Partially Coherent Wave Scattering and Radiative Transfer: An Integral Equation Approach

Thesis by

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In partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1986

(Submitted March 17, 1986)

ACKNOWLEDGEMENTS

It is my pleasure to acknowledge these individuals who have assisted me in performing this research. I am particularly grateful to Professor Yuk L. Yung, who as my thesis adviser has provided much valuable advice and inspiration.

I thank all my fellow graduate students, the faculty of planetary science, and the faculty of the physics department, who have made Caltech a pleasant and rewarding place for me to learn and to do research.

The assistance of Kay Campbell and the efforts of Donna Lathrop in typing this manuscript are greatly appreciated.

Finally, I thank my parents and my family for their encouragement, their support and especially their love throughout my life.

ABSTRACT

This thesis consists of two parts. The first one is "Scattering of Waves in a Random Medium," and the second one is "Radiative Transfer in a Sphere Illuminated by a Parallel Beam: An Integral Equation Approach."

In the first part, a new formalism for partially coherent wave scattering in a random medium is developed. In this formalism the coherent wave is the solution of a phenomenological wave equation, and the mutual coherence function of the wave field satisfies a simple integral equation. Using this formalism, the Peierls equation can be readily derived. Also, an improved version of the Peierls equation is derived in which the intensity of the wave field and the first-order derivative of the mutual coherence function are calculated at the same time. A simple problem is solved to find the mutual coherence function produced by a laser beam in the atmosphere. The similarity between the mutual coherence function and the density matrix of quantum mechanics is explored and a measure of the randomness is defined for the partially coherent wave field.

In the second part of this work, the problem of multiple scattering of non-polarized light in a planetary body of arbitrary shape illuminated by a parallel beam is formulated using the integral equation approach. There exists a simple functional whose stationarity condition is equivalent to solving the equation of radiative transfer and whose value at the stationary point is proportional to the differential cross section. Our analysis reveals a direct relation between the microscopic symmetry of the phase function for each scattering event and the macroscopic symmetry of the differential cross section for the entire planetary body, and the intimate connection between these symmetry relations and the variational principle. The case of a homogeneous sphere containing isotropic scatterers is investigated in detail. It is shown that the solution can be expanded in a multipole series such that the general spherical problem is reduced to solving a set of decoupled integral equations in one dimension. Computations have been performed for a range of parameters of interest, and illustrative examples of applications to planetary problems are provided.

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Part One

Scattering of Waves in a Random Medium

I. INTRODUCTION

Different kinds of waves, such as electromagnetic waves, elastic waves and sound waves, are widely used to explore the properties and structures of different substances. In remote sensing, information carried by visible, ultraviolet and infrared radiation from remote objects is collected and analyzed. In recent years, using the space technology and coherent light sources, a tremendous amount of accurate observation data has been collected. In order to deduce the correct information from the observed data we must know how the electromagnetic wave interacts with the objects and how it propagates in a medium.

Unfortunately, most media such as the atmosphere of earth and the ocean, and planetary atmosphere are random in the sense that the optical characters of these objects are subjected to fluctuations from time to time and from point to point. It is impossible to find the exact solution for the problem of wave propagation in a random medium, which is described mathematically by a set of partial differential equations with random parameters.

During the past several decades, different kinds of approximate methods have been developed to solve this problem. Some of them treat randomly distributed scatterers; others deal with continuous random media. In fact, this distinction is not fundamental, and they are closely related to each other from both a physical and a mathematical point of view. After all, microscopically, any medium is made of discrete scatterers.

Historically, the single scattering approximation was developed to calculate the scattering of electromagnetic waves from random medium (Booker and

Gordon, 1950; Vallars and Weisskopf, 1954; Salpeter and Trieman, 1964; Tatarskii, 1961; Wheelon, 1959; Sheffield, 1975; Hardy and Katz, 1969; Ishimaru, 1978a; Bremmer, 1964; Batten, 1973). In many cases, this approximation is sufficient. But there are some situations where the multiple scattering effects must be taken into account. For example, the data on lidar scattering from clouds (Milton et al., 1972; Anderson and Browell, 1972; Cohen, 1975) have indicated that the single scattering approximation is inadequate.

The problem of multiple scattering of electromagnetic wave in a random medium has been investigated from two points of view, "phenomenological" and "statistical". The phenomenological method is the radiative transfer theory. It deals with the propagation of the specific intensity of the radiation field and has been successfully used in many physical problems (Chandrasekhar, 1960). It does not, however, deal directly with fluctuations and correlation functions of the field. This is rather unsatisfectary, since the problem of wave propagation in a random medium is essentially a statistical one. Also, in the phenomenological theory it is impossible to find the region of validity for the radiative transfer equation or to incorporate the diffraction. Because the statistical method starts from first principles, the Maxwell equation, and uses the well-established ensemble averaging method, it is more fundamental. Several methods have been used to solve the moment equations, among which the so-called Bethe-Salpeter equation is of second moment. The diagram method (Barabanenkov, 1967; Barabanenkov and Finkel'berg, 1968; Tatarskii, 1971; Furutsu, 1972, 1975, 1985a, 1985b and 1985c); the operator method (Furutsu, 1975 and 1980; Fante, 1982), and the asymptotic method (Barabanenkov et al., 1972; Fante, 1975; Prokhorov et al., 1975) are commonly used. Instead of working with moment equations, some workers prefer the parabolic equation (Beran, 1975; Shishov, 1968; Beran and Ho, 1969; Tatarskii,1971; Molyneux, 1971a and 1971b; McCoy, 1972;

Furutsu, 1972), and others use the extended Huygens-Fresnel principle (Tur and Beran, 1983; Feizulin and Kravtsov, 1967 and 1969; Lutominski and Yura, 1971). As we have mentioned above, all these methods are approximate. They can usually give good results in their narrow region of validity. Some of them have been used to clarify the relation of the radiative transfer equation to Maxwell equation and give the former a more sound foundation (Watson, 1969; Stott, 1968; Fante, 1973; Walther, 1968). Even the attempts have been made of putting certain diffraction effects of wave field in random medium into radiative transfer equation (Lau and Watson, 1970; Barabanenkov et al., 1972). Since the basic quantity in radiative transfer theory, the specific intensity of radiation, cannot be defined properly in statistical theory except in a vacuum (Wolf, 1976 and 1978, Zubairy and Wolf, 1977), a rigorous derivation of radiative transfer equation directly from Maxwell equation is still unavailable. This is just a minor problem of the theory of the wave propagation in a random medium. Although much progress has been made in the statistical approach, the user of these different methods still faces the problem of large number of parameters and conditions of validity.

Because of the intrinsic complexity of the problem, it is not possible in the near future to establish a theory of wave propagation in a random medium that is comprehensive but still simple enough to be used by non-experts. But progress can be made along this direction using the new approach based on better understanding about the physical processes inside the medium when waves are propagating through it.

The relationship between the multiple scattering, radiative transfer and wave propagation in a random medium becomes really clear only from the coherent theory point of view. In classical electromagnetic theory the refraction and diffraction of light in regular medium comes from the coherent sum of all scattered waves (R. P. Feynman et al., 1963, Vol. 1, Ch. 30 and 31). What we need to do is to generalize this idea in the case of wave propagation in a random medium. The randomness of the medium produces the fluctuation of the electromagnetic field by wave scattering, which is not totally coherent. So the wave propagation in a random medium is a partially coherent phenomenon. When the coherence between different scattered waves is totally destroyed, the electromagnetic field loses its wave character and behaves just like a set of particles. In this extreme case the radiative transfer theory should be a good approximation.

What we try to do in this research is to determine how the randomness of a medium destroys the coherence of the scattered waves. With better understanding of the basic physical processes, a more efficient method for partially coherent scattering problems can be developed afterwards.

To simplify the problem, we use the scalar wave equation instead of Maxwell equation. Our working model is a finite volume of random medium with a light beam incident on it. Also we assume that the incident beam from the outside source is coherent, monochromatic and stationary. There is no fundamental difficulty in adding the vector character of the electromagnetic wave and the randomness of light source to the problem. We add the unscattered wave to the coherent scattering wave to form the coherent wave, which would be the whole wave field if there were no randomness in the medium. The coherent wave is not a random field. In a weakly random medium, we derive a phenomenological wave equation for it. The index of refraction in this wave equation is almost equal to the average of the index of refraction of the medium. Because the coherent wave satisfies a wave equation with a regular index of refraction, it has all the wave properties, such as diffraction, refraction and interference. We can use the well-developed electromagnetic field theory to find the solution for this equation. In the limit of geometric optics, the light ray follows a curve in an inhomogeneous medium.

The total wave field is a random function, so we are interested only in the ensemble average quantities. For both practical and theoretical purposes, the most important one is the mutual coherence function (see Born and Wolf, 1980, Ch.10). We use the transformation between the background and source term to derive an integral equation for the mutual coherence function. This equation shows explicitly that the mutual coherence function consists of two parts, one of which comes from the coherent wave; the other comes from the incoherent wave. From this integral equation, the Peierls equation, the radiative transfer equation for the homogeneous medium with isotropic scatterers, is derived naturally. We also derive a higher order version of the Peierls equations. A simple example is given to show the application of the integral equation to the mutual coherence function. Although the multidimensionality of the equation makes it difficult to solve in general, we think the simplicity of the mathematical form and the clarity of the physical processes will ensure the usefulness of this formalism in providing theoretical insight and numerical results.

How to define the entropy for the partially coherent light beams has long been an interesting problem in physics. Jones (1953) used entropy to determine the reversibility of some optical experiments. Gamo (1964) gave the definition of the entropy of a partially coherent radiation field. When we use the coherence theory point of view to unify the radiative transfer theory and wave theory in random media, we need a quantity to measure the randomness of the wave field. If we can define this quantity correctly, it must be the entropy itself. In Section IV, we use the similarity between the mutual coherence function and the density matrix in quantum mechanics to define a quantity which can measure the randomness of the partially coherent wave field in general. It is not exactly the entropy as defined in thermodynamics . More work should be done before we can define a correct thermodynamic entropy for the partially coherent wave field.

II. A New Formalism for Wave Scattering

1. The Wave Equation and Associated Physical Variables

The steady state equation for the monochromatic wave scattered in a random medium is

$$\nabla^2\psi(\vec{r}) + k^2(\vec{r})\psi(\vec{r}) = 0 \quad , \tag{II-1}$$

where $\psi(\vec{r})$ is the random wave field, and $k^2(\vec{r}) \equiv (\frac{\omega}{v(\vec{r})})^2$, with $\omega =$ the frequency of the wave, and $v(\vec{r})$ = the local phase speed of the wave. In electromagnetic wave scattering,

$$v(\vec{r}) = \left[\frac{c}{n(\vec{r})}\right] \quad .$$

So

$$k^2(\vec{r}) = \left[\frac{\omega}{c}\right]^2 n^2(\vec{r}) = k_o^2 n^2(\vec{r}) \quad , \tag{II-2}$$

where $k_o \equiv \frac{\omega}{c}$ and $n(\vec{r})$ is the index of refraction. $n(\vec{r})$ is a random variable in the problem.

As in any statistical problem, we are interested only in the ensemble averaged quantities (Huang, 1963, Ch.7). This is because the index of refraction is a random variable. Talking about the state of a single system is meaningless, physically. All systems which are identical macroscopically can have a quite different index refraction; so do the other physical variables. Only the ensemble averaged quantities can exclude the fluctuations and have real physical contents. In our problem they are the mean-wave function $<\psi(\vec{r})>$, the mutual coherence function $<\psi(\vec{r}_1)\psi^*(\vec{r}_2)>$ and higher-order correlation functions $<\psi(\vec{r}_1)\psi(\vec{r}_2)\ \cdots\ \psi^*(\vec{r}_1)\psi^*(\vec{r}_2)\ \cdots >$ (Bohn and Wolf, 1980, Ch.10). Here $< \cdots >$ means ensemble average, and the upper index * means complex conjugation. Of all these quantities, the mutual coherence function $<\psi(\vec{r}_1)\psi^*(\vec{r}_2)>$, denoted by $\Gamma(\vec{r}_1,\vec{r}_2)$, is most important in optics and photometry. The mutual coherence function at the same points $\Gamma(\vec{r}_1,\vec{r}_1)$ is the mean intensity of the wave field $<|\psi(\vec{r}_1)|^2>$, and the Fourier transformation of $\Gamma(\vec{r}_1,\vec{r}_2)$ is related to the angular distribution of the wave energy flux (Tatarskii, 1971). If $\Gamma(\vec{r}_1,\vec{r}_2)$ is known, most optical problems can be solved. Fourth-order correlation functions are needed only when wave intensity fluctuations are important (Fante, 1983, Majumdar, 1984).

The scattering properties of the medium depend not only on the mean of the index of refraction $<n(\vec{r})>$, but also on the autocorrelation of the random part of the index of refraction,

$$N(\vec{r}_1,\vec{r}_2) \equiv <n_1(\vec{r}_1)\, n_1^*(\vec{r}_2)> \quad , \tag{II-3}$$

where $n_1(\vec{r})$ is the random part of the index of refraction

$$n_1(\vec{r}) \equiv n(\vec{r}) - <n(\vec{r})> \quad . \tag{II-4}$$

If the medium is statistically uniform and isotropic,

$$<n(\vec{r})> = n_o \quad ,$$

where n_o is a constant, and

$$N(\vec{r_1}, \vec{r_2}) = N(\rho) \quad , \tag{II-5}$$

with

$$\vec{\rho} \equiv \vec{r_1} - \vec{r_2} \quad , \quad \rho = |\vec{\rho}| \quad . \tag{II-6}$$

The autocorrelation function $N(\rho)$ can have a different form, depending on the nature of the randomness of the medium. The following are some examples:

$$N_1(\rho) = a^2 \exp\left\{-\rho^2 / r_o^2\right\} \tag{II-7}$$

is the Gaussian form,

$$N_2(\rho) = a^2 \exp\left\{-\rho / r_o\right\} \tag{II-8}$$

is the exponential form, and

$$N_3(\rho) = \frac{a^2}{2^l (l-1)!} (\rho / r_o)^l \ K_l(\rho / r_o) \tag{II-9}$$

is the von Karman form, which was proposed by von Karman for turbulent media (Tatarskii, 1961). Here l is a real number, e.g., 1/3, and K_l is a Bessel function of the second kind with imaginary argument.

The r_o is known as the "scale size" of the irregularities and can be thought of as the average distance over which the fluctuations of the index of refraction remain correlated. The a is a measure of the fluctuation intensity.

2. Differential Equation for the Coherent Wave Function $\psi_c(\vec{r})$

We divide the wave function $\psi(\vec{r})$ into a coherent part and an incoherent part,

$$\psi(\vec{r}) = \psi_c(\vec{r}) + \psi_1(\vec{r}) \quad . \tag{II-10}$$

The coherent wave is defined by

$$\psi_c(\vec{r}) \equiv <\psi(\vec{r})> \quad , \tag{II-11}$$

which comes from unscattered incident wave and coherently scattered wave. This is the generalization of the classical electromagnetic theory, where there is only the coherent wave which consists of unscattered incident wave and coherently scattered wave. For the time being, we suppose that the incident wave is coherent. The two parts of the coherent wave produce all coherent phenomena of waves such as deflection, diffraction, reflection, and interference. The incoherent wave $\psi_1(\vec{r})$ comes from incoherently scattered waves. This will be shown clearly in the integral equation for the wave function, Eqn. (II-22).

By taking the ensemble average of Eqn. (II-1), we have

$$\nabla^2<\psi(\vec{r})> + <k^2(\vec{r})\psi(\vec{r})> = 0 \quad . \tag{II-12}$$

Obviously, $k^2(\vec{r})$ and $\psi(\vec{r})$ are not statistically independent. This is because the fluctuation of the index of refraction causes the fluctuation of wave function. But this is done through the wave Equ. (II-1). So the correlation length of the wave function should be of the same order as the wave length $\lambda \sim 1/k$, which can be quite different from the correlation length of the index of refraction, r_o. For stationary and statistically uniform problems, the ensemble average, the time average and the volume average are all the same. In volume average it is easy to prove that the average of the product of two variables with quite different correlation lengths is almost equal to the product of two average variables. So, if $\lambda \gg r_o$ or $r_o \gg \lambda$,

$$<k^2(\vec{r})\psi(\vec{r})> \approx <k^2(\vec{r})><\psi(\vec{r})> .$$

We can write

$$<k^2(\vec{r})\psi(\vec{r})> = <k^2(\vec{r})><\psi(\vec{r})>\left[1 + \delta(\vec{r})\right] , \tag{II-13}$$

where δ is a small correction, and

$$|\delta(\vec{r})| \ll 1 . \tag{II-14}$$

We suppose that this small correction is the characteristic of the medium and is independent of the wave field. Otherwise, we would have to deal with a nonlinear wave problem. Define

$$k_c^2(\vec{r}) \equiv <k^2(\vec{r})>\left[1 + \delta(\vec{r})\right] . \tag{II-15}$$

In a simple problem we can derive the imaginary part of $\delta(\vec{r})$ which is important in the complete formulation. From Eqn. (II-12) and (II-13), we find the differential equation for coherent wave $\psi_c(\vec{r})$,

$$\left[\nabla^2 + k_c^2(\vec{r})\right]\psi_c(\vec{r}) = 0 . \tag{II-16}$$

This is the Helmholtz equation with $k_c(r)$ as a wave number. Neither $k_c^2(\vec{r})$, nor $\psi_c(\vec{r})$ is a random variable.

$\delta(\vec{r})$ should be a complex function. If the medium is a pure scattering medium, $k^2(\vec{r})$ is a real function. In this case, the imaginary part of $\delta(\vec{r})$ supplies the only decay factor for a coherent wave as it propagates inside the medium. The decaying part of the coherent wave is not really absorbed by the medium; instead it is transformed into an incoherent wave.

The imaginary part of $\delta(\vec{r})$ is crucial to our formalism. Except for some simple cases, it is very difficult to derive it from statistical properties of media. But we think it is not very difficult to measure it by experiment.

3. Integral Equation for the Wave Function $\psi(\vec{r})$

In the last section we have established the equation for the higher-order correlation functions of the wave field. We need the integral equation for the wave function $\psi(\vec{r})$. The integral equation,

$$\psi(\vec{r}) = \psi^{(0)}(\vec{r}) + \int G^{(0)}(\vec{r},\vec{r}')\left[k_o^2 - k^2(\vec{r}')\right]\psi(\vec{r}')\, d\vec{r}' \quad , \tag{II-17}$$

is equivalent to the Helmholtz equation,

$$\left[\nabla^2 + k^2(\vec{r})\right]\psi(\vec{r}) = 0 \quad , \tag{II-18}$$

with appropriate boundary conditions. In Eqn. (II-17), $\psi^{(0)}(\vec{r})$ is the solution of equation

$$\left[\nabla^2 + k_o^2\right]\psi^{(0)}(\vec{r}) = 0 \quad , \tag{II-19}$$

and $G^{(0)}(\vec{r},\vec{r}')$ is the Green's function which satisfies

$$\left[\nabla^2 + k_o^2\right]G^{(0)}(\vec{r},\vec{r}') = \delta(\vec{r} - \vec{r}') \quad . \tag{II-20}$$

The boundary conditions of Eqn. (II-20) and (II-19) must be matched correctly with that of Eqn. (II-18), so that the solution of integral equation (II-20) is the solution of the original Helmholtz equation and boundary condition. If we write Eqn. (II-18) as

$$(\nabla^2 + k_o^2)\psi(\vec{r}) = \left[k_o^2 - k^2(\vec{r})\right]\psi(\vec{r}) \quad , \tag{II-21}$$

the equivalence of Eqn. (II-21) and integral Eqn. (II-17) is obvious. Equation (II-17) states that the solution of Eqn. (II-21) is the sum of the solution of a homogeneous equation and a special solution of an inhomogeneous equation. We can call k_o^2 the background and [k_o^2 - $k^2(\vec{r})$] the scattering source term, because the first term of the R. H. S. of Eqn. (II-17) is the wave field propagating in a medium with k_o as the index of refraction, and the second term of R. H. S. of Eqn. (II-17) is the scattered wave. The scattered wave comes from every point of the medium, $\vec{r}$. The intensity of the scattered wave is proportional to [k_o^2 - $k^2(\vec{r})$]. This wave also propagates in the same background medium described by the Green's function $G^{(0)}(\vec{r},\vec{r}')$, which is the solution of Eqn. (II-20). But there are an infinite number of ways to separate $k^2(r)$ into a background and scattering source term. Usually the vacuum is chosen as the background in scattering problems. But choosing other backgrounds may be more convenient in some cases. For example, the problem of wave propagation in the water with bubbles can be solved using the results of wave propagation in the air with water drops after the background of water is separated. If we want to separate the coherent wave from the incoherent wave, a more natural way is to choose $k_c^2(\vec{r})$ of (II-15) as a background. Then the integral equation for the wave function is

$$\psi(\vec{r}) = \psi_c(\vec{r}) + \int d\vec{r}'\, G_c(\vec{r},\vec{r}')\left[k_c^2(\vec{r}) - k^2(\vec{r})\right]\psi(\vec{r}) \quad , \tag{II-22}$$

where $\psi_c(\vec{r})$ = $<\psi(\vec{r})>$ is the solution of Eqn. (II-16) and the Green's function $G_c(\vec{r},\vec{r}')$ is the solution of

$$\left[\nabla^2 + k_c^2(\vec{r})\right]G_c(\vec{r},\vec{r}') = \delta(\vec{r} - \vec{r}') \quad . \tag{II-23}$$

The incoherent wave is defined by the second term of R. H. S. of Eqn. (II-22),

$$\psi_1(\vec{r}) \equiv \int d\vec{r}\,' G_c(\vec{r},\vec{r}\,')\Big[k_c^2(\vec{r}\,') - k^2(\vec{r}\,')\Big]\psi(\vec{r}\,') \quad . \tag{II-24}$$

Because of Eqn. (II-13) and (II-15), the ensemble average of the incoherent wave $\psi_1(\vec{r})$ is

$$<\psi_1(\vec{r})> = \int d\vec{r}\,' \ G_c(\vec{r},\vec{r}\,')\Big[k_c^2(\vec{r}\,')<\psi(\vec{r}\,')> - <k^2(\vec{r}\,')\psi(\vec{r}\,')>\Big]$$

$$= \int d\vec{r}\,' \ G_c(\vec{r},\vec{r}\,')\Big[k_c^2(\vec{r}\,')<\psi(\vec{r}\,')> - k_c^2(\vec{r}\,')<\psi(\vec{r}\,')>\Big] = 0 \quad , \tag{II-25}$$

as required.

In the Appendix, a simple one dimensional scattering problem is solved to show that the integral equations with different backgrounds are equivalent. Also, it is found that the boundary condition must be treated with caution in order to get the correct solutions.

4. Integral Equation for the Mutual Coherence Function $\Gamma(\vec{r}_1,\vec{r}_2)$

The most important quantity in optics and photometry is the mutual coherence function of the wave field, defined by

$$\Gamma(\vec{r}_1,\vec{r}_2) \equiv <\psi(\vec{r}_1)\psi^*(\vec{r}_2)> \quad . \tag{II-26}$$

The intensity of the wave field

$$J(\vec{r}) \equiv <\psi(\vec{r})\psi^*(\vec{r})> = <|\psi(\vec{r})|^2> \quad , \tag{II-27}$$

is $\Gamma(\vec{r},\vec{r})$.

$\Gamma(\vec{r}_1,\vec{r}_2)$, which itself is the correlation function of the wave function, can be measured and used to provide information about the random wave field $\psi(\vec{r})$

and the medium through which the wave propagates. So we need the equation which governs the behavior of $\Gamma(\vec{r}_1,\vec{r}_2)$.

The conjugate of the integral equation for the wave function (II-22)

$$\psi(\vec{r}_1) = \psi_c(\vec{r}_1) + \psi_1(\vec{r}_1) \quad , \tag{II-28}$$

is

$$\psi^*(\vec{r}_2) = \psi_c^*(\vec{r}_2) + \psi_1^*(\vec{r}_2) \quad . \tag{II-29}$$

From Eqn. (II-28) and (II-29), we find

$$\psi(\vec{r}_1)\psi^*(\vec{r}_2) = \left[\psi_c(\vec{r}_1) + \psi_1(\vec{r}_1)\right]\left[\psi_c^*(\vec{r}_2) + \psi_1^*(\vec{r}_2)\right] \tag{II-30}$$

The ensemble average of Eqn. (II-30) is

$$<\psi(\vec{r}_1)\psi^*(\vec{r}_2)> = \psi_c(\vec{r}_1)\psi_c^*(\vec{r}_2) + <\psi_1(\vec{r}_1)\psi_1^*(\vec{r}_2)> \quad . \tag{II-31}$$

In deriving Eqn. (II-31), Eqn. (II-25) has been used. The definition of $\psi_1(\vec{r})$ is Eqn. (II-24). So,

$$<\psi_1(\vec{r}_1)\psi_1^*(\vec{r}_2)> =$$

$$\int\int d\vec{r}_1{}' d\vec{r}_2{}' \; G_c(\vec{r}_1,\vec{r}_1{}')G_c^*(\vec{r}_2,\vec{r}_2{}')<S(\vec{r}_1{}')S^*(\vec{r}_2{}')\psi(\vec{r}_1{}')\psi^*(\vec{r}_2{}')> \quad , \tag{II-32}$$

where

$$S(\vec{r}) \equiv k_c^2(\vec{r}) - k^2(\vec{r}) \quad . \tag{II-33}$$

By the same argument which was used in Section 2 of this chapter to get Eqn. (II-13), we have

$$<S(\vec{r_1}')S^*(\vec{r_2}')\psi(\vec{r_1}')\psi^*(\vec{r_2}')> \approx <S(\vec{r_1}')S^*(\vec{r_2}')><\psi(\vec{r_1}')\psi^*(\vec{r_2}')> \quad . \qquad \text{(II-34)}$$

Substituting Eqn. (II-34) into Eqn. (II-32) and using the definition (II-26), we find the integral equation for the mutual coherence function,

$$\Gamma(\vec{r_1},\vec{r_2}) = \psi_c(\vec{r_1})\psi_c^*(\vec{r_2})$$

$$+ \int\int d\vec{r_1}'d\vec{r_2}'G_c(\vec{r_1},\vec{r_1}')G_c^*(\vec{r_2},\vec{r_2}')<S(\vec{r_1}')S^*(\vec{r_2}')>\Gamma(\vec{r_1}',\vec{r_2}') \quad . \qquad \text{(II-35)}$$

Although Eqn. (II-35) looks simple, it contains a multi-dimensional integral term which is very difficult to handle. Only in some simple cases Eqn (II-35). can be used to get interesting results.

Now we should calculate $<S(\vec{r_1})S^*(\vec{r_2})>$, starting from the definition of $S(\vec{r})$, Eqn. (II-33),

$$<S(\vec{r_1})S^*(\vec{r_2})> = <[k_c^2(\vec{r_1}) - k^2(\vec{r_1})][k_c^2(\vec{r_2}) - k^2(\vec{r_2})]^*>$$

$$= k_c^2(\vec{r_1})\Big[k_c^2(\vec{r_2})\Big]^* - k_c^2(\vec{r_1})<k^2(\vec{r_2})>^* - <k^2(\vec{r_1})>\Big[k_c(\vec{r_2})\Big]^*$$

$$+ <k^2(\vec{r_1})\Big[k^2(\vec{r_2})\Big]^*> \quad . \qquad \text{(II-36)}$$

The definition of $k_c^2(\vec{r})$ is

$$k_c^2(\vec{r}) = <k^2(\vec{r})>\Big[1 + \delta(\vec{r})\Big] \quad .$$

Substituting this definition in Eqn. (II-36) we find

$$<S(\vec{r_1})S^*(\vec{r_2})> =$$

$$<k^*(\vec{r_1})\left[k^2(\vec{r_2})\right]^*> - <k^2(\vec{r_1})><k^2(\vec{r_1})>^*[1 - \delta(\vec{r_1})\delta^*(\vec{r_2})]$$

$$\approx <k^2(\vec{r_1})\left[k^2(\vec{r_2})\right]^*> - <k^2(\vec{r_1})><k^2(\vec{r_2})>^*$$

$$= <k_1^2(\vec{r_1})\left[k_1^2(\vec{r_2})\right]^*> \quad , \qquad \text{(II-37)}$$

where

$$k_1^2(\vec{r}) \equiv k^2(\vec{r}) - <k^2(\vec{r})> \qquad \text{(II-38)}$$

is the fluctuation of $k^2(\vec{r})$. Now the integral equation for the mutual coherence function is

$$\Gamma(\vec{r_1},\vec{r_2}) = \psi_c(\vec{r_1})\psi_c^*(\vec{r_2}) +$$

$$+ \int\int d\vec{r_1}' d\vec{r_2}' \; G_c(\vec{r_1},\vec{r_1}') \; G_c^*(\vec{r_2},\vec{r_2}') <k_1^2(\vec{r_1}')\left[k_1^2(\vec{r_2}')\right]^*> \Gamma(\vec{r_1}',\vec{r_2}') \quad . \qquad \text{(II-39)}$$

This equation can be explained as follows. The mutual coherence of the wave has two sources, the coherent wave and the incoherent wave. The two sources are independent. Obviously, $\psi_c(\vec{r_1})\psi_c^*(\vec{r_2})$ is the correlation inside the coherent wave. The second term on the R.H.S. of Eqn. (II-39) is the correlation inside the incoherent wave. It sums all contributions from the mutual coherence function at $(\vec{r_1}',\vec{r_2}')$, which is scattered coherently by $<k_1^2(\vec{r_1}')\left[k_1^2(\vec{r_2}')\right]^*>$, then propagates coherently through $G_c(\vec{r_1},\vec{r_1}') \, G_c^*(\vec{r_2},\vec{r_2}')$ from $(\vec{r_1}',\vec{r_2}')$ to $(\vec{r_1},\vec{r_2})$.

Equation (II-39) is most important in our formalism. We will discuss its applications in the next chapter.

III. Applications of the Formalism

1. Derivation of the Peierls Equation

For a long time the theory of radiative transfer had been used in various applications, while there had been no sound basis for the theory. The radiative transfer equation had been constructed from a self-evident picture in which the radiation field was seen as composed of a set of independent particles moving with the speed of light. The particles, photons, can be scattered and absorbed in the medium. This picture completely ignored the wave and statistical properties of light and the statistical properties of the media. The situation has changed during the past 20 years. Many attempts have been made to derive the radiative transfer equation from first principles, i.e., from Maxwell equations for the electromagnetic field in a random medium. In almost all these attempts, the first step is to give a more rigorous definition to the usual photometric concepts such as specific intensity of radiation (or radiance), mean energy flux, or mean-energy density, by connecting them to the mutual coherence function and its Fourier transformation. After that, different workers used different mathematical forms to derive radiative transfer equations, which include differential equations, integro-differential equations, and integral equation methods. Although the relationship between the theory of radiative transfer and the random wave theory in the free space case is quite clear now, the status of the same problem in arbitrary inhomogeneous random media is not so satisfactory. In what follows we use the integral equation for the mutual coherence function to derive the integral equation of radiative transfer in a homogeneous medium with isotropic scatters, the Peierls equation.

In this problem we are interested in the intensity of the wave field,

$$J(\vec{r}) \equiv \langle|\psi(\vec{r})|^2\rangle = \langle\psi(\vec{r})\psi^*(\vec{r})\rangle = \Gamma(\vec{r},\vec{r}) \quad . \tag{III-1}$$

Setting two arguments in Eqn. (II-39) equal to each other, we have

$$\Gamma(\vec{r},\vec{r}) = |\psi_c(\vec{r})|^2$$

$$+ \int\int d\vec{r}_1{}' d\vec{r}_2{}' \; G_c(\vec{r},\vec{r}_1{}') \; G_c^*(\vec{r},\vec{r}_2{}') \langle k_1^2(\vec{r}_1{}') \left[k_1^2(\vec{r}_2{}')\right]^* \rangle \; \Gamma(\vec{r}_1{}',\vec{r}_2{}') \tag{III-2}$$

We assume that the correlation length of the medium is much smaller than that of the wave field. Then the autocorrelation of $k_1^2(\vec{r})$ in Eqn. (III-2), $\langle k_1^2(\vec{r}_1)\left[k_1^2(\vec{r}_2)\right]^*\rangle$, can be replaced by $C\,\delta(\vec{r}_1 - \vec{r}_2)$, where the constant C can be calculated from the following equation,

$$\int\int d\vec{r}_1 d\vec{r}_2 \langle k_1^2(\vec{r}_1)\left[k_1^2(\vec{r}_2)\right]^*\rangle = \int\int d\vec{r}_1 d\vec{r}_2 \; C\,\delta(\vec{r}_1 - \vec{r}_2)$$

$$= \int d\vec{R} \cdot C \quad , \tag{III-3}$$

where

$$\vec{R} \equiv (\vec{r}_1 + \vec{r}_2)/2 \quad .$$

So Eqn. (III-2) becomes

$$\langle|\psi(\vec{r})|^2\rangle = |\psi_c(\vec{r})|^2 + \int d\vec{r}' \; |G_c(\vec{r},\vec{r}')|^2 \; C \langle|\psi(\vec{r}')|^2\rangle \quad . \tag{III-4}$$

Suppose the medium fills half-space $x > 0$ and an incident plane wave $e^{ik_0 x}$ comes from $x = -\infty$. By solving Eqn. (II-16) we find the coherent wave inside the medium,

$$\psi_c(\vec{r}) = A e^{ik_c x} \quad , \tag{III-5}$$

where

$$k_c^2 = \langle k^2(\vec{r})\rangle(1 + \delta) \equiv \left[k_r + ik_i\right]^2 \quad . \tag{III-6}$$

Because $|\delta| \ll 1$,

$$k_r \approx \left[\langle k^2(\vec{r})\rangle\right]^{1/2} \quad , \qquad k_i = \frac{1}{2}k_r \mathrm{Im}(\delta) \quad . \tag{III-7}$$

The imaginary part of δ, $\mathrm{Im}(\delta)$, must be positive so that we have a decaying coherent wave. During propagation the coherent wave becomes incoherent due to random scattering,

$$\psi_c(\vec{r}) = A e^{ik_r x} e^{-k_i x} \quad . \tag{III-8}$$

The Green's function can be chosen as the solution of Eqn. (II-23) with radiation condition,

$$G_c(\vec{r},\vec{r}\,') = \frac{1}{4\pi}\frac{e^{ik_c|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} = \frac{1}{4\pi}\frac{e^{ik_r|\vec{r}-\vec{r}\,'|}e^{-k_i|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \quad . \tag{III-9}$$

Substituting Eqn. (III-8) and (III-9) in Eqn. (III-4), we have

$$\langle|\psi(\vec{r_1})|^2\rangle =$$

$$= A^2 e^{-2k_i x} + \int d\vec{r_2}\, \frac{C}{16\pi^2}\frac{e^{-2k_i r_{12}}}{r_{12}^2}\langle|\psi(\vec{r_2})|^2\rangle \quad , \tag{III-10}$$

where

$$r_{12} \equiv |\vec{r_1} - \vec{r_2}| \quad .$$

The radiative transfer equation for homogeneous medium with isotropic scatters is (Peierls, 1939)

$$J(\vec{r}_1) = J_o(\vec{r}_1) + \frac{\alpha\omega}{4\pi}\int \frac{e^{-\alpha r_{12}}}{r_{12}^2}\, d\vec{r}_2 \quad . \tag{III-11}$$

Apparently, $A^2 e^{-2k_i x}$ in Eqn. (III-10) is equivalent to the source term $J_o(r_{\vec{1}})$ in Eqn. (III-11). If we define

$$2k_i \equiv \alpha \quad ,$$

then Eqn. (III-10) is exactly the same as Eqn. (III-11) with

$$\omega = \frac{C}{4\pi}\frac{1}{\alpha} \quad . \tag{III-12}$$

For the pure scattering medium, $\omega = 1$ to conserve the energy of the wave field. This gives

$$k_i = \frac{C}{8\pi} = \frac{1}{8\pi}\int d\vec{\rho} < k_1^2\left(\vec{R} + \frac{\vec{\rho}}{2}\right)\left[k_1^2\left(\vec{R} - \frac{\vec{\rho}}{2}\right)\right]^* > \quad . \tag{III-13}$$

This equation relates the decay rate of the coherent wave to the statistical properties of the medium.

2. A Simple Example

In this section we calculate the mutual coherent function in a very simple case using the formalism developed in the last chapter.

Suppose a laser beam is projected into a homogeneous sphere, which can be served as the model of the atmosphere. The mutual coherence function of the scattering wave is measured outside of the beam trajectory. All the geometry is shown schematically in Figure 1.

Also we assume the correlation length of the medium is very small just as in the last section. Then from Eqn. (II-35), we have

$$\Gamma(\vec{r}_1,\vec{r}_2) = \psi_c(\vec{r}_1)\psi_c^*(\vec{r}_2) + \int d\vec{r}_1{}' \, G_c(\vec{r}_1,\vec{r}_1{}') \, G_c^*(\vec{r}_2,\vec{r}_1{}') \, C \, \Gamma(\vec{r}_1{}',\vec{r}_1{}') \quad . \qquad \text{(III-14)}$$

Suppose the geometric optics is a good approximation for coherent waves in this problem. Then we have

$$\psi_c(\vec{r}_l) = e^{ik_c l} \quad , \quad \textit{on the beam trajectory},$$

where

$$l \equiv |\vec{AE}| \quad ,$$

and

$$\psi_c(\vec{r}) = 0 \quad , \quad \textit{otherwise}. \qquad \text{(III-15)}$$

We assume that the intensity of the incident beam outside the sphere is one. The trajectory is a straight line with cross section S. Here we have neglected the dispersion effect and the reflection effect, both of which can be included into Eqn. (III-15) without difficulties.

The mutual coherence function outside the beam trajectory is

$$\Gamma(\vec{r}_1,\vec{r}_2) = \int d\vec{r}' \, G_c(\vec{r}_1,\vec{r}_1{}') G_c^*(\vec{r}_2,\vec{r}_1{}') C \, \Gamma(\vec{r}_1{}',\vec{r}_1{}') \quad . \qquad \text{(III-16)}$$

As a first approximation, we use

$$\Gamma(\vec{r}_1{}',\vec{r}_1{}') = |\psi_c(\vec{r}_1{}')|^2 \quad . \tag{III-17}$$

Then

$$\Gamma(\vec{r}_1,\vec{r}_2) = \int_0^L dl\, G_c(\vec{r}_1,\vec{r}_l)\, G_c^*(\vec{r}_2,\vec{r}_l)\, C\, S\, e^{-2k_i l} \quad , \tag{III-18}$$

where

$$L \equiv |\vec{AB}| .$$

Let

$$\vec{r}_{12} \equiv \vec{r}_1 - \vec{r}_2 \quad , \tag{III-19}$$

and let O is the center of $\vec{FG}$. For the simplicity of calculation, we assume $\vec{r}_{12}$ is in OAB plane and $|\vec{r}_{12}| \ll r_1$ and r_2. Suppose that the angle between $\vec{r}_{12}$ and $\vec{OD}$ is α. Using the condition

$$|\vec{OE}| \gg |\vec{r}_{12}| \quad , \tag{III-20}$$

we have

$$G_c(\vec{r}_1,\vec{r}_l)\, G_c^*(\vec{r}_2,\vec{r}_l) \approx \frac{e^{-k_i R}}{R} e^{ik_r\left|\vec{r}_1-\vec{r}_l\right|} \frac{e^{-k_i R}}{R} e^{-ik_r\left|\vec{r}_2-\vec{r}_l\right|}$$

$$\approx \frac{e^{-2k_i R}}{R^2} e^{-ik_r \sin\vartheta\, r_{12}} \quad , \tag{III-21}$$

where

$$R \equiv |\vec{OE}| \quad ,$$

$$\vartheta \equiv \alpha \pm \beta \quad ,$$

$$\beta = tg^{-1}\frac{|\vec{DE}|}{|\vec{OD}|} \quad .$$

Here β is the angle between $\vec{OE}$ and $\vec{OD}$. Substituting Eqn. (III-2) in Eqn. (III-18), we have

$$\Gamma(\vec{r}_1\vec{r}_2) = \int_0^L dl \;\; C\, S\, e^{-2k_i l}\, \frac{e^{-2k_i R}}{R^2} e^{-ik_r \sin\vartheta\, r_{12}} \quad , \qquad \text{(III-22)}$$

$$L = |\vec{AB}| \quad ,$$

and

$$\vartheta = \alpha + \beta \quad , \quad when \quad l > |\vec{AD}| \quad ,$$

$$\vartheta = \alpha - \beta \quad , \quad when \quad l < |\vec{AD}| \quad .$$

From Eqn. (III-22), $\Gamma(\vec{r}_1,\vec{r}_2)$ is not difficult to calculate.

If the sphere is not homogeneous, the light beam is no longer a straight line. We need to solve the differential equation for the coherent wave to find the trajectory and the Green's function first. Then we can calculate the mutual coherence function as above.

3. An Improved Version of the Peierls Equation

When we derived Peierls equation in Section 1, using the integral equation for the mutual coherence function, only the first term of the expansion of the Green's function and of the mutual coherence function in R.H.S. of Eqn. (II-

39) is kept. Now we will keep the second term of the expansion to improve the Peierls equation.

Define

$$\vec{R} \equiv (\vec{r}_1{}' + \vec{r}_2{}') \,/\, 2 \quad ,$$

$$\vec{\rho} \equiv \vec{r}_1{}' - \vec{r}_2{}' \quad ,$$

$$\vec{R}_1 \equiv \vec{R} - \vec{r}_1 \quad ,$$

$$\hat{R}_1 \equiv \vec{R}_1 \,/\, R_1 \quad . \qquad \text{(III-23)}$$

Then

$$G_c(\vec{r}_1,\vec{r}_1{}') \approx \frac{e^{-k_i R_1}}{4\pi R_1} e^{i(k_r R_1 + \frac{1}{2} k_r \vec{\rho}\cdot\hat{R}_1)} \quad ,$$

$$G_c^*(\vec{r}_1,\vec{r}_2{}') \approx \frac{e^{-k_i R_1}}{4\pi R_1} e^{-i(k_r R_1 - \frac{1}{2} k_r \vec{\rho}\cdot\hat{R}_1)} \quad ,$$

$$\Gamma(\vec{r}_1{}',\vec{r}_2{}') \approx \Gamma(\vec{R},\vec{R}) + \frac{\partial\Gamma}{\partial\vec{\rho}}\Big|_{\vec{\rho}=0} \cdot \vec{\rho} \quad , \qquad \text{(III-24)}$$

Because the medium is homogeneous and isotropic, we can write

$$<k_1^2(\vec{r}_1{}')\left[k_1^2(\vec{r}_2{}')\right]^* > = f(\rho) \quad . \qquad \text{(III-25)}$$

Using Eqn. (III-23), (III-24), and (III-25) we change Eqn. (III-2) to

$$\Gamma(\vec{r}_1,\vec{r}_1) = \left|\psi_c(\vec{r}_1)\right|^2 +$$

$$+ \int\int d\vec{R}d\vec{\rho}\frac{1}{16\pi^2}\frac{e^{-2k_iR_1}}{R_1^2}f(\rho)\left[\Gamma(\vec{R},\vec{R}) + \left(\frac{\partial\Gamma(\vec{R}+\frac{\vec{\rho}}{2},\vec{R}-\frac{\vec{\rho}}{2})}{\partial\vec{\rho}}\right)_{\vec{\rho}=0}\cdot\vec{\rho}\right] \quad ,\text{(III-26)}$$

where the relation,

$$\int f(\rho)\vec{R_1}\cdot\vec{\rho}\, d\vec{\rho} = 0 \quad , \qquad \text{(III-27)}$$

has been used. Simplify the notation by

$$\tilde{\Gamma}(\vec{R},\vec{\rho}) \equiv \Gamma\left(\vec{R}+\frac{\vec{\rho}}{2}, \vec{R}-\frac{\vec{\rho}}{2}\right) \quad ,$$

and

$$\frac{\partial\tilde{\Gamma}}{\partial\vec{\rho}}(\vec{R}) \equiv \left(\frac{\partial\Gamma(\vec{R}+\frac{\vec{\rho}}{2},\vec{R}-\frac{\vec{\rho}}{2})}{\partial\vec{\rho}}\right)_{\vec{\rho}=0} \quad . \qquad \text{(III-28)}$$

Now Eqn. (III-26) is

$$\Gamma(\vec{r_1},\vec{r_1}) = |\psi_c(\vec{r_1})|^2 + \int d\vec{R}\frac{C}{16\pi^2}\frac{e^{-2k_iR_1}}{R_1^2}\Gamma(\vec{R},\vec{R})$$

$$+ \int d\vec{R}\int d\vec{\rho}\frac{1}{16\pi^2}\frac{e^{-2k_iR_1}}{R_1^2}f(\rho)\left(\frac{\partial\tilde{\Gamma}(\vec{R})}{\partial\vec{\rho}}\cdot\vec{\rho}\right) \quad . \qquad \text{(III-29)}$$

So we need another equation for $\frac{\partial\tilde{\Gamma}(\vec{R})}{\partial\vec{\rho}}$. Starting from Eqn. (III-39), using Eqn. (III-3), we find

$$\Gamma(\vec{r_1},\vec{r_2}) = \psi_c(\vec{r_1})\psi_c^*(\vec{r_2}) + \int d\vec{r_1}'\, G_c(\vec{r_1},\vec{r_1}')G_c^*(\vec{r_2},\vec{r_1}')C\,\Gamma(\vec{r_1}',\vec{r_1}') \quad . \qquad \text{(III-30)}$$

We change the variables in Eqn. (III-30) and differentiate it,

$$\frac{\partial \tilde{\Gamma}(\vec{R})}{\partial \vec{\rho}} = \frac{\partial \tilde{\psi}_c^2(\vec{R})}{\partial \vec{\rho}} + \int d\vec{r}_1 \frac{\partial \tilde{G}_c^2(\vec{R},\vec{r}_1)}{\partial \vec{\rho}} C\,\Gamma(\vec{r}_1,\vec{r}_1) \quad . \qquad \text{(III-31)}$$

We define

$$\vec{R} \equiv (\vec{r}_1 + \vec{r}_2)/2 \quad ,$$

$$\vec{\rho} \equiv \vec{r}_1 - \vec{r}_2 \quad ,$$

$$\frac{\partial \tilde{\psi}_c^2(\vec{R})}{\partial \vec{\rho}} \equiv \left[\frac{\partial}{\partial \vec{\rho}}\left\{\psi_c\left[\vec{R} + \frac{\vec{\rho}}{2}\right]\psi_c^*\left[\vec{R} - \frac{\vec{\rho}}{2}\right]\right\}\right]_{\vec{\rho}=0}$$

$$\frac{\partial \tilde{G}_c^2(\vec{R},\vec{r}_1)}{\partial \vec{\rho}} \equiv \left[\frac{\partial}{\partial \vec{\rho}}\left\{G_c\left[\vec{R} + \frac{\vec{\rho}}{2}, \vec{r}_1\right] G_c^*\left[\vec{R} - \frac{\vec{\rho}}{2}, \vec{r}_1\right]\right\}\right]_{\vec{\rho}=0} \quad . \qquad \text{(III-32)}$$

Equations (III-29) and (III-32) need to be solved simultaneously to find $\Gamma(\vec{r},\vec{r})$ and $\frac{\partial \tilde{\Gamma}(\vec{R})}{\partial \vec{\rho}}$. The vector function $\frac{\partial \tilde{\Gamma}(\vec{R})}{\partial \vec{\rho}}$ itself is a useful physical quantity.

IV. Mutual Coherence Function, Density Matrix and the Entropy of the Random Wave Field

The mutual coherence function is quite similar to the density matrix in quantum mechanics (Gamo, 1964). If we can use a complete set of orthonormal functions to expand the wave function, this similarity is even more apparent. Let $\{\varphi_n(\vec{r})\}$ be this kind of set. Because our functional space is complex, $\{\varphi_n^*(\vec{r})\}$ must be a complete set, too. The mutual coherence function $\Gamma(\vec{r}_1,\vec{r}_2)$ has two variables, $\vec{r}_1$ and $\vec{r}_2$. First, we expand $\Gamma(\vec{r}_1,\vec{r}_2)$ by $\{\varphi_n(\vec{r}_1)\}$, keeping $\vec{r}_2$ as a parameter,

$$\Gamma(\vec{r}_1,\vec{r}_2) = \sum_n F_n(\vec{r}_2)\varphi_n(\vec{r}_1) \quad . \tag{IV-1}$$

If the orthonormality relation is

$$\int_v \varphi_m^*(\vec{r})\varphi_n(\vec{r})\,d\vec{r} = \delta_{mn} \quad , \tag{IV-2}$$

then

$$F_n(\vec{r}_2) = \int_v \varphi_n^*(\vec{r}_1)\Gamma(\vec{r}_1,\vec{r}_2)\,d\vec{r}_1 \quad . \tag{IV-3}$$

Now we can expand $F_n(\vec{r}_2)$ in $\{\varphi_n^*(\vec{r}_2)\}$. So

$$F_n(\vec{r}_2) = \sum_m f_{nm}\varphi_n^*(\vec{r}_2) \quad , \tag{IV-4}$$

where

$$f_{nm} = \int_v \varphi_m(\vec{r}_2) F_n(\vec{r}_2)\, d\vec{r}_2 \quad . \tag{IV-5}$$

Putting Eqn. (IV-1) and (IV-4) together, we find

$$\Gamma(\vec{r}_1,\vec{r}_2) = \sum_{nm} f_{nm}\varphi_n(\vec{r}_1)\varphi_m^*(\vec{r}_2) \tag{IV-6}$$

and

$$f_{nm} = \int_v \int_v \Gamma(\vec{r}_1,\vec{r}_2)\varphi_n^*(\vec{r}_1)\varphi_m(\vec{r}_2)\, d\vec{r}_1\, d\vec{r}_2 \quad . \tag{IV-7}$$

By definition,

$$\Gamma(\vec{r}_1\vec{r}_2) = \Gamma^*(\vec{r}_2,\vec{r}_1) \quad . \tag{IV-8}$$

It is easy to prove that the matrix $F = (f_{mn})$ is Hermitian, because

$$f_{mn}^* = \left[\int_v \int_v \Gamma(\vec{r}_2\vec{r}_1)\varphi_m^*(\vec{r}_2)\varphi_n(\vec{r}_1)\, d\vec{r}_1\, d\vec{r}_2\right]^*$$

$$= \int_v \int_v \Gamma^*(\vec{r}_2,\vec{r}_1)\varphi_n^*(\vec{r}_1)\varphi_m(\vec{r}_2)\, d\vec{r}_1\, d\vec{r}_2$$

$$= \int_v \int_v \Gamma(\vec{r}_1,\vec{r}_2)\varphi_n^*(\vec{r}_1)\varphi_m(\vec{r}_2)\, d\vec{r}_1\, d\vec{r}_2 = f_{nm} \quad . \tag{IV-9}$$

So we can diagonalize the matrix F by a unitary transformation

$$P = UFU^{-1} \quad , \tag{IV-10}$$

where U is a unitary matrix,

$$U^{-1} = U^+ \quad , \tag{IV-11}$$

and P is a diagonal matrix. Now the mutual coherence function can be written as

$$\Gamma(\vec{r}_1,\vec{r}_2) = \sum_m p_m \Phi_m(\vec{r}_1)\Phi_m^*(\vec{r}_2) \quad , \tag{IV-12}$$

where $\{p_m\}$ are the diagonal elements of matrix P, and

$$\Phi_m(\vec{r}) \equiv \sum u_{mn}^* \varphi_n(\vec{r}) \quad ,$$

and $\{u_{mn}\}$ are the elements of matrix U. In quantum mechanics, the density matrix is

$$\hat{\rho} = \sum_i q_i |i\rangle\langle i| \quad , \tag{IV-13}$$

which describes a statistical state (Schiff, 1968). The normalization condition of the density matrix is

$$\sum_i q_i = 1 \quad . \tag{IV-14}$$

Compare Eqn. (IV-12) and (IV-13). The similarity between the mutual coherence function and the density matrix is quite obvious except for the normalization condition.

Also, we know that if

$$q_i = \begin{cases} 1 & \text{for } i = s \\ 0 & others \end{cases} \quad , \tag{IV-15}$$

then the density matrix corresponds to a pure state.

Similarly, if

$$p_m = \begin{cases} p & \text{for } m = i \\ 0 & others \end{cases} \quad , \tag{IV-16}$$

we have a totally coherent wave field,

$$\Gamma(\vec{r}_1,\vec{r}_2) = \sum_i p_i \Phi_i(\vec{r}_1)\Phi_i^*(\vec{r}_2) \quad . \tag{IV-17}$$

In this case, the complex degree of coherence (see Born and Wolf, 1980, for the definition) is

$$\gamma(\vec{r}_1,\vec{r}_2) \equiv \frac{\Gamma(\vec{r}_1,\vec{r}_2)}{\left[\Gamma(\vec{r}_1,\vec{r}_1)\Gamma(\vec{r}_2,\vec{r}_2)\right]^{\frac{1}{2}}} = \frac{p\,\Phi_i(\vec{r}_1)\,\Phi_i^*(\vec{r}_2)}{\left[p\left|\Phi_i(\vec{r}_1)\right|^2 p\left|\Phi_i(\vec{r}_2)\right|^2\right]^{\frac{1}{2}}} \quad , \tag{IV-18}$$

and

$$\left|\gamma(\vec{r}_1,\vec{r}_2)\right| = 1 \quad . \tag{IV-19}$$

It means the wave field is totally coherent and the wave function is

$$\psi(\vec{r}) = p^{\frac{1}{2}}\Phi_i(\vec{r}) \quad . \tag{IV-20}$$

Because of Eqn. (IV-7), the matrix F is semi positive as defined by

$$A^+FA \geq 0 \quad . \tag{IV-21}$$

where $A = \{a_n\}$ is an arbitrary vector. The eigenvalues of a semi positive definite Hermitian matrix are greater or equal to zero,

$$p_m \geq 0 \quad . \tag{IV-22}$$

The trace of the matrix F is proportional to the total energy of the field,

$$E \propto \int d\vec{r}\,\Gamma(\vec{r},\vec{r}) = \sum_{mn}\int d\vec{r}\, f_{mn}\varphi_m(\vec{r})\varphi_n^*(\vec{r}) = \sum_m f_{mm} = \sum_m p_m \quad . \tag{IV-23}$$

Usually, if we want to compare two wave fields to see which one distributes the total energy between all possible modes more randomly, these two fields should have the same total energy. That means we will treat different wave fields with the same trace of the matrix F. So we can normalize the eigenvalues of the matrix F by defining

$$p_i' = p_i / \sum_i p_i \quad .$$

Then we can define a measure of randomness for the wave fields, which have the same total energy,

$$W \equiv \sum_i p_i' \log p_i' \quad . \tag{IV-22}$$

For the totally coherent wave, i.e., $p_i' = 1$ for $i = 1$, and $p_i' = 0$ for others, then

$$W = 0 \quad .$$

If the energy is evenly distributed between all modes, i.e., all p_i are equal, W attains maximum.

V. Conclusions

Our original motivation was to find a method to calculate the light reflection from natural surfaces without using brute force. Because usually natural materials have similar structures of different scales, the wave fields inside the materials are partially coherent. Although we have not yet reached this goal, we think that our new formalism is the correct approach to solve this problem. By separating the coherent scattering wave from the incoherent scattering wave and putting it into a coherent wave, the wave that is left is random and is more easily treated.

Still there are many places for improvement, for example, how to find a better expression of $\delta(\vec{r})$ in Eqn. (II-13) for the general case. Also, we need a better numerical codes to solve the three-dimensional integral equations. We will use this formalism for more realistic problems in remote sensing. And we hope that better understanding of entropy in partially coherent wave field will be found in the future.

APPENDIX

Calculations of Wave Scattering

In this appendix we show that in the problem of one-dimensional wave scattering by a layer of dielectric, both the vacuum background and the arbitrary background can reach the same solution. In the latter case the iteration method has been used to find the solution.

1. One-Dimensional Wave Scattering

The one dimentional wave equation is

$$\left[\frac{d^2}{dx^2} + k^2(x)\right]\psi(x) = 0 \quad , \tag{1}$$

where

$$k^2(x) = \begin{cases} m^2 k_o^2 & , \; at \; 0 \le x \le a \\ k_o^2 & , \; elsewhere \end{cases} \quad , \tag{2}$$

which is schematically shown in Figure 2 on page 56. Suppose the solution is

$$\psi(x) = \begin{cases} \psi_I = e^{ik_o x} + B_1 e^{-ik_o x} & in \; I \\ \psi_{II} = A_2 e^{ik_o m x} + B_2 e^{-ik_o m x} & in \; II \\ \psi_{III} = A_3 e^{ik_o x} & in \; III \end{cases} \quad . \tag{3}$$

It must satisfy the following boundary conditions,

$$\psi_I(x=0) = \psi_{II}(x=0) \implies 1 + B_1 = A_2 + B_2 \quad ,$$

$$\frac{d}{dx}\psi_I(x=0) = \frac{d}{dx}\psi_{II}(x=0) \implies (1 - B_1) = m(A_2 - B_2) \quad ,$$

$$\psi_{II}(x=a) = \psi_{III}(x=a) \implies A_2 e^{i\alpha} + B_2 e^{-i\alpha} = A_3 e^{i\gamma} \quad ,$$

$$\frac{d}{dx}\psi_{II}(x=a) = \frac{d}{dx}\psi_{III}(x=a) \implies m(A_2 e^{i\alpha} - B_2 e^{-i\alpha}) = A_3 e^{i\gamma} \quad , \tag{4}$$

where

$$\alpha \equiv m k_o a \quad ,$$

$$\gamma \equiv k_o a \quad .$$

The solution of Eqn. (4) is

$$A_2 = \frac{-2}{D}(m+1)e^{-i\alpha} \quad ,$$

$$B_2 = \frac{2}{D}(1-m)e^{i\alpha} \quad ,$$

$$B_1 = \frac{1}{D}(1-m^2)(e^{i\alpha} - e^{-i\alpha}) \quad ,$$

$$A_3 = \frac{-4}{D} m e^{-i\gamma} \quad , \tag{5}$$

where

$$D \equiv (1-m)^2 e^{i\alpha} - (1+m)^2 e^{-i\alpha} \quad . \tag{6}$$

2. Background Separation Method

In region II the wave equation (1) is

$$\left[\frac{d^2}{dx^2} + m^2k_o^2\right]\psi_{II}(x) = 0 \quad , \tag{7}$$

or, by separating a background nk_o,

$$\left[\frac{d^2}{dx^2} + n^2k_o^2\right]\psi_{II}(x) = (n^2 - m^2)k_o^2\psi_{II}(x) \quad , \tag{8}$$

where n is a arbitrary constant. Therefore,

$$\psi_{II}(x) = \Phi_o(x) + \int_0^a G^{(0)}(x,x')(n^2 - m^2)k_o^2\psi_{II}(x')\, dx'$$

$$\equiv \Phi_o(x) + \Phi(x) \quad , \tag{9}$$

where $\Phi_o(x)$ is the solution of the equation

$$\left[\frac{d^2}{dx^2} + n^2k_o^2\right]\Phi_o(x) = 0 \quad , \tag{10}$$

and

$$G^{(0)}(x,x') \equiv \frac{1}{2ink_o}e^{ink_o|x-x'|} \tag{11}$$

is the Green's function of equation (10). Therefore,

$$\Phi_o(x) = C_1e^{ink_ox} + C_2e^{-ink_ox} \quad , \tag{12}$$

where C_1 and C_2 are constants which can be fixed by boundary conditions at x = 0 and $x = a$.

Suppose we use the iterative method to solve this integral equation. First, we choose a test solution $\psi_{II}^{(1)}$. Define

$$\Phi^{(i+1)} \equiv \int G^{(0)}(x,x')(n^2 - m^2)k_o^2 \psi_{II}^{(i)}, \quad i = 1, 2, \cdots \quad , \tag{13}$$

and

$$\psi_{II}^{(i+1)}(x) = \Phi_o^{(i+1)}(x) + \Phi^{(i+1)}(x), \quad i = 1, 2, \cdots \quad . \tag{14}$$

The solution of Eqn. (9) is

$$\psi_{II}(x) = \lim_{i \to \infty} \psi_{II}^{(i+1)}(x) \quad , \tag{15}$$

where

$$\Phi_o^{(i+1)}(x) = C_1^{(i+1)} e^{ink_o x} + C_2^{(i+1)} e^{-ink_o x} \quad , \tag{16}$$

and $C_1^{(i+1)}$ and $C_2^{(i+1)}$ are determined by the requirement that

$$\psi_{II}^{(i+1)}(x) = C_1^{(i+1)} e^{ink_o x} + C_2^{(i+1)} e^{ink_o x} + \Phi^{(i+1)}(x) \tag{17}$$

satisfies the boundary conditions:

$$\begin{aligned} \psi_{II}^{(i+1)}(0) = \psi_I(0) \quad &, \quad \frac{d}{dx}\left[\psi_{II}^{(i+1)}(0)\right] = \frac{d}{dx}\left[\psi_I(0)\right] \quad , \\ \psi_{II}^{(i+1)}(a) = \psi_{III}(a) \quad &, \quad \frac{d}{dx}\left[\psi_{II}^{(i+1)}(a)\right] = \frac{d}{dx}\left[\psi_{III}(a)\right] \quad . \end{aligned} \tag{18}$$

So the iteration scheme is

$$\psi_{II}^{(1)} => \Phi^{(2)} => \begin{bmatrix} C_1^{(2)} & , & C_2^{(2)} \\ A_3^{(2)} & , & B_1^{(2)} \end{bmatrix} => \psi_{II}^{(2)} => \Phi^{(3)} \ldots.$$

We will show below that if we choose

$$\psi_{II}^{(1)} = A_2 e^{imk_o x} + B_2 e^{imk_o x} \tag{19}$$

with A_2 and B_2 having the values in Eqn. (5), then

$$\psi_{II}^{(2)} = A_2^{(2)} e^{imk_o x} + B_2^{(2)} e^{imk_o x}$$

and $A_2^{(2)}$, $B_2^{(2)}$ are exactly the same as in Eqn. (5). This gives a prove of the equivalence between the normal method and the background separation method.

From Eqn. (13), we have

$$\Phi_{(x)}^{(2)} = \int_0^a G^{(0)}(x,x')(n^2 - m^2)k_o^2 \psi_{II}^{(1)}(x')\, dx'$$

$$\Phi_{(0)}^{(2)} = (n^2 - m^2)k_o^2 \int_0^a \frac{1}{2ink_o} e^{ink_o(x'-x)} \psi_{II}^{(1)}(x')\, dx'$$

$$= \frac{(n^2 - m^2)k_o}{2in} \int_0^a e^{ink_o x'} \left[A_2 e^{imk_o x'} + B_2 e^{-imk_o x'}\, dx'\right]$$

$$= \frac{(n^2 - m^2)k_o}{2in} \left[A_2 \frac{e^{i(m+n)k_o a} - 1}{i(m+n)k_o} + B_2 \frac{e^{-i(m-n)k_o a} - 1}{-i(m-n)k_o}\right]$$

$$= \frac{(n^2 - m^2)}{2n} \left[A_2 \frac{1 - e^{i\alpha + i\beta}}{(m+n)} + B_2 \frac{e^{-i\alpha + i\beta} - 1}{(m-n)}\right] , \qquad (20)$$

where

$$\alpha \equiv mk_a \quad ,$$

$$\beta \equiv nk_o a \quad ,$$

and

$$\Phi^{(2)}(a) = (n^2 - m^2)k_o^2 \int_0^a \frac{1}{2ink_o} e^{ink(a-x')} \psi_{II}^{(1)}(x')\, dx'$$

$$= \frac{(n^2 - m^2)}{2in} k_o \int_0^a e^{i\beta} e^{-ink_o x'} \left[A_2 e^{imk_o x'} + B_2 e^{-imk_o x'} \right] dx'$$

$$= \frac{(n^2 - m^2)}{2in} k_o e^{i\beta} \left[A_2 \frac{e^{i(m-n)k_o a} - 1}{i(m-n)k_o} + B_2 \frac{e^{-i(m+n)k_o a} - 1}{-i(m+n)k_o} \right]$$

$$= \frac{(n^2 - m^2)}{2n} e^{i\beta} \left[A_2 \frac{1 - e^{i\alpha - i\beta}}{(m-n)} + B_2 \frac{e^{-i\alpha - i\beta} - 1}{(m+n)} \right] . \qquad (21)$$

We know that

$$\frac{d}{dx} f(|x|) = \frac{d}{dx}\left[f(x)\vartheta(x) + f(-x)\vartheta(-x) \right]$$

$$= f'(x)\vartheta(x) - f'(-x)\vartheta(-x) + f(x)\delta(x) - f(-x)\delta(x)$$

$$= f'(x)\vartheta(x) - f'(-x)\vartheta(-x) \quad ,$$

where

$$f'(-x) = \frac{d}{dy}[f(y)]|_{y=-x} \quad ,$$

and ϑ is the step function. Therefore,

$$\frac{d}{dx} G^{(0)}(x - x') = \frac{d}{dx}\left(\frac{1}{2ink_o} e^{ink_o|x-x'|} \right)$$

$$= \frac{1}{2}\left[e^{ink_o(x-x')} \vartheta(x - x') - e^{ink_o(x'-x)} \vartheta(x' - x) \right] .$$

The differential of Φ at $x = 0$ is

$$\frac{d}{dx}\Phi^{(2)}_{(0)}(0) = \int_0^a \left[\frac{d}{dx}G^{(0)}(x,x')\right]_{x=0} (n^2 - m^2)k_o^2\, \psi_{II}^{(1)}(x')\, dx'$$

$$= \frac{-1}{2}(n^2 - m^2)k_o^2 \int_0^a e^{ink_o x'}\left[A_2 e^{imk_o x'} + B_2 e^{-imk_o x'}\right] dx'$$

$$= -\frac{(n^2 - m^2)}{2}k_o^2\left[\frac{e^{i(m+n)k_o a} - 1}{i(m+n)k_o}A_2 + \frac{e^{-i(m-n)k_o a} - 1}{-i(m-n)k_o}B_2\right]$$

$$= \frac{i(n^2 - m^2)}{2}k_o\left[\frac{e^{i\alpha+i\beta} - 1}{(m+n)}A_2 - \frac{e^{-i\alpha+i\beta} - 1}{(m-n)}B_2\right] .$$

The differential of Φ at x = a is

$$\frac{d}{dx}\Phi^{(2)}(a) = \int_0^a \left[\frac{d}{dx}G^{(0)}(x,x')\right]_{x=a} (n^2 - m^2)k_o^2\, \psi_{II}^{(1)}(x')\, dx'$$

$$= \frac{e^{ink_o a}}{2}(n^2 - m^2)k_o^2 \int_0^a e^{-ink_o x'}\left[A_2 e^{imk_o x'} + B_2 e^{-imk_o x'}\right] dx'$$

$$= \frac{(n^2 - m^2)}{2}k_o^2 e^{i\beta}\left[\frac{e^{i(m-n)k_o a} - 1}{i(m-n)k_o}A_2 + \frac{e^{-i(m+n)k_o a} - 1}{-i(m+n)k_o}B_2\right]$$

$$= \frac{i(n^2 - m^2)}{2}k_o e^{i\beta}\left[\frac{e^{i\alpha-i\beta} - 1}{(n-m)}A_2 - \frac{e^{-i\alpha-i\beta} - 1}{(m+n)}B_2\right] . \qquad (23)$$

The boundary conditions at $x = 0$ are

$$\psi_{II}^{(2)}(x = 0) = C_1^{(2)} + C_2^{(2)} + \Phi^{(2)}_{(0)} =$$

$$= \psi_I(0) = 1 + B_1^{(2)}$$

$$C_1 + C_2 + \frac{(n^2 - m^2)}{2n}\left[\frac{1 - e^{i\alpha + i\beta}}{(m + n)}A_2 + \frac{e^{-i\alpha + i\beta} - 1}{(m - n)}B_2\right] = 1 + B_1^{(2)} \quad , \qquad (24)$$

and

$$\frac{d}{dx}\left[\psi_{II}^{(2)}(x = 0)\right] = ink_o C_1^{(2)} - ink_o C_2^{(2)} + \frac{d}{dx}\left[\Phi^{(2)}(0)\right] =$$

$$= \frac{d}{dx}\left[\psi_I(0)\right] = ik_o(1 - B_1^{(2)}) \quad ,$$

$$nC_1^{(2)} - nC_2^{(2)} + \frac{(n^2 - m^2)}{2}\left[\frac{e^{i\alpha + i\beta} - 1}{(m + n)}A_2 - \frac{e^{-i\alpha + i\beta} - 1}{(m - n)}B_2\right] = 1 - B_1^{(2)} \quad . \qquad (25)$$

The boundary conditions at $x = a$ are

$$\psi_{II}^{(2)}(a) = C_1^{(2)}e^{ink_o a} + C_2^{(2)}e^{-ink_o a} + \Phi^{(2)}(a) =$$

$$= \psi_{III}(a) = A_3^{(2)}e^{ik_o a} \quad ,$$

$$C_1^{(2)}e^{i\beta} + C_2^{(2)}e^{-i\beta} +$$

$$+ \frac{(n^2 - m^2)}{2n}e^{i\beta}\left[\frac{1 - e^{i\alpha - i\beta}}{(m - n)}A_2 + \frac{e^{-i\alpha - i\beta} - 1}{(m + n)}B_2\right] = A_3^{(3)}e^{ik_o a} \quad , \qquad (26)$$

and

$$\frac{d}{dx}\left[\psi_{II}^{(2)}(a)\right] = ink_o\left[C_1^{(2)}e^{ink_o a} - C_2^{(2)}e^{-ink_o a}\right] + \frac{d}{dx}\left[\Phi^{(2)}(a)\right] =$$

$$= \frac{d}{dx}\left[\psi_{II}(a)\right] = ik_o e^{-ink_o a}A_3^{(2)} \quad ,$$

$$n\left[C_1^{(2)}e^{i\beta} - C_2^{(2)}e^{-i\beta}\right] +$$

$$+ \frac{(n^2 - m^2)}{2}e^{i\beta}\left[\frac{1 - e^{i\alpha - i\beta}}{(m - n)}A_2 + \frac{e^{-i\alpha - i\beta} - 1}{(m + n)}B_2\right] = A_3^{(2)}e^{ik_0 a} \quad . \tag{27}$$

Using $A_2 = (-2/D)(m + 1)e^{-i\alpha}$ and $B_2 = (2/D)(1 - m)e^{i\alpha}$, Eqn. (24) becomes

$$C_1^{(2)} + C_2^{(2)} + \frac{-1}{nD}\left[\left[e^{i\beta} - e^{-i\alpha}\right](m + 1)(m - n) + \left[e^{i\beta} - e^{-i\alpha}\right](1 - m)(m + n)\right]$$

$$= 1 + B_1^{(2)} \quad .$$

Eqn. (25), (26), and (27) can be written into

$$nC_1^{(2)} - nC_2^{(2)} + \frac{1}{D}\left[(e^{i\beta} - e^{-i\alpha})(m + 1)(m - n) + (e^{i\beta} - e^{i\alpha})(1 - m)(m + n)\right] =$$

$$= 1 - B_1^{(2)} \quad ,$$

$$C_1^{(2)}e^{i\beta} + C_2^{(2)}e^{-i\beta} +$$

$$+ \frac{-e^{i\beta}}{nD}\left[\left[e^{-i\beta} - e^{-i\alpha}\right](m + 1)(m + n) + \left[e^{-i\beta} - e^{-i\alpha}\right](1 - m)(m - n)\right]$$

$$= A_3^{(2)}e^{ik_0 a} \quad ,$$

and

$$n\left[C_1^{(2)}e^{i\beta} - C_2^{(2)}e^{-i\beta}\right] +$$

$$+ \frac{-e^{i\beta}}{D}\left[\left[e^{-i\beta} - e^{-i\alpha}\right](m + 1)(m + n) + \left[e^{-i\beta} - e^{-i\alpha}\right](1 - m)(m - n)\right]$$

$$= A_3^{(2)} e^{ik_o a} \quad .$$

Therefore,

$$C_1^{(2)} + C_2^{(2)} - I = 1 + B_1^{(2)} \tag{28}$$

$$C_1^{(2)} - C_2^{(2)} + I = \frac{1}{n}\left(1 - B_1^{(2)}\right) \tag{29}$$

$$C_1^{(2)} e^{i\beta} + C_2^{(2)} e^{-i\beta} - J = A_3^{(2)} e^{ik_o a} \tag{30}$$

$$C_1^{(2)} e^{i\beta} - C_2^{(2)} e^{i\beta} - J = \frac{A_3^{(2)}}{n} e^{ik_o a} \quad , \tag{31}$$

where

$$I \equiv \frac{1}{nD}\left[\left(e^{-i\beta} - e^{-i\alpha}\right)(m+1)(m-n) + \left(e^{i\beta} - e^{i\alpha}\right)(1-m)(m+n)\right] ,$$

and

$$J \equiv \frac{e^{i\beta}}{nD}\left[\left(e^{-i\beta} - e^{-i\alpha}\right)(m+1)(m+n) + \left(e^{-i\beta} - e^{i\alpha}\right)(1-n)(m-n)\right] \quad .$$

$$(28) + (29) \Rightarrow 2C_1^{(2)} = \left(1 + \frac{1}{n}\right) + B_1^{(2)}\left(1 - \frac{1}{n}\right) \quad . \tag{32}$$

$$(28) + (29) \Rightarrow 2C_2^{(2)} - 2I = \left(1 - \frac{1}{n}\right) + B_1^{(2)}\left(1 + \frac{1}{n}\right) \quad . \tag{33}$$

$$(30) + (31) \Rightarrow 2C_1^{(2)} e^{i\beta} - 2J = A_3^{(2)} e^{ik_o a}\left(1 + \frac{1}{n}\right) \quad . \tag{34}$$

$$(30) - (31) \Rightarrow 2C_2^{(2)} e^{-i\beta} = A_3^{(2)} e^{ik_o a}\left(1 - \frac{1}{n}\right) \quad . \tag{35}$$

From (32) and (34), we find

$$\left(1+\frac{1}{n}\right)+B_1^{(2)}\left(1-\frac{1}{n}\right)=\left[A_3^{(2)}e^{ik_o a}\left(1+\frac{1}{n}\right)+2J\right]e^{-i\beta} \quad . \tag{36}$$

From (33) and (35), we find

$$\left(1-\frac{1}{n}\right)+B_1^{(2)}\left(1+\frac{1}{n}\right)=A_3^{(2)}e^{ik_o a}e^{i\beta}\left(1-\frac{1}{n}\right)-2I \quad . \tag{37}$$

(36) $\times \left(1+\frac{1}{n}\right)$ − (37) $\times \left(1-\frac{1}{n}\right)$ gives

$$\left(1+\frac{1}{n}\right)^2-\left(1-\frac{1}{n}\right)^2=$$

$$=A_3^{(2)}e^{ik_o a}\left[e^{-i\beta}\left(1+\frac{1}{n}\right)^2-e^{i\beta}\left(1-\frac{1}{n}\right)^2\right]+2J\,e^{-i\beta}\left(1+\frac{1}{n}\right)-2I\left(1-\frac{1}{n}\right) \quad . \tag{38}$$

But,

$$J\,e^{-i\beta}\left(1+\frac{1}{n}\right)-I\left(1-\frac{1}{n}\right)=$$

$$=\frac{1}{n^2D}\Big\{\left(e^{-i\beta}-e^{i\alpha}\right)(m+1)(m+n)(n+1)+$$

$$+\left(e^{-i\beta}-e^{i\alpha}\right)(1-m)(m-n)(n+1)+$$

$$+\left(e^{i\beta}-e^{i\alpha}\right)(m+1)(m-n)(n-1)+\left(e^{i\beta}-e^{i\alpha}\right)(1-m)(m+n)(n-1)\Big\}$$

$$=2\frac{1}{n}+\frac{2m}{D}\left[e^{-i\beta}\left(1+\frac{1}{n}\right)^2-e^{i\beta}\left(1-\frac{1}{n}\right)\right] \quad . \tag{39}$$

Substituting (39) in (38), we find

$$A_3^{(2)} = \frac{-4m}{D}e^{-ik_o a} = \frac{-4}{D}me^{-i\gamma} \quad . \tag{40}$$

Substitute (40) in (37)

$$\left(1 - \frac{1}{n}\right) + B_1^{(2)}\left(1 + \frac{1}{n}\right) = \frac{-4m}{nD}e^{i\beta}(n - 1)$$

$$- \frac{2}{nD}\left[\left(e^{-i\beta} - e^{-i\alpha}\right)(m + 1)(m - n) + \left(e^{-i\beta} - e^{i\alpha}\right)(1 - m)(m + n)\right]$$

$$= \frac{2}{nD}\left[e^{-i\alpha}(m + 1)(m - n) + e^{i\alpha}(1 - m)(m + n)\right] \quad .$$

Therefore,

$$B_1^{(2)}\left(1 + \frac{1}{n}\right) = \frac{1}{nD}\left\{e^{-i\alpha}2(m + 1)(m - n) + e^{i\alpha}2(1 - m)(m + n)\right.$$

$$\left. - (n - 1)\left[e^{i\alpha}(1 - m)^2 - (1 + m)^2 e^{-i\alpha}\right]\right\}$$

$$= \frac{1}{nD}\left[e^{-i\alpha}(m^2 - 1)(n + 1) + e^{i\alpha}(1 - m^2)(n + 1)\right] \quad .$$

From this equation, we find

$$B_1^{(2)} = \frac{1}{D}(1 - m^2)\left[e^{i\alpha} - e^{-i\alpha}\right] \quad . \tag{41}$$

$A_3^{(2)}$ and $B_1^{(2)}$ in Eqn. (40) and (41) are the same as A_3 and B_1 in Eqn. (5). By substituting Eqn.(40) and (41) in Eqn.(35) and (32) we can calculate C_2 and C_1.

$$C_1^{(2)} = \frac{1}{2n}\Big[(n + 1) + B_1(n - 1)\Big] =$$

$$= \frac{1}{2n}\Big[(n + 1) + (n - 1)\frac{1}{D}(1 - m^2)\Big(e^{i\alpha} - e^{-i\alpha}\Big)\Big] =$$

$$= \frac{1}{nD}\Big[e^{i\alpha}(m - 1)(m - n) - e^{-i\alpha}(m + 1)(m + n)\Big] \quad . \tag{42}$$

So,

$$C_2^{(2)} = \frac{1}{2}A_3 e^{ik_o a} e^{i\beta}\left(1 - \frac{1}{n}\right) =$$

$$= \frac{1}{2n}(n - 1)\frac{-4}{D}me^{i\beta} = \frac{-2}{nD}me^{i\beta}(n - 1) \tag{43}$$

Now we can calculate $\Phi^{(2)}$ in Eqn. (14):

$$\Phi^{(2)}(x) = \int_0^a G^{(0)}(x,x')(n^2 - m^2)k_o^2\psi_{II}^{(1)}(x')\,dx' =$$

$$= \int_0^x \frac{1}{2ink_o}e^{-ink_o(x'-x)}(n^2 - m^2)k_o^2\psi_{II}^{(2)}(x')\,dx' +$$

$$+ \int_x^a \frac{1}{2ink_o}e^{ink_o(x'-x)}(n^2 - m^2)k_o^2\psi_{II}^{(1)}(x')\,dx' =$$

$$= \frac{(n^2 - m^2)}{2ink_o}k_o^2 e^{ink_o x}\int_0^x e^{-ink_o x'}\Big[A_2 e^{imk_o x'} + B_2 e^{-imk_o x'}\Big]\,dx' +$$

$$+ \frac{(n^2 - m^2)}{2ink_o}k_o^2 e^{-ink_o x}\int_x^a e^{ink_o x'}\Big[A_2 e^{imk_o x'} + B_2 e^{-imk_o x'}\Big]\,dx' =$$

$$= \frac{(n^2 - m^2)}{2in} k_o e^{ink_o x} \left[\frac{e^{i(m-n)k_o x} - 1}{i(m-n)k_o} A_2 + \frac{e^{-i(m+n)k_o x} - 1}{-i(m+n)k_o} B_2 \right] +$$

$$+ \frac{(n^2 - m^2)}{2in} k_o e^{-ink_o x} \left[\frac{e^{i(m+n)k_o a} - e^{i(m+n)k_o x}}{i(m+n)k_o} A_2 + \frac{e^{-i(m-n)k_o a} - e^{-i(m-n)k_o x}}{-i(m-n)k_o} B_2 \right]$$

$$= \frac{1}{2n} e^{ink_o x} \left[\left(e^{i(m-n)k_o x} - 1 \right)(m-n)A_2 - \left(e^{-i(m+n)k_o x} - 1 \right)(m-n)B_2 \right] +$$

$$+ \frac{1}{2n} e^{-ink_o x} \left[\left(e^{i\alpha + i\beta} - e^{i(m+n)k_o x} \right)(m-n)A_2 - \left(e^{-i\alpha + i\beta} - e^{-i(m-n)k_o x} \right)(m+n)B_2 \right] =$$

$$= \frac{1}{2n} \left\{ e^{imk_o x} \left[(m+n) - (m-n) \right] A_2 - e^{-imk_o x} \left[(m-n) - (m+n) \right] B_2 + \right.$$

$$\left. e^{ink_o x} \left[-(m+n)A_2 + (m-n)B_2 \right] + e^{-ink_o x} \left[e^{i\alpha + i\beta}(m-n)A_2 - e^{-i\alpha + i\beta}(m+n)B_2 \right] \right\}$$

$$= \frac{1}{2n} \left\{ 2nA_2 e^{imk_o x} + 2nB_2 e^{-imk_o x} + \right.$$

$$+ e^{ink_o x} \left[\frac{2}{D}(m+n)(m+1)e^{-i\alpha} + \frac{2}{D}(1-m)(m-n)e^{i\alpha} \right] +$$

$$\left. + e^{ink_o x} e^{i\beta} \left[(m-n)\frac{-2}{D}(m+1) - (m+n)\frac{2}{D}(1-m) \right] \right\} =$$

$$= A_2 e^{imk_o x} + B_2 e^{-imk_o x} - e^{ink_o x} C_1^{(2)} - e^{-ink_o x} C_2^{(2)} \quad . \qquad (44)$$

Therefore,

$$\psi_{II}^{(2)}(x) = \Phi_0^{(2)}(x) + \Phi^{(2)}(x)$$

$$= C_1^{(2)} e^{ink_0 x} + C_2^{(2)} e^{-ink_0 x} + A_2 e^{imk_0 x} + B_2 e^{-imk_0 x} - C_1^{(2)} e^{ink_0 x} - C_2^{(2)} e^{-ink_0 x} =$$

$$= A_2 e^{imk_0 x} + B_2 e^{-imk_0 x} = \psi_{II}^{(1)}(x) =$$

$$= \psi_{II}(x) \; .$$

So we get the same solution using the background separation method.

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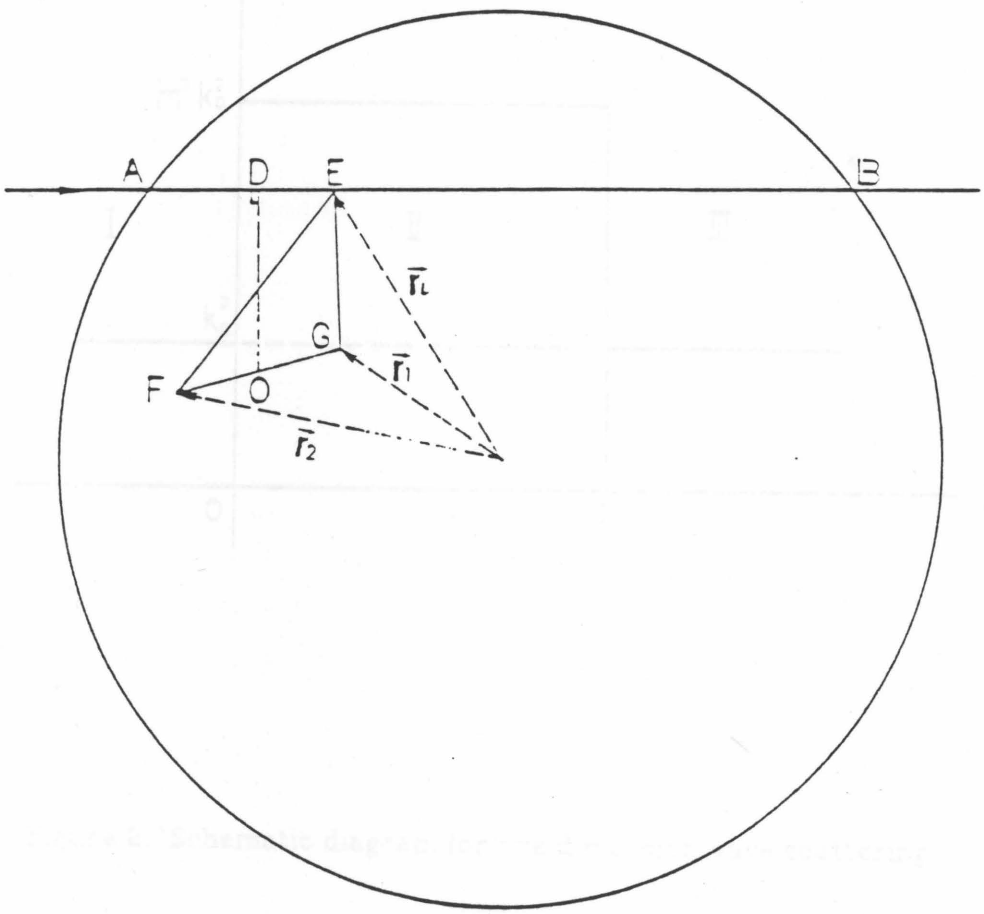


Figure 1. Schematic diagram for laser beam scattering in the atmosphere.

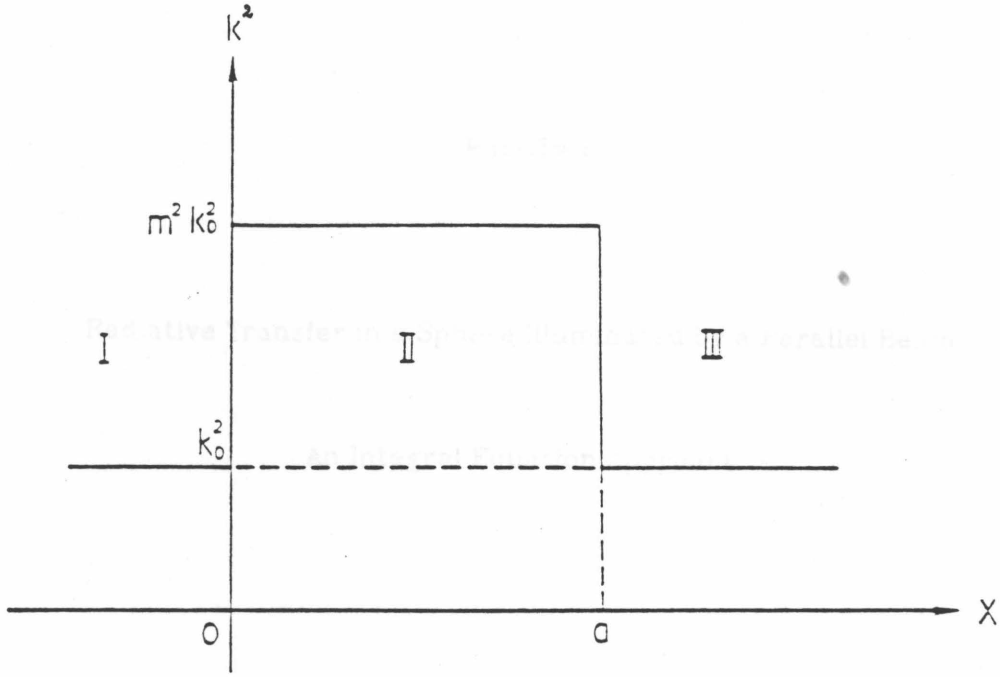


Figure 2. Schematic diagram for one dimension wave scattering.

Part Two

Radiative Transfer in a Sphere Illuminated by a Parallel Beam:

An Integral Equation Approach

I. INTRODUCTION

The problem of radiative transfer in an object illuminated by a parallel beam is interesting both theoretically and practically. Extensive work has been performed for the plane parallel atmosphere, and the results are summarized in the treatises by Chandrasekhar (1960), Sobolev (1975) and van de Hulst (1980). But the problem with other geometries is more complicated (see review by Fouquart, Irvine and Lenoble, 1980). Recently, Flannery, Roberge and Rybicki (1980) studied the radiative transfer of ultraviolet photons in a sphere. Chang and Kylafis (1983) investigated the scattering of X rays in a spherical shell. In both cases the boundary conditions are spherically symmetric. The planetary problem with incident radiation from infinity has advanced only as far as Monte Carlo simulations (Modali, Brandt and Kastner, 1975; Anderson and Hord, 1977; Adams and Kattawar, 1978).

The approach we adopt is to use the integral equation and systematically exploit its symmetry properties to simplify the mathematics. In this paper we report two initial successes of this approach. In the first part of the paper we provide the first complete formulation of the most general radiative transfer problem (except for polarization) in the form of a variational principle. We construct a functional and show that the equation of radiative transfer can be obtained by imposing the stationarity condition on this functional and that the differential cross section is given by the extremum value of the functional. This result generalizes the work of Stokes and DeMarcus (1971) and Chow, Friedson and Yung (1984). It extends and adds new insight to the work of Cheyney and Arking (1976). The symmetry of the phase function $P(\Omega_1, \Omega_2) = P(-\Omega_2, -\Omega_1)$ (see detailed discussion in §3 and Appendix B) plays a fundamental role in the variational principle.

In the second part of the paper the integral equation is applied to the simple case of a homogeneous sphere containing isotropic scatterers. We note that the symmetry of this problem is the same as that of the scattering of electromagnetic waves by a dielectric sphere (Mie, 1908; Debye, 1909; van de Hulst, 1957). The solution can, therefore, be expanded in a similar multipole series. This provides a natural generalization of the work of Sobolev (1972), whose solution corresponds to the first term of our series. Representative numerical results are presented, along with brief discussions of applications to planetary atmospheres.

II. The Integral Equation

The integral equation for the specific intensity inside an object of volume V and surface S illuminated by a parallel beam is

$$I(\mathbf{r}_1,\Omega_1) = F(\mathbf{r}_1,\Omega_1)$$

$$+ \int_v \int_{4\pi} d\mathbf{r}_2 d\Omega_2 \, \delta(\Omega_1 - \mathbf{r}_{12}) \frac{e^{-\tau(\mathbf{r}_1,\mathbf{r}_2)}}{4\pi r_{12}^2} \alpha(\mathbf{r}_2) \, P(\mathbf{r}_2;\Omega_1,\Omega_2) \, I(\mathbf{r}_2,\Omega_2) \quad , \qquad (1)$$

where $I(\mathbf{r},\Omega)$ is the specific intensity at $\mathbf{r}$, in the direction Ω in units of photons cm^{-2} s^{-1} sr^{-1}; $P(\mathbf{r};\Omega_2,\Omega_1)$ is the phase function which gives the fraction of photons scattered at $\mathbf{r}$ from Ω_1 to Ω_2 and is dimensionless;

$\tau(\mathbf{r}_1,\mathbf{r}_2) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} ds \, \alpha(\mathbf{r})$ is the optical depth between $\mathbf{r}_1$ and $\mathbf{r}_2$ and is dimensionless.

Here $ds = |d\mathbf{r}|$, $\alpha(\mathbf{r})$ is the extinction coefficient in units of cm^{-1}, $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$ and $\mathbf{r}_{12} = (\mathbf{r}_1 - \mathbf{r}_2)/r_{12}$. As illustrated in Figure 1a, (1) states that the specific intensity at $\mathbf{r}_1$ consists of two contributions. The first is derived from the primary solar radiation propagating from infinity to V in direction $\mathbf{k}_o$,

$$F(\mathbf{r}_1,\Omega_1) = \pi F_o \, \delta(\Omega_1 - \mathbf{k}_o) \, \varepsilon(\mathbf{r}_1,\mathbf{k}_o) \quad , \qquad (2)$$

where πF_o is the solar flux in units of photons cm^{-2} s^{-1} (as defined in Chandrasekhar, 1960). $\delta(\Omega)$ is a two-dimensional delta function, which has the charecter

$$\int d\Omega \, \delta(\Omega - \Omega_o) \, F(\Omega) = F(\Omega_o) \quad ,$$

and the attenuation factor is given by

$$\varepsilon(\mathbf{r}_1,\mathbf{k}_o) = e^{-\tau(\mathbf{r}_1,\mathbf{R}(\mathbf{r}_1,\mathbf{k}_o))} \quad , \tag{3}$$

with $\mathbf{R}(\mathbf{r}_1,\mathbf{k}_o)$ being the point on S where the solar beam that intercepts $\mathbf{r}_1$ first enters V. Inspection of Figure 1a reveals that

$$(\mathbf{r}_1 - \mathbf{R}(\mathbf{r}_1,\mathbf{k}_o)) \cdot \mathbf{k}_o = |\mathbf{r}_1 - \mathbf{R}(\mathbf{r}_1,\mathbf{k}_o)| \quad . \tag{4}$$

The second term in (1) arises from photons which are scattered from all other points $\mathbf{r}_2$ into $\mathbf{r}_1$. The structure and meaning of (1) are either obvious or can be easily understood by referring to standard texts on transport theory (Davison, 1957; Case and Zweifel, 1967). A complete solution of (1) provides the most detailed information of the internal and external radiation field. For observations made at asymptotic distance from V, it is convenient to define a differential cross section for scattering of radiation from the initial direction $\mathbf{k}_o$ into final direction $\mathbf{k}$,

$$\frac{d\sigma}{d\Omega}(\mathbf{k},\mathbf{k}_o) = \frac{1}{4\pi^2 F_o} \int_V \int_{4\pi} d\mathbf{r}\, d\Omega\, \alpha(\mathbf{r})\, \varepsilon(\mathbf{r},-\mathbf{k})\, P(\mathbf{r};\mathbf{k},\Omega)\, I(\mathbf{r},\Omega) \tag{5}$$

Note that the attenuation factor is $\varepsilon(\mathbf{r},-\mathbf{k})$ for a beam that leaves V in the direction $\mathbf{k}$ (see Figure 1a). The differential cross section is a fundamental physical quantity, and its relation to the more familiar quantities such as reflectivity and phase variation is discussed in Appendix A.

For most problems of practical interest, the phase function admits of a simple expansion,

$$P(\mathbf{r};\Omega_1,\Omega_2) = \sum_{l=0}^{\infty} \omega_l(\mathbf{r})\, P_l(\Omega_1 \cdot \Omega_2)$$

$$= \sum_{l=0}^{\infty} \omega_l(\mathbf{r}) \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\Omega_1)\, Y_{lm}^*(\Omega_2) \quad .$$

$$= 4\pi \sum_{lm} \frac{\omega_l(\mathbf{r})}{2l + 1} Y_{lm}(\Omega_1) \; Y^*_{lm}(\Omega_2) \quad , \tag{6}$$

where P_l and Y_{lm} are Legendre polynomials and spherical harmonics, respectively , and $\omega_o(\mathbf{r})$ is single scattering albedo. Unless otherwise stated, the summation over the indices l and m are over the range 0 to ∞ for l and $-l$ to l for m. Without loss of generality, the specific intensity and the solar term can be expanded as follows,

$$I(\mathbf{r},\Omega) = \sum_{lm} \left[\frac{\alpha(\mathbf{r})\,\omega_l(\mathbf{r})}{2l + 1} \right]^{-1/2} J_{lm}(\mathbf{r}) \; Y_{lm}(\Omega) \tag{7}$$

$$F(\mathbf{r},\Omega) = \sum_{lm} \left[\frac{\alpha(\mathbf{r})\,\omega_l(\mathbf{r})}{2l + 1} \right]^{-1/2} F_{lm}(\mathbf{r}) \; Y_{lm}(\Omega) \tag{8}$$

Substituting (6), (7), and (8) in (1), multiplying (1) by $Y^*_{l_1m_1}(\Omega_1)$ and integrating over $d\Omega_1$, we obtain a set of coupled integral equations:

$$J_{l_1m_1}(\mathbf{r}_1) = F_{l_1m_1}(\mathbf{r}_1)$$

$$+ \int_v d\mathbf{r}_2 \sum_{l_2m_2} K_{l_1m_1l_2m_2}(\mathbf{r}_1,\mathbf{r}_2) \; J_{l_2m_2}(\mathbf{r}_2) \quad , \tag{9}$$

where

$$F_{l_1m_1}(\mathbf{r}_1) = \left[\frac{\alpha(\mathbf{r}_1)\,\omega_{l_1}(\mathbf{r}_1)}{2l_1 + 1} \right]^{1/2} \pi F_o \, \varepsilon(\mathbf{r}_1,\mathbf{k}_o) \; Y^*_{l_1m_1}(\mathbf{k}_o) \tag{10}$$

$$K_{l_1m_1l_2m_2}(\mathbf{r}_1,\mathbf{r}_2) = \left[\frac{\omega_{l_1}(\mathbf{r}_1)\,\omega_{l_2}(\mathbf{r}_2)\,\alpha(\mathbf{r}_1)\,\alpha(\mathbf{r}_2)}{(2l_1 + 1)(2l_2 + 1)} \right]^{1/2} \frac{e^{-\tau(\mathbf{r}_1,\mathbf{r}_2)}}{r_{12}^2} Y^*_{l_1m_1}(\mathbf{r}_{12}) \; Y_{l_2m_2}(\mathbf{r}_{12}) \; . \tag{11}$$

Following Cheyney and Arking (1976), we rewrite (9) in a compact operator notation

$$(1 - K)J = F \quad , \tag{12}$$

where

$$J = \left[J_{00}(\mathbf{r}) \, , \; J_{11}(\mathbf{r}) \, , \; J_{10}(\mathbf{r}) \, , \; J_{1-1}(\mathbf{r}) \, , \; \cdots \; J_{lm}(\mathbf{r}) \, , \; \cdots \right]$$

$$F = \left[F_{00}(\mathbf{r}) \, , \; F_{11}(\mathbf{r}) \, , \; F_{10}(\mathbf{r}) \, , \; F_{1-1}(\mathbf{r}) \, , \; \cdots \; F_{lm}(\mathbf{r}) \, , \; \cdots \; . \right]$$

and K is the corresponding matrix. The operation of K on J involves both matrix multiplication and integration over spatial variables. Note that (12) is equivalent to (1). No approximation has been made.

III. Variational Method and Principle of Reciprocity

To employ the variational method for solving the integral equation of radiative transfer, the kernel of the equation must be self-adjoint. By definition the adjoint operator K^+ is

$$K^+_{l_1 m_1 l_2 m_2}(\mathbf{r}_1,\mathbf{r}_2) = K^*_{l_2 m_2 l_1 m_1}(\mathbf{r}_2,\mathbf{r}_1) = (-1)^{l_1+l_2} K_{l_1 m_1 l_2 m_2}(\mathbf{r}_1,\mathbf{r}_2) \quad , \tag{13}$$

where the upper asterisk denotes complex conjugation. In general, K is not self-adjoint; i.e., $K^+ \neq K$. However, for phase functions with symmetry properties given by (6) (we shall later show how to relax this requirement), it is possible to define an extended equation which includes (12), and whose operator is self-adjoint. In the following discussion the mathematics is adapted from Cheyney and Arking (1976), but the physical interpretation and insight are new.

Suppose we adopt a new coordinate system to describe the direction of the radiation. In the new system the direction Ω in the original system becomes $-\Omega$. The position vector $\mathbf{r}$ is unchanged. We will use the tilde sign to denote quantities in the new system; for example:

$$\tilde{I}(\mathbf{r},\Omega) = I(\mathbf{r},-\Omega) \tag{14}$$

$$\tilde{F}(\mathbf{r},\Omega) = F(\mathbf{r},-\Omega) \quad . \tag{15}$$

Expending $\tilde{I}$ and $\tilde{F}$ we have

$$\tilde{I}(\mathbf{r},\Omega) = \sum_{lm} \left[\frac{\alpha(\mathbf{r})\,\omega_l(\mathbf{r})}{2l+1} \right]^{-\frac{1}{2}} \tilde{\mathcal{I}}_{lm}(\mathbf{r})\, Y_{lm}(\Omega)$$

$$\tilde{F}(\mathbf{r},\Omega) = \sum_{lm} \left[\frac{\alpha(\mathbf{r})\,\omega_l(\mathbf{r})}{2l+1} \right]^{-\frac{1}{2}} \tilde{F}_{lm}(\mathbf{r})\, Y_{lm}(\Omega) \quad .$$

Using (7), (8), (14) and (15) we have

$$\tilde{J}_{lm}(\mathbf{r}) = (-1)^l J_{lm}(\mathbf{r}) \quad . \tag{16}$$

$$\tilde{F}_{lm}(\mathbf{r}) = (-1)^l F_{lm}(\mathbf{r}) \quad . \tag{17}$$

From (9), (13), (16) and (17) we can prove that $\tilde{J}$ is the solution of equation

$$(1 - K^+)\tilde{J} = \tilde{F} \quad , \tag{18}$$

where

$$\tilde{J} = \left[\tilde{J}_{00}(\mathbf{r}) \, , \; \tilde{J}_{11}(\mathbf{r}) \, , \; \tilde{J}_{10}(\mathbf{r}) \, , \; \tilde{J}_{1-1}(\mathbf{r}) \, , \; \cdots \; \tilde{J}_{lm}(\mathbf{r}) \, , \; \cdots\right]$$

$$\tilde{F} = \left[\tilde{F}_{00}(\mathbf{r}) \, , \; \tilde{F}_{11}(\mathbf{r}) \, , \; \tilde{F}_{10}(\mathbf{r}) \, , \; \tilde{F}_{1-1}(\mathbf{r}) \, , \; \cdots \; \tilde{F}_{lm}(\mathbf{r}) \, , \; \cdots\right] \tag{19}$$

and K^+ is as defined in (13). Therefore (18) is nothing but (12) written in the new system! In summary, we have,

$$(1 - K)J_{\mathbf{k}_o} = F_{\mathbf{k}_o} \tag{20}$$

$$(1 - K^+)\tilde{J}_{\mathbf{k}_o} = \tilde{F}_{\mathbf{k}_o} \quad . \tag{21}$$

These equations are different expressions of the same physical process, and we have provided a simple physical interpretation of the adjoint operator K^+. Note that in going from (12) and (18) to (20) and (21) we have put in the explicit dependence of the parameter $\mathbf{k}_o$. With the help of the new coordinate system, we can express the differential cros section more concisely. Using (5) and substituting expressions (6) and (7) for $P(\mathbf{r};\mathbf{k},\Omega)$ and $I(\mathbf{r},\Omega)$, we have

$$\frac{d\sigma}{d\Omega}(\mathbf{k},\mathbf{k}_o) = \frac{1}{\pi F_o}\int_v d\mathbf{r} \sum_{lm} J_{lm}(\mathbf{r}) \left[\frac{\alpha(\mathbf{r})\,\omega_l(\mathbf{r})}{2l + 1}\right]^{+1/2} \varepsilon(\mathbf{r},-\mathbf{k})\, Y_{lm}(\mathbf{k}) \quad . \tag{22}$$

This can be further simplified using (10), and the final result is

$$\frac{d\sigma}{d\Omega}(\mathbf{k},\mathbf{k}_o) = \frac{1}{\pi^2 F_o^2} \langle F_{-\mathbf{k}}, J_{\mathbf{k}_o} \rangle \quad , \tag{23}$$

where the scalar product is defined by

$$\langle X,Y \rangle = \int_v d\mathbf{r} \sum_{lm} X_{lm}^{*}(\mathbf{r})\, Y_{lm}(\mathbf{r}) = \langle Y,X \rangle^{*} \tag{24}$$

for any two vectors of the type as in (12). It is easy to show that $\frac{d\sigma}{d\Omega}(\mathbf{k},\mathbf{k}_o)$ as given by (23) is real even though complex functions are involved.

It is of interest to consider the simultaneous solution of a forward scattering problem and its reverse. As shown in Figure 1b, in the forward problem the incident and emergent directions are $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. The corresponding directions in the reverse problem are $-\mathbf{k}_2$ and $-\mathbf{k}_1$, respectively. The appropriate field equations are

$$(1 - K)J_{\mathbf{k}_1} = F_{\mathbf{k}_1} \tag{25}$$

$$(1 - K^{+})\tilde{J}_{-\mathbf{k}_2} = \tilde{F}_{-\mathbf{k}_2} \ . \tag{26}$$

The corresponding differential cross-sections are

$$\frac{d\sigma}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) = \frac{1}{\pi^2 F_o^2} \langle F_{-\mathbf{k}_2}, J_{\mathbf{k}_1} \rangle \tag{27}$$

$$\frac{d\sigma}{d\Omega}(-\mathbf{k}_1,-\mathbf{k}_2) = \frac{1}{\pi^2 F_o^2} \langle F_{\mathbf{k}_1}, J_{-\mathbf{k}_2} \rangle \ . \tag{28}$$

Equations (25) and (26) can be written as a single equation in an extended double Hilbert space

$$LJ = S \ , \tag{29}$$

where

$$L = \begin{pmatrix} 0 & 1 - K \\ 1 - K^{+} & 0 \end{pmatrix} \tag{30}$$

$$J = (\tilde{J}_{-\mathbf{k}_2}, J_{\mathbf{k}_1}) \tag{31}$$

$$S = (F_{\mathbf{k}_1}, \tilde{F}_{-\mathbf{k}_2}) \quad . \tag{32}$$

Since L is a self-adjoint operator, we can construct a functional

$$F(Q) = \langle Q, LQ \rangle - \langle Q, S \rangle - \langle S, Q \rangle \quad , \tag{33}$$

where $Q = (\tilde{q}, q)$ is an arbitrary vector in the double Hilbert space, and the scalar product is defined as the straightforward generalization of (24). On imposing the stationarity condition on (33) and using $L^+ = L$ we have

$$\delta F(Q) = \langle \delta Q, LQ - S \rangle - \langle LQ - S, \delta Q \rangle \quad , = 0 \tag{34}$$

which implies

$$LQ = S \quad . \tag{35}$$

We recognize (35) as equivalent to (29) and, hence (33) is the correct functional for the radiative transfer problem. We can use (33) and apply the usual variational techniques to obtain approximate solutions to (20) and (21). But there is an additional advantage, as we shall show in the following.

Let us evaluate (33) for some special choices of Q: $Q_o = (\tilde{J}_{-\mathbf{k}_2}, J_{\mathbf{k}_1})$, $Q_1 = (\tilde{q}, J_{\mathbf{k}_1})$ and $Q_2 = (\tilde{J}_{-\mathbf{k}_2}, q)$, where $\tilde{q}$ and q are arbitrary, but $J_{\mathbf{k}_1}$ and $\tilde{J}_{-\mathbf{k}_2}$ are the solutions of (25) and (26), respectively. After some simple algebra we have,

$$F_o = F(Q_o) = -\langle F_{\mathbf{k}_1}, \tilde{J}_{-\mathbf{k}_2} \rangle - \langle \tilde{F}_{-\mathbf{k}_2}, J_{\mathbf{k}_1} \rangle$$

$$= -\left[\frac{d\sigma}{d\Omega}(-\mathbf{k}_1, -\mathbf{k}_2) + \frac{d\sigma}{d\Omega}(\mathbf{k}_2, \mathbf{k}_1) \right] \pi^2 F_o^2 \tag{36}$$

$$F_1 = F(Q_1) = -\langle J_{\mathbf{k}_1}, \tilde{F}_{-\mathbf{k}_2} \rangle - \langle \tilde{F}_{-\mathbf{k}_2}, J_{\mathbf{k}_1} \rangle$$

$$= -2\frac{d\sigma}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1)\pi^2 F_o^2 \tag{37}$$

$$F_2 = F(Q_2) = -\langle \tilde{J}_{-\mathbf{k}_2}, F_{\mathbf{k}_1} \rangle - \langle F_{\mathbf{k}_1}, \tilde{J}_{-\mathbf{k}_2} \rangle$$

$$= -2\frac{d\sigma}{d\Omega}(-\mathbf{k}_1,-\mathbf{k}_2)\pi^2 F_o^2 \ . \tag{38}$$

Since $\tilde{q}$ and q are arbitrary, we must have

$$F_o = F_1 = F_2 = -2\pi^2 F_o^2 \frac{d\sigma}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) = -2\pi^2 F_o^2 \frac{d\sigma}{d\Omega}(-\mathbf{k}_1,-\mathbf{k}_2) \ . \tag{39}$$

This proves that the differential cross section satisfies the principle of reciprocity (Minnaert, 1941) and that the extremum value of the functional (33) is proportional to the differential cross section.

The results summarized in (33) and (39) are new and provide a generalization of previous work on solving the planetary problem in radiative transfer, using variational methods (Huang, 1953; Stokes and DeMarcus, 1971; Cheyney and Arking, 1976; Sze, 1976; Yung, 1976; Chow, Friedson and Yung, 1984). Our work reveals the intimate connection between the microscopic symmetry of the phase function $P(\mathbf{r};\Omega_1,\Omega_2)$, the macroscopic symmetry of the differential cross section $\frac{d\sigma}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1)$ and the variational principle. Actually, the symmetry given by (6) is unnecessarily strong. A weaker microscopic symmetry $P(\mathbf{r};\Omega_1,\Omega_2) = P(\mathbf{r};-\Omega_2,-\Omega_1)$ is sufficient to ensure the macroscopic symmetry $\frac{d\sigma}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) = \frac{d\sigma}{d\Omega}(-\mathbf{k}_1,-\mathbf{k}_2)$ (Case, 1957), and the existence of a variational principle. In Appendix B we give an alternative proof of our results using the weaker microscopic symmetry assumption. We also show that, if we use the method of successive orders to compute the radiation field to order n, then the differential cross section evaluated using (33) is accurate to order $2n$. The usefulness of symmetry is appreciated by van de Hulst (1980, p. 17): "Such symmetry relations are of great practical help in checking the consistency of analytic formulas or of computational results." But he did not recognize the full potential of such relations. The recognition of the importance of variational methods in radiative transfer is overdue, especially in light of notable advances in using variational methods in related fields such as neutron transport (Francis et al., 1959; Pomraning and Clark, 1963a,b), kinetic theory (Cercignani and Pagani, 1966; Cercignani, 1969), electromagnetic wave scattering (Levine and Schwinger, 1950), acoustic wave scattering (Gerjuoy and Saxon, 1954) and quantum theory of scattering (Lippmann and Schwinger, 1950). Viewed in the broader context, our results are not only reasonable; they are inevitable (Gerjuoy, Rau, and Spruch, 1983).

IV. Homogeneous Sphere With Isotropic Scatterers

We will now apply the integral equation to the particularly simple case of a homogeneous sphere of radius a filled with isotropic scatterers of single scattering albedo ω_o. In this case only $l = 0$ equation is needed. From Eqn.(9) we have

$$J_{00}(\mathbf{r}_1) = \frac{\alpha\omega_o}{4\pi}\int d\mathbf{r}_2 \frac{e^{-\tau(\mathbf{r}_1,\mathbf{r}_2)}}{r_{12}^2} J_{00}(\mathbf{r}_2) + F_{00}(\mathbf{r}_1) \quad . \tag{1}$$

Since the medium is homogeneous, we can replace the distance variable by an optical variable

$$\mathbf{t} = \alpha\mathbf{r} \quad .$$

Note that $\tau_o = \alpha a$ and $\tau_{12} = \tau(\mathbf{r}_1,\mathbf{r}_2) = \alpha|\mathbf{r}_1 - \mathbf{r}_2|$. Then Eqn.(40) becomes

$$J(\mathbf{t}_1) = J_0(\mathbf{t}_1) + \omega_o \int d\mathbf{t}_2\, G(\mathbf{t}_1,\mathbf{t}_2)\, J(\mathbf{t}_2) \quad , \tag{2}$$

where the mean intensity is

$$J(\mathbf{t}) = \frac{1}{4\pi}\int d\Omega\, I(\mathbf{t},\Omega) = \left[\frac{1}{4\pi\alpha\omega_o}\right]^{\frac{1}{2}} \int J_{oo}(\mathbf{t}) \quad . \tag{3}$$

($J(\mathbf{t})$ is not the same as the J in Eqn.(12) or $J_{00}(\mathbf{t})$ in Eqn.(40)); the solar term is

$$J_0(\mathbf{t}) = \frac{1}{4\pi}\int d\Omega\, F(\mathbf{t},\Omega) = \frac{F_o}{4}\varepsilon(\mathbf{t},\mathbf{k}_o) = \frac{F_o}{4} e^{-[\tau_o^2 - \tau^2 + (\mathbf{t}\cdot\mathbf{k}_o)^2]^{\frac{1}{2}} - \mathbf{t}\cdot\mathbf{k}_o} \quad , \tag{4}$$

and the Green's function of the integral equation is

$$G(\mathbf{t}_1,\mathbf{t}_2) = \frac{e^{-\tau_{12}}}{4\pi\tau_{12}^2} \quad . \tag{5}$$

The integration of the $\mathbf{t}_2$ variable is over the entire optical sphere of radius τ_o. This equation is known in literature as the Peierls equation (Peierls, 1939).

Eqn.(41) bears striking similarity to the integral equation that describes the scattering of a scalar wave of wavenumber **k** by a dielectric sphere with the index of refraction equal to m (Jackson, 1975, Chapter 16),

$$\psi(\mathbf{r}_1) = e^{i\mathbf{k}\cdot\mathbf{r}_1} + k^2(m^2 - 1)\int d\mathbf{r}_2 G(\mathbf{r}_1,\mathbf{r}_2;k)\psi(\mathbf{r}_2) \quad , \tag{45}$$

where the Green's function is

$$G(\mathbf{r}_1,\mathbf{r}_2;k) = \frac{e^{ikr_{12}}}{4\pi r_{12}}$$

$$= ik\sum_{l=0}^{\infty}(2l + 1)j_l(kr_<)h_l^{(1)}(kr_>)\sum_{m=-l}^{l} Y_{lm}^*(\vartheta_1\varphi_1)\,Y_{lm}(\vartheta_2,\varphi_2) \tag{46}$$

where $r_< = \min(r_1,r_2)$, $r_> = \max(r_1,r_2)$, j_l and $h_l^{(1)}$ are spherical Bessel functions of the first and the third kind, respectively. Note that

$$G(\mathbf{t}_1,\mathbf{t}_2) = \int_1^{\infty} d\sigma\, G(\mathbf{t}_1,\mathbf{t}_2;i\sigma) \quad . \tag{47}$$

This suggests a close connection between Eqn.(41) and (45), and that multipole expression method first developed for solving Eqn.(45) would be very useful for solving Eqn.(41). Because of the cylindrical symmetry around the sun-sphere axis (see Figure 10 b), we can expand $J(\mathbf{t})$ and $J_o(\mathbf{t})$, using the Legendre polynomials.

$$J(\mathbf{t}) = J(\tau,\mu) = \sum_{l=0}^{\infty}\tau^{-1}\psi_l(\tau)P_l(\mu) \tag{48}$$

$$J_o(\mathbf{t}) = J_o(\tau,\mu) = \sum_{l=0}^{\infty}\tau^{-1}S_l(\tau)P_l(\mu) \quad , \tag{49}$$

where $\tau = |\mathbf{t}|$ and $\mu = \mathbf{k}_o\cdot\mathbf{t}/\tau$ is the cosine of the angle between the direction of incident radiation $\mathbf{k}_o$ and the position vector $\mathbf{t}$. Substituting Eqn.(45), (46), (47), and (48) in Eqn.(41) and integrating with respect to φ_1 and φ_2, we get a set of uncoupled integral equations for l = 0, 1, 2 ...

$$\psi_l(\tau_1) = S_l(\tau_1) + \omega_o \int_0^{\tau_0} d\tau_2 \, G_l(\tau_1,\tau_2)\psi_l(\tau_2) \quad , \tag{50}$$

where

$$G_l(\tau_1,\tau_2) = -\tau_1\tau_2 \int_1^{\infty} d\sigma \, \sigma j_l(i\sigma\tau_<) h_l^{(1)}(i\sigma\tau_>)$$

$$= \sum_{m=0}^{2l} \frac{1}{2}\Big[(-1)^{l+1} A_{lm} E_{m+1}(\tau_> + \tau_<) + B_{lm} E_{m+1}(\tau_> - \tau_<)\Big] \tag{51}$$

with

$$A_{lm} = \sum_{k=0}^{l}\sum_{k'=0}^{l} \delta_{k+k',m} \frac{(l+k)!(l+k')!}{2^m \tau_<^k \tau_>^{k'} k! k'! (l-k)!(l-k')!}$$

$$B_{lm} = \sum_{k=0}^{l}\sum_{k'=0}^{l} \delta_{k+k',m} \frac{(-1)^k (l+k)!(l+k')!}{2^m \tau_<^k \tau_>^{k'} k! k'! (l-k)!(l-k')!}$$

$$E_n(x) \equiv \int_1^{\infty} \frac{e^{-xt}}{t^n} dt \quad .$$

A detailed derivation of (50) and an examination of the properties of the Green's function are referred to Appendix C. To gain insight into the meaning of (50), let us investigate the special case $l = 0$,

$$\psi_o(\tau_1) = S_o(\tau_1) + \frac{1}{2}\omega_o \int_0^{\tau_0} d\tau_2 \Big[E_1(|\tau_1 - \tau_2|) - E_1(\tau_1 + \tau_2)\Big]\psi_o(\tau_2) \quad . \tag{52}$$

This is the same as Eqn.(3) in Sobolev (1972), and is the Milne integral equation for a homogeneous sphere (see Appendix of Sobolev, 1975). However, to solve the complete planetary problem with the source of illumination placed at infinity we need the entire series in Eqn.(48). Thus, our results are a generalization of Sobolev (1972). In practice we have to truncate the series in Eqn.(48) at some finite $l = l_{max}$, and the series is useful only if it converges rapidly. That this is so will be discussed in the following section.

The differential cross section, according to Eqn.(5), is

$$\frac{d\sigma}{d\Omega}(\mathbf{k},\mathbf{k}_o) = \frac{\omega_o}{\pi F_o \alpha^2} \int d\mathbf{t}\,\varepsilon(\mathbf{t},-\mathbf{k})\,J(\mathbf{t}) \quad . \tag{53}$$

But the expression can be further simplified using Eqn.(48), (49), and Appendix A,

$$\frac{d\sigma}{d\Omega}(\alpha_s) = \frac{\omega_o}{\alpha^2} \sum_{l=0}^{\infty} (-1)^l f_l P_l(\cos\alpha_s) \tag{54}$$

with

$$f_l = \frac{16}{F_o^2} \frac{1}{2l+1} \int_0^{\tau_o} d\tau\, S_l(\tau)\psi_l(\tau) \quad , \tag{55}$$

and α_s = scattering angle defined by $\cos\alpha_s = \mathbf{k}\cdot\mathbf{k}_o$ (see Figure 10b). There is a host of interesting functions that can be given in terms of the differential cross section. Two distinct sets of nomenclature exist depending on whether the overall scattering is viewed as due to a particle or a planet. In the former case we are interested in the extinction and absorption cross section, the g-factor and the scattering phase function. In the latter case, the relevant quantities are the geometric albedo, the phase vibration, the phase integral and the bond albedo. Table 1 provides a listing of the important "particle" and "planetary" photometric functions taken from standard references (van de Hulst, 1957; Horak, 1950; Harris, 1961) and their evaluation in terms of the multipole functions in Eqn.(54) and (55).

V. Numerical Results and Discussion

In our numerical model we evaluate the solar term in Eqn.(49) using 36 points Gaussian quadrature. The Green's function in Eqn.(51) is computed using a procedure described in Appendix C. The integral Eqn.(50) is discretized with 100-200 points between 0 and τ_0, and the resulting matrix equation is solved as in Sze (1976) and Yung (1976). A large number of cases have been studied for τ_0 in the range 0 to 8 and for ω_0 from 0 to 1. For the requirement of an overall accuracy of order 3%, it turns out that the maximum number of multipole equations, $l_{\max} + 1$, is less than 12. We will present detailed results for two representative cases, $\tau_0 = 1$ and $\tau_0 = 8$, both with $\omega_0 = 1$. These results will serve to illustrate the character of our solutions for spheres of small to moderate optical thickness. A large amount of interesting information deduced from our numerical studies is summarized in the graphs of photometric functions.

The multipole expansions of the solar term $S_l(\tau)$ are shown in Figures 2a and 2b for the cases $\tau_0 = 1$ and 8, respectively. Note that the higher multipoles decrease rapidly as l approaches and exceeds τ_0. Figures 3a and 3b show the corresponding multipole solutions $\psi_l(\tau)$. The solutions display similar properties as the solar terms. The mean intensity of the internal radiation field (as defined by Eqn.(48)) on the sun-sphere axis $J(\tau,-1)$, and its comparison with the primary solar term $J_0(\tau,-1)$ are shown in Figure 4a for $\tau_0 = 1$. Similar results for the case $\tau_0 = 8$ are presented in Figure 4b. The comparison clearly demonstrates the importance of multiple scattering for determining the internal radiation field inside a sphere. Table 2 provides a summary of the integrals of the multipole radiation fields f_l for $\tau_0 = 1$ and 8. This table gives a proper assessment of the relative contribution of the higher multipoles.

Values for the extinction efficiency Q_{ext} are presented in Figure 5. This is a measure of the efficiency of the sphere for absorbing incident light relative to a disk of area πa^2. The limiting values can be easily shown to be

$$\lim_{\tau_o \to 0} Q_{ext} = \frac{4}{3}\tau_o$$

$$\lim_{\tau_o \to \infty} Q_{ext} = 1 \; .$$

We note that as τ_o exceeds 8, Q_{ext} exceeds 99% and, hence, the sphere is essentially opaque. The scattering efficiency Q_{sca} is shown as a function of ω_o in Figure 6. Q_{sca} by definition is the same as the Bond albedo. The case $\tau_o = \infty$ is taken from Irvine's (1975) Figure 5 for Bond albedo (or spherical albedo). By conservation of energy, $Q_{sca} = Q_{ext}$ for $\omega_o = 1$. The limiting values of Q_{sca} are (van de Hulst, 1980, Chapter 12)

$$\lim_{\tau_o \to \infty} Q_{sca} \approx (1-s)\frac{1-0.139s}{1+1.170s}$$

$$\lim_{\tau_o \to 0} Q_{sca} = \frac{4}{3}\omega_o \tau_o \quad ,$$

where $s = (1-\omega_o)^{1/2}$. The first expression reflects the importance of multiple scattering, as is obvious from the appearance of the factor $1-(1-\omega_o)^{1/2}$. (For an illuminating discussion of this factor the reader is referred to McElroy, 1971). The second expression includes only single scattering. The asymmetry factor g is shown in Figure 7. For a sphere with isotropic scatterers, g is always negative. The limiting values of g are,

$$\lim_{\tau_0 \to \infty} g = -0.45$$

$$\lim_{\tau_o \to 0} g = 0 \quad .$$

The large sphere value is approximately computed using the tabulation of Harris (1961) and should be compared with $g = -\frac{4}{9}$ for a Lambert sphere (van de Hulst, 1980, Chapter 18).

Figure 8 shows values of the geometric albedo of a sphere as a function of ω_o for a range of values of τ_o. The case of $\tau_o = \infty$ is taken from Harris (1961). In the small sphere limit we have

$$\lim_{\tau_o \to 0} p = \frac{1}{3}\omega_o \tau_o \quad .$$

The values of the phase integral q can be deduced from the relation

$$pq = A = Q_{sca}$$

and are not separately plotted. The phase variation $\varphi(\alpha_p)$ as a function of the phase angle α_p is shown in Figure 9 for $\tau_o = 1$, 8, and ∞ with $\omega_o = 1$. For comparison we also present the case of a Lambert sphere. It is clear that as τ_o increases, the scattering becomes more backward-peaked and the phase variation approaches that of a Lambert sphere (van de Hulst, 1980, Chapter 18),

$$\varphi(\alpha_p) = \frac{1}{\pi}\left[(\pi - \alpha_p)\cos\alpha_p + \sin\alpha_p\right] \quad .$$

Note that $\varphi(\alpha_p)$ is related to the scattering phase function $p(\alpha_s)$ by a normalization constant and a change of variable.

VI. Applications

The results we have presented in this initial report tend to emphasize the theoretical rather than the practical aspects of our work. More realistic modeling is not difficult in principle. However, before we build more complicated models we will qualitatively discuss the possible applications of our results to a number of interesting problems.

The photometry of Jupiter has been investigated by Cochran (1977); Tomasko, West and Castillo (1978); Sato and Hansen (1979); and Smith and Tomasko (1984); and there is general agreement that the cloud phase function can be represented by a two-term Henyey-Greenstein phase function,

$$P(\alpha_s) = f_1 P(g_1, \alpha_s) + (1 - f_1) P(g_2, \alpha_s)$$

with

$$P(g, \alpha_s) = \frac{1 - g^2}{(1 + g^2 - 2g\cos\alpha_s)^{3/2}} \quad ,$$

where α_s = scattering angle, g = asymmetry parameter. The best choices of the parameters are: f_1 = 0.938, g_1 = 0.80, g_2 = -0.65. The first part of the phase function with forward-scattering g_1 = 0.80 poses no problem, since this is typical of most cloud particles (Hansen, 1969). The origin of the second part of the phase function with backward-scattering is obscure. The phase function of ammonia crystals, believed to be what the cloud particles are composed of, do not exhibit a pronounced backward peak (Tomasko and Doose, 1984). We propose an explanation. Figure 7 shows that the asymmetry factor for patches of thick spherical clouds is always negative. In fact, if the clouds of Jupiter were not horizontally homogeneous, but possessed small scale structures, negative values of g would be inevitable. But, of course, to provide a realistic interpretation of the photochemistry of Jupiter, we have to make our spherical clouds out of anisotropic scatterers. This quantitative modeling must be deferred to the future.

We can apply our method to the study of planetary coronae arising from resonance scattering of solar photons (Modali, Brandt and Kastner, 1975; Anderson and Hord, 1977), and the radiation field in the terrestrial stratosphere and troposphere at twilight. However, a complication arises in these problems due to the presence of the solid planetary body, which is opaque to radiation. In this case, Eqn.(1) must be modified so that it becomes the appropriate equation of radiative transfer in the spherical shell atmosphere.

In a cometary atmosphere the geometry is roughly spherical. The scattering of sunlight in the continuum is dominated by micron-sized dust grains with low single scattering albedo $\omega_0 \sim 0.1$-0.4 (Ney and Merrill, 1976; Hanner, 1979). The total optical depth is at most of order a few (Hellmich, 1981), since greater optical depths would shield the incident sunlight from reaching the nucleus and cut off the production of gas and dust. In this problem, the best approach would be to use Eqn.(1) and apply the method of successive orders and the variational principle (Appendix B). However, $I(\mathbf{r},\mathbf{\Omega})$ is a function of five variables, and, therefore, solving (1) by any numerical scheme is a formidable task. In this case, it is desirable to find a transformation (such as (48)) that would allow us to separate at least one or two independent variables, using the intrinsic symmetry properties of the sphere, the scattering geometry and the phase function.

The method developed in 4 is useful as a first approximation. The scattering phase function of the dust grains in a comet is not isotropic, but can be approximated by

$$P(\mathbf{r};\Omega_1,\Omega_2) = \frac{1}{4\pi}\left[4\pi f\,\delta(\Omega_1 - \Omega_2) + (1 - f)\right] \quad ,$$

where $0 \leq f \leq 1$. In this case as in the case of a plane parallel atomosphere (Sobolev, 1975 8.3), the problem can be reduced to the isotropic one, using a suitable transformation.

VII. Conclusions

In the past thirty years the theory of radiative transfer in slab geometry has reached the level of maturity comparable to other branches of mathematical physics (Chandrasekhar, 1960; van de Hulst, 1980). However, the theory is still primitive for other geometries of practical interest. The integral equation approach holds great potential for advancing the subject. This method has at least three advantages: (a) there are no boundary conditions; the integral equation contains the complete formulation of the radiative transfer problem, (b) the important photometric quantity, the differential cross-section, can be computed using a functional expression that is highly accurate for reasonably accurate radiation fields, and (c) in many cases the symmetry of the equation is such as to suggest a simple transformation that results in the separation of variables for the radiation field.

In this paper we explored and exploited some of the beauties and subtleties of the integral equation. A complete variational formulation of the radiative transfer of non-polarized light was given in III. The derivation reveals a profound connection, hitherto unsuspected, between the variational principle, the principle of reciprocity, and the differential scattering cross section. The use of Feynman diagrams in Appendix B provides an intuitive understanding of some of these connections.

The integral equation for a homogeneous sphere with isotropic scatterers is shown to be separable, and have the solution which can be expanded in a multipole series. Detailed computations of the internal and photometric functions have been carried out. For all practical purposes, the optical properties of the homogeneous sphere of arbitrary size with isotropic scatterers are now known (or easily calculable). Our technique can be used to study the photometry of planetary atmospheres, cometary atmospheres and the twilight problem in the Earth's atmosphere. These studies will be pursued and reported in future publications.

Acknowledgements. We thank K.S.K. Chow, L.W. Esposito, A.J. Friedson, R.M. Goody and N.D. Sze for helpful discussions. The search for a variational principle for the planetary problem was assigned as one of the research topics in Professor R.M. Goody's class on atmospheric radiation at Harvard in 1972. The length of time it has taken to solve the problem is perhaps a measure of the sagacity of the master and the stupidity of the pupil (YLY). The research is supported by a NASA grant NSG 7376 under the Planetary Atmospheres Program.

Appendix A: Differential Cross-section

The definition of the differential cross section given by (5) is motivated by similar definitions in the electromagnetic and quantum theories of scattering (Jackson, 1975, Chapter 9; Landau and Lifshitz, 1965, Chapter 17). Since this is the first time that this quantity is introduced in radiative transfer, we shall give examples demonstrating the connection between the differential cross-section and related physical quantities that may be more familiar to the reader.

In an optically thin homogeneous medium, Eqn.(5) can be evaluated in the single scattering approximation

$$\frac{d\sigma}{d\Omega}(\mathbf{k},\mathbf{k}_o) \approx \frac{1}{4\pi}\alpha V P(\mathbf{k},\mathbf{k}_o) \quad . \tag{A1}$$

The total cross section in this case is

$$\sigma \approx \alpha \omega_o V$$

$$= N_o \omega_o \sigma_o \quad , \tag{A2}$$

where N_o is the total number of scatterers each with cross section σ_o.

For scattering of sunlight by a plane parallel atmosphere of area A, A being large, Eqn.(5) gives

$$\lim_{A\to\infty} \frac{\pi}{A}\frac{d\sigma}{d\Omega}(\mu,\mu_o) = \mu\mu_o R(\mu,\mu_o) \quad , \tag{A3}$$

where the reflectivity $R(\mu_1\mu_o)$ is a defined by van de Hulst (1980), and the meaning of the relevant angles is as shown in Figure 10a.

For scattering of sunlight by a sphere of radius a (see Figure 10b for description of geometry), evaluation of Eqn.(5) yields

$$\frac{d\sigma}{d\Omega}(\alpha_s) = \frac{a^2}{\pi F_o} j(\alpha_p) \quad , \tag{A4}$$

where α_s is the scattering angle given by $\cos\alpha_s = \mathbf{k}\cdot\mathbf{k}_o$, $\alpha_p = \pi - \alpha_s$ and $j(\alpha_p)$ is a well-known photometric function ,the flux (Horak, 1950; Harris, 1961).

It is of special interest to evaluate Eqn.(A4) for the simple case of a Lambert sphere with surface reflectivity λ. From van de Hulst (1980) we have

$$j(\alpha_p) = \frac{2}{3}\lambda F_o\left[\sin\alpha_p + (\pi - \alpha_p)\cos\alpha_p\right]$$

$$= \frac{2}{3}\lambda F_o\left[\sin\alpha_s - \alpha_s\cos\alpha_s\right] \quad . \tag{A5}$$

Substituting Eqn.(A5) in Eqn.(A4), we obtain

$$\frac{d\sigma}{d\Omega}(\alpha_s) = \frac{2\lambda}{3\pi} a^2 (\sin\alpha_s - \alpha_s\cos\alpha_s) \quad . \tag{A6}$$

On integrating Eqn.(A6) we get the expected total cross section

$$\sigma_{sca} = \lambda\pi a^2 \quad . \tag{A7}$$

Appendix B: Method of Successive Orders

To gain insight into the connection between the microscopic symmetry of $P(\mathbf{r};\Omega_1,\Omega_2)$ and the macroscopic symmetry of $\frac{d\sigma}{d\Omega}(\mathbf{k}_1,\mathbf{k}_2)$, we consider the method of successive orders for solving (1) (van de Hulst, 1980, Chapter 4). The idea is simple. We can regard Eqn.(1) as an iterative equation for keeping track of photons that have been scattered once, twice, $\cdots$, i times, $\cdots$. By summing over the contribution from all orders, we obtain the correct solution to Eqn.(1). The advantage of this approach is the intuitive understanding it offers. The main disadvantage seems to be slow convergence (Irvine, 1975), but that is not the issue here, since we are primarily interested in gaining theoretical insight.

Since radiative transfer is a simple many-body problem, it is instructive to represent the physical processes using Feynman diagrams. Equations Eqn.(1) and (5) contain four fundamental processes: the input factor $\varepsilon(\mathbf{r},\mathbf{k})$, the output factor $\varepsilon(\mathbf{r},-\mathbf{k})$, the distributor $\frac{\alpha(\mathbf{r})}{4\pi}P(\mathbf{r};\Omega_1,\Omega_2)$ and the propagator $\pi(\mathbf{r}_1,\mathbf{r}_2)$, as summarized in Table 3, using symbols adapted from the quantum theory of many-body problems (Mahan, 1981). Applying the method of successive orders, we can write the differential cross section as a series of partial cross-sections

$$\frac{d\sigma}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) = \sum_{i=1}^{\infty} \frac{d\sigma^{(i)}}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) \quad , \tag{B1}$$

where the superscript i refers to the number of times the photon has been scattered. The expressions for the partial cross sections can be easily obtained with the aid of Feynman diagrams. For single scattering we have

$$\frac{d\sigma^{(1)}}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) = \langle \varepsilon(\mathbf{r},-\mathbf{k}_2)\Lambda(\mathbf{r},\mathbf{k}_2,\mathbf{k}_1)\varepsilon(\mathbf{r},\mathbf{k}_1)\rangle \quad , \tag{B2}$$

where the pointed brackets imply integration over internal variables. The reverse process is

$$\frac{d\sigma^{(1)}}{d\Omega}(-\mathbf{k}_1,-\mathbf{k}_2) = <\varepsilon(\mathbf{r},\mathbf{k}_1)\Lambda(\mathbf{r},-\mathbf{k}_1,-\mathbf{k}_2)\varepsilon(\mathbf{r},-\mathbf{k}_2)> \quad . \tag{B3}$$

The expressions (B2) and (B3) are represented by the Feynman diagrams in Figure 11a. For photons that are scattered twice, we have

$$\frac{d\sigma^{(2)}}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) = <\varepsilon(\mathbf{r}_2,-\mathbf{k}_2)\Lambda(\mathbf{r}_2,\mathbf{k}_2,\mathbf{r}_{21})\pi(\mathbf{r}_2,\mathbf{r}_1)\Lambda(\mathbf{r}_1,\mathbf{r}_{21},\mathbf{k}_1)\varepsilon(\mathbf{r}_1,\mathbf{k}_1)> \quad , \tag{B4}$$

where the pointed brackets have the same meaning as before. The reverse is

$$\frac{d\sigma^{(2)}}{d\Omega}(-\mathbf{k}_1,-\mathbf{k}_2) = <\varepsilon(\mathbf{r}_1,\mathbf{k}_1)\Lambda(\mathbf{r}_1,-\mathbf{k}_1,\mathbf{r}_{12})\pi(\mathbf{r}_1,\mathbf{r}_2)\Lambda(\mathbf{r}_2,\mathbf{r}_{12},-\mathbf{k}_2)\varepsilon(\mathbf{r}_2,-\mathbf{k}_2)> \quad , \tag{B5}$$

where $\mathbf{r}_{12} = -\mathbf{r}_{21}$. The corresponding Feynman diagrams are shown in Figure 11b. Inspection of Eqn.(B2)-(B5) reveals that

$$\frac{d\sigma^{(i)}}{d\Omega}(-\mathbf{k}_1,-\mathbf{k}_2) = \frac{d\sigma^{(i)}}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) \tag{B6}$$

for $i = 1$ and 2, if

$$\Lambda(\mathbf{r},-\Omega_2,-\Omega_1) = \Lambda(\mathbf{r},\Omega_1,\Omega_2) \tag{B7}$$

and

$$\pi(\mathbf{r}_2,\mathbf{r}_1) = \pi(\mathbf{r}_1,\mathbf{r}_2) \quad . \tag{B8}$$

Eqn.(B8) is true from its definition. Eqn.(B7) holds if

$$P(\mathbf{r};-\Omega_2,-\Omega_1) = P(\mathbf{r};\Omega_1,\Omega_2) \quad . \tag{B9}$$

We can now see the direct connection between Eqn.(B9) and (B6). The argument can be obviously generalized to all orders and, hence, Eqn.(B9) implies the reciprocity symmetry of the differential cross section in Eqn.(B1). The results obtained here are stronger than those in §3 for two reasons: the symmetry requirement in Eqn.(B9) is weaker than in Eqn.(6), and the reciprocity relation, Eqn.(B6), is true for each order of scattering as well as for the sum total given by Eqn.(39).

The method of successive orders offers a demonstration of the advantage of the functional expression of Eqn.(39) over the direct expression of Eqn.(27) for approximate evaluation of the differential cross section. Let J_n and $\tilde{J}_n$ be approximate solutions of (25) and (26) that are correct to n^{th} order; i.e.

$$J_n = \sum_{i=1}^{n} J^{(n)} \tag{B10}$$

$$\tilde{J}_n = \sum_{i=1}^{n} \tilde{J}^{(n)} \quad , \tag{B11}$$

where $J^{(i)}$ and $\tilde{J}^{(i)}$ refer to the radiation field of photons which are scattered i times. It can be shown, after some tedious but straightforward algebra, that expressions in Eqn.(27) and (39), respectively, give

$$\frac{d\sigma}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) = \sum_{i=1}^{n} \frac{d\sigma^{(i)}}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) \tag{B12}$$

$$\frac{d\sigma}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) = \sum_{i=1}^{2n} \frac{d\sigma^{(i)}}{d\Omega}(\mathbf{k}_2,\mathbf{k}_1) \quad . \tag{B13}$$

Thus, the functional expression achieves an accuracy equivalent to $2n$ orders of scattering for approximate radiation fields that are correct only to n orders of scattering.

Appendix C: Green's Function

To derive Eqn.(51), all we need is to use the following formulas (Abramowitz and Stegun, 1972, Chapter 10):

$$j_l(z) = h_l^{(1)}(z) + h_l^{(2)}(z) \tag{C1}$$

$$h_l^{(1)}(z) = i^{-l-1} z^{-1} e^{iz} \sum_{k=0}^{l} \frac{(l+k)!}{k!(l-k)!}(-2iz)^{-k} \tag{C2}$$

$$h_l^{(2)}(z) = i^{l+1} z^{-1} e^{-iz} \sum_{k=0}^{l} \frac{(l+k)!}{k!(l-k)!}(2iz)^{-k} \quad . \tag{C3}$$

The resulting expression in Eqn.(51) follows from simplifying and collecting the coefficients for the exponential integral functions. Representative values for the coefficients A_{lm} and B_{lm} are shown in Table 4. Eqn.(51) is unfortunately not suitable for numerical computation of $G_l(\tau_1,\tau_2)$ when l is large and τ_1 and τ_2 are $\lesssim 1$. The reason is that in this situation the individual terms in Eqn.(51) are extremely large, even though they eventually cancel each other to yield a much smaller sum. Indeed, round-off errors render Eqn.(51) useless for $l \geq 10$ even for computations carried out with 32 significant figures. In practice we use Eqn.(51) only for $l = 0$ to 2. For other multipoles we use another formula:

$$G_l(\tau_1,\tau_2) = \frac{\tau_1\tau_2}{2} \int_{-1}^{1} \frac{e^{-(\tau_1^2+\tau_2^2-2\tau_1\tau_2\mu)^{1/2}}}{\tau_1^2+\tau_2^2-2\tau_1\tau_2\mu} P_l(\mu)\, d\mu \tag{C4}$$

and carry out the numerical evaluation with 36 points Gaussian quadrature. The effect of the singularity $G_l(\tau,\tau)$ is properly assessed by its integral over a grid box (Yung, 1976).

Figure 12 shows the values of $G_l(\tau_1,\tau_2)$ for two representative cases: $\tau_1 = 1$ and $\tau_1 = 8$. It is of interest to note that as l increases the Green's functions become more concentrated near $\tau_1 = \tau_2$, and that the areas under the curves become smaller. The net effect of this behavior is to make the first term more important relative to the second term on the right hand side of Eqn.(50).

Hence, it would be practical to use the method of successive orders for solving Eqn.(50) for higher multipoles.

Table 1

List of important photometric functions for the scattering of a parallel light beam by a particle or a planet. These functions are evaluated in terms of functions defined in this work (see Figure 10b for explanation of scattering angle α_s and phase angle α_p). Note that $Q_{sca} = A$.

Particle Photometric Functions:

Differential cross section

$$\frac{d\sigma}{d\Omega}(\alpha_s) = \frac{\omega_o}{\alpha^2}\sum_{l=0}^{\infty}(-1)^l f_l P_l(\cos\alpha_s)$$

Scattering cross-section

$$\sigma_{sca} = 2\pi\int_0^{\pi}\frac{d\sigma}{d\Omega}\sin\alpha_s\, d\alpha_s = \frac{4\pi\omega_o}{\alpha^2}f_o$$

Scattering efficiency

$$Q_{sca} = \frac{\sigma_{sca}}{\pi a^2} = \frac{4\omega_o}{\tau_o^2}f_o$$

Extinction efficiency

$$Q_{ext} = \frac{16}{\tau_o^2 F_o}\int_0^{\tau_o} d\tau\,\tau\, S_o(\tau)$$

Asymmetry factor

$$g = \frac{2\pi}{\sigma_{sca}}\int_0^{\pi}\frac{d\sigma}{d\Omega}\sin\alpha_s\,\cos\alpha_s\, d\alpha_s = -\frac{f_1}{3f_o}$$

Phase function

$$p(\cos\alpha_s) = \frac{4\pi\omega_o}{\sigma_{sca}}\frac{d\sigma}{d\Omega}(\alpha_s) = \frac{\omega_o}{f_o}\sum_{l=0}^{\infty}(-1)^l f_l P_l(\cos\alpha_s)$$

Planetary Photometric Functions:

Flux

$$j(\alpha_p) = \frac{\pi F_o}{a^2}\frac{d\sigma}{d\Omega}(\alpha_s)$$

$$= \frac{\omega_o \pi F_o}{\tau_o^2}\sum_{l=0}^{\infty} f_l P_l(\cos\alpha_p)$$

Geometric albedo

$$p = \frac{j(o)}{\pi F_o} = \frac{\omega_o}{\tau_o^2}\sum_{l=0}^{\infty} f_l$$

Phase variation

$$\varphi(\alpha_p) = \frac{j(\alpha_p)}{j(o)} = \frac{\sum_{l=0}^{\infty} f_l P_l(\cos\alpha_p)}{\sum_{l=0}^{\infty} f_l}$$

Phase integral

$$q = 2\int_0^{\pi} d\alpha_p \varphi(\alpha_p) = \frac{4f_o}{\sum_{l=0}^{\infty} f_l}$$

Bond albedo

$$A = pq = \frac{4\omega_o f_o}{\tau_o^2}$$

Table 2

Table 2

Values of f_l (see equation (55) for definition)
for the cases τ_o = 1 and 8.

l	f_l (τ_o = 1)	f_l (τ_o = 8)
0	4.39×10^{-2}	3.49
1	5.48×10^{-3}	3.22
2	1.30×10^{-4}	0.12
3	4.8×10^{-5}	0.05
4	1.4×10^{-5}	9.4×10^{-3}
5	2.0×10^{-6}	1.1×10^{-2}
6	2.1×10^{-6}	1.5×10^{-3}
7	1.9×10^{-7}	3.5×10^{-3}
8	4.5×10^{-7}	3.3×10^{-4}
9	2.9×10^{-8}	1.4×10^{-4}
10	1.3×10^{-7}	9.6×10^{-5}

Table 3

Graphical representation of fundamental processes in radiative transfer.
These symbols are used in the construction of Feynman diagrams in Figure 11.

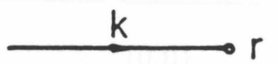
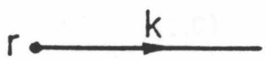
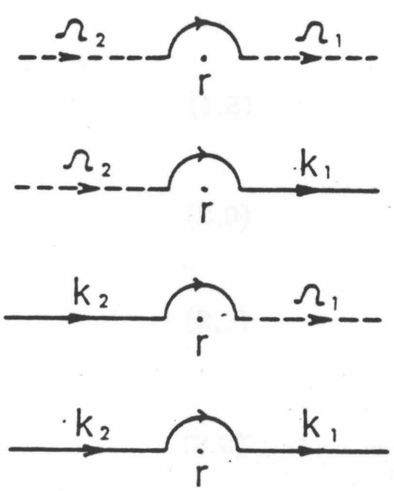
Symbol	Factor in $\frac{d\sigma}{d\Omega}$
	$\varepsilon(\mathbf{r},\mathbf{k})$, incoming photon
	$\varepsilon(\mathbf{r},-\mathbf{k})$, outgoing photon
	$\Lambda(\mathbf{r},\mathbf{n}_1,\mathbf{n}_2) = \frac{\alpha(\mathbf{r})}{4\pi} P(\mathbf{r},\mathbf{n}_1,\mathbf{n}_2)$ with $\mathbf{n}_i = \Omega$ or $\mathbf{k}$ angular distributor
	$\pi(\mathbf{r}_1,\mathbf{r}_2) = \frac{e^{-\tau(\mathbf{r}_1,\mathbf{r}_2)}}{r_{12}^2}$ internal propagator

Table 4

Values for the coefficients A_{lm} and B_{lm} as defined in (51)

for the first three multipoles l = 0, 1, and 2.

Multipole (l)	Index (l,m)	A_{lm}	B_{lm}
0	(0,0)	1	1
1	(1,0)	1	1
	(1,1)	$\frac{1}{\tau_<\tau_>}(\tau_> + \tau_<)$	$-\frac{1}{\tau_<\tau_>}(\tau_> - \tau_<)$
	(1,2)	$\frac{1}{\tau_<\tau_>}$	$-\frac{1}{\tau_<\tau_>}$
2	(2,0)	1	1
	(2,1)	$\frac{3}{\tau_<\tau_>}(\tau_> + \tau_<)$	$-\frac{3}{\tau_<\tau_>}(\tau_> - \tau_<)$
	(2,2)	$\frac{3}{\tau_<^2\tau_>^2}(2\tau_>^2 + 3\tau_>\tau_< + 2\tau_<^2)$	$\frac{3}{\tau_<^2\tau_>^2}(2\tau_>^2 - 3\tau_>\tau_< + 2\tau_<^2)$
	(2,3)	$\frac{9}{\tau_<^2\tau_>^2}(\tau_> + \tau_<)$	$\frac{9}{\tau_<^2\tau_>^2}(\tau_> - \tau_<)$
	(2,4)	$\frac{9}{\tau_<^2\tau_>^2}$	$\frac{9}{\tau_<^2\tau_>^2}$

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Figure Captions

Figure 1(a). Schematic diagram for radiative transfer in an object of volume v and surface S. The incident and emergent beams are in the directions $\mathbf{k}_o$ and $\mathbf{k}$, respectively. The points $\mathbf{r}_1$ and $\mathbf{r}_2$ are interior points, and $d\mathbf{r}_2$ is an infinitesmal volume at $\mathbf{r}_2$. The points $R(\mathbf{r}_1,\mathbf{k}_o)$ and $R(\mathbf{r}_1,-\mathbf{k})$ are on the surface defined such that $R(\mathbf{r}_1,\mathbf{k}_o) - \mathbf{r}_1$ is antiparallel to $\mathbf{k}_o$, and $R(\mathbf{r}_1,-\mathbf{k}) - \mathbf{r}_1$ is parallel to $\mathbf{k}$. (b) Schematic diagram for scattering of sunlight by an object of volume v. In the forward process the incident and observation directions are $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. In the reverse process the corresponding directions are $-\mathbf{k}_2$ and $-\mathbf{k}_1$, respectively.

Figure 2. Source function in Eqn. (50). (a) S_o, S_1, and S_2 for τ_o = 1 case. (b) S_o, S_1, S_2, S_8 for τ_o = 8 case.

Figure 3. Solution of Eqn.(50). (a) ψ_o, ψ_1, ψ_2 for τ_o = 1 case. (b) ψ_o, ψ_1, ψ_2, ψ_8 for τ_o = 8 case.

Figure 4. Internal mean intensity of radiation as a function of τ. (a) $J_o(\tau,\mu=-1)$, $J(\tau,\mu=-1)$ for τ_o = 1 case. (b) $J_o(\tau,\mu=-1)$, $J(\tau,\mu=-1)$ for τ_o = 8 case.

Figure 5. Extinction efficiency Q_{ext} as a function of τ for τ = 1 to 8.

Figure 6. Scattering efficiency Q_{sca} as a function of ω_o for τ_o = 1 and τ_o = 4 cases. The case $\tau_o = \infty$ is taken from Irvine's (1975) Figure 5.

Figure 7. Asymmetry factor g as a function of τ for ω_o = 1 and ω_o = 0.5 cases.

Figure 8. Geometric albedo of a sphere p as a function of ω_o for τ_o = 1 and τ_o = 4 cases. The case of τ_o ∞ is taken from Harris (1961).

Figure 9. Phase variation $\varphi(\alpha_p)$ as a function of phase angle α_p for τ_o = 1 and τ_o = 8 cases. The case of $\tau_o = \infty$ is taken from van de Hulst (1980).

Figure 10. (a) Schematic diagram for scattering of sunlight by a plane parallel atmosphere of area A and local normal $\mathbf{n}$. The cosine of the zenith angles μ_o and μ are given by $-\mathbf{k}_o \cdot \mathbf{n}$ and $\mathbf{k} \cdot \mathbf{n}$, respectively. (b) Schematic diagram for scattering of sunlight by a sphere of radius a. The phase angle α_p and the scattering angle α_s are defined by $\cos\alpha_p = -\mathbf{k}_o \cdot \mathbf{k}$ and $\cos\alpha_s = \mathbf{k}_o \cdot \mathbf{k}$.

Figure 11(a). Feynman diagrams for photons that are scattered once. See Table 3 for explanation of symbols. (b) Same as (a) for photons that are scattered twice.

Figure 12. Green's function $G_l(\tau_1,\tau_2)$ as a function of τ_2. (a) $l = 0, 4, 8$ and $\tau_1 = 1.0$. (b) $l = 0, 4, 8$ and $\tau_1 = 8.0$.

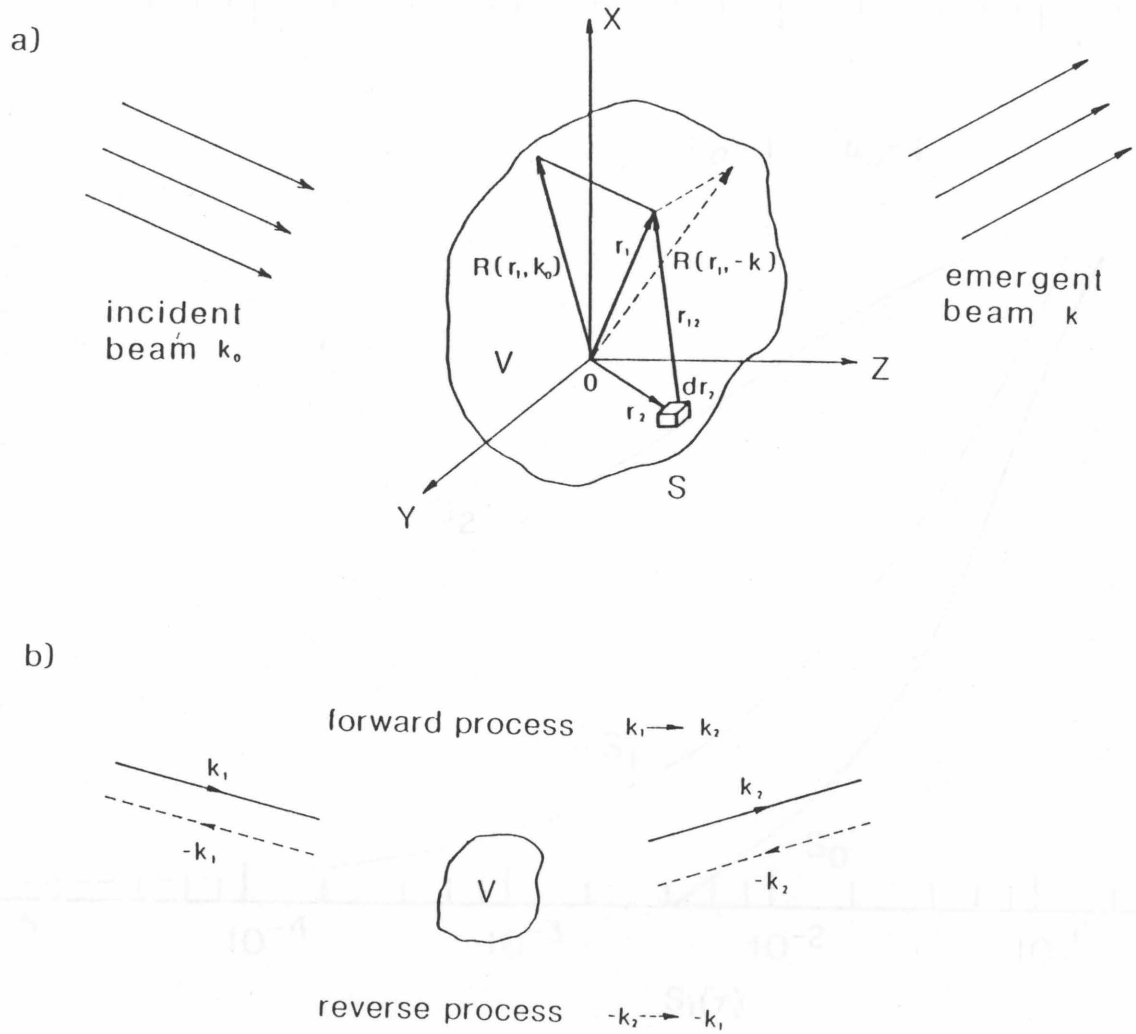

a)
X
incident
beam k_0
$R(r_1, k_0)$
r_1
$R(r_1, -k)$
r_{12}
emergent
beam k
V
0
Z
dr_2
r_2
S
Y
b)
forward process $k_1 \rightarrow k_2$
k_1
$-k_1$
k_2
$-k_2$
V
reverse process $-k_2 \dashrightarrow -k_1$

Fig. 1

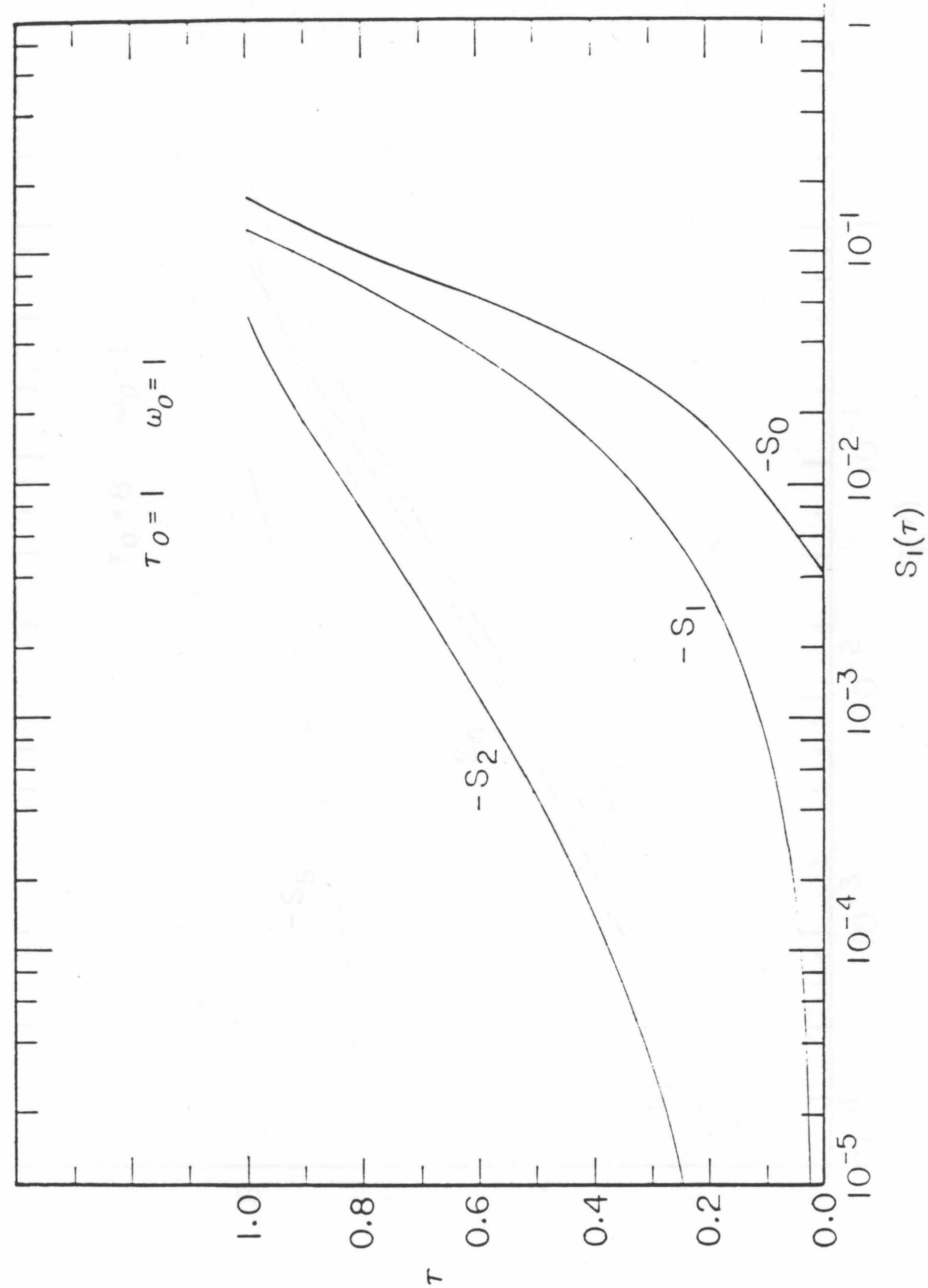

τ
1.0
0.8
0.6
0.4
0.2
0.0
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
$S_I(\tau)$
$\tau_0 = 1$ $\omega_0 = 1$
$-S_0$
$-S_1$
$-S_2$

Fig.2(a)

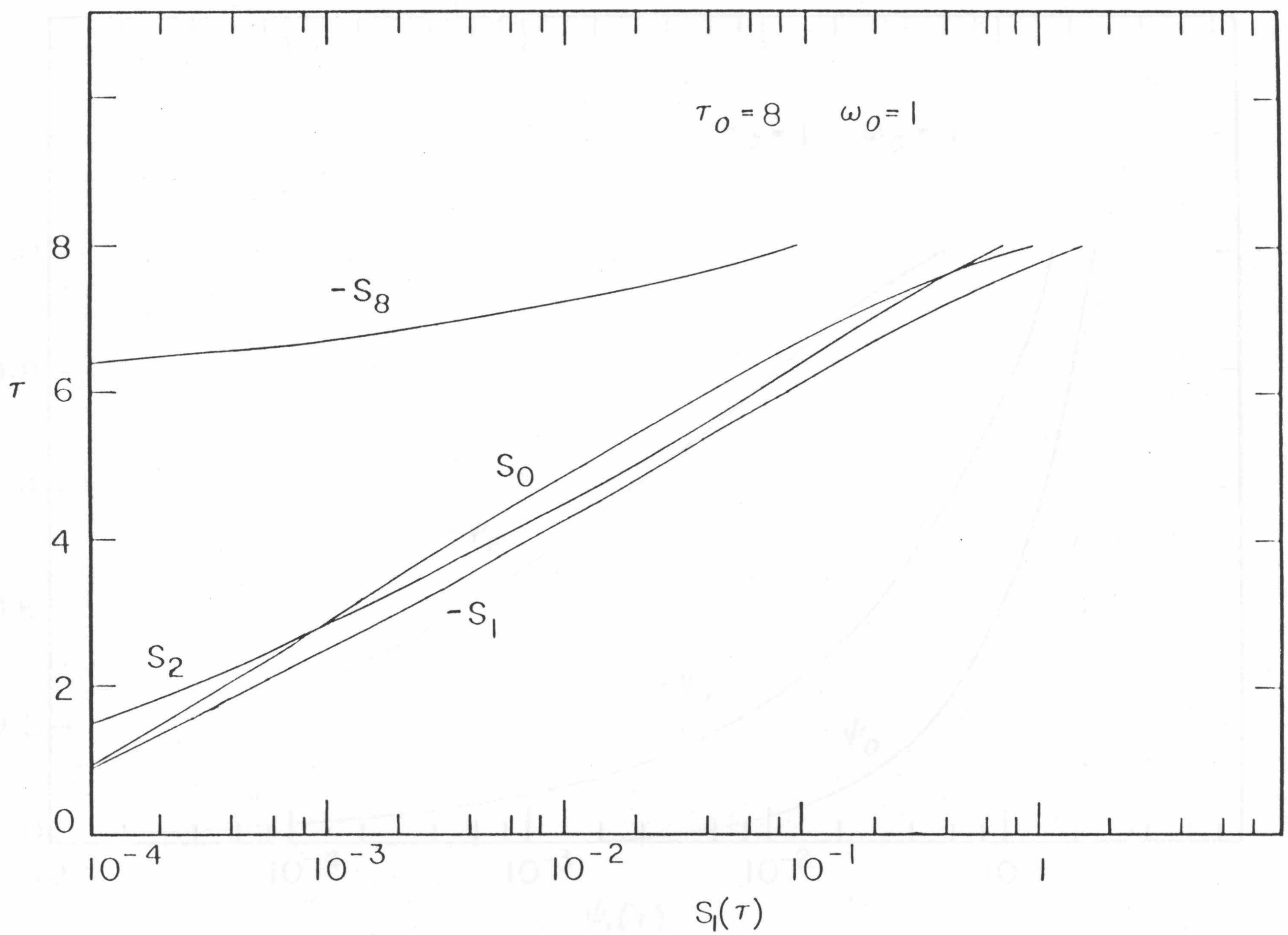


Fig.2(b)

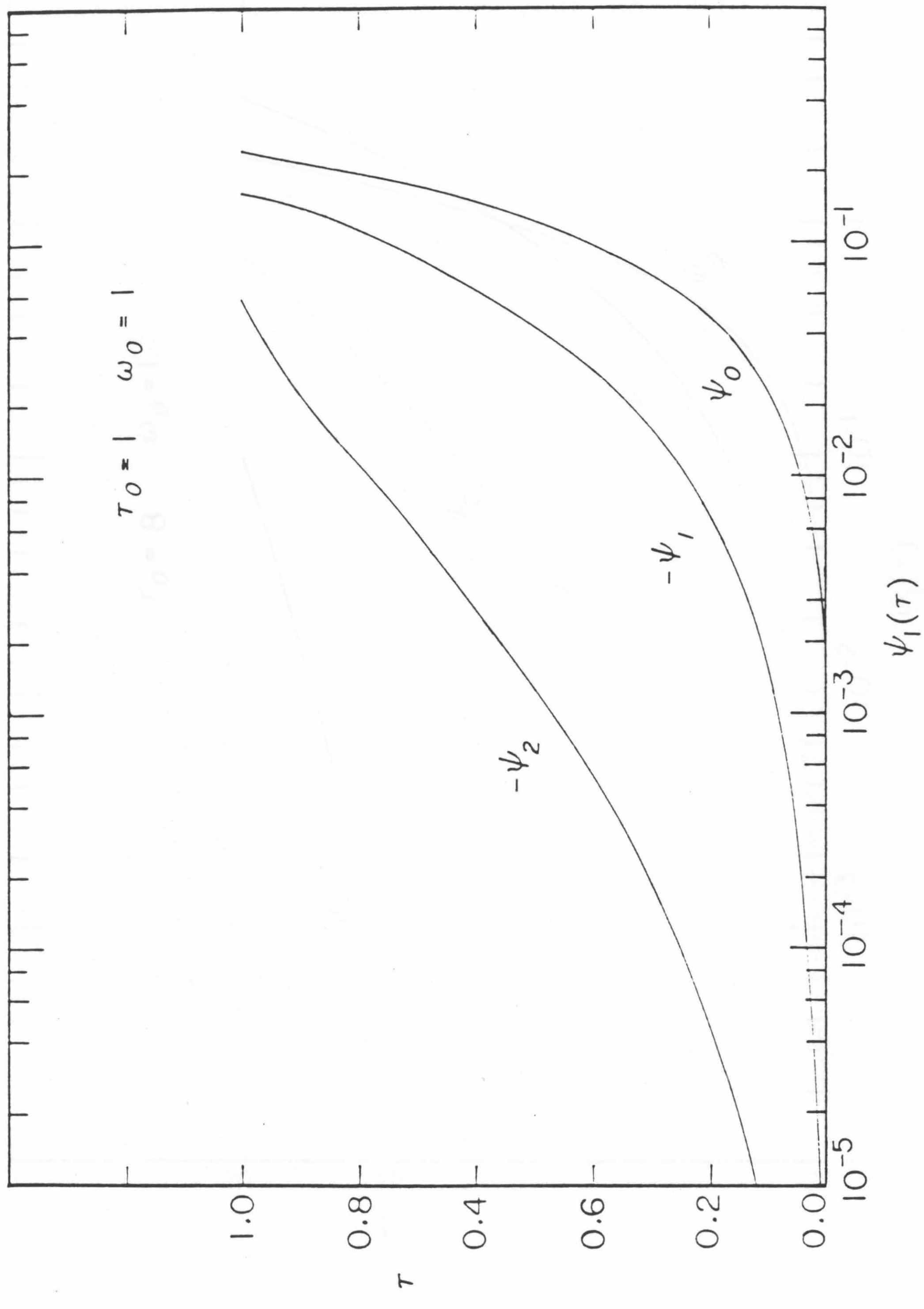

$\tau_0 = 1 \quad \omega_0 = 1$
$-\psi_2$
$-\psi_1$
ψ_0
$\psi_l(\tau)$
τ
1.0
0.8
0.4
0.6
0.2
0.0
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}

Fig.3(a)

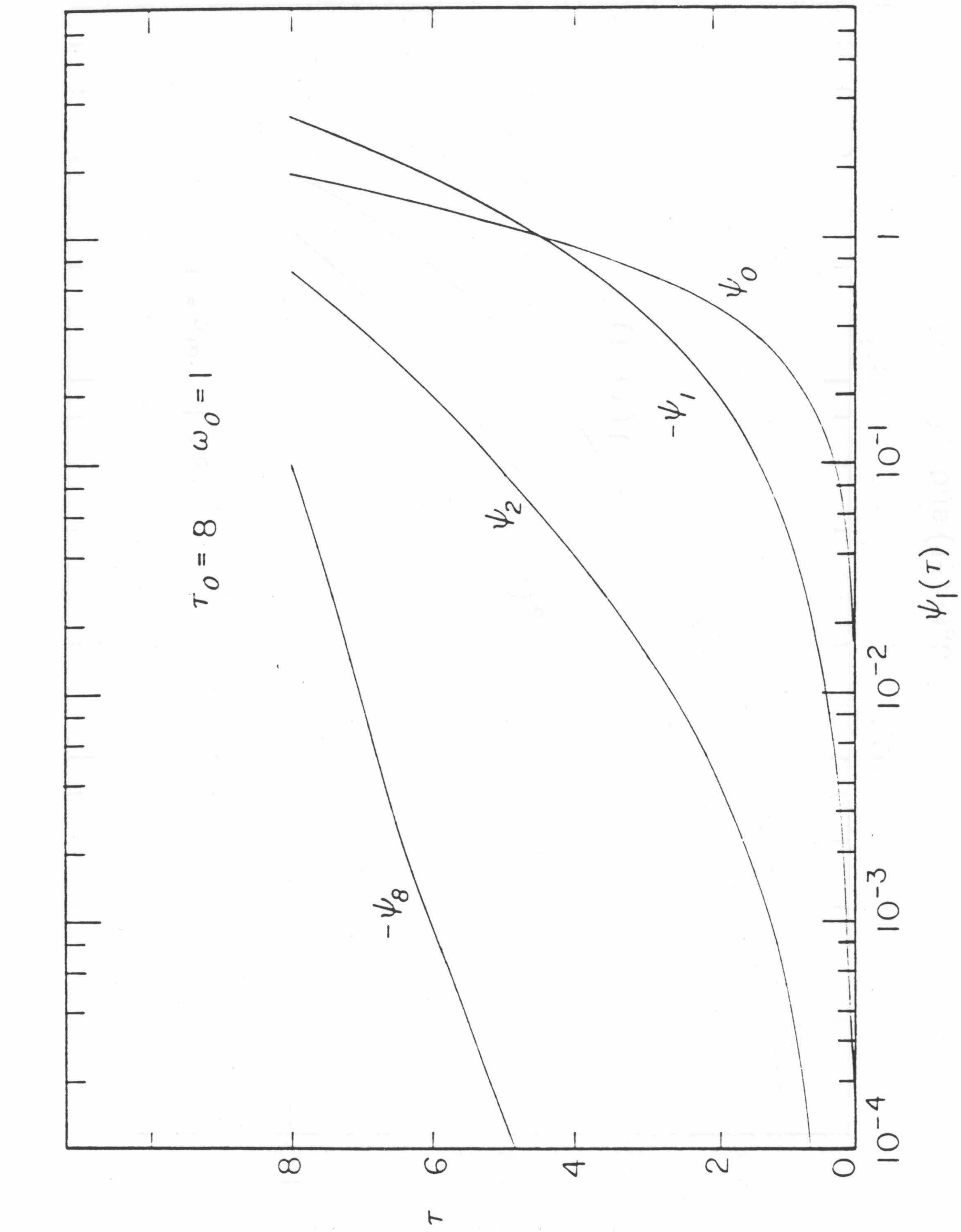
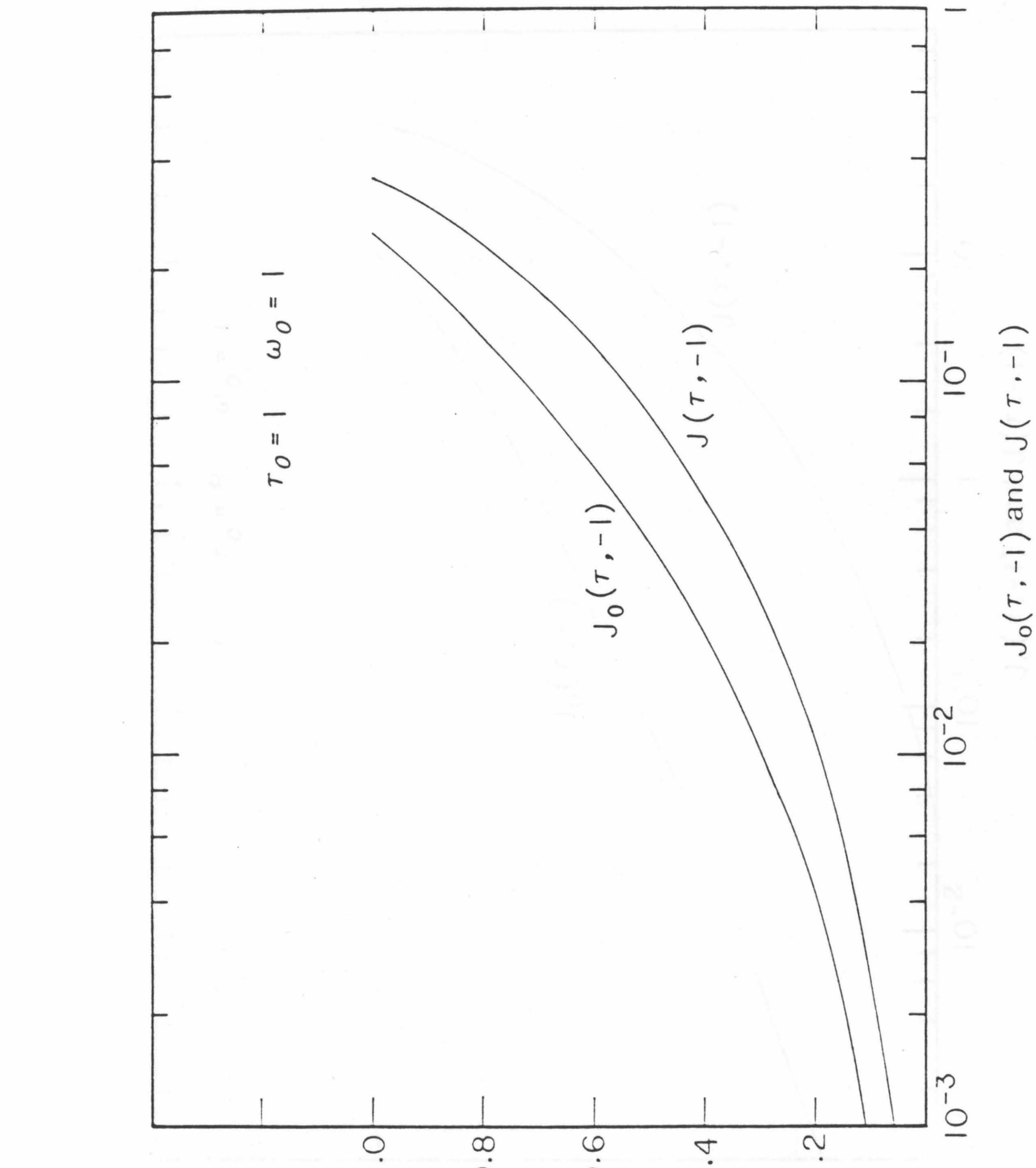
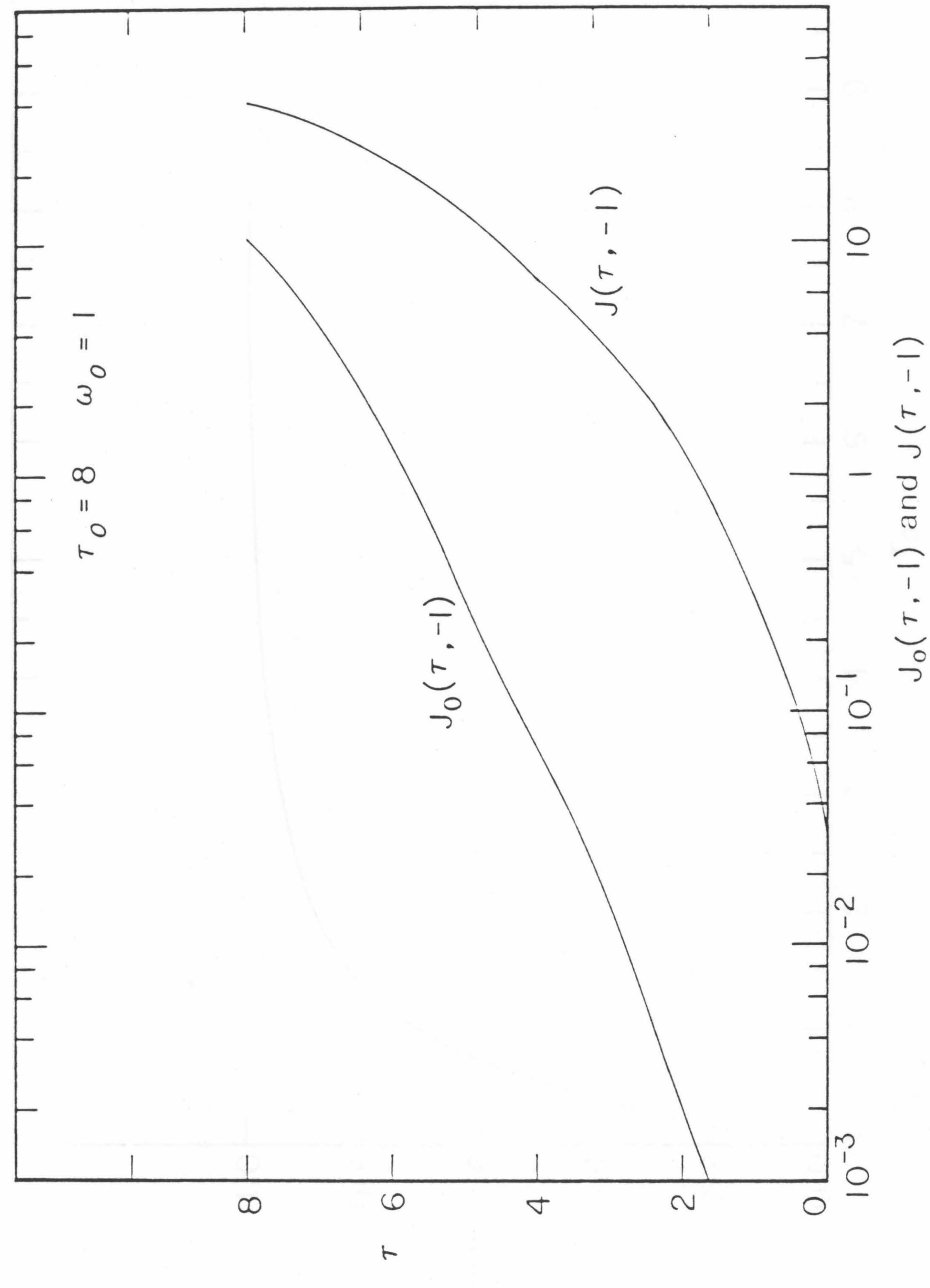

$\tau_0 = 8$
$\omega_0 = 1$
ψ_0
$-\psi_1$
ψ_2
$-\psi_8$
τ
0
2
4
6
8
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
$\psi_l(\tau)$

Fig.3(b)



$\tau_0 = 1$ $\omega_0 = 1$
$J_0(\tau,-1)$
$J(\tau,-1)$
1.0
0.8
τ 0.6
0.4
0.2
10^{-3}
10^{-2}
10^{-1}
1
$J_0(\tau,-1)$ and $J(\tau,-1)$

Fig.4(a)



$\tau_0 = 8$ $\omega_0 = 1$
$J(\tau, -1)$
$J_0(\tau, -1)$
τ
0
2
4
6
8
10^{-3}
10^{-2}
10^{-1}
1
10
$J_0(\tau, -1)$ and $J(\tau, -1)$

Fig. 4(b)

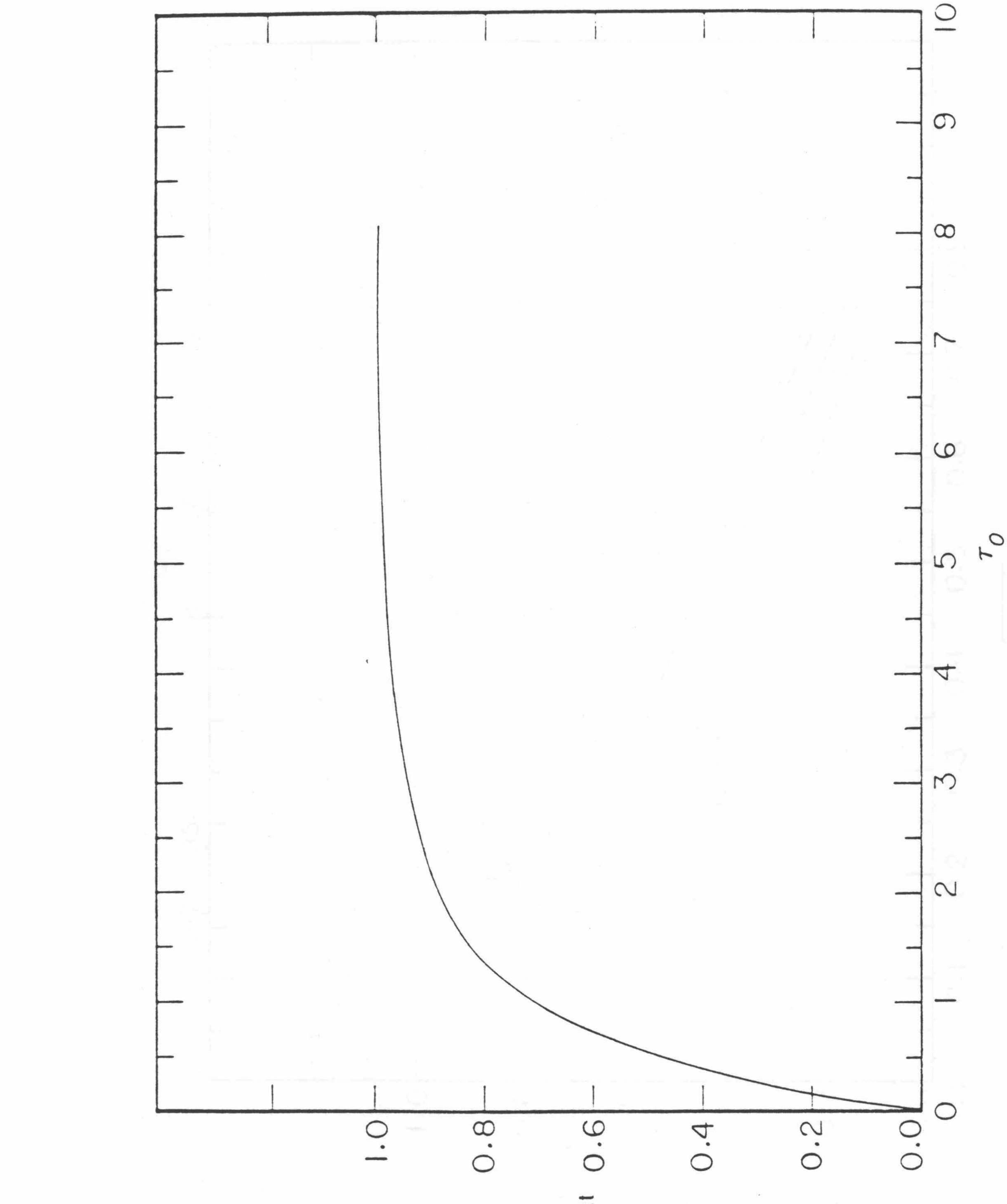


Fig. 5

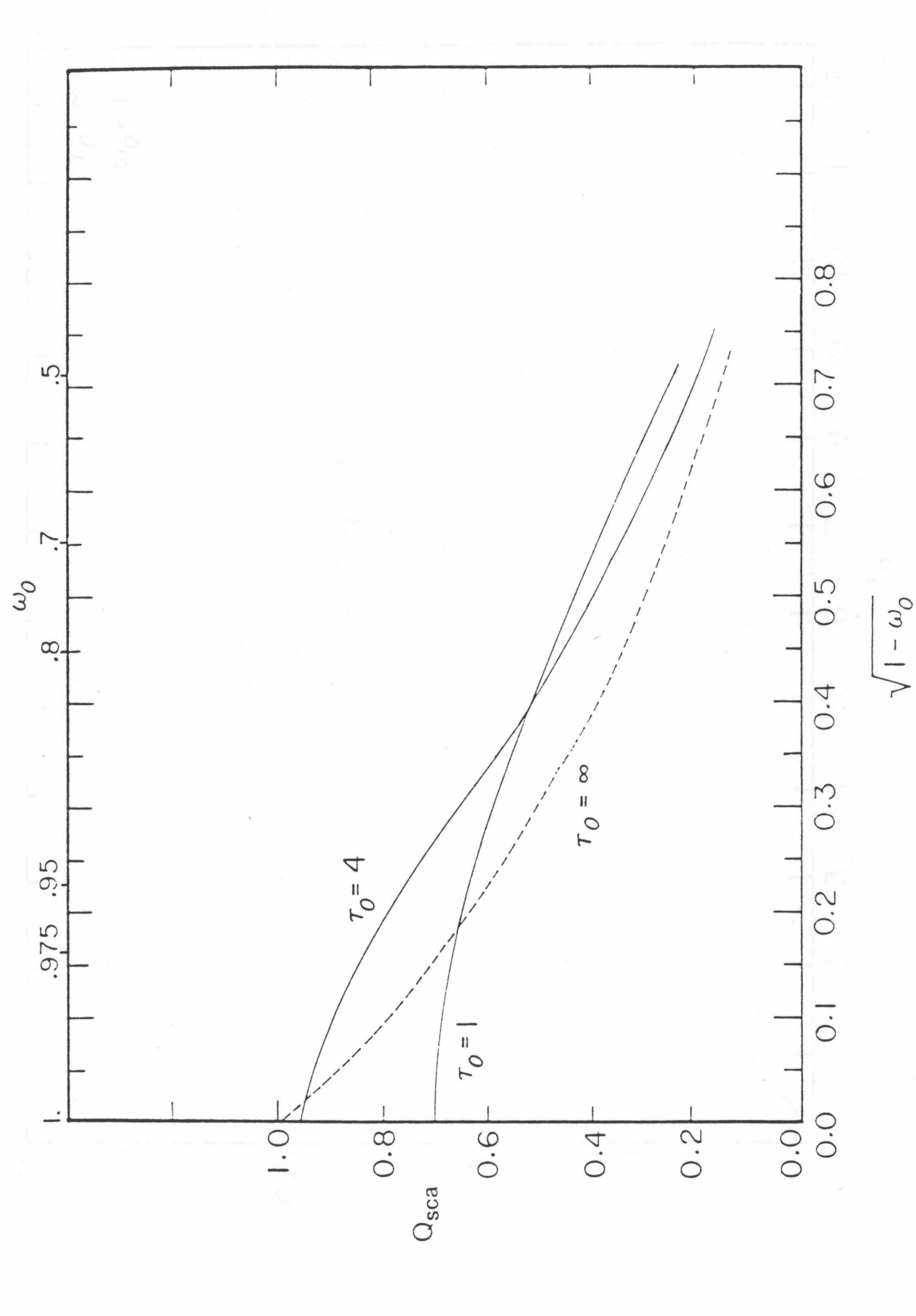

ω_0
1.
.975
.95
.8
.7
.5
Q_{sca}
1.0
0.8
0.6
0.4
0.2
0.0
0.1
0.3
0.5
0.7
$\tau_0 = 4$
$\tau_0 = 1$
$\tau_0 = \infty$
$\sqrt{1 - \omega_0}$

Fig. 6

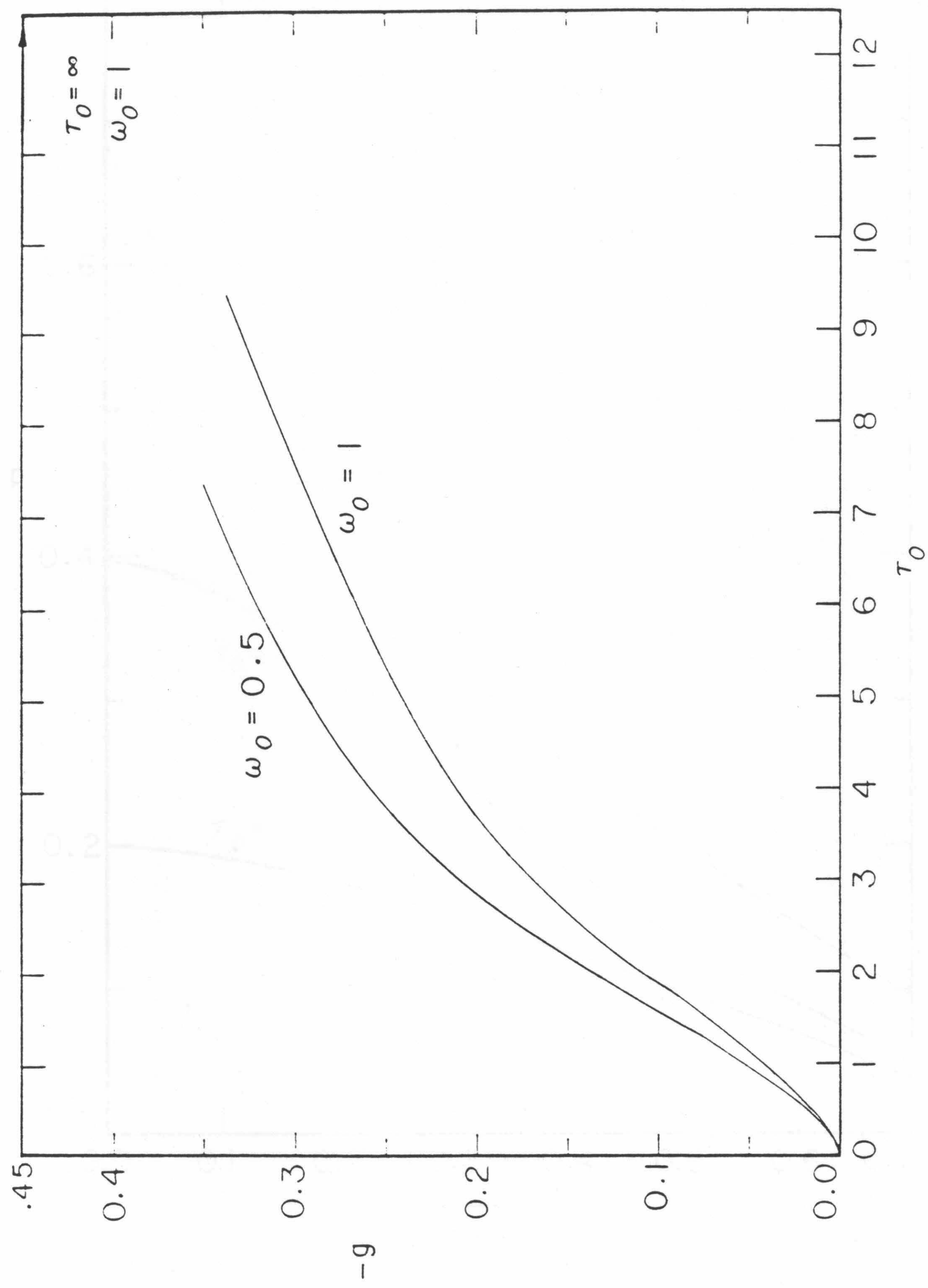

$\tau_0 = \infty$
$\omega_0 = 1$
$\omega_0 = 0.5$
$\omega_0 = 1$
-g
.45
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
6
7
8
9
10
11
12
τ_0

Fig. 7

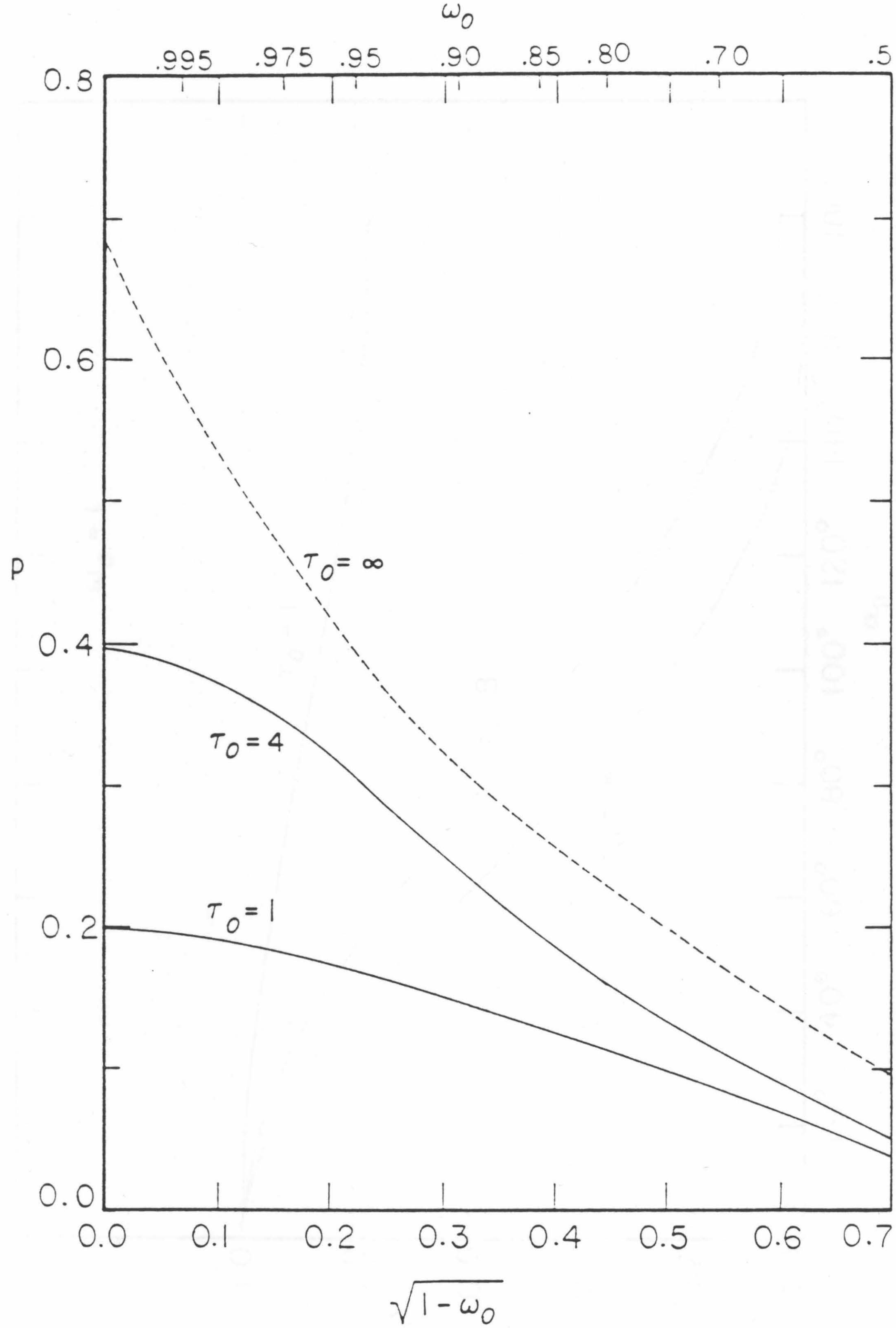

ω_o
.995
.975
.95
.90
.85
.80
.70
.5
0.8
0.6
0.4
0.2
0.0
p
$\tau_o = \infty$
$\tau_o = 4$
$\tau_o = 1$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
$\sqrt{1-\omega_o}$

Fig. 8

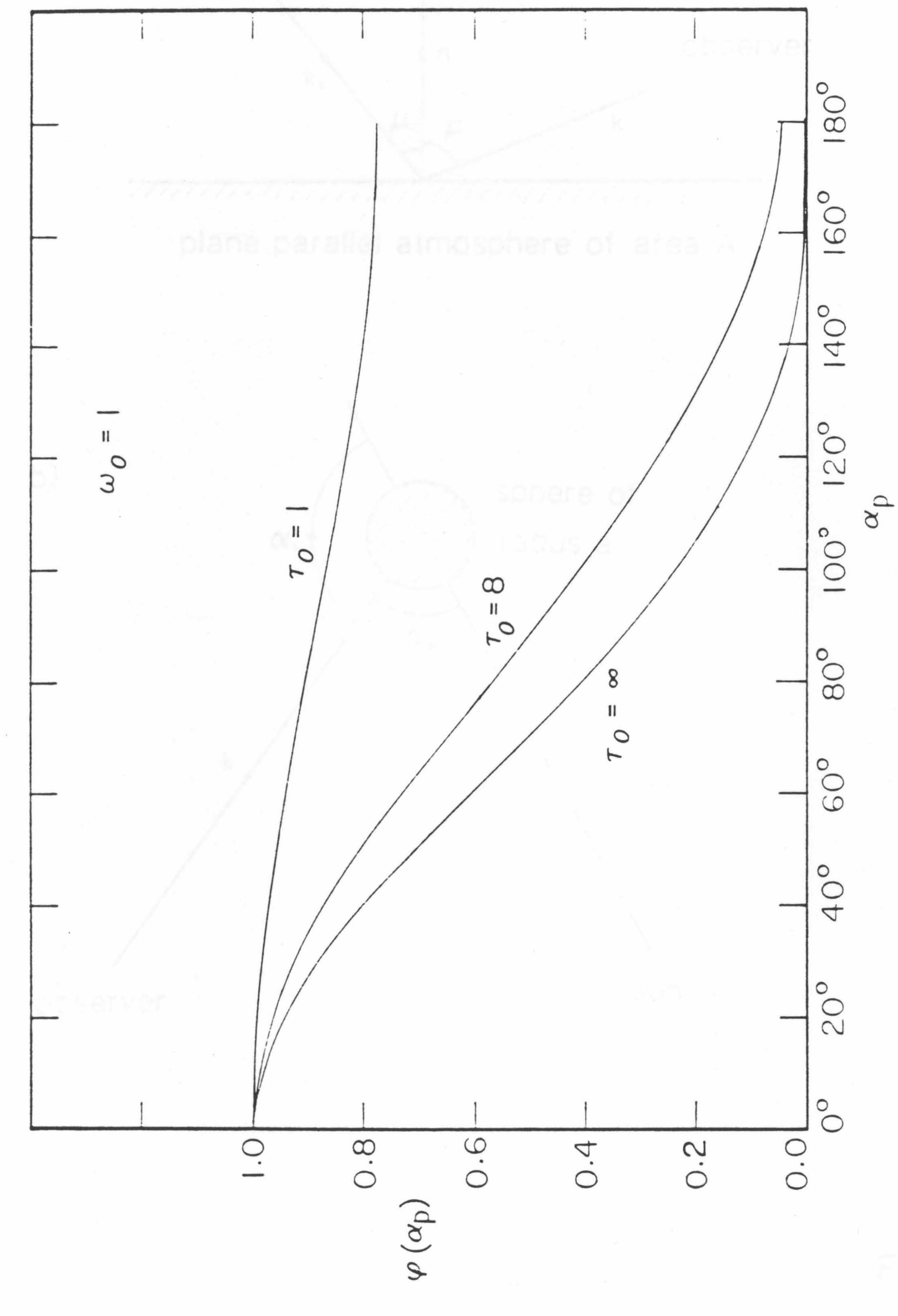
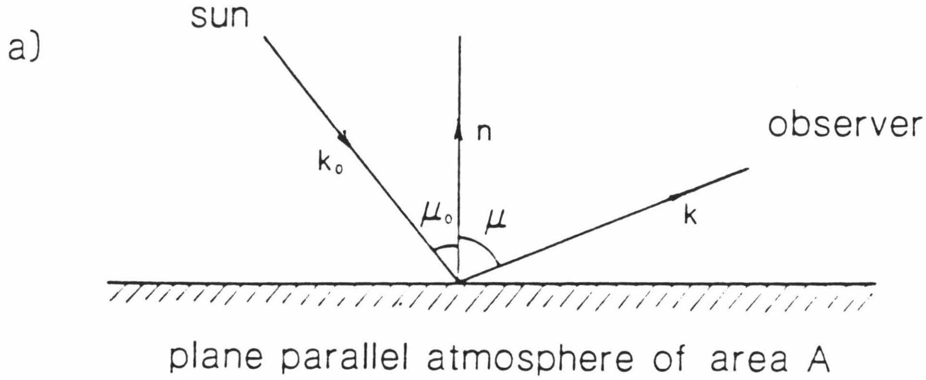

φ(α_p)
1.0
0.8
0.6
0.4
0.2
0.0
0°
20°
40°
60°
80°
100°
120°
140°
160°
180°
α_p
ω_0 = 1
τ_0 = 1
τ_0 = 8
τ_0 = ∞

Fig. 9


a)
sun
n
observer
k₀
μ_0
μ
k
plane parallel atmosphere of area A

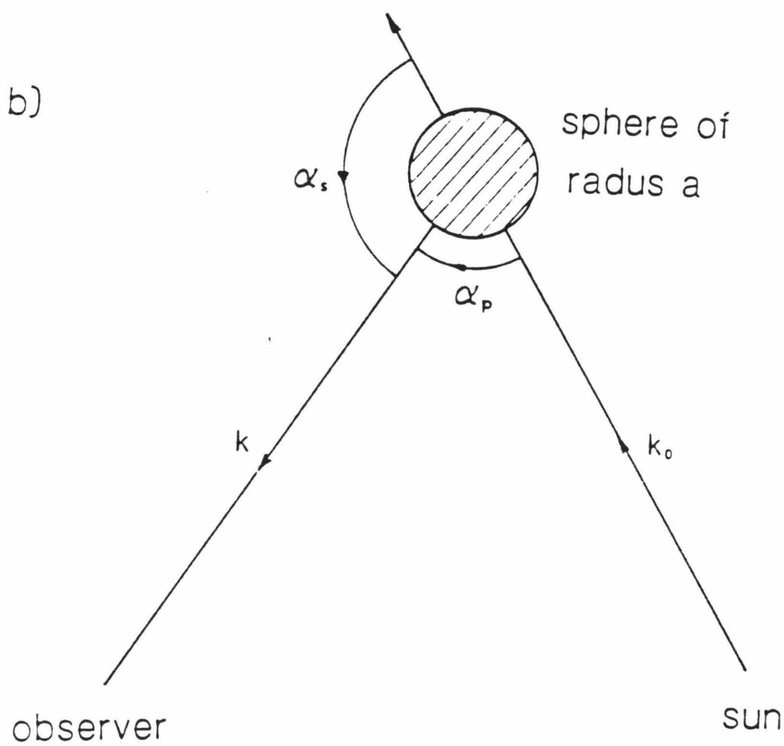
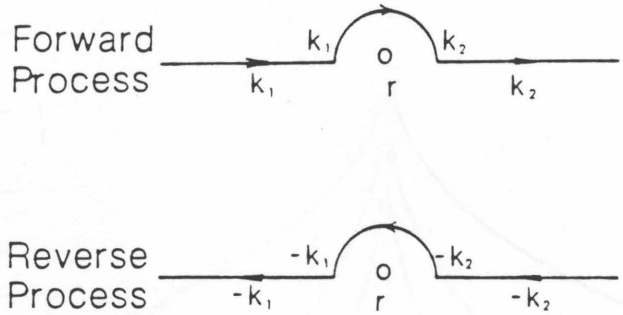
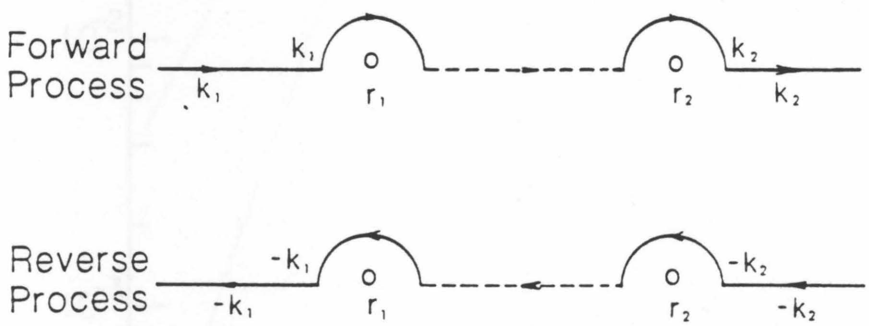

b)
sphere of
radus a
α_s
α_p
k
k_0
observer
sun

Fig. 10

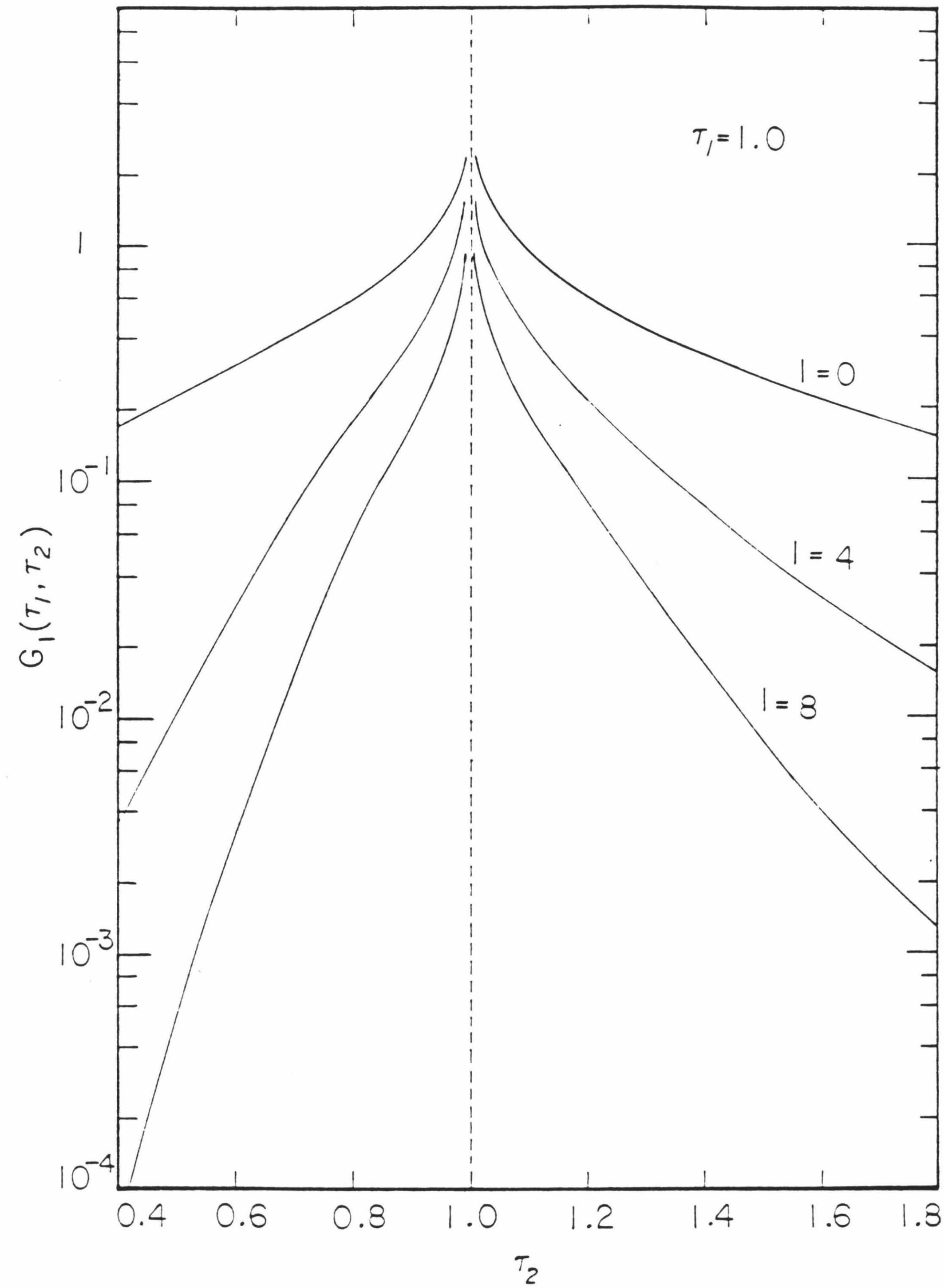


a) photon scattered once



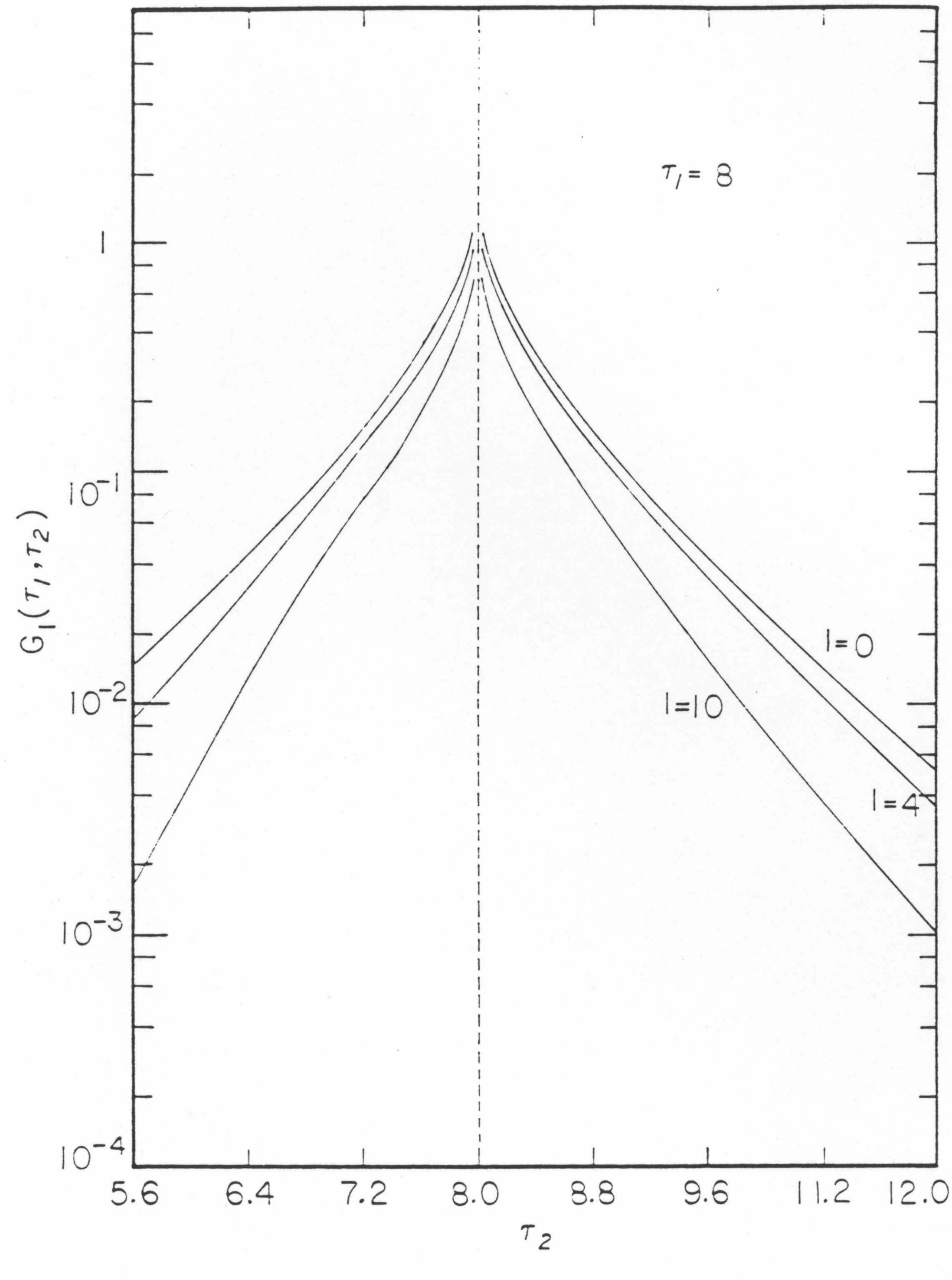
b) photon scattered twice

Fig. 11



$\tau_1 = 1.0$
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$G_1(\tau_1, \tau_2)$
l = 0
l = 4
l = 8
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
τ_2

Fig.12(a)



$\tau_1 = 8$
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$G_l(\tau_1, \tau_2)$
l=0
l=10
l=4
5.6
6.4
7.2
8.0
8.8
9.6
11.2
12.0
τ_2

Fig.12(b)