Supergravity Theory from Ten Dimensions

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This thesis is dedicated to Marina, for making it all worth while.

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Abstract

This work is concerned with the study of several ten-dimensional field theories intimately associated with superstring theories, and possibilities for obtaining realistic four-dimensional theories from them.

Three chapters follow the N = 2b supergravity from ten to five, then to four dimensions. First of all, compactifications to five dimensions on various manifolds are studied. Then the entire mass spectrum for the compactification on S^5 is derived using techniques of harmonic analysis on spheres. A particular set of modes corresponds to a gauged maximal supergravity theory in five dimensions; this theory, with Yang-Mills group SO(6), is constructed in detail. By a process similar to analytic continuation, noncompact versions of this theory are also obtained, gauging all the semisimple real forms of SO(6). One particular form, with gauge group $SO^*(6) \approx SU(3,1)$, compactifies to flat four-dimensional spacetime and offers attractive phenomenological possibilities.

The final chapter is concerned with candidates for effective low-energy theories for N = 1 superstrings with gauge group SO(32) or $E_8 \times E_8$. These effective theories contain curvature squared terms, and require unusual gravitational interactions to cancel anomalies. The field equations are derived and found to admit compactifications to flat four dimensional spacetime, with the possibility of accommodating many phenomenological considerations.

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Chapter 1

Introduction

Much of the incentive for studying supergravity theories has come from two basic sources. The first is the difficulty of incorporating general relativity, enormously successful at the classical level, into the framework of quantum field theory. Attempting to quantize the gravitational field coupled to matter leads to ultraviolet divergences too severe to admit sensible interpretation in any known way [1]. Even pure gravitation suffers from this problem [2]. The renormalization programme so successful for Yang-Mills theories is inapplicable to gravity; the reason may be traced to the presence of a dimensional coupling constant for gravity, Newton's constant. A promising approach for eliminating the divergences is provided by the principle of supersymmetry, a type of symmetry which relates bosons and fermions. With such a symmetry one may hope to cancel divergences from bosonic fields against divergences from fermions, taking advantage of the intrinsic minus sign required for fermionic quantum loop diagrams. Supergravity theories, which contain the graviton and its fermionic partners under supersymmetry, the spin-$\frac{3}{2}$ gravitinos, indeed exhibit dramatically improved quantum properties [3]. Unfortunately, there is no real evidence that any supergravity theory is finite to all orders, though finite supersymmetric models have been constructed containing particles of spin ≤ 1 [4].

Besides their improved quantum behavior, a second more aesthetic reason for studying supersymmetric theories, and especially supergravity, is the great predictive power inherent in theories with a high degree of symmetry. Supersymmetry is the only known type of symmetry which can relate particles of different spin, and hence can hope to completely account for the observed particle spectrum from some fundamental principle. In addition to restricting the combinations of particles that can occur, supersymmetry places severe restrictions on the possible interactions between particles. Supersymmetry even restricts the dimensionality of spacetime; it is believed that consistent supersymmetrical theories exist only in eleven or fewer dimensions. The eleven-dimensional supergravity [5] thus occupies a special position among supergravity theories and it is natural to address the possibility that it may provide an ultimate theory of nature. (The way that a field theory can yield an effective low-energy theory in fewer dimensions is discussed in Chapters 3 and 4.) Unfortunately, the theory is almost certainly badly divergent at the quantum level, and even at the classical level appears unable to reproduce the observed phenomenology, aside from the Yang-Mills gauge group [6].

It has recently become evident that a fundamentally different type of field theory, superstring theory [7], may provide considerably more promising alternatives than any of the supergravity theories in both of these arenas. String theories are field theories of extended one-dimensional objects, as opposed to traditional theories of zero-dimensional points. A string theory may be considered as a point-particle field theory, with a spectrum containing an infinite number of particles of unbounded spin and mass. There is a purely bosonic string theory in 26 dimensions, but it contains a tachyon in its spectrum which spoils attempts at physical interpretation. Luckily, the supersymmetric versions of string theories are much better behaved.

The one-loop finiteness of superstring theories is a particularly stunning and suggestive result, requiring many more types of cancellation than point-particle field theories require. There are arguments to suggest that quantum finiteness persists to all orders. Furthermore, superstring theories are fantastically restrictive: there are only five known (barring essentially degenerate examples in two dimensions), all requiring spacetime to have ten dimensions. To begin to characterize these theories will require some general knowledge of ten-dimensional field theory.

In ten dimensions one may impose both Majorana and Weyl conditions upon a spinor. The Weyl (chirality) condition exists for any even dimension, since there is an analogue of the four-dimensional γ_5. In ten dimensions this matrix is denoted Γ_{11}. Its crucial properties are $(\Gamma_{11})^2 =$

1 and $\{\Gamma_{11}, \Gamma_A\} = 0$, where Γ_A are the ten-dimensional Dirac matrices. ($A = 0,1,\ldots,9$ is a ten-dimensional vector index.) The projectors onto left- and right-handed chiral components are then $\frac{1}{2}(1 \pm \Gamma_{11})$. The Majorana condition can be taken to be a reality condition on the spinors. In order for this to be consistent with the Lorentz transformation properties of the spinors, the Lorentz generators $\Gamma_{AB} \equiv \frac{1}{2}[\Gamma_A, \Gamma_B]$ must all be real. Thus the gamma matrices may be taken to be either all real or all imaginary.

In four dimensions one may define both Majorana and Weyl conditions, but it is impossible to impose both simultaneously. This is because γ_5 is imaginary, so the chirality condition is inconsistent with the reality condition (unless the spinor itself vanishes). In ten dimensions, however, Γ_{11} is real and one may impose both conditions at once, for a "Majorana-Weyl" spinor.

The minimal (N = 1) supersymmetry expresses invariance under transformations parametrized by a single Majorana-Weyl spinor. There are three distinct superstring theories known with N = 1 supersymmetry, type I superstrings with gauge group SO(32) [8], and heterotic superstrings with gauge group SO(32) or $E_8 \times E_8$ [9].

At the classical level, the known point particle field theories with N = 1 supersymmetry are super-Yang-Mills gauge theories with arbitrary gauge group, simple supergravity, and the coupled supergravity-Yang-Mills system. It turns out that the simple supergravity is inconsistent because of gravitational anomalies, and simple supergravity coupled to Yang-Mills is afflicted with mixed gauge and gravitational anomalies, unless the gauge group is either SO(32) or ee8 [8]. Even then, one must alter the theory in order to cancel the anomalies, in a way which naïvely breaks the supersymmetry. Chapter 5 contains a more complete discussion of these N = 1 theories and their relationship to superstrings, including the relevant references.

If one looks for ten-dimensional theories invariant under a general number N of supersymmetries, to describe particles of maximum spin two one can have at most N = 2 supersymmetry. This corresponds to the fact that in four dimensions, in order to have spin ≤ 2 (i.e., at most a helicity spread of -2 to 2), one must have $N \leq 8$, since (at least in Minkowski space) each independent supersymmetry transformation can raise or lower helicity by one-half. Upon dimensional reduction from ten to four dimensions, in which all fields are taken to be independent of the extra six coordinates, each ten-dimensional Majorana-Weyl spinor yields four four-dimensional Majorana or Weyl spinors. Thus N = 2 in ten dimensions directly corresponds to N = 8 in four dimensions.

In going from N = 1 to N = 2 in ten dimensions, one has a choice for the relative chirality of the two spinorial supersymmetry parameters, leading to two fundamentally different N = 2 superalgebras. In type 2a the two spinorial charges have opposite chirality, so they can be combined into a single Majorana spinor with no definite chirality. Alternatively, relaxing the Majorana condition leads to the type 2b superalgebra, in which the two charges have the same chirality and may be considered as the real and imaginary parts of a single complex (i.e., non-Majorana) spinor.

Supergravity theories exhibiting N = 2a and N = 2b supersymmetry have been constructed; the theories are completely different, even having different field content. Both the 2a and 2b supergravities are the zero-mass sectors of superstrings, the IIa and IIb theories, respectively. This completes the list of the five known superstring theories.

The N = 2a theory may be obtained by simple dimensional reduction of the eleven-dimensional supergravity [10], by taking all fields to be independent of the eleventh coordinate. The eleven-dimensional theory contains a graviton, a gravitino and a gauge three-index antisymmetric tensor. These reduce to d = 10 as follows:

$$g_{M'N'} \rightarrow (g_{MN}) + (g_{M,10} \equiv A_M) + (g_{10,10} \equiv \varphi) \qquad (1.1.1)$$

$$\psi_{M'} \rightarrow (\psi_M) + (\psi_{10} \equiv \chi) \qquad (1.1.2)$$

$$A_{M'N'P'} \rightarrow (A_{MNP}) + (A_{MN,10} \equiv B_{MN}) \qquad (1.1.3)$$

where primed indices run from 0 to 10 and unprimed indices from 0 to 9. Thus the N = 2a theory contains in the bosonic sector a graviton, a vector, a scalar and two antisymmetric tensor gauge fields, one with three indices and one with two. All these fields are real. The fermionic sector consists of one gravitino and one spinor, both Majorana and non-chiral.

The identification of dimensionally reduced fields above, though physically well motivated, is somewhat heuristic in that complicated field redefinitions are in general required to identify the correct mass eigenstates. For example, a Weyl rescaling involving the scalar φ relates the ten-dimensional graviton and the corresponding components of the eleven-dimensional graviton.

The fact that the ten-dimensional fermionic fields in (1.1.2) are Majorana but not Weyl corresponds to the fact that chirality cannot be defined in eleven dimensions. In general, dimensional reduction results in non-chiral theories [11], and it is hence not surprising that there is no known way to relate the chiral N = 2b theory to higher dimensions. In this sense the N = 2b theory is on the same footing as the eleven-dimensional supergravity for providing a fundamental maximal supergravity.

All other maximal supergravities, gauged or ungauged, are believed to be derivable from one or both of these theories by a process of compactification and truncation. The ungauged supergravities in less than ten dimensions may be derived from either theory by simple dimensional reduction,a process which may be considered to be compactification on a torus and subsequent truncation to the zero-mass sector. The previously constructed gauged maximal supergravities, in four and seven dimensions, are derivable from the eleven-dimensional theory through compactification on the seven- and four-sphere, respectively. As we shall see, the gauged N = 8 supergravity in five dimensions, constructed in Chapter 4, is derivable from compactification of the ten-dimensional N = 2b theory on the five-sphere.

The N = 2b theory is extremely interesting from a purely field-theoretical point of view; some of its unusual aspects will be discussed in Chapter 2. It is intimately related both to superstrings and to gauged supergravities. As we shall see in Chapter 4, an "analytic continuation" of the gauged supergravity resulting from compactification on S^5, in which the isometry group SO(6) of S^5 is replaced with the noncompact form $SO^*(6) \approx SU(3,1)$, has a truly remarkable property: it admits compactifications to flat four-dimensional Minkowski space, with gauge symmetry SU(3) × U(1) × U(1), global symmetry SU(2), and N = 2 supersymmetry spontaneously broken at any scale desirable. It would be extremely interesting to know whether the structure of the associated IIB superstring exhibits any phenomena analogous to the noncompact gaugings. It is hoped that a study of the superstring theory can provide a better geometrical understanding of this most interesting compactification.

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Chapter 2

Chiral N = 2 Supergravity in Ten Dimensions and its Spontaneous Compactification*

2.1 Chiral N = 2b supergravity

Much of the structure of the ten-dimensional supergravity theories could be inferred from the associated superstring theories, but the actual construction of the N = 2a and the N = 1 theories proceeded indirectly through dimensionally reducing the eleven-dimensional supergravity. This gave the N = 2a supergravity [3], which could be truncated to N = 1 supergravity coupled to one Maxwell multiplet [4]. The latter theory could either be truncated further to pure N = 1 supergravity, or extended by generalizing the gauge group [5].

The N = 2b theory was first formulated in the (noncovariant) light-cone gauge in terms of a single unconstrained superfield containing all the physical fields of the theory [6]. The field content could be read off

*The original material contained herein appears in Ref. [1], with a minor modification given in Ref. [2]. I acknowledge valuable discussions with Nick Warner.

from this superfield, and consisted of a graviton, a real four-index antisymmetric tensor gauge field A_{MNPQ}, a complex scalar B, a complex two-index antisymmetric tensor A_{MN}, a complex Weyl gravitino ψ_M and a complex Weyl spinor λ of opposite chirality. The four-index tensor has a five-index field strength which must be self-dual on shell. The gravitino, the spinor and the four-index tensor are all chiral fields contributing to gravitational anomalies, but for this particular field content all anomalies cancel [7]. (The N = 2a theory, being non-chiral, has no problem with gravitational anomalies.)

The construction of the theory beyond leading order in the gravitational coupling constant turned out to be impractical in the light-cone gauge. It would have been desirable to implement the standard Noether procedure to construct the theory in a Lorentz covariant formalism, but it proved to be quite difficult to obtain the self-duality condition for F_{MNPQR} from a manifestly covariant action principle [8]. Although the necessary techniques are believed to be available [9], they have not been applied successfully to the interacting theory.

The windfall for the construction of the N = 2b theory was the realization that the complete set of covariant field equations and supersymmetry transformation rules could be deduced by an indirect procedure [10]. (An alternative approach using superspace techniques is given in Ref. [11].) It is generic for a supersymmetric theory that in order for the commutator of two supersymmetries to close upon symmetries of the theory, the first-order equations of motion must be imposed. (In order for the supersymmetry to close off shell, one usually has to introduce auxiliary fields.) Since field equations vary into other field equations under supersymmetry, all of the supersymmetry transformations and equations of motion must be interrelated. For the N = 2b theory, an important ingredient for the analysis, besides these general principles, was the identification of a global SU(1,1) symmetry which acts nonlinearly upon the complex scalar field B, so that the scalar self-interaction could be identified as a nonlinear sigma model over the coset space SU(1,1)/U(1) [12]. This symmetry restricts the possible forms of the supersymmetry transformations sufficiently to eventually determine all transformation rules and field equations.

The resulting classical equations of motion, as derived in Ref. [10], are (to leading order in fermionic fields):

$$R_{MN} = P_M P_N^* + P_M^* P_N + \frac{1}{6} F^{PQRS}{}_M F_{PQRSN} + $$

$$+ \frac{1}{4}(G^{PQ}{}_{(M} G^*_{N)PQ} - \frac{1}{12} g_{MN} G^{PQR} G^*_{PQR}) \qquad (2.1.1)$$

$$F_{MNPQR} = \frac{1}{120} \varepsilon_{MNPQRSTUVW} F^{STUVW} \qquad (2.1.2)$$

$$(\nabla^P - iQ^P)\, G_{MNP} = P^P G^*_{MNP} - \frac{2}{3} i F_{MNPQR} G^{PQR} \qquad (2.1.3)$$

$$(\nabla^M - 2iQ^M)\, P_M = \frac{1}{24}\, G^{PQR} G_{PQR} \qquad (2.1.4)$$

$$\Gamma^{MNP} \tilde{\nabla}_N \psi_P = \frac{1}{2} \Gamma^N P_N \Gamma^M \lambda^* - \frac{1}{48} \Gamma^{NPQ} G^*_{NPQ} \Gamma^M \lambda \qquad (2.1.5)$$

$$\Gamma^M(\nabla_M - \frac{3}{2}iQ_M)\lambda = \frac{i}{240}\,\Gamma^{MNPQR}F_{MNPQR}\lambda, \qquad (2.1.6)$$

where the following definitions have been made:

$$f^2 \equiv \frac{1}{1 - B^* B} \qquad (2.1.7)$$

$$P_M \equiv f^2 \partial_M B \qquad (2.1.8)$$

$$Q_M \equiv f^2 \mathrm{Im}(B\partial_M B^*) \qquad (2.1.9)$$

$$F_{MNP} \equiv 3\,\partial_{[M}A_{NP]} \qquad (2.1.10)$$

$$G_{MNP} \equiv f(F_{MNP} - BF^*_{MNP}) \qquad (2.1.11)$$

$$F_{MNPQR} \equiv 5\,\partial_{[M}A_{NPQR]} - \frac{5}{4}\mathrm{Im}(A_{[MN}F^*_{PQR]}) \qquad (2.1.12)$$

$$\tilde{\nabla}_M \equiv \nabla_M - \frac{1}{2}iQ_M + \frac{i}{480}\Gamma^{NPQRS}F_{NPQRS}\Gamma_M. \qquad (2.1.13)$$

The supersymmetry transformations which transform (2.1.1-6) into one another, and for which the first-order field equations (2.1.2,5,6) are required for the superalgebra to close, are

$$\delta e_M{}^A = 2\,\mathrm{Re}\ \bar{\varepsilon}\Gamma^A\psi_M \qquad (2.1.14)$$

$$\delta\psi_M = \tilde{\nabla}_M\varepsilon - \frac{1}{96}(\Gamma_M{}^{NPQ} - 9\,\delta_M{}^N\Gamma^{PQ})\,G_{NPQ}\varepsilon^* \qquad (2.1.15)$$

$$\delta A_{MN} = f(\bar{\varepsilon}\Gamma_{MN}\lambda + 4\,\bar{\varepsilon}^*\Gamma_{[M}\psi_{N]} +$$

$$+ B\bar{\varepsilon}^*\Gamma_{MN}\lambda^* + 4\,\bar{\varepsilon}\Gamma_{[M}\psi^*_{N]}) \qquad (2.1.16)$$

$$\delta A_{MNPQ} = -2\,\mathrm{Im}\,\bar{\varepsilon}\Gamma_{[MNP}\psi_{Q]} + \frac{3}{4}\mathrm{Im}\,A^{*}_{[MN}\delta A_{PQ]} \qquad (2.1.17)$$

$$\delta\lambda = P_M\Gamma^M\varepsilon^* + \frac{1}{24}G_{MNP}\Gamma^{MNP}\varepsilon \qquad (2.1.18)$$

$$\delta B = -f^{-2}\bar{\varepsilon}^{*}\lambda. \qquad (2.1.19)$$

The physical scalar degrees of freedom are carried in the field strength P_M of (2.1.8). The composite vector field Q_M defined in (2.1.9) acts as a connection for the U(1) within the global SU(1,1) symmetry. The charges of the fields with respect to this composite symmetry, as read off the field equations, are 2 for the scalar B, $\frac{3}{2}$ for the spinor λ, 1 for the tensor A_{MN} and $\frac{1}{2}$ for the gravitino. The graviton and the four-index tensor are neutral.

2.2 Freund-Rubin type Solutions

As proposed in Ref. [10], we will take as an ansatz for the five-index field strength

$$F_{\mu\nu\rho\sigma\tau} = -e\,\varepsilon_{\mu\nu\rho\sigma\tau}\,, \quad F_{mnpqr} = e\,\varepsilon_{mnpqr} \tag{2.2.1}$$

(where the index sets M, μ, m run over 0-9, 0-4 and 5-9, respectively); this is analogous to that taken by Freund and Rubin [13] to give solutions to the eleven-dimensional supergravity. All field equations may then be satisfied by taking $B = 0$, $A_{MN} = 0$ and the Ricci curvatures

$$R_{\mu\nu} = -4\,e^2 g_{\mu\nu}\ , \quad R_{mn} = 4\,e^2 g_{mn}. \tag{2.2.2}$$

Thus the spacetime may be taken to be the direct product of a negatively curved five-dimensional Einstein spacetime (for convenience we will take the maximally symmetric case, anti-de Sitter space AdS^5 with radius e^{-1}) and a positively curved spacelike Einstein manifold M^5 with cosmological constant $4e^2$. Recall that we use signature $(-{+}{+}{+}{+})({+}{+}{+}{+}{+})$.

The degenerate case, $e = 0$, allows compactification to a Ricci flat spacetime Min^d (e.g., Minkowski space) for any dimension $d < 10$, by taking for the internal manifold a torus T^{10-d}. This procedure yields the maximal ungauged supergravity for d dimensions plus, as always, infinite towers of massive states for each field.

There are also compactifications with $e = 0$ for any Ricci-flat compact internal manifold, for example, K3 or a product of K3 with a torus.

The compactification on K3 × T^{6-d} × Min^d for $d \leq 6$ (discussed in [14] for $d = 6$) always yields a half-maximal ungauged supergravity coupled to matter. The recently discovered "Calabi-Yau" spaces [15] provide six-dimensional Ricci-flat internal manifolds leading to quarter-maximal supergravity theories in four dimensions, or less if products with tori are considered. Both K3 × K3 and the eight-dimensional generalizations of Calabi-Yau spaces will provide compactification to two spacetime dimensions.

As in Ref. [16] we restrict our search for five-dimensional positively curved compact Einstein manifolds to coset spaces, as a matter of convenience. The only candidates are of the form:

(a) S^5

(b) $\dfrac{SU(3)}{SO(3)_{maximal}}$

(c) $\dfrac{SU(2) \times SU(2) \times U(1)}{U(1) \times U(1)}$.

Case (a) is discussed in Chapter 3 and found to admit the full N = 8 supersymmetry. In five dimensions one may have N = 2,4,6,8 supersymmetry, where the N symplectic-Majorana spinorial supercharges transform under the invariance group USp(N). The massless sector can be truncated to a gauged N = 8 supergravity with gauge group SO(6) (see Section 2.4 below).

As usual for such compactifications, the supersymmetry content is given by the number of independent solutions ε of the equations

$$\delta_\varepsilon \psi_M = 0 \,, \quad \delta_\varepsilon \lambda = 0 \,, \tag{2.2.3}$$

where the variations under supersymmetry parametrized by the spinorial parameter ε are given in eqns. (2.1.15,18) and in general depend on the background. In other words, the unbroken symmetry is simply the stabilizer of the vacuum expectation value of the (fermionic) fields. It is unnecessary to check the variations of the bosonic fields, since such variations must contain fermionic factors which vanish in the vacuum. For the present ansatz, equation (2.2.3) for $\delta\lambda$ is trivially satisfied, and for the gravitino we are led to search for Killing spinors on the internal manifold, that is, spinors η satisfying

$$\tilde{\nabla}_m \eta \equiv (D_m - i\mu\Gamma_m)\eta = 0 \tag{2.2.3}$$

for a constant μ, which is a distance scale for the compactification. The integrability condition for the existence of Killing spinors is then

$$[\tilde{\nabla}_m, \tilde{\nabla}_n]\eta = \frac{1}{4}\, C_{mn}{}^{ab}\Gamma_{ab}\eta \;=\; 0 \tag{2.2.4}$$

where $C_{mn}{}^{ab}$ is the Weyl tensor on the internal manifold. Thus the Weyl operator considered as acting on the space of Spin(5) spinors must have zeroes.

The symmetric space (b) with SU(3) isometry is discussed in appendix A of [13], where its curvature tensor is given. A simple calculation shows that the space (b) has no solutions to the condition (2.2.5) and hence yields no supersymmetry.

The manifolds included in case (c) are all U(1) bundles over

$$\frac{SU(2) \times SU(2)}{U(1) \times U(1)} = S^2 \times S^2.$$

They have as covering space the coset spaces of the form

$$\frac{SU(2) \times SU(2)}{U(1)},$$

and hence we need only study these in detail. Let SU(2) × SU(2) be generated by σ_i, τ_I; then the generator of the U(1) in the denominator may be written as $p\sigma_3 + q\tau_3$ where p and q are integers determining the space T^{pq}, and may be taken to be relatively prime. The most general space of the form (c) is T^{pq} modulo a finite cyclic group $\mathbb{Z}_r$. Therefore define $T^{pqr} = T^{pq}/\mathbb{Z}_r$ for $r = 1,2, \cdots$.

The coset space directions may then be taken to be generated by σ_a, τ_A and $Z \equiv q\sigma_3 - p\tau_3$, where a, A run from 1 to 2. Allowing rescalings in the σ_a, τ_A, Z directions of α, β, γ, respectively, the results of Ref. [17] allow one to rewrite the Einstein condition $R_{mn} = -\Lambda g_{mn}$ in the algebraic form

$$\begin{aligned} \frac{4\Lambda}{\gamma^2}(p^2 + q^2) &= 2a\,(p^2 + q^2) - q^2 a^2 \\ &= 2b\,(p^2 + q^2) - p^2 b^2 \qquad (2.2.5) \\ &= q^2 a^2 + p^2 b^2, \end{aligned}$$

where $a \equiv \frac{\alpha^2}{\gamma^2}$, $b \equiv \frac{\beta^2}{\gamma^2}$. One finds for each pair (p^2, q^2) exactly one solution for a and b. For $p^2 = 0$, $q^2 = 1$ the solution has $a = 1$, $b = \frac{1}{2}$ (of course, interchanging p and q merely interchanges a and b), yielding the space

$$T^{01} = \frac{SU(2)}{U(1)} \times SU(2) = S^2 \times S^3$$

with isometry SU(2) × SU(2) × SU(2). All other T^{pq} have isometry group SU(2) × SU(2) × U(1), the "extra" U(1) coming from right multiplication by the coset generator Z (see ref. [17]).

The integrability condition (2.2.4) implies that supersymmetry is only possible for $p^2 = q^2 = 1$. In that case, $a = b = \frac{4}{3}$ and we can prove that there is exactly one complex Killing spinor (for any signs of p and q), giving N = 2 supersymmetry. This combines with the bosonic gauge symmetry SU(2) × SU(2) × U(1) and the five-dimensional anti-de Sitter group SO(4,2) ≈ SU(2,2) for a full symmetry of SU(2,2|2) × SU(2) × U(1).

2.3 Pope-Warner type Solutions

We have obtained a further type of solution in which the internal space M^5 is a U(1) bundle over a four-dimensional Kähler space M^4. Kähler spaces have been extensively studied in both the physics and the mathematics literature (see for example [18,19] and references therein); we will review here only those properties relevant for our solutions.

In terms of an orthonormal vierbein $\bar{e}^i$ for M^4 (i runs from 1 to 4; we use signature (+ + + +)), the Kähler form $J_{ij} = J_{[ij]}$ satisfies

$$J_i{}^j J_j{}^k = -\delta_i{}^k \quad , \quad \bar{D}_i J_j{}^k = 0, \tag{2.3.1}$$

where barred objects refer to the intrinsic geometry of M^4. We may take J_{ij} to be self-dual, i.e., $J_{ij} = \frac{1}{2}\varepsilon_{ijkl} J^{kl}$. The Ricci two-form is given by

$$P_{ij} = \frac{1}{2} \bar{R}_{ij}{}^{kl} J_{kl} = \bar{R}_i{}^k{}_j{}^l J_{kl}. \tag{2.3.2}$$

From eqn. (2.3.1) , $[\bar{D}_i, \bar{D}_j] J_{kl} = 0$ and we have

$$P_{ij} = J_{ik} \bar{R}_{kj} = \bar{R}_{ik} J_{kj}, \tag{2.3.3}$$

from which we see that P_{ij} is proportional to J_{ij} if and only if the metric on M^4 is Einstein. Taking the curl of (2.3.2), a Bianchi identity for $\bar{R}_{ijkl}$ gives $\bar{D}_{[i} P_{jk]} = 0$; hence we can write locally $P_{ij} = 2\bar{D}_{[i} A_{j]}$.

Every Kähler manifold possesses a gauge-covariantly constant spinor η satisfying

$$D_i \eta \equiv (\bar{D}_i - i\mu A_i)\eta = 0, \qquad (2.3.4)$$

and hence the integrability condition

$$[D_i, D_j]\eta = (\tfrac{1}{4}\bar{R}_{ij}{}^{kl}\,\Gamma_{kl} - i\mu P_{ij})\eta = 0. \qquad (2.3.5)$$

The matrices Γ_i satisfy $\{\Gamma_i, \Gamma_j\} = 2g_{ij}$, and we define $\gamma_5 \equiv \frac{1}{24}\,\varepsilon_{ijkl}\,\Gamma^{ijkl}$. There is a charge conjugation matrix C with

$$C\,\Gamma_i C^{-1} = \Gamma_i{}^T = -\,\Gamma_i{}^* \ , \ C^T = -\,C. \qquad (2.3.6)$$

Contracting eqn. (2.3.5) with Γ^{ij} and J^{ij} gives $\mu^2 = \frac{1}{4}$. Taking $\mu = \frac{1}{2}$ for η, we see that the charge conjugate spinor $\chi \equiv C\eta^*$ satisfies eq. (2.3.4) with $\mu = -\,\frac{1}{2}$. Both spinors have negative chirality:

$$\gamma_5 \eta = -\,\eta \ , \ \gamma_5 \chi = -\,\chi. \qquad (2.3.7)$$

Contracting eq. (2.3.5) with Γ^j gives (when $\det \bar{R}_{ij} \neq 0$)

$$J^{ij}\Gamma_j \eta = i\Gamma^i \eta \quad , \quad J^{ij}\Gamma_j \chi = -\,i\Gamma^i \chi. \qquad (2.3.8)$$

Choosing the normalization $\eta^\dagger \eta = 1 = \chi^\dagger \chi$, we may identify

$$J_{ij} = -\,i\eta^\dagger \Gamma_{ij}\eta = i\chi^\dagger \Gamma_{ij}\chi. \qquad (2.3.9)$$

Furthermore, the complex two-form defined by

$$K_{ij} \equiv \chi^\dagger \Gamma_{ij}\eta \quad , \quad K_{ij}^* = -\,\eta^\dagger \Gamma_{ij}\chi \qquad (2.3.10)$$

satisfies

$$\overline{D}_i K_{jk} = iA_i K_{jk}\ , \quad J_i^{\ k} K_{kj} = iK_{ij}\ , \quad K_{ij} = \frac{1}{2}\varepsilon_{ijkl}K^{kl}, \qquad (2.3.11)$$

and

$$K_{ik}K_j^{\ k} = 0\ , \quad K_{ik}K_j^{*\,k} = 2(g_{ij} - iJ_{ij}). \qquad (2.3.12)$$

We have made use of the Fierz identity

$$\Gamma^i \eta_1 \varepsilon^\dagger \Gamma_i \eta_2 = -\,\Gamma^i \eta_2 \varepsilon^\dagger \Gamma_i \eta_1 \qquad (2.3.13)$$

for commuting spinors η_1, η_2 of the same chirality, and an arbitrary spinor ε.

We now construct our space M^5, a U(1) bundle over M^4, by taking for the metric

$$ds^2 = \eta_{ab}\, e^a e^b \qquad a,b = 1,\ldots,5\ ,\ \eta_{ab} = (+++++) \qquad (2.3.14)$$

where the orthonormal fünfbein is taken to be $e^a = (\overline{e}^i\ ,\ e^5)$ with $e^5 = c\,(d\tau - A_i\overline{e}^i)$ for constant c. The Ricci tensor for M^5 is then

$$R_{ij} = \overline{R}_{ij} + \frac{1}{2}c^2 \overline{R}_i^{\ k}\, \overline{R}_{jk}$$

$$R_{55} = \frac{1}{4}c^2 \overline{R}^{ij}\, \overline{R}_{ij} \qquad (2.3.15)$$

$$R_{i5} = \frac{1}{4}c\ J_i^{\ j}\partial_j \overline{R}.$$

We will take as an ansatz for the complex three-index field strength

$$G_{ijk} = 0 \quad , \quad G_{ij5} = \alpha\, e^{i\beta\tau} K_{ij}. \tag{2.3.16}$$

G_{abc} then satisfies the Bianchi identity for the ten-dimensional field strength if we take $\beta = -1$. The two-index potential which gives (2.3.16) is then

$$A_{ij} = i\alpha\, e^{-i\tau} K_{ij} \quad , \quad A_{i5} = 0. \tag{2.3.17}$$

Note that we may take α real since any phase may be absorbed into a shift of τ.

For the five-index field strength we use the ansatz (2.2.1) with the parameter e to be determined. This satisfies the self-duality condition (2.1.2) and is consistent with the appropriate Bianchi identity, since the definition (2.1.12) relates the curl of A_{mnpq} to

$$F_{abcde} + \frac{5}{4}\,\mathrm{Im}\, A_{[ab} F^{*}_{cde]} = (e + \frac{1}{2}\alpha^2)\, \varepsilon_{abcde} \tag{2.3.18}$$

and the curl of this expression manifestly vanishes.

The scalar B has been set to zero, which is consistent with its field equation (2.1.4) since $G^{abc} G_{abc} = 0$. The remaining field equations are then the Einstein equation (2.1.1):

$$R_{ij} = (4e^2 + \frac{1}{2}\alpha^2) g_{ij}$$

$$R_{55} = 4e^2 + \frac{3}{2}\alpha^2$$

$$R_{i5} = 0 \tag{2.3.19}$$

$$R_{\mu\nu} = -(4e^2 + \frac{1}{2}\alpha^2)g_{\mu\nu}$$

and from eqn. (2.1.3):

$$-\frac{i}{c}\, G_{ij5} = 4ie\, G_{ij5}. \tag{2.3.20}$$

For $\alpha = 0$ we of course recover the usual Freund-Rubin type solutions. For $\alpha \neq 0$, (2.3.20) gives $ec = -\frac{1}{4}$.

Since M^4 is Kähler, the eigenvalues of $\bar{R}_{ij}$ must come in two pairs, and we may write

$$\bar{R}_{ij} = \mathrm{diag}(\mu, \mu, \lambda, \lambda), \tag{2.3.21}$$

for a suitable choice of basis. Then eqs. (2.3.15) along with (2.3.19) give

$$\begin{aligned} \mu - \frac{1}{2}c^2\mu^2 \ &= \ \lambda - \frac{1}{2}c^2\lambda^2 \ = \ 4e^2 + \frac{1}{2}\alpha^2 \\ \frac{1}{2}c^2(\mu^2 + \lambda^2) \ &= \ 4e^2 + \frac{3}{2}\alpha^2 \\ \partial_i(\mu + \lambda) \ &= \ 0. \end{aligned} \tag{2.3.22}$$

This system of equations, given $e = -\frac{1}{4c}$, has exactly one solution: $\mu = \lambda = 2\alpha^2 = c^{-2}$, so M^4 must be Einstein. Thus, for a given size of M^4, the solution with $\alpha \neq 0$ has its U(1) fibers stretched by a factor $\frac{\sqrt{3}}{2}$ with respect to the solution with $\alpha = 0$ and M^5 Einstein, which has

$\mu = \lambda = \frac{2}{3}c^{-2}$.

There are only two known four-dimensional Kähler-Einstein spaces with positive curvature: $S^2 \times S^2$ with spheres of equal size, and $\mathbb{CP}^2$. Because G_{abc} is charged with respect to the U(1) of the bundle, the bosonic gauge symmetry of a solution with $\alpha \neq 0$ is broken from the isometry group of M^5 to that of M^4 (SU(2) × SU(2) and SU(3), respectively). These solutions break all supersymmetry as may be seen by explicitly verifying that $\delta\lambda = 0$, $\delta\psi_M = 0$ have no solutions $\varepsilon(x^\mu, y^m)$.

On the other hand, given a Kähler-Einstein space M^4, the U(1) bundle M^5 over M^4 with the Einstein metric (and hence $\alpha = 0$) always yields at least N = 2 supersymmetry. It is easy to show that $e^{-i\tau/2}\eta$ (or $e^{-i\tau/2}\chi$, depending on the orientation of M^5) is a Killing spinor on M^5. In particular cases, M^5 may have more supersymmetry: for $M^4 = \mathbb{CP}^2$, M^5 is S^5 and there are eight supersymmetries; for M^5 = K3 × U(1) there are four. For $M^4 = S^2 \times S^2$, we recover for M^5 the N = 2 supersymmetric coset space T^{11} discussed in Section 1.3.

The solutions discussed here preserve the noncompact global SU(1,1) symmetry of the ten-dimensional theory. Since the field strength G_{abc} is charged with respect to the original U(1) of the ten-dimensional theory as well as the U(1) of the fiber, neither of these can survive as a symmetry of the compactification. However, some linear combination of the two U(1)'s must leave the expectation value of G_{abc} fixed. This combination generates the U(1) subgroup of the SU(1,1) symmetry.

2.4 Discussion

The ten-dimensional chiral N = 2 supergravity has been found to admit a rich variety of compactifications to five dimensions. If there should exist any further Kähler-Einstein spaces M^4 with positive curvature, the construction given in section 2.3 will give both a Freund-Rubin solution with at least N = 2 supersymmetry, and a Pope-Warner solution (with N = 0 supersymmetry, as always) for each such space. The problem of classifying such manifolds M^4, or Einstein manifolds M^5, in general, is an open question; the classification here is complete if one restricts oneself to coset spaces.

As stated in Section 2.2, K3 × U(1) gives a compactification with N = 4 supersymmetry. One can construct a number of nontrivial U(1) bundles over K3; unfortunately, the methods given here do not produce any new solutions of either the Freund-Rubin or the Pope-Warner type.

The present solution with SU(3) symmetry on a stretched five-sphere is quite analogous to that found for eleven-dimensional supergravity on a stretched seven-sphere [20]. In addition, there is another type of solution that the ten- and eleven-dimensional theories have in common, namely the de Wit-Nicolai type solutions, in which the space-time metric is given an internal space dependent Weyl rescaling [21,22]. Both of these solutions for the eleven-dimensional theory appear to correspond to particular extrema of the scalar potential of gauged N = 8 d = 4 supergravity (see [23] and references therein for discussion). It is natural to believe that similar relations should exist between the ten-and five-dimensional theories; in fact, this conjecture was made in reference [22] for the solution found there.

More precisely, the issue is whether the scalar potential of the $N = 8$ supergravity in five dimensions with non-Abelian gauge symmetry SO(6) (discussed in Chapter 4) possesses extrema for which the supersymmetry is completely broken and the gauge symmetry is broken to SU(3) and SO(5). Since this is indeed the case, it is natural to conclude that the extrema correspond to the Pope-Warner and de Wit-Nicolai type solutions of the $d = 10$ theory, respectively.

The question of the precise relationship between a higher-dimensional theory and lower-dimensional counterparts, especially in the context of spontaneously compactified supergravity, is as fascinating as it is difficult. Eleven-dimensional supergravity admits geometrically nontrivial solutions in which it appears that the values of all fields may lie within the restriction of the full theory corresponding to a complete interacting theory of the massless supermultiplet. If this is indeed the case, as all available evidence seems to indicate, an understanding of this phenomenon would contribute a great deal to our knowledge of the dynamics of compactification and the structure of extended supergravities in general. It is fortunate that such a qualitatively different theory as the ten-dimensional supergravity provides another laboratory for the investigation of these issues.

Of course, it would be interesting to have compactifications of the ten-dimensional theory to four dimensions, for comparison with phenomenology. As we shall see in Chapter 4, certain gaugings of N = 8 supergravity indirectly, but quite closely, related to the compactification on S^5 offer very exciting phenomenological prospects. The present chapter does not appear to be directly applicable to such goals, but one should remember that there are several lines of investigation that have gone practically unexplored. For example, a nontrivial interaction with the scalar field might drop a five-dimensional compactification to four dimensions, or give completely new compactifications. In addition, if one takes seriously the full superstring of which this theory is only a particular limiting case (an option which may be forced if one is to obtain a consistent quantum theory), the possibilities for compactification may be substantially different.

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Chapter 3

The Spectrum of Chiral N = 2 d = 10 Supergravity Compactified on the Five-sphere*

3.1 Introduction

This chapter will be concerned with studying in some detail one of the compactifications of the chiral N = 2b supergravity discussed in Chapter 1, the Freund-Rubin type compactification in which the background geometry describes the product of a five-sphere S^5 and five-dimensional anti-de Sitter spacetime AdS^5. Recall that for this solution the five-index field strength is given the expectation value

$$F_{\mu\nu\rho\sigma\tau} = -e\,\varepsilon_{\mu\nu\rho\sigma\tau}\,, \quad F_{mnpqr} = e\,\varepsilon_{mnpqr} \qquad (3.1.1)$$

where the parameter e is an arbitrary overall mass scale for the compactification. Assuming that only the four-index tensor and the metric are nonvanishing in the background, the Einstein equations read

*Most of the material contained herein appears in Ref. [1]. We acknowledge an illuminating discussion with Krzysztof Pilch.

$$R_{\mu\nu} = -4e^2 g_{\mu\nu}, \quad R_{mn} = 4e^2 g_{mn} \tag{3.1.2}$$

while all other field equations are automatically satisfied. Here we are interested in the maximally symmetric solution to (3.1.2), in which $g_{\mu\nu}$ describes AdS^5 and g_{mn} describes S^5, both of radius e^{-1}. The Riemann curvatures are given by

$$R_{\mu\nu\rho\sigma} = -e^2(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{3.1.3a}$$

$$R_{mnpq} = e^2(g_{mp}g_{nq} - g_{mq}g_{np}). \tag{3.1.3b}$$

We will be concerned with finding the spectrum of masses for the effective theory defined in the five-dimensional spacetime. In general, a theory in a spacetime background which factorizes into the product of a lower-dimensional spacetime and a compact spacelike manifold M may be considered as a field theory on the lower-dimensional spacetime. The dependence of each of the original fields upon the coordinates of M may be expressed in a harmonic expansion, the coefficients providing the propagating modes on the lower-dimensional spacetime.

A simple example of this mechanism is provided by a complex scalar Φ, which satisfies the field equation

$$\Box\Phi(x,y) = 0. \tag{3.1.4}$$

We will assume that the field theory is formulated on a d-dimensional spacetime which is the direct product of a $(d-1)$-dimensional spacetime described by coordinates x, and a circle of radius e^{-1} with coordinate y

$(0 < y < 2\pi e^{-1})$. The harmonic expansion of Φ on the circle reads

$$\Phi(x,y) = \Sigma \varphi^k(x)\, Y^k(y)\,, \quad Y^k(y) \equiv \exp(ieky) \qquad (3.1.5)$$

where $k = 0, \pm 1, \pm 2, \ldots$ Substituting the expansion (3.1.5) into the field equation (3.1.4) yields field equations for an infinite tower of modes propagating in the $(d-1)$-dimensional spacetime:

$$(\Box_x + \frac{\partial^2}{\partial y^2})\varphi^k(x)\, Y^k(y) = \left[(\Box_x - k^2 e^2)\varphi^k(x) \right] Y^k(y) = 0. \qquad (3.1.6)$$

The spectrum is discrete with masses consistent with unitarity because the internal manifold (the circle, in this case) is compact and spacelike.

The computations in this chapter are essentially more complicated versions of the foregoing. Both bosonic and fermionic fields may be expanded in harmonics of the relevant differential operators on the five-sphere S^5; for convenience bose and fermi fields are handled separately in Sections 2.2 and 2.3. Much of the work consists of properly diagonalizing coupled modes of various spin. Sensible gauge choices for the gauge fields of the original theory simplify much of the analysis, but sometimes such choices require great care to implement, as we shall see.

The model raises several interesting questions. A sector of the compactification is expected to contain a maximal (N = 8) supergravity with the non-Abelian isometry group SO(6) of S^5 gauged by fifteen vectors. This is in analogy to the compactification of eleven-dimensional supergravity on S^7, which yields an N = 8 supergravity in four dimensions with gauge group SO(8) [2]. In the latter case, the gauged theory contains 28 vectors, just enough to gauge SO(8), and the gauging could be done by simply adding terms of order g to the Lagrangian and transformation rules, and a scalar potential of order g^2 to the Lagrangian [3]. The ungauged maximal supergravity in five dimensions [4] has 27 vectors, however, 12 more than necessary to gauge SO(6). As we shall see, there are no vectors arising from the compactification which could correspond to these extra twelve vectors. Rather, they must be replaced by a complex sextet of two-index antisymmetric tensor fields satisfying self-dual field equations of the type discussed in [5].

Based in part upon the work here, the gauged N = 8 supergravity in five dimensions with Yang-Mills group SO(6) has been constructed, along with associated noncompact gaugings; the construction is presented in Chapter 4.

Another question concerns the four-dimensional singleton [6] and the seven-dimensional doubleton [7] supermultiplets. These multiplets are ultra-short oscillator-like unitary irreducible representations of the pertinent superalgebras. Although there are no corresponding propagating modes in the compactified spectrum, they may be identified with geometrically interesting sets of gauge modes. In fact, a similar phenomenon occurs for the five-dimensional doubleton [8] representation.

Perhaps most interesting is the question of masslessness. We recall that in four dimensions the scalars and spinors in the "massless" supermultiplet (i.e., the supermultiplet corresponding to the gauged $N = 8$ supergravity, containing all the gauge fields) satisfy a conformally invariant field equation. This was considered hardly surprising, since massless particles should propagate on the light cone. Furthermore, being in a supermultiplet with so many clearly massless particles (namely, particles possessing gauge invariances) would lead one to expect a particle to be "massless," for any reasonable definition of the word.

In seven dimensions, however, it was found in several models [9,10] that the scalars in the same supermultiplet as the massless graviton do not have a conformal field equation, although for several models these field equations were the same (to linear order). Thus one seemed to have a choice of definition of masslessness for scalars: either the scalar had a conformally invariant field equation, or was related by supersymmetry to a field guaranteed to be massless by gauge invariance. The same issues hold for spinor fields.

In five dimensions we find a further surprise: scalars and spinors in different SO(6) representations, but all within the massless supermultiplet, have different mass terms! Thus supersymmetry seems to have no relation to any inherent quality of masslessness for scalars and spinors. The relationship in four dimensions appears to be quite special.

Closely related to the doubleton issue is a subtlety concerning the removal of modes by fixing the general coordinate gauges. After imposing de Donder-like gauge choices, one is still left with a residual gauge symmetry whose spherical harmonics are conformal scalars [11,12] (scalar harmonics Y on spheres for which $D_m D_n Y$ is proportional to $g_{mn} Y$). In the sector of these conformal scalars one can either algebraically eliminate certain nonpropagating modes, or remove them by a conformal gauge choice. Counting degrees of freedom seems to present a problem, because in the latter procedure one naïvely ends up with more modes than in the former. The resolution, as we shall see, is that one must use the gauge freedom to eliminate twice as many modes as there are gauge parameters, as in the case of electromagnetism.

Our work makes contact with the group-theoretical analysis of Ref. [8] insofar as the entire set of modes we find fit precisely into the supermultiplets constructed there. In addition, we have obtained the spectrum of masses.

3.2 Bosonic modes

In this section we will determine the bosonic modes for the compactification of N = 2b supergravity on S^5. The procedure will be to linearize the field equations in terms of excitations about the background configuration given by eqns. (3.1.1-3b). Diagonalization of the resulting system will then give the spectrum of bosonic modes.

The ten-dimensional theory possesses a number of gauge invariances, many of which may be fixed for easier identification of the propagating modes. This is purely for convenience (the gauge modes would decouple otherwise) but imposing a particular set of covariant gauge conditions will considerably simplify the analysis. As we will see, the most natural covariant gauge choices will leave unfixed a particular set of pure gauge modes which may be identified with the bosonic part of the "doubleton" representation of U(2,2|4) [8].

The bosonic field equations, to first order in excitations, are

$$D^M \partial_M B = 0 \tag{3.2.1}$$

$$D^M \partial_{[M} A_{NP]} + \frac{2}{3} i \bar{F}_{NPQRS} D^Q A^{RS} = 0 \tag{3.2.2}$$

$$F_{MNPQR} = \frac{1}{120} \varepsilon_{MNPQRSTUVW} F^{STUVW} \tag{3.2.3}$$

$$R_{MN} - \frac{1}{6} F_{MPQRS} F_N{}^{PQRS} = 0 . \tag{3.2.4}$$

Raising and lowering of indices is done with respect to the background metric $\bar{g}_{MN}$. All covariant derivatives are with respect to the background spacetime geometry; the U(1) connection Q_M and its first variation vanish in the background. To leading order in fluctuations,

$$R_{MN} = -\frac{1}{2}(D_P D^P h_{MN} + D_M D_N h_P{}^P) + D_P D_{(M} h_{N)}{}^P \qquad (3.2.5)$$

$$F_{MNPQR} = \bar{F}_{MNPQR} + 5\,\partial_{[M} a_{NPQR]} \qquad (3.2.6)$$

where $g_{MN} \equiv \bar{g}_{MN} + h_{MN}$ and $A_{MNPQ} \equiv \bar{A}_{MNPQ} + a_{MNPQ}$. The internal graviton may be rewritten in terms of its traceless component plus a trace part:

$$h_{mn} \equiv H_{mn} + \frac{1}{5}\bar{g}_{mn}\pi\,, \quad \text{where} \quad H_m{}^m = 0 \qquad (3.2.7)$$

and

$$\pi \equiv h_m{}^m. \qquad (3.2.8)$$

The propagating spacetime graviton is related to $h_{\mu\nu}$ by a Weyl rescaling, so anticipating the necessary field redefinition we define

$$H_{\mu\nu} \equiv h_{\mu\nu} + \frac{1}{3}\bar{g}_{\mu\nu}\pi\,, \qquad (3.2.9)$$

to give the linearized Weyl shift. (For d spacetime dimensions the coefficient $\frac{1}{3}$ is replaced by $\frac{1}{(d-2)}$.) Furthermore, we label the spacetime graviton trace

$$\varphi \equiv H_\mu{}^\mu. \qquad (3.2.10)$$

Inserting eqns. (3.2.5-10) into the Einstein equation (3.2.4) gives the set of three field equations

$$\frac{1}{2}(\Box_x + \Box_y + 2e^2)H_{\mu\nu} + 3e^2\bar{g}_{\mu\nu}\varphi - D_{(\mu}D^{\rho}H_{\nu)\rho} +$$

$$+ \frac{1}{2}D_{\mu}D_{\nu}\varphi - D_{(\mu}D^{p}h_{\nu)p} - \frac{1}{6}\bar{g}_{\mu\nu}(\Box_x + \Box_y - \frac{16}{3}e^2)\pi$$

$$+ \frac{1}{3}e\,\bar{g}_{\mu\nu}\varepsilon^{\rho\sigma\tau\kappa\lambda}\partial_{\rho}a_{\sigma\tau\kappa\lambda} = 0 \qquad (3.2.11)$$

$$\frac{1}{2}(\Box_x + \Box_y)h_{\mu n} - \frac{1}{2}D_{\mu}D^{p}h_{np} - \frac{1}{2}D_nD^{\rho}H_{\mu\rho} + D_{\mu}D_n(\frac{1}{2}\varphi - \frac{4}{15}\pi) -$$

$$- \frac{1}{2}D_{\mu}D^{p}H_{np} - \frac{1}{2}D_nD^{p}h_{\mu p} + \frac{1}{6}e\,\varepsilon_{\mu}{}^{\nu\rho\sigma\tau}(\partial_n a_{\nu\rho\sigma\tau} - 4\,\partial_{\nu}a_{n\rho\sigma\tau})$$

$$+ \frac{1}{6}e\,\varepsilon_n{}^{pqrs}(\partial_{\mu}a_{pqrs} - 4\,\partial_p a_{\mu qrs}) = 0 \qquad (3.2.12)$$

$$\frac{1}{2}(\Box_x + \Box_y - 2e^2)H_{mn} + \frac{1}{10}\bar{g}_{mn}(\Box_x + \Box_y - 32e^2)\pi + D_mD_n(\frac{1}{2}\varphi - \frac{8}{15}\pi) -$$

$$- D_{(m}D^{\mu}h_{n)\mu} - D_{(m}D^{p}H_{n)p} + \frac{1}{3}e\,\bar{g}_{mn}\varepsilon^{pqrst}\partial_p a_{qrst} = 0, \quad (3.2.13)$$

where $\Box_x \equiv D_{\mu}D^{\mu}$ and $\Box_y \equiv D_mD^m$. These equations are coupled to the field equations resulting from the self-duality condition (3.2.3), which are given by

$$5\,\partial_{[\mu}a_{\nu\rho\sigma\tau]} = \frac{5}{120}\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon^{mnpqr}\partial_m a_{npqr} + e(\frac{1}{2}\varphi - \frac{4}{3}\pi)\varepsilon_{\mu\nu\rho\sigma\tau} \qquad (3.2.14)$$

$$4\,\partial_{[\mu}a_{\nu\rho\sigma]m} + \partial_m a_{\mu\nu\rho\sigma} =$$

$$= -\frac{5}{120}\varepsilon_{\mu\nu\rho\sigma}{}^{\tau}\varepsilon_m{}^{npqr}(\partial_{\tau}a_{npqr} + 4\,\partial_n a_{pqr\tau}) + e\,\varepsilon_{\mu\nu\rho\sigma}{}^{\tau}h_{m\tau} \quad (3.2.15)$$

$3\,\partial_{[\mu}a_{\nu\rho]mn} + 2\,\partial_{[m}a_{n]\mu\nu\rho} =$

$$= \frac{10}{120}\varepsilon_{\mu\nu\rho}{}^{\sigma\tau}\varepsilon_{mn}{}^{pqr}(2\,\partial_{\sigma}a_{\tau pqr} + 3\,\partial_{p}a_{qr\sigma\tau}). \qquad (3.2.16)$$

The other three components of eqn. (3.2.3), corresponding to $F_{\mu\nu mnp}$, $F_{\mu mnpq}$ and F_{mnpqr}, are simply the duals of eqns. (3.2.14-16).

On the n-dimensional sphere with radius e^{-1} the scalar harmonics $Y^{i_s}(y)$ are defined for the index $k = i_s = 0,1,2,\ldots$ (The label i_s means i_{scalar}; the label k is a conventional designation.) The Y^{i_s} are eigenfunctions of the Laplacian $\Box_y$ with eigenvalue $-Ke^2$, where we define $K \equiv k(k+n-1)$. For $k = 0$, Y is constant, and for $k = 1$ the Y^i are the conformal scalars, transforming in the $(n+1)$-dimensional vector representation of SO(n+1) and satisfying

$$D_m D_n Y^i = -e^2\bar{g}_{mn}Y^i, \quad \text{so} \quad D_{(m}D_{n)|}Y^i = 0. \qquad (3.2.17)$$

The symbol $(mn)|$ means that this index pair is symmetrized with trace removed; that is, $X_{(mn)|} \equiv \frac{1}{2}(X_{mn} + X_{nm}) - \frac{1}{5}g_{mn}X_p{}^p$. For S^5 the Y^{i_s} transform in the (0,k,0) representation of SO(6) ≈ SU(4), in the Dynkin notation.

The transverse vector spherical harmonics $Y_m^{i_v}$ are defined for $k = 1,2,3,\ldots$ For $k = 1$, the Y_m^i are the Killing vectors satisfying $D_{(m}Y_{n)}^i = 0$ and transforming in the adjoint of SO(n+1). The vector harmonics are eigenfunctions of the Laplacian with eigenvalue $(1-K)e^2$ and on S^5 transform in the $(1,k-1,1)$ representation of SO(6).

The transverse traceless tensor spherical harmonics $Y^{i_t}_{mn} = Y^{i_t}_{(mn)|}$, defined for $k = 2,3,4,...$, are also eigenfunctions of $\Box_y$, with eigenvalues $(2-K)e^2$. On S^5 they transform in the $(2,k-2,2)$ representation of SO(6).

Expanding $H_{\mu\nu}$, $h_{\mu n}$, H_{mn}, π and $\varphi(x,y)$ into a complete set of spherical harmonics on S^5 we have

$$H_{\mu\nu}(x,y) = \Sigma \varphi^{i_s}_{\mu\nu}(x) Y^{i_s}(y) \tag{3.2.18}$$

$$h_{\mu n}(x,y) = \Sigma B^{i_v}_{\mu}(x) Y^{i_v}_{n}(y) + \Sigma B^{i_s}_{\mu}(x) D_n Y^{i_s}(y) \tag{3.2.19}$$

$$H_{mn}(x,y) = \Sigma H^{i_t}(x) Y^{i_t}_{mn}(y) + \Sigma H^{i_v}(x) D_{(m} Y^{i_v}_{n)|}(y) +$$

$$+ \Sigma H^{i_s}(x) D_{(m} D_{n)|} Y^{i_s}(y) \tag{3.2.20}$$

$$\pi(x,y) = \Sigma \pi^{i_s}(x) Y^{i_s}(y) \tag{3.2.21}$$

$$\varphi(x,y) = \Sigma \varphi^{i_s}(x) Y^{i_s}(y) \tag{3.2.22}$$

where of course $\varphi^{i_s} = \bar{g}^{\mu\nu} \varphi^{i_s}_{\mu\nu}$ follows directly from the definition (3.2.10).

We impose the following internal de Donder and Lorentz-type gauge conditions

$$D^m h_{(mn)|} = 0 , \quad D^m h_{m\mu} = 0 . \tag{3.2.23}$$

Under diffeomorphisms, one has $\delta h_{MN} = 2 D_{(M}\xi_{N)}$ and by expanding ξ_M into spherical harmonics it becomes clear that one can gauge away all x-space fields which correspond to gradients of spherical harmonics in eqns. (3.2.19,20). This yields

$$h_{\mu n}(x,y) \;=\; \Sigma B^{i_v}_{\mu}(x)\, Y^{i_v}_{n}(y) \qquad (3.2.24)$$

$$H_{mn}(x,y) \;=\; \Sigma H^{i_t}(x)\, Y^{i_t}_{mn}(y). \qquad (3.2.25)$$

Those diffeomorphisms which respect the conditions (3.2.23) are given by

$$D_{(m}\xi_{n)|} \;=\; 0 \quad \text{and} \quad D^{m}(D_m\xi_\mu + D_\mu\xi_m) \;=\; 0. \qquad (3.2.26)$$

They consist of (i) ordinary SO(6) Yang-Mills symmetries for which $\xi_m = \Lambda^i(x)\, Y^i_m(y)$ where Y^i_m are the Killing vectors on S^5, (ii) ordinary five-dimensional spacetime diffeomorphisms for which $\xi_\mu = \xi_\mu(x)$, and finally (iii) what have been called conformal diffeomorphisms [11,12]:

$$\xi_m \;=\; \lambda^i(x) D_m Y^i(y)\,, \quad \xi_\mu \;=\; -\partial_\mu\lambda^i(x)\, Y^i(y) \qquad (3.2.27)$$

where $Y^i(y)$ are the $k = 1$ conformal scalars transforming in the six-dimensional vector representation of SO(6). These three classes of diffeomorphisms also respect the form of the expansion (3.2.18,21,22,24,25) although the x-space coefficient fields will in general be transformed. The appearance of the extra conformal diffeomorphisms is not surprising, because in eqn. (3.2.23) the terms with $D_{(m}D_{n)|}Y^{i_s}$ cancel when Y^{i_s} is a $k = 1$ scalar harmonic, so that no gauge parameter has been fixed to eliminate these modes. We shall come back to the role of these conformal diffeomorphisms later, and use the unfixed gauge freedom.

We now repeat the foregoing analysis for the antisymmetric tensor fluctuations $a_{MNPQ} = A_{MNPQ} - \bar{A}_{MNPQ}$. Taking the internal Lorentz-type gauge conditions

$$D^m a_{mnpq} = D^m a_{mnp\mu} = D^m a_{mn\mu\nu} = D^m a_{m\mu\nu\rho} = 0 \qquad (3.2.28)$$

again removes terms with gradients from the harmonic expansions for the various fluctuations. One ends up with the expressions

$$a_{\mu\nu\rho\sigma}(x,y) = e^{-1}\Sigma\, b^{i_s}_{\mu\nu\rho\sigma}(x)\, Y^{i_s}(y) \qquad (3.2.29a)$$

$$a_{\mu\nu\rho m}(x,y) = e^{-1}\Sigma\, b^{i_v}_{\mu\nu\rho}(x)\, Y^{i_v}_m(y) \qquad (3.2.29b)$$

$$a_{\mu\nu mn}(x,y) = e^{-1}\Sigma\, b^{i_2}_{\mu\nu}(x)\, Y^{i_2}_{mn}(y) \qquad (3.2.29c)$$

$$a_{\mu mnp}(x,y) = e^{-1}\Sigma\, b^{i_v}_{\mu}(x)\, Y^{i_v}_{mnp}(y) \qquad (3.2.29d)$$

$$a_{mnpq}(x,y) = e^{-1}\Sigma\, b^{i_s}\, Y^{i_s}_{mnpq} \qquad (3.2.29e)$$

where the factors of e^{-1} are for future notational convenience.

On S^n, the transverse antisymmetric tensor harmonics with m indices ($0 < m < n$) are defined for the index $k = 1,2,3,\ldots$ and are eigenfunctions of $\Box_y$ with eigenvalue $(m-K)e^2$, where as before $K \equiv k(k+n-1)$. On S^5 we can take

$$Y^{i_v}_{mnp} \equiv \varepsilon_{mnp}{}^{qr} D_q Y^{i_v}_r \qquad (3.2.30)$$

$$Y^{i_s}_{mnpq} \equiv \varepsilon_{mnpq}{}^{r} D_r Y^{i_s}. \qquad (3.2.31)$$

The two-index antisymmetric tensor harmonics may be taken to satisfy first-order self-duality type equations

$$\varepsilon_{mn}{}^{pqr} D_p Y^{i_2}_{qr} = \beta Y^{i_2}_{mn} \tag{3.2.32}$$

where for consistency with the second-order equation $\Box_y Y^{i_2}_{mn} = (2-K)e^2 Y^{i_2}_{mn}$ the constant β must be $\pm 2ie(k+2)$. The harmonics $Y^{i_2}_{mn}$ are then split into the complex conjugate tensors Y^{i+}_{mn} and Y^{i-}_{mn}, which for $k = 1$ transform in the 10 and $\overline{10}$ of SO(6), in general the $(0,k-1,2)$ and $(2,k-1,0)$ representations.

We can now substitute the expansions of the graviton and the four-index tensor into the linearized field equations (3.2.11-16), and collect in each field equation the coefficients of a given spherical harmonic. One thus obtains from (3.2.11)

$$\Big[\tfrac{1}{2}(\Box_x + \Box_y + 2e^2)\varphi^{i_s}_{(\mu\nu)|} - D_{(\mu}D^{\rho}\varphi^{i_s}_{\nu)|\rho} + \\ + \tfrac{1}{2}D_{(\mu}D_{\nu)|}\varphi^{i_s} \Big] Y^{i_s} = 0 \tag{3.2.33}$$

and

$$\Big[-\tfrac{1}{5}D^{\mu}D^{\nu}\varphi^{i_s}_{\mu\nu} + \tfrac{1}{5}(\Box_x + \tfrac{1}{2}\Box_y + 16e^2)\varphi^{i_s} - \\ - \tfrac{1}{6}(\Box_x + \Box_y + 32e^2)\pi^{i_s} + \tfrac{1}{3}\varepsilon^{\mu\nu\rho\sigma\tau}\partial_{\mu} b^{i_s}_{\nu\rho\sigma\tau} \Big] Y^{i_s} = 0. \tag{3.2.34}$$

From (3.2.12) one finds

$$\left[\frac{1}{2}(\mathrm{Max} + \Box_y)B_\mu^{i_v} + \frac{2}{3}\varepsilon_\mu{}^{\nu\rho\sigma\tau}\partial_\nu b_{\rho\sigma\tau}^{i_v} + 4\Box_y b_\mu^{i_v}\right]Y_n^{i_v} = 0 \qquad (3.2.35)$$

and

$$\left[-\frac{1}{2}D^\rho\varphi_{\rho\mu}^{i_s} + D_\mu\left(\frac{1}{2}\varphi^{i_s} - \frac{4}{15}\pi^{i_s} + 4b^{i_s}\right) + \right.$$

$$\left. + \frac{1}{6}\varepsilon_\mu{}^{\nu\rho\sigma\tau}b_{\nu\rho\sigma\tau}^{i_s}\right]D_n Y^{i_s} = 0; \qquad (3.2.36)$$

and (3.2.13) yields

$$\left[\frac{1}{2}(\Box_x + \Box_y - 2e^2)H^{i_t}\right]Y_{mn}^{i_t} = 0, \qquad (3.2.37)$$

$$\left[\frac{1}{2}\varphi^{i_s} - \frac{8}{15}\pi^{i_s}\right]D_{(m}D_{n)|}Y^{i_s} = 0, \qquad (3.2.38)$$

$$\left[D^\mu B_\mu^{i_v}\right]D_{(m}Y_{n)|}^{i_v} = 0 \qquad (3.2.39)$$

and

$$\left[\frac{1}{10}(\Box_x - \frac{1}{15}\Box_y - 32e^2)\pi^{i_s} + \frac{1}{10}\Box_y\varphi^{i_s} + 8\Box_y b^{i_s}\right]Y^{i_s} = 0. \qquad (3.2.40)$$

The self-duality equation yields from (3.2.14)

$$\left[5\,\partial_{[\mu}b_{\nu\rho\sigma\tau]}^{i_s} - \varepsilon_{\mu\nu\rho\sigma\tau}\left(\frac{1}{2}e^2\varphi^{i_s} - \frac{4}{3}e^2\pi^{i_s} + \Box_y b^{i_s}\right)\right]Y^{i_s} = 0; \qquad (3.2.41)$$

from (3.2.15)

$$\left[4\,\partial_{[\mu}b_{\nu\rho\sigma]}^{i_v} + \varepsilon_{\mu\nu\rho\sigma}{}^\tau\left((\Box_y - 4e^2)b_\tau^{i_v} - e^2 B_\tau^{i_v}\right)\right]Y_m^{i_v} = 0 \qquad (3.2.42)$$

and

$$\left[b^{i_s}_{\mu\nu\rho\sigma} + \varepsilon_{\mu\nu\rho\sigma}{}^{\tau} D_\tau b^{i_s} \right] D_m Y^{i_s} = 0 \qquad (3.2.43)$$

and finally, from (3.2.16)

$$3\,\partial_{[\mu} b^{i_2}_{\nu\rho]} Y^{i_2}_{mn} - \frac{1}{4}\varepsilon_{\mu\nu\rho}{}^{\sigma\tau} b^{i_2}_{\sigma\tau} \varepsilon_{mn}{}^{pqr} D_p Y^{i_2}_{qr} = 0 \qquad (3.2.44)$$

and

$$\left[b^{i_v}_{\mu\nu\rho} + \varepsilon_{\mu\nu\rho}{}^{\sigma\tau} \partial_\sigma b^{i_v}_\tau \right] D_{[m} Y^{i_v}_{n]} = 0 . \qquad (3.2.45)$$

We have defined the Maxwell operator $(\mathrm{Max}) b_\mu \equiv D^\rho (D_\rho b_\mu - D_\mu b_\rho)$.

Consider the equations (3.2.41-45) arising from the self-duality condition (3.2.3). For $k > 0$, equations (3.2.43) and (3.2.45) allow us to algebraically eliminate $b^{i_s}_{\mu\nu\rho\sigma}$ and $b^{i_v}_{\mu\nu\rho}$, respectively. The curl of eqn. (3.2.42) reads

$$(K+3) D^\tau b^{i_v}_\tau + D^\tau B^{i_v}_\tau = 0 . \qquad (3.2.46)$$

Substituting these results into eqns. (3.2.41) and (3.2.42), and decomposing the tensor harmonics $Y^{i_2}_{mn}$ into self-dual components, one obtains the following three equations which summarize the content of (3.2.3).

$$(\Box_x - Ke^2) b^{i_s} + e^2 \left(\frac{1}{2}\varphi^{i_s} - \frac{4}{3}\pi^{i_s} \right) = 0 \qquad (3.2.47)$$

(for $k > 0$), and

$$(\mathrm{Max} - (K+3)e^2) b^{i_v}_\mu - e^2 B^{i_v}_\mu = 0 \qquad (3.2.48)$$

$$3\,\partial_{[\mu} b^{i\pm}_{\nu\rho]} \mp \frac{i}{2} e\,(k+2)\,\varepsilon_{\mu\nu\rho}{}^{\sigma\tau} b^{i\pm}_{\sigma\tau} = 0 \qquad (3.2.49)$$

where we have substituted the eigenvalues for differential operators on S^5.

For the special case $k = 0$, Y^{i_s} is constant and eqn. (3.2.43) vanishes. Then (3.2.46) must be replaced by

$$5\,\partial_{[\mu} b^0_{\nu\rho\sigma\tau]} = e^2 \varepsilon_{\mu\nu\rho\sigma\tau}\left(\frac{1}{2}\varphi^0 - \frac{4}{3}\pi^0\right). \qquad (3.2.50)$$

We now divide all equations arising from the Einstein and self-duality equations into three classes, which we shall discuss separately: (i) Maxwell-Proca equations, (ii) coupled scalar equations and (iii) diagonal equations.

Maxwell-Proca equations

These consist of the field equations (3.2.48) and (3.2.35), where the latter yields

$$\frac{1}{2}(\mathrm{Max} - (K+3)e^2) B^{i_v}_\mu - 4\,(\mathrm{Max} + (K+3)e^2)\, b^{i_v}_\mu = 0 \qquad (3.2.51)$$

upon substitution of (3.2.45). The resulting 2×2 system is easily diagonalized, yielding two branches of vector fields

$$v^k_\mu = B^k_\mu - 4\,(k+3)\, b^k_\mu\,; \quad w^k_\mu = B^k_\mu + 4\,(k+1)\, b^k_\mu \qquad (3.2.52)$$

with masses

$$M_v^2 = (k-1)(k+1)e^2\,, \quad M_w^2 = (k+3)(k+5)e^2 \qquad (3.2.53)$$

where v_μ satisfies the field equation $(\mathrm{Max} - M_v^2)v_\mu = 0$ and similarly for w_μ.

These field equations must be supplemented with the transversality conditions eqn. (3.2.46) and (3.2.39), the latter of which gives $D^\mu B_\mu^k = 0$ for $k > 1$, but vanishes for $k = 1$. Thus all the v_μ and w_μ satisfy transversality conditions except for v_μ^1. These latter, having $M_v^2 = 0$, are clearly the SO(6) gauge fields; as mentioned before, their gauge invariance has not been fixed.

Coupled scalar equations

There are five equations involving the three scalars φ^{i_s}, π^{i_s} and b^{i_s}:

$$\varphi^k = \frac{16}{15}\pi^k \qquad (3.2.54)$$

for $k > 1$ from (3.2.38);

$$D^\rho \varphi_{\rho\mu}^k = D_\mu(\varphi^k - \frac{8}{15}\pi^k + 16\,b^k) \qquad (3.2.55)$$

for $k > 0$ from (3.2.36) and (3.2.43);

$$(\Box_x - (K+32)e^2)\pi^k - Ke^2(80\,b^k + \varphi^k - \frac{16}{15}\pi^k) = 0 \qquad (3.2.56)$$

from (3.2.40) and (3.2.43);

$$(\Box_x - Ke^2)b^k + e^2(\frac{1}{2}\varphi^k - \frac{4}{3}\pi^k) = 0 \qquad (3.2.57)$$

from (3.2.47). The fifth scalar equation, (3.2.34), is linearly dependent upon the other four so may be dropped. Equation (3.2.55) will supply the transversality condition for the massive gravitons, as we shall see.

In the case that $k > 1$, eqn. (3.2.54) allows us to eliminate φ^k yielding from (3.2.56) and (3.2.57) a simple 2×2 coupled system with mass eigenvalues

$$M^2 = k(k-4)e^2 , \quad M^2 = (k+4)(k+8)e^2. \qquad (3.2.58)$$

In the absence of an unambiguous definition of mass for scalar fields, M^2 is simply taken to be the eigenvalue of $\Box_x$; i.e., a generic scalar field σ satisfies a field equation $(\Box_x - M^2)\sigma = 0$. As we shall see, the second branch in (3.2.58) may be extended down to $k = 0$, but the first branch has only $k = 2,3,4,\ldots$

For $k = 1$, (3.2.54) is no longer a field equation, but one could in principle still use the unfixed conformal diffeomorphisms to obtain $\varphi^1 = \frac{16}{15}\pi^1$, and proceed exactly as before. One would then naively obtain two modes, with masses as in eqn. (3.2.58) for $k = 1$. However, one could use eqn. (3.2.56) to directly eliminate φ^1 and insert the result into (3.2.57). This reasoning leads to one field equation for only one mode, namely,

$$(\Box_x - 45e^2)(\pi^1 + 10b^1) = 0. \qquad (3.2.59)$$

The resolution of this apparent inconsistency requires a closer examination of the conformal diffeomorphisms.

Under the conformal diffeomorphisms of eqn. (3.2.27), the scalars transform as follows

$$\delta\varphi^1 = (2\Box_x + \frac{50}{3})\lambda\,, \quad \delta\pi^1 = 10\lambda\,, \quad \delta b^1 = -\lambda \qquad (3.2.60)$$

where the factor of $\frac{50}{3}$ comes from the Weyl shift. The result for δb^1 follows from the general result

$$\delta A_{mnpq} = \xi^\tau F_{\tau mnpq} - 4\,\partial_{[m}(\xi^\tau A_{npq]\tau}) \qquad (3.2.61)$$

by taking the lowest-order terms and making a compensating gauge transformation. Under the sum of the conformal diffeomorphism and the compensating gauge transformation with parameter $\Lambda_{mnp} = -4\,\xi^q \bar{A}_{qmnp}$, the other components of A_{MNPQ} are all inert except, as required by self-duality, $A_{\mu\nu\rho\sigma}$. Under conformal diffeomorphisms, $A_{\mu\nu\rho\sigma}$ transforms as

$$\delta A_{\mu\nu\rho\sigma} = \xi^\tau F_{\tau\mu\nu\rho\sigma} - 4\,\partial_{[\mu}(\xi^\tau A_{\nu\rho\sigma]\tau}) \qquad (3.2.62)$$

so that $\delta b^1_{\mu\nu\rho\sigma} = -\varepsilon_{\mu\nu\rho\sigma\tau} D^\tau \lambda$, if one adds a further compensating gauge transformation with parameter $\Lambda_{\mu\nu\rho} = -4\,\xi^\sigma \bar{A}_{\sigma\mu\nu\rho}$. One may verify that all field equations are invariant under these combined conformal diffeomorphisms and gauge transformations.

With these preliminaries we will now argue that fixing the conformal gauge leads to the same result as direct elimination of φ^1. Consider as a simplified model the field equation

$$\Box A + B = 0 \tag{3.2.63}$$

with gauge invariance

$$\delta A = \lambda\,, \quad \delta B = -\Box\lambda. \tag{3.2.64}$$

One can either directly eliminate B by $B = -\Box A$, in which case there are no surviving propagating modes, or one can use the local symmetry to fix $B = \alpha A$. In the latter case, one obtains the field equation $(\Box + \alpha)A = 0$, which seems to indicate a propagating mode with gauge-dependent mass. However, we can use the local symmetry once more to gauge A completely away. To see this, note that we can still make gauge transformations which respect $B = \alpha A$ by using a parameter λ which satisfies $(\Box + \alpha)\lambda = 0$. Thus "the gauge shoots twice" and no modes remain; A has been shown to be pure gauge.

Returning to our original model, we conclude that in the $k = 1$ sector there is only one mode propagating, namely, the mode in (3.2.59). This is the $k = 1$ mode in the second branch of (3.2.58). Thus, although in the spherical harmonic expansion one does find a second mode at $k = 1$, it drops from the theory. In the discussion, Section 2.4, we shall argue that this set of six scalars is part of the doubleton multiplet.

The necessity of utilizing a residual gauge invariance for a correct identification of physical modes is familiar in the context of Maxwell theory. There, the field equations read (in flat spacetime)

$$\partial^\rho F_{\rho\mu} = \Box A_\mu - \partial_\mu(\partial^\rho A_\rho) = 0. \tag{3.2.65}$$

One may use the gauge invariance $\delta A_\mu = \partial_\mu \Lambda$ to gauge $\partial^\rho A_\rho = 0$, thus reducing the number of degrees of freedom carried by A_μ from four to three (in four dimensions); the wave equation for these modes is then

$$\Box A_\mu = 0. \tag{3.2.66}$$

In order to restrict A_μ to the two physical polarizations, one must utilize the remaining invariance of (3.2.66) under gauge transformations parametrized by Λ satisfying $\Box\Lambda = 0$.

In the $k = 0$ sector, there is no b_{mnpq} term from the expansion of a_{mnpq}, and (3.2.57) is replaced by (3.2.50). Equation (3.2.56) reduces to

$$(\Box_x - 32e^2)\pi^0 = 0 \tag{3.2.67}$$

which describes the dilatational mode of the internal metric. Again, this mode can be found in the continuation of the second branch of (3.2.58) to $k = 0$.

Diagonal equations

The remaining fields, $b^{i\pm}_{\mu\nu}$, H^{i_t} and the gravitons $\varphi^{i_s}_{(\mu\nu)|}$, have diagonal field equations. From (3.2.44) the antisymmetric tensor fields satisfy first order self-dual field equations:

$$[(*D)^{\sigma\tau}_{\mu\nu} - iM\delta^{\sigma\tau}_{\mu\nu}]b^{i\pm}_{\sigma\tau} = 0 \tag{3.2.68}$$

where $(*D)^{\sigma\tau}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma\tau}\partial_\rho$. Upon iteration this takes the more familiar form

$$(\text{Max} - M^2)\,b^{i\pm}_{\mu\nu} \;=\; 0 \tag{3.2.69}$$

where the generalization of the Maxwell operator to tensor fields is $(\text{Max})\,b_{\mu\nu} \equiv D^\rho(3\,D_{[\rho}b_{\mu\nu]})$. Here, the mass parameter M takes on the values $\pm(k+2)e$ for $k = 1,2,3,\ldots$

The scalars H^{it} have masses $Ke^2 = k(k+4)e^2$, as can be read off from eqn. (3.2.37), for $k = 2,3,4,\ldots$

The graviton field equations appear in (3.2.33). For $k > 0$ the field redefinition

$$\Phi^k_{\mu\nu} \;\equiv\; \varphi^k_{(\mu\nu)|} - \frac{1}{(k+1)(k+3)}D_{(\mu}D_{\nu)|}(\tfrac{2}{5}\pi^k - 12\,b^k) \tag{3.2.70}$$

gives

$$(\text{Ein} - k(k+4)e^2)\,\Phi^k_{\mu\nu} \;=\; 0, \tag{3.2.71}$$

where the Einstein operator is defined by

$$(\text{Ein})\,\Phi_{\mu\nu} \;\equiv\; (\Box_x + 2e^2)\,\Phi_{\mu\nu} - 2\,D_{(\mu}D^\rho\Phi_{\nu)|\rho}. \tag{3.2.72}$$

Furthermore, eqn. (3.2.55) shows that $\Phi^k_{\mu\nu}$ is transversal on shell.

For $k = 0$, there is no transversality condition (we have not fixed the general coordinate invariance) and upon imposition of the scalar equations (3.2.50) and (3.2.67) the field equation (3.2.33) reads $(\text{Ein})\,\varphi^0_{\mu\nu} = 0$, the field equation for a massless graviton. Thus we have a tower of gravitons of $(\text{mass})^2 = k(k+4)e^2$, $k = 0,1,2,\ldots$

Other bosonic fields

We now discuss the modes contained in the fields A_{MN} and B. These fields are purely fluctuations; that is, they vanish in the vacuum. Again we choose internal Lorentz-type gauge conditions

$$D^m A_{mn} = 0 \,, \quad D^m A_{m\mu} = 0 . \tag{3.2.73}$$

The spherical harmonic expansions are then

$$A_{\mu\nu}(x,y) = \Sigma a_{\mu\nu}^{i_s}(x)\, Y^{i_s}(y) \tag{3.2.74a}$$

$$A_{\mu n}(x,y) = \Sigma a_{\mu}^{i_v}(x)\, Y_n^{i_v}(y) \tag{3.2.74b}$$

$$A_{mn}(x,y) = \Sigma a^{i_2}(x)\, Y_{mn}^{i_2}(y) \tag{3.2.74c}$$

and for the scalar field

$$B(x,y) = \Sigma B^{i_s}(x)\, Y^{i_s}(y) . \tag{3.2.75}$$

Substituting these expansions into the field equations (3.2.1,2) yields

$$\left[(\Box_x + \Box_y) B^{i_s} \right] Y^{i_s} = 0 , \tag{3.2.76}$$

$$\left[(\mathrm{Max} + \Box_y) a_{\mu\nu}^{i_s} + 2ie\, \varepsilon_{\mu\nu}{}^{\sigma\tau\kappa} \partial_\sigma a_{\tau\kappa}^{i_s} \right] Y^{i_s} = 0 , \tag{3.2.77}$$

$$\left[(\mathrm{Max} + \Box_y - 4e^2) a_{\mu}^{i_v} \right] Y_n^{i_v} = 0 , \tag{3.2.78}$$

$$\left[D^\rho a_{\rho\mu}^{i_s} \right] D_n Y^{i_s} = 0 , \tag{3.2.79}$$

$$(\Box_x + \Box_y - 6e^2)a^{i_2} Y^{i_2}_{mn} + 2ie\, a^{i_2} \varepsilon_{mn}{}^{pqr} D_p Y^{i_2}_{qr} = 0\,, \qquad (3.2.80)$$

and

$$\left[D^\mu a^{i_v}_\mu \right] D_{[m} Y^{i_v}_{n]} = 0\,. \qquad (3.2.81)$$

We immediately identify a tower of complex scalars B^k satisfying

$$(\Box_x - k(k+4)e^2)B^k = 0\,. \qquad (3.2.82)$$

Substituting the decomposition of $Y^{i_2}_{mn}$ into $Y^{i\pm}_{mn}$, equation (3.2.80) yields

$$(\Box_x - (k+2)^2 e^2 \mp 4(k+2)e^2)a^{k\pm} = 0\,, \qquad (3.2.83)$$

describing two towers of complex scalars of masses

$$M^2 = k(k-4)e^2\,, \quad M^2 = (k+2)(k+6)e^2 \qquad (3.2.84)$$

for $k = 1,2,3,\ldots$ There are also complex massive vectors of $(\text{mass})^2 = (k+1)(k+3)e^2$, $k = 1,2,3,\ldots$ described by (3.2.78) and satisfying the transversality condition (3.2.81)

The equation of motion for the tensor fields a^k_{mn}, $k = 0,1,2,\ldots$, factorizes into the product of two first-order field equations:

$$(\text{Max} - k(k+4)e^2)a^k_{\mu\nu} + 2ie\, \varepsilon_{\mu\nu}{}^{\rho\sigma\tau}\partial_\rho a^k_{\sigma\tau} =$$

$$= \left[({*D})^{\sigma\tau}_{\mu\nu} - ie(k+4)\delta^{\sigma\tau}_{\mu\nu} \right]\left[({*D})^{\kappa\lambda}_{\sigma\tau} + iek\,\delta^{\kappa\lambda}_{\sigma\tau} \right] a^k_{\kappa\lambda} = 0\,. \qquad (3.2.85)$$

Thus the propagating modes in $a^k_{\mu\nu}$ split into two sets of complex tensor fields satisfying self-dual equations, with masses ke and $(k+4)e$. One can show that the first mode (for $k = 0$) is pure gauge and can be considered part of the doubleton representation. The $k > 0$ modes are all transversal due to eqn. (3.2.79).

3.3 Fermionic modes

The ten-dimensional gravitino and spinor field equations, linearized in fluctuations, read

$$\Gamma^{MNP}(D_N + \frac{i}{480}\Gamma^{QRSTU}\bar{F}_{QRSTU}\Gamma_N)\Psi_P = 0 \qquad (3.3.1)$$

$$(\not{D} - \frac{i}{240}\Gamma^{MNPQR}\bar{F}_{MNPQR}) = 0. \qquad (3.3.2)$$

We will take a representation of the 32×32 ten-dimensional Dirac matrices which is manifestly Weyl but not Majorana:

$$\Gamma^{\mu} = \gamma^{\mu} \otimes 1_{4\times 4} \otimes \sigma^1 , \quad \Gamma^m = 1_{4\times 4} \otimes \tau^m \otimes \sigma^2 \qquad (3.3.3)$$

where the σ^i are the usual Pauli matrices and γ^{μ}, τ^m are 4×4 Dirac matrices in the five-dimensional spacetime and internal space, respectively. The "γ_5" matrix in ten dimensions is

$$\Gamma_{11} \equiv \Gamma^0\Gamma^1 \cdots \Gamma^9 = 1_{4\times 4} \otimes 1_{4\times 4} \otimes (-\sigma^3). \qquad (3.3.4)$$

The chirality conditions $\Gamma_{11}\Psi_M = -\Psi_M$ and $\Gamma_{11}\Lambda = \Lambda$ imply that we can write

$$\Psi_M = \begin{bmatrix} \psi_M \\ 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}. \qquad (3.3.5)$$

We decompose the spinor field into spinor spherical harmonics on S^5:

$$\lambda(x,y) = \Sigma\lambda^{\dot{u}}(x) \otimes \Xi^{\dot{u}}(y) \qquad (3.3.6)$$

where both $\lambda^{ii}(x)$ and $\Xi^{ii}(y)$ are four-component spinors.

The $\Xi^{ii}(y)$ are eigenfunctions of $\not{D}_y \equiv \tau^m D_m$, and can be easily expressed in terms of the bosonic scalar harmonics. On the n-sphere with radius e^{-1}, there are two sets of Killing spinors η^+ and η^- (on S^5, complex conjugates of each other) satisfying

$$D_m \eta^{\pm} \pm \frac{i}{2} e \tau_m \eta^{\pm} = 0. \qquad (3.3.7)$$

The eigenspinors of $\not{D}_y$ are then

$$\Xi^{k\pm} = \Big[(k + n - 1)e \pm i\not{D}_y \Big] Y^k \eta^{\pm} \qquad (3.3.8)$$

for $k = 0,1,2,...$, and satisfy

$$\not{D}_y \Xi^{k\pm} = m^{k\pm} \Xi^{k\pm} \quad \text{with} \quad m^{k\pm} = \mp ie(k + \tfrac{1}{2}n). \qquad (3.3.9)$$

An alternative expression for Ξ^{k-} in terms of η^+ can also be obtained; it is given by

$$\Xi^{k-} = \Big[(k + 1)e + i\not{D}_y \Big] Y^{k+1} \eta^+ , \quad k \geq 0. \qquad (3.3.10)$$

On S^5, the η^+ and the η^- transform in the spinor 4 and $\bar{4}$ of Spin(6) ≈ SU(4). In general, the Ξ^{k+} and the Ξ^{k-} transform in the $(1,k,0)$ and $(0,k,1)$ representations, in the Dynkin notation.

Similarly, the vector-spinor spherical harmonics satisfy

$$\not{D}_y \Xi^{k\pm}_m = m^{k\pm} \Xi^{k\pm}_m \quad \text{with} \quad m^{k\pm} = \mp ie(k + \tfrac{1}{2}n) \qquad (3.3.11)$$

where now $k = 1,2,3,\ldots$ The $\Xi_m^{k\pm}$ are both gamma- and space-transversal:

$$\tau^m \Xi_m^{k\pm} = D^m \Xi_m^{k\pm} = 0, \qquad (3.3.12)$$

and can be expressed in terms of the bosonic vector harmonics $(Y^k)^p$ and Killing spinors as follows:

$$\begin{aligned} \Xi_m^{k\pm} = \Big[& (n-2)e\, g_{mp}\{ (n+k)(n+k-2)e \mp i(n+k-1)\not{D}_y \} - \\ & -(n+k-1)(n+k-2)e^2\tau_{mn} \mp i(n+k)e\,\tau_{mnp}D^n \pm \\ & \pm i(n-2)e\,\tau_p D_m - \tau_{np} D_m D^n \Big] (Y^k)^p \eta^{\pm}. \end{aligned} \qquad (3.3.13)$$

Again one can express the Ξ_m^{k-} in terms of η^+:

$$\begin{aligned} \Xi_m^{k-} = \Big[& (n-2)e\, g_{mp}\{ k(k+2)e + i(k+1)\not{D}_y \} - \\ & -(k+1)(k+2)e^2\tau_{mp} + ike\,\tau_{mnp}D^n + \\ & + i(n-2)e\,\tau_p D_m - \tau_{np} D_m D^n \Big] (Y^k)^p \eta^{+}. \end{aligned} \qquad (3.3.14)$$

On S^5 the $\Xi_m^{k\pm}$ transform in the $(2,k-1,1)$ and the $(1,k-1,2)$ representations of SU(4).

It is important to stress that in all of these equations, the derivatives do not act on the Killing spinors but only on the bosonic harmonics. The basic relations to obtain the fermionic spectrum are given in

(3.3.9) and (3.3.11).

Inserting the expansion (3.3.6) into the spinor field equation (3.3.2) gives immediately

$$(\not{D}_x + i\not{D}_y + e)\lambda^{k\pm} = (\not{D}_x + e \pm (k + \tfrac{5}{2})e)\lambda^{k\pm} = 0 \qquad (3.3.15)$$

where $\not{D}_x \equiv \gamma^\mu D_\mu$. Thus we can immediately identify two towers of complex spinors with masses $(k + \frac{3}{2})e$ and $(k + \frac{7}{2})e$, for $k = 0,1,2,...$ In the absence of an unambiguous definition of mass for spinors, we simply label the eigenvalue of $\not{D}_x$ the "mass" (only defined up to a sign).

We now turn to the gravitino field equation. The two cases $M = \mu$ and $M = m$ yield, respectively,

$$\gamma^{\mu\nu\rho} D_\nu \psi_\rho - i\gamma^{\mu\nu} D_\nu(\tau^m \psi_m) + i\gamma^{\mu\nu}\not{D}_y \psi_\nu +$$

$$+ \gamma^\mu(\tau^{mn} D_m \psi_n) - \gamma^{\mu\nu}\psi_\nu = 0 \qquad (3.3.16)$$

$$\tau^{mnp} D_n \psi_p + i\tau^{mn} D_n(\gamma^\mu \psi_\mu) - i\tau^{mn}\not{D}_x \psi_n +$$

$$+ \tau^m(\gamma^{\mu\nu} D_\mu \psi_\nu) + i\tau^{mn}\psi_n = 0. \qquad (3.3.17)$$

We partially fix the ten-dimensional local supersymmetries by trying to achieve $\Gamma^m \Psi_m = 0$. However, since the supersymmetry in the background gives

$$\delta\Psi_m = (D_m + \tfrac{i}{2} e\, 1_{4\times 4} \otimes \tau_m)\varepsilon(x,y) \qquad (3.3.18)$$

upon decomposing the supersymmetry parameter

$$\varepsilon(x,y) = \Sigma \varepsilon^{i_l}(x)\Xi^{i_l}(y) \tag{3.3.19}$$

it becomes clear that $\Gamma^m \Psi_m$ is invariant under those supersymmetry modes proportional to η^+. Thus the nearest one can come to fixing $\Gamma^m \Psi_m = 0$ is

$$\psi_m = \psi_{(m)} + \chi \tau_m \eta^+ \tag{3.3.20}$$

where the notation (m) indicates that the gamma-trace has been removed; that is, $\psi_{(m)} \equiv \psi_m - \frac{1}{5}\tau_m \tau^n \psi_n$ so $\tau^m \psi_{(m)} = 0$. Note that the coefficients of the other Killing spinors, η^-, can be gauged away along with all the higher modes. (Of course, the situation is reversed for Ψ_m^*.) We expand the gravitino fields as follows

$$\psi_\mu(x,y) = \Sigma \psi_\mu^{i_l}(x)\Xi^{i_l}(y) \tag{3.3.21a}$$

$$\psi_{(m)}(x,y) = \Sigma \alpha^{i_l}(x)\Xi_m^{i_l}(y) + \Sigma \beta^{i_l}(x) D_{(m)}\Xi^{i_l}. \tag{3.3.21b}$$

We will first analyze the gravitino field equations in the sectors excluding η^+, and separately consider the η^+ sector. From eqn. (3.3.16) we find

$$\Big[\gamma^{\mu\nu\rho} D_\nu \psi_\rho^{i_l} + (im^{i_l} - e)\gamma^{\mu\nu}\psi_\nu^{i_l} -$$

$$- (5e^2 + \tfrac{4}{5}(m^{i_l})^2)\gamma^\mu \beta^{i_l} \Big] \Xi^{i_l} = 0 \tag{3.3.22}$$

where the last term arises from $\gamma^\mu D^m \psi_{(m)}$. From eqn. (3.3.17) we have

$$\Big[(\not{D}_x - im^{i_t} + e)\alpha^{i_t}\Big]\Xi^{i_t}_m = 0 \qquad (3.3.23)$$

$$\Big[-3(5e^2 + \tfrac{4}{5}(m^{i_t})^2)\beta^{i_t} + 5\gamma^{\mu\nu}D_\mu\psi^{i_t}_\nu +$$

$$+ 4im^{i_t}\gamma^\mu\psi^{i_t}_\mu\Big]\Xi^{i_t} = 0 \qquad (3.3.24)$$

$$\Big[(\not{D}_x - \tfrac{3}{5}im^{i_t} - e)\beta^{i_t} - \gamma^\mu\psi^{i_t}_\mu\Big]D_{(m)}\Xi^{i_t} = 0. \qquad (3.3.25)$$

Here, (3.3.24) is obtained by contracting (3.3.17) with τ_m, while (3.3.25) is the τ_m–transverse part. From (3.3.23) we immediately see that the α^{i_t} describe two towers of complex spinors, of masses $(k + \frac{5}{2} \pm 1)e = (k + \frac{3}{2})e$, $(k + \frac{7}{2})e$, for $k = 1,2,3,\ldots$

Eliminating $\gamma^\mu\psi^{i_t}_\mu$ in terms of β^{i_t} from (3.3.24) and the γ_μ contraction of (3.3.22) gives

$$\gamma^\mu\psi^{i_t}_\mu = -\tfrac{4}{5}(5e + 2im^{i_t})\beta^{i_t} \qquad (3.3.26)$$

Inserting this into (3.3.25) yields

$$(\not{D}_x + im^{i_t} + 3e)\beta^{i_t} = 0, \qquad (3.3.27)$$

which describes two towers of complex spinors, with masses $(k + \frac{5}{2} \pm 3)e = (k - \frac{1}{2})e$, $(k + \frac{11}{2})e$. Normally, the index k would start at zero, but since we will handle the η^+ harmonic separately, for now we take $k = 1,2,3,\ldots$

For the gravitino modes, eqn. (3.3.22) yields, upon eliminating β^{i_l},

$$\gamma^{\mu\nu\rho}D_\nu\psi_\rho^{i_l} + (im^{i_l} - e)\gamma^{\mu\nu}\psi_\nu^{i_l} +$$

$$+ \tfrac{1}{4}(5e - 2im^{i_l})\gamma^\mu(\gamma^\nu\psi_\nu^{i_l}) = 0. \qquad (3.3.28)$$

In order to find the physical propagating gravitino modes, we make the field redefinition

$$\varphi_\mu^{i_l} \equiv \psi_{(\mu)}^{i_l} + aD_{(\mu)}(\gamma^\nu\psi_\nu^{i_l}). \qquad (3.3.29)$$

We can guarantee that $D^\mu\varphi_\mu^{i_l} = 0$ on shell by taking

$$a = \frac{3}{4(2im^{i_l} + e)}. \qquad (3.3.30)$$

Note that the denominator only vanishes for $\Xi^{i_l} = \eta^+$, which we will discuss later. The field equation for φ_μ then reads

$$\begin{aligned} 0 &= \gamma^{\mu\nu\rho}D_\nu\varphi_\rho^{i_l} + (im^{i_l} - e)\varphi^{\mu i_l} \\ &= (\text{R.-S.} + im^{i_l} + \tfrac{5}{2}e)\varphi^{\mu i_l}, \end{aligned} \qquad (3.3.31)$$

where we have defined the Rarita-Schwinger operator by

$$(\text{R.-S.})\varphi^\mu \equiv \gamma^{\mu\nu\rho}(D_\nu + \tfrac{1}{2}e\gamma_\nu)\varphi_\rho \qquad (3.3.32)$$

which describes a gauge-invariant massless gravitino mode in our anti-de Sitter background. The field equations (3.3.31) hence describe two (complex) towers of massive gravitini, of masses $(k + \frac{5}{2} \pm \frac{5}{2})e = ke$,

$(k+5)e$, for $k = 1,2,3,\ldots$ The η^- mode, which we can include, supplies the missing gravitino mode with mass = $10e$ for the second branch.

We must now analyze the η^+ sector. One finds the following two field equations

$$\gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{3}{2} e \gamma^{\mu\nu} \psi_\nu + 5i \gamma^{\mu\nu} D_\nu \chi + 2ie \gamma^\mu \chi = 0 \tag{3.3.33}$$

$$\gamma^{\mu\nu} D_\mu \psi_\nu + 2e \gamma^\mu \psi_\mu + 4i \not{D}_x \chi + 2ie \chi = 0. \tag{3.3.34}$$

Contracting the first equation and combining with the second gives a diagonal result for χ:

$$(\not{D}_x + \frac{11}{2} e)\chi = 0. \tag{3.3.35}$$

Thus χ has the correct mass to fill the $k = 0$ position in the upper branch of modes arising from β^{ii}. The $k = 0$ slot in the lower branch remains vacant, and may be identified with the spinorial part of the doubleton.

Upon defining the shifted field

$$\varphi_\mu \equiv \psi_\mu + \frac{5}{3} i \gamma_\mu \chi \tag{3.3.36}$$

the gravitino field equation becomes simply

$$(\text{R.-S.}) \varphi^\mu = 0 \tag{3.3.37}$$

and hence φ_μ (and its complex conjugate) describe the massless gravitino modes consistent with N = 8 supersymmetry.

3.4 Discussion

We have obtained the complete mass spectrum for the compactification of chiral N = 2 d = 10 supergravity on S^5. The results may be compared with the U(2,2|4) supermultiplet structure obtained in Ref. [8]. Naturally, all the modes given here may be fitted into supermultiplets. Furthermore, all of the representations found in [8] occur in the compactification, with one important exception: the "doubleton" representation, which may instead be identified with a set of gauge modes, as stated in Sections 3.2 and 3.3. These modes are (i) the six conformal scalars in the $k = 1$ sector of π^{i_s} and b^{i_s}, (ii) the $k = 0$ component of $a^{i_s}_{\mu\nu}$ which is pure gauge on shell, and (iii) the Fourier coefficients of the η^+ terms in $\psi^{i_t}_m$.

Now that the linearized field equations of the modes found in [8] are known, one can give physical meaning to the quantity E_0, which helps to characterize the SU(4) representation and should correspond to some sort of "energy." In fact, for all of the scalar, spinor, vector, tensor and graviton modes there is a very simple relationship between the mass parameter obtained here and the parameter E_0 of Ref. [8]. Define ε to be $\frac{1}{2}eE_0$. Then the linearized field equations that all the scalars satisfy may be written as

$$[\Box_x - \varepsilon(\varepsilon - 4)]\varphi = 0. \qquad (3.4.1)$$

For all the spinors

$$[\not{D}_x - i(\varepsilon - 2)]\chi = 0. \tag{3.4.2}$$

For the vectors

$$[\mathrm{Max} - (\varepsilon - 1)(\varepsilon - 3)]A_\mu = 0. \tag{3.4.3}$$

For the antisymmetric tensors

$$[({*D}) - i(\varepsilon - 2)]a_{\mu\nu} = 0 \tag{3.4.4}$$

where the operator $({*D})$ has been defined in Section 3.2. The gravitons satisfy

$$[\mathrm{Ein} - \varepsilon(\varepsilon - 4)]h_{\mu\nu} \tag{3.4.5}$$

with Ein the operator defined in Section 3.2. The gravitini require separate treatment for each tower; for the tower including the massless gravitini the field equations are

$$(\text{R.-S.})\psi_\mu - (\varepsilon - \tfrac{7}{2})\gamma^{\mu\nu}\psi_\nu = 0 \tag{3.4.6a}$$

while for the higher tower we have

$$(\text{R.-S.})\psi_\mu - (\varepsilon - \tfrac{1}{2})\gamma^{\mu\nu}\psi_\nu = 0. \tag{3.4.6b}$$

The Rarita-Schwinger operator describing propagation of massless gravitons has been given in Section 3.3.

The linearized field equations for the scalars and spinors in the massless supermultiplet, along with their SU(4) representations, are

$$(\Box_x + 4e^2)\pi^{k=2} = 0 \qquad (20') \tag{3.4.7a}$$

$$(\Box_x + 3e^2)a^{k=1} = 0 \qquad (10 + \overline{10}) \tag{3.4.7b}$$

$$(\Box_x)B^{k=0} = 0 \qquad (1 + 1) \tag{3.4.7c}$$

$$(\not{D}_x \pm \frac{3}{2}ie)\lambda^{k=0} = 0 \qquad (4 + \bar{4}) \tag{3.4.8a}$$

$$(\not{D}_x \pm \frac{1}{2}ie)\beta^{k=1} = 0 \qquad (20 + \overline{20}). \tag{3.4.8b}$$

As mentioned in Section 3.1, the scalars in the massless supermultiplet have several different masses, as do the spinors.

All forty-two of the scalars in (3.4.7a-c) lie inside or on the boundary of the perturbative stability region, for which is defined by $(\Box_x + \alpha)\varphi = 0$ with $\alpha \leq 4e^2$ [13]. Note that the conformally invariant field equation reads $(\Box_x + \frac{15}{4}e^2)\varphi = 0$, which is also within the stability region. The conformally invariant wave equation for spinors is $\not{D}_x\chi = 0$, as in all dimensions. None of the scalars or the spinors satisfy conformally invariant field equations.

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Chapter 4

Gauged N = 8 Supergravity in Five Dimensions*

4.1 Introduction

The ungauged N = 8 supergravity in five dimensions was constructed several years ago [4] following the analogous construction in four dimensions [5,6]. There is a gauged version of the four-dimensional theory [6], in which the 28 Abelian vector fields of the ungauged theory become the non-Abelian gauge fields of the $SO(8)_{\mathrm{local}}$ symmetry of the gauged theory. The five-dimensional theory contains 27 vector fields transforming in the fundamental representation of the non-compact symmetry group $E_{6(6)}$. Since there is no 27-dimensional semi-simple subgroup of $E_{6(6)}$ under which all the vector fields transform in the adjoint representation, it was unclear whether a gauged version of this theory could possibly exist.

*Most of the original material in this chapter has appeared in letter form as Ref. [1]. Reference [2] is a more complete version with additional results. An independent derivation of much of the structure of one of the theories discussed here appears in Ref. [3]

Historically, most of the structure of the ungauged N = 8 theory in four dimensions was first obtained by dimensional reduction from the eleven-dimensional supergravity [7]. It has been conjectured that the gauged N = 8 theory of Reference [6] can be obtained by a consistent truncation to the massless supermultiplet of the S^7 compactification of the eleven-dimensional theory; the evidence that such a truncation exists is by now practically overwhelming (see Ref. [8] and references within).

Before considering the five-dimensional theory, it is instructive to make a digression on the subject of supergravity in seven dimensions. The ungauged maximal supergravity in seven dimensions [9], which may be obtained by dimensional reduction from eleven dimensions, contains ten vector fields, suggesting a possible SO(5) gauging. Such a gauging would correspond to the massless sector of the S^4 compactification of the eleven-dimensional theory which is formally quite similar to the S^7 compactification, but with the spacetime and internal space interchanged. However, the explicit S^4 compactification [10] produced something of a surprise: the massless supermultiplet in seven-dimensional anti-de Sitter space indeed contained ten vector fields in the adjoint of the isometry group SO(5), but whereas the ungauged theory contained five two-index antisymmetric tensor fields, the compactification led to five three-index antisymmetric tensor fields. Even though, in seven dimensions, three-index antisymmetric tensor fields are equivalent to two-index antisymmetric tensor fields at the free-field level in flat spacetime (via duality transformations), one finds that once one tries to construct the interacting gauged SO(5) theory this equivalence disappears. In fact, the difficulties encountered at first in gauging the maximal supergravity [11] did not arise at all when three-index instead of two-index tensor fields were used [12]. The problem with the two-index fields is that it is difficult, if not impossible, to render their tensor gauge invariance consistent with the local SO(5) gauge invariance. The three-index fields satisfy first-order self-dual field equations of the form

$$\varepsilon_{\mu\nu\rho}{}^{\sigma\tau\kappa\lambda} D_\sigma a_{\tau\kappa\lambda} = M a_{\mu\nu\rho} \qquad (4.1.1)$$

(where M is some mass parameter) and hence do not need any gauge invariance to yield the correct number of propagating modes. Analogous field equations have been encountered in Chapter 2 and are the subject of Ref. [13]. In fact, an explicit construction of the oscillatorlike unitary irreducible representations of SO(6,2), the anti-de Sitter group in seven dimensions, shows that the two types of tensor field are fundamentally different, even having different numbers of degrees of freedom [14] in anti-de Sitter space.

While the eleven-dimensional supergravity theory naturally compactifies to seven and four dimensions [15], we have seen in Chapter 1 that the ten-dimensional chiral N = 2b theory favors compactification to five dimensions. The maximally symmetric compactification of the ten-dimensional theory on the five-sphere S^5 admits N = 8 supersymmetry. Obtaining the spectrum for this compactification was the subject of Chapter 3, where it was seen that the full spectrum could be fitted into the oscillatorlike unitary irreducible supermultiplets of the N = 8 anti-de Sitter supergroup SU(2,2|4) [16]. It was conjectured in Chapter 3 (see also [16,17]) that the massless anti-de Sitter supermultiplet provides the fields of a gauged N = 8 supergravity theory in five dimensions. Note that twelve of the 27 vector fields in the Poincaré supermultiplet of Ref. [4] are replaced by two-index antisymmetric tensor fields. These tensor fields again satisfy first-order self-dual field equations.

The presence of only fifteen vectors transforming in the adjoint representation of the isometry group SO(6) suggests an obvious choice of gauge group for at least one gauging of N = 8 supergravity in five dimensions. Thus the problem of gauging in five dimensions is somewhat analogous to that in seven dimensions, in that the appropriate field content differs from what one would naïvely guess from the ungauged theory. In this chapter we construct the gauged N = 8 supergravity in five dimensions with local non-Abelian gauge group SO(6), and in addition non-compact gaugings having gauge group SO(5,1), SO(4,2), SO(3,3) and SO*(6) ≈ SU(3,1). As conjectured the fields of the SO(6) theory are precisely those of the massless N = 8 anti-de Sitter supermultiplet. Interestingly, these theories have, in addition, a global symmetry analogous to the SU(1,1) of the chiral ten-dimensional theory; this symmetry is SU(2) for the SO*(6) gauging and SU(1,1) for the others. Furthermore, the theories can be formulated in a USp(8) covariant form similar to the SU(8) covariant formulation of the gauged N = 8 theory in four dimensions.

4.2 Compact and non-compact symmetries

In the ungauged five-dimensional N = 8 theory of [4] the 42 scalar fields were shown to parametrize the noncompact symmetric space $E_{6(6)}/USp(8)$. Although $E_{6(6)}$ will not be a symmetry of the gauged theories, it plays a crucial role in the formulation. The obvious gauge subgroup of $E_{6(6)}$, SO(6) ≈ SU(4), may be embedded in a maximal SL(6,ℝ) × SL(2,ℝ) subgroup of $E_{6(6)}$. The construction of the SO(6) gauging then generalizes trivially to the noncompact gaugings of SO(p,6-p), p = 1, 2, 3, by taking suitable subgroups of SL(6,ℝ). This is accomplished by using an invariant metric η^{IJ}, with appropriate signature, in the minimal couplings which break the $E_{6(6)}$ invariance. The group SL(2,ℝ) ≈ SU(1,1) will continue to be a symmetry irrespective of the gauge group.

One may also consider the maximal SU*(6) × SU(2) subgroup of $E_{6(6)}$. Introducing the invariant metric δ^{IJ} then breaks the SU*(6) down to SO*(6) ≈ SU(3,1), with an SU(2) now remaining unbroken. The formal manipulations to follow are identical to those for SL(6,ℝ) × SL(2,ℝ), though the tensors will have different reality conditions.

The fundamental 27-dimensional representation of $E_{6(6)}$ decomposes under either the SL(6,ℝ) × SL(2,ℝ) or the SU*(6) × SU(2) subgroup as

$$27 = (\overline{15},1) + (6,2)$$

and the adjoint representation decomposes as

$$78 = (35,1) + (1,3) + (20',2)\ .$$

Let Z^{AB} denote a vector in the fundamental representation of $E_{6(6)}$, and let Z_{IJ} and $Z^{I\alpha}$ denote the decompositions under SL(6,$\mathbb{R}$) × SL(2,$\mathbb{R}$) or SU*(6) × SU(2). We adopt the convention that index pairs AB belong to E_6, indices $I, J, K, \cdots$ run from 1 to 6 and transform under SL(6,$\mathbb{R}$) or SU*(6) with raised indices transforming in the 6 and lowered indices transforming in the $\bar{6}$. The indices $\alpha, \beta, \gamma, \cdots$ take the values 1 and 2 and transform under SL(2,$\mathbb{R}$) or SU(2). The vector Z_{IJ} is antisymmetric in $[IJ]$, and therefore transforms in the $\overline{15}$ of SL(6,$\mathbb{R}$) or SU*(6).

The infinitesimal action of $E_{6(6)}$ on its fundamental representation is given by

$$\delta Z^{AB} = \begin{pmatrix} \delta Z_{IJ} \\ \delta Z^{K\alpha} \end{pmatrix} = \begin{pmatrix} -2\Lambda^{[M}{}_{[I}\delta^{N]}{}_{J]} & \Sigma_{IJP\beta} \\ \Sigma^{MNK\alpha} & \Lambda^{K}{}_{P}\delta^{\alpha}{}_{\beta} + \Lambda^{\alpha}{}_{\beta}\delta^{K}{}_{P} \end{pmatrix} \begin{pmatrix} Z_{MN} \\ Z^{P\beta} \end{pmatrix} , \qquad (4.2.1)$$

where $\Lambda^{I}{}_{J}$, $\Lambda^{\alpha}{}_{\beta}$ and $\Sigma_{IJK\alpha}$ are all real in the SL(6,$\mathbb{R}$) × SL(2,$\mathbb{R}$) basis,

$$\Sigma_{IJK\alpha} = \Sigma_{[IJK]\alpha} = \frac{1}{6}\varepsilon_{IJKMNP}\varepsilon_{\alpha\beta}\Sigma^{MNP\beta} , \qquad (4.2.2)$$

and

$$\Lambda^{I}{}_{I} = \Lambda^{\alpha}{}_{\alpha} = 0 . \qquad (4.2.3)$$

The reality requirement on these quantities selects the non-compact real form $E_{6(6)}$, of E_6.

The scalars parametrizing the symmetric space $E_{6(6)}$ / USp(8) can be represented in terms of a 27-bein, $V_{AB}{}^{ab}$ [4]. This may be defined by exponentiating the infinitesimal transformation (4.2.1), to give

$$(Z')^{ab} = V_{AB}{}^{ab} Z^{AB} . \tag{4.2.4}$$

As in Reference [4] we will view $V_{AB}{}^{ab}$ as transforming in the $\overline{27}$ of E_6 and in the 27 of USp(8). The indices $a, b, c, \cdots$ run from 1 to 8, transform in the fundamental representation of USp(8), and can be raised and lowered with the symplectic metric Ω_{ab} as follows [4]:

$$X_a = \Omega_{ab} X^b , \quad X^a = \Omega^{ab} X_b . \tag{4.2.5}$$

The 27-bein is skew symmetric and symplectic traceless in the indices $[ab]$.

The inverse 27-bein, $\tilde{V}_{ab}{}^{AB}$, is defined by

$$\tilde{V}_{ab}{}^{AB} V_{AB}{}^{cd} \equiv \tilde{V}_{abIJ} V^{IJcd} + \tilde{V}_{ab}{}^{I\alpha} V_{I\alpha}{}^{cd} = I_{ab}^{cd} , \tag{4.2.6}$$

where

$$I_{ab}^{cd} \equiv \delta_{ab}^{cd} + \Omega_{ab} \Omega^{cd} , \tag{4.2.7a}$$

and for later convenience we define

$$I_{abc}^{def} \equiv \delta_{abc}^{def} + \frac{1}{2} \Omega_{[ab} \delta_{c]}^{[d} \Omega^{ef]} \tag{4.2.7b}$$

$$I_{abcd}^{efgh} \equiv \delta_{abcd}^{efgh} + \frac{3}{2} \Omega_{[ab} \delta_{cd]}^{[ef} \Omega^{gh]} + \frac{1}{8} \Omega_{[ab} \Omega_{cd]} \Omega^{[ef} \Omega^{gh]} \tag{4.2.7c}$$

$$I^{fghij}_{abcde} \equiv \delta^{fghij}_{abcde} + 5\,\Omega_{[ab}\,\delta^{[fgh}_{cde]}\Omega^{ij]} + \frac{15}{8}\Omega_{[ab}\,\Omega_{cd}\,\delta^{[f}_{e]}\Omega^{gh}\Omega^{ij]}. \qquad (4.2.7d)$$

These quantities are all projection operators, symplectic traceless in any pair of indices either both upper or both lower. We define $X_{[ab]|} \equiv I^{cd}_{ab}\,X_{cd}$, the antisymmetrization with all symplectic traces removed, for any USp(8) tensor X_{ab}, and similarly for $X_{[abc]|}$ etc. In fact, I^{cd}_{ab}, I^{def}_{abc} and I^{efgh}_{abcd} are projectors onto the 27, 48 and 42-dimensional representations of USp(8), respectively. The symplectic Schouten identity [4] states that $I^{fghij}_{abcde} = 0$; that is, $X_{[abcde]|}$ always vanishes.

The inverse relation to eqn. (4.2.6) yields

$$V^{IJab}\,\tilde{V}_{abKL} = \delta^{IJ}_{KL} \qquad (4.2.8a)$$

$$V_{I\alpha}{}^{ab}\,\tilde{V}_{ab}{}^{J\beta} = \delta^J_I\delta^\beta_\alpha \qquad (4.2.8b)$$

$$V^{IJab}\,\tilde{V}_{ab}{}^{K\beta} = V_{I\alpha}{}^{ab}\,\tilde{V}_{abKL} = 0\,. \qquad (4.2.8c)$$

The cubic invariant $J(Z)$ of $E_{6(6)}$ can be expressed in either the SL(6,ℝ) × SL(2,ℝ) or the SU*(6) × SU(2) basis as

$$J(Z^{AB}) = \frac{1}{8}\varepsilon^{IJKLMN}Z_{IJ}Z_{KL}Z_{MN} - \frac{3}{2}\varepsilon_{\alpha\beta}Z_{IJ}Z^{I\alpha}Z^{J\beta}\,. \qquad (4.2.9)$$

Since the 27-bein is an element of the group $E_{6(6)}$ the quantity $\tilde{V}_{cd}{}^{AB}\partial_\mu V_{AB}{}^{ab}$ is an element of the Lie algebra of $E_{6(6)}$ and hence can be decomposed with respect to the USp(8) subgroup as the adjoint plus the 42-dimensional representation [4], as follows

$$\tilde{V}_{cd}{}^{AB}\partial_\mu V_{AB}{}^{ab} = 2\,Q_{\mu[c}{}^{[a}\delta_{d]}{}^{b]} + P_\mu{}^{ab}{}_{cd}\,, \tag{4.2.10}$$

where $Q_{\mu ab} \equiv Q_{\mu(ab)}$ is the composite USp(8) connection and $P_{\mu abcd} \equiv P_{\mu[abcd]|}$ describes the physical scalar degrees of freedom. This equation may be rewritten

$$\tilde{V}_{cd}{}^{AB} D_\mu V_{AB}{}^{ab} = P_\mu{}^{ab}{}_{cd} \tag{4.2.11}$$

where the USp(8) covariant derivative acts as follows:

$$D_\mu X_a \equiv \partial_\mu X_a + Q_{\mu a}{}^b X_b. \tag{4.2.12}$$

4.3 Ungauged N = 8 supergravity

In this section we rewrite the ungauged N = 8 supergravity of Ref. [4] in a SL(6,ℝ) × SL(2,ℝ) or SU*(6) × SU(2) basis, to facilitate comparison with the gauged theory.

The twenty-seven vector fields are $A_{\mu IJ}$, $A_\mu{}^{I\alpha}$, with corresponding field strengths $F_{\mu\nu}{}^{AB} \equiv 2\partial_{[\mu}A_{\nu]}{}^{AB}$. The transformation rules of the fields under supersymmetry are then, to leading order in fermionic fields,

$$\delta e_\mu{}^r = \bar{\varepsilon}^a \gamma^r \psi_{\mu a} \tag{4.3.1}$$

$$\delta A_{\mu IJ} = \vartheta_\mu{}^{ab}\, \tilde{V}_{ab\, IJ} \tag{4.3.2}$$

$$\delta A_\mu{}^{I\alpha} = \vartheta_\mu{}^{ab}\, \tilde{V}_{ab}{}^{I\alpha} \tag{4.3.3}$$

$$\delta \psi_{\mu a} = D_\mu \varepsilon_a - \frac{i}{6} H_{\nu\rho ab}\left(\gamma^{\nu\rho}\gamma_\mu + 2\gamma^\nu \delta_\mu^\rho\right)\varepsilon^b \tag{4.3.4}$$

$$\delta \chi_{abc} = \sqrt{2}\, i \gamma^\mu P_{\mu abcd}\, \varepsilon^d + \frac{3}{2\sqrt{2}} \gamma^{\mu\nu} H_{\mu\nu[ab}\, \varepsilon_{c]|} \tag{4.3.5}$$

$$\begin{pmatrix} \delta V^{IJab} \\ \delta V_{I\alpha}{}^{ab} \end{pmatrix} = -2\sqrt{2}\, i \begin{pmatrix} V^{IJ}{}_{cd} \\ V_{I\alpha cd} \end{pmatrix} \bar{\varepsilon}^{[a} \chi^{bcd]|} \tag{4.3.6}$$

where the USp(8) covariant derivative is as in (4.2.12), and we have defined

$$H_{\mu\nu}{}^{ab} \equiv F_{\mu\nu IJ} V^{IJab} + F_{\mu\nu}{}^{I\alpha} V_{I\alpha}{}^{ab} \tag{4.3.7}$$

and

$$\vartheta_\mu{}^{ab} \equiv 2i\,\bar{\varepsilon}^a \psi^b_\mu + \frac{1}{\sqrt{2}} \bar{\varepsilon}_c \gamma_\mu \chi^{abc} . \qquad (4.3.8)$$

The Lagrangian is

$$e^{-1}L = \frac{1}{4}R - \frac{1}{2}\bar{\psi}^a_\mu \gamma^{\mu\nu\rho} D_\nu \psi_{\rho a} - \frac{1}{12}\bar{\chi}^{abc}\gamma^\mu D_\mu \chi_{abc}$$

$$- \frac{1}{24} P_{\mu abcd} P^{\mu abcd} - \frac{1}{8} H_{\mu\nu ab} H^{\mu\nu ab} - \frac{i}{3\sqrt{2}} P_{\nu abcd} \bar{\psi}^a_\mu \gamma^\nu \gamma^\mu \chi^{bcd}$$

$$+ \frac{i}{4} H_{\mu\nu}{}^{ab}\Big[\bar{\psi}^\rho_a \gamma_{[\rho}\gamma^{\mu\nu}\gamma_{\sigma]}\psi^\sigma_b - \frac{i}{\sqrt{2}}\bar{\psi}^c_\rho \gamma^{\mu\nu}\gamma^\rho \chi_{abc} - \frac{1}{2}\bar{\chi}_{acd}\gamma^{\mu\nu}\chi_b{}^{cd}\Big]$$

$$- \frac{1}{96}\varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon^{IJKLMN} F_{IJ\mu\nu} F_{KL\rho\sigma} A_{MN\tau}$$

$$+ \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma\tau}\, \varepsilon_{\alpha\beta}\, A_{\mu IJ}\, F_{\nu\rho}{}^{I\alpha} F_{\sigma\tau}{}^{J\beta}. \qquad (4.3.9)$$

The last two terms are the decomposition of the $E_{6(6)}$ invariant Chern-Simons form

$$-\frac{1}{12}\varepsilon^{\mu\nu\rho\sigma\tau}\,(F_{\mu\nu})^A{}_B (F_{\rho\sigma})^B{}_C (A_\tau)^C{}_A \qquad (4.3.10)$$

where the cubic invariant of $E_{6(6)}$ expressed in a USp(8) basis is

$$J(Z^{AB}) \;=\; Z^A{}_B Z^B{}_C Z^C{}_A \qquad (4.3.11)$$

with A, B, C indices in the fundamental representation of USp(8).

4.4 *T*-tensor identities

As in the four-dimensional gauged N = 8 supergravity, a fourth-rank tensor will play a crucial role in our construction. This tensor is defined in terms of the 27-bein as

$$T^{a}{}_{bcd} \equiv Y^{ae}{}_{becd}, \tag{4.4.1}$$

where

$$Y^{ab}{}_{cdef} \equiv (2\, V^{IKab}\, \tilde{V}_{cdJK} - V_{J\alpha}{}^{ab}\, \tilde{V}_{cd}{}^{I\alpha})\, \eta^{JL}\, \tilde{V}_{efIL}\,, \tag{4.4.2}$$

and η^{IJ} is the SO'(6) invariant metric, where for convenience SO'(6) will refer to any of the real forms SO(p,6-p) or SO*(6). (For the SO(6) gauging one takes $\eta^{IJ} = \delta^{IJ}$). Note that the presence of the metric explicitly breaks the SL(6,ℝ) or SU*(6) invariance down to that of the SO'(6) gauge group. Both the T- and Y-tensors carry USp(8) indices; $T^{a}{}_{bcd}$ is analogous to the SU(8) tensor $T^{i}{}_{jkl}$ of the gauged N = 8 theory in four dimensions [6].

By similar methods to those of Reference [6], one can use the $E_{6(6)}$ structure, though broken, to derive several useful properties of the T- and Y-tensors. By construction, the first four USp(8) indices of Y_{abcdef} are those of an element of the Lie algebra of $E_{6(6)}$. In particular, if we define

$$Y^{\pm}_{ab\,cdef} \equiv \frac{1}{2}(Y_{ab\,cdef} \pm Y_{cdab\,ef}), \tag{4.4.3}$$

so that $Y = Y^{+} + Y^{-}$, then

$$Y^{+}_{abcdef} = Y_{[abcd]|ef} \tag{4.4.4}$$

corresponds to the 42-dimensional representation and

$$Y^{-ab}{}_{cdef} = \frac{2}{3}\delta^{[a}{}_{[c}\, T^{b]}{}_{d]ef} \tag{4.4.5}$$

corresponds to the 36-dimensional adjoint of USp(8). From this it follows that T_{abcd} is symmetric in its first two indices.

From the definition (4.4.2) one sees directly that

$$Y_{ab\,cdef} + Y_{ab\,efcd} = V_{Jaab}\,\eta^{JK}(\tilde{V}^{Ia}{}_{cd}\,\tilde{V}_{IKef} + (cd \leftrightarrow ef)). \tag{4.4.6}$$

At this point, considerable simplifications can be made using the cubic invariant of $E_{6(6)}$. Expressing the cubic invariant (4.2.9) in the USp(8) basis, one obtains the $E_{6(6)}$ invariant quantity [4]:

$$J(Z^{ab}) = Z^{a}{}_{b}\,Z^{b}{}_{c}\,Z^{c}{}_{a}\,. \tag{4.4.7}$$

By identifying Z^{AB} with $Z^{ab}\,\tilde{V}_{ab}{}^{AB}$, substituting this into (4.2.9) and comparing it with (4.4.7), one obtains cubic identities for the inverse 27-bein $\tilde{V}_{ab}{}^{AB}$. In particular, equation (4.4.6) may be rewritten as

$$Y_{ab\,cdef} + Y_{ab\,efcd} =$$

$$= \frac{1}{2}(\Omega_{ce}\,W_{abdf} + 3\text{ terms}) - \frac{1}{4}\Omega_{cd}\,W_{abef} - \frac{1}{4}\Omega_{ef}\,W_{abcd}\,, \tag{4.4.8}$$

where the "3 terms" are to effect antisymmetry in $c \leftrightarrow d$ and $e \leftrightarrow f$ and the last two terms are to render the expression symplectic traceless in the index pairs $[cd]$ and $[ef]$. The tensor W_{abcd} is defined by

$$W_{abcd} \equiv \varepsilon^{\alpha\beta}\eta^{IJ} V_{I\alpha ab} V_{J\beta cd} \tag{4.4.9}$$

and satisfies $W_{abcd} = -W_{cdab}$.

By using equations (4.4.6) and (4.4.8) and the trivial identity

$$\begin{aligned} Y^{+}_{ab\,cdef} = \tfrac{1}{2}(Y_{ab\,cdef} + Y_{ab\,efcd}) - \tfrac{1}{2}(Y_{ab\,efcd} - Y_{ef\,ab\,cd}) - \\ - \tfrac{1}{2}(Y_{ef\,ab\,cd} + Y_{ef\,cdab}) + \tfrac{1}{2}(Y_{ef\,cdab} - Y_{cdef\,ab}) + \\ + \tfrac{1}{2}(Y_{cdef\,ab} + Y_{cdab\,ef}) , \end{aligned} \tag{4.4.10}$$

one may express Y^{+} in terms of the T- and W-tensors. Then by using the fact that Y^{+}_{abcdef} is completely antisymmetric, as well as symplectic traceless, in its first four indices (i.e., equation (4.4.4)), one may derive a number of nontrivial identities involving the T- and W-tensors. Define

$$A_{abcd} \equiv T_{a[bcd]|} \tag{4.4.11}$$

and

$$T_{ab} \equiv T^{c}{}_{abc} \; . \tag{4.4.12}$$

Then by taking the $\Omega^{bc}\Omega^{de}$ trace of (4.4.10) one finds that

$$T_{ab} = T_{ba} = \frac{15}{4} W^{c}{}_{acb} \; . \tag{4.4.13}$$

From this and the vanishing of the Ω^{bc} trace of (4.4.10) one can solve for W_{abcd} in terms of T_{abcd}:

$$W_{abcd} = \frac{2}{3} T_{acbd} + \frac{2}{5} \Omega_{ac} T_{bd} \,, \qquad (4.4.14)$$

where the expression on the right must be antisymmetrized in $a \leftrightarrow b$ and $c \leftrightarrow d$. From (4.4.14) we also have $A_{abcd} = -3\, W_{a[bcd]|}$. Finally, by taking the Ω^{de} trace of (4.4.10) and using the fact that it is antisymmetric in $[abc]$ one obtains an expression for the T-tensor itself:

$$T_{abcd} = \frac{3}{4} A_{abcd} - \frac{4}{15} \Omega_{a[c}\, T_{d]|b} + (a \leftrightarrow b) \,. \qquad (4.4.15)$$

Thus we have shown that the T-tensor has only two USp(8) irreducible components. The tensor $A_{abcd} = A_{a[bcd]|}$ is symplectic traceless in all index pairs with the total antisymmetrization $A_{[abcd]}$ vanishing, so it transforms in the irreducible 315-dimensional representation of USp(8), while T_{ab}, being symmetric in its indices, transforms in the 36-dimensional adjoint.

From the definitions (4.4.2) and (4.4.9), and the orthogonality property (4.2.8c) we immediately see that

$$Y^{ab}{}_{cdef}\, W^{ef}{}_{gh} \;=\; 0 \,. \qquad (4.4.16)$$

This fact, along with the given decompositions of the tensors involved, allow one to derive a set of quadratic T-tensor identities analogous to those of the four-dimensional theory. Substituting the expressions for T_{abcd} and W_{abcd} into eqn. (4.4.10) gives

$$Y^{+}_{abcd}{}^{ef} \;=\; 2\,\delta^{[e}{}_{[a} A^{f]}{}_{bcd]|} \,. \qquad (4.4.17)$$

Substituting this and the expression (4.4.5) for Y^- into eqn. (4.4.16) gives, after some straightforward manipulation, the two identities

$$\frac{64}{225} T_{[a}{}^{c} T_{b]|c} = A_{[a}{}^{cde} A_{b]|cde} \tag{4.4.18}$$

and

$$\frac{8}{5} T^{cd} A_{cdab} + A_{[a}{}^{cde} A_{b]|cde} + 9 A_{cde[a} A^{cde}{}_{b]|} = 0 \,. \tag{4.4.19}$$

In addition, using (4.2.10) one can derive the differential identities for the T_{ab}- and A-tensors:

$$D_\mu T_{ab} = \frac{5}{2} P_{\mu(a}{}^{cef} A_{b)cef} \tag{4.4.20}$$

$$D_\mu A_{abcd} = 3 P_\mu{}^{ef}{}_{a[b} W_{cd]|ef} + 3 P_\mu{}^{ef}{}_{[bc} W_{d]|aef} \,. \tag{4.4.21}$$

The entire set of T-tensor identities is important for proving the supersymmetry of the theory and provides numerous nontrivial consistency checks for our results.

4.5 Lagrangian and transformation rules

As discussed in Section 4.1, the field content of the SO(6) gauged theory is expected to differ from that of the ungauged theory by the substitution of twelve two-index tensor fields $B_{\mu\nu}{}^{I\alpha}$ for the vector fields $A_\mu{}^{I\alpha}$ of Section 4.3. We will take this field content and construct gauged supergravities for all the real forms SO'(6) of SO(6). This will prove to be possible even though the direct geometrical motivation for the field content provided in Chapter 3 does not apply to the noncompact gaugings.

The starting point for gauging SO'(6) is the introduction of minimal couplings to $A_{\mu IJ}$ for all objects transforming linearly under SO'(6), by covariantizing the derivatives acting upon them. The transformation properties of fields under SO'(6) are induced by the embedding SO'(6) $\in$ { SL(6,$\mathbb{R}$) or SU*(6) } $\in$ $E_{6(6)}$ as discussed in Section 4.2. For example, for a spacetime scalar X_{aI} which transforms both with respect to global SL(6,$\mathbb{R}$) or SU*(6) and composite local USp(8), the covariant derivative is

$$D_\mu X_{aI} \equiv \partial_\mu X_{aI} + Q_{\mu a}{}^b X_{bI} - g A_{\mu IJ} \eta^{JK} X_{aK} \,. \qquad (4.5.1)$$

The $E_{6(6)}$/USp(8) coset space structure for the scalars is preserved throughout the covariantation by taking

$$\tilde{V}^{abAB} D_\mu V_{AB}{}^{cd} = P_\mu{}^{abcd} \equiv P_\mu{}^{[abcd]|} , \qquad (4.5.2)$$

where D_μ is now given by (4.5.1) and not by (4.2.12). Thus (4.5.2) is the generalization of (4.2.11). This equation determines the composite USp(8) connection to be:

$$Q_{\mu a}{}^{b} = -\frac{1}{3}\Big[\, \tilde{V}^{bcAB}\partial_\mu V_{ABac} +$$

$$+ g A_{\mu IL}\eta^{JL}(2V_{ac}{}^{IK}\tilde{V}^{bc}{}_{JK} - V_{Jaac}\tilde{V}^{bcIa})\Big]. \qquad (4.5.3)$$

The piece proportional to $A_{\mu IL}$ supplies the minimal couplings to the fermions, which naïvely transform only under USp(8).

If we take variations of the scalar and vector fields of the form

$$\delta A_{\mu IJ} = \vartheta_\mu{}^{ab}\tilde{V}_{abIJ}, \quad \tilde{V}^{abAB}\delta V_{AB}{}^{cd} = \Theta^{abcd} \equiv \Theta^{[abcd]|} \qquad (4.5.4)$$

where the form of Θ^{abcd} is to preserve the coset space structure, then one finds that

$$\delta Q_{\mu ab} = \frac{1}{3}g\,T_{abcd}\vartheta_\mu{}^{cd} - \frac{2}{3}P_{\mu(a}{}^{cde}\Theta_{b)cde} \qquad (4.5.5)$$

$$\delta P_{\mu abcd} = g\,Y^{+}_{abcdef}\vartheta_\mu{}^{ef} + D_\mu\Theta_{abcd}, \qquad (4.5.6)$$

where T_{abcd} and Y^{+}_{abcdef} are precisely the tensors defined in (4.3.1), (4.3.3). The form of eqns. (4.5.5,6), in fact, provided the motivation for those definitions. Corresponding to the transformation laws (4.5.5,6) are several differential identities. The composite USp(8) field strength is

$$Q_{\mu\nu a}{}^{b} \equiv \partial_\mu Q_{\nu a}{}^{b} + Q_{\mu a}{}^{c}Q_{\nu c}{}^{b} - (\mu \leftrightarrow \nu)$$

$$= \frac{1}{3}g\,T_a{}^{b}{}_{cd}F_{\mu\nu}{}^{cd} + \frac{2}{3}P_{[\mu}{}^{bcde}P_{\nu]acde}. \qquad (4.5.7)$$

In addition, we have

$$D_{[\mu}P_{\nu]abcd} = \frac{1}{2}g\,Y^{+}_{abcdef}F_{\mu\nu}{}^{ef}, \qquad (4.5.8)$$

where we have defined

$$F_{\mu\nu}{}^{ab} \equiv F_{\mu\nu IJ} V^{IJab} . \qquad (4.5.9)$$

Naturally, $F_{\mu\nu IJ}$ is the SO'(6) covariant Yang-Mills field strength. Making the further definitions

$$B_{\mu\nu}{}^{ab} \equiv B_{\mu\nu}{}^{I\alpha} V_{I\alpha}{}^{ab} \qquad (4.5.10)$$

$$H_{\mu\nu}{}^{ab} \equiv F_{\mu\nu}{}^{ab} + B_{\mu\nu}{}^{ab} \qquad (4.5.11)$$

the orthogonality relationship (4.2.8c) gives both F and B in terms of H:

$$F_{\mu\nu IJ} = H_{\mu\nu}{}^{ab} \tilde{V}_{abIJ} \qquad (4.5.12a)$$

$$B_{\mu\nu}{}^{I\alpha} = H_{\mu\nu}{}^{ab} \tilde{V}_{ab}{}^{I\alpha} \qquad (4.5.12b)$$

and allows us to replace $F_{\mu\nu}{}^{ab}$ with $H_{\mu\nu}{}^{ab}$ in eqns. (4.5.7,8).

The covariantizations with respect to SO'(6) have introduced pieces of order g to most of the scalar identities which are necessary for demonstrating the supersymmetry of the ungauged theory. On the other hand, the order g^0 parts have the same formal structure, suggesting that the gauged and ungauged theories will share many aspects. We will first present the results of the gauging, then discuss its derivation.

The transformation rules of the fields under supersymmetry, to leading order in fermi fields, are given by

$$\delta e_\mu{}^r = \bar{\varepsilon}^a \gamma^r \psi_{\mu a} \qquad (4.5.13a)$$

$$\delta A_{\mu IJ} = \vartheta_\mu{}^{ab}\,\tilde{V}_{abIJ} \qquad (4.5.13b)$$

$$\delta B^{I\alpha}_{\mu\nu} = 2\,D_{[\mu}(\vartheta_{\nu]}{}^{ab}\,\tilde{V}_{ab}{}^{I\alpha})$$

$$+\,2g\,\eta^{IJ}\varepsilon^{\alpha\beta}\,V_{J\beta ab}(\bar{\psi}^a_{[\mu}\gamma_{\nu]}\varepsilon^b - \frac{i}{4\sqrt{2}}\bar{\chi}^{abc}\gamma_{\mu\nu}\varepsilon_c) \qquad (4.5.13c)$$

$$\delta\psi_{\mu a} = D_\mu\varepsilon_a - \frac{2}{15}ig\,T_{ab}\gamma_\mu\varepsilon^b - \frac{i}{6}H_{\nu\rho ab}(\gamma^{\nu\rho}\gamma_\mu + 2\gamma^\nu\delta^\rho_\mu)\varepsilon^b \qquad (4.5.13d)$$

$$\delta\chi_{abc} = \sqrt{2}i\gamma^\mu P_{\mu abcd}\,\varepsilon^d - \frac{1}{\sqrt{2}}g\,A_{dabc}\,\varepsilon^d + \frac{3}{2\sqrt{2}}\gamma^{\mu\nu}H_{\mu\nu[ab}\,\varepsilon_{c]|} \qquad (4.5.13e)$$

$$\begin{pmatrix} \delta\,V^{IJab} \\ \\ \delta\,V_{I\alpha}{}^{ab} \end{pmatrix} = -2\sqrt{2}i \begin{pmatrix} V^{IJ}{}_{cd} \\ \\ V_{I\alpha cd} \end{pmatrix} \bar{\varepsilon}^{[a}\chi^{bcd]|} \qquad (4.5.13f)$$

where

$$\vartheta_\mu{}^{ab} \equiv 2i\,\bar{\varepsilon}^a\psi^b_\mu + \frac{1}{\sqrt{2}}\bar{\varepsilon}_c\gamma_\mu\chi^{abc}. \qquad (4.5.14)$$

The Lagrangian for the N = 8 supergravity in five dimensions, excluding four fermion terms, is then determined to be

$$e^{-1}L = \frac{1}{4}R - \frac{1}{2}\bar{\psi}^a_\mu\gamma^{\mu\nu\rho}D_\nu\psi_{\rho a} - \frac{1}{12}\bar{\chi}^{abc}\gamma^\mu D_\mu\chi_{abc}$$

$$-\,\frac{1}{24}P_{\mu abcd}P^{\mu abcd} - \frac{1}{8}H_{\mu\nu ab}\,H^{\mu\nu ab} - \frac{i}{3\sqrt{2}}P_{\nu abcd}\bar{\psi}^a_\mu\gamma^\nu\gamma^\mu\chi^{bcd}$$

$$+\,\frac{i}{4}H_{\mu\nu}{}^{ab}\Big[\,\bar{\psi}^\rho_a\gamma_{[\rho}\gamma^{\mu\nu}\gamma_{\sigma]}\psi^\sigma_b - \frac{i}{\sqrt{2}}\bar{\psi}^c_\rho\gamma^{\mu\nu}\gamma^\rho\chi_{abc} - \frac{1}{2}\bar{\chi}_{acd}\gamma^{\mu\nu}\chi_b{}^{cd}\,\Big]$$

$$+\,\frac{i}{15}g\,T_{ab}\bar{\psi}^a_\mu\gamma^{\mu\nu}\psi^b_\nu - \frac{1}{6\sqrt{2}}g\,A_{dabc}\bar{\chi}^{abc}\gamma^\mu\psi^d_\mu$$

$$+ \frac{i}{2} g \overline{\chi}^{abc} \left(\frac{1}{2} A_{bcde} - \frac{1}{45} \Omega_{bd} T_{ce} \right) \chi_a{}^{de}$$

$$+ \frac{1}{96} g^2 \left[\frac{64}{225} (T_{ab})^2 - (A_{abcd})^2 \right]$$

$$- \frac{1}{96} \varepsilon^{\mu\nu\rho\sigma\tau} \varepsilon^{IJKLMN} \Big[F_{IJ\mu\nu} F_{KL\rho\sigma} A_{MN\tau} + g \eta^{PQ} F_{IJ\mu\nu} A_{KL\rho} A_{MP\sigma} A_{QN\tau}$$

$$+ \frac{2}{5} g^2 \eta^{PQ} \eta^{RS} A_{IJ\mu} A_{KP\nu} A_{QL\rho} A_{MR\sigma} A_{SN\tau} \Big]$$

$$- \frac{1}{8g} \varepsilon^{\mu\nu\rho\sigma\tau} \eta_{IJ} \, \varepsilon_{\alpha\beta} \, B_{\mu\nu}{}^{I\alpha} D_\rho B_{\sigma\tau}{}^{J\beta}. \qquad (4.5.15)$$

Clearly, the formal structure of the theory is quite similar to that of the ungauged theory, especially if one notes that the final term, the kinetic term for the tensor fields $B_{\mu\nu}{}^{I\alpha}$, may be written as

$$+ \frac{1}{8} \varepsilon^{\mu\nu\rho\sigma\tau} \varepsilon_{\alpha\beta} A_{\mu IJ} B_{\nu\rho}{}^{I\alpha} B_{\sigma\tau}{}^{J\beta}$$

$$- \frac{1}{8g} \varepsilon^{\mu\nu\rho\sigma\tau} \eta_{IJ} \, \varepsilon_{\alpha\beta} B_{\mu\nu}{}^{I\alpha} \partial_\rho B_{\sigma\tau}{}^{J\beta}. \qquad (4.5.16)$$

In fact, many of the manipulations used to verify the supersymmetry are identical; we will concentrate on the essentially different features of the present theory. In fact, we have allowed much more deviation from the structure of the ungauged theory than is apparent here, but it has turned out that a great many of the restrictions imposed by supersymmetry operate identically in the two theories.

The first term in the variation of $B_{\mu\nu}{}^{I\alpha}$ is analogous to $\delta F_{\mu\nu}^{I\alpha} = 2\,\partial_{[\mu}(\vartheta_{\nu]}{}^{ab}\,\tilde{V}_{ab}{}^{I\alpha})$ in the ungauged theory. The second term, upon substitution in the kinetic term, directly cancels the new variations proportional to $D_{[\mu}B_{\rho\sigma]}{}^{I\alpha}$. The corresponding terms for the ungauged theory vanish due to the Bianchi identity for $F_{\mu\nu}{}^{I\alpha}$.

The variation of the kinetic term for $B_{\mu\nu}{}^{I\alpha}$ contributes two more terms of order g^0,

$$\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon_{\alpha\beta}B_{\mu\nu}{}^{I\alpha}F_{\rho\sigma IJ}\vartheta_\tau{}^{ab}\,\tilde{V}_{ab}{}^{J\beta} \tag{4.5.17}$$

from the first term in $\delta B_{\mu\nu}{}^{I\alpha}$, and

$$\frac{1}{8}\varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon_{\alpha\beta}B_{\mu\nu}{}^{I\alpha}\vartheta_\rho{}^{ab}\,\tilde{V}_{ab\,IJ}B_{\sigma\tau}{}^{J\beta} \tag{4.5.18}$$

from the variation of $A_{\mu IJ}$ in the gauge covariant derivative. These combine with the variation of the vector Chern-Simons term,

$$-\frac{1}{32}\varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon^{IJKLMN}F_{IJ\mu\nu}F_{KL\rho\sigma}\vartheta_\tau{}^{ab}\,\tilde{V}_{ab\,MN} \tag{4.5.19}$$

to give

$$-\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma\tau}H_{\mu\nu}{}^{ab}H_{\rho\sigma}{}^{cd}\vartheta_\tau{}^{ef}\times$$

$$\times\left(\frac{1}{8}\varepsilon^{IJKLMN}\tilde{V}_{IJab}\,\tilde{V}_{KLcd}\,\tilde{V}_{MNef}-\frac{3}{2}\varepsilon_{\alpha\beta}\tilde{V}_{IJ(ab}\,\tilde{V}_{cd}{}^{I\alpha}\,\tilde{V}_{ef)}{}^{J\beta}\right). \tag{4.5.20}$$

The last factor, with the symmetrization in the three index pairs ab, cd, ef, is precisely the cubic invariant, or rather the explicit translation of the cubic invariant from the SL(6,$\mathbb{R}$) × SL(2,$\mathbb{R}$) or the SU*(6) × SU(2) basis to the USp(8) basis. Hence the expression (4.5.20) reduces to

$$-\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma\tau}H_{\mu\nu}{}^{a}{}_{b}H_{\rho\sigma}{}^{b}{}_{c}\vartheta_{\tau}{}^{c}{}_{a}, \tag{4.5.21}$$

in complete analogy to the result one obtains immediately by varying the Chern-Simons term of the ungauged theory. The rest of the order g^0 calculation is the same as for the ungauged theory.

The verification of supersymmetry to order g and g^2 in many ways parallels that of the gauged four-dimensional theories. For the order g calculation the complete set of linear T-tensor identities of Section 3.4 is needed, that is, the decomposition of all USp(8) tensors involved into irreducible components. (The tensor W_{abcd} appears in the contribution of the second term of $\delta B_{\mu\nu}{}^{I\alpha}$ to $\delta H_{\mu\nu}{}^{ab}$.) In addition, the differential identities (4.4.20,21) and (4.5.7,8) are required, as well as the symplectic Schouten identity discussed in Section 3.2. The determination of the order g terms in the Lagrangian and the transformation rules is then straightforward.

The remaining uncancelled variations are all of order g^2, and arise from the order g variations of the spinor fields in the order g terms of the form $\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu$, $\bar{\chi}\gamma^\mu\psi_\mu$ and $\bar{\chi}\chi$. These variations are either of the form $\bar{\varepsilon}\gamma^\mu\psi_\mu$ or $\bar{\varepsilon}\chi$. Those of the first sort may be written

$$-\frac{1}{12}g^2\bar{\varepsilon}^a\gamma^\mu\psi^b_\mu\left(\frac{64}{225}T_a{}^c\,T_{bc} - A_a{}^{cde}A_{bcde}\right). \tag{4.5.22}$$

From the quadratic T-identity (4.4.18), we see that the expression in parentheses is proportional to its symplectic trace, times Ω_{ab}. Since $\bar{\varepsilon}^a \gamma^\mu \psi_{\mu a}$ is simply $e^{-1}\delta e$, the expression (4.5.22) may be cancelled by that part of the variation of a scalar potential $eP(\Phi)$ which is due to the variation of the vielbein determinant e.

The remaining terms, which may be written

$$-\frac{i}{\sqrt{2}} g^2 \bar{\chi}^{abc} \varepsilon^d \left[\frac{1}{27} A_{eabc} T^e{}_d + A_{da}{}^{fg} \left(\frac{1}{2} A_{bcfg} - \frac{1}{45} \Omega_{bf} T_{cg} \right) \right] \quad (4.5.23)$$

must cancel against the variation of the purely scalar part of the potential term. The variations of the T-tensors are given by

$$\delta T_{ab} = \frac{5}{2} \Theta_{(a}{}^{cef} A_{b)cef} \quad (4.5.24)$$

$$\delta A_{abcd} = 3\,\Theta^{ef}{}_{a[b} W_{cd]|ef} + 3\,\Theta^{ef}{}_{[bc} W_{d]|aef} \quad (4.5.25)$$

corresponding to the differential identities (4.2.20,21). The tensor Θ^{abcd} parametrizing the scalar variations is defined in (4.5.4) and from (4.5.13f) may be identified as $2\sqrt{2} i \bar{\chi}^{[abc} \varepsilon^{d]|}$. Note that in order for the variation of the potential, involving (4.5.24,25), to cancel against (4.5.23), the expression in (4.5.23) multiplying $\bar{\chi}^{abc} \varepsilon^d$ must project out the antisymmetric and symplectic component $\bar{\chi}^{[abc} \varepsilon^{d]|}$. The vanishing of the trace terms requires a further quadratic T identity, precisely eqn. (4.4.19). The antisymmetry in $[abcd]$ and finally the actual cancellation against the variation of the scalar potential then follow from straightforward manipulations.

This concludes our summary of the demonstration of supersymmetry for the gauged N = 8 supergravity theories.

The Lagrangian (4.5.15) is analogous to that of the four-dimensional gauged theory, with the composite local SU(8) replaced by a composite local USp(8), and the propagating vector fields now gauging SO'(6) instead of SO(8). However, one outstanding difference between the four- and five-dimensional gauged N = 8 theories is that the latter has a global invariance of either SL(2,$\mathbb{R}$) ≈ SU(1,1) or SU(2), depending on the gauging. For the compact SO(6) gauging, we believe that the SU(1,1) may be interpreted as the descendant of the global SU(1,1) symmetry of the ten-dimensional theory.

In the foregoing we have only gauged the fifteen parameter simple subgroups of SL(6,$\mathbb{R}$). One might ask whether one can also gauge the standard contractions of SO(p,6-p). While it may be possible to mimic the construction of reference [18], it is not clear how this may be done in our formulation. For these group contractions the invariant metric η^{IJ} has zero eigenvalues, and consequently the kinetic term for the $B_{\mu\nu}{}^{I\alpha}$ fields, which involves the inverse metric η_{IJ}, will be undefined in the corresponding directions. In order to gauge the contractions of SO(p,6-p), it may be necessary to replace with vector fields those antisymmetric two-index tensor fields which would not have well-defined kinetic terms.

The terms involving the $\varepsilon^{\mu\nu\rho\sigma\tau}$ and the vector fields alone constitute the Chern-Simons form for the gauge group SO'(6). Such terms were also found in the gauging of a special class of N = 2 Maxwell-Einstein supergravity theories in five dimensions [19,20]. As was observed for these N = 2 theories, the Chern-Simons form, at least for SO(6) ≈ SU(4), leads to a quantization of the dimensionless ratio $\frac{\kappa}{g^3}$, since the fifth homotopy group of SU(4) is $\mathbb{Z}$.

The scalar potential in (4.5.15) has exactly the same form as that of the four-dimensional theory, the tensors T_{ab} and A_{abcd} being the analogues of A_{1ij} and $A_{2i}{}^{jkl}$ of Reference [6]. One can also show that under left multiplication of the 27-bein by an element of $E_{6(6)}$, the tensors T_{ab} and A_{abcd} transform in the 351 of $E_{6(6)}$. This is analogous to A_1 and A_2 transforming in the 912 of $E_{7(7)}$ [21].

The commutator of two supersymmetries is given by

$$[\delta_{s.s.}(\varepsilon_2), \delta_{s.s.}(\varepsilon_1)] = \delta_{g.c.}(\xi^\mu) + \delta_{s.s.}(\varepsilon') + \delta_{l.l.}(\Sigma^{rs}) + \\ + \delta_{USp(8)}(\lambda_a{}^b) + \delta_{SO'(6)}(\Lambda_{IJ}) \qquad (4.5.26)$$

where to leading order in fermi fields

$$\xi^\mu = \bar{\varepsilon}_1^c \gamma^\mu \varepsilon_{2c} \qquad (4.5.27a)$$

$$\varepsilon'^a = -\xi^\mu \psi_\mu^a - \frac{i\sqrt{2}}{4}(3\chi^{abc}\bar{\varepsilon}_{1b}\varepsilon_{2c} - \gamma_r \chi^{abc}\bar{\varepsilon}_{1b}\gamma^r \varepsilon_{2c}) \qquad (4.5.27b)$$

$$\Sigma^{rs} = \xi^\mu \omega_\mu{}^{rs} + 2i\bar{\varepsilon}_1^a \Big[-\frac{2}{45} g\, T_{ab}\gamma^{rs} + \\ + \frac{1}{6} H_{tu\,ab}(\gamma^{rstu} + 4g^{rt}g^{su}) \Big] \varepsilon_2^b \qquad (4.5.27c)$$

$$\lambda_a{}^b \;=\; \xi^\mu Q_{\mu a}{}^b - \frac{2}{3} ig\, T_a{}^b{}_{cd} \bar{\varepsilon}_1^c \varepsilon_2^d \tag{4.5.27d}$$

$$\Lambda_{IJ} \;=\; -\xi^\mu A_{\mu IJ} + 2i\,\bar{\varepsilon}_1^a \varepsilon_2^b \tilde{V}_{abIJ}. \tag{4.5.27e}$$

The results are quite analogous to those of the ungauged theory [4], as expected, though there are some interesting manifestations of the SO'(6) structure. To determine the supersymmetry parameter ε'^a and verify the closure of the algebra on χ^{abc}, we have assumed that there are no g-dependent terms in either the Lagrangian or the transformation rules among the terms of higher order in fermionic fields than those presented here. The corresponding result has been demonstrated for the four-dimensional gauging [6], and the cancellation of the g-dependent terms under this assumption provided a great many consistency checks for the present theory. However, we have not as yet obtained a complete proof that the gauged theory has the same structure as the ungauged theory of Ref. [4] to higher order in fermionic fields.

At any rate, the closure of the algebra on either χ^{abc} or ψ_μ^a requires the field equations of both, which we give here, to leading order:

$$\gamma^{\mu\nu\rho} D_\nu \psi_{\rho a} + \frac{i}{3\sqrt{2}} P_{\nu abcd} \gamma^\nu \gamma^\mu \chi^{bcd} - \frac{i}{2} H^{\rho\sigma}_{ab} \gamma^{[\mu} \gamma_{\rho\sigma} \gamma^{\nu]} \psi_\nu^b -$$

$$- \frac{1}{4\sqrt{2}} H^{bc}_{\rho\sigma} \gamma^{\rho\sigma} \gamma^\mu \chi_{abc} - \frac{2}{15} ig\, T_{ab} \gamma^{\mu\nu} \psi_\nu^b +$$

$$+ \frac{1}{6\sqrt{2}} g\, A_{abcd} \gamma^\mu \chi^{bcd} \;=\; 0 \tag{4.5.28}$$

$$\frac{1}{6}\gamma^\mu D_\mu \chi_{abc} - \frac{i}{3\sqrt{2}} P_{\nu abcd}\gamma^\mu\gamma^\nu\psi^d_\mu - \frac{i}{4\sqrt{2}}\gamma^\rho\gamma_{\mu\nu}\psi_{\rho[a}H^{\mu\nu}_{bc]|} +$$

$$+ \frac{i}{4}\gamma_{\mu\nu}\chi^d{}_{[ab}H^{\mu\nu}_{c]|d} + \frac{1}{6\sqrt{2}} g\, A_{dabc}\gamma^\mu\psi^d_\mu -$$

$$- ig\left(\frac{1}{2}A_{[ab}{}^{ef} + \frac{1}{45}\delta_{[a}{}^e T_b{}^f \right)\chi_{c]|ef} = 0. \qquad (4.5.29)$$

The closure of the algebra on $B_{\mu\nu}{}^{I\alpha}$ requires its own field equation; this is generic for fields with first-order equations of motion, bosonic as well as fermionic. The field equation may be written, to leading order in fermionic fields,

$$D_{[\mu}B_{\nu\rho]}{}^{I\alpha} + \frac{1}{12} g\,\eta^{IJ}\varepsilon^{\alpha\beta}\varepsilon_{\mu\nu\rho\sigma\tau}V_{J\beta}{}^{ab}H^{\sigma\tau}_{ab} = 0. \qquad (4.5.30)$$

Up to the scalar factors and the interactions with the vectors, this is exactly the sort of self-duality field equation discussed in Sections (3.2) and (4.1), and references therein.

4.6 Discussion

There is much interesting structure in the scalar potentials for the various gaugings of N = 8 supergravity in five dimensions. The most interesting scalar potential is that of the SO*(6) ≈ SU(3,1) gauging. For this gauging, the origin in the scalar manifold is a critical point, with zero cosmological constant. This means that any Ricci-flat five-dimensional spacetime, in particular the product of four-dimensional Minkowski space with a circle, satisfies the field equations. The maximally symmetric solution breaks the gauge symmetry to its maximal compact subgroup SU(3) × U(1), which acquires another U(1) factor upon reduction to four dimensions, from the isometry group of the circle. (Contrary to what occurs for compact gaugings, even the maximally symmetric vacuum, if one exists, must break the original gauge symmetry. A noncompact group may be a gauge symmetry of the theory, but it cannot be preserved in a sensible vacuum.) The supersymmetry is broken to N = 2. A global SU(2) symmetry acting upon the scalars also survives, and to the extent that one believes in composite scenarios the phenomenological gauge group (with an extra U(1) factor) may be generated from the

$$SU(3) \times U(1) \times U(1) \times SU(2)_{\text{global}}$$

symmetry of the four-dimensional theory.

Another remarkable aspect of this gauging is that the scalar potential is completely flat in the directions of 14 of the 42 scalars. This means that these scalars may take any value whatsoever, and flat spacetime will still solve the equations of motion, as long as the other 28 scalars are still zero. This provides a mechanism similar to that of Ref. [20] for breaking the N = 2 supersymmetry at any mass scale desired.

In the same way that one finds a relationship between compactification of the eleven-dimensional theory based on S^7 and critical points in four dimensions [8], we expect the various vacua of the gauged SO(6) theory to correspond to compactifications of the ten-dimensional chiral N = 2b theory. There are two known non-trivial compactifications of the latter theory based on S^5. One solution, obtained in Section 2.3, has a non-vanishing value for the complex three-index field strength, with the metric obtained from S^5 by stretching in the direction of the U(1) fiber over $\mathbb{CP}^2$. This solution has SU(3) gauge symmetry. The other solution [22], similar to that of Reference [23], assigns all the field strengths the standard values for the trivial compactification, but gives the space-time metric a Weyl rescaling which depends on the internal coordinates. This solution has SO(5) gauge symmetry. Both of these compactifications break all supersymmetries.

We have indeed found critical points of the potential in the local SO(6) invariant Lagrangian (4.5.15), breaking all the supersymmetry and reducing the gauge symmetry to SO(5) and SU(3), respectively. The SO(5) invariant critical point corresponds to giving an expectation value to a scalar in the 20′ of SO(6). Similarly, the SU(3) critical point corresponds to an expectation in the $10 + \overline{10}$ of SO(6). This agrees with the identification in Chapter 3 of the ten-dimensional origins of these modes, and suggests that the $20'$ and $10 + \overline{10}$ play similar roles to the scalar and the pseudoscalar fields, respectively, in the four-dimensional theory.

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Chapter 5

Ten plus R squared goes into Four*

5.1 Introduction

There has been a recent surge of interest in ten-dimensional theories of N = 1 supergravity coupled to super-Yang-Mills matter, due to the intimate connection these theories are believed to have with superstring theories. The coupled field theory was first formulated for arbitrary gauge group [1] (but see [2] for a correction to the four-fermion sector) following the coupling of the pure Maxwell multiplet [3]. Unfortunately, it was soon realized that none of these theories, as they stood then, admitted compactifications to a maximally symmetric four-dimensional spacetime [4] (see [5], however, for solutions with non-maximally symmetric four-dimensional cosmologies).

Meanwhile, it has been shown that all the ten-dimensional field theories with N = 1 supergravity are inconsistent because of either

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purely gravitational or mixed gauge and gravitational anomalies, unless the Yang-Mills matter multiplet has gauge group SO(32) or $E_8 \times E_8$. For those gauge groups all anomalies cancel (at the one-loop level) upon the addition of a suitable local counterterm, if one takes the proper gauge and Lorentz transformation properties for the fields of the theory [6]. This statement generalizes to the entire SO(32) Type I superstring theory [6], and motivated the construction of the $E_8 \times E_8$ and SO(32) heterotic superstring theories [7].

The anomaly cancellation for the field theory requires the two-index tensor gauge field B_{mn} in the supergravity multiplet to have an unusual Lorentz transformation property. Naïvely, one might have expected a tensor B_{mn} with two world indices to be inert under local Lorentz transformations, transforming only under general coordinate transformations. However, under Lorentz transformations with parameter Θ_{ab}, it is necessary to impose

$$\delta B_{mn} = \omega_{[m}{}^{ab} \partial_{n]} \Theta^{ba}, \qquad (5.1.1)$$

where $a, b, \cdots$ are flat (local Lorentz) indices and $m, n, p, \cdots$ are curved (world) indices. The field strength for B_{mn}, H_{mnp}, must not transform under Lorentz transformations; this is accomplished by modifying H_{mnp} by subtracting away a term which varies into the curl of the expression in equation (5.1.1), namely, the Lorentz Chern-Simons form

$$(\omega_{3L}^{0})_{mnp} \;\equiv\; \frac{1}{2}R_{[mn}{}^{ab}\,\omega_{p]}{}^{ba} - \frac{1}{3}\omega_{[m}{}^{ab}\,\omega_{n}{}^{bc}\,\omega_{p]}{}^{ca}. \qquad (5.1.2)$$

As pointed out in [6], these modifications are exactly analogous to the appearance of the Yang-Mills Chern-Simons form

$$(\omega_{3Y}^{0})_{mnp} \;\equiv\; \mathrm{Tr}\,(\,\frac{1}{2}F_{[mn}A_{p]} - \frac{1}{3}A_{[m}A_{n}A_{p]}) \qquad (5.1.3)$$

in the three-index field strength which now reads

$$H_{mnp} \;=\; \partial_{[m}B_{np]} + \frac{1}{30}(\omega_{3Y}^{0})_{mnp} - (\omega_{3L}^{0})_{mnp}. \qquad (5.1.4)$$

The fields A_m and F_{mn} and the Yang-Mills gauge parameter Λ are antihermitian matrices in the adjoint representation of the gauge group G = SO(32) or $E_8 \times E_8$. The following transformation rules under local Yang-Mills and Lorentz transformations establish our normalizations:

$$\delta A_m \;=\; D_m\Lambda \;\equiv\; \partial_m\Lambda + [A_m,\Lambda] \qquad (5.1.5)$$

$$\delta\omega_m{}^{ab} \;=\; D_m\Theta^{ab} \;\equiv\; \partial_m\Theta^{ab} + \omega_m{}^{ac}\,\Theta^{cp} + \omega_m{}^{bc}\,\Theta^{ac} \qquad (5.1.6)$$

$$\delta B_{mn} \;=\; -\frac{1}{30}\mathrm{Tr}\,(A_{[m}\partial_{n]}\Lambda) + \omega_{[m}{}^{ab}\,\partial_{n]}\Theta^{ba} \qquad (5.1.7)$$

$$\delta F_{mn} \;=\; [F_{mn},\Lambda]\,, \quad \delta R_{mn}{}^{ab} \;=\; 2\,R_{mn}{}^{[a}{}_{c}\,\Theta_{c}{}^{b]}\,, \quad \delta H_{mnp} \;=\; 0\,. \qquad (5.1.8)$$

In order to cancel the gauge and gravitational anomalies, the Lagrangian itself must have a local counterterm S added. This counterterm is expressible as the sum of terms of the form constant $\times$ $\varepsilon^{mnpqrstuvw} \times Z_{mnpqrstuvw}$, where $Z_{mnpqrstuvw}$ is either

$$B_{mn}\mathrm{Tr}(F_{pq}F_{rs}F_{tu}F_{vw}), \qquad (5.1.9a)$$

$$B_{mn}R_{pq}{}^{ab}R_{rs}{}^{bc}R_{tu}{}^{cd}R_{vw}{}^{da}, \qquad (5.1.9b)$$

$$B_{mn}X_{pqrs}X'_{tuvw}, \qquad (5.1.9c)$$

$$(\omega^0_{3L})_{mnp}(\omega^0_{3Y})_{qrs}X''_{tuvw} \qquad (5.1.9d)$$

or

$$(\omega^0_{3L,Y})_{mnp}(\omega^0_{7L,Y})_{qrstuvw}, \qquad (5.1.9e)$$

where the X_{mnpq}'s are either $\mathrm{Tr}(F_{mn}F_{pq})$ or $R_{mn}{}^{ab}R_{pq}{}^{ba}$, and

$$(\omega^0_{7Y})_{mnpqrst} = \mathrm{Tr}(F_{[mn}F_{pq}F_{rs}A_{t]} + + FFA^3 + FA^5 + A^7 \quad \text{terms}) \qquad (5.1.10)$$

and similarly for ω^0_{7L}.

The field theory of Ref. [1] is invariant under the global scale transformation

$$L \to \Omega L, \quad g^{mn} \to \Omega g^{mn}, \quad \varphi \to \Omega\varphi,$$

$$\psi_\mu \to \Omega^{-\frac{1}{4}}\psi_\mu, \quad \lambda \to \Omega^{\frac{1}{4}}\lambda, \quad \chi \to \Omega^{\frac{1}{4}}\chi \qquad (5.1.11)$$

corresponding to an invariance of string theories which is expected to survive in the classical low-energy effective theories [8]. The counter-term S would require an overall scalar factor of φ^{-4} in order to preserve this invariance. However, such a factor would spoil the gauge invariances of the theory (for example, invariance under $\delta B_{mn} = \partial_{[m}\zeta_{n]}$) and hence cannot appear, indicating that the scale invariance is broken at the one-loop order.

5.2 R squared terms

In addition to these necessary modifications for anomaly cancellation, it is known that the effective theories corresponding to superstring theories must contain terms quadratic in curvatures (see Ref. [9] and references therein). In general one expects R^2 terms (and higher-order terms, in a general background metric) to lead to badly behaved theories, in which the graviton inverse propagator is fourth-order in derivatives. However, it has been shown [10] that a particular combination of R^2 terms is "safe," having only two-derivative inverse propagators. This combination,

$$y^4 \equiv \frac{4!}{2^2}\delta^{mnpq}_{rstu} R_{mn}{}^{rs} R_{pq}{}^{tu}$$

$$= R^2 - 4R^{mn}R_{mn} + R^{mnpq}R_{mnpq}, \qquad (5.2.1)$$

is a topological invariant in four dimensions, the Euler class, and hence leads to no dynamics. (As usual, $\delta^{mnpq}_{rstu} \equiv \delta^m_{[r}\delta^n_s\delta^p_t\delta^q_{u]}$ with all antisymmetrizations of unit weight. In four dimensions $\delta^{mnpq}_{rstu} = \frac{1}{24}\varepsilon^{mnpq}\varepsilon_{rstu}$.) However, in more dimensions y^4 has dynamical content with sensible propagators. So far, explicit calculations in string theories have yielded only pieces of y^4, but, as argued in [10], the fact that string theories are ghost-free strongly suggests that the R^2 terms appear exactly as in eqn. (5.2.1).

As an aside, we remark that there is an extremely easy proof that y^4 and its generalizations

$$y^6 \equiv \frac{6!}{2^3}\delta^{mnpqrs}_{tuvwxy}\, R_{mn}{}^{tu} R_{pq}{}^{vw} R_{rs}{}^{xy}$$

$$= R^3 + 3R(R^{mnpq}R_{mnpq} - 4R^{mn}R_{mn}) + 16 R_m{}^n R_n{}^p R_p{}^m +$$

$$+ 2R^{mn}{}_{pq}(R^{pq}{}_{rs}R^{rs}{}_{mn} - 4R^{ps}{}_{mr}R^{rq}{}_{sn} + 12 R_m{}^p R_n{}^q - 12 R_m{}^r R_{rn}{}^{pq}),$$

$$y^8 \equiv \frac{8!}{2^4}\delta^{mnpqrstu}_{vwxyzjkl}\, R_{mn}{}^{vw} R_{pq}{}^{xy} R_{rs}{}^{zj} R_{tu}{}^{kl} = R^4 + \cdots \qquad (5.2.2)$$

and so forth, all lead to ghost-free theories.* Consider the Lagrangian

$$L = f(\Phi)y^6 \qquad (5.2.3)$$

where $f(\Phi)$ is an arbitrary scalar function of the other fields of the theory (but not their derivatives). Under a metric variation $\delta g^{mn} = h^{mn}$ we have

$$\delta R_{mn}{}^{pq} = R_{mn}{}^{[p}{}_r h^{q]r} + 2D_{[m}D^{[p}h_{n]}{}^{q]}\ , \qquad (5.2.4)$$

so

$$\delta(eL) = 90\, e f \delta^{mnpqrs}_{tuvwxy}\, 3\left[R_{mn}{}^t{}_z h^{uz} + 2D_m D^t h_n{}^u\right] R_{pq}{}^{vw} R_{rs}{}^{xy}$$

$$- \frac{1}{2} e h^{mn} g_{mn} L \qquad (5.2.5)$$

* Essentially the same argument was found independently by B. Zumino [11].

where e is the vielbein determinant. The term with two derivatives of h^{mn} may be integrated by parts, but when either of the derivatives acts upon a Riemann curvature, the result vanishes due to the Bianchi identity

$$D_{[m} R_{pq]}{}^{vw} = 0 = D^{[t} R_{pq}{}^{vw]} \qquad (5.2.6)$$

(which holds in our formalism, since the connections do not contain torsion) and the antisymmetry imposed by the delta symbol. Thus the contribution to the graviton field equation is polynomial in curvatures, and in fact is

$$270\, \delta^{mnpqrs}_{tuvwxy} \Big[f R_{mn}{}^{t}{}_{(k} \delta_{l)}{}^{u} + 2\,(D^t D_m f)\, g_{n(k} \delta_{l)}{}^{u} \Big] R_{pq}{}^{vw} R_{rs}{}^{xy}$$

$$- \frac{1}{2} g_{kl} L \,. \qquad (5.2.7)$$

The inverse propagator for h^{mn} is given by the variation of the field equation (i.e., the piece of the Lagrangian quadratic in h^{mn}) and clearly can have only two derivatives acting on h^{mn}, for arbitrary background curvatures. The same argument obviously applies to all the y^i ($i \geq 4$).

The introduction of the Lorentz Chern-Simons form in the field strength H_{mnp} certainly spoils the supersymmetry of the theory in Ref. [1]. One way to restore supersymmetry is to couple towers of fields of arbitrarily high spin so as to reproduce an entire superstring theory, but it is still uncertain whether there is a well-defined effective theory containing a finite number of particles which describes low-energy superstring interactions and is supersymmetric and anomaly free as a field theory. If one simply tries to modify the Lagrangian and transformation rules to restore the supersymmetry, one is led to consider at least R^2 type terms, but there is no compelling reason to believe that there are not many more types of interactions, for example, the y^6 and y^8 terms in the series of equation (5.2.2). In ten dimensions y^{10} is a total derivative and the higher-order y^i vanish.

We will take the approach advocated in much of Ref. [9] and study the best candidates for effective theories we have so far, namely, the theories of [1] with gauge group G = $E_8 \times E_8$ or SO(32), the Chern-Simons modifications to H_{mnp}, and plausible R^2 terms. As will be discussed, there is reason to suspect that some of the classical solutions we consider will solve the string field equations and provide sensible backgrounds for string theories.

5.3 Field equations

Motivated by these considerations, we proceed to an analysis of the Lagrangian given by

$$L = -\frac{1}{2\kappa^2}R - \frac{1}{\kappa^2}\varphi^{-2}D^m\varphi\, D_m\varphi - \frac{3\kappa^2}{2}\varphi^{-2}H^{mnp}H_{mnp} +$$

$$+ \frac{1}{4}\varphi^{-1}\left\{ \frac{1}{30}\mathrm{Tr}(F^{mn}F_{mn}) + y^4 \right\} + S, \qquad (5.3.1)$$

where H_{mnp} is given in (5.1.4) and y^4 in (5.2.1), and we have set the fermionic fields to zero to look for compactifications. The sign of the vector kinetic term is positive since the trace is negative definite (recall that the F_{mn} are antihermitian matrices). The gravitational coupling κ has dimensions of (length)^4 and the (strictly positive) scalar field φ has dimensions of (length)^6. The form of the y^4 coupling is taken from [9] as generalized in [10]. The scalar factor multiplying y^4 is to preserve the classical scale invariance of eqn. (5.1.11); in general, y^{2i} requires a factor of φ^{1-i} in front.

Several of the field equations are completely straightforward to read off from (5.3.1). The equation of motion for the scalar field is

$$\frac{2}{\kappa^2}\varphi^{-1}D^m(\varphi^{-1}\partial_m\varphi) + 3\kappa^2\varphi^{-3}H^{mnp}H_{mnp} -$$

$$- \frac{1}{4}\varphi^{-2}\left\{ \frac{1}{30}\mathrm{Tr}(F^{mn}F_{mn}) + y^4 \right\} = 0. \qquad (5.3.2)$$

For the tensor field B_{mn} we have

$$D_p(\varphi^{-2}H^{mnp}) + \frac{1}{3\kappa^2}\frac{\delta S}{\delta B_{mn}} = 0. \qquad (5.3.3)$$

The counterterm S should be properly considered part of the one-loop effective potential, ignorable at the level of the present discussion. However, it will turn out that for the background configurations we will consider (i.e., factorized "4 + 6" spacetime geometry and our ansätze for F_{mn} and B_{mn}) the general form of expressions (5.1.9a-e) is sufficient to show that the contributions of S to the field equations all vanish. For now we will formally include these contributions (which are completely straightforward to evaluate) but not discuss them in any detail.

To determine the field equation for A_m, we need to know how H_{mnp} varies under a general change in A_m. For the Yang-Mills Chern-Simons form

$$\delta(\omega^0_{3Y})_{mnp} = -\partial_{[m}\mathrm{Tr}(A_n\delta A_{p]}) + \mathrm{Tr}(F_{[mn}\delta A_{p]}). \qquad (5.3.4)$$

The presence of the first term leads to noncovariant field equations, but it is of the same form as a variation of B_{mn},

$$\delta B_{mn} = -\frac{1}{30}\mathrm{Tr}(A_{[m}\delta A_{n]}). \qquad (5.3.5)$$

By definition, the right-hand side of eqn. (5.3.5) will simply multiply the B_{mn} field equation when the Lagrangian is varied with respect to A_m. Hence if we are only interested in the whole system of field equations, we are free to ignore the first term in the variation of $(\omega^0_{3Y})_{mnp}$ given in (5.3.4). Equivalently, we may consider the variation of A_m as inducing a compensating variation of B_{mn} to cancel the unwanted term. The remaining part of the variation of $(\omega^0_{3Y})_{mnp}$, the last term in (5.3.4), leads to perfectly gauge-covariant field equations for A_m.

The fact that the Lagrangian varies into a combination of field equations suggests that one has not properly identified the true independently propagating modes. The formalism in which a variation of A_m induces a compensating transformation of B_{mn} is reminiscent of what generically occurs when the physical fields of a theory form a non-linear realization of a symmetry group. The Yang-Mills gauge symmetry is indeed nonlinearly realized here, as reflected in its action on B_{mn} (see eqn. (5.1.7)). These issues will be clarified by a better geometrical understanding of the unusual gauge transformations of B_{mn}; such an understanding may come from a study of the superstrings containing this field.

The resulting field equation for A_p (modulo the field equation for B_{mn}) is then

$$D_n(\varphi^{-1}F^{np}) + 3\kappa^2\varphi^{-2}H^{mnp}F_{mn} - \frac{\delta S}{\delta A_p} = 0. \qquad (5.3.6)$$

For the graviton we have the analogous problem that the Chern-Simons variation is not covariant:

$$\delta(\omega^0_{3L})_{mnp} = -\partial_{[m}(\omega_n{}^{ab}\,\delta\omega_{p]}{}^{ba}) + R_{[mn}{}^{ab}\,\delta\omega_{p]}{}^{ba}. \qquad (5.3.7)$$

($\delta\omega_m{}^{ab}$ is covariant since D_m is, but, of course, $\omega_m{}^{ab}$ itself is not.) Again one notes that under arbitrary vielbein variations $\delta e_m{}^a$, the first term has the form of a variation of B_{mn}. One then obtains a contribution to the graviton field equation due to the last term of eqn. (5.3.7), and exactly the same conceptual issues as for the A_m field.

It is interesting to note that by defining

$$\delta B_{mn} \;=\; -\omega_{[m}{}^{ab}\,\delta\omega_{n]}{}^{ba} + R_{mna}{}^{p}\,\delta e_p{}^a - 2\,\partial_{[m}(\omega_{n]p}{}^a\delta e_p{}^a) \qquad (5.3.8)$$

one not only cancels the noncovariant part of $\delta(\omega_{3L}^0)$, but also exactly reproduces the local Lorentz transformation of B_{mn}, since the antisymmetric part of δe_{ma} acts as a Lorentz transformation. Note that the last term of eqn. (5.2.8) is simply a gauge transformation. For the symmetric part $e_{(m}{}^a\delta e_{n)a} \;=\; -\frac{1}{2}h_{mn}$ (recall $h^{mn} \;\equiv\; \delta g^{mn}$), only the last term of eqn. (5.3.7) contributes, yielding

$$\delta H_{mnp} \;=\; R_{rs[mn}D^r h^s{}_{p]}. \qquad (5.3.9)$$

The analogous interpretation of the Yang-Mills gauge transformation of B^{mn} as a special case of its dependence of A_m (as reflected in eqn. (5.3.5)) cannot be realized since A_m is an elementary field, as opposed to $\omega_m{}^{ab}$.

The contribution from the R^2 terms to the graviton field equation is easily evaluated as

$$\delta(e\varphi^{-1}y^4) \;=\; e\varphi^{-1}h^{kl}\Bigg\{\,12\,\delta^{mnpq}_{rstu}\Big[\,R_{mn}{}^r{}_k\,\delta^s_l \;+$$

$$+ 2(\varphi D^r D_m \varphi^{-1}) g_{nk} \delta^s_l \Big] R_{pq}{}^{tu} - \frac{1}{2} g_{kl} L_2 \Big\}. \qquad (5.3.10)$$

Using the expressions (5.3.9) and (5.3.10) to vary the H^2 and y^4 terms, the covariant equation of motion for the graviton, modulo the equation of motion for B_{mn}, is then

$$0 = -\frac{1}{2\kappa^2} R_{mn} - \frac{1}{\kappa^2} \varphi^{-2} \partial_m \varphi \partial_n \varphi - \frac{9\kappa^2}{2} \varphi^{-2} H_m{}^{pq} H_{npq} -$$

$$- 3 D^r (\varphi^{-2} H^{pq}{}_{(m} R_{n)rpq}) +$$

$$+ \frac{1}{4} \varphi^{-1} \Big\{ \frac{1}{30} \mathrm{Tr}(2 F_m{}^p F_{np}) + 24 g_{mv} \delta^{vstu}_{npqr} R_{tu}{}^{qr} (\varphi D^p D_s \varphi^{-1}) +$$

$$+ 2 (R R_{mn} - 2 R_m{}^p R_{np} - 2 R^{pq} R_{mpnq} + R_m{}^{pqr} R_{npqr}) \Big\} +$$

$$+ \frac{\delta S}{\delta h^{mn}} - \frac{1}{2} g_{mn} L. \qquad (5.3.11)$$

As pointed out in Ref. [8], the scale invariance of eqn. (5.1.11) implies that the Lagrangian vanishes on shell (at least when the anomalous contribution from S vanishes, as it will in our case). In fact, the Lagrangian itself is a linear combination of the scalar field equation and the trace of the graviton field equation. Thus we may impose eqn. (5.3.11) without the last term $-\frac{1}{2} g_{mn} L$, and upon imposing the scalar field equation (5.3.2), $L = 0$ will emerge as a consistency check.

The Bianchi identity following from eqn. (5.1.4),

$$4\,\partial_{[m} H_{npq]} \;=\; \frac{1}{30}\mathrm{Tr}\,(F_{[mn} F_{pq]}) - R_{[mn}{}^{ab} R_{pq]ba}, \qquad (5.3.12)$$

serves as the integrability condition for the field B_{mn}.

5.4 Compactification to four dimensions

We may now look for solutions of the system of equations (5.3.2,3,6,11,12). We are interested in solutions with maximal four-dimensional spacetime symmetry, so we take

$$R_{\mu\nu}{}^{\rho\sigma} = 2\mu\delta^{\rho\sigma}_{\mu\nu}\,, \quad \text{which gives} \quad R_{\mu}{}^{\nu} = 3\mu\delta^{\nu}_{\mu} \tag{5.4.1}$$

and we require all other fields to only have components in the internal six dimensions. From now on, indices μ, ν, $\cdots$ run over the four spacetime directions and $m, n,$ $\cdots$ run over the six internal directions. The parameter μ is negative for anti-de Sitter space and positive for de Sitter space. We will take for an ansatz φ = constant, which, as stated in Ref. [4], would require F_{mn} and H_{mnp} to vanish if the y^4 term were absent. This is easy to see in this context. Since the metric is positive definite in the internal six directions, the scalar field equation (5.3.2) forces both the $(H)^2$ and the $(F)^2$ terms to vanish, unless, of course, the y^4 term can cancel them.

We will follow Ref. [9] (and references within) and consider an embedding of the holonomy group H within the gauge group G, either SO(32) or $E_8 \times E_8$. Let the generators of the internal local Lorentz group SO(6) be $M_{ab} = M_{[ab]}$, where $a, b, ,\cdots$ are flat internal vector indices. Then the holonomy group is the subgroup of SO(6) spanned by the curvature operator $R_{mn}{}^{ab} M_{ab}$ as it sweeps over the internal manifold. If $a_m{}^{ab} = a_m{}^{[ab]}$ is any spacetime tensor with $a_m{}^{ab} M_{ab}$ taking values within the holonomy group, the same will hold for

$$A_m{}^{ab} \equiv \omega_m{}^{ab} + a_m{}^{ab}. \qquad (5.4.2)$$

The embedding of H in G gives a prescription for considering $A_m{}^{ab}$ as a gauge field in the algebra of G. We then have for the usual Yang-Mills field strength

$$F_{mn}{}^{ab} = R_{mn}{}^{ab} + 2\,(D_{[m}a_{n]}{}^{ab} + a_{[m}{}^{ac}a_{n]}{}^{cb}). \qquad (5.4.3)$$

In eqn. (5.4.3), $D_m a_n{}^{ab}$ is the covariant derivative of $a_n{}^{ab}$, considered to be a tensor with three spacetime indices (one curved, two flat). The "extra" spacetime connections come from the $\omega_m{}^{ab}$ part of $A_m{}^{ab}$, as is generic for this type of ansatz.

If we take $R_{mn} = 0 = R_{\mu\nu\rho\sigma}$ and $H_{mnp} = 0$, our field equations reduce to those considered in Ref. [9], and we recover the result stated there that by taking the ansatz of eqn. (5.4.2) with $a_m{}^{ab} = 0$, all the field equations may be satisfied for a certain class of embeddings of H in G. If the embedding is such that the trace of a matrix in the adjoint representation of G is 30 times the trace in the vector representation of SO(6), then the scalar field equation and the Bianchi identity (5.3.12) are automatically satisfied. All contributions to the field equations due to the counterterm S vanish. The Maxwell equation (5.3.6), which from (5.4.3) concerns the divergence of the Riemann tensor, is satisfied since a contraction of the gravitational Bianchi identity (5.2.6) gives

$$D_n R^{npab} = 2\,D^{[a}R^{b]p} \qquad (5.4.4)$$

which vanishes in this case.

Besides Ricci-flat examples, a promising manifold to attempt compactification on is the six-sphere. As discussed in Ref. [12], S^6 admits an almost-complex structure F_{mn} which is not covariantly constant (as a true complex structure must be) but rather satisfies

$$D_m F_{np} \equiv T_{mnp} = T_{[mnp]}. \tag{5.4.5}$$

The tensor T_{mnp} and its dual S_{mnp} satisfy a number of nice algebraic and differential identities and provide natural candidates for background values for H_{mnp} and $a_m{}^{ab}$. Unfortunately, if we take H_{mnp} and $a_m{}^{ab}$ to be arbitrary linear combinations of T_{mnp} and S_{mnp}, there are no nontrivial solutions to the field equations and the Bianchi identity.

Another natural possibility is the product of two three-spheres. Since $S^3 \times S^3$ may be considered as the group manifold $SU(2) \times SU(2)$ with the standard (Killing) metric for the group, the structure constants (in a local orthonormal frame) provide a natural three-index antisymmetric tensor on the manifold. This tensor is, in fact, a parallelizing torsion, which means that the curvature constructed from the modified connection $\Gamma_{mn}{}^p + S_{mn}{}^p$, where S_{mnp} is a tensor proportional to the structure constants of the group, vanishes identically. That is,

$$\begin{aligned} R_{mnp}{}^q(\Gamma + S) &= R_{mnp}{}^q(\Gamma) + \left[D_m S_{np}{}^q + S_{mp}{}^r S_{nr}{}^q - (m \leftrightarrow n) \right] \\ &\equiv 0. \end{aligned} \tag{5.4.6}$$

This property may be sufficient to ensure the consistency of a superstring theory with the given manifold as background; we will return to this point in Section 5.5.

Let the two spheres have inverse radii m_1 and m_2, so the Riemann and Ricci curvatures for the first S^3 of the internal manifold are given by

$$R_{ij}{}^{kl} = 2m_1^2\delta_{ij}^{kl}\,, \quad \text{and} \quad R_i^k = 2m_1^2\delta_i^k. \qquad (5.4.7)$$

Throughout most of the discussion, it will only be necessary to explicitly consider the first S^3 (with indices $i, j, \cdots$); the second S^3 behaves exactly analogously. The spin connection leading to eqn. (5.4.7) may be taken to be

$$\omega_{ijk} = m_1\varepsilon_{ijk} \qquad (5.4.8)$$

(where ε_{ijk} is the covariantly constant tensor). We take for the vector potential

$$A_i^{jk} = (1 + \lambda_1)\omega_i^{jk}, \qquad (5.4.9)$$

which is indeed of the form (5.4.2) with $a_m{}^{ab} = \lambda_1\omega_m{}^{ab}$. The field strength is then

$$F_{ij}{}^{kl} = 2f_1\delta_{ij}^{kl}\,, \quad \text{where} \quad f_1 \equiv (1-\lambda_1^2)m_1^2. \qquad (5.4.10)$$

Comparing eqns. (5.4.10) and (5.4.6), we see that $F_{ij}{}^{kl}$ is simply the Riemann curvature in the presence of torsion $S_m{}^{ab} = a_m{}^{ab}$, so the manifold is indeed parallelized for $S_{ijk} = \pm m_1 \varepsilon_{ijk}$.

As previously remarked, for this particular type of ansatz for F_{mn} and the form of the curvatures, all of the terms in S, eqns. (5.1.9a-e), have at least two factors which vanish. Thus there is no contribution to any of the field equations from S.

The ansatz

$$H_{ijk} = h_1 \varepsilon_{ijk} \tag{5.4.11}$$

automatically satisfies both the field equation for B_{mn} eqn. (5.3.3) and its Bianchi identity (5.3.12). The Maxwell equation becomes

$$(\lambda_1^2 - 1)(h_1 - \frac{\varphi}{3\kappa^2} m_1 \lambda_1) = 0 \tag{5.4.12}$$

with the analogous statement for the second S^3. The scalar field equation may be written

$$x_1 f_1^2 + x_2 f_2^2 = \frac{1}{12} y^4 - 6\kappa^2 \varphi^{-1} h^2 \tag{5.4.13}$$

where here the R^2 terms are

$$y^4 = 24(\mu^2 + 6\mu m^2 + 3 m_1^2 m_2^2), \tag{5.4.14}$$

$m^2 \equiv m_1^2 + m_2^2$ and $h^2 \equiv h_1^2 + h_2^2$.

In (5.4.13) we have defined $30 x_i$ to be the ratio of a trace taken over matrices in the adjoint representation of G, to the trace taken in the adjoint of $SU(2)_i$. This ratio does not depend upon which matrices one uses, but is a characteristic of the embedding $SU(2)_i \subset G$. For example, $x_1 = x_2 = \frac{1}{2}$ for all three of the embeddings

$$SU(2) \times SU(2) \times SO(12) \times E_8 \subset E_8 \times E_8,$$

$$SU(2) \times E_7 \times SU(2) \times E_7 \subset E_8 \times E_8$$

and

$$SU(2) \times SU(2) \times SO(28) \approx SO(4) \times SO(28) \subset SO(32),$$

while $x_1 = x_2 = 1$ for the embedding

$$SU(2) \times SU(2) \times SO(26) \approx SO(3) \times SO(3) \times SO(28) \subset SO(32),$$

and $x_1 = x_2 = 2$ for the embedding

$$SU(2) \times E_6 \times SU(2) \times E_6 \subset E_8 \times E_8$$

with each SU(2) embedded maximally within the SU(3) commuting with the $E_6 \subset E_8$.

The graviton field equations reduce to

$$\mu(\mu + 3m^2 - \frac{\varphi}{2\kappa^2}) = 0 \qquad (5.4.15)$$

and

$$\frac{1}{\kappa^2}m_1^2 + 9\kappa^2\varphi^{-2}h_1^2 + 2\varphi^{-1}x_1 f_1^2 +$$

$$+ 6\varphi^{-1}m_1^2(2\mu + m^2 - m_1^2) = 0 \qquad (5.4.16)$$

and similarly for the other S^3. Adding and subtracting eqn. (5.4.16) for subscripts "1" and "2" and using the other field equations we have

$$m^2 + 2\mu - \frac{3\kappa^4}{\varphi^2}h^2 = -\frac{\kappa^2}{3}L = 0 \qquad (5.4.17)$$

$$(\frac{1}{\kappa^2} - 12\varphi^{-1}\mu)(m_1^2 - m_2^2) + 2\varphi^{-1}(x_1 f_1^2 - x_2 f_2^2) +$$

$$+ 9\kappa^2\varphi^{-1}(h_1^2 - h_2^2) = 0. \qquad (5.4.18)$$

Note that the condition $L = 0$ has indeed emerged from the field equations, as promised.

The system of equations (5.4.12,13,15,17,18) has a wide class of solutions for various values of the group theoretical factors x_1, x_2 and relative sizes of the two three-spheres; in fact, there are no necessary restrictions on the embedding of H at all. If we look for solutions with vanishing four-dimensional cosmological constant ($\mu = 0$) and for simplicity take $m_1^2 = m_2^2 = \frac{1}{2}m^2$ and $h_1^2 = h_2^2 = \frac{1}{2}h^2$, we find that if we take $\lambda_1^2 = \lambda_2^2 = 3$, we always have a solution with $\varphi = \kappa^2 m^2(\frac{3}{4} - x_1)$, $h^2 = \frac{\varphi^2 m^2}{3\kappa^4}$, provided that the embedding of the holonomy group in G is such that $x_1 = x_2 < \frac{3}{4}$.

5.5 Discussion

We can easily obtain compactifications to flat four-dimensional Minkowski space, since we have seen that there are embeddings of $SU(2) \times SU(2)$ in either $SO(32)$ or $E_8 \times E_8$ with $x_1 = x_2 < \frac{3}{4}$. The resulting gauge symmetry will have two factors, the first of which is the unbroken part of the ten-dimensional Yang-Mills group, namely, the little group for the embedding of the holonomy group H in G. As we have seen, this can be (for example) $SO(12) \times E_8$, $E_7 \times E_7$ or $SO(28)$. The second factor is the isometry group of $S^3 \times S^3$, $SO(4) \times SO(4) \approx [SU(2)]^4$.

The resulting four-dimensional field theory (in Minkowski space) can certainly accommodate a range of phenomenological considerations, especially via a number of the techniques developed in Refs. [2,9] and elsewhere. The gauge symmetry for some of our models may be easily broken to realistic low-energy grand unification groups, with the needed chiral couplings. However, due to the inherent limitations of our approach, we feel that it is most appropriate to postpone a full-scale investigation of phenomenological implications. The field theories we study here, while anomaly-free, have not been rigorously shown to be related to theories of more fundamental interest, such as versions of these theories in which supersymmetry is restored, or of course the full superstring theories themselves.

The results we present are interesting, however, in that they naturally lead to the consideration of background geometries which are likely to be relevant for superstring theories. The fact that the manifolds are absolutely parallelizable may be sufficient to ensure the consistency of a superstring theory with the given manifold as background [13]. In general, a string theory may be formulated as a sigma-model defined on the two-dimensional world sheet of the string. One expects that conformal invariance of this sigma model is necessary for a well-behaved quantum theory of strings (but see for example Ref. [14] for a possible approach circumventing this requirement). This property has been demonstrated to two-loop order in Ref. [13] for absolutely parallelizable manifolds such as $S^3 \times S^3$. In Ref. [15] this property was demonstrated to all orders for Ricci-flat Kähler manifolds (which are relevant for Ref. [9]), if one assumes that no supersymmetry anomalies arise.

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Appendix: Conventions for Spacetime Geometry

We always take as the spacetime signature η_{ab} = diag (- + + + ...); that is, positive in all spacelike directions. Letters near the beginning of the alphabet denote local Lorentz (flat) indices; letters near the middle denote global (curved) indices. The Levi-Civita symbol $\varepsilon^{mnp\cdots}$ is taken to be the covariantly constant true tensor regardless of the type of indices, and is only numerically constant when all indices are flat.

The sign conventions for the Christoffel and spin connections are fixed by

$$D_m X_n \equiv \partial_m X_n + \Gamma_{mn}{}^p X_p \tag{A.1.1}$$

$$D_m X_a \equiv \partial_m X_a + \omega_{ma}{}^b X_b . \tag{A.1.2}$$

The curvature is given by

$$[D_m, D_n] X_p \equiv R_{mnp}{}^q X_q \tag{A.1.3}$$

$$[D_m, D_n] X_a \equiv R_{mna}{}^b X_b , \tag{A.1.4}$$

or explicitly

$$R_{mnp}{}^q = \partial_m \Gamma_{np}{}^q + \Gamma_{mp}{}^r \Gamma_{nr}{}^q - (m \leftrightarrow n) \tag{A.1.5}$$

$$R_{mna}{}^b = \partial_m \omega_{na}{}^b + \omega_{ma}{}^c \omega_{nc}{}^b - (m \leftrightarrow n) . \tag{A.1.6}$$

The metricity condition on the vielbein is

$$D_m e_{na} \;=\; \partial_m e_{na} + \Gamma_{mn}{}^p e_{pa} + \omega_{ma}{}^b e_{nb} \equiv 0\,, \qquad \text{(A.1.7)}$$

so we have

$$[D_m, D_n] e_{pa} \;=\; R_{mnp}{}^q e_{qa} + R_{mna}{}^b e_{pb} \;=\; 2R_{mn(pa)} \;=\; 0\,, \qquad \text{(A.1.8)}$$

which establishes the consistency of our two definitions for the curvature. All (anti-)symmetrization is with weight one; for example, $X_{[m}Y_{n]} \equiv \frac{1}{2!}(X_m Y_n - X_n Y_m)$.

The condition (A.1.7) allows one to solve for the connections, and hence the curvature, in terms of the vielbein $e_m{}^a$ (for a general torsion $T_{mn}{}^p \equiv \Gamma_{[mn]}{}^p$, which we always take to be zero). We do not need the explicit expressions, but only their variations with respect to the vielbein. Decompose a general $\delta e_m{}^a \equiv \varepsilon_m{}^a$ into its symmetric and antisymmetric parts:

$$h_{ab} \equiv 2\,\varepsilon_{(ab)} \quad \text{and} \quad \lambda_{ab} \equiv \varepsilon_{[ab]}\,, \qquad \text{(A.1.9)}$$

so $\delta e_m{}^a \;=\; \frac{1}{2} h_m{}^a + \lambda_m{}^a$, and $\delta g_{mn} \;=\; h_{mn}$. Then it turns out that

$$\delta\omega_{mab} \;=\; D_m \lambda_{ab} - D_{[a} h_{b]m} \qquad \text{(A.1.10)}$$

and

$$\delta\Gamma_{mn}{}^p \;=\; \frac{1}{2} D^p h_{mn} - D_{(m} h_{n)}{}^p\,. \qquad \text{(A.1.11)}$$

Thus λ_{ab} plays the role of a local Lorentz transformation, and h_{mn} carries the dynamical information. The variation of the curvature is given by

$$\begin{aligned} \delta R_{mn}{}^{ab} &= 2\,D_{[m}(\delta\omega_{n]}{}^{ab}) \\ &= 2\lambda^{c[a}R_{mnc}{}^{b]} - 2\,D_{[m}D^{[a}h_{n]}{}^{b]}. \end{aligned} \qquad (A.1.12)$$

If we define the Ricci tensor and scalar as $R_{mn} \equiv R_{mpn}{}^{p}$ and $R \equiv R_{m}{}^{m}$, then we have

$$\delta R_{mn} = -\frac{1}{2}(D_pD^ph_{mn} + D_mD_nh_p{}^p) + D_pD_{(m}h_{n)}{}^p \qquad (A.1.13)$$

and

$$\delta R = -h^{mn}R_{mn} + D_p(D_qh^{pq} - D^ph_q{}^q). \qquad (A.1.14)$$