ON POLYNOMIAL INVARIANTS FOR KNOTS AND LINKS

Thesis by

Chi Fai Ho

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To my parents.

Acknowledgments

I am grateful to my advisor, Professor Brock Fuller, who suggested the problem of knots and supervised the research, and corrected a number of errors. I would like to thank Professor Herb Ryser and Professor Richard Wilson for their suggestions during our conversations. Also, Professor Alain Martin has supervised my research in the field of Computer Science.

During my stay at Caltech, the Tran family has been nice company to me. I went to dinner in their house almost every night during the first four years of my Caltech life. I enjoyed a family atmosphere there which I had missed since I came to the States.

Tai Yung House, as we call it, consists of several Caltech students, who are a great gang to live with. The Tai Yung members have been helpful in my study of Chinese computing. I would like to thank them, Kwok-wai Cheung, Fai-ho Mok, Alex Ho, Ricky Ng, Peter Tong, Kenneth Leung, for their great companionship.

Abstract

This thesis presents an investigation of many known polynomial invariants of knots and links. Following Alexander's original idea, we define another multi-indeterminant polynomial for links and show that it satisfies some of Torres' conditions. We conjecture that they are equivalent.

Conway polynomials have been known since the sixties. In this paper, we show that the polynomials of various orientations of a link are related, at least in the first and second coefficients. The relationship can be expressed as a function of the Conway polynomials of all sublinks.

A new invariant polynomial of knots and links has been discovered which is independent of the orientation. This polynomial is also invariant of link inverses. Moreover, it is different from the Conway polynomial and the newly discovered HOMFLY polynomial. It distinguishes the trivial 3-unlink and the Borremean ring of 3 components. Various properties of the polynomial are studied.

Contents

Chapter 1. Development of polynomial invariants of knots and links.

1. Introduction.

The aim of this chapter is to give a brief account of the development of polynomial invariants of knots and links. We will also discuss a possible next step in the combinatorial study of the knot problem. For a survey of the knot problem, we recommend the articles of Fox [F4], McA Gordon [M2], Thistlethwaite [T2], and Rolfsen [R2]. Bibliographies in those articles are fairly complete up to the current development of knot theory with the exception of the newly discovered polynomials.

2. Preliminaries.

A *knot* is a homeomorphic image of S^1 in S^3. It is oriented if S^1 and S^3 are oriented. Two oriented knots K and L are *ambient equivalent* if there exists an orientation preserving homeomorphism of S^3 mapping K onto L, preserving their orientations. K is then said to be *isotopic* to L. Weaker equivalences are defined when the orientation of S^1 or S^3 is dropped.

If K is isotopic to its mirror image, then K is *amphicheiral*. The figure eight knot is amphicheiral but the trefoil is not. Let K' be K with the orientation reversed. If K is isotopic to K', the knot K is said to be *invertible*. Figure-eight knot and trefoil are both invertible. The existence of non-invertible knots was found by Trotter [T4].

The equivalent class, in any sense of equivalence, of K is its *knot type*. There are two main streams of knot types in the course of knot theory, the *tame knots* and the *wild knots*. A tame knot is one which is equivalent to a piecewise linear knot, while a wild knot is not equivalent to any piecewise linear knot. We shall consider exclusively the tame knots.

Regard S^3 as $R^3 + \infty$. Any knot can be assumed to lie in R^3. Take any projection of R^3 onto a plane. Under an arbitrarily small perturbation, the image of K contains no triple points, i.e., no three points of K are mapped into a single point. This projection is referred as a *regular projection* or a *diagram.* A double point, image of two points of K, is sometimes called a *crossing.*

For links, we have essentially the same definitions.

3. Knot tables.

It seems that the first systematic study on knots was made in the middle of eighteen century by Tait [T1], Little [L4,L5,L6,L7], and Kirkman [K3].

Inspired by the theory of vortex atoms, Tait imitiated a study of knots, using the following algebraic description of a knot projection. Trace the knot projection in any direction and put down in order the sequence of crossings encountered. If a crossing is passed over, the symbol, V, of the crossing is put down; if it is passed under, we put down the inverse of the symbol, V^{-1}. The resulting word represents the knot. It is not difficult to reconstruct the knot if a word is given. Word representation of a knot is not unique for we can trace in the reversed direction, or starting from any point of the knot. Therefore, we can define an equivalence among all possible word representations of knots.

Because of the structure of knot diagrams, not every word of crossing symbols comes from a knot. Between V and V^{-1} there must be an even number of symbols. Also, certain patterns in the word can be eliminated. If VV^{-1} appears in the word, we have the figure 1.1 and the link is equivalent to a simpler one with a simpler word. If the word is of form $PVQV^{-1}$ where P and Q are subwords that have no symbol in common, we have a *nullgitory loop* (fig. 1.2) and the diagram can be simplified.

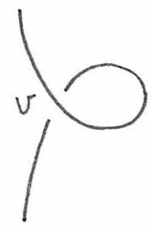


fig.1.1

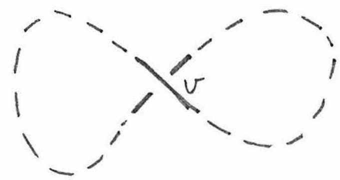


fig.1.2

For any finite number of crossings, there is only a finite number of words. One can eliminate inadmissible words and those that can be simplified. For the rest, construct the knot, if possible, and eliminate those representing equivalent knots. This is essentially the method used by Tait and Little in constructing their knot tables. However, their tables contain only prime knots up to 10 crossings and alternating prime knots of 11 crossings. Their tables contain very few errors.

4. Alexander polynomials and knot groups.

During the early twentieth century, Alexander and Reidemeister made major contribution to the study of knots and links.

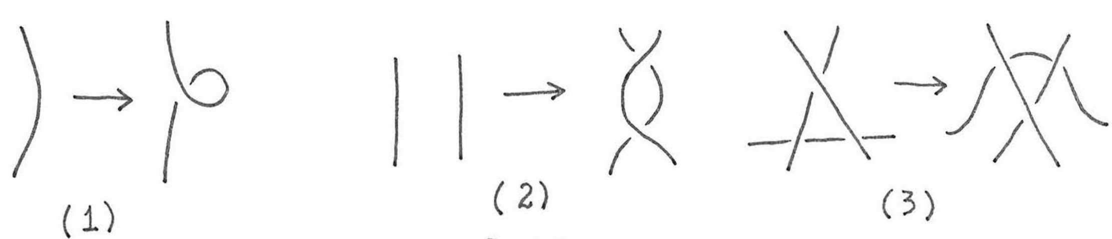


fig. 1.3

One sees that the operations in figure 1.3 do not change the knot type of a diagram. These are the *Reidemeister's moves.* Reidemeister proved that if two diagrams correspond to an equivalent knot, one of them can be obtained by a finite number of moves applied to the other. This gives a combinatorial definition of knot equivalence.

Using Reidemeister's combinatorial definition, Alexander [A2] discovered a polynomial invariant of knots. For any diagram of n crossings, number the $n+2$

regions, into which the knot projection divides the plane, and form an *Alexander matrix* A. Assign at each crossing the symbol structure (fig. 1.4), and define A_{ij} to be the symbol of region i incident with crossing j. If the crossing is met more than once, A_{ij} would be the sum of the symbols. A is an $n \times n+2$ matrix of polynomials in t. For a diagram of a link, we may not have $n+2$ regions unless the diagram is connected. In that case, either use Reidemeister's move of type 2 to make the diagram connected, or simply add columns of zero entries so that A has $n+2$ columns.

fig. 1.4

Delete two columns corresponding to two adjacent regions from A, compute the determinant, and multiply by a suitable power of t so that the result is an integral polynomial with non-zero constant. This polynomial is the *Alexander polynomial* of K. In general, if K is a link, it is the *reduced Alexander polynomial.*

This polynomial distinguishes all knots up to 8 crossings and all except 6 pairs up to 9 crossings. It remained one of the most powerful tool in distinguishing knots until the appearance of the HOMFLY polynomial.

In his paper, Alexander showed that his polynomial can also be obtained from the fundamental group of the knot complement, which is always referred as the *knot group.*

5. Fox's algebra.

Fox is probably the most influential mathematician in the study of knot group. His theory dominated the study of the knot problem for thirty years. Exposition of his work can be found in [C3], [N1], [F1,F2,F3], [T3], etc. We will give just a very brief account of his algebra.

Let ϕ be the knot group of K, $(\lambda, \mu) \in \phi$ be the longitude-meridian pair. The triple (ϕ, λ, μ) is the knot group with peripheral structure of K. Using this additional peripheral information, Dehn [D1] showed that the trefoil is not equivalent to its mirror-image. In fact with a suitable choice of peripheral structure, Conway [C2] showed that the knot is uniquely determined.

If we consider the Q-module of $H_1(X_\infty; Q)$ where X_∞ is the infinite cyclic cover of K, we can find [C3] the *elementary ideals* and the i^{th} *Alexander polynomial.* The first Alexander polynomial is the conventional Alexander polynomial. The other polynomials provide additional information. They distinguish the knots 9_{46} and 6_1 which have same Alexander polynomial.

Another study is that of the finite cyclic cover and the differential calculus [F1] [F2] [F3] of the knot group. It gives a better understanding the algebraic nature of the group which reflects some geometric features of the knot.

6. Conway polynomial and tangles.

Conway invented a new approach to knots. In [C3], he introduced a notion called *tangle*. A tangle (fig. 1.5) is a pair (D, t) where D is a 3-ball and t is a union of two strings in D intersecting the boundary of D at four points. Conway showed that any link is a combination of tangles. With this notion, he constructed a table of knots and links up to 11 crossings.

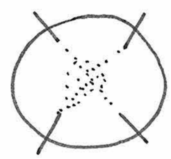


fig. 1.5

Besides tangles, Conway also defined recursively a polynomial invariant $\nabla_K(z)$, of knots and links as follows.

1. $\nabla_{unknot}(z) = 1$,
2. If K_1, K_2 and L are knots with diagrams differing only at a crossing indicated in figure 1.6, then

$$\nabla_{K_1}(z) - \nabla_{K_2}(z) = z\nabla_L(z).$$

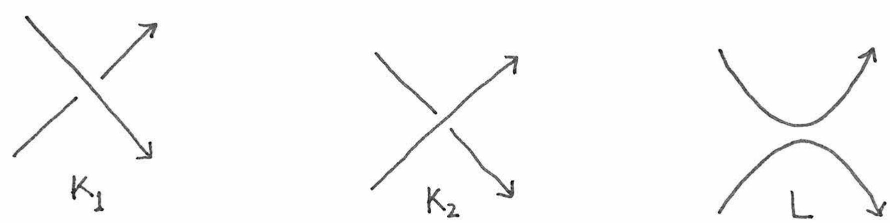


fig. 1.6

This definition is purely combinatorial. Like the discovery of Reidemeister's moves, a totally new approach toward the study of knots and links was under way. The existence of the polynomial is in fact not an accident. A similar recursive nature of the Alexander polynomial had been known already. The Alexander polynmial satisfies $t^{1/2}\Delta_{K_1}(t) - t^{-1/2}\Delta_{K_2}(t) = t\Delta_L(t)$, where K_1, K_2 and L are as in figure 1.5. It turns out that Conway polynomial is equivalent up to sign to the reduced Alexander polynomial under a change of variable.

Although the two polynomials are equivalent, the philosophies behind each are very different. First, the Conway polynomial is combinatorial in nature and can be computed by a recursive procedure suggested by the definition. Second, the geometric meanings of some coefficients of the polynomial are easier to understand. For example, the linking number and Arf invariant are reflected in the first and second coefficients. Moreover, when we turn to the study of braids, one can find bounds on the coefficients [H2]. The situation in Alexander polynomials is much more complicated. Murasugi [M3] computed the Alexander polynomials for 3-braids, but there are no known bounds on the polynomial for an arbitrary braid.

7. New polynomials.

Since the introduction of Conway polynomial and the tangle algebra, a combinatorial study of knots and links has taken place. At the end of 1984, Jone [J1] found another polynomial invariant for knots and links through the Von Neumann algebra applied to braids. A combinatorial version of his polynomial is the following. There is a polynomial $Q_K(x)$ for each link K such that

1. $Q_{unknot}(x) = 1$,
2. $xQ_{K_1}(x) - x^{-1}Q_{K_2}(x) = (x^{1/2} - x^{-1/2})Q_L(x)$.

where K_1, K_2 and L are as in figure 1.5.

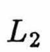
Jones polynomial was found to be different from the Conway polynomial.

The similarity of the Jones and Conway polynomials fascinated four different groups of mathematicians. During the Topology workshop held in Berkeley in 1985, they announced a new **two-variable** polynomial invariant. It is sometimes referred as the *HOMFLY polynomial.* If $F_K(x, y)$ denotes the polynomial for K, then $F_{unknot}(x, y)$ is 1, as before. But for the recursive relation, we have

$$xF_{K_1}(x, y) + yF_{K_2}(x, y) = F_L(x, y).$$

This polynomial contains the Conway and Jone's polynomials as special cases. The real power of the new polynomial is demonstrated by its ability to distinguish the left hand trefoil from the right hand trefoil, which has the same knot group.

However, all polynomial invariants discovered so far depend heavily on the orientation of the knot or link. Though the polynomials for the two different orientations of a knot are the same, the corresponding statement is false for links with more than one component. In fact, if the link has c components, there are in general 2^{c-1} different polynomials for the 2^c possible orientations of the link.

Suppose K is not oriented, and consider a crossing and figure 1.6. We can switch v from K_1 to K_2 as in the figure. When we split, there are two possible ways L_1 () and L_2 (). If we define a polynomial $H_K(x)$ such that $H_{unknot}(x) = 1$ and $H_{K_1}(x) + H_{K_2}(x) = x(H_{L_1}(x) + H_{L_2}(x))$. This defines a polynomial invariant for knots and links which is independent of the orientation. This invariant is different from the HOMFLY polynomial.

These new polynomial invariants have a common feature: they are defined using combinatorial properties of knot and link projection. Unlike the Alexander polynomial which can be derived from the knot group, there is no known relationship between the polynomials and the group, although the new polynomials are found not to be obtainable from the group. An obstacle in studying these invariants is the lack of any geometric interpretation of them. Although some geometric invariant such as the Arf invariant and the signature can be obtained from the polynomial, the geometric nature of the invariants is yet to be studied.

Although we have many invariants of knots and links, they do not completely solve the basic knot classification problem. They fail to distinguish composite links. If $L = K_1 \# K_2$, $P(L) = P(K_1)P(K_2)$ for any polynomial P. Even with prime knots and links considered, they fail to distinguish all different types. Links within a mutant class have the same polynomial. If A is a tangle embedded in a diagram of K, rotate A by 180 degree and let L be the resulting link. Then $P(K) = P(L)$. This is understandable, for K and L have essentially the same abelian properties [T2]; in particular, they have the same 2-branched covering.

Another interesting phenomenon is that there are infinitely many knots with Conway polynomial 1, which is the polynomial for the unknot. But there is no known non-trivial knot with HOMFLY or H polynomial 1. I suspect that there is no such knot. These new polynomials seem to be more complicated than the Conway polynomial. For split links, the Conway polynomial is always zero, while the new polynomials are non-zero unless the link is a union of separated unknots. The degrees of the new polynomials are much greater than the corresponding Conway polynomial. The degree of the Conway polynomial is bounded above by the rank of the homology group of a spanning surface, whereas that of the H polynomial is bounded above by 1 less than the crossing number. Prime alternating links seem to be the only class that achieves the upper bound.

8. Computability of the polynomials.

One can easily compute by hand the polynomial by a recursive procedure. However, the complexity of the procedure will be a main issue if one uses a computer. An algorithm for computing these polynomials is to unlink a given link. At each switching or splitting, the link becomes simpler and we can apply the procedure recursively. But there is no known smart unlinking algorithm. A general algorithm requires at most $n/2$ switchings if the diagram has n crossings with the worst case requiring exactly $n/2$ switchings. In the worst case, a link diagram with 10 crossings needs approximately 500 switchings in computing the $HOMFLY$ polynomial, and more than a million switchings for the H polynomial. This is an exponential algorithm, and a very undesirable one.

However, the Alexander polynomial is the determinant of some matrix with polynomial entries. Since the maximum degree of the Alexander polynomial is known, say d, we can plug in $d+1$ values for the indeterminant and calculate the determinant in n^3 time, where n is the size of the matrix, or number of crossings. As $d \leq n$, to find the Alexander polynomial, we interpolate on the $d+1$ values for a best fit polynomial of degree d. This is another d^3 algorithm. Alexander polynomial has integral coefficients, so we can allow fairly rough accuracy for the interpolation. Thus, for the Alexander polynomial, there is an n^4 algorithm, but the described ones for HOMFLY and H are exponential.

It would be nice to find a polynomial algorithm for the new polynomials but it seems very difficult. Conway [C1] claimed he had an efficient algorithm for his polynomial. But it depends heavily on the tangle representation of a link diagram. In general, to apply his method, one needs to write manually a link into a tangle form, which is of course another issue.

9. Next step towards the knot problem.

Since the discovery of the new polynomial invariants, we might hope that more invariants can be found along the line of manipulating the diagrams as in figures 1.5, and that sufficiently many invariants will solve the classification problem. This is unlikely until more is understood about those existing invariants, especially how they reflect both the geometric and algebraic properties of links. To achieve a complete classification along this line, one must find something to attack the mutancy problem. This seems impossible by considering only the local structure of a crossing. A global picture must be put into consideration.

A good invariant is one that can be easily calculated. It is better if there is an efficient algorithm that can be implemented by a computer. The new polynomials lack this capability; at least no good algorithm is yet known. Efficiency should be a main factor if we hope to solve the knot problem practically.

There is another fundamental problem associated with knots — the enumeration problem. Since Tait's table, not much effort was spent on construction of a larger table until Dowker and Thistlewaite [T2]. They use a similar method to Tait and have employed a computer to generate all possible knots and links of a given number of crossings. So far, a table of knots up to 13 crossings is found, but it is not known whether it is complete. The number of knots of a given number of crossings seems to grow exponentially. Although it may be too early to say, the enumeration problem is also a very difficult problem.

Chapter 2. Alexander Polynomials

1. Introduction.

In this chapter we introduce a multi-variable polynomial invariant for links. The definition is a modification of Alexander's original definition of his polynomial of knots. We verify part of Torres' conditions. In fact, we believe that our polynomial is the same as Torres' Alexander polynomial. Although Torres used the link group to define his polynomial, our definition is combinatorial and simpler.

2. Definition of Alexander polynomials.

Following Alexander's idea of assigning symbols to crossings, we will define a multi-indeterminant integral polynomial for a link.

Let L be an oriented link of $c > 1$ components $K_1, \dots, K_c$, and D be a diagram of L. We assign to each crossing of D symbols according to the following scheme (fig. 2.1)

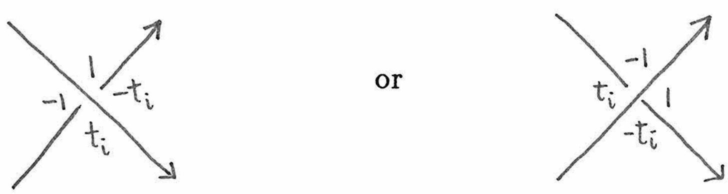


fig. 2.1

where the underneath segment belongs to K_i. Construct an Alexander matrix A as usual. Delete columns i and j of A which correspond to adjacent regions of D. Let A^{ij} be the resulting matrix and $p(t_1, \dots, t_c)$ the greatest common divisor of $det(A^{ij})$ where (i, j) varies over all pairs of columns of adjacent regions. Extract all factors $t_1, \dots, t_c$ from p, and denote the resulting polynomial by $\overline{\Delta}_L(t_1, \dots, t_c)$

and call it an *Alexander polynomial* of L. Two observations immediately follow from the definition. First, $\overline{\Delta}_L(t_1,\ldots,t_c)$ is an integral polynomial. Second, since $-p(t_1,\ldots,t_c)$ is also a greatest common divisor of those determinants, $\overline{\Delta}_L(t_1,\ldots,t_c)$ is defined only up to a sign. Let us agree that our polynomial means either $\overline{\Delta}_L(t_1,\ldots,t_c)$ or $-\overline{\Delta}_L(t_1,\ldots,t_c)$.

We defer the proof of the invariance of $\overline{\Delta}_L(t_1,\ldots,t_c)$ under Reidemeister's moves and the following theorem to later sections, and we would like to examine some of their consequences here.

Theorem 2.2.1. *$p(t_1,\ldots,t_c) \doteq det(A^{ij})/(t_k-1)$ where regions corresponding to columns i and j share a common edge belonging to the k^{th} component. Here $\doteq$ means equal up to a product of powers of t_i's.*

If $\widetilde{\Delta}_L(t)$ denotes the reduced Alexander polynomial, then

$$\widetilde{\Delta}_L(t) \doteq det(A^{ij})|_{t_1=\cdots=t_c=t} \;\doteq\; p(t,\ldots,t)(t-1).$$

Hence we deduce

Corollary 2.2.2. *Suppose L has more than one component, then*

$$\widetilde{\Delta}_L(t) \doteq \overline{\Delta}_L(t,\ldots,t)(t-1).$$

Next we will prove

Theorem 2.2.3. *Let L_1 be an oriented link with more than one component, and L_2 be the link obtained by reversing the orientation of the i^{th} component of L_1. Then*

$$\overline{\Delta}_{L_1}(t_1,\ldots,t_c) = \epsilon\, t_i^{\lambda}\, \overline{\Delta}_{L_2}(t_1,\ldots,t_i^{-1},\ldots,t_c), \qquad \epsilon = \pm 1.$$

Proof. L_1 and L_2 have the same diagram except for the i^{th} component of both links. Let D_1, D_2 be diagrams of L_1 and L_2 respectively. Then symbols assigned to the crossings of D_2 are the same as those of D_1 except when the under segment belongs to K_i, the i^{th} component of both L_1 and L_2. In that case, 1 is replaced by t_i and t_i is replaced by 1. Let A_1 and A_2 be the Alexander matrices of D_1 and D_2 respectively. In A_2, for the rows corresponding to those crossings, divide by t_i and let A' be the resulting matrix. So $A_1 = A'|_{t_i=t_i^{-1}}$. Hence $det(A_1^{ij}) = det((A')^{ij})|_{t_i=t_i^{-1}} = t_i^{\lambda} det(A_2^{ij})|_{t_i=t_i^{-1}}$. ∎

Returning to the reduced Alexander polynomial, we have

Theorem 2.2.4. *Suppose L is an oriented link and $\overline{\Delta}_L(t_1, \ldots, t_c)$ its Alexander polynomial. Suppose L' is equivalent to L except that the orientation of L' differs from L in the $i_1, \ldots, i_m$ components. Then*

$$\widetilde{\Delta}_{L'}(t) \doteq (t-1)\overline{\Delta}_L(t, \ldots, t^{-1}, \ldots, t^{-1}, \ldots, t^{-1}, \ldots, t),$$

where t is replaced by t^{-1} in the i_j ordinates, $1 \leq j \leq m$.

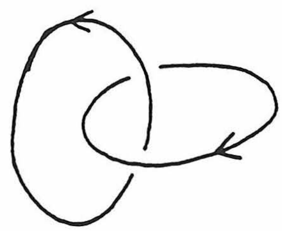
Proof. This follows immediately from Corollary 2.2.2 and Theorem 2.2.3. ∎

Corollary 2.2.5. *If L has c components then L has 2^{c-1} reduced Alexander polynomials.*

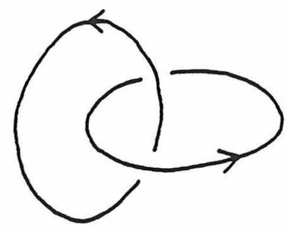
Proof. L has 2^c reduced Alexander polynomials by Theorem 2.2.4. But we know from Alexander [A1] that $\widetilde{\Delta}_L(t) \doteq \widetilde{\Delta}_L(t^{-1})$. So if L_1 and L_2 are equivalent links with reversed orientation, then they have the same reduced Alexander polynomial. Hence L has only 2^{c-1} reduced polynomials. ∎

Corollary 2.2.5 explains why we obtain more than one reduced polynomial for some examples in section 1. We conclude this section by computing the Alexander polynomials of some oriented links.

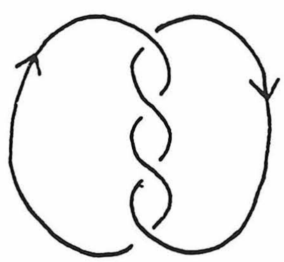
Examples.



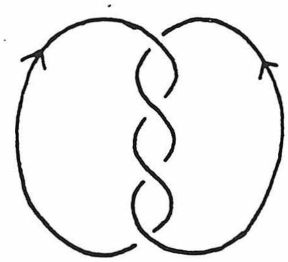
$\overline{\Delta}_L(t_1, t_2) = 1$



$\overline{\Delta}_L(t_1, t_2) = 1$



$\overline{\Delta}_L(t_1, t_2) = 1 + t_1 t_2$



$\overline{\Delta}_L(t_1 t_2) = t_1 + t_2$

3. Proof of Theorem 2.2.1.

In this section our main concern is to prove

Theorem 2.2.1. *$p(t_1, \ldots, t_c) \doteq det(A^{ij})/(t_k - 1)$ where $\doteq$ means equal up to powers of $t_1, \ldots, t_c$ and sign, and regions corresponding to columns i and j sharing a common edge belonged to K_k.*

To each region R of a diagram D, we associate a c-tuple $(a_1, \ldots, a_c)$, called the *index* of R, as followed:

(1) The exterior region is indexed with $(0, \ldots, 0)$.

(2) If two regions R_l and R_m share a common edge as indicated by figure 2.2, then R_l has index $(a_1, \ldots, a_c)$ iff R_m has index $(a_1, \ldots, a_j - 1, \ldots, a_c)$.

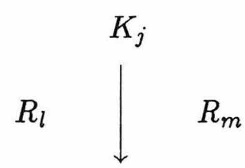


fig. 2.2

The theorem can be restated:

Theorem 2.2.1′. *$p(t_1, \ldots, t_c) \doteq det(A^{ij})/(t_k - 1)$ where regions corresponding to columns i and j have indices differed only in the k^{th} ordinate by 1.*

Before we continue, we need some notations. Let Λ_{ij} be the determinant of A^{ij}, C_i be the i^{th} column of A and R_i the region corresponding to C_i. We can assume R_1 is the exterior region. Denote the index of R_i or C_i by $\iota(R_i)$ or $\iota(C_i) = (a_{i,1}, \ldots, a_{i,c})$.

We also need some lemmas.

Lemma 2.3.1.

(a) $\sum_{i=1}^{n+2} C_i = 0,$

(b) $\sum_{i=1}^{n+2} t_1^{-a_{i,1}} \cdots t_c^{-a_{i,c}} C_i = 0.$

Proof. (a) follows immediately from the fact that all row sums of A are zero.

Multiply all symbols in region R_i by $t_1^{-a_{i,1}} \cdots t_c^{-a_{i,c}}$. Then the sum of all four symbols of a crossing is zero, that is, row sums of the resulting matrix are zero. ∎

Lemma 2.3.2. $\Lambda_{1\alpha}(t_1^{-a_{\beta,1}} \cdots t_c^{-a_{\beta,c}} - 1) = \epsilon\Lambda_{1\beta}(t_1^{-a_{\alpha,1}} \cdots t_c^{-a_{\alpha,c}} - 1)$ *where* $\epsilon = \pm 1$.

Proof. By $\widehat{C}$, we mean column C is deleted from a matrix. So by definition,

$$\begin{aligned}\Lambda_{1\alpha} &= det(\widehat{C}_1, C_2, \ldots, C_{\alpha-1}, \widehat{C}_\alpha, \ldots, C_\beta, \ldots, C_{n+2})\\ &= \frac{1}{\Gamma_\beta}\, det(\widehat{C}_1, \ldots, \widehat{C}_\alpha, \ldots, \Gamma_\beta C_\beta, \ldots)\\ &= \frac{(-1)^{\alpha-\beta}}{\Gamma_\beta}\, det(\widehat{C}_1, \ldots, C_{\alpha-1},\ \Gamma_\beta C_\beta,\ C_{\alpha+1}, \ldots, \widehat{C}_\beta, \ldots).\end{aligned}$$

where $\Gamma_\beta = t_1^{-a_{\beta,1}} \cdots t_c^{-a_{\beta,c}} - 1$. Add to $\Gamma_b C_\beta$ all multiples C_i by $t_1^{-a_{i,1}} \cdots t_c^{-a_{i,c}} - 1$ where $i \neq 1$, α or β. Using Lemma 2.3.1, we obtain

$$\begin{aligned}\Lambda_{1\alpha} &= \frac{(-1)^{\alpha-\beta+1}}{\Gamma_\beta}\, det(\widehat{C}_1, \ldots, \widehat{C}_{\alpha-1}, \Gamma_\alpha C_\alpha, C_{\alpha+1}, \ldots, \widehat{C}_\beta, \ldots)\\ &= \frac{(-1)^{\alpha-\beta+1}}{\Gamma_\beta}\Gamma_\alpha \Lambda_{1\beta}. \quad ∎\end{aligned}$$

If we consider the plane as a 2-sphere omitting a point, we can redraw our diagram so that any region of the initial diagram can be made into the exterior region in the new diagram. So if region R of which the index was $(r_1, \ldots, r_c)$, is made into the exterior region with new index $(0, \ldots, 0)$, a region Q with old index $(q_1, \ldots, q_c)$ has new index $(q_1 - r_1, \ldots, q_c - r_c)$. Following from Lemma 2.3.2, we get

$$(t_1^{a_{\alpha,1}-a_{\beta,1}} \cdots t_c^{a_{\alpha,c}-a_{\beta,c}} - 1)\Lambda_{\alpha\gamma} = \epsilon(t_1^{a_{\alpha,1}-a_{\gamma,1}} \cdots t_c^{a_{\alpha,c}-a_{\gamma,c}} - 1)\Lambda_{\alpha\beta}, \quad \epsilon = \pm 1.$$

Thus if

$$(t_1^{a_{\beta,1}-a_{\delta,1}} \cdots t_c^{a_{\beta,c}-a_{\delta,c}} - 1)\Lambda_{\alpha\beta} = \epsilon(t_1^{a_{\beta,1}-a_{\alpha,1}} \cdots t_c^{a_{\beta,c}-a_{\alpha,c}} - 1)\Lambda_{\delta\beta},$$

then

$$\Lambda_{\alpha\gamma} = \epsilon\Lambda_{\delta\beta} t_1^{a_{\alpha,1}-a_{\beta,1}} \cdots t_c^{a_{\alpha,c}-a_{\beta,c}} \frac{t_1^{a_{\alpha,1}-a_{\delta,1}} \cdots t_c^{a_{\alpha,c}-a_{\delta,c}} - 1}{t_1^{a_{\beta,1}-a_{\delta,1}} \cdots t_c^{a_{\beta,c}-a_{\delta,c}} - 1}.$$

So if regions corresponding to α and γ have indices differing in the i^{th} ordinate and those of β and δ have indices differing in the j^{th} ordinate and both differences are 1, then

$$\Lambda_{\alpha\gamma} \doteq \Lambda_{\delta\beta} \frac{t_i - 1}{t_j - 1} \quad ,$$

Thus $(t_j - 1)\Lambda_{\alpha\gamma} \doteq (t_i - 1)\Lambda_{\delta\beta}$. Therefore the greatest common divisor of $\Lambda_{\alpha\beta}$ where $\iota(C_\alpha)$ and $\iota(C_\beta)$ differ only in the same ordinate by one is equal up to a product of powers of t_i's to $\Lambda_{ij}/(t_k - 1)$ where $\iota(C_i)$ and $\iota(C_j)$ differ in the k^{th} ordinate by 1. This completes the proof of Theorem 2.2.1. ∎

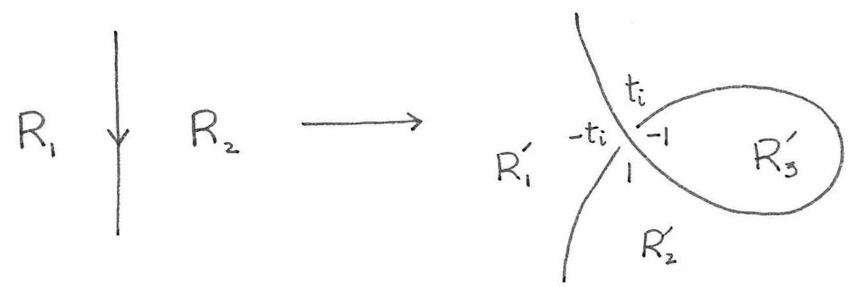
4. Invariance of Alexander polynomials.

So far the Alexander polynomial is defined for a link diagram. In this section we will convince ourselves that it is invariant under ambient equivalence. To do this, recall in Section 1 that two links are ambient equivalent iff there is a finite number of Reidemeister's moves transforming a diagram of one link into a diagram of the other. So it suffices to show that after a Reidemeister's move, the two diagrams produce the same polynomial, up to sign.

For the move T_0 and its inverse, there is nothing to prove since both diagrams are essentially the same, thus give the same Alexander matrix.

For the moves T_1, T_2 and T_3, there are many cases. We will only consider one case from each type. The other cases can be easily verified similarly.

Case 1. T_1.



Let $A = (C_1, \ldots, C_{n+2})$ denote an Alexander matrix obtained from the diagram before the move. Then

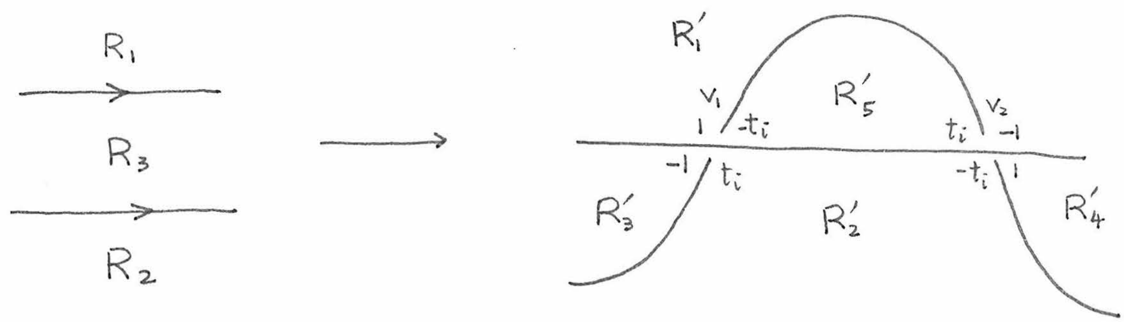
$$A' = \begin{pmatrix} -t_1 & t_i + 1 & -1 & 0 \\ C_1 & C_2 & 0 & B \end{pmatrix},$$

where $B = (C_3, \ldots, C_{n+2})$ is an Alexander matrix for the diagram after the move. Delete any two columns of B,

$$\begin{aligned} det((A')^{ij}) &= det \begin{pmatrix} -t_1 & t_i + 1 & -1 & 0 \\ C_1 & C_2 & 0 & B^{ij} \end{pmatrix} \\ &= \epsilon \; det(C_1, C_2, B^{ij}) \\ &= \epsilon \; det(A^{ij}), \qquad \epsilon = \pm 1. \end{aligned}$$

So, by Theorem 2.2.1, both polynomials coincide.

Case 2. T_2.



Let D_1 be the diagram before the move and D_2 after. Suppose $A = (C_1, C_2, C_3, B)$ is an Alexander matrix of D_1 and C_i corresponds to R_i. Then

$$A_2 = \begin{pmatrix} 1 & t_i & -1 & 0 & -t_i & 0 \\ -1 & -t_i & 0 & 1 & t_i & 0 \\ C_1 & C_2 & U & V & 0 & B \end{pmatrix},$$

where U, V are some column vectors summed to C_3, is an Alexander matrix of D_2. Delete two columns from B.

$$A_1^{ij} = (C_1, C_2, C_3, B^{ij}),$$

$$A_2^{ij} = \begin{pmatrix} 1 & t_i & -1 & 0 & -t_i & 0 \\ -1 & -t_i & 0 & 1 & t_i & 0 \\ C_1 & C_2 & U & V & 0 & B^{ij} \end{pmatrix}.$$

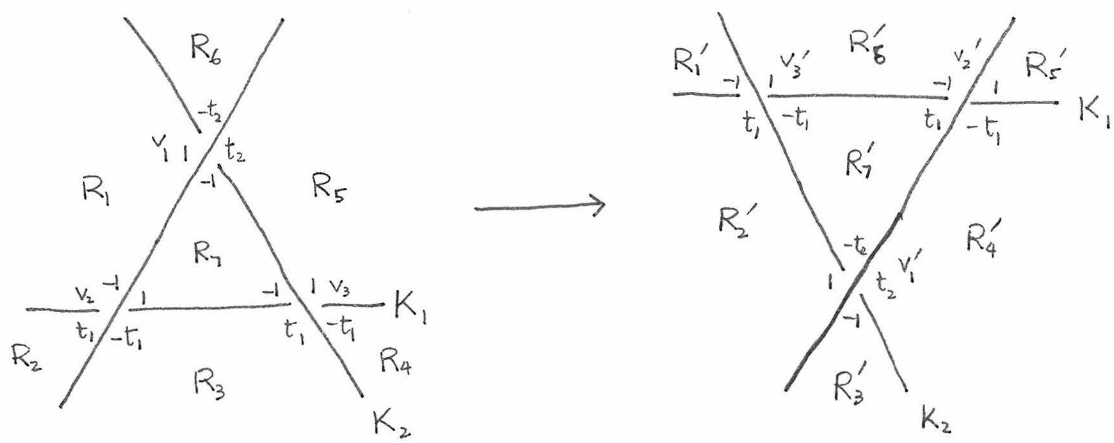
In A_2^{ij}, add the fifth column to the second, and divide it by t_i and add to the first. Next add the fourth column to the third and subtract the resulting column by the fifth.

$$det(A_2^{ij}) = t_i\, det \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ C_1 & C_2 & C_3 & V & 0 & B^{ij} \end{pmatrix} = t_i\, det(A_1^{ij}).$$

Hence we are done.

Case 3. T_3.

Suppose we have



Suppose also D_1 is the diagram before the move and D_2 the diagram after, A_1 is an Alexander matrix of D_1 and A_2 that of D_2. Let

$$A_1 = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \\ \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & t_2 & -t_2 & -1 & 0 \\ -1 & t_1 & -t_1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & t_1 & -t_1 & 1 & 0 & -1 & 0 \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & 0 & B \end{pmatrix}$$

and

$$A_2 = \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ \\ \end{array} \begin{pmatrix} 0 & 1 & -1 & t_2 & 0 & 0 & -t_2 & 0 \\ 0 & 0 & 0 & -t_1 & 1 & -1 & t_1 & 0 \\ -1 & t_1 & 0 & 0 & 0 & 1 & -t_1 & 0 \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & 0 & B \end{pmatrix}.$$

Delete two columns i, j from columns containing B, we have

$$det(A_1^{ij}) = det \begin{pmatrix} 1 & 0 & 0 & 0 & t_2 & -t_2 & -1 & 0 \\ -1 & t_1 & -t_1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & t_1 & -t_1 & 1 & 0 & -1 & 0 \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & 0 & B^{ij} \end{pmatrix}$$

Adding various multiples of the seventh column to others,

$$= det \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & t_1 & -t_1 & 0 & t_2 & -t_2 & 1 & 0 \\ -1 & 0 & t_1 & -t_1 & 1-t_2 & t_2 & -1 & 0 \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & 0 & B^{ij} \end{pmatrix}$$

$$= -det \begin{pmatrix} 0 & t_1 & -t_1 & 0 & t_2 & -t_2 & 0 \\ -1 & 0 & t_1 & -t_1 & 1-t_2 & t_2 & 0 \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & B^{ij} \end{pmatrix}.$$

Also,

$$det(A_2^{ij}) = det \begin{pmatrix} 0 & 1 & -1 & t_2 & 0 & 0 & -t_2 & 0 \\ 0 & 0 & 0 & -t_1 & 1 & -1 & t_1 & 0 \\ -1 & t_1 & 0 & 0 & 0 & 1 & -t_1 & 0 \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & 0 & B^{ij} \end{pmatrix}$$

Add suitable multiples of the seventh column to other columns to clear out the second row,

$$= \frac{1}{t_1^2} det \begin{pmatrix} 0 & 1 & -1 & 0 & t_2 & -t_2 & -t_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0 \\ -1 & t_1 & 0 & -t_1 & t_1 & 0 & -t_1 & 0 \\ C_1 & C_2 & C_3 & C_4 & t_1C_5 & t_1C_6 & 0 & B^{ij} \end{pmatrix}.$$

Expand along second row

$$= \frac{1}{t_1} det \begin{pmatrix} 0 & 1 & -1 & 0 & t_2 & -t_2 & 0 \\ -1 & t_1 & 0 & -t_1 & t_1 & 0 & 0 \\ C_1 & C_2 & C_3 & C_4 & t_1C_5 & t_1C_6 & B^{ij} \end{pmatrix}.$$

Multiply all columns except the fifth and the sixth by t_1 and extract a factor of t_1 from the second row onward. We get

$$det(A_2^{ij}) = t_1^{\lambda} det \begin{pmatrix} 0 & t_1 & -t_1 & 0 & t_2 & -t_2 & 0 \\ -1 & t_1 & 0 & -t_1 & 1 & 0 & 0 \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & B^{ij} \end{pmatrix} = t_1^{\lambda} det(A_1^{ij}).$$

for some integer λ.

Hence both polynomials are similar. ∎

4. Torres' conditions.

Torres [T3] established two properties for his polynomial. Although we cannot prove these two properties here, we will obtain some partial results which are special cases of those properties. However, examples show that the properties also are held by our polynomials.

Theorem 2.5.1. *Let L be an oriented link with $c > 1$ components. Then $\overline{\Delta}_L(t_1, \ldots, t_c)$ satisfies :*

(1) $\overline{\Delta}_L(t_1, 1, \ldots, 1) \doteq \Delta_{K_1}(t_1)(t_1^{l_2} - 1) \cdots (t_1^{l_c} - 1)/(t_1 - 1)$ where K_1 is the first component of L and l_i, , $i \in \{2, \ldots, c\}$, is the linking number between K_1 and the i^{th} component of L. Here, linking numbers are positive.

(2) $\overline{\Delta}_{L'}(t_1, \ldots, t_{c-1})$ is a factor of $\overline{\Delta}_L(t_1, \ldots, t_{c-1}, 1)$ where L' is the proper sublink of L by deleting the c^{th} component.

(3) $\overline{\Delta}_L(1, \ldots, 1) = 0$ if $c > 2$ and $|\overline{\Delta}_L(1, 1)| = l_2$ if $c = 2$.

Proof. Observe that (3) is a consequence of (1) and the fact that $|\Delta_K(1)| = 1$ for any knot K.

Suppose K is a proper sublink of L. Let D be a diagram of L and D_1 be part of D corresponding to K. Any region of D is contained in some region of D_1. Let A be the Alexander matrix of D and A_1 the matrix of D_1. For each region R_i in D_1, sum all columns of A which correspond to regions in D contained in R_i and replace one of them by the sum in A. Renumber regions in D or permute rows in A if necessary and leave the summed column in the i^{th} column. Observe that a region of D cannot be in two different regions of D_1. Calling the new matrix A^*, we have

$$A^* = \begin{pmatrix} A_1 & 0 \\ \hline 0 & \\ B & C \end{pmatrix} \begin{array}{l} \{ \text{ rows correspond to crossings of } D_1 \\ \{ \text{ rows correspond to crossings not involving } D_1 \\ \{ \text{ other crossings} \end{array}$$

where B is a 0 matrix if $t_i = 1$ and the i^{th} component of L is not a component of K.

Observe also that C does not depend on the structure of crossings of D_1, that is, does not change if we change a crossing from to in D_1. So $det(C)$ is independent of the link type of K. Therefore

$$det(A^{12}) = det((A^*)^{12}).$$

Assume $K = K_1$, and $t_2 = t_3 = \cdots = t_c = 1$, then B is the 0 matrix. So

$$\begin{aligned} det(A^{12}) &= det(A_1^{12})\, det(C)|_{t_2=\cdots=t_c=1} \\ &= \Delta_{K_1}(t_1)\, det(C)|_{t_2=\cdots=t_c=1}. \end{aligned}$$

Since $t_2 = \cdots = t_c = 1$, $det(C)|_{t_2=\cdots=t_c=1}$ does not depend on the link type of $L - K_1$. From a previous remark, it does not depend on knot type of K_1 neither. Thus $det(C) = (det(A^{12})/det(A_1^{12}))|_{t_2=\cdots=t_c=1}$, where $L - K$ and K_1 can be any link type. Using Reidemeister's moves and allowing ourselves to switch any crossing, from to which does not involve either $L - K$ or K_1, we can deform any diagram of L into figure 2.3. with K_1 and K_i having linking number l_i.

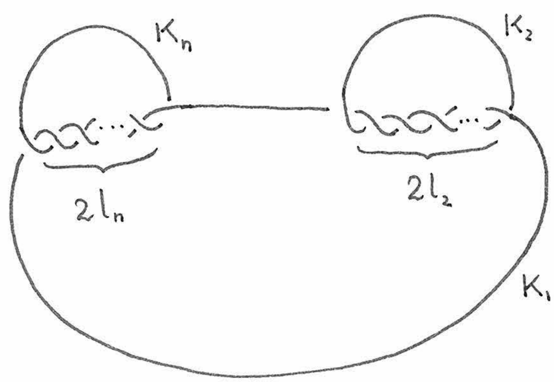


fig. 2.3

This link has Alexander polynomial $\prod_{i=2}^{c}(t_1^{l_i}t_i^{l_i}-1)/(t_1-1)$. As $\Delta_{K_1}(t) = 1$ $det(C) = \prod_{i=2}^{c}(t_1^{l_i}-1)/(t_1-1)$ and $\overline{\Delta}_L(t_1, 1, \ldots, 1) = \overline{\Delta}_{K_1}(t) \times (t_1^{l_2}-1)\cdots(t_1^{l_c}-1)/(t_1-1)$. This proves (1).

To prove (2), let K be the sublink of L by removing the c^{th} component; we get

$$det(A^{12})|_{t_c=1} = det\begin{pmatrix} A_1^{12} & 0 \\ 0 & C|_{t_c=1} \end{pmatrix}$$

and thus $det(A_1^{12})$ is a factor of $det(A^{12})$. ∎

Chapter 3. Conway Polynomials.

1. Introduction.

Since Conway's discovery, a lot of effort has spent to understand his polynomial. Although Conway polynomial is found to be equivalent to the reduced Alexander polynomial, study of the polynomial is not at all futile. For example, Conway polynomials of braids are easier to compute than their Alexander polynomials.

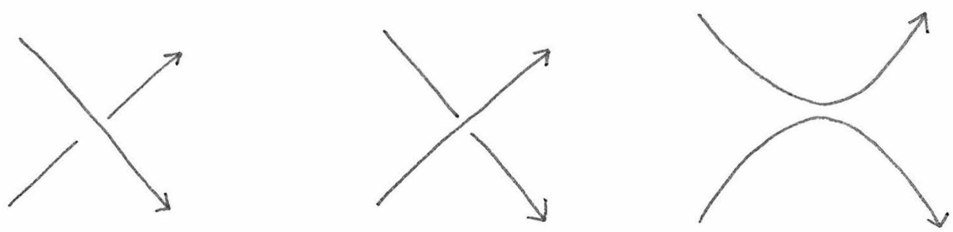
Alexander polynomial is known to depend on the orientation of the link. Relationship among the polynomials correspond to various orientations of a link was not known. In this chapter, we study the effect on the Conway polynomial by a change of the orientation of a link. We obtain an explicit relationship between the first and second coefficients of the Conway polynomials of two similar links differed by the orientation of one component. This relation is a combination of the Conway polynomials of all proper sublinks of the given link. We prove the result when c, the number of components, is 2 or 3. Similar results are obtained when c is 4 or 5, but the proof is lengthy and is not included. I believe a simpler proof can be found and the result is true for any c. Moreover I think it is true that the polynomials of two similar links with different orientations are related explicitly.

2. Definition of Conway polynomials.

The following is an axiomatic definition of the polynomial.

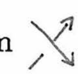
Axioms for the Conway Polynomial.

1. If L is an unknotted circle, then $\nabla_L(z) = 1$.
2. Suppose K_1 and K_2 are two oriented links and D_1, D_2 are some diagrams of K_1 and K_2 respectively. Let L be a link with a diagram D such that D_1, D_2 and D are similar but differ only at a crossing as indicated (fig. 3.1): Then $\nabla_{K_1}(z) - \nabla_{K_2}(z) = z\nabla_L(z)$.



D_1 D_2 D

fig. 3.1

By *switching* a crossing v, we mean by changing v from  to , or  to  ; by *unknotting* a knot we mean changing a knot diagram to a diagram of an unknotted circle by switching crossings; by *unlinking* a link we mean changing a link diagram to a diagram of a *split link*, a separated union of knotted components, by switching crossings. It is well known that we can always unknot or unlink a diagram. An algorithm is as follows.

Let L be an oriented link and D its diagram. Order the components. On each component choose a *starting point*. Starting from the first component, traverse the i^{th} component from the starting point along the direction of the orientation. When a crossing v is encountered, do nothing if v has been travelled before or if it is an *overpass*, that is, when the travelled segment lies above the other segment. Do nothing if it is an *inter-crossing*, a crossing involving two different components, with the other component having a smaller subscript. Otherwise switch v. Let n be the number of crossings of D, and r the number of switchings of the process. Call the pair (n, r) the *complexity* of D.

This procedure provides an inductive procedure to compute the Conway polynomial of a diagram. Denote the diagram D in Axiom 2 as the *split* of the switching and L as its link type. And denote the diagram D_2 the *switched diagram* of the switching. It should be clear from the context what we mean by a split and a switched diagram.

If L is a split link, a separated union of its sublinks, $\nabla_L(z) = 0$ [K2]. Diagrams with complexity $(0, r)$ or $(n, 0)$ are split links and have zero polynomial. Order the complexities lexicographically; we can, by induction of the complexity, compute the polynomial of a link L as follows. Unlink L as described and let v be the first switched crossing. Let D' be the switched diagram and E the split diagram. Let L' and K be their link types respectively. Then by Axiom 2, we have

$$\nabla_L(z) = \nabla_{L'}(z) + \epsilon z \nabla_K(z),$$

where $\epsilon = 1$ if v is of structure  and $\epsilon = -1$ if v is . By induction on the complexity, $\nabla_{L'}(z)$ and $\nabla_K(z)$ are known and therefore $\nabla_L(z)$ can be computed.

Remarks.

1. In Axiom 1, we define $\nabla_{unknot}(z) = 1$. Suppose we replace 1 by any other arbitrary polynomial $p(z)$; we still obtain a unique polynomial $\nabla^*_L(z)$ for each oriented link L. Indeed, we have $\nabla^*_L(z) = p(z)\nabla_L(z)$.

2. In practice we will apply Reidemeister's moves to reduce the number of crossings to find an efficient way to unlink a diagram. However, there is no known algorithm to minimize the number of switchings to unlink a

diagram. It would be nice if we could find an efficient way to program a computer to compute the polynomial.

3. First coefficient of the polynomial.

Let D be a diagram of an oriented link L. Associate to each crossing 1 or -1, called the *index*, according to whether it has structure  or  . Let K_1 and K_2 be two proper sublinks of L. A *self-crossing* of K_1 is a crossing belonging to the projected image of K_1 in D. An *inter-crossing* between K_1 and K_2 is a crossing in D involving both images of K_1 and K_2. If we add indices of all inter-crossings between K_1 and K_2 when they are knotted components of L, the sum is twice their linking number [K2].

Suppose K' is a link obtained by a crossing switch of a diagram of K, and L is the split of the switch. If s is the index of the crossing, by Axiom 2,

$$\nabla_K(z) - \nabla_{K'}(z) = sz\nabla_L(z). \tag{3.1}$$

Denote by $\nabla(K)$ the Conway polynomial of K instead of using $\nabla_K(z)$. Define the codegree of $\nabla(K)$, *cod*(K), to be the order of zero of $\nabla(K)$, that is, the minimal index of z such that its coefficient is nonzero. We also consider the zero polynomial being an odd and even polynomial with codegree ∞. The following theorem is known [K2].

Theorem 3.3.1. *Let K be a c-link. $\nabla(K)$ is even if c is odd, and is odd if c is even. Moreover, cod$(K) \geq c-1$. If c =1, cod$(K) = 0$ and the constant term of $\nabla(K)$ is 1.*

Since $cod(\nabla(K)) \leq c-1$, define the *order* of $\nabla(K)$ to be $c-1$ and the i^{th} coefficient of $\nabla(K)$, $coef_i(\nabla(K))$ to be the coefficient of $z^{c-1+2(i-1)}$. The first coefficient, $coef_1(\nabla(K))$ is the coefficient of z^{c-1}. The previous theorem states that if K is a knot, $coef_1(\nabla(K)) = 1$. We will compute $coef_1(\nabla(K))$ when $c \geq 1$.

For convenience, we will describe another procedure to unlink a link K of c components. If $c = 1$ unlink as usual. If $c > 1$, order the components and choose starting points as usual. When we traverse component C_i, do nothing on all self-crossings but switch as usual on inter-crossings. When all components are traversed, the resulting link K^* is a separated union of knotted components. K^* has Conway polynomial 0 if $c > 1$ and 1 if $c = 1$ regardless of the orientation. Moreover,

$$\nabla(K) = \nabla(K^*) + z \sum_{i=1}^{m} s_i \nabla(L_i) \tag{3.2}$$

where s_i is the index of the i^{th} switched crossing, and m is the total number of switchings.

Theorem 3.3.2 [K2]. *If $c = 2$, then $coef_1(\nabla(K))$ is the linking number.*

Suppose K has $c \geq 2$ components $K_1, K_2, \cdots, K_c$. Denote $l_{ij}(K)$ or l_{ij} be the linking number between K_i and K_j. Notice that $l_{ij} = l_{ji}$.

Define the linking number matrix A by $A_{ij} = l_{ij}$ if $i \neq j$ and $A_{ii} = -\sum_{i \neq j} l_{ij}$. A is symmetric and all column sums and row sums are 0. If $(A)_{ij}$ is any minor of A then $(-1)^{i+j} det(A)_{ij}$ is independent of i and j and we have

Theorem 3.3.3. *$coef_1(\nabla(K)) = (-1)^{c-1+i+j} det(A)_{ij}$ if $c > 1$ and equals 1 if $c = 1$.*

Proof. First we use induction on c. If c=1 or 2 the result follows from Theorems 3.3.1 and 3.3.2.

Suppose K has $c > 2$ components. We next use induction on $\sum_{i<j} |l_{ij}|$.

First assume all linking numbers are zero. Let $K_1, \cdots, K_c$ be the knot components of K. Push K_1 above K_2 and let M_2 be the resulting link. At the split of each switching, if K' denotes the link, all linking numbers of K' are zero and since K' has only $c-1$ components, $coef_1(\nabla(K')) = 0$. Thus $coef_1(\nabla(K)) = coef_1(\nabla(M_2))$.

For each $i > 1$, push K_1 above K_{i+1} in M_i and let the resulting link be M_{i+1}. The split at each switching has zero linking numbers and hence we get $coef_1(\nabla(M_i)) = coef_1(\nabla(M_{i+1}))$. But in M_c, K_1 lies above all other components and is separable from them. Thus M_c is a split link and $\nabla(M_c) = 0$. Hence $coef_1(\nabla(K)) = coef_1(\nabla(M_c)) = 0$.

Suppose $coef_1(\nabla(K))$ is given by the Theorem. Let M be a link of c components with same linking numbers as K except that $l_{ij}(M) = l_{ij}(K) + \epsilon$, $\epsilon = \pm 1$. We can assume, without loss of generality, $l_{1c}(M) = l_{1c}(K) + \epsilon$, and $l_{1c}(M)$ and ϵ are of the same sign. Denote $l_{ij}(K)$ by l_{ij}. Switch an inter-crossing between K_1 and K_c in M with index ϵ. Such a crossing exists since $l_{1c}(M)$ is of same sign as ϵ. Let K' and K'' be the switched and split link types respectively. Then

$$coef_1(\nabla(M)) = coef_1(\nabla(K')) + \epsilon\, coef_1(\nabla(K'')).$$

K' has same linking numbers as K. K'' has $c-1$ components $K_1', K_2, \ldots, K_{c-1}$, and $l_{1i}(K'') = l_{1i} + l_{ic}$, $i \neq 1$ and $l_{ij}L'' = \; l_{ij}$, $i \neq 1$. Hence

$$(A(M))_{1,1} = \begin{pmatrix} -\sum_{i\neq 2} l_{2i} & l_{23} & \cdots & l_{2c} \\ l_{23} & -\sum_{i\neq 3} l_{3i} & \cdots & l_{3c} \\ \vdots & \vdots & \ddots & \vdots \\ l_{2c} & . & \cdots & -\sum_{i\neq c} l_{ci} - \epsilon \end{pmatrix},$$

$$(A(K'))_{1,1} = \begin{pmatrix} -\sum_{i\neq 2} l_{2i} & l_{23} & \cdots & l_{2c} \\ l_{23} & -\sum_{i\neq 3} l_{3i} & \cdots & l_{3c} \\ \vdots & \vdots & \ddots & \vdots \\ l_{2c} & . & \cdots & -\sum_{i\neq c} l_{ci} \end{pmatrix},$$

and

$$(A(K''))_{1,1} = \begin{pmatrix} -\sum_{i\neq 2} l_{2i} & l_{23} & \cdots & l_{2,c-1} \\ l_{23} & -\sum_{i\neq 3} l_{3i} & \cdots & l_{3,c-1} \\ \vdots & \vdots & \ddots & \vdots \\ l_{2,c-1} & \cdot & \cdots & -\sum_{i\neq c} l_{c-1,i} \end{pmatrix}.$$

Hence

$$\begin{aligned}(-1)^{c-1+2} det(A(M))_{1,1} &= (-1)^{c+1} det(A(K'))_{1,1} - \epsilon(-1)^{c+1} det(A(K''))_{1,1} \\ &= (-1)^{c+1} det(A(K'))_{1,1} + \epsilon(-1)^{c} det(A(K''))_{1,1} \\ &= coef_1(\nabla(K')) + \epsilon\, coef_1(\nabla(K'')) \\ &= coef_1(\nabla(K)). \quad \blacksquare\end{aligned}$$

Remark. This result was also discovered independently by Hoste [H3].

4. Second coefficient of the polynomial.

Definition of the Conway polynomial depends heavily on the orientation. Our next task is to investigate this dependence.

Examples show that a link of c components has 2^{c-1} polynomials for various orientations, and the polynomial for any orientation is identical to the one of reversed orientation, that is, if orientation of each component is reversed. The last statement is in fact true.

Theorem 3.4.1. *If K_1 and K_2 are equivalent links with reversed orientations, then $\nabla(K_1) = \nabla(K_2)$.*

Proof. This can be easily proved by induction on the complexity of a diagram. If D_1 is a diagram of K_1 then a diagram of K_2 is D_2 which is similar to D_1 except all arrows of D_1 are reversed. Let (n, r) be the complexity of D_1. If $n = 0$ or $r = 0$ then the theorem is trivial since both K_1 and K_2 are trivial unlinks or unknots.

Otherwise, let v be the any crossing to be switched when we unlink D_1. Let D_1' and E_1 be the switched and split diagrams, and K_1' and L_1 their link types respectively. Let K_2' and L_2 be the reversed oriented links of K_1' and L_1 respectively. Then as the complexities of D_1' and E_1 are simpler than that of D_1, $\nabla(K_2') = \nabla(K_1')$ and $\nabla(L_2) = \nabla(L_1)$. Also if v' is the crossing in D_2 corresponding to v, then v' has the same index as v. Let D_2' and E_2 be switched and split diagrams of D_2' at v'. By Axiom (3), $\nabla(K_1) = \nabla(K_2)$. ∎

There are, as follows from the above theorem, 2^{c-1} polynomials for a link of c components. It is natural to ask whether these polynomials are related. Theorems 3.3.2, 3.3.3 provide some partial answer. Suppose L_1 and L_2 are equivalent links of c components with same orientation except the first component. Consider $coef_1(\nabla(L_1))$ and $coef_1(\nabla(L_2))$. We have the following relation between them for different values of c, all linking numbers involved are linking numbers among compnents of L_1.

$$c = 2, \qquad coef_1(\nabla(L_1)) - coef_1(\nabla(L_2)) = 2l_{12}$$

$$c = 3, \qquad coef_1(\nabla(L_1)) - coef_1(\nabla(L_2)) = 2l_{23}(l_{12} + l_{13})$$

$$c = 4, \qquad coef_1(\nabla(L_1)) - coef_1(\nabla(L_2)) = 2((l_{12} + l_{13} + l_{14})$$
$$(l_{23}l_{24} + l_{23}l_{34} + l_{24}l_{34}) + l_{12}l_{13}l_{14})$$

etc.

As we just saw, at least the first coefficients of all polynomials of any link are related in some way. How about the other coefficients? The following theorems provide an answer when $c = 2$ or 3.

Before we state our theorems, we need some notations. Let D be a diagram of a 2-component link L and K_1, K_2 be the components of L. A *segment* S of K_1 is an interval of K_1 and PS is, a segment of D, the image of S of the projection. Suppose S_1 and S_2 are two segments of some link. We say that we *push* S_1 above S_2 if we switch all necessary inter-crossings between S_1 and S_2 so that every inter-crossing between them is of the types 1 and 3, or 2 and 4 as in figure 3.2.

Suppose we switch an inter–crossing between S_1 and S_2 and let D_1 be the split diagram. So for each $i = 1,\ 2$, S_i is *cut* into two segments S_{i1} and S_{i2}. Although strictly speaking, S_{i1} and S_{i2} do not union to form S_i, by abuse of language we would say so (fig. 3.3). The readers are advised to draw some pictures to understand our notations.

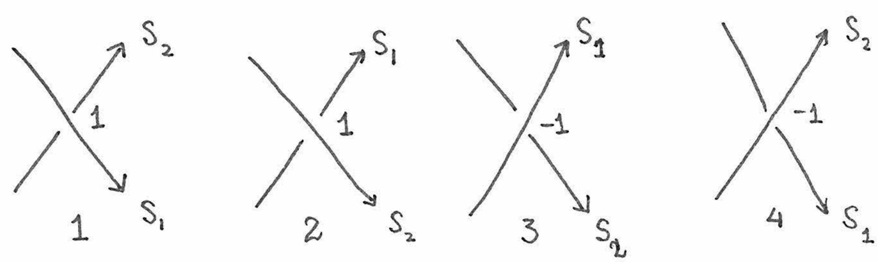


fig. 3.2

All links and knots are considered oriented for the rest of the chapter. If K is a knot, $\overline{K}$ denotes the same knot with reversed orientation.

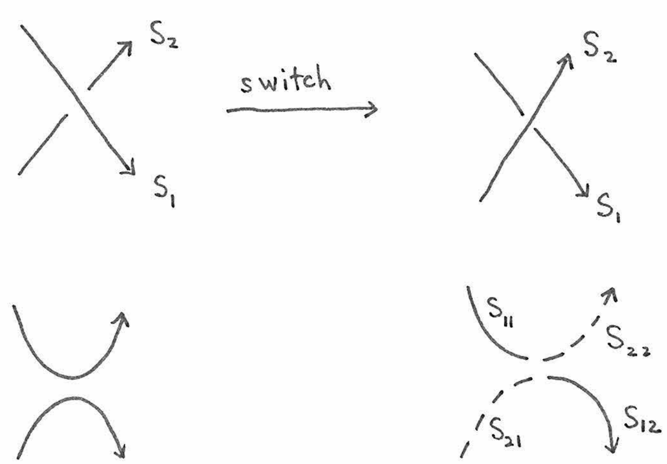


fig. 3.3

Theorem 3.4.2. *Suppose M_1 and M_2 are two equivalent 2- component links* *with components* K_1, K_2 *and* $\overline{K_1}$, K_2 *respectively, and let* l *be the linking number between* K_1 *and* K_2*, so* $-l$ *is the linking number between* $\overline{K_1}$ *and* K_2*. Then*

$$coef_2(\nabla(M_1)) - coef_2(\nabla(M_2)) = 2l(coef_1(\nabla(K_1)) + coef_1(\nabla(K_2))) + \frac{l(l^2-1)}{6}.$$

Proof. Without loss of generality, we can assume $l \geq 0$ for if not, we interchange the role of M_1 and M_2.

We will prove by induction on l. Suppose $l = 0$. We need to show $coef_2(\nabla(M_1)) = coef_2(\nabla(M_2))$. Let D_1 be a diagram of M_1. First we claim that if we permit ourselves to switch any self-crossing of projected image of K_1 or K_2 in D_1, we can unlink M_1. The same process certainly unlinks M_2.

First unknot K_1 and we can then assume it has an image of a circle in D_1. Then traverse K_2 starting from any point in either direction and write down the indices of encountered inter-crossings into a sequence S. Since l is 0, the sum of the sequence is 0. So S either alternates between -1 and 1 or has two consecutive similar index entries. In the latter case, the two crossings have the following diagram (fig. 3.4), where S_1 and S_2 are the two arcs involved.

Switch all necessary crossings on S_2 so that we can apply Reidemeister's move of type inverse of T_2 and obtain the diagram (fig. 3.5).

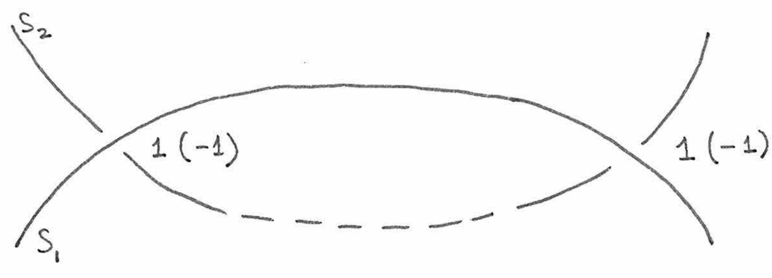


fig. 3.4

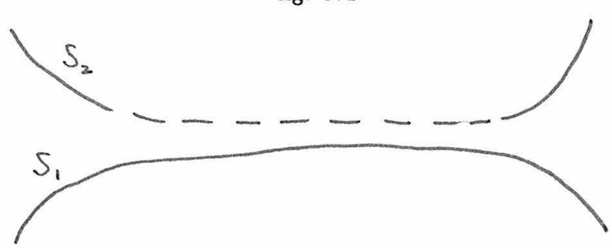


fig. 3.5

Notice that crossings on S_2 are all self-crossings, for the two inter-crossings are consecutive in sequence. The new link has similar linking number as the initial one but has two less inter-crossings. Proceeding inductively on the number of crossings, we can assume S is alternating between 1 and -1. But then all inter-crossings are of type 1 and 4, or type 2 and 3 as in (fig. 3.2), hence K_1 is separable from K_2.

Now, unlink M_1 and M_2 using the described algorithm, by (3.2) we have

$$\nabla(M_1) = \nabla(M_1^*) + z\sum s_i \nabla(L_i)$$

and

$$\nabla(M_2) = \nabla(M_2^*) + z\sum s_i' \nabla(L_i').$$

where $s_i = s_i'$ since corresponding self-crossings between diagrams of M_1 and M_2 bear the same indices; M_1^* and M_2^* are the resulting split links. Thus $\nabla(M_1^*) =$

$\nabla(M_2^*) = 0$. So

$$coef_2(\nabla(M_1)) - coef_2(\nabla(M_2)) = \sum s_i(coef_1(\nabla(L_i)) - coef_1(\nabla(L_i')))$$

as each L_i has 3 components. At the i^th switch L_i has components K_j^i, obtained from K_j, A_{1i}, and A_{2i} where A_{1i} and A_{2i} union to form K_k^i, obtained from K_k, $k \neq j$, k, $j \in \{1,2\}$. So L_i' has components K_j^i, $\overline{A_{1i}}$, and $\overline{A_{2i}}$ if $j=1$, or $\overline{K_j}$, A_{1i} and A_{2i} if $j=2$. Let l_{jm} be the linking number between K_j^i and A_{mi}, and l_{ai} the linking number between A_{1i} and A_{2i}. So linking numbers of L_i' are $-l_{j1}$, $-l_{j2}$ and l_{ai} and thus

$$coef_1(\nabla(L_i)) - coef_1(\nabla(L_i')) = 2l_{ai}(l_{j1} + l_{j2}).$$

But $l_{j1} + l_{j2} = l = 0$. Hence for each i, $coef_1(\nabla(L_i)) = coef_1(\nabla(L_i'))$ and we conclude that $coef_2(\nabla(M_1)) = coef_2(\nabla(M_2))$.

Next, suppose we have proved the theorem when K_1 is an unknotted circle. Then we can show that the theorem holds in general. Assume K_1 is an arbitrary knot. Unknot K_1 and $\overline{K_1}$ with a same process. We have

$$\nabla(M_1) = \nabla(M_1^*) + z\sum s_i \nabla(L_i)$$

and

$$\nabla(M_2) = \nabla(M_2^*) + z\sum s_i \nabla(L_i')$$

where

$$coef_2(\nabla(M_1^*)) - coef_2(\nabla(M_2^*)) = 2l coef_2(\nabla(K_2)) + l(l^2-1)/6$$

by assumption since first components of M_1^* and M_2^* are unknotted circles. Each L_i has three components K_2, A_{1i} and A_{2i} where the union A_{1i} and A_{2i} forms K_1^i. So L_i' has components K_2, $\overline{A_{1i}}$ and $\overline{A_{2i}}$. Thus as seen previously,

$coef_1(\nabla(L_i)) - coef_1(\nabla(L_i')) = 2l_{a,i}(l_{j,1} + l_{j,2}) = 2l\, l_{a,i}$. Consider the knot K_1, unknotting it to get

$$\nabla(K_1) = \nabla(K_1^*) + z \sum s_i \nabla(N_i)$$

where N_i has precisely components A_{1i} and A_{2i}. So

$$coef_2(\nabla(K_1)) = \sum s_i\, coef_1(\nabla(N_i)) = \sum s_i\, l_{a,i}.$$

Thus

$$\sum s_i(coef_1(\nabla(L_i)) - co1L_1') = 2l\, coef_2(\nabla(K_1))$$

and so

$$\begin{aligned} coef_2(\nabla(M_1)) - coef_2(\nabla(M_2)) &= 2l(coef_2(\nabla(K_1)) + coef_2(\nabla(K_2))) \\ &\quad + l(l^2 - 1)/6. \end{aligned}$$

Lastly we need to show that the theorem is true when K_1 is an unknotted circle. Let D_1 be a diagram for M_1 with projection of K_1 being a cicle, D_2 be the corresponding diagram for M_2. Suppose we are done when the linking number is less than l and $l > 0$. There is an inter-crossing v with index 1 in D_1. Switch v. Let M_{11} be the resulting link and L_1 be the split. Similarly, switching v in D_2 gives M_{21} and L_2. Note that in M_2, v has index -1. M_{11} has linking number $l-1$. By induction hypothesis, we can compute $coef_2(\nabla(M_{11})) - coef_2(\nabla(M_{21}))$. Moreover, we have

$$\begin{aligned} coef_2(\nabla(M_1)) - coef_2(\nabla(M_2)) &= coef_2(\nabla(M_{11})) - coef_2(\nabla(M_{21})) \\ &\quad + coef_2(\nabla(L_1)) + coef_2(\nabla(L_2)) \end{aligned}$$

since L_1 and L_2 are knots and v in D_2 has index -1. As

$$\frac{l}{6}(l^2 - 1) - \frac{l-1}{6}((l-1)^2 - 1) = \frac{l}{2}(l - 1),$$

it suffices to show that $coef_2(\nabla(L_1))$ - $coef_2(\nabla(L_2)) = \frac{l}{2}(l-1)$ to conclude the proof of the theorem.

Let S_i, $i = 1, 2$ be the segment of L_1 corresponding to K_i in M_1 ($\overline{S}_1$, S_2 in case of L_2) and PS_i be their projections ($P\overline{S}_1, PS_2$ in L_2) (fig. 3.6). Push S_2 above S_1 according to the following procedure so that the resulting knot L_1^* is equivalent to $K_1 \# K_2$ and thus $\nabla(L_1^*) = \nabla(K_1)\nabla(K_2) = \nabla(K_2)$. Similarly, we have $\nabla(L_2^*) = \nabla(\overline{K}_1)\nabla(K_2) = \nabla(K_2)$ by the same procedure.

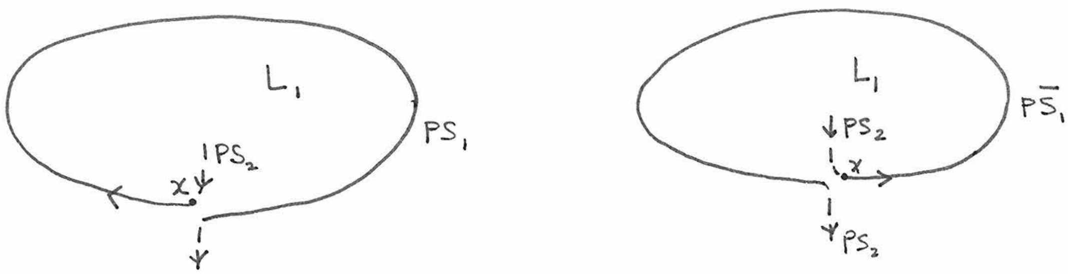


fig. 3.6

First traverse from x (fig. 3.6) along PS_2 on L_1 (L_2) according to the orientation and list indices of all inter-crossings between $S_1(\overline{S}_1)$ and S_2 in sequence. Switch those crossings with positive (negative) indices and then switch alternately starting from the first inter-crossing. The resulting knot L_1^* (L_2^*) has required property, that is, S_2 lies totally above S_1 (below $\overline{S}_1$).

Suppose there are $2(n+l)$ inter-crossings in D_1, then there are $2(n+l) - 1$ intercrossings between PS_1 and PS_2, $n + 2l - 1$ positively indexed and n negatively indexed. Let N_i (N_i') be the split at the i^th switch of L_1 (L_2), then

$$\nabla(L_1) = \nabla(L_1^*) + z \sum_{i=1}^{n+2l-1} \nabla(N_i) - z \sum_{i=n+2l}^{2n+3l-1} \nabla(N_i)$$

and

$$\nabla(L_2) = \nabla(L_2^*) - z \sum_{i=1}^{n+2l-1} \nabla(N_i') + z \sum_{i=n+2l}^{2n+3l-1} \nabla(N_i').$$

In $N_i(N_i')$, S_j ($\overline{S}_1$ in case) is cut into two segments S_{j1} and S_{j2} ($\overline{S}_{j1}$ and $\overline{S}_{j2}$ in the case of $\overline{S}_1$) and there are $2(n+l-1)$ inter-crossings between PS_{1j} and PS_{2k}. Among these crossings, there are $n+2l-i-1$ positively (negatively) indexed and $n+i-1$ negatively (positively) indexed when $1 \le i \le n+sl-1$. If $n+2l \le i \le 2n+3l-1$ then there are $i-1-(n+2l-1)$ positively (negatively) indexed and $3n+4l-i-2$ negatively (positively) indexed. Without loss of generality, N_i (N_i') is composed of the two components $S_{11} \cup S_{21}$ ($\overline{S}_{11} \cup S_{22}$) and $S_{12} \cup S_{22}$ ($\overline{S}_{12} \cup S_{21}$). Denote $\|A \cap B\|$ be the sum of indices of all inter-crossings between A and B. The linking number l_i (l_i') of N_i (N_i') equals

$$\frac{1}{2}\left(\|PS_{11} \cap PS_{12}\| + \|PS_{11} \cap S_{22}\| + \|PS_{21} \cap PS_{12}\| + \|PS_{21} \cap PS_{22}\|\right)$$

or

$$\frac{1}{2}\left(\|P\overline{S}_{11} \cap P\overline{S}_{12}\| + \|P\overline{S}_{11} \cap PS_{21}\| + \|P\overline{S}_{12} \cap PS_{22}\| + \|PS_{21} \cap PS_{22}\|\right)$$

in the case of N_i'. But $\|PS_{11} \cap PS_{12}\| = \|P\overline{S}_{11} \cap P\overline{S}_{12}\| = 0$. As K_1 is a circle, for all possible values of j, k, $\|PS_{1j} \cap PS_{2k}\| = -\|P\overline{S}_{1j} \cap PS_{2k}\|$. Therefore,

$$\begin{aligned} coef_1(\nabla(N_i)) &- coef_1(\nabla(N_i')) \\ &= \frac{1}{2}(\|PS_{11} \cap PS_{21}\| + \|PS_{11} \cap PS_{22}\| \\ &\quad + \|PS_{12} \cap PS_{21}\| + \|PS_{12} \cap PS_{22}\|) \\ &= \begin{cases} l - i & \text{if } 0 \le i \le n+2l-1; \\ i - (2n+3l) & \text{if } n+2l \le i \le 2n-3l-1. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=1}^{2n+3l-1} s_i(coef_1(\nabla(N_i)) - coef_1(\nabla(N_i'))) &= \sum_{i=1}^{n+2l-1} (l-i) - \sum_{i=1}^{n+l} (i - (n+l)) \\ &= \frac{l}{2}(l-1). \end{aligned}$$

Therefore,

$$coef_2(\nabla(L_1)) - coef_2(\nabla(L_2)) = coef_2(\nabla(L_1^*)) - coef_2(\nabla(L_2^*)) + \frac{l}{2}(l-1)$$
$$= 2coef_2(\nabla(K_2)) + \frac{l}{2}(l-1).$$

This completes the proof of the Theorem. ∎

Suppose now M_1, M_2 are two equivalent 3-component links with components K_1, K_2, K_3 for M_1 and $\overline{K}_1$, K_2 and K_3 for M_2. Suppose also K_2 is separable from K_3. We want to study $coef_2(\nabla(M_1)) - coef_2(\nabla(M_2))$.

Let D_1 be a diagram of M_1 such that the image of K_2 does not intersect that of K_3, and D_2 the corresponding diagram for M_2. Push K_2 above K_1 ($\overline{K}_1$), we get

$$\nabla(M_1) = \nabla(M_1^*) + z\sum s_i\nabla(L_i), \qquad \sum s_i = l_{12}$$

and

$$\nabla(M_2) = \nabla(M_2^*) - z\sum s_i\nabla(L_i'). \tag{3.7}$$

since corresponding inter-crossings between K_1 and K_2 in D_1 and D_2 have opposite indices. M_1^* and M_2^* are split links since K_2 is separable from the other two components. Also, each L_i (L_i') has two components K_3 and $S_1 \cup S_2$ ($\overline{S}_1 \cup S_2$) where S_j corresponds to K_j, $j = 1,\ 2$ of M_1. Consider $L_i(L_i')$, push S_2 above S_1 ($\overline{S}_1$), and get

$$\begin{aligned} \nabla(L_i) &= \nabla(L_i^*) + z\sum t_{i,j}\nabla(L_{ij}), \\ \nabla(L_i') &= \nabla(L_i'^*) - z\sum t_{ij}\nabla(L_{ij}') \end{aligned} \tag{3.8}$$

By construction, L_i^* ($L_i'^*$) is equivalent to $K_2\#(K_1 \cup K_3)$ ($K_2\#(\overline{K}_1 \cup K_3)$) where $A \cup B$ denotes the sublink composed of A and B. Let $M_i^{(j,k)}$ be the sublink of M_i by deleting all components except K_j and K_k. Then $\nabla(L_i^*) = \nabla(K_2)\nabla(M_1^{(1,3)})$ and $\nabla(L_i'^*) = \nabla(K_2)\nabla(M_2^{(1,3)})$.

Moreover, each L_{ij} (L'_{ij}) has three components K_3, $S_{11} \cup S_{21}$ and $S_{12} \cup S_{22}$ (K_3, $\overline{S}_{11} \cup S_{22}$, $\overline{S}_{12} \cup S_{21}$) where S_{11}, S_{12}, S_{21}, and S_{22} are as in the proof of previous theorem. But in this case K_1 is arbitrary instead. Let $a_1 = \frac{1}{2}\|S_{11} \cap S_{12}\|$, $a_2 = \frac{1}{2}\|S_{21} \cap S_{22}\|$, $b_{1m} = \frac{1}{2}\|S_{11} \cap S_{2m}\|$, and $b_{2m} = \frac{1}{2}\|S_{12} \cap S_{2m}\|$, and $c_m = \frac{1}{2}\|K_3 \cap S_{1m}\|$ for each $m = 1, 2$. So $c_1 + c_2 = l_{13}$. The linking numbers of L_{ij} are $a_1 + a_2 + b_{12} + b_{21}$, c_1, c_2 and those of L'_{ij} are $a_1 + a_2 - b_{11} - b_{22}$, $-c_1$, and $-c_2$. By Theorem 3.3.2,

$$\begin{aligned}
coef_1(\nabla(L_{ij})) - coef_1(\nabla(L'_{ij})) &= (c_1c_2 + (c_1 + c_2)(a_1 + a_2 + b_{12} + b_{21})) \\
&\quad - (c_1c_2 - (c_1 + c_2)(a_1 + a_2 - b_{11} - b_{22})) \\
&= (c_1 + c_2)(2a_1 + 2a_2 - b_{11} + b_{12} + b_{21} - b_{22}) \\
&= l_{13}(2a_1 + 2a_2 - b_{11} + b_{12} + b_{21} - b_{22}) \\
&= l_{13}\left(coef_1(\nabla(N_{ij})) + coef_1(\nabla(N'_{ij}))\right),
\end{aligned}$$

where N_{ij} (N'_{ij}) is the link of which the components are $S_{11} \cup S_{21}$ ($\overline{S}_{11} \cup S_{22}$) and $S_{12} \cup S_{22}$ ($\overline{S}_{12} \cup S_{21}$). Combining (3.7) and (3.8), we have

$$\begin{aligned}
& coef_2(\nabla(M_1)) - coef_2(\nabla(M_2)) \\
= \;& coef_2(\nabla(M_1^*)) - coef_2(\nabla(M_2^*)) + \sum s_i(coef_2(\nabla(L_i)) + coef_2(\nabla(L'_i))) \\
= \;& coef_2(\nabla(M_1^*)) - coef_2(\nabla(M_2^*)) + \sum s_i(coef_2(\nabla(L_i^*)) + coef_2(\nabla(L_i'^*)) \\
& + \sum_{ij} s_it_{ij}(coef_1(\nabla(L_{ij})) - coef_1(\nabla(L'_{ij}))) \\
= \;& \sum s_i(coef_2(\nabla(K_2)\nabla(M_1^{(1,3)})) + coef_2(\nabla(K_2)\nabla(M_2^{(1,3)}))) \\
& + l_{13}\sum s_it_{ij}(coef_1(\nabla(N_{ij})) + coef_1(\nabla(N'_{ij}))) \\
= \;& l_{12}(coef_2(\nabla(M_1^{(1,3)})) + coef_2(\nabla(M_2^{(1,3)}))) \\
& + l_{13}\sum s_it_{ij}(coef_1(\nabla(N_{ij})) - coef_1(\nabla(N'_{ij}))).
\end{aligned}$$

Now consider $M_1^{(1,2)}$ and $M_2^{(1,2)}$. Apply same procedure to them. It is easy to check that

$$\nabla(M_1^{(1,2)}) + \nabla(M_2^{(1,2)}) = z^2 \sum s_i t_{ij}(\nabla(N_{ij}) + \nabla(N'_{ij})).$$

Therefore,

$$\begin{aligned} coef_2(\nabla(M_1^{(1,2)}))+&coef_2(\nabla(M_2^{(1,2)})) \\ &= \sum s_i t_{ij}(coef_1(\nabla(N_{ij})) + coef_1(\nabla(N'_{ij}))). \end{aligned}$$

Thus

$$\begin{aligned} coef_2(\nabla(M_1)) - coef_2(\nabla(M_2)) =& l_{12}(coef_2(\nabla(M_1^{(1,3)})) + coef_2(\nabla(M_2^{(1,3)}))) \\ &+ l_{13}(coef_2(\nabla(M_1^{(1,2)})) + coef_2(\nabla(M_2^{(1,2)}))). \end{aligned}$$

Next suppose K_2 and K_3 are not separable. Let D_1 be a diagram for M_1 and D_2 the corresponding diagram for M_2. Push K_2 above K_3 with same procedure applied to both D_1 and D_2. Then

$$\nabla(M_1) = \nabla(M_1^*) + z\sum s_i \nabla(N_i), \qquad \sum s_i = l_{23}$$

and

$$\nabla(M_2) = \nabla(M_2^*) + z\sum s_i \nabla(N'_i)$$

where N_i and N'_i are equivalent links with different orientations for K_1. So $coef_2(\nabla(N_i))$ - $coef_2(\nabla(N'_i))$ is given by Theorem 3.4.2, $2l(coef_2(\nabla(K_1)) + coef_2(\nabla(L_i))) + l(l^2-1)/6$ where l is the linking number of N_i, and L_i is the other component. But l=l_{12}+l_{13}. Push K_2 above K_3 in $M_1^{(2,3)}$, we obtain $\nabla(M_1^{(2,3)}) = z\sum s_i \nabla(L_i)$. Therefore,

$$\begin{aligned} &coef_2(\nabla(M_1)) - coef_2(\nabla(M_2)) \\ =&coef_2(\nabla(M_1^*)) - coef_2(\nabla(M_2^*)) + \sum s_i(coef_2(\nabla(N_i)) - coef_2(\nabla(N'_i))) \end{aligned}$$

$$
\begin{aligned}
=&l_{12}(coef_2(\nabla(M_1^{(1,3)})) + coef_2(\nabla(M_2^{(1,3)})))\\
&+ l_{13}(coef_2(\nabla(M_1^{(1,2)})) + coef_2(\nabla(M_2^{(1,2)})))\\
&+ 2(l_{12} + l_{13}) \sum s_i(coef_2(\nabla(K_1)) + coef_2(\nabla(L_i)))\\
&+ \frac{l_{23}}{6}(l_{12} + l_{13})(l_{12} + l_{13} - 1)(l_{12} + l_{13} + 1)\\
=&l_{12}(coef_2(\nabla(M_1^{(1,3)})) + coef_2(\nabla(M_2^{(1,3)})))\\
&+ l_{13}(coef_2(\nabla(M_1^{(1,2)})) + coef_2(\nabla(M_2^{(1,2)})))\\
&+ 2(l_{12} + l_{13})coef_2(\nabla(M_1^{(2,3)})) + 2l_{23}(l_{12} + l_{13})coef_2(\nabla(K_1))\\
&+ \frac{l_{23}}{6}(l_{12} + l_{13})(l_{12} + l_{13} - 1)(l_{12} + l_{13} + 1). \qquad (3.9)
\end{aligned}
$$

Hence we have proved

Theorem 3.4.3. *If M_1 and M_2 are equivalent 3-component links with same orientation except the first component, then (3.9) holds.*

Theorem 3.4.2 and 3.4.3 show polynomials of equivalent links with different orientations are related at least when c is small. In fact a similar result is found when $c = 4, 5$. The relation is a combination of the first and second coefficients of all of its sublinks. The proof is tedious and extremely technical. Nevertheless, we believe that an explicit relation can be found for all c, and that the polynomials are in fact related in some explicit way.

Chapter 4. A new polynomial invariant for knots and links

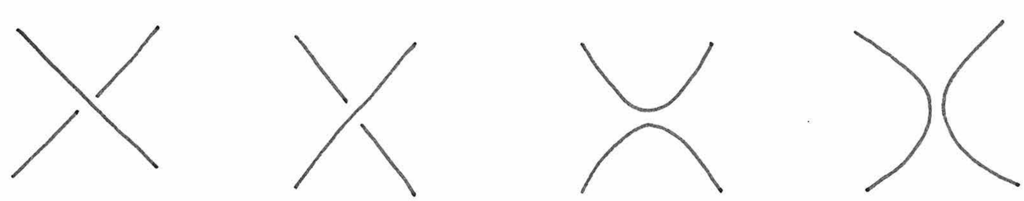
1. Introduction.

The classical theory of knots and links has concentrated on oriented links. For the Alexander or the HOMFLY polynomial, different orientations of a link yield different polynomials. For knots, all polynomial invariants produce the same results for the two distinct orientations. However, the following definition results in a polynomial invariant for knots and links which is indifferent to the orientation of the link.

Suppose K_1 is not oriented, that is, all arrows of the diagram are removed. We can switch a crossing as usual. But when we split at v, there are two possibilities L_1 and L_2 (fig. 4.1). Suppose a polynomial function F is defined on knots and links satisfying

1. $H_{unknot}(x) = 1.$
2. $H_{K_1}(x) + H_{K_2}(x) = x(H_{L_1}(x) + H_{L_2}(x)).$

This indeed defines a non-trivial polynomial invariant for non–oriented knots and links. Properties and proofs will be discussed in later sections. First we compute some examples. These examples show that the trivial 3-unlink is different from the Borremean ring of 3 components.



K_1 K_2 L_1 L_2

fig. 4.1

Examples.

1. Let T_c denote the trivial c-unlink.

unknot *unknot* *unknot* T_2

$$H(unknot) + H(unknot) = x(H(unknot) + H(T_2)),$$
$$H(T_2) = 2x^{-1} - 1.$$

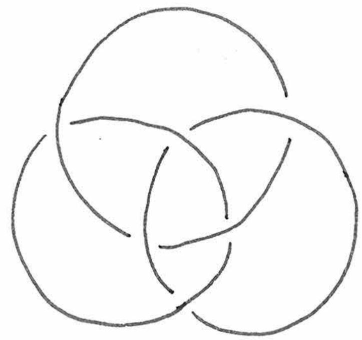
2.

T_{c-1} T_{c-1} T_{c-1} T_c

$$H(T_{c-1}) + H(T_{c-1}) = x(H(T_{c-1}) + H(T_c)),$$
$$H(T_c) = (2x^{-1} - 1)H(T_{c-1}).$$

By induction on c, $H(T_c) = (2x^{-1} - 1)^{c-1}$.

3. Borremean ring.



$$H(B) = 4x^{-2} - 4x^{-1} - 3 + 10x - 6x - 16x^2 + 12x^4 + 4x^5$$
$$\neq H(T_3).$$

Thus F distinguishes B from the trivial 3-unlink.

2. Existence of the invariant polynomial.

In this section we will prove the existence of an invariant polynomial for non-oriented links that satisfies equation (4) of section 1. All links will not be oriented in this chapter.

Let D be a diagram of some link K. D can be changed to a diagram of a trivial unlink by switching some crossings from underpasses to overpasses. We will use the following procedure to unlink D.

Order the components of K. For each component C choose a point a, called the *starting point* of C, and a *direction* from a to traverse C. Starting from the first component, traverse the i^{th} component C_i from the starting point in the chosen direction. Switch all *inter-crossings*, those with segments from two different components, when C_i underpasses the other component which is of a higher order; but switch all *self-crossings*, segments from the same component, from an underpass to an overpass if the crossing has not been encountered before. After the last component is traversed, the diagram is that of a trivial unlink and the components are in descending order.

Let n be the number of crossings of D and R be the number of switchings required to unlink D. If, for some choice of component order and starting points, r is 0, D is said to be *descending.* The order pair (n, r) will denote the *complexity* of D. It depends on the choice of component order and starting points. Observe that $r \leq n$.

We first define a polynomial function on diagrams with certain choice of component order, directions and starting points.

In case of a descending diagram D, define the polynomial $H_D(x) = \mu^{c-1}$ where c is the number of components and $\mu = 2x^{-1}-1$. Suppose we have assigned to each diagram with n crossings with a polynomial and let D have complexity $(n+1, r)$. If $r = 0$, D is descending and hence $H_D(x)$ is defined. If $r > 0$, unlink

D and let v be the first crossing switched. Let D' be the switched diagram, E_1 and E_2 be the two splits at v as in figure 4.2. D' has complexity $(n+1, r-1)$ with same choice of component order, direction and starting points; and each E_i has n crossings, $i = 1, 2$. So by induction on r and n, $H_{D'}(x)$, $H_{E_1}(x)$, $H_{E_2}(x)$ are known. $H_D(x)$ is defined by

$$H_D(x) + H_{D'}(x) = x(H_{E_1}(x) + H_{E_2}(x)). \tag{4.1}$$

In order to show F is indeed a link invariant we need to verify that it is independent of the choice of component order, starting points and directions, and diagrams of the same link type. To proceed, we first show F is independent of the choice of starting points. Then we prove F is invariant under any Reidemeister's moves. Also we will show that the polynomials for trivial unlinks (link type of descending diagrams) are consistent with equation (4) of section 1.

For convenience we also denote $H_D(x)$ by $H(D)$.

First we show that H is independent of the sequence of switchings applied. To do so we will show that same polynomial is obtained if we reverse the order of two immediate switchings. Let u and v be the two crossings involved, and δ_x denotes the switching at crossing x. If $u = v$ there is nothing to prove. So assume u and v are distinct. Let σ_{1x} and σ_{2x} deonte the two splits at x (fig. 4.2). First apply δ_u before δ_v. To compute $H_D(x)$, we employ (4.1),

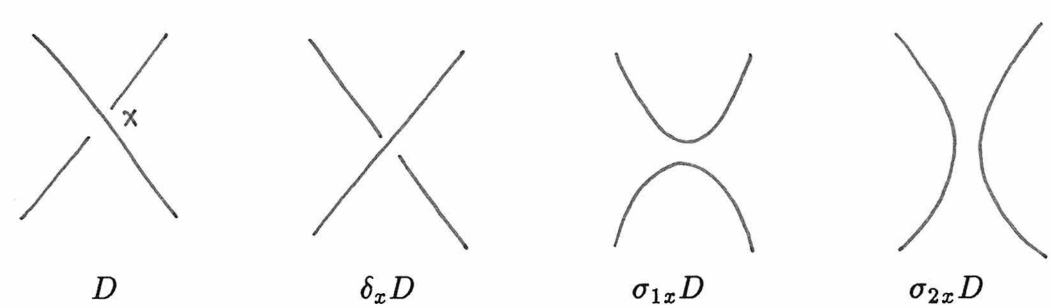


fig. 4.2

$$H(D) = -H(\delta_u D) + x(H(\sigma_{1u}D) + H(\sigma_{2u}D))$$
$$= H(\delta_v\delta_u D) - x(H(\sigma_{1v}\delta_u D) + H(\sigma_{2v}\delta_v D)) + x(H(\sigma_{1u}D) + H(\sigma_{2u}D))$$

Let $G(D)$ be the polynomial obtained by applying δ_v before δ_u.

$$G(D) = -H(\delta_v D) + x(H(\sigma_{1v}D) + H(\sigma_{2v}D))$$
$$= H(\delta_u\delta_v D) - x(H(\sigma_{1u}\delta_v D) - H(\sigma_{2u}\delta_v D)) + x(H(\sigma_{1v}D) + H(\sigma_{2v}D)).$$

Since $\delta_u\delta_v = \delta_v\delta_u$ the first two terms are equal. For each $\sigma_{ix}D$, we apply equation (4.1) to find

$$H(\sigma_{iu}D) = -H(\delta_v\sigma_{iu}D) + x(H(\sigma_{1v}\sigma_{iu}D) + H(\sigma_{2v}\sigma_{iu}D))$$

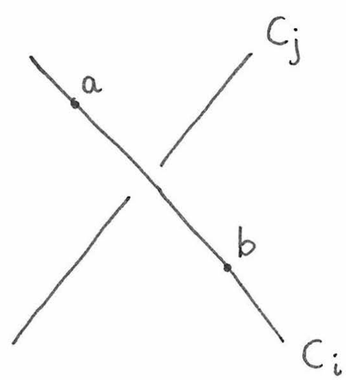
and

$$H(\sigma_{iv}D) = -H(\delta_u\sigma_{iv}D) + x(H(\sigma_{1u}\sigma_{iv}D) + H(\sigma_{2u}\sigma_{iv}D)).$$

Substituting these expressions into $H(D)$ and $G(D)$ and noticing that for all $i, j, \delta_x\sigma_{iy}D = \sigma_{iy}\delta_x D$ and $\sigma_{ix}\sigma_{jy}D = \sigma_{jy}\sigma_{ix}D$ as long as $x \neq y$, we see that $H(D) = G(D)$.

Next, we show that $H(D)$ is independent of the choice of the point we start with on C_i, the i^{th} component. Suppose a and b are two choices of starting points on C_i, $H(D)$ and $G(D)$ are the corresponding polynomials obtained. It suffices to show that if a and b are on a crossing, then $H(D) = G(D)$. There are two cases.

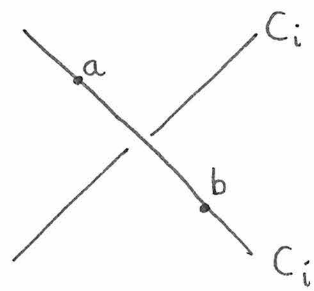
<u>Case 1. $i \neq j$.</u>



Suppose $j > i$, v will not be switched and $H(D) = G(D)$. If $j < i$, v

will be switched despite the choice of starting point. Since both polynomials are independent of the switching order and switchings needed for both choices of starting points are the same, $H(D) = G(D)$.

<u>Case 2. $i = j$.</u>



The crossing v will be switched for one choice of starting point and not for the other. Except this crossing, both choices require similar switchings on other crossings. Since the polynomial is independent of switching order, we can assume v is switched last. Without loss, let E be the diagram with starting point $b, H(E)$ the polynomial of E, and E' the diagram just before v is switched. Let x and y be the preimages of v on the component C, S the segment on C joining x and y by traversing along the given direction, and T be the remaining segment. By construction, S always lies above T. If we consider S and T as knots by joining x and y with a vertical line segment, they form a trivial 2-unlink. Hence the two splits of C at v are an unknot and a trivial 2-unlink. Thus one of $\sigma_{iv}E'$, say $\sigma_{1v}E'$ is a trivial c-unlink and the other a trivial $(c+1)$-unlink. We then have

$$H(D) = \epsilon H(E') + Q(x)$$

and

$$H(E) = -\epsilon(H(\delta_v E') - x(H(\sigma_{1v}E') + F\sigma_{2v}E')) + Q(x),$$

where $\epsilon = \pm 1$ and $Q(x)$ is some Laurent polynomial in x. But E' and $\delta_v E'$ are trivial c-unlinks, they have polynomials μ^{c-1}. Also, $H(\sigma_{2v}E') = \mu^c$ and

$H(\sigma_{1v}E') = \mu^{c-1}$. Substituting into above two equations, we get $H(D) = H(E)$.

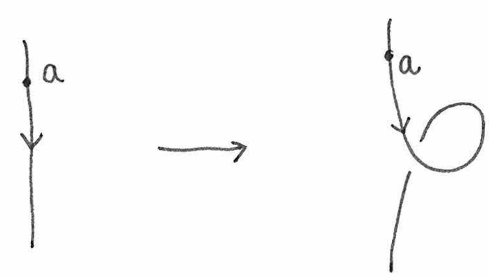
Next we will show that $H(D)$ is invariant under Reidemeister moves.

Two diagrams are n-equivalent if there is a finite sequence of Reidemeister moves which, when applied to a diagram, will result in the other, such that no diagram before or after each move has more than n crossings. Note that two diagrams of n crossing, can be equivalent, that is, correspond to the same link type, but not n-equivalent. We will show that $H(D) = H(E)$ if D and E are n-equivalent for some n.

Since any two diagrams of a link are n-equivalent for some n, we can verify that F is an invariant of the link type, up to choice of component order and directions. As descending diagrams receive polynomials which are independent of component order and directions, a simple induction argument on the complexity of a diagram will conclude that the polynomial of any diagram is independent of the component order and directions. This completes the proof of existence and uniqueness of the polynomial F for any link type.

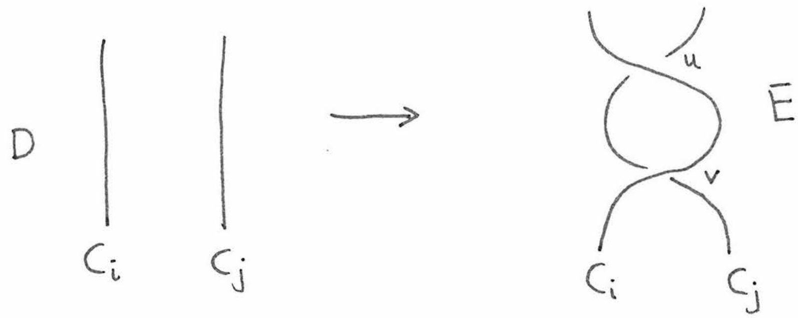
We will use induction on n. If $n = 0$ there is nothing to prove. Suppose we are done for diagrams with n crossings and let D have $n + 1$ crossings.

Case 1.

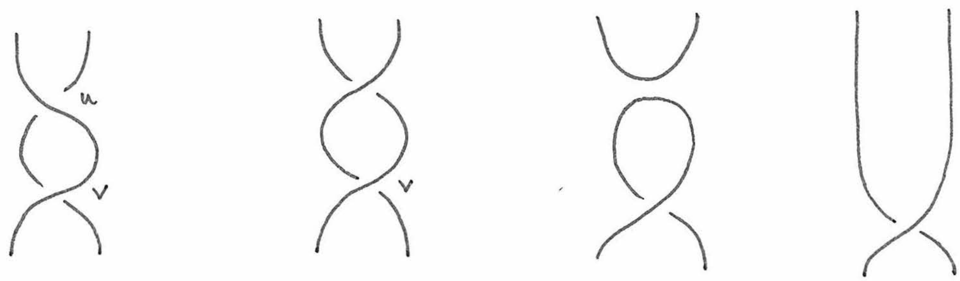


We start at the indicated point and the traversing direction.

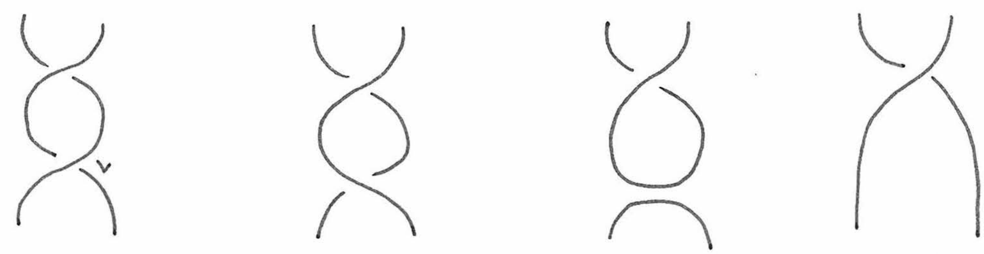
Case 2.



In the case $i \leq j$, we start at a point such that no switching occurs at u or v. If $i > j$, both crossings will be switched. We have the following diagrams (fig. 4.3) and equations, and find that H(D) is independent of the move.



E $\delta_u E$ $\sigma_{1u} E$ $\sigma_{2u} E$



$\delta_u E$ $\delta_v \delta_u E$ $\sigma_{2x} \delta_u E$ $\sigma_{1x} \delta_u E$

fig. 4.3

$$H(E) = H(\delta_v \delta_u E) + x(H(\sigma_{1u} E) + H(\sigma_{2u} E) - H(\sigma_{1v} \delta_u E) - H(\sigma_{2v} \delta_u E)).$$

But $\sigma_{2v}\delta_u = \sigma_{1u}, H(\sigma_{1v}\delta_u E) = H(\sigma_{2u} E)$ by case 1, and $H(D) = H(\delta_v \delta_u E)$ by case $2(i < j)$. Hence $H(D) = H(E)$.

Case 3.

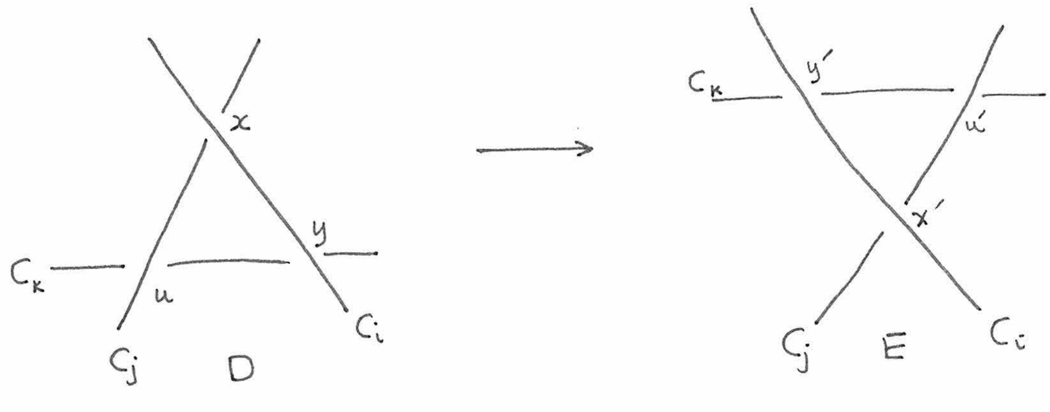


fig. 4.4.a

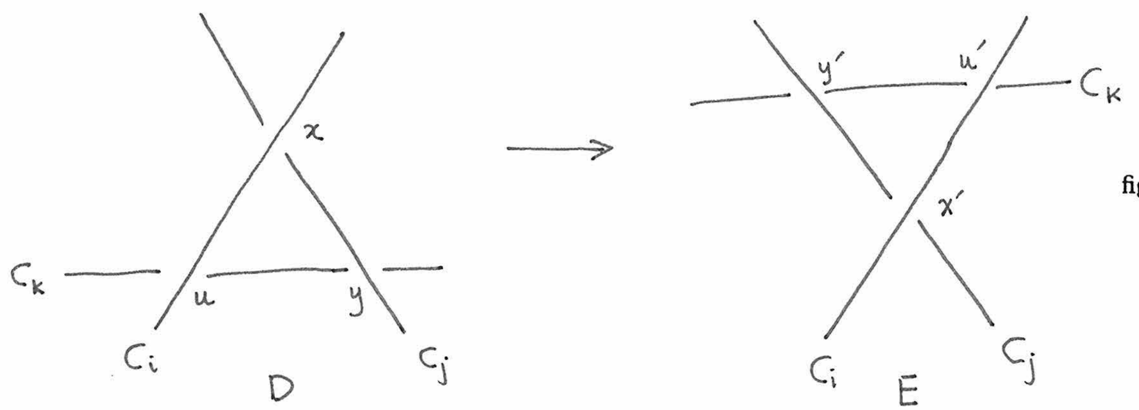


fig. 4.4.b

Choose a component order of D such that either $i \leq j \leq k$, or $i < j$ and $i = k$. In the first case, no switching on u, x, y is needed for both D and E. In the later case, y and y' of figure 4.4.a, and u and u' of figure 4.4.b will be switched. Other than this crossing, both D and E require the same set of crossings to be switched. Independence of switching order allows us to switch them last. Let πD and πE denote the resulting diagrams of D and E just before the switching. By induction hypothesis on the number of crossings, $H(D) = H(E)$ iff $H(\pi D)$=$H(\pi E)$. Following pictures (fig. 4.5) should convince the readers that $H(\pi D) = H(\pi E)$ for the case of figure 4.4.a. The other case is similar.

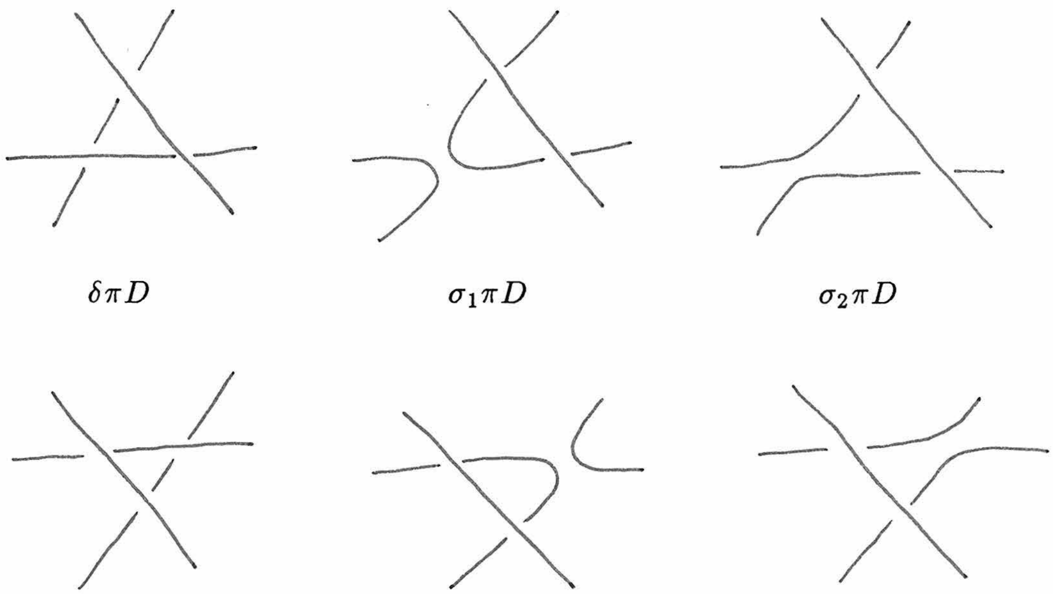


fig. 4.5

$$H(\pi D) = -H(\delta\pi D) + x(H(\sigma_1\pi D) + H(\sigma_2\pi D))$$

and

$$H(\pi E) = -H(\delta\pi E) + x(H(\sigma_1\pi E) + H(\sigma_2\pi E)).$$

But by assumption $\delta\pi D$ and $\delta\pi E$ are trivial unlinks. $\sigma_1\pi E$ can be obtained from $\sigma_1\pi D$ by two T_2 moves and by case 2, they have same polynomials. Also, $\sigma_2\pi E = \sigma_2\pi D$. Hence $H(\pi D) = H(\pi E)$.

3. Basic properties of the polynomial.

The recursive definition of Conway polynomial provides an inductive argument to prove properties about the polynomial. This argument arrives from the fact that a link can be unlinked by switching some set of vertices, and the switched links have smaller unlinking numbers while the splits have less crossing numbers. If (n, r), the complexity of a diagram of some choices of component order, orientation, and starting points is ordered lexicographically, then both the switched and the split diagrams have simpler complexity than the initial diagram if the crossing is chosen to be one of the required switched crossing of the unlinking process. This induction argument can be easily employed in proving properties of both the Conway and the F polynomials.

Denote c_i the coefficient of x^i in $H_K(x)$.

Proposition 4.3.1. *Let K be a link of c components,*

1. *If K is a trivial link, then $H_K(x) = \mu^{c-1}$.*
2. *If $K=K_1\#K_2$ then $H_K(x) = H_{K_1}(x)H_{K_2}(x)$.*
3. *If K is a separated union of K_1 and K_2 then $H_K(x) = \mu H_{K_1}(x)H_{K_2}(x)$.*
4. *If L is the inverse of K, that is, inverting all crossings of K, then $H_L(x) = H_K(x)$.*
5. *$H_K(x) - 1$ is divisible by $2(x-1)$.*
6. *$H_K(x) - \mu^{c-1}$ is divisible by $H_T(x) - 1$ where T is the trefoil.*
7. *The minimum power of x of $H_K(x)$ is $1-c$.*
8. *$Deg(H_K(x)) \leq m - 1$ where m is the crossing number of K.*
9. *$H_K(1) = 1$, and $H_K(-2) = (-2)^{c-1}$.*

Proof. Use induction on the complexity pair (n, r) and regard the unknot and the trivial unlinks as the base cases. We prove (5) to illustrate the method of induction. This property also leads to another interesting result.

Certainly, (5) is true for a trivial unlink since its polynomial is μ^{c-1}. Let K be a diagram with complexity (n, r), and switched a crossing v which reduces complexities of the switched and splitted diagrams. Suppose these new diagrams satisfy (5), then define $Q_L(x)$ the quotient of $H_L(x) - 1$ by $2(x-1)$,

$$\begin{aligned} H_K(x) - 1 &= -(H_{K_2}(x) - 1) + x(H_{L_1}(x) - 1 + H_{L_2}(x) - 1) + 2(x-1) \\ &= 2(x-1)\big(-Q_{K_2}(x) + x(Q_{L_1}(x) + Q_{L_2}(x)) + 1\big). \end{aligned}$$

Thus $H_K(x) - 1$ is divisible by $2(x-1)$ and the quotient, $Q_K(x)$, satisfies

$$Q_K(x) + Q_{K_2}(x) = x(Q_{L_1}(x) + Q_{L_2}(x)) + 1. \quad \blacksquare$$

Notice that $Q_{unknot}(x) = 0$ and Q can be derived from F, hence we proved

Theorem 4.3.2. *There is a uniquely defined polynomial for each link type such that*

1. $Q_{unknot}(x) = 0$,
2. $Q_{K_1}(x) + Q_{K_2}(x) = x(Q_{L_1}(x) + Q_{L_2}(x)) + 1$.

Property 4 gives us some idea about the coefficients. The constant term is odd while others are even. In fact, we have

Property 4.3.1.10. *If l denote the sum of all linking numbers of K,*

a. $c_0 \equiv (-1)^{l+c+1} \bmod 4$ *where $l = 0$ if K is a knot.*

b. c_{-i} *is divisible by* 2^{i+1}, *for* $1 \le i \le c-1$.

c. $c_{-1} \equiv 0 \bmod 4$ *if c is odd and* $c_{-1} \equiv 1 \bmod 4$ *if c is even.*

d. All other coefficients are even.

Proof. Observe that for all $0 \leq i \leq c-1$,

$$c_{-i}(K_1) + c_{-i}(K_2) = c_{-i-1}(L_1) + c_{-i-1}(L_2)$$

where K_1, K_2, L_1, and L_2 are diagrams differing only at crossing v. The property can be proved by induction on number of crossings. If $n = 0$, the conclusion is trivial, since the polynomial is $(2x^{-1} - 1)^{c-1}$ and the coefficient of x^{-i} is $(-1)^i \binom{c-1}{i+1} 2^{i+1}$. If $c = 1$, then one of L_1 or L_2 is a link and the other is a knot and by property (c), and Proposition 4.3.1.7, we are done. If $c > 1$, there are two cases to be considered. First if v is a self-crossing, then say L_1 has $c+1$ components and L_2 has c components. Either $c_{-i-1}(L_2)$ is 0, or is divisible by 2^i. Thus

$$c_{-i}(K_1) + c_{-i}(K_2) \equiv 0 \mod 2^{i+1}.$$

If v is an inter-crossing, then both L_1 and L_2 have $c-1$ components and each $c_{-i-1}(L)$ is divisible by 2^i and so

$$c_{-i}(K_1) + c_{-i}(K_2) \equiv 0 \mod 2^{i+1}.$$

In either case, the result follows from the base case when the number of crossings is 0.

(d) follows from Property 4.3.1.5. ∎

Other than the ten properties listed above, there are nice features that relate the polynomial and some geometric and algebraic properties of the link. Following properties are found by [L3] and [M1]. The first one gives the signature of a link. Though the proof is elementary, I am not able to find a proof that does not use a spanning surface of a link. The second result gives a lower bound for the unknotting number of a knot. Also, the proof requires an understanding of the fundamental group of the knot complement. A combinatorial proof has not been found.

Property 4.3.2.11. *$F_K(2) = (\delta_K)^2$, where δ_K is the signature of K.*

Proof. See [L3].

Property 4.3.2.12.

(a) $F_K(-1) = (-3)^d$, where $d = \dim H_1(M; Z_3)$ and M is the double cover of S^3 branched over K.

(b) $u(K) \geq d - c + 1$, where $u(K)$ is the unknotting number of K.

Proof. For (a), see [L3], and for (b) see Murakami[M1].

4. Arf invariant.

The Arf invariant of a knot K, Arf(K), can be defined by the coefficient of x^2 of its Conway polynomial. In fact, $Arf(K) \equiv d_2(K) \bmod 2$ where $d_2(K)$ denotes the coefficient of z^2 [K1]. Let K be oriented and v a crossing of some diagram of K. Switch v to obtain another knot K' and a link L (fig. 4.1) . As $d_2(K)$ satisfies

$$d_2(K) - d_2(K') = \pm Lk(L),$$

where $Lk(L)$ is the linking number of L, we have

Proposition 4.4.1. *$Arf(K) + Arf(K') \equiv Lk(L) \quad \bmod 2.$*

Returning to our polynomial F_K, we have

Theorem 4.4.2. *$\frac{1}{4}(c_0 - 1) \equiv \frac{c_1}{2} \equiv Arf(K) \bmod 2$ if K is a knot.*

Proof. Again our proof will be based on an induction on (n, r), the complexity of a diagram. If $r = 0$ or $n = 0$, then

$$\frac{1}{4}(c_0 - 1) = \frac{c_1}{2} = 0 = Arf(unknot).$$

Otherwise let v be a crossing and K', L_1, and L_2 be the link types of the switched and the two split diagrams respectively. Suppose further that L_1 has two components and so L_2 has only one. All three diagrams have simpler complexities than that of K. Moreover, by (5.1)

$$\begin{aligned} c_0(K) + c_0(K') &= c_{-1}(L_1), \\ c_1(K) + c_1(K') &= c_0(L_1) + c_0(L_2). \end{aligned}$$

Consider any two component link L with components N and M. Unlink L by pushing N above M, that is, switch all inter-crossings between N and M from underpasses to overpasses. If m is the number of switchings, as the resulting link is a separated union of N and M, by Theorem 4.2.3,

$$H_L(x) = (-1)^m \mu\, H_N(x) H_M(x) + q(x)$$

where $q(x) \in Z[x]$. So $c_{-1}(L) \equiv (-1)^m 2 \mod 8$. Since $c_0(unknot) \equiv 1 \mod 4$. As $m \equiv Lk(L) \mod 2$, and by induction hypothesis on the complexity,

$$\begin{aligned} \frac{1}{4}(c_0(K') - 1) &\equiv \frac{c_1(K')}{2} \equiv Arf(K') \mod 2, \\ c_0(K') &= 1 + 4Arf(K') + 8k, \end{aligned}$$

and

$$c_1(K') = 2Arf(K') + 4k'.$$

So

$$\begin{aligned} c_0(K) &= (-1)^m 2 + 8l - 1 - 4Arf(K') - 8k \\ &= \begin{cases} 1 - 4Arf(K') - 8l', & \text{if } m \text{ is even;} \\ 1 - 4(Arf(K') + 1) + 8l', & \text{if } m \text{ is odd.} \end{cases} \end{aligned}$$

Hence,

$$\frac{1}{4}(c_0(K) - 1) \equiv \begin{cases} Arf(K') \mod 2, & \text{if } Lk(L_1) \text{ is even;} \\ Arf(K') + 1 \mod 2, & \text{if } Lk(L_1) \text{ is odd.} \end{cases}$$

implying

$$\frac{1}{4}(c_0(K)-1)+Arf(K')\equiv Lk(L_1)\mod 2.$$

Thus $\frac{1}{4}(c_0(K)-1)\equiv Arf(K)\mod 2$. This gives part of the Theorem.

Suppose N_1 and N_2 are the two knot components of L_1, then

$$c_0(L_1)=(-1)^m\Big(2\big(c_1(N_1)c_0(N_2)+c_0(N_1)c_1(N_2)\big)-c_0(N_1)c_0(N_2)\Big).$$

As $c_1(N_i)$ is even and $c_0(N_i)=1+4k_i$, $i=1,2$, we have

$$\begin{aligned}c_0(L_1)&=(-1)^m\,(4k''-(1+4k_1+4k_2+16k_1k_2))\\&=(-1)^m(-1+4k_3)\\&=(-1)^{m+1}+4k_4,\\c_0(L_2)&=1+4k_5\end{aligned}$$

for L_2 is a knot. Hence

$$\begin{aligned}c_1(K)&=-c_1(K')+(-1)^{m+1}+1+4k_6,\\\frac{c_1(K)}{2}&=\frac{1+(-1)^{m+1}}{2}-\frac{c_1(K')}{2}+2k_6\\&\equiv m+Arf(K')\mod 2\\&\equiv Lk(L_1)+Arf(K')\mod 2.\end{aligned}$$

So $\frac{c_1(K)}{2}\equiv Arf(K)\mod 2$. ∎

5. Polynomials of tangles.

In this section, we will show another property enjoyed by both HOMFLY and our polynomial.

In our discussion, the definition of a tangle is different from the usual notion of a tangle [L1, C1]. A tangle, A, is a portion of a link diagram that can be drawn in a disc with four intersections on the boundary. As different from the usual notion of tangles, we do not require that there are only two strings in the disc. In fact, we allow knot components inscribed in the disc.

If A is a tangle, let A^D and A^N denote the links (fig. 4.6):

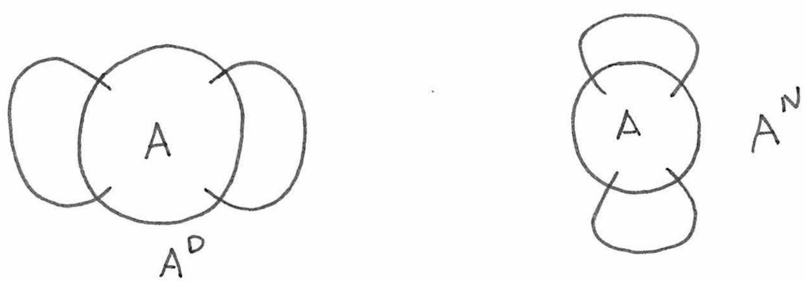


fig. 4.6

If B is another tangle, then $(A+B)^N$ is the link (fig. 4.7):

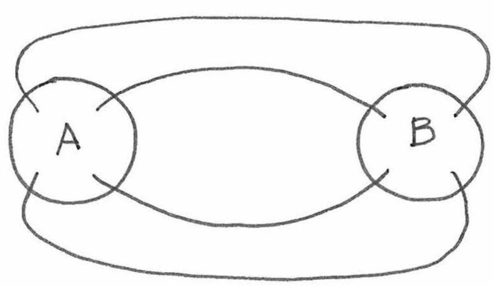


fig. 4.7

Denote the link polynomial also by the tangle link. The following property is satisfied by F and HOMFLY polynomial [L3].

Theorem 4.5.1. *If μ is the polynomial of a separated union of two unknots, then*

$$(1-\mu^2)(A+B)^N = A^N B^D + A^D B^N - \mu(A^N B^N + A^D B^D). \tag{4.2}$$

Let A be a tangle with n crossings. Orient A and order its components, treating the two loose strings as individual components (fig. 4.3). Choose a starting point on each component, except on the two strings; starting points will always be a and c or b depending on whether a and c are in the same string (fig. 4.8). Unlink A as usual. The resulting tangle is a separated union of two unknotted strings and some unknotted components. Let (n, r) denote the complexity of A where r is the number of switchings of the unlinking process.

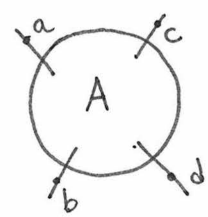


fig. 4.8

We will prove the Theorem by inducting on (n, r) ordered lexicographically. Suppose A has c components.

The base case of the induction consists of two cases :

Case 1. a is connected to b.

If $n = 0$ or $r = 0$, then A^N and A^D are trivial $(c-1)$-unlink and c-unlink, $(A+B)^N$ is a separated union of B^D and $c-2$ unknots. So

$$(1-\mu^2)(A+B)^N = (1-\mu^2)\mu^{c-2}B^D,$$

and

$$\begin{aligned}
& A^N B^D + A^D B^N - \mu(A^N B^N + A^D B^D) \\
= \; & \mu^{c-2}B^D + \mu^{c-1}B^N - \mu^{c-1}B^N - \mu^c B^D \\
= \; & (1-\mu^2)\mu^{c-2}B^D \\
= \; & (1-\mu^2)(A+B)^N.
\end{aligned}$$

Case 2. a is connected to d.

If $n = 0$ or $r = 0$, then A^D and A^N are trivial $(c-1)$-unlink and c-unlink, $(A+B)^N$ is a separated union of B^N and $c-2$ unknots. So

$$(1-\mu^2)(A+B)^N = (1-\mu^2)\mu^{c-2}B^N,$$

and

$$\begin{aligned} & A^N B^D + A^D B^N - \mu(A^N B^N + A^D B^D) \\ = \;& \mu^{c-1}B^D + \mu^{c-2}B^N - \mu^c B^N - \mu^{c-1}B^D \\ = \;& (1-\mu^2)\mu^{c-2}B^N \\ = \;& (1-\mu^2)(A+B)^N. \end{aligned}$$

If A has complexity (n, r), where none of n nor r is zero. Let v be a crossing switched in the unlinking process. Let M be switched tangle, M_1 and M_2 be the two split tangles. The by induction hypothesis, these tangles satisfy (4.2) when they are added to B. Hence

$$\begin{aligned} & (1-\mu^2)(A+B)^N \\ = \;& (1-\mu^2)\left(-(M+B)^N + x((M_1+B)^N + (M_2+B)^N)\right) \\ = \;& B^D(-M^N + x(M_1^N + M_2^N)) + B^N(-M^D + x(M_1^D + M_2^D)) \\ & - \mu\Big((B^N(-M_N + x(M_1^N + M_2^N)) + B^D(-M^D + x(M_1^D + M_2^D))\Big) \\ = \;& A^N B^D + A^D B^N - \mu(A^N B^N + A^D B^D). \end{aligned}$$

This completes our proof of the Theorem. ∎

6. Degree of the polynomial.

One of the properties in Section 3 gives an upper bound of the degree. In this section, we will improve the bound.

Let D be a diagram of a link K with n crossings. On D, each non-broken arc contains a certain number of crossings, or overpasses. Let m be the maximum number and S be such an arc.

Theorem 4.6.1. *$Deg(H_K(x)) \leq n - m$.*

Proof. Let (n, r) be a complexity of D and use induction on the complexity.

Certainly this is true when $n = 0$ or $r = 0$ for K is a trivial unlink which has degree zero for its polynomial.

The choice of complexity depends on the ordering of components, orientation and starting points. Let us choose the component containing S as the first component and one of the terminal points of S as the starting point; the orientation of the component will be from the starting point towards the other terminal point of S. Choices for other components are arbitrary.

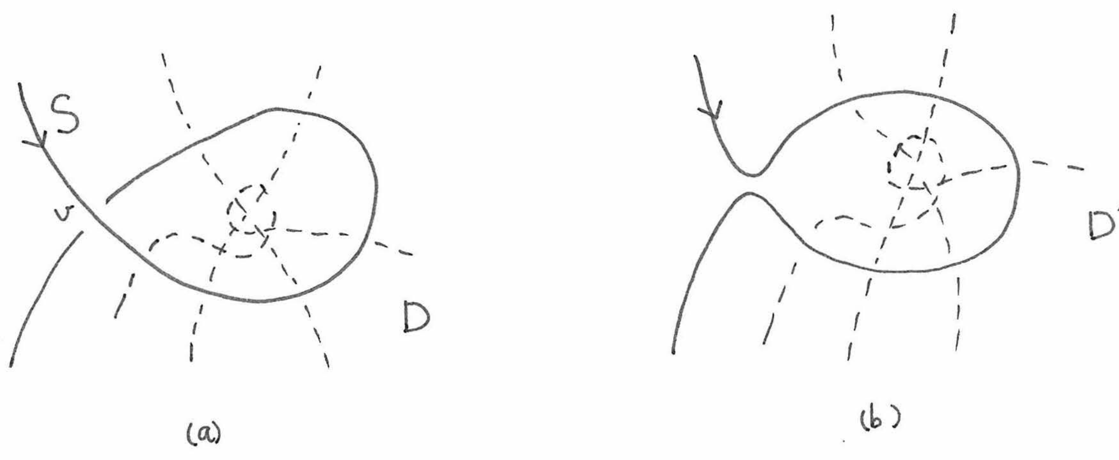


fig. 4.9

Let v be the first switching occurring in the unlinking process. If v is on S, then we have the situation as in figure 4.9a. So D represents the same link type as some diagram D' split at v (fig. 4.9b). D' has $n-1$ crossings and $m(D')$ is at least $m(D)-1$. Hence $H_K(x)$ has degree at most $n-1-m(D')-1 \leq n-m(D)$.

If v is not on S, let D', E_1 and E_2 be the switched and the two split diagrams respectively. We have $m(D') > m(D)$, and each $m(E_i) \geq m(D)$. Also E_i has $n-1$ crossings. By induction hypothesis on the complexity of diagrams, $deg(H_{D'}(x)) < n - m(D)$, and $deg(H_{E_i}(x)) \leq n - 1 - m(E_i) \leq n - 1 - m(D)$. Hence $deg(H_K(x)) \leq n - m(D)$ as $H_D(x) = -H_{D'}(x) + x(H_{E_1}(x) + H_{E_2}(x))$. ∎

Other than the degree of the polynomial, we would like to investigate the leading coefficient of the polynomial. Suppose K has *maximal degree*, that is, for some diagram D of K, $deg(H_K(x)) = n(D) - m(D)$. If K is a separated union of unknots, then the leading coefficient is clearly $(-1)^{c-1}$ where c is the number of components of K.

Suppose K is not a union of separated unknots but it has p separated unknotted components. If C is the diagram of a separated unknotted component in D, then C cannot have any overpass crossing. For if it has, let G be a disk spanned by C in R^3 such that G does not intersect with the rest of the link. Shrink C on G to a circle intersecting no part of the rest of the diagram. The resulting diagram D' has $n(D') < n(D)$ and $m(D') \leq m(D)$, and so K cannot have maximal degree since by Theorem 4.6.1, $deg(H_K(x)) \leq n(D') - m(D')$. This means that C does not have any self-crossing and it lies below all other components. Without loss of generality, all separated unknotted components are non-intersecting in D.

Let S be the longest non-broken arc on D and unlink D as in the proof of the previous theorem. Let v be the first underpass encountered. If v is already on S then we can reduce v to obtain an equivalent diagram D' of K which has fewer crossings but $m(D') \geq m(D)$. This contradicts that K has maximal degree in D.

Otherwise switch v and let E_1 and E_2 be the two split diagrams and L_1 and L_2 be their link types respectively. Certainly every non-intersecting unknot in D is non-intersecting in E_i. If E_i has a new unknotted component C', and if C' has any overpass, then $deg(H_{L_i}(x)) \leq n(D) - m(D) - 1$, and so the polynomial does not contribute to the leading coefficient of $H_K(x)$. Thus C' has a diagram of a circle in L_i and all of its crossings are underpasses. If C' does not involve the two arcs at v, then it is also a separated unknotted component in K and so C' is non-intersecting in D. Otherwise let G be a disk in 3-space spanned by C' such that G does not intersect with the rest of the diagram of L_i. Connect the two preimages of v in K by a vertical line segment; G is also a non-intersecting disk in K (fig. 4.10). We can then assume C' is the boundary of G in K. Shrink C' on G to a small non-intersecting circle near v (fig. 4.10) and we can reduce D to D' by eliminating v (fig. 4.10).

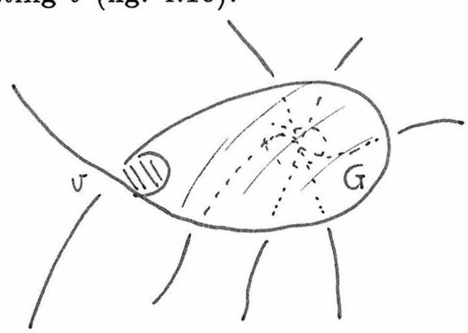


fig. 4.10

Otherwise, L_1 and L_2 do not introduce any separated unknotted component. If K has maximal degree in D then one or both of L_i has maximal degree in their diagrams. Hence a simple induction proof on a complexity of a diagram will give the following theorem.

Theorem 4.6.2. *Let K be a link of c components and have p separated unknotted components and D a diagram in which $deg(H_K(x)) = n(D) - m(D)$. Then the leading coefficient of $H_K(x)$ has a sign of $(-1)^p$, i.e., positive if p is* *even, negative if p is odd; except when $p = c$, the leading coefficient is* $(-1)^{p-1}$.

Proof. The case when $n = 0$ or $r = 0$ is trivial. In the general case, as L_1 and L_2 have the same number of separated unknot components, the result follows immediately by applying the recursive relation of their polynomials. ∎

Chapter 5. Alternating knots and links

1. Introduction.

The main result of this chapter is to show that a certain type of link achieves the maximal degree for its H polynomial.

The notation used in this chapter can be traced back to Tait [T1].

Let L be a link of c components and D a diagram of L with n crossings. We will identify D with L. When we say a link we refer also to some diagram of it.

Order and orient all components $C_1, \dots, C_c$, and choose starting points on each component as usual. Label the crossings by $V_1, \dots, V_n$. For each component C_i associate a word W_i in $\{V_j^{\pm 1}\}$ by recording the crossing symbol V or V^{-1} according to whether it is an overpass or underpass, by traversing the component starting from the chosen point along the given orientation. A *non-intersecting unknotted component*, one that has no crossing with itself or with other components, will receive a null word. The set $\{W_1, \dots, W_c\}$ is called a *cycloton* of L. Figure 5.1 is a cycloton of some link.

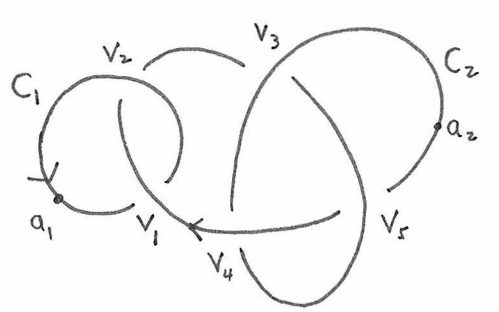


fig. 5.1

$$W_1 : V_1^{-1} V_2$$

$$W_2 : V_5^{-1} V_4 V_1 V_2^{-1} V_3^{-1} V_5 V_4^{-1} V_3.$$

Suppose a different starting point is chosen on C_i, the new word W_i' is a cyclic permutation of W_i. If C_i is reversedly oriented, then W_i' is obtained by reversing the order of symbols of W_i, which we will denote by the *reverse* of W_i or denoted by W_i^{-1}. If components are ordered differently, we get a permutation of the set of words $W_1, \ldots, W_c$. Hence cycloton of a link is defined up to permutation of words, reverses and cyclic permutations of the words.

Imagine that each word of a component is written on a closed ribbon. The word can be read starting from any symbol in any direction.

Introduction on an non-intersecting unknotted component will increase the degree of $H(L)$ by one, and removal of such a component decreases the degree by one. As we are interested only in the degree of the polynomial in this chapter, we can assume that L has no non-intersecting unknotted component. In this case the cycloton of L contains no null word.

A set $\{W_1, \ldots, W_c\}$ of words of symbols $V_1, \ldots, V_n$ is a *cycloton* if each V and V^{-1} occurs exactly once in one of W_i and none of the words is null. A cycloton is defined up to equivalence of permutation of all words, reverse and cyclic permutation of each word.

From the definition, not every cycloton is a cycloton of a link. For example, $\{V_1 V_2 V_3^{-1} V_2^{-1} V_3 V_1^{-1}\}$ is not a cycloton of any knot. In fact, if a cycloton originates from a link, then if V and V^{-1} occur in the same word, there must be an even number of symbols between them. Also each word is of even length.

Definitions

1) Two words W_1 and W_2 are the same, $W_1 = W_2$, if they have the same sequence of symbols.

2) W_1 is a *prefix* of W_2 if $W_2 = W_1 W_3$ for some word W_3.

3) W_1 is a *suffix* of W_2 if $W_2 = W_3 W_1$ for some word W_3.

4) $W_1 \equiv W_2$ if W_1 is a cyclic permutation of W_2 or its reverse.

5) W_1 is a subword of W_2, $W_1 \lesssim W_2$, if $W_2 \equiv W_1 W_3$ for some word W_3. W_1 is *proper* if W_3 is non-null.

6) If $S = \{W_1, \ldots, W_\mu\}$ and $S' = \{Z_1, \ldots, Z_\nu\}$ are cyclotons, S' is a sub-cycloton of S, $S' \lesssim S$, if there is a 1-1 mapping $f : \{1, \ldots, \mu\} \longrightarrow \{1, \ldots, \nu\}$ such that $Z_i \lesssim W_{f(i)}$, and either $\mu < \nu$, or when $\mu = \nu$ there is uniquely an i such that Z_i a proper subword of $W_{f(i)}$.

(7) W_1 is related to W_2 if there is some symbol in W_1 whose inverse is in W_2. W_1 is eventually related to W_2 if there there is a finite sequence $W_1 = Z_1, \ldots, Z_n = W_2$ so that Z_i is related to Z_{i+1}. If P is a word in cycloton S then R_P is the set of all words except P in S eventually related to P.

2. Degree of polynomials of alternating diagrams.

A cycloton is *loop free* if $\{VV^{-1}\}$ does not occur in any word of the cycloton.

If L is alternating, then the words in its cycloton alternate in symbols and their inverses. If L has an alternating cycloton then L is certainly alternating.

Denote $H(L)$ by $H(S)$ where S is the cycloton of L.

Theorem 5.2.1. *If S is alternating and loop free and has no sub-cycloton, then $deg(H(S)) = n - 1$ where n is the number of crossings of L.*

Proof. We will induct on n.

Since L has no non-intersecting unknotted component, it is at least one crossing. When $n = 1$, S has a loop since $S = \{VV^{-1}\}$. When $n = 2$, there is only one alternating, loop free cycloton (fig. 5.2) and its polynomial has degree 1. Hence we are done in this case.

fig. 5.2

Suppose $n > 2$, and let V be any crossing. Switch at V (fig. 5.3).

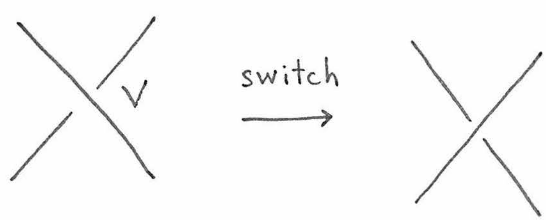


fig. 5.3

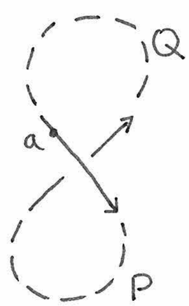
Let L' be the switched link and S' be its cycloton. L_1 and L_2 are the split link type at V, S_1 and S_2, their cyclotons respectively. Without loss of generality, all four cyclotons use the same set of crossing symbols.

L' has a non-broken arc of 2 overpasses. By Theorem 5.6.1 the degree of its polynomial is less than $n-1$. Thus $deg(H(L)) = n-1$ if and only if one or both of $H(L_1)$ and $H(L_2)$ has degree $n-2$ and sum of the coefficients of x^{n-2} is not zero. But in Section 5.6, both L_1 and L_2 have the same number of non-intersecting unknot components and hence their leading coefficients are of the same sign. Hence $deg(H(L)) = n-1$ if and only if one or both of $H(L_1))$ and $H(L_2))$ has degree $n-2$.

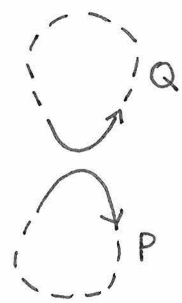
Since L_1 and L_2 are alternating, S_1 and S_2 are both alternating. Each L_1 or L_2 has $n-1 \geq 2$ crossings. Since a diagram free of sub-cycloton cannot contain any loop, and as either S_1 or S_2 has no sub-cycloton, the induction hypothesis would imply the conclusion. Assume now both S_1 and S_2 have sub-cyclotons T_1 and T_2 respectively.

Since orientation is irrelevant, we can assume V has structure ⤨. There are two cases.

Case 1. V is a self-crossing.

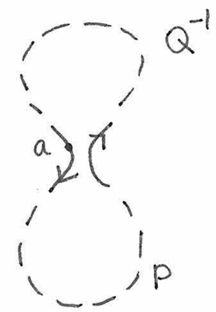


$W_c : VPV^{-1}Q$



$U_c : P$

$U_{c+1} : Q$



$Z_c : PQ^{-1}$

and $W_i = U_i = Z_i$ for each $1 \leq i < c$.

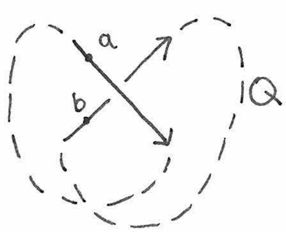
T_2 must have a proper subword Y of Z_c or else T_2 or $T_2 \setminus \{Y\} \cup \{W_c\}$ is a sub-cycloton of S. Y cannot be a subword of P or Q or else T_2 is a sub-cycloton of S. Hence we must have $Y \equiv P_1 Q_1^{-1}$ where P_1 and Q_1 are suffixes of P and Q respectively; or they are prefixes of P and Q. If $P_1 = P$ then $T_2 \setminus \{Y\} \cup \{VPV^{-1}Q_1^{-1}\}$ is a sub-cycloton of S. Similarly $Q_1 = Q$ implies $T_2 \setminus \{Y\} \cup \{P_1 V^{-1} Q V\}$ is a sub-cycloton of S. Thus P_1 and Q_1 are proper in P and Q.

Moreover, T_1 has subwords X_1, X_2, possibly null, of U_c and U_{c+1} respectively. One of the X's must be proper or null. If one of them is null then the other is not proper, for if not, T or $T_1 \setminus \{X_1, X_2\} \cup \{W_c\}$ is a sub-cycloton of S. Suppose $X_1 = P$, and X_2 is proper. Then T_1 has no other proper subwords of S_1. If $X_2 \equiv Q_2 Q_3$ where Q_2 is a suffix and Q_3 a prefix of Q, then $\{Q_2VPV^{-1}Q_3\} \cup T_1 \setminus \{X_1, X_2\}$ is a sub-cycloton of S. Otherwise choose Q_2 a suffix, if Q_1 is a suffix, and Q_2 a prefix if Q_1 is a prefix, of Q containing X_2 and Q_1 such that Q_2 is proper in Q. Then the cycloton $\{Q_2VPV^{-1}\} \cup T_1 \setminus \{X_1, X_2\} \cup T_2 \setminus Y$ is a sub-cycloton of S.

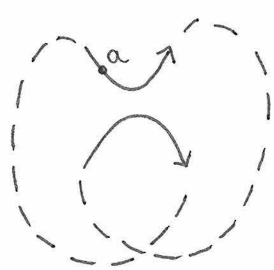
Similarly if $X_2 \equiv Q$ and X_1 is proper, we get a contradiction.

Therefore one of X_1 or X_2 is null and without loss of generality, X_2 is null and $X_1 = P$. Consider R_P, the set of words in S_1 eventually related to P, $R_{P_1} \subseteq R_P \subseteq S$ since P_1 is a subword of P. Let $\overline{U_i}$ in T_1 be the proper subword of U_i. Then $R_P \subseteq T_1 \setminus \{\overline{U_i}\} \cup \{U_i\}$ and $R_{P_1} \subseteq T_2$. If R_{P_1} is a proper subset of R_P, then $R_{P_1} \cup \{P_1\}$ is a sub-cycloton of S_2, hence a sub-cycloton of S_2. If $R_{P_1} = R_P$ then $R_P \cup \{P_1\}$ is a sub-cycloton of S.

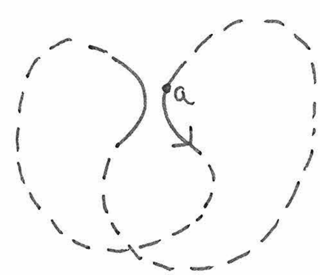
Case 2. V is an intercrossing of C_c and C_{c-1}.



$W_{c-1} : VP$
$W_c : V^{-1}Q$



$U_{c-1} : PQ$



$Z_{c-1} : PQ^{-1}$

and $W_i = U_i = Z_i$ for each $1 \leq i < c-1$.

T_1 and T_2 must have non-null subwords X, Y of U_{c-1} and Z_{c-1} repectively, for if not, T_1 or T_2 is a sub-cycloton of S.

If X is not proper, then $T_1 \setminus \{X\} \cup \{W_{c-1}, W_c\}$ is a sub-cycloton of S. If $P \subsetneqq X$ then $X \equiv Q_1PQ_2$ where Q_2 is a prefix of Q and Q_1 a suffix of Q, and $T_1 \setminus \{X\} \cup \{W_{c-1}, Q_2V^{-1}Q_1\}$ is a sub-cycloton of S. Similarly $Q \subsetneqq X$ induces a sub-cycloton of S.

Hence $X \equiv P_1Q_1$ where P_1 is a suffix of P and Q_1 is a prefix of Q or $X \equiv Q_1P_1$ where P_1 is a prefix of P and Q_1 a suffix of Q.

Similarly $Y \equiv P_2Q_2^{-1}$ where P_2 and Q_2 are suffixes of P and Q respectively, or $Y \equiv Q_2^{-1}P_2$ where P_2 and Q_2 are prefixes of P and Q respectively.

Case $X \equiv P_1Q_1, Y \equiv P_2Q_2^{-1}$, ($X \equiv Q_1P_1, Y \equiv Q_2^{-1}P_2$ is similar.)

Since P_1 and P_2 are both suffixes of P, one of them is a suffix of the other. We can assume $P_1 \lesssim P_2$. Then $R_{P_1} \subseteq T_1$, T_2 and R_{P_2}. Choose T_1 among all possible sub-cyclotons of S_1 containing P_1 and no other symbol of P with Q_1 having the shortest length.

Q_1 cannot be null for this gives a sub-cycloton of S as explained before. So some symbol of P_1 or words in R_{P_1} has its inverse in Q_1 and hence in Q_2^{-1}. Thus Q_1 and Q_2^{-1} has symbols in common. But Q_1 is a prefix and Q_2 is a suffix of Q. Hence $Q_1 \cup Q_2 = Q$ and so $T_1 \setminus \{P_1Q_1\} \cup T_2 \setminus \{P_2Q_2^{-1}\} \cup \{P_2V, V^{-1}Q\}$ is a sub-cycloton of S, a contradiction.

Case $X \equiv P_1Q_1, Y \equiv Q_2^{-1}P_2$, ($X \equiv Q_1P_1, Y \equiv P_2Q_2^{-1}$ is similar.)

Q_1 and Q_2 are both prefixes of Q and so we can assume Q_1 is a prefix of Q_2. Therefore $R_{Q_1} \subseteq R_{Q_2} \cap T_1 \cap T_2$. Choose T_1 among all sub-cyclotons of S_1 containing Q_1 but no other symbol of Q with P_1 having the shortest length. So some symbol in Q_1 or words in R_{Q_1} has its inverse in P_1 and hence in P_2. But P_1 is a suffix and P_2 is a prefix of P, $P_1 \cup P_2 = P$. Thus, $T_1 \setminus \{P_1Q_1\} \cup T_2 \setminus \{Q_2^{-1}P_2\} \cup \{PV, V^{-1}Q_2\}$ is a sub-cycloton of S, a contradiction. ∎

On a diagram, if there is a simple closed curve intersecting the link in two points and the segments in the interior and exterior are both knotted, then the link has sub-cyclotons, namely, the cyclotons corresponding to the link factors. In this case the link does not achieve a maximal degree. Experience shows that if a link is prime and alternating, it has no sub-cycloton. It is conjected that alternating and prime links achieve maximal degrees. When a braid has an alternating diagram, it has maximal degree.

An alternating diagram with maximal degree is minimal, since any diagram with less crossing number has a smaller degree. The theorem provides a crude test for minimal diagrams. An alternating braid is minimal.

If L_1 and L_2 are two alternating and prime links, the connected sum $L{=}L_1\#L_2$ has degree $deg(H(L_1)) + deg(H(L_2))$ and so the crossing number, $c(L)$ of L is at least $c(L_1) + c(L_2) - 1$ and is bounded above by $c(L_1) + c(L_2)$. Since it is an open problem whether crossing numbers are additive, in case of alternating and prime links, when we connecte two such links, the best we can do is reduce one crossing in the sum from given links.

3. Remarks.

It is not true that any alternating link achieves the maximal degree, for a composite link of two alternating links does not have a maximal degree. However, from all examples known so far, including those in the table in this paper, all prime alternating links do not possess subcycloton, and hence have maximal degree. We suspect that primeness is the same as there is a subcycloton free diagram. If this is true, there is a way to test for primeness.

Unlike the Conway polynomial, the degree of the H polynomial is bounded above by 1 less than the crossing number. Thus we can get a lower bound of the crossing number if we know the polynomial. In particular, let $L = K_1\#K_2$, a composite sum of two prime links. It is a conjecture that crossing numbers are additive with respect to composite sum. If the summands are both alternating and subcycloton free and their crossing numbers are n and m respectively, the degree of H_L is $n+m-2$ and so L has a crossing number of at least $n+m-1$. If a diagram for L has that many crossings, it must be alternating, or else H_L

does not have the given degree. The general case is more complicated, for the degree of the polynomial does not reflect much information.

H polynomial of knots with 9 or fewer crossings and links with 8 or fewer crossings.

The following knots and links are ordered as in Rolfsen's book [R2]. Only the coefficients of the polynomial are displayed. They are written in ascending order of the index of x. The first coefficient is the constant term for a knot, and it is that of x^{-1} if it is a 2-component link. Those starred knots and links are non-alternating. The two daggered knots, 9_{25} and 9_{26} are the only pair that have the same polynomial.

Knots with 9 or fewer crossings:

3_1 –3 2 2

4_1 –3 –2 4 2

5_1 5 –2 –6 2 2

5_2 1 –4 –2 4 2

6_1 1 4 –6 –4 4 2

6_2 5 –2 –10 0 6 2

6_3 5 –6 –12 4 8 2

7_1 –7 4 16 –6 –10 2 2

7_2 –3 6 8 –10 –6 4 2

7_3 –3 2 6 –6 –4 4 2

7_4 1 8 –4 –12 0 6 2

7_5 1 0 –4 –6 2 6 2

7_6 5 2 –12 –10 6 8 2

7_7 5 6 –18 –14 10 10 2

8_1 –3 –6 14 12 –14 –8 4 2

8_2 –7 0 22 2 –20 –4 6 2

8_3 1 –8 4 12 –8 –6 4 2

8_4 –3 2 14 –2 –16 –2 6 2

8_5 –11 14 26 –16 –24 2 8 2

8_6 1 0 0 –8 –8 6 8 2

8_7 –7 4 20 –8 –20 2 8 2

8_8 1 4 6 –10 –14 4 8 2

8_9 –7 4 16 –10 –16 4 8 2

8_{10} –11 14 22 –22 –22 8 10 2

8_{11} –3 6 4 –12 –10 6 8 2

8_{12} 5 2 –8 –12 –4 8 8 2

8_{13} –3 10 10 –22 –16 10 10 2

8_{14} 1 8 0 –22 –10 12 10 2

8_{15} –7 16 10 –32 –16 16 12 2

8_{16} –3 10 18 –22 –30 8 16 4

8_{17} –3 6 12 –20 –24 10 16 4

8_{18} 5 2 12 –26 –36 14 24 6

8_{19}* –11 10 20 –10 –12 2 2

8_{20}* –7 12 12 –14 –8 4 2

8_{21}* –7 8 6 –12 –2 6 2

9_1 9 –4 –32 14 34 –10 –14 2 2

9_2	1 –8 –12 24 18 –18 –10 4 2
9_3	5 –6 –12 16 14 –14 –8 4 2
9_4	5 –2 –18 12 18 –12 –8 4 2
9_5	1 –12 2 28 0 –22 –4 6 2
9_6	–3 –2 4 14 –2 –16 –2 6 2
9_7	5 –6 –8 14 4 –14 –2 6 2
9_8	1 –8 8 22 –12 –22 2 8 2
9_9	1 0 0 4 –2 –10 0 6 2
9_{10}	1 0 –4 2 –6 –8 6 8 2
9_{11}	–7 0 18 12 –18 –18 4 8 2
9_{12}	–3 –6 10 14 –12 –16 4 8 2
9_{13}	–3 –2 12 6 –16 –12 6 8 2
9_{14}	–3 –10 20 24 –26 –24 8 10 2
9_{15}	1 4 2 –4 –18 –8 12 10 2
9_{16}	–7 12 16 –12 –20 –6 8 8 2
9_{17}	–7 4 24 6 –30 –18 10 10 2
9_{18}	1 4 2 –8 –12 –4 8 8 2
9_{19}	1 –4 10 18 –22 –22 8 10 2
9_{20}	–7 4 28 0 34 –14 12 10 2
9_{21}	–3 –2 16 4 –26 –12 12 10 2
9_{22}	–11 –2 42 12 –48 –22 16 12 2
9_{23}	–3 10 14 –12 –28 –6 14 10 2
9_{24}	–11 10 24 –12 –30 –6 14 10 2
9_{25}†	–7 0 30 2 –42 –14 18 12 2

9_{26}†	–7 0 30 2 –42 –14 18 12 2
9_{27}	–7 8 30 –12 –44 –8 20 12 2
9_{28}	–11 18 32 –26 –46 –2 22 12 2
9_{29}	–11 10 40 4 –50 –30 16 18 4
9_{30}	–11 6 42 –10 –58 –10 26 14 2
9_{31}	–7 12 36 –22 –58 –4 28 14 2
9_{32}	–3 2 30 6 –52 –28 22 20 4
9_{33}	–3 2 26 0 –50 –22 24 20 4
9_{34}	–3 –2 28 18 –58 –42 26 28 6
9_{35}	–3 –18 28 34 –24 –28 2 8 2
9_{36}	–11 –2 38 14 –38 –22 10 10 2
9_{37}	–3 –14 30 18 –40 –20 16 12 2
9_{38}	–7 4 24 2 –40 –22 18 18 4
9_{39}	–7 –4 28 14 –40 –28 16 18 4
9_{40}	1 8 16 10 –54 –48 26 34 8
9_{41}	–7 –8 38 32 –46 –42 12 18 4
9_{42}*	–7 –4 24 12 –20 –10 4 2
9_{43}*	–11 2 32 2 –26 –6 6 2
9_{44}*	–7 0 22 2 –20 –4 6 2
9_{45}*	–7 4 20 –8 –20 2 8 2
9_{46}*	1 –12 18 16 –18 –10 4 2
9_{47}*	–3 –10 28 8 –32 –6 12 4
9_{48}*	5 –10 2 0 –10 4 8 2
9_{49}*	–7 –4 16 0 –16 2 8 2

Links with 8 or fewer crossings:

2_1^2	–2 1 2
4_1^2	2 –1 –4 2 2
5_1^2	2 –1 –8 0 6 2
6_1^2	–2 1 10 –4 –8 2 2
6_2^2	–2 1 2 –4 –2 4 2
6_3^2	2 –1 –4 –6 2 6 2
7_1^2	–2 1 10 0 –14 –2 6 2
7_2^2	–2 1 10 –8 –14 4 8 2
7_3^2	2 –1 0 0 –8 0 6 2
7_4^2	–6 7 18 –10 –20 2 8 2
7_5^2	–6 7 14 –16 –18 8 10 2
7_6^2	2 –1 8 –8 –22 4 14 4
7_7^2*	–6 7 14 –8 –10 2 2
7_8^2*	–6 7 10 –10 –6 4 2
8_1^2	2 –1 –16 8 24 –8 –12 2 2
8_2^2	2 –1 –8 8 10 –10 –6 4 2
8_3^2	2 –1 2 4 –2 –10 0 6 2
8_4^2	2 –1 0 0 –4 –6 2 6 2
8_5^2	–2 1 10 4 –16 –12 6 8 2
8_6^2	2 –1 –4 10 2 –14 –2 6 2
8_7^2	–2 1 18 0 –28 –12 12 10 2

8^2_8	−2 1 18 0 −36 −12 18 12 2
8^2_9	−6 −1 22 10 −28 −18 10 10 2
8^2_{10}	−6 −1 26 4 −32 −14 12 10 2
8^2_{11}	−6 7 14 −8 −18 −6 8 8 2
8^2_{12}	−6 7 18 −10 −28 −6 14 10 2
8^2_{13}	2 −1 8 8 −30 −24 16 18 4
8^2_{14}	−6 7 14 8 −26 −26 10 16 4
8^2_{15}*	−6 −1 18 8 −16 −8 4 2
8^2_{16}*	−6 −1 22 2 −20 −4 6 2

References

[A1] *J. W. Alexander*, **Topological Invariants of Knots and Links**, Tran. Amer. Math. Soc. 30 (1923), 275–306.

[A2] *J. W. Alexander, G. B. Briggs*, **On types of knotted curves**, Annals of Math. 28 (1927), 562–586.

[A3] *E. Artin*, **Theorie der Zöpfe**, Hamb. Abh. 4 (1925), 47–72.

[A4] *E. Artin*, **Theory of Braids**, Annals of Math. 48 (1947), 101–126.

[B1] *J. S. Birman*, **Braids, Links, and Mapping Class Groups**, Annals of Math. Studies. Princeton Univ. Press.

[B2] *W. Burau*, **Über Zöpfgruppen und gleichsinning verdrillte Verkettungen**, Abh. Math. Sem. Hanischen Univ. 11, 171–178.

[C1] *J. H. Conway*, **An enumeration of knots and links, and some of their algebraic properties**, Computational Problems in Abstract Algebra, 329–358.

[C2] *J. H. Conway and C. McA Gordon*, **A group to classify knots**, Bull. London Math. Soc. 7(1975) 84–86.

[C3] *R. H. Crowell and R. H. Fox*, **Introduction to Knot Theory**, Springer-Verlag.

[D1] *M. Dehn*, **Die beiden Kleeblattschlingen**, Math. Ann. 75(1914), 402–413.

[F1] *R. H. Fox*, **Free Differential Calculus I**, Annals of Math. 57 (1953), 547–560.

[F2] *R. H. Fox*, **Free Differential Calculus II**, Annals of Math. 59 (1954), 196–210.

[**F3**] *R. H. Fox*, **Free Differential Calculus III**, Annals of Math. 64 (1956), 407–419.

[**F4**] *R. H. Fox*, **A quick trip through knot theory**, Topology of 3-Manifolds and Related Topics, Prentice-Hall, Englewood Cliffs, New Jersey, 1962, 120–167.

[**F5**] *P. Freyd; D. Yetter, J. Hoste; W. B. R. Lickorish, K. C. Millett; and A. Ocneanu*, **A new polynomial invariant of knots and links**, Bull. Amer. Math. Soc. 12 (1985), 239–246.

[**G1**] *C. A. Giller*, **A Family of Links and the Conway Calculus**, Tran. Amer. Math. Soc. 270 (1982), 75–109.

[**H1**] *R. Hartley*, **The Conway potential functions for links**, Comment. Math. Helvetici 58 (1983), 365–378.

[**H2**] *C. F. Ho*, **On the second coefficient of the Conway polynomial**, preprint.

[**H3**] *J. Hoste*, **The first coefficient of the Conway polynomial**, Proc. Amer. Math. Soc. 95 (1985), No. 2, 299–302.

[**J1**] *V. F. R. Jones*, **A polynomial invariant for knots via von Neumann algebras**, Bull. Amer. Math. Soc. 12 (1985), 103–111.

[**K1**] *L. H. Kauffman*, **The Conway Polynomial**, Topology 20 (1980), No. 1, 101–108.

[**K2**] *L. H. Kauffman*, **Formal Knot Theory**, Mathematical Notes. Princeton Univ. Press.

[**K3**] *T. P. Kirkman*, **The enumeration, description, and construction of knots with fewer than 10 crossings**, Tran. Roy. Soc. Edin. 32(1885), 281–309.

[**L1**] *W. B. R. Lickorish*, **Prime knots and tangles**, Tran. Amer. Math. Soc. 267 (1981), No. 1, 321–332.

[**L2**] *W. B. R. Lickorish, K. C. Millett*, **A polynomial invariant of oriented link**, preprint.

[**L3**] *W. B. R. Lickorish, K. C. Millett, and R. D. Brandt*, **A polynomial invariant for unoriented knots and links**, preprint.

[**L4**] *C. N. Little*, **On knots, with a census for order ten**, Trans. Conn. Acad. Sci. 18(1885), 374–378.

[**L5**] *C. N. Little*, **Non-alternate ± knots of order eight and nine**, Trans. Roy. Soc. Edin. 35(1889), 663–664.

[**L6**] *C. N. Little*, **Non-alternate ± knots of order eleven**, Trans. Roy. Soc. Edin. 36(1890), 253–255.

[**L7**] *C. N. Little*, **Non-alternate ± knots**, Trans. Roy. Soc. Edin. 39(1900), 771–778.On knots, with a census for order ten Trans. Conn. Acad. Sci. 18(1885), 374–378

[**M1**] *H. Marusami*, **Unknotting number and polynomial invariants of a link**, preprint.

[**M2**] *C. McA Gordon*, **Some aspects of classical knot theory**, Springer lecture nots in math. 685(1978), 1–60.

[**M3**] *K. Murasugi*, **On closed 3-braids**, Memoirs Amer. Math. Soc. No. 151, 1974.

[**N1**] *L. Neuwirth*, **Knot Groups**, Ann. of Math. Studies 56, Princeton Univ. Press, 1965.

[**R1**] *K. Reidemeister*, **Knotentheorie**, (1984) Chelsea.

[**R2**] *D. Rolfsen*, **Knots and links**, (1976) Publish or Perish Inc..

[S1] *H. Seifert*, **Über das Geschlecht von Knoten**, Math. Annal 110 (1934), 571–592.

[T1] *P. G. Tait*, **On knots I, II, III**, Scientific Papers Vol. I, Cambridge University Press, London, 1898, 273–347.

[T2] *M. B. Thistlewaite*, **Knot tabulations and related topics**, .

[T3] *G. Torres*, **On the Alexander polynomial**, Annals of Math. 57 (1953), 57–89.

[T4] *H. F. Trotter*, **Noninvertible knots exist**, Topology 2(1964), 275–280.