CHIRAL ANOMALIES AND THE CHIRAL LAGRANGIAN

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Hidenori Sonoda

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Dedicated to My Parents,

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ABSTRACT

The subject of this thesis is some implications of chiral anomalies for chiral Lagrangians. The thesis consists of three parts:

In the first part, a somewhat heuristic discussion of the topological meaning of anomalies is given in the framework recently introduced by Alvarez. Its application to the sigma model anomalies is also given.

In the second part, the incorporation of chiral anomalies into the chiral Lagrangian is discussed in a simple manner. The Wess-Zumino term and the sigma model anomalies for the effective theory are explained.

Finally in the third part, as an implication of chiral anomalies, the chiral soliton model is described. Its relation to QCD in large N is discussed in detail. Quantization of the soliton is done in the path integral formalism.

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Chapter 1

Introduction

The study of chiral anomalies has been very fruitful in the past decade in helping our understanding of quantum field theory. The purpose of this thesis is to give a more or less self-contained account of chiral anomalies (mainly gauge anomalies) including some of the original work done by the author (and collaborators).

The history of chiral anomalies itself is somewhat anomalous (see p. 167 of [1] for this part). They were first encountered by Steinberger, who computed the amplitude of $\pi^0 \to 2\gamma$ in the pseudo-scalar coupling model and the pseudo-vector coupling model [2]. He noticed that only the first model gives a nonvanishing answer (which agrees with experiment). A puzzle arises, since one gets one theory from another by using the equation of motion. (According to Jackiw [1], this puzzle made Steinberger quit theoretical physics!)

The resolution of the puzzle was given a little later by Schwinger, who realized the subtleness of the calculations and introduced a careful regularization to get the same answer from the two models [3]. (This is explained in the appendix to chapter 2 from a modern point of view.)

Schwinger's work had been forgotten for a long time until Adler [4] and Bell and Jackiw [5] rediscovered the puzzle of Steinberger. This time chiral anomalies entered the mainstream of theoretical physics. The general form of the anomalies was soon calculated by Bardeen [6]. Chiral anomalies in the chiral Lagrangian were discussed by Wess and Zumino, who then gave a consistency condition for the gauged chiral Lagrangian [7].

One of the most interesting applications of chiral anomalies has been made by 't Hooft [8]. He discussed that the anomaly is a low energy phenomenon and that both fundamental and phenomenological theories should have the same anomaly. This condition has been found useful to determine the patterns of chiral symmetry breaking for some fermionic theories [9].

In the past two or three years, we have seen many progresses in two aspects of chiral anomalies: the structure of anomalies and the incorporation of anomalies into the chiral Lagrangian. These are the subjects of this thesis. The organization of the thesis is given below. In chapter 2, an overview of chiral anomalies is given, followed by a pedagogical account of the topological meaning of the gauge anomalies. Nonlinear sigma model anomalies are also explained. In chapter 3, a systematic way to introduce chiral anomalies into the chiral Lagrangian and the general effective theory is explained. In chapter 4, we discuss the chiral soliton model as an example of the implications of chiral anomalies. The path integral quantization of the soliton is shown to give a quantization of the Wess-Zumino term. The relation of the model to QCD in large N is discussed in detail. Some of the predictions from the chiral soliton model are described in the last section. Finally we conclude the thesis in chapter 5.

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Chapter 2

Chiral Anomalies

2-1. Overview of Chiral Anomalies

In quantum field theory, "anomaly" means a violation of a classical symmetry due to quantum effects. Chiral symmetry is a symmetry of chiral fermions, left and right handed. Anomalies in chiral symmetry are called chiral anomalies [1].

As a concrete example, let us consider quantum chromodynamics (QCD, SU(3) gauge theory) with three massless flavors, u, d and s. The Lagrangian is given by

$$L_{QCD} = -\frac{1}{4g^2} F_{\mu\nu}^2 + i\,\bar{\psi}\gamma^\mu D_\mu \psi \quad , \tag{2.1}$$

where $\psi^a = (u^a , d^a , s^a)^T$, $(a = 1,2,3)$. g is the QCD coupling constant, $F_{\mu\nu}$ is the field strength of the gluon field and D_μ is the covariant derivative. The Lagrangian (2.1) is invariant under the transformation

$$\psi_L \to L\,\psi_L \quad , \quad \psi_R \to R\,\psi_R \quad , \tag{2.2}$$

where $L \in U(3)_L$, $R \in U(3)_R$ are global and act on flavors. There are three types of chiral anomalies, all of which appear in QCD:

(i) $U(1)_A$ anomaly

(ii) perturbative gauge anomaly

(iii) SU(2) anomaly

The invariance of the Lagrangian (2.1) under the transformation (2.2) implies that the currents defined by

$$J_L^{\mu,a} = \bar{\psi}_L \gamma^\mu T^a \psi_L \ , \ J_R^{\mu,a} = \bar{\psi}_R \gamma^\mu T^a \psi_R \quad , \tag{2.3}$$

where T^a are generators of U(3), are conserved:

$$\partial_\mu J_L^{\mu,a} = 0 \ , \ \partial_\mu J_R^{\mu,a} = 0 \quad . \tag{2.4}$$

In fact due to one-loop quantum effects, the subgroup $U(1)_A$ is broken:

$$\partial_\mu J_A^\mu = \frac{1}{16\pi^2} \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \quad , \tag{2.5}$$

where

$$J_A^\mu = \bar{\psi}_R \gamma^\mu \psi_R - \bar{\psi}_L \gamma^\mu \psi_L \quad . \tag{2.6}$$

This is called the $U(1)_A$ anomaly [1]. One of the interesting applications of this phenomenon is the resolution of the U(1) problem [2]. The actual chiral symmetry of QCD is $G = SU(3)_L \times SU(3)_R \times U(1)_V$ and there appear eight Goldstone bosons upon the spontaneous breaking of G to $H = SU(3)_V \times U(1)_V$ [3].

Let us introduce the external gauge fields A_L , A_R(antihermitian):

$$L = L_{QCD} + i\, \bar{\psi}_L \gamma^\mu A_{L\mu} \psi_L + i\, \bar{\psi}_R \gamma^\mu A_{R\mu} \psi_R \quad . \tag{2.7}$$

Let the corresponding effective action be $\Gamma[A_L , A_R]$. In order to define a gauge theory of G, it is necessary to have a gauge invariance of Γ.

The perturbative gauge anomaly is a noninvariance of Γ under an infinitesimal gauge transformation. The infinitesimal change of Γ (for arbitrary G) has been explicitly evaluated by Bardeen [4]:

$$\Gamma[A_L^{1+v_L} , A_R^{1+v_R}] - \Gamma[A_L , A_R]$$

$$= \frac{-i}{24\pi^2} \int \mathrm{tr} \left[dv_L(A_L dA_L + \frac{1}{2} A_L^3) - dv_R(A_R dA_R + \frac{1}{2} A_R^3)\right] , \quad (2.8)$$

where $g_L = 1 + v_L$, $g_R = 1 + v_R$ are infinitesimal gauge transformations, and

$$A^g \equiv g^{-1}(A + d)g \quad . \quad (2.9)$$

Differential forms are used for notational convenience [5]. The anomaly only comes from one-loop diagrams. (2.8) is unremovable by a local counter term (i.e. a polynomial of A_L , A_R) if and only if

$$\mathrm{tr}\, T^a\{T^b , T^c\} \neq 0 \quad , \quad (2.10)$$

where T^a are generators of G. This leaves only U(1) and SU(n>2) potentially anomalous [6]. In the case of QCD the perturbative gauge anomaly forbids us to gauge the entire G but it is still possible to gauge any anomaly free subgroup of G, for example, $U(3)_V$.

What about $SU(2)_L$? There is no perturbative anomaly. However, this does not guarantee the gauge invariance of the effective action Γ.

This is because there is a class of SU(2) gauge transformations which cannot be reached continuously from the identity, as can been seen from

$$\pi_4(SU(2)) = Z_2 \quad . \tag{2.11}$$

In section 4, it will be shown that under these nontrivial gauge transformations, $e^{i\Gamma}$ changes its sign. This is called the SU(2) anomaly, first found by Witten [7]. This sign ambiguity forbids the gauging of $SU(2)_L$.

Although we will not discuss it in the following sections, it should be mentioned that the gauge anomaly (the perturbative gauge anomaly and the SU(2) anomaly) can be also seen in the Hamiltonian formalism (in the $A_0 = 0$ gauge) [8,9]. The physical states are defined by the Gauss equation

$$G^a(\vec{x};A_i)|\Psi_{phys.}\rangle = 0 \quad , \tag{2.12}$$

where

$$G^a(\vec{x};A_i) = (D_iE^i)^a - J_0^a(\vec{x};A) \quad . \tag{2.13}$$

$E^{i,a}$ is the canonical momentum conjugate to A_i^a. The gauge anomaly arises as impossibility of defining the physical states by (2.12). In the case of the perturbative anomaly, the integrability of (2.12) breaks down even locally. For the SU(2) anomaly the integrability condition breaks down only globally [9].

2-2. The Topological Meaning of the Gauge Anomalies

In this section a formalism is developed to understand the topological meaning of the gauge anomalies. It will be very useful for nonlinear sigma model anomalies, discussed in the next section. The topological meaning of the gauge anomalies has been discussed in [10]. In the following, a technique which was first introduced into physics literature by Alvarez is used [11].

Let us consider a fermionic theory with a chiral symmetry G. The external gauge field A is introduced. Let the fermion functional integral be P[A]. Then the effective action $\Gamma[A]$ can be formally defined by

$$P[A] = e^{i\Gamma[A]} \quad . \tag{2.14}$$

The purpose of this section is to examine a necessary condition to have a gauge invariant P[A].

Some definitions follow. Let $A^{(4)}$ be the space of all possible gauge field configurations in the space-time, and $G^{(4)}$ be the space of all gauge transformations. Then a point in the coset space $C = A^{(4)}/G^{(4)}$ corresponds to a gauge equivalent class of gauge fields.

The space C has a nontrivial topology although $A^{(4)}$ has a trivial one. Let $\{U_\alpha\}$ be a covering of C. In U_α, each point is represented by a gauge field $A_\alpha(x)$. It is assumed that only one gauge field is necessary to cover the entire space-time, i.e. the instanton number of the gauge field is zero. In $U_\alpha \cap U_\beta$, there is a relationship

$$A_\alpha = g_{\beta\alpha}^{-1}(d + A_\beta)g_{\beta\alpha} \quad , \tag{2.15}$$

where we note that the exterior derivative d is taken with respect to x, the coordinates of the space-time. Under the gauge transformation (2.15), the fermion integral changes by a phase:

$$P[A_\alpha] = e^{i\Gamma_{\alpha\beta}} P[A_\beta] \quad . \tag{2.16}$$

To see this, note that $|P|^2$ corresponds to the functional integral of the corresponding theory obtained by replacing the left-handed fermions by Dirac fermions. It can be regularized in a gauge invariant way by the Pauli-Villars regulator [12]. Hence the gauge noninvariance always arises in the phase.

In $\mathcal{A}^{(4)}$, there are configurations for which P[A] vanishes. This gives singularities in $\Gamma[A]$, so it is not well defined globally in $\mathcal{A}^{(4)}$.[f1] Therefore, we only discuss P[A] and $\Gamma[A]$ is not considered in this section.

In $U_\alpha \cap U_\beta \cap U_\gamma$, there are two additional relations:

$$P[A_\beta] = e^{i\Gamma_{\beta\gamma}} P[A_\gamma] \tag{2.17}$$

$$P[A_\gamma] = e^{i\Gamma_{\gamma\alpha}} P[A_\alpha] \quad . \tag{2.18}$$

Combining (2.16), (2.17) and (2.18), we find

$$e^{i(\Gamma_{\alpha\beta} + \Gamma_{\beta\gamma} + \Gamma_{\gamma\alpha})} = 1 \quad , \tag{2.19}$$

which implies

(f1) However, the gauge variation $\Gamma[A_\alpha] - \Gamma[A_\beta]$ is always well defined.

$$(\delta\Gamma)_{\alpha\beta\gamma} \equiv \Gamma_{\alpha\beta} + \Gamma_{\beta\gamma} + \Gamma_{\gamma\alpha} = 2\pi n_{\alpha\beta\gamma} \quad , \quad n_{\alpha\beta\gamma} \in Z \ . \tag{2.20}$$

The set $\{n_{\alpha\beta\gamma}\}$ contains information necessary to understand the topological meaning of the anomaly.

Now suppose that the fermion functional integral P[A] can be made gauge invariant by a counter term φ_α. Namely, if we redefine the fermion integral by

$$P'[A_\alpha] = e^{-i\varphi_\alpha} P[A_\alpha] \quad , \tag{2.21}$$

then in $U_\alpha \cap U_\beta$,

$$P'[A_\alpha] = P'[A_\beta] \tag{2.22}$$

and P' is defined globally on C. From (2.16), (2.22) implies that

$$\Gamma_{\alpha\beta} = \varphi_\alpha - \varphi_\beta + 2\pi k_{\alpha\beta} \quad , \text{ where } \ k_{\alpha\beta} \in Z \ . \tag{2.23}$$

This gives

$$(\delta\Gamma)_{\alpha\beta\gamma} = 2\pi(k_{\alpha\beta} + k_{\beta\gamma} + k_{\gamma\alpha}) = 2\pi n_{\alpha\beta\gamma} \ . \tag{2.24}$$

(2.23) or equivalently (2.24) is a necessary condition for defining P globally on C(i.e. the gauge invariance of P).

There is a well established procedure to get a closed two-form in C which contains the same amount of information as $\{n_{\alpha\beta\gamma}\}$. It goes as follows. First we take an exterior derivative of $\Gamma_{\alpha\beta}$ in C, $d_C\Gamma_{\alpha\beta}$. Since

$$(\delta(d_C\Gamma))_{\alpha\beta\gamma} \equiv d_C\Gamma_{\alpha\beta} + d_C\Gamma_{\beta\gamma} + d_C\Gamma_{\gamma\alpha}$$

$$= d_C(2\pi n_{\alpha\beta\gamma}) = 0 \quad , \tag{2.25}$$

there is a one-form T_α defined on each patch U_α, which satisfies

$$d_C\Gamma_{\alpha\beta} = T_\alpha - T_\beta \quad . \tag{2.26}$$

This is due to an analogue of Poincaré's lemma for differential forms in R^n .(f2) Take an exterior derivative of T_α , d_CT_α . This is globally defined on C, since in $U_\alpha \cap U_\beta$,

$$d_CT_\alpha - d_CT_\beta = d_C^2\Gamma_{\alpha\beta} = 0 \quad , \tag{2.27}$$

from (2.26). Let us call d_CT_α as F. This is obviously a closed form ($d_CF = 0$). The above procedure can be summarized in the "tic-tac-toe box" [11]:

	0			
F	d_CT_α	0		
	T_α	$d_C\Gamma_{\alpha\beta}$	0	
		$\Gamma_{\alpha\beta}$	$(\delta\Gamma)_{\alpha\beta\gamma}$	0
			$2\pi n_{\alpha\beta\gamma}$	

(f2) Let ρ_α be a partition of unity ($\sum_\alpha \rho_\alpha = 1$, $0 \le \rho_\alpha \le 1$, $\rho_\alpha = 0$ outside U_α). Then $T_\alpha = \sum_\beta \rho_\beta d_C\Gamma_{\alpha\beta}$.

It has been shown that a necessary condition for the gauge independence of P[A] is (2.23) or equivalently (2.24). What is the corresponding condition for F? (2.23) implies that

$$d_C\Gamma_{\alpha\beta} = d_C\varphi_\alpha - d_C\varphi_\beta \quad . \tag{2.28}$$

This determines T_α as

$$T_\alpha = d_C\varphi_\alpha + \tau \quad , \tag{2.29}$$

where τ is a globally defined one-form. This gives

$$F = d_C\tau \quad , \tag{2.30}$$

namely F is an exact form. Similarly it can be shown that if F is an exact form, $\Gamma_{\alpha\beta}$ can be written as (2.23).[f3] Therefore, for the gauge invariance of P[A], F is necessary to be exact. This is equivalent to

$$\int_Y F = 0 \quad , \tag{2.31}$$

where Y is an arbitrary two-dimensional surface without boundary in C.

Now let us obtain the explicit formulae for F, T_α and $\Gamma_{\alpha\beta}$. For this purpose, we introduce a connection $\bar{A}$ on $C \times M$.(M is the space-time.) A one-form $\bar{A}_\alpha$ is defined on each patch $U_\alpha \times M$. In $(U_\alpha \cap U_\beta)\times M$,

$$\bar{A}_\alpha = g_{\beta\alpha}^{-1}(d_C + d + \bar{A}_\beta)g_{\beta\alpha} \quad . \tag{2.32}$$

[f3] For this to be true, it is necessary that the cover $\{U_\alpha\}$ be a good cover, i.e. any multiple intersection of U_α can be contractible to a point.

If y represents coordinates of C, $\bar{A}_\alpha$ can be symbolically written as

$$\bar{A}_\alpha = f_\alpha(x,y)d_C y + A_\alpha \quad , \tag{2.33}$$

i.e. its restriction to M gives the ordinary gauge field A_α. Let g be a function on $C\times M$, taking a value in G. It depends on parameters, with respect to which an exterior derivative d_G can be taken. We define

$$\bar{A}^g = g^{-1}(\bar{A} + d_C + d)g \quad , \tag{2.34a}$$

$$\bar{F} = (d_C + d)\bar{A} + \bar{A}^2 \quad . \tag{2.34b}$$

The following formulae are well known in physics literature [5]:

$$-\frac{i}{24\pi^2}\,\mathrm{tr}\,\bar{F}^3 = (d_C + d)\omega_5(\bar{A}^g) \tag{2.35a}$$

$$d_G\omega_5(\bar{A}^g) = -\ (d_C + d)\omega_4^1(\bar{A}^g\ ,\ g^{-1}d_G g) \quad , \tag{2.35b}$$

where

$$\omega_5(A) = -\frac{i}{24\pi^2}\,\mathrm{tr}\,(A(dA)^2 + \frac{3}{5}A^5 + \frac{3}{2}A^3dA) \tag{2.36a}$$

$$\omega_4^1(A,v) = -\frac{i}{24\pi^2}\,\mathrm{tr}\,(dv(AdA + \frac{1}{2}A^3)) \quad . \tag{2.36b}$$

The trace is taken in the representation of the fermions considered. We note that using (2.36b), Bardeen's formula (2.8) is written as

$$\Gamma[A^{1+v}] - \Gamma[A] = \int_M \omega_4^1(A,v) \quad . \tag{2.37}$$

We are now ready to construct F, T_α and $\Gamma_{\alpha\beta}$. $\Gamma_{\alpha\beta}$ can be obtained as a line integral over (2.37). Let L be a line in $G^{(4)}$ which connects the identity and $g_{\beta\alpha}$.[f4] Then

$$\Gamma_{\alpha\beta} = \int_L \int_M \omega_4^1(A_\beta^g, g^{-1}d_G g)$$

$$= \int_L \int_M \omega_4^1(\bar{A}_\beta^g, g^{-1}d_G g) \quad . \tag{2.38}$$

In the second line, the integral over M is meant to pick up only forms proportional to the volume form of M. The difference between A_β^g and $\bar{A}_\beta^g$ is $g^{-1}d_C g$ from (2.34a), which gives a term vanishing upon the integral over M.

Next take an exterior derivative of $\Gamma_{\alpha\beta}$,

$$d_C\Gamma_{\alpha\beta} = \int_L \int_M d_C\omega_4^1(\bar{A}_\beta^g, g^{-1}d_G g)$$

$$= \int_L \int_M (d + d_C)\omega_4^1$$

$$= \int_L \int_M - d_G\omega_4^1$$

$$= \int_M \omega_5(\bar{A}_\beta) - \int_M \omega_5(\bar{A}_\alpha) \quad . \tag{2.39}$$

Note that d_C commutes with integrals. In the second line, the integral of $d\omega_4^1$ over M gives zero. To go from the second line to the third line, (2.35b) is used. (2.39) implies

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(f4) $\pi_0(G^{(4)}) = \pi_4(G) = 0$ is assumed.

$$T_\alpha = -\int_M \omega_5(\bar{A}_\alpha) \quad , \tag{2.40}$$

modulo a global form. Finally F is obtained:

$$\begin{aligned} F &= d_C T_\alpha \\ &= -\int_M (d_C + d)\omega_5(\bar{A}_\alpha) \\ &= \frac{i}{24\pi^2}\int_M \mathrm{tr}\,\bar{F}_\alpha^3 \quad , \end{aligned} \tag{2.41}$$

where (2.35a) is used.

It is known [10] that (2.41) gives a nontrivial closed form only for SU(n>2) if and only if

$$\mathrm{tr}T^a\{T^b , T^c\} \neq 0 \quad . \tag{2.42}$$

This agrees with the result of perturbation theory (2.10). In the case of U(1), (2.41) vanishes. However, in perturbation theory there is a gauge anomaly. This discrepancy comes from the requirement of quantum field theory that the counter term φ_α in (2.21) be a local (in space-time) function of A_α. Although there exists φ_α which gives a gauge invariant $P[A_\alpha]$, it is not a local function of the gauge field.(f5) The SU(2) anomaly cannot be seen from the differential form F. The anomaly appears as a sign ambiguity of P and this cannot be incorporated into F. A simple derivation of the SU(2) anomaly will be given in section 4.

(f5) However, it has recently been shown that there is a topological obstruction even for U(1) if the gauge field has a twist over the space-time [13].

2-3. Nonlinear Sigma Model Anomalies

In this section, we digress to the nonlinear sigma model anomalies which have been recently discussed by Moore and Nelson [14]. The formalism developed in the previous section is directly applicable to this case.

We start with a brief account of the nonlinear sigma models [14]. Consider an arbitrary manifold N. It is covered by patches $\{U_\alpha\}$. The scalar field $\varphi(x)$ takes values on N, i.e. φ is a map from the space-time M to N. For simplicity we assume that $\varphi(M)$ is entirely in U_α if it intersects with U_α. At each point φ of N there is a k-dimensional vector space V_φ. The fermion field $\psi(x)$ is an element of $V_{\varphi(x)}$. A connection is given on N. On each patch U_α, a k×k matrix valued one-form Θ_α is defined. In $U_\alpha \cap U_\beta$, Θ_α and Θ_β are related by

$$\Theta_\alpha = g_{\beta\alpha}^{-1}(d + \Theta_\beta)g_{\beta\alpha} \quad . \tag{2.43}$$

In the background field $\varphi(x)$ contained in U_α, the Lagrangian of the fermion is given by

$$L_\alpha = i\,\bar{\psi}_\alpha \gamma^\mu(\partial_\mu + \Theta_{\alpha\,,\,a}\frac{\partial\varphi^a}{\partial x^\mu})\psi_\alpha \quad , \tag{2.44}$$

where φ^a are local coordinates of N and $\Theta_\alpha = \Theta_{\alpha\,,\,a}d\varphi^a$. In $U_\alpha \cap U_\beta$, the Lagrangian can be also given as

$$L_\beta = i\,\bar{\psi}_\beta \gamma^\mu(\partial_\mu + \Theta_{\beta\,,\,a}\frac{\partial\varphi^a}{\partial x^\mu})\psi_\beta \quad . \tag{2.45}$$

From (2.43), L_β can be obtained from L_α by the change of variables:

$$\psi_\beta = g_{\beta\alpha}\,\psi_\alpha \quad . \tag{2.46}$$

The structure of the theory is identical to the one of the gauge theory if we substitute the gauge potential A_α by

$$A_\alpha = \Theta_{\alpha\,,\,a}\frac{\partial\varphi^a}{\partial x^\mu}\,dx^\mu \quad . \tag{2.47}$$

Let the fermion functional integral of L_α be P_α. Likewise for the gauge theories, P_α is in general dependent on which patch we use:

$$P_\alpha = e^{i\Gamma_{\alpha\beta}}\,P_\beta \quad . \tag{2.48}$$

Let C be the space of all possible space-time configurations of $\varphi(x)$. The question is whether the fermion functional can be redefined on C. The necessary condition for the gauge theories was given by (2.31) and (2.41). The corresponding condition for nonlinear sigma models is

$$\int_Y\int_M \frac{i}{24\pi^2}\,\mathrm{tr}\,\overline{F}^3 = 0 \quad , \tag{2.49}$$

where Y is an arbitrary two-dimensional closed surface in C. $\overline{F}$ is defined as follows:

$$\overline{F} = (d + d_C)\overline{A} + \overline{A}^2 \quad , \tag{2.50a}$$

$$\overline{A} = \Theta_a\left(\frac{\partial\varphi^a}{\partial x^\mu}\,dx^\mu + \frac{\partial\varphi^a}{\partial y^i}\,d_C y^i\right) \quad , \tag{2.50b}$$

where d is an exterior derivative in the space-time and y^i are coordinates of C. The condition (2.49) has been obtained in [14], where some interesting examples are also discussed.

2-4. The SU(2) Anomaly

In this last section of chapter 2, we derive the SU(2) anomaly in a simple way [15].

We consider $SU(2)_L$ gauge theory with a doublet of left-handed fermions. Let the corresponding functional integral be $P_2[A]$, where A is an external $SU(2)_L$ gauge field. By adding an $SU(2)_L$ singlet, a triplet of $SU(3)_L$ can be formed. The functional integral of the $SU(3)_L$ theory is defined to be $P_3[A']$, where A' is an external $SU(3)_L$ gauge field. If A' is restricted to an $SU(2)_L$ gauge field, the $SU(2)_L$ singlet is free, and it does not contribute to P_3. This implies

$$P_2[A] = P_3[A] \tag{2.51}$$

for any $SU(2)_L$ gauge field A. Let g(x) be a nontrivial element of $\pi_4(SU(2))$. Since $\pi_4(SU(3)) = 0$, A^g can be obtained from A by repeating infinitesimal SU(3) gauge transformations. Let $g(s) \in SU(3)$ be an interpolation between $g(0) = 1$ and $g(1) = g$. Then recalling that the perturbative anomaly is given by (2.38), we find

$$P_2[A^g] = \exp[\int_0^1 ds \int_M \omega_4^1(A^{g(s)}, g(s)^{-1}\partial_s g(s))]\, P_2[A]$$

$$= \exp[\int_0^1 ds \int_M \frac{-i}{48\pi^2} \, \mathrm{tr}\,(g^{-1}\partial_s g (g^{-1}dg)^4)]\, P_2[A]$$

$$= - P_2[A] \quad . \tag{2.52}$$

The above calculation can be done explicitly for a particular g(x):

$$g(\vec{x}\,,\, t) = \Sigma_0(R_t\vec{x})\, \Sigma_0^\dagger(\vec{x}) \quad , \tag{2.53}$$

where $\Sigma_0(\vec{x})$ is an element of SU(2) with wrapping number 1 and R_t denotes a rotation around the z-axis by $2\pi t$. The calculation (2.52) will be necessary to determine the statistics of a chiral soliton in section 4-4 and the calculation is done in Appendix 1 to chapter 4.

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Chapter 3

Chiral Anomalies in the Chiral Lagrangian

3-1. The Chiral Lagrangian

Quantum chromodynamics (QCD) is a theory based upon the SU(3) gauge symmetry. In the limit of massless u, d and s quarks, the theory possesses the chiral symmetry $G = SU(3)_L \times SU(3)_R \times U(1)_V$ ($U(1)_A$ is broken by the anomaly). At the scale $\Lambda_{\chi SB} \sim 1$ GeV, the symmetry G dynamically breaks down to its subgroup $H = SU(3)_V \times U(1)_V$. As a consequence, an octet of Goldstone bosons(pions, kaons,eta) arise [1].

The chiral Lagrangian [2] describes an effective theory for the interactions of the Goldstone bosons, which are incorporated into an SU(3) matrix $\Sigma(x)$:

$$\Sigma(x) = \exp[\frac{2i\,M}{f}] \quad , \tag{3.1}$$

where

$$M = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix} \quad . \tag{3.2}$$

f is the pion decay constant($\sim$ 134 MeV).

The chiral Lagrangian is constructed such that it is consistent with the symmetry G and parity and charge conjugation. The field M is a nonlinear realization of G, and only H is linearly represented [3]. Under G, Σ transforms as

$$\Sigma \to L\,\Sigma\,R^{\dagger} \quad , \tag{3.3}$$

where L and R are global SU(3) matrices. Σ is invariant under $U(1)_V$. The most general Lagrangian invariant under (3.3) is written as

$$L = \frac{f^2}{8}\,\mathrm{tr}\,\partial_\mu\Sigma\partial^\mu\Sigma^\dagger + \cdots \quad . \tag{3.4}$$

Here higher order derivative terms are suppressed.

The Lagrangian (3.4) is constructed so that it correctly describes the low energy interactions of the Goldstone bosons. It has been found useful in quantitative analyses of meson scattering, meson weak decays, etc. [4].

3-2. Chiral Anomalies in QCD

Let us imagine that a set of external gauge fields A_L, A_R are introduced for $G_0 = SU(3)_L \times SU(3)_R$. The Lagrangian is written as

$$L = -\frac{1}{4}\,F_{\mu\nu}^2 + i\,\bar{\psi}\gamma^\mu D_\mu\psi + i\,\bar{\psi}_L\gamma^\mu A_\mu^L\psi_L + i\,\bar{\psi}_R\gamma^\mu A_\mu^R\psi_R\,, \tag{3.5}$$

where $\psi^a = (u^a , d^a , s^a)^T$ $(a = 1,2,3)$. The first term is a kinetic term for the gluons. D_μ is a QCD covariant derivative. $A_\mu^{L,R}$ are antihermitian matrices.

The effective action is defined as follows:

$$e^{i\Gamma[A_L , A_R]} = \int [dA_\mu^{QCD}][d\psi d\bar{\psi}] e^{i\int d^4x L} \quad , \tag{3.6}$$

$e^{i\Gamma}$ is the vacuum to vacuum amplitude in the presence of the external gauge fields A_L , A_R .

Γ, defined by (3.6) is not invariant under the gauge transformations of A_L , A_R . The change of Γ under the infinitesimal gauge transformations $g_L = 1 + v_L$, $g_R = 1 + v_R$ is given as follows [5]:

$$\Gamma[A_L^{1+v_L} , A_R^{1+v_R}] - \Gamma[A_L , A_R]$$

$$= - \frac{iN}{24\pi^2} \int [\, \mathrm{tr}\, dv_L(A_L dA_L + \frac{1}{2} A_L^3) - \mathrm{tr}\, dv_R(A_R dA_R + \frac{1}{2} A_R^3)] \quad , \tag{3.7}$$

where $A^g = g^{-1}(A + d)g$ (cf. eqn. (2.8)). The form notation has been used for simplicity. N is the number of colors. For QCD, N = 3.

If the scale of the space-time variations of A_L and A_R is large compared to $\Lambda_{\chi SB}^{-1}$, the effective theory is valid. Therefore, the transformation property (3.7) must be possessed by the chiral Lagrangian. A simple replacement of ∂_μ by $\partial_\mu + A_{\mu L} + A_{\mu R}$ will not do. It gives a gauge invariant Lagrangian. Unlike fermions, scalar fields do not induce any chiral anomaly. Whatever gauge noninvariance the chiral Lagrangian has, it must be explicitly introduced at the classical level.

The "incorporation" of chiral anomalies into the chiral Lagrangian is the subject of the next section.

3-3. The Construction of the Wess-Zumino Term

The goal here is to construct a functional $\Gamma[\Sigma, A_L, A_R]$ which has the required anomalous transformation property (3.7).

First let us consider an effective action $\Gamma_1[\Sigma, A_L, A_R]$ for the theory defined by

$$L_1 = i\,\bar{\psi}\gamma^\mu\partial_\mu\psi - m(\bar{\psi}_L\Sigma\psi_R + \bar{\psi}_R\Sigma^\dagger\psi_L)$$

$$+ i\bar{\psi}_L\gamma^\mu A^L_\mu\psi_L + i\bar{\psi}_R\gamma^\mu A^R_\mu\psi_R \quad , \tag{3.8}$$

where $\psi = (u\ , d\ , s)^T$ does not carry the color quantum number, but it transforms in the same way as the quark fields under G. The gauge noninvariance of $N\,\Gamma_1$ is also given by (3.7). Somehow the anomalous part of Γ_1 should be extracted.

There is a trick to achieve this. It is a generalization of the resolution of the famous puzzle of the π^0 decay first encountered by Steinberger [6] (see Appendix).

We introduce the following change of variables:

$$\psi'_L = \Sigma^\dagger\psi_L \quad , \quad \psi'_R = \psi_R \quad . \tag{3.9}$$

The Lagrangian (3.8) changes to

$$L_2 = i\bar{\psi}'\gamma^\mu\partial_\mu\psi' - m\bar{\psi}'\psi' + i\bar{\psi}_L'\gamma^\mu A^\Sigma_{\mu L}\psi_L' + i\bar{\psi}_R'\gamma^\mu A_{\mu R}\psi_R' \quad . \qquad (3.10)$$

Under G_0 , the field A^Σ_L transforms in the same way as the field A_R :

$$A^\Sigma_L \rightarrow R(A^\Sigma_L + d)R^\dagger \quad . \qquad (3.11)$$

Therefore, the corresponding effective action Γ_2 of L_2 can be made gauge invariant by adding a counter term [5]:

$$\Delta\Gamma = \frac{1}{96\pi^2}\int tr[(A^\Sigma_L A_R - A_R A^\Sigma_L)(F_R + \Sigma^\dagger F_L \Sigma)$$

$$+ i\,(-2A^\Sigma_L(A_R)^3 + 2A_R(A^\Sigma_L)^3 - (A_R A^\Sigma_L)^2)] \, , \qquad (3.12)$$

where $F = dA + A^2$.

The two theories defined by (3.8) and (3.10) are of course inequivalent due to the Jacobian e^{iW} of the change of variables (3.9) [7]:

$$\Gamma_1 = \Gamma_2 + W \quad . \qquad (3.13)$$

Since $\Gamma_2 + \Delta\Gamma$ is gauge invariant, the functional defined by

$$\Gamma_{W-Z} = W - \Delta\Gamma \qquad (3.14)$$

has the same transformation property as Γ_1 . Γ_{W-Z} is called the (gauged) Wess-Zumino term [8,9]. W can be readily evaluated by integrating Bardeen's formula (3.7) [10]:

$$W = \frac{-i}{24\pi^2}\int_0^1 ds \int tr\, \Sigma_s^\dagger \partial_s \Sigma_s (dA^{\Sigma_s}_L dA^{\Sigma_s}_L + \frac{1}{2}\, d(A^{\Sigma_s}_L)^3)$$

$$= -\frac{i}{48\pi^2}\int_0^1 ds \int \mathrm{tr}\Sigma_s^\dagger \partial_s \Sigma_s (\Sigma_s^\dagger d\Sigma_s)^4 \tag{3.15}$$

$$-\frac{i}{24\pi^2}\int \mathrm{tr}[\frac{1}{2}(d\Sigma^\dagger \cdot \Sigma)^3 A_L + \frac{1}{4}(d\Sigma^\dagger \cdot \Sigma)A_L(d\Sigma^\dagger \cdot \Sigma)A_L$$

$$-\frac{1}{2}(d\Sigma^\dagger \cdot \Sigma)^2 A_L^2 - d\Sigma^\dagger \cdot \Sigma(A_L dA_L + \frac{1}{2}A_L^3)] \quad ,$$

where Σ_s interpolates between $\Sigma_0 = 1$ and $\Sigma_1 = \Sigma$.

The construction of the effective theory can now be stated as follows. First we replace ∂_μ by the covariant derivative $\partial_\mu + A_{\mu L} + A_{\mu R}$ in the ordinary chiral Lagrangian (3.4). Add gauge invariant terms which vanish when $F_L = F_R = 0$. Finally add $N\,\Gamma_{W-Z}$ given by (3.14). Now we note that Γ_{W-Z} is nonvanishing even for the vanishing gauge fields [(f)]:

$$\Gamma_{W-Z}[\Sigma] = \Gamma_{W-Z}[\Sigma\ , A_L = A_R = 0]$$

$$= -\frac{i}{48\pi^2}\int_0^1 ds \int \mathrm{tr}\Sigma_s^\dagger \partial_s \Sigma_s (\Sigma_s^\dagger d\Sigma_s)^4 \ . \tag{3.16}$$

This is the ordinary Wess-Zumino term. Some of the implications of this term are the subjects of the next chapter.

(f) This is not really surprising. As has been discussed in [9], the ordinary chiral Lagrangian without the Wess-Zumino term has a redundant conservation rule, i.e. the conservation of the number of mesons modulo two. This is broken by the Wess-Zumino term which contains, e.g. the $KK\pi\pi\pi$ vertex.

3-4. Generalization of the Wess-Zumino Term

The construction of the Wess-Zumino term discussed in the previous section can be easily generalized to any effective theory [11].

Let us start with a summary of the general effective theory [3]. The modern notation adopted here is from the reference [12]. Let G be a group of chiral symmetries possessed by the fundamental fermionic theory. Suppose G breaks down dynamically to H at a low energy scale. For each generator of the broken part of G, there is a Goldstone boson [1].

The Goldstone boson fields $\varphi(x)$ take values on the coset space M = G/H. Mathematically speaking φ is a map from the space-time to M. In general M cannot be covered by only one coordinate patch. Let $\{U_\alpha\}$ be a covering of M. It is assumed that if the image of the space-time under φ intersects with U_α, it lies entirely in U_α. This is for the sake of simplicity, since otherwise we need to use several patches to cover the whole configuration of $\varphi(x)$.

In each U_α, a point φ of M is represented by a group element $s_\alpha(\varphi)$ of G. In each $U_\alpha \cap U_\beta$, there is a relation

$$s_\alpha(\varphi) = s_\beta(\varphi) h_{\beta\alpha}(\varphi) \quad , \tag{3.17}$$

where $h_{\beta\alpha}(\varphi) \in H$, since both s_α and s_β belong to the same equivalence class. This coordinate transformation (3.17) will be important in the next section. Only one patch U_α is necessary for the discussions of this section and the suffix α will be omitted hereafter.

An element g of G induces a transformation of the Goldstone boson field φ:

$$g : \varphi \to \varphi'$$

$$g : s(\varphi) = s(\varphi')h(g;\varphi) \quad . \tag{3.18}$$

A matter field ψ transforms as

$$g : \psi \to D_H(h(g;\varphi))\psi \quad . \tag{3.19}$$

D_H is a representation matrix of H. A connection on M is given by $s^{-1}\partial_\mu s|_H$ ($|_H$ means a projection into the algebra of H) which transforms under (3.18) as

$$g : \ s^{-1}\partial_\mu s|_H \to h^{-1}(\partial_\mu + s^{-1}\partial_\mu s|_H)h \quad . \tag{3.20}$$

The covariant derivative of ψ can be defined using the connection:

$$D_\mu\psi = (\partial_\mu + D_H(s^{-1}\partial_\mu s|_H))\psi \quad . \tag{3.21}$$

The covariant derivative of s is given by

$$D_\mu s = s^{-1}\partial_\mu s|_{G/H} \tag{3.22}$$

which transforms covariantly under (3.18):

$$g : D_\mu s \to h^{-1}\, D_\mu s\, h \quad . \tag{3.23}$$

The effective Lagrangian can be constructed out of ψ , $D_\mu\psi$, s and $D_\mu s$ such that it is invariant under the G transformations.

The fundamental fermionic theory has chiral anomalies associated with the group G. Let the external gauge field for G be A_μ. The effective action $\Gamma[A]$ is not gauge invariant in general.

It is demanded that the effective Lagrangian have the same transformation property as $\Gamma[A]$. The noninvariance of the Lagrangian is given at the classical level, since the scalar fields cannot produce anomalies due to quantum effects.

The most straightforward way to gauge the effective Lagrangian $L_{\text{eff}}(\psi , D_\mu\psi , s , D_\mu s)$ is as follows. The covariant derivative $D_\mu\psi$ is replaced by

$$D^A_\mu\psi = (\partial_\mu + (s^{-1}\partial_\mu s + s^{-1}A_\mu s)_H)\psi \quad , \tag{3.24}$$

and the covariant derivative of s is redefined by

$$D^A_\mu s = (s^{-1}\partial_\mu s + s^{-1}A_\mu s)_{G/H} = A^s|_{G/H} \quad . \tag{3.25}$$

The effective Lagrangian $L_{\text{eff}}(\psi , D^A_\mu\psi , s , D^A_\mu s)$ thus constructed is gauge invariant under G. The puzzle arises again if we notice that the transformation of $A^s_\mu|_H$ is the same as the H-gauge field. Therefore, L_{eff} can only produce the H-anomaly but not the whole G-anomaly. A systematic construction of the term, called the generalized Wess-Zumino term, which provides the rest of the G-anomaly is discussed in the remaining part of this section.

Let us consider a free theory with the same fermion contents as the fundamental theory. The Lagrangian is

$$L_0 = i\bar{\chi}\gamma^\mu \partial_\mu \chi + i\bar{\chi}\gamma^\mu A_\mu \chi \quad , \tag{3.26}$$

where the G-gauge field A_μ is introduced. Its effective action $\Gamma_0[A]$ obviously has the same anomaly as the effective action of the original theory $\Gamma[A]$. χ linearly represents G via a representation matrix D_G. A nonlinear realization of G of the type (3.19) can be obtained from χ by a change of variables:

$$\psi = D_G(s^{-1})\chi \quad . \tag{3.27}$$

In terms of ψ, the Lagrangian is written as

$$L_1 = i\bar{\psi}\gamma^\mu(\partial_\mu + A^s_\mu)\psi \quad , \tag{3.28}$$

where $A^s_\mu = s^{-1}(A_\mu + \partial_\mu)s$. The Jacobian of (3.27) gives a counter term

$$\Delta L = -\frac{i}{24\pi^2}\int_0^1 d\rho \; \mathrm{tr}\; s_\rho^{-1}\frac{\partial s_\rho}{\partial \rho}\left(dA^{s_\rho}dA^{s_\rho} + \frac{1}{2}\, d(A^{s_\rho})^3\right) \quad , \tag{3.29}$$

where s_ρ interpolates $s_0 = 1$ and $s_1 = s$. The trace is taken in the representation D_G. The two theories defined by L_0 and $L_1 + \Delta L$ are equivalent. The Lagrangian L_1 only provides the H-anomaly , since A^s_μ transforms as a H-gauge field. The rest of the G-anomaly is given explicitly by ΔL. We note that ΔL can be written as

$$\Delta L = \Gamma[A] - \Gamma[A^s] \quad . \tag{3.30}$$

This expression will be very helpful in the next section.

Now the 't Hooft anomaly matching condition should be recalled [13]. Both the fundamental and the low energy theories have the exact chiral symmetry H. The anomaly matching condition states that the chiral fermions of the low energy theory must give the same H-anomaly as the fundamental theory. This amounts to the fact that the massless fermion is the only source of the anomaly. The Goldstone bosons, for example, which are linear representations of H cannot induce any H-anomaly [14].

Whatever phenomenological fermions Ψ we have, they must have the same H-anomaly as ψ given by (3.27). This is the consequence of the 't Hooft anomaly matching condition. Therefore, as far as the anomaly is concerned, L_1 in (3.28) is equivalent to

$$L_2 = i\overline{\Psi}\gamma^{\mu}(\partial_{\mu} + A^{s}_{\mu})\Psi \quad . \tag{3.31}$$

As a conclusion, the most general effective Lagrangian which has the same transformation property as $\Gamma[A]$ is given by

$$L = L_{eff}(\Psi \, , \, D^{A}_{\mu}\Psi \, , \, s \, , \, D^{A}_{\mu}s) + \Gamma[A] - \Gamma[A^{s}] \quad . \tag{3.32}$$

3-5. Absence of Nonlinear Sigma Model Anomalies

The nonlinear sigma model anomalies have been discussed in section 3 of chapter 2. It has been shown in [12] that the theory defined by (3.32) does not suffer from nonlinear sigma model anomalies. We end this chapter with this short section by rederiving the result of [12].

Let us suppose that the field configuration $\varphi(x)$ of the Goldstone bosons is entirely in $U_\alpha \cap U_\beta$. Then there are two possible ways of writing down the Lagrangian:

$$L_\alpha = i\overline{\Psi}_\alpha \gamma^\mu (\partial_\mu + A_\mu^{s_\alpha}) \Psi_\alpha + \Gamma[A] - \Gamma[A^{s_\alpha}] \quad , \tag{3.33a}$$

$$L_\beta = i\overline{\Psi}_\beta \gamma^\mu (\partial_\mu + A_\mu^{s_\beta}) \Psi_\beta + \Gamma[A] - \Gamma[A^{s_\beta}] \quad , \tag{3.33b}$$

where only the terms relevant to the G-anomaly have been kept. Recall that $\Gamma[A]$ is the effective action for the fundamental theory. The question to ask is if the two theories defined by L_α , L_β are equivalent.

The first term of (3.33b) is obtained from the first term of (3.33a) by the change of variables:

$$\Psi_\beta = D_H(s_\beta^{-1} s_\alpha) \Psi_\alpha = D_H(h_{\beta\alpha}) \Psi_\alpha \quad . \tag{3.34}$$

This induces a counter term due to the H-anomaly. Because of the 't Hooft anomaly matching condition, the counter term can be written in terms of the H-anomaly of the fundamental theory:

$$\Gamma_{\beta\alpha} = \Gamma[A^{s_\alpha}] - \Gamma[A^{s_\beta}] \quad . \tag{3.35}$$

Therefore, L_α is equivalent to

$$i\overline{\Psi}_\beta \gamma^\mu(\partial_\mu + A_\mu^{s_\beta})\Psi_\beta + \Gamma[A] - \Gamma[A^{s_\alpha}] + \Gamma[A^{s_\alpha}] - \Gamma[A^{s_\beta}] \quad . \qquad (3.36)$$

This is precisely equal to L_β, proving that L_α and L_β are equivalent. Thus, there is no nonlinear sigma model anomaly associated with the coordinate transformation from s_α to s_β .

The above result can be also seen from the condition (2.49) derived in the previous chapter. We denote the trace in the representation D_H by tr_H and the trace in D_G by tr_G . Then for

$$\overline{A} = s^{-1}(d + d_C)s|_H \quad , \qquad (3.37)$$

we find

$$tr_H\, \overline{F}^3 = tr_G\, \overline{F}^3 \qquad (3.38)$$

by the 't Hooft condition. Now we notice that both $\overline{A}$ in (3.37) and $A' \equiv s^{-1}(d + d_C)s$ give connections on C×(space-time) with the gauge group G. For A', the corresponding F' vanishes. The difference between $tr_G\, \overline{F}^3$ and $tr_G\, F'^3 = 0$ is an exact form [15]:

$$tr_G\, \overline{F}^3 = -3(d + d_C) \int_0^1 dt\; tr_G\, [(s^{-1}(d + d_C)s)_{G/H}\, \overline{F}_t^2\,] \, , \qquad (3.39)$$

where

$$\overline{A}_t = \overline{A} + t(s^{-1}(d + d_C)s)_{G/H} \quad , \qquad (3.40a)$$

$$\overline{F}_t = (d + d_C)\overline{A}_t + +\overline{A}_t^2 \quad . \qquad (3.40b)$$

Since the integral in (3.39) is a global form, $\mathrm{tr}_G\, \overline{\mathrm{F}}^3$ is exact. Therefore, the two-form F defined by

$$F = \frac{i}{24\pi^2} \int_M \mathrm{tr}_G \overline{\mathrm{F}}^3 \qquad (3.41)$$

(see (2.41)) is exact, and there is no anomaly.

Appendix. The Puzzle of $\pi^0 \to 2\gamma$ Decay

The interaction of π^0 and u, d quarks can be described in two ways.

Theory 1 (pseudo-scalar coupling)

$$L_1 = i\bar{\psi}\gamma^\mu(\partial_\mu + QA_\mu)\psi - m\bar{\psi}\psi - i\frac{\sqrt{2}m}{f}\bar{\psi}\pi^0\tau^3\gamma^5\psi \quad . \tag{A.1}$$

Theory 2 (pseudo-vector coupling)

$$L_2 = i\bar{\psi}\gamma^\mu(\partial_\mu + QA_\mu)\psi - m\bar{\psi}\psi + \frac{m}{\sqrt{2f}}\bar{\psi}\partial_\mu\pi^0\gamma^\mu\gamma^5\tau^3\psi \quad . \tag{A.2}$$

Here $\psi^a = (u^a, d^a)^T$ $(a = 1,2,3)$, $Q = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3})$. L_1 gives the $\pi^0 \to 2\gamma$ decay from the graph:

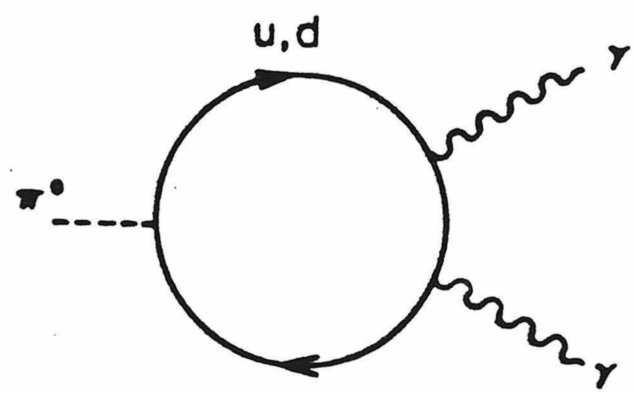

Fig. $\pi^0 \to 2\gamma$ decay

The amplitude is $-\frac{ie^2\sqrt{2}}{32\pi^2 f}\pi^0\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$. However, in the massless limit of π^0, the decay amplitude due to L_2 is zero. This is because the loop of a massive quark does not give any singularity to cancel the momentum of π^0 from $\partial_\mu \pi^0$.

The puzzle arises [6] when we notice that L_2 can be obtained from L_1 by the change of a variable:

$$\psi'_L = e^{-i\frac{\sqrt{2}}{f}\tau^3\pi^0}\psi_L \quad . \tag{A.3}$$

The resolution to this puzzle is the anomaly in the transformation (A.3). The corresponding Jacobian gives a counter term

$$\Delta L = -\frac{ie^2\sqrt{2}}{32\pi^2 f}\,\pi^0 \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \quad . \tag{A.4}$$

L_1 and $L_2 + \Delta L$ are now physically equivalent.

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Chapter 4

The Chiral Soliton Model

4-1. Solitons in the Chiral Lagrangian

Consider the chiral Lagrangian for QCD with massless u, d and s quarks. There are an infinite number of possible higher order terms. A particular example of the chiral Lagrangian is given by

$$L = \frac{f^2}{8} \operatorname{tr} \partial_\mu \Sigma \partial^\mu \Sigma^\dagger + \frac{1}{32e^2} \operatorname{tr} [\partial_\mu \Sigma \cdot \Sigma^\dagger , \partial_\nu \Sigma \cdot \Sigma^\dagger]^2 \quad . \tag{4.1}$$

It was first noticed by Skyrme (in the case of two flavors) [1] that (4.1) has a soliton solution:

$$\Sigma_0(\vec{x}) = \exp[if(r)\hat{\lambda}] \quad , \tag{4.2}$$

where $\hat{\lambda} = \lambda^i \frac{r^i}{r}$. The function f of radius vanishes at the origin and goes to π at infinity. Since Σ_0 approaches -1 at infinity, Σ_0 has a finite energy. Σ_0 represents a nontrivial element of $\pi_3(SU(3)) = Z$ and its wrapping number is 1:

$$w = \frac{1}{24\pi^2} \int \operatorname{tr} (\Sigma_0^\dagger d\Sigma_0)^3 = 1 \quad . \tag{4.3}$$

Here we note that the higher derivative terms are necessary in (4.1) in order to stabilize the classical solution. (This is known as Derrick's theorem [2].)

Skyrme tried to model a baryon as a soliton of the Lagrangian (4.1). The justification of his idea in the framework of QCD is the subject of the next section.

4-2. Baryons in Large N QCD

The picture of a baryon being a soliton emerges from QCD in the limit of large N (N being the number of colors). In order to understand this statement, a brief summary of the $\frac{1}{N}$ expansions for QCD is necessary.

As has been discussed by 't Hooft and Witten [3,4], the most natural expansion parameter of QCD is $\frac{1}{N}$. The gauge coupling constant is scaled such that

$$g^2N = \text{constant} \quad . \tag{4.4}$$

Thereby the self-energy graphs of a gluon have a smooth limit as $N \to \infty$. For the meson interactions, the power counting can be done simply by drawing Feynman diagrams. It can been shown that the leading contributions come from the so-called planar diagrams. There are two important conclusions:

(a) The meson masses and mass splittings are of order 1.

(b) The n-meson interactions are suppressed by $\frac{1}{N^{\frac{1}{2}(n-2)}}$.

These can be incorporated into the chiral Lagrangian by determining the N dependence of the parameters as follows:

$$L = N(a_1 \operatorname{tr} \partial_\mu \Sigma \partial^\mu \Sigma^\dagger + a_2 \operatorname{tr} [\partial_\mu \Sigma , \partial_\nu \Sigma^\dagger]^2 + \cdots) \quad . \tag{4.5}$$

All parameters $a_1, a_2, \ldots$ in the parenthesis are independent of N. The N dependence is factored out. The properties (a), (b) are reproduced immediately from (4.5), if we substitute

$$\Sigma \sim e^{\frac{i\pi}{\sqrt{N}}} \quad , \tag{4.6}$$

where the normalization of the meson field π is independent of N .

What about baryons? This is more subtle, since we cannot use diagrams for the power counting. The baryon contains N quarks and the mass is of order N. The Feynman diagrams which represent the corrections to the amplitudes involve powers of N, and they do not provide a convergent series.

Witten got around this difficulty by introducing the Hartree-Fock approximation for the baryon which is a system of N quarks [4]. The approximation amounts to a mean field theory, and it gets better as N becomes larger. (The corrections are suppressed by $\frac{1}{N}$.) For large N, each quark feels a common potential field of order 1. The important conclusions are as follows:

(c) The baryon masses are of order N. The baryon mass splittings are of order 1.

(d) The baryon size is of order 1.

(e) The baryon-meson scattering amplitudes are of order 1.

The excited baryons can be obtained by putting some of the quarks into excited states. Since the excited energies of the quarks are of order 1, the mass splittings are of order 1. The size of the baryon is determined by the size of the quark wave function, which is set by the common potential of order 1. (e) implies that for a baryon the interactions with mesons are suppressed by $\frac{1}{N}$, but the effects of a baryon on mesons are of order 1.

Does a soliton of the Lagrangian (4.5) have the properties (c), (d) and (e)? First we note that the classical solution of (4.5) is independent of N. Therefore, the size of the soliton is of order 1 and (d) is satisfied. The baryon masses are obviously of order N. The mass splittings of the baryons are of order 1 as we shall see later when we discuss the baryon mass spectrum (section 6). Therefore, (c) is satisfied. The quadratic term in the normalized meson field in (4.5) is independent of N, however, it depends on the classical soliton solution. Therefore, the scattering amplitude of a meson from a soliton is of order 1. (e) is satisfied.

In this section we have seen that the baryons in the large N QCD have the same N dependence as the solitons of the chiral Lagrangian.

4-3. Baryon Number of the Soliton

The justification of the soliton being a baryon will be complete if we derive its baryon number [5,6].

Let us consider QCD with N colors first. One sensible way to define the baryon number is to introduce an external field B_μ coupled to the quark current $\frac{1}{N}\bar{\psi}\gamma^\mu\psi$. Calculate the effective action $\Gamma[B]$. The baryon current can be defined as $\frac{\delta\Gamma}{\delta B_\mu}$, and the baryon number is defined as the space integral over the time component of the baryon current.

This definition can be introduced to the effective theory which is required to provide the same effective action $\Gamma[B]$ if the momentum of B_μ is small compared to $\Lambda_{\chi SB}$. In the chiral Lagrangian, at first glance there does not seem to be anything to which the field B_μ can be coupled. The field Σ is a singlet under $U(1)_V$. However, the requirement that Γ should have the correct anomaly content demands the appearance of B_μ in Γ. For simplicity let us consider the two-flavor case. The whole chiral symmetry is $G = SU(2)_L \times SU(2)_R \times U(1)_V$ and the only anomalies are of mixed types: $SU(2)_L^2 \cdot U(1)_V$ and $SU(2)_R^2 \cdot U(1)_V$. Again for the sake of simplicity only the $SU(2)_L^2 \cdot U(1)_V$ anomaly will be considered. The anomalous transformation property of Γ is given by

$$\Gamma[A_L^{1+v_L}, B + d\varepsilon] - \Gamma[A_L, B]$$

$$= \frac{-i}{24\pi^2} \int \mathrm{tr}\,[d\varepsilon(A_L dA_L + \frac{1}{2} A_L^3) + dv_L(2A_L dB + \frac{1}{2} A_L^2 B)] \quad . \quad (4.7)$$

The corresponding Wess-Zumino term can be calculated as follows:

$$\Gamma_{W-Z} = \frac{1}{24\pi^2} \int B \, \mathrm{tr} \, [(\Sigma^\dagger(d + A_L)\Sigma)^3$$

$$- 3\Sigma^\dagger F_L(d + A_L)\Sigma + A_L F_L - \frac{1}{2} A_L^3] \quad . \qquad (4.8)$$

This implies that in the absence of A_L, the baryon number current is given by

$$J^\mu = \frac{1}{24\pi^2} \varepsilon^{\mu\nu\alpha\beta} \, \mathrm{tr} \, [\Sigma^\dagger\partial_\nu\Sigma \cdot \Sigma^\dagger\partial_\alpha\Sigma \cdot \Sigma^\dagger\partial_\beta\Sigma] \quad . \qquad (4.9)$$

This expression has also been obtained by Goldstone and Wilczek by a one-loop calculation [7]. From (4.3), it is seen that the wrapping number coincides with the baryon number.

We have seen that the coupling of the field B_μ is uniquely determined by the anomaly.(f1) The formula (4.9) is correct as long as the spatial variations of the soliton solution are smooth compared to $\Lambda_{\chi SB}^{-1}$. (Or equivalently it is valid as long as the chiral Lagrangian makes sense.)

The reasoning which led to (4.9) can be applied to an arbitrary number of flavors.

(f1) The gauge invariant terms of the form $(\partial_\mu B_\nu - \partial_\nu B_\mu) \times (\text{gauge invariants})^{\mu\nu}$ can be added to the Lagrangian. But they do not contribute to the total charge.

4-4. Statistics of the Soliton

Is the soliton a boson or a fermion? This was first examined by Witten who looked at the amplitude for the soliton under a 2π rotation [8]. The soliton is a boson or a fermion according as the amplitude is ± 1.

Consider the three-flavor case. When the field Σ changes adiabatically, only the Wess-Zumino term gives a nonvanishing contribution besides an inessential constant term in the Lagrangian. This is because the Wess-Zumino term is linear in the time derivative while the rest of the terms in the Lagrangian involve at least two time derivatives. Therefore, it is the Wess-Zumino term only which determines the statistics of the soliton. This is somewhat puzzling, since the Wess-Zumino term is a consequence of the chiral anomaly, and it has , at a first glance, nothing to do with the rotational symmetry of the soliton. The resolution to this puzzle is given by the observation that

$$\Sigma_0(R_t\vec{x}) = e^{2\pi i t\lambda_3}\Sigma_0(\vec{x})e^{-2\pi i t\lambda_3} \quad , \tag{4.10}$$

where R_t denotes a rotation around the z-axis by $2\pi t$. (4.10) is a consequence of the symmetry of the classical solution Σ_0 given in (4.2). Σ_0 is invariant under simultaneous rotations in spin and isospin. Thus this symmetry gives a connection between the chiral symmetry and the rotational property of the soliton.

Now consider a field configuration:

$$\Sigma(\vec{x},t) = \Sigma_0(R_t\vec{x})\Sigma_0^\dagger(\vec{x}) \quad . \tag{4.11}$$

The baryon number of this configuration is zero. Σ corresponds to a series of events. At $t = 0$, a pair of a soliton and an antisoliton is created. Between $t = 0$ and $t = 1$, only the soliton is rotated by angle $2\pi t$. Finally at $t = 1$, the pair is annihilated. The Wess-Zumino term for this process is calculated in Appendix 1:

$$N\,\Gamma_{W-Z}[\Sigma(\vec{x},t)] = N\pi \quad . \tag{4.12}$$

Therefore, the corresponding amplitude is $(-1)^N$. and the soliton is a fermion or a boson depending on whether N is odd or even. This agrees with the quark model in which a baryon is made of N fermions(quarks).

The above argument does not simply apply to the two-flavor case. There is no Wess-Zumino term. However, this does not imply that the soliton is a boson. The reason for this is essentially that

$$\pi_4(SU(2)) = Z_2 \quad . \tag{4.13}$$

The field configurations which have zero baryon number can be classified by $\pi_4(SU(2))$. The nontrivial π_4 implies that there are two classes of configurations which cannot be smoothly deformed from one another. In the functional integral over the field Σ, it is possible to multiply a weight -1 for the nontrivial configurations.[f2]

For SU(2), (4.11) belongs to the nontrivial class of $\pi_4(SU(2))$. Therefore, the soliton is a fermion or a boson depending on whether or

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[f2] $\pi_4(SU(2)) = Z_2$ restricts this weight to be ± 1 due to the cluster property.

not we multiply the extra weight -1 [8].

In fact a further argument can show that it is necessary to multiply the weight -1 [9]. Let $\Gamma_{QCD}[A_L , A_R]$ be the effective action of QCD with two flavors. As has been discussed in section 2-4 (the SU(2) anomaly), under a nontrivial gauge transformation g the effective action changes:

$$\exp[i\Gamma_{QCD}[A_L^g , A_R]] = - \exp[i\Gamma_{QCD}[A_L , A_R]] \quad . \tag{4.14}$$

Let $\Gamma_{\mathbf{chiral}}[\Sigma , A_L , A_R]$ be the space-time integral of the gauged chiral Lagrangian. The eqn. (4.14) implies that we must have

$$\exp[i\Gamma_{\mathbf{chiral}}[g\Sigma , A_L^g , A_R]] = - \exp[i\Gamma_{\mathbf{chiral}}[\Sigma , A_L , A_R]] \quad . \tag{4.15}$$

However, Γ_{chiral} is invariant under any gauge transformations, including the nontrivial g :

$$\Gamma_{\mathbf{chiral}}[g\Sigma , A_L^g , A_R] = \Gamma_{\mathbf{chiral}}[\Sigma , A_L , A_R] \quad . \tag{4.16}$$

The equations (4.15) and (4.16) look contradictory but this can be reconciled in the following way. When we quantize the field Σ using the path integral, the integral over the field configurations consists of two parts, the integral over Σ belonging to the trivial element of $\pi_4(SU(2))$ and the one over Σ belonging to the nontrivial element of $\pi_4(SU(2))$. The factor -1 in (4.15) can be introduced by hand, since Σ and $g\Sigma$ belong to different elements of the homotopy group. This implies that for non-trivial configurations we are forced to multiply the weight -1 in order to recover the SU(2) anomaly.

4-5. The Path Integral Quantization of the Soliton

The purpose of this section is to construct the wave functions of the baryons, ignoring meson excitations [10]. In this approximation, the configuration of Σ can be parametrized by the collective coordinates as follows:

$$\Sigma(\vec{x}\,,\,t) = A(t)\Sigma_0(\vec{x})A^\dagger(t) \quad , \tag{4.17}$$

where $A \in SU(3)$. For a static A, Σ is another classical solution with the same value $\Sigma = -1$ at spatial infinity. The transformation property of A is given as follows:

$$A \rightarrow V\,A\,R^\dagger \quad . \tag{4.18}$$

A multiplication of an SU(3) matrix V from the left of A corresponds to an $SU(3)_V$ transformation. A multiplication of an SU(2) matrix $R^\dagger$ from the right is a spatial rotation of the soliton.

We should note that in fact A lives in the coset space $M = SU(3)/U(1)$, since $A(t)$ and $A(t)e^{i\vartheta(t)\lambda_8}$ gives the same $\Sigma(\vec{x},t)$. The Lagrangian must be also invariant under the λ_8 rotation. Therefore, up to second derivatives in time, the general form of the Lagrangian, invariant under (4.18), is

$$L = \frac{m}{2}\sum_{a=1}^{3}(\mathrm{tr}\,T^a iA^\dagger\partial_0 A)^2 + \frac{M}{2}\sum_{a=4}^{7}(\mathrm{tr}\,T^a iA^\dagger\partial_0 A)^2$$

$$+\, i\,\frac{N}{\sqrt{6}}\int_0^1 d\rho\;\mathrm{tr}\,T^8[A^\dagger\partial_\rho A\,,\,A^\dagger\partial_t A] \quad , \tag{4.19}$$

where the generators are normalized so that tr $T^aT^b = \delta^{ab}$. The last term is the Wess-Zumino term. N is the number of colors. In order to derive this form, we have imposed the periodic boundary condition on the field Σ. $A(\rho,t)$ is an interpolating field of A(t) which satisfies

$$A(1,t) = A(t) \quad . \tag{4.20}$$

The periodic boundary condition on Σ implies:

$$A(\rho,t_i) = A(\rho,t_f)e^{i\vartheta(\rho)T^8} \quad . \tag{4.21}$$

Therefore, $A(\rho,t)$ is a map from a two dimensional disc to M.

In general it is necessary to have many patches $\{U_\alpha\}$ to cover the manifold M. In each U_α , a point of M is represented by an element A_α of SU(3). In $U_\alpha \cap U_\beta$, there is a relation:

$$A_\alpha = A_\beta e^{i\psi_{\beta\alpha}\sqrt{6}T^8} \quad . \tag{4.22}$$

In $U_\alpha \cap U_\beta \cap U_\gamma$, there are two other relations similar to (4.22). By combining the three relations together, the following consistency condition is obtained:

$$\psi_{\alpha\gamma} + \psi_{\gamma\beta} + \psi_{\beta\alpha} = 2\pi n_{\alpha\beta\gamma} \quad , \tag{4.23}$$

where $n_{\alpha\beta\gamma}$ is an integer. The Wess-Zumino term can be simplified as follows:

$$- i \int_0^1 d\rho \, \mathrm{tr} \, T^8[A^\dagger \partial_\rho A \, , \, A^\dagger \partial_t A]$$

$$= i\,\mathrm{tr}\,A^{\dagger}\partial_t A T^8 - i\frac{d}{dt}\int_0^1 d\rho\,\mathrm{tr}\,A^{\dagger}\partial_\rho A T^8 \quad . \tag{4.24}$$

The total time derivative term can be discarded, since it does not affect the equation of motion. The Lagrangian is now written as

$$L = \frac{m}{2}\sum_{a=1}^{3}(\mathrm{tr}\,T^a i A^{\dagger}\partial_0 A)^2 + \frac{M}{2}\sum_{a=4}^{7}(\mathrm{tr}\,T^a i A^{\dagger}\partial_0 A)^2$$

$$+ i\frac{N}{\sqrt{6}}\,\mathrm{tr}\,A^{\dagger}\partial_t A T^8 \quad . \tag{4.25}$$

We now discard the periodic boundary condition (4.21) we imposed before. The Lagrangian (4.25) is certainly unambiguous as far as the trajectory of A(t) stays in a particular patch U_α. The question is whether it is well defined globally on M. The procedure to test this is originally due to Alvarez [11].

Consider a path starting from U_α and ending in U_β :

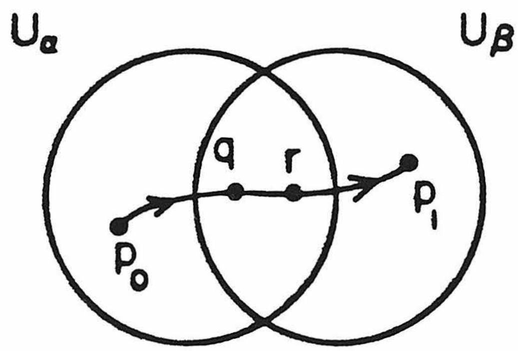

Fig. 1 A path from U_α to U_β .

The quadratic term in (4.25) is globally defined, so only the Wess-Zumino term needs be considered. A natural choice for the action in this case would be

$$\Gamma_q = i\frac{N}{\sqrt{6}}\left[\int_{p_0}^{q} \mathrm{tr}\, A_\alpha^\dagger dA_\alpha T^8 + \int_q^{p_1} \mathrm{tr}\, A_\beta^\dagger dA_\beta T^8\right] \quad . \tag{4.26}$$

It is also possible to take another point r in $U_\alpha \cap U_\beta$:

$$\Gamma_r = i\frac{N}{\sqrt{6}}\left[\int_{p_0}^{r} \mathrm{tr}\, A_\alpha^\dagger dA_\alpha T^8 + \int_r^{p_1} \mathrm{tr}\, A_\beta^\dagger dA_\beta T^8\right] \quad . \tag{4.27}$$

The difference is

$$\Gamma_q - \Gamma_r = i\frac{N}{\sqrt{6}} \int_q^r \mathrm{tr}\,(A_\beta^\dagger dA_\beta - A_\alpha^\dagger dA_\alpha)T^8$$

$$= N \int_q^r d\psi_{\beta\alpha}$$

$$= N\,(\psi_{\beta\alpha}(r) - \psi_{\beta\alpha}(q)) \quad , \tag{4.28}$$

where (4.22) has been used. Therefore, the action

$$\Gamma = \Gamma_q + N\,\psi_{\beta\alpha}(q) \tag{4.29}$$

is independent of q . Now is this Γ well defined globally? Not quite. An ambiguity arises from a triple intersection:

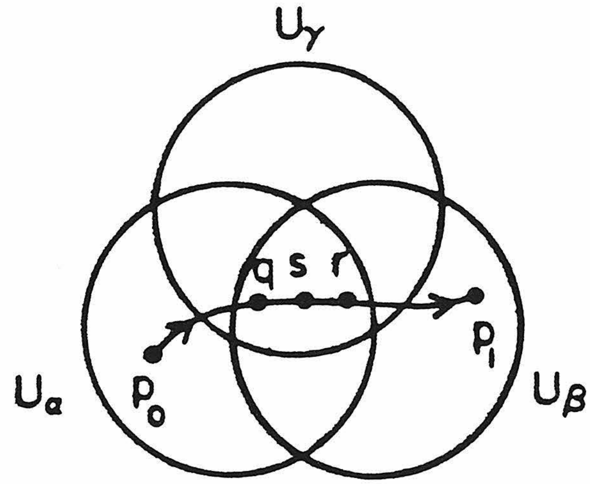

Fig. 2 A triple intersection.

There are two ways to write down the action:

$$\Gamma_1 = i\frac{N}{\sqrt{6}} \int_{p_0}^{r} \mathrm{tr}\, A_\alpha^\dagger dA_\alpha T^8 + N\,\psi_{\beta\alpha}(r) + i\frac{N}{\sqrt{6}} \int_{r}^{p_1} \mathrm{tr}\, A_\beta^\dagger dA_\beta T^8 \quad , \tag{4.30a}$$

$$\Gamma_2 = i\frac{N}{\sqrt{6}} \int_{p_0}^{q} \mathrm{tr}\, A_\alpha^\dagger dA_\alpha T^8 + N\,\psi_{\gamma\alpha}(q) + i\frac{N}{\sqrt{6}} \int_{q}^{s} \mathrm{tr}\, A_\gamma^\dagger dA_\gamma T^8$$

$$+ i\frac{N}{\sqrt{6}} \int_{s}^{p_1} \mathrm{tr}\, A_\beta^\dagger dA_\beta T^8 \quad . \tag{4.30b}$$

The difference between these two expressions is a constant from (4.23):

$$\Gamma_1 - \Gamma_2 = N(\psi_{\alpha\gamma} + \psi_{\gamma\beta} + \psi_{\beta\alpha})$$

$$= 2\pi N n_{\alpha\beta\gamma} \quad . \tag{4.31}$$

At the classical level this constant is totally harmless. However, for quantum mechanics $e^{i\Gamma}$ must be single-valued. It is a weight for each path in the path integral. Therefore, we conclude that N must be an integer. In this particular case N is the number of colors, and this quantization condition is satisfied.

A wave function of a baryon is a collection of wave functions $\{\psi_\alpha(A_\alpha)\}$ defined on patches. The time evolution of the wave functions is given by the path integral:

$$\psi_\alpha(A_\alpha,t) = \int_{\text{path}} e^{i\Gamma}\,\psi_\beta(A_\beta,0) \quad , \tag{4.32}$$

where the integral is over all the paths from A_β to A_α in the coset space M. A_β is also integrated over. In order that (4.32) be well defined, a certain relation between ψ_α and ψ_β must be imposed. Let us consider a path starting from the intersection $U_\alpha \cap U_\beta$.

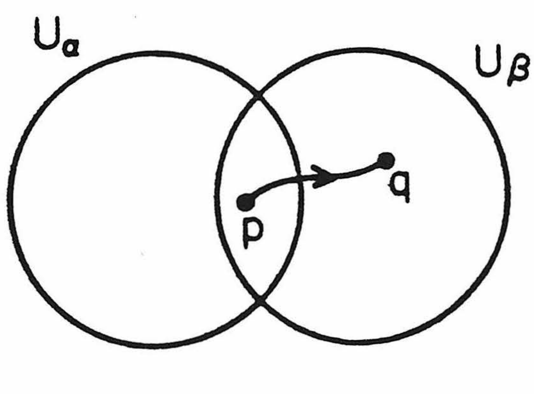

Fig. 3 A path starting from $U_\alpha \cap U_\beta$.

If the point p is regarded as belonging to U_α , the action is

$$\Gamma_1 = N\,\psi_{\beta\alpha}(p) + \int_p^q \mathrm{tr}\, A_\beta^\dagger dA_\beta\, T^8 \quad . \tag{4.33}$$

Its contribution to the path integral can be written symbolically as

$$\psi_\beta(A_\beta(q)) = e^{i\Gamma_1}\psi_\alpha(A_\alpha(p)) \quad . \tag{4.34}$$

If p is regarded as belonging to U_β , the action is

$$\Gamma_2 = \int_p^q \mathrm{tr}\, A_\beta^\dagger dA_\beta T^8 \quad . \tag{4.35}$$

This gives

$$\psi_\beta(A_\beta(q)) = e^{i\Gamma_2}\psi_\beta(A_\beta(p)) \quad . \tag{4.36}$$

$\psi_\beta(A_\beta(q))$ must be independent of how we look at p. The two expressions (4.34) and (4.36) are equivalent if and only if we impose the consistency condition

$$e^{iN\psi_{\beta\alpha}}\,\psi_\alpha(A_\beta e^{i\psi_{\beta\alpha}\sqrt{6}T^8}) = \psi_\beta(A_\beta) \quad . \tag{4.37}$$

This condition provides a rule of pasting together the wave functions ψ_α and ψ_β. Thus for the entire SU(3) space the wave function $\psi(A)(A \in SU(3))$ can be defined. It satisfies

$$\psi(Ae^{i\vartheta\sqrt{6}T^8}) = e^{-iN\vartheta}\psi(A) \quad . \tag{4.38}$$

In the path integral, all possible paths in the whole SU(3) space are taken into account. The weight of a path is determined by the action

$$\Gamma = \int dt[\frac{m}{2}\sum_{a=1}^{3}(\mathrm{tr}\, T^a\, iA^\dagger\partial_0 A)^2 + \frac{M}{2}\sum_{a=4}^{7}(\mathrm{tr}\, T^a\, iA^\dagger\partial_0 A)^2]$$

$$+ i\frac{N}{\sqrt{6}} \int \mathrm{tr}\, A^{\dagger} dA T^{8} \quad . \tag{4.39}$$

A path in SU(3)/U(1) given in the left-hand side of Fig. 4 is now replaced by a path in SU(3):

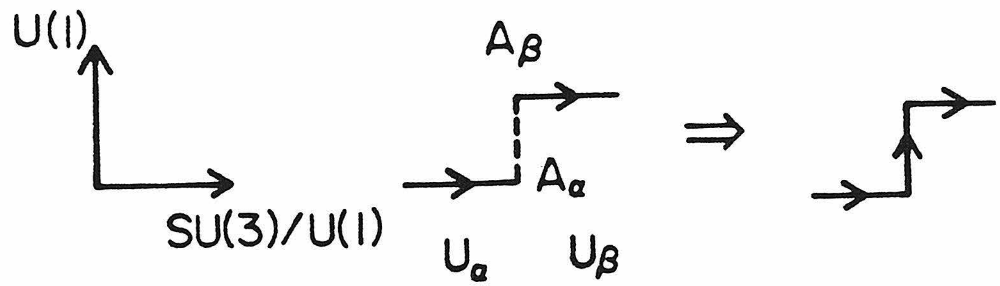

Fig. 4 Replacement of a path.

The extra path between A_α and A_β does not contribute to the first two terms. The third term gives N $\psi_{\beta\alpha}$, reproducing (4.29).

4-6. Derivation of the Schrödinger Equation

In this section we will derive the Schrödinger equation from the path integral [12] and find the energy eigenmodes. The path integral formula for the time evolution of the wave function is

$$\psi(A,t) = \int_{\text{paths}} [dB] e^{i\Gamma} \psi(B,0) \quad , \tag{4.40}$$

where the integral is taken over all paths connecting B at t = 0 and A at t in the SU(3) space. The integral is also taken over B. The action Γ is given by (4.39). Let us suppose $t = \varepsilon$ is infinitesimally small. Then the paths which give dominant contributions to the path integral do not fluctuate much around A . Therefore, the following local coordinates are useful:

$$B(x_a) = Ae^{i\sum_{a=1}^{8} x_a T^a} \quad . \tag{4.41}$$

The path integral (4.40) can be approximated as

$$\psi(A,\varepsilon) = \frac{1}{C} \int \prod_a dx_a \exp[i\varepsilon(\frac{m}{2} \sum_{a=1}^{3} (\frac{x_a}{\varepsilon})^2 + \frac{M}{2} \sum_{a=4}^{7} (\frac{x_a}{\varepsilon})^2 + \frac{N}{\sqrt{6}} x_8)]$$

$$\times (\psi(A,0) + x_a \frac{\partial\psi(A,0)}{\partial x_a} + \frac{1}{2} x_a x_b \frac{\partial^2\psi(A,0)}{\partial x_a \partial x_b}) \quad , \tag{4.42}$$

where C is a normalization constant. The integral over x_8 can be discarded, since it only forces the condition (4.38) on the wave function. Therefore, in (4.42) x_8 can be put to zero. As a result of the integral, we find

$$\psi(A,\varepsilon) - \psi(A,0) = i\varepsilon[\frac{1}{2m} \sum_{a=1}^{3} \frac{\partial^2\psi}{\partial x_a^2} (A,0) + \frac{1}{2M} \sum_{a=4}^{7} \frac{\partial^2\psi}{\partial x_a^2} (A,0)] \; . \tag{4.43}$$

It is convenient to introduce the operators which generate right multiplications:

$$[R^a , A] = -A\, T^a \quad , \tag{4.44a}$$

$$[R^a\ ,\ R^b] = -i\sqrt{2}\, f^{abc}\, R^c \qquad . \tag{4.44b}$$

From (4.43), the Hamiltonian can be written in terms of R^a as follows:

$$H = \frac{1}{2m} \sum_{a=1}^{3} (R^a)^2 + \frac{1}{2M} \sum_{a=4}^{7} (R^a)^2 \quad . \tag{4.45}$$

In fact the Hamiltonian should have a constant term E_0 of order N (the energy of the classical solution) which is omitted from (4.45).

The baryon wave function is given by

$$\psi(A) = \frac{1}{\sqrt{\dim(\rho)}} D^{(\rho)}_{ab}(A) \qquad , \tag{4.46}$$

where (ρ) is an irreducible representation of SU(3) which has a state b with the right hypercharge $\frac{N}{3}$ [10]. It is required by (4.38). The SU(2) quantum numbers of b correspond to spin, therefore,

$$b = (J\ ,\ -J_z\ ,\ \frac{N}{3}) \qquad . \tag{4.47}$$

a is the $SU(3)_V$ quantum numbers:

$$a = (I\ ,\ I_3\ ,\ Y) \qquad . \tag{4.48}$$

Let us examine the mass spectrum for each J. The Hamiltonian (4.45) can be rewritten as follows:

$$H = \frac{1}{2}\left(\frac{1}{m} - \frac{1}{M}\right) \sum_{a=1}^{3} (R^a)^2 + \frac{1}{2M} \sum_{a=1}^{8} (R^a)^2 - \frac{N^2}{12M}$$

$$= (\frac{1}{m} - \frac{1}{M})J(J+1) + \frac{1}{M} C_2(\rho) - \frac{N^2}{12M} \quad , \qquad (4.49)$$

where $C_2(\rho)$ is the quadratic Casimir invariant of the representation (ρ). If $(\rho) = (p,q)$,

$$C_2(p,q) = \frac{1}{3}(p^2 + 3p + pq + 3q + q^2) \quad . \qquad (4.50)$$

For each J, the lowest energy states belong to the representation $(2J \,, \frac{N}{2} - J)$ given by the Young tableau below [13]:

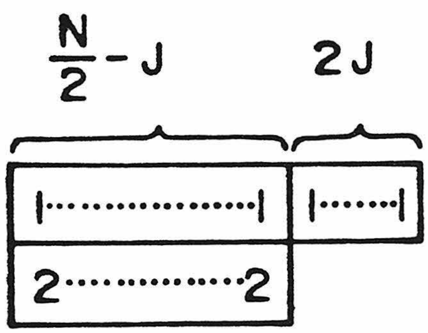

Fig. 5 Young tableau for the lowest energy baryons.

The numbers are put in to show the state b given by (4.47) (J_z is taken to be $-J$). From (4.49) and (4.50), the energy of this multiplet is calculated as

$$E = \frac{1}{m} J(J+1) + \frac{N}{2M} \quad . \qquad (4.51)$$

It should be noted that m and M are of order N. Therefore, the mass splittings of the various spin multiplets are of order $\frac{1}{N}$.

What about excited states? In general the excited states are given by the representation $(2(J-s)+r\,,\ \frac{N}{2}-(J-s)+r)$ where $0 \leq s \leq r\,,\ 2J$. The corresponding Young tableau is given below: (Again the state b is shown.)

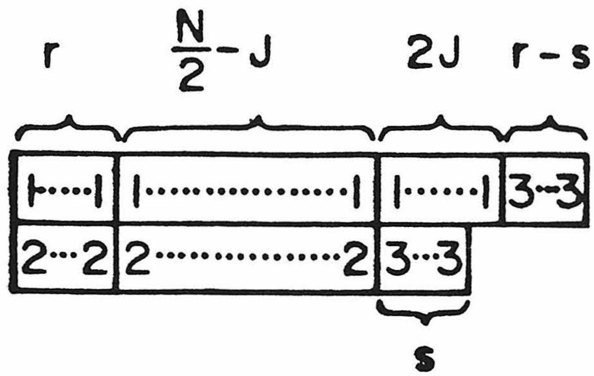

Fig. 6 Young tableau for excited states.

The energy of this multiplet is

$$E = \frac{1}{m}\,J(J+1) + \frac{N}{2M}\,(1+r) + \frac{1}{M}\,(r^2 + 2r + Jr - s(r+1+2J-s)) \ . \tag{4.52}$$

The change of energy due to s is only of order $\frac{1}{N}$, while r gives the mass splittings of order 1, which is expected from the $\frac{1}{N}$ expansions of QCD (section 2).

Finally let us look at the lowest energy multiplet $(\rho) = (1\,,\ \frac{N-1}{2})$ with $J = \frac{1}{2}$. For N = 3, this is the familiar $J^P = (\frac{1}{2})^+$ octet. The states are shown in Fig. 7. The states which survive at N = 3 are given names accordingly. Modulo a normalization constant the wave functions of this multiplet are given as follows:

$$\psi(A) = D_{(I\, I_3\, Y)\, n}{}^{(p)}(A) \qquad \text{for } J_z = \frac{1}{2} \quad , \tag{4.53a}$$

$$D_{(I\, I_3\, Y)\, p}{}^{(p)}(A) \qquad \text{for } J_z = -\frac{1}{2} \quad . \tag{4.53b}$$

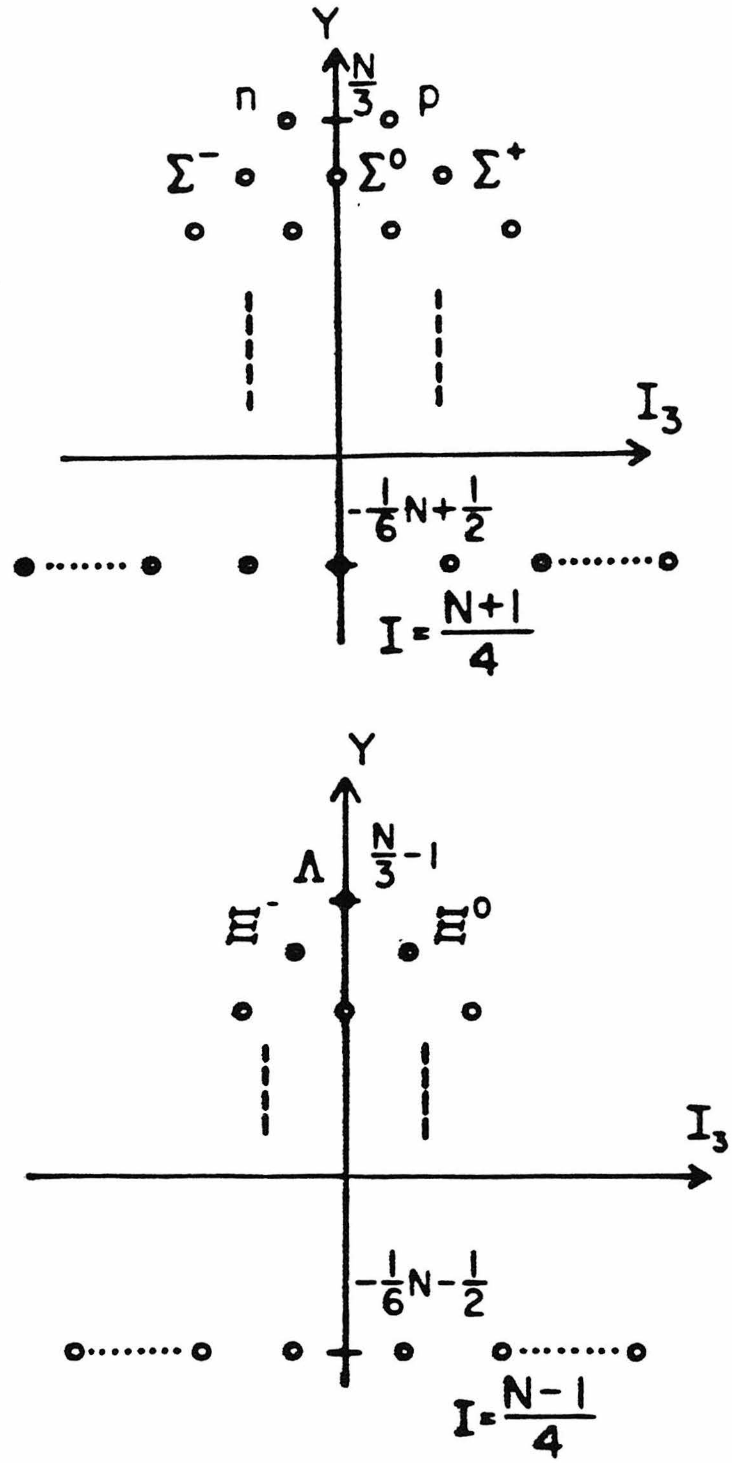

Fig. 7 Baryons with $J = \frac{1}{2}$.

4-7. Power Counting in $\frac{1}{N}$

One advantage of the chiral soliton model over other phenomenological models like the nonrelativistic quark model [14] is that it provides a systematic expansion in $\frac{1}{N}$ for a matrix element.

Suppose we are interested in some physical observable, say, a weak decay amplitude. The same operator which gives the weak decays of the mesons also gives the weak decays of the baryons. The N dependence of the operator can be written as follows:

$$O = \text{const.}\ (f_1(\Sigma) + \frac{1}{N} f_2(\Sigma) + \frac{1}{N^2} f_3(\Sigma) + \cdots)\ . \tag{4.54}$$

The constant involves an appropriate powers of N. The operators f_1 , f_2 , $\cdots$ are local functions of Σ and they do not possess any N dependence. The form of (4.54) can be determined by counting powers of N for the relevant Feynman diagrams. f_2 , f_3 ,... all come from non-planar diagrams. If we want to obtain matrix elements for a baryon, the same operator (4.54) should be evaluated. Neglecting the meson fluctuations, which will be considered later, (4.54) is written as

$$O = \text{const.}\ (f_1(A\ ,\ \partial_0 A) + \frac{1}{N} f_2(A\ ,\ \partial_0 A) + \cdots)\ . \tag{4.55}$$

f_1 in general involves terms with many derivatives. Now note that

$$\partial_0 A = i\,[H\ ,\ A]$$

$$= \frac{i}{2} \left(\frac{1}{m} \sum_{a=1}^{3} [(R^a)^2 , A] + \frac{1}{M} \sum_{a=4}^{7} [(R^a)^2 , A] \right)$$

$$= i \left(\frac{1}{m} \sum_{a=1}^{3} T^a R^a + \frac{1}{M} \sum_{a=4}^{7} T^a R^a - \frac{3}{4m} - \frac{3}{2M} \right) . \qquad (4.56)$$

Since both m and M are of order N, (4.56) implies that the time derivative of A is suppressed by a power of $\frac{1}{N}$. For large N, the heavy soliton rotates very slowly. Therefore, from each time derivative, one power of $\frac{1}{N}$ is obtained. In order to calculate a matrix element to leading order in $\frac{1}{N}$ (we call this a semiclassical approximation), only the operators with no time derivative need be considered. The operators with time derivatives give contributions to the matrix element suppressed by powers of $\frac{1}{N}$. Some of the calculations in the semiclassical approximation are shown in the next section [15, 16]. If we wish to go to the next order in $\frac{1}{N}$, it is necessary to consider all possible operators with a time derivative. This has been done for some observables in [17].

Two remarks are in order:

1. In fact R^a has off-diagonal matrix elements of order $\sqrt{N}$. Namely, the matrix elements of R^a between the states with the right hypercharge different by unity are of order $\sqrt{N}$. We can see this from the fact that $\sum_{a=1}^{7} (R^a)^2$ is of order N. Therefore, the power counting given above is too naive. However in Appendix 3, it is shown that this naive counting is

actually correct.

2. The meson fluctuations are another source of corrections to the matrix elements. The amplitude of emission or absorption of a meson is suppressed by $\frac{1}{\sqrt{N}}$. (This is because the pion decay constant f is of order $\sqrt{N}$.) The emitted meson must be absorbed, so the corrections due to mesons are suppressed by powers of $\frac{1}{N}$. The meson interactions give corrections to the coefficients of the operators in (4.55). Namely they induce operators f_2, f_3, $\cdots$. Therefore, the power counting is not affected by them.

4-8. The $\frac{F}{D}$ Ratios in the Semiclassical Approximation

Let us consider operators belonging to the octet representation of $SU(3)_V$. Since we are interested in evaluating the matrix elements to leading order in $\frac{1}{N}$, only operators with no time derivative need be considered. If the operator is a scalar, there is a unique expression:

$$O_s^a = \mathrm{tr}\, T^8 A^\dagger T^a A = D_{a\Lambda}^{(8)}(A) \quad . \tag{4.57}$$

If it is a vector,

$$O_i^a = \mathrm{tr}\, T^i A^\dagger T^a A = D_{ai}^{(8)}(A) \quad . \tag{4.58}$$

Both operators transform in the correct way under $SU(3)_V$ and the spatial rotation. They are also invariant under

$$A \rightarrow A e^{i\vartheta(t)T^8} \quad , \tag{4.59}$$

which is required due to the invariance of Σ under (4.59).

For N = 3, the matrix elements of (4.57) and (4.58) for the baryon octet can be incorporated by the following expressions:

$$O_s^a = F_s \, \mathrm{tr}\, B^\dagger[T^a , B] + D_s \, \mathrm{tr}\, B^\dagger[T^a , B] \quad , \tag{4.60}$$

$$O_i^a = F_V \, \mathrm{tr}\, B^\dagger\sigma^i[T^a , B] + D_V \, \mathrm{tr}\, B^\dagger\sigma^i[T^a , B] \quad , \tag{4.61}$$

where the spinor field B is given by

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\sqrt{\frac{2}{3}}\,\Lambda \end{pmatrix} . \tag{4.62}$$

Therefore, the $\frac{F}{D}$ ratios can be predicted from the chiral soliton model in the semiclassical limit [16].

The matrix elements of **(4.57)** can be calculated in terms of the Clebsch-Gordan coefficients [18]:

$$<B_f|O_s^a|B_i>$$

$$= \int dA \; D^{(8)*}_{B_f n}(A) D^{(8)}_{a\Lambda}(A) D^{(8)}_{B_i n}(A)$$

$$= \begin{pmatrix} 8 & 8 & 8 \\ a & B_i & B_f \end{pmatrix} \begin{pmatrix} 8 & 8 & 8 \\ \Lambda & n & n \end{pmatrix} + \begin{pmatrix} 8 & 8 & 8' \\ a & B_i & B_f \end{pmatrix} \begin{pmatrix} 8 & 8 & 8' \\ \Lambda & n & n \end{pmatrix} \quad . \tag{4.63}$$

Here the baryons with spin up are considered. By comparing (4.63) with the matrix elements of (4.60), we find

$$\frac{F_s}{D_s} = -\frac{5}{3} \quad . \tag{4.64a}$$

Similarly we find

$$\frac{F_V}{D_V} = \frac{5}{9} \quad . \tag{4.64b}$$

There are several interesting octet operators in nature. The Hamiltonian for the nonleptonic decays (in s-wave) of the hyperons is a scalar operator in the octet. Therefore, the $\frac{F}{D}$ ratio is predicted to be $-\frac{5}{3}$. A least squares fit to the experimental data for the seven decay modes [19] gives $\frac{F}{D} \sim -2.3$. The predicted value is 30 % off the observed value.

The Hamiltonian for the semileptonic decays of the hyperons has the current-current form:

$$H^{\Delta s = -1} = \text{const.}\ J^p_\mu \cdot J^{\mu,\,\Xi^-}_{\text{lepton}} \quad . \tag{4.65}$$

Here J_μ is a V - A current. Since the vector SU(3) is conserved, the matrix element of the vector current is known. The time component of the axial part of J^p_μ can be ignored for nonrelativistic hyperons. The axial part of J^p_i has $\frac{F}{D} = \frac{5}{9}$ in the semiclassical approximation. A least squares fit to the experimental data [19] gives $\frac{F}{D} \sim .51$. The predicted value has an excellent agreement with it.

Since the electromagnetic charge

$$Q = \mathrm{diag}\,(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) = \sqrt{\frac{2}{3}}\, T^8 + \sqrt{2}\, T^3 \qquad (4.65)$$

is a generator of SU(3), the magnetic moment can be written as

$$\mu_i = \sqrt{\frac{2}{3}}\, O_i^8 + \sqrt{2}\, O_i^3 \quad . \qquad (4.67)$$

The $\frac{F}{D}$ ratio is again $\frac{5}{9}$ which agrees reasonably with the observed ratio .7 obtained by a least squares fit to the data [19].

In the above examples, the semiclassical approximation gives reasonably good predictions to the $\frac{F}{D}$ ratios. However, the approximation does not always work well. The semiclassical approximations for the mass splitting of the baryon octet [10] and the parton distribution function [20] are known to give predictions quite far from the observed values. The accuracy of the semiclassical approximation varies depending on what we calculate. This implies that in the real world N = 3 seems to be somewhat too small for the semiclassical approximation to be uniformly valid.

Appendix 1. Calculation of the Wess-Zumino Term

The Wess-Zumino term for the rotating soliton (4.11) is considered in this appendix. The calculation is originally due to Witten [6].

Since the classical solution Σ_0 commutes with λ_8,

$$\Sigma_0(R_t\vec{x}) = \begin{bmatrix} e^{i\pi t} & 0 & 0 \\ 0 & e^{-i\pi t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Sigma_0(\vec{x}) \begin{bmatrix} e^{-i\pi t} & 0 & 0 \\ 0 & e^{i\pi t} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-2\pi it} & 0 \\ 0 & 0 & e^{-2\pi it} \end{bmatrix} \Sigma_0(\vec{x}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi it} & 0 \\ 0 & 0 & e^{-2\pi it} \end{bmatrix} . \tag{A1.1}$$

Choose

$$A(t\,,\rho) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho e^{-2\pi it} & \sqrt{1-\rho^2} \\ 0 & -\sqrt{1-\rho^2} & \rho e^{2\pi it} \end{bmatrix} , \tag{A1.2}$$

and define

$$\Sigma(\vec{x}\,,t\,,\rho) = A^\dagger(t\,,\rho)\Sigma_0(\vec{x})A(t\,,\rho)\Sigma_0^\dagger(\vec{x}) . \tag{A1.3}$$

The Wess-Zumino term is calculated as follows:

$$\Gamma[\Sigma(\vec{x}\,,t)] = \frac{-i}{48\pi^2}\int_0^1 d\rho \int \mathrm{tr}\,[\Sigma^\dagger\partial_\rho\Sigma\,(\Sigma^\dagger d\Sigma)^4\,]$$

$$= -\frac{i}{\sqrt{6}}\int_0^1 d\rho\,\mathrm{tr}\,T^8\,[A^\dagger\partial_\rho A\,,\,A^\dagger\partial_t A]$$

$$= \pi \quad . \tag{A1.4}$$

Appendix 2. Notation

A brief summary of the notation which is originally due to de Swart [17] is given in this appendix.

Let (ρ) be an irreducible representation of SU(3). Each state of (ρ) is uniquely specified by three quantum numbers (I, I_3, Y), the magnitude of isospin, the isospin third component and the hypercharge. The D-function is defined by

$$D_{\beta\alpha}^{(\rho)*}(A) = <\beta|A|\alpha> \quad . \tag{A2.1}$$

For the adjoint representation, this implies

$$A^{\dagger}T^{a}A = D^{(8)}_{ab}(A)T^{b} \quad . \tag{A2.2}$$

The generators of left, right multiplications are defined by

$$[L^{a}, A] = -T^{a}A \ , \ [L^{a}, L^{b}] = -i\sqrt{2}\, f^{abc}L^{c} \quad , \tag{A2.3}$$

$$[R^{a}, A] = -T^{a}A \ , \ [R^{a}, R^{b}] = -i\sqrt{2}\, f^{abc}R^{c} \quad . \tag{A2.4}$$

Consider the following octet operators:

$$O^{a} = A^{\dagger}T^{a}A \quad . \tag{A2.5}$$

This has the quantum numbers of T^{a} with respect to the left multiplication($SU(3)_V$). The commutation relation

$$[L^{a}, O^{b}] = i\sqrt{2}\, f^{abc}O^{c} \tag{A2.6}$$

implies, for example, that O^{Σ^+} increases the eigenvalue of L^{Σ^0} by $\sqrt{2}$.

The D-functions satisfy

$$L^{\Sigma^0} D_{(I\, I_3 Y)\beta}{}^{(\rho)}(A) = D_{(I\, I_3 Y)\beta}{}^{(\rho)}(-T^3 A) = \sqrt{2}\, I_3\, D_{(I\, I_3 Y)\beta}{}^{(\rho)}(A)\,, \quad \text{(A2.7a)}$$

$$L^{\Lambda} D_{(I\, I_3 Y)\beta}{}^{(\rho)} = D_{(I\, I_3 Y)\beta}{}^{(\rho)}(-\, T^8 A) = \sqrt{\frac{3}{2}}\, Y\, D_{(I\, I_3 Y)\beta}{}^{(\rho)}(A)\,. \quad \text{(A2.7b)}$$

The analogous formulae for the right multiplication are

$$R^{\Sigma^0} D_{\alpha(I\, I_3 Y)}{}^{(\rho)}(A) = D_{\alpha(I\, I_3 Y)}{}^{(\rho)}(-\, A\, T^3) = \sqrt{2}\, I_3 D_{\alpha(I\, I_3 Y)}{}^{(\rho)}(A), \quad \text{(A2.8a)}$$

$$R^{\Lambda} D_{\alpha(I\, I_3 Y)}{}^{(\rho)}(A) = D_{\alpha(I\, I_3 Y)}{}^{(\rho)}(\, -\, A\, T^8) = \sqrt{\frac{3}{2}}\, Y\, D_{\alpha(I\, I_3 Y)}{}^{(\rho)}(A)\;. \quad \text{(A2.8b)}$$

Appendix 3. Matrix Elements of Time Derivative Operators

First consider an example of an operator with one time derivative:

$$O^a = \frac{1}{N} \sum_{b=1}^{7} D^{(8)}_{ab}(A) R^b \quad . \qquad \text{(A3.1)}$$

This operator is a scalar belonging to the octet of $SU(3)_Y$. It is invariant under the T^8 rotations. Its matrix element in the lowest energy multiplet $(\rho) = (1 \, , \, \frac{N-1}{2})$ is calculated as follows:

$$< I'I'_3Y' \, , \, J_z = -\frac{1}{2} \mid \sum_{b=1}^{7} D^{(8)}_{ab} R^b \mid I \, I_3 Y \, , \, J_z = -\frac{1}{2} >$$

$$= \int dA \, D_{(I'I'_3Y')p}{}^{(\rho)*}(A) \sum_{b=1}^{7} D^{(8)}_{ab}(A) \, R^b \, D_{(I\,I_3Y)p}{}^{(\rho)}(A)$$

$$= \int dA \, D_{(I'I'_3Y')p}{}^{(\rho)*}(A) \sum_{b=1}^{7} D^{(8)}_{ab}(A) \, D_{(I\,I_3Y)\gamma}{}^{(\rho)}(A) \, (T^b)_{p\gamma}$$

$$= \sum_{\rho' \sim \rho} \begin{pmatrix} 8 & \rho & \rho' \\ a & I\,I_3Y & I'I'_3Y' \end{pmatrix} \sum_{b=1}^{7} \begin{pmatrix} 8 & \rho & \rho' \\ b & \gamma & p \end{pmatrix} (T^b)_{p\gamma} \quad . \qquad \text{(A3.2)}$$

The matrix elements of R^b give $(T^b)_{p\gamma}$. p has the hypercharge $\frac{N}{3}$. If γ has the hypercharge $\frac{N}{3} - 1$, the matrix element is of order $\sqrt{N}$. All nonvanishing elements are

$$\frac{1}{\sqrt{2}} T^{1+i2}_{pn} = 1 \quad , \quad T_{pp} = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} T_{p\Sigma^0} = -\frac{1}{2} \sqrt{N-1} \quad , \quad \frac{1}{\sqrt{2}} T^{4+i5}_{p\Lambda} = -\frac{1}{2} \sqrt{N+3} \qquad \text{(A3.3)}$$

$$\frac{1}{2}\, T^{6+i7}_{p\Sigma^+} = -\sqrt{\frac{N-1}{2}} \quad .$$

The Clebsch-Gordan coefficients $\begin{pmatrix} 8 & \rho & \rho' \\ b & \gamma & p \end{pmatrix}$ are of order 1 if γ has the hypercharge $\frac{N}{3}$ and of order $\frac{1}{N}$ if γ has the hypercharge $\frac{N}{3} - 1$. $(\rho)\times(8)$ contains two irreducible representations (ρ_1) and (ρ_2) equivalent to (ρ). The relevant Clebsch-Gordan coefficients are given below:

$$\begin{pmatrix} 8 & \rho & \rho_1,\rho_2 \\ \Sigma^+ & n & p \end{pmatrix} = c_1 \quad , \quad \sqrt{2}\, n(n+5)c_2$$

$$\begin{pmatrix} 8 & \rho & \rho_1,\rho_2 \\ \Sigma^0 & p & p \end{pmatrix} = \frac{c_1}{\sqrt{2}} \quad , \quad n(n+5)c_2$$

$$\begin{pmatrix} 8 & \rho & \rho_1,\rho_2 \\ p & \Sigma^0 & p \end{pmatrix} = 0 \quad , \quad \sqrt{n}\,(5n+16)c_2 \qquad \text{(A3.4)}$$

$$\begin{pmatrix} 8 & \rho & \rho_1,\rho_2 \\ p & \Lambda & p \end{pmatrix} = -\sqrt{\frac{2}{N+2}}\, c_1 \quad , \quad 3n\sqrt{n+2} \quad c_2$$

$$\begin{pmatrix} 8 & \rho & \rho_1,\rho_2 \\ n & \Sigma^+ & p \end{pmatrix} = 0 \quad , \quad \sqrt{2n}\,(5n+16)c_2 \quad ,$$

where

$$c_1 = \sqrt{\frac{3(n+2)}{5n+16}}\ , \ c_2 = \frac{1}{\sqrt{6n(n+2)(n+4)(5n+16)}}\ , \ n = \frac{N-1}{2}\ .$$

As a result, the matrix elements of (A3.1) are of order $\frac{1}{N}$.

This result can be generalized easily. In general, higher time derivative operators are written as

$$O = \frac{1}{N^k} D_{\alpha\beta}(A) R^{a_1} \cdots R^{a_k} t^{\alpha\beta}{}_{a_1 \cdots a_k} \,, \qquad \text{(A3.5)}$$

where t is a constant of order 1. The invariance under the T^8 rotations imposes that O carry zero right hypercharge. If $R^{a_1} \cdots R^{a_k}$ carries $Y_R = m$, then the state β must carry $Y_R = -m$. When we evaluate the matrix elements of O between the two states with the same right hypercharge, the power counting goes as follows. The state

$$R^{a_1} \cdots R^{a_k} |(I\, I_3 Y) Y_R> \qquad \text{(A3.6)}$$

has the right hypercharge $Y_R + m$ and its amplitude is proportional to $N^{\frac{m}{2}}$ due to the off-diagonal elements of R^a. The matrix elements of $D_{\alpha\beta}$ between the states with the right hypercharge Y_R and $Y_R + m$ are, however, suppressed by $N^{-\frac{m}{2}}$ due to the off-diagonal Clebsch-Gordan coefficients. Therefore, the matrix elements of O are of order $\frac{1}{N^k}$.

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Chapter 5

Conclusion

Recently it has been discovered that anomalies have a rich topological structure. This was the subject of chapter 2. We have derived (or rederived) many previous results in a simpler manner.

What makes the study of anomalies so interesting is their nontrivial physical implications. As described in chapter 1, the calculation of the $\pi^0 \to 2\gamma$ decay amplitude initiated the study of anomalies. This is still the clearest example of the implications of chiral anomalies.

In chapter 3, we have described how to incorporate chiral anomalies into chiral Lagrangians. This incorporation gave us another implication of chiral anomalies, namely, the chiral soliton model, which was the subject of chapter 4. We have seen how essential anomalies are in determining the spin and statistics of a soliton.

One feature which makes the chiral soliton model special is that it gives a systematic expansion of physical quantities in $\frac{1}{N}$. This is unlike other phenomenological models, e.g. the nonrelativistic quark model. The expansion in $\frac{1}{N}$ was explained in sections 4-2 and 7. A semiclassical approximation was introduced as a calculation to leading order in $\frac{1}{N}$. Unfortunately the semiclassical approximation did not turn out to be very successful.

Historically anomalies have been a source of new physics. Other implications of anomalies should be looked for in the future.