

STOCHASTIC ANALYSIS OF THE SEISMIC  
RESPONSE OF SECONDARY SYSTEMS

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ABSTRACT

This thesis is concerned with the earthquake response of light equipment in structures. The motion of the ground during an earthquake is represented as a stochastic process in order to reflect uncertainty in the prediction of such motion. A number of different stochastic earthquake models are considered, and analytical methods are described for these models.

The response of equipment in a structure to stochastic ground motion is derived, in the case of a single-degree-of-freedom secondary system (equipment) attached to a single-degree-of-freedom structure. The distribution of the envelope of the secondary system displacement is obtained for general transient ground motion. Closed form expressions are computed for the transient response to stationary ground motion.

The effect of the interaction of equipment with the structure is described by the introduction of an equivalent non-interacting system. However, this method applies only to classically damped systems.

The results are applied in a simple way to the problem of the computation of floor spectra. It is found that the ground spectrum is amplified in a simple way, except near resonance, where special considerations must be addressed.

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## CHAPTER I

### INTRODUCTION

A problem which has received increased attention recently is the design of secondary systems to withstand seismic loads. A secondary system may be a piece of equipment or a structure which is distinguished from the supporting primary structure. Secondary systems are usually characterized by a mass which is small in comparison with the mass of the structure on which they are supported. Frequently, such substructures are essential for the safety of the occupants of the primary structure and may have even greater importance. This is true especially in the design of crucial facilities such as a nuclear reactor.

The motion of a secondary system during an earthquake is influenced primarily by resonance effects between the equipment and the primary structure. In order to account for these effects, the dynamic response of the system must be modeled.

In this thesis, the dynamic response of secondary systems to earthquakes is considered. The first step in such an analysis is the modeling of earthquake ground motion. The use of a stochastic process to model ground motion is widely accepted. In Chapter II, a firm basis for stochastic modeling of ground motion is given.

The transient nature of an earthquake is an essential feature, but is often neglected or disguised by the use of stationary earthquake models. In Chapter II, transient earthquake models are discussed, and

methods of spectral analysis for transient models are described. A convenient way to characterize transient ground motion is by its evolutionary power spectral density, a generalization of the power spectral density for a stationary random process. In order to calibrate model parameters for a specific earthquake, however, a more restricted model must be selected. Two different models, the modulated stationary process, and the filtered, modulated white noise process, are discussed in Chapter II. Some considerations in the numerical simulation of ground motion are also discussed.

The response of a secondary system to transient ground motion is considered in Chapter III. In order to simplify the analysis, both the primary structure and the secondary system are modeled as single-degree-of-freedom linear oscillators. An understanding of this system is essential to the understanding of more complicated systems. The ground motion is characterized by its evolutionary power spectral density, in order to allow for a general stochastic model.

Initially, the effect of the secondary system on the primary system is assumed to be negligible. The response quantity considered in Section 3.2 is the time-varying envelope of the relative displacement of the secondary system. The use of the envelope is advantageous when the response fails to be narrow-band, since it is certain to be nearly Rayleigh distributed. (It is known to be such for a stationary process.) [36]

The mean square value of the envelope as a function of time is derived as the convolution in time of the evolutionary power spectral density of the ground motion with a known function, which depends on the parameters of the system. In the special case of ground motion which is a finite portion of a stationary process, the mean square envelope is derived in closed form. Simulation studies indicate the validity of the equations.

The effect of interaction is considered in Section 3.4. It is found that a simple transformation allows an interacting system to be described in terms of an equivalent non-interacting system. This approach is valid only if the system parameters are such that the two-degree-of-freedom primary/secondary system is classically damped.

In Chapter IV, the results of Chapter III are applied to the computation of floor response spectra. A simple approximation is shown to give good results except very near resonance. Near resonance, the approximation may be improved considerably by accounting for the increased narrow-bandedness of the system response.

References to previous work in stochastic modeling and the analysis of secondary systems may be found in the body of the thesis. Chapter II is primarily a synthesis of current practice and theory in a coherent form. Chapter III presents new results on the envelope of a secondary system. The goal of Chapter IV is a critical analysis of some basic assumptions sometimes used in floor spectrum calculations. A simple system is chosen in order to evaluate the validity of these assumptions.

## CHAPTER II

### STOCHASTIC MODELS OF EARTHQUAKE GROUND MOTION

#### 2.1 Introduction

In this chapter, stochastic models of strong ground motion are discussed. The development of earthquake models is of obvious importance in the design and evaluation of engineering structures. A realistic model of ground motion, together with an accurate model of structural response, allows the structure to be subjected to earthquake forces in advance of the actual destructive event.

In theory, the earthquake process is deterministic. If the state of stress, material properties, location of fractures, etc., were known throughout a region, and tectonic processes were completely understood, then the ground motion  $z(t)$  at a given site would be known in advance. Of course, such complete information will never be available, so engineers and seismologists must accept a level of uncertainty. One objective of earthquake research is to reduce this uncertainty.

A useful earthquake model will incorporate as much knowledge as possible of the state of the earth and the physical processes involved, and at the same time will account for the uncertainty that remains. At present, models tend to be formulated as deterministic processes at lower frequencies, and as stochastic processes at higher frequencies. (The breakpoint seems to be at periods of several seconds.) The reason

for this dichotomy may be seen by exploring the sensitivity of measured ground motion to uncertainty in knowledge about the state of the earth.

Long period waves are generated primarily by large-scale source mechanisms, such as average rupture velocity and stress drop. Variations in stress and material properties produce variations in long period waves only through spatial averages. Similarly, long period waves respond primarily to large-scale averages of material properties as they propagate from source to site. Thus, low frequency recordings at the site are relatively insensitive to local variations in the state of the earth. Source models and earth models seem to be sufficiently accurate at present to allow deterministic modeling at these frequencies.

High frequency recordings, which are of most significance to many engineering structures, are much more sensitive to local variations. High frequency energy is generated at the source primarily by local stress concentrations. These waves are strongly affected by variations in material properties on the scale of the wavelengths involved. Thus, it is seen that prediction of high frequency ground motion requires detailed knowledge of the state of the earth.

A stochastic model is appropriate to this situation in much the same way that probabilistic models apply to errors in measurements of physical quantities. Although the actual earthquake process is understood to be deterministic, it is replaced by a stochastic process in order to reflect the uncertainty of our knowledge. The present state of knowledge is embodied in theoretical descriptions of the earthquake

process, including some understanding of the effects of soil behavior at the site, together with ground motion from past events recorded on strong-motion and teleseismic instruments.

Consider, in principle, the ensemble of earth states and tectonic conditions which are consistent with currently available information. Each member of the ensemble will have averaged properties which agree with models developed from long period data and theoretical studies. Samples from the ensemble will differ, however, on a smaller scale. Each sample will therefore give rise to a different prediction of the strong ground motion at a particular site. These differences will be much more pronounced at high frequencies than at low frequencies. The resulting ensemble of predicted ground motion will represent a stochastic process. The low frequency part of the motion, though a stochastic process, exhibits enough consistency throughout the ensemble to be well described as a deterministic function.

It should be reiterated that the actual earthquake process is deterministic. However, variations in the source and earth properties -- on a scale which significantly affects ground motion -- cannot be distinguished with currently available data. Thus, an appropriate model for strong motion must be described as an ensemble of records, based on an ensemble of earth states, any of which must be considered equally likely. The stochastic model is therefore seen to be a mechanism for reflecting uncertainty. This motivation for using stochastic processes in strong-motion modeling leads to methodologies already in use. However, such methods are typically motivated by observation of the

"random" appearance of recorded accelerograms. The motivation given here may be of more value in clarifying the nature of the ensemble which underlies the random processes involved.

A stochastic earthquake model is defined by specifying the nature of the ensemble of potential ground motions. In principle, this may be accomplished by producing an ensemble of earth states, and deducing the resulting ensemble of ground motions. In practice, such a procedure is impossible for two reasons. For one thing, "current knowledge" is not packaged in mathematical form to allow the ensemble of possible states to be produced. Secondly, the generation of ground motion for a sufficiently detailed source and earth model would be computationally infeasible. Thus, the stochastic ground motion must be constructed on some other basis.

There are several conflicting objectives in the selection of a stochastic model. Primarily, the model should realistically reflect the features of an actual earthquake. These features include the frequency content of the ground motion (which may vary with time) and the duration of shaking. At the same time, it should be possible to calibrate the parameters of the model to observed data with reasonable accuracy. Otherwise, the model will be of no practical value. The more parameters there are in the model, the more difficult the calibration process will become. Yet, if insufficient parameters are included, the model cannot be expected to be realistic. Finally, the model should be mathematically tractable. For analytical work, this means the mathematical description should be relatively simple, and operations such as



filtering of the motion should not be overly involved. For numerical work, it is desirable that an efficient scheme for generating an ensemble of ground motion records be available.

## 2.2 Review of Stochastic Models

As strong-motion records have accumulated over the past 50 years, a number of stochastic models have been proposed. Most of these models fall into a few broad categories. A brief review of these models is presented here.

### 2.2.1 Stationary Models

One of the simplest of random processes is a zero-mean, Gaussian stationary process. Because stationary processes are convenient and well-understood, much effort has gone into attempts to model earthquakes as stationary processes. Clearly, such a model can only be used to represent the central high-intensity part of a strong-motion record. It cannot be expected to model short-duration earthquakes, or the buildup or tail of the ground motion.

In 1947, Housner [14] suggested a model which was essentially white noise -- a large number of impulses arriving at random times. Others used the white noise model [15,18] to investigate structural behavior in earthquakes.

As more data became available, it was possible to propose non-white stationary models with frequency content matching observed records. Housner [16] used a superposition of one-cycle sine pulses arriving at

random times, with the average number of pulses depending on frequency. Tajimi [17] proposed a computationally simple form for the power spectral density of a stationary model.

The primary motivation for the use of a stationary model is the high level of development of the theory of stationary processes. A zero-mean, Gaussian stationary process  $u(t)$  is completely determined by its autocorrelation function, defined as

$$R(\tau) = \langle u(t)u(t+\tau) \rangle \quad (2.1)$$

$R(\tau)$  is independent of  $t$  because  $u(t)$  is stationary. Equivalently, the process may be specified by its power spectral density

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau \quad (2.2)$$

The autocorrelation may be recovered from the power spectral density by Fourier inversion.

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \quad (2.3)$$

Setting  $\tau=0$ ,

$$\langle u^2(t) \rangle = \int_{-\infty}^{\infty} S(\omega) d\omega \quad (2.4)$$

Hence,  $S(\omega)$  may be interpreted as a frequency decomposition of the total energy.

Clearly, a stationary model allows complete freedom in specification of the frequency content of the earthquake. However, this frequency content must be consistent throughout the earthquake. Also, the duration of the ground motion is not explicitly included in the model, and must be artificially accounted for. This has often been done by considering a finite portion of the stationary process. For long-duration earthquakes, the beginning and ending phases may be expected to be unimportant. The effect of ignoring these phases is not clear, however, especially in the case of medium- and short-duration earthquakes.

#### 2.2.2 Modulated Stationary Models

The transient nature of the earthquake process may be modeled explicitly by modulating a stationary process with a deterministic function of time. If the modulating function is reasonably smooth, it will not significantly affect the frequency content of the resulting record. Thus, standard methods may be used to generate the stationary part of the model. The envelope is generally chosen empirically.

The stationary model of Section 2.2.1 is actually a special case of a modulated stationary process. A finite segment of stationary ground motion is produced by an envelope which has a "boxcar" shape. It seems preferable to explicitly include the modulating function in the model, even in this simple case.

Several forms of envelopes have been proposed. Jennings, Housner, and Tsai [19] used an envelope composed of a quadratic buildup phase, a constant phase, and an exponentially decaying tail. Shinozuka [20]

proposed the form

$$\theta(t) = A \left( e^{-\alpha t} - e^{-\beta t} \right) \quad (\beta > \alpha > 0) \quad (2.5)$$

Saragoni and Hart [21] suggested an envelope of the form

$$\theta(t) = At^{\alpha} e^{-\beta t} \quad (2.6)$$

In general, envelopes are chosen with only a few parameters to be estimated, such as intensity, duration, and buildup time.

Modulated stationary processes produce artificial accelerograms which resemble strong-motion records. The parameters of the model are the stationary frequency content and the modulating function. This type of model is therefore able to represent the major features of strong-motion -- average frequency content, intensity, and duration. It is unable to reproduce time-varying frequency content. A general characteristic of strong-motion records is a shift from higher frequencies to lower frequencies toward the end of the record. This is probably of secondary importance for many structures, but may be significant for yielding structures.

### 2.2.3 Filtered, Modulated White Noise Models

Modulated stationary processes are used in modeling primarily because of their computational convenience. One drawback in their use is the lack of physical significance of the envelope. A model which has a more solid basis physically is the filtered, modulated white noise (FMWN) process. A FMWN is defined to be the result of passing

deterministically modulated Gaussian white noise through a linear, time-invariant filter. As in the case of a modulated stationary process, the parameters available are the modulating function and the frequency characteristics of the filter. The two processes are distinguished by the order in which the operations of filtering and modulation are performed. Since a stationary process may be conceived as filtered white noise, a modulated stationary process is expressible as white noise which is first filtered, then modulated. A FMWN process is white noise which is first modulated, then filtered.

A FMWN model visualizes the earthquake process as a white noise "source" which is deterministically modulated, and then filtered by the transmission path. (Actually, the filter characteristics are determined in part by source properties as well as the transmission path.) The envelope in this case is associated with the source mechanism. Although this logic is plausible, the model does not directly allow for randomness in either the transmission path or the envelope. If variations in the path produce only phase changes in the arriving waves, these may be absorbed effectively into the phase of the white noise. Amplitude effects due to transmission path variations are not allowed in the model, however, since the path is assumed to be deterministic.

In spite of the assumed lack of randomness in the amplitude magnification characteristics of the transmission path, the FMWN model remains a reasonable compromise between physical authenticity and computational or mathematical convenience. It has the same parameters as a modulated stationary process model, and is rather more plausible on physical

grounds. The artificial records produced by the model exhibit a slight frequency shift with time due to transient behavior of the filter, but it is unlikely that this is related to the actual phenomenon observed in strong-motion records.

A FMWN process is defined mathematically as

$$u(t) = \int_{-\infty}^t h(t-\tau) \theta(\tau) w(\tau) d\tau \quad (2.7)$$

where  $\theta(t)$  is the deterministic envelope,  $h(t)$  is the impulse response of the filter, and  $w(t)$  is white noise with power spectral density equal to unity. By comparison, a modulated stationary process may be written as

$$u(t) = \theta(t) \int_{-\infty}^t h(t-\tau) w(\tau) d\tau \quad (2.8)$$

In this case,  $\theta(t)$  is the envelope, and

$$S(\omega) = |H(\omega)|^2 \quad (2.9)$$

is the power spectral density of the stationary process, where

$$H(\omega) = \int_0^{\infty} h(t) e^{-i\omega t} dt \quad (2.10)$$

$H(\omega)$  is the transfer function of the filter.

If the filter characteristics and envelope functions are identical, the difference between a modulated stationary process and FMWN process may be seen to depend on the smoothness of  $\theta(t)$ . If  $\theta(t)$  is very flat, then it may essentially be assumed constant in equation (2.7) for the FMWN process, and taken out of the integral, resulting in a modulated stationary process. Thus, for earthquakes with a long section of quasi-stationary motion, both models will have similar characteristics. The differences will appear in the beginning and tail of the record. The two models are thus in greatest contrast in the modeling of shorter duration earthquakes.

Shinozuka and Sato [20] performed numerical simulation of a long duration earthquake using both models, and found the two indistinguishable for their choice of  $\theta(t)$  and  $h(t)$ . Boore [10] has used a FMWN model successfully in reproducing peak velocity, Wood-Anderson response, and response spectra of strong-motion records over a wide range of durations.

### 2.3 Spectral Analysis of Transient Stochastic Processes

For purposes of characterizing and calibrating stochastic models, it is desirable to use frequency domain analytical methods. The power spectral density is a powerful tool in the analysis of stationary random processes because it completely defines the process while describing the frequency decomposition of the total energy. In this section, a number of frequency domain descriptions are discussed which apply to transient stochastic processes.

### 2.3.1 Fourier Analysis

One advantage of transient models (such as modulated stationary models and FMWN models) over stationary models is the possibility of direct Fourier analysis, without having to resort to delicate limiting arguments. If the sample functions of a transient model are  $u(t)$ , then each will possess a Fourier transform

$$U(\omega) = \frac{1}{2\pi} \int_0^{\infty} u(t) e^{-i\omega t} dt \quad (2.11)$$

Taken as an ensemble,  $U(\omega)$  represents a complex random process. If  $u(t)$  is Gaussian, then the real and imaginary parts of  $U(\omega)$  will be Gaussian also. If  $u(t)$  is zero-mean, then so is  $U(\omega)$ .

The amplitude spectrum of  $u(t)$ , defined for each sample as  $|U(\omega)|$ , is a common measure of the frequency content of an earthquake. The root-mean-square (RMS) amplitude spectrum, defined as

$$A(\omega) = \langle |U(\omega)|^2 \rangle \quad (2.12)$$

measures, in an average sense, the frequency content in a given model. The RMS amplitude spectrum is an imperfect description of the process, since variation of frequency content with time is lost when the phase is ignored.

For a modulated stationary process, defined in equation (2.8), it may be seen that



$$U(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta(t) \int_{-\infty}^t h(t-\tau)w(\tau)d\tau e^{-i\omega t} dt \quad (2.13)$$

$$= \int_{-\infty}^{\infty} w(\tau) \left\{ e^{-i\omega\tau} \frac{1}{2\pi} \int_0^{\infty} h(t)\theta(\tau+t)e^{-i\omega t} dt \right\} d\tau \quad (2.14)$$

Since  $w(t)$  is Gaussian white noise, it follows that

$$\langle w(t) \rangle = 0 \quad (2.15)$$

$$\langle w(t_1)w(t_2) \rangle = 2\pi\delta(t_1-t_2) \quad (2.16)$$

Therefore, if  $\phi(t)$  is any complex-valued function,

$$\begin{aligned} \left\langle \left| \int_a^b w(\tau)\phi(\tau)d\tau \right|^2 \right\rangle &= \int_a^b \int_a^b \langle w(\tau_1)w(\tau_2) \rangle \phi(\tau_1)\phi^*(\tau_2)d\tau_1d\tau_2 \\ &= 2\pi \int_a^b |\phi(\tau)|^2 d\tau \end{aligned} \quad (2.17)$$

This formula may be applied to equation (2.14) to yield

$$\langle |U(\omega)|^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \int_0^{\infty} h(t)\theta(\tau+t)e^{-i\omega t} dt \right|^2 d\tau \quad (2.18)$$

Applying Parseval's formula, this integral may be expressed as

$$\langle |U(\omega)|^2 \rangle = \int_{-\infty}^{\infty} |\Phi(\omega, \omega')|^2 d\omega' \quad (2.19)$$

where

$$\Phi(\omega, \omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega'\tau} \int_0^{\infty} h(t)\theta(\tau+t)e^{-i\omega t} dt d\tau \quad (2.20)$$

$$= \int_0^{\infty} h(t)e^{-i(\omega-\omega')t} \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta(\tau+t)e^{-i\omega'(\tau+t)} d\tau dt$$

$$= H(\omega-\omega')\Theta(\omega') \quad (2.21)$$

$H(\omega)$  is defined in equation (2.10), and

$$\Theta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta(t)e^{-i\omega t} dt \quad (2.22)$$

Therefore, combining equations (2.9), (2.19), and (2.21), the RMS amplitude spectrum of a modulated stationary process is found to be

$$A(\omega) = \sqrt{\int_{-\infty}^{\infty} S(\omega-\omega') |\Theta(\omega')|^2 d\omega'} \quad (2.23)$$

where  $S(\omega)$  is the power spectral density of the underlying stationary process.

The RMS amplitude spectrum of a FMWN process is much simpler to

obtain. From equation (2.7), it may be seen that

$$U(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^t h(t-\tau)\theta(\tau)w(\tau)d\tau e^{-i\omega t} dt \quad (2.24)$$

$$= \int_{-\infty}^{\infty} w(\tau) \left\{ \frac{1}{2\pi} \theta(\tau) e^{-i\omega\tau} \int_0^{\infty} h(t) e^{-\omega t} dt \right\} d\tau \quad (2.25)$$

$$\int_{-\infty}^{\infty} w(\tau) \left\{ \frac{1}{2\pi} \theta(\tau) e^{-i\omega\tau} H(\omega) \right\} d\tau \quad (2.26)$$

Applying the formula of equation (2.17) in this case leads to

$$\langle |U(\omega)|^2 \rangle = |H(\omega)|^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta^2(\tau) d\tau \quad (2.27)$$

Therefore, for a FMWN process,

$$A(\omega) = |H(\omega)| \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \theta^2(\tau) d\tau} \quad (2.28)$$

Alternatively, Parseval's formula may be applied to the integral in equation (2.28) to give

$$A(\omega) = |H(\omega)| \sqrt{\int_{-\infty}^{\infty} |\theta(\omega)|^2 d\omega} \quad (2.29)$$

A comparison may be made between equation (2.23) for a modulated stationary process and equation (2.29) for a FMWN process. Denote the former by  $A_1(\omega)$  and the latter by  $A_2(\omega)$ . Since  $S(\omega) = |H(\omega)|^2$ , it may

be seen that

$$A_1^2(\omega) = \int_{-\infty}^{\infty} A_2^2(\omega-\omega') r(\omega') d\omega' \quad (2.30)$$

where

$$r(\omega') = \frac{|\theta(\omega')|^2}{\int_{-\infty}^{\infty} |\theta(\omega'')|^2 d\omega''} \quad (2.31)$$

$A_1^2(\omega)$  is therefore a weighted average of  $A_2^2(\omega)$ . If there exists a  $\Delta\omega$  for which  $r(\omega) \ll 1$  for  $|\omega| > \Delta\omega$ , and  $A_2(\omega)$  varies slowly as  $\omega$  changes by  $\Delta\omega$ , then it may be asserted that

$$A_1(\omega) \approx A_2(\omega) \quad (2.32)$$

The general conditions for this assertion are that  $\theta(t)$  vary slowly, and that  $A(\omega)$  be smooth.

Fourier analysis is more delicate for stationary processes, since the sample functions do not possess Fourier transforms. Some insight is provided by considering a stationary process to be the limit of a FMWN process, as  $\theta(t) \rightarrow 1$ . [For example,  $\theta(t)$  may be taken as  $e^{-a|t|}$ , with

$\alpha \rightarrow 0^+$ .] From equation (2.26),

$$\begin{aligned} \langle U(\omega_1)U^*(\omega_2) \rangle &= H(\omega_1)H^*(\omega_2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle w(\tau_1)w(\tau_2) \rangle \cdot \\ &\cdot \frac{1}{4\pi^2} \theta(\tau_1)\theta(\tau_2)e^{-i(\omega_1\tau_1-\omega_2\tau_2)} d\tau_1d\tau_2 \end{aligned} \quad (2.33)$$

$$= H(\omega_1)H^*(\omega_2) \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta^2(\tau)e^{-i(\omega_1-\omega_2)\tau} d\tau \quad (2.34)$$

Letting  $\theta(t) \rightarrow 1$ , the integral in equation (2.34) approaches  $\delta(\omega_1-\omega_2)$ .

Thus, in a limiting sense, for a stationary process

$$\langle U(\omega_1)U^*(\omega_2) \rangle = S(\omega_1)\delta(\omega_1-\omega_2) \quad (2.35)$$

For the FMWN process, Fourier inversion of equation (2.11) gives

$$u(t) = \int_{-\infty}^{\infty} \bar{U}(\omega)e^{i\omega t} d\omega \quad (2.36)$$

In the limiting case of a stationary process, equation (2.36), together with equation (2.35), motivates the spectral representation of a stationary process as

$$u(t) = \int_{-\infty}^{\infty} H(\omega)e^{i\omega t} dZ(\omega) \quad (2.37)$$

In equation (2.37),  $H(\omega)$  is the transfer function describing the frequency content of  $u(t)$ , and  $dZ(\omega)$  is a complex orthogonal process such

that

$$\langle dZ(\omega) \rangle = 0 \quad (2.38)$$

$$\langle dZ(\omega_1) dZ^*(\omega_2) \rangle = \delta(\omega_1 - \omega_2) d\omega \quad (2.39)$$

The power spectral density of  $u(t)$  may be recovered directly from the spectral representation as

$$S(\omega) = |H(\omega)|^2 \quad (2.9)$$

### 2.3.2 Response Spectrum

A frequently used measure of the spectral content of strong ground motion is the response spectrum. The response spectrum arises in structural engineering through the consideration of a linear oscillator with viscous damping. Let  $u(t)$  be the ground acceleration. Define  $x(t)$  by the differential equation

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = u(t) \quad (2.40)$$

$$x(0) = \dot{x}(0) = 0 \quad (2.41)$$

$x(t)$  is the relative displacement of a linear, viscously damped oscillator with undamped natural frequency  $\omega_0$ , and fraction of critical damping  $\zeta$ , in response to ground motion  $u(t)$ . Studying the response of such oscillators as a function of frequency is one means of assessing the frequency content of  $u(t)$ .

For a deterministic  $u(t)$ , the response spectrum is defined as a function of  $\zeta$  and  $\omega_0$  as

$$SD(\zeta, \omega_0) = \max_{0 \leq t < \infty} |x(t)| \quad (2.42)$$

where  $x(t)$  is the oscillator response defined by equations (2.40)-(2.41). Response spectra are routinely reported for strong-motion recordings. The utility of the response spectrum is due to the fact that it gives directly the maximum response of a structure to an earthquake, which is of primary importance in assessing its safety. The response of many structures is well described either by equation (2.40) or as a linear combination of such oscillators.

Because of the utility of the response spectrum description of an earthquake, a common practice is the prescription of earthquake excitation in terms of the anticipated response spectrum. If a stochastic model is used to characterize an earthquake, an extension to the deterministic response spectrum is necessary. If  $u(t)$  is a stochastic process, then the oscillator response  $x(t)$  will be a stochastic process as well. The mean response spectrum may be defined as

$$\langle SD(\zeta, \omega_0) \rangle = \langle \max_{0 \leq t < \infty} |x(t)| \rangle \quad (2.43)$$

The mean response spectrum indicates the maximum displacement attained by the oscillator, on the average.

A common design question is whether a given structure will safely sustain the ground motion of an earthquake. For a stochastic earthquake

model, which reflects uncertainty, such a question must be answered in probabilistic terms. Far more important than the mean response is a level which, with some probability, will not be exceeded. The design spectrum for a level of confidence  $p$  is defined to be that value  $RS(\zeta, \omega_0; p)$  for which

$$\text{Prob}\left\{SD(\zeta, \omega_0) \leq RS(\zeta, \omega_0; p)\right\} = p \quad (2.44)$$

For example, an oscillator with frequency  $\omega_0$  and damping  $\zeta$  will have maximum response of no more than  $RS(\zeta, \omega_0; 0.9)$  for 90% of the sample earthquakes  $u(t)$ .

The problem of actually computing the design spectrum for a given stochastic model is extremely difficult, even for the simplest models. Most analytical work has focused on the closely related first passage problem in random vibration, which seeks the distribution of the time at which a stationary process first exceeds a given level. The form of first passage problem which relates most closely to the determination of the design spectrum is the so-called stationary-start, type-D problem [22]. If  $x(t)$  is a zero-mean stationary process, it is desired to find  $W(t)$ , the probability that in an interval of length  $t$ ,  $|x(t)|$  does not exceed a barrier level  $b$ .

For large  $t$ , it may be argued heuristically that

$$W(t) \sim Ae^{-\alpha t} \quad (2.45)$$

for some  $A, \alpha$ . Attention is often focused on the limiting decay rate  $\alpha$ . A number of approximations to  $\alpha$  have been proposed. A conservative



estimate results from assuming that excursions above the barrier level occur independently. This assumption leads to the Poisson estimate

$$\alpha = 2\nu_0 e^{-\frac{1}{2} \frac{b^2}{\sigma^2}} \quad (2.46)$$

where  $\nu_0$  is the mean rate of zero-upcrossings by  $x(t)$ , and

$$\sigma^2 = \langle x^2(t) \rangle \quad (2.47)$$

Vanmarcke [23] developed a semi-empirical formula for  $\alpha$  which takes into account the dependence between barrier crossings.

The design spectrum may be computed from an estimate of  $\alpha$  by assuming: (i) that equation (2.45) holds even for small  $t$ ; and (ii) that the response at all times is quasi-stationary. Such assumptions motivate the assertion that

$$\text{Prob}\{|x(t)| \leq b \text{ for } 0 \leq t < \infty\} \approx e^{-\int_0^{\infty} \alpha(\tau) d\tau} \quad (2.48)$$

where  $\alpha(\tau)$  is the limiting decay rate corresponding to the quasi-stationary response at time  $t$ . Note that  $A$  has been set equal to unity because of the certainty of safety at  $t=0$ . The design spectrum value, defined in equation (2.44), is seen to be that value of  $b$  for which

$$\int_0^{\infty} \alpha(\tau) d\tau = -\log p \quad (2.49)$$

This value must be determined iteratively.

The above methodology has been used by Corotis [25] to compute first-passage probabilities for a zero-start type-D problem, reportedly with good results. In a design spectrum computation using a modulated stationary process, Mason [24] reported reasonable agreement with simulation results for  $\zeta = 0.02$ , if the earthquake duration was at least ten periods of the oscillator. Presumably, smaller values of the damping coefficient would require longer durations to achieve the quasi-stationarity required by this method.

In view of the assumptions made in reaching this approximation, the results achieved are surprisingly good. However, more systematic study of the range of validity of the approximation is probably needed. The most reliable check is provided by Monte Carlo simulation.

Although response spectrum methods are widely used, their limitations are also recognized. As a description of ground motion, the response spectrum is incomplete. Any number of models may be proposed which will match a given design spectrum, with great variation in duration and intensity. The application of response spectrum information to multiple-degree-of-freedom structures introduces significant unknown errors, since the response spectrum does not include the time at which the peaks occur.

Trifunac [9] proposed an extension to the response spectrum which addresses both of these limitations. He defined the response envelope spectrum (RES)  $A(\zeta, \omega_0, t)$  of a deterministic  $u(t)$  to be the envelope of the oscillator response  $x(t)$  in equation (2.40). The envelope of  $x(t)$

may be defined in a number of ways. For numerical computation,  $A(\zeta, \omega_0, t)$  may be set equal to  $|x(t)|$  at the extrema of  $x(t)$ , and interpolated linearly between these values. The RES of  $u(t)$  contains its standard response spectrum, since

$$SD(\zeta, \omega_0) = \max_{0 \leq t < \infty} A(\zeta, \omega_0, t) \quad (2.50)$$

Knowledge of the RES of an earthquake record would allow more precise modal combination for multiple-degree-of-freedom structures.

Furthermore, for a zero-mean, broad-band Gaussian model, the mean square RES,  $\langle A^2(\zeta, \omega, t) \rangle$ , is closely related to the evolutionary power spectral density (defined in the following section), as seen in equation (3.16). Thus, the mean square RES of such a model provides a complete description of the process.

The practical value of the RES is limited by the difficulties involved in plotting  $A(\zeta, \omega, t)$ . For a fixed level of damping,  $A(\zeta, \omega, t)$  must be visualized as a surface over the  $(\omega, t)$  plane. Another difficulty arises when the RES is extended to stochastic models. The mean square RES is easily obtained, but is not simply related to the design spectrum, or even to the mean or mean square response spectrum of the model.

### 2.3.3 Evolutionary Power Spectral Density

By far the most useful spectral description of a stationary random process is the power spectral density. A widely accepted extension of the power spectral density to nonstationary processes is due to

Priestley [6]. Recall that a stationary zero-mean Gaussian random process  $u(t)$  may be represented as

$$u(t) = \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} dZ(\omega) \quad (2.37)$$

where  $H(\omega)$  is deterministic, and  $dZ(\omega)$  is an orthogonal process by equations (2.38) and (2.39). The stationarity of  $u(t)$  is seen to be due to the "stationarity" of the elements  $H(\omega) e^{i\omega t} dZ(\omega)$  of which it is composed. A nonstationary process may be generated from the spectral representation

$$u(t) = \int_{-\infty}^{\infty} H(\omega, t) e^{i\omega t} dZ(\omega) \quad (2.51)$$

Here  $dZ(\omega)$  is as before, but  $H(\omega, t)$  may vary with time. The evolutionary power spectral density (EPSD) of such a process is defined [by analogy with equation (2.9)] to be

$$S(\omega, t) = |H(\omega, t)|^2 \quad (2.52)$$

If  $u(t)$  is defined by equation (2.51), its autocorrelation function

is

$$R(t_1, t_2) = \langle u(t_1)u(t_2) \rangle \quad (2.53)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H^*(\omega_1, t_1) H(\omega_2, t_2) e^{i(\omega_2 t_2 - \omega_1 t_1)}$$

$$\langle dZ^*(\omega_1) dZ(\omega_2) \rangle \quad (2.54)$$

$$= \int_{-\infty}^{\infty} H^*(\omega, t_1) H(\omega, t_2) e^{i\omega(t_2 - t_1)} d\omega \quad (2.55)$$

There is no simple transformation between  $R(t_1, t_2)$  and  $S(\omega, t)$ . However, putting  $t_1 = t_2 = t$  in equation (2.55) gives

$$\langle u^2(t) \rangle = \int_{-\infty}^{\infty} S(\omega, t) d\omega \quad (2.56)$$

The EPSD therefore retains an interpretation as a frequency decomposition of the total energy in  $u(t)$ .

Equation (2.55) can be simplified in some cases. If for some  $\Delta t$ ,  $R(t_1, t_2)$  is negligible for  $|t_1 - t_2| \geq \Delta t$ , and  $H(\omega, t)$  varies little as  $t$

varies by  $\Delta t$ , then

$$\begin{aligned} \langle u(t_1)u(t_2) \rangle &= \int_{-\infty}^{\infty} H^*(\omega, t_1)H(\omega, t_2)e^{i\omega(t_2-t_1)} d\omega \\ &\approx \int_{-\infty}^{\infty} H^*(\omega, t_1)H(\omega, t_1)e^{i\omega(t_2-t_1)} d\omega \end{aligned} \quad (2.57)$$

$$= \int_{-\infty}^{\infty} S(\omega, t_1)e^{i\omega(t_2-t_1)} d\omega \quad (2.58)$$

The conditions for equation (2.58) to hold are that  $S(\omega, t)$  be slowly varying with time and broad-band in frequency. These conditions are generally appropriate for earthquake models.

For a stationary process, the power spectral density constitutes a complete description of the process. In general, the EPSD of a nonstationary process is not a complete description of the process, since the crucial quantity  $R(t_1, t_2)$  depends on the phase of  $H(\omega, t)$  in equation (2.55). The EPSD specifies the magnitude, but not the phase, of  $H(\omega, t)$ . However, in the special case of a broad-band process, equation (2.58) holds, and  $S(\omega, t)$  completely prescribes the process  $u(t)$ , in an approximate sense.

The EPSD of a modulated stationary process may be found by using

the spectral representation of white noise,

$$w(t) = \int_{-\infty}^{\infty} e^{i\omega t} dZ(\omega) \quad (2.59)$$

Substituting this representation into equation (2.8) and interchanging the order of integration gives

$$u(t) = \int_{-\infty}^{\infty} \theta(t) \int_{-\infty}^t h(t-\tau) e^{i\omega\tau} d\tau dZ(\omega) \quad (2.60)$$

$$= \int_{-\infty}^{\infty} \theta(t) H(\omega) e^{i\omega t} dZ(\omega) \quad (2.61)$$

Thus, the EPSD of  $u(t)$  is seen to be

$$S(\omega, t) = \theta^2(t) |H(\omega)|^2 \quad (2.62)$$

Because of the form of equation (2.62), a modulated stationary process is sometimes referred to as a separable process.

In the case of a filtered, modulated white noise process, substitu-

tion of equation (2.59) into equation (2.7) yields

$$u(t) = \int_{-\infty}^{\infty} \int_{-\infty}^t h(t-\tau)\theta(\tau)e^{-\omega\tau} d\tau dZ(\omega) \quad (2.63)$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} h(\tau)\theta(t-\tau)e^{-i\omega\tau} d\tau e^{i\omega t} dZ(\omega) \quad (2.64)$$

Thus, for a FMWN process,

$$S(\omega, t) = \left| \int_0^{\infty} h(\tau)\theta(t-\tau)e^{-i\omega\tau} d\tau \right|^2 \quad (2.65)$$

Alternatively, equation (2.65) may be written

$$S(\omega, t) = \left| \int_{-\infty}^{\infty} H(\omega+\omega')\Theta(\omega')e^{i\omega't} d\omega' \right|^2 \quad (2.66)$$

where  $H(\omega)$  and  $\Theta(\omega)$  are defined in equations (2.10) and (2.23), respectively. Note that equation (2.65) reduces to equation (2.62) for a modulated stationary process if  $\theta(t)$  is essentially constant over the decay time of  $h(t)$ .

Because of the generality of the representation of equation (2.51), it would seem reasonable to propose an earthquake model based on this representation. Certainly, equation (2.51) is capable of representing such earthquake phenomena as frequency variation with time. However, a model which allowed complete freedom in  $S(\omega, t)$  would be practically impossible to calibrate to actual data. Whereas both a modulated



stationary process and a FMWN process have  $H(\omega)$  and  $\theta(t)$  as parameters, a general nonstationary model would have  $S(\omega, t)$  as a parameter to be calibrated. The uncertainty in estimating  $S(\omega, t)$  would more than offset the added generality of the model. Nevertheless, since both a modulated stationary model and a FMWN model may be represented by equation (2.51), this model will be used to represent a general stochastic model.

#### 2.4 Calibration of Stochastic Models

A modulated stationary process or FMWN process provides the framework for a stochastic model of earthquake ground motion. At some point, however, the parameters in a model must be set to some values in order for the model to predict actual ground motion. A number of approaches to the calibration problem are summarized in this section.

The frequency content of stationary models has often been determined empirically from observation of strong-motion records. Based on spectral analysis of several records and a simple model of local soil properties, Kanai [26] and Tajimi [17] proposed the form

$$S(\omega) = S_0 \frac{\omega_g^4 + 4\zeta_g^2 \omega_g^2 \omega^2}{(\omega_g^2 - \omega^2)^2 + 4\zeta_g^2 \omega_g^2 \omega^2} \quad (2.67)$$

for the power spectral density of ground acceleration. The parameters  $\zeta_g$  and  $\omega_g$  are sometimes related to properties of the soil at the recording site.

Often, the design spectrum for a site is estimated empirically from response spectra obtained at similar sites. It is common to prescribe earthquake forces entirely by empirical design spectra. In turn, models are sometimes calibrated to these design spectra. The approximate methods mentioned in Section 2.3.2 allow a stationary model of given duration to be calibrated to a design spectrum. Mason [24] outlined a method for calibrating a modulated stationary model to a smooth design spectrum, when  $\theta(t)$  is assumed to be known in advance.

Recently, digital signal processing techniques have been applied to single accelerograms in an attempt to estimate the parameters of a FMWN model consistent with the record. Nau and others [7] described a method for determining the white noise envelope  $\theta(t)$ , and filter characteristics  $H(\omega)$  (in the form of an auto-regressive, moving average filter), directly from a given record. In fact, some time variation of the filter characteristics was included in the model, making it somewhat more flexible than the FMWN process described previously. These estimated parameters can then be used with digitally simulated white noise to produce an ensemble of records similar to the original record.

All of the above methods of calibration rely essentially on observation and analysis of strong motion records. Models may also be calibrated from theoretical considerations. Trifunac [13] proposed a model based on the assumption that the near field ground motion is dominated by surface waves. He used experimentally determined surface wave dispersion curves for the Imperial Valley region of California to predict local ground motion, with limited success.

Boore [10] calibrated a FMWN model to Brune's theoretical amplitude spectra [27]. The parameters of the spectrum were related to the seismic moment  $M_0$ , using empirically determined relationships. The duration and intensity of the envelope  $\theta(t)$  were also related to  $M_0$ . The resulting model has been shown to reproduce many of the characteristics of strong-motion records, including peak velocity, Wood-Anderson instrument response, and response spectra.

It should be noted that modulated stationary models are well suited to response spectrum calibration, because of the presence of the stationary process. FMWN models are well suited to amplitude spectrum calibration, because of the simplicity of equation (2.28).

## 2.5 Numerical Simulation of Ground Motion

An important objective of stochastic modeling is the generation of an ensemble of sample records which is consistent with a given model. One consideration in choosing a model is the ease with which sample records can be generated numerically. These sample records are used in analysis of various aspects of earthquake ground motion, including dynamic structural response.

The spectral representation of many stochastic processes gives rise to several standard numerical methods. Consider the process defined by

$$u(t) = \int_{-\infty}^{\infty} H(\omega, t) e^{i\omega t} dZ(\omega) \quad (2.68)$$

Let the interval  $(-\infty, \infty)$  be divided into a finite or infinite number of

intervals, each centered at  $\omega_n$ , of width  $\Delta\omega_n$ . If the intervals are sufficiently small, equation (2.68) may be approximated as

$$u(t) \approx \sum_n H(\omega_n, t) U_n(t) \quad (2.69)$$

where

$$U_n(t) = \int_{\omega_n - \frac{1}{2}\Delta\omega_n}^{\omega_n + \frac{1}{2}\Delta\omega_n} e^{i\omega t} dZ(\omega) \quad (2.70)$$

Recall that

$$\langle dZ(\omega) \rangle = 0 \quad (2.38)$$

$$\langle dZ(\omega_1) dZ^*(\omega_2) \rangle = \delta(\omega_1 - \omega_2) d\omega \quad (2.39)$$

$U_n(t)$  is seen to be a complex-valued, stationary random process, with

$$\langle U_n(t) \rangle = 0 \quad (2.71)$$

$$\langle U_m(t_1) U_n^*(t_2) \rangle = 0 \quad \text{if } m \neq n \quad (2.72)$$

$$\langle U_n(t) U_n^*(t+\tau) \rangle = R_n(\tau) \quad (2.73)$$

where

$$R_n(\tau) = \frac{2}{\tau} \sin \frac{1}{2}\Delta\omega_n \tau e^{-i\omega_n \tau} \quad (2.74)$$

The term  $e^{-i\omega_n \tau}$  may be accounted for by letting

$$U_n(t) = V_n(t) e^{-i(\omega_n t + \phi_n)} \quad (2.75)$$

where  $V_n(t)$  is a stationary process, and the  $\phi_n$  are independent random variables, uniformly distributed on  $[0, 2\pi]$ . The  $\phi_n$  are also assumed to be independent of the  $V_n(t)$ . It may be seen that equations (2.71) and (2.72) are satisfied, and

$$\langle U_n(t) U_n^*(t+\tau) \rangle = e^{-i\omega_n \tau} \langle V_n(t) V_n^*(t+\tau) \rangle \quad (2.76)$$

Therefore  $V_n(t)$  must satisfy only

$$\langle V_n(t) V_n^*(t+\tau) \rangle = \frac{2}{\tau} \sin \frac{1}{2} \Delta\omega_n \tau \quad (2.77)$$

The numerical approximation to equation (2.51) now takes the form

$$u(t) \approx \sum_n H(\omega_n, t) e^{-i(\omega_n t + \phi_n)} V_n(t) \quad (2.78)$$

where the  $\phi_n$  are independent and uniform on  $[0, 2\pi]$ , and the  $V_n(t)$  satisfy equation (2.77). It is now possible to examine some common implementations of this result.

For  $\Delta\omega_n \tau \ll 1$ , equation (2.77) reduces to

$$\langle V_n(t) V_n^*(t+\tau) \rangle \approx \Delta\omega_n \quad (2.79)$$

The approximation

$$V_n(t) = \sqrt{\Delta\omega_n} \quad (2.80)$$

will be adequate over a time interval  $T \ll \frac{1}{\Delta\omega_n}$ . Over larger time

intervals, equation (2.80) will introduce too much correlation into  $u(t)$ . This approximation gives

$$u(t) = \sum_n H(\omega_n, t) \sqrt{\Delta\omega_n} e^{-i(\omega_n t + \phi_n)} \quad (2.81)$$

and is seen to be reasonable, provided  $\Delta\omega_n \ll \frac{1}{T}$ , where  $T$  is the duration of the simulation. Note that  $u(t)$  in equation (2.81) is not Gaussian. If the number of terms is large, however, the Central Limit Theorem guarantees that  $u(t)$  will be nearly Gaussian. Alternatively,  $V_n(t)$  may be taken as a random constant whose value is normally distributed with zero mean, and with variance  $\Delta\omega_n$ . Then,  $u(t)$  will be Gaussian.

If  $u(t)$  is stationary, then

$$H(\omega, t) = H(\omega) \quad (2.82)$$

and equation (2.81) may be rewritten as

$$u(t) = \sum_n \sqrt{S(\omega_n) \Delta\omega_n} \cos(\omega_n t + \phi_n') \quad (2.83)$$

where the real part of  $u(t)$  has been taken, and

$$\phi_n' = \phi_n + \arg [H(\omega_n)] \quad (2.84)$$

That is, the phase of  $H(\omega)$  has been absorbed into  $\phi_n'$ . The  $\phi_n'$  may be generated as uniform on  $[0, 2\pi]$ . In the nonstationary case, the corresponding expression is

$$u(t) = \sum_n \sqrt{S(\omega_n, t)\Delta\omega_n} \cos [\omega_n t + \phi_n + \theta_n(t)] \quad (2.85)$$

where

$$\theta_n(t) = \arg [H(\omega_n, t)] \quad (2.86)$$

The nonstationary simulation depends critically on the phase of  $H(\omega, t)$ , since the phase is time-dependent. Since a description of the process by its evolutionary power spectral density does not include phase information,  $\theta_n(t)$  is unknown. Normally, it is taken to be independent of time, so that

$$u(t) = \sum_n \sqrt{S(\omega_n, t)\Delta\omega_n} \cos [\omega_n t + \phi_n'] \quad (2.87)$$

This assumption is probably justified when  $S(\omega, t)$  is smooth and broadband, since equation (2.58) indicates that the phase is unimportant in this case. The implications of this simplification are unclear, however, for more general nonstationary processes.

In the case of a modulated stationary process, the stationary process is first generated, and the envelope is then applied. The stationary part,  $v(t)$ , may be produced by the technique described above, or by a variety of other methods unique to stationary processes. For example, a method proposed by Shinozuka [28] is

$$v(t) = A \sum_{k=1}^N \cos (\omega_k t + \phi_k) \quad (2.88)$$

where  $\phi_k$  are independent random variables, uniformly distributed on

$[0, 2\pi]$ ; and  $\omega_k$  are independent random variables distributed according to the probability density function

$$p(\omega) = \frac{S(\omega)}{\int_{-\infty}^{\infty} S(\omega') d\omega'} \quad (2.89)$$

The normalizing constant A is found to be given by

$$A^2 = \frac{2}{N} \int_{-\infty}^{\infty} S(\omega) d\omega \quad (2.90)$$

The process described by equation (2.88) has power spectral density exactly equal to  $S(\omega)$ , and is Gaussian in the limit as  $N \rightarrow \infty$ . It is unclear how large N must be to give good results. Also, the realization of the  $\omega_k$  is problematical.

Another method for generating a stationary  $v(t)$  is matching of the parameters of an auto-regressive, moving average (ARMA) digital filter to the desired frequency characteristics. The output values,

$$v_n = v(n\Delta t) \quad (2.91)$$

are given recursively by

$$v_n = \sum_{k=1}^p a_k v_{n-k} + \sum_{k=1}^q b_k w_{n-k} \quad (2.92)$$

where  $w_k$  are independent, normally distributed random variables with mean zero, and variance unity. The filter parameters  $a_k$  and  $b_k$  are



chosen to give a "best fit" to the desired characteristics of  $v(t)$ .

This technique has been used successfully in the modeling of ocean waves [29]. It is generally very efficient, since no transcendental functions need to be evaluated.

The generation of a FMWN process may be accomplished by carrying out the filtering in the frequency domain. A time series corresponding to  $\theta(t)w(t)$  is produced as a sequence of scaled, normally distributed random variables. The discrete Fourier Transform of this sequence (computed by the FFT algorithm) is then multiplied by  $H(\omega)$ . The transformed result gives the desired record.

Often, it is not the ground motion itself that is of interest, but rather its effect on a structure. Considerable savings in computation time are sometimes possible in this case.

## CHAPTER III

### RESPONSE STATISTICS OF SECONDARY SYSTEMS

#### 3.1 Introduction

In complex engineering structures, one is frequently concerned with the dynamic response of a piece of equipment attached to a heavy structure. When the mass of the equipment is much less than the mass of the structure, the former is often referred to as a secondary system. Although the secondary system is small, its survival in an earthquake is often essential, and a dynamic analysis is called for.

The general problem of the analysis of secondary systems has received increasing attention recently, as it has become clear that special considerations must be addressed. Although it is relatively easy to write a solution for the dynamic response of a complete primary/secondary system, it is more difficult to illuminate the principal features contained in the solution. Simplification of the numerical computations involved is also important for practical application.

Penzien and Chopra [31] showed that the response of a single-degree-of-freedom secondary system attached to a multiple-degree-of-freedom, classically damped structure could be approximated as a combination of two-degree-of-freedom systems (one for each mode of the structure). This approximation breaks down if some natural frequencies of the structure are closely spaced. Based on the same assumption, Biggs and Roesset [30] proposed an empirical rule for obtaining the floor

spectrum directly from the response spectrum of the ground motion. (The floor spectrum is defined as the response spectrum corresponding to the motion of a point in the structure in response to an earthquake.)

Several researchers have investigated the response of secondary systems to stochastic ground motion. Primarily, attention has been focused on obtaining the stationary response of a primary/secondary system to stationary excitation. It may be argued that this will represent the response to a long duration earthquake with a nearly stationary segment.

Singh [32] attempted to simplify the computation of the stationary mean square response of a single-degree-of-freedom secondary system attached to a multiple-degree-of-freedom primary system. His results were valid if the secondary system's natural frequency was not near a natural frequency of the primary system. Expressions for the case of exact tuning were later obtained [33].

Igusa and Der Kiureghian [12] used perturbation methods to simplify expressions for the stationary response statistics of a multiple-degree-of-freedom secondary system attached arbitrarily to a multiple-degree-of-freedom primary system. Their results include the effect of interaction, which is often neglected.

Very little work has been done on the response of secondary systems to transient ground motion. Chakravorty and Vanmarcke [34] obtained the mean square relative displacement of a single-degree-of-freedom secondary system attached to a single-degree-of-freedom primary system, in response to suddenly applied white noise. Vanmarcke [35] suggested a

procedure for obtaining the floor spectrum in a multiple-degree-of-freedom structure, using an approximation for the time-varying response of a two-degree-of-freedom system to stationary ground motion.

The approach taken in this thesis leads to a general expression for the time-varying statistics of the envelope of the response of a secondary system to ground motion represented as a general stochastic process. The primary system and secondary system are each represented as single-degree-of-freedom systems as in Figure 3.1. The results derived here may be combined, as in reference [31], to approximate the behavior of a more complicated primary system.

### 3.2 Review of Primary System Response

As a first step in analyzing the response of a secondary system, this section reviews some features of the response of the primary system in the absence of a secondary system. If interaction effects are ignored, then the results presented herein will describe the motion of the primary system even with a light secondary system attached.

There are many approaches to analyzing the response of a single-degree-of-freedom linear oscillator to stochastic excitation. The approach chosen here is analogous to that used for the secondary system in the following section. Only a few results will be obtained which will be useful later.

The relative displacement  $x(t)$  of the primary system is described by the equation

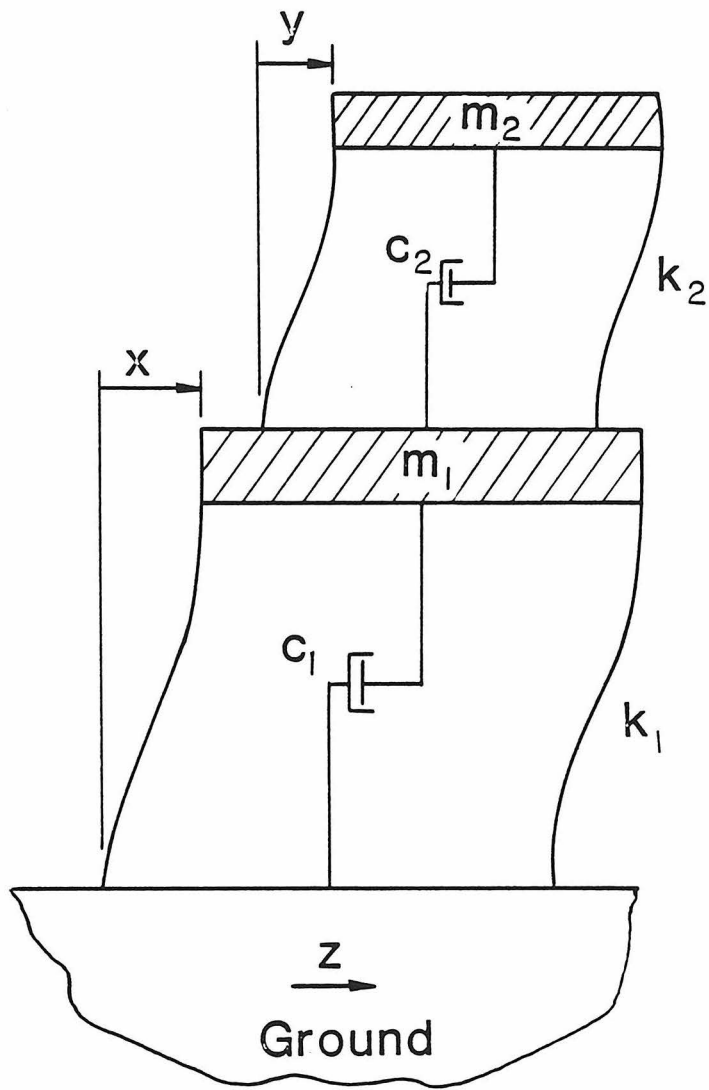


Fig. 3.1 Single-degree-of-freedom secondary system attached to single-degree-of-freedom primary system

$$\ddot{x} + 2\zeta_1 \omega_1 \dot{x} + \omega_1^2 x = -\ddot{z}(t) \quad (3.1)$$

$$x(0) = \dot{x}(0) = 0 \quad (3.2)$$

where  $\ddot{z}(t)$  is a zero-mean Gaussian stochastic process described by the evolutionary power spectral density  $S(\omega, t)$  (defined in Section 2.3.1).  $z(t)$  is the absolute ground displacement.

The response  $x(t)$  to any ground motion  $z(t)$  may be written as the Duhammel integral

$$x(t) = \int_0^t \frac{1}{\omega_{d1}} e^{-\zeta_1 \omega_1 (t-\tau)} \sin \omega_{d1} (t-\tau) [-\ddot{z}(\tau)] d\tau \quad (3.3)$$

where  $\omega_{d1} = \omega_1 \sqrt{1-\zeta_1^2}$ . This integral may be decomposed in harmonics of the frequency  $\omega_{d1}$  as

$$x(t) = x_1(t) \cos \omega_{d1} t + x_2(t) \sin \omega_{d1} t \quad (3.4)$$

where

$$x_1(t) = - \int_0^t \frac{1}{\omega_{d1}} e^{-\zeta_1 \omega_1 (t-\tau)} \sin \omega_{d1} \tau [-\ddot{z}(\tau)] d\tau \quad (3.5)$$

$$x_2(t) = \int_0^t \frac{1}{\omega_{d1}} e^{-\zeta_1 \omega_1 (t-\tau)} \cos \omega_{d1} \tau [-\ddot{z}(\tau)] d\tau \quad (3.6)$$

For structural systems,  $\zeta_1$  will generally be small, and the response  $x(t)$  will have the appearance of a harmonic wave of frequency  $\omega_{d1}$  which is modulated by a slowly varying envelope. Thus

$$x(t) = a_1(t) \cos [\omega_{d1}t + \phi_1(t)] \quad (3.7)$$

where  $a_1(t)$  and  $\phi_1(t)$  are nearly constant over a period  $2\pi/\omega_{d1}$ .  $a_1(t)$  may be interpreted as half the mechanical energy in the oscillator.

Equation (3.7) may be decomposed in the form of equation (3.4) to give

$$x_1(t) = a_1(t) \cos \phi_1(t) \quad (3.8)$$

$$x_2(t) = -a_1(t) \sin \phi_1(t) \quad (3.9)$$

From these equations, the envelope  $a_1(t)$  may be expressed as

$$a_1^2(t) = x_1^2(t) + x_2^2(t) \quad (3.10)$$

$$= \int_0^t \int_0^t \frac{1}{2} e^{-\zeta_1 \omega_{d1} (2t - \tau_1 - \tau_2)} \cos \omega_{d1} (\tau_1 - \tau_2) [-\ddot{z}(\tau_1)] [-\ddot{z}(\tau_2)] d\tau_1 d\tau_2$$

$$\quad (3.11)$$

Equation (3.11) expresses the stochastic process  $a_1(t)$  in terms of the stochastic process  $-\ddot{z}(t)$ . Its utility arises from the fact that it linearly relates  $a_1^2(t)$  to the cross product  $[-\ddot{z}(\tau_1)] [-\ddot{z}(\tau_2)]$ .

Taking the ensemble average of  $a_1^2(t)$  gives

$$\langle a_1^2(t) \rangle = \frac{1}{2} \int_{\omega_{d1}}^t \int_0^t e^{-\zeta_1 \omega_1 (2t - \tau_1 - \tau_2)} \cos \omega_{d1} (\tau_1 - \tau_2) \langle [-\ddot{z}(\tau_1)] [-\ddot{z}(\tau_2)] \rangle d\tau_1 d\tau_2 \quad (3.12)$$

Recall that for broad-band ground motion,

$$\langle [-\ddot{z}(t_1)] [-\ddot{z}(t_2)] \rangle \approx \int_{-\infty}^{\infty} S(\omega, t_1) e^{i\omega(t_2 - t_1)} d\omega \quad (2.58)$$

Substituting equation (2.58) into equation (3.12) and interchanging the order of integration gives

$$\langle a_1^2(t) \rangle = \frac{1}{2} \int_{\omega_{d1}}^t \int_0^{\infty} e^{-2\zeta_1 \omega_1 (t - \tau_1)} S(\omega, \tau_1) \xi(\omega, t) d\omega d\tau_1 \quad (3.13)$$

where

$$\xi(\omega, t) = \int_0^t e^{\zeta_1 \omega_1 \tau} \cos \omega_{d1} \tau e^{i\omega \tau} d\tau \quad (3.14)$$

But  $\xi(\omega, t)$  is sharply peaked at  $\omega = \omega_{d1}$ , and

$$\int_{-\infty}^{\infty} \xi(\omega, t) d\omega = 2\pi \quad (3.15)$$

Thus, taking  $S(\omega, t)$  as nearly constant over the peak of  $\xi$ ,



$$\langle a_1^2(t) \rangle \approx \frac{2\pi}{\omega_{d1}} \int_0^t e^{-2\zeta_1 \omega_1 (t-\tau_1)} S(\omega_{d1}, \tau_1) d\tau_1 \quad (3.16)$$

This result is identical to that obtained by Spanos [5], except that  $\omega_1$  is replaced here by  $\omega_{d1}$ .

The complete distribution of  $a_1(t)$  can be deduced from equation (3.16), since the distribution is known to be approximately Rayleigh. The probability density of  $a_1(t)$  is

$$p_1(a; t) = \frac{a}{\sigma_1^2(t)} e^{-\frac{1}{2} \frac{a^2}{\sigma_1^2(t)}} \quad (3.17)$$

Thus

$$\begin{aligned} \langle a_1^2(t) \rangle &= \int_0^\infty a^2 p_1(a; t) da \\ &= 2\sigma_1^2(t) \end{aligned} \quad (3.18)$$

The density of  $a_1(t)$  is therefore given by equation (3.17), with

$$\sigma_1^2(t) = \frac{\pi}{\omega_{d1}} \int_0^t e^{-2\zeta_1 \omega_1 (t-s)} S(\omega_{d1}, s) ds \quad (3.19)$$

In particular,

$$\begin{aligned}\langle a_1(t) \rangle &= \int_0^{\infty} a p_1(a; t) da \\ &= \sqrt{\frac{\pi}{2}} \sigma_1(t)\end{aligned}\tag{3.20}$$

### 3.3 Secondary System Envelope Statistics vs. Time

In this section, the statistics of the envelope of the response of the secondary system are derived in terms of the evolutionary power spectral density of the ground motion. Initially, the effect of the secondary system on the primary system will be neglected to simplify the analysis. Interaction effects will be considered separately in the following section.

The envelope of the secondary system response will be shown to have statistics expressible in a form very similar to those of the primary system. In particular, the mean square value of the envelope as a function of time is shown to be the convolution of a known function of time with the evolutionary power spectral density of the ground acceleration.

The statistics of the secondary system envelope are derived in closed form in the case of a suddenly applied stationary ground motion, and in the case of stationary ground motion of finite duration. The results are verified by simulation studies.

#### 3.3.1 Integral Form of Secondary Envelope

The relative displacement  $y(t)$  of a non-interacting secondary system is governed by the equation

$$\ddot{y} + 2\zeta_2\omega_2\dot{y} + \omega_2^2y = -\ddot{z}(t) - \ddot{x}(t) \quad (3.21)$$

$$y(0) = \dot{y}(0) = 0 \quad (3.22)$$

where  $z(t)$  is the absolute ground displacement and  $x(t)$  is the relative displacement of the primary system.

The right-hand side of equation (3.21) may be expressed more simply by using equation (3.1) for the primary system.

$$\begin{aligned} -\ddot{z} - \ddot{x} &= 2\zeta_1\omega_1\dot{x} + \omega_1^2x \\ &\approx \omega_{d1}^2 a_1(t) \cos [\omega_{d1}t + \phi_1(t)] \end{aligned} \quad (3.23)$$

where equation (3.7) has been used together with the assumption that  $\zeta_1 \ll 1$ . The error introduced by this assumption is negligible in view of subsequent simplifications. Now  $y(t)$  may be expressed in terms of the Duhammel integral representation

$$y(t) = \int_0^t \frac{1}{\omega_{d2}} e^{-\zeta_2\omega_2(t-\tau)} \sin \omega_{d2}(t-\tau) \omega_{d1}^2 a_1(\tau) \cos [\omega_{d1}\tau + \phi_1(\tau)] d\tau \quad (3.24)$$

where  $\omega_{d2} = \omega_2\sqrt{1-\zeta_2^2}$ .

It will be assumed that  $y(t)$  has the appearance of a harmonic oscillation with slowly varying amplitude, and possibly slowly varying frequency. That is,

$$y(t) = a_2(t) \cos [\omega^*(t)t + \phi_2(t)] \quad (3.25)$$

where  $a_2(t)$ ,  $\omega^*(t)$ , and  $\phi_2(t)$  are slowly varying random functions of

time. The envelope  $a_2(t)$  is of most interest for analysis purposes. The representative frequency  $\omega^*(t)$  would be important for an analysis of fatigue of the secondary structure. The following analysis will produce results for  $a_2(t)$  which are independent of the value of  $\omega^*(t)$ .

Equation (3.24) exhibits two characteristic frequencies,  $\omega_{d1}$  and  $\omega_{d2}$ . However, one of these frequencies may dominate the other. Equations (3.1) and (3.21) are characterized by the transfer function

$$H_y(\omega) = \frac{1}{(\omega_2^2 - \omega^2) + 2i\zeta_2\omega_2\omega} \frac{2i\zeta_1\omega_1\omega + \omega_1^2}{(\omega_1^2 - \omega^2) + 2i\zeta_1\omega_1\omega} \quad (3.26)$$

which gives the steady-state amplitude of  $y(t)$  in response to excitation of the form

$$-z(t) = e^{i\omega t} \quad (3.27)$$

$H_y(\omega)$  is characterized by peaks near  $\omega_1$  and  $\omega_2$ . If  $\zeta_1$  and  $\zeta_2$  are small,

$$H_y(\omega_1) \approx \frac{i}{2\zeta_1\omega_1^2} \frac{1}{(\omega_2^2 - \omega_1^2) + 2i\zeta_2\omega_1\omega_2} \quad (3.28)$$

$$H_y(\omega_2) \approx \frac{i}{2\zeta_2\omega_2^2} \frac{\omega_1^2 + 2i\zeta_1\omega_1\omega_2}{(\omega_1^2 - \omega_2^2) + 2i\zeta_1\omega_1\omega_2} \quad (3.29)$$

If  $\omega_2 \ll \omega_1$ , then it may be verified that  $|H_y(\omega_1)| \ll |H_y(\omega_2)|$ .

Similarly, if  $\omega_1 \ll \omega_2$ , then  $|H_y(\omega_2)| \ll |H_y(\omega_1)|$ . Thus, when the two frequencies are well separated, the response of the secondary system is dominated by the lower frequency, although a small amount of the higher frequency is still present.

Equation (3.24) will first be decomposed in harmonics of  $\omega_{d1}$ . This procedure may be expected to yield good results for  $\omega_1 \leq \omega_2$ , since the frequency  $\omega_{d1}$  will certainly be present in  $y(t)$  in this case. The decomposition takes the form

$$y(t) = y_1(t) \cos \omega_{d1} t + y_2(t) \sin \omega_{d1} t \quad (3.30)$$

where

$$y_1(t) = \int_0^t \frac{1}{\omega_{d2}} e^{-\zeta_2 \omega_2 (t-\tau)} \sin \omega_{d2} (t-\tau) \omega_{d1}^2 a_1(\tau) \cos [\omega_{d1} (t-\tau) + \phi_1(\tau)] d\tau \quad (3.31)$$

$$y_2(t) = \int_0^t \frac{1}{\omega_{d2}} e^{-\zeta_2 \omega_2 (t-\tau)} \sin \omega_{d2} (t-\tau) \omega_{d1}^2 a_1(\tau) \sin [\omega_{d1} (t-\tau) + \phi_1(\tau)] d\tau \quad (3.32)$$

If equation (3.25) is decomposed in a similar way, it may be seen that

$$y_1(t) = a_2(t) \cos[\omega^*(t)t - \omega_{d1} t + \phi_2(t)] \quad (3.33)$$

$$y_2(t) = a_2(t) \sin[\omega^*(t)t - \omega_{d1} t + \phi_2(t)] \quad (3.34)$$

Thus,

$$a_2^2(t) = y_1^2(t) + y_2^2(t) \quad (3.35)$$

$$= \int_0^t \int_0^t \frac{1}{2} e^{-\zeta_2 \omega_2 (2t - \tau_1 - \tau_2)} \sin \omega_{d2}(t - \tau_1) \sin \omega_{d2}(t - \tau_2)$$

$$\omega_{d1}^4 \chi(\tau_1, \tau_2) d\tau_1 d\tau_2 \quad (3.36)$$

where

$$\chi(\tau_1, \tau_2) = a_1(\tau_1)a_1(\tau_2) \cos [\omega_{d1}(\tau_1 - \tau_2) + \phi_1(\tau_1) - \phi_1(\tau_2)] \quad (3.37)$$

The function  $\chi(\tau_1, \tau_2)$  is a random function which depends only on the primary response  $x(t)$ .  $\chi(\tau_1, \tau_2)$  can in turn be written in terms of the ground motion  $z(t)$  by using the random functions  $x_1(t)$  and  $x_2(t)$  defined in equations (3.8) and (3.9). It may be verified that

$$\begin{aligned} \chi(t_1, t_2) &= [x_1(t_1)x_1(t_2) + x_2(t_1)x_2(t_2)] \cos \omega_{d1} (t_1 - t_2) \\ &+ [x_1(t_2)x_2(t_1) - x_1(t_1)x_2(t_2)] \sin \omega_{d1} (t_1 - t_2) \end{aligned} \quad (3.38)$$

Using equations (3.5) and (3.6),

$$\begin{aligned}
 & x_1(t_1)x_1(t_2) + x_2(t_1)x_2(t_2) \\
 &= \int_0^{t_1} \int_0^{t_2} \frac{1}{2} e^{-\zeta_1 \omega_1 (t_1+t_2-\tau_1-\tau_2)} \\
 & \quad \cos \omega_{d1}(\tau_1-\tau_2) [-\ddot{z}(\tau_1)] [-\ddot{z}(\tau_2)] d\tau_1 d\tau_2
 \end{aligned} \tag{3.39}$$

$$\begin{aligned}
 & x_1(t_2)x_2(t_1) - x_1(t_1)x_2(t_2) \\
 &= \int_0^{t_1} \int_0^{t_2} \frac{1}{2} e^{-\zeta_1 \omega_1 (t_1+t_2-\tau_1-\tau_2)} \\
 & \quad \sin \omega_{d1}(\tau_1-\tau_2) [-\ddot{z}(\tau_1)] [-\ddot{z}(\tau_2)] d\tau_1 d\tau_2
 \end{aligned} \tag{3.40}$$

Thus,

$$\begin{aligned}
 \chi(t_1, t_2) &= \int_0^{t_1} \int_0^{t_2} \frac{1}{2} e^{-\zeta_1 \omega_1 (t_1+t_2-\tau_1-\tau_2)} \cos \omega_{d1}(t_1-t_2-\tau_1+\tau_2) \\
 & \quad [-\ddot{z}(\tau_1)] [-\ddot{z}(\tau_2)] d\tau_1 d\tau_2
 \end{aligned} \tag{3.41}$$

Equations (3.36) and (3.41) express  $a_2^2(t)$  as a four-fold integral involving the random function  $[-\ddot{z}(\tau_1)] [-\ddot{z}(\tau_2)]$ .

A similar calculation may be made in the case that  $\omega_2 < \omega_1$ . In this case, it will always be appropriate to decompose equations (3.24) and (3.25) in harmonics of  $\omega_{d2}$ . This leads to the expression

$$a_2^2(t) = \int_0^t \int_0^t \frac{1}{2} e^{-\zeta_2 \omega_2 (2t - \tau_1 - \tau_2)} \cos \omega_{d2} (\tau_1 - \tau_2) \hat{x}(\tau_1) \hat{x}(\tau_2) d\tau_1 d\tau_2 \quad (3.42)$$

where

$$\hat{x}(t) = 2\zeta_1 \omega_1 \dot{x}(t) + \omega_1^2 x(t) \quad (3.43)$$

From equation (3.3),  $\hat{x}(t)$  may be expressed in terms of the ground excitation as

$$\hat{x}(t) = \int_0^t e^{-\zeta_1 \omega_1 (t-\tau)} \left[ \sin \omega_{d1} (t-\tau) + \frac{\zeta_1 \omega_1}{\omega_{d1}} \cos \omega_{d1} (t-\tau) \right] [-\ddot{z}(\tau)] d\tau \quad (3.44)$$

### 3.3.2 Closed Form Solution for $\langle a_2^2(t) \rangle$

The equations derived above provide a complete description of the stochastic process  $a_2(t)$  in terms of the ground acceleration  $\ddot{z}(t)$ . In principle, the probability distribution of  $a_2(t)$  could be derived. However, only the second moment  $\langle a_2^2(t) \rangle$  is readily computed from these equations.

Consider first the case  $\omega_1 \leq \omega_2$ . Taking ensemble averages in equations (3.36) and (3.41) gives



$$\langle a_2^2(t) \rangle = \int_0^t \int_0^t \frac{\omega_{d1}^4}{\omega_{d2}^2} e^{-\zeta_2 \omega_2 (2t - \tau_1 - \tau_2)} \sin \omega_{d2}(t - \tau_1) \sin \omega_{d2}(t - \tau_2) \langle \lambda(\tau_1, \tau_2) \rangle d\tau_1 d\tau_2 \quad (3.45)$$

where

$$\langle \lambda(\tau_1, \tau_2) \rangle = \int_0^{\tau_1} \int_0^{\tau_2} \frac{1}{\omega_{d1}^2} e^{-\zeta_1 \omega_1 (\tau_1 + \tau_2 - \tau_3 - \tau_4)} \cos \omega_{d1}(\tau_1 - \tau_2 - \tau_3 + \tau_4) \langle [-\ddot{z}(\tau_3)] [-\ddot{z}(\tau_4)] \rangle d\tau_3 d\tau_4 \quad (3.46)$$

For broad-band ground motion,

$$\langle [-\ddot{z}(t_1)] [-\ddot{z}(t_2)] \rangle \approx \int_{-\infty}^{\infty} S(\omega, t_1) e^{i\omega(t_2 - t_1)} d\omega \quad (2.58)$$

Substituting equation (2.58) into equation (3.46) and interchanging the order of integration yields

$$\langle \lambda(\tau_1, \tau_2) \rangle = \int_0^{\tau_1} \frac{1}{\omega_{d1}^2} e^{-\zeta_1 \omega_1 (\tau_1 + \tau_2 - \tau_3)} \int_{-\infty}^{\infty} S(\omega, \tau_3) \xi(\tau_1, \tau_2, \tau_3, \omega) d\omega d\tau_3 \quad (3.47)$$

where

$$\xi(\tau_1, \tau_2, \tau_3, \omega) = \int_0^{\tau_2} e^{\zeta_1 \omega_1 \tau_4} \cos \omega_{d1}(\tau_1 - \tau_2 - \tau_3 + \tau_4) e^{i\omega(\tau_4 - \tau_3)} d\tau_4 \quad (3.48)$$

But  $\xi(\tau_1, \tau_2, \tau_3, \omega)$  is sharply peaked at  $\omega = \omega_{d1}$ , and

$$\int_{-\infty}^{\infty} \xi(\tau_1, \tau_2, \tau_3, \omega) d\omega = \begin{cases} 2\pi e^{\zeta_1 \omega_1 \tau_3} \cos \omega_{d1}(\tau_1 - \tau_2) & \text{if } 0 < \tau_3 < \tau_2 \\ 0 & \text{otherwise} \end{cases} \quad (3.49)$$

If  $S(\omega, \tau_3)$  is assumed to be nearly constant over the peak of  $\xi$ , then equation (3.47) may be simplified as

$$\langle \chi(\tau_1, \tau_2) \rangle = \int_0^{\min(\tau_1, \tau_2)} \frac{2\pi}{\omega_{d1}} e^{-\zeta_1 \omega_1 (\tau_1 + \tau_2 - 2\tau_3)} \cos \omega_{d1}(\tau_1 - \tau_2) S(\omega_{d1}, \tau_3) d\tau_3 \quad (3.50)$$

Substitution of equation (3.50) into equation (3.45) leads to the three-fold integral

$$\begin{aligned}
 \langle a_2^2(t) \rangle &= \int_0^t \int_0^t \int_0^{\min(\tau_1, \tau_2)} \frac{2\pi\omega_{d1}^2}{\omega_{d2}^2} e^{-\zeta_1\omega_1(\tau_1+\tau_2-2\tau_3)} e^{-\zeta_2\omega_2(2t-\tau_1-\tau_2)} \\
 &\quad \cos \omega_{d1}(\tau_1-\tau_2) \sin \omega_{d2}(t-\tau_1) \\
 &\quad \sin \omega_{d2}(t-\tau_2) S(\omega_{d1}, \tau_3) d\tau_3 d\tau_2 d\tau_1 \tag{3.51}
 \end{aligned}$$

Because the integrand is symmetric in  $\tau_1$  and  $\tau_2$ ,

$$\begin{aligned}
 \langle a_2^2(t) \rangle &= \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} \frac{4\pi\omega_{d1}^2}{\omega_{d2}^2} e^{-\zeta_1\omega_1(\tau_1+\tau_2-2\tau_3)} e^{-\zeta_2\omega_2(2t-\tau_1-\tau_2)} \\
 &\quad \cos \omega_{d1}(\tau_1-\tau_2) \sin \omega_{d2}(t-\tau_1) \\
 &\quad \sin \omega_{d2}(t-\tau_2) S(\omega_{d1}, \tau_3) d\tau_3 d\tau_2 d\tau_1 \tag{3.52}
 \end{aligned}$$

where the integration is over  $\tau_2 < \tau_1$  only.

Interchanging the order of integration in equation (3.52),

$$\begin{aligned} \langle a_2^2(t) \rangle &= \int_0^t S(\omega_{d1}, \tau_3) \left\{ \frac{4\pi\omega_{d1}^2}{\omega_{d2}^2} \int_{\tau_3}^t \int_{\tau_2}^t e^{-\zeta_1\omega_1(\tau_1+\tau_2-2\tau_3)} \right. \\ &\quad e^{-\zeta_2\omega_2(2t-\tau_1-\tau_2)} \cos \omega_{d1}(\tau_1-\tau_2) \\ &\quad \left. \sin \omega_{d2}(t-\tau_1) \sin \omega_{d2}(t-\tau_2) d\tau_1 d\tau_2 \right\} d\tau_3 \end{aligned} \quad (3.53)$$

$$\begin{aligned} &= \int_0^t S(\omega_{d1}, \tau_3) \left\{ \frac{4\pi\omega_{d1}^2}{\omega_{d2}^2} \int_0^{t-\tau_3} \int_0^{\tau_2'} e^{-\zeta_1\omega_1(2t-2\tau_3-\tau_1'-\tau_2')} \right. \\ &\quad e^{-\zeta_2\omega_2(\tau_1'+\tau_2')} \cos \omega_{d1}(\tau_1'-\tau_2') \\ &\quad \left. \sin \omega_{d2}\tau_1' \sin \omega_{d2}\tau_2' d\tau_1' d\tau_2' \right\} d\tau_3 \end{aligned} \quad (3.54)$$

where  $\tau_1' = t - \tau_1$ ,  $\tau_2' = t - \tau_2$ . The term in brackets is a function of  $t - \tau_3$  only, so equation (3.54) may be written

$$\langle a_2^2(t) \rangle = \int_0^t \mu_1(t-s) S(\omega_{d1}, s) ds \quad (3.55)$$

where

$$\mu_1(t) = \frac{4\pi\omega_{d1}^2}{\omega_{d2}} e^{-2\zeta_1\omega_1 t} \int_0^t \int_0^{\tau_2} e^{(\zeta_1\omega_1 - \zeta_2\omega_2)(\tau_1 + \tau_2)} \cos \omega_{d2}(\tau_1 - \tau_2) \sin \omega_{d2}\tau_1 \sin \omega_{d2}\tau_2 d\tau_1 d\tau_2 \quad (3.56)$$

Equation (3.56) is analogous to equation (3.16) for the second moment of the primary system envelope. For the primary system,  $\mu_1(t)$  is replaced by  $\frac{2\pi}{\omega_{d1}} e^{-2\zeta_1\omega_1 t}$ .

The case where  $\omega_2 < \omega_1$  is treated in much the same way. Beginning with equations (3.42) and (3.44), it may be shown that

$$\langle a_2^2(t) \rangle = \int_0^t \mu_2(t-s) S(\omega_{d2}, s) ds \quad (3.57)$$

where

$$\mu_2(t) = \frac{4\pi\omega_{d1}^2}{\omega_{d2}} e^{-2\zeta_1\omega_1 t} \int_0^t \int_0^{\tau_2} e^{(\zeta_2\omega_2 - \zeta_1\omega_1)(\tau_1 + \tau_2)} \cos \omega_{d2}(\tau_1 - \tau_2) \sin \omega_{d1}\tau_1 \sin \omega_{d1}\tau_2 d\tau_1 d\tau_2 \quad (3.58)$$

The similarity of equations (3.56) and (3.58) indicates that a general expression may be written which will be applicable to all  $\omega_1$  and  $\omega_2$ . Let the subscript "a" refer to the system with the lower frequency, so that

$$\omega_a = \min(\omega_1, \omega_2) \quad (3.59)$$

and  $\zeta_a$  is the corresponding damping coefficient. Similarly, let the subscript "b" refer to the system with the higher frequency. Then equations (3.55)-(3.58) may be generalized as

$$\langle a_2^2(t) \rangle = \int_0^t \mu(t-s) S(\omega_a, s) ds \quad (3.60)$$

where

$$\mu(t) = \frac{4\pi\omega_{d1}^2}{\omega_{d2}^2} e^{-2\zeta_a\omega_a t} \int_0^t \int_0^{\tau_2} e^{(\zeta_a\omega_a - \zeta_b\omega_b)(\tau_1 + \tau_2)} \cos \omega_{da}(\tau_1 - \tau_2) \sin \omega_{db}\tau_1 \sin \omega_{db}\tau_2 d\tau_1 d\tau_2 \quad (3.61)$$

Note that in equation (3.60) it has been assumed that

$$S(\omega_{da}, t) \approx S(\omega_a, t) \quad (3.62)$$

since  $S(\omega, t)$  is smooth.

The convolving function  $\mu(t)$  may be obtained in closed form. Let

$$\nu = \zeta_b\omega_b - \zeta_a\omega_a \quad (3.63)$$

$$\Delta\omega = \sqrt{(\omega_{db} - \omega_{da})^2 + \nu^2} \quad (3.64)$$

Note that  $\Delta\omega \neq 0$  unless  $\omega_1 = \omega_2$  and  $\zeta_1 = \zeta_2$ . The closed form expression for  $\mu(t)$  when  $\Delta\omega = 0$  must be computed separately. For  $\Delta\omega \neq 0$ , it may be shown

that

$$\mu(t) = \frac{\frac{\pi \omega_{d1}^2}{\omega_{d2}^2}}{(\Delta\omega)^2 [(\omega_{da} + \omega_{db})^2 + \nu^2]} \left\{ 2\omega_{db}^2 e^{-2\zeta_a \omega_a t} \right.$$

$$+ e^{-2\zeta_b \omega_b t} \left[ 2\omega_{db}^2 + (\omega_{da}^2 - \omega_{db}^2 + \nu^2) (1 - \cos 2\omega_{db} t) + 2\nu\omega_{db} \sin 2\omega_{db} t \right]$$

$$+ 2\omega_{db} e^{-(\zeta_a \omega_a + \zeta_b \omega_b) t} \left[ (\omega_{da} - \omega_{db}) \cos (\omega_{da} + \omega_{db}) t - \nu \sin (\omega_{da} + \omega_{db}) t \right.$$

$$\left. \left. - (\omega_{da} + \omega_{db}) \cos (\omega_{da} - \omega_{db}) t + \nu \sin (\omega_{da} - \omega_{db}) t \right] \right\} \quad (3.65a)$$

For  $\Delta\omega=0$ , the corresponding result is

$$\mu(t) = \pi e^{-2\zeta_1 \omega_1 t} \left[ \frac{1}{2} t^2 - \frac{1}{2\omega_{d1}} t \sin 2\omega_{d1} t + \frac{1}{4\omega_{d1}^2} (1 - \cos 2\omega_{d1} t) \right] \quad (3.65b)$$

Finally, note from equation (3.61) that

$$\mu(t) = \frac{2\pi\omega_{d1}^2}{\omega_{d2}^2} e^{-2\zeta_a \omega_a t} \int_0^t \int_0^t e^{-\nu(\tau_1 + \tau_2)} \cos \omega_{da}(\tau_1 - \tau_2) \sin \omega_{db} \tau_1 \sin \omega_{db} \tau_2 d\tau_1 d\tau_2 \quad (3.66)$$

by symmetry in  $\tau_1$  and  $\tau_2$ . Hence, expanding  $\cos \omega_{da}(\tau_1 - \tau_2)$  yields

$$\mu(t) = \frac{2\pi\omega_{d1}^2}{\omega_{d2}^2} e^{-2\zeta_a \omega_a t} \left[ \int_0^t e^{-\nu\tau} \cos \omega_{da} \tau \sin \omega_{db} \tau d\tau \right]^2 + \left[ \int_0^t e^{-\nu\tau} \sin \omega_{da} \tau \sin \omega_{db} \tau d\tau \right]^2 \quad (3.67)$$

Thus, it may be seen that  $\mu(t) \geq 0$ .

In summary, the second moment  $\langle a_2^2(t) \rangle$  is seen to be expressible as a convolution of the evolutionary power spectral density  $S(\omega, t)$  of the ground acceleration  $\ddot{z}(t)$  with the non-negative  $\mu(t)$ , given in closed form in equations (3.65). In order to obtain the probability distribution function of  $a_2(t)$ , moments of all orders must be computed. However, it is reasonable to suppose that the secondary envelope  $a_2(t)$  is nearly Rayleigh distributed, as is the primary system. The Rayleigh distribution arises as the distribution of the peaks of a narrow-band random process. Since the Rayleigh distribution has only a single parameter, it is determined completely by its second moment. Specifi-



cally, the probability density of  $a_2(t)$  will be assumed to be

$$p_2(a) = \frac{a}{\sigma_2^2(t)} e^{-\frac{1}{2} \frac{a^2}{\sigma_2^2(t)}} \quad (3.68)$$

where

$$\sigma_2^2(t) = \frac{1}{2} \langle a_2^2(t) \rangle \quad (3.69)$$

In particular, the mean value of the envelope will be

$$\langle a_2(t) \rangle = \sqrt{\frac{\pi}{2}} \sigma_2(t) \quad (3.70)$$

$$= \frac{1}{2} \sqrt{\pi \langle a_2^2(t) \rangle} \quad (3.71)$$

### 3.3.3 Response to Finite Duration Stationary Ground Motion

One application of the above results is to the case where the ground acceleration is derived from a stationary process which is modulated by a boxcar function of time. The evolutionary power spectral density of the ground acceleration will be taken as

$$S(\omega, t) = \begin{cases} S_0(\omega) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \quad (3.72)$$

$S_0(\omega)$  is the power spectral density of the underlying stationary process, and  $T$  is the duration of shaking.

Substitution of equation (3.72) into equation (3.60) for  $t \geq 0$  leads to

$$\langle a_1^2(t) \rangle = S_0(\omega_a) \rho(t) \quad (3.73)$$

where

$$\rho(t) = \begin{cases} \int_0^t \mu(s) ds & 0 \leq t \leq T \\ 0 & \\ \int_{t-T}^t \mu(s) ds & t \geq T \end{cases} \quad (3.74)$$

Let

$$\rho^*(t) = \int_0^t \mu(s) ds \quad (t \geq 0) \quad (3.75)$$

Then

$$\rho(t) = \begin{cases} \rho^*(t) & 0 \leq t \leq T \\ \rho^*(t) - \rho^*(t-T) & t > T \end{cases} \quad (3.76)$$

$\rho^*(t)$  describes the buildup of the secondary response from rest to stationarity, when  $T$  is made arbitrarily large.

The integral in equation (3.75) may be carried out in closed form.

For  $\Delta\omega \neq 0$ ,

$$\begin{aligned}
 \rho^*(t) = & \frac{\frac{\pi\omega_{d1}^2}{\omega_{d2}^2}}{(\Delta\omega)^2 [(\omega_{da} + \omega_{db})^2 + \nu^2]} \left\{ \frac{\omega_{db}^2}{\zeta_a \omega_a} \left( 1 - e^{-2\zeta_a \omega_a t} \right) \right. \\
 & + \frac{\omega_{da}^2 + \omega_{db}^2 + \nu^2}{2\zeta_b \omega_b} \left( 1 - e^{-2\zeta_b \omega_b t} \right) \\
 & + \frac{1}{2\omega_b} \left[ \left( \zeta_b \omega_b (\omega_{da}^2 - \omega_{db}^2 + \nu^2) + 2\nu\omega_{db}^2 \right) \left( 1 - e^{-2\zeta_b \omega_b t} \cos 2\omega_{db} t \right) \right. \\
 & \left. \left. + \left( -\omega_{db} (\omega_{da}^2 - \omega_{db}^2 + \nu^2) + 2\nu\zeta_b \omega_b \omega_{db} \right) e^{-2\zeta_b \omega_b t} \sin 2\omega_{db} t \right] \right\} \\
 & + \frac{2\omega_{db}}{(\zeta_a \omega_a + \zeta_b \omega_b)^2 + (\omega_{da} + \omega_{db})^2} \left[ 2(\zeta_a \omega_a \omega_{db} - \zeta_b \omega_b \omega_{da}) \right. \\
 & \left. \left( 1 - e^{-(\zeta_a \omega_a + \zeta_b \omega_b)t} \cos (\omega_{da} + \omega_{db})t \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & - (\omega_a^2 - \omega_b^2) e^{-(\zeta_a \omega_a + \zeta_b \omega_b) t} \sin (\omega_{da} + \omega_{db}) t \Big] \\
 & - \frac{2\omega_{db}}{(\zeta_a \omega_a + \zeta_b \omega_b)^2 + (\omega_{da} - \omega_{db})^2} \left[ 2(\zeta_a \omega_a \omega_{db} + \zeta_b \omega_b \omega_{da}) \right. \\
 & \left. \left( 1 - e^{-(\zeta_a \omega_a + \zeta_b \omega_b) t} \cos (\omega_{da} - \omega_{db}) t \right) \right. \\
 & \left. + (\omega_a^2 - \omega_b^2) e^{-(\zeta_a \omega_a + \zeta_b \omega_b) t} \sin (\omega_{da} - \omega_{db}) t \right] \quad (3.77a)
 \end{aligned}$$

For  $\Delta\omega=0$ ,

$$\begin{aligned}
 \rho^*(t) & = \frac{\pi}{8\zeta_1^3 \omega_1^3} \left[ 1 - (1 + 2\zeta_1 \omega_1 t + 2\zeta_1^2 \omega_1^2 t^2) e^{-2\zeta_1 \omega_1 t} \right] \\
 & + \frac{\pi}{8\zeta_1 \omega_1 \omega_{d1}^2} \left( 1 - e^{-2\zeta_1 \omega_1 t} \right) \\
 & - \frac{\pi}{4\omega_1^3 \omega_{d1}} \left( \zeta_1 \omega_{d1} - (\zeta_1 \omega_{d1} + \omega_1 \omega_{d1} t) e^{-2\zeta_1 \omega_1 t} \cos 2\omega_{d1} t \right. \\
 & \left. + [(\frac{1}{2} - \zeta_1^2) \omega_1 - \zeta_1 \omega_1^2 t] e^{-2\zeta_1 \omega_1 t} \sin 2\omega_{d1} t \right)
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{\pi}{8\omega_1^2 \omega_{d1}^2} [\zeta_1 \omega_1 (1 - e^{-2\zeta_1 \omega_1 t} \cos 2\omega_{d1} t) \\
 & + \omega_{d1} e^{-2\zeta_1 \omega_1 t} \sin 2\omega_{d1} t] \quad (3.77b)
 \end{aligned}$$

These formidable expressions may be simplified considerably by enforcing the assumption that  $\zeta_1, \zeta_2 \ll 1$ . To order  $\zeta_i$  ( $i=1,2$ ), for  $\Delta\omega \neq 0$ , equation (3.77a) becomes

$$\begin{aligned}
 \rho^*(t) \approx & \frac{\pi\omega_1^2}{(\Delta\omega)^2 \omega_2^2 (\omega_a + \omega_b)^2} \left\{ \frac{\omega_b^2}{\zeta_a \omega_a} (1 - e^{-2\zeta_a \omega_a t}) + \frac{\omega_a^2 + \omega_b^2}{2\zeta_b \omega_b} (1 - e^{-2\zeta_b \omega_b t}) \right. \\
 & - \frac{2\omega_b}{(\omega_a - \omega_b)^2 + \omega_a \omega_b (\zeta_a + \zeta_b)^2} \\
 & \left. \left[ 2(\zeta_a \omega_a^2 + \zeta_b \omega_b^2) \left( 1 - e^{-(\zeta_a \omega_a + \zeta_b \omega_b)t} \cos (\omega_a - \omega_b)t} \right) \right. \right. \\
 & \left. \left. + (\omega_a^2 - \omega_b^2) e^{-(\zeta_a \omega_a + \zeta_b \omega_b)t} \sin (\omega_a - \omega_b)t \right] \right\} \quad (3.78a)
 \end{aligned}$$

The oscillating terms in equation (3.78a) are significant only if  $\omega_1 \sim \omega_2$ . It may be seen that the terms in equation (3.77a) which oscillate rapidly are of order  $\zeta_i$  and may be neglected. This is convenient, since these terms would have been inappropriate for the slowly varying envelope defined as  $a_2(t)$ .

If  $\Delta\omega=0$ , then equation (3.78a) is indeterminate. The corresponding simplification of equation (3.77b) for this case gives

$$\rho^*(t) \approx \frac{\pi}{8\zeta_1^3\omega_1^3} \left[ 1 - \left( 1 + 2\zeta_1\omega_1 t + 2\zeta_1^2\omega_1^2 t^2 \right) e^{-2\zeta_1\omega_1 t} \right] \quad (3.78b)$$

Note that in equations (3.65), the approximation

$$\Delta\omega \approx \sqrt{(\omega_1 - \omega_2)^2 + \omega_1\omega_2(\zeta_1 - \zeta_2)^2} \quad (3.79)$$

is correct to order  $\zeta_i$ .

If T is large, the system will eventually achieve stationarity. Equations (3.73) and (3.76) indicate that at stationarity

$$\langle a_2^2 \rangle_{\text{stat}} = \rho^*(\infty) S_0(\omega_a) \quad (3.80)$$

The stationary variance of the displacement of a secondary system driven by white noise may be shown to be [4]

$$\langle y^2 \rangle_{\text{stat}} = \frac{\pi S_0 \left[ \omega_1^4 + \omega_1\omega_2^3(\zeta_1/\zeta_2) + A \right]}{2\zeta_2^3\omega_2^3 \left[ (\omega_1 - \omega_2)^2 + A \right]} \quad (3.81)$$

where

$$A = 4\omega_1\omega_2 \left[ \zeta_1\zeta_2(\omega_1^2 + \omega_2^2) + \omega_1\omega_2(\zeta_1^2 + \zeta_2^2) \right] \quad (3.82)$$

$S_0$  is the constant power spectral density of the ground acceleration.

If the secondary response is narrow-band, then at stationarity

$$y(t) = a_2(t) \cos [\omega^*t + \phi(t)] \quad (3.83)$$

where  $a_2(t)$  and  $\phi(t)$  may be assumed to be independent, and  $\omega^*$  is the dominant frequency. Then

$$\langle y^2 \rangle_{\text{stat}} \approx \frac{1}{2} \langle a_2^2 \rangle_{\text{stat}} \quad (3.84)$$

Equations (3.84) and (3.81) lead to an approximation for  $\langle a_2^2 \rangle_{\text{stat}}$  based on an assumption of narrow-bandedness. A comparison shows asymptotic agreement as  $\omega_2/\omega_1 \rightarrow 0$ , as  $\omega_2/\omega_1 \rightarrow 1$ , and as  $\omega_2/\omega_1 \rightarrow \infty$ . In these cases, the assumption of narrow-bandedness is valid. For intermediate values of  $\omega_2/\omega_1$ , however,  $y(t)$  has a bimodal behavior. Figures 3.2 and 3.3 show a comparison of the actual stationary response and that predicted by the narrow-bandedness assumption. It may be seen that the narrow-bandedness assumption underestimates the mean square envelope.

### 3.3.4 Comparison of Results to Simulation Studies

In order to verify the results of this section, a series of numerical simulations was performed. The ground motion was taken to be white noise modulated by a boxcar of finite duration  $T$ . The number of parameters may be reduced by letting

$$\xi = \omega_1 t \quad (3.85)$$

$$x(t) = \frac{S_0^{1/2}}{\omega_1^{3/2}} \hat{x}(\xi) \quad (3.86)$$

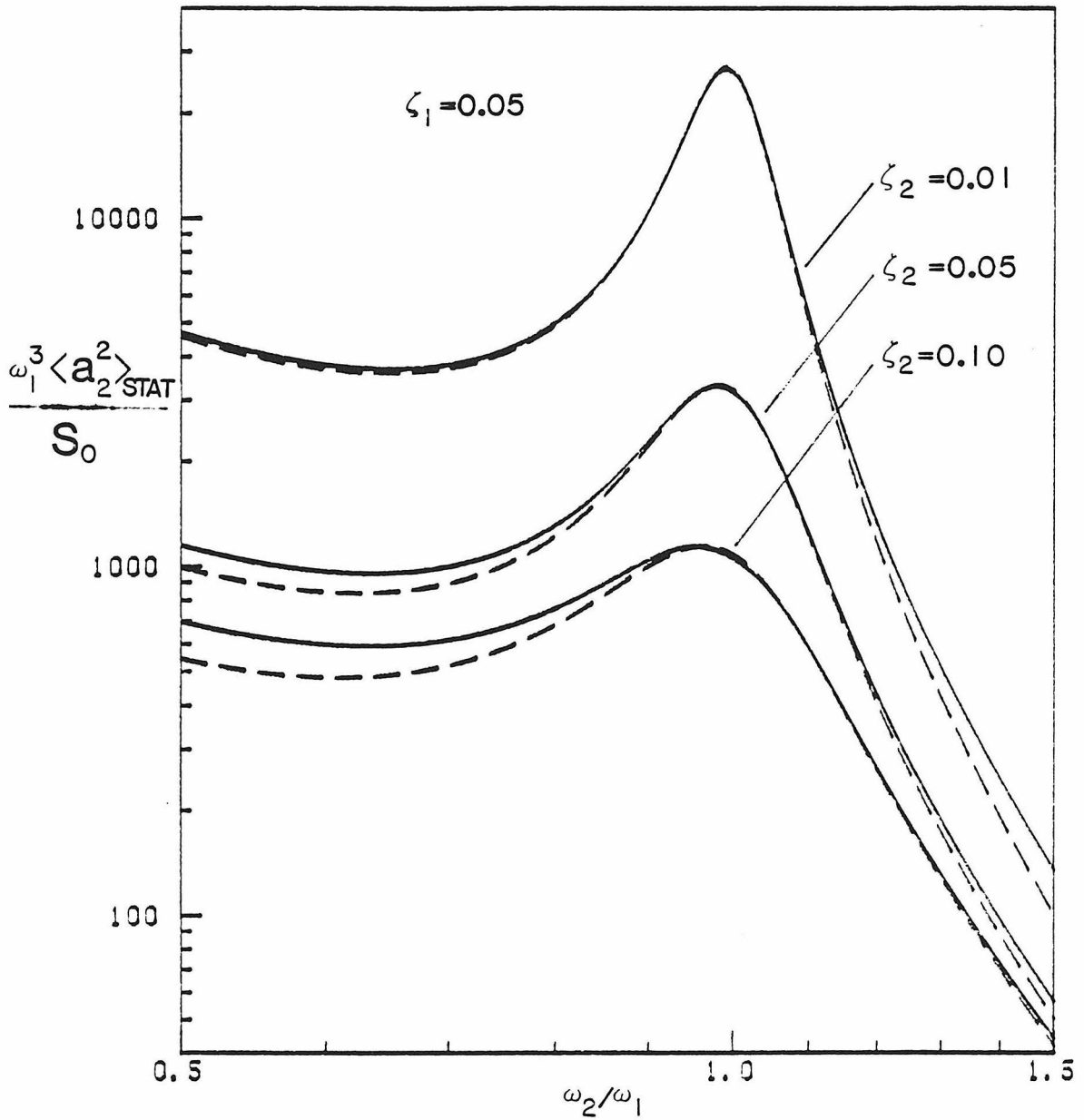


Fig. 3.2 Comparison of stationary mean square secondary envelope (solid) with narrow-band prediction (dashed).  $\zeta_1 = 0.05$ ,  $\zeta_2 = 0.01, 0.05, 0.10$ .



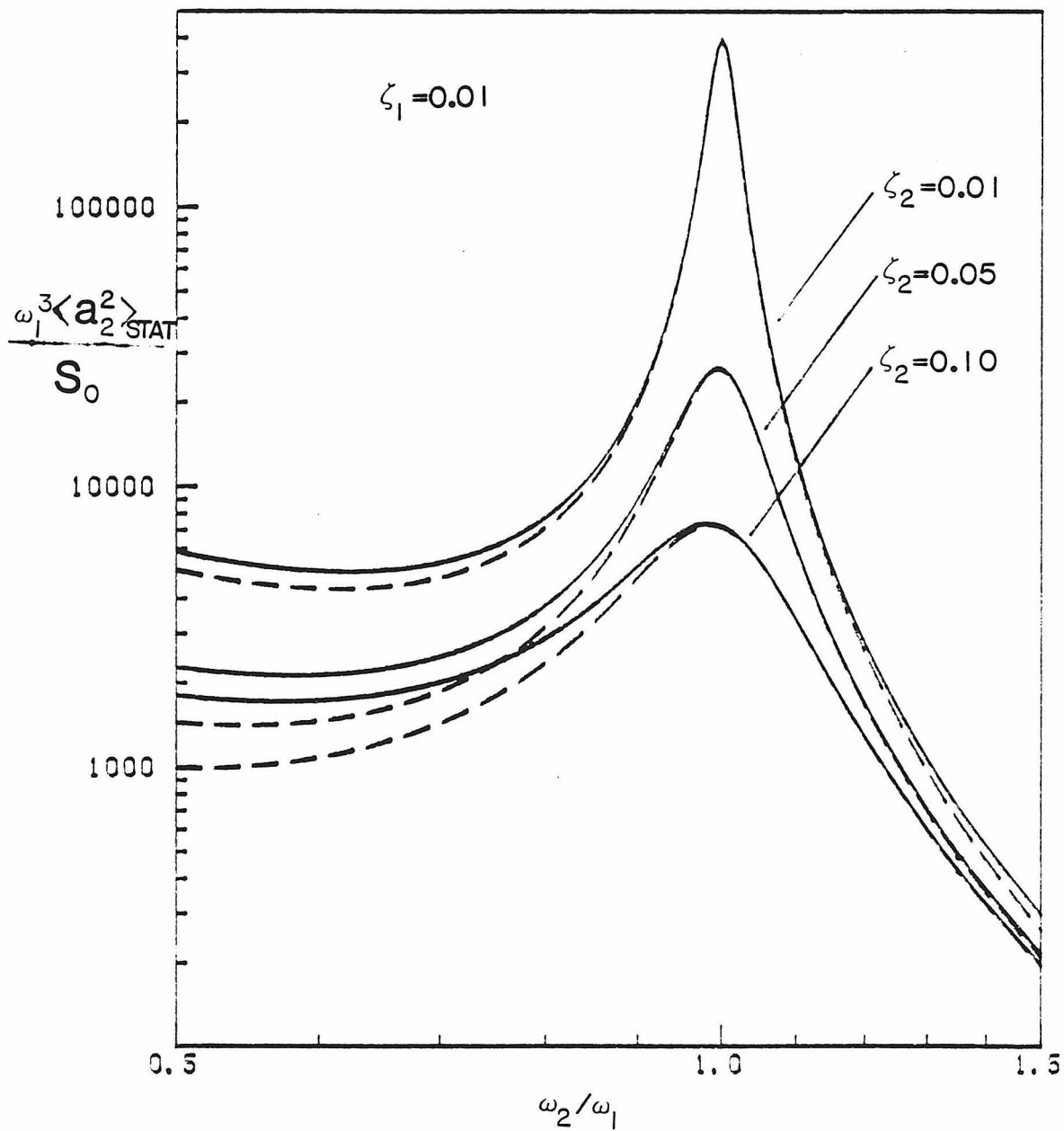


Fig. 3.3 Comparison of stationary mean square secondary envelope (solid) with narrow-band prediction (dashed).  $\zeta_1 = 0.01$ ,  $\zeta_2 = 0.01, 0.05, 0.10$ .

$$y(t) = \frac{S_0^{3/2}}{\omega_1^{3/2}} \hat{y}(\xi) \quad (3.87)$$

where  $S_0$  is the intensity of the white noise. The equations of motion may then be written in dimensionless form as

$$\hat{x}'' + 2\zeta_1 \hat{x}' + \hat{x} = \theta(\xi)w(\xi) \quad (3.88)$$

$$\hat{y}'' + 2\zeta_2 \beta \hat{y}' + \beta^2 \hat{y} = \theta(\xi)w(\xi) - \hat{x}'' \quad (3.89)$$

$$\hat{x}(0) = \hat{x}'(0) = \hat{y}(0) = \hat{y}'(0) = 0 \quad (3.90)$$

where

$$\beta = \omega_2/\omega_1 \quad (3.91)$$

$$\theta(\xi) = \begin{cases} 1 & \text{if } 0 \leq \xi \leq \omega_1 T \\ 0 & \text{otherwise} \end{cases} \quad (3.92)$$

$w(t)$  is white noise with unit power spectral density. The parameters to be varied are seen to be  $\zeta_1$ ,  $\zeta_2$ ,  $\omega_2/\omega_1$ , and  $\omega_1 T$ .

The simulations were performed using the algorithm described in the Appendix. During each simulated earthquake, the amplitude was set equal  $|y(t)|$  at the peaks of  $y$ , and interpolated linearly for all other  $t$ . This assumes that  $|y(t)|$  coincides with the envelope at each peak, which may not be true when  $y(t)$  is strongly bimodal.

The average value of  $a_2(t)$  over 250 samples is shown in Figures 3.4 and 3.5 for various values of the parameters. The theoretical mean envelope value is shown for comparison. In general, the theory gives

conservative results. The difference is most pronounced at higher levels of damping and away from resonance, when the bimodality of  $y(t)$  is greatest.

The intensity of response may be characterized by the maximum achieved by the mean value of the envelope. Figures 3.6 and 3.7 show a comparison of simulated and theoretical values for  $\max_{0 \leq t < \infty} \langle a_2(t) \rangle$ , plotted as a function of  $\omega_2/\omega_1$  for fixed duration. Figures 3.8 and 3.9 show the same comparison as a function of duration for fixed  $\omega_2/\omega_1$ . In both cases, the theoretical values correspond closely to those obtained by simulation.

### 3.3.5 Probability Distributions of $a_1(t)$ and $a_2(t)$

It has been assumed that  $a_1(t)$  and  $a_2(t)$  are approximately Rayleigh distributed. This assumption is reasonable, since the distribution of the peaks of a stationary zero-mean Gaussian process can be shown to approach a Rayleigh distribution as the process becomes more and more narrow-banded [1]. In this section, the actual distributions of  $a_1(t)$  and  $a_2(t)$  are discussed.

Equation (3.11) for the primary system and equations (3.36) and (3.41) for the secondary system are of the form

$$a_i^2 = \int_0^t \int_0^t f_i(t, \tau_1, \tau_2) [-\ddot{z}(\tau_1)] [-\ddot{z}(\tau_2)] d\tau_1 d\tau_2 \quad (3.93)$$

For the primary system,

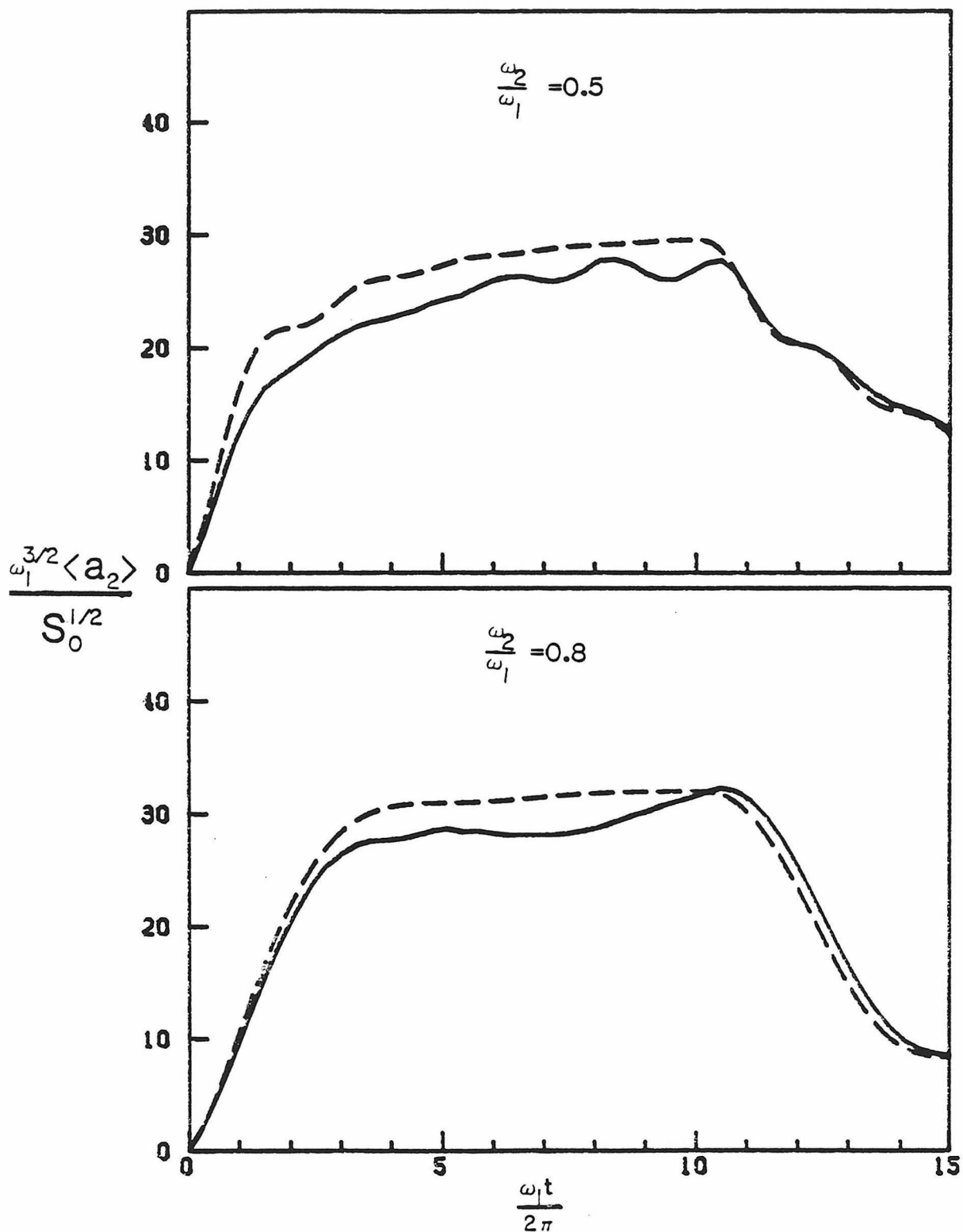


Fig. 3.4a Comparison of mean secondary envelope (dashed) with simulation (solid).  $\zeta_1 = \zeta_2 = 0.05$ ,  $\omega_1 T = 20\pi$

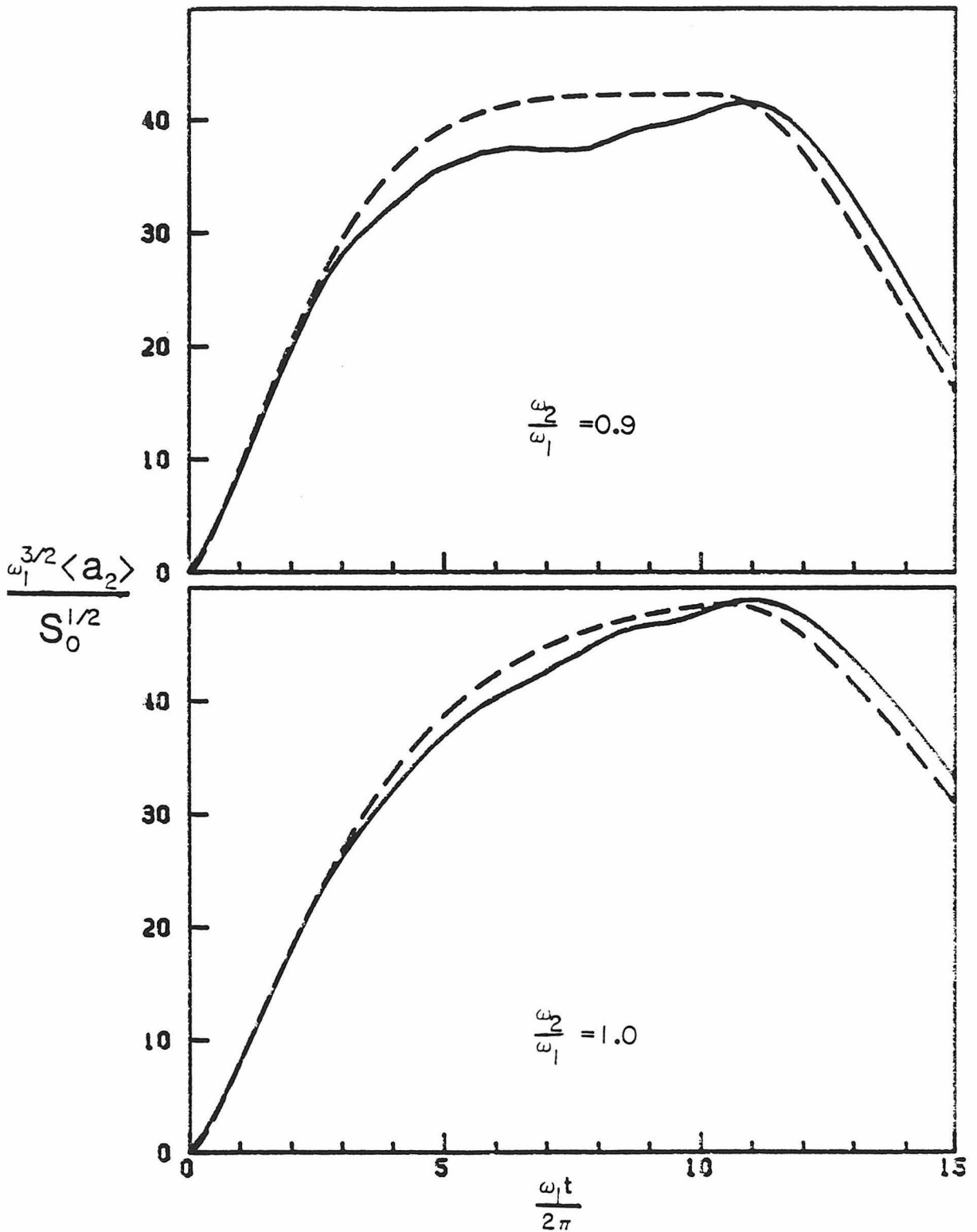


Fig. 3.4b Comparison of mean secondary envelope (dashed) with simulation (solid).  $\zeta_1 = \zeta_2 = 0.05$ ,  $\omega_1 T = 20\pi$

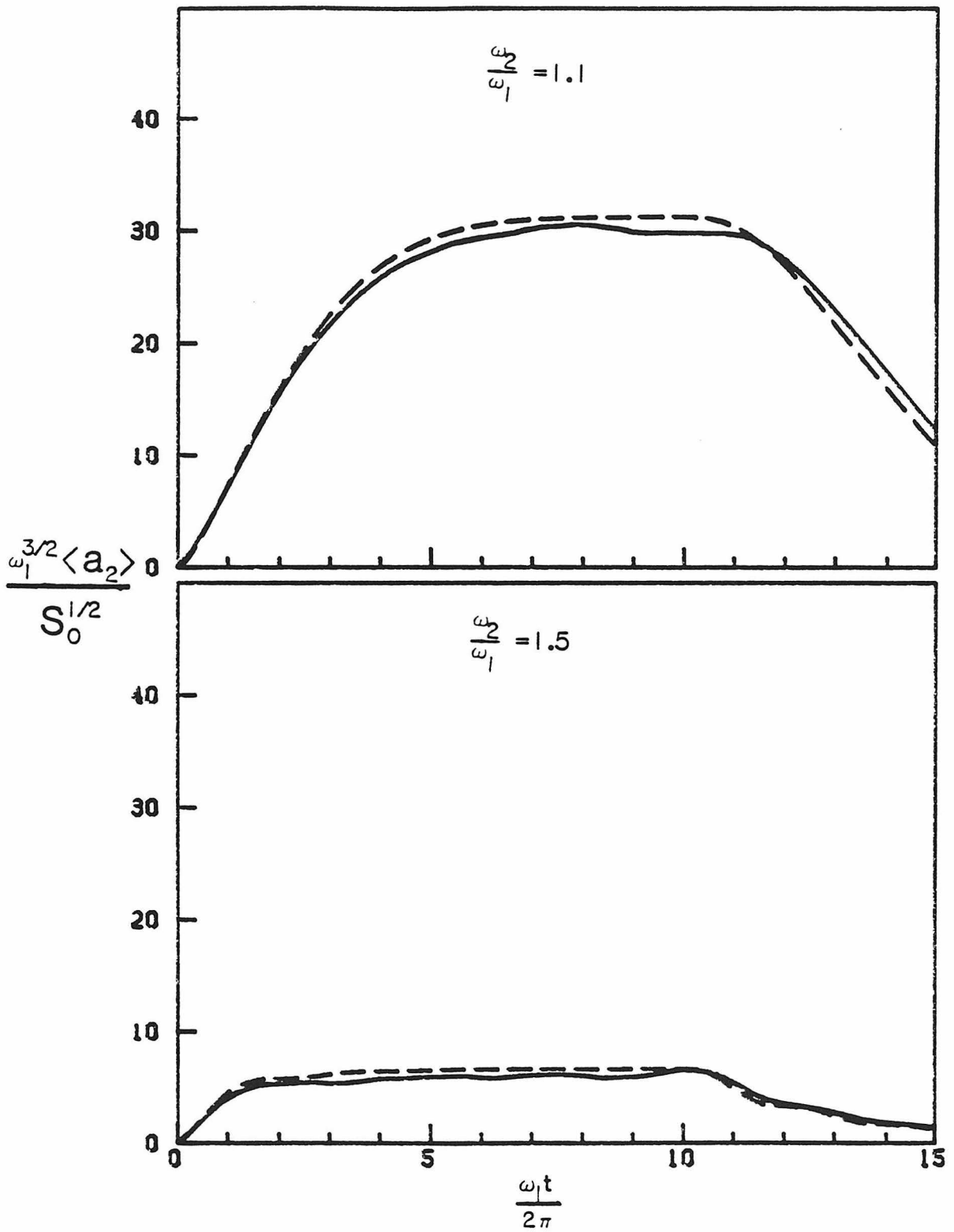


Fig. 3.4c Comparison of mean secondary envelope (dashed) with simulation (solid).  $\zeta_1 = \zeta_2 = 0.05$ ,  $\omega_1 T = 20\pi$

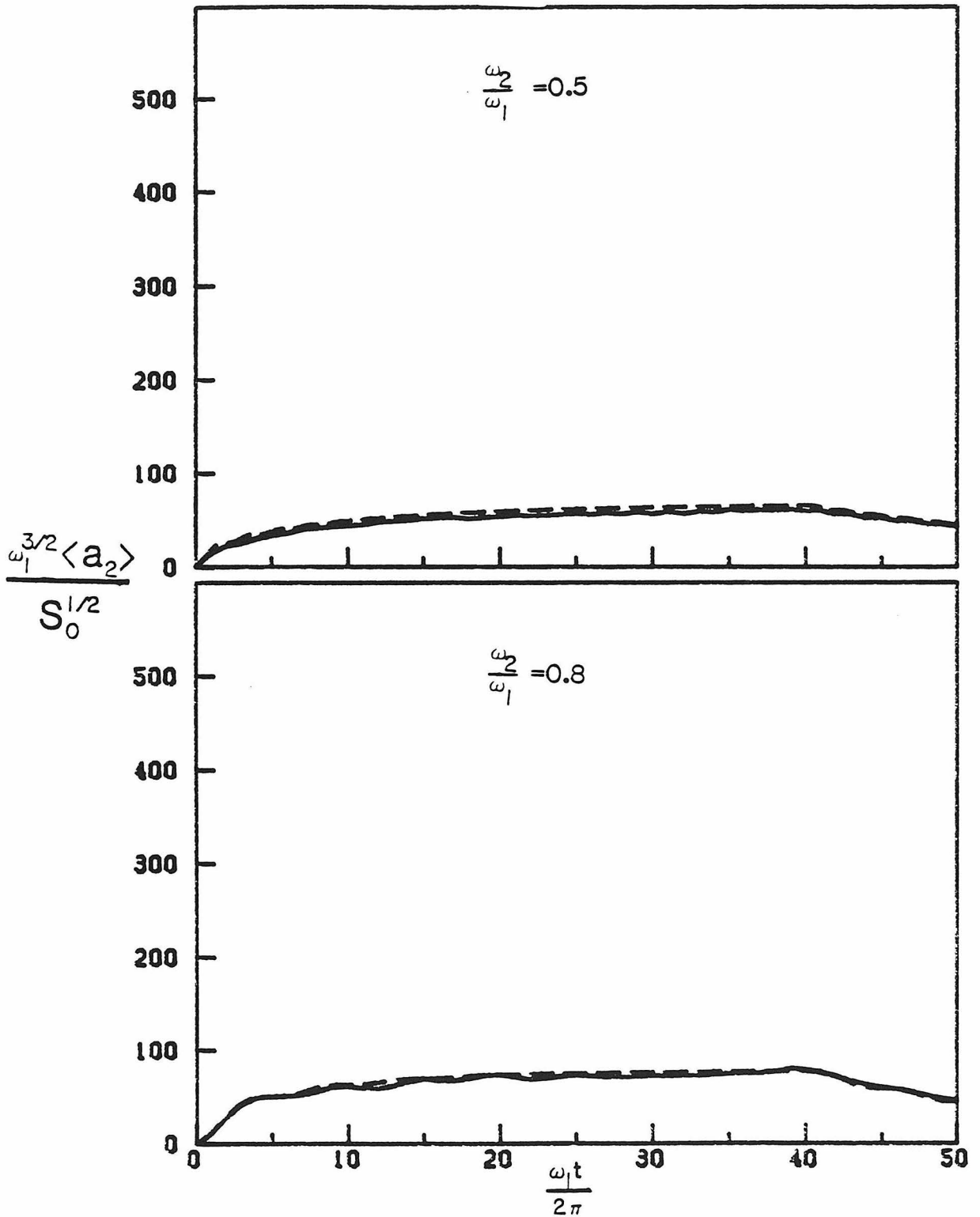


Fig. 3.5a Comparison of mean secondary envelope (dashed) with simulation (solid).  $\zeta_1 = \zeta_2 = 0.01$ ,  $\omega_1 T = 80\pi$

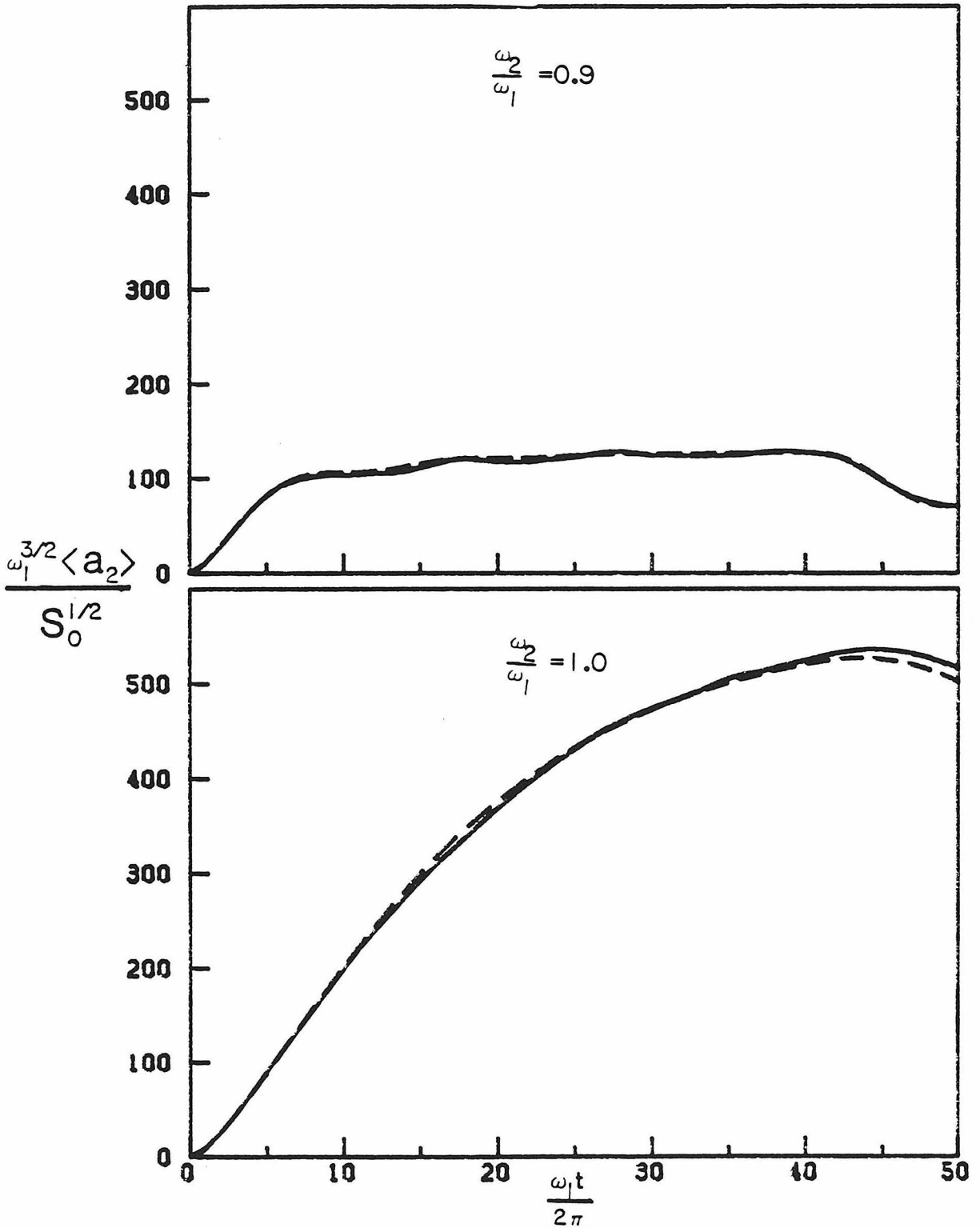


Fig. 3.5b Comparison of mean secondary envelope (dashed) with simulation (solid).  $\zeta_1 = \zeta_2 = 0.01$ ,  $\omega_1 T = 80\pi$



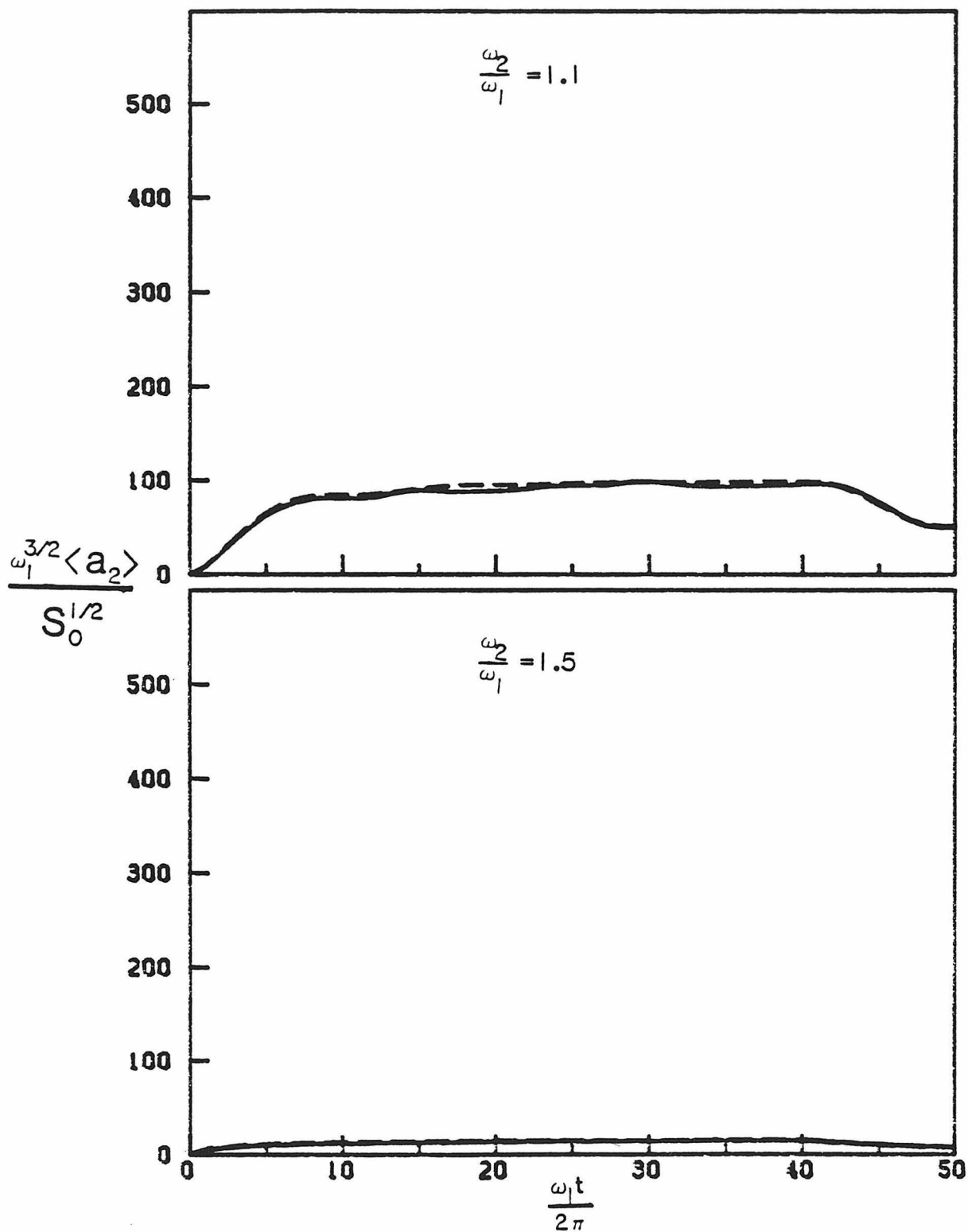


Fig. 3.5c Comparison of mean secondary envelope (dashed) with simulation (solid).  $\zeta_1 = \zeta_2 = 0.01$ ,  $\omega_1 T = 80\pi$

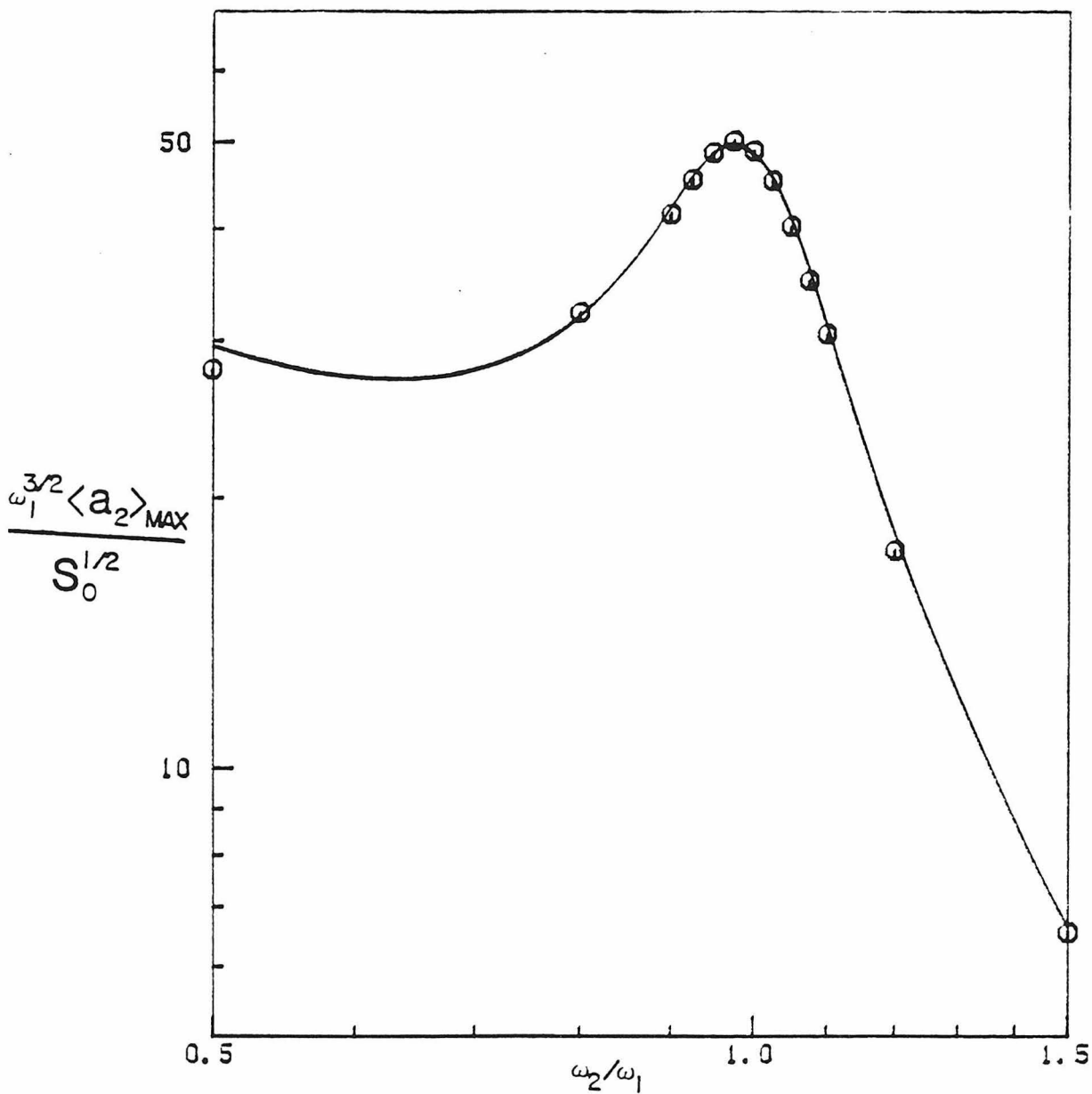


Fig. 3.6 Comparison of  $\max_{0 \leq t < \infty} \langle a_2(t) \rangle$  with simulation (circles).  
 $\zeta_1 = \zeta_2 = 0.05, \omega_1 T = 20\pi$

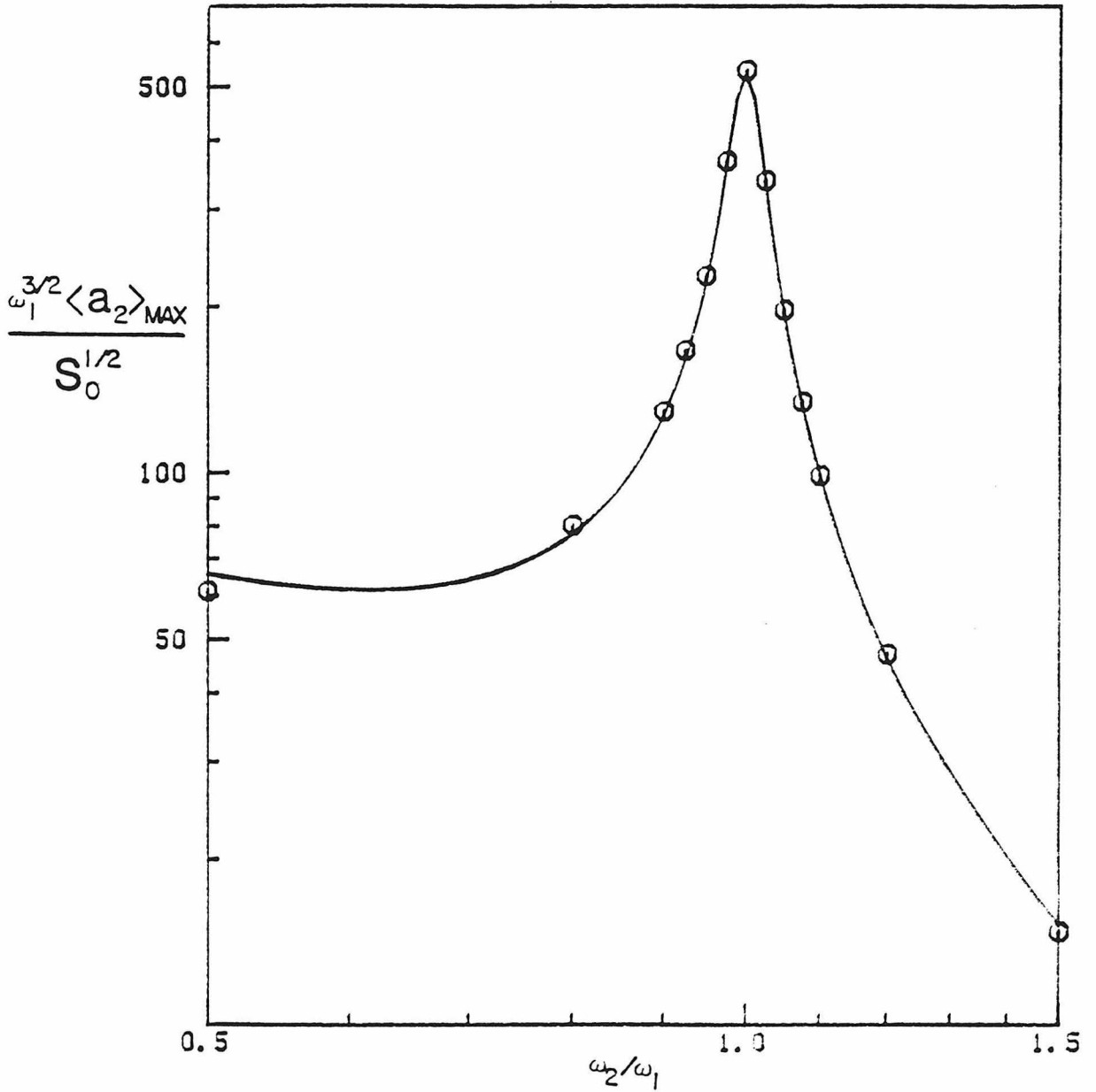


Fig. 3.7 Comparison of  $\max_{0 \leq t < \infty} \langle a_2(t) \rangle$  with simulation (circles).  
 $\zeta_1 = \zeta_2 = 0.01, \omega_1 T = 80\pi$

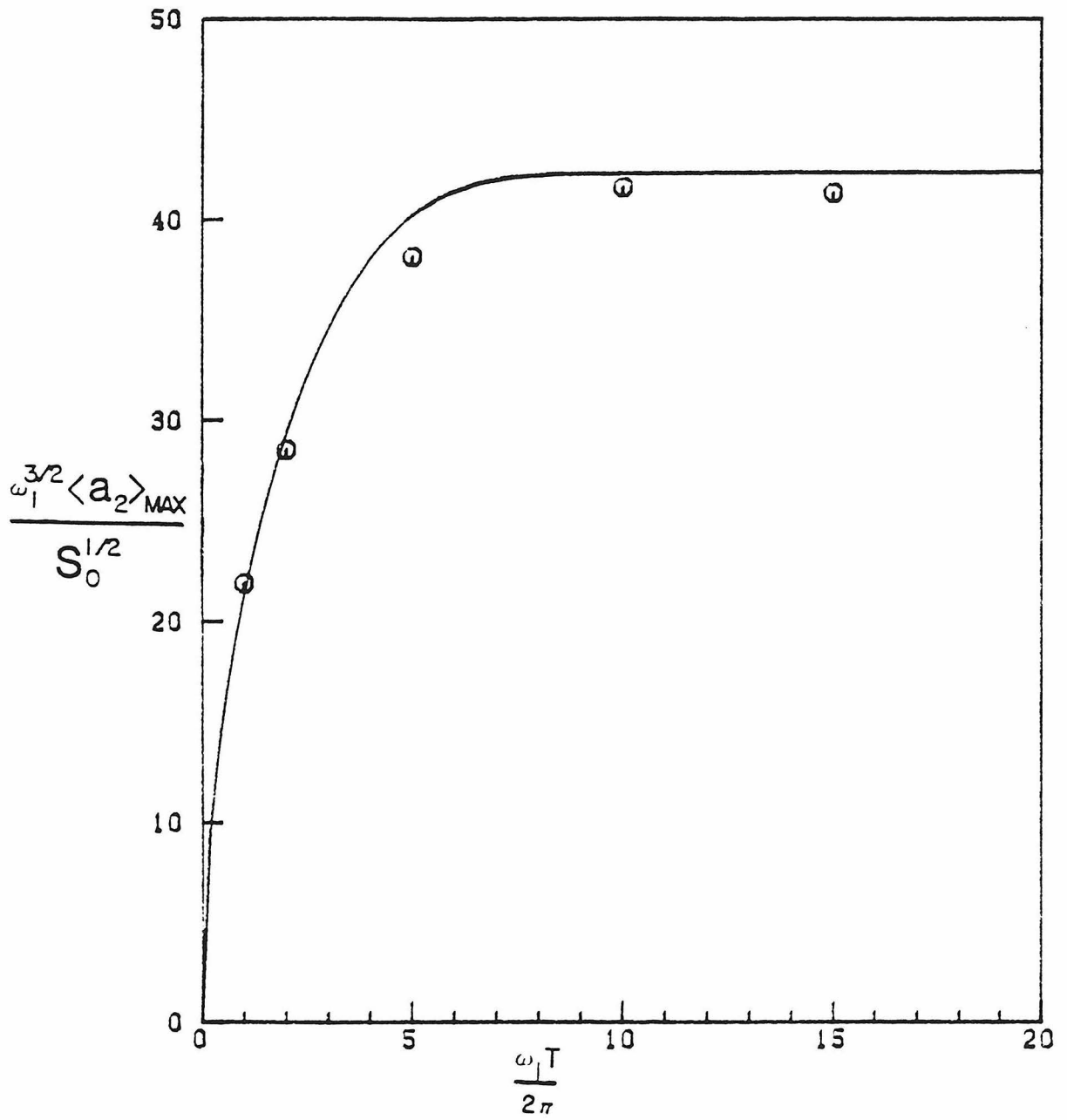


Fig. 3.8 Comparison of  $\max_{0 \leq t < \infty} \langle a_2(t) \rangle$  with simulation (circles)  
 $\zeta_1 = \zeta_2 = 0.05, \omega_2/\omega_1 = 0.9$

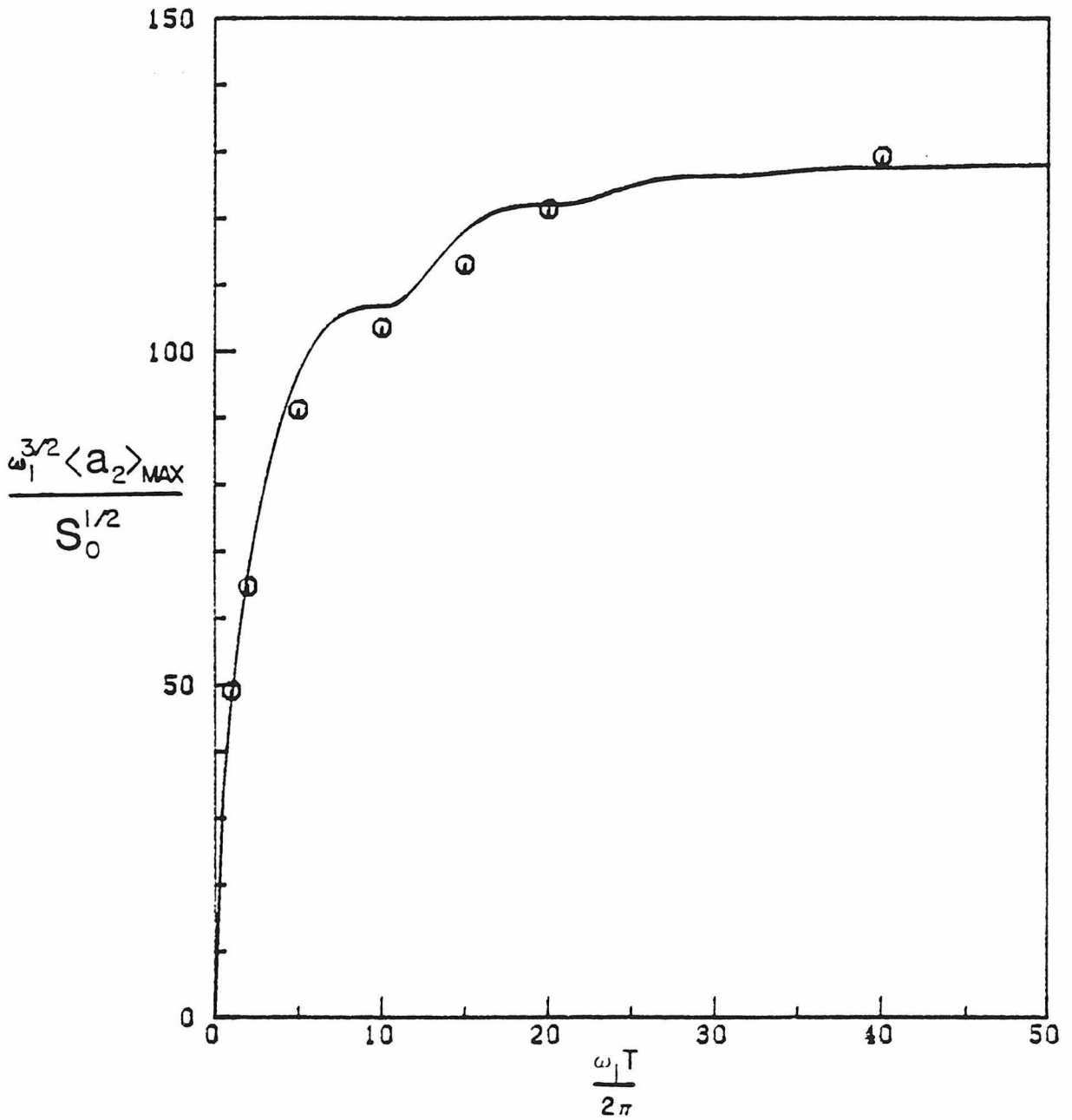


Fig. 3.9 Comparison of  $\max_{0 \leq t < \infty} \langle a_2(t) \rangle$  with simulation (circles)  
 $\zeta_1 = \zeta_2 = 0.01, \omega_2/\omega_1 = 0.9$

$$f_1(t, t_1, t_2) = \frac{1}{\omega_{d1}} e^{-\zeta_1 \omega_1 (2t - t_1 - t_2)} \cos \omega_{d1} (t_1 - t_2) \quad (3.94)$$

For the secondary system, if  $\omega_1 \leq \omega_2$ , equations (3.36) and (3.41) may be combined, and the order of integration changed to show that

$$f_2(t, t_1, t_2) = \int_{t_1}^t \int_{t_2}^t \frac{\omega_{d1}^2}{\omega_{d2}} e^{-\zeta_1 \omega_1 (\tau_1 + \tau_2 - t_1 - t_2)} e^{-\zeta_2 \omega_2 (2t - \tau_1 - \tau_2)} \cos \omega_{d1} (t_1 - t_2 - \tau_1 + \tau_2) \sin \omega_{d2} (t - \tau_1) \sin \omega_{d2} (t - \tau_2) d\tau_1 d\tau_2 \quad (3.95)$$

A similar expression may be derived for  $\omega_2 < \omega_1$ . Note that both  $f_1$  and  $f_2$  are symmetric in  $t_1$  and  $t_2$ .

If  $z(t)$  is broad-band with a smoothly varying spectrum, then the excitation terms may be approximated by white noise with time-varying intensity  $S(\omega_1, t)$ . It has been seen that the primary and secondary system response may be approximated as the response to white noise.

With this motivation, consider the random variable

$$Y = \int_0^t \int_0^t g(\tau_1, \tau_2) w(\tau_1) w(\tau_2) d\tau_1 d\tau_2 \quad (3.96)$$

where  $w(t)$  is Gaussian white noise with power spectral density equal to unity. The random variable  $Y$  will represent the random variable  $a_i^2(t)$  if

$$g(t_1, t_2) = f_i(t, t_1, t_2) \sqrt{S(\omega_1, t_1) S(\omega_1, t_2)} \quad (3.97)$$

The integral in equation (3.96) may be discretized as

$$Y = \sum_{j=1}^N \sum_{k=1}^N g(j\Delta, k\Delta) X_j X_k \quad (3.98)$$

where

$$\Delta = t/N \quad (3.99)$$

$$X_j = \int_{j\Delta}^{(j+1)\Delta} w(\tau) d\tau \quad (3.100)$$

Equation (3.98) will provide a good approximation to  $Y$  if  $\Delta$  is small compared with the scale of fluctuations of  $g(t_1, t_2)$ . The  $X_j$  ( $j=1, \dots, n$ ) are independent, normally distributed random variables with mean zero and variance  $2\pi/\Delta$ .

The distribution of the quadratic form in equation (3.98) is most easily expressed in terms of its characteristic function. Let the probability density of  $Y$  be  $p_Y(y)$ . Its characteristic function  $\phi_Y(\xi)$  is then defined to be

$$\phi_Y(\xi) = \langle e^{i\xi Y} \rangle \quad (3.101)$$

$$= \int_{-\infty}^{\infty} p_Y(y) e^{i\xi y} dy \quad (3.102)$$

If  $\phi_Y(\xi)$  is known, then  $p_Y(y)$  may be recovered by Fourier inversion as

$$p_Y(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_Y(\xi) e^{-i\xi y} d\xi \quad (3.103)$$

It can be shown [2] that if  $\tilde{U}$  is a vector of  $N$  independent, normally distributed random variables with zero mean and unit variance, and  $A$  is a real, symmetric  $N \times N$  matrix, then the characteristic function of the distribution of  $Z = \tilde{U}^T A \tilde{U}$  is

$$\phi_Z(\xi) = [\det (I - 2i\xi A)]^{-1/2} \quad (3.104)$$

where  $I$  is the  $N \times N$  identity matrix and  $\det( )$  denotes the determinant of the matrix. Equivalently,

$$\phi_Z(\xi) = \prod_{j=1}^N (1 - 2i\mu_j \xi)^{-1/2} \quad (3.105)$$

where  $\mu_j$  are the eigenvalues of  $A$ .

By analogy with equation (3.105), the distribution of  $Y$  in equation (3.98) is seen to be

$$\phi_Y(\xi) = \prod_{j=1}^N [1 - (4\pi\Delta) i\mu_j \xi]^{-1/2} \quad (3.106)$$

where  $\mu_j$  are the eigenvalues of the matrix  $G$  whose entries are

$$[G]_{ij} = g(i\Delta, j\Delta) \quad (3.107)$$

This result may be expressed in terms of the continuous function  $g(t_1, t_2)$  by considering the limit as  $N \rightarrow \infty$ . It may be seen that  $\phi_Y(\xi)$  approaches the limit



$$\phi_Y(\xi) = \prod_{j=1}^{\infty} (1 - 4\pi i \lambda_j \xi)^{-1/2} \quad (3.108)$$

where  $\lambda_j$  are the eigenvalues of  $g(t_1, t_2)$ . Strictly speaking,  $\lambda_j$  are the discrete values of  $\lambda$  for which the equation

$$L_t(\phi) = \int_0^t g(t_1, t_2) \phi(t_2) dt_2 = \lambda \phi(t_1) \quad (3.109)$$

possesses non-trivial solutions  $\phi_j(t)$ .  $g(t_1, t_2)$  is real, symmetric, and positive definite [since it assigns a positive value  $a_i^2$  for any  $w(t)$ ].

Thus  $L_t$  has only positive real eigenvalues which may be ordered [3]

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq 0 \quad (3.110)$$

Equation (3.108) expresses  $\phi_Y(\xi)$  in terms of these eigenvalues.  $p_Y(y)$  may then be recovered from equation (3.103).

Note that  $\lambda_i$  will vary with  $t$ , so the distribution of  $a_i(t)$  will depend on  $t$ . Clearly, the result derived here is of theoretical rather than practical importance. For practical purposes, the Rayleigh approximation seems to give reasonably accurate results.

### 3.4 Interaction Effects

So far, it has been assumed that the mass of the secondary system is negligible in comparison with the mass of the primary system, so that the response of the primary system is unaffected by the presence of the secondary system. This assumption simplifies the analysis considerably by reducing the combined system to two chained single-degree-of-freedom

systems.

As the mass ratio increases, the non-interaction approximation becomes progressively worse, particularly if  $\omega_1 \sim \omega_2$ . Ignoring interaction effects generally leads to conservative estimates of system response, since the secondary system will actually absorb energy from the primary system. However, in some cases (especially near resonance), the assumption of non-interaction leads to gross overestimates of system response.

In this section, the effect of primary/secondary interaction is examined. A particularly simple method of analysis is developed for a classically damped system.

### 3.4.1 Equivalent Non-Interacting System (Classical Damping)

The equations of motion for the two-degree-of-freedom primary/secondary system, including interaction effects, are:

$$\ddot{x} + 2\zeta_1\omega_1\dot{x} + \omega_1^2x - \varepsilon(2\zeta_2\omega_2\dot{y} + \omega_2^2y) = -\ddot{z} \quad (3.111)$$

$$\ddot{y} + 2\zeta_2\omega_2\dot{y} + \omega_2^2y = -\ddot{z} - \ddot{x} \quad (3.112)$$

$$x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 0 \quad (3.113)$$

where  $\varepsilon = m_2/m_1$  is the mass ratio.

Introduce the change of variables

$$x_e(t) = x(t) - \theta y(t) \quad (3.114)$$

$$y_e(t) = \frac{1}{1+\theta} y(t) \quad (3.115)$$

where  $\theta$  is a constant to be determined. Then equations (3.111)-(3.113) become

$$\begin{aligned} \ddot{x}_e + 2(1+\theta)\zeta_1\omega_1\dot{x}_e + (1+\theta)\omega_1^2x_e \\ + 2(1+\theta)\left[\theta\zeta_1\omega_1 - \left(\frac{\theta}{1+\theta} + \varepsilon\right)\zeta_2\omega_2\right]\dot{y}_e \\ + (1+\theta)\left[\theta\omega_1^2 - \left(\frac{\theta}{1+\theta} + \varepsilon\right)\omega_2^2\right]y_e = -z \end{aligned} \quad (3.116)$$

$$\ddot{y}_e + 2\frac{1}{1+\theta}\zeta_2\omega_2\dot{y}_e + \frac{1}{1+\theta}\omega_2^2y_e = -z - \ddot{x}_e \quad (3.117)$$

$$x_e(0) = \dot{x}_e(0) = y_e(0) = \dot{y}_e(0) = 0 \quad (3.118)$$

Suppose it is possible to choose  $\theta$  so that

$$\theta\zeta_1\omega_1 - \left(\frac{\theta}{1+\theta} + \varepsilon\right)\zeta_2\omega_2 = 0 \quad (3.119)$$

$$\theta\omega_1^2 - \left(\frac{\theta}{1+\theta} + \varepsilon\right)\omega_2^2 = 0 \quad (3.120)$$

Then, equations (3.116)-(3.118) reduce to the same form as equations (3.111)-(3.113), with  $\varepsilon=0$ . In other words, if equations (3.119) and (3.120) can be satisfied, then  $x_e$  and  $y_e$  as defined in equations (3.114)-(3.115) satisfy the equations of motion of the non-interacting primary/secondary system

$$\ddot{x}_e + 2\zeta_1'\omega_1'\dot{x}_e + (\omega_1')^2x_e = -z \quad (3.121)$$

$$\ddot{y}_e + 2\zeta_2' \omega_2' \dot{y}_e + (\omega_2')^2 y_e = -z \ddot{x}_e \quad (3.122)$$

$$x_e(0) = \dot{x}_e(0) = y_e(0) = \dot{y}_e(0) = 0 \quad (3.123)$$

where

$$\zeta_1' = \sqrt{1+\theta} \zeta_1 \quad (3.124a)$$

$$\omega_1' = \sqrt{1+\theta} \omega_1 \quad (3.124b)$$

$$\zeta_2' = \sqrt{\frac{1}{1+\theta}} \zeta_2 \quad (3.124c)$$

$$\omega_2' = \sqrt{\frac{1}{1+\theta}} \omega_2 \quad (3.124d)$$

Since

$$y(t) = (1+\theta) y_e(t) \quad (3.125)$$

the actual (interacting) secondary envelope is obtained as  $(1+\theta)$  times the secondary envelope of the equivalent non-interacting system of equations (3.121)-(3.123), with system parameters modified as in equations (3.124). Thus, the results already obtained for non-interacting systems can be carried over to the interacting system simply by modifying the system parameters and scaling the response.

Unfortunately, the simultaneous solution of equations (3.119)-(3.120) requires

$$\frac{\zeta_1}{\zeta_2} = \frac{\omega_1}{\omega_2} \quad (3.126)$$

This is exactly the condition that the original two-degree-of-freedom

system be classically damped. Because of the simplicity of the above approach, it will be assumed temporarily that the system is classically damped, to guarantee the existence of an equivalent non-interacting system.

If equation (3.126) holds, two roots to equations (3.119)-(3.120) are available:

$$\theta = \frac{1}{2} \left[ (1+\varepsilon) \frac{\omega_2^2}{\omega_1^2} - 1 \right] \pm \left\{ \frac{1}{4} \left[ (1+\varepsilon) \left( \frac{\omega_2}{\omega_1} \right)^2 - 1 \right]^2 + \varepsilon \left( \frac{\omega_2}{\omega_1} \right)^2 \right\}^{1/2} \quad (3.127)$$

There are therefore two equivalent non-interacting systems. It is natural to choose  $\theta$  so that  $\theta \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Then, it may be seen from equations (3.114)-(3.115) that the equivalent non-interacting system approaches the actual system as  $\varepsilon \rightarrow 0$ . This leads to

$$\theta = \frac{1}{2} \left[ (1+\varepsilon) \frac{\omega_2^2}{\omega_1^2} - 1 \right] + \operatorname{sgn}(\omega_1^2 - \omega_2^2) \left\{ \frac{1}{4} \left[ (1+\varepsilon) \left( \frac{\omega_2}{\omega_1} \right)^2 - 1 \right]^2 + \varepsilon \left( \frac{\omega_2}{\omega_1} \right)^2 \right\}^{1/2} \quad (3.128)$$

where the positive square root is taken.

Figure 3.10 shows  $\theta$  as a function of the mass ratio  $\varepsilon$ , for several values of  $\omega_2/\omega_1$ . It may be seen that  $\theta$  is positive when  $\omega_2 < \omega_1$ , and negative when  $\omega_2 > \omega_1$ .

The jump in  $\theta$  at  $\omega_2 = \omega_1$  corresponds to a bifurcation in the choice of the equivalent system. The solution will be continuous, however. If  $\theta$  is small, then equations (3.124) give

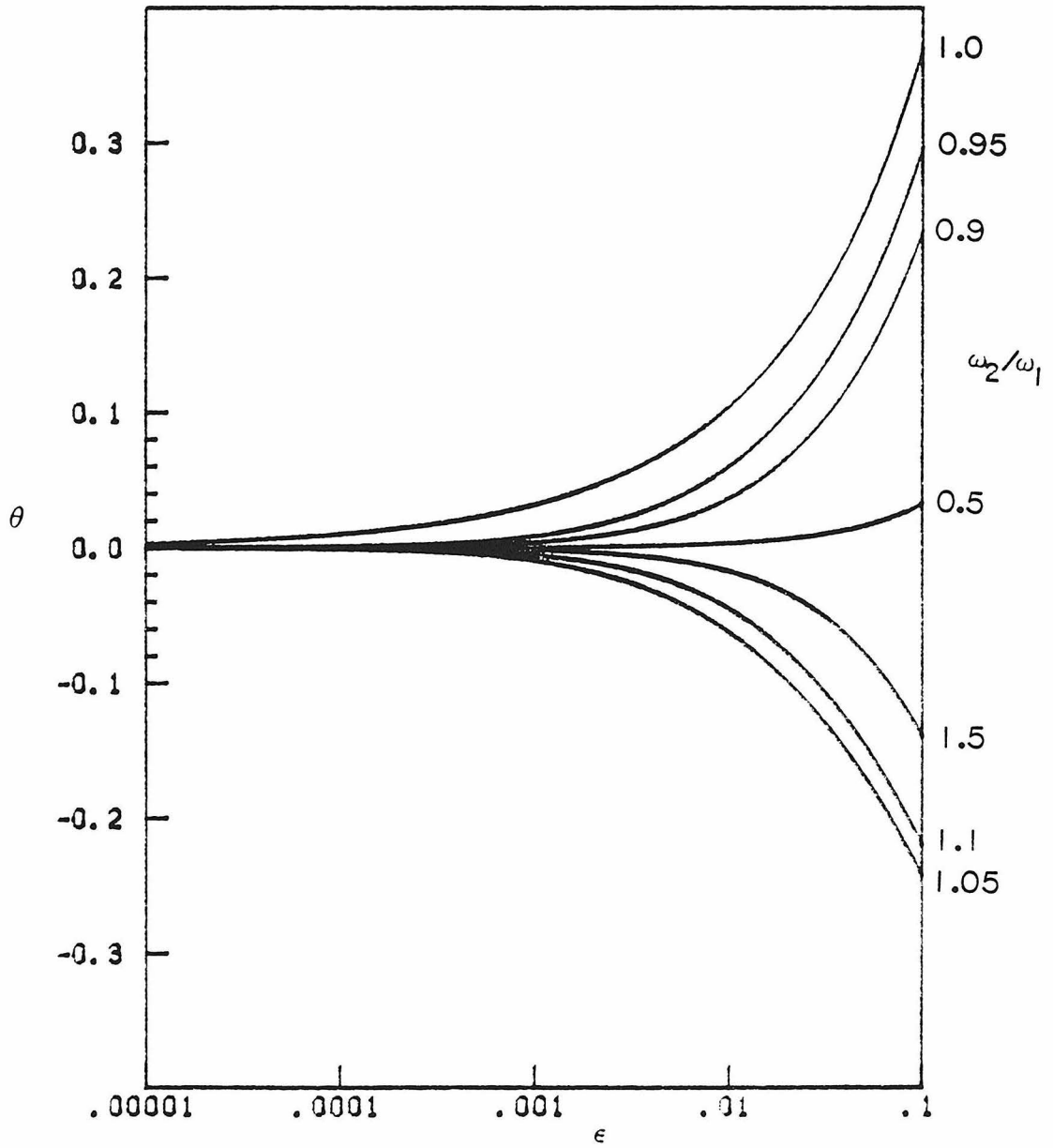


Fig. 3.10 Interaction parameter  $\theta$  vs. mass ratio  $\epsilon$  for several values of  $\omega_2/\omega_1$ .

$$\omega_1' - \omega_2' = \sqrt{1+\theta} \omega_1 - \sqrt{\frac{1}{1+\theta}} \omega_2 \quad (3.129)$$

$$= (\omega_1 - \omega_2) + \theta \frac{\omega_1 + \omega_2}{2} + O(\theta^2) \quad (3.130)$$

Thus, the natural frequencies of the equivalent non-interacting system are more widely spaced than those of the original system. This effect is most prominent near resonance, where a slight "detuning" may have a large influence on the response. By comparison, the changes in damping and the scaling of  $y(t)$  are much less significant.

Near resonance, for small  $\theta$ , equation (3.130) simplifies to

$$\omega_1' - \omega_2' \approx \theta \omega_1 \quad (3.131)$$

The width of the resonance peak is of order  $\zeta_1 \omega_1$ , so the interaction effect may be neglected if

$$|\theta| \ll \zeta_1 \quad (3.132)$$

At perfect resonance, equation (3.128) gives

$$\theta = \pm \sqrt{\varepsilon} [1 + O(\varepsilon)] \quad (3.133)$$

Thus, interaction may be neglected if

$$\varepsilon \ll \zeta_1^2 \quad (3.134)$$

Note that the restriction to classical damping requires  $\zeta_1 \approx \zeta_2$  when

$$\omega_1 \approx \omega_2.$$

Away from resonance, the detuning is less significant, so equation (3.134) remains a conservative criterion. If  $\zeta_1 = \zeta_2 = 0.05$ , this requires that the secondary system's mass be much less than 1/400 of the primary system's mass for interaction effects to be ignored.

### 3.4.2 Non-Classically Damped Systems

If the two-degree-of-freedom system is not classically damped, the above analysis does not apply. In this case, there seems to be no simplification of the interacting system, and the complete system must be considered.

The stationary analysis of Igusa and Der Kiureghian [12] indicates that the effects of interaction may be neglected if

$$\varepsilon \ll \zeta_1 \zeta_2 \quad (3.135)$$

In any event, the assumption of non-interaction gives conservative results when  $\varepsilon$  is small.



CHAPTER IV

APPLICATION TO FLOOR RESPONSE SPECTRA

4.1 Introduction

An important application of stochastic analysis techniques for secondary systems is the development of floor response spectra. These spectra serve the same function in the design of secondary systems as the ground response spectrum does for primary systems.

A floor response spectrum is defined for a given ground motion record, primary structure, and attachment point within the structure. Let  $y(t)$  be the relative displacement of a single-degree-of-freedom oscillator with negligible mass, which is attached at the specified point, with natural frequency  $\omega$  and damping  $\zeta$ . Then the floor response spectrum value for this oscillator is defined as

$$SD_2(\zeta, \omega) = \max_{0 \leq t < \infty} |y(t)| \quad (4.1)$$

While the ground response spectrum depends only on the ground motion, the floor response spectrum also depends on the primary structure. It is a convenient representation of the combined effect of earthquake and structure on a range of secondary systems.

If the ground motion is modeled as a stochastic process, then  $y(t)$  will also be stochastic. An extension may be made to the floor response spectrum in the same way as for the ground response spectrum. The mean floor response spectrum is simply

$$\langle SD_2(\zeta, \omega) \rangle = \langle \max_{0 \leq t < \infty} |y(t)| \rangle \quad (4.2)$$

Alternatively, the design floor spectrum may be defined for a confidence level  $p$  as that level  $FS(\zeta, \omega; p)$  for which

$$\text{Prob} \{SD_2(\zeta, \omega) \leq FS(\zeta, \omega; p)\} = p \quad (4.3)$$

Knowledge of the design floor spectrum for the point at which a secondary system is to be attached greatly simplifies the design problem. The floor spectrum contains the essential information about the primary structure which is needed for evaluation of the safety of proposed secondary systems. Most importantly, the design of the secondary system may proceed without further resort to dynamic analysis of the primary system.

A number of methods have been proposed for the approximate computation of floor spectra. In nearly every case, the design ground spectrum is assumed known. Biggs and Roeset [30] developed an approximate relationship between the ground spectrum and the floor spectrum, based on a semi-empirical amplification curve. The curve was defined for a single-degree-of-freedom primary system, then extended to multiple-degree-of-freedom structures by a root-sum-squares rule.

Singh [32,33] and Vanmarcke [35] took similar approaches to floor spectrum computation. The RMS acceleration of the secondary system in response to stationary ground motion was computed. (Singh obtained the stationary response, while Vanmarcke took an approximation to the transient response.) This value was expressed in terms of the RMS

acceleration of the ground and each mode of the primary system. Assuming the ground spectrum to be a constant multiple of the RMS acceleration of the appropriate oscillator allowed the floor spectrum to be expressed in terms of ground spectrum values.

In this chapter, some simple approximations for the computation of design floor spectra are discussed. In order to clarify the nature of the approximations involved, a single-degree-of-freedom primary system is considered.

#### 4.2 Simple Analytical Approximation for Floor Design Spectra

Consider a primary system which may be modeled as a single-degree-of-freedom oscillator with damping  $\zeta_1$  and natural frequency  $\omega_1$ . For the calculation of the floor spectrum for such a primary system, a secondary system with arbitrary damping  $\zeta_2$  and natural frequency  $\omega_2$  is attached as in Figure 3.1. If the primary structure has more than one degree of freedom, the present analysis will apply to each of the modes of the primary system.

The analytical difficulties which arise in regard to design floor spectra are similar to those associated with ground response spectra. Although the time-varying distribution of the secondary system response may be developed (as, for example, in Chapter 3), the determination of the maximum response over a period of time is a much more formidable problem.

It may first be recognized that the amplitude of response is an important value which scales the corresponding maximum response. The maximum response will occur most probably when the amplitude tends to be largest. With this motivation, define the dimensionless peak factor  $r(\zeta_1, \omega_1, \zeta_2, \omega_2)$  by

$$r(\zeta_1, \omega_1, \zeta_2, \omega_2) = \frac{\max_{0 \leq t < \infty} |y(t)|}{\max_{0 \leq t < \infty} \langle a_2(t) \rangle} \quad (4.4)$$

where  $y(t)$  is the relative displacement of the secondary system, and  $a_2(t)$  is its envelope, as defined in Chapter 3. The peak factor depends explicitly on the primary and secondary system parameters, and also depends implicitly on the nature of the ground motion. Note that  $r(\zeta_1, \omega_1, \zeta_2, \omega_2)$  is a stochastic function. It should be noted that other measures of the intensity of  $y(t)$  could be used in place of

$\max_{0 \leq t < \infty} \langle a_2(t) \rangle$ . For example, the maximum RMS displacement of  $y$ ,  $\max_{0 \leq t < \infty} \langle y^2(t) \rangle$ , could be used. In order to be generally applicable, the measure must be appropriate for transient ground motion. The maximum mean envelope is used here because expressions are available from Chapter 3 for this quantity.

A closely related function is the peak factor for the maximum primary response. Let

$$r_1(\zeta_1, \omega_1) = \frac{\max_{0 \leq t < \infty} |x(t)|}{\max_{0 \leq t < \infty} \langle a_1(t) \rangle} \quad (4.5)$$

where  $x(t)$  is the relative displacement of the primary system, and  $a_1(t)$  is its envelope.  $r_1(\zeta_1, \omega_1)$  depends implicitly on the nature of the

ground motion. Although  $\zeta_1$  and  $\omega_1$  are fixed parameters of the primary system being considered, the function  $r_1(\zeta, \omega)$  may be defined for all  $\zeta$  and  $\omega$  in terms of the response of corresponding primary systems.

Since  $\langle a_1(t) \rangle$  and  $\langle a_2(t) \rangle$  are known for general stochastic ground motion, the determination of the peak factors is the crucial analytical challenge in the development of ground response spectra and floor spectra. The ground spectrum is expressible in terms of  $r_1(\zeta, \omega)$ , and the floor spectrum is expressible in terms of  $r(\zeta_1, \omega_1, \zeta_2, \omega_2)$ .

If the ground motion is broad-band, the two peak factors  $r_1$  and  $r$  will be related. In Section 3.3.1, it was shown that if  $\omega_1$  and  $\omega_2$  differ greatly, the lower frequency dominates the secondary response. If  $\omega_2 \ll \omega_1$ , the primary system essentially transmits the ground motion unchanged in the frequency range of the secondary system. Thus, if  $\omega_2 \ll \omega_1$ , the secondary system responds as if it were on the ground. That is,

$$r(\zeta_1, \omega_1, \zeta_2, \omega_2) \approx r_1(\zeta_2, \omega_2) \quad \text{if } \omega_2 \ll \omega_1 \quad (4.6)$$

If  $\omega_2 \gg \omega_1$ , the secondary system responds quasi-statically to the primary system, so that

$$y(t) \approx - \frac{\omega_1^2}{\omega_2} x(t) \quad (4.7)$$

In this case, therefore,  $y$  differs from  $x$  only by a scale factor, which is removed in the definition of the peak factor. Thus,

$$r(\zeta_1, \omega_1, \zeta_2, \omega_2) \approx r_1(\zeta_1, \omega_1) \quad \text{if } \omega_2 \gg \omega_1 \quad (4.8)$$

Hereafter, all spectra will be assumed to be mean spectra, in order to simplify notation. A similar analysis may be carried out in the case of spectra defined by a confidence value  $p$ . Assume temporarily that the mean ground response spectrum is known for all  $\zeta$  and  $\omega$ , and that the corresponding stochastic model is known. Then  $\langle r_1(\zeta, \omega) \rangle$  may be computed as the ratio of the mean ground spectrum to the maximum mean primary envelope, as

$$\langle r_1(\zeta, \omega) \rangle = \langle SD(\zeta, \omega) \rangle / \max_{0 \leq t < \infty} \langle a_1(t; \zeta, \omega) \rangle \quad (4.9)$$

where  $a_1(t; \zeta, \omega)$  is the envelope of the oscillator with natural frequency  $\omega$ , and damping  $\zeta$ . Equation (4.4) allows the mean floor spectrum to be expressed as

$$\langle SD_2(\zeta_2, \omega_2) \rangle = \langle r(\zeta_1, \omega_1, \zeta_2, \omega_2) \rangle \cdot \max_{0 \leq t < \infty} \langle a_2(t) \rangle \quad (4.10)$$

Combining equations (4.9) and (4.10) with equations (4.6) and (4.8) gives the following limiting values for the mean floor spectrum:

$$\langle SD_2(\zeta_2, \omega_2) \rangle \approx \frac{\max_{0 \leq t < \infty} \langle a_2(t) \rangle}{\max_{0 \leq t < \infty} \langle a_1(t; \zeta_2, \omega_2) \rangle} \langle SD(\zeta_2, \omega_2) \rangle \quad \text{if } \omega_2 \ll \omega_1 \quad (4.11)$$

$$\langle SD_2(\zeta_2, \omega_2) \rangle \approx \frac{\max_{0 \leq t < \infty} \langle a_2(t) \rangle}{\max_{0 \leq t < \infty} \langle a_1(t; \zeta_1, \omega_1) \rangle} \langle SD(\zeta_1, \omega_1) \rangle \quad \text{if } \omega_2 \gg \omega_1 \quad (4.12)$$

Note that in both cases  $a_2(t)$  refers to the envelope of the secondary

system attached to the actual primary system. The quantity  $a_1(t; \zeta, \omega)$  refers to the envelope of a fictitious single-degree-of-freedom system attached to the ground.

Equations (4.11) and (4.12) may be combined as

$$\langle SD_2(\zeta_2, \omega_2) \rangle = \left[ \frac{\max_{0 \leq t < \infty} \langle a_2(t) \rangle}{\max_{0 \leq t < \infty} \langle a_1(t; \zeta_e, \omega_e) \rangle} \right] \langle SD(\zeta_e, \omega_e) \rangle \quad (4.13)$$

where  $\zeta_e$  and  $\omega_e$  are "equivalent" damping and frequency values exhibited by the secondary system values. These parameters take on the extreme values

$$(\zeta_e, \omega_e) = \begin{cases} (\zeta_2, \omega_2) & \text{if } \omega_2 \ll \omega_1 \\ (\zeta_1, \omega_1) & \text{if } \omega_2 \gg \omega_1 \end{cases} \quad (4.14)$$

For intermediate values of  $\omega_2$ , equation (4.13) may hold in an approximate sense, if  $\zeta_e$  and  $\omega_e$  are chosen appropriately. The problem of determining the floor spectrum then reduces to the choice of  $\zeta_e$  and  $\omega_e$ , identification of  $\langle SD(\zeta_e, \omega_e) \rangle$  from the ground spectrum, and computation of the amplification factor in equations (4.13) (the term in brackets) from the stochastic model.

The choice of  $\omega_e$  may be made on the basis of the transfer function of the primary/secondary system. Recall that

$$H_y(\omega) = \frac{1}{(\omega_2^2 - \omega^2) + 2i\zeta_2\omega_2\omega} \frac{2i\zeta_1\omega_1\omega + \omega_1^2}{(\omega_1^2 - \omega^2) + 2i\zeta_1\omega_2\omega} \quad (3.26)$$

If  $\omega_1$  and  $\omega_2$  are reasonably separated, there will be peaks at  $\omega_1$  and  $\omega_2$ .

Near  $\omega=\omega_1$ , if  $\zeta_1, \zeta_2 \ll 1$ ,

$$H_y(\omega) \approx \frac{1}{(\omega_2^2 - \omega_1^2)} \frac{\omega_1^2}{(\omega_1^2 - \omega^2) + 2i\zeta_1\omega_1\omega} \quad (4.15)$$

Near  $\omega=\omega_2$ ,

$$H_y(\omega) \approx \frac{1}{(\omega_2^2 - \omega^2) + 2i\zeta_2\omega_2\omega} \frac{\omega_1^2}{(\omega_1^2 - \omega_2^2)} \quad (4.16)$$

Thus, the area under the peak at  $\omega_1$  is approximately

$$A_1 = \int_{\omega_1 - \Delta\omega}^{\omega_1 + \Delta\omega} |H_y(\omega)|^2 d\omega \approx \frac{1}{(\omega_2^2 - \omega_1^2)^2} \frac{\pi\omega_1^4}{2\zeta_1\omega_1^3} \quad (4.17)$$

Similarly,

$$A_2 = \int_{\omega_2 - \Delta\omega}^{\omega_2 + \Delta\omega} |H_y(\omega)|^2 d\omega \approx \frac{\pi}{2\zeta_2\omega_2^3} \frac{\omega_1^4}{(\omega_1^2 - \omega_2^2)^2} \quad (4.18)$$

The ratio of the two areas is

$$\frac{A_2}{A_1} = \frac{\zeta_1\omega_1^3}{\zeta_2\omega_2^3} \quad (4.19)$$

It seems appropriate to choose  $\omega_e$  as  $\omega_1$  or  $\omega_2$ , depending on which part of  $H_y(\omega)$  is dominant. This leads to the result



$$\omega_e = \begin{cases} \omega_1 & \text{if } \omega_2 > \left(\frac{\zeta_1}{\zeta_2}\right)^{1/3} \omega_1 \\ \omega_2 & \text{if } \omega_2 \leq \left(\frac{\zeta_1}{\zeta_2}\right)^{1/3} \omega_1 \end{cases} \quad (4.20)$$

The choice of  $\zeta_e$  probably has less influence on the resulting prediction than  $\omega_e$ . A simple choice for  $\zeta_e$  is

$$\zeta_e = \begin{cases} \zeta_1 & \text{if } \omega_2 > \left(\frac{\zeta_1}{\zeta_2}\right)^{1/3} \omega_1 \\ \zeta_2 & \text{if } \omega_2 \leq \left(\frac{\zeta_1}{\zeta_2}\right)^{1/3} \omega_1 \end{cases} \quad (4.21)$$

Note that equations (4.20) and (4.21) have the correct limiting behavior given by equation (4.14).

#### 4.3 Comparison of Approximate Theory to Simulation Studies

A series of simulations was performed to test the ability of equation (4.13) to predict actual floor spectra. The ground motion was chosen to be white noise modulated by a boxcar envelope. There were a number of reasons for this selection. An extremely efficient algorithm for computing the response of a linear system to modulated white noise is described in the Appendix. The use of this algorithm made possible a greater number of simulations. The use of a boxcar envelope is convenient, since closed form expressions are available in Chapter 3 for the mean envelope in this case. Also, as explained in Section 3.3.4, the number of parameters to be varied in the study may be reduced to

four:  $\zeta_1$ ,  $\zeta_2$ ,  $\omega_2/\omega_1$ , and  $\omega_1 T$ , where  $T$  is the duration of shaking.

A preliminary step was the estimation of the ground spectrum to be used in equation (4.13). For each set of parameters considered, from 100 to 400 sample ground motions were generated. The maximum oscillator response to each motion was computed, and these maxima were averaged to estimate the corresponding ground spectrum value. In all cases, the sample standard deviation of the estimate was from 1% to 4% of the sample mean. A smooth spectrum was produced by interpolating logarithmically between the sample points. The resulting spectra are shown in Figure 4.1.

The algorithm described in the Appendix made computation of the ground spectra very efficient. On a PRIME 500 computer, it was possible to simulate approximately 75 cycles of oscillator response to enveloped white noise per second of real time. The time step was taken to be one-fifth of a period, and peak values were determined by cubic interpolation, using the displacement and derivative at each time step.

Three sets of parameters were considered for floor spectrum simulations: i)  $\zeta_1 = 0.05$ ,  $\zeta_2 = 0.05$ ,  $\omega_1 T = 20\pi$ ; ii)  $\zeta_1 = 0.05$ ,  $\zeta_2 = 0.01$ ,  $\omega_1 T = 40\pi$ ; iii)  $\zeta_1 = 0.01$ ,  $\zeta_2 = 0.01$ ,  $\omega_1 T = 80\pi$ . The average floor spectrum was found as a function of  $\omega_2/\omega_1$ . In each case, the theoretical value of  $\langle a_2(t) \rangle$  was obtained from equations (3.71), (3.73), (3.76), and (3.78), with  $S(\omega) = S_0$ . The value of  $\langle a_1(t; \zeta_e, \omega_e) \rangle$  was computed from equations (3.19) and (3.20). Equation (4.13) was then used to predict the floor spectra. Figures 4.2, 4.3 and 4.4 show a comparison of the simulated floor spectra with the prediction. The dashed curves

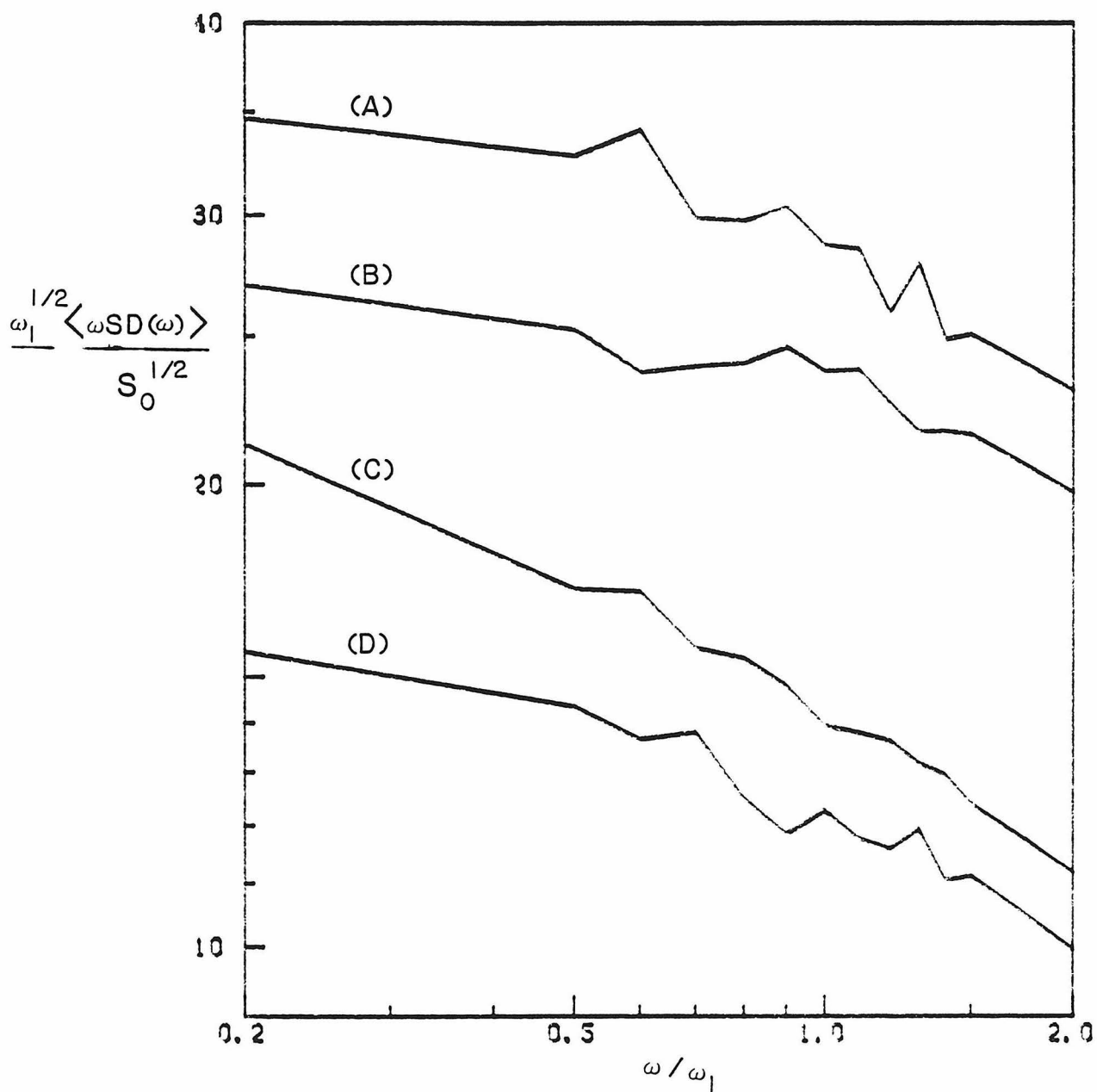


Fig. 4.1 Mean ground spectrum for boxcar-modulated white noise.  
(A)  $\zeta=0.01$ ,  $\omega_1 T=80\pi$ ;  
(B)  $\zeta=0.01$ ,  $\omega_1 T=40\pi$ ;  
(C)  $\zeta=0.05$ ,  $\omega_1 T=40\pi$ ;  
(D)  $\zeta=0.05$ ,  $\omega_1 T=20\pi$ .  
 $\omega_1$  is a reference value.

were obtained by putting  $\zeta_e = \zeta_1$  and  $\omega_e = \omega_1$  in equation (4.13). The solid curves result if  $\zeta_e = \zeta_2$  and  $\omega_e = \omega_2$ .

The simplified theory accurately predicts the floor spectra except near resonance, where it overestimates the response by 20% to 40%. The limiting behavior predicted by equation (4.14) is observed in the simulations. The choice of  $\omega_e$  in equation (4.20) is not well tested by these simulations. In the one case where  $\zeta_1 \neq \zeta_2$ , Figure 4.3 indicates that  $\omega_e = \omega_2$  more accurately predicts the floor spectrum for the entire range of  $\omega_2$  considered. Equation (4.20) would select  $\omega_e = \omega_1$  for  $\omega_2/\omega_1 > 1.7$  in this case.

The failure of the simple theory to predict the resonance peak of the floor spectrum may be attributed to the choice of  $\zeta_e$ . When  $\omega_1$  and  $\omega_2$  are well separated, the secondary system behaves essentially as a single-degree-of-freedom oscillator on the ground, with the appropriate frequency and damping. At resonance, however, the two resonance peaks at  $\omega_1$  and  $\omega_2$  in  $H_y(\omega)$  coalesce into one peak, which is then sharper than either separate peak. A reduction of  $\zeta_e$  near resonance would reflect the narrowing of the resonance peak.

In order to study the effect of  $\zeta_e$  on the height of the floor spectrum peak, values of the ground spectrum were obtained for several values of damping. These values were used in equation (4.13) for the floor spectra. The values of the predicted floor spectra at resonance are shown in Figure 4.5 as a function of  $\zeta_e$ , for the three sets of parameters. In general, it is seen that a reduction of  $\zeta_e$  results in a lower predicted floor spectrum value. This tendency reflects the fact

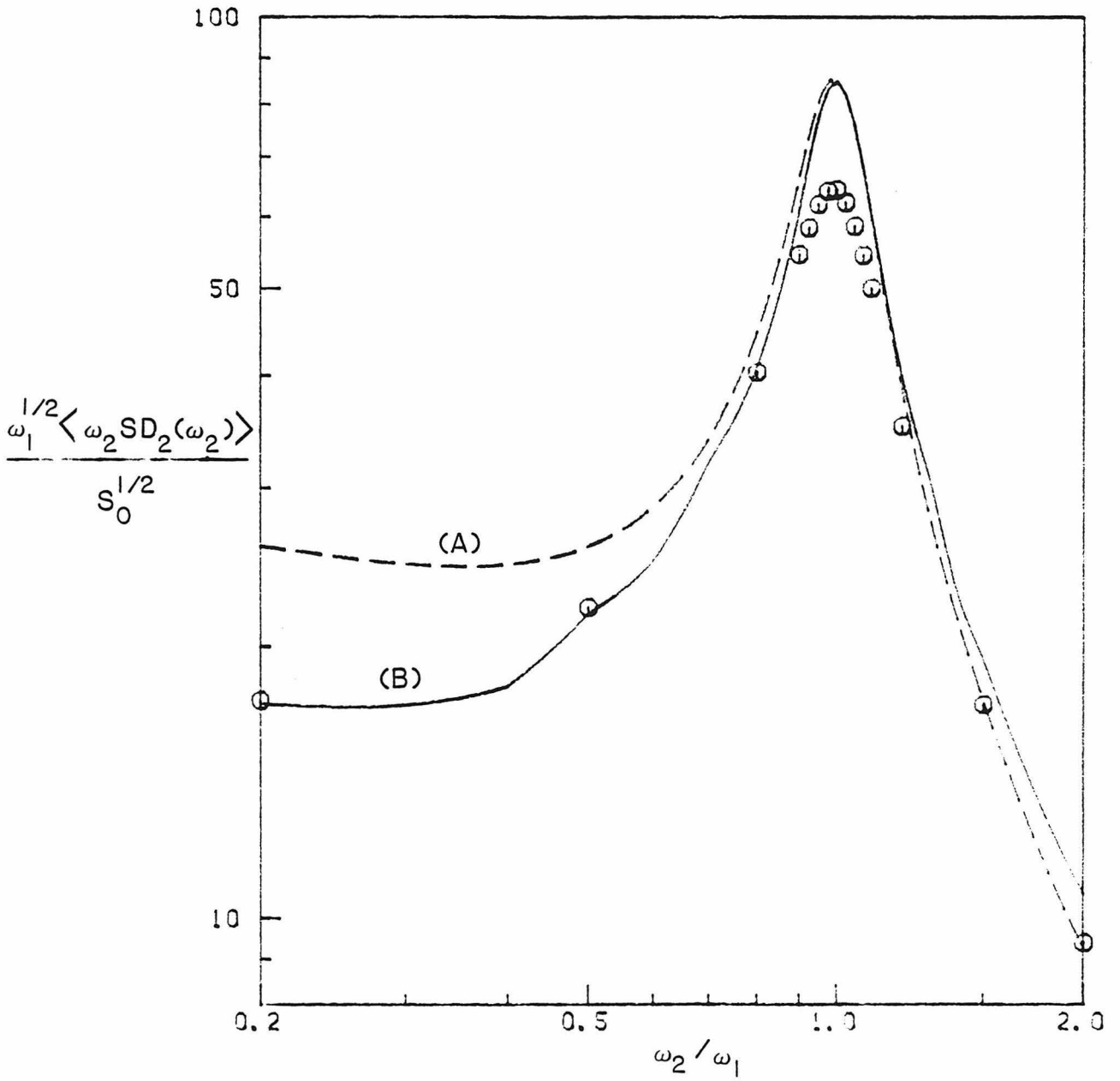


Fig. 4.2 Comparison of approximate and simulated mean floor spectrum (circles) for  $\zeta_1 = \zeta_2 = 0.05$ ,  $\omega_1 T = 20\pi$ .  
 (A)  $\zeta_e = 0.05$ ,  $\omega_e = \omega_1$ ; (B)  $\zeta_e = 0.05$ ,  $\omega_e = \omega_2$ .

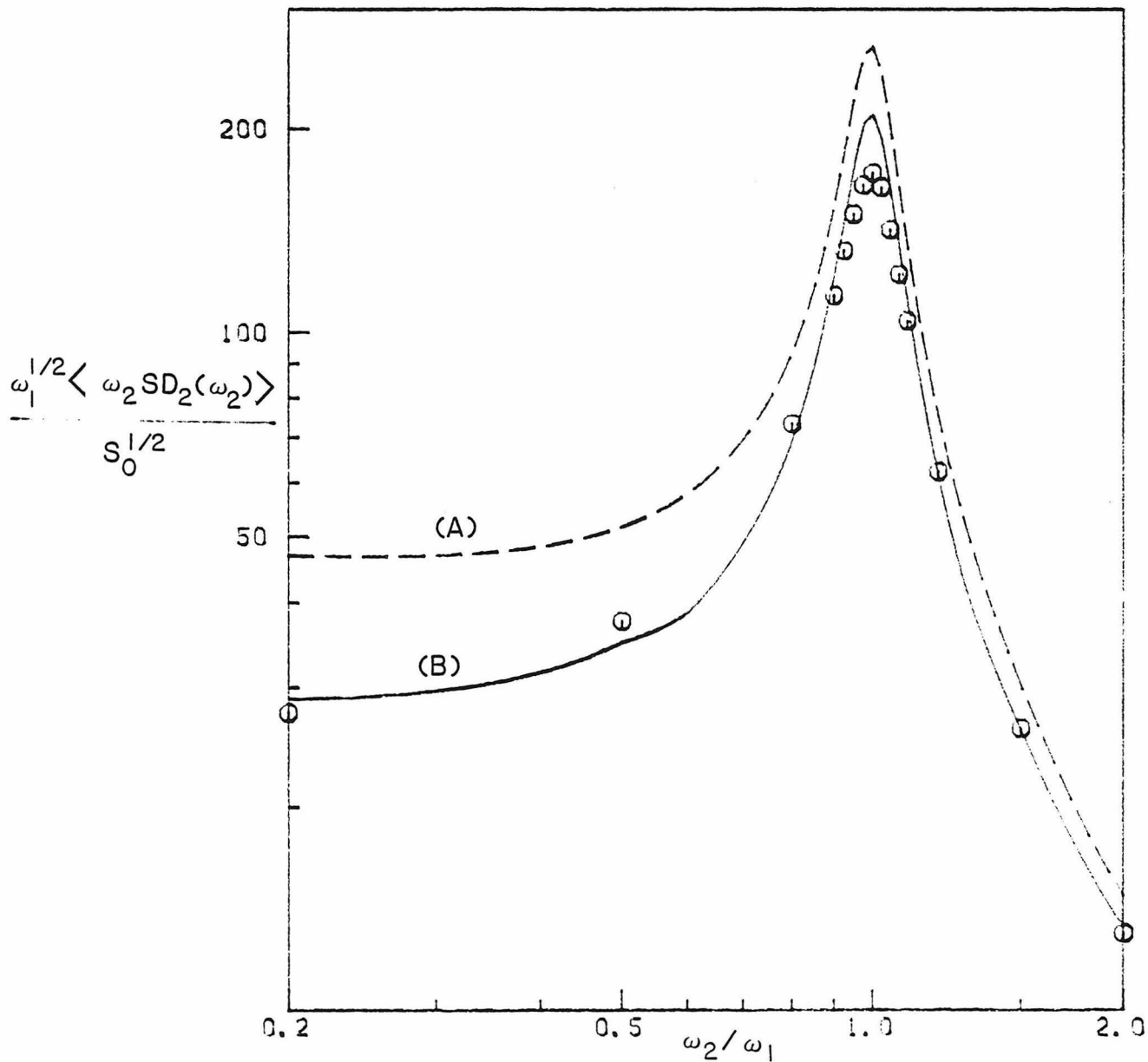


Fig. 4.3 Comparison of approximate and simulated mean floor spectrum (circles) for  $\zeta_1=0.05$ ,  $\zeta_2=0.01$ ,  $\omega_1 T=40\pi$ . (A)  $\zeta_e=0.05$ ,  $\omega_e=\omega_1$ ; (B)  $\zeta_e=0.01$ ,  $\omega_e=\omega_2$ .

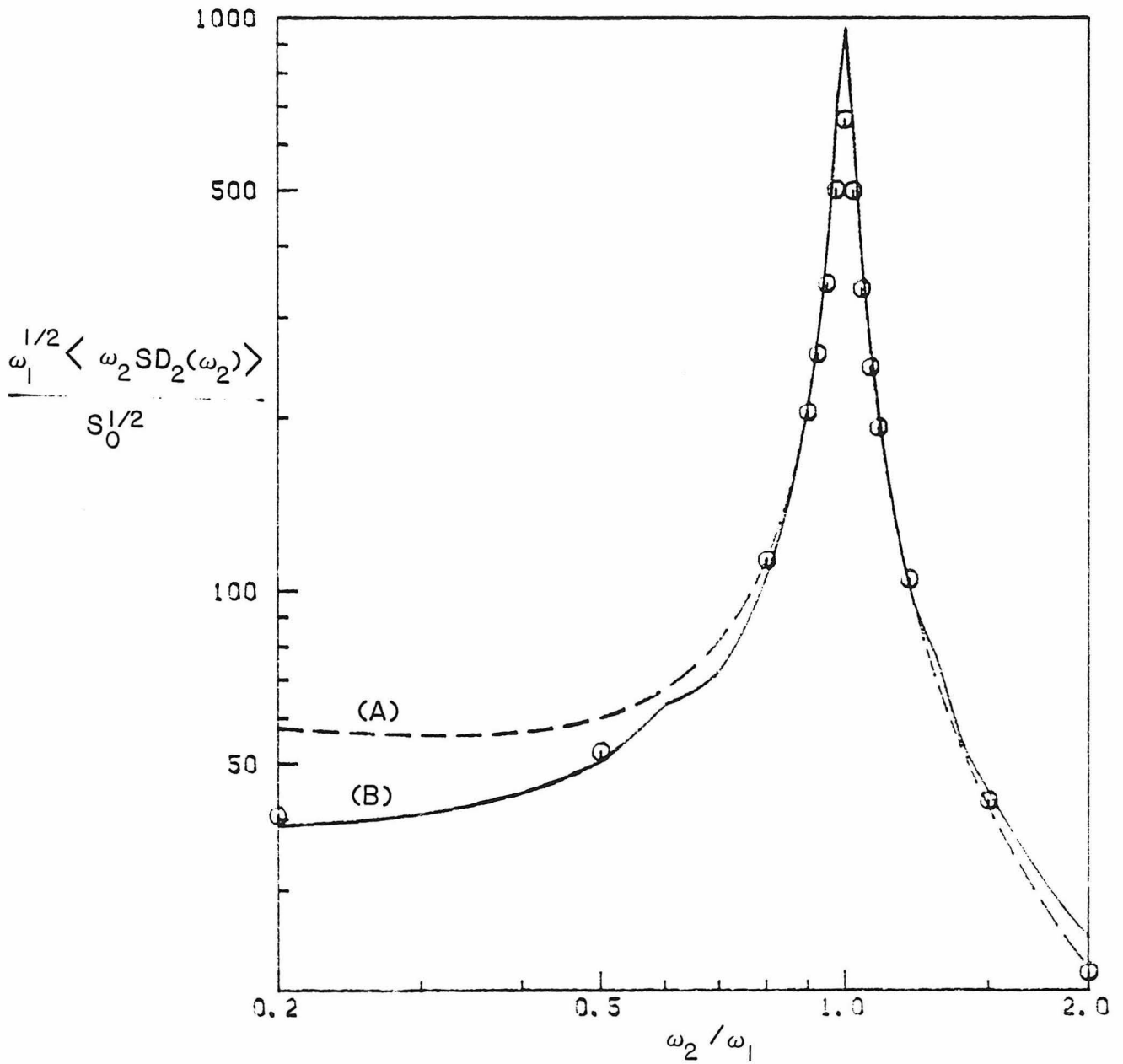


Fig. 4.4 Comparison of approximate and simulated mean floor spectrum (circles) for  $\zeta_1 = \zeta_2 = 0.01$ ,  $\omega_1 T = 80\pi$ . (A)  $\zeta_e = 0.01$ ,  $\omega_e = \omega_1$ ; (B)  $\zeta_e = 0.01$ ,  $\omega_e = \omega_2$ .

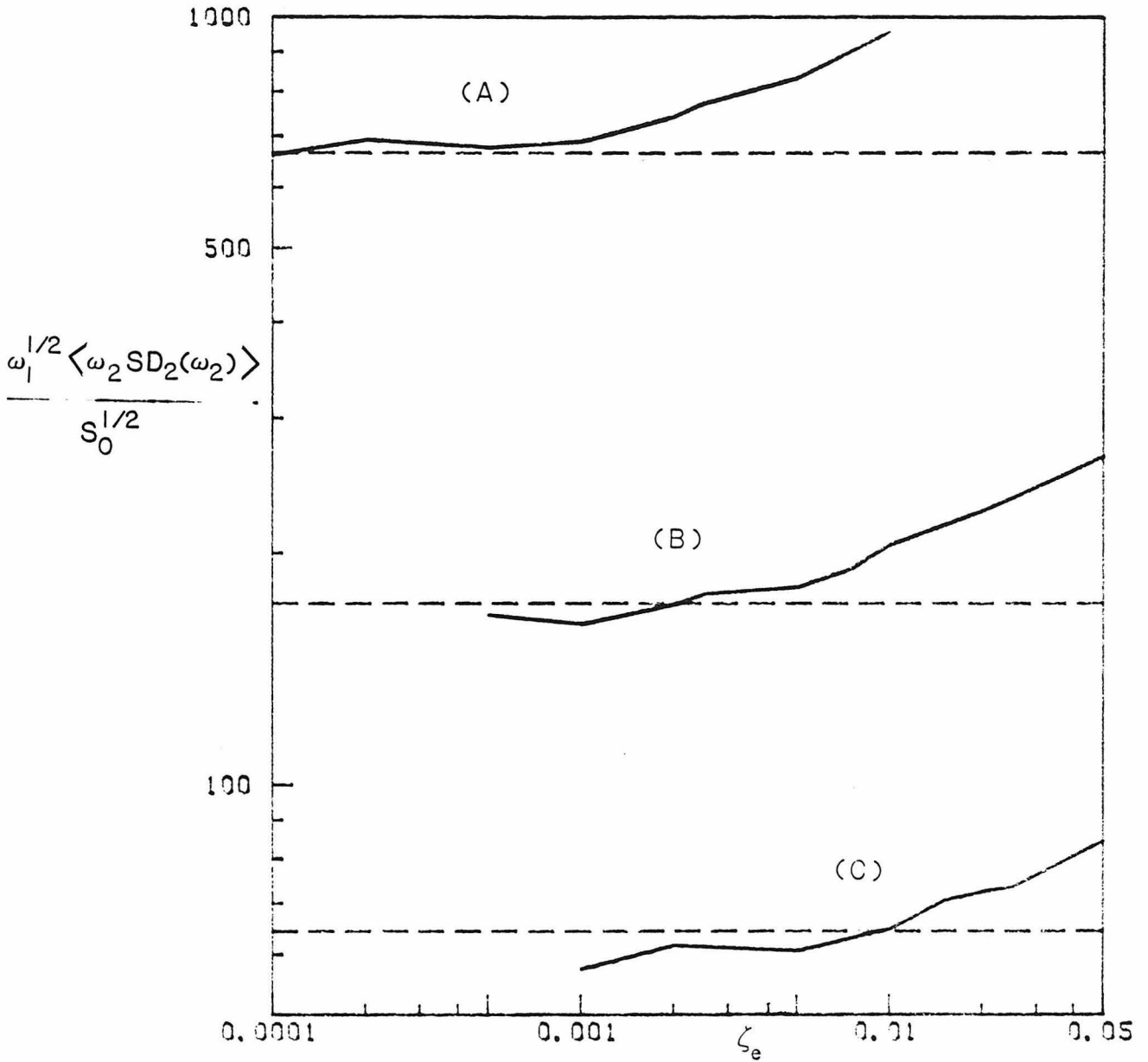


Fig. 4.5 Dependence of approximate floor spectrum on  $\zeta_e$  at resonance.  $\omega_e = \omega_1 = \omega_2$ . Dashed lines are simulated spectral values.  
 (A)  $\zeta_1 = \zeta_2 = 0.01$ ,  $\omega_1 T = 80\pi$ ; (B)  $\zeta_1 = 0.05$ ,  $\zeta_2 = 0.01$ ,  $\omega_1 T = 40\pi$ ;  
 (C)  $\zeta_1 = \zeta_2 = 0.05$ ,  $\omega_1 T = 20\pi$ .



that as damping decreases, the maximum response increases more slowly than the mean amplitude, at least for the small damping levels considered here. The empirical result

$$\zeta_e = 4\zeta_1\zeta_2 \quad (4.22)$$

gives good results at resonance for the three cases considered, but it is not necessarily justified theoretically.

#### 4.4 Further Approximations for Floor Spectrum Computation

The approximate theory developed in Section 4.2 depends on a knowledge of both the ground spectrum and the stochastic model of ground motion which leads to that spectrum. Normally, earthquakes are prescribed by the design ground spectrum, without reference to a stochastic model. In this case, a stochastic model may be inferred from the ground spectrum. The resulting model will not be unique. Mason [24] described an approximate procedure for determining a modulated stationary process which is consistent with a given spectrum. The modulating function may be specified arbitrarily, indicating the non-uniqueness of the inversion procedure.

The stochastic model is needed to compute the amplification factor in equation (4.13). It may be hoped that the amplification factor is not strongly dependent on the particular choice of model. In fact, the computationally difficult inversion of the ground spectrum is sometimes ignored, and a stochastic model is used which is not necessarily consistent with the given ground response spectrum.

If the stochastic model used for the computation of the amplification factor is not required to relate to the ground spectrum, it is possible to choose a model which simplifies the computation of the factor. For example, a long-duration stationary model may be chosen, and the amplification factor approximated as

$$\frac{\langle SD_2(\zeta_2, \omega_2) \rangle}{\langle SD(\zeta_e, \omega_e) \rangle} \approx \sqrt{\frac{\langle y^2 \rangle_{\text{stat}}}{\langle x^2 \rangle_{\text{stat}}}} \quad (4.23)$$

where  $\langle x^2 \rangle_{\text{stat}}$  is the stationary variance of the equivalent linear oscillator. Equation (4.23) becomes particularly simple if the ground motion is white noise. However, the cumulative effect of these simplifications may be considerable. An investigation of this approximation is beyond the scope of the present work.

The dependence of the amplification factor on the stochastic model may be seen in part by varying the duration of a boxcar-modulated white noise model. Figures 4.6, 4.7 and 4.8 show the dependence of the amplification factor on duration for the three sets of parameters considered in this study, at resonance. In all three cases, there is a significant dependence on duration until the primary/secondary system essentially reaches stationarity. Thus, if a shorter duration model is appropriate, amplification factors derived from a stationary model may introduce large errors into the calculation.

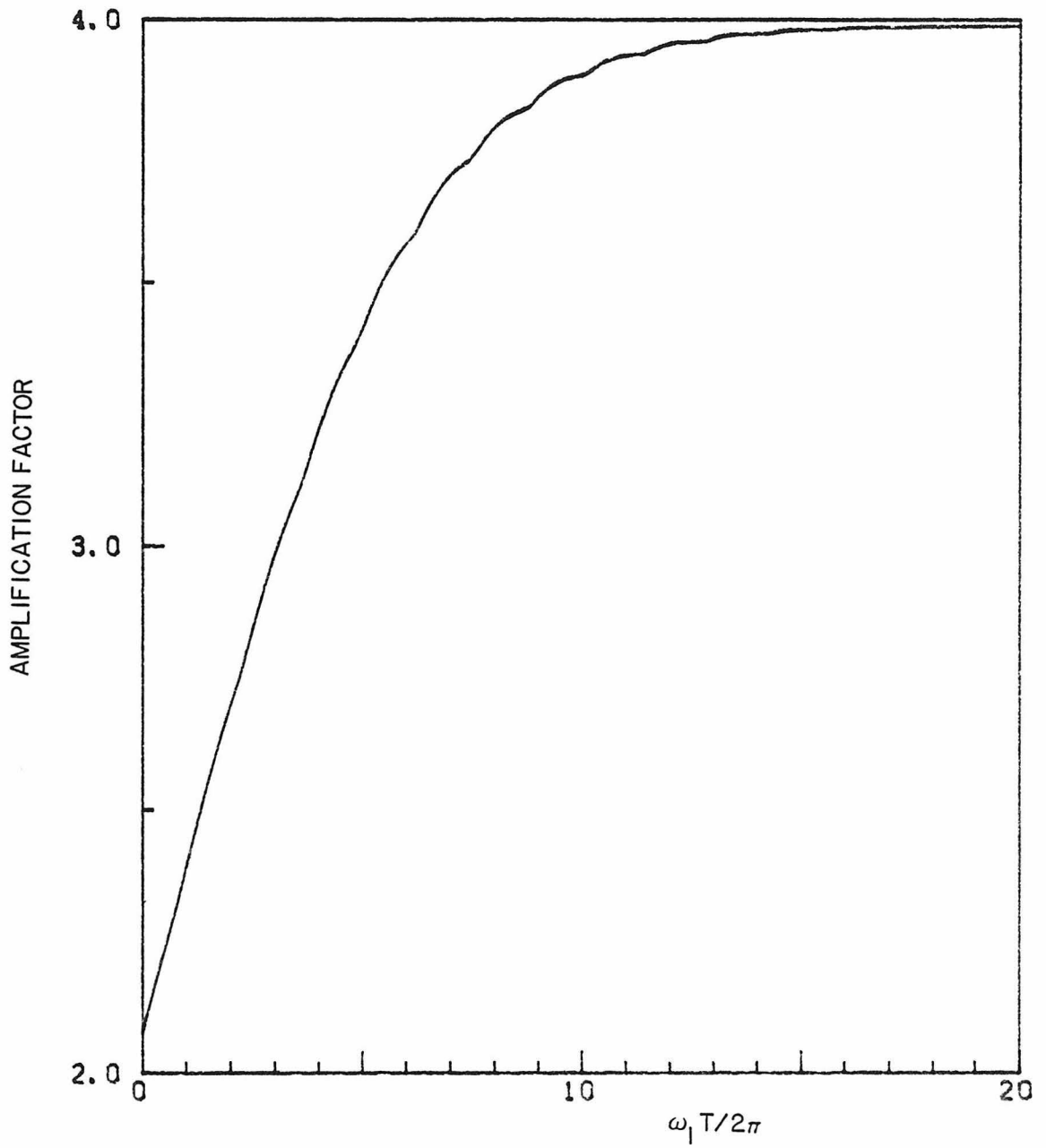


Fig. 4.6 Amplification factor vs. duration for  $\zeta_1=\zeta_2=0.05$ ,  $\omega_1=\omega_2$ ,  $\zeta_e=0.05$ .

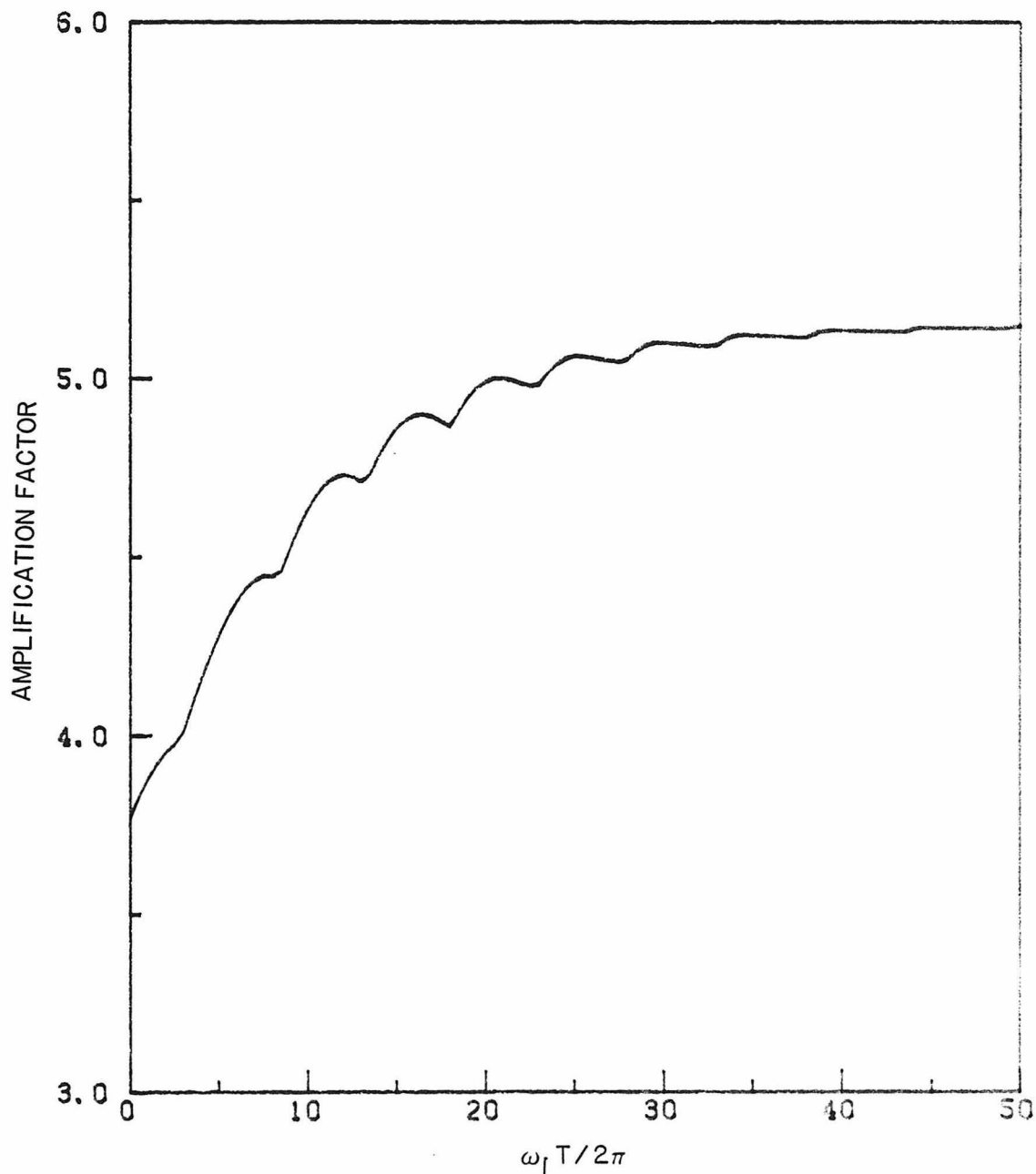


Fig. 4.7 Amplification factor vs. duration for  $\zeta_1=0.05$ ,  $\zeta_2=0.01$ ,  $\omega_1=\omega_2$ ,  $\zeta_e=0.01$ .

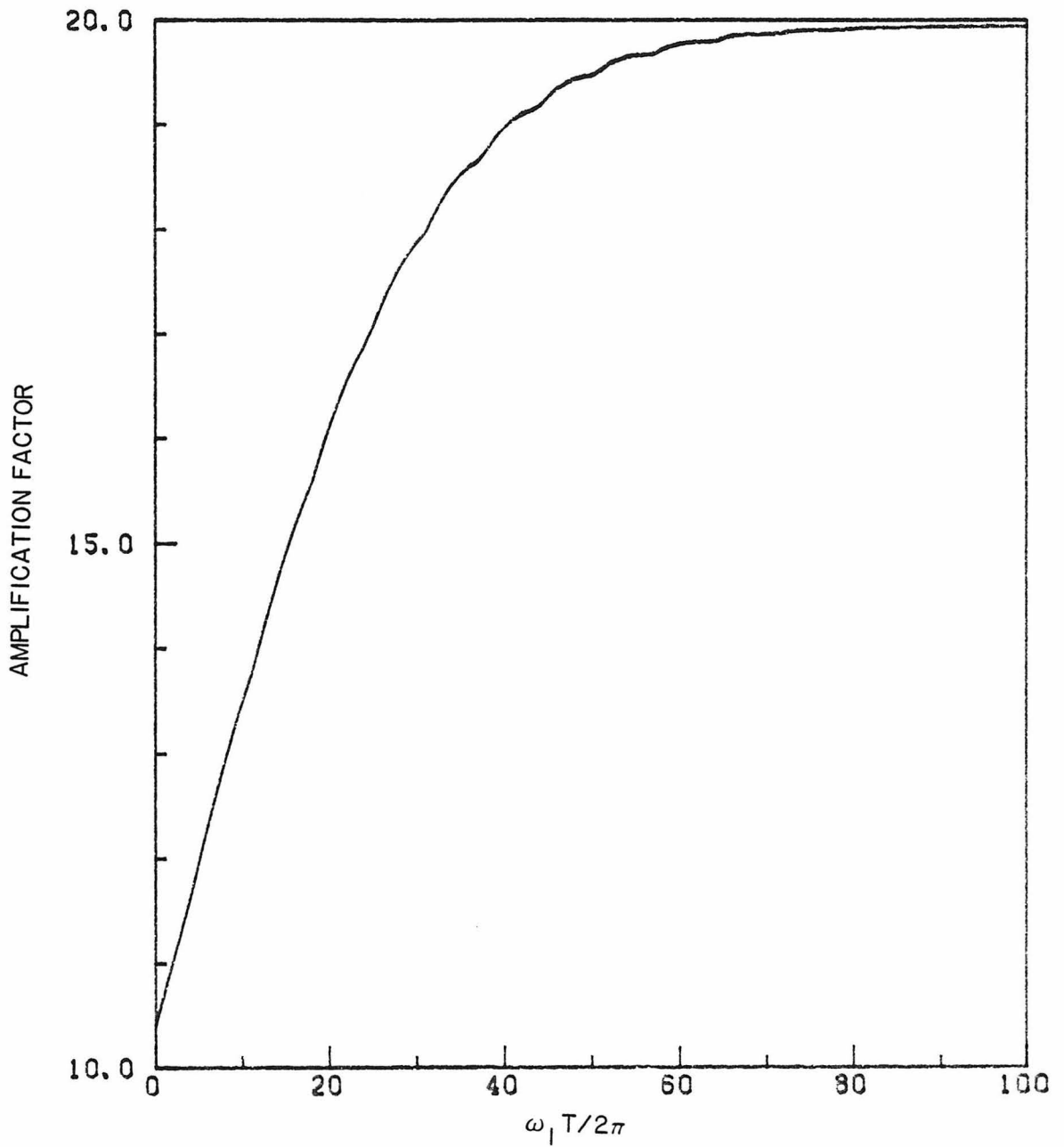


Fig. 4.8 Amplification factor vs. duration for  $\zeta_1=\zeta_2=0.01$ ,  $\omega_1=\omega_2$ ,  $\zeta_e=0.01$ .

#### 4.5 Summary and Discussion

In this chapter, the results of Chapter 3 have been applied in a simple way to the calculation of floor spectra. The use of the mean envelope is not necessarily superior to the use of the time-varying RMS value of  $y$ . However, this chapter presents only one potential application of the envelope statistics derived in Chapter 3.

Simulation studies have demonstrated that a simple formula is adequate for predicting floor spectra, except near resonance, where the predicted spectra are overly conservative by up to 40%. This conservatism may be reduced by more realistic selection of equivalent damping near resonance.

It is possible that simpler formulas may be adequate for the prediction of floor spectra in some special cases. An examination of these simplifications was deemed beyond the scope of the present investigation. The formula developed in this chapter is applicable to very general stochastic models of ground motion.

## CHAPTER V

### SUMMARY AND CONCLUSIONS

In Chapter II, stochastic models of earthquake ground motion were discussed. Although an earthquake is, in principle, a deterministic process, a level of uncertainty must be accepted when predicting strong ground motion. This uncertainty is incorporated in the randomness of the stochastic model.

While an ideal stochastic model may be based on an ensemble of potential earth conditions, this is not practical. However, it is possible to define reasonably simple models of ground motion which may be expected to contain the essential features of uncertain earthquakes. In Section 2.2, the modulated stationary process and the filtered, modulated white noise process were suggested as potential models. In Section 2.3, methods of analysis were described for transient stochastic earthquake models. Of particular importance is the evolutionary power spectral density, which both defines and characterizes a nonstationary stochastic process.

In Section 2.4, the problem of the calibration of a stochastic model to actual data was addressed. It was seen that modulated stationary models are well suited to calibration by response spectrum. Filtered, modulated white noise models are especially convenient when it is desired to match a given Fourier amplitude spectrum.

Some aspects of numerical simulation of ground motion were considered in Section 2.5. The spectral representation of a nonstationary

process was used to develop numerical methods for simulation of a general model. Modulated stationary processes and filtered, modulated white noise processes may be simulated more efficiently.

Chapter III was concerned with the response of a secondary system to ground motion of the type described in Chapter II. Attention was focused on the envelope of the displacement of the secondary system. In Section 3.2, the envelope of the primary system was derived. A similar method was used in Section 3.3 to obtain the time-varying distribution of the secondary envelope. It was found that for both the primary and secondary systems, the mean square envelope, as a function of time, is given by the convolution of a known function with the evolutionary power spectral density of the ground acceleration. The convolving function for the secondary system is given in Section 3.3.2.

The special case of boxcar-modulated stationary ground motion was considered in Section 3.3.3. The distribution of the secondary envelope was obtained in closed form. A number of simulations were performed in Section 3.3.4. These simulations indicate that the results give a conservative measure of the system response, but not overly so. The envelope tends to be most conservative when the system has a strongly bimodal behavior.

In Section 3.4, the question of the effect of interaction was addressed. A particularly simple method of accounting for interaction effects was described for a system which is classically damped. In this case, it is possible to define an equivalent non-interacting system which may be analyzed by the previous methods. It was shown that the



dominant effect of interaction is an effective detuning of the system, which has special significance if the system was tuned to begin with. For a detuned system, the effect of interaction was seen to be small.

One application of the non-interacting results from Chapter III was considered in Chapter IV. There, the problem of the computation of floor spectra was considered. In Section 4.2, it was postulated that the effect of the structure on the spectrum was mainly contained in the amplification of the envelope of response. An approximation was developed which was highly accurate except at resonance. This method requires a knowledge of both the ground spectrum and the stochastic model for the ground motion. A suggestion was made for improvement of the approximation near resonance. It was assumed that the original method failed at resonance because it ignored the narrowing of the resonance peak of the secondary system transfer function.

Even simpler methods were discussed in Section 4.4. These methods involve a number of untested assumptions. An evaluation of the validity of these assumptions could be the focus of another investigation.

It is concluded that stochastic modeling of earthquakes leads to analytically tractable solutions for the response of secondary systems. It is therefore feasible to undertake a dynamic analysis of the seismic loads on a secondary system using a realistic earthquake model. Before the results of this thesis can be used practically, they would have to be extended to multi-degree-of-freedom primary structures. If the frequencies of the structure are separated, this could be carried out by

modal analysis [31]. The emphasis of the present work was on developing a clearer understanding of the simple system.

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APPENDIX

EFFICIENT NUMERICAL SIMULATION OF A VECTOR MARKOV PROCESS

Consider the  $n$ -dimensional vector process  $\tilde{X}(t)$  defined by

$$\dot{\tilde{X}}(t) = \tilde{A}\tilde{X}(t) + \tilde{r}\theta(t)w(t) \quad (\text{A.1})$$

$$\tilde{X}(0) = \tilde{\xi} \quad (\text{A.2})$$

where  $A$  is a constant  $n \times n$  matrix,  $\tilde{r}$  and  $\tilde{\xi}$  are constant  $n$ -vectors,  $\theta(t)$  is a deterministic function, and  $w(t)$  is a stationary Gaussian white noise process with unit power spectral density.  $\tilde{X}(t)$  represents an  $n^{\text{th}}$  order Markov process.

If  $\tilde{X}(t)$  is known for some  $t$ , then the solution at any later time may be written

$$\tilde{X}(t+\Delta t) = Q(\Delta t)\tilde{X}(t) + \int_0^{\Delta t} Q(\Delta t-\tau)\tilde{r}\theta(t+\tau)w(t+\tau)d\tau \quad (\text{A.3})$$

where  $Q(t)$  is the fundamental matrix solution.  $Q(t)$  satisfies the matrix equation

$$\dot{Q}(t) = AQ(t) \quad (\text{A.4})$$

$$Q(0) = I \quad (\text{A.5})$$

In a numerical simulation, the process  $\tilde{X}(t)$  is evaluated at discrete time intervals. Hence, let

$$\tilde{X}_{\tilde{k}} = \tilde{X}(k\Delta t) \quad (k=0,1,2,\dots) \quad (\text{A.6})$$

From equation (A.3), it may be seen that

$$\tilde{X}_{\tilde{k}+1} = Q(\Delta t)\tilde{X}_{\tilde{k}} + \tilde{Y}_{\tilde{k}} \quad (\text{A.7})$$

where

$$\tilde{Y}_{\tilde{k}} = \int_0^{\Delta t} Q(\Delta t - \tau) \tilde{r} \theta(k\Delta t + \tau) w(k\Delta t + \tau) d\tau \quad (\text{A.8})$$

The random vectors  $\tilde{Y}_{\tilde{k}}$  are normally distributed. From equation (A.8), it may be seen that

$$\langle \tilde{Y}_{\tilde{k}} \rangle = 0 \quad (\text{A.9})$$

$$\begin{aligned} \langle \tilde{Y}_{\tilde{j}} \tilde{Y}_{\tilde{k}}^T \rangle &= \int_0^{\Delta t} \int_0^{\Delta t} Q(\Delta t - \tau_1) \tilde{r} \tilde{r}^T Q^T(\Delta t - \tau_2) Q(j\Delta t + \tau_1) \theta(k\Delta t + \tau_2) \\ &\quad \cdot \langle w(j\Delta t + \tau_1) w(k\Delta t + \tau_2) \rangle d\tau_1 d\tau_2 \end{aligned} \quad (\text{A.10})$$

$$= \begin{cases} 0 & \text{if } j \neq k \\ 2\pi \int_0^{\Delta t} Q(\Delta t - \tau) \tilde{r} \tilde{r}^T Q^T(\Delta t - \tau) \theta^2(k\Delta t + \tau) d\tau & \text{if } j = k \end{cases} \quad (\text{A.11})$$

Thus, the  $Y_{\tilde{k}}$  are seen to be independent, normally distributed random vectors with zero mean, and covariance matrices

$$\langle Y_{\tilde{k}} Y_{\tilde{k}}^T \rangle = 2\pi \int_0^{\Delta t} Q(\Delta t - \tau) \underset{\sim}{r} r^T Q(\Delta t - \tau) \theta^2(k\Delta t + \tau) d\tau \quad (A.12)$$

Suppose  $\theta(t)$  may be approximated by a piecewise constant function of the form

$$\theta(t) \approx \hat{\theta}(t) = \theta([t/\Delta t] \cdot \Delta t) \quad (A.13)$$

where  $[x]$  denotes the largest integer  $\leq x$ . In order for  $\hat{\theta}(t)$  to be a reasonable approximation to  $\theta(t)$ ,  $\Delta t$  must be chosen sufficiently small. Then, equation (A.12) may be simplified as

$$\langle Y_{\tilde{k}} Y_{\tilde{k}}^T \rangle = 2\pi \theta^2(k\Delta t) \int_0^{\Delta t} Q(\tau) \underset{\sim}{r} r^T Q(\tau) d\tau \quad (A.14)$$

The problem of generating a sequence  $X_{\tilde{k}}$  which is consistent with  $X(t)$  is now reduced to the problem of generating independent normally distributed random vectors  $Y_{\tilde{k}}$  which have the properties of equations (A.9) and (A.14). Then,  $X_{\tilde{k}}$  may be generated through the recursive relationship of equation (A.7), with the initial condition

$$X_{\tilde{0}} = \xi_{\tilde{~}} \quad (A.15)$$

A convenient method for generating the random vectors  $\tilde{Y}_k$  is to find a matrix  $C$  for which

$$CC^T = 2\pi \int_0^{\Delta t} Q(\tau) \tilde{r} \tilde{r}^T Q^T(\tau) d\tau \quad (\text{A.16})$$

Then, if  $\tilde{Z}$  is a vector of  $n$  independent, normally distributed random variables with zero mean and unit variance,

$$\tilde{Y}_k = \theta(k\Delta t) C \tilde{Z} \quad (\text{A.17})$$

will have the desired properties. This is because

$$\langle \tilde{Z} \rangle = 0 \quad (\text{A.18})$$

and

$$\langle \tilde{Z} \tilde{Z}^T \rangle = I \quad (\text{A.19})$$

The procedure for generating  $\tilde{X}_k$  may be summarized as follows:

$$1. \quad \tilde{X}_0 = \tilde{\xi} \quad (\text{A.15})$$

$$2. \quad \tilde{X}_{k+1} = B \tilde{X}_k + \theta(k\Delta t) C \tilde{Z}_k \quad (\text{A.20})$$

for  $k=0,1,2,\dots$ , where

$$B = Q(\Delta t) \tag{A.21}$$

$$CC^T = 2\pi \int_0^{\Delta t} Q(\tau) \underbrace{r r^T}_{\sim} Q^T(\tau) d\tau \tag{A.16}$$

Thus, if B and C are known, the computation required at each time step is:

1. Generation of n new unit normal variables;
2. Two n×n matrix-vector multiplications;
3. Addition of two vectors.

Furthermore, the method provides statistically correct sample functions  $\tilde{X}_k$  regardless of the size of the time step.

The computation of the matrices B and C need be done once only for a given  $(A, \tilde{r})$  combination. Any numerical or analytical scheme may be used to solve equations (A.4) and (A.5) for  $Q(\Delta t)$ . The integral on the right-hand side of (A.16) may be evaluated numerically as the solution  $Q(t)$  is developed for  $0 \leq t \leq \Delta t$ . Since this integral is seen to be a symmetric and positive definite matrix, it may be factored by Gaussian elimination (without need for row exchanges) into the form  $LDL^T$ , where L is lower triangular, and D is diagonal. Then, C is taken to be the lower triangular matrix

$$C = LD^{1/2} \tag{A.22}$$

where  $D^{1/2}$  is easily generated. The initial computation of B and C may be time-consuming, but in comparison with the time required for the genera-

tion of a large number of samples of  $\tilde{X}(t)$ , the extra time is negligible.

The advantage of the present technique over traditional numerical simulation methods lies primarily in the ability to lengthen the time step  $\Delta t$ . Currently, a standard method for the generation of white noise is to let  $w(t)$  be approximated by a piecewise constant function. For  $k\Delta t \leq t < (k+1)\Delta t$ ,  $w(t)$  is taken to be  $W_k$ , a zero-mean normally distributed random variable with variance  $w\pi/\Delta t$ . The power spectral density of such an approximation is

$$S(\omega) = \frac{2}{(\omega\Delta t)^2} (1 - \cos \omega\Delta t) \quad (\text{A.23})$$

It may be verified that

$$0.95 \leq S(\omega) \leq 1 \quad \text{for} \quad 0 \leq \omega\Delta t \leq 0.78 \quad (\text{A.24})$$

$$0.9 \leq S(\omega) \leq 0.95 \quad \text{for} \quad 0.78 \leq \omega\Delta t \leq 1.12 \quad (\text{A.25})$$

Thus, to simulate the response of a narrow-band oscillator of frequency  $\omega_0$ ,  $\Delta t$  is often taken to be  $T/50$  or even  $T/100$ , where  $T = 2\pi/\omega_0$ , the natural period of the oscillator. This small time step is necessary to ensure that the excitation is sufficiently "white" over the necessary range of frequencies. At each time step,  $w(t)$  is generated, and  $\tilde{X}_{k+1}$  is computed by a numerical scheme based on equation (A.1).

In the present scheme,  $w(t)$  is never computed. The time step  $\Delta t$  is constrained only by the following considerations:

1.  $\Delta t$  must be small enough to resolve  $\theta(t)$ .
2.  $\Delta t$  must be small enough to monitor the features of  $X(t)$  which are being observed.

For example, if the peaks of a narrow-band oscillator are being observed, it is possible to choose  $\Delta t$  so that only five data points per period are computed. (The interpolation of the peak value must be carried out with care with this broad time step.) This results in a speed-up of an order of magnitude over standard simulation techniques, while maintaining exactly the white noise nature of the excitation.