based on the momentum consideration of the far flow field, are brought together to produce a theory for Oseen-flow ship resistance.

1. Modelling the Flow Around Thin Ships

Consider the problem of the uniform flow past a thin ship with a given hull shape. The flow has a constant velocity U far upstream, and is steady in the frame of reference moving with the ship. A Cartesian coordinate system is fixed to the body with x pointing in the direction of the uniform stream and z positive upwards. The wetted surface area of the hull S_B has the projected area S_o on the x-z plane. With a total waterline length L, maximum beam B, and draft T, the hull form is symmetric and prescribed by

$$y = \pm h(x, z) \qquad (x, z \text{ in } S_o) \quad , \tag{6.1}$$

where the centerplane area S_o is contained within the bounding rectangle $0 < x < L$, $-T < z < 0$.

The fundamental physical parameters of this flow are the Froude number and Reynolds number based on the length L,

$$\text{Froude number:} \quad F_L = U/\sqrt{gL} \quad \text{and} \quad \sigma_L = gL/U^2 = \kappa_o L = F_L^{-2},$$
$$\text{where} \quad \kappa_o = g/U^2 \quad , \tag{6.2}$$
$$\text{Reynolds number:} \quad R_L = UL/\nu \quad .$$

Surface tension effects are ignored, and our interest here centers on the case $R_L \gg 1$.

Suppose now that the singularity systems* discussed in Chapters II and III are distributed on centerplane area S_o. We denote these dimensional distributions by subscript zero, with $M_o(x,z)$ as the mass source distribution. The distributions $M_o(x,z)$, $X_o(x,z)$, and $Z_o(x,z)$ represent symmetrical flow disturbances with respect to y. But a simple yawlet distribution $Y_o(x,z)$ causes an antisymmetrical flow disturbance, and hence cannot be used directly for a symmetrical ship form. A satisfactory distribution is obtained by arranging two yawlet singularities antisymmetrically and letting the distance between them approach zero. The resulting double-yawlet distribution is denoted by $Y_o^D(x,z)$ and is represented schematically in Fig. (6.1). Velocity components for the symmetric flow disturbance $Y_o^D(x,z)$ are derived by replacing $Y_o(x,z)$ by $-Y_o^D(x,z)\frac{\partial}{\partial y}$ in all of the previous results.

The problem is made nondimensional, using as reference quantities the velocity U and the length L. Pressure and stress are nondimensionalized by ρU^2. Nondimensional forms of the singularity distribution functions are denoted by a small bold $\mathbf{c}$ with the appropriate subscript m, x, y, or z.

$$\mathbf{c}_m(x,z) = \frac{M_o(x,z)}{\frac{1}{2}U} \quad , \tag{6.3}$$

*One might be tempted to call the fundamental solutions 'Havelock Oseenlets,' from their similarity to the classical Havelock sources.

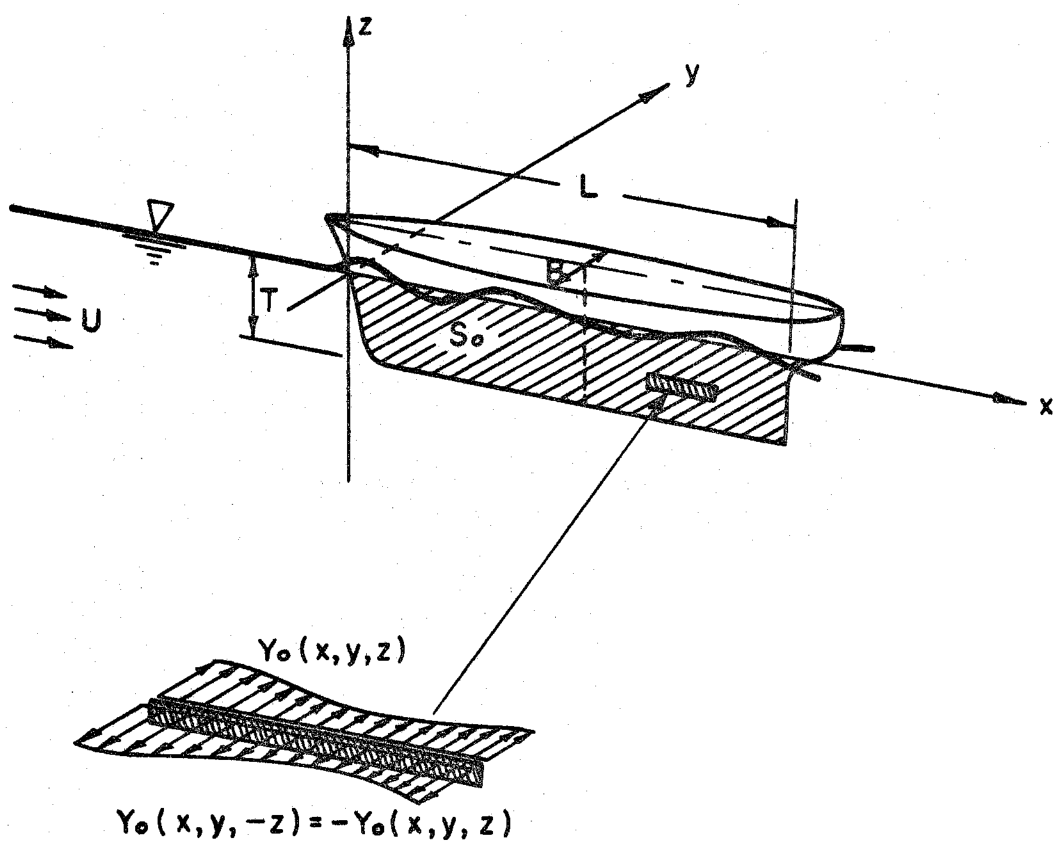


Fig. (6.1) Schematic representation of the symmetric double-yawlet flow disturbance $Y_0^D(x,z)$. The hull shape to be modelled by all four distributions is symmetric, given by $y = \pm h(x,z)$.

$$\begin{bmatrix} \mathfrak{c}_x(x,z) \\ \mathfrak{c}_y^D(x,z) \\ \mathfrak{c}_z(x,z) \end{bmatrix} = \frac{1}{\frac{1}{2}\rho U^2} \begin{bmatrix} X_o(x,z) \\ L^{-1} Y_o^D(x,z) \\ Z_o(x,z) \end{bmatrix} . \qquad (6.4)$$

The centerplane area S_o in the nondimensional problem is bounded by the rectangle $0 < x < 1$, $-T/L < z < 0$. The underlines for dimensionless variables are omitted.

In terms of integrals over the centerplane area S_o, the nondimensional perturbation velocities are written out immediately from the results of Chapter III. We use the splitting $\vec{q} = \vec{q}_o + \vec{q}_1^{(o)} + \vec{q}_s$. The basic flow $\vec{q}_o$ and the image flow $\vec{q}_1^{(o)}$ are separated into their longitudinal and solenoidal components.

Basic flow: $\vec{q}_o = \vec{q}_{o_L} + \vec{q}_{o_T}$

$$\vec{q}_{o_L} = \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi} \nabla\left\{ -\frac{\mathfrak{c}_m}{R} - (\vec{\mathfrak{c}}_F \cdot \nabla)\,\ell n(R-(x-\xi)) \right\} , \qquad (6.5)$$

$$\vec{q}_{o_T} = \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi} \left\{ R_L\, \vec{\mathfrak{c}}_F \left(\frac{e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]}}{R} \right) + \right.$$
$$\left. + \nabla\left[e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]} (\vec{\mathfrak{c}}_F \cdot \nabla)\,\ell n\,(R-(x-\xi)) \right] \right\} , \qquad (6.6)$$

where now

$$\vec{\mathfrak{c}}_F = \left(\mathfrak{c}_x,\ -\mathfrak{c}_y^D \frac{\partial}{\partial y},\ \mathfrak{c}_z \right) , \qquad (6.7)$$

and

$$R = \sqrt{(x-\xi)^2 + y^2 + (z-\zeta)^2} \quad , \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) .$$

The image flow $\vec{q}_1^{(o)}$ is represented by the same equations with $(z-\zeta)$ replaced by $(z+\zeta)$ and R replaced by R_1. Appendix D contains a listing of the three components of $\vec{q}_o$ with the indicated differentiations in (6.5) and (6.6) carried out in full.

The wave elevation ζ and the $\vec{q}_s$-velocity components to be used in calculating the linearized ship resistance are

$$\zeta(x,y) = \int_{-\pi}^{\pi} d\theta \int_{o}^{\infty} dk \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi^2} \frac{e^{ik[(x-\xi)\cos\theta + y\sin\theta]}}{\Delta(k,\theta)} \left\{ e^{k\zeta} A_{\zeta} + e^{K\zeta} B_{\zeta} \right\}, \tag{6.8}$$

$$\begin{bmatrix} u_{s_L} \\ v_{s_L} \\ w_{s_L} \end{bmatrix} = \int_{-\pi}^{\pi} d\theta \int_{o}^{\infty} dk \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi^2} \frac{e^{ik[(x-\xi)\cos\theta + y\sin\theta]}}{\Delta(k,\theta)}$$

$$\times \left\{ e^{k(z+\zeta)} \begin{bmatrix} A_{u_s} \\ A_{v_s} \\ A_{w_s} \end{bmatrix} + e^{kz+K\zeta} \begin{bmatrix} B_{u_s} \\ B_{v_s} \\ B_{w_s} \end{bmatrix} \right\}, \tag{6.9}$$

$$u_{s_T}(x,y,z) = \int_{-\pi}^{\pi} d\theta \int_{o}^{\infty} dk \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi^2} \frac{e^{ik[(x-\xi)\cos\theta + y\sin\theta]}}{\Delta(k,\theta)}$$

$$\times \left\{ e^{Kz+k\zeta} C_{u_s} + e^{K(z+\zeta)} D_{u_s} \right\}, \tag{6.10}$$

where

$$K = \sqrt{k^2 + iR_L k\cos\theta}, \tag{6.11}$$

$$\Delta(k,\theta) = \sigma_L - k\left(\cos^2\theta - \frac{4ik\cos\theta}{R_L}\right) + \frac{4k^3}{R_L^{\,2}} - \frac{4k^2K}{R_L^{\,2}} \quad . \tag{6.12}$$

The coefficients in the integral representations are

$$A_\zeta = \left(ik\cos\theta + \frac{2k^2}{R_L}\right)\left[\,\mathfrak{c}_m - \mathfrak{c}_x + i(k\sin^2\theta\sec\theta\;\mathfrak{c}_y^D - \sec\theta\;\mathfrak{c}_z)\right] \quad , \tag{6.13}$$

$$B_\zeta = \frac{2k}{R_L}\,K\left[\,\mathfrak{c}_x - ik\sin^2\theta\sec\theta\;\mathfrak{c}_y^D\right] + \frac{2ik^2\sec\theta}{R_L}\;\mathfrak{c}_z \quad , \tag{6.14}$$

$$\begin{bmatrix} A_{u_s} \\ A_{v_s} \\ A_{w_s} \end{bmatrix} = \begin{bmatrix} 1 \\ \tan\theta \\ -i\sec\theta \end{bmatrix}\left(ik^2\cos^3\theta + \frac{4k^3\cos^2\theta}{R_L}\right)$$

$$\times\;\left[-\,\mathfrak{c}_m + \mathfrak{c}_x - i(k\sin^2\theta\;\mathfrak{c}_y^D - \sec\theta\;\mathfrak{c}_z)\right] \quad , \tag{6.15}$$

$$\begin{bmatrix} B_{u_s} \\ B_{v_s} \\ B_{w_s} \end{bmatrix} = \begin{bmatrix} 1 \\ \tan\theta \\ -i\sec\theta \end{bmatrix}\left\{\frac{2k^2K\cos^2\theta}{R_L}\left[-\,\mathfrak{c}_x + ik\sin^2\theta\sec\theta\;\mathfrak{c}_y^D\right] - \frac{4ik^3\cos\theta}{R_L}\;\mathfrak{c}_z\right\} \quad , \tag{6.16}$$

$$C_{u_s} = \frac{2k^2K\cos^2\theta}{R_L}\left[\,\mathfrak{c}_m - \mathfrak{c}_x + i(k\sin^2\theta\sec\theta\;\mathfrak{c}_y^D - \sec\theta\;\mathfrak{c}_z)\right] \quad , \tag{6.17}$$

$$D_{u_s} = \frac{4k^3\cos^2\theta}{R_L}\left[\,\mathfrak{c}_x - ik\sin^2\theta\sec\theta\;\mathfrak{c}_y^D\right] \quad . \tag{6.18}$$

Applying the general resistance formulae (5.20) and (5.21) to the particular case at hand, the nondimensional versions of wavemaking resistance R_{w_t} and viscous resistance R_v are

$$\frac{R_{w_t}}{\frac{1}{2}\rho U^2 L^2} = \sigma_L \int_{-\infty}^{\infty} \zeta^2(x_D, y)dy + \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \left[v^2_{s_L} + w^2_{s_L} - u^2_{s_L}\right]_{x=x_D} dy \quad , \tag{6.19}$$

$$\frac{R_v}{\frac{1}{2}\rho U^2 L^2} = -2\int_{-\infty}^{0} dz \int_{-\infty}^{\infty} \left[u_{o_T} + u^{(o)}_{1_T} + u_{s_T}\right]_{x=x_D} dy \; + $$

$$-2 \iint_{\Sigma_{wake}} \left[2u_T u_L + u^2_T\right]_{x=x_D} dy\,dz \quad , \tag{6.20}$$

where the shear stress τ_{11} in (5.21) is neglected in these calculations. The wave elevation and longitudinal velocity components appearing in (6.19) are the 'free' or wavemaking flow quantities with the superscript f omitted.

2. Calculation of the Wave Resistance

When computing the wave resistance from (6.19), the squares of the flow quantities $\zeta, u_{s_L}, v_{s_L}, w_{s_L}$ are to be integrated with respect to y and z over the infinite half space S_2 at the downstream station $x = x_D$. The details of these tedious calculations are somewhat repetitious. So as an example, only the integral

$\int_{-\infty}^{0} dz \int_{-\infty}^{\infty} u^2_{s_L} dy$ is discussed. The remaining three integrals follow in similar fashion.

First, we use a result described in Chapter III to rewrite u_{s_L} as

$$u_{s_L}(x,y,z) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \, \mathrm{Re} \int_0^\infty dk \iint_{S_o} d\xi \, d\zeta \, \frac{e^{ik[(x-\xi)\cos\theta + y\sin\theta]}}{4\pi^2 \Delta(k,\theta)}$$

$$\times \left\{ e^{k(z+\zeta)} A_{u_s} + e^{kz+K\zeta} B_{u_s} \right\} \quad . \tag{6.21}$$

Far downstream we have that $(x-\xi) \gg 1$. Also, only the free-wave disturbance part of u_{s_L} (from the residue contribution of the pole k_p) is required to compute the wave resistance. The function u_{s_L} is squared and integrated with respect to y. After the order of integrations is interchanged, there results

$$\int_{-\infty}^{\infty} u^2_{s_L} dy =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \, \mathrm{Re} \frac{2\pi i}{4\pi^2 D_1(k_p)} \iint_{S_o} d\xi \, d\zeta \, e^{ik_p(x-\xi)\cos\theta}$$

$$\times \left\{ e^{k_p(z+\zeta)} A_{u_s} + e^{k_p z + \zeta K(k_p)} B_{u_s} \right\}_{k_p} \times$$

$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_1 \, \mathrm{Re} \frac{2\pi i}{4\pi^2 D_1(k_{p_1})} \iint_{S_o} d\xi_1 \, d\zeta_1 \, e^{ik_{p_1}(x-\xi_1)\cos\theta_1}$$

$$\left\{ e^{k_{p_1}(z+\zeta_1)} A_{u_s} + e^{k_{p_1} z + \zeta_1 K(k_{p_1})} B_{u_s} \right\}_{k_{p_1}} \cdot I_o \, ,$$

$$\tag{6.22}$$

where

$$I_o = \int_{-\infty}^{\infty} e^{i(k_p \sin\theta + k_{p_1} \sin\theta_1)y} \, dy \tag{6.23}$$

$$k_p(\theta) = \sigma_L \sec^2\theta + i \frac{4\sigma_L^2 \sec^5\theta}{R_L} \quad , \quad k_{p_1}(\theta_1) = \sigma_L \sec^2\theta_1 + i \frac{4\sigma_L^2 \sec^5\theta_1}{R_L} \quad ,$$

$$K(k_p) = \sqrt{k_p^2 + iR_L k_p \cos\theta} \quad , \quad K(k_{p_1}) = \sqrt{k_{p_1}^2 + iR_L k_{p_1} \cos\theta_1} \quad , \tag{6.24}$$

$$1/D_1(k_p) \simeq -\sec^2\theta \left(1 + i \frac{8\sigma_L^2 \sec^3\theta}{R_L}\right) \quad , \quad 1/D_1(k_{p_1}) \simeq -\sec^2\theta_1 \left(1 + i \frac{8\sigma_L^2 \sec^3\theta_1}{R_L}\right).$$

After substituting $\lambda = \sigma_L \sec^2\theta_1 \sin\theta_1$, we rewrite the exponent in (6.23) as

$$i(k_p \sin\theta + k_{p_1} \sin\theta_1) = i(\lambda - \lambda_o) \quad , \tag{6.25}$$

where

$$\lambda_o = -\sigma_L \sec^2\theta \sin\theta \quad . \tag{6.26}$$

The integral (6.23) simplifies to

$$I_o = \int_{-\infty}^{\infty} e^{i(\lambda - \lambda_o)y} \, dy = 2\pi\delta(\lambda - \lambda_o) \quad , \tag{6.27}$$

where $\delta(\lambda - \lambda_o)$ is the delta function of Dirac. Considering $\theta_1 = \theta_1(\lambda)$, we find that

$$d\theta_1 = \frac{\cos^3\theta_1(\lambda)}{\sigma_L(1 + \sin^2\theta_1(\lambda))} \, d\lambda \quad . \tag{6.28}$$

With $\cos\theta_1(\lambda_o) = \cos\theta$, $\sin\theta_1(\lambda_o) = -\sin\theta$, we have

$$k_{p_1}(\lambda_o) = k_p(\theta) \quad ,$$

and

$$[D_1(k_p(\lambda_o), \theta_1(\lambda_o)]^{-1} = -\sec^2\theta\left(1 + i\,\frac{8\sigma_L \sec^3\theta}{R_L}\right) .$$

We use the known property of the delta function to compute the θ_1-integration in (6.22). The result is

$$\int_{-\infty}^{\infty} u^2_{s_L}\, dy =$$

$$= \frac{1}{2\pi\sigma_L}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\,\frac{\cos^3\theta}{[1+\sin^2\theta]}\ \mathrm{Re}\iint_{S_o} d\xi\, d\zeta\, i\,\frac{e^{ik_p(x-\xi)\cos\theta}}{D_1(k_p)}$$

$$\times\left\{e^{k_p(z+\zeta)}A_{u_s}(k_p)+e^{k_p z+\zeta K(k_p)}B_{u_s}(k_p)\right\}$$

$$\times\ \mathrm{Re}\iint_{S_o} d\xi_1\, d\zeta_1\, i\,\frac{e^{ik_p(x-\xi_1)\cos\theta}}{D_1(k_p)}$$

$$\times\left\{e^{k_p(z+\zeta_1)}A_{u_s}(k_p) + e^{k_p z+\zeta_1 K(k_p)}B_{u_s}(k_p)\right\} . \qquad (6.29)$$

In carrying out the indicated operations in (6.29), it is convenient to organize the results in terms of orders of magnitude of the factor $R_L^{-\frac{1}{2}}$, where R_L is assumed large throughout. The exponential $\exp\{\zeta\sqrt{k^2+iR_L k\cos\theta}\}$ is treated as in the fundamental solution of Chapter III, with σ_ℓ, R_ℓ replaced by σ_L, R_L respectively. After simplification, the integral (6.29) becomes

$$\int_{-\infty}^{\infty} u_{s_L}^2 \, dy = \frac{\sigma_L^3}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sec^3\theta}{[1+\sin^2\theta]} \, e^{2z\sigma_L \sec^2\theta}$$

$$\times \Big\{ P\cos x_\sigma + Q\sin x_\sigma + \sqrt{\frac{2\sigma_L}{R_L}}\Big[(J_x - g_1 K_y)\cos x_\sigma + (K_x + g_1 J_y)\sin x_\sigma\Big] + O(R_L^{-1})\Big\}^2 \quad , \qquad (6.30)$$

where

$$x_\sigma = x\sigma_L \sec\theta \quad , \quad g_1 = \sigma_L \sin^2\theta \sec^3\theta \quad . \qquad (6.31)$$

The $O(R_L^{-1})$ terms in these equations involve long expressions with many cross products, but these are negligible compared with the terms retained above. In Eq. (6.30), the quantities P, Q, J_x, K_x, J_y and K_y are all functions of (θ, x):

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \iint_{S_o} d\xi \, d\zeta \; e^{\frac{4\sigma_L^2 \sec^4\theta}{R_L}(\xi - x) + \zeta\,\sigma_L \sec^2\theta}$$

$$\times \Big\{ [\,\mathfrak{c}_m - \mathfrak{c}_x] \begin{pmatrix} \cos\xi_\sigma \\ \sin\xi_\sigma \end{pmatrix} + \sec\theta [\,\mathfrak{c}_z - \sigma_L \sec^2\theta \sin^2\theta \, \mathfrak{c}_y^D] \begin{pmatrix} -\sin\xi_\sigma \\ \cos\xi_\sigma \end{pmatrix} \Big\} \quad , \qquad (6.32)$$

$$\begin{bmatrix} J_j \\ K_j \end{bmatrix} = \iint_{S_o} d\xi \, d\zeta \; e^{\frac{4\sigma_L^2 \sec^4\theta}{R_L}(\xi - x) + \zeta \sqrt{\frac{\sigma_L R_L}{2}} \sec\theta}$$

$$\times \, \mathfrak{c}_j \Big\{ \begin{pmatrix} \cos\zeta_R + \sin\zeta_R \\ \cos\zeta_R - \sin\zeta_R \end{pmatrix} \cos\xi_\sigma + \begin{pmatrix} \sin\zeta_R - \cos\zeta_R \\ \sin\zeta_R + \cos\zeta_R \end{pmatrix} \sin\xi_\sigma \Big\} \, , \qquad (6.33)$$

where

$$j = x, y \qquad \text{and} \qquad \mathfrak{c}_{j=y} = \mathfrak{c}_y^D(\xi, \zeta) \quad ,$$

and

$$\xi_\sigma = \xi \sigma_L \sec\theta \quad , \quad \zeta_R = \zeta \sqrt{\frac{\sigma_L R_L}{2}} \sec\theta \quad .$$

The distribution functions $\mathfrak{c}_m(\xi,\zeta)$, $\mathfrak{c}_x(\xi,\zeta)$, $\mathfrak{c}_y^D(\xi,\zeta)$, and $\mathfrak{c}_z(\xi,\zeta)$ are as yet unknown.

Similar calculations are performed for $\int_{-\infty}^{\infty} \zeta^2 dy$, and $\int_{-\infty}^{\infty} v_{s_L}^2 dy$, and $\int_{-\infty}^{\infty} w_{s_L}^2 dy$ with results analogous to (6.30). After performing the indicated operations in (6.19) for each of the four integrals, then expanding the result about the limit for R_L very large, we obtain the result for the nondimensional wave resistance

$$R_{w_t} = R_w^{(o)} + R_w^{(1)} + R_{w_D}^{(o)} + O(R_L^{-1}) \quad . \tag{6.34}$$

The basic wave resistance components are

$$\frac{R_w^{(o)}}{\frac{1}{2}\rho U^2 L^2} = \frac{\sigma_L^2}{2\pi} \int_0^{\frac{\pi}{2}} \sec^3\theta [P_o^2 + Q_o^2] d\theta \quad , \tag{6.35}$$

$$\frac{R_w^{(1)}}{\frac{1}{2}\rho U^2 L^2} = \frac{\sigma_L^2}{\pi} \sqrt{\frac{2\sigma_L}{R_L}} \int_0^{\frac{\pi}{2}} \sec^{9/2}\theta \Big\{ P_o [J_{x_o} - \sigma_L \sec^3\theta \sin^2\theta K_{y_o}] + $$

$$+ Q_o [K_{x_o} + \sigma_L \sec^3\theta \sin^2\theta J_{y_o}] \Big\} \quad , \tag{6.36}$$

where

$$\begin{bmatrix} P_o \\ Q_o \end{bmatrix} = \iint_{S_o} d\xi \, d\zeta \, e^{\zeta\sigma_L \sec^2\theta} \left\{ [\mathfrak{c}_m - \mathfrak{c}_x] \begin{pmatrix} \cos\xi_\sigma \\ \sin\xi_\sigma \end{pmatrix} + \sec\theta [\mathfrak{c}_z - \sigma_L \sec^2\theta \sin^2\theta \mathfrak{c}_y^D] \begin{pmatrix} -\sin\xi_\sigma \\ \cos\xi_\sigma \end{pmatrix} \right\} , \tag{6.37}$$

$$\begin{bmatrix} J_{j_o} \\ K_{j_o} \end{bmatrix} \approx \sqrt{\frac{2}{\sigma_L R_L \sec\theta}} \int_{S_o} d\xi \, \mathfrak{c}_j(\xi, 0) \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos\xi_\sigma + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin\xi_\sigma \right\} , \tag{6.38}$$

where $j = x, y$ and $\mathfrak{c}_{j=y} = \mathfrak{c}_y^D(x, z)$.

The damped wave resistance component is

$$\frac{R_{w_D}^{(o)}}{\frac{1}{2}\rho U^2 L^2} = \frac{\sigma_L^2}{2\pi} \int_{\theta_v}^{\frac{\pi}{2}} \sec^3\theta \, [P^2 + Q^2]_{x=x_d} d\theta \quad , \tag{6.39}$$

where the functions P and Q are given in Eqs. (6.32), and the angle θ_v is a function of Reynolds number, Froude number, and the distance x_D

$$\theta_v = \sec^{-1}\left[\frac{C_o F_L^4 R_L}{4x_D} + 1\right]^{\frac{1}{4}} \tag{6.40}$$

The constant C_o is typically less than 1. Viscous dissipation of the wave system is contained in the damped component $R_{w_D}^{(o)}$, and it is principally the diverging wave system that is most affected.

It is noted that $R_w^{(o)}$ has the same positive functional form as Havelock's (1963, p. 374) familiar form for thin ships. Converting to dimensional variables, with $\sigma_L = \kappa_o L$, we find that the result for

$R_w^{(o)}$ from Eq. (6.35) is

$$R_w^{(o)} \rightarrow \frac{\rho g^2}{\pi U^4} \int_0^{\frac{\pi}{2}} \sec^3\theta [P_{o_m}^2 + Q_{o_m}^2] d\theta \quad , \tag{6.41}$$

where from (6.37), in dimensional variables,

$$\begin{bmatrix} P_{o_m} \\ Q_{o_m} \end{bmatrix} = \iint_{S_o} M_o(\xi, \zeta) e^{\zeta \kappa_o \sec^2\theta} \begin{matrix} \cos \\ \sin \end{matrix} (\kappa_o \xi \sec\theta) d\xi \, d\zeta \quad . \tag{6.42}$$

For potential flow, the hull kinematic boundary condition for thin ships gives

$$M_o(x, z) = 2U \frac{\partial h}{\partial x} \quad , \tag{6.43}$$

where $y = \pm h(x, z)$ is the hull shape and $M_o(x, z)$ is the distribution of sources and sinks on the centerplane S_o. Therefore in the limit of potential flow, $R_w^{(o)}$ correctly yields the classical wave resistance.

3. Calculation of the Viscous Resistance

Turning now to the nondimensional viscous resistance formula of Eq. (6.20), we use the fact that the u-velocity is even in y to obtain

$$\frac{R_v}{\frac{1}{2}\rho U^2 L^2} = -4\int_{-\infty}^{o} dz \int_{o}^{\infty} \left[u_{o_T} + u^{(o)}_{1_T} + u_{s_T} \right]_{x=x_D} dy \; +$$

$$-2 \iint_{\Sigma_{wake}} \left[2u_T u_L + u_T^2 \right]_{x=x_D} dy\, dz \quad , \qquad (6.44)$$

where u_{o_T} and $u^{(o)}_{1\,T}$ are given in Eq. (D.5) of Appendix D, and u_{s_T} is given in (6.10). The total viscous resistance is then separated into four components

$$R_v = R_{v_o} + R_{v_1} + R_{v_2} + R_{v_3} \quad , \qquad (6.45)$$

where

$$\frac{R_{v_o}}{\frac{1}{2}\rho U^2 L^2} = -4\int_{-\infty}^{o} dz \int_{o}^{\infty} \left[u_{o_T} + u^{(o)}_{1\,T} \right]_{x=x_D} dy \quad , \qquad (6.46)$$

$$\frac{R_{v_1}}{\frac{1}{2}\rho U^2 L^2} = -4 \iint_{\Sigma_{wake}} \left[u_T u_L \right]_{x=x_D} dy\, dz \quad , \qquad (6.47)$$

$$\frac{R_{v_2}}{\frac{1}{2}\rho U^2 L^2} = -2 \iint_{\Sigma_{wake}} \left[u_T^2 \right]_{x=x_D} dy\, dz \quad , \qquad (6.48)$$

$$\frac{R_{v_3}}{\frac{1}{2}\rho U^2 L^2} = -4\int_{-\infty}^{o} dz \int_{o}^{\infty} \left[u_{s_T} \right]_{x=x_D} dy \quad . \qquad (6.49)$$

We consider the R_{v_o} part first. The fact that R_L is assumed large is used to good advantage in reducing (6.46) to a relatively compact expression. As these calculations are long and somewhat repetitious, only a sample is included here. Taking the first term in

u_{o_T} of Eq. (D.5), we consider the integral

$$I_a = \int_{-\infty}^{o} dz \int_{o}^{\infty} dy \iint_{S_o} d\xi\, d\zeta \; \frac{R_L\, \mathfrak{c}_x}{8\pi R} \; e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]} \quad . \tag{6.50}$$

By expanding for large $(x-\xi)$, and interchanging orders of integration, this becomes

$$I_a \simeq \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi} \; R_L\, \mathfrak{c}_x \int_{-\infty}^{o} dz \int_{o}^{\infty} dy \; \frac{e^{-\frac{R_L}{4(x-\xi)}\left(y^2+(z-\zeta)^2\right)}}{(x-\xi)} \left[1 - \frac{1}{2}\,\frac{y^2+(z-\zeta)^2}{(x-\xi)^2}\right] \quad . \tag{6.51}$$

Performing the necessary integrals, we find that

$$I_a \simeq \iint_{S_o} \frac{\mathfrak{c}_x}{8}\, d\xi\, d\zeta - \iint_{S_o} \frac{3\, \mathfrak{c}_x}{8R_L(x-\xi)}\, d\xi\, d\zeta + J_a \quad , \tag{6.52}$$

where

$$J_a = \iint_{S_o} \frac{R_L\, \mathfrak{c}_x}{(x-\xi)} \left[\pi \left(\frac{x-\xi}{R_L}\right)^{\frac{1}{2}} I_o - \frac{\sqrt{\pi}}{(x-\xi)^2} \left(\frac{x-\xi}{R_L}\right)^{3/2} I_o + \right.$$

$$\left. - \frac{\sqrt{\pi}}{(x-\xi)^2} \left(\frac{x-\xi}{R_L}\right)^{3/2} I_1 \right] d\xi\, d\zeta \quad , \tag{6.53}$$

where

$$I_o = \int_{o}^{-\zeta} e^{-\alpha^2 p^2}\, dp \quad , \qquad I_1 = \int_{o}^{-\zeta} e^{-\alpha^2 p^2}\, p^2\, dp \quad .$$

The analogous term from $u^{(o)}_{1_T}$ is

$$I a_1 = \int_{-\infty}^{0} dz \int_{0}^{\infty} dy \iint_{S_o} d\xi\, d\zeta \; \frac{R_L\, \mathbb{C}_x}{R_1} \; e^{\left(\frac{R_L}{2}\right)\left[(x-\xi)R_1\right]}$$

$$\simeq \iint_{S_o} \frac{\mathbb{C}_x}{8}\, d\xi\, d\zeta - \iint_{S_o} \frac{3\, \mathbb{C}_x\, d\xi\, d\zeta}{8R_L(x-\xi)} - J_a \quad , \tag{6.54}$$

where J_a is given in (6.57). Then

$$I_a + I_{a_1} \simeq \iint_{S_o} \frac{\mathbb{C}_x}{4}\, d\xi\, d\zeta + O\left(\frac{1}{R_L x_D}\right) \quad . \tag{6.55}$$

When these tedious, but straightforward calculations are performed for all the terms of u_{o_T} and $u^{(o)}_{1_T}$, we find that the $\mathbb{C}^D_y$ terms cancel out, and the viscous drag component in (6.46) becomes approximately

$$\frac{R_{v_o}}{\frac{1}{2}\rho U^2 L^2} \simeq -4 \iint_{S_o} d\xi\, d\zeta \left\{ \frac{\mathbb{C}_x(\xi,\zeta)}{4} - \frac{\mathbb{C}_z(\xi,\zeta)}{4\sqrt{\pi}\sqrt{R_L(x-\xi)}}\; e^{-\frac{R_L}{4(x-\xi)}\zeta^2} \right\}_{x=x_D} + O\left(\frac{1}{R_L x_D}\right) \quad . \tag{6.56}$$

This is further simplified by expanding $(x-\xi)^{\frac{1}{2}}$ for large x, and expanding $\mathbb{C}_z(\xi,\zeta)$ around $\zeta = 0$ so that the ζ-integral for the $\mathbb{C}_z$-term can be estimated. The final result for the basic viscous resistance in nondimensional variables is

$$\frac{R_{v_o}}{\frac{1}{2}\rho U^2 L^2} \simeq -\iint_{S_o} \mathfrak{C}_x(\xi,\zeta)d\xi d\zeta + O\left(\frac{\mathfrak{C}_z}{R_L}\right) + O\left(\frac{1}{R_L x_D}\right) \quad . \qquad (6.57)$$

So the $\mathfrak{C}_x$-term is the most important term of R_{v_o}, and the minus sign is correct, since a positive drag is represented by a negative X-force on the fluid. In terms of dimensional variables, Eq. (6.62) becomes

$$R_{v_o} \simeq -\iint_{S_o} X_o(\xi,\zeta)d\xi d\zeta + O\left(\frac{Z_o}{R_L}\,,\ \frac{1}{R_L x_D}\right) \quad . \qquad (6.58)$$

Next we deal with R_{v_1} and R_{v_2} of Eqs. (6.47) and (6.48), respectively. These terms arise solely because of the presence of the free surface, and might be better termed as the two parts of a viscous 'free surface wake-drag.' In an unbounded flow these components would be zero. Keeping only the most important terms in u_L and u_T, and noting that u_T is exponentially small outside the wake, we approximate the integration over Σ_{wake} in (6.47) and (6.48) as follows

$$\iint_{\Sigma_{wake}} [u_L u_T]\,dy\,dz \simeq 2\int_{-\infty}^{0} dz \int_{0}^{\infty} \left[u_{s_L}(u_{o_r} + u^{(o)}_{1_r})\right]_{x=x_D} dy \quad , \qquad (6.59)$$

$$\iint_{\Sigma_{wake}} [u_T^2]\,dy\,dz \simeq 2\int_{-\infty}^{0} dz \int_{0}^{\infty} \left[u_{o_r} + u^{(o)}_{1_r}\right]^2_{x=x_D} dy \quad . \qquad (6.60)$$

The u_{o_r} and $u^{(o)}_{1_r}$ are strictly rotational parts of u_{o_T} and $u^{(o)}_{1_T}$, (see Eq. (7.28)).

Expanding the individual terms for large $(x-\xi)$ and performing the indicated integrations approximately, we obtain the non-dimensional expressions for the free surface wake-drag

$$\frac{R_{v_1}}{\frac{1}{2}\rho U^2 L^2} \simeq \frac{2\sigma_L}{\pi^{3/2}}\sqrt{\frac{R_L}{x_D}}\int_0^{\frac{\pi}{2}} d\theta \sec\theta \int_{S_o} d\xi \int_{S_o} d\xi_1 \, c_x(\xi_1, 0)\Big\{ [c_m(\xi,0)-c_x(\xi,0)]\cos(\sigma_L(x_D-\xi)\sec\theta) + \sec\theta[c_z(\xi,0)-(\sigma_L\sec^2\theta\sin^2\theta)c_y^D(\xi,0)]\sin(\sigma_L(x_D-\xi)\sec\theta)\Big\}, \quad (6.61)$$

and

$$\frac{R_{v_2}}{\frac{1}{2}\rho U^2 L^2} \simeq -(4\sqrt{2\pi})^{-1}\sqrt{\frac{R_L}{x_D}}\left[\int_{S_o} c_x(\xi,0)d\xi\right]^2 \quad . \quad (6.62)$$

The fourth component of viscous resistance is R_{v_3} from Eq. (6.49) It is also clearly a free surface contribution. Using (6.10), and after considerable simplification, the expression reduces to

$$\frac{R_{v_3}}{\frac{1}{2}\rho U^2 L^2} \simeq \frac{2\sigma_L}{R_L}\iint_{S_o} d\xi\, d\zeta\, e^{\sigma_L\zeta - \frac{4\sigma_L^2 x_D}{R_L}}\Big\{[-(c_m-c_x)\sin(\sigma_L x) + c_z\cos(\sigma_L x)]\cos(\sigma_L\xi) + [(c_m-c_x)\cos(\sigma_L x)+c_z\sin(\sigma_L x)]\sin(\sigma_L\xi)\Big\}_{x=x_D} \quad . \quad (6.63)$$

Being of order $O(R_L^{-1})$, this term may be neglected in comparison with the contribution of the other terms.

4. Discussion

The principal results of the foregoing analysis indicate that the total resistance of a thin ship in Oseen-flow can be expressed approximately as

$$R \simeq [R_w^{(o)} + R_w^{(1)} + R_{w_D}^{(o)}] + [R_{v_o} + R_{v_1} + R_{v_2}] \qquad (6.64)$$

where the main wave resistance terms $R_w^{(o)}$, $R_w^{(1)}$, and $R_{w_D}^{(o)}$ are are given in Eqs. (6.35), (6.36), and (6.39). The three most important viscous resistance components R_{v_o}, R_{v_1}, and R_{v_2} are given in Eqs. (6.57), (6.61), and (6.62) respectively. R_{v_o} represents the basic form of the viscous dissipation drag which the ship hull would experience even in the absence of the free surface. All the remaining terms in (6.64) display explicitly the influence of the free surface. Of these, the most important term is $R_w^{(o)}$ of Eq. (6.35), which represents the drag due to the propagating free surface disturbances as viewed at a far distance from the ship hull. In the limit $x_D \to \infty$, the total resistance is represented by the constant $R \simeq R_w^{(o)} + R_{v_o}$. In any actual experimental determination of the resistance components (e.g., by wake surveys and by transverse wave height analysis) the physical measurements are always taken at some finite distance aft of body. The terms $R_{w_D}^{(o)}$, R_{v_1}, and R_{v_2} indicate the possible influence of the distance x_D on the measured quantities.

Boundary layer and wake interaction effects are included formally in the wave resistance terms by the presence of the forcelet distributions $\mathfrak{c}_x$, $\mathfrak{c}_y^D$, and $\mathfrak{c}_z$ in the functions P_o, Q_o, J_{j_o}, and K_{j_o}.

Within the context of this linearized theory, the physical meaning of the forcelet distributions can be inferred from their relationship to the shear stresses τ_{yx} and τ_{yz}. It can be shown that, in terms of non-dimensional variables

$$\tau_{yx} = -\frac{1}{4}\left[c_x(x,z) - \frac{\partial}{\partial x} c_y^D(x,z)\right] \tag{6.65}$$

$$\tau_{yz} = -\frac{1}{4} c_z(x,z) \quad . \tag{6.66}$$

Of course all four distribution functions are interrelated, and they depend ultimately on the shape of the hull through the physical boundary conditions. The boundary conditions themselves are discussed in Chapter VII.

VII. OSEEN-FLOW BOUNDARY VALUE PROBLEM

The physical boundary conditions are used here to set up integral equations for the four unknown distributions $M_o(\xi,\zeta)$, $X_o(\xi,\zeta)$, $Y_o^D(\xi,\zeta)$, and $Z_o(\xi,\zeta)$. Certain primary features have already been assumed about these functions. All four of them are nonzero only on the centerplane area S_o. This means that the total flow disturbance, in addition to being very thin laterally, is of finite extent in the x - and z-directions. If any of the functions have singularities within the region S_o, these are assumed to integrable. Thus, all the functions P, Q, J_j, K_j of Eqs. (6.32) and (6.33) remain bounded and well defined.

It is important that the boundary conditions be consistent with the flow linearization as it applies near the body. The equations presented here represent a linear approximation to flow that is laminar and attached. Of course the actual boundary layer on a ship is turbulent for most of the length of the hull, and flow separation often occurs near the stern, especially for large block coefficient ship hulls. Even though the approximate conditions developed here do not apply to the turbulent ship flow directly they can provide approximate results that form a guide for an 'indirect solution.' In an indirect calculation, the mass source and forcelet distributions are prescribed, and then the general resistance formulae are used to compute the resulting drag. This approach is useful for investigating the relative importance of the various forcelet distributions; and an example of such a computation is discussed in Chapter VIII.

Ideally, the system of boundary conditions on the ship hull should be sufficient to solve for M_o, X_o, Y_o^D, Z_o, using as the only input the shape of the body. This is called the 'direct problem;' and an approximate zeroth order solution is given below in Section 4.

1. Physical Boundary Conditions

Two types of boundary conditions are to be satisfied on and near the surface of the hull.

(a) No-slip Conditions. In a viscous flow, the fluid immediately adjacent to the body has zero velocity relative to the solid surface. If the hull shape given by $y = \pm h(x,z)$ is thin, then to the desired order of accuracy the linearized boundary conditions are to be satisfied on the centerplane region, in the limit as $y \to 0$. The no-slip conditions are

$$u = u_{o_L} + u_{o_T} + u_{1_L}^{(o)} + u_{1_T}^{(o)} + u_s = -U \qquad (y = 0) , \qquad (7.1)$$

$$v = v_{o_L} + v_{o_T} + v_{1_L}^{(o)} + v_{1_T}^{(o)} + v_s = 0 \qquad (y = 0) , \qquad (7.2)$$

$$w = w_{o_L} + w_{o_T} + w_{1_L}^{(o)} + w_{1_T}^{(o)} + w_s = 0 \qquad (y = 0) , \qquad (7.3)$$

for (x, z) in the region S_o. In these, the $\vec{q}_s$ components are known in terms of Fourier integral representations, and the components of $\vec{q}_o$, $\vec{q}_1^{(o)}$ are given in detail in Appendix D.

(b). Kinematic Condition. The conditions of (7.1)-(7.2) are determined by Taylor expansions of the velocities about $y = 0$. The linearized no-slip condition (7.2) has a second consequence that may be regarded as a singular expansion or boundary layer limit as $y \to 0$. We define a displacement thickness δ^* in the usual way (Schlichting (1960))

$$\delta^{*} = \int_{o}^{y_o} \left(1 - \frac{U+u}{U} \right) d\eta \quad , \tag{7.4}$$

where η is defined in the sketch of Fig. (7.1), y_o is a value of η greater than the boundary layer thickness, and the x-perturbation velocity is $u = u_o + u_1^{(o)} + u_s$. Then for the flow to be tangential to the virtual body shape $y_v = h(x,z) + \delta^*$, the kinematic condition reads

$$\frac{v(x, y_v, z)}{U+u(x, y_v, z)} = \frac{\partial}{\partial x} \left[h(x,z) + \delta^{*} \right] \quad . \tag{7.5}$$

This may be simplified in several ways. We neglect the perturbation quantity $u(x, y_v, z)$ compared with U in the denominator of the left hand side. Also, the linearized version of this condition is to be satisfied on the centerplane $y = 0$. However, the velocity v is evaluated at $y_v > 0$. This is a point which lies outside the viscous-dominated region, where the flow is governed mainly by the longitudinal velocity component $\vec{q}_L$. Therefore we approximate the $v(x, y_v, z)$ in (7.5) by the component $v_L(x, 0, z)$.

The expression for δ^* in (7.4) is also simplified by exploiting the splitting of the flow quantities into longitudinal and solenoidal components. The upper limit y_o of the integral in (7.4) is extended to infinity after replacing the total perturbation velocity u by the rotational part of the solenoidal component. This is permissable because the solenoidal component is exponentially small outside the viscous-dominated region.

The linearized kinematic boundary condition at the outer edge

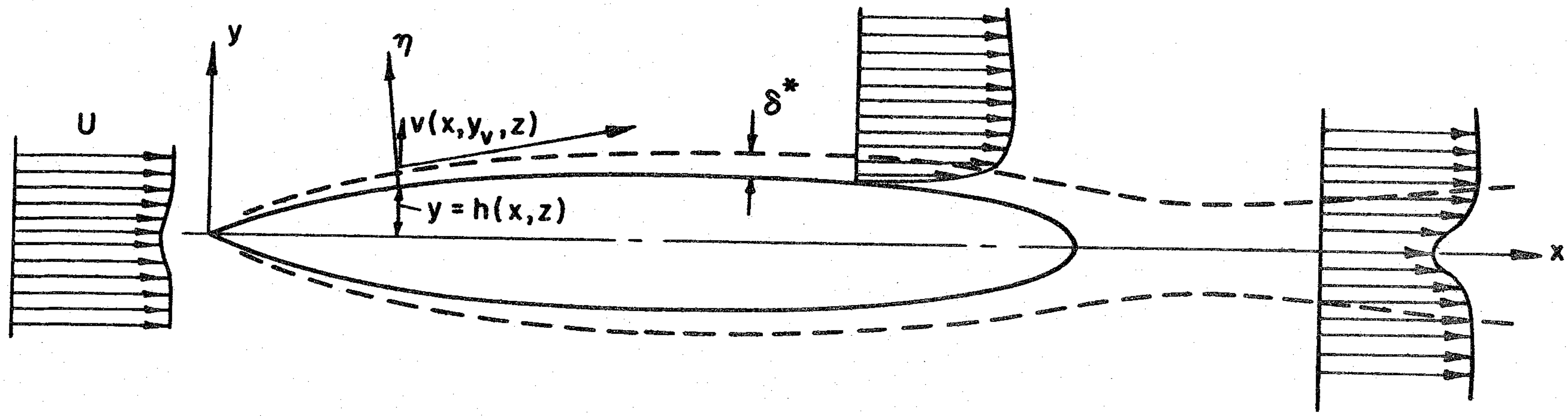


Fig. (7.1) Definition sketch for the kinematic condition applied to the flow just outside the boundary layer.

of the boundary layer becomes

$$v_L(x, y \to 0, z) = U \frac{\partial}{\partial x} [h(x,z) + \delta^*(x,z)] \quad , \qquad (7.6)$$

where the linearized displacement thickness is

$$\delta^*(x,z) \simeq - \frac{1}{U} \int_0^\infty [u_{o_r} + u_{1_r}^{(o)} + u_{s_T}]_{y=0} \, dy \quad . \qquad (7.7)$$

2. The Integral Equations

For convenience, the conditions (7.1) - (7.3) and (7.6) are made dimensionless using the quantities described in Chapter VI, Section 1. The underlines are omitted.

(a) No-slip Conditions. We consider first the details of the conditions in Eqs. (7.1) - (7.3). The linearizing approximation $y \to 0$ affords some simplification in the rather long equations of the no-slip condition. Referring to Appendix D for the full expressions of u_o, v_o, and w_o, we focus our attention on the terms that appear to go to zero as $y \to 0$. If these same terms contain $R = \sqrt{(x-\xi)^2 + y^2 + (z-\zeta)^2}$ or $b^2 = y^2 + (z-\zeta)^2$ in the denominator, then they are singular at $y = 0$, $\xi = x$, $\zeta = z$ and must be treated more carefully.

In the u-equation, (7.1), only the c_y^D-terms have singular parts. As an example of the technique for handling these terms, we consider the following integral from one of the terms of u_{o_T}

$$I = \lim_{y\to 0} \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi} \; c_y^D(\xi,\zeta) \left(\frac{R_L}{2}\right) \frac{2y^2}{b^2R^2} \; e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]} . \quad (7.8)$$

As $y \to 0$, the contribution to this integral comes entirely from an infinitesimal patch centered at the point $\xi = x$, $\zeta = z$ within the region S_o. Therefore the function $c_y^D(\xi,\zeta)$ is expanded about the point $\xi = x$, $\zeta = z$ and $c_y^D(x,z)$ is taken outside the integral. Then it is convenient to change the variables of integration, using the substitution

$$(\xi - x) = y\lambda \cos\theta \quad , \qquad (\zeta - z) = y\lambda \sin\theta \qquad (7.9)$$

The integral in (7.8) becomes

$$I = \lim_{y\to 0} R_L \frac{c_y^D(x,z)}{8\pi} \int_0^{a/y} \frac{\lambda\, e^{-\left(\frac{R_L}{2}\right) y\sqrt{\lambda^2+1}}}{(\lambda^2+1)}\, d\lambda$$

$$\times \int_{-\pi}^{\pi} \frac{e^{-\left(\frac{R_L}{2} y\lambda\right)\cos\theta}}{(1+\lambda^2 \sin^2\theta)}\, d\theta \quad , \qquad (7.10)$$

where a is the small finite radius of a circular patch about the singular point. In the limit $y \to 0$, the upper limit of the λ-integral becomes infinite, and after performing the elementary integrals we obtain

$$I = \frac{R_L}{4} \; c_y^D(\xi,\zeta) \quad . \qquad (7.11)$$

This same procedure is also used to obtain the results

$$\lim_{y\to 0}\iint_{S_o}\frac{d\xi\, d\zeta}{8\pi}\ \mathfrak{c}_y^D(\xi,\zeta)\left(\frac{R_L}{2}\right)\frac{(x-\xi)y^2}{b^2 R^3}\left(1-\frac{(x-\xi)}{R}\right)e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]}$$

$$= \frac{(\pi+1)}{32}\ R_L \mathfrak{c}_y^D(x,z)\quad , \qquad (7.12)$$

$$\lim_{y\to 0}\iint_{S_o}\frac{d\xi\, d\zeta}{8\pi}\ \mathfrak{c}_y^D(\xi,\zeta)\ \frac{3y^2}{R^5}\left(1-e^{\frac{R_L}{2}[(x-\xi)-R]}\right) = 0\quad . \qquad (7.13)$$

The contribution from the $u_1^{(o)}$ term corresponding to (7.8) is obtained by replacing R by R_1 and b by b_1. Then the result for I_1 is

$$I_1 = \lim_{y\to 0}\iint_{S_o}\frac{d\xi\, d\zeta}{8\pi}\ \mathfrak{c}_y^D(\xi,\zeta)\left(\frac{R_L}{2}\right)\frac{2y^2}{b_1^2 R_1^2}\ e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R_1]} = 0\quad , \qquad (7.14)$$

since $R_1 = \sqrt{(x-\xi)^2+y^2+(z+\zeta)^2}$ and $b_1^2 = [\,y^2+(z+\zeta)^2\,]$ do not vanish for values of ξ, ζ ranging inside of S_o. The $u_1^{(o)}$ terms corresponding to (7.12) and (7.13) also give zero contributions.

In the v-equation of (7.2), every term of v_{o_L}, v_{o_T}, $v_{1_L}^{(o)}$, and $v_{1_T}^{(o)}$ must be treated using a procedure similar to that outlined above. Only those terms giving non-zero contributions are listed here

$$\lim_{y\to 0}\iint_{S_o}\frac{d\xi\, d\zeta}{8\pi}\ \mathfrak{c}_m(\xi,\zeta)\left(\frac{y}{R^3}\right) = \frac{\mathfrak{c}_m(x,z)}{4} \qquad (7.15)$$

$$\lim_{y\to 0}\iint_{S_o}\frac{d\xi\, d\zeta}{8\pi}\ R_L \mathfrak{c}_y^D(\xi,\zeta)\left(\frac{y}{R^3}\right) e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]} = \frac{R_L}{4}\ \mathfrak{c}_y^D(x,z)\ . \qquad (7.16)$$

$$\lim_{y\to 0} \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi} \; \mathbb{c}_y^D(\xi,\zeta)\left(\frac{y}{b^2 R}\right)\left(1+\frac{(x-\xi)}{R}\right)\left(1-\frac{2y^2}{b^2}\right) e^{\frac{R_L}{2}[(x-\xi)-R]}$$

$$= -\frac{R_L}{16}\, \mathbb{c}_y^D(x,z) \quad . \tag{7.17}$$

The $v_s(x,y,z)$ term in (7.2) is expressed in terms of a double Fourier transform, and it can be shown to give a vanishing contribution on the centerplane $y = 0$ because it is odd in y.

In the w-equation of (7.3), all the terms appearing to go to zero as $y \to 0$ give vanishing contributions.

The complete nondimensional no-slip conditions of (7.1) - (7.3) become

u-equation:

$$\iint_{S_o} \frac{d\xi\, d\zeta}{8\pi}\left\{ \mathbb{c}_m\left[\frac{(x-\xi)}{R^3} + \frac{(x-\xi)}{R_1^3}\right] - \mathbb{c}_x \frac{(x-\xi)}{R^3}\left(1-e^{\frac{R_L}{2}[(x-\xi)-R]}\right) + \right.$$

$$- \mathbb{c}_x \frac{(x-\xi)}{R_1^3}\left(1-e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R_1]}\right) +$$

$$+ \mathbb{c}_y^D \frac{1}{R^3}\left(1-e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]}\right) + \mathbb{c}_y^D \frac{1}{R_1^3}\left(1-e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R_1]}\right) +$$

$$- \mathbb{c}_z \frac{(z-\zeta)}{R^3}\left(1-e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]}\right) - \mathbb{c}_z \frac{(z+\zeta)}{R_1^3}\left(1-e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R_1]}\right) +$$

$$+ e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]}\left[\frac{R_L \mathbb{c}_x}{R} - \mathbb{c}_x\left(\frac{R_L}{2}\right)\frac{1}{R}\left(1-\frac{(x-\xi)}{R}\right) - \mathbb{c}_y^D\left(\frac{R_L}{2}\right)\frac{1}{R^2} + \right.$$

$$\left. + \mathbb{c}_z\left(\frac{R_L}{2}\right)\frac{(z-\zeta)}{R^2}\right] + \qquad \text{(cont'd)}$$

$$+ e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R_1]}\left[\frac{R_L \mathfrak{c}_x}{R_1} - \mathfrak{c}_x\left(\frac{R_L}{2}\right)\frac{1}{R_1}\left(1-\frac{(x-\xi)}{R_1}\right) - \mathfrak{c}_y^D\left(\frac{R_L}{2}\right)\frac{1}{R_1^2} + \right.$$

$$\left. + \mathfrak{c}_z\left(\frac{R_L}{2}\right)\frac{(z+\zeta)}{R_1^2}\right]\Bigg\} +$$

$$+\left(\frac{9+\pi}{32}\right) R_L \mathfrak{c}_y^D(x,z) + u_s(x,0,z) = -1 \quad , \tag{7.18}$$

<u>v-equation:</u>

$$\frac{\mathfrak{c}_m(x,z)}{4} + \left(\frac{3R_L}{16}\right)\mathfrak{c}_y^D(x,z) = 0 \quad , \tag{7.19}$$

<u>w-equation:</u>

$$\iint_{S_o} \frac{d\xi\, d\zeta}{8\pi}\left\{\mathfrak{c}_m\left[\frac{(z-\zeta)}{R^3} + \frac{(z+\zeta)}{R_1^3}\right] + \mathfrak{c}_x \frac{(z-\zeta)}{R^3}\left(1-e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]}\right) + \right.$$

$$+ \mathfrak{c}_x \frac{(z+\zeta)}{R_1^3}\left(1-e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R_1]}\right) +$$

$$+ \mathfrak{c}_y^D\left[-\frac{2}{(z-\zeta)^3}\left(1+\frac{(x-\xi)}{R}\right) - \frac{(x-\xi)}{(z-\zeta)R^3}\right]\left(1-e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]}\right) +$$

$$+ \mathfrak{c}_y^D\left[-\frac{2}{(z+\zeta)^3}\left(1+\frac{(x-\xi)}{R_1}\right) - \frac{(x-\xi)}{(z+\zeta)R_1^3}\right]\left(1-e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R_1]}\right) +$$

$$+ \mathfrak{c}_z\left[\frac{1}{(z-\zeta)^2}\left(1+\frac{(x+\xi)}{R}\right)\right]\left(1-e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]}\right) +$$

$$+ \mathfrak{c}_z\left[\frac{1}{(z+\zeta)^2}\left(1+\frac{(x-\xi)}{R_1}\right)\right]\left(1-e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R_1]}\right) +$$

(cont'd)

$$+e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]}\left[\frac{R_L \mathfrak{c}_z}{R} + \mathfrak{c}_x\left(\frac{R_L}{2}\right)\frac{(z-\zeta)}{R^2} + \mathfrak{c}_y^D\left(\frac{R_L}{2}\right)\frac{1}{(z-\zeta)R}\left(1+\frac{(x-\xi)}{R}\right)+ \right.$$

$$\left. - \mathfrak{c}_z\left(\frac{R_L}{2}\right)\frac{1}{R}\left(1+\frac{(x-\xi)}{R}\right)\right] +$$

$$+e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R_1]}\left[\frac{R_L \mathfrak{c}_z}{R_1} + \mathfrak{c}_x\left(\frac{R_L}{2}\right)\frac{(z+\zeta)}{R_1^2} + \mathfrak{c}_y^D\left(\frac{R_L}{2}\right)\frac{1}{(z+\zeta)R_1}\left(1+\frac{(x-\xi)}{R_1}\right) + \right.$$

$$\left. - \mathfrak{c}_z\left(\frac{R_L}{2}\right)\frac{1}{R_1}\left(1+\frac{(x-\xi)}{R_1}\right)\right]\Bigg\} +$$

$$+ w_s(x,0,z) = 0 \quad . \tag{7.20}$$

The velocity components $u_s(x,0,z)$ and $w_s(x,0,z)$ have the Fourier integral representations

$$\begin{pmatrix} u_s \\ w_s \end{pmatrix} = \int_{-\pi}^{\pi} d\theta \int_0^{\infty} dk \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi^2} \frac{e^{ik(x-\xi)\cos\theta}}{\Delta(k,\theta)} \left\{ e^{k(z+\zeta)} \begin{pmatrix} A_{u_s} \\ A_{w_s} \end{pmatrix} + e^{kz+\zeta K} \begin{pmatrix} B_{u_s} \\ B_{w_s} \end{pmatrix} + \right.$$

$$\left. + e^{zK+k\zeta} \begin{pmatrix} C_{u_s} \\ C_{w_s} \end{pmatrix} + e^{(z+\zeta)K} \begin{pmatrix} D_{u_s} \\ D_{w_s} \end{pmatrix} \right\} , \tag{7.21}$$

where the notation is identical with that given previously in Eqs. (6.15)-(6.18).

<u>(b) Kinematic Condition.</u> Now we consider the linearized condition of Eq. (7.6). The longitudinal component of the v-velocity has three parts

$$v_L = v_{o_L} + v_{1_L}^{(o)} + v_{s_L} \quad , \tag{7.22}$$

where the v_{o_L} is written out in Eq. (D.3) of Appendix D. When use is made of the substitutions (7.9), the following results are obtained

$$\lim_{y \to 0} \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi} [\mathfrak{c}_m(\xi,\zeta) - \mathfrak{c}_x(\xi,\zeta)] \frac{y}{R^3} = \frac{1}{4} [\mathfrak{c}_m(x,z) - \mathfrak{c}_x(x,z)] \ , \tag{7.23}$$

$$\lim_{y \to 0} \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi} \mathfrak{c}_z(\xi,\zeta) \left[\frac{2y(z-\zeta)}{b^4} \left(1 + \frac{(x-\xi)}{R} \right) + \frac{y(x-\xi)(z-\zeta)}{b^2 R^3} \right] = 0 \quad . \tag{7.24}$$

The contributions of the $\mathfrak{c}_y^D$ terms to the component $v_{o_L}(x, y \to 0, z)$ are obtained by integrating by parts, and can be shown to be equal to zero provided that the function $\mathfrak{c}_y^D(x,z)$ vanishes at the endpoints of the region S_o

$$\mathfrak{c}_y^D(x_{min}, z) = \mathfrak{c}_y^D(x_{max}, z) = 0 \quad . \tag{7.25}$$

All the terms of $v_{1_L}^{(o)}$ vanish as $y \to 0$, as does the contribution from $v_{s_L}(x, 0, z)$. Hence the nondimensional kinematic boundary condition (7.6) reduces to

$$\frac{1}{4}[\mathfrak{c}_m(x,z) - \mathfrak{c}_x(x,z)] = \frac{\partial}{\partial x}[h(x,z) + \delta^*(x,z)] \quad , \tag{7.26}$$

which bears a strong resemblance to the familiar kinematic condition in potential theory. The displacement thickness δ^* is also a function of the distributions $\mathfrak{c}_x$, $\mathfrak{c}_y^D$, $\mathfrak{c}_z$ through the velocity components u_{o_r}, $u_{1_r}^{(o)}$, and u_{s_T}. In the nondimensional form,

$$\delta^*(x,z) = -\int_0^\infty [u_{o_r} + u_{1_r}^{(o)} + u_{s_T}]\,dy \quad , \tag{7.27}$$

where the rotational velocity $(u_{o_r} + u_{1_r}^{(o)})$ is

$$(u_{o_r} + u_{1_r}^{(o)}) = \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi}\, e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]} \Bigg\{ R_L \mathbb{c}_x \left(\frac{1}{R} + \frac{1}{R_1}\right) +$$

$$-\left(\frac{R_L}{2}\right) \mathbb{c}_x \left[\frac{1}{R}\left(1 - \frac{(x-\xi)}{R}\right) + \frac{1}{R_1}\left(1 - \frac{(x-\xi)}{R_1}\right)\right] +$$

$$+\left(\frac{R_L}{2}\right) \mathbb{c}_y^D \frac{1}{R^2}\left[\left(\frac{2y^2}{b^2} - 1\right) + \frac{(x-\xi)y^2}{b^2 R}\left(1 - \frac{(x-\xi)}{R}\right)\right] +$$

$$+\left(\frac{R_L}{2}\right) \mathbb{c}_y^D \frac{1}{R_1^2}\left[\left(\frac{2y^2}{b_1^2} - 1\right) + \frac{(x-\xi)y^2}{b_1^2 R_1}\left(1 - \frac{(x-\xi)}{R_1}\right)\right] +$$

$$+\left(\frac{R_L}{2}\right) \mathbb{c}_z \left[\frac{(z-\zeta)}{R^2} + \frac{(z+\zeta)}{R_1^2}\right] \Bigg\} , \tag{7.28}$$

and u_{s_T} is given in Eq. (6.14).

3. Shear Stress and the Forcelet Distributions

The forcelet distributions $\mathbb{c}_x$, $\mathbb{c}_y^D$, and $\mathbb{c}_z$ are shown here to have definite physical interpretations. Intuitively, one feels that they should be related to the fluid shear stresses acting over the surface of the body, and this is indeed the case. The linearized approximations to the horizontal and vertical shear stress components are obtained in the limit as $y \to 0$, and for x, z ranging inside the centerplane area S_o. When τ_{yx} and τ_{yz} are made dimensionless by ρU^2, the general expressions in the nondimensional variables are

$$\tau_{yx} = \frac{1}{R_L}\left[\frac{\partial}{\partial y}\ (u_o+u_1^{(o)}+u_s) + \frac{\partial}{\partial x}\ (v_o+v_1^{(o)}+v_s)\right] \qquad (y=0) \quad , (7.29)$$

$$\tau_{yz} = \frac{1}{R_L}\left[\frac{\partial}{\partial y}\ w_o+w_1^{(o)}+w_s) + \frac{\partial}{\partial z}\ (v_o+v_1^{(o)}+v_s)\right] \qquad (y=0) \quad . (7.30)$$

Using the substitutions (7.9), and performing the limiting operations for all the derivatives indicated in (7.29) and (7.30), we obtain the following nondimensional results

$$\tau_{yx} = -\frac{1}{4}\left[\mathfrak{c}_x(x,z) - \frac{\partial}{\partial x}\ \mathfrak{c}_y^D(x,z)\right] , \qquad (7.31)$$

$$\tau_{yz} = -\frac{1}{4}\ \mathfrak{c}_z(x,z) \quad , \qquad (7.32)$$

where use has again been made of the provision that $\mathfrak{c}_y^D$ vanishes at the endpoints of the region S_o (see Eq. (7.25)).

4. Approximate Solution for M_o, X_o, Y_o^D, and Z_o

The Oseen-flow boundary value problem for the four non-dimensional distributions $\mathfrak{c}_m, \mathfrak{c}_x, \mathfrak{c}_y^D$, and $\mathfrak{c}_z$ is contained in the four integral equations (7.18), (7.19), (7.20), and (7.26). As they stand, these equations are too complicated to solve by any other means than by an extensive numerical procedure. However, it is questionable whether the result from such a calculation would actually justify the cost and effort involved. For the present, some interesting results can be obtained from an approximate solution.

In this section it is useful to convert all the flow quantities back to their dimensional forms. The conversion is based on the reference length L, the reference velocity U, and the coefficients defined in Eqs. (6.3) and (6.4). For example, the dimensional version of the kinematic condition (7.26) becomes

$$\left[M_o(x,z) - \frac{X_o(x,z)}{\rho U}\right] = 2U \frac{\partial}{\partial x}\left[h(x,z) + \delta^*(x,z)\right] \quad ,(7.33)$$

where the singularities are distributed over the region S_o, for which $0 < x < L$, $-T < z < 0$. The solution obtained here is essentially an iterative one, and is begun by assuming that (7.33) can be split as follows

$$M_o(x,z) = 2U \frac{\partial h}{\partial x} \quad (7.34)$$

and

$$-\left[\frac{X_o(x,z)}{\rho U}\right] = 2U \frac{\partial}{\partial x} \delta^* \quad . \quad (7.35)$$

Further, we assume that the zeroth order solution for $X_o(x,z)$ is equal to twice the laminar shear stress function (to agree with Eq. (6.67) for laminar flow). Then

$$\left[\frac{X_o(x,z)}{\rho U}\right] = A_x \sqrt{\frac{L}{R_L}} \frac{1}{\sqrt{x}} \qquad (0 \leq x \leq L) \quad , \quad (7.36)$$

where $A_x = -(.664)U$ for the flat plate shear stress. From the dimensional version of Eq. (7.19), we can solve immediately for the $Y_o^D(x,z)$ distribution

$$\left[\frac{Y_o^D(x,z)}{\rho UL}\right] = -\frac{4}{3R_L} M_o(x,z) \quad (7.37)$$

Now all that remains for the completion of the approximate solution is to determine the vertical force distribution $Z_o(x,z)$. This can be accomplished by considering an approximate version of the no-slip condition (7.18). In that equation, we neglect the contributions from the terms involving $\left(1 - e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]}\right)$, $\left(1 - e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R_1]}\right)$, $\mathbb{C}_x\left(1 - \frac{(x-\xi)}{R}\right)$, and $\mathbb{C}_x\left(1 - \frac{(x-\xi)}{R_1}\right)$ as being small compared with the remaining terms. Also, the velocity component $u_s(x,0,z)$ is omitted for this zeroth order solution. Among the remaining terms, there are groups of functions containing the exponential factors $e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R]}$ and $e^{\left(\frac{R_L}{2}\right)[(x-\xi)-R_1]}$. We note that for these particular terms, the range of the integration variable $\xi > x$ gives an exponentially small contribution because R_L is so large. Therefore for the viscous dominated terms, the full ξ-integration is approximated by the partial range $0 < \xi < x$. Converting to dimensional quantities and using the results and approximations discussed above, Eq. (7.18) reduces to

$$\int_o^L d\xi \int_{-T}^{o} d\zeta \Bigg\{ M_o(\xi,\zeta)\left[\frac{(x-\xi)}{R^3} + \frac{(x-\xi)}{R_1^3}\right] +$$
$$+ \int_o^x d\xi \int_{-T}^{o} d\zeta \, e^{\frac{-R_L(z-\zeta)^2}{4L(x-\xi)}} \left[\frac{R_L}{LR}\left[\frac{X_o}{\rho U}\right] - \left(\frac{R_L}{2}\right)\frac{1}{R^2}\left[\frac{Y_o^D}{\rho U L}\right] + \left(\frac{R_L}{2}\right)\frac{(z-\zeta)}{LR^2}\left[\frac{Z_o}{\rho U}\right]\right] +$$
$$+ e^{\frac{-R_L(z+\zeta)^2}{4L(x-\xi)}} \left[\frac{R_L}{LR_1}\left[\frac{X_o}{\rho U}\right] - \left(\frac{R_L}{2}\right)\frac{1}{R_1^2}\left[\frac{Y_o^D}{\rho U L}\right] + \left(\frac{R_L}{2}\right)\frac{(z+\zeta)}{LR_1^2}\left[\frac{Z_o}{\rho U}\right]\right]\Bigg\} +$$
$$- \frac{\pi(9+\pi)}{3} M_o(x,z) \simeq -4\pi U \quad . \qquad (7.38)$$

Upon substitution of the known functions $X_o(\xi,\zeta)$ and $Y_o^D(\xi,\zeta)$ into (7.38), expanding for large $(x-\xi)$, and after performing the ζ-integrations on the approximated integrands, we find that the integral equation reduces to

$$F_M(x,z) - \frac{\pi(9+\pi)}{3} M_o(x,z) + 4\pi U +$$

$$+\sqrt{\pi}\, A_x \int_o^x \frac{d\xi}{\sqrt{\xi}\sqrt{x-\xi}} \left[1 + \operatorname{erf}\sqrt{\frac{\beta}{(x-\xi)}}\right] +$$

$$- \frac{2}{3}\sqrt{\frac{\pi L}{R_L}} \int_o^x \frac{d\xi}{(x-\xi)^{3/2}} M_o(\xi,z) \operatorname{erfc}\sqrt{\frac{\beta}{(x-\xi)}} - \int_o^x \frac{d\xi}{(x-\xi)}\left[\frac{Z_o(\xi,z)}{\rho U}\right] e^{-\frac{\beta}{(x-\xi)}} = 0 \,, \tag{7.39}$$

where

$$F_M(x,z) = \int_o^L d\xi \int_{-T}^o d\zeta\, M_o(\xi,\zeta)\left[\frac{(x-\xi)}{R^3} + \frac{(x-\xi)}{R_1^3}\right] \,, \tag{7.40}$$

$$\beta = \frac{(T+z)^2 R_L}{4L} \gg 1, \quad \text{except for } z \text{ very near } -T \,.$$

It has been necessary to assume that $Y_o^D(x,-z) = -Y_o^D(x,z)$ and $Z_o(x,-z) = Z_o(x,z)$. Equation (7.39) is a Volterra integral equation containing integrals of the convolution type. After applying the Laplace transform (Churchill (1958)), Eq. (7.39) reduces to an algebraic expression in the transform variable S. The transform of the unknown function $Z_o(x,z)$ is $\bar{Z}_o(s,z) = \mathcal{L}[Z_o(x,z)] = \int_o^\infty e^{-sx} Z_o(x,z)dx$.

The transforms of the functions involving $\operatorname{erfc}\sqrt{\frac{\beta}{x}}$ are performed approximately for large β. After solving for $\bar{Z}_o(s,z)$ and applying the inverse transform, we find the result valid for $-T < z < 0$,

$$\left[\frac{Z_o(x,z)}{\rho U}\right] \simeq -\frac{4LM_o(x,z)}{R_L(T+z)} + \mathcal{L}^{-1}\left\{\frac{1}{K_o(\gamma\sqrt{s})}\left[\frac{1}{2}\bar{F}_M(s,z) + \frac{2\pi U}{s} - \frac{\pi(9+\pi)}{6}\bar{M}_o(s,z) + \sqrt{\pi}A_x(\pi - K_1(\gamma\sqrt{s}))\right]\right\}, \tag{7.41}$$

where $\mathcal{L}^{-1}\{\ \}$ indicates the inverse Laplace transform; $K_o(\gamma\sqrt{s})$, $K_1(\gamma\sqrt{s})$ are modified Bessel functions; $\gamma = 2\sqrt{\beta} = (T+z)\sqrt{R_L/L}$; A_x is a coefficient associated with $X_o(x,z)$ (see Eq. (7.36)); the functions $\bar{F}_M(s,z)$ and $\bar{M}_o(s,z)$ are Laplace transforms of (7.40) and the $M_o(x,z)$ distribution respectively. This approximate solution for $Z_o(x,z)$ is not valid for $z = -T$, because near that point the approximations used earlier for large $\beta = \frac{(T+z)^2R_L}{4L}$ would fail. A more detailed and complicated solution for z near $-T$ is not undertaken here. Even the approximate solution of (7.41) cannot be fully inverted in terms of elementary functions.

VIII. AN EXAMPLE OF THE COMPUTATION OF WAVE RESISTANCE

Results from an example of the numerical calculation of the wave resistance are presented in this chapter. This computation has been undertaken in the spirit of an indirect problem. That is, two of the four singularity distributions have been prescribed and varied somewhat arbitrarily. This means that the flow being modelled here is not necessarily the flow about the hull shape indicated by the mass source distribution. The sole purpose of these calculations is to indicate in a preliminary way the qualitative effects of the forcelet distributions upon the wave resistance results derived in Chapter VI.

For this example, we consider only the wave resistance component $R_w^{(o)}$ given in the form Eq. (6.35). Rewriting the formula in dimensional variables, we have that

$$R_w^{(o)} = \frac{\rho g^2}{\pi U^4} \int_0^{\frac{\pi}{2}} \sec^3\theta [\mathbb{P}_o^2 + \mathbb{Q}_o^2]\, d\theta \quad , \tag{8.1}$$

where

$$\begin{bmatrix} \mathbb{P}_o \\ \mathbb{Q}_o \end{bmatrix} = \iint_{S_o} d\xi\, d\zeta\; e^{\zeta \kappa_o \sec^2\theta} \left\{ \left[M_o - \left(\frac{X_o}{\rho U}\right) \right] \begin{pmatrix} \cos \xi_\kappa \\ \sin \xi_\kappa \end{pmatrix} + \right.$$
$$\left. + \left[\sec\theta \left(\frac{Z_o}{\rho U}\right) - \sigma_L \sec^3\theta \sin^2\theta \left(\frac{Y_o^D}{\rho U L}\right) \right] \begin{pmatrix} -\sin \xi_\kappa \\ \cos \xi_\kappa \end{pmatrix} \right\}, \tag{8.2}$$

where
$$\xi_\kappa = (\xi \kappa_o \sec\theta) \quad . \tag{8.3}$$

We recall that the result in (8.1) has been determined from a momentum consideration of the various flow quantities measured at the downstream station $x = x_D$, where x_D is supposed to be large but finite. As they appear in the expression (8.2), the forcelet distribution functions $X_o(\xi, \zeta)$, $Y_o^D(\xi, \zeta)$, and $Z_o(\xi, \zeta)$ merely represent some flow disturbances occurring upstream of the control surface S_2 (cf. Chapter V). Nowhere in the derivation of (8.1) is there a limitation imposed on the detailed near-field behavior of these functions. Therefore, it is interesting to consider forcelet distributions that might approximate turbulent flow shear stresses along the body. It should be noted that such distributions would not arise from the boundary conditions discussed in Chapter VII. Those equations apply strictly to the Oseen approximation of laminar flow.

The mass source distribution $M_o(x, z)$ is determined principally from the slope of the hull function, although in fact the four distribution functions are all interrelated as indicated in the equations (7.18), (7.19), (7.20), and (7.26). To provide some common ground for comparison between computed and experimental wave resistance, the present calculations are based on a simple mathematical hull form. The simplest of Wigley's model shapes is appropriate because it has been used in numerous experimental investigations (e.g., more recently Gadd and Hogben (1963), Shearer and Cross (1965), Lackenby (1965)). It has also been used in some theoretical comparisons of interest here (Gadd (1968) and Beck (1970)). The offsets of the Wigley hull form are given by the equation

$$y = h(\xi,\zeta) = \pm \frac{B}{2}\left(1 - \frac{\xi^2}{\ell^2}\right)\left(1 - \frac{\zeta^2}{T^2}\right) , \quad (-\ell < \xi < \ell \ , \ -T < \zeta < 0) \quad (8.4)$$

where B is the total beam, ℓ is the half length L/2, and T is the draft.

The prescribed singularity distributions used for this example are given as follows

$$M_o(\xi,\zeta) = 2U \frac{\partial h}{\partial \xi} = - \frac{8UB}{L^2} \xi\left(1 - \frac{\zeta^2}{T^2}\right) , \qquad (8.5)$$

$$\left[\frac{X_o(\xi,\zeta)}{\rho U}\right] = - \frac{(.0753)U}{R_L^{1/5}}\left(\frac{L}{\xi+\ell}\right)^{1/5} , \qquad (8.6)$$

$$\left[\frac{Y_o^D(\xi,\zeta)}{\rho UL}\right] = - \frac{\left|A_y\right|}{R_L^{1/5}} M_o(\xi,\zeta) = + \frac{\left|A_y\right|}{R_L^{1/5}}\left(\frac{8UB}{L^2}\right) \xi\left(1 - \frac{\zeta^2}{T^2}\right) , \qquad (8.7)$$

$$\left[\frac{Z_o(\xi,\zeta)}{\rho U}\right] = - \frac{\left|A_z\right|}{R_L^{1/5}}\left(\frac{8UB}{L^2}\right)\frac{\xi^2}{T^2} (T-\zeta) , \qquad (8.8)$$

for $-\ell < \xi < \ell$, $-T < \zeta < 0$, and where A_y and A_z are constants that can be varied in numerical experiments with a computer program used to calculate the wave resistance. The form of Eq. (8.6) is based on a simple splitting of the kinematic boundary condition of Eq. (7.33), such that the X_o distribution may be thought of as being related to the approximate turbulent boundary layer displacement thickness δ^* by

$$-\left[\frac{X_o}{\rho U}\right] = 2U \frac{\partial}{\partial \xi} \delta^*(\xi,\zeta) . \qquad (8.9)$$

The constant (0.0753) appearing in (8.6) has been determined by using the approximation in Eq. (6.58) for the viscous resistance.

R_{v_o} is given here in dimensional variables

$$R_{v_o} \simeq -\int_{-T}^{0} d\zeta \int_{-\ell}^{\ell} X_o(\xi,\zeta)d\xi = \frac{(.094)\rho U^2 TL}{R_L^{1/5}} \quad . \tag{8.10}$$

Choosing the constant as indicated gives fairly good agreement between (8.10) and the viscous resistance curve in Fig. 7 of Lackenby (1965) for the Reynolds number range considered. For the Y_o^D and Z_o distributions, convenient simple expressions are assumed here for ease of computation of the integrals $\mathbb{P}_o$ and $\mathbb{Q}_o$. The Y_o^D distribution resembles the result of (7.37) with R_L^{-1} replaced by $R_L^{-1/5}$.

For purposes of this discussion, the functions $\mathbb{P}_o$ and $\mathbb{Q}_o$ are written in the abbreviated form

$$\mathbb{P}_o(\theta) = -\left(\frac{8UB}{L^2}\right)[H_c] + \frac{(.0753)U}{R_L^{1/5} L^{1/5}}[J_1] + \frac{A_y}{R_L^{1/5}}\left(\frac{8UB}{L^2}\right)\sigma_L \sec^3\theta \sin^2\theta[H_s] + \frac{A_z}{R_L^{1/5}}\left(\frac{8UB}{L^2}\right)\sec\theta[HSZ] \; , \tag{8.11}$$

$$\mathbb{Q}_o(\theta) = -\frac{8UB}{L^2}[H_s] + \frac{(.0753)U}{R_L^{1/5} L^{1/5}}[J_2] + - \frac{A_y}{R_L^{1/5}}\left(\frac{8UB}{L^2}\right)\sigma_L \sec^3\theta \sin^2\theta[H_c] - \frac{A_z}{R_L^{1/5}}\left(\frac{8UB}{L^2}\right)\sec\theta[HCZ] \quad . \tag{8.12}$$

The functions H_c, H_s, J_1, J_2, HCZ, and HSZ are given by the following integrals

$$\begin{bmatrix} H_c \\ H_s \end{bmatrix} = \int_{-T}^{0} d\zeta\, e^{\zeta\kappa_o \sec^2\theta} \left(1 - \frac{\zeta^2}{T^2}\right) \int_{-\ell}^{\ell} \xi \begin{pmatrix} \cos\xi_\kappa \\ \sin\xi_\kappa \end{pmatrix} d\xi \quad , \tag{8.13}$$

$$\begin{bmatrix} J_1 \\ J_2 \end{bmatrix} = \int_{-T}^{0} d\zeta\, e^{\zeta\kappa_o \sec^2\theta} \int_{-\ell}^{\ell} \frac{1}{(\xi+\ell)^{1/5}} \begin{pmatrix} \cos\xi_\kappa \\ \sin\xi_\kappa \end{pmatrix} d\xi \quad , \tag{8.14}$$

$$\begin{bmatrix} HCZ \\ HSZ \end{bmatrix} = \int_{-T}^{0} d\zeta\, e^{\zeta\kappa_o \sec^2\theta} \frac{(T-\zeta)}{T^2} \int_{-\ell}^{\ell} \xi^2 \begin{pmatrix} \cos\xi_\kappa \\ \sin\xi_\kappa \end{pmatrix} d\xi \quad , \quad , \tag{8.15}$$

where ξ_κ is given in (8.3).

In the inviscid limit $R_L \to \infty$, the function $H_c \to 0$ and thus $\mathbb{P}_o(\theta, x) \to 0$. Hence, for the hull function of Eq. (8.4), only the $\mathbb{Q}_o$ function would contribute to the inviscid wave resistance because $\partial h/\partial\xi$ is odd in ξ.

After squaring the functions $\mathbb{P}_o$ and $\mathbb{Q}_o$, we split the wave resistance component $R_w^{(o)}$ into a sum of three parts,

$$R_w^{(o)} = R_{w_o} + R_{w_i} + R_{w_w} \quad , \tag{8.16}$$

where

$$R_{w_o} = \frac{\rho\sigma_L^2 A_m^2}{\pi L^2} \int_0^{\frac{\pi}{2}} \sec^3\theta [H_c^2 + H_s^2]\, d\theta \quad , \tag{8.17}$$

$$R_{w_i} = -\frac{2\rho\sigma_L^2 A_m A_x}{\pi L^2 R_L^{1/5}} \int_0^{\frac{\pi}{2}} d\theta \sec^3\theta \left\{ (H_c \cdot J_1 + H_s \cdot J_2) + \right.$$

$$\left. + \frac{A_z A_m}{A_x} \sec\theta\, (H_c \cdot HSZ - H_s \cdot HCZ) + \right.$$

(cont'd)

$$+ \frac{1}{R_L^{1/5}} \left[A_y \sigma_L \sec^3\theta \sin^2\theta (J_2 \cdot H_c - J_1 \cdot H_s) + A_z \sec\theta (J_2 \cdot HCZ - J_1 \cdot HSZ) + \right.$$

$$\left. - \frac{A_y A_z A_m}{A_x} \sigma_L \sec^4\theta \sin^2\theta (H_s \cdot HSZ + H_c \cdot HCZ) \right] \Bigg\} , \qquad (8.18)$$

$$R_{w_w} = \frac{\rho \sigma_L^2 A_x^2}{\pi L^2 R_L^{2/5}} \int_0^{\frac{\pi}{2}} d\theta \sec^3\theta \Bigg\{ (J_1^2 + J_2^2) +$$

$$+ \frac{A_y^2 A_m^2}{A_x^2} (\sigma_L \sec^3\theta \sin^2\theta)^2 (H_c^2 + H_s^2) + \frac{A_z^2 A_m^2}{A_x^2} [(HCZ)^2 + (HSZ)^2] \Bigg\} , \qquad (8.19)$$

where $A_m = 8UB/L^2$, $A_x = (.0753)U/L^{1/5}$; and where the forcelet strength parameters A_y and A_z are dimensionless constants. The R_{w_o} is essentially the potential flow wave resistance due to the mass source distribution. R_{w_i} is the interaction wave resistance due to cross products of the mass source term and the forcelet terms in the functions $\mathbb{P}_o$ and $\mathbb{Q}_o$. R_{w_w} is the wave resistance due to the squares of all the forcelet terms. While R_{w_o} and R_{w_w} are always positive, the interaction component R_{w_i} displays an undulatory behavior with Froude number, and can take on positive or negative values depending on the choice of the constants A_y and A_z.

To facilitate the numerical computation of the integrals in Eqs. (8.17) - (8.19), it is convenient to use a new variable of integration $t = \tan\theta$. A computer program has been written to calculate the wave resistance integrals for a variety of test cases. The results presented here are for the Froude number range $0.2 < F_L < 0.35$. For comparison purposes, the same hull dimensions as those used by Gadd (1968) and Beck (1970) are also used here, namely

$L = 2\ell = 20.0$ ft.

$B = 2.0$ ft.

$T = 1.25$ ft.

S_B = wetted surface area = $59.52\ ft^2$.

The kinematic viscosity was taken as $\nu = 1.22 \times 10^{-5}\ ft^2/sec$.

Nondimensional drag coefficients C_{w_o}, C_{w_i} and C_{w_w} are formed by dividing respectively the drag quantities (8.17), (8.18), and (8.19) by $\frac{1}{2}\rho U^2 S_B$.

It is interesting to display some of the results of this present work in a form used by Sharma (1960) for comparing his experimental data with potential theory results. With the substitution $u = \tan\theta \sec\theta$ in (8.1) and (8.2) the dimensional functions $\mathbb{P}_o$ and $\mathbb{Q}_o$ can be shown to be related to Sharma's free-wave spectra functions G(u) and F(u) by the following expressions

$$G[u(\theta)] = \frac{4\kappa_o^2 \sec^2\theta}{\pi U(2\sec^2\theta - 1)} \mathbb{P}_o(\theta) \quad , \tag{8.20}$$

$$F[u(\theta)] = \frac{4\kappa_o^2 \sec^2\theta}{\pi U(2\sec^2\theta - 1)} \mathbb{Q}_o(\theta) \quad . \tag{8.21}$$

Figure (8.1) shows a comparison between (a) experimental wave resistance (cf. Lackenby (1965) or Gadd (1968)), (b) potential theory wave resistance, and (c) the present results for $C_w^{(o)}$ calculated from Eqs. (8.15) - (8.18) with values of $A_y = 0.05$ and $A_z = +1.0$. In Fig. (8.2), the wave resistance components C_{w_i} and C_{w_w} are plotted, showing that when A_z remains constant, the interaction wave resistance C_{w_i} oscillates with Froude number. It is noted that in Fig. (8.2) the C_{w_i} curve reaches local maxima at Froude numbers 0.21, 0.25, and 0.32. If at these same three points the sign of A_z is reversed (so that $A_z = -1.0$), the results are indicated by the points ⊡ in both Fig. (8.1) and (8.2). The interesting feature of these graphs is not so much the comparison in magnitude of the wave resistance curves, but rather the qualitative variation with the Froude number. The shifts in the locations of the humps and hollows of $C_w^{(o)}$ as a function of the Froude number seem to accord well with the experimental curve.

Figures (8.3) - (8.6) are plots of the free-wave spectra G(u) and F(u) at selected Froude numbers (see Eqs. (8.20) and (8.21)). Both Fig. (8.3) and (8.4) are for $F_L = 0.25$. The case $A_z = 1.0$ is shown in Fig. (8.3), and the case $A_z = -1.0$ is plotted in Fig. (8.4). The two remaining figures for $F_L = 0.275$ and 0.35 are both for $A_z = 1.0$. In all of these graphs of free wave spectra, the potential theory would predict that $G(u) = 0$.

There is no particular benefit from investigating the indirect problem much further. Extensive computation of all the resistance components for the case of laminar flow must await the complete solution of the boundary value problem of Chapter VII. For the more interesting case of turbulent flow, the experimental determination of shear stress distributions can supply definitive information about the X_o, Y_o^D, and Z_o functions. The usefulness of this theory can only be judged conclusively on how well the derived resistance formulae provide a framework for interpreting experimental results.

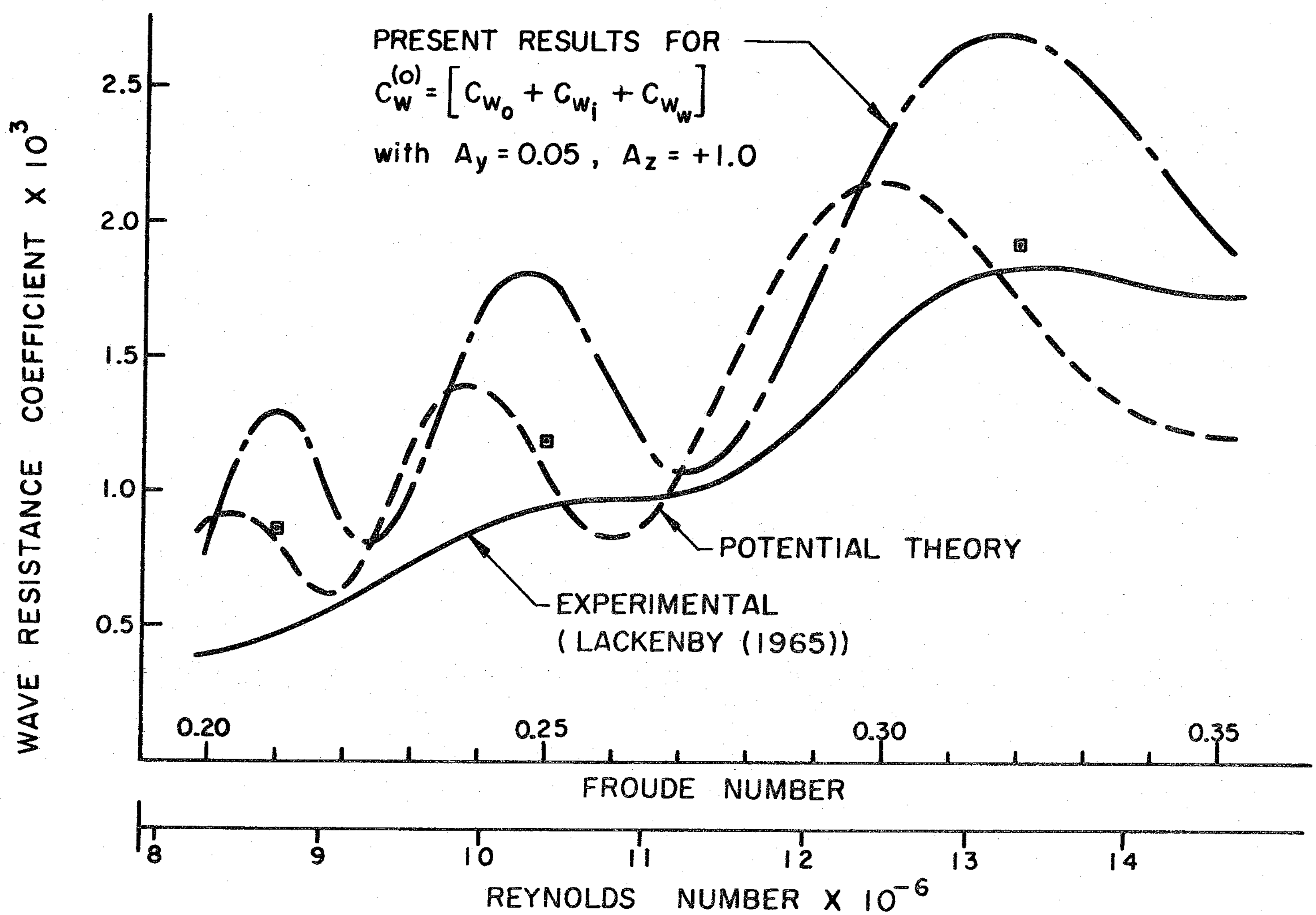


Fig. (8.1) Comparison plot of coefficients of wave resistance. The points indicated by ⊡ are for $C_W^{(o)}$ with $A_z = 1.0$.

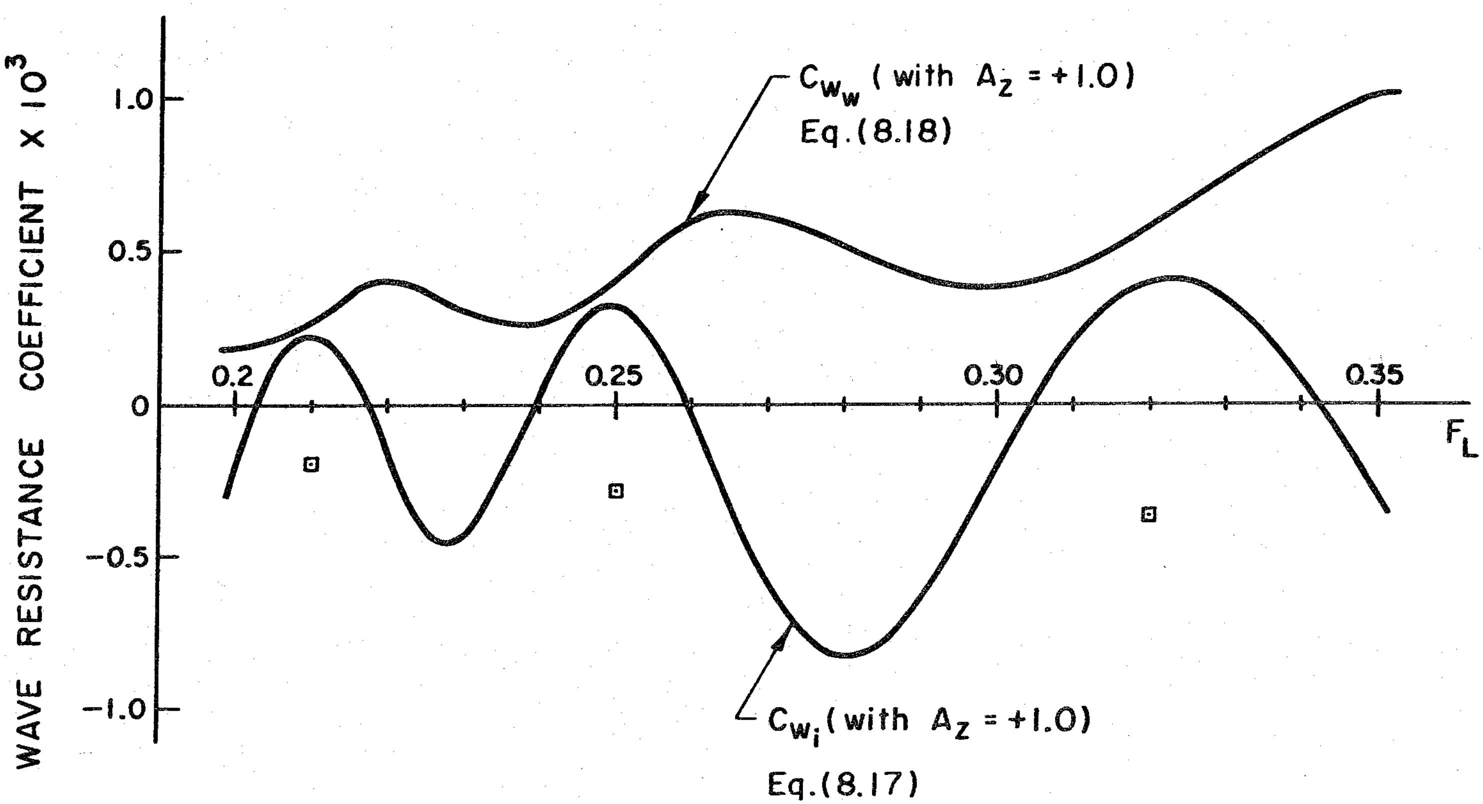


Fig. (8.2) The coefficients of wave resistance components C_{w_i} and C_{w_w} versus Froude number. The points indicated ⊡ are for the C_{w_i} with the forcelet strength parameter $A_z = -1.0$.

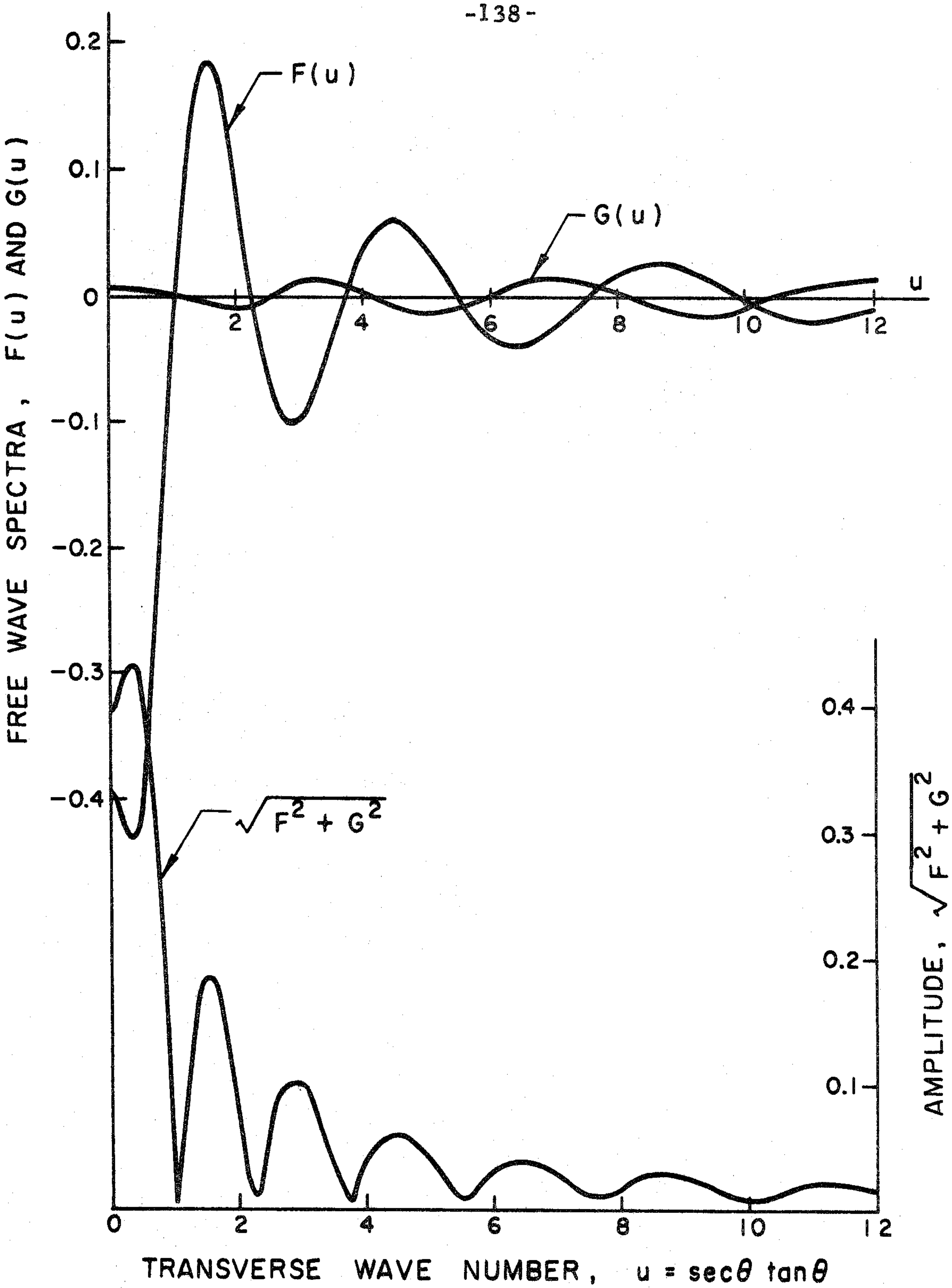


Fig. (8.3) Free-wave spectra at $F_L = 0.25$ ($\sigma_L = 16.0$) with the forcelet strength parameters $A_y = 0.05$, $A_z = 1.0$.

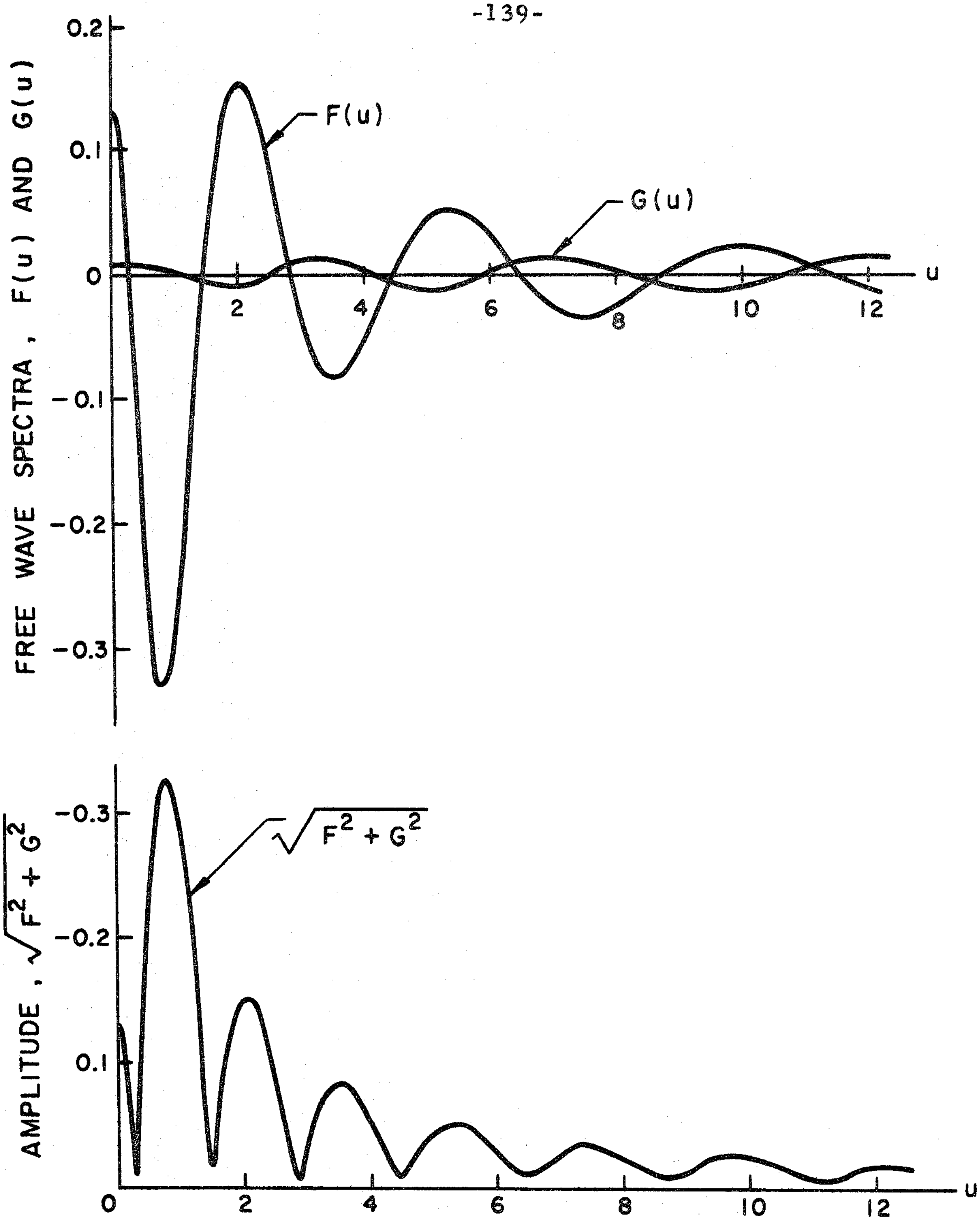
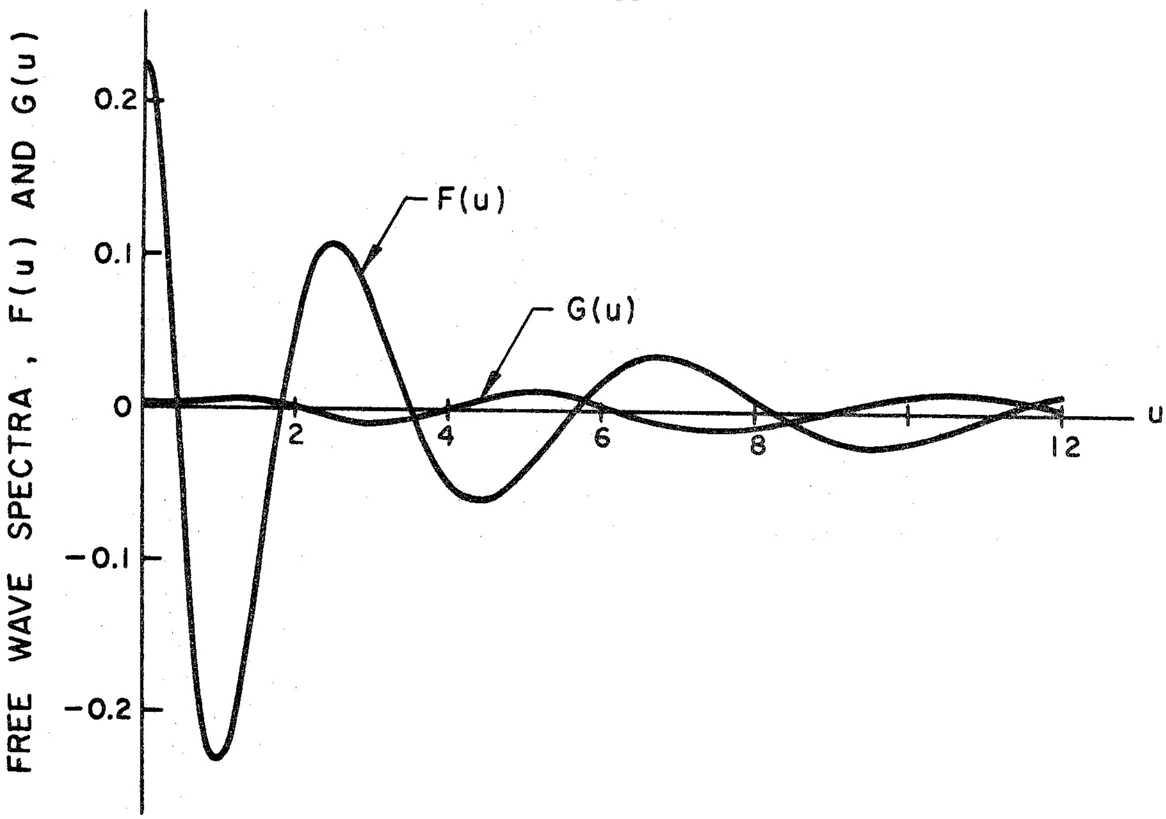


Fig. (8.4) Free-wave spectra at $F_L = 0.25$ ($\sigma_L = 16.0$) with the forcelet strength parameters $A_y = 0.05$, $A_z = -1.0$.



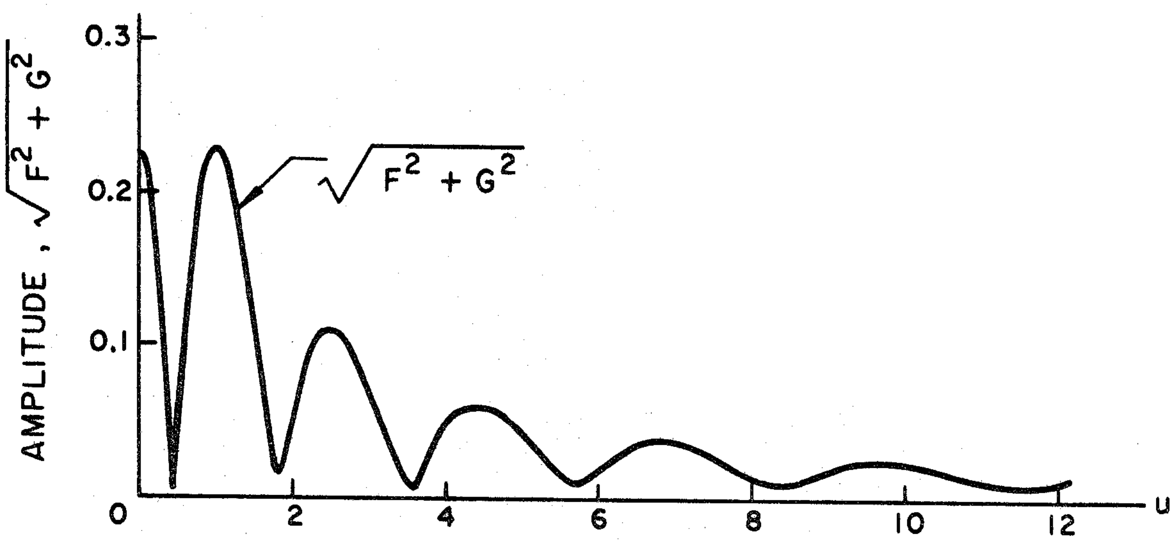


Fig. (8.5) Free-wave spectra at $F_L = 0.275$ ($\sigma_L = 13.2$) with the forcelet strength parameters $A_y = 0.05$, $A_z = 1.0$.

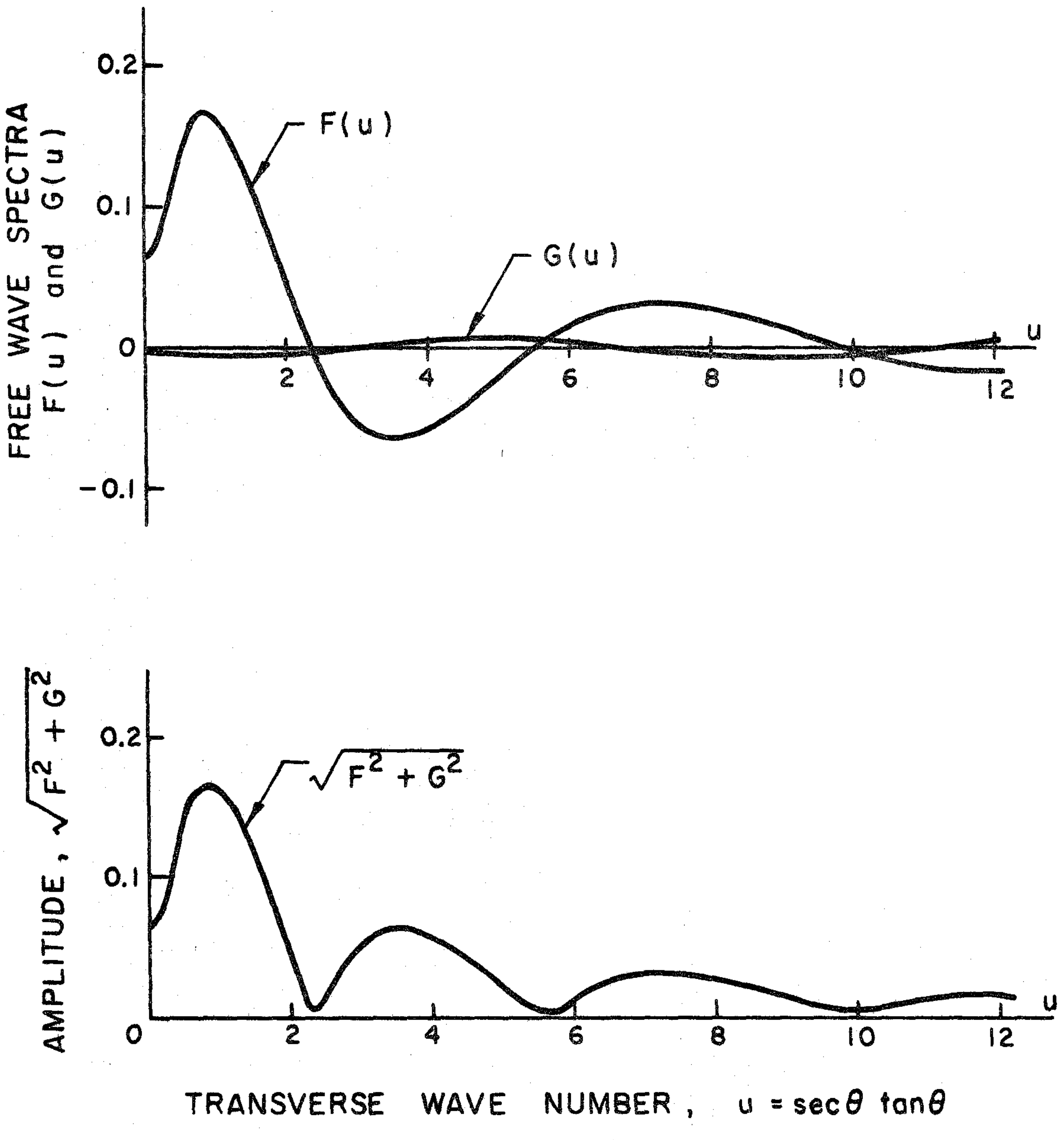


Fig. (8.6) Free-wave spectra at $F_L = 0.35$ ($\sigma_L = 8.16$) with the forcelet strength parameters $A_y = 0.05$, $A_z = 1.0$.

SUMMARY AND DISCUSSION

The work presented in this thesis is divided into two basic parts. In the first part, the complete linearized problem of free surface viscous flow past submerged disturbances is formulated and solved formally by use of double Fourier transforms. The fundamental solutions thus obtained are in themselves interesting because they represent the general Oseen-flow analogy to Havelock's submerged sources. Expressions for the free surface flow quantities $\tilde{u}_1, \tilde{v}_1, \tilde{w}_1, \tilde{p}_1$, and $\tilde{\zeta}$ in transform variables are given in Chapter III in Eqs. (3.40) - (3.55). In those equations the complete transformed solutions for the velocity and pressure components ($\tilde{\vec{q}}_s$, $\tilde{p}_s$) contain terms up to and including order R_ℓ^{-2}.

Inversion of the transformed solution, valid for the far flow regime, is performed approximately using the method of stationary phase. This provides results in terms of the physical variables of the problem for the propagating parts of the wave height and velocity components. All the terms smaller than $O(R_\ell^{-1})$ are neglected, where R_ℓ is assumed large throughout. It is found that the total free wave height $\zeta^{(f)}$ may be split into two parts $\zeta_1^{(f)}$ and $\zeta_2^{(f)}$ corresponding to the two oscillatory functions $\exp[i\sigma_\ell r\psi_o]$ and $\exp[i\sigma_\ell r\Phi]$ respectively. At least in the far field ($r \gg 1$, $\hat{h} < 1$), it is not surprising that the major features of the viscous flow are perturbations about the classical potential flow results.

For $\zeta_1^{(f)}$, the familiar phase function $\psi_o(\theta, \omega)$ of free surface potential theory (Appendix B) provides the stationary points.

Asymptotic formulae for $\zeta_1^{(f)}$ are given in Eqs. (4.7) and (4.29).

For $\zeta_2^{(f)}$, the influence of the depth of submergence and viscosity interact to give the modified phase function $\Phi(\theta, \omega; \hat{h})$ (see Appendix C). Results are presented that show that the $\zeta_2^{(f)}$ terms suffer a very rapid diminution when the depth of submergence is increased, because of the factor $\exp[-h\sqrt{\sigma_\ell R_\ell \sec\theta/2}\,]$. The asymptotic results for $\zeta_2^{(f)}$ are found in Eqs. (4.39) and (4.48). The solutions for both ζ_1 and ζ_2 indicate that there will be a region along the trace of the disturbance where the diverging wave especially is severely damped by viscosity.

Analogous asymptotic results for the propagating parts of the velocities $\vec{q}_{s_L}$ are summarized in Eqs. (4.53) and (4.70); and for $\vec{q}_{s_T}$, the results are given in Eq. (4.60).

Further work on the inversion of the fundamental solutions could deal with, for example, the details of the near field and local disturbance flow quantities, and possibly with a solution for the ζ_2 wave component for moderate or large $\hat{h}$.

The second basic part of the thesis begins in Chapter V where the momentum theorem is used to obtain a result for the total drag experienced by a ship in a viscous fluid. The final expression is written in a form appropriate to the Oseen-flow velocity splitting. The resistance formula contains terms that can be identified with the wave resistance, viscous dissipation drag, and components that may be grouped under the heading of free surface wake-drag.

Modelling the flow around thin ships is approximated by centerplane distributions of the mass source and Oseenlet singularities. Formulae are derived for each of the components of drag in terms of the strengths of the four distribution functions. The principal result of these calculations is the wave resistance formulae for R_w presented in Eqs. (6.35) and (6.36). From (6.35) it is evident that the strengths of the forcelet distributions enter into the first order wave resistance. $R_w^{(1)}$ can usually be neglected compared to $R_w^{(o)}$.

Integral equations for the four distribution functions are derived using the physical boundary conditions. The forcelet distributions X_o, Y_o^D, Z_o are shown to be related directly to the fluid shear stresses acting on the body surface (Eqs. (7.31) and (7.32)). An approximate solution is obtained for the four distributions, although the result for Z_o cannot be completed in terms of elementary functions.

Finally, in the example of the numerical computation of the wave resistance component $R_w^{(o)}$, reasonable functions for the distribution functions have been supplied, and the strengths were varied in computer calculations. The results of these numerical experiments should be viewed as preliminary. It is a promising indication that the positions of the humps and hollows of the computed $R_w^{(o)}$ curve as a function of the Froude number are shifted to a close correspondence with the known experimental curve. Also, by varying the sign of the vertical forcelet parameter A_z, the exaggerated humps of the $R_w^{(o)}$ curve can essentially be eliminated (see Fig. (8.1)). Apparently the vertical forcelet exerts a strong influence on the behavior of R_w. This adds some evidence to the general feeling that the three-dimensional character of the viscous flow has important consequences in terms of ship wave resistance.

Extensions to this work could include: (1) a numerical solution for M_o, X_o, Y_o^D, and Z_o from the boundary conditions of Chapter VII. At best this calculation would be an approximate solution of laminar attached flow. (2) By placing forcelet distributions on the body surface, a second order theory could be developed that would include the possibility of modelling asymmetrical hull shapes. Work along these lines would have to involve second order corrections to the free surface conditions as well.

APPENDIX A

Analytic Properties of the Function $\Delta(k,\theta)$

The function Δ appears in the denominator of all the integrands of the Fourier integral representations of the flow quantities $(\vec{q}_s, p_s; \zeta)$. In its general form,

$$\Delta(\alpha,\beta) = \sigma' - \frac{1}{k_o}\left(\alpha - \frac{2ik_o^2}{R_\ell}\right)^2 - \frac{4k_o^2 k_1}{R_\ell} \quad , \qquad \text{(A.1)}$$

where

$$\sigma' = \sigma_\ell + \frac{k_o^2}{W_\ell} \quad , \quad k_o = \sqrt{\alpha^2+\beta^2} \quad , \quad k_1 = \sqrt{k_o^2 + i R_\ell \alpha} \quad , \quad \sigma_\ell = g\ell / U^2$$

$$W_\ell = \rho\ell U^2 / T \quad , \quad \ell = \text{convenient reference length} \quad .$$

Cumberbatch (1965) has investigated the singularities of this function in the case of zero surface tension $(W_\ell = \infty)$.

In the present work, and for the time being keeping $T \neq 0$, it is more convenient to deal with Δ in terms of the polar coordinates $\alpha = k\cos\theta$, $\beta = k\sin\theta$. Then we have

$$\Delta(k_1\theta) = \sigma_\ell - \frac{k^2}{W_\ell} - k\left(\cos^2\theta - \frac{4ik}{R_\ell}\cos\theta - \frac{4k^2}{R_\ell^2}\right) - \frac{4k^2}{R_\ell^2}\sqrt{k^2 + i R_\ell k\cos\theta} \quad . \qquad \text{(A.2)}$$

There are two branch points at $k = 0$ and at $k = -iR_\ell\cos\theta$, and the k-plane is cut along the imaginary axis between these two points to keep Δ a single-valued function of the complex variable k (see Fig. (A.1)).

The function $\Delta(k,\theta)$ has the same analytic form as the analogous function studied first by Wu and Messick (1958) for the

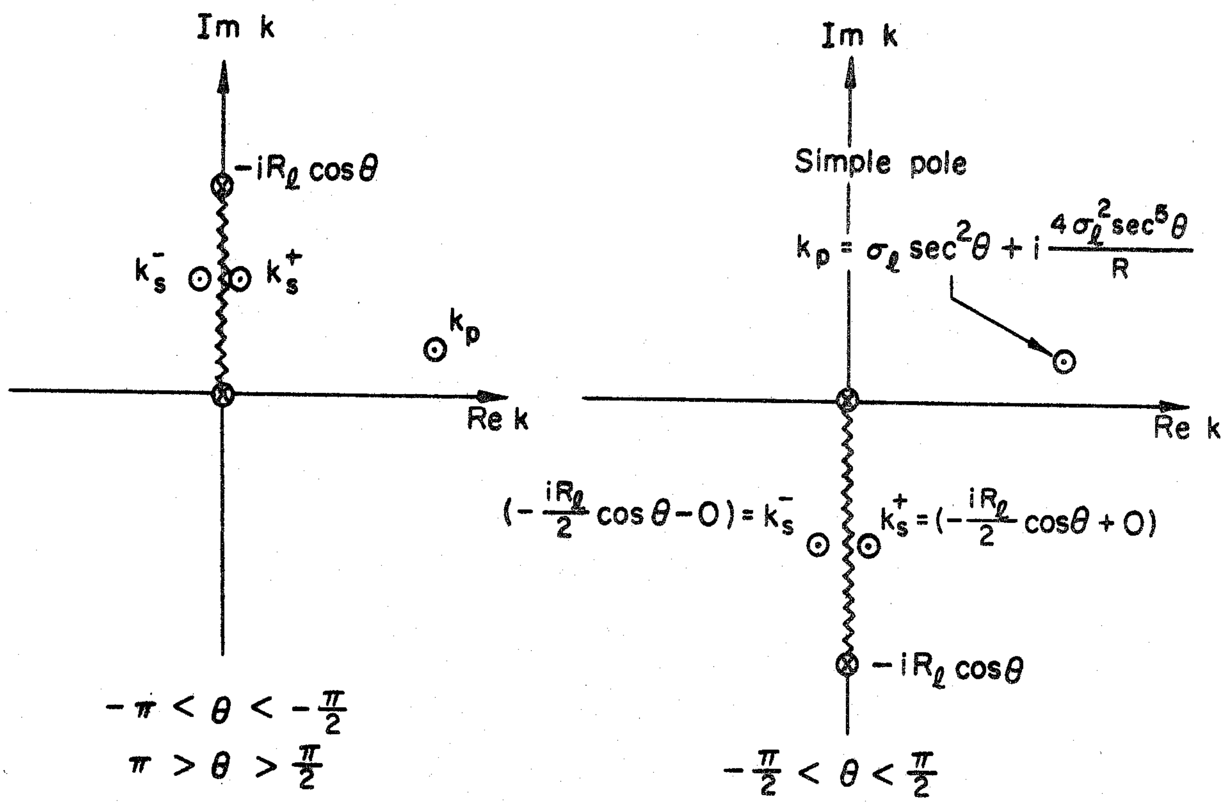


Fig. (A.1) The singularities of the function $\Delta(k, \theta)$ in the finite k-plane, including the effects of surface tension. Branch points are indicated by ⊗ , poles are indicated by ⊙ . The special roots $k_s^{\mp}$ are on the branch cut.

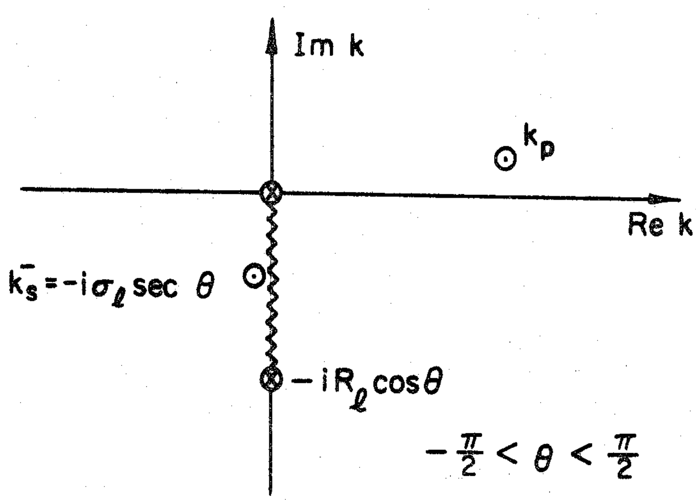


Fig. (A.2) The singularities of the function $\Delta(k, \theta)$ for zero surface tension.

two-dimensional case. That reference is used here as a guide to show that $\Delta(k, \theta)$ has exactly two simple zeros $k_1(\theta)$ and $k_2(\theta)$ in the entire cut k-plane provided that $\frac{4\sigma_\ell \sec^3\theta}{R_\ell}$ is not equal to $\sqrt{\frac{4\sigma_\ell}{W_\ell}\sec^2\theta + 1} \pm 1$. It is assumed that $\sigma_\ell, R_\ell, W_\ell$ are all real and positive. The branch cut is removed by using the transformation

$$k = \frac{R_\ell \cos\theta(\tau - i)^2}{4\tau} . \tag{A.3}$$

This maps the entire cut k-plane conformally into the region $|\tau| \geqslant 1$ of the complex τ-plane. Substituting (A.3) into (A.2) leads to

$$\Delta(k(\tau), \theta) = \frac{1}{\left(\frac{16\sec^2\theta\,\tau^3}{R_\ell^2}\right)}\left\{\frac{16\sigma_\ell}{R_\ell^2}\sec^2\theta\,\tau^3 + \frac{\tau(\tau-i)^4}{W_\ell} + \frac{2i\cos\theta}{R_\ell}(\tau-i)^2(\tau^3+i\tau^2+\tau-1)\right\} . \tag{A.4}$$

Just as in Wu and Messick (1958) this plainly shows that Δ has a triple pole at $\tau = 0$ and five zeros in the entire τ-plane.

On the unit circle $|\tau| = 1$, the equation $\Delta(k(\tau), \theta) = 0$ has two particular solutions. When $\tau = \pm 1$, Eq. (A.4) gives

$$16\sigma_\ell \sec^2\theta \left(\frac{1}{R_\ell}\right)^2 \pm 8\cos\theta\,\frac{1}{R_\ell} - \frac{4}{W_\ell} = 0 , \tag{A.5}$$

so that

$$\frac{1}{R_\ell} = \frac{\cos^3\theta}{4\sigma_\ell}\left[\sqrt{1 + \frac{4\sigma_\ell}{W_\ell}\sec^4\theta} \mp 1\right] , \tag{A.6}$$

where the signs are taken to keep R_ℓ^{-1} positive and real. These particular solutions represent a special relationship between σ_ℓ, R_ℓ, W_ℓ, and θ. It may be shown that these solutions are simple zeros of Δ by expanding (A.4) around $\tau = \pm 1$. Transferring back to the cut k-plane, these points are found to be located on either side of the branch cut at

$$k_s^{\mp} = -i\,\frac{R_\ell \cos\theta}{2} \mp 0 \quad , \tag{A.7}$$

corresponding to

$$\frac{4\sigma_\ell \sec^3\theta}{R_\ell} = \sqrt{1 + \frac{4\sigma_\ell \sec^4\theta}{W_\ell}} \pm 1 \quad .$$

These special roots are indicated in Fig. (A.1).

Aside from the special values of $\sigma_\ell, R_\ell, W_\ell, \theta$ determined by (A.6), the parameters are such that $\Delta(k(\tau), \theta)$ has no zeros on the circle $|\tau| = 1$ and hence no zeros on the cut in the k-plane. Using the properties of analytic functions (Churchill (1960)) it may be shown that because of the pole of $\Delta(\tau, \theta)$ of order three at $\tau = 0$, there are also three zeros inside $|\tau| = 1$. Then since there is a total of five zeros of $\Delta(k(\tau), \theta)$ in the entire τ-plane, there must be only two zeros outside $|\tau| = 1$. Transforming back into the k-plane, this means that there are just two roots in the entire cut k-plane except when $\sigma_\ell, R_\ell, W_\ell$, and θ are related by (A.6).

The two roots of $\Delta(k_1 \theta) = 0$ cannot be determined exactly, but are found approximately in the present case by expanding around the inviscid roots. For $R_\ell \to \infty$, Eq. (A.2) becomes

$$\sigma_\ell + \frac{k^2}{W_\ell} - k\cos^2\theta = 0 \quad , \tag{A.8}$$

for which the two roots are denoted by $\lambda_1(\theta;\sigma_\ell, W_\ell)$ and $\lambda_2(\theta;\sigma_\ell, W_\ell)$

$$\lambda_1 = \frac{W_\ell}{2}\left[\cos^2\theta - \sqrt{\cos^4\theta - \frac{4\sigma_\ell}{W_\ell}}\right] ,$$
$$\lambda_2 = \frac{W_\ell}{2}\left[\cos^2\theta + \sqrt{\cos^4\theta - \frac{4\sigma_\ell}{W_\ell}}\right] . \tag{A.9}$$

We may restrict our attention to the case $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, since the values of θ outside this interval merely give the same information in complex conjugate form. Provided the surface tension $T \neq 0$, the roots λ_1 and λ_2 will remain purely real for the case $\frac{4\sigma_\ell}{W_\ell} \leqslant 1$ when $-\theta_o < \theta < \theta_o$, where the critical angle θ_o is

$$\theta_o = \cos^{-1}\left(\frac{4\sigma_\ell}{W_\ell}\right)^{\frac{1}{4}} . \tag{A.10}$$

If surface tension is the dominant influence, then $\frac{4\sigma_\ell}{W_\ell} > 1$, and the θ interval for real roots λ_1, λ_2 is reduced to zero because $\theta_o = 0$.

The roots $k_1(\theta;\sigma_\ell, W_\ell, R_\ell)$ and $k_2(\theta;\sigma_\ell, W_\ell, R_\ell)$ are expanded about λ_1, λ_2 for R_ℓ large as follows

$$k_1 \simeq \lambda_1 + \alpha_1 \frac{1}{R_\ell} + \alpha_2 \frac{1}{R_\ell^{3/2}} + O\left(\frac{1}{R_\ell^2}\right)$$
$$k_2 \simeq \lambda_2 + \beta_1 \frac{1}{R_\ell} + \beta_2 \frac{1}{R_\ell^{3/2}} + O\left(\frac{1}{R_\ell^2}\right) , \tag{A.11}$$

After substitution of these expansions into $\Delta = 0$ and solving for α_1, $\alpha_2, \beta_1, \beta_2$ we find that

$$k_1 \simeq \lambda_1 + \frac{i4\lambda_1^2\cos\theta}{R_\ell\left(\cos^2\theta - \frac{2\lambda_1}{W_\ell}\right)} + \frac{4e^{i\frac{\pi}{4}}\lambda_1^{5/2}\sqrt{\cos\theta}}{R_\ell^{3/2}\left(\cos^2\theta - \frac{2\lambda_1}{W_\ell}\right)} + O\left(\frac{1}{R_\ell^2}\right) \tag{A.12}$$

$$k_2 \simeq \lambda_2 + \frac{i4\lambda_2^2 \cos\theta}{R_\ell\left(\cos^2\theta - \frac{2\lambda_2}{W_\ell}\right)} + \frac{4e^{i\frac{\pi}{4}}\lambda_2^{5/2}\sqrt{\cos\theta}}{R_\ell^{3/2}\left(\cos^2\theta - \frac{2\lambda_2}{W_\ell}\right)} + O\left(\frac{1}{R_\ell^2}\right) \quad . \qquad (A.13)$$

The root k_1 is in the first quadrant of the complex k-plane and k_2 is in the fourth.

Now, in the limit of vanishing surface tension ($W_\ell \to \infty$), the inviscid roots reduce to

$$\lim_{T\to 0} \lambda_1 = \sigma_\ell \sec^2\theta \qquad \text{(gravity dominated)} \quad , \qquad (A.14)$$

$$\lim_{T\to 0} \lambda_2 = W_\ell \cos^2\theta \qquad \text{(surface tension dominated)} \quad . \qquad (A.15)$$

In the main body of the thesis we are interested in the case of negligible surface tension. The results above indicate that for the case of $W_\ell \to \infty$, the two simple zeros of (A.12) and (A.13) reduce to one simple finite zero in the cut k-plane.

$$k_p(\theta;\sigma_\ell, R_\ell) \simeq \sigma_\ell \sec^2\theta + i\,\frac{4\sigma_\ell^2 \sec^5\theta}{R_\ell} - \frac{4e^{i\frac{\pi}{4}}\sigma_\ell^{5/2}\sec^{13/2}\theta}{R_\ell^{3/2}} + O\left(\frac{1}{R_\ell^2}\right) . \qquad (A.16)$$

For all the main results of the thesis work, it is sufficient to use only the first two terms of this expansion.

It is interesting to observe what happens to the special roots $k_s^{\mp}$ when $W_\ell \to \infty$. From (A.6) and (A.7), we see that in the limit of zero surface tension, the root k_s^+ associated with $\frac{4\sigma_\ell \sec^2\theta}{R_\ell} = \sqrt{1 + \frac{4\sigma_\ell}{W_\ell}\sec^4\theta} - 1$ corresponds to $R_\ell^{-1} = 0$ or the inviscid case, for which there is one simple zero in the cut k-plane. The root k_s^- in the same limit corresponds to $\frac{1}{R_\ell} = \frac{\cos^3\theta}{2\sigma_\ell}$, and it is located along the left side of the cut at

$$k_s^- = -i\sigma_\ell \sec^2\theta - 0 \quad . \tag{A.17}$$

APPENDIX B

Phase Function $\psi_o(\theta,\omega)$ and Related Asymptotic Representations

The phase function $\psi_o(\theta,\omega)$ is

$$\psi_o(\theta,\omega) = \sec^2\theta \cos(\theta-\omega) \quad , \tag{B.1}$$

where $x = r\cos\omega$, $y = r\sin\omega$. Points of stationary phase are determined from the solution of $\frac{\partial\psi_o}{\partial\theta} = 0$, and are therefore given by

$$(2\tan\omega)\tan^2\theta + \tan\theta + \tan\omega = 0 \quad . \tag{B.2}$$

The roots are

$$\begin{pmatrix} \tan\theta_1 \\ \\ \tan\theta_2 \end{pmatrix} = \frac{-1 \pm \sqrt{1-8\tan^2\omega}}{4\tan\omega} \quad , \tag{B.3}$$

and these are real provided

$$1 - 8\tan^2\omega \geqslant 0 \quad . \tag{B.4}$$

Hence for the stationary points to remain on the path of integration in (4.5), the angle ω must be inside the wedge $0 < \omega < \omega_c$, where ω_c is the Kelvin angle

$$\omega_c = \tan^{-1}\frac{1}{2\sqrt{2}} \cong 19°28' \quad . \tag{B.5}$$

When $\omega = \omega_c$, the two roots θ_1, θ_2 coalesce to the same value

$$\theta_1 = \theta_2 = \theta_c \quad , \tag{B.6}$$

where

$$\theta_c = \tan^{-1}\left(\frac{-\sqrt{2}}{2}\right) = -35°16' \quad . \tag{B.7}$$

Rewriting Eq. (B.2), we have that for $\theta = \theta_1, \theta_2$ the locus of points corresponding to stationary phase is given by

$$\tan\omega = -\frac{\tan\theta}{1+2\tan^2\theta} \quad , \tag{B.8}$$

and thus

$$\frac{\partial}{\partial\theta}(\tan\omega) = \frac{-\sec^2\theta}{(1+2\tan^2\theta)^2}\,[1-2\tan^2\theta] \quad . \tag{B.9}$$

When $\theta = \theta_c$, we find that $\frac{\partial}{\partial\theta}(\tan\omega) = 0$.

The second derivative of the phase function, evaluated at the stationary points is

$$\frac{\partial^2\psi_o}{\partial\theta^2}(\theta) = \sec^2\theta\cos(\theta-\omega)[1-2\tan^2\theta] \qquad \text{at } \theta_1, \theta_2 \quad . \tag{B.10}$$

This function appears explicitly in certain of the asymptotic expansion formulas, and the sign of $\frac{\partial^2\psi_o}{\partial\theta^2}(\theta_1 \text{ or } \theta_2)$ has a direct bearing on the character of the constant phase lines of the wave system. We find that

$$\frac{\partial^2\psi_o}{\partial\theta^2}(\theta_1) > 0 \qquad \text{and} \qquad \frac{\partial^2\psi_o}{\partial\theta^2}(\theta_2) < 0 \quad . \tag{B.11}$$

The point θ_1 is associated with the transverse free surface wave system (the phase function is concave upward) and θ_2 gives the diverging wave system ($\psi_o(\theta_2,\omega)$ is concave downward). The principal features of these results are displayed in Figs. (B.1) and (B.2).

For values of ω within or on the wedge $0 \leqslant \omega \leqslant \omega_c$, there are

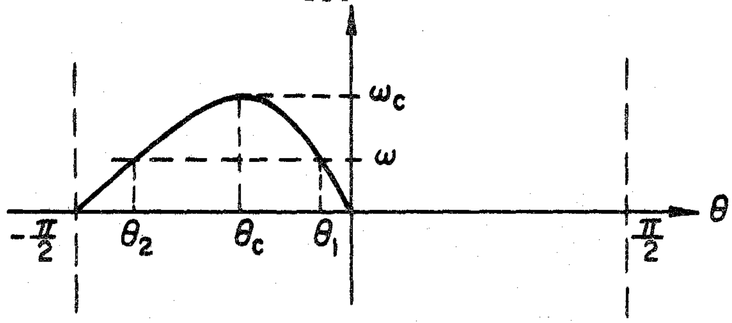


Fig. (B.1) Locus of stationary phase points from Eq. (B.8). For a given ω, the stationary points are θ_1 and θ_2. These points coalesce when $\omega = \omega_c$.

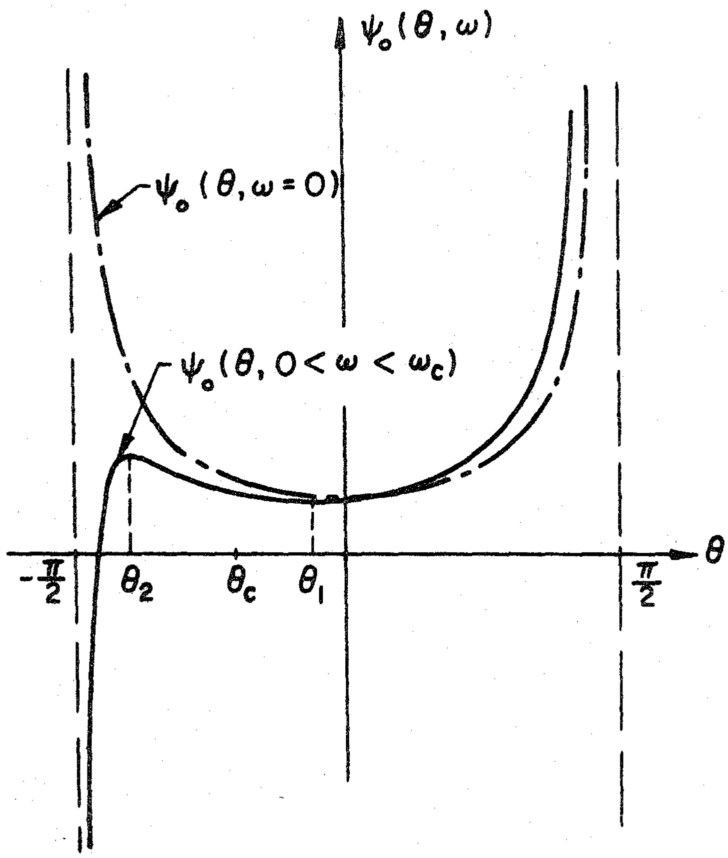


Fig. (B.2) The phase function $\psi_o(\theta,\omega) = \sec^2\theta\cos(\theta-\omega)$. The stationary phase points appear as local extrema of the $\psi_o(\theta)$ curve. In the special case $\omega = 0$, $\psi_o(\theta) = \sec\theta$.

two interesting limits: $\omega \to 0$ and $\omega \to \omega_c$.

(a) As $\omega \to 0$

$$\begin{aligned}
&\tan\theta_1 \to -\omega \quad, && \tan\theta_2 \to -\frac{1}{2\omega} \\
&\theta_1 \to 0- \quad, && \theta_2 \to -\frac{\pi}{2} \\
&\psi_o(\theta_1,\omega) \to 1 \quad, && \psi_o(\theta_2,\omega) \to \infty \\
&\frac{\partial^2\psi_o}{\partial\theta^2}(\theta_1,\omega) \to 1 \quad, && \frac{\partial^2\psi_o}{\partial\theta^2} \to -\infty \quad .
\end{aligned} \tag{B.12}$$

(b) As $\omega \to \omega_c$

$$\begin{aligned}
&\tan\theta_1 \to \tan\theta_2 = \tan\theta_c = -\frac{\sqrt{2}}{2} \\
&\psi_o(\theta_c,\omega_c) = \frac{\sqrt{3}}{2} \\
&\frac{\partial^2\psi_o}{\partial\theta^2}(\theta_c,\omega_c) = 0 \quad .
\end{aligned} \tag{B.13}$$

(1) When applying the method of stationary phase for $0 \leqslant \omega < \omega_c$, the phase function $\psi_o(\theta,\omega)$ is expanded in a Taylor series about the first order stationary points θ_1 and θ_2.

$$\theta \text{ near } \theta_1,\ \psi_o(\theta,\omega) = \psi_o(\theta_1,\omega) = \frac{1}{2}\frac{\partial^2\psi_o}{\partial\theta^2}(\theta_1,\omega)(\theta-\theta_1)^2 \tag{B.14}$$

$$\theta \text{ near } \theta_2,\ \psi_o(\theta,\omega) = \psi_o(\theta_2,\omega) + \frac{1}{2}\frac{\partial^2\psi_o}{\partial\theta^2}(\theta_2,\omega)(\theta-\theta_2)^2 \quad , \tag{B.15}$$

where $\dfrac{\partial^2\psi_o}{\partial\theta^2}(\theta,\omega)$ is given by (B.10). If the original integral to be evaluated has the form

$$I = \mathrm{Re}\int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\theta) e^{i\sigma_\ell r\psi_o(\theta,\omega)} d\theta \quad , \qquad \text{(B.16)}$$

where $F(\theta)$ is a slowly varying function of θ, and $\psi_o(\theta,\omega)$ is the function discussed above, it can be shown* that for large $\sigma_\ell r$

$$I \sim \mathrm{Re}\sum_{j=1}^{2}\left\{F(\theta_j)\sqrt{\frac{2\pi}{|\psi_{o_{\theta\theta}}(\theta_j)|}}\;\frac{e^{i\sigma_\ell r\psi_o(\theta_j)+i\frac{\pi}{4}\mathrm{sgn}(\psi_{o_{\theta\theta}}(\theta_j))}}{\sqrt{\sigma_\ell r}}\right\} + O\left(\frac{1}{\sigma_\ell r}\right) \; . \qquad \text{(B.17)}$$

(2) Now when $\omega = \omega_c$, the first order stationary phase analysis breaks down because $\frac{\partial^2\psi_o}{\partial\theta^2}(\theta_1) \to \frac{\partial^2\psi_o}{\partial\theta^2}(\theta_2) \to \frac{\partial^2\psi_o}{\partial\theta}(\theta_c) = 0$ in the asymptotic result (B.17). The point $\theta = \theta_c$ is a double root of Eq. (B.2) and represents a stationary point of order two. The situation is remedied by expanding the phase function about θ_c and ω_c, including terms up to order $(\omega-\omega_c)^2$ and $(\theta-\theta_c)^3$.

From Appendix C, the result for the phase function

$$\Phi(\theta,\omega;\hat{h},R_\ell) = \psi_o(\theta,\omega) - \hat{h}\sqrt{\sec\theta}$$

is specialized to the present case by putting $\hat{h} = 0$ in Eq. (C.14). The correct expansion for $\psi_o(\theta,\omega)$ in the neighborhood of $\omega = \omega_c$, $\theta = \theta_c$ is

$$\psi_o(\theta,\omega) = \frac{\sqrt{3}}{2}\left(1-\sqrt{2}\,\bar{\omega} - \frac{1}{2}\bar{\omega}^2\right) + 3\bar{\omega}\bar{\theta} - 3\sqrt{2}\,\bar{\omega}\,\bar{\theta}^2 + \frac{1}{\sqrt{2}}\bar{\theta}^3 \quad , \qquad \text{(B.18)}$$

where $\bar{\theta} = \theta - \theta_c$, $\bar{\omega} = \omega - \omega_c$.

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*see, for example Erdelyi (1956).

This result is the same as the formula obtained by Plesset and Wu (1960). If the original integral to be evaluated for ω near ω_c has the form (B.16), the asymptotic expansion for $\sigma_\ell r$** large is

$$I \sim \operatorname{Re} F(\theta_c)\sqrt{\frac{2}{3}}\,\frac{2\pi}{(\sigma_\ell r)^{1/3}}\,A_i(Z_\omega^\circ)\,e^{i(\sigma_\ell r)\frac{\sqrt{3}}{2}\left[1-\sqrt{2}\,\bar{\omega}+\frac{11}{2}\bar{\omega}^2\right]} + O\left(\frac{1}{\sigma_\ell r}\right)^{2/3} , \qquad \text{(B.19)}$$

where $A_i(Z_\omega^\circ)$ is the Airy function*** with argument

$$Z_\omega^\circ = \frac{3}{\sqrt{2}}(\sigma_\ell r)^{2/3}\,\bar{\omega}\,(1-2\sqrt{2}\,\bar{\omega}) \quad . \qquad \text{(B.20)}$$

(3) For $\omega_c < \omega \leqslant \pi$ there are no stationary phase points of $\psi_o(\theta,\omega)$ on the path of integration, and the asymptotic behavior of integrals of the form (B.16) can be calculated by integration by parts. As in Eq. (4.5), the type of integrand functions represented by $F(\theta)$ in (B.16) are such that $F\left(\theta = \frac{\pi}{2}\right) = 0$, and we find that

$$I \sim \operatorname{Re}\left[\frac{i}{(\sigma_\ell r)}\;\frac{F(\theta=\omega-\frac{\pi}{2})}{\csc^2\omega}\right] + O\left(\frac{1}{\sigma_\ell r}\right)^2 , \qquad \text{(B.21)}$$

since $\frac{\partial\psi_o}{\partial\theta}\left(\theta=\omega-\frac{\pi}{2}\right) = \csc^2\omega$, $e^{i(\sigma_\ell r)\psi_o(\theta=\omega-\frac{\pi}{2},\,\omega)} = 1$. Hence for ω outside the cusp line $\omega = \omega_c$, the wave amplitudes are at most $O\left(\frac{1}{r}\right)$.

** see Plesset and Wu (1960).

*** see Abramowitz and Stegun (1964), p. 447.

APPENDIX C

Phase Function $\Phi(\theta,\omega;\hat{h})$ and Related Asymptotic Representations

When viscosity is included in the analysis of free surface flows, certain of the terms in the integrands of the θ-integrals contain the oscillating function $\exp[i(\sigma_\ell r)\Phi]$. The phase function

$$\Phi(\theta,\omega;\hat{h}) = \psi_o(\theta,\omega) - \hat{h}\sqrt{\sec\theta} \tag{C.1}$$

contains the parameter $\hat{h}$ which depends on depth of submergence, Reynolds number, and Froude number

$$\hat{h} = \frac{h}{r}\sqrt{\frac{R_\ell}{2\sigma_\ell}} \quad . \tag{C.2}$$

We shall assume that although the Reynolds number R_ℓ is very large, the interesting range of h is so small and r is to be taken far enough downstream that $\hat{h}$ is assumed to be a small number, $\hat{h} < 1$.

The points of stationary phase are determined from the zeros of $\frac{\partial\Phi}{\partial\theta} = 0$:

$$(2\tan\omega)\tan^2\theta + \tan\theta + \tan\omega - \frac{\hat{h}}{2}\,\frac{\tan\theta}{\cos\omega(1+\tan^2\theta)^{1/4}} = 0 \quad . \tag{C.3}$$

For $0 \leqslant \omega < \omega_c$, it is reasonable to expect that Φ will have two first order stationary points similar to ψ_o. To find these roots approximately, we expand around the point θ_{12} (cf. Appendix B) as follows:

$$\tan\theta = \tan\theta_{1,2} + \hat{h}\tan\varphi_{1,2} \quad . \tag{C.4}$$

Substituting (C.4) into (C.3), expanding powers of $\hat{h}$, and neglecting terms of order $\hat{h}^2$ or smaller, we obtain the functions $\tan\varphi_{1,2}$ by equating like powers of $\hat{h}$. The two real roots of Eq. (C.3) are the stationary points of Φ, and are denoted by $\theta = t_{1,2}$. We find that

$$\tan t_{1,2} = \tan\theta_{1,2}\,[\,1 + \hat{h}\,\gamma_{1,2}\,] \quad , \tag{C.5}$$

where

$$\tan\theta_{1,2} = \frac{-1 \pm \sqrt{1-8\tan^2\omega}}{4\tan\omega} \quad , \tag{C.6}$$

$$\gamma_{1,2} = \frac{1}{2\cos\omega\sqrt{\sec\theta_{1,2}}\,[\pm\sqrt{1-8\tan^2\omega}\,]} \quad , \tag{C.7}$$

with the subscript 1 corresponding to the (+) and 2 corresponds to the (-). These expressions are valid inside $0 < \omega < \omega_c$; and they fail when $\omega \to \omega_c$. A separate approximation must be undertaken for ω near ω_c.

The second derivative of the phase function can be simplified to the form

$$\frac{\partial^2}{\partial\theta^2}\Phi(\theta,\omega;\hat{h}) = \psi_o(\theta,\omega)\{1+6\tan^2\theta - 4\tan\theta\tan(\theta-\omega)\} - \frac{\hat{h}}{2}\sqrt{\sec\theta}\left(1+\frac{3}{2}\tan^2\theta\right), \tag{C.8}$$

and it may be shown that

$$\frac{\partial^2\Phi}{\partial\theta^2}(t_1,\omega;\hat{h}) > 0 \quad \text{and} \quad \frac{\partial^2\Phi}{\partial\theta^2}(t_2,\omega;\hat{h}) < 0 \tag{C.9}$$

parallel to the results for $\dfrac{\partial^2\psi_o}{\partial\theta^2}(\theta,\omega)$ at the points θ_1 and θ_2 respectively. The root t_1 is associated with the transverse wave system, and t_2 corresponds to the diverging wave system.

(1) For $0 \leqslant \omega < \omega_c$, we use Taylor series expansions about the stationary points $\theta = t_1$ and t_2 as in (B.14) and (B.15).

Application of the method of stationary phase to an integral of the form

$$J = \mathrm{Re}\int_{\omega-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\theta) e^{i(\sigma_\ell r)\Phi(\theta,\omega;\hat{h})} d\theta \quad , \tag{C.10}$$

result in an asymptotic representation for J given by the formula (B.17) of Appendix B with $\psi_{o_{\theta\theta}}(\theta_j)$ replaced by $\Phi_{\theta\theta}(t_j)$, θ_j replaced by t_j, etc.

(2) In the neighborhood of the cusp line $\omega = \omega_c$, the first order stationary phase approximation for Φ is no longer valid. The expansions of the stationary points $t_{1,2}$ become infinite. This is resolved by forcing $\Phi_{\theta\theta}(\theta, \omega_c;\hat{h}) = 0$ at some new point $\theta = t_c$. To approximate the root, we expand about the critical point θ_c

$$\tan\theta = \tan\theta_c + \hat{h}\tan\varphi_c \quad . \tag{C.11}$$

The new second order stationary point is found to be

$$\tan t_c = -\frac{\sqrt{2}}{2}\left[1 - \hat{h}\,\frac{7\sqrt{2}}{16}\left(\frac{2}{3}\right)^{1/4}\right] . \tag{C.12}$$

For ω near ω_c we expand $\Phi(\theta,\omega;\hat{h})$ about $\theta = t_c$ and $\omega = \omega_c$

$$\begin{aligned}\Phi(\theta,\omega;\hat{h}) &= \Phi(t_c,\omega_c;\hat{h}) + \left.\frac{\partial\Phi}{\partial\theta}\right|_c(\theta-t_c) + \left.\frac{\partial\Phi}{\partial\omega}\right|_c(\omega-\omega_c) + \frac{1}{2}\left.\frac{\partial^2\Phi}{\partial\theta^2}\right|_c(\theta-t_c)^2 + \\ &+ \left.\frac{\partial^2\Phi}{\partial\theta\partial\omega}\right|_c(\theta-t_c)(\omega-\omega_c) + \frac{1}{2}\left.\frac{\partial^2\Phi}{\partial\omega^2}\right|_c(\omega-\omega_c)^2 + \frac{1}{3!}\left.\frac{\partial^3\Phi}{\partial\theta^3}\right|_c(\theta-t_c)^3 + \\ &+ \frac{3}{3!}\left.\frac{\partial^3\Phi}{\partial\theta^2\partial\omega}\right|_c(\theta-t_c)^2(\omega-\omega_c) + \frac{3}{3!}\left.\frac{\partial^3\Phi}{\partial\theta\partial\omega^2}\right|_c(\theta-t_c)(\omega-\omega_c)^2 \quad ,\end{aligned} \tag{C.13}$$

neglecting terms of order $(\omega-\omega_c)^3$, $(\theta-t_c)^4$, and higher.

It is found that

$$\Phi(\theta,\omega;h) \simeq \frac{\sqrt{3}}{2}\left\{\left[1-\hat{h}\sqrt{2}\left(\frac{2}{3}\right)^{\frac{1}{4}} - \sqrt{2}\,\bar{\omega}\left(1-\hat{h}\;\frac{7\sqrt{2}}{16}\left(\frac{2}{3}\right)^{\frac{1}{4}}\right) - \frac{1}{2}\,\bar{\omega}^2\right] + \right.$$

$$\left. + [\,3\bar{\omega}\,\alpha_1 + \alpha_2]\,\bar{\theta} + \beta_1(-3\sqrt{2})\bar{\omega}\,\theta^2 + c_1\frac{\bar{\theta}^3}{\sqrt{2}}\right\} , \qquad \text{(C.14)}$$

where

$$\begin{aligned} \alpha_1 &= \left(1 - \hat{h}\;\frac{7\sqrt{2}}{12}\left(\frac{2}{3}\right)^{\frac{1}{4}}\right) \\ \alpha_2 &= \frac{\hat{h}}{2}\left(\frac{2}{3}\right)^{\frac{1}{4}} \\ \beta_1 &= \left(1 - \hat{h}\;\frac{21\sqrt{2}}{24}\left(\frac{2}{3}\right)^{\frac{1}{4}}\right) \\ c_1 &= \left(1 - \hat{h}\;\frac{83\sqrt{2}}{96}\left(\frac{2}{3}\right)^{\frac{1}{4}}\right) , \end{aligned} \qquad \text{(C.15)}$$

and $\qquad \bar{\theta} = \theta - t_c \quad , \qquad \bar{\omega} = \omega - \omega_c \quad .$

When $h = 0$, the result in Eq. (C.14) reduces to the correct expansion of the phase function $\psi_o(\theta,\omega)$ about $\omega = \omega_c$ and $\theta = \theta_c$. The formula for $h = 0$ is given in Eq. (B.18) of Appendix B.

We assume that the integral to be evaluated asymptotically has the form (C.10). It can be shown using results of (B.19) that

$$J \sim \operatorname{Re} F(t_c)\sqrt{\frac{2}{3}}\,\frac{2\pi}{(\sigma_\ell r)^{1/3}}\left[1 + \hat{h}\;\frac{83\sqrt{2}}{288}\left(\frac{2}{3}\right)^{\frac{1}{4}}\right]A_i(Z_\omega)$$

$$\times \exp\left\{i\sigma_\ell r\;\frac{\sqrt{3}}{2}\left[\left(1-\sqrt{2}\,\bar{\omega} + \frac{11}{2}\,\bar{\omega}^2\right) - \hat{h}\sqrt{2}\left(\frac{2}{3}\right)^{\frac{1}{4}}\left(1 - \frac{15\sqrt{2}}{16}\,\bar{\omega} + \frac{57}{16}\,\bar{\omega}^2\right)\right]\right\}$$

$$+ O\left(\frac{1}{\sigma_\ell r}\right)^{\frac{2}{3}} , \qquad \text{(C.16)}$$

where

$$Z_\omega = \frac{3}{\sqrt{2}}(\sigma_\ell r)^{\frac{2}{3}}\left\{\bar{\omega}(1-2\sqrt{2}\,\bar{\omega}) + \frac{\hat{h}}{6}\left(\frac{2}{3}\right)^{\frac{1}{4}}\left[1 - \frac{85\sqrt{2}}{48}\,\bar{\omega} + \frac{43\sqrt{2}}{3}\,\bar{\omega}^2\right]\right\} \qquad \text{(C.17)}$$

When $\omega_c < \omega < \pi$, the asymptotic behavior of the integrals containing $\exp[i(\sigma_\ell r)\Phi]$ can be found by integration by parts. As with $\psi_o(\theta,\omega)$, in this range of ω there are no stationary phase points on the path of integration. The result is

$$J \sim \mathrm{Re}\left[\frac{i}{(\sigma_\ell r)}\,\frac{F\left(\theta = \omega - \frac{\pi}{2}\right)}{\csc^2\omega}\left(1 - \frac{\hat{h}}{2}\cot\omega\csc^{-3/2}\omega\right)\right] + O\left(\frac{1}{\sigma_\ell r}\right)^2 \quad . \qquad \text{(C.18)}$$

APPENDIX D

Velocity Components $\vec{q}_o$, $\vec{q}_1^{(o)}$ due to Distributed Singularities

The components of the basic unbounded flow velocity $\vec{q}_o(x, y, z)$ for a symmetrical hull shape are listed here for reference. These equations contain the four nondimensional distributions $c_m(\xi, \zeta)$, $c_x(\xi, \zeta)$, $c_y^D(\xi, \zeta)$, $c_z(\xi, \zeta)$ (cf. Chapter VI). All the indicated differentations in Eqs. (6.9) - (6.10) are carried out in full, and the expressions are dimensionless with the underlines omitted.

The velocities are divided into longitudinal and solenoidal components

$$\vec{q}_o = \vec{q}_{o_L} + \vec{q}_{o_T} \quad . \tag{D.1}$$

Longitudinal components $\vec{q}_{o_L}$:

$$u_{o_L}(x, y, z) = \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi} \left\{ c_m \frac{(x-\xi)}{R^3} - c_x \frac{(x-\xi)}{R^3} + \right.$$

$$\left. + c_y^D \frac{1}{R^3} \left(1 - \frac{3y^2}{R^2}\right) - c_z \frac{(z-\zeta)}{R^3} \right\} \tag{D.2}$$

$$v_{o_L}(x, y, z) = \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi} \left\{ c_m \frac{y}{R^3} - c_x \frac{y}{R^3} + c_y^D \left[\left(1 + \frac{(x-\xi)}{R}\right) \left(- \frac{6y}{b^4} + \frac{8y^3}{b^6} \right) + \right. \right.$$

$$\left. \left. - \frac{3y(x-\xi)}{b^2 R^3} + \frac{4y^3(x-\xi)}{b^4 R^3} + \frac{3y^3(x-\xi)}{b^2 R^5} \right] + c_z \left[\frac{2y(z-\zeta)}{b^4} \left(1 + \frac{(x-\xi)}{R}\right) + \frac{y(x-\xi)(z-\zeta)}{b^2 R^3} \right] \right\}$$

(D.3)

$$w_{o_L}(x,y,z) = \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi} \left\{ c_m \frac{(z-\zeta)}{R^3} - c_x \frac{(z-\zeta)}{R^3} \right. +$$

$$+ c_y^D \left[\left(1 + \frac{(x-\xi)}{R}\right)\left(- \frac{2(z-\zeta)}{b^4} + \frac{8y^2(z-\zeta)}{b^6}\right) + \right.$$

$$\left. + \frac{4(x-\xi)y^2(z-\zeta)}{b^4 R^3} - \frac{(x-\xi)(z-\zeta)}{b^2 R^3} + \frac{3(x-\xi)y^2(z-\zeta)}{b^2 R^5} \right] +$$

$$\left. + c_z \left[\left(1 + \frac{(x-\xi)}{R}\right)\left(\frac{2(z-\zeta)^2}{b^4} - \frac{1}{b^2}\right) + \frac{(x-\xi)(z-\zeta)^2}{b^2 R^3} \right] \right\}$$

(D.4)

<u>Solenoidal components $\vec{q}_{o_T}$:</u>

$$u_{o_T}(x,y,z) = \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi}\, e^{\frac{R_L}{2}[(x-\xi)-R]} \left\{ \frac{R_L c_x}{R} + c_x \left[\frac{(x-\xi)}{R^3} \right. \right. +$$

$$\left. - \left(\frac{R_L}{2}\right) \frac{1}{R} \left(1 - \frac{(x-\xi)}{R}\right) \right] +$$

$$+ c_y^D \left[\left\langle - \frac{1}{R^3}\left(1 - \frac{3y^2}{R^2}\right) \right\rangle + \left(\frac{R_L}{2}\right) \frac{1}{R^2} \left[\left(\frac{2y^2}{b^2} - 1\right) + \frac{(x-\xi)y^2}{b^2 R}\left(1 - \frac{(x-\xi)}{R}\right) \right] \right] +$$

$$\left. + c_z \left[\frac{(z-\zeta)}{R^3} + \left(\frac{R_L}{2}\right) \frac{(z-\zeta)}{R^2} \right] \right\} \qquad \text{(D.5)}$$

$$v_{o_T}(x,y,z) = \iint_{S_o} \frac{d\xi\, d\zeta}{8\pi}\, e^{\frac{R_L}{2}[(x-\xi)-R]} \left\{ R_L c_y^D \frac{y}{R^3} \right. +$$

$$+ c_x \left[\frac{y}{R^3} + \left(\frac{R_L}{2}\right) \frac{y}{R^2} \right] +$$

(cont'd)

$$+ \mathfrak{c}_y^D\left[\left\langle\left(1+\frac{(x-\xi)}{R}\right)\left(\frac{6y}{b^4}-\frac{8y^3}{b^6}\right)+\frac{3y(x-\xi)}{b^2R^3}-\frac{4y^3(x-\xi)}{b^4R^3}-\frac{3y^3(x-\xi)}{b^2R^5}\right\rangle+\right.$$

$$\left.+\left(\frac{R_L}{2}\right)\frac{y}{b^2R}\left[\left(1+\frac{(x-\xi)}{R}\right)\left(1-\frac{2y^2}{b^2}\right)-\frac{(x-\xi)y^2}{R^3}\right]+\left(\frac{R_L}{2}\right)^2\frac{2y}{R^2}\right]+$$

$$\left.+\mathfrak{c}_z\left[-\frac{2y(z-\zeta)}{b^4}\left(1+\frac{(x-\xi)}{R}\right)-\frac{y(x-\xi)(z-\zeta)}{b^2R^3}-\left(\frac{R_L}{2}\right)\frac{y(z-\zeta)}{b^2R}\left(1+\frac{(x-\xi)}{R}\right)\right]\right\} \tag{D.6}$$

$$\mathbf{w}_{o_T}(x,y,z)=\iint_{S_o}\frac{d\xi\, d\zeta}{8\pi}\, e^{\frac{R_L}{2}[(x-\xi)-R]}\left\{\frac{R_L\mathfrak{c}_z}{R}+\right.$$

$$+\mathfrak{c}_x\left[\frac{(z-\zeta)}{R^3}+\left(\frac{R_L}{2}\right)\frac{(z-\zeta)}{R^2}\right]+$$

$$+\mathfrak{c}_y^D\left[\left\langle\left(1+\frac{(x-\xi)}{R}\right)\left(\frac{2(z-\zeta)}{b^4}-\frac{8y^2(z-\zeta)}{b^6}\right)-\frac{4(x-\xi)y^2(z-\zeta)}{b^4R^3}+\right.\right.$$

$$\left.+\frac{(x-\xi)(z-\zeta)}{b^2R^3}-\frac{3(x-\xi)y^2(z-\zeta)}{b^2R^5}\right\rangle+$$

$$\left.+\left(\frac{R_L}{2}\right)\frac{(z-\zeta)}{b^2R}\left[\left(1+\frac{(x-\xi)}{R}\right)\left(1-\frac{2y^2}{b^2}\right)-\frac{(x-\xi)y^2}{R^3}\right]\right]+$$

$$+\mathfrak{c}_z\left[\left(1+\frac{(x-\xi)}{R}\right)\left(-\frac{2(z-\zeta)^2}{b^4}+\frac{1}{b^2}\right)-\frac{(x-\xi)(z-\zeta)^2}{b^2R^3}+\right.$$

$$\left.\left.-\left(\frac{R_L}{2}\right)\frac{(z-\zeta)^2}{b^2R}\left(1+\frac{(x-\xi)}{R}\right)\right]\right\}. \tag{D.7}$$

In these equations, $R^2 = (x-\xi)^2+y^2+(z-\zeta)^2$ and $b^2 = y^2+(z-\zeta)^2$, so that $R^2 = (x-\xi)^2 + b^2$.

The image flow system $\vec{q}_1^{(o)}$ is represented by the same equations with $(z-\zeta)$ replaced by $(z+\zeta)$, and thus with R replaced by $R_1^2 = (x-\xi)^2+y^2+(z+\zeta)^2$ and b^2 replaced by $b_1^2 = y^2+(z+\zeta)^2$.

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