The Ultraviolet Divergences of Einstein Gravity

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1986

(Submitted December 18, 1985)

Acknowledgements

This work was performed in collaboration with Augusto Sagnotti, whom I would like to thank for a long and fruitful partnership. I am also grateful to Neil Marcus for many useful discussions in the early stages of this work.

I would like to thank John H. Schwarz for his guidance and help throughout my years at Caltech.

A large portion of the calculation described here was performed on the A.C.A.L. VAX 11/784 of the Lawrence Berkeley Laboratory.

ABSTRACT

We discuss a two–loop calculation showing that the S matrix of Einstein's theory of gravity contains nonrenormalizable ultraviolet divergences in four dimension. We discuss the calculation in both background field and normal field theory. We describe a new method for dealing with ghost fields in gauge theories by combining them with suitable extensions of the gauge fields in higher dimensions. We show how using subtracted integrals in the calculation of higher loop graphs simplifies the calculation in the background field method by eliminating the need for "mixed" counterterms. Finally, we make some remarks about the implications of our result for supergravity theories.

Table of Contents

Figure Captions

1. Introduction

Quantum field theory has proven to be spectacularly successful in describing the interactions between elementary particles at the limits of present day experiments. The first relativistic quantum field theory, quantum electrodynamics, has been verified to high accuracy through many different experimental tests. Nevertheless, there are many shortcomings in our present day understanding of fundamental interactions.

It was discovered very early on in the development of quantum field theory that divergences arose in perturbation theory when loops were calculated. These divergences would render quantum field theory useless without some prescription for dealing with them. The solution to these infinites is renormalization theory. This provides us with a prescription for eliminating divergences in any physically meaningful quantity. In the case of quantum electrodynamics, renormalization can be heuristically understood by saying the electron has a "bare" mass and charge and is shielded by quantum fluctuations of the surrounding vacuum. At high energies, we start to penetrate this shielding and we see the bare charge and mass. However, at low energies, we measure a charge and mass which are the renormalized quantities.

Renormalization is implemented by isolating the divergent pieces of loop diagrams and then realizing that these divergent parts can be absorbed by a redefinition of charge and mass. The renormalized quantities are then what we measure at low energies. This approach only works because the divergences are of the right form to be absorbed into a redefinition of a limited number of parameters, the values of which can be derived from experiments. Such theories are called renormalizable. The commonly accepted $SU(2)\times U(1)$ electro-weak theory and $SU(3)$ QCD are renormalizable Yang-Mills theories.

For all the success of renormalizable theories, there has been one long standing failure. Although Einstein's General Theory of Relativity is widely accepted as being the correct classical theory of gravity, all of the many attempts to form a quantum field theory which reduces to classical Einstein gravity in the low energy limit have so far been incomplete [1].

It may seem somewhat puzzling at first why so much effort has been expended on attempting to quantize gravity. After all, the effects of quantum gravity would presumably only come into play around the Planck mass of 10^{19} GeV. Such energies may be present in some very exotic cosmological situations, but certainly they are far removed from present and probably future particle experiments. Nevertheless, the study of quantum gravity has led to at least two major formal developments. The first originated from the work of Feynman [2]. In the early sixties, while studying the one–loop behavior of Einstein gravity quantized in a covariant gauge, he noticed that diagrams would satisfy the proper unitarity conditions only at the price of having, besides the graviton, additional anticommuting bosonic particles circulating in the loop. This work was elaborated upon in the late sixties by DeWitt and others [3], and finally Faddeev and Popov [4] gave an elegant prescription for quantizing gauge theories in general gauges. The extra fictitious particles are now commonly referred to as Faddeev–Popov ghosts.

The second formal development was stimulated by the exceeding complexity of the Einstein–Hilbert action, and is originally due to DeWitt [3,5]. It is the background field method, an elegant alternative approach to quantum field theory. In the background field method, the fields are expanded with respect to background values, according to

$$\phi \Longrightarrow \phi_B + \phi_{qu} \,, \tag{1.1}$$

and only the quantum fields ϕ_{qu} are integrated over in the path integral. The background fields ϕ_B are effectively external sources. A proper choice of the gauge–fixing term then leads to an effective action that maintains the gauge invariance of the classical action even off–shell, insofar as transformations of the ϕ_B are concerned.

Aside from purely formal developments, there is another reason one might wish to attempt to quantize gravity. Although it is certainly true that we cannot reach the Plank energy in the conceivable future, it may well be that a consistent theory of quantum gravity is unique. If this is true, then it may be possible to calculate the consequences at low energies and thus shed new light on modern day particle physics, possibly eliminating the plethora of arbitrary dimensionless parameters which plague particle theory* .

There are many different ways of looking at the difficulties involved in quantum gravity. The point of view most in keeping with the rest of quantum field theory, is to view gravity as a theory of a spin–2 field in Minkowski space. This spin–2 field is represented by a symmetric tensor, $h_{\mu\nu}$, which is the deviation of the spacetime metric from flat Minkowski space. In this point of view, the difficulties of quantum gravity are due to the dimensionality of the coupling constant, κ. The space–time metric $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \, . \tag{1.2}$$

The classical action is just the ordinary Einstein–Hilbert action,

$$S = -\frac{2}{\kappa^2}\int d^4x \sqrt{-g} \;\, g^{\mu\nu} R_{\mu\nu} \, . \tag{1.3}$$

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* A similar example is Grand Unified Theories, which predict θ_W, while their predictions at the unification scale are still untestable.

Appendix A contains a list of our conventions. The coupling constant κ is proportional to the square root of Newton's constant, G, and thus has negative mass dimension. This implies that the S–matrix *may* contain nonrenormalizable divergences. It is just this question, whether nonrenormalizable divergences occur at two loops for pure gravity, which is settled in this work.

The negative dimensionality of κ is not by itself enough to prove that a naive approach to quantum gravity is hopelessly diseased. It only means that there is no available proof that physically meaningful divergences do not exist in the S–matrix. In order to decide whether such divergences are present, one must perform loop calculations with the theory. If this approach to quantizing General Relativity is to succeed, all such divergences must be absent. It is encouraging that such cancellations do occur at the one–loop order for pure gravity, *i.e.*, gravity not coupled to matter. This fact was first demonstrated by explicit calculation in 1974 [6]. However, as will be shown later, this result can be derived by general coordinate invariance and the Gauss–Bonnet theorem, without resorting to a calculation. Shortly afterwards, it was found that coupling arbitrary matter to gravity spoiled the one–loop cancellations and introduced divergences [7]. For a time, it was thought that all matter couplings would spoil the one–loop finiteness of pure gravity.

However, there are special couplings of matter fields which maintain the one–loop finiteness. These theories are the pure supergravity theories, in which all the fields reside in a super–multiplet with the graviton [8]. Thus, the criteria of finiteness of a theory including gravity at one loop highly constrains the available couplings. This is a very encouraging feature in the search for a theory of quantum gravity.

The question still remains, however, whether higher loops will diverge. For supergravity theories, the requirement that the effective action be supersymmetric postpones the onset of possible divergences to three loops for the maximally extended N=8 theory. However, there are no known symmetry arguments to decide the question of pure gravity at two loops. Although a theory of quantum gravity alone, with no matter fields, cannot be regarded as physically relevant by itself, it can help shed light on the possible problems with supergravity theories.

Possible cancellations of divergences in supergravity theories are expected to come from two possible sources. First is the requirement that the on-shell effective action be supersymmetric (this is only true if we assume there are no anomalies in the supersymmetry currents). The second possible mechanism is cancellations in the pure gravity sector, which by virtue of the fields lying in supermultiplets, will be carried over to the entire theory.

The strongest constraints on possible divergences of supergravity theories so far derived are the superspace arguments of ref. [9]. These arguments are based upon certain assumptions about the structure of extended superfields. If these assumptions are correct, then divergences in the N=8 theory are excluded up to and including six loops. However, these extended superfields have yet to be constructed and certain other implications of the assumptions of ref. [9] have been explicitly tested and found to be wrong [10]. Thus, it appears that the assumptions about the structure of extended superfields in ref. [9] are incorrect, and therefore N=8 may diverge at three loops. It seems that the only hope would be for some "miraculous" cancellation in the pure gravity sector at higher loops. Thus, the lack of two-loop divergences of pure gravity would be encouraging for supergravity theories, and such a calculation is far easier than the three-loop calculation necessary to directly test the N=8 theory. As will be shown in this thesis, a direct calculation of pure gravity at two loops displays the presence of nonrenormalizable divergences in the on–shell effective action, and therefore in the S–matrix of pure gravity, of the form,

$$\Gamma_{\infty}^{(2)} = \frac{209}{2880(4\pi)^4} \frac{1}{\varepsilon} \int d^4x \sqrt{-g}\, R^{\alpha\beta}{}_{\gamma\delta} R^{\gamma\delta}{}_{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta} , \tag{1.4}$$

where $\varepsilon = (4-d)$ is the familiar parameter of dimensional regularization. We take this to be a discouraging indication for supergravity theories.

Although we have said the calculation of two–loop pure gravity is easier than three–loop supergravity, it should be said that it is far from trivial. Since the one–loop calculation of 't Hooft and Veltman in 1974, there have been numerous attempts to perform the two–loop calculation [11]. Nevertheless, the problem had remained unsolved until this work in 1985 [12].

The exceeding complexity of a perturbative expansion of the Einstein–Hilbert action requires that huge numbers of terms must be manipulated. It is this fact which has been the major obstacle to performing the two–loop calculation. In order to overcome this difficulty, we had to develop our own algebraic manipulation programs so that the calculation could be performed with the aid of available computers. Although we will describe many techniques which were extremely helpful during the course of this work, it is nevertheless true that without the use of computers to deal with the huge numbers of terms which arise, this calculation would have been impossible.

Another important issue that arises in a calculation of such complexity is the correctness of the result. We have, of course, taken all steps we could to assure ourselves that we have the correct result. Past experience with computer aided calculations has shown that the most likely cause of error is human intervention in the steps of the calculation. In this calculation, all steps were performed by computer after thoroughly checking the programs. Another important feature was that all programs were kept as small as possible and were very limited in their functions. This meant that programs were easily debugged and optimized for their task. This also meant the number of programs was rather large (about 50).

The most convincing checks on the final result, however, are that the two-loop calculation was performed in three different ways, all of which yielded the same answer. The calculation was performed in background field method on-shell and then off-shell. The on-shell part of the off-shell calculation agreed with the on-shell calculation. Furthermore, the fact that the answer must be general coordinate invariant in background field, even off-shell, provides a good check. In our case, we had 5 possible counterterms in the off-shell case after dropping the divergence and trace of the graviton field, and more than 30 equations determining the coefficients of these 5 terms. This highly over-determined system was consistently solved and the coefficient of the on-shell invariant could, in fact, be determined from a subsystem of 5 equations, all of which vanish on-shell.

Finally, the third method of performing the calculation was to use normal field theory and calculate on-shell. This changes the vertices used in the calculation. As shown in Table 1, each graph gives a different on-shell result from the on-shell background field calculation, yet the sum of all graphs gives the same result. This is an explicit check of the gauge-independence of the result.

The remainder of this thesis is divided into 6 different sections. In section 2, we give a brief review of regularization and renormalization, as well as some formalism of the background field method of quantum field theory. Section 3 shows the method we used for determining the pole parts of dimensionally regularized integrals. In section 4, we give the lagrangian for gravity, including gauge fixing and ghost terms, and we describe a new technique for dealing with ghost fields. This technique allows us to combine ghost and gauge fields into a single field in higher dimensions, thus eliminating the need for calculating separate ghost diagrams. Section 5 reviews some results of one–loop gravity, while section 6 gives the results for the two–loop calculation. Finally, section 7 contains some conclusions.

2. Regularization and Renormalization

As we have said before, loop calculations in quantum field theory invariably give rise to divergences. The process of renormalization is used to give a physically meaningful result from a divergent theory. However, in dealing with infinities in a calculation, it is first necessary to carefully isolate the divergent pieces in a consistent manner. This is the process of regularization. A theory may have a finite S–matrix, *i.e.*, one not requiring renormalization, and yet it is still necessary to regularize during the steps of a calculation so that all cancellations may be properly seen.

The first method of regularization was introduced long ago and is simply to insert a cutoff parameter in loop momentum integrals [13]. The ultraviolet divergences of a theory are those that come from the large momentum portions of loop integrals. A cutoff simply puts an upper limit on the integration momenta. Any final physical result must be independent of the cutoff parameter. This method has the advantage of conceptual simplicity, but it ruins the manifest symmetries of the original integral and it is very difficult to actually perform the resulting integrals.

Another old, but popular method of regularization is to introduce extra massive fields with special couplings. This is Pauli–Villars regularization and it also has the disadvantage of being very difficult to actually calculate with. There are many other regularization schemes one can use, but the most convenient method for our purposes is that of dimensional regularization [14].

Dimensional regularization is based on analytically continuing the number of space–time dimensions, n, to $d = n - \varepsilon$ dimensions, and then examining the limit as $\varepsilon \to 0$. The infinite parts of integrals are then poles in $\frac{1}{\varepsilon}$. The primary advantages of this method are that it manifestly preserves all symmetries which are independent of the dimensionality of space–time, and that the pole parts of integrals are fairly easy to calculate. As general coordinate invariance is independent of the dimensionality of space–time, general coordinate invariance will not be broken by this regularization scheme. Thus, we could work in a background field gauge and maintain explicit general coordinate invariance.

In a theory with fermions, however, one must be very cautious about using dimensional regularization, as γ–matrix algebras cannot be analytically continued to noninteger dimensions. The presence of anomalies in these theories can be thought of as a failure of the regularization scheme. This is a source of confusion with regard to regularizing supergravity theories [15], but can be ignored in the present context, as gravity is a purely bosonic theory (with a slight caveat, which will appear later).

Once the pole parts of the effective action have been calculated, the divergences can be removed by adding counterterms to cancel only the poles in ε. This is the process of minimal subtraction and was used throughout the course of this work [14]. If the counterterms that must be added to render a theory finite are of the right form, they can be absorbed into redefinitions of the basic parameters of the theory, such as mass and charge, or into a redefinition of the basic fields. The renormalization of coupling constants is physically meaningful, while the renormalization of fields via local field redefinitions can be shown not to affect the S–matrix of the theory. If a theory is not renormalizable and not finite, then the divergences cannot be removed by redefining a finite number of parameters or by redefining the basic fields, and the theory is void of any meaningful predictive power.

To see that local field redefinitions of the form,

$$\Phi'(x) = \Phi(x) + O(\hbar) = f[\Phi(x)] , \tag{2.1}$$

are not physically meaningful, one simply notices that the effect of this local field redefinition in the path integral is (apart from source terms) merely to introduce a Jacobian factor of the form,

$$det(1+X) = \int d\bar{c}\, dc\, e^{\bar{c}(1+X)c} \,, \tag{2.2}$$

where c and $\bar{c}$ are anticommuting ghost fields. Since X is a local operator the ghosts have local interactions. Furthermore, the ghost propagator is one in momentum space, so all ghost loop diagrams are integrals of local polynomials and therefore vanish in dimensional regularization. Thus, the Jacobian in eq. (2.2) is one in dimensional regularization.

The field redefinition also changes the source terms according to,

$$J\Phi(x) = Jf^{-1}[\Phi'(x)] \,. \tag{2.3}$$

The effect of eq. (2.3) is to introduce additional source couplings of J to several fields Φ' at the same point in spacetime. However, it is well known that such additional couplings do not affect the S–matrix [16]. Hence, local field redefinitions are seen not to affect the S–matrix of the theory.

The question still remains of how to recognize counterterms which are field redefinitions in perturbation theory. The answer is that these counterterms vanish when the classical field equations are applied. To see this, we note that the classical field equations are given by

$$\frac{\delta S}{\delta \phi} = 0 \,. \tag{2.4}$$

A term which vanishes when the field equations are used is proportional to the field equations. Therefore, we have to first order in $\hbar$,

$$S'[\Phi] = S[\Phi] + \hbar \Delta S[\Phi] = S[\Phi] + \hbar \frac{\delta S}{\delta \Phi} \Delta \Phi \,. \tag{2.5}$$

This can then be expressed as,

$$S'[\Phi] = S[\Phi + \hbar \Delta \Phi] \,, \tag{2.6}$$

and so is seen to be a field redefinition. The same arguments hold true for higher orders of $\hbar$, though one then needs to expand eq. (2.6) to higher order in $\hbar$ as well.

As an example of a renormalizable theory, let us briefly consider the case of pure Yang–Mills theory. We start with a vector gauge field, A^i_μ and form the Maxwell tensor,

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + C^i_{jk} A^j_\mu A^k_\nu \,, \tag{2.7}$$

where the C^i_{jk} are the structure constants of the gauge group. We then form the Yang–Mills action,

$$S = \int d^4x \, \frac{-1}{4g^2} F^i_{\mu\nu} F^i_{\mu\nu} \,. \tag{2.8}$$

To this action, one would need to add a gauge fixing term and Faddeev–Popov ghost terms. However, we will ignore those terms for the moment. A simple proof of the renormalizability of this theory then follows if we assume that the divergences in loop calculations will be local, gauge invariant functions. Making these assumptions, it is a simple matter to see that the only possible divergences of the proper dimensionality go as $F \cdot F$, or as the topological term $F \cdot \tilde{F}$, where $\tilde{F}$ is the dual of F. This second term is a total divergence, and so can be ignored in simple discussions of perturbation theory. Therefore, the divergences of the quantum theory can be absorbed by a redefinition of the dimensionless coupling constant

g.

Unfortunately, this simple proof of renormalizability is flawed. In order to quantize a theory via perturbation theory, it is necessary to break the gauge invariance in order to define a propagator. Thus, the assumption that the divergences are gauge invariant is called into question. There are two solutions to this dilemma. First, one can concern oneself with only the on–shell effective action. This means that external legs are assumed to obey their classical equations of motion. This is all that is required to calculate S–matrix elements. Since an S–matrix element is a physically measurable quantity, it must be gauge invariant. Therefore the on–shell divergences of the effective action are also gauge invariant. The only remaining question is then whether the divergences are guaranteed to be local objects.

The second method for dealing with the problem of gauge fixing is to use the background field method of DeWitt, as mentioned in the previous section. In order to be somewhat self–contained, we now briefly review some of the more notable features of the background field method.

For a general theory the method proceeds via an expansion of the fields with respect to background values, which in effect act like sources, according to

$$\phi \Longrightarrow \phi_B + \phi_{qu} \; . \tag{2.9}$$

Only the "quantum" field ϕ_{qu} is integrated over in the path integral. The generating functional of the Green functions in the background field method for a general gauge theory is then, schematically,

$$e^{iW[J,\phi_B]} = \int [d\phi_{qu}][d\phi_{gh}]\, e^{i\int d^4x\, L(\phi_B+\phi_{qu}) + L_{g.f.}(\phi_B,\phi_{qu}) + L_{gh}(\phi_B,\phi_{qu},\phi_{gh}) + J\phi_{qu}} \tag{2.10}$$

In eq. (2.10) it is implicit that one is using a gauge–fixing term that is invariant under the gauge transformations of the background fields, which at the same time fixes the gauge invariance associated with the quantum fields. The relation with normal field theory then follows upon a change of variable in the integral, whereby the integration variable is redefined by undoing the shift in eq. (2.10). The result is

$$W[J,\phi_B] = -\int d^4x\, J\,\phi_B + W'[J,\phi_B]\,. \tag{2.11}$$

Eq. (2.11) yields the relation between the classical fields ϕ_{cl} and ϕ'_{cl}, which are arguments of the background field effective action and of the ordinary field theory effective action, respectively. Then, performing the usual Legendre transform gives

$$\Gamma[\phi_{cl},\phi_B] = \Gamma'[\phi_{cl} + \phi_B,\phi_B]\,. \tag{2.12}$$

Thus, the effective action in the background field method is equivalent to the effective action in ordinary field theory, computed with a "shifted" argument and in a peculiar gauge that depends on the background field [17]. In fact, the gauge fixing term in eq. (2.10) depends on the background fields and on the quantum fields individually, and not only on their sum. Therefore, the inverse redefinition produces a gauge fixing term which still depends on the background fields.

As a result, in general the effective action depends on two kinds of fields, the background fields ϕ_B and the classical fields ϕ_{cl}, obtained by averaging over the quantum fields ϕ_{qu}. This corresponds to the possibility of defining 1PI Green functions with the two kinds of external fields. If one sets the fields ϕ_{cl} to zero, *i.e.*, if one considers Green functions for background fields only, it is clear from the construction that these exhibit the same gauge invariance as the classical theory, even off–shell, since the coupling to the background in eq. (2.10) is manifestly covariant. The corresponding Ward identities for the effective action are just the statement of its background gauge invariance. However, it should be emphasized that, off–shell, the effective action depends on the choice of gauge–fixing function made in defining the functional integral in eq. (2.10). A prescription for defining a "unique" effective action in the off–shell case has actually been advocated in ref. [18]. However, the quantities of physical relevance are the on–shell amplitudes which are independent of the gauge choice anyway. The familiar normal field theory expansion is recovered in the case of vanishing background, where, of course, one computes Green functions with external ϕ_{qu} fields.

The other assumption used in our simple "proof" of renormalizability of Yang–Mills theory is that divergences are local. This is a rather tricky issue and no completely rigorous proof of locality has been put forth. Nevertheless, there do exist somewhat less than rigorous proofs, and we shall assume they are correct [19].

If we assume the validity of the proofs of locality of divergences, we can use this fact to develop a very simple method for evaluating the pole parts of integrals. We shall return to this point in the next section.

3. Method of Integration

In this section we outline the procedure for calculating the pole parts of Euclidean loop integrals in dimensional regularization and minimal subtraction [20]. Because of the importance of this technique for this calculation, we will proceed in some detail. This method is based upon the observation that the pure n–loop divergence of a loop integral in dimensional regularization is a local function of external momenta and masses. Any non–local contributions to the pole part of an n–loop integral are due to the well known phenomena of overlapping subdivergences. If an n–loop diagram is calculated using the lower loop counterterms, the resulting function is just a polynomial in momenta and masses. This provides a simple method to determine the pole parts.

A little thought shows that the effect of the lower loop counterterms is simply to subtract out subdivergent parts of individual integrals. This is because the lower loop counterterms are, by definition, the divergent parts of lower loop Green functions, which is precisely what the subdivergent integrals are. The point can be made more transparent by considering a two–loop Feynman graph, such as figure 1. In the standard methods, one calculates the two–loop graph, figure 1a and subtracts the 3 one–loop counterterm insertion graphs shown in figures 1b, 1c, and 1d. However, the one–loop counterterms are just the infinite parts of the corresponding one–loop subgraphs, which are just the infinite parts of the one–loop subdivergent parts of figure 1a. Figure 1b would correspond to taking the infinite part the k loop integral of 1a, with l being held constant, as represented in figure 2a. Similarly, figure 1c is the infinite part of the l loop integral with k held fixed, as represented in figure 2b. And finally, figure 1d corresponds to letting $k \to k+l$, $l \to -l$, and then taking the infinite part of the resulting l loop integral, as shown in figure 2c. We can therefore incorporate the effects of the lower loop counterterms by subtracting from the two–loop graph the three possible subdivergent subgraphs, which we will refer to as the k, l and $k-l$ subdivergent parts. Although we have illustrated the effects of subdivergences with a two–loop graph, it can be readily seen that the same arguments apply to any number of loops.

In practice, one subtracts the subdivergent parts of individual integrals without any need for explicitly calculating lower loop counterterms. The major advantage of following this procedure is that the resulting integrals minus their subdivergent parts, are local functions and are thus simple to calculate.

We must stress that a higher loop diagram is only local after the lower loop subdivergent diagrams have been removed. This can be seen to correspond to saying that the pole parts of individual higher loop Feynman integrals are local, provided one subtracts out the lower loop subdivergences. These lower loop subdivergent parts precisely correspond to the lower loop counterterms used to remove lower loop divergences. This can be seen by noting that these lower loop counterterms are just the infinite parts of lower loop graphs and that the subdivergent integrals also correspond to the lower loop graphs.

Once we only need to calculate local functions, a simple trick can be used to calculate the divergent part of all the needed integrals. Given a particular subtracted integral, I, divergent of order N, and dependent on momenta and masses q_i, it is evident from the homogeneity condition,

$$I = \frac{1}{N}\sum_i q_{i\mu}\frac{\partial I}{\partial q_{i\mu}} , \tag{3.1}$$

that $\frac{\partial I}{\partial q_{i\mu}}$ is at most divergent of order $(N-1)$. Therefore, by repeated differentiation with respect to their parameters, all subtracted integrals can be reduced to simple subtracted logarithmic integrals, though the process of differentiation can create a large number of terms. The divergent parts of these logarithmic integrals cannot depend on masses or external momenta, which can then effectively be set to zero. This allows one to avoid the need for many Feynman parameter integrals to combine denominators. The resulting integrals should be regarded as limits of corresponding ones with, say, massive propagators, and can be computed very simply. At two loops, the problem is reduced to evaluating integrals of the form

$$\int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{k_\mu k_\nu \cdots l_\rho l_\sigma \cdots}{(k-l)^{2a} k^{2b} l^{2c}} , \tag{3.2}$$

where all masses and external momenta have been set to zero. There is, however, one subtlety left. To illustrate matters, consider for instance

$$I_{\mu\nu\rho\sigma} = \int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{k_\mu k_\nu l_\rho l_\sigma}{(k-l)^4 k^6 l^2} , \tag{3.3}$$

This integral is *not* a tadpole, and must not be set to zero, as it comes from differentiating an integral which originally contained external momenta and masses. One would write the pole part as,

$$I_{\mu\nu\rho\sigma} = \frac{A}{4} \eta_{\mu\nu}\eta_{\rho\sigma} + \frac{B}{4} \eta_{\mu\rho}\eta_{\nu\sigma} + \text{symmetrizations in } \mu\,\nu \text{ and } \rho\,\sigma\,. \tag{3.4}$$

The symmetrizations are left out in eq. (3.4) to illustrate the procedure one would follow in practice. There is no need to produce a large number of terms via symmetrizations from the beginning. The terms can be symmetrized in stages when the integrals are substituted in the graphs and indices are then contracted as soon as possible to avoid generating excess terms.

Eq. (3.4) is not quite correct, as the subtractions require special care. The integral in eq. (3.3) receives contributions both from the subtractions and from the two–loop part. Let us consider first the two–loop contribution. This is done by treating the two–loop part as local, which is actually incorrect. However, the sum of the two–loop part and the subtractions is local. Therefore, although the individual parts will not be correct, the sum including subtractions is. There are two independent ways of contracting indices in the integral, resulting in two equations that determine A and B in terms of scalar integrals. We stress that the general two–loop contribution to the pole part can always be reduced to the case of scalar integrals. This takes care of the spurious infrared divergences that are introduced by the elimination of masses and external momenta. Contracting $\mu \to \nu$ and $\rho \to \sigma$, eq. (3.3) becomes,

$$\int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{1}{(k-l)^4 k^4} . \tag{3.5}$$

The other independent contraction is $\mu \to \rho$ and $\nu \to \sigma$. Then eq. (3.3) becomes,

$$\int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{(k \cdot l)^2}{(k-l)^4 k^6 l^2} . \tag{3.6}$$

Letting $l \to k-l$ in eq. (3.5) gives

$$\int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{1}{k^4 l^4} = \frac{1}{(4\pi)^4} \frac{4}{\varepsilon^2} , \tag{3.7}$$

ignoring terms with Euler's constant, γ_E, which cancel in subtracted integrals. Again, this integral is not a tadpole, as it originated from differentiating a divergent integral which originally contained external momenta. One could imagine introducing in eq. (3.7) a mass in the denominator and then taking the limit as the mass goes to zero. This gives the correct result. Similarly, the other integral in eq. (3.6) (and any other scalar logarithmic integral) can be reduced to integrals of the form (3.7) and

$$\int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{1}{k^2 l^2 (k-l)^4} = \frac{1}{(4\pi)^4} \left(\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\right) \tag{3.8}$$

Thus, the right hand side of eq. (3.3) can be easily evaluated for the two independent sets of contractions. The resulting equations for A and B can then be solved to give,

$$A = \frac{1}{(4\pi)^4} \left(\frac{5}{24}\frac{1}{\varepsilon^2} + \frac{31}{288}\frac{1}{\varepsilon}\right), \tag{3.9}$$

$$B = \frac{1}{(4\pi)^4} \left(\frac{1}{6}\frac{1}{\varepsilon^2} + \frac{1}{36}\frac{1}{\varepsilon}\right). \tag{3.10}$$

One must now calculate the subdivergent pieces. Simple power counting reveals that the integral in eq. (3.3) is logarithmically divergent in the l subintegral and the $(k-l)$ subintegral, while the k subintegral is convergent. The $(k-l)$ subintegral is defined by letting $k \to (k+l)$ and then $l \to -l$, and by examining the resulting l subintegral. To perform the subtraction integrals, one first combines the two (or one) denominators containing the subdivergent momenta using Feynman parameters, and then the pole part of the integral follows from the standard formula

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^{2a}}{(k^2+p^2)^b} = \frac{1}{(4\pi)^d} \frac{\Gamma(a+d/2)}{\Gamma(d/2)} \frac{\Gamma(b-a-d/2)}{\Gamma(b)} (p^2)^{d/2+a-b}. \tag{3.11}$$

This is a straightforward exercise. The only subtlety has to do with Lorentz indices. The method is tailored for minimal subtraction, and makes implicit use of the gauge–invariance of the regulator. Minimal subtraction requires that only the pole parts of the divergent amplitudes be subtracted. In working with subtracted integrals, this implies that one must be careful, when expressing the uncontracted integral in terms of metric tensors, to distinguish indices coming from a subdivergent channel. In order to enforce minimal subtraction, the trace of metric tensors originating from subdivergent loops is to be set to 4, not to $(4-\varepsilon)$. This is what one would obtain from calculating separate counterterm graphs. This can be implemented by using "barred" indices to distinguish η's coming from a subdivergent loop, and then defining

$$\eta_{\overline{\mu}\nu} = \eta_{\mu\nu} \tag{3.12}$$

$$\eta_{\overline{\mu\nu}}\eta^{\overline{\mu\nu}} = 4 \tag{3.13}$$

$$\eta_{\mu\nu}\eta^{\mu\nu} = \eta_{\overline{\mu\nu}}\eta^{\mu\nu} = \eta_{\mu\nu}\eta^{\overline{\mu\nu}} = 4 - \varepsilon \; . \tag{3.14}$$

For an N-loop integral, minimal subtraction implies that contracting η's while calculating an M-loop subdivergence can only be allowed to produce $\varepsilon^{(M-1)}$. In this way, we will always be assured that we are subtracting only the pole part of the subdivergent loop, as the M-loop subdivergence can produce a $\frac{1}{\varepsilon^M}$. With this in mind, the final result for (3.3), including subtractions, is,

$$\begin{aligned} I = \frac{1}{4}\frac{1}{(4\pi)^4}\Big(&(\frac{5}{24\varepsilon^2} + \frac{13}{288\varepsilon})\eta_{\mu\nu}\eta_{\rho\sigma} + (\frac{1}{6\varepsilon^2} + \frac{1}{36\varepsilon})\eta_{\mu\rho}\eta_{\nu\sigma} \\ &-\frac{1}{4\varepsilon^2}\eta_{\mu\nu}\eta_{\overline{\rho\sigma}} - \frac{1}{2\varepsilon^2}\eta_{\overline{\mu\rho}}\eta_{\overline{\nu\sigma}} - \frac{1}{6\varepsilon^2}\eta_{\overline{\mu\nu}}\eta_{\overline{\rho\sigma}} \\ &+ \text{symmetrizations in } \mu\,\nu \text{ and } \rho\,\sigma\Big) \; . \end{aligned} \tag{3.15}$$

The techniques used in this example can be readily extended to the case of an arbitrary number of indices, as well as to arbitrary loop order. Although they are rather cumbersome by hand, they are well suited to being implemented on computers.

There is another major advantage of using the technique of subtracting out subdivergent parts of integrals. As mentioned previously, there are two types of fields present when working with the background field method. There is the background field, ϕ_B, and the classical field ϕ_{cl}, which is the average over the quantum field ϕ_{qu}. The presence of two types of field poses some difficulties at higher loops [21]. The S–matrix for a theory is given by calculating Green functions with only the fields ϕ_B on the external legs. These Green functions then have the advantage of being gauge invariant off–shell. However, in order to calculate higher loop diagrams in the standard background field method, it is necessary to calculate counterterms for use in subtraction graphs which contain ϕ_{qu} fields on external legs as well. As we can see from figure 3b, we require one–loop counterterms with, for example, one ϕ_B and two ϕ_{qu}'s. Such Green functions do not have the advantage of being gauge invariant off–shell, and considerably increase the number of different Green functions one needs to calculate. In the case of Yang–Mills, one can do without actually calculating these "mixed" Green functions, as they are related to the Green functions with only ϕ_B in a simple way [17]. This is essentially due to the fact that the difference between the "pure" and "mixed" counterterms is just a multiplicative factor, as wave function renormalizations are just multiplicative factors in this case. Unfortunately, this is not true for the case of gravitation. However, if we calculate graphs by using subtracted integrals, we never need to explicitly mention lower loop counterterms. Thus, using this procedure, we circumvent the problem of calculating the mixed Green functions altogether.

4. The Lagrangian and Ghost Terms

In this section, we give the gauge fixing terms and ghost terms used in the calculation of graviton graphs in the background field method. We also describe in some detail a new technique for treating ghost fields which allows a substantial reduction in the number of diagrams one needs to calculate.

We start with the Einstein–Hilbert action (in natural units),

$$L = -2\sqrt{-g}\; g^{\mu\nu} R_{\mu\nu} , \tag{4.1}$$

which is written here in convenient natural units, such that the propagator has the standard normalization (see Appendix A for our conventions). The metric is shifted according to

$$g_{\mu\nu} \Longrightarrow g_{\mu\nu} + h_{\mu\nu} , \tag{4.2}$$

where $h_{\mu\nu}$ is the variable of integration in the path integral. Correspondingly, in the shifted action $g_{\mu\nu}$ will denote the background gravitational field. The exceeding complexity of the Einstein action (see Appendix A for more details) makes it convenient to choose a gauge leading to the simplest possible propagator. This is the De Donder gauge, the gravitational analogue of the familiar Feynman gauge for quantum electrodynamics. One adds to the action the (background) general coordinate invariant gauge–fixing term

$$L_{g.f.} = -\sqrt{-g}\,(h^{\mu\nu}{}_{;\nu} - \frac{1}{2} h_{\nu}{}^{\nu;\mu})(h^{\rho}{}_{\mu;\rho} - \frac{1}{2} h^{\rho}{}_{\rho;\mu}) , \tag{4.3}$$

where semicolons denote covariant derivatives with respect to the background metric, $g_{\mu\nu}$. In terms of the classical equations of motion, this gauge–fixing term implies,

$$h^{\mu\nu}{}_{;\mu} = \frac{1}{2} h_\mu{}^{\mu;\nu} \tag{4.4}$$

Then, with the familiar Faddeev–Popov prescription, the ghost action is

$$L_{gh} = \sqrt{-g}\,\big(-\bar{c}^{\mu;\nu} c_{\mu;\nu} - R_{\mu\nu}\bar{c}^{\mu} c^{\nu} - \bar{c}^{\mu;\nu} c^{\rho}{}_{;\mu} h_{\nu\rho} - \bar{c}^{\mu;\nu} c^{\rho}{}_{;\nu} h_{\mu\rho}$$

$$- \bar{c}^{\mu;\nu} c^{\rho} h_{\mu\nu;\rho} + \bar{c}^{\mu}{}_{;\mu} c^{\rho;\nu} h_{\nu\rho} + \frac{1}{2}\bar{c}^{\mu}{}_{;\mu} c^{\rho} h^{\nu}{}_{\nu;\rho}\big)\,. \tag{4.5}$$

The resulting Lagrangian to be used in deriving the Feynman rules is simply the sum of the terms in eqs. (4.1), (4.3) and (4.5). To derive the Feynman rules for the normal field theory approach, one simply takes the background metric to be the Minkowski metric. Thus, covariant derivatives become ordinary derivatives, and we would calculate diagrams with external $h_{\mu\nu}$. Thus, one needs to deal with the real metric field and the complex vector anticommuting ghost (and the corresponding antighost). Moreover, ghost loops are endowed with the "minus" sign characteristic of anticommuting fields. As we will show, it is possible to conveniently combine the ghosts with the gauge fields. This has the advantage of only requiring one type of line inside loop diagrams, thus reducing the number of distinct diagrams. For simplicity, we illustrate the details of the procedure in the simpler case of Yang–Mills theory, and at the end of the section we give the corresponding results for background–field Einstein gravity.

The quantum action for the pure Yang–Mills theory in the Feynman gauge is

$$L = -\frac{1}{4} F^{\,a}_{\mu\nu} F^{a\,\mu\nu} - \frac{1}{2}(\partial_\mu A^{\,a\mu})^2 - \partial^\mu \bar{c}^{\,a} D_\mu c^{\,a}\,. \tag{4.6}$$

In eq. (4.6) D_μ denotes the gauge–covariant derivative, $A^{\,a\mu}$ denotes the quantum gauge field, and $c^{\,a}$ denotes the ghost, which in this case is a complex scalar. To simplify eq. (4.6) further, one proceeds as follows. We write the complex ghosts in terms of real ghosts, which are to be treated as *commuting*, according to

$$c^a = \frac{1}{\sqrt{2}}(c_1^a + i\, c_2^a) \,, \tag{4.7}$$

and similarly for the antighost. Then, for example,

$$\bar{c}^a c^b = \frac{1}{2}(1-\sigma_2)_{ij}\, c_i^a c_j^b \,, \tag{4.8}$$

where σ_2 is the familiar Pauli matrix. Clearly, the matrix in eq. (4.8) is a projection operator, is orthogonal to its transpose and has trace equal to one. One can now define the projection operators Γ, γ, Π and Υ as follows. Let Γ_{IJ} be a matrix obtained by adjoining to the $(4-\varepsilon)$-dimensional background metric $g_{\mu\nu}$ a 2×2 matrix of zeros, *i.e.*,

$$\Gamma_{IJ} = \begin{Bmatrix} g_{\mu\nu} & 0 \\ 0 & 0 \end{Bmatrix} , \tag{4.9}$$

and similarly for Γ^{IJ} in terms of the inverse metric $g^{\mu\nu}$. One can also define the Minkowski space analog of Γ_{IJ},

$$\gamma_{IJ} = \begin{Bmatrix} \eta_{\mu\nu} & 0 \\ 0 & 0 \end{Bmatrix} , \tag{4.10}$$

and extend the projection operator in eq. (4.8) to

$$\Pi_{IJ} = \begin{Bmatrix} 0 & 0 \\ 0 & \frac{1}{2}(1-\sigma_2) \end{Bmatrix} . \tag{4.11}$$

Finally, Υ is the analog of the metric tensor in the two extra dimensions:

$$\Upsilon_{IJ} = \begin{Bmatrix} 0 & 0 \\ 0 & 1 \end{Bmatrix} . \tag{4.12}$$

It is defined as a separate matrix for convenience, even though it is actually the sum of Π and its transpose. Raising and lowering indices is irrelevant as far as Π and Υ are concerned. In all these cases, it is understood that the indices I and J run over $(6-\varepsilon)$ values. These matrices commute with one another and satisfy the following algebra:

$$\Gamma\cdot\Gamma = \Gamma \, , \, \Gamma\cdot\Pi = 0 \, , \, \Gamma\cdot\Upsilon = 0 \, , \tag{4.13a}$$

$$\Pi\cdot\Gamma = 0 \, , \, \Pi\cdot\Pi = \Pi \, , \, \Pi\cdot\Upsilon = \Pi \, , \tag{4.13b}$$

$$\Upsilon\cdot\Gamma = 0 \, , \, \Upsilon\cdot\Pi = \Pi \, , \, \Upsilon\cdot\Upsilon = \Upsilon \, , \tag{4.13c}$$

together with

$$\Pi\cdot\Pi^T = 0 \, . \tag{4.13d}$$

Moreover,

$$tr(\Gamma) = 4-\varepsilon \, , \tag{4.14}$$

$$tr(\Pi) = 1 \text{ and } tr(\Upsilon) = 2 \, . \tag{4.15}$$

In eqs. (4.13) – (4.15) all operations involving Γ are implicitly done with the proper raising and lowering of indices needed to ensure covariance. Clearly, the Minkowski space projector γ satisfies relations similar to those satisfied by Γ.

One can then rewrite the quantum action for Yang–Mills theory in terms of a *single* field, the six-dimensional vector A_I^a, where the 2 extra components correspond to the two real commuting ghost fields. The result is,

$$L = -\frac{1}{2}F_{IJ}^a F^{a\,IJ} - \frac{1}{2}(\partial_I A^{a\,I})^2 - \Pi_{KL}\partial^I A_K^a D_I A_L^a \, , \tag{4.16}$$

where indices are contracted with γ matrices, unless otherwise specified. The only problem left is that the ghosts, being anticommuting fields, need a "minus" sign for every closed loop. But even this is simply taken care of, to all orders of perturbation theory, by *redefining* the trace relations for Π_{IJ} and Υ_{IJ}, so that

$$tr(\Pi) = -1 \text{ and } tr(\Upsilon) = -2 , \tag{4.15'}$$

while maintaining all the other relations in eqs. (4.13) and (4.14). Every ghost loop is associated with the trace of a Π_{IJ} or Υ_{IJ} matrix, and the "minus" sign rule is automatically enforced in this way. Formally, one is replacing a *positive* number of anticommuting fields with a *negative* number of commuting fields. The action now contains only one field, and the number of distinct diagrams to be considered is correspondingly reduced. Strictly speaking, with the modified definition for the trace, the matrices Π_{IJ} and Υ_{IJ} could not be built out of ordinary numbers. This is the remnant of the anticommuting nature of the ghosts. However, the algebra of the matrices is well defined, and the compact description in terms of a single field is clearly more convenient than the usual one obtained by considering separately the Faddeev–Popov ghosts in the complex representation. The simplification is particularly effective when use is made of computers, as the algebra in eqs. (4.13), (4.14) and (4.15') is quite easy to implement.

We wish to emphasize that all the fields have been recast into a single six-dimensional vector. In this notation, the ghost fields are reminiscent of the scalar modes arising in a reduction *á la* Kaluza–Klein of the vector potential. This suggests the way to proceed in general. For example, in the case of direct interest to us, the metric tensor $g_{\mu\nu}$ will combine with the two *real* vector ghosts coming from fixing the general–coordinate gauge freedom into a six-dimensional metric tensor. Of course, the scalar modes that would originate in a corresponding reduction must be decoupled. Thus, the kinetic term for graviton and ghosts in this combined notation takes the following form:

$$-\frac{1}{2}\partial^\mu h_{IJ}\Big(\frac{1}{2}\big[\gamma_{IK}\gamma_{JL} + \gamma_{IL}\gamma_{JK} - \gamma_{IJ}\gamma_{KL}\big]$$

$$+\frac{1}{4}\big[\gamma_{IK}\Upsilon_{JL} + \gamma_{IL}\Upsilon_{JK} + \gamma_{JK}\Upsilon_{IL} + \gamma_{JL}\Upsilon_{IK}\big]\Big)\partial_\mu h_{KL}\ , \qquad (4.17)$$

where h_{IJ} denotes all the quantum fields, assembled in a symmetric matrix as said above. Correspondingly, the Euclidean propagator is

$$\frac{1}{p^2}\Big(\frac{1}{2}\big[\gamma_{IK}\gamma_{JL} + \gamma_{IL}\gamma_{JK} - \frac{1}{(2-\varepsilon)}\gamma_{IJ}\gamma_{KL}\big]$$

$$+\gamma_{IK}\Upsilon_{JL} + \gamma_{IL}\Upsilon_{JK} + \gamma_{JK}\Upsilon_{IL} + \gamma_{JL}\Upsilon_{IK}\Big)\ . \qquad (4.18)$$

The ghost terms in the lagrangian can all be recast in the form

$$\sqrt{-g}\,\Upsilon^{IJ}\Gamma^{KL}\Gamma^{PQ}\Big(-\frac{1}{2}h_{IK;P}h_{JL;Q} - \frac{1}{2}R_{KP}h_{IL}h_{JQ}\Big)$$

$$+\sqrt{-g}\,\Pi^{IJ}\Gamma^{KL}\Gamma^{PQ}\Gamma^{RS}\Big(-h_{IK;P}h_{JR;L}h_{QS} - h_{IK;P}h_{JR;Q}h_{LS}$$

$$-h_{IK;P}h_{JR}h_{LQ;S} + h_{IK;L}h_{JP;R}h_{QS} + \frac{1}{2}h_{IK;L}h_{JP}h_{RS;Q}\Big)\ . \qquad (4.19)$$

These terms can then be combined with the remaining ones in Appendix A, rewritten in the six-dimensional notation, and where all $g_{\mu\nu}$ should be replaced with Γ_{IJ}. Clearly, with minor modifications, the discussion above applies to space-time dimensions other than four, as well as to other gauge theories.

Since this method of dealing with ghosts is quite different from the standard methods, it is perhaps worth emphasizing that they do indeed give identical results. In the standard methods of dealing with complex, anti–commuting ghosts, one has oriented closed loops. When going to real ghosts, the two projection operators, Π and Π^T replace the oriented loops. Effectively, all loops with Π correspond to one orientation, while those with Π^T correspond to the opposite orientation. Finally, the anticommuting nature of the standard ghosts has as its only real effect, the introduction of a minus sign for closed loops. As we have seen, this can be accounted for by redefining the traces of projection operators as in eq. (4.15'). The other effect of the anticommuting standard ghosts is to introduce a sign change in every vertex and propagator when the orientation of the loop is changed. However, as all ghost loops are closed, there are always an even number of vertices plus propagators. Thus, the sign change is irrelevant and we see our method of dealing with real commuting ghosts will give the identical answer, but with the advantage of being able to combine ghosts and gauge fields such that there are no separate ghost diagrams to calculate.

We should now clarify some details about the actual calculation of graphs. We are interested in calculating the effective action for gravity. From this, one could apply reduction formulas and obtain the S–matrix. In order to calculate vertices, we expand the lagrangian to the required order in fields and then differentiate with respect to quantum fields. The resulting quantity is then symmetrized with respect to interchange of the quantum fields in order to obtain a vertex. This step shows another small advantage of using the background field method. We only need to symmetrize vertices with respect to interchange of quantum lines, whereas in normal field theory, we would symmetrize with respect to all lines. Thus, the background field vertices may be smaller due to fewer symmetrizations.

Finally, vertices and propagators are combined to form graphs. The usual procedure is to include separately all graphs which differ by interchange of external lines. However, this is unnecessary. The reason for symmetrizing over external lines is that we normally calculate in momentum space. However, if we perform all loop integrals in momentum space and then transform back to x–space, we see there is no need for explicit symmetrization of the external lines. One needs to be somewhat careful about combinatoric factors, but it is actually very simple in this approach. One associates with each graph a combinatoric factor which is one over the product of the number of symmetries of internal legs and the number of symmetries of external legs. So for example, the graph in figure 3 has a factor of $\frac{1}{2}$ for the internal symmetries of swapping the upper and lower lines and a factor of $\frac{1}{2}\times\frac{1}{2}$ for the symmetries of the two pairs of external legs. Thus, the overall factor to be associated with this graph is $\frac{1}{8}$ and one just calculates the single graph of figure 3 and can symmetrize in external lines later. This procedure eliminates calculating different graphs which are just symmetrizations of one another.

5. One–Loop Gravity

The first step in discussing the ultraviolet behavior of a theory involves producing a list of all possible on–shell divergences that are compatible with the symmetries of the classical action. If there are no such possible on–shell divergences, the theory is recognized to be finite, without the need for any explicit calculation. This applies both to background field quantization, and to quantization in ordinary field theory. In the case of Einstein gravity at one loop, this is actually all one needs to do. If one works in the background field method, one can readily discuss matters starting from the off–shell case, *i.e.*, considering all invariants.

First, we must determine the required dimensionality of the counterterms needed for any given loop order. We do this by noting that the Einstein–Hilbert action for gravity is quadratic in derivatives. Thus, when we perform the expansion of the metric in powers of the graviton field, $h_{\mu\nu}$, we find that all orders of expansion contain 2 momenta. This means that all vertices in diagrams have 2 momenta associated with them. The propagator goes as the usual $\frac{1}{p^2}$.

If we now look at any one–loop graph such as figure 4, we see that there are N vertices and N propagators and one integral over d^4p. Therefore, the superficial degree of divergence of any one–loop graviton graph is 4. Thus, the possible counterterms must be of dimension 4 and in a background field gauge they must also be general coordinate invariant off–shell. The only such structures are easily seen to be various quadratic combinations of Riemann tensors. We can therefore express the possible divergences at one loop as,

$$\Gamma_\infty^{(1)} = \frac{\hbar}{\varepsilon} \int \sqrt{-g}\, d^4x \left(c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) . \tag{5.1}$$

The first two terms in eq.(5.1) clearly vanish when the classical field equations, $R_{\mu\nu} = 0$ are used. Therefore, as was pointed out previously, they are proportional to a variation of the action and can be considered as nonlinear field redefinitions. Specifically, for the case at hand,

$$-2\int\sqrt{-g}\, d^4x\, (R - \frac{\hbar}{2\varepsilon}(c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu})) = -2\int\sqrt{-g'}\, d^4x\, R(g'_{\mu\nu})\,, \quad (5.2)$$

where,

$$g'_{\mu\nu} = g_{\mu\nu} + \frac{4\hbar}{\varepsilon}(c_1 R_{\mu\nu} - (c_2 + \frac{1}{2}c_1)g_{\mu\nu}R)\,. \quad (5.3)$$

As has been previously stated, such field redefinitions do not affect the S-matrix of the theory and are thus "harmless". However, the last term in eq. (5.1) does not vanish when the field equations are applied and thus does not correspond to a field redefinition. This term, therefore, seems to be a possible physical divergence at one loop. However, in four dimensions, there is a topological relationship involving the three terms on the right-hand side of eq. (5.1), known as the Gauss-Bonnet theorem. The quantity,

$$\int d^4x\sqrt{-g}\;\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}R^{\mu\nu}{}_{\alpha\beta}R^{\rho\sigma}{}_{\gamma\delta}\,, \quad (5.4)$$

defines the topological Euler number in four dimensions and, to each order in perturbation theory, the integrand appears to be a total divergence (modulo some subtleties to be discussed later). If we now apply the relationship,

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta} = \delta\begin{bmatrix}\alpha\beta\gamma\delta\\ \mu\nu\rho\sigma\end{bmatrix}, \quad (5.5)$$

where square brackets represent antisymmetrization with strength one, we see that the third term in eq. (5.1) is linearly dependent on the first two, modulo a harmless renormalization of the Euler number. The final conclusion is that pure

Einstein gravity has a finite S–matrix at the one–loop level. Notice that we have derived this result without any explicit calculations. The finiteness of one–loop gravity was first shown by explicit calculation [6] and only later was it realized that it could be derived on the basis of gauge invariance of the effective action.

There is a somewhat subtle point that has been already mentioned. The quantity in eq. (5.4), is only a topological invariant in four spacetime dimensions. One then may ask how will this manifest itself in a perturbation expansion? The answer, is that the higher orders in a perturbation expansion of the integrand of eq. (5.4) are not total divergences until one requires that the antisymmetrization over 5 or more indices is identically zero. However, the lowest order of expansion is a total divergence. That can be seen by simply counting the number of vector indices. To lowest order, eq. (5.4) contains two $h_{\mu\nu}$ fields and four derivatives. Since all indices are contracted, this gives only four uncontracted vector indices to lowest order; not enough to apply the vanishing of 5 or more indices. At the next order in expansion, there are three $h_{\mu\nu}$ fields and four derivatives, for a total of five indices; this is enough to apply such identities.

This simple counting argument can also be applied to other dimensions. In $2n$ dimensions, the analogous topological invariant is,

$$\int \sqrt{-g}\, d^{2n}x\, \varepsilon_{a_1\cdots a_{2n}} \varepsilon^{b_1\cdots b_{2n}} R^{a_1 a_2}{}_{b_1 b_2} \cdots R^{a_{(2n-1)} a_{2n}}{}_{b_{(2n-1)} b_{2n}} \,. \qquad (5.6)$$

To lowest order, this has n $h_{\mu\nu}$ fields and $2n$ derivatives, for a total of $2n$ uncontracted indices. Again, this is not enough to apply the identity that greater antisymmetrization of greater than $2n$ indices is zero. Therefore, the lowest order expansion of the topological relation in $2n$ dimensions is always a manifest total divergence. This result has also been derived using topological arguments and may have important implications for superstring theories [22].

Returning to 4 dimensions, this means that any one–loop propagator diagram will not be able to determine the topological term, as it is the lowest order expansion. However, if one were to calculate a one–loop vertex correction and *not* apply these Fierz–like antisymmetric identities, the renormalization of the topological density can be determined. We now describe the calculation of one–loop divergences in the De Donder gauge, including the renormalization of the topological invariant.

The one–loop divergences of Einstein gravity can be expressed as,

$$\Gamma^{(1)}_{\infty} = \int d^4x \sqrt{-g} \left(c_{1}' R^2 + c_{2}' R^{\mu\nu} R_{\mu\nu} \right.$$
$$\left. + c_{3}' \left(-4R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + 16 R^{\mu\nu} R_{\mu\nu} - 4R^2 \right) \right) , \qquad (5.7)$$

where the last term is just the expansion of eq. (5.4). Thus, modulo a renormalization of the topological term, all one–loop divergences cancel in the S–matrix of Einstein gravity.

The next question is how to compute the coefficients in eq. (5.7). Actually, only the coefficient of the Euler number density is physically meaningful, whereas the other two are gauge dependent, and disappear altogether after a suitable gauge choice [23]. One can compute c_{1}' and c_{2}' in eq. (5.7) by considering the propagator diagram in figure 5, where the internal line denotes the graviton and the ghost, combined in a 6×6 symmetric matrix as explained in Section 4. One can also extract the coefficient of the topological term in eq. (5.7) from the two vertex diagrams of figure 6, provided one does not apply Fierz–like identities as explained previously.

If we expand the background metric, $g_{\mu\nu}$ with respect to flat space according to,

$$g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} \,, \tag{5.8}$$

then, at the cubic level in $H_{\mu\nu}$ and fully on–shell (see the discussion below for the proper definition of "fully on–shell"), one finds the residual term, arising from the Riemann–squared term in eq. (5.7),

$$H_{\alpha\beta}\,\partial_\gamma\partial_\delta H^{\alpha\sigma}\,\partial^\beta\partial_\sigma H^{\gamma\delta} \,, \tag{5.9}$$

ignoring the trace and divergence of $H_{\mu\nu}$. This structure is easily seen to vanish if one demands that antisymmetrizations of five indices in it vanish identically, as is proper for four dimensions. Resorting to such identities in the context of a dimensionally regularized theory can cause some concern. However, in this case of a purely bosonic model, the continuation in the number of spacetime dimensions can be achieved by thinking of the tensors as having some (ε) vanishing components, and then the antisymmetric identities can be formally applied in an integer number of dimensions. No inconsistent manipulations result from this prescription, and actually one never needs to enforce antisymmetric identities in the course of the calculation, but can reserve them for the final answer. Alternatively, one can note that all the inconsistent manipulations associated with such identities in dimensional regularization require the presences of $\delta_{\mu\nu}$. This can be supplied by γ–matrix algebras in the presence of fermions, but in the case of a purely bosonic theory, everything is contracted into fields and there are no $\delta_{\mu\nu}$ with which to form inconsistent relationships.

Computing the divergent part of the propagator diagram in figure 5 reproduces the result of ref. [6],

$$\frac{1}{(4\pi)^2\varepsilon}\int d^4x\,\sqrt{-g}\left[\frac{1}{120}\,\frac{1}{\varepsilon}R^2 + \frac{7}{20}\,\frac{1}{\varepsilon}R^{\mu\nu}R_{\mu\nu}\right] . \tag{5.10}$$

The vertex diagrams in figure 6, computed fully on–shell, then suffice to determine the coefficient $c_{3'}$ in eq. (5.8).

We wish to stress that, both here and in the subsequent discussion of the two–loop divergences in the next section, a fully on–shell vertex diagram is defined with momenta that satisfy

$$p \cdot p = q \cdot q = p \cdot q = 0 \; , \tag{5.11}$$

but are *not* collinear. This corresponds to effectively continuing the momenta to complex values.

Only diagram 6a contributes on–shell, because the vertex for emission of two graviton lines from the same point is not capable of giving rise to a tensor with the right structure to survive the conditions (5.11). This is analogous to what happens for the vertex correction at two loops which will be discussed in the next section, and can be simply recognized by looking at the form of the vertex. Of course, one could skip the calculation of the propagator correction altogether, and determine the coefficients of the three terms from the vertex correction off–shell. The final result for the one–loop case in the gauge (4.3) is then:

$$\Gamma_{\infty}^{(1)} = \frac{1}{(4\pi)^2 \varepsilon} \int d^4x \sqrt{-g} \left[-\frac{53}{180} \varepsilon^{\alpha\beta\gamma\delta} \varepsilon_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\alpha\beta} R^{\rho\sigma}{}_{\gamma\delta} \right.$$

$$\left. + \frac{7}{20} R^{\mu\nu} R_{\mu\nu} + \frac{1}{120} R^2 \right] , \tag{5.12}$$

where the topological term has been separated out. The coefficient of the topological term in eq. (5.12) agrees with a previous result [24], obtained by computing the one–loop functional determinant of the background field method in ζ–function regularization. Clearly, the conceptual simplicity of the approach discussed in this section is achieved at the price of some algebraic complications. However, these can be simply dealt with, once the suitable computer software has been developed.

Previous work on one–loop gravity made extensive use of an algorithm introduced by 't Hooft [25]. This allows one to derive a general one–loop counterterm formula for a theory that, in the background field expansion, can be written at quadratic level in the quantum fields as

$$L_2 = -\frac{1}{2}\sqrt{-g}\; g^{\mu\nu} W^{ij} \partial_\mu \phi_i \partial_\nu \phi_j + \sqrt{-g}\; g^{\mu\nu} N_\mu^{ij}\, \phi_i \partial_\nu \phi_j + \frac{1}{2}\sqrt{-g} M^{ij} \phi_i \phi_j \;, \tag{5.13}$$

with W and M symmetric, and N antisymmetric. This encompasses the case of gravity, provided one identifies the "internal" indices i and j in eq. (5.13) with a symmetric pair of space–time indices on the quantum field of eq. (4.2). The one–loop calculation is then reduced to a substitution into the general counterterm formula which can be derived for the lagrangian in eq. (5.13). The lagrangian in eq. (5.13) has an $SO(N)$ gauge invariance associated with rotations of the fields ϕ_i. This allows one to determine the one–loop counterterm formula for this lagrangian by considering a few simple graphs. Substituting for the functions W, M, and N allows one to determine the one–loop divergences of gravity with minimal effort. The extension of this method to two loops has been considered for the case of renormalizable theories [26]. The result is a general counterterm formula in terms of some fifty independent structures. However, the method becomes quite impractical in the general case of a theory in curved space–time with nonrenormalizable couplings, which is needed in its entirety to encompass the case of gravity at two loops. The number of invariants grows enormously, and the determination of the coefficients of some of them requires rather complicated calculations. Moreover, the problem of two–loop gravity is quite specific, and can be treated conveniently using the approach discussed in this section. As an exercise, we have in fact calculated the two–loop counterterm formula for a nonrenormalizable theory in flat space–time. There are thousands of independent invariants in this case, but nevertheless, they can all be determined with the appropriate computer software.

6. Two-Loop Gravity

We now proceed to examine the possible counterterms for the two-loop case. As in the one-loop case, each vertex is dimension 2. However, an N point diagram now has N, $N+1$, or $N+2$ vertices, $N+1$, $N+2$, or $N+3$ propagators and 2 loop integrations over d^4p (figure 7). Therefore, all two-loop diagrams have a superficial degree of divergence of 6. We must therefore list all possible gauge invariant quantities of degree 6. The answer has been given in ref. [27]. A bit of work (done in our case by computer) shows that there are 10 possible counterterms which are linearly independent when the Bianchi identities have been accounted for. The 10 structures are

$$R^{;\mu} R_{;\mu} \ ; \ R^3 \ ; \ R_{\alpha\beta;\mu} R^{\alpha\beta;\mu} \ ; \ R\, R_{\alpha\beta} R^{\alpha\beta} \tag{6.1a}$$

$$R_{\alpha\gamma} R_{\beta\delta} R^{\alpha\beta\gamma\delta} \ ; \ R_{\alpha}{}^{\beta} R_{\beta}{}^{\gamma} R_{\gamma}{}^{\alpha} \ ; \ R R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \ ; \ R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\varepsilon} R_{\delta}{}^{\varepsilon} \tag{6.1b}$$

$$R^{\alpha\beta}{}_{\gamma\delta} R^{\gamma\delta}{}_{\varepsilon\zeta} R^{\varepsilon\zeta}{}_{\alpha\beta} \ ; \ R_{\alpha\beta\gamma\delta} R^{\alpha}{}_{\varepsilon}{}^{\gamma}{}_{\zeta} R^{\beta\varepsilon\delta\zeta} \, . \tag{6.1c}$$

Clearly, the 8 structures in eqs. (6.1a) and (6.1b) all vanish when the classical equations of motion are applied. Thus, in complete analogy with the one-loop case, they represent field redefinitions and do not contribute to the S-matrix. The remaining 2 structures in eq.(6.1c) do not vanish, however, and represent possible physical divergences. However, these 2 structures are not independent. This is not a consequence of the symmetries of the Riemann tensor, nor of the Bianchi identities, but can be seen from examining the topological invariant in 6 dimensions. In 6 dimensions, the topological Euler number is given by,

$$\int d^6x \sqrt{-g} \ \varepsilon_{\alpha\beta\gamma\delta\varepsilon\theta} \, \varepsilon^{\mu\nu\rho\sigma\tau\zeta} R_{\mu\nu}{}^{\alpha\beta} R_{\rho\sigma}{}^{\gamma\delta} R_{\tau\zeta}{}^{\varepsilon\theta} \, . \tag{6.2}$$

Expanding the product of epsilon tensors in eq. (6.2), we obtain a relation between the two terms in eq. (6.1c), modulo terms in eqs. (6.1a) and (6.1b). Although this is a topological invariant only in 6 dimensions, one can easily see that this relationship becomes a local identity in less than 6 dimensions. Since antisymmetrization over 6 indices is identically zero in less than 6 dimensions, the integrand in eq. (6.2) vanishes identically in less than 6 dimensions. This implies that in 4 dimensions,

$$R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\varepsilon\zeta}R^{\varepsilon\zeta}{}_{\alpha\beta} = 2\, R_{\alpha\beta\gamma\delta}R^{\alpha}{}_{\varepsilon}{}^{\gamma}{}_{\zeta}R^{\beta\varepsilon\delta\zeta}\,, \tag{6.3}$$

modulo terms in eqs (6.1a) and (6.1b), when Fierz–like identities are applied. However, as we previously argued, the lowest order expansion contains this relationship without the use of the Fierz–like identities, as there are not enough indices to allow such identities to be applied in 6 dimensions. This can be explicitly verified by expanding the two terms in eq. (6.1c) to cubic order in $H_{\mu\nu}$ fields. One then finds the relationship in eq. (6.3). Hence, if we restrict ourselves to calculating vertex corrections at two loops, we do not need to actually implement any Fierz–like identities.

We can therefore conclude that the nonrenormalizable divergences of gravity at two loops can be parameterized as,

$$\int d^4x\,\sqrt{-g}\;\; c\, R_{\alpha\beta}{}^{\gamma\delta}R_{\gamma\delta}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}\,. \tag{6.4}$$

The question then becomes what is the simplest Green function one can find which will determine the coefficient of this structure? Clearly this structure starts at the cubic level in $H_{\mu\nu}$, and it can be shown by direct expansion of the other 7 terms in (6.1a) and (6.1b) that all 9 structures are independent at the cubic level. Thus, we need only calculate a two–loop vertex correction.

The discussion of the one–loop case presented suggests how to proceed in the two–loop case. If we examine the invariants at cubic order and fully on–shell (as previously defined), we see that there is one structure which can determine the coefficient of eq. (6.3) by itself. The structure is,

$$H^{\alpha\beta}\partial_\alpha\partial_\varepsilon\partial_\zeta H^{\gamma\delta}\partial_\beta\partial_\gamma\partial_\delta H^{\varepsilon\zeta}\,, \tag{6.5}$$

where, as in eq. (5.8), $H_{\mu\nu}$ denotes the difference between the background metric and the flat Minkowski metric. Clearly, the term in eq. (6.5) is a very convenient one to track, as it does not contain the trace of H or its divergence. Therefore, both the trace and the divergence of H can be ignored altogether. As in the one–loop case, the fully on–shell amplitude is computed for momenta which are not collinear (and are thus effectively continued to complex values), otherwise the term in eq. (6.5) is pure gauge. Moreover, the two–loop propagator diagrams cannot contribute to the structure in eq. (6.5), even after the nonlinear field equations are used in them. At the cubic level, the relation in eq. (6.3) is manifest. At the higher levels, one would need a Fierz–like identity to relate the expansions of the two invariants.

Resorting to the combined notation for graviton and ghosts introduced in Section 4 one needs, in principle, to compute the 14 diagrams in figures 8 and 9. However, it can be seen that the diagrams in figure 8 cannot contribute to the structure in eq. (6.5). The reason is that the two–graviton emission vertex originates from a second–derivative interaction, and the corresponding Feynman rule contributes to the graph terms with two $H_{\mu\nu}$ fields and two, one or zero external momenta. Correspondingly, the number of free indices coming from the vertex is four, five or six. These terms would contract, in the pole part of the graph, with another $H_{\mu\nu}$ field and with four, five or six powers of its momentum, which is the only one circulating in the graph. It is then a matter of simple index counting to see that, as a consequence, the structure in eq. (6.5) cannot be generated. Thus, one needs only the diagrams in figure 9 to decide about the on-shell divergences of Einstein gravity at two loops, and actually, with a bit more effort, one can see that the last two diagrams in figure 9 are also irrelevant, as the first one only contributes to the double pole, whereas the second one vanishes identically. It should be clear from figures 8 and 9 that *no* counterterm diagrams are explicitly calculated, as explained in section 3.

We must now look to see what the effect of the one-loop field redefinition is at the two-loop level. We may rewrite the one-loop field redefinition to order $(\Delta\phi)^2$ as,

$$S[\phi] + \frac{1}{\varepsilon}\frac{\delta S}{\delta\phi}\Delta\phi = S[\phi + \frac{1}{\varepsilon}\Delta\phi] - \frac{1}{2\varepsilon^2}\frac{\delta^2 S}{\delta\phi^2}(\Delta\phi)^2 \,. \tag{6.6}$$

Thus, the one-loop redefinition results, in general, in the introduction of a spurious term in the double-pole part at two loops, which must be properly identified in order to interpret the result correctly. On the other hand, the double pole is completely determined from the one-loop subtractions and, if properly calculated, vanishes identically for a theory which is one-loop finite. For the case of pure gravity in background field this was originally pointed out in ref. [28]. As pointed out in ref. [29], this can be seen by noting that if a theory is finite at $N-1$ loops, then one can calculate at N loops without the need of counterterms, after making the needed field redefinitions. In this case, the N-loop divergence will have the form,

$$\mu^{N\varepsilon}\left[\frac{C_1}{\varepsilon} + \frac{C_2}{\varepsilon^2} + \cdots + \frac{C_N}{\varepsilon^N}\right], \tag{6.7}$$

where μ is the regularization mass needed to maintain the proper dimensionality of integrals in $4-\varepsilon$ dimensions. However, one may also calculate with counter–terms. In that case, locality assures us that there will be no factors of μ^{ε} in the answer. The answer for that case would take the form,

$$\left[\frac{C_1{}'}{\varepsilon} + \frac{C_2{}'}{\varepsilon^2} + \cdots + \frac{C_N{}'}{\varepsilon^N}\right] . \tag{6.8}$$

However, both ways of calculating must provide the same answer on–shell. Thus, eqs. (6.7) and (6.8) must be identical. If we expand $\mu^{N\varepsilon}$ in eq. (6.7) and equate it with (6.8), we find,

$$(1 + N\varepsilon \log\mu + \cdots)\left[\frac{C_1}{\varepsilon} + \cdots + \frac{C_N}{\varepsilon^N}\right] = \left[\frac{C_1{}'}{\varepsilon} + \cdots + \frac{C_N{}'}{\varepsilon^N}\right] . \tag{6.9}$$

Equating powers of ε in eq. (6.9), we find that only the $\frac{1}{\varepsilon}$ pole does not vanish at N loops, on–shell, since $C_A{}'$ is a local function and cannot contain $\log\mu$. For the case of gravity at two loops, this means the $\frac{1}{\varepsilon^2}$ pole must vanish on–shell. In this case actually the last term in eq. (6.6) is absent, as the field redefinitions vanish on–shell. For the case of gravity, one can therefore ignore the last term in eq. (6.6), and conclude that the subtracted diagrams do not contribute to the double pole. The vanishing of the double pole is a reassuring check on a calculation of such complexity. However, only the $\frac{1}{\varepsilon}$ – part is sensitive to the subtleties of the subtraction, and these require special care. The advantage of the background field method is evident in this case. If one computes the vertex correction fully off–shell, the result is guaranteed to be general coordinate invariant.

Calculating the vertex diagrams of figures 8 and 9 fully on–shell, we find the two–loop divergences of the effective action for pure gravity to be,

$$\Gamma_\infty^{(2)} = \frac{209}{2880(4\pi)^4}\frac{1}{\varepsilon}\int d^4x\sqrt{-g}\; R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta} \tag{6.10}$$

Given the approach discussed previously of subtracting subdivergences from integrals, there is very little difference between computing in background field and in the normal field theory approach. Thus, with minor additional effort we actually repeated the calculation of the vertex correction on-shell in the normal field theory approach.

The results for the pole parts of the individual on-shell graphs are given in Table 1. The entries are the contributions to the invariant in eq. (6.4). As reviewed in Section 2, the background field effective action, computed restricting one's attention to Green functions for external background fields only, is the normal field theory effective action in a peculiar gauge [17]. Therefore, the identity of the corresponding results in the table is an explicit verification of the gauge independence of the effective action on-shell, and is thus a very good check on the calculation.

We have also computed the vertex correction off-shell in the background field method, while still dropping the trace of H and its divergence. This still determines five of the nine invariants, and the result in the gauge (4.3) is:

$$\Gamma_\infty^{(2)} = \frac{1}{(4\pi)^4}\int d^4x\sqrt{-g}\left[\frac{209}{2880}\frac{1}{\varepsilon}R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}\right.$$

$$-\left(\frac{5}{18}\frac{1}{\varepsilon^2}+\frac{5771}{4800}\frac{1}{\varepsilon}\right)R^{\alpha\beta}D^2R_{\alpha\beta}$$

$$+\left(\frac{1255}{54}\frac{1}{\varepsilon^2}-\frac{703049}{64800}\frac{1}{\varepsilon}\right)R_{\alpha\beta}R_{\gamma\delta}R^{\alpha\gamma\beta\delta}$$

$$-\left(\frac{551}{27}\frac{1}{\varepsilon^2}-\frac{833}{16200}\frac{1}{\varepsilon}\right)R^{\alpha}{}_{\beta}R^{\beta}{}_{\gamma}R^{\gamma}{}_{\alpha}$$

$$+\left(\frac{1033}{108}\frac{1}{\varepsilon^2}-\frac{47417}{8100}\frac{1}{\varepsilon}\right)R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\sigma}R^{\delta}{}_{\sigma}$$

$$\left.+\text{ terms involving the scalar curvature}\right\} . \tag{6.11}$$

The first term in eq. (6.11) is the only one surviving on–shell, and agrees with the on–shell results in the Table 1. The system of equations determining eq. (6.11) consists of over 30 equations for the 5 unknowns. It is gratifying to see that this overdetermined system does admit a consistent solution, and that it is possible to choose a particular subsystem of equations, all of which vanish on–shell, which determines eq. (6.11).

7. Conclusions

The problem of the divergences of Einstein gravity has long aroused the interest of physicists, especially after 't Hooft and Veltman discovered that the theory is one–loop finite more than ten years ago. This result has motivated the hope of arriving at a finite theory of gravity in the context of rather conventional generalizations of Einstein's theory, the extended supergravities. We have shown that Einstein's theory of gravity does not display any unexpected cancellation mechanism beyond the long–recognized one responsible for its one–loop finiteness. Cancellations of divergences in supergravity theories are expected to derive from two sources, the properties of the pure gravity theory and supersymmetry. Supersymmetry alone can postpone divergences to the third order of perturbation theory in four dimensions, while going to higher dimensions almost certainly makes things worse. At three loops and beyond we have no reasons to expect further cancellations, since we know that the superspace arguments of ref. [9] are in explicit contradiction with the ultraviolet behavior of N=4 Yang–Mills in more than four dimensions [10]. Thus, with gravity diverging at the two–loop order, it seems very unlikely that the divergence problems of gravity can be solved within extended supergravity. It is tempting then to look for a more radical approach to quantizing gravity. Superstring theories seem to be the most promising departure from the conventional approach [30]. We should also point out that the introduction of a cosmological constant in the classical theory would not change matters, as far as the divergence in eq. (1.4) is concerned. However, in that case additional terms of lower dimensionality would appear in the divergent part and, by virtue of the modified field equations, all previously "harmless" terms would turn, on–shell, into renormalizations of the cosmological constant.

We have shown that the procedure of working with subtracted integrals is very convenient, especially when working in the background field method. The distinction between Green functions with external background or quantum fields becomes immaterial, as no counterterm diagrams are computed explicitly. Moreover, it is very simple to extract pole parts from dimensionally regularized Feynman integrals, since the overlapping divergences are removed *term by term*. With this approach, there is very little difference between calculating in the background field method and in normal field theory. We have also shown that considering the Faddeev–Popov ghosts as separate fields is an unnecessary complication. They can be conveniently embedded into extensions of the gauge fields *á la* Kaluza–Klein. The only remnant of the ghosts is the need to consider, in addition to the extended metric tensor, a few projection operators that satisfy a very simple commutative algebra.

Although these techniques are clearly very effective, they do not by themselves, make the problem of the divergences of quantum gravity tractible by hand. The large number of indices and momenta present in a perturbative expansion of gravity make it impractical to apply the methods so far described without extensive use of computers. Moreover, it has long been recognized that the problem of quantum gravity at two loops lies beyond the power of existing standard algebraic manipulators (see ref. [31] for a recent attempt following a more conventional approach). The limitations have to do both with speed and memory requirements.

In this calculation, nearly every step was done by computer using a number of relatively small programs (typically not exceeding 1,000 lines of code) written by us in the C language. The C language is convenient for manipulations of characters, and the relatively small size of the programs makes them relatively simple to write and debug. By limiting the applicability of the programs, one can easily gain a factor of about 1,000 in speed with respect to the general purpose programs. This is the main difference between our approach and previous ones. Of course, there are a number of techniques one needs to master, some of which are familiar to computer scientists [32]. For instance, one of the most difficult steps in the calculation is the construction of the graphs, especially in the off–shell case. A large number of terms is generated at intermediate stages, and it is essential to be very efficient with the memory allocation. This was done by working with encoded representations for the terms. This corresponds to finding the minimal set of bits that describe a typical structure, thus gaining a nontrivial factor in the amount of memory required. The programs made it possible to perform the entire on–shell calculation in less than three days on a single VAX 11/780. Resorting to computers for specific algebraic problems may well become common practice in Theoretical Physics in the years to come.

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Appendix A

Our conventions are as follows. We use the space–time signature $(-+++)$, and we define the Riemann tensor in terms of the Christoffel symbols as

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\varepsilon}_{\beta\delta}\Gamma^{\alpha}_{\varepsilon\gamma} - \Gamma^{\varepsilon}_{\beta\gamma}\Gamma^{\alpha}_{\varepsilon\delta} \, . \tag{A.1}$$

Then, the Ricci tensor is

$$R_{\alpha\beta} = \delta^{\varepsilon}_{\gamma} R^{\gamma}{}_{\alpha\beta\varepsilon} \, , \tag{A.2}$$

and we write the Einstein–Hilbert action as

$$L = -2\sqrt{-g}\; g^{\mu\nu} R_{\mu\nu} \, . \tag{A.3}$$

For completeness, we give below the expansion of the gravity lagrangian up to quartic order in quantum fields, including the gauge fixing term in eq. (4.3). The quantum field is denoted by $h_{\mu\nu}$, and it is implicitly assumed that all other quantities are constructed out of the background metric, $g_{\mu\nu}$. Indices are raised and lowered using the background metric. The terms quadratic in the quantum field are:

$$L_2 = \sqrt{-g}\,\Big[-\tfrac{1}{2} h^{\alpha\beta}{}_{;\gamma} h_{\alpha\beta}{}^{;\gamma} + \tfrac{1}{4} h^{\alpha}{}_{\alpha;\gamma} h_{\beta}{}^{\beta;\gamma} + h_{\alpha\beta} h_{\gamma\delta} R^{\alpha\gamma\beta\delta} - h_{\alpha\beta} h^{\beta}{}_{\gamma} R^{\delta\alpha\gamma}{}_{\delta}$$

$$+ h^{\alpha}{}_{\alpha} h_{\beta\gamma} R^{\beta\gamma} - \tfrac{1}{2} h_{\alpha\beta} h^{\alpha\beta} R + \tfrac{1}{4} h^{\alpha}{}_{\alpha} h^{\beta}{}_{\beta} R \Big] \, . \tag{A.4}$$

The terms cubic in the quantum field are:

$$L_3 = \sqrt{-g}\,\Big[-\tfrac{1}{2} h^{\alpha\beta} h^{\gamma\delta}{}_{;\alpha} h_{\gamma\delta;\beta} + 2 h^{\alpha\beta} h^{\gamma\delta}{}_{;\alpha} h_{\beta\gamma;\delta} - h^{\alpha\beta} h^{\gamma}{}_{\gamma;\alpha} h^{\delta}{}_{\beta;\delta}$$

$$- \tfrac{1}{2} h^{\alpha}{}_{\alpha} h^{\beta\gamma;\delta} h_{\beta\delta;\gamma} + \tfrac{1}{4} h^{\alpha}{}_{\alpha} h^{\beta\gamma;\delta} h_{\beta\gamma;\delta} - h^{\alpha\beta} h^{\gamma}{}_{\gamma;\delta} h^{\delta}{}_{\alpha;\beta} + \tfrac{1}{2} h^{\alpha\beta} h^{\gamma}{}_{\gamma;\alpha} h^{\delta}{}_{\delta;\beta}$$

$$- h^{\alpha\beta}h_{\alpha\beta;\gamma}h^{\gamma\delta}{}_{;\delta} + \tfrac{1}{2}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta;\gamma}h^{\gamma\delta}{}_{;\delta} + h^{\alpha\beta}h_{\alpha\beta;\gamma}h_{\delta}{}^{\delta;\gamma} + \tfrac{1}{4}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta;\gamma}h_{\delta}{}^{\delta;\gamma}$$

$$- h^{\alpha\beta}h^{\gamma}{}_{\alpha;\delta}h_{\beta\gamma}{}^{;\delta} + h^{\alpha\beta}h^{\gamma}{}_{\alpha;\delta}h^{\delta}{}_{\beta;\gamma} + R_{\alpha\beta}\big(2h^{\alpha\gamma}h_{\gamma\delta}h^{\beta\delta} - h^{\gamma}{}_{\gamma}h^{\alpha\delta}h^{\beta}{}_{\delta}$$

$$- \tfrac{1}{2}h^{\alpha\beta}h^{\gamma\delta}h_{\gamma\delta} + \tfrac{1}{4}h^{\alpha\beta}h^{\gamma}{}_{\gamma}h^{\delta}{}_{\delta}\big) + R\,\big(-\tfrac{1}{3}h^{\alpha\beta}h_{\beta\gamma}h^{\gamma}{}_{\alpha} + \tfrac{1}{4}h^{\alpha}{}_{\alpha}h^{\beta\gamma}h_{\beta\gamma}$$

$$- \tfrac{1}{24}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta}h^{\gamma}{}_{\gamma}\big)\Bigg]\,. \qquad \text{(A.5)}$$

Finally, the terms quartic in the quantum field are:

$$L_4 = \sqrt{-g}\;\Bigg[\big(h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta} - 2h^{\alpha\beta}h_{\alpha\beta}\big)\big(\tfrac{1}{16}h^{\gamma\delta;\sigma}h_{\gamma\delta;\sigma} - \tfrac{1}{8}h^{\gamma\delta;\sigma}h_{\gamma\sigma;\delta} + \tfrac{1}{8}h^{\gamma}{}_{\gamma;\delta}h^{\delta\sigma}{}_{;\sigma}$$

$$- \tfrac{1}{16}h^{\gamma}{}_{\gamma;\delta}h_{\sigma}{}^{\sigma;\delta}\big) + h^{\alpha}{}_{\alpha}h^{\beta\gamma}\big(-\tfrac{1}{2}h_{\beta\gamma;\delta}h^{\delta\sigma}{}_{;\sigma} + \tfrac{1}{2}h_{\beta\gamma;\delta}h_{\sigma}{}^{\sigma;\delta} - \tfrac{1}{2}h^{\delta}{}_{\delta;\beta}h^{\sigma}{}_{\sigma;\gamma}$$

$$+ \tfrac{1}{4}h^{\delta}{}_{\delta;\beta}h^{\sigma}{}_{\sigma;\gamma} + h^{\delta}{}_{\beta;\sigma}h^{\sigma}{}_{\delta;\gamma} - \tfrac{1}{4}h^{\delta\sigma}{}_{;\beta}h_{\delta\sigma;\gamma} - \tfrac{1}{2}h^{\delta}{}_{\beta;\sigma}h_{\delta\gamma}{}^{;\sigma} - \tfrac{1}{2}h^{\delta}{}_{\delta;\sigma}h^{\sigma}{}_{\beta;\gamma}$$

$$+ \tfrac{1}{2}h_{\beta\delta;\sigma}h_{\gamma}{}^{\sigma;\delta}\big) + h^{\alpha}{}_{\beta}h^{\beta\gamma}\big(h^{\delta}{}_{\delta;\sigma}h^{\sigma}{}_{\alpha;\gamma} - h_{\alpha\gamma;\delta}h_{\sigma}{}^{\sigma;\delta} + \tfrac{1}{2}h^{\delta\sigma}{}_{;\alpha}h_{\delta\sigma;\gamma}$$

$$- h^{\delta}{}_{\alpha;\sigma}h^{\sigma}{}_{\gamma;\delta} - 2h^{\delta}{}_{\alpha;\sigma}h^{\sigma}{}_{\delta;\gamma} + h_{\alpha\gamma;\delta}h^{\delta\sigma}{}_{;\sigma} + h^{\delta}{}_{\delta;\alpha}h^{\sigma}{}_{\gamma;\sigma} - \tfrac{1}{2}h^{\delta}{}_{\delta;\alpha}h^{\sigma}{}_{\sigma;\gamma}$$

$$+ h^{\delta}{}_{\alpha;\sigma}h_{\gamma\delta}{}^{;\sigma}\big) + h^{\alpha\gamma}h^{\beta\delta}\big(h_{\alpha\gamma;\beta}h^{\sigma}{}_{\delta;\sigma} - h_{\alpha\gamma;\delta}h^{\sigma}{}_{\sigma;\beta} + \tfrac{1}{2}h_{\alpha\beta;\sigma}h_{\gamma\delta}{}^{;\sigma}$$

$$- \tfrac{1}{2}h_{\alpha\gamma;\sigma}h_{\beta\delta}{}^{;\sigma} + h^{\sigma}{}_{\alpha;\beta}h_{\gamma\sigma;\delta} - h^{\sigma}{}_{\alpha;\beta}h_{\delta\sigma;\gamma} + h_{\alpha\beta;\delta}h^{\sigma}{}_{\sigma;\gamma} - 2h^{\sigma}{}_{\alpha;\beta}h_{\delta\gamma;\sigma}$$

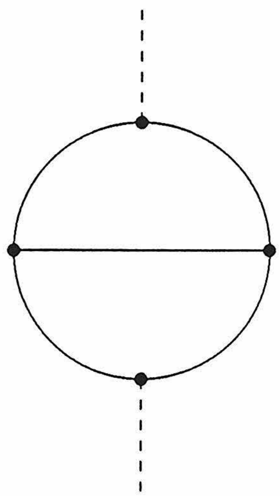
$$+ h_{\alpha\gamma;\sigma}h^{\sigma}{}_{\beta;\delta}\big) + R_{\alpha\beta}\big(-2h^{\alpha\gamma}h_{\gamma\delta}h^{\delta\sigma}h_{\sigma}{}^{\beta} + h^{\gamma}{}_{\gamma}h^{\alpha\delta}h_{\delta\sigma}h^{\sigma\beta} + \tfrac{1}{2}h^{\alpha\gamma}h_{\gamma}{}^{\beta}h^{\delta\sigma}h_{\delta\sigma}$$

$$- \tfrac{1}{4}h^{\alpha\gamma}h_{\gamma}{}^{\beta}h^{\delta}{}_{\delta}h^{\sigma}{}_{\sigma} + \tfrac{1}{3}h^{\alpha\beta}h^{\gamma\delta}h_{\delta\sigma}h^{\sigma}{}_{\gamma} - \tfrac{1}{4}h^{\alpha\beta}h^{\gamma}{}_{\gamma}h^{\delta\sigma}h_{\delta\sigma}$$

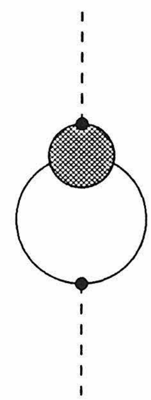
$$+ \tfrac{1}{24}h^{\alpha\beta}h^{\gamma}{}_{\gamma}h^{\delta}{}_{\delta}h^{\sigma}{}_{\sigma}\big) + R\,\big(-\tfrac{1}{192}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta}h^{\gamma}{}_{\gamma}h^{\delta}{}_{\delta} + \tfrac{1}{16}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta}h^{\gamma\delta}h_{\gamma\delta}$$

$$+ \frac{1}{4} h^{\alpha\beta} h_{\beta\gamma} h^{\gamma\delta} h_{\delta\alpha} - \frac{1}{16} h^{\alpha\beta} h_{\alpha\beta} h^{\gamma\delta} h_{\gamma\delta} - \frac{1}{6} h^{\alpha}{}_{\alpha} h^{\beta\gamma} h_{\gamma\delta} h^{\delta}{}_{\beta} \Big) \Bigg] . \qquad (A.6)$$

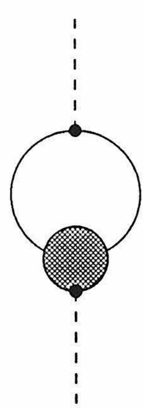
These terms are sufficient for the background field calculation. For the calculation in normal field theory one needs, in addition, quintic terms.



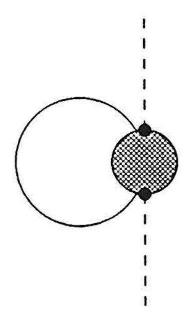
(1a)



(1b)

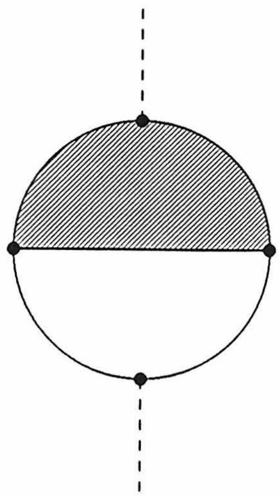


(1c)

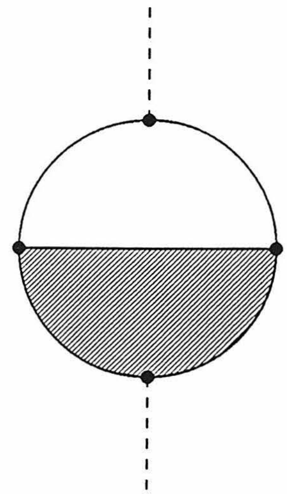


(1d)

Figure 1



(2a)



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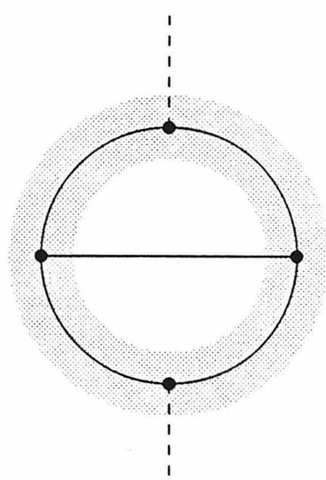


(2c)

Figure 2

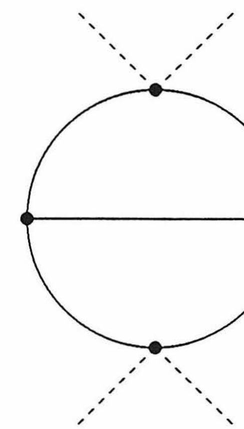


Figure 3

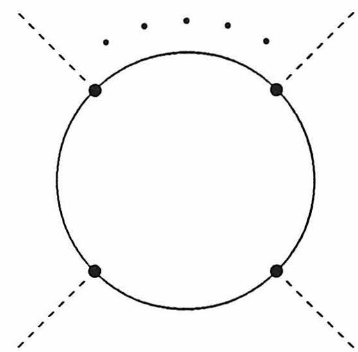


Figure 4

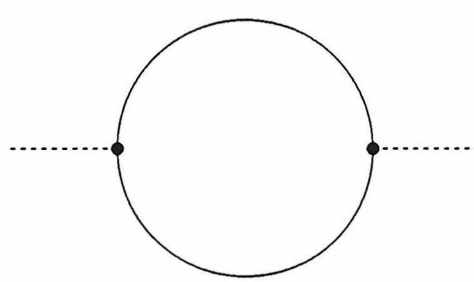
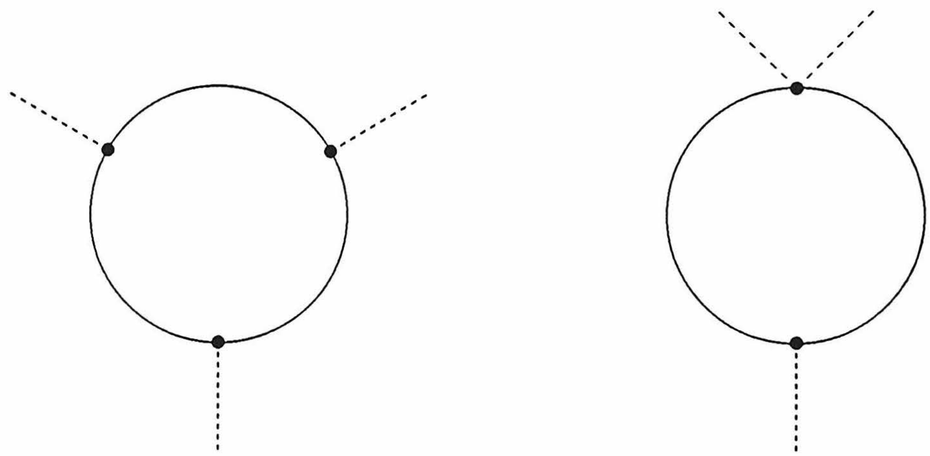


Figure 5



(6a) (6b)

Figure 6

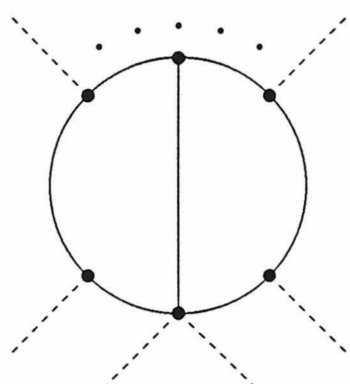
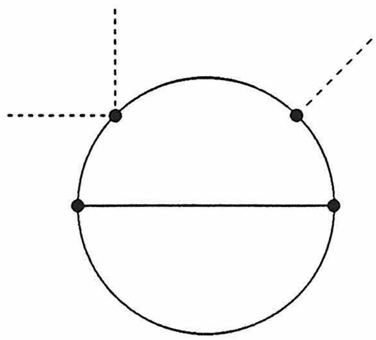
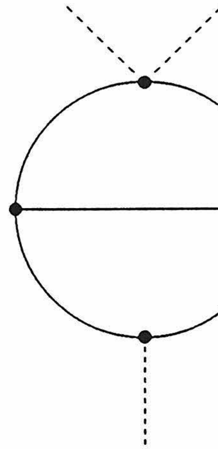


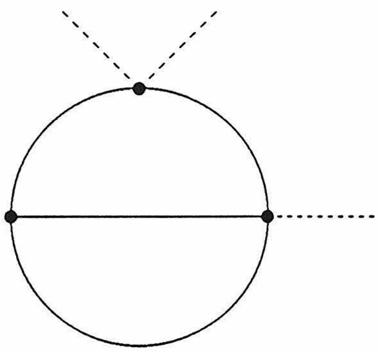
Figure 7



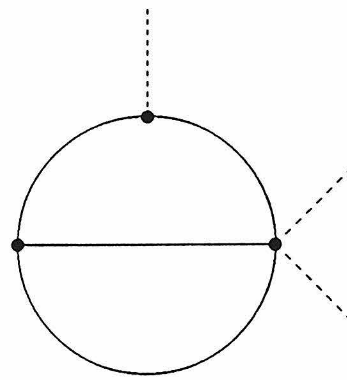
(8a)



(8b)



(8c)



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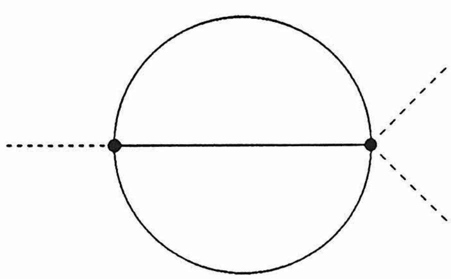
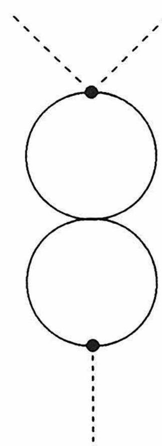
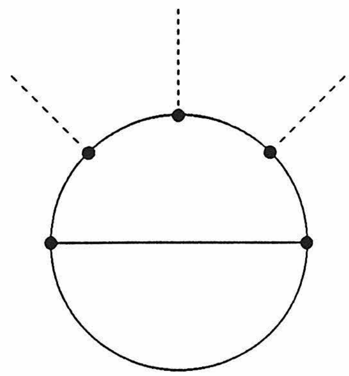


(8e)

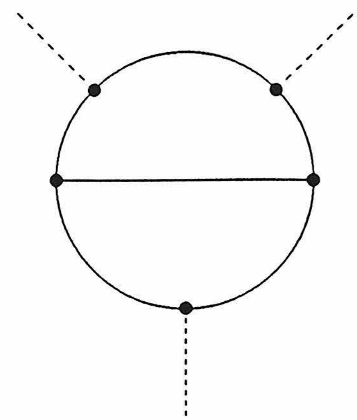


(8f)

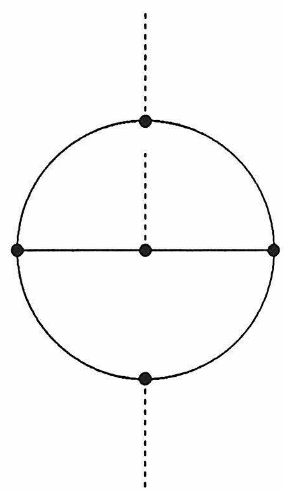
Figure 8



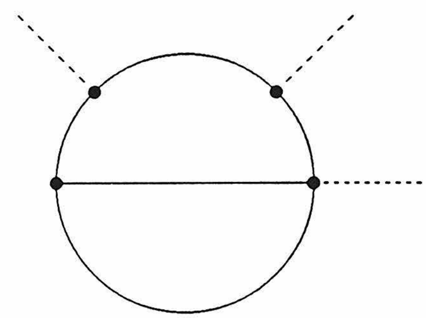
(9a)



(9b)



(9c)



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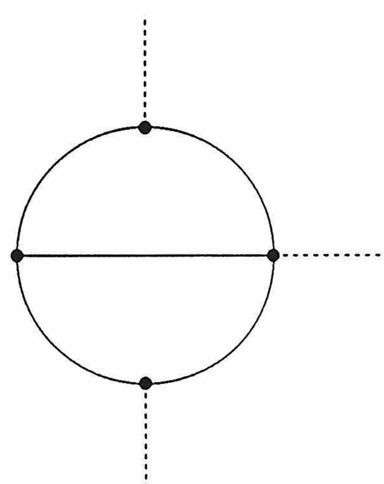

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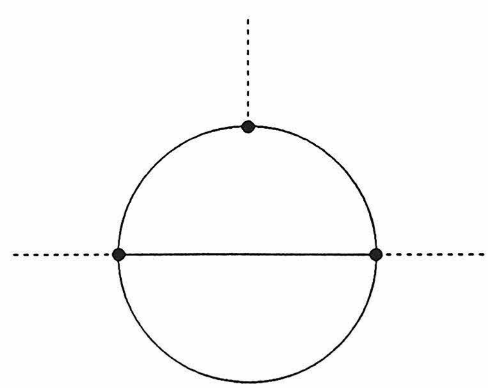

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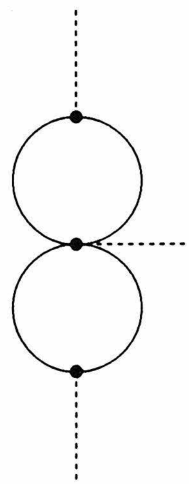
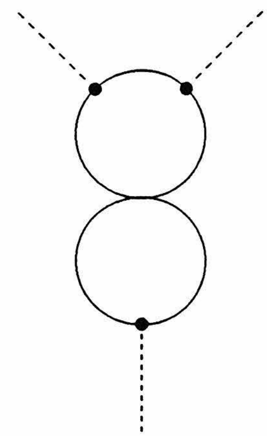


(9g)



(9h)

Figure 9

Table 1

Results for the On–Shell Graphs

On–Shell Graphs $*(4\pi)^4$		
Graph	Background Field	Non–Background Field
8	0	0
9a	$-\frac{7}{120}\frac{1}{\varepsilon^2}-\frac{21307}{302400}\frac{1}{\varepsilon}$	$-\frac{7}{150}\frac{1}{\varepsilon^2}-\frac{59167}{756000}\frac{1}{\varepsilon}$
9b	$\frac{25}{18}\frac{1}{\varepsilon^2}-\frac{5471}{5600}\frac{1}{\varepsilon}$	$\frac{761}{600}\frac{1}{\varepsilon^2}-\frac{178781}{216000}\frac{1}{\varepsilon}$
9c	$-\frac{68}{45}\frac{1}{\varepsilon^2}-\frac{38299}{20160}\frac{1}{\varepsilon}$	$-\frac{3119}{3600}\frac{1}{\varepsilon^2}-\frac{4222229}{3024000}\frac{1}{\varepsilon}$
9d	$-\frac{37}{36}\frac{1}{\varepsilon^2}+\frac{294199}{302400}\frac{1}{\varepsilon}$	$-\frac{49}{40}\frac{1}{\varepsilon^2}+\frac{10511}{10800}\frac{1}{\varepsilon}$
9e	$\frac{17}{6}\frac{1}{\varepsilon^2}+\frac{23293}{8400}\frac{1}{\varepsilon}$	$\frac{2447}{1200}\frac{1}{\varepsilon^2}+\frac{6162691}{3024000}\frac{1}{\varepsilon}$
9f	$-\frac{1}{8}\frac{1}{\varepsilon^2}-\frac{1307}{1800}\frac{1}{\varepsilon}$	$-\frac{23}{180}\frac{1}{\varepsilon^2}-\frac{192449}{302400}\frac{1}{\varepsilon}$
9g	$-\frac{3}{2}\frac{1}{\varepsilon^2}$	$-\frac{25}{24}\frac{1}{\varepsilon^2}$
9h	0	0
Total	$\frac{209}{2880}\frac{1}{\varepsilon}$	$\frac{209}{2880}\frac{1}{\varepsilon}$