ENERGY TRANSPORT BY COMBINED RADIATION AND CONDUCTION

Thesis by

Lon Edward Bell

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In memory of my mother,

Anna Paula Bell

(1907-1966)

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ABSTRACT

The problem of one-dimensional radiative and conductive heat transfer in a medium bounded by two infinite parallel walls is treated. The medium isotropically absorbs, scatters, and emits radiation. The walls absorb and reflect isotropically, and transmit diffusely. Thermal conductivity, as well as all radiative properties, are assumed independent of temperature and wavelength. Steady state solutions are sought.

Approximate solutions are developed for the temperature distribution and the heat transfer rate. The temperature distribution is expanded in a power series in a dimensionless variable that is a measure of the ratio of radiative to conductive energy transport. The first two terms are retained in the expansion. This allows the radiation integrals that appear in the basic equations to be expressed in a simple, readily evaluated form. Values for the heat transfer rate and temperature distribution are computed and compared with those of other investigators, and the range of validity of the approximation is examined.

The problem of two adjacent slabs with different optical properties is discussed. The analysis shows the interaction between the radiative and conductive transport mechanisms and displays the relative importance of each with changes in temperature. The problem of a coating of finite optical depth irradiated by an external source is also treated. The results indicate the conditions under which a coating may be characterized by an emissivity and the conditions under which it must be described in terms of its conductive and optical properties.

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I. INTRODUCTION

The study of radiative energy transport had its first important developments in connection with astrophysics, where an understanding of the radiation emitted by celestial objects or the portion of radiation reflected from other emitting bodies was required. More recently, the ability to describe radiative energy transport has become a necessary adjunct to accurate astronomical measurements.

Pioneering work in astrophysics was done by Rosseland[1], who treated the energy transport in stars. Through a simplified model, he showed that radiative transport was several orders of magnitude greater than conduction, and further, that the photon mean free path was much smaller than the diameter of the body. Hence, radiative flux could be described as a diffusion process, the resulting equation being known as the Rosseland Approximation.

Several astrophysicists subsequently have treated the problem of the photon emission from a star. Since the emission comes from a zone near the surface of a star and not from the surface itself, the intensity is a function of the angle from normal to the surface. As a result, the intensity observed from the entire disk of a star is the average intensity and must be interpreted correctly. Eddington[2] has given an approximate solution to the intensity distribution and the heat transfer rate; Kourganoff[3] has improved upon the approximate solution and has presented the exact solution of Chandrasekhar[4]. Chandrasekhar has also treated the problem of radiative transport through a semi-infinite slab which absorbed and scattered radiation, thus enabling one to take into account in astronomical measurements the effect of an intervening atmosphere.

The production, scattering, and absorption of neutrons in atomic piles, nuclear bombs, and atomic instruments is analogous to the process of photon transport. But unlike astrophysical problems, in nuclear engineering the dimensions of the system are often of the order of a neutron mean free path. This, coupled with the need for high accuracy in neutron flux calculations, has led to rather sophisticated treatment of the transport equation. Modifications have been made to the Rosseland approximation to include boundary effects[5], and the Monte Carlo method has been developed to give numerical results[6].

In earlier engineering problems involving modest temperatures, conduction was the dominant mode of energy transport in solid materials. However, recent technological developments in high temperature structures, such as rocket nozzles, gas-cooled nuclear reactors, transpiration-cooled surfaces, and ablating surfaces, as well as cryogenic systems utilizing porous or fibrous insulations, have led to a need for better understanding of heat transfer with the addition of radiation as a transport mechanism. As a result, heat transfer by conduction and radiation simultaneously has become important.

The radiative flux at a point within a medium depends on the intensity of radiation passing through the point from all directions. The intensity, in turn, results from emission, absorption, and scattering throughout the medium plus the flux contributions from boundaries. This leads to a representation of the energy absorbed at a point by an integral over the medium and the bounding surfaces, the radiative intensity usually being proportional to the fourth power of the temperature. Conduction, on the other hand, is described by a differential relation. Thus, when both mechanisms are required to describe the net heat-transfer rate, a nonlinear integro-differential equation results.

The most extensive and recent work on simultaneous heat transfer by radiation and conduction has been done by engineers specifically interested in determining the heat transfer in various media. Recently, Walther, Dorr, and Eller[7] treated the equation as it applies to heat transfer in hot glass. Numerical methods suitable for digital computer computations were used. Hamaker[8] treated a scattering and absorbing medium using a two flux model, where it was assumed that scattering and absorption occurred only in the forward and backward directions. In this approximation, the radiative term reduced to two coupled differential expressions for the forward and backward intensities. This model, first proposed by Schuster[9], and extended by Chen and Churchill[10], was applied to porous insulations by Larkin and Churchill[11]. Larkin and Churchill's paper also presented experimental determinations of the forward and backward scattering and absorption cross section for polystyrenes and various glasses.

Goulard and Goulard[12] studied a plane layer of stagnant gas between gray and transparent walls. The problem was greatly simplified by ignoring the absorption and reradiation of emission originating within the gas.

An extensive review article by Viskanta and Grosh[13] covered the literature in radiant heat transfer through the middle of 1963. The full spectrum of heat transfer mechanisms which interact with radiation was surveyed.

In the following sections, the problem of one-dimensional radiative and conductive heat transfer in a medium bounded by two infinite parallel walls is treated. The medium isotropically emits, absorbs, and scatters radiation. The walls isotropically absorb and reflect, and transmit diffusively. Thermal conductivity as well as all radiative properties are independent of temperature and wavelength. Steady state solutions are sought.

Under these conditions, the general equation becomes

$$\theta''(\tau, \alpha) + \frac{\alpha N}{2}\left[S_1''(\tau, \alpha) + S_2''(\tau, \alpha) + \int_0^{\tau_o} \theta^4(t, \alpha) E_1 |t-\tau| dt - 2\theta^4(\tau, \alpha)\right]$$

$$- \frac{1-\alpha}{2} \int_0^{\tau_o} \theta''(t, \alpha) E_1 |t-\tau| dt = 0 ,$$

where θ is the dimensionless temperature, τ_o is the total optical depth between the walls, α is the ratio of the radiative absorption cross section to the total cross section, N is a dimensionless measure of radiative to conductive transport, and the kernel E_1 is a form of the exponential integral and will be discussed in greater detail later. The source functions, S_1'' and S_2'', are contributions to the radiative flux from the boundaries. It is sufficient to note that the source functions are integrals over θ'' and θ^4. They are similar in nature to the other terms on the right hand side and need not be written explicitly.

Several authors have treated the above equation with a variety of boundary conditions. Rather than describe their contributions at this point, it is better to wait until their respective problems are taken up in the text.

As a particular facet of radiative and conductive energy transport is studied in the text, the approaches used by others are described. Then the present results are developed, and finally, a comparison is made.

In the following pages, an approximate solution to the general equation is developed in terms of the linear conduction temperature distribution which allows the radiation integrals to be expressed in a simple, readily evaluated form and to be tabulated. This is accomplished by restricting the range of N for which the solution is useful. But in this range, wall emissivity, reflectivity, and transmissivity, and external source strength as well as N enter in a simple way. In the case of pure absorption, the solution is given as continuous functions of these variables, so that the effect of each can be independently evaluated. With scattering included, the solution is given as a readily computed integral over a Green's function.

The treatment is divided into six parts, starting with Section II. In Section II, the radiative transport equations are developed along with general forms of the source functions. Allowance is made for external sources, with the result that the general equation describes a broad spectrum of conditions, including, for instance, a coating of finite optical thickness irradiated by an external source. An exact solution for the energy transport for pure radiation, from Chandrasekhar[4], applicable to the general problem with scattering, is also given.

Solutions for the temperature distribution for absorption and conduction, but no scattering, are developed in Section III. The temperature profile is expanded in N , about N = 0 , so that

$$\theta(\tau) = \theta_o(\tau) + N\theta_1(\tau) + \dots .$$

The first term in the expansion is the linear temperature profile of pure conduction, since N equal to zero corresponds to the total absence of radiative transport. The next term, $N\theta_1$, is the first order correction to the temperature profile. It can be computed exactly as an integral over θ_o. The result involves three rather complex functions. Two depend solely on $\theta(0)$ and τ_o, and the third depends on τ in addition. These functions are tabulated for a wide spectrum of values of $\theta(0)$, τ_o, and τ. As previously mentioned, the other parameters, emissivity, reflectivity, transmissivity, external source strength, and N appear as simple algebraic functions so that θ is given as a continuous function of these variables. Thus, the effect of each parameter on the temperature profile can be independently studied.

In Section IV, the heat transfer rate for absorption and conduction is developed in a similar expansion, and is given in terms of three functions, two of which appear in the expression for θ_1. Again, the solution is a continuous function of the same parameters as θ_1.

Scattering is included in the temperature distribution in Section V. The equation no longer reduces to a linear integral equation; derivatives of θ either appear as boundary conditions or explicitly under an integral sign. This, plus the addition of the critical parameter, α, prevents using the previous methods. Instead, the integrals over θ

are shown to be nearly equal to a simple term involving θ_1. The resulting approximate equation is a linear integral equation:

$$\theta''(\tau,\alpha) \approx \alpha N \theta_1''(\tau,\alpha) + [1-\alpha][T_1''(\tau,\alpha)+T_2''(\tau,\alpha)] + \int_0^{\tau_o} \theta''(t,\alpha) E_1 |t-\tau| dt \ .$$

The functions T_1'' and T_2'' are integrals over θ''. The kernel is difficult to treat, and so a substitute kernel is used for E_1. With this approximation, θ can be found in terms of Green's functions that can be evaluated numerically. The limiting cases $\tau_o \gg 1$ and $\tau_o \ll 1$ are considered and θ reduces to a form similar to that for pure absorption.

The heat transfer rate with scattering is then derived in Section VI as an integral of θ'' plus the result for pure absorption. For $\tau_o \leq 1$, the result reduces to a simple form.

The last section, Section VII, is concerned with an application of the previous methods. First, a problem of two adjacent slabs is studied. Both slabs are assumed to conduct, emit, and absorb. At a common boundary between them, radiation is diffusely transmitted, so that the radiative flux from one region appears as an isotropic source in the other. The diffuse transmission at the interface distinguishes the adjacent slab problem treated in the text from that of a single medium with an abrupt optical density change at an interior boundary. In the latter, radiation crossing the boundary would retain its angular dependence and not be diffused.

Solutions are found by first computing the heat transfer across each region for assumed values of the interface temperature and then determining the temperature which makes the heat transfer the same in both media. Results are presented for a spectrum of optical depths in both zones. They show the influence of optical depth on energy transport in a configuration where the radiative properties vary in specific regions. In addition, the solution for the particular case of identical properties in both regions is compared with the approximate solution for a single medium of similar properties but without the diffusing intermediate boundary.

A second application describes effects of an external source with one boundary transparent and the other opaque. Varying the source strength leads to the unexpected results that under some circumstances the heat transfer rate may reach a maximum for pure scattering, and under others reach a maximum for pure absorption. Also, the component of the heat transfer rate induced by radiation can be a strong function of the degree of scattering. If the above system is considered to be an emitting coating on an opaque surface, the results indicate which properties are important to determine its absorption and heat transfer properties. It is shown that great care must be taken in assigning an emissivity to the coating. Indeed, under some circumstances, the concept of emissivity is not appropriate, and the heat transfer properties become a complex function of several parameters.

Pertinent integrations, an error analysis, and properties of the generalized exponential integrals are included as Appendices. In the error analysis, an upper bound for the absolute error in the expansions for pure absorption is developed. The results are used to place limits on the range of N for which the expansions have demonstrable validity. The greatest use is found as the estimate applies to the heat transfer rate, since the calculated error bounds tend to overestimate greatly the actual errors for the temperature profile. As applied to the heat transfer rate, the form developed is shown to be useful for relatively large radiative fluxes.

The functions required to compute numerical results for the heat transfer rate and temperature profile are given in Tables I, II, and III; also included are functions related to the error analysis.

II. DEVELOPMENT OF THE BASIC EQUATIONS

Basic to the treatment of simultaneous radiative and conductive energy transport is the radiative term. Its development requires several steps. An expression for the radiative intensity at any point within the medium is needed. Then the result is used in an energy balance expression in terms of the temperature profile. Finally, the heat transfer must be expressed in terms of the temperature profile. In the first part of this section, an expression for the radiative intensity at a point is developed. It is used in the next three parts to formulate the equation for the temperature profile and the heat transfer rate. In the fifth part, the particular case of pure radiation is discussed.

The treatment in this section parallels the development of the intensity given in Chandrasekhar[4] and Kourganoff[3], while the final expressions for the transport equation are given in a slightly modified form by Viskanta and Grosh[14]. The results for pure radiation follow Heaslet and Warming[15]. Further details of other authors' concern with the latter problem are discussed as the problem is taken up.

1. The Intensity

It is convenient to describe the radiative intensity, I_ν, in terms of the amount of radiant energy, dE_ν, in a specific frequency interval, $(\nu, \nu + d\nu)$, traveling in directions confined to the solid angle $d\Omega$ about Ω and incident on a surface area ds at a point ℓ in a time dt. With θ the angle between Ω and the normal to ds, dE_ν can be expressed in terms of the intensity as

$$dE_\nu(\Omega, \ell, t) = I_\nu(\Omega, \ell, t)d\nu ds d\Omega dt ,$$

or, the intensity is

$$I_\nu(\Omega, \ell, t) = \frac{dE_\nu(\Omega, \ell, t)}{\cos\theta d\nu ds d\Omega dt} .$$

Thus I_ν may vary from point to point and in direction as well.

The intensity can be attenuated by absorption while passing through matter; it is reduced by conversion into other forms of energy, including radiation of other frequencies. The change in the intensity dI_{ν_a} due to absorption from traveling an infinitesimal distance $d\ell$ in a continuous medium is

$$dI_{\nu_a}(\Omega, \ell, t) = -\sigma_{\nu_a}(\Omega, \ell)I_\nu(\Omega, \ell, t)d\ell .$$

The attenuation factor, σ_{ν_a}, is called the macroscopic absorption cross section. It is generally a function of frequency, direction, and position. Similarly, the energy flux in a specific direction can be reduced by true scattering; that is, scattering of the beam into new directions without a change of frequency. An expression analogous to the one for absorption describes the change in intensity caused by scattering, dI_{ν_s}:

$$dI_{\nu_s}(\Omega, \ell, t) = -\sigma_{\nu_s}(\Omega, \ell)I_\nu(\Omega, \ell, t)d\ell .$$

Further, σ_{ν_s}, the macroscopic scattering cross section, can be also a function of direction, position, and frequency.

The total attenuation is the sum of the above two effects:

$$dI_\nu(\Omega, \ell, t) = -\sigma_{\nu_a}(\Omega, \ell)I_\nu(\Omega, \ell, t)d\ell - \sigma_{\nu_s}(\Omega, \ell)I_\nu(\Omega, \ell, t)d\ell$$

$$= -[\sigma_{\nu_a}(\Omega, \ell) + \sigma_{\nu_s}(\Omega, \ell)]I_\nu(\Omega, \ell, t)d\ell$$

$$= -\sigma_{\nu_t}(\Omega, \ell) I_{\nu_t}(\Omega, \ell, t) d\ell ,$$

where σ_{ν_t} is the extinction coefficient and equals the sum of σ_{ν_a} and σ_{ν_s}. Obviously, it can be a function of the frequency, position, and direction.

The intensity can be increased by two sources; the emission from within an element and scattering from all directions into the direction of I_ν. For local thermodynamic equilibrium (so that a local temperature T can be defined), the emission in a direction Ω can be written in terms of an emission coefficient j_ν. The emission coefficient is related to the Planck function $B_\nu(T)$ through σ_{ν_a} by Kirchhoff's law,

$$j_{\nu_a}(\Omega, \ell, t) = \sigma_{\nu_a}(\Omega, \ell) B_\nu(T)$$

where

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/\kappa T} - 1} .$$

In the Planck function, h is Planck's constant, κ is Boltzmann's constant, and c is the velocity of light. Similarly, the intensity scattered into a direction Ω is:

$$j_{\nu_s}(\Omega, \ell, t) = \sigma_{\nu_s}(\Omega, \ell) \frac{1}{4\pi} \int_{\Omega'} P_\nu(\Omega, \Omega', \ell) I_\nu(\Omega', \ell, t) d\Omega' ,$$

where P_ν is the phase function defined as the amount of radiation coming from a direction Ω' that is scattered into the direction Ω. In the above expression, P_ν is assumed normalized so that

$$\frac{1}{4\pi} \int_{\Omega'} P_\nu(\Omega, \Omega', \ell) d\Omega' = 1$$

for all Ω. If σ_{ν_a} were zero, the scattering would be conservative; that is, all of the intensity at a frequency ν entering a volume, whether scattered or passed through unaffected, would leave the volume at the same frequency. No portion would be converted to other frequencies or otherwise removed.

If the above production and attenuation terms are combined, the intensity change in a particular direction and at a particular point can be written as

$$\frac{dI_\nu(\Omega, \ell, t)}{d\ell} = -\sigma_{\nu_t}(\Omega, \ell)I_\nu(\Omega, \ell, t)+j_{\nu_a}(\Omega, \ell, t)+j_{\nu_s}(\Omega, \ell, t) \ .$$

In the following treatment, steady state conditions with local thermodynamic equilibrium and isotropic scattering and absorption are assumed, so P_ν is unity and Kirchhoff's law holds. Thus, with the variable Ω suppressed in the cross sections and the time dependence removed throughout:

$$\frac{dI_\nu(\Omega, \ell)}{d\ell} = -\sigma_{\nu_t}(\ell)I_\nu(\Omega, \ell)+\sigma_{\nu_a}B_\nu(T)+\frac{\sigma_{\nu_s}(\ell)}{4\pi}\int_{\Omega'} I_\nu(\Omega', \ell)d\Omega'$$

$$= -\sigma_{\nu_t}(\ell)I_\nu(\Omega, \ell)+F_\nu(\ell) \qquad (2.1)$$

where

$$F_\nu(\ell) = \sigma_{\nu_a}(\ell)B_\nu(T) + \frac{\sigma_{\nu_s}(\ell)}{4\pi}\int_{\Omega'} I_\nu(\Omega', \ell)d\Omega' \ . \qquad (2.2)$$

The production terms in the last expression are considered as a single source function F_ν. Since the scattering term and σ_{ν_a} are independent of Ω, F_ν is also. With σ_t independent of Ω, it is convenient to change the independent variable ℓ to τ_ν, where

$$\tau_\nu(\ell, \ell') = \int_{\ell'}^{\ell} \sigma_{\nu_t}(\ell'')d\ell'' \ .$$

Now, in terms of τ_ν, I_ν can be found from (2. 1):

$$I_\nu(\Omega, \ell) = I_\nu(\Omega, \ell')e^{-\tau_\nu(\ell, \ell')} + \int_{\ell'}^{\ell} F_\nu(\ell'')e^{-\tau_\nu(\ell, \ell'')} d\ell'' \ . \qquad (2.\ 3)$$

The function τ_ν is the optical thickness between ℓ' and ℓ at the frequency ν. Further, by the choice of signs in the exponents, ℓ must be greater than ℓ', with the path of integration a straight line between ℓ and ℓ'.

2. Statement of the Problem and Reduction of the Intensity

The problem of concern in the following sections is that of two isotropically emitting, transmitting, and reflecting walls bounding a uniformly distributed scattering, absorbing, and conducting medium. All properties, σ_{ν_s}, σ_{ν_a}, and σ_{ν_t}, thermal conductivity, k, as well as the wall emmisivity, transmissivity, and reflectivity, are assumed independent of frequency and temperature. Steady state, one-dimensional solutions are sought.

With all properties independent of the frequency, I_ν can be integrated over all ν to give the total intensity. From (2. 3),

$$\int_0^\infty I_\nu(\Omega, \ell)d\nu = I(\Omega, \ell) = \int_0^\infty [I_\nu(\Omega, \ell)e^{-\tau_\nu(\ell, \ell')} + \int_{\ell'}^{\ell} F_\nu(\ell'')\ell^{-\tau_\nu(\ell, \ell'')} d\ell'']d\nu$$

$$= I(\Omega, \ell')e^{-\tau_\nu(\ell, \ell')} + \int_{\ell'}^{\ell} F(\ell'')e^{-\tau_\nu(\ell, \ell'')} d\ell'' \ , \qquad (2.\ 4)$$

where I and F stand for, respectively, the total intensity in the direction Ω at ℓ and the total source function at ℓ. Since σ_{ν_t} and

τ_ν are now independent of ν, their subscripts can be dropped. From the definition of F_ν in (2. 2),

$$F(\ell) = \int_0^\infty F_\nu(\ell)d\nu = \int_0^\infty [\sigma_{\nu_a}(\ell)B_\nu(T) + \frac{\sigma_{\nu_s}(\ell)}{4\pi} \int_{\Omega'} I_\nu(\Omega', \ell)d\Omega']d\nu$$

$$= \sigma_a B(T) + \frac{\sigma_s}{4\pi} \int_{\Omega'} I(\Omega', \ell)d\Omega' \ , \qquad (2.5)$$

where B(T) is the blackbody intensity

$$B(T) = \int_0^\infty B_\nu(T)d\nu = \frac{\sigma T^4(\tau)}{\pi} \ . \qquad (2.6)$$

For the planar case, it is convenient to call τ the (dimensionless) distance from one wall and θ the angle between the wall normal and the direction under consideration. The third coordinate, ϕ, the azimuthal angle, need not be specified, since the scalar properties vary only with τ, and hence I varies only with the angle θ. Now call μ the cosine of θ, x the geometrical distance from one wall, and replace $\tau(\ell, \ell')$ by

$$\tau(\ell, \ell') = \int_{\ell'}^{\ell} \sigma_t\,(\ell)d\ell = \int_{x'}^{x} \sigma_t\,\frac{dx}{\mu} = \frac{\tau}{\mu} - \frac{\tau'}{\mu} \ . \qquad (2.7)$$

For convenience, choose μ to be positive; $+\mu$ indicates the intensity from directions for $0 \le \theta \le 90$, so that the termination boundary is always the plane at $\tau = \tau_o$. Similarly, $-\mu$ includes only the intensity from $90 \le \theta \le 180$, and the corresponding boundary plane at $\tau = 0$. A diagram of the system is shown in Figure 1. The angle θ is measured from the inward normal of the wall at $\tau = 0$. The geometrical distance between the boundaries is x.

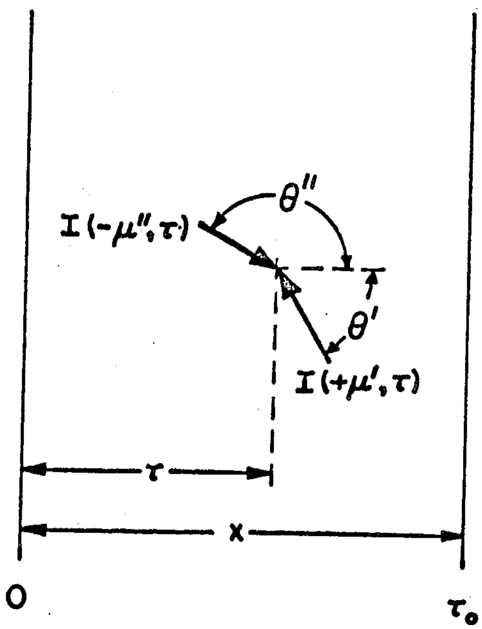

Figure 1. Coordinate System for Two Infinite Parallel Walls.

With these definitions and equations (2.4), (2.5), and (2.6), the intensity at any point is given by

$$I(+\mu, \tau) = I(+\mu, \tau_o)e^{-(\tau_o-\tau)/\mu} + \int_{\tau}^{\tau_o} \left[\frac{\sigma_a \sigma T^4(t)}{\pi} + \frac{\sigma_s}{2} \int_{-1}^{1} I(\mu', t)d\mu' \right] e^{-(t-\tau)/\mu} \frac{dt}{\mu\sigma_t}$$

$$(0 \le \mu \le 1) \qquad (2.8)$$

$$I(-\mu, \tau) = I(-\mu, 0)e^{-\tau/\mu} + \int_{0}^{\tau} \left[\frac{\sigma_a \sigma T^4(t)}{\pi} + \frac{\sigma_s}{2} \int_{-1}^{1} I(\mu', t)d\mu' \right] e^{-(\tau-t)/\mu} \frac{dt}{\mu\sigma_t}$$

$$(0 \le \mu \le 1) \qquad (2.9)$$

In the above expressions, the azimuthal integration in the contribution from scattering has been performed. Equations (2.8) and (2.9) express the intensity at an interior point in terms of the material properties, the temperature profile, and the influx of radiation from the boundaries. These expressions will now be used to derive the basic transport equations. However, it should be noted that the expressions are not completely reduced. The boundary intensities $I(+\mu, \tau_o)$ and $I(-\mu, 0)$ are composed of external source transmission, emission from the boundaries themselves, and flux reflected from within. The reflected portion, in turn, is made up of the radiation from the interior as well as the opposite boundary. The final expressions for $I(+\mu, \tau_o)$ and $I(-\mu, 0)$ are best treated after some fundamental relations for the temperature profile throughout the medium are derived.

3. The Temperature Distribution

The transport equation for a conducting medium with a radiative source, Q_R, is Poisson's Equation:

$$\nabla\cdot(k\nabla T(x)) + Q_R = 0\ . \tag{2.10}$$

With the assumption that k is constant and the problem is one dimensional, (2.10) can be rewritten in terms of τ, using (2.7):

$$\nabla\cdot(k\nabla T(x)) = \frac{\partial}{\partial x}\, k \frac{\partial}{\partial x}\, T(x) = \sigma_t^2 \frac{\partial^2}{\partial \tau^2}\, T(\tau)\ ,$$

so that (2.10) becomes

$$\sigma_t^2 k \frac{\partial^2 T(\tau)}{\partial \tau^2} + Q_R = 0\ , \tag{2.11}$$

where Q_R is the radiative source term and is equal to the net absorption of radiation at a point. In terms of the intensity, it is the intensity absorbed minus the emission at a point; using (2.6) for the emission, $B(T)$,

$$Q_R = \int_\Omega \sigma_a I(\mu,\tau)d\Omega - \int_\Omega \sigma_a B(T)d\Omega = 2\pi\sigma_a \int_{-1}^{1} I(\mu,\tau)d\mu - 2\pi\sigma_a \int_{-1}^{1} \frac{\sigma T^4(\tau)}{\pi}\, d\mu$$

$$= 2\pi\sigma_a \int_{-1}^{1} I(\mu,\tau)d\mu - 4\sigma_a \sigma T^4(\tau)\ .$$

The last step follows from noting that T is independent of μ. Thus, (2.11) becomes:

$$\sigma_t^2 k \frac{\partial^2 T(\tau)}{\partial \tau^2} + 2\pi\sigma_a \int_{-1}^{1} I(\mu,\tau)d\mu - 4\sigma_a \sigma T^4(\tau) = 0\ , \tag{2.12}$$

or, solving for the integral expression:

$$\int_{-1}^{1} I(\mu,\tau)d\mu = \frac{2\sigma T^4(\tau)}{\pi} - \frac{\sigma_t^2 k}{2\pi\sigma_a} \frac{\partial^2 T(\tau)}{\partial \tau^2}\ . \tag{2.13}$$

The left hand side can be expressed in terms of the intensities from the boundaries and the temperature profile by using equations (2.8) and (2.9) with the integrals over $I(\mu', t)$ appearing in the equations replaced by the right hand side of (2.13):

$$\int_{-1}^{1} I(\mu, \tau)d\mu = \int_{0}^{1} I(-\mu, \tau)d\mu + \int_{0}^{1} I(+\mu, \tau)d\mu , \qquad (2.14)$$

where, upon noting that the intensities from the walls, $I(-\mu, 0)$ and $I(+\mu, \tau_o)$ are isotropic and hence independent of μ,

$$\int_{0}^{1} I(-\mu,\tau)d\mu = \int_{0}^{1} I(-\mu,0)e^{-\tau/\mu}d\mu + \int_{0}^{1}\int_{0}^{\tau}\left[\frac{\sigma T^4(t)}{\pi} - \frac{\sigma_t\sigma_s k}{4\pi\sigma_a}\frac{\partial^2 T(t)}{\partial t^2}\right]e^{-(\tau-t)/\mu}\frac{dt}{\mu}d\mu$$

$$= I(-\mu,0)\int_{0}^{1} e^{-\tau/\mu}d\mu + \int_{0}^{\tau}\left[\frac{\sigma T^4(t)}{\pi} - \frac{\sigma_t\sigma_s k}{4\pi\sigma_a}\frac{\partial^2 T(t)}{\partial t^2}\right]\int_{0}^{1} e^{-(\tau-t)/\mu}\frac{d\mu}{\mu}\,dt . \qquad (2.15)$$

In the last step, the order of integration has been reversed; this is possible since T is independent of μ. Similarly,

$$\int_{0}^{1} I(+\mu,\tau)d\mu = I(+\mu,\tau_o)\int_{0}^{1} e^{-(\tau_o-\tau)/\mu}d\mu +$$

$$+ \int_{\tau}^{\tau_o}\left[\frac{\sigma T^4(t)}{\pi} - \frac{\sigma_t\sigma_s k}{4\pi\sigma_a}\frac{\partial^2 T(t)}{\partial t^2}\right]\int_{0}^{1} e^{-(t-\tau)/\mu}\frac{d\mu}{\mu}\,dt . \qquad (2.16)$$

The integrals over μ are the exponential integrals, $E_n(\tau)$. The properties of the functions and the forms appearing in (2.15) and (2.16) are discussed in Appendix I (see equation (I-2) in particular). In terms of the exponential integrals, (2.15) and (2.16) can be written:

$$\int_{0}^{1} I(-\mu, \tau)d\mu = I(-\mu, 0)E_2(\tau) + \int_{0}^{\tau}\left[\frac{\sigma T^4(t)}{\pi} - \frac{\sigma_t\sigma_s k}{4\pi\sigma_a}\frac{\partial^2 T(t)}{\partial t^2}\right]E_1(\tau-t)dt \qquad (2.17)$$

$$\int_0^1 I(+\mu,\tau)d\mu = I(+\mu,\tau_o)E_2(\tau_o-\tau)+\int_\tau^{\tau_o}[\frac{\sigma T^4(t)}{\pi} - \frac{\sigma_t\sigma_s k}{4\pi\sigma_a}\frac{\partial^2 T(t)}{\partial t^2}]E_1(t-\tau)dt\ , \tag{2. 18}$$

and finally, using (2. 14), (2. 17), and (2. 18), the transport equation (2. 12) becomes:

$$\sigma_t^2 k\frac{\partial^2 T(\tau)}{\partial\tau^2} + 2\pi\sigma_a[I(-\mu,0)E_2(\tau)+I(+\mu,\tau_o)E_2(\tau_o-\tau)]$$

$$+ 2\sigma_a\int_0^{\tau_o}[\sigma T^4(t) - \frac{\sigma_t\sigma_s k}{4\sigma_a}\frac{\partial^2 T(t)}{\partial t^2}]E_1|t-\tau|dt - 4\sigma_a\sigma T^4(\tau) = 0\ . \tag{2. 19}$$

This equation is expressed more conveniently in dimensionless form. Let

$$\theta(\tau,\alpha) = \frac{T(\tau)}{T(\tau_o)}\ ; \qquad \theta(\tau_o,\alpha) = 1 \tag{2. 20}$$

$$\alpha = \sigma_a/\sigma_t\ ; \qquad 1-\alpha = \sigma_s/\sigma_t \tag{2. 21}$$

$$N = \frac{4\sigma T^4(\tau_o)}{\sigma_t k T(\tau_o)}\ . \tag{2. 22}$$

Now (2. 19) becomes

$$\theta''(\tau,\alpha)+\frac{\alpha N}{2}[\frac{\pi I(-\mu,0)}{\sigma T^4(\tau_o)}E_2(\tau)+\frac{\pi I(+\mu,\tau_o)}{\sigma T^4(\tau_o)}E_2(\tau_o-\tau)]$$

$$+\frac{\alpha N}{2}\int_0^{\tau_o}[\theta^4(t,\alpha) - \frac{(1-\alpha)}{\alpha N}\theta''(t,\alpha)]E_1|t-\tau|dt-\alpha N\theta^4(\tau,\alpha) = 0 \tag{2. 23}$$

where the derivatives are with respect to τ. The expression will be complete once $I(-\mu,0)$ and $I(+\mu,\tau_o)$ are evaluated.

The contributions to the intensities from the boundaries arise from wall emission, internal reflection, and transmitted radiation from beyond the boundaries. The contribution from scattering and conduction appears in the portion of the intensity arising from reflections at the boundaries. If (2.8) and (2.9) are evaluated at $\tau = 0$ and τ_o respectively,

$$I(-\mu, \tau_o) = I(-\mu, 0)e^{-\tau_o/\mu} + \int_0^{\tau_o} [\frac{\sigma_a \sigma T^4(t)}{\pi} + \frac{\sigma_s}{2} \int_{-1}^{1} I(\mu', t) d\mu'] e^{-(\tau_o - t)/\mu} \frac{dt}{\mu\sigma_t}$$

and

$$I(+\mu, 0) = I(+\mu, \tau_o)e^{-\tau_o/\mu} + \int_0^{\tau_o} [\frac{\sigma_a \sigma T^4(t)}{\pi} + \frac{\sigma_s}{2} \int_{-1}^{1} I(\mu', t) d\mu'] e^{-t/\mu} \frac{dt}{\mu\sigma_t} \ .$$

Expression (2.13) can now be used to remove the integrals over μ':

$$I(-\mu, \tau_o) = I(-\mu, 0)e^{-\tau_o/\mu} + \int_0^{\tau_o} [\frac{\sigma T^4(t)}{\pi} - \frac{\sigma_t \sigma_s k}{4\pi\sigma_a} \frac{\partial^2 T(t)}{\partial t^2}] e^{-(\tau_o - t)/\mu} \frac{dt}{\mu} \quad (2.24)$$

$$I(+\mu, 0) = I(+\mu, \tau_o)e^{-\tau_o/\mu} + \int_0^{\tau_o} [\frac{\sigma T^4(t)}{\pi} - \frac{\sigma_t \sigma_s k}{4\pi\sigma_a} \frac{\partial^2 T(t)}{\partial t^2}] e^{-t/\mu} \frac{dt}{\mu} \quad (2.25)$$

If the contributions to the radiative flux from external isotropic sources plus wall emission are added to the portion of the incoming flux reflected at the wall at $\tau = 0$, it equals the total flux radiated, or $I(-\mu, 0)\mu$ integrated over a hemisphere. The factor μ is the component of the unit area in the direction of $I(-\mu, 0)$. Thus,

$$\int_{\text{hemisphere}} I(-\mu, 0)\mu d\Omega = \int_{\text{hemisphere}} [\frac{\epsilon_1 \sigma T^4(0)}{\pi} + \frac{t_1 \sigma T_x^4(0)}{\pi}] \mu d\Omega + (1 - \epsilon - t_1) \int_{\text{hemisphere}} I(+\mu, 0)\mu d\Omega \ . \quad (2.26)$$

The term in brackets under the integral sign includes emission from the boundary as well as flux passing through the wall. The wall has an emissivity ϵ_1, and a transmissivity t_1. The external source is at an apparent temperature $T_x(0)$. With transmission, the reflectivity becomes $(1-\epsilon_1-t_1)$.

Equation (2. 26) can be written in terms of $I(+\mu, \tau_o)$ by substituting (2. 25) for $I(+\mu, 0)$ and performing the integration over the hemisphere, while keeping in mind that $I(-\mu, 0)$ is independent of μ.

$$\pi I(-\mu, 0) = \epsilon_1 \sigma T^4(0) + t_1 \sigma T_x^4(0) + 2(1-\epsilon_1-t_1)\Big(I(+\mu, \tau_o)E_3(\tau_o) + \int_0^{\tau_o} [\sigma T^4(t) - \frac{\sigma_t \sigma_s k}{4\sigma_a} \frac{\partial^2 T(t)}{\partial t^2}] E_2(t) dt\Big) . \tag{2. 27}$$

Similarly, $I(+\mu, \tau_o)$, the intensity from the opposite wall, can be expressed in terms of $I(-\mu, 0)$:

$$\pi I(+\mu, \tau_o) = \epsilon_2 \sigma T^4(\tau_o) + t_2 \sigma T_x^4(\tau_o) + 2(1-\epsilon_2-t_2)\Big(I(-\mu, 0)E_3(\tau_o) + \int_0^{\tau_o} [\sigma T^4(t) - \frac{\sigma_t \sigma_s k}{4\sigma_a} \frac{\partial^2 T(t)}{\partial t^2}] E_2(\tau_o - t) dt\Big) . \tag{2. 28}$$

These expressions are reduced to dimensionless form by dividing (2. 27) and (2. 28) by $\sigma T^4(\tau_o)$ and using the following definitions:

$$\alpha_1 = \epsilon_1 \theta^4(0) + t_1 \theta_x^4(0) \tag{2. 29}$$

$$\alpha_2 = \epsilon_2 \theta^4(\tau_o) + t_2 \theta_x^4(\tau_o) \tag{2. 30}$$

$$\beta_1 = 2(1-\epsilon_1-t_1) \tag{2. 31}$$

$$\beta_2 = 2(1-\epsilon_2-t_2) \tag{2. 32}$$

$$\gamma_1 = \int_0^{\tau_o} [\theta^4(t, \alpha) - \frac{1-\alpha}{\alpha N} \theta''(t, \alpha)] E_2(t) dt \tag{2. 33}$$

$$\gamma_2 = \int_0^{\tau_o} [\theta^4(t, \alpha) - \frac{1-\alpha}{\alpha N} \theta''(t, \alpha)] E_2(\tau_o - t) dt . \tag{2. 34}$$

Now (2. 27) and (2. 28) become

$$\frac{\pi I(-\mu, 0)}{\sigma T^4(\tau_o)} = \alpha_1 + \beta_1 \left(\frac{I(+\mu, \tau_o)}{\sigma T^4(\tau_o)} E_3(\tau_o) + \gamma_1 \right) ,$$

$$\frac{\pi I(+\mu, \tau_o)}{\sigma T^4(\tau_o)} = \alpha_2 + \beta_2 \left(\frac{I(-\mu, 0)}{\sigma T^4(\tau_o)} E_3(\tau_o) + \gamma_2 \right) .$$

Solving for $I(-\mu, 0)$ and $I(+\mu, \tau_o)$,

$$\frac{\pi I(-\mu, o)}{\sigma T^4(\tau_o)} = \frac{\alpha_1 + \beta_1 [\alpha_2 E_3(\tau_o) + \gamma_1 + \beta_2 \gamma_2 E_3(\tau_o)]}{1 - \beta_1 \beta_2 E_3^2(\tau_o)} , \tag{2. 35}$$

$$\frac{\pi I(+\mu, \tau_o)}{\sigma T^4(\tau_o)} = \frac{\alpha_2 + \beta_2 [\alpha_1 E_3(\tau_o) + \gamma_2 + \beta_1 \gamma_1 E_3(\tau_o)]}{1 - \beta_1 \beta_2 E_3^2(\tau_o)} . \tag{2. 36}$$

The products of the right hand side of (2. 35) and (2. 36) times $E_2(\tau)$ and $E_2(\tau_o - \tau)$ are called source functions. They are denoted by S_1'' and S_2'':

$$S_1''(\tau, \alpha) = \frac{\alpha_1 + \beta_1 [\alpha_2 E_3(\tau_o) + \gamma_1 + \beta_2 \gamma_2 E_3(\tau_o)]}{1 - \beta_1 \beta_2 E_3^2(\tau_o)} E_2(\tau) , \tag{2. 37}$$

$$S_2''(\tau, \alpha) = \frac{\alpha_2 + \beta_2 [\alpha_1 E_3(\tau_o) + \gamma_2 \beta_1 \gamma_1 E_3(\tau_o)]}{1 - \beta_1 \beta_2 E_3^2(\tau_o)} E_2(\tau_o - \tau) . \tag{2. 38}$$

With this notation, the primes on S_1 and S_2 can be treated as the differential operator with respect to τ. Later, S_1 and S_2 will be used when approximate solutions are developed. From Appendix I, equation (I-5), S_1 and S_2 are

$$S_1(\tau, \alpha) = \frac{\alpha_1 + \beta_1 [\alpha_2 E_3(\tau_o) + \gamma_1 + \beta_2 \gamma_2 E_3(\tau_o)]}{1 - \beta_1 \beta_2 E_3^2(\tau_o)} E_4(\tau) \tag{2. 39}$$

$$S_2(\tau, \alpha) = \frac{\alpha_2 + \beta_2 [\alpha_1 E_3(\tau_o) + \gamma_2 + \beta_1 \gamma_1 E_3(\tau_o)]}{1 - \beta_1 \beta_2 E_3^2(\tau_o)} E_4(\tau_o - \tau) \tag{2. 40}$$

where the various terms making up S_1 and S_2 are defined in (2. 29) through (2. 34).

In terms of the source functions, the transport equation (2. 23) becomes

$$\theta''(\tau, \alpha) + \frac{\alpha N}{2} [S_1''(\tau, \alpha)+S_2''(\tau, \alpha)+ \int_0^{\tau_o} \theta^4(t, \alpha)E_1 |t-\tau| dt-2\theta^4(\tau, \alpha)]$$

$$= \frac{1-\alpha}{2} \int_0^{\tau_o} \theta''(t, \alpha)E_1 |t-\tau| dt \ . \qquad (2.\ 41)$$

4. The Heat Transfer Rate

The heat transfer rate per unit area, q_d, is the contribution from conduction and the next radiative flux crossing a unit area parallel to the boundaries:

$$q_d(\tau, \alpha) = \frac{k\partial T(x)}{\partial x} + 2\pi \left[\int_0^1 I(+\mu, \tau)\mu d\mu - \int_0^1 I(-\mu, \tau)\mu d\mu \right]. \qquad (2.\ 42)$$

The expression in brackets is the net radiant flux passing through a unit area in a positive sense. The first part of the expression is the radiative flux passing through a unit area in a positive sense, and the last term is the radiation passing through the opposite side and hence reduces q_d. With (2. 24) and (2. 25) used for the intensity, q_d becomes

$$q_d(\tau, \alpha) = \sigma_t k \frac{\partial T(\tau, \alpha)}{\partial \tau} + 2\pi[I(+\mu, \tau_o)E_3(\tau_o - \tau) - I(-\mu, 0)E_3(\tau)]$$

$$+2\int_{\tau}^{\tau_o}[\sigma T^4(t) - \frac{\sigma_t \sigma_s k}{4\sigma_a}\frac{\partial^2 T(t)}{\partial t^2}]E_2(t-\tau)dt$$

$$-2\int_0^{\tau}[\sigma T^4(t) - \frac{\sigma_t \sigma_s k}{4\sigma_a}\frac{\partial^2 T(t)}{\partial t^2}]E_2(\tau - t)dt \qquad (2.43)$$

This expression can be written in dimensionless form, with the dimensionless heat transfer rate, q, defined by

$$\frac{q(\tau, \alpha)}{\tau_o} = \frac{q_d(\tau, \alpha)}{\sigma_t k T(\tau_o)} \ . \qquad (2.44)$$

With this definition and (2. 29) through (2. 34), (2. 43) becomes:

$$\frac{q(\tau, \alpha)}{\tau_o} = \theta'(\tau, \alpha) + \frac{N}{2}[S_1'(\tau, \alpha) + S_2'(\tau, \alpha) + \int_{\tau}^{\tau_o}\theta^4(t, \alpha)E_2(t-\tau)dt$$

$$-\int_0^{\tau}\theta^4(t, \alpha)E_2(\tau-t)dt] - \frac{1-\alpha}{2\alpha}\left[\int_{\tau}^{\tau_o}\theta''(t, \alpha)E_2(t-\tau)dt - \int_0^{\tau}\theta''(t, \alpha)E_2(\tau-t)dt\right] \qquad (2.45)$$

where S_1' and S_2' are the derivatives with respect to τ of S_1 and S_2 which in turn are defined in (2. 39) and (2. 40). For steady state, q is independent of τ. Nevertheless, the functional form is retained. It will prove convenient to evaluate q by an appropriate choice of τ.

5. The Temperature Distribution and Heat Transfer Rate for Radiation

Before treating the general transport equation with conduction, the equation and solutions for pure radiative transport should be discussed. The results for the temperature profile and heat transfer rates will prove useful in the following sections.

The equation for the temperature profile can be derived from (2.41) by dividing by N and letting k, in the definition of N given in (2.22), approach zero. Hence, if all of the other coefficients remain constant, $N \to \infty$. This corresponds to a decrease in conductivity so that radiation dominates. In the limit as $N \to \infty$, (2.41) yields

$$0 = \tfrac{1}{2}[S_1''(\tau) + S_2''(\tau) + \int_0^{\tau_o} \theta^4(t)E_1|t-\tau|dt - 2\theta^4(\tau)] . \qquad (2.46)$$

As can be seen in (2.33), (2.34), (2.37), and (2.38), the source functions are independent of α, and thus θ is also. The solution is a function of the parameters ϵ_1, ϵ_2, t_1, t_2, and τ_o, as well as the boundary temperatures and external source strengths but not α.

Similarly, the heat transfer rate is found from (2.45). It must be noted that when this equation is divided by N, and $k \to 0$, q remains finite:

$$\lim_{k \to 0} \frac{q}{N\tau_o} = \frac{q_{R_d}}{4\sigma T^4(\tau_o)}$$

so that (2.45) becomes

$$\frac{q_{R_d}}{4\sigma T^4(\tau_o)} = \tfrac{1}{2}[S_1'(\tau)+S_2'(\tau)+\int_\tau^{\tau_o} \theta^4(t)E_2(t-\tau)dt - \int_0^{\tau} \theta^4(t)E_2(\tau-t)dt] . \qquad (2.47)$$

Again, S_1' and S_2' as well as θ (from (2.46)) are independent of α, so q_{R_d} is not a function of the degree of scattering.

Chandrasekhar[4] gave an exact expression for q_{R_d} of the form

$$\frac{q_{R_d}}{\sigma T^4(\tau_o) - \sigma T^4(0)} = \frac{B_o(\tau_o)}{1 + \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 2\right)B_o(\tau_o)} \qquad (2.48)$$

for arbitrary diffuse wall emissivities. In Heaslet and Warming's paper[15], the values of B_o as a function of τ_o were given to four figures, and are reproduced in Table I. Expressions for the temperature profile were also given in their paper in graphical form. Typical profiles will appear later when the solutions of temperature profile with radiation and conduction are developed.

III. APPROXIMATE SOLUTION FOR THE TEMPERATURE DISTRIBUTION IN AN ABSORBING AND CONDUCTING MEDIUM

The temperature profile for a medium which absorbs and conducts is treated in this section. The more general problem, allowing scattering, is best studied later; the present results apply directly to the more general problem and are needed in its solution. In addition, the absorption problem is itself of interest and has held the attention of several authors.

Most authors have been primarily interested in the heat transfer rate; their treatment of the temperature distribution is only an intermediate step in the heat transfer computation. Viskanta and Grosh[14, 16] have been somewhat of an exception in that they presented curves for the temperature distribution under several boundary conditions. They approached the problem by starting with the basic equation (2.41) with $\alpha = 1$:

$$\theta''(\tau) = -\frac{N}{2}[S_1''(\tau)+S_2''(\tau)+\int_0^{\tau_o}\theta^4(t)E_1|t-\tau|dt-2\theta^4(\tau)] \qquad (3.1)$$

and, following the method suggested by Lichtenstein[17] for treating such nonlinear integro-differential equations, integrated it twice;

$$\theta(\tau) = A+B\tau - \frac{N}{2}[S_1(\tau)+S_2(\tau)+\int_0^{\tau_o}\theta^4(t)E_3|t-\tau|dt] \ . \qquad (3.2)$$

The constants were removed by evaluating the equation at $\tau = 0$ and τ_o. The solution was found by successive iterations. An initial guess, θ_o, for the temperature distribution, was put in the right hand side of (3. 2). The resulting expression, numerically evaluated, led to a new value, θ_1, for the temperature profile. This value was again inserted in the right hand side, and the process repeated until the successive θ_i differed by a sufficiently small amount. The resulting numerical expression for θ was considered the solution. Later, in 1963, Viskanta[18] presented a second method. He expanded the term θ^4 on the right hand side of (3. 1) in a Taylor series about τ. The integral term could then be evaluated, and the result yielded a nonlinear differential equation in θ. He found that the latter method was more suitable for numerical computations but gave less accurate results.

Of the other authors who have treated the temperature profile in order to compute the heat transfer rate, Grief[19] has done the most to delineate the various methods. One approach, first proposed by Lick[21], again reduced (3. 1) to a nonlinear differential equation which could be treated numerically. He integrated the equation once with respect to τ. The kernel, E_2, appearing in the result, was approximated by a substitute kernel, $3/4\, e^{-3/2\,\tau}$, which has the same area and the same first moment as the exact kernel. With this substitution, the integral term was removed by differentiating the approximate equation twice and subtracting the result from the approximate equation. The result is

$$\theta^{(4)}(\tau) - \tau_o^2\theta''(\tau) - \frac{N}{3}\left(\theta^4(\tau)\right)'' = 0 \quad . \qquad (3.3)$$

It is a nonlinear differential equation in θ that can be solved numerically for θ. For very small N (so that conduction dominates), Lick found it useful to expand θ in a power series in τ. The coefficients were evaluated by substituting the expansion into (3. 3). For large optical depth, Grief ignored the first term in (3. 3). This reduced the equation to a diffusion approximation given by Cess[22]. Cess, in turn, based his formulation on Rosseland's approximation for pure radiation.[1] Rosseland suggested that an optically thick medium would diffuse radiation so that the energy transport equation would be

$$\frac{N}{3}\nabla^2(\sigma T^4) = 0 .$$

This, plus the normal conduction term and the present boundary conditions, led directly to Grief's result; that is,

$$\tau_o^2\theta''(\tau) + \frac{N}{3}\left(\theta^4(\tau)\right)'' = 0 .$$

1. The Solution

The present approach expresses the temperature distribution as an expansion in powers of N about N = 0. The validity of the expansion is discussed in Appendix II.

Before turning to the solution, it is important to discuss the expansion parameter, N. From its definition;

$$N = \frac{4\sigma T^4(\tau_o)}{\sigma_t k T(\tau_o)} . \qquad (3.4)$$

N is a measure of the relative magnitude of radiation to condition. However, care must be taken to interpret its exact meaning. If the denominator in (3. 4) is multiplied and divided by x , the geometrical distance between the boundaries, N, becomes:

$$N = \frac{4\sigma T^4(\tau_o)}{\sigma_t x\left(\frac{kT(\tau_o)}{x}\right)} = \frac{4\sigma T^4(\tau_o)}{\tau_o\left(\frac{kT(\tau_o)}{x}\right)} .$$

The numerator is four times the radiant flux from a black surface at temperature $T(\tau_o)$. The denominator is the optical depth times the conductive heat transfer rate in a slab with a temperature differential, $T(\tau_o)$, and thickness x. The product $N\tau_o$ is a measure of the relative magnitudes of the two transport mechanisms. Thus, if τ_o is small, N must be quite large for radiation to be important, and conversely, if τ_o is large, even modest N will correspond to large radiative fluxes.

The following result is required to treat the equation for the temperature profile. Consider an equation of the form

$$f(\tau) = A + B\tau .$$

To solve for A and B, the equation can be evaluated at $\tau = 0$ to give A, and then at $\tau = \tau_o$ to give B:

$$A = f(0)$$
$$B = \frac{1}{\tau_o}\left(f(\tau_o) - f(0)\right)$$

so that

$$f(\tau) = f(0) + \frac{\tau}{\tau_o}\left(f(\tau_o) - f(0)\right)$$

or

$$f(\tau) - f(0) - \frac{\tau}{\tau_o}\left(f(\tau_o) - f(0)\right) = 0 .$$

If a linear operator G, operating on functions of τ, is defined by

$$G\left(f(\tau)\right) = f(\tau) - f(0) - \frac{\tau}{\tau_o}\left(f(\tau_o) - f(0)\right), \qquad (3.5)$$

it has the effect of replacing the constants A and B by the function evaluated at the boundaries.

The operator G can be used in the equation for the temperature profile (3.2),

$$G\left(\theta(\tau)\right) = -\frac{N}{2}\left[G\left(S_1(\tau)+S_2(\tau)\right)+\int_0^{\tau_o}\theta^4(t)G(E_3|t-\tau|)dt\right] \quad . \qquad (3.6)$$

This expression has the advantage of eliminating the constants A and B while not burdening the expression with unnecessary complexity.

If θ is considered a function of the parameter N, and expanded about $N = 0$, that is,

$$\theta(\tau) = \theta_o(\tau) + N\theta_1(\tau) + \ldots \quad ,$$

and this form is used in (3.6), the latter becomes

$$G\left(\theta_o(\tau)+N\theta_1(\tau)+\ldots\right) = -\frac{N}{2}\left[G\left(S_1(\tau)+S_2(\tau)\right)+\right.$$
$$\left.+\int_0^{\tau_o}\left(\theta_o(t)+N\theta_1(t)+\ldots\right)^4 G(E_3|t-\tau|)dt\right] . \qquad (3.7)$$

Since, in general, the source functions are functions of θ, let S_{1i} and S_{2i} correspond to the terms in S_1 and S_2 multiplied by the i^{th} power of N. Then, equating powers of N in (3.7),

$$G\left(\theta_o(\tau)\right) = 0 \qquad (3.8)$$

$$G\left(\theta_1(\tau)\right) = -\tfrac{1}{2}\left[G\left(S_{10}(\tau)+S_{20}(\tau)\right)+\int_0^{\tau_o}\theta_o^4(t)G(E_3|t-\tau|)dt\right] \qquad (3.9)$$

. . .

Note that $\theta(0)$ and $\theta(\tau_o)$ are specified, and hence independent of N, so for i greater than zero,

$$\theta_i(0) = \theta_i(\tau_o) = 0 .$$

With this result, (3.8) yields

$$\theta_o(\tau) = \theta(0) + \frac{\tau}{\tau_o}\left(\theta(\tau_o) - \tau(0)\right) \qquad (3.10)$$

and

$$\theta_1(\tau) = -\tfrac{1}{2}[G(S_{10}(\tau)+S_{20}(\tau))+\int_0^{\tau_o}\theta_o^4(t)G(E_3|t-\tau|)dt] \qquad (3.11)$$

. . . .

The relations represent a self-generating sequence for θ. This is evident from (3.7). Since N multiplies the entire right hand side, the highest order of θ_k that can appear for the i^{th} order term is $k = i-1$, so that $\theta_i(\tau)$ is a function of θ_{i-1}, θ_{i-2}, ..., θ_o. Hence, given θ_o, θ_1 can be found; then using θ_1, θ_2 is determined, and so on.

The expression for θ_o corresponds to the solution for $N = 0$ and is the expected linear conduction temperature profile. Higher order terms, $\theta_1, \theta_2, \ldots,$ represent corrections to the temperature profile for N near zero.

Consider approximating θ by

$$\theta(\tau) \sim \theta_o(\tau) + N\theta_1(\tau) \ . \qquad (3.12)$$

With (3.11), (3.12) can be written as

$$\theta(\tau) \sim \theta_o(\tau) - \frac{N}{2}[G(S_{10}(\tau)+S_{20}(\tau))+\int_0^{\tau_o}\theta_o^4(t)G(E_3|t-\tau|)dt] \ . \qquad (3.13)$$

The source functions, S_{10} and S_{20}, are computed from (2.39) and (2.40) with θ replaced by θ_o, and α set equal to unity in γ_1 and γ_2 given in (2.33) and (2.34). Writing the integrals which involve θ_o in $G(S_{10})$ explicitly,

$$G\big(S_{10}(\tau)\big) = G\big(E_4(\tau)\big)\frac{\alpha_1+\beta_1\left[\alpha_2E_3(\tau_o)+\int_0^{\tau_o}\theta_o^4(t)E_2(t)dt+\beta_2E_3(\tau_o)\int_0^{\tau_o}\theta_o^4(t)E_2(\tau_o-t)dt\right]}{1-\beta_1\beta_2E_3^2(\tau_o)} . \qquad (3.14)$$

Similarly,

$$G\big(S_{20}(\tau)\big) = G\big(E_4(\tau_o-\tau)\big)\frac{\alpha_2+\beta_2\left[\alpha_1E_3(\tau_o)+\int_0^{\tau_o}\theta_o^4(t)E_2(\tau_o-t)dt+\beta_1E_3(\tau_o)\int_0^{\tau_o}\theta_o^4(t)E_2(t)dt\right]}{1-\beta_1\beta_2E_3^2(\tau_o)} . \qquad (3.15)$$

Since θ_o is a function of τ, τ_o, and $\theta(0)$ only, the notation can be simplified by defining the following functions:

$$F_1(\tau,\tau_o) = -G\big(E_4(\tau)\big) , \qquad (3.16)$$

$$I\big(\tau,\tau_o,\theta(0)\big) = \int_0^{\tau_o}\theta_o^4(t)G(E_3|t-\tau|)dt , \qquad (3.17)$$

$$J_1\big(\tau_o,\theta(0)\big) = \int_0^{\tau_o}\theta_o^4(t)E_2(t)dt , \qquad (3.18)$$

$$J_2\big(\tau_o,\theta(0)\big) = \int_0^{\tau_o}\theta_o^4(t)E_2(\tau_o-t)dt . \qquad (3.19)$$

The integrations necessary to compute I, J_1, and J_2 are carried out in Appendix III and appear as linear sums of the exponential functions $E_2, \ldots, E_8$.

Finally, in terms of (3.13) through (3.19), the first order approximation to the temperature distribution is:

$$\theta(\tau) \sim \theta_o(\tau) + N\theta_1(\tau) =$$

$$= \theta_o(\tau) + \frac{N}{2}\left(\frac{\alpha_1+\beta_1[\alpha_2 E_3(\tau_o)+J_1+\beta_2 E_3(\tau_o)J_2]}{1-\beta_1\beta_2 E_3^2(\tau_o)} F_1(\tau,\tau_o) + \right.$$

$$\left. + \frac{\alpha_2+\beta_2[\alpha_1 E_3(\tau_o)+J_2+\beta_1 E_3(\tau_o)J_1]}{1-\beta_1\beta_2 E_3^2(\tau_o)} F_1(\tau_o-\tau,\tau_o) - I(\tau)\right) . \quad (3.20)$$

The temperature distribution can be computed once the boundary conditions and the material properties are specified. In (3.10), $\theta_o(\tau)$ is given, and α_1, α_2, β_1, and β_2 are computed from (2.29) through (2.32). The exponential integral E_3 is presented in Table I. The functions J_1 and J_2 are found in Table II, while F_1 and I are given in Table III. In the tables, the variable $\theta(0)$ ranges from 0.1 to 0.9 in intervals of 0.1; τ goes from 0 to τ_o in intervals of $\tau_o/20.0$; and τ_o equals 0.1, 0.2, 0.5, 1.0, 2.0, 5.0, and 10.0. In Appendix II, an absolute bound for the error in (3.20) is developed. The result, equation (II-20), tends to overestimate the actual error. However, the expression is simple in form and indicates when the error may be large, and in problems with unusual boundary conditions, it can be used to guarantee the validity of the approximation for sufficiently small N.

Before discussing the physical meaning of F_1, I, J_1, and J_2, it is necessary to mention some properties of some terms appearing in them. From Appendix I, equation (I-9),

$$E_n(\tau) \sim \frac{e^{-\tau}}{\tau} \qquad \tau > 1 \qquad \text{all } n ,$$

and from (I-4) and (I-6),

$$E_1(\tau) \sim -\ln\tau \qquad \tau << 1 ,$$

$$E_n(0) = 1/(n-1) \qquad n > 1 .$$

Thus, all E_n decrease rapidly for $\tau > 1$. As a second remark, note that I, J_1, and J_2 are integrals of θ_o. As such, they are evaluated using the conductive temperature profile and not the exact temperature distribution.

The function F_1, (3. 16), is a measure of the portion of the doubly integrated boundary intensities reaching a point τ. The second derivative of F_1, i.e.,

$$F_1''(\tau, \tau_o) = -E_2(\tau) ,$$

is the negative of the actual portion. As τ or $\tau_o - \tau$ becomes greater than one, F_1'' becomes very small. As seen from the asymptotic form of E_n, for a medium with $\tau_o >> 1$, most of the flux comes from within one optical depth of τ, and so F_1'' main contribution is near boundaries. Thus, the effect of external sources and wall conditions is mainly limited to the region within one optical depth of the boundaries. Of course, the radiation absorbed in this region can be reradiated or conducted into the interior. By this process, the actual temperature profile is modified by the intensities at the boundaries.

The next function, I, can best be discussed in terms of its second derivative, I''. The latter appears in the original expression for the temperature profile, while I is a consequence of the particular method chosen to treat the equation. The function, I, is the doubly integrated net flux of radiant energy from the interior medium to a point, τ, while its second derivative is twice the actual net influx:

$$I'' = \int_0^{\tau_o} \theta_o^4(t) E_1 |t-\tau| dt - 2\theta_o^4(\tau) \quad .$$

In this expression, the integral term represents the portion of the emission at t that reaches τ. The term θ_o^4 is the dimensionless emission from τ, and so the entire term is the net influx of radiant energy from the interior medium.

The two functions J_1 and J_2 are one-half the flux from the interior medium to the boundaries at $\tau = 0$ and $\tau = \tau_o$, respectively. Since the exponential integrals decrease rapidly for large arguments, for large optical depths, the main contribution comes from a zone of about one optical depth in thickness. Thus, for $\tau_o >> 1$, J_1 and J_2 depend only on conditions near the boundaries.

2. Examples

As a particular example of the effect of radiation interacting with conduction, let $\theta(0) = 0.5$ and $\tau_o = 0.1$. This corresponds to a medium that is optically quite thin, since a beam normal to one wall is only attenuated by $\exp(-0.1)$ in passing between the walls. Thus, most of the radiation from one boundary interacts with the other without being absorbed by the intervening medium, and the coupling between radiation and conduction is relatively weak even for modest values of N. As a consequence, the temperature profile should be close to that for pure conduction. Figure 2 shows the profile for various values of N, using the approximation in (3. 20) with $\varepsilon_1 = \varepsilon_2 = 1$. Also included for the same wall conditions is the result for pure radiation from Heaslet and Warming[15]. The results show that even for

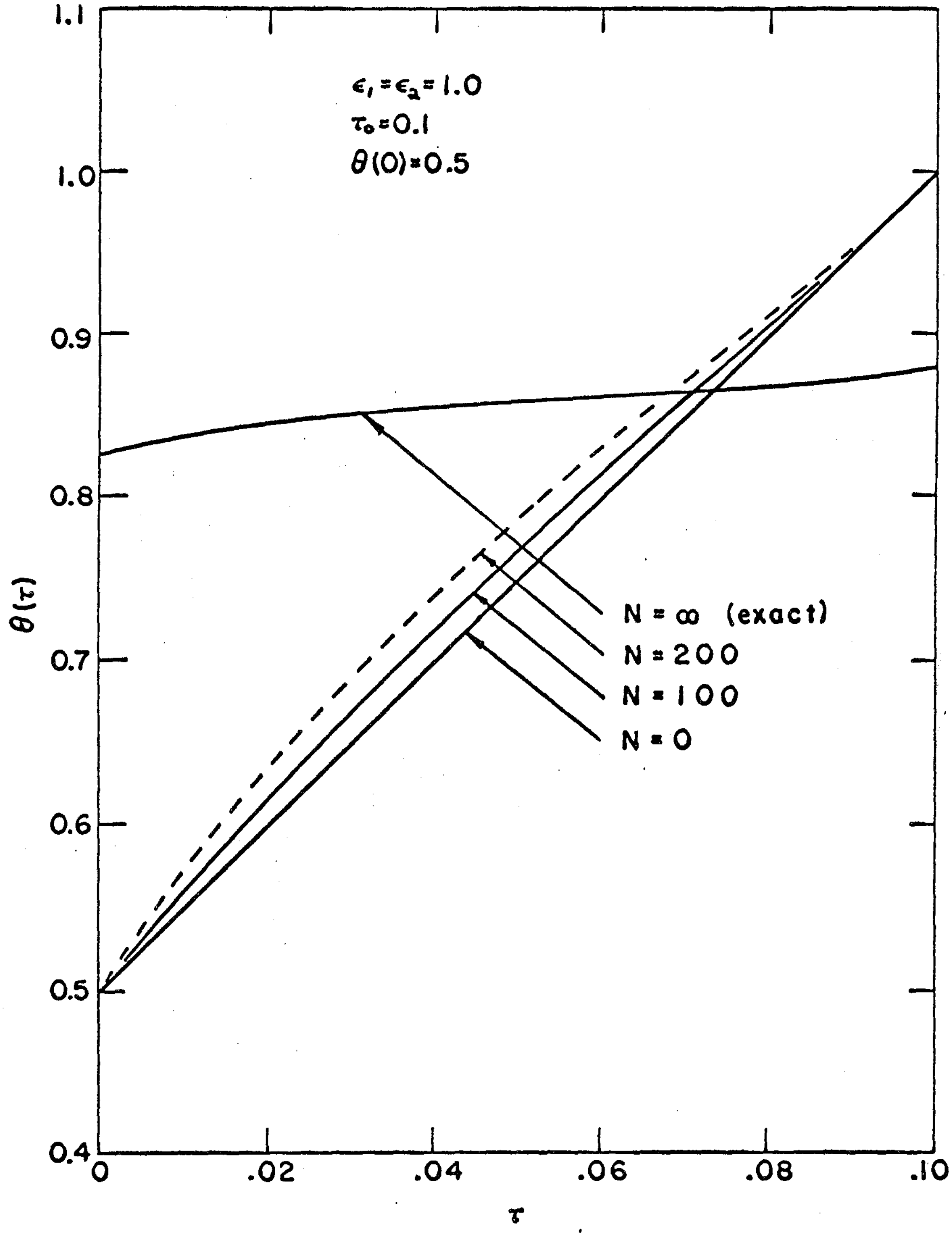

Figure 2. The Temperature Distribution.

$N = 200$ the temperature profile does not change much. The value $N = 200$ exceeds the demonstrable usefulness of the linear expansion given in Appendix II. To indicate this, it is included as a dashed line. Aside from reasoning physically that most of the radiation does not interact with the interior medium, it must be remembered that the product $N\tau_o$ is the ratio of radiative transport to conduction. But τ_o is one-tenth in the case chosen, so N must be relatively large for radiation to be important.

For arbitrary wall emissivities, the temperature profile varies in shape. As the emissivities become small, reflections occur causing the intensities from the opposite boundaries to become similar. The quantitative effect on θ_1, for various wall emissivities with $\epsilon_1 = \epsilon_2$, is shown in Figure 3. From these results, it is seen that varying the emissivities changes the magnitude of θ_1 as well as its shape.

Obviously, the inclusion of high temperature external sources along with high transmission will affect θ_1. Even a modest value of N may correspond to large changes in the temperature profile, as well as limit the approximation's range of validity.

Now consider $\theta(0) = 0.5$ and $\tau_o = 1.0$. The optical depth is large enough for the intervening medium to absorb a significant portion of the radiation from the boundaries. Since $\tau_o = 1$, N is order of the ratio of radiative transport to conduction. The results for various N and black walls are presented in Figure 4. Also included is the case of pure radiation.

If the wall emissivities are allowed to vary, the profile changes shape once again. But the medium close to one wall is

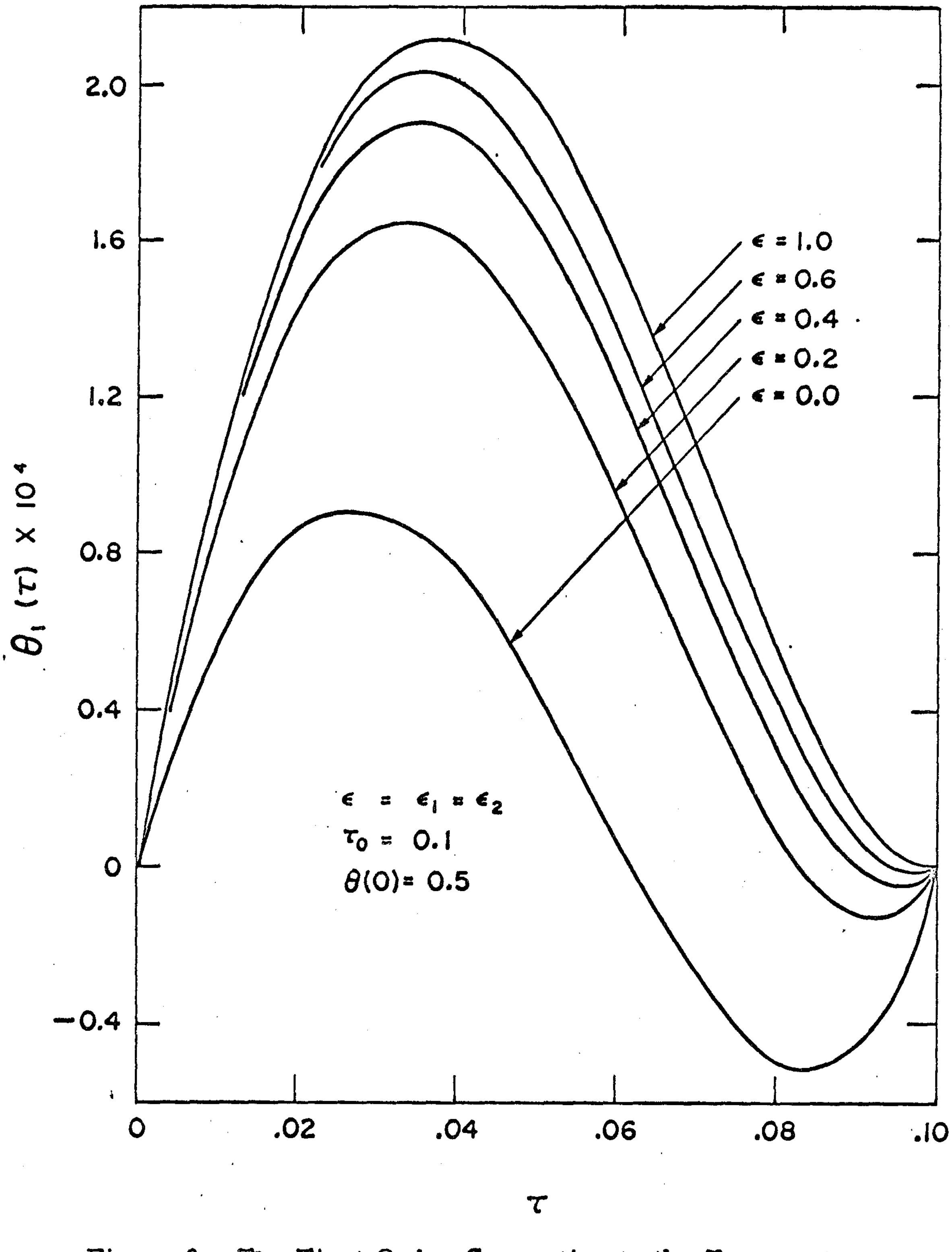

Figure 3. The First Order Correction to the Temperature Distribution.

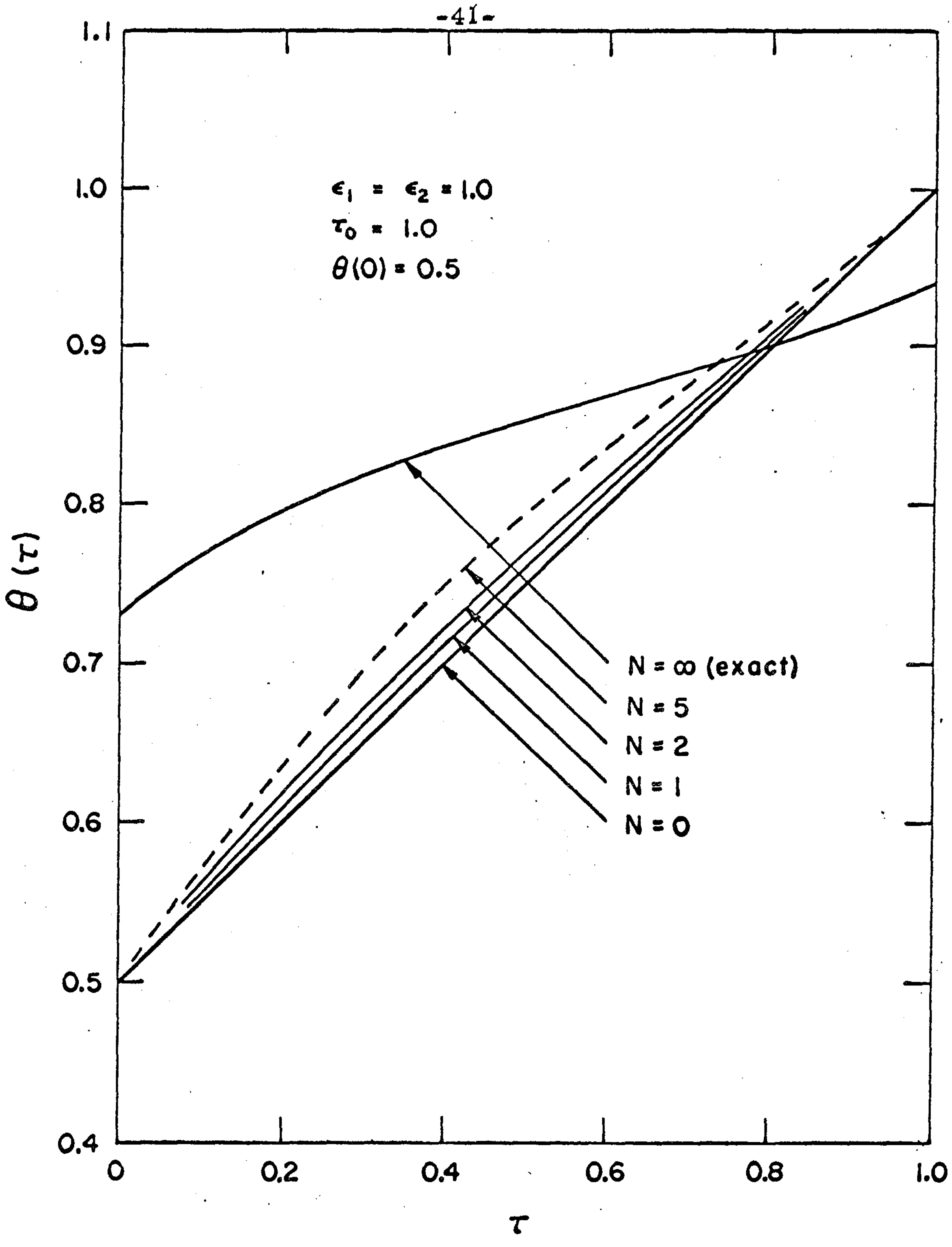

Figure 4. The Temperature Distribution.

partially shielded from the other wall by the intervening medium, so the profile is more dependent on the local wall property, as opposed to the previous case where the average over both boundaries dominated. Results for θ_1 with various wall emissivities ($\epsilon_1 = \epsilon_2$) are given in Figure 5.

Finally, with $\theta(0) = 0.5$ and $\tau_o = 10$, the medium is optically thick. The radiation appears to diffuse, and on the average it is absorbed and re-emitted many times in traversing the medium. From the definition of N, it is apparent that N is now one-tenth the ratio of radiation to conduction, so that even modest N corresponds to large radiative fluxes. The results for wall emissivities equal to one and various N are given in Figure 6 along with the pure radiation result.

As expected, the wall emissivity has little effect on θ except close to the boundary at $\tau = \tau_o$. Near the other, cooler wall, the radiative intensity is much lower and the wall's emissive properties are unimportant. In the interior, radiative transport is very much a diffusion process. Boundary conditions only determine the radiative influx, but the temperature profile is governed by diffusion. These characteristics are shown in Figure 7. In the figure, θ_1 is given for several wall emissivities, $\theta(0) = 0.5$ and $\tau_o = 10$.

In Figures 2 - 7, $\theta(0)$ was taken equal to one-half. Little qualitative variation would have occurred for other values. The shape of the temperature profile is mainly dependent on the optical depth and the flux from the boundaries. The flux, in turn, is determined by external sources and wall emissivity.

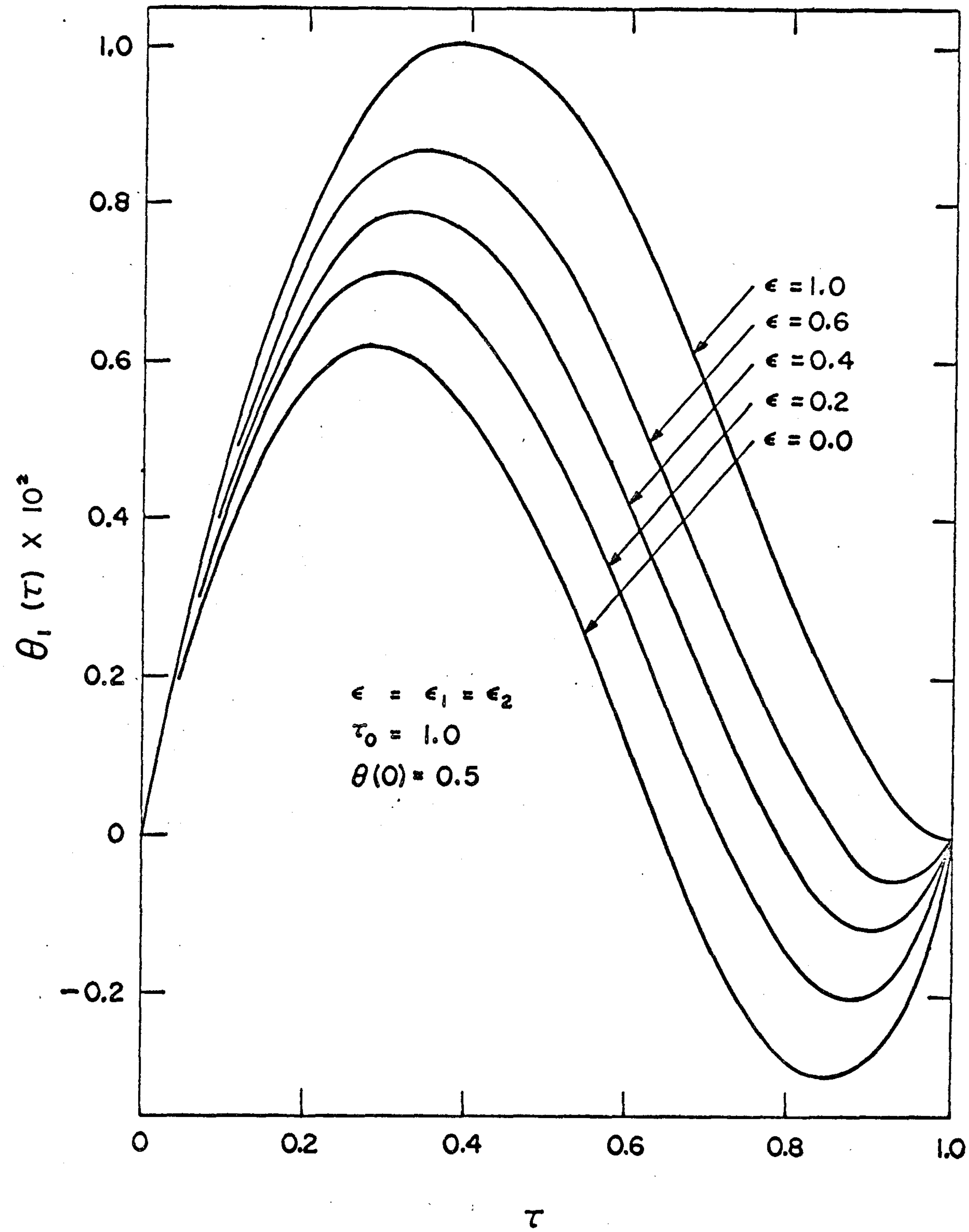

Figure 5. The First Order Correction to the Temperature Distribution.

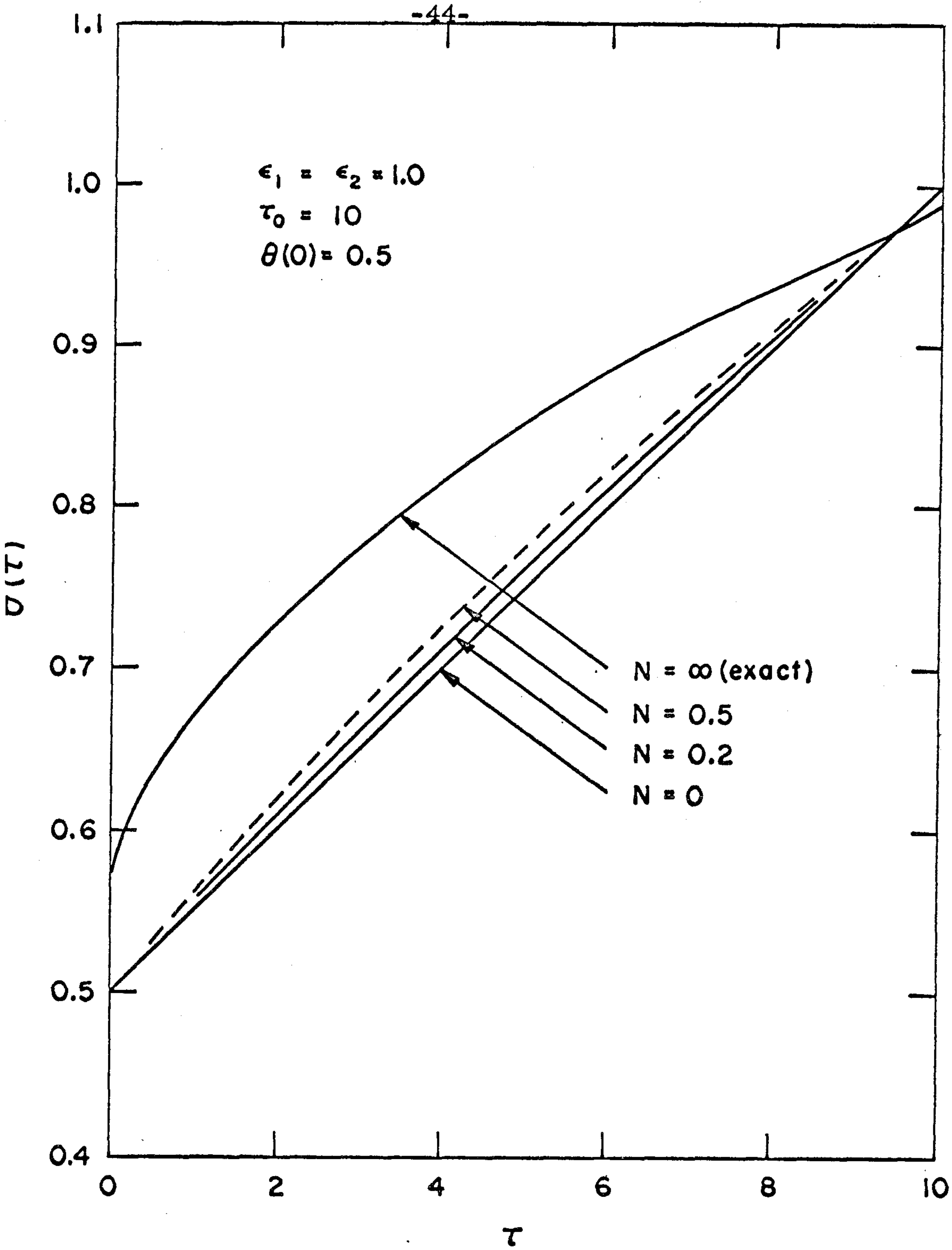

Figure 6. The Temperature Distribution.

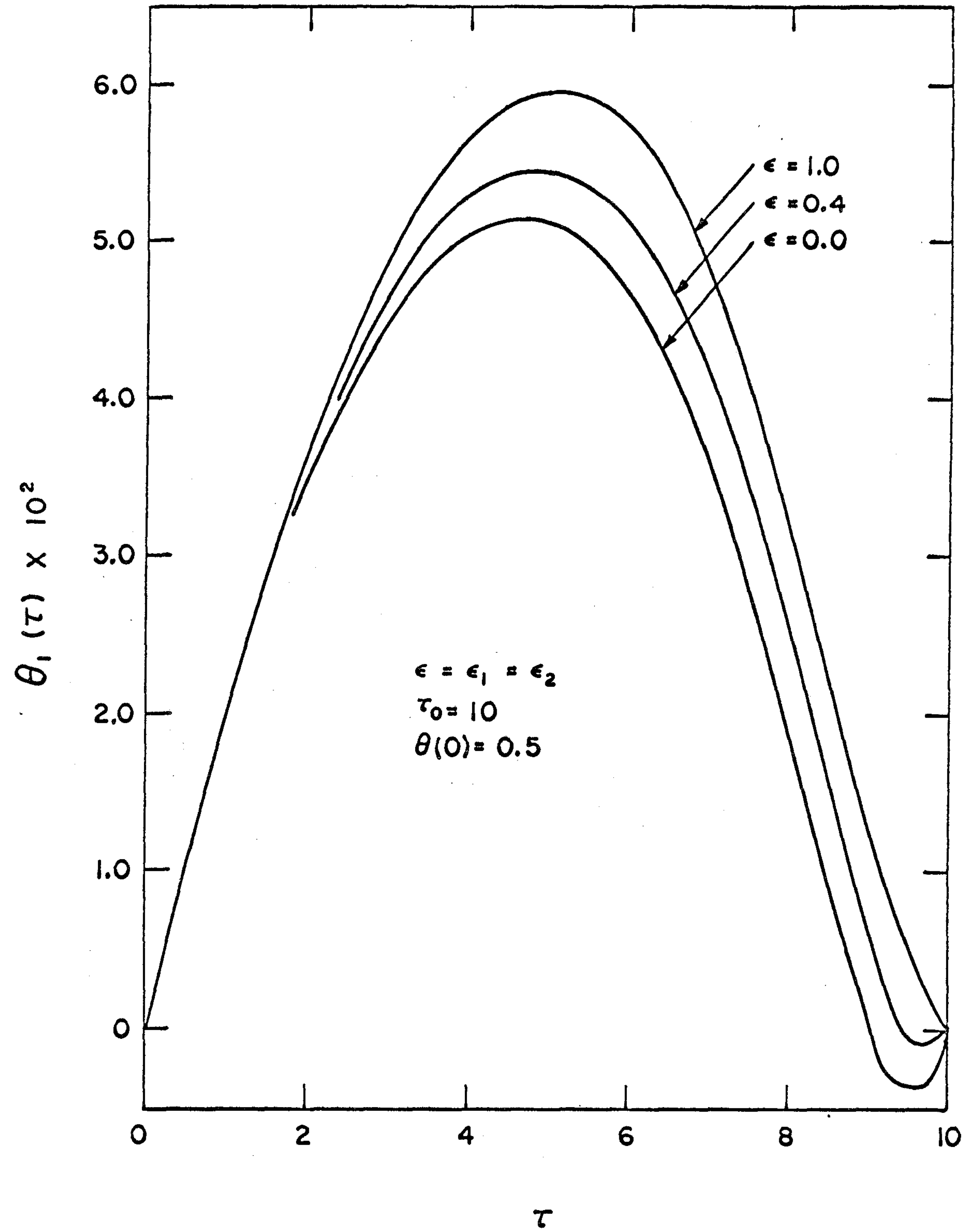

Figure 7. The First Order Correction to the Temperature.

IV. APPROXIMATE SOLUTION FOR THE HEAT TRANSFER RATE IN AN ABSORBING AND CONDUCTING MEDIUM

The heat transfer rate in an absorbing and conducting medium has held the attention of several authors. As mentioned in the previous section, diverse methods have been employed to find the temperature distribution from which the heat transfer rate could be computed.

From its definition in equation (2.45), with $\alpha = 1$, q is given by

$$\frac{q(\tau)}{\tau_o} = \theta'(\tau) + \frac{N}{2}\left[S_1'(\tau) + S_2'(\tau) + \int_{\tau}^{\tau_o} \theta^4(t)E_2(t-\tau)dt - \int_0^{\tau} \theta^4(t)E_3(\tau-t)dt\right]. \quad (4.1)$$

It can be seen that computations are complicated by the appearance of the derivative of θ as well as integrals over θ^4. Since the temperature profile cannot be given in an exact analytical form, θ' and the integrals must be computed numerically.

The methods used to find the temperature distribution have been discussed in Section III. Relatively little more need be added, except to note that Lick found q without explicitly solving for θ. His final approximate equation, (3.3), can be considered a function of θ' and, with the substitute kernel, the integrals of θ^4 do not appear.

An entirely different approach to the heat transfer problem was made by Einstein[23]. He chose to uncouple radiation and conduction and to consider the independent sum of the separate heat transfer rates as the solution. This formulation has the advantage of reducing the analysis to a simple algebraic computation in terms of the results for pure radiation given in (2.48), and that of conduction. However, the method is only of value if the wall emissivities are near one and the walls are opaque. If these conditions do not exist, a significant portion of the radiation is generated or absorbed within the medium. This contribution is ignored in his formulation.

A comparison of results obtained by the various approaches is discussed in Part 2 of this section after the present approximate solution has been developed.

1. The Solution

The form for q given in (4. 1) is inappropriate for the present approach since the expression for the conduction term involves a derivative of θ, which, in terms of the previously developed approximation for the temperature distribution, involves the derivative of θ_1. However, if the equation is integrated with respect to τ and satisfied at the endpoints, the term involving θ is exact, since only $\theta(0)$ and $\theta(\tau_o)$ appear. Carrying out the integration, (4. 1) becomes:

$$\frac{\tau q(\tau)}{\tau_o} + A = \theta(\tau) + \frac{N}{2}\left[S_1(\tau) + S_2(\tau) + \int_0^{\tau_o} \theta^4(t) E_3|t-\tau|\,dt\right] ,$$

since q is independent of τ. A can be eliminated by first evaluating the expression at $\tau = \tau_o$ and then subtracting from it the expression evaluated at $\tau = 0$. This can be done in terms of an operator, G_1, defined by

$$G_1\big(f(\tau)\big) = f(\tau_o) - f(0) . \tag{4. 2}$$

The expression for q becomes:

$$q = G_1\big(\theta(\tau)\big) + \frac{N}{2}\left[G_1\big(S_1(\tau)+S_2(\tau)\big) + \int_0^{\tau_o} \theta^4(t) G_1(E_3|t-\tau|)\,dt\right]$$

or

$$q = \theta(\tau_o)-\theta(0)+\frac{N}{2}[G_1\big(S_1(\tau)+S_2(\tau)\big)+\int_0^{\tau_o}\theta_4(t)G_1(E_3|t-\tau|)dt] . \qquad (4.3)$$

The first two terms are the dimensionless expression for the heat transfer by pure conduction, q_c,

$$q_c = \theta(\tau_o)-\theta(0) = \frac{k[T(\tau_o)-T(0)]}{x}\,\frac{x}{kT(\tau_o)}\,\frac{\sigma_t}{\sigma_t} = \frac{k[T(\tau_o)-T(0)]}{x}\,\frac{\tau_o}{\sigma_t kT(\tau_o)} . \qquad (4.4)$$

The remaining terms are the modification to q associated with radiation. They include a contribution to conductive transfer as well as the net radiative flux.

If now θ appearing in the integral terms is replaced by the linear profile, θ_o, q is approximated by

$$q \sim q_c + \frac{N}{2}[G_1\big(S_{10}(\tau)+S_{20}(\tau)\big) + \int_0^{\tau_o}\theta_o^4 G_1(E_3|t-\tau|)dt] . \qquad (4.5)$$

Let the factor multiplied by N be q_o, that is,

$$q_o = \tfrac{1}{2}G_1\big(S_{10}(\tau)+S_{20}(\tau)\big)+\tfrac{1}{2}\int_0^{\tau_o}\theta_o^4(t)G_1(E_3|t-\tau|)dt . \qquad (4.6)$$

In addition to the functions J_1 and J_2 which appeared in the expression for θ_1, two new functions, k and $G_1(E_4)$, are required to represent q_o. Let

$$K\big(\tau_o, \theta(0)\big) = \int_0^{\tau_o}\theta_o^4(t)G_1(E_3|t-\tau|)dt = \int_0^{\tau_o}\theta_o^4(t)[E_3(\tau_o-t)-E_3(t)]dt . \qquad (4.7)$$

The last equality follows from the definition of G_1 in (4.2). The integration necessary to compute K is performed in Appendix III.

With this notation, and the functions S_1 and S_2 in (3.14) and

(3.15), q_o can be written in a form analogous to θ_1 :

$$q_o = \frac{G_1(E_4(\tau_o-\tau))}{2}\left[\frac{\alpha_2-\alpha_1+(\beta_2\alpha_1-\beta_1\alpha_2)E_3(\tau_o)+\beta_2J_2-\beta_1J_1+\beta_1\beta_2E_3(\tau_o)(J_2-J_1)}{1-\beta_1\beta_2E_3^2(\tau_o)}\right]$$

$$+\frac{K}{2}, \tag{4.8}$$

and from (4.5) and (4.6),

$$q \sim q_c + Nq_o \;. \tag{4.9}$$

Now q can be computed using (4.8) and (4.9); q_c is given in (4.4), and α_1, α_2, β_1, and β_2 in (2.29) through (2.32). The functions $G_1(E_4(\tau_o-\tau))$ and $E_3(\tau_o)$ are given in Table I, while J_1, J_2, and K are presented in Table II.

2. Comparison of Results

In Figures 8, 9, 10, and 11, the present results given by equation (4.9) are compared with those of Lick[21], Viskanta and Grosh[14,16], and Einstein[23]. Lick's substitute kernel solutions are only available for wall emissivities equal to unity and $\theta(0) = 0.1$, while Viskanta and Grosh's iterative values have been presented for several wall emissivities and $\theta(0) = 0.1$ and 0.5. Einstein's results for uncoupled radiation and conduction have been computed using equation (2.48) for the radiative term.

Little can be said about the accuracy of the approximate solutions of Lick and Einstein, except to compare them, along with the present results, to the values of Viskanta and Grosh. Viskanta and Grosh estimate a maximum error of 0.5 per cent in their values. Deviations beyond this value could be considered the error, except that the error estimate is further complicated by Viskanta's publication of

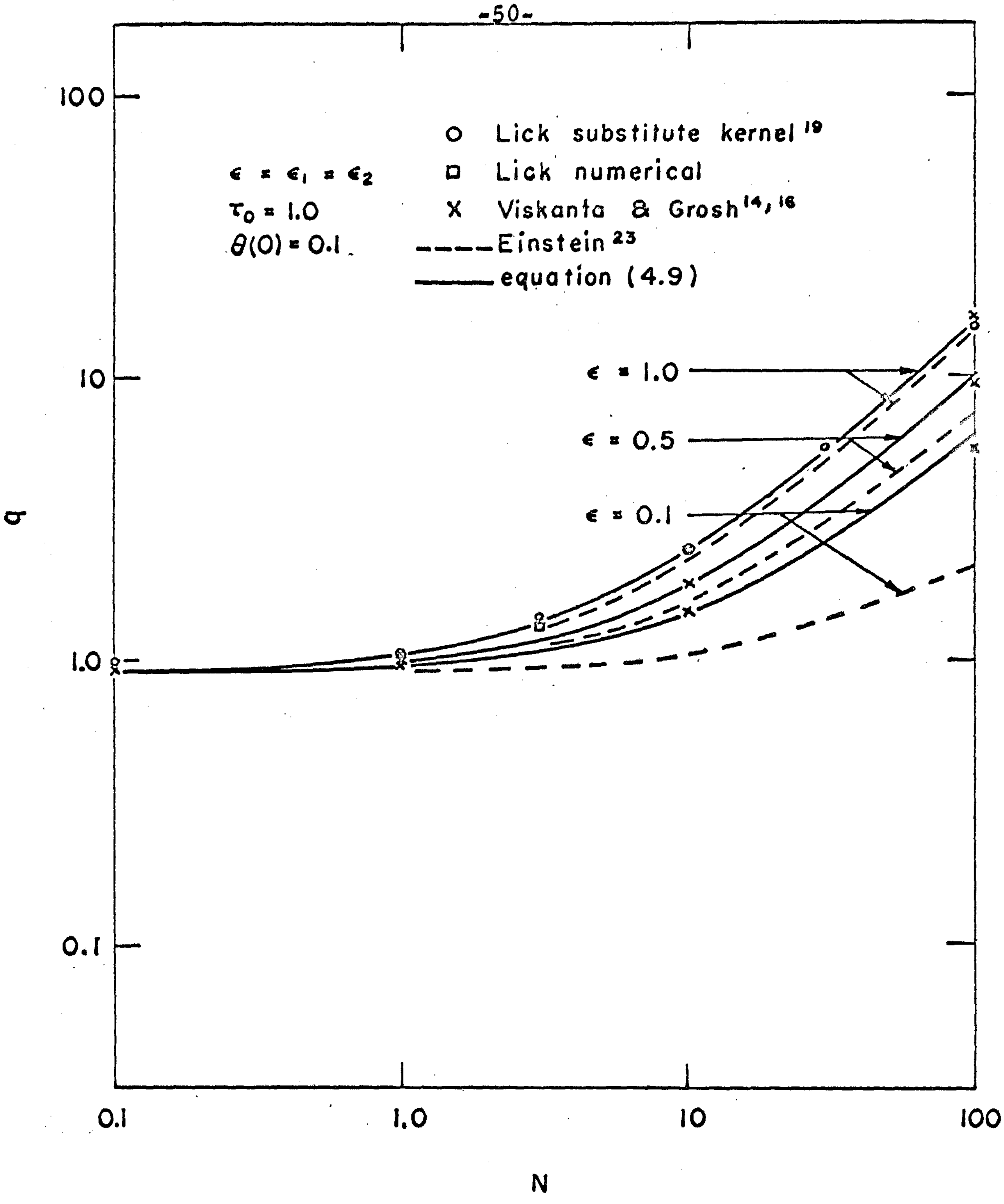

Figure 8. The Heat Transfer Rate.

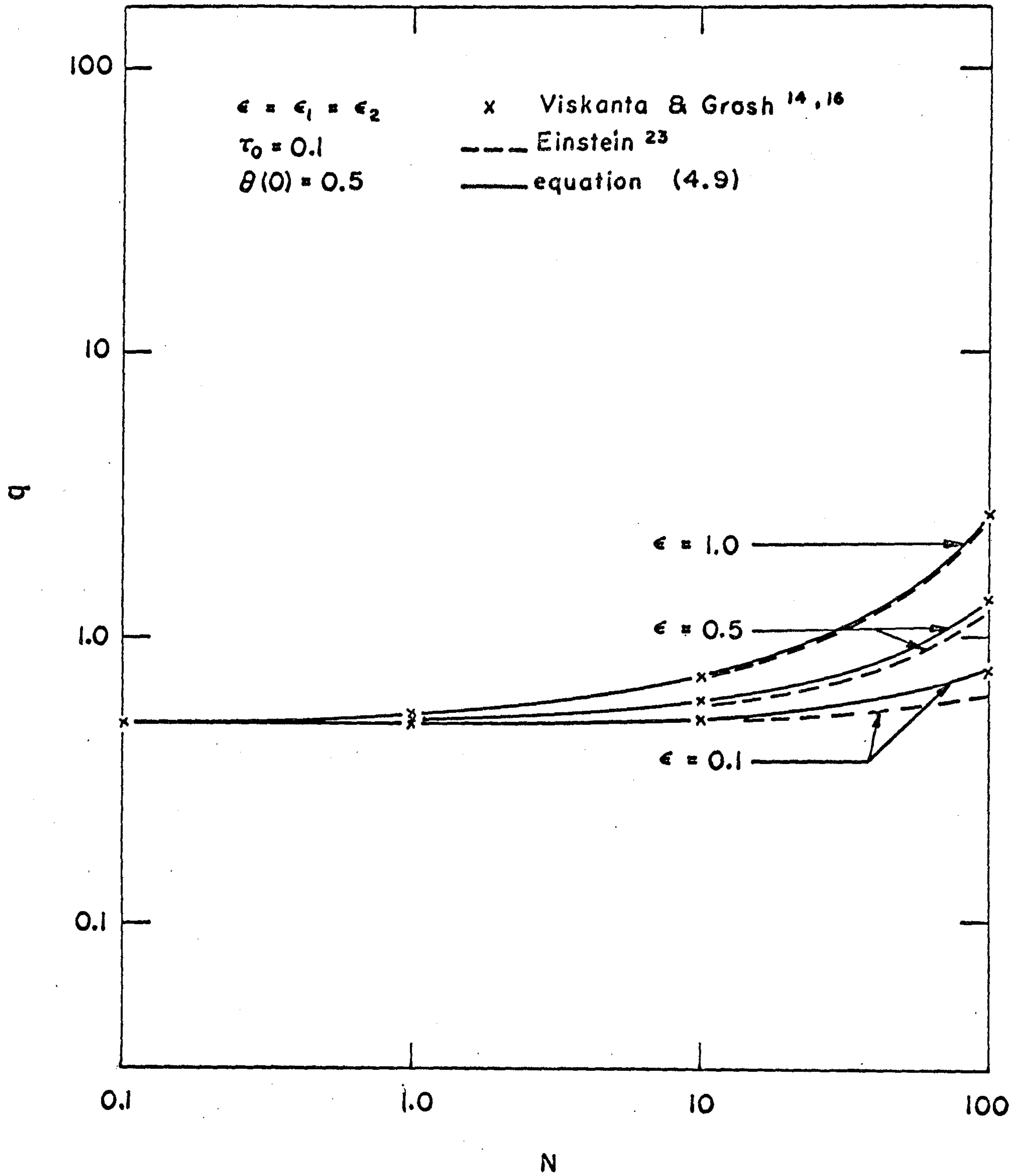

Figure 9. The Heat Transfer Rate.

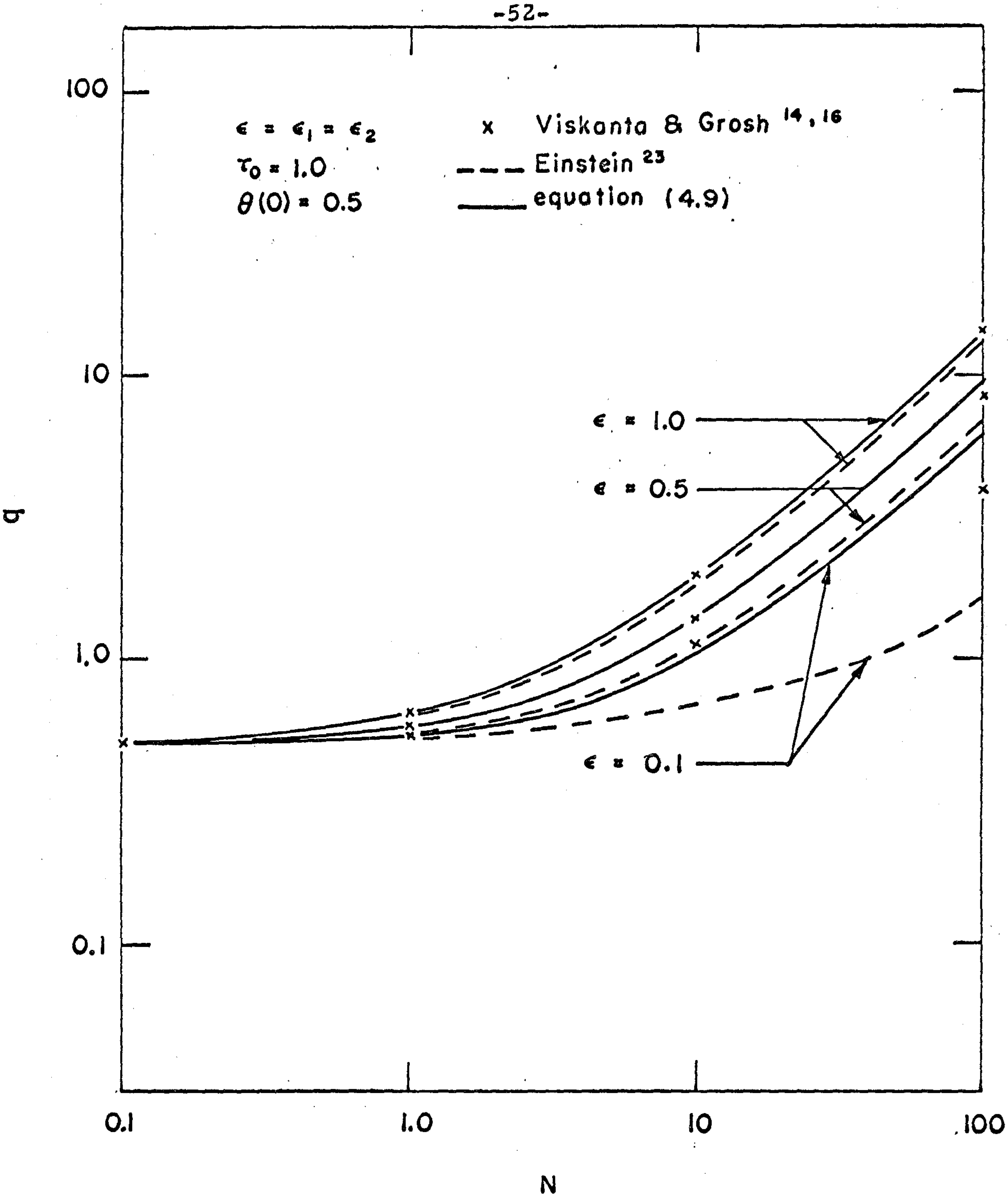

Figure 10. The Heat Transfer Rate.

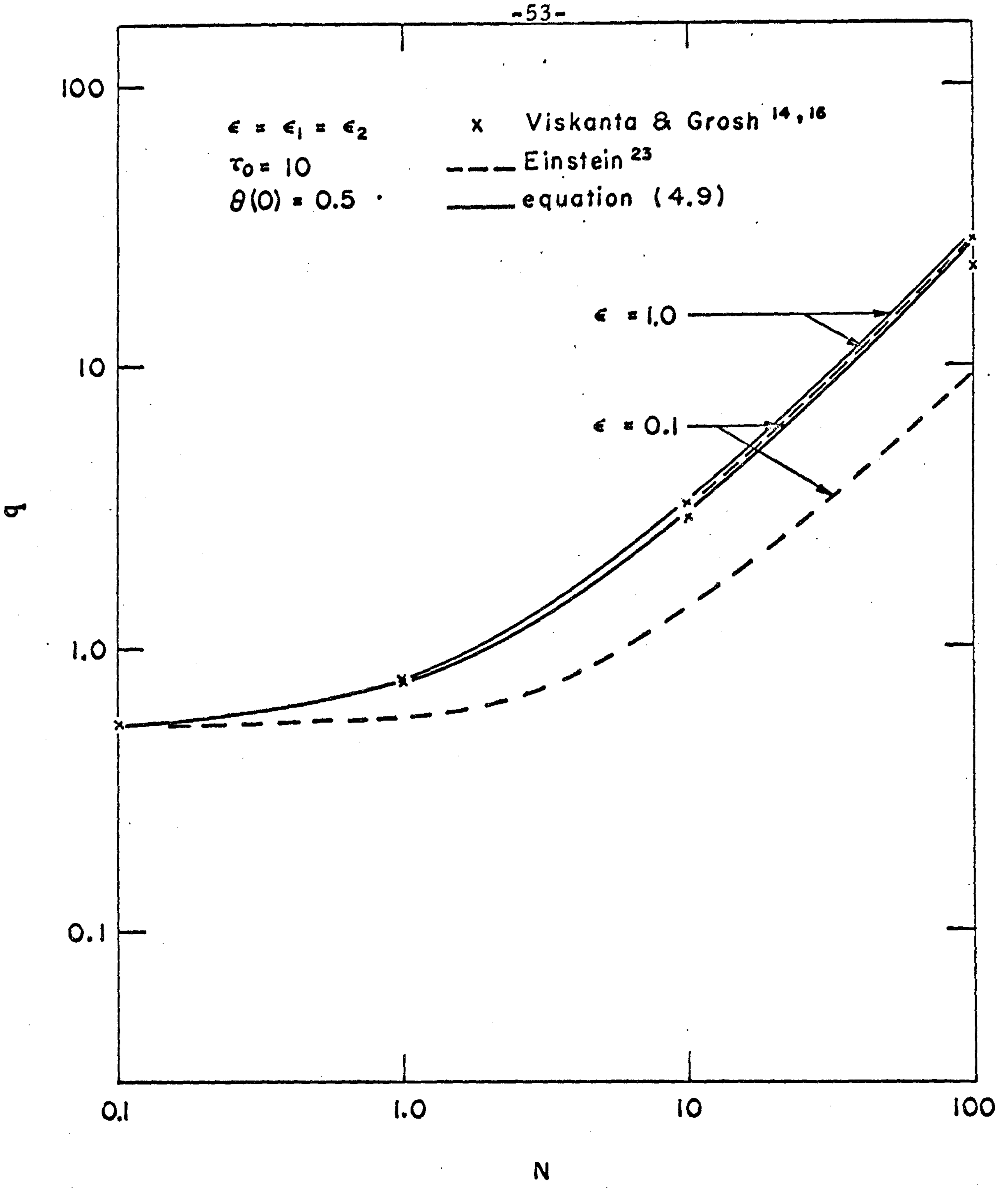

Figure 11. The Heat Transfer Rate.

additional results[20] which overlap to some extent, but do not agree with, his earlier values. In general, the differences are of the order of 1.3 per cent (however, in one case the deviation is 17 per cent). Where duplications occur, the more recent values have been used.

In Figure 8, the heat transfer rate is given for $\theta(0) = 0.1$, $\epsilon_1 = \epsilon_2 = 0.1$, 0.5, and 1.0, and $\tau_o = 1.0$. The comparisons are extended to $N = 100$, a value much greater than the bounds placed on it in the error analysis of Appendix II. The range of N shows the error trend for the various methods and indicates the range of validity of the various solutions.

All results are in good agreement for black walls, but as the emissivity decreases, Einstein's values are low. This is expected, since with low emissivity, there is strong coupling between radiation and conduction, and so his simplified form would not be expected to hold. A similar trend in the error appears in Figures 9 - 11, where $\theta(0) = 0.5$ and $\tau_o = 0.1$, 1.0, and 10.

Comparison of the present results with the tabulated data of Viskanta and Grosh shows good agreement for small values of N, with deviations up to 0.1 per cent for $N = 0.1$ and 2.7 per cent with $N = 1.0$. For larger N, a discernable error appears and is up to 10 per cent for $N = 10$. The error shows the expected pattern of increasing with N and, for a fixed N, increasing with τ_o. The dependence on τ_o occurs because the product $N\tau_o$ rather than N is the ratio of radiation to conduction. Thus, for fixed N, an increase in τ_o corresponds to a proportional increase in the radiative flux. The error also becomes larger as the wall emissivity decreases. This again is expected. With reflections, the source functions depend on θ so the heat transfer rate is a more complex function of the actual temperature profile. Thus, the conduction profile used in the present form for the source function introduces an additional possibility of error. The actual computed errors are smaller than those computed using the error bounds of Appendix II, equation (II-26), especially for moderate to large optical depths. It should be noted that the error bounds hold in general, while results for only two values of $\theta(0)$ and four wall emissivities have been compared. Under certain circumstances, then, errors can be expected to increase beyond those characteristics of the examples available for comparison.

3. Discussion of the Present Method

The proposed approximation is accurate for a wide range of N because the contributions from the higher order terms in the expansion for the temperature profile are generally quite small. To show the contributions are small, two limiting cases are considered; that of an optically thin ($\tau_o \ll 1$) and an optically thick ($\tau_o \gg 1$) medium.

In an optically thin medium, radiation interacts only slightly with conduction. The contribution to the radiative flux is mainly due to radiation passing directly from one boundary to the other. Conduction is only affected when the radiative intensity is so high that the radiant energy absorbed by the medium measurably modifies the slope of the temperature profile. Even for this condition, the overall effect is not very great, since radiation would then dominate and the error due to the change in conduction would be relatively small.

How this is manifested in the expression for q, equation (4.3), can be seen by studying the integral term. It will become apparent that the overall behavior is similar for all of the components; thus, only consider the term

$$\int_0^{\tau_o} \theta^4(t) G_1(E_3|t-\tau|)dt \sim \int_0^{\tau_o} (\theta_o(t)+N\theta_1(t)+\ldots)^4 G_1(E_3|t-\tau|)dt$$

$$= \int_0^{\tau_o} \theta_o^4(t) G_1(E_3|t-\tau|)dt + N\int_0^{\tau_o} 4\theta_o^3(t)\theta_1(t) G_1(E_3|t-\tau|)dt + O(N^2) \ .$$

The second integral can be written in the form

$$N\int_0^{\tau_o} 4\theta_o^3(t)\theta_1(t) G_1(E_3|t-\tau|)dt = 4N\int_0^{\tau_o} \theta_o^4(t)\left(\frac{\theta_1(t)}{\theta_o(t)}\right) G_1(E_3|t-\tau|)dt$$

$$\leq \frac{4N \max_{\tau_o}|\theta_1|}{\theta(0)} \int_0^{\tau_o} \theta_o^4(t) G_1(E_3|t-\tau|)dt \ . \qquad (4.10)$$

The ratio of (4.10) to the term retained in the expansion for q is just

$$\frac{4N \max_{\tau_o}|\theta_1|}{\theta(0)} \ .$$

But $\max_{\tau_o}|\theta_1|$ is generally much smaller than $\theta(0)$, as can be seen in Figure 3, so the relative error is not very large even for large N. Analysis of the other integrals over θ^4 does not change the conclusion; the relative error associated with them has exactly the same form.

In an optically thick medium, τ_o is much greater than one. The radiant flux from the interior is shielded from the boundaries, and only the flux from a layer about one optical depth in thickness reaches the walls. Thus, if the temperature profile is not described as accurately as possible in the interior, it should have little effect on the heat transfer rate as long as the approximation near the walls is good. Only for large N will the change in the profile be great enough near the boundaries to materially alter the results and necessitate a more precise description of the temperature profile. Also, to repeat, $N\tau_o$ is of the order of magnitude of radiative to conductive heat transfer. Even small values of N correspond to appreciable radiative fluxes for an optically thick medium.

The shielding effect described above can be seen in equation (4.3). First, from Appendix I, equation (I-9), the exponential integral can be approximated by

$$E_n(\tau) \sim \frac{e^{-\tau}}{\tau} \qquad \tau > 1 \ , \quad \text{all } n \ .$$

Since the integral kernel multiplying the temperature profile is

$$G_1(E_3|t-\tau|) = E_3(\tau_o - t) - E_3(t) \ ,$$

the contribution to the integral term mainly comes from the values of the temperature distribution near the boundaries (where it approaches θ_o). Again, the discussion has ignored the contributions from the source functions. However, they contain a similar kernel; it is apparent that the same considerations will hold for them.

For τ_o the order of one, there will be partial shielding near the walls, and a portion of the radiative flux from one wall will pass unattenuated to the opposite wall. A trade off between the dominant mechanisms of the optically thin and optically thick cases will govern the heat transfer properties. It is not apparent physically that the results of this interaction will serve to make the approximation accurate. However, the error analysis in Appendix II shows that the accuracy does not deteriorate to any great extent for moderate optical depths.

4. An Example of Coupling Between Radiation and Conduction

Before turning to the scattering problem, a graphic example of the interaction between radiation and conduction appears in studying the effect of optical depth and boundary emissivity on q.

To start with, consider $\epsilon_1 = \epsilon_2 = 0$ and $t_1 = t_2 = 0$. The boundaries are perfect reflectors, so conduction is the only mode of energy transport at the walls. For modest optical depths, the radiation produced near one wall is partially absorbed near the other, increasing the energy transport. However, as the optical depth becomes even greater, the radiative transport is retarded by being absorbed and re-emitted many times in traversing the medium. The result reduces the heat transfer rate from the corresponding value for moderate optical depths. Thus, q first increases with τ_o, reaches a maximum for moderate optical depths, and finally decreases in the optically thick limit. Results for perfectly reflecting walls and boundaries with emissivities other than zero are given in Figure 12 for $N\tau_o = 2$. As expected, with higher wall emissivities, the maxima in the heat transfer rate shift toward smaller optical depths. For $\epsilon_1 = \epsilon_2 > 0.6$, the maximum is at $\tau_o = 0$, since now the boundaries efficiently absorb and emit radiation and the intervening medium can only serve to resist the radiative transport.

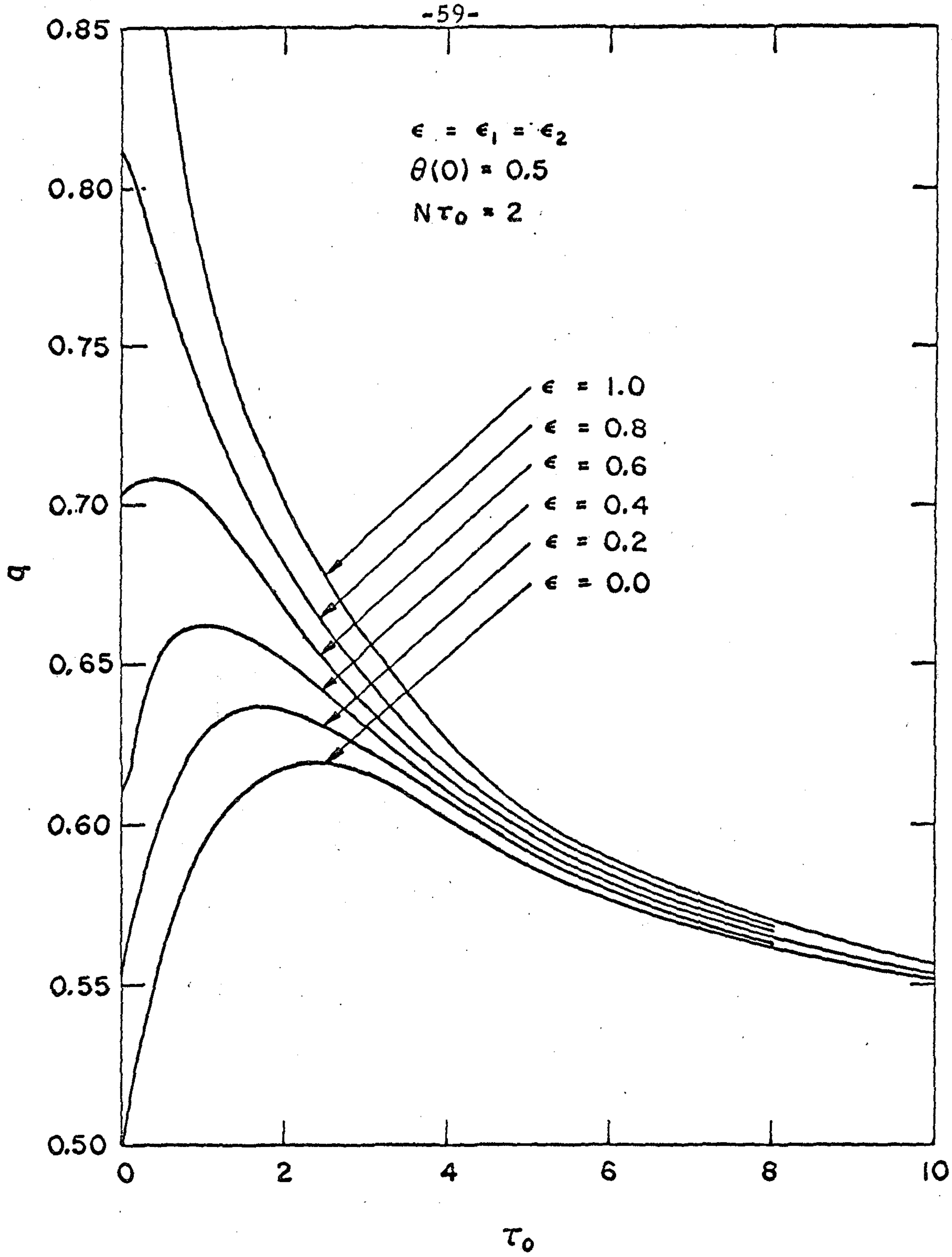

Figure 12. The Heat Transfer Rate.

V. APPROXIMATE SOLUTION FOR THE TEMPERATURE DISTRIBUTION IN AN ABSORBING, SCATTERING, AND CONDUCTING MEDIUM

When scattering is included, the character of the general equation changes radically. Even if the equation is twice integrated, it no longer reduces to a nonlinear integral equation in θ , since the derivatives of the function appear. This precludes the straightforward application of the method suggested by Lichtenstein and used by Viskanta and Grosh in the pure absorption case.

In a recent extension of his earlier work, Viskanta[20] included scattering. An iterative procedure was again used, after equation (2.41) was rewritten in the form

$$\theta''(\tau, \alpha) = \alpha N[\theta^4(\tau, \alpha) - \frac{1}{4}\zeta(\tau, \alpha)] ,$$

$$\zeta(\tau, \alpha) = 2[S_1''(\tau, \alpha)+S_2''(\tau, \alpha)+\int_0^{\tau_o}[\alpha\theta^4(t, \alpha)+\frac{\zeta(t, \alpha)}{4}]E_1|t-\tau|dt] .$$

This expression follows directly from (2.12) and (2.19), if ζ is defined as the dimensionless form of the integral over μ appearing in (2.12), and (2.19) is rewritten in terms of ζ . An iterative scheme was carried out over the two coupled equations. However, convergence was quite slow and only limited results were given. Further, no values for temperature profile suitable for comparison were presented, as the heat transfer rate was of principal interest.

1. The Solution

In the previous analysis with scattering excluded, an expansion in N for the temperature profile produced satisfactory results for a wide range of N. The temperature profile was shown to vary only slightly from the conduction results even for relatively large N. With scattering, then, the temperature profile is again expected to be close to that of pure conduction, since the inclusion of scattering (while holding the total optical depth constant) tends to uncouple radiative effects.

This suggests simplifying the basic equation by ignoring higher order terms in N, with the hope of more readily treating the simplified equation. Consider equation (2.41), and note that if θ^4 is replaced by θ_o^4, it is integrated twice, and G is used;

$$\frac{\alpha N}{2}\left[G\Big(S_{10}(\tau)+S_{20}(\tau)\Big)+\int_0^{\tau_o}\theta_o^4(t)G(E_3|t-\tau|)dt\right]+(1-\alpha)G\Big(T_{1e}(\tau,\alpha)+T_{2e}(\tau,\alpha)\Big)$$

$$+\alpha G(\theta(\tau,\alpha)) \approx \frac{1-\alpha}{2}\int_0^{\tau_o}\theta''(t,\alpha)G(E_3|t-\tau|)dt \qquad (5.1)$$

where

$$T_{1e}(\tau,\alpha) = -\frac{1}{2}\frac{E_4(\tau)}{1-\beta_1\beta_2E_3^2(\tau_o)}\int_0^{\tau_o}\beta_1\theta''(t,\alpha)[E_2(t)+\beta_2E_2(\tau_o-t)E_3(\tau_o)]dt\ , \qquad (5.2)$$

$$T_{2e}(\tau,\alpha) = -\frac{1}{2}\frac{E_4(\tau_o-\tau)}{1-\beta_1\beta_2E_3^2(\tau_o)}\int_0^{\tau_o}\beta_2\theta''(t,\alpha)[E_2(\tau_o-\tau)+\beta_1E_2(t)E_3(\tau_o)]dt\ . \qquad (5.3)$$

From the analysis for pure absorption,

$$\theta_1(\tau) = -\frac{1}{2}\left[G\Big(S_{10}(\tau)+S_{20}(\tau)\Big)+\int_0^{\tau_o}\theta_o^4(t)G(E_3|t-\tau|)dt\right]\ ,$$

so that

$$\alpha G(\theta(\tau,\alpha) \approx \alpha N\theta_1(\tau)+(1-\alpha)G(T_{1e}(\tau,\alpha)+T_{2e}(\tau,\alpha))+\frac{1-\alpha}{2}\int_0^{\tau_o}\theta''(t,\alpha)G(E_3|t-\tau|)dt. \tag{5.4}$$

The terms neglected are integrals over

$$4N\theta_o^3(\tau)\theta_\alpha(\tau,\alpha)+O(N^2),$$

where θ_α is a function of α and equals θ_1 when α is unity. It is reasonable to assume that θ_α is always the magnitude of θ_1, since for α less than one, the temperature profile tends toward θ_o. Also, there is no physical mechanism to make θ_α significantly greater than θ_1. Assume, then, that the correction is roughly θ_1. The error involved in the above approximation is the order of

$$\tfrac{1}{2}[G(S_{11}(\tau)+S_{21}(\tau))+\int_0^{\tau_o}4\theta_o^3(t)\theta_1(t)G_1(E_3|t-\tau|)dt] = -\theta_2(\tau) .$$

For N sufficiently small, the product $N^2\theta_2$ can be neglected compared to $N\theta_1$, as with pure absorption. Thus, the result for $\theta(\tau,\alpha)$ has approximately the same range of validity as the linear expression for $\theta(\tau)$ in the pure absorption case.

The approximate equation for θ can be reduced to a Fredholm integral equation of the second kind in θ'', if (5.4) is twice differentiated with respect to τ:

$$\theta''(\tau,\alpha) \approx \alpha N\theta_1''(\tau)+[1-\alpha][T_{1e}''(\tau,\alpha)+T_{2e}''(\tau,\alpha)]+\frac{1-\alpha}{2}\int_0^{\tau_o}\theta''(t,\alpha)E_1|t-\tau|dt. \tag{5.5}$$

This, in itself, is not very useful, since the integral kernel is not readily inverted. The solution kernel is, however, given by Chandrasekhar[4] as a function of the solution of two coupled linear integral equations. The numerical values can be found in Sobouti's analysis[24]. Unfortunately, the numerical results only yield θ'' and must be twice numerically integrated to give θ.

An alternate approach is the substitute kernel method used by Lick[21]. He replaced the difficult kernel by a simpler, approximate kernel of the general form $Ae^{-B\tau}$. The constants A and B were chosen so that the substitute kernel has some of the characteristics of the one it replaced.

In the present problem, it is natural to require that A and B satisfy

$$\int_0^\infty E_1(t)dt = A\int_0^\infty e^{-Bt}dt \quad ,$$

$$\int_0^\infty tE_1(t)dt = A\int_0^\infty te^{-Bt}dt \quad .$$

Thus, the integrals and their first moments are equal. The constants A and B are found with the aid of equations (I-5) and (I-6) of Appendix I. The result is

$$A = B = 2 \quad .$$

With these values, (5.5) becomes

$$\alpha\theta''(\tau,\alpha) \approx \alpha N\theta_1''(\tau)+[1-\alpha][T_{1e}''(\tau,\alpha)+T_{2e}''(\tau,\alpha)]+(1-\alpha)\int_0^{\tau_o}\theta''(t,\alpha)e^{-2|t-\tau|}dt \quad . \tag{5.6}$$

If this is differentiated twice with respect to τ, the integral expression remains

$$\theta^{(4)}(\tau,\alpha) \approx \alpha N\theta_1^{(4)}(\tau)+[1-\alpha][T_{1e}^{(4)}(\tau,\alpha)+T_{2e}^{(4)}(\tau,\alpha)]+4(1-\alpha)\int_0^{\tau_o}\theta''(t,\alpha)e^{-2|t-\tau|}dt. \tag{5.7}$$

(5.7) can be added to (-4) times equation (5.6),

$$\theta^{(4)}(\tau,\alpha)-4\alpha\theta''(\tau,\alpha) \approx \alpha N[\theta_1^{(4)}(\tau)-4\theta_1''(\tau)]+[1-\alpha][T_{1e}^{(4)}(\tau,\alpha)+T_{2e}^{(4)}(\tau,\alpha) - 4T_{1e}''(\tau,\alpha) - 4T_{2e}''(\tau,\alpha)] \ .$$

Finally, integrating twice,

$$\theta''(\tau,\alpha)-4\alpha\theta(\tau,\alpha) \approx \alpha N[\theta_1''(\tau)-4\theta_1(\tau)]+[1-\alpha][T_{1e}''(\tau,\alpha)+T_{2e}''(\tau,\alpha) -4T_{1e}(\tau,\alpha) - 4T_{2e}(\tau,\alpha)]+C_1' + C_2'\tau \ . \qquad (5.8)$$

In this equation, θ is an approximation to the temperature profile for arbitrary α. A change of variables to

$$\theta(\tau,\alpha) \approx \theta_o(\tau)+N\ {}_1\theta(\tau,\alpha) \qquad (5.9)$$

simplifies the expression. The approximately equals sign is used to denote that $N\ {}_1\theta$ satisfies (5.8) and hence only approximates the exact equation, (5.1). With this change of variables, T_{1e} and T_{2e}, defined in (5.2) and (5.3),are replaced by T_1 and T_2, the corresponding functions with θ'' replaced by ${}_1\theta''$:

$$T_1(\tau,\alpha) = -\frac{1}{2}\frac{E_4(\tau)}{1-\beta_1\beta_2E_3^2(\tau_o)}\int_0^{\tau_o} {}_1\theta''(t,\alpha)\beta_1[E_2(t)+\beta_2E_2(\tau_o-t)E_3(\tau_o)]dt \ , \qquad (5.10)$$

$$T_2(\tau,\alpha) = -\frac{1}{2}\frac{E_4(\tau_o-\tau)}{1-\beta_1\beta_2E_3^2(\tau_o)}\int_0^{\tau_o} {}_1\theta''(t,\alpha)\beta_2[E_2(\tau_o-t)+\beta_1E_2(t)E_3(\tau_o)]dt \ . \qquad (5.11)$$

The substitution also modifies the boundary conditions to ${}_1\theta(0,\alpha) = {}_1\theta(\tau_o,\alpha) = 0$. If C_1 and C_2 are defined by

$$N[C_1+C_2\tau] = C_1' + C_2'\tau + 4\alpha\theta_o(\tau) \ ,$$

(5.8), with θ replaced by $\theta_o+N\ {}_1\theta$, becomes

$$ {}_1\theta''(\tau,\alpha)-4\alpha\,{}_1\theta(\tau,\alpha) = \alpha[\theta_1''(\tau)-4\theta_1(\tau)]+[1-\alpha][T_1''(\tau,\alpha)+T_2''(\tau,\alpha)- -4T_1(\tau,\alpha)-4T_2(\tau,\alpha)]+\alpha[C_1+C_2\tau] \tag{5.12} $$

which is independent of N. Thus, the approximate expression (5.9) for the temperature distribution is linear in N as was its pure absorption counterpart. Let

$$ f(\tau,\alpha) = \theta_1(\tau) + \frac{1-\alpha}{\alpha}\,[T_1(\tau,\alpha)+T_2(\tau,\alpha)] \tag{5.13} $$

$$ \varkappa^2 = 4\alpha \tag{5.14} $$

so that (5.12) becomes

$$ {}_1\theta''(\tau,\alpha) - \varkappa^2\,{}_1\theta(\tau,\alpha) = \alpha[f''(\tau,\alpha)-4f(\tau,\alpha)+C_1+C_2\tau] . $$

Again, the operator G, defined in (3.5), can be used to remove the constants C_1 and C_2.

$$ G({}_1\theta''(\tau,\alpha)) - \varkappa^2\,{}_1\theta(\tau,\alpha) = \alpha G(f''(\tau,\alpha) - 4f(\tau,\alpha)) \tag{5.15} $$

The operator does not affect the term containing ${}_1\theta$, since it equals zero at $\tau = 0$ and $\tau = \tau_o$, and hence $G({}_1\theta) = {}_1\theta$.

The function $G({}_1\theta'')$ contains the unknowns ${}_1\theta''(0,\alpha)$ and ${}_1\theta''(\tau_o,\alpha)$, which can be removed using the relation (5.5) with $\theta''(\tau,\alpha)$ replaced by $N\,{}_1\theta''$:

$$ {}_1\theta''(\tau,\alpha) = \alpha\theta_1''(\tau)+[1-\alpha][T_1''(\tau,\alpha)+T_2''(\tau,\alpha)]+\frac{1-\alpha}{2}\int_0^{\tau_o}{}_1\theta''(t,\alpha)E_1|t-\tau|\,dt $$

$$ = \alpha f''(\tau,\alpha)+\frac{1-\alpha}{2}\int_0^{\tau_o}{}_1\theta''(t,\alpha)E_1|t-\tau|\,dt . $$

If this is now evaluated at $\tau = 0$ and $\tau = \tau_o$,

$$ {}_1\theta''(0,\alpha) = \alpha f''(0,\alpha) + (1-\alpha)u(0,\alpha) , \tag{5.16} $$

$$ {}_1\theta''(\tau_o,\alpha) = \alpha f''(\tau_o,\alpha) + (1-\alpha)u(\tau_o,\alpha) , \tag{5.17} $$

where

$$u(\tau, \alpha) = \frac{1}{2} \int_0^{\tau_o} {}_1\theta''(t, \alpha) E_1 |t-\tau| dt \ .$$

It is evident from the definitions of T_1 and T_2 (which are both integrals over ${}_1\theta''$) that introducing $u(0, \alpha)$ and $u(\tau_o, \alpha)$ does not increase the complexity of the expression.

Equation(5. 15) can now be cast in a more convenient form by replacing the terms involving ${}_1\theta''(0, \alpha)$ and ${}_1\theta''(\tau_o, \alpha)$ by (5. 16) and (5. 17),

$$\begin{aligned} {}_1\theta''(\tau, \alpha) - \varkappa^2 \, {}_1\theta(\tau, \alpha) = \alpha[f''(\tau, \alpha) - 4f(\tau, \alpha)] + (1-\alpha)\Big(u(0, \alpha) + \\ + \frac{\tau}{\tau_o}[u(\tau_o, \alpha) - u(0, \alpha)]\Big) + \alpha\Big(4f(0, \alpha) + \frac{\tau}{\tau_o}[f(\tau_o, \alpha) - f(0, \alpha)]\Big) \ . \end{aligned} \quad (5.\ 18)$$

Let

$$g(\tau, \alpha) = 4f(\tau, \alpha) + \frac{1-\alpha}{\alpha} u(\tau, \alpha) \ . \quad (5.\ 19)$$

Thus, (5. 18) can be written as

$${}_1\theta''(\tau, \alpha) - \varkappa^2 \, {}_1\theta(\tau, \alpha) = \alpha\Big(f''(\tau, \alpha) - 4f(\tau, \alpha) + g(0, \alpha) + \frac{\tau}{\tau_o}[g(\tau_o, \alpha) - g(0, \alpha)]\Big) \ .$$

This has the solution

$${}_1\theta(\tau, \alpha) = \alpha \int_0^{\tau_o} \Big(f''(t, \alpha) - 4f(t, \alpha) + g(0, \alpha) + \frac{t}{\tau_o}[g(\tau_o, \alpha) - g(0, \alpha)]\Big) \mathcal{G}(t, \tau, \alpha) dt \quad (5.\ 20)$$

where $\mathcal{G}$ is the Green's function

$$\mathcal{G}(t, \tau, \alpha) = \frac{1}{\varkappa \sinh \varkappa\tau_o} \begin{cases} \sinh \varkappa t \sinh \varkappa(\tau - \tau_o) & 0 < t < \tau \\ \sinh \varkappa\tau \sinh \varkappa(t - \tau_o) & \tau < t < \tau_o \end{cases} \quad (5.\ 21)$$

with $\varkappa^2 = 4\alpha$. The expression given for θ_1 in (5. 20) can be reduced by integrating by parts twice the term containing f'' and carrying out the integration involving the function g. The result is

$$
{}_1\theta(\tau,\alpha) = \alpha f(\tau,\alpha) - 4\alpha(1-\alpha)\int_0^{\tau_o} f(t,\alpha)\mathcal{G}(t,\tau,\alpha)dt
$$

$$
- \frac{1}{4\sinh \varkappa\tau_o}\Big([4\alpha f(0,\alpha)-g(0,\alpha)]\sinh\varkappa(\tau_o-\tau)+
$$

$$
+[4\alpha f(\tau_o,\alpha)-g(\tau_o,\alpha)]\sinh\varkappa\tau\Big) - \frac{1}{4}\Big(g(0,\alpha)+\frac{\tau}{\tau_o}[g(\tau_o,\alpha)-g(0,\alpha)]\Big) \quad . \tag{5.22}
$$

This form for ${}_1\theta$ is not completely reduced since the functions f and g contain T_1, T_2, and u , which in turn are functions of ${}_1\theta''$. But ${}_1\theta''$ can be expressed in terms of f and g by substituting the expression for ${}_1\theta$ in (5.20) into (5.18):

$$
{}_1\theta''(\tau,\alpha) = \alpha[f''(\tau,\alpha)-4f(\tau,\alpha)+4\alpha\int_0^{\tau_o}[f''(t,\alpha)-4f(t,\alpha)]\mathcal{G}(t,\tau,\alpha)dt]
$$

$$
+\frac{\alpha}{\sinh\varkappa\tau_o}[g(0,\alpha)\sinh\varkappa(\tau_o-\tau)+g(\tau_o,\alpha)\sinh\varkappa\tau] \; . \tag{5.23}
$$

Equation (5.23) expresses ${}_1\theta''$ as a linear function of f and g , which in turn are linear functions of T_1, T_2, $u(0,\alpha)$ and $u(\tau_o,\alpha)$. But each can be expressed as a function of ${}_1\theta''$:

$$
T_1(\tau,\alpha) = -\frac{1}{2}\frac{E_4(\tau)}{1-\beta_1\beta_2E_3^2(\tau_o)}\int_0^{\tau_o}{}_1\theta''(t,\alpha)\beta_1[E_2(t)+\beta_2E_2(\tau_o-t)E_3(\tau_o)]dt \tag{5.24}
$$

$$
T_2(\tau,\alpha) = -\frac{1}{2}\frac{E_4(\tau_o-\tau)}{1-\beta_1\beta_2E_3^2(\tau_o)}\int_0^{\tau_o}{}_1\theta''(t,\alpha)\beta_2[E_2(\tau_o-t)+\beta_1E_2(t)E_3(\tau_o)]dt \tag{5.25}
$$

$$
u(0,\alpha) = \frac{1}{2}\int_0^{\tau_o}{}_1\theta''(t,\alpha)E_1(t)dt \tag{5.26}
$$

$$u(\tau_o, \alpha) = \frac{1}{2} \int_0^{\tau_o} {}_1\theta''(t, \alpha) E_1(\tau_o - t) dt \quad . \tag{5. 27}$$

If now (5. 23) is used to evaluate (5. 24) through (5. 27), four linear algebraic equations in $u(0, \alpha)$ and $u(\tau_o, \alpha)$ and the integral portions of T_1 and T_2 result. By solving these, ${}_1\theta$ can be written explicitly, and so, with (5. 22), θ can be expressed as

$$\theta(\tau, \alpha) \approx \theta_o(\tau) + N_1\theta(\tau, \alpha)$$

$$= \theta_o(\tau) + \alpha N f(\tau, \alpha) - 4\alpha(1-\alpha) N \int_0^{\tau_o} f(t, \alpha) \mathcal{G}(t, \tau, \alpha) dt$$

$$- \frac{N}{4 \sinh \varkappa\tau_o} \Big([4\alpha f(0,\alpha) - g(0,\alpha)] \sinh\varkappa(\tau_o - \tau) +$$

$$+ [4\alpha f(\tau_o, \alpha) - g(\tau_o, \alpha)] \sinh\varkappa\tau \Big) - \frac{N}{4} \Big(g(0,\alpha) + \frac{\tau}{\tau_o} [g(\tau_o,\alpha) - g(0,\alpha)] \Big) \quad . \tag{5. 28}$$

The functions f and g are defined in (5. 13) and (5. 19). As just mentioned, they involve T_1 , T_2 , $u(0, \alpha)$, and $u(\tau_o, \alpha)$, which are found using (5. 24) through (5. 27). Finally, the Green's function, $\mathcal{G}$, is defined in (5. 21).

The form of the solution for θ given in (5. 28) is excessively complex to present results in tabulated form. Nevertheless, all integrations required can be carried out using the functions given in Tables II and III for θ_1.

2. Analysis of the Solution and Further Approximations

The accuracy of (5. 28) depends on the accuracy of the substitute kernel solution for ${}_1\theta$. The only other error comes from approximating θ^4 by θ_o^4 in the linearization of the transport equation. Since the form of the equation Lick treated is similar to the above, the error in the substitute kernel can be estimated qualitatively by comparing the results of Lick (for no conduction) with the exact results of Chandrasekhar. These errors are the order of a few per cent and so the present results probably have the same general accuracy.

The complexity of the above result masks the temperature profile characteristics as a function of α. Consider, then, some limiting cases; the first, an optically thin medium. As before, attenuation of the radiation by the interior medium can be neglected, and so the coupling between the radiative field and conduction is proportional to the absorptivity at each point. In the analysis for pure absorption, this coupling resulted in a linear change in the temperature profile for N sufficiently small. In the case at hand, the radiative contribution will be decreased since the total cross section includes scattering. The portion absorbed is α times that of pure absorption. Therefore, the resultant temperature profile should have the form:

$$\theta(\tau, \alpha) \approx \theta_o(\tau) + \alpha N \theta_1(\tau) .$$

This form can also be deduced from the basic equations for θ. For convenience, consider the expression for θ'' given in equation (5.5). Momentarily set $T''_{1e} = T''_{2e} = 0$. The equation reduces to

$$\theta''(\tau, \alpha) \approx \alpha N \theta''_1(\tau) + \frac{1-\alpha}{2} \int_0^{\tau_o} \theta''(t, \alpha) E_1 |t-\tau| dt \quad . \tag{5.29}$$

From Appendix I, equation (I-4), for small τ the asymptotic form of

E_1 is

$$E_1(\tau) \sim -\ell n\, \tau \qquad \tau \ll 1 .$$

Thus, the integral is order of $\tau_o \ell n\, \tau_o |\theta''|$, and can be neglected compared to θ_1''. Thus, (5. 29) is approximately

$$\theta''(\tau, \alpha) \approx \alpha N \theta_1''(\tau) , \tag{5. 30}$$

or, using the definition $\theta \approx \theta_o + N_1\theta$,

$$_1\theta''(\tau, \alpha) \approx \alpha \theta_1''(\tau) . \tag{5. 31}$$

This form will be used later.

Integrating (5. 30) twice and using the boundary condition that $\theta(0, \alpha) = \theta_o(0)$ and $\theta(\tau_o, \alpha) = \theta_o(\tau_o)$,

$$\theta(\tau, \alpha) \approx \theta_o(\tau) + \alpha N \theta_1(\tau) , \tag{5. 32}$$

and from (5. 31) and $\theta \approx \theta_o + N_1\theta$,

$$_1\theta(\tau, \alpha) \approx \alpha \theta_1(\tau) , \tag{5. 33}$$

which are the expected forms. If T_{1e}'' and T_{2e}'' had been included, the result would have been the same, since T_{1e}'' and T_{2e}'' are composed of integrals of $\theta'' E_2(\tau)$ and $\theta'' E_2(\tau_o - \tau)$. Again appealing to Appendix I, equation (I-7), the asymptotic form of E_2 is

$$E_2(\tau) \sim 1 \qquad \tau \ll 1 ,$$

so that the integrals are order of $\tau_o|\theta|$ and are negligible compared to θ.

Now consider $\tau_o \gg 1$. In the region away from the boundaries, T_{1e}'' and T_{2e}'' can be again neglected, since the terms contain $E_2(\tau)$ or $E_2(\tau_o - \tau)$ and go to zero as $\frac{e^{-\tau}}{\tau}$ for large τ, as shown in Appendix I, equation (I-9). Thus, (5. 5) again reduces to

$$\theta''(\tau, \alpha) \approx \alpha N\theta_1''(\tau) + \frac{1-\alpha}{2}\int_0^{\tau_o} \theta''(t, \alpha)E_1|t-\tau|dt \quad . \tag{5.34}$$

If θ'' is assumed to have the general character of θ_1'', it will be a slowly varying function of τ in the interior. Therefore, the integral term in (5.34) can be approximated by

$$\frac{1-\alpha}{2}\int_0^{\tau_o} \theta''(t, \alpha)E_1|t-\tau|dt \approx \frac{1-\alpha}{2}\,\theta''(\tau, \alpha)\int_0^{\tau_o} E_1|t-\tau|dt$$

$$\approx \frac{1-\alpha}{2}\,\theta''(\tau, \alpha)\int_{-\infty}^{\infty} E_1|t-\tau|dt = (1-\alpha)\theta''(\tau, \alpha) \quad .$$

This is valid since first, the kernel heavily weights the integrand near τ, and second, for large τ_o, and τ in the interior, the limits of integration can be extended from $0, \tau_o$ to $\pm\infty$ with little loss of accuracy. If this approximation for the integral term is used in (5.34),

$$\alpha\theta''(\tau, \alpha) \approx \alpha N\theta_1''(\tau) , \tag{5.35}$$

or, integrating twice,

$$\theta(\tau, \alpha) \approx A + B\tau + N\theta_1(\tau) .$$

This shows that in the interior $\theta(\tau, \alpha)$ assumes a profile within a linear term of the pure absorption profile. The constants A and B can not be evaluated since the approximation is invalid near the walls.

3. Examples

Numerical calculations have been made for $\tau_o = 0.1$, 1.0, and 10 ; $\theta(0) = 0.5$; $\epsilon_1 = \epsilon_2 = 1$; and a range of α to show the character of θ. Figure 13 shows the first order contribution to θ, i.e., ${}_1\theta$, for $\tau_o = 0.1$. It is evident that the profile increases nearly linearly

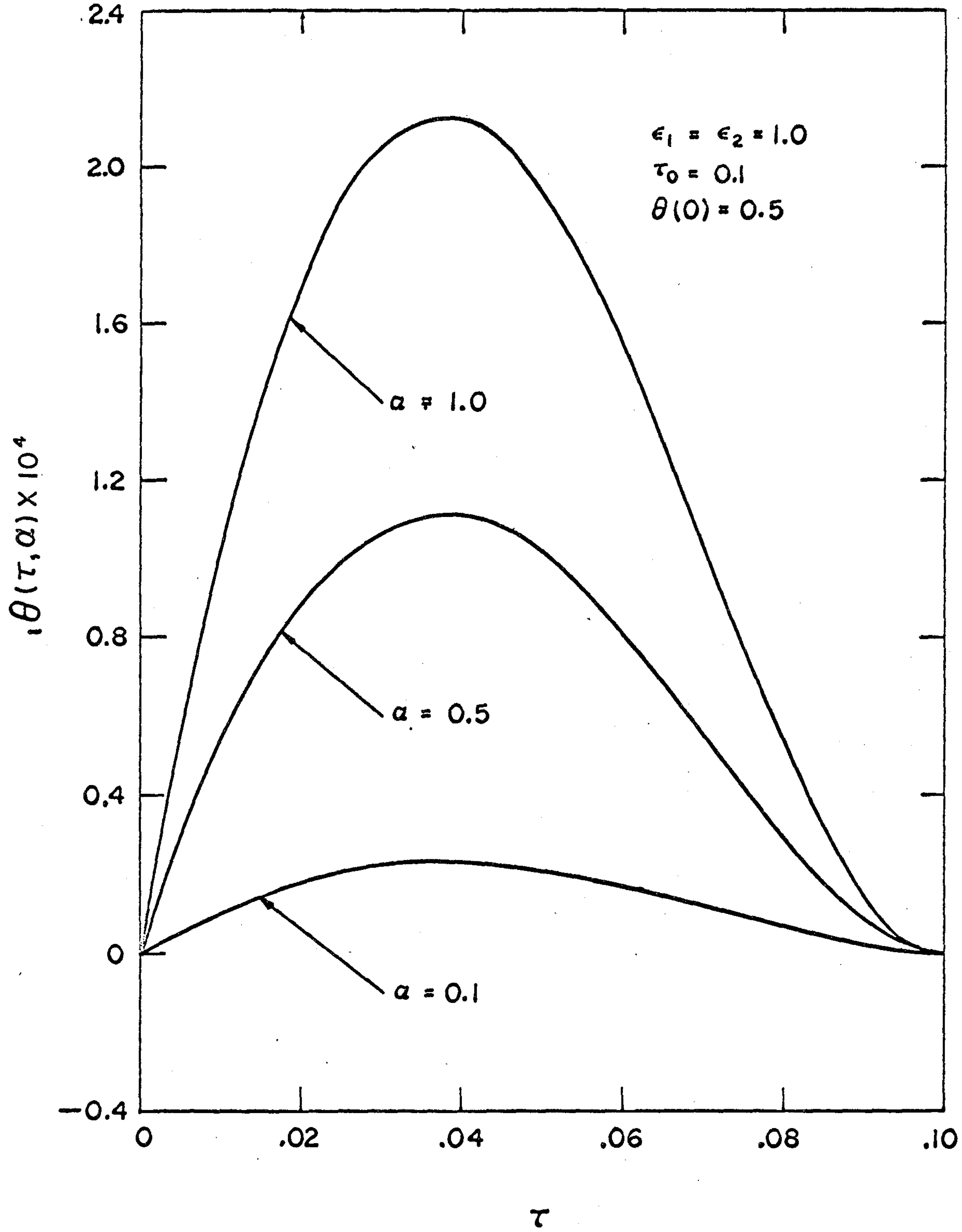

Figure 13. The First Order Correction to the Temperature Distribution.

with α. This can be seen in Figure 14, where the results are replotted as a function of α for various values of τ. Included as dashed lines are results from the linear approximation developer earlier in equation (5.33),

$$ {}_1\theta(\tau, \alpha) \approx \alpha\theta_1(\tau) \ . $$

Agreement is quite good and indicates the usefulness at the above approximation. Figure 15 gives similar results for $\tau_o = 1.0$. Here, the first order term, ${}_1\theta$, is a more complex function of α. Again, the results are replotted as a function of α in Figure 16. The dashed curves again give the linear approximation. The representation is not quite as accurate as for the optically thin medium. A similar set of results for $\tau_o = 10$ are given in Figures 17 and 18. For the optically thick medium, the maximum value of ${}_1\theta$ shifts to lower values of τ as α decreases. In addition, for α near 0.1 and $9 < \tau < 10$, the term becomes slightly negative. Thus, the characteristics of ${}_1\theta$ have changed from the previous cases, but the changes are quite subtle. In Figure 14, the dashed curve shows that the linear approximation is still of value. However, the curves do deviate from the straight line for τ near τ_o. This difference will become important in the derivation of the heat transfer rate; as with pure absorption, the radiative contribution to q will appear as an integral weighted near the boundaries.

Thus, ${}_1\theta$ is similar in nature to θ_1, the pure absorption result. The equation for ${}_1\theta$ has been reduced to a linear integral equation, and even though a final expression has been developed in terms of the approximate kernel, actual numerical results are more difficult

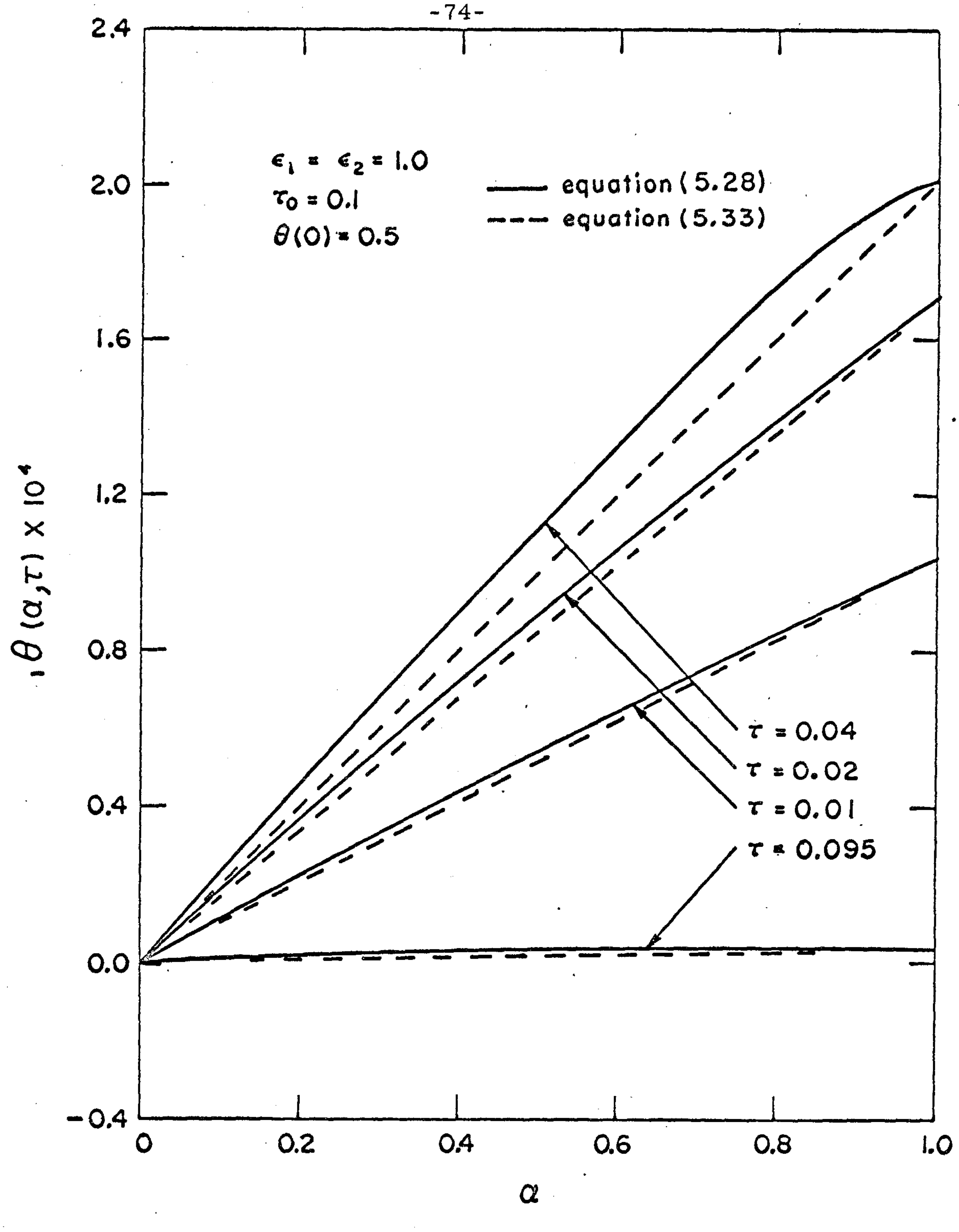

Figure 14. The First Order Correction to the Temperature Distribution Replotted as a Function of α.

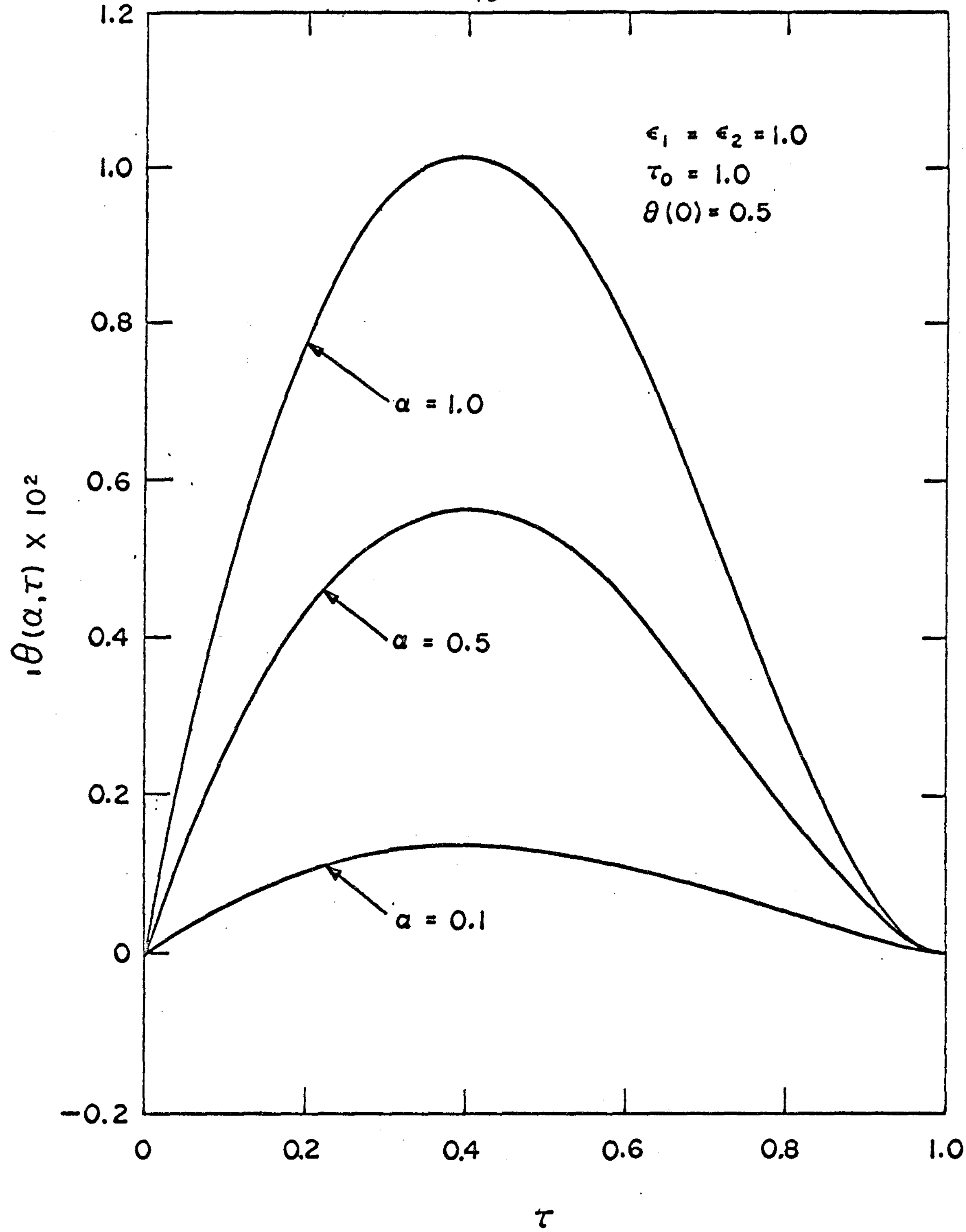

Figure 15. The First Order Correction to the Temperature Distribution.

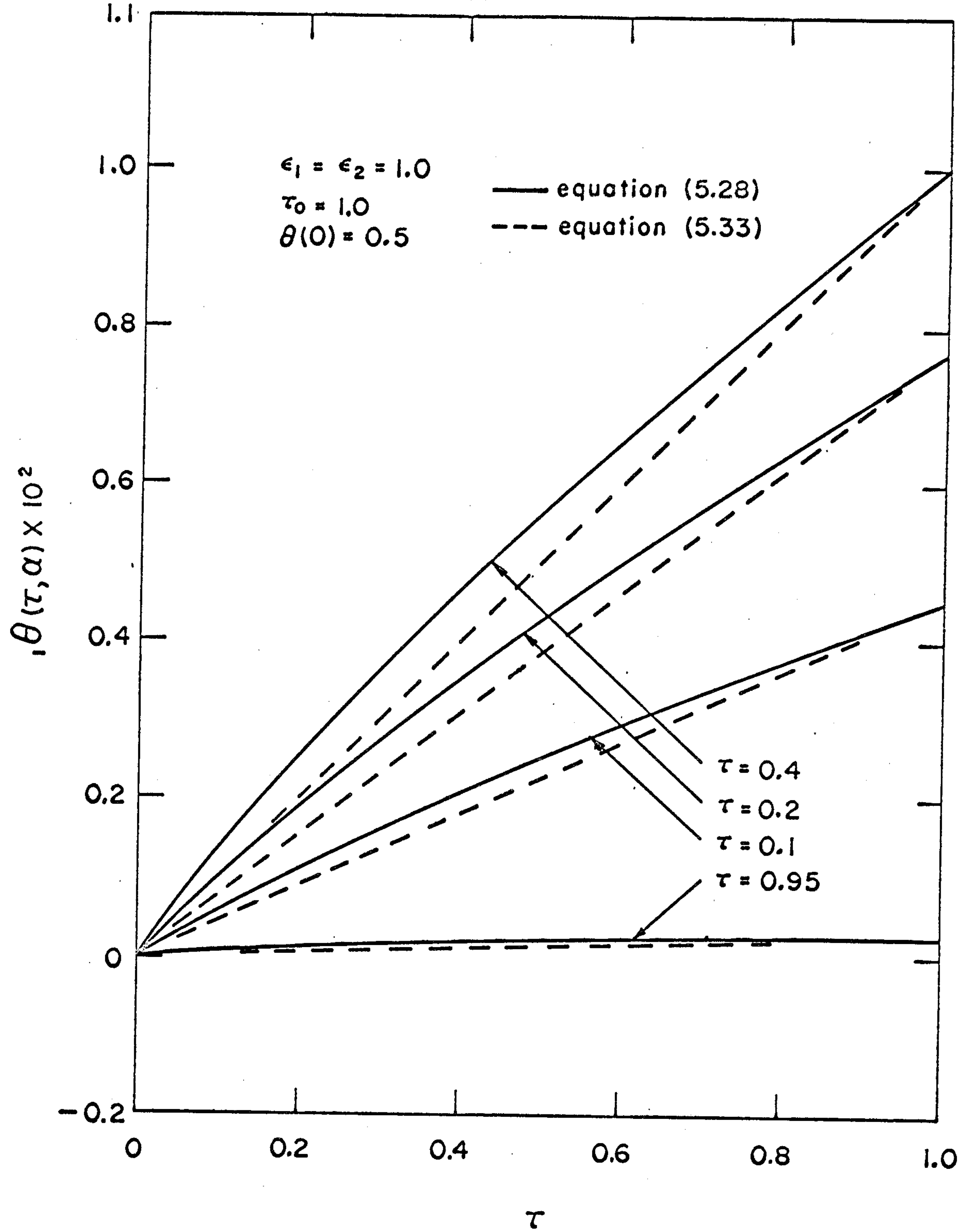

Figure 16. The First Order Correction to the Temperature Distribution Replotted as a Function of α.

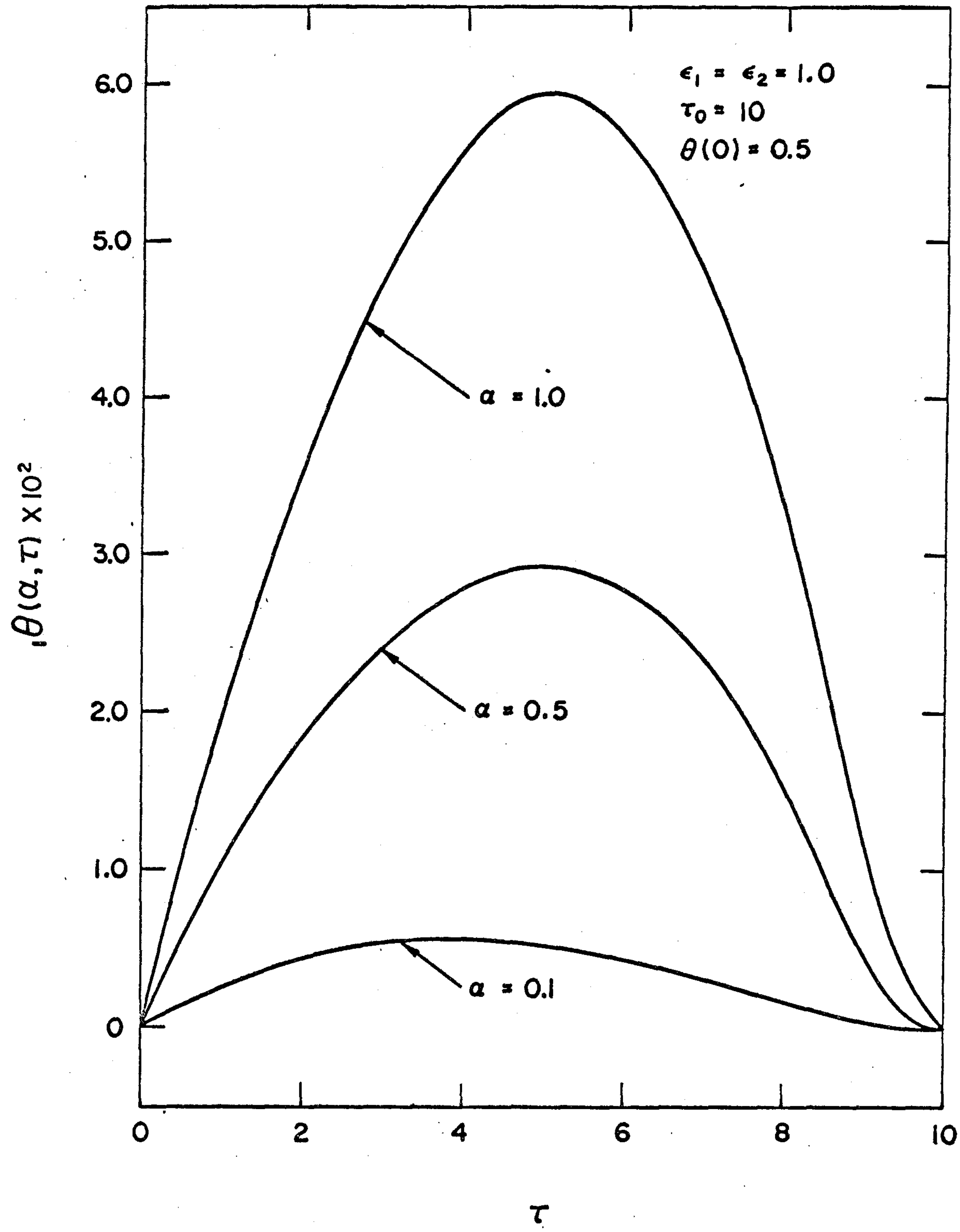

Figure 17. The First Order Correction to the Temperature Distribution.

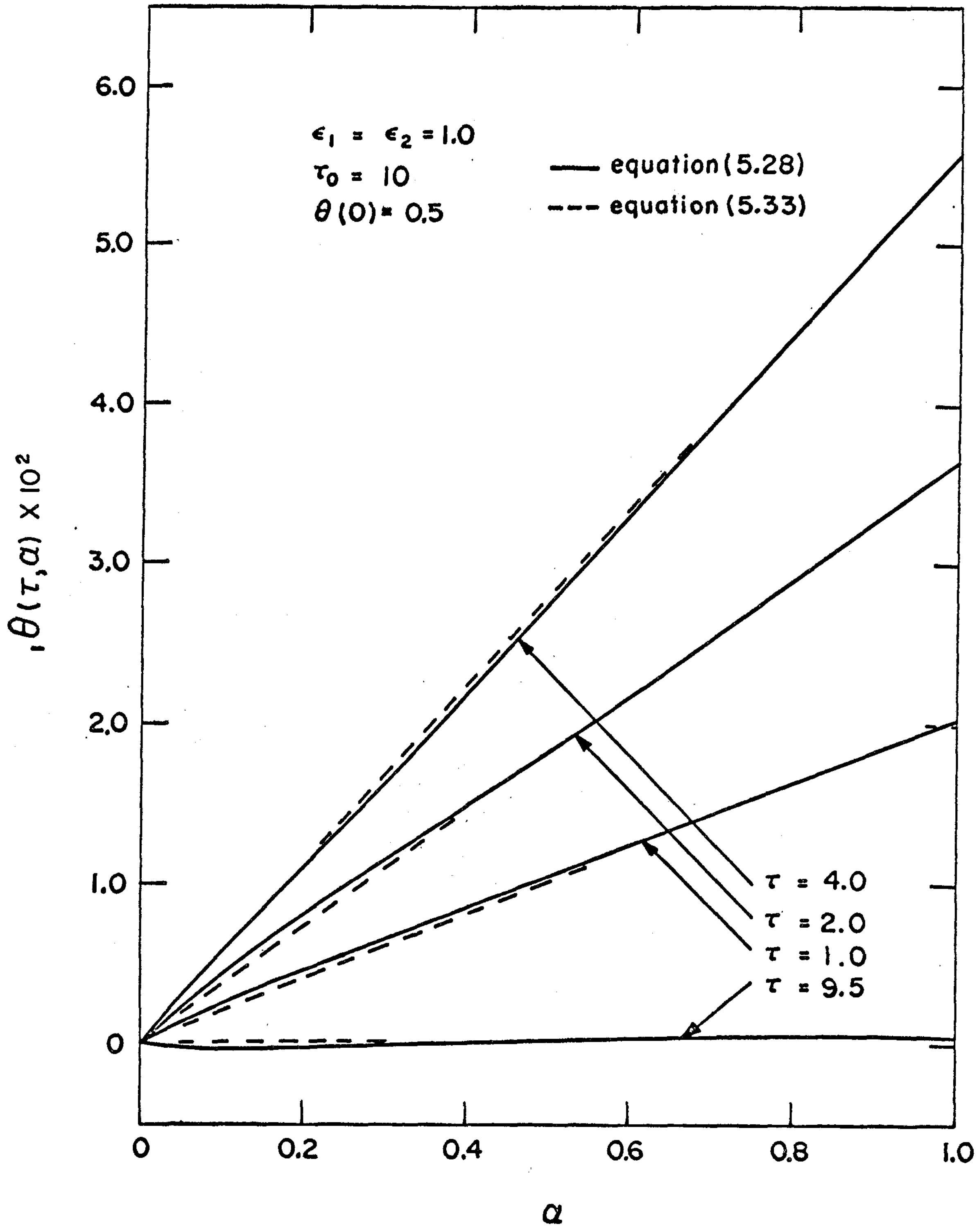

Figure 18. The First Order Correction to the Temperature Distribution Replotted as a Function of α.

to compute than before. However, ${}_1\theta$ may be replaced by $\alpha\theta_1$ to describe the temperature distribution for all α.

VI. APPROXIMATE SOLUTION FOR THE HEAT TRANSFER RATE OF AN ABSORBING, SCATTERING, AND CONDUCTING MEDIUM

The only study of the heat transfer rate with absorption and scattering based on the exact equation has been the recent work of Viskanta[20]. As mentioned in Section V, he used an iterative procedure to compute the temperature profile, and then found the heat transfer rate by numerical integration. In Part 3 of this section, Viskanta's values will be compared with the present solution, and where possible, exact results. The exact solutions are available for $\alpha = 0$ (which corresponds to pure scattering) since radiation and conduction uncouple. The heat transfer is simply the sum of the two separate modes of energy transport.

1. The Solution

As in the pure absorption solution, q will be expressed as the sum of conduction and the net radiative flux crossing a unit area parallel to the boundaries.

From Section II, q is given by

$$\frac{q(\tau,\alpha)}{\tau_o} = \theta'(\tau,\alpha) + \frac{N}{2}\Big[S_1'(\tau,\alpha) + S_2'(\tau,\alpha) + \int_{\tau}^{\tau_o}\theta^4(t,\alpha)E_2(t-\tau)dt - \int_0^{\tau}\theta^4(t,\alpha)E_2(\tau-t)dt\Big] - \frac{1-\alpha}{2\alpha}\Big[\int_{\tau}^{\tau_o}\theta''(t,\alpha)E_2(t-\tau)dt - \int_0^{\tau}\theta''(t,\alpha)E_2(\tau-t)dt\Big]\,. \tag{6. 1}$$

For the purposes at hand, it is convenient to temporarily follow the procedure used for pure absorption; that is, integrate (6. 1) with respect to τ and evaluate the result at $\tau = 0$ and $\tau = \tau_o$. In addition, if the operator G_1 defined in (4. 2) is used, and the heat transfer rate is is denoted by $q(\alpha)$, (6. 1) becomes

$$q(\alpha) = G_1\big(\theta(\tau,\alpha)\big) + \frac{N}{2}\Big[G_1\big(S_1(\tau,\alpha)+S_2(\tau,\alpha)\big) + \int_0^{\tau_o} \theta^4(t,\alpha) G_1(E_3|t-\tau|)dt\Big]$$

$$- \frac{1-\alpha}{2\alpha} \int_0^{\tau_o} \theta''(t,\alpha) G_1(E_3|t-\tau|)dt \quad . \tag{6. 2}$$

Again,

$$\begin{aligned} G_1\big(\theta(\tau,\alpha)\big) &= \theta(\tau_o,\alpha) - \theta(0,\alpha) \\ &= \theta(\tau_o) - \theta(0) \\ &= q_c \quad . \end{aligned} \tag{6. 3}$$

The second to the last equality holds, because the temperature at both boundaries is independent of α. The last equality is a restatement of the definition of q_c.

If only θ_o^4 is retained in the integrals over θ^4, the terms in brackets in (6. 2) can be split into two portions. Then, S_1 and S_2 can be written with one portion independent of α. From (2. 39) and (2. 40),

$$\frac{N}{2}\Big[G_1\big(S_1(\tau,\alpha)+S_2(\tau,\alpha)\big) + \int_0^{\tau_o} \theta^4(t,\alpha) G_1(E_3|t-\tau|)dt\Big] \sim \frac{N}{2} G_1\big(S_{10}(\tau)+S_{20}(\tau)\big) +$$

$$+ \frac{1-\alpha}{\alpha} G_1\big(T_{1e}(\tau,\alpha)+T_{2e}(\tau,\alpha)\big) + \frac{N}{2}\int_0^{\tau_o} \theta_o^4(t) G_1(E_3|t-\tau|)dt$$

$$= Nq_o + \frac{1-\alpha}{\alpha} G_1\big(T_{1e}(\tau,\alpha)+T_{2e}(\tau,\alpha)\big) \quad . \tag{6. 4}$$

Here, q_o is the result for $\alpha = 1$ in (4. 8), and T_{1e} and T_{2e} are terms defined in the previous section in (5. 2) and (5. 3).

To the first order in N, (6. 2) is

$$q \sim q_c + Nq_o + \frac{1-\alpha}{\alpha}\left[G_1\left(T_{1e}(\tau,\alpha)+T_{2e}(\tau,\alpha)\right) - \frac{1}{2}\int_0^{\tau_o} \theta''(t,\alpha)G_1(E_3|t-\tau|dt)\right] . \tag{6.5}$$

In the previous section, an approximate solution, $N_1\theta''$ was given for θ''; it can now be used in (6.5). Again, if T_{1e} and T_{2e} are replaced by T_1 and T_2, the corresponding functions of $_1\theta''$ defined in (5.24) and (5.25), equation (6.5) can be rewritten in terms of q_s where

$$q_s(\alpha) = \frac{1-\alpha}{\alpha}\left[G_1\left(T_1(\tau,\alpha)+T_2(\tau,\alpha)\right) - \frac{1}{2}\int_0^{\tau_o} {}_1\theta''(t,\alpha)G_1(E_3|t-\tau|)dt\right] . \tag{6.6}$$

With (6.6), (6.5) becomes

$$q(\alpha) = q_c + Nq_o + Nq_s(\alpha) . \tag{6.7}$$

The first two terms on the right hand side are the expression for q with pure absorption, defined in (4.9). The remaining term, q_s, can be computed with the aid of equations (5.23) through (5.27).

2. Examples and Further Approximations

As with θ, the expression for q masks its character and limiting forms should be studied. A simple approximation can be developed for q_s defined in (6.6), if the medium is optically thin. For algebraic simplicity, set T_1 and T_2 equal to zero and note, from (5.31), that for $\tau_o \ll 1$

$${}_1\theta''(\tau,\alpha) \approx \alpha\theta_1''(\tau) .$$

So, from (6.6), q_s can be written:

$$\frac{q_s(\alpha)}{1-\alpha} \approx -\frac{1}{2}\int_0^{\tau_o} \theta_1''(t)G_1(E_3|t-\tau|)dt . \tag{6.8}$$

This is an important result. The integral term is independent of α and thus, $q_s/1-\alpha$ must be also. If T_1 and T_2 had been retained in (6.8), the conclusion would have been the same. Both terms are multiplied by the critical quantity $1-\alpha/\alpha$, and are integrals over ${}_1\theta''$.

Under these conditions, q_s can be readily evaluated. If α is identically zero, the heat transfer rate is exactly the sum of the independent contribution from conduction, q_o, and radiation, Nq_R, where q_R is expressed in terms of q_{R_d}

$$\frac{q_R}{\tau_o} = \frac{q_{R_d}}{4\sigma T^4(\tau_o)} = \left[\frac{\sigma T^4(\tau_o)-\sigma T^4(0)}{4\sigma T^4(\tau_o)}\right]\left[\frac{B_o(\tau_o)}{1+\left(\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}-2\right)B_o(\tau_o)}\right]$$

$$= \left[\frac{1-\theta^4(0)}{4}\right]\left[\frac{B_o(\tau_o)}{1+\left(\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}-2\right)B_o(\tau_o)}\right]. \qquad (6.9)$$

Within the order of the approximation, $q_c + Nq_R$ is equal to (6.7) with α set equal to zero. Thus

$$q_c + Nq_R \approx q_c + Nq_o + Nq_s(0) ;$$

so

$$q_s(0) \approx q_R - q_o .$$

But from (6.8),

$$q_s(0) \approx -\tfrac{1}{2}\int_0^{\tau_o} \theta_1''(t)G_1(E_3|t-\tau|)dt \approx \frac{q_s(\alpha)}{1-\alpha}$$

or

$$q_s(\alpha) \approx (1-\alpha)q_s(0) \approx (1-\alpha)(q_R-q_o) . \qquad (6.10)$$

Substituting this into (6.7),

$$q(\alpha) \approx q_c + \alpha Nq_o + (1-\alpha)Nq_R , \qquad (6.11)$$

where again q_c, q_o, and q_R are defined in (4.4), (4.8), and (6.9). Therefore, with an optically thin medium, q can be approximated by linear interpolation between the pure absorption and pure scattering results.

For arbitrary τ_o, q_s is not a linear function of α, as can be seen from Figure 19, where q_s is shown as a function of τ_o for $\tau_o = 0.1$, 1.0, and 10. Nevertheless, for τ_o as large as one, q_s is still relatively linear.

3. Comparison of Results

In Figure 20, values of the heat transfer rate from Viskanta[20] are compared with the present substitute kernel results (6.7) and the interpolation formula of equation (6.11). In the figure, two curves are given for $\theta(0) = 0.5$, $\tau_o = 1.0$, and $N = 1$ and 10.

All three methods lead to similar results. With τ_o equal to one, the interpolation form, shown as a solid line, is expected to hold. The values appear to deviate from Viskanta's by about the same amount as that of the more complex substitute kernel. At $\alpha = 0$, the interpolation form is exact, and the errors in the other methods become apparent. The second set of results, with $N = 1$, indicate that the various solutions are nearly identical. This is mainly due to the small contribution of radiation with $N = 1$, and the small difference between the heat transfer rate for pure absorption and pure scattering.

The results show that accuracy is comparable for all values of α, with $N = 1.0$ and 10 and τ_o equal to unity. However, the limited range of the critical parameters N and τ_o does not permit the con-

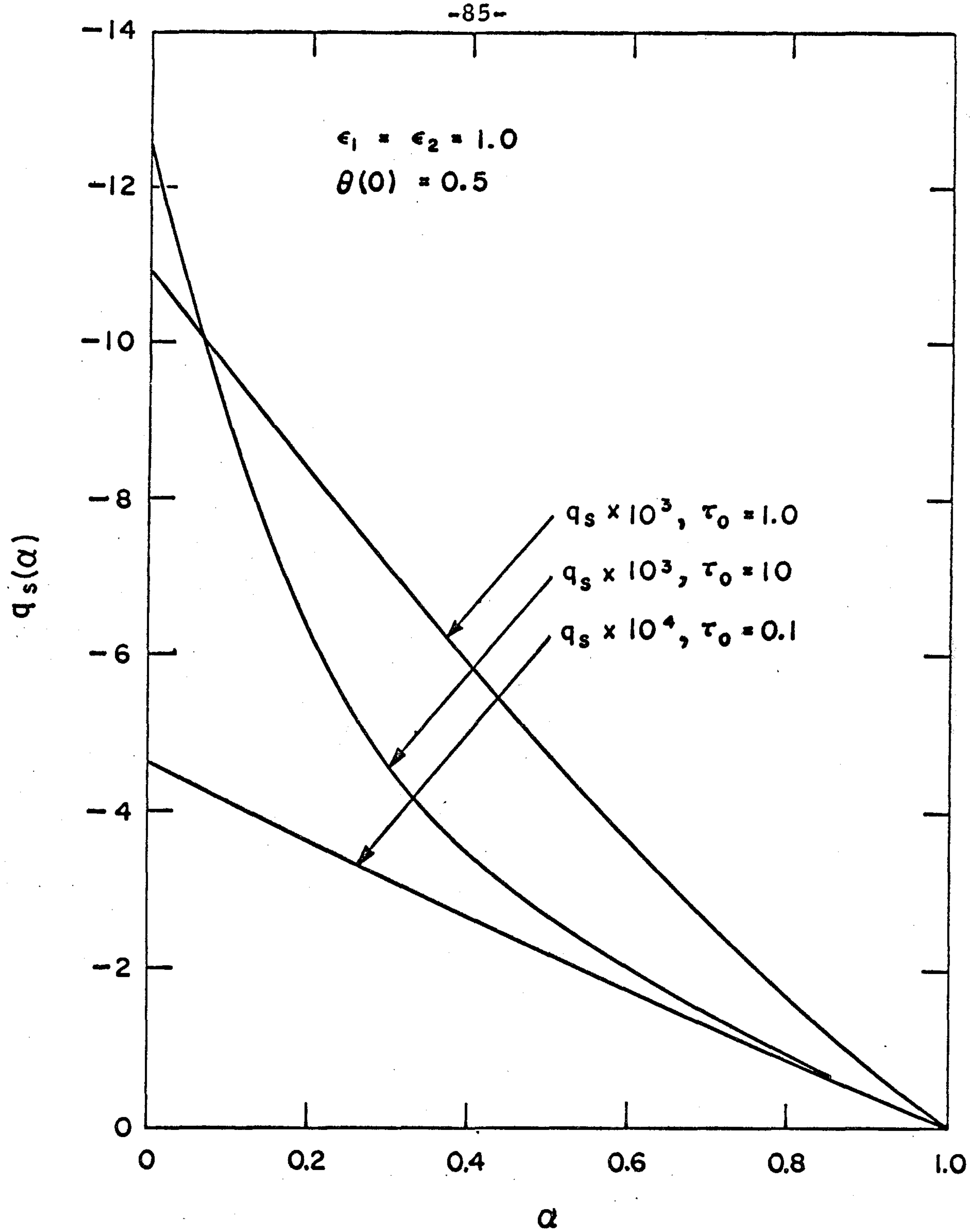

Figure 19. The Scattering Component of the Heat Transfer Rate.

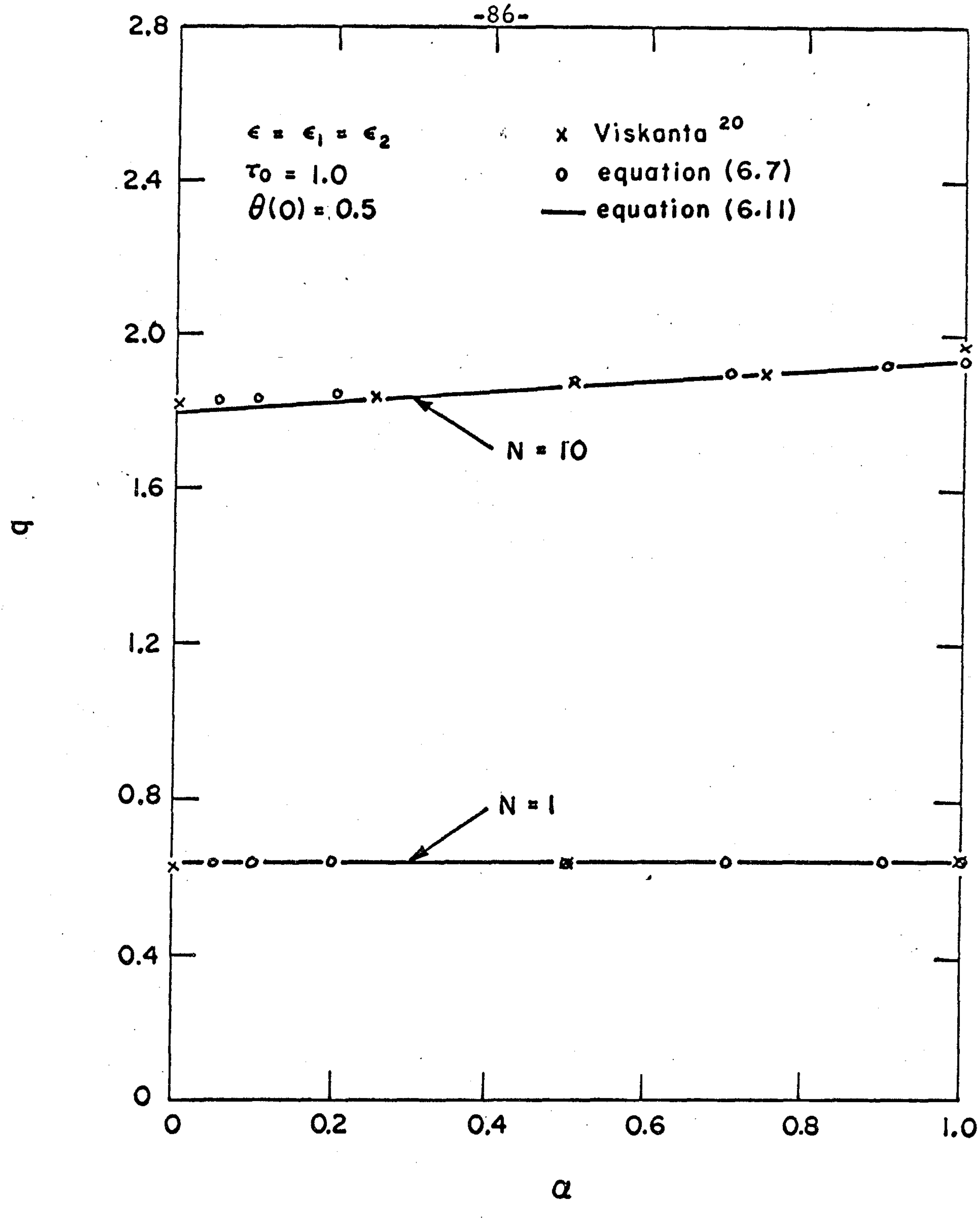

Figure 20. The Heat Transfer Rate, with Scattering.

clusion that the error will be of this magnitude under more general conditions.

VII. TWO ADJACENT SLABS AND EXTERNAL SOURCE PROBLEMS

1. The Solution to the Two Adjacent Slabs Problem

The basic solution to the general equation can be applied to physical systems more complex than the one studied to this point. In particular, the problem of two adjacent, infinite slabs, with different radiative and conductive properties, can be treated. This problem is of physical interest and shows the influence of optical depth on energy transport in situations where the radiative properties vary in specific regions.

A single medium, with an abrupt change in optical depth, is distinguished from the present problem by the type of radiative transmission across the interior boundary. In the single medium, the radiation retains its angular dependence, while in the other case, the radiation is transmitted diffusely. To illustrate the difference, the heat transfer rate for two adjacent slabs, with identical optical properties in both regions, is compared with that of a single medium with similar uniform properties. The results are presented in Part 2 of this section.

A diagram of the adjacent slabs configuration is given in Figure 21. The geometrical thickness and thermal conductivities of both regions are assumed equal, so that the pure conduction temperature profile is a straight line. With this choice, the influence of radiation can be readily studied. Properties of the region to the left are denoted by a subscript ℓ and those to the right by r. The boundaries at $\tau_\ell = 0$ and $\tau_r = \tau_{or}$ are black. The interface is a diffuse transmitter. Let $\theta_\ell(0) = 0.5$, $\theta_r(\tau_{or}) = 1.0$, and $\theta_\ell(\tau_{o\ell}) = \theta_r(0) = \theta_i$,

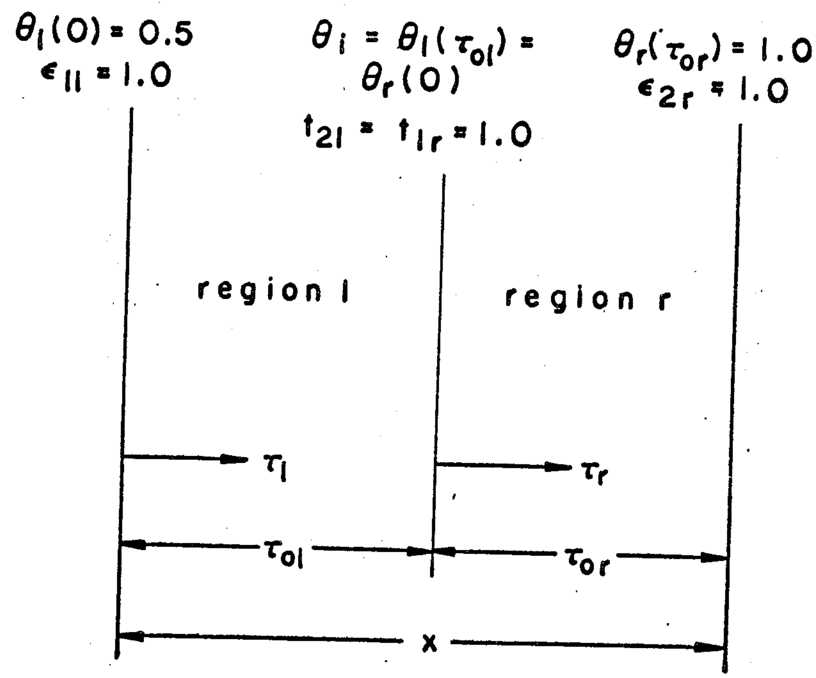

Figure 21. Diagram of the Two Adjacent Slabs Problem.

so that the temperature is continuous across the interface and equal to θ_i. The purpose of these assumptions is to limit the variables in the final solutions, rather than to simplify the problem. Reflections or external sources would not significantly complicate the treatment.

Each region has its own value of N, defined in terms of the radiative flux emitted at $\theta_r(\tau_{or})$ and the conductive heat transfer rate across the entire system. Let

$$\frac{\sigma T^4(\tau_{or})}{\frac{kT(\tau_{or})}{x}} = 1 , \tag{7.1}$$

where x is the geometrical distance across the system. This condition makes the radiative transport for zero optical depth nearly equal to twice that of pure conduction, as can be seen from the explicit representation of each:

$$\begin{aligned} q_{R_d} &= \sigma[T_r^4(\tau_{or}) - T_\ell^4(0)] \\ &= \sigma T_r^4(\tau_{or})[1 - 0.5^4] \\ &= \frac{15\sigma T_r^4(\tau_{or})}{16} \end{aligned}$$

and

$$q_{c_d} = \frac{k[T_r(\tau_{or}) - T_\ell(0)]}{x} = \frac{1}{2}\,\frac{kT_r(\tau_{or})}{x} .$$

Also, with (7.1), N_ℓ and N_r are determined, within the optical depth, in each region;

$$N_\ell = \frac{4\sigma T_\ell^4(\tau_{o\ell})}{\sigma_{a\ell} k T_\ell(\tau_{o\ell})} = \left[\frac{4}{\frac{\sigma_{a\ell} x}{2}}\right]\left[\frac{\sigma T_r^4(\tau_{or})}{\frac{kT_r(\tau_{or})}{x}}\right]\left[\frac{T_\ell^3(\tau_{o\ell})}{2T_r^3(\tau_{or})}\right] = \frac{2}{\tau_{o\ell}}\,\theta_i^3 ,$$

and

$$N_r = \frac{4\sigma T_r^4(\tau_{or})}{\sigma_{ar} k T_r(\tau_{or})} = [\tfrac{1}{2}]\left[\frac{4}{\frac{\sigma_{ar} x}{2}}\right]\left[\frac{\sigma T_r^4(\tau_{or})}{\frac{kT_r(\tau_{or})}{x}}\right] = \frac{2}{\tau_{or}} .$$

Once $\tau_{o\ell}$ and τ_{or} are chosen, N_ℓ and N_r are uniquely defined.

The solution is found by computing q_ℓ and q_r as functions of θ_i, and satisfying the condition

$$q_\ell(\theta_i) = q_r(\theta_i) \ .$$

This can best be done by assuming several values for θ_i, computing both heat transfer rates from (4. 9), and graphically determining the intersection of the two curves $q_\ell(\theta_i)$ and $q_r(\theta_i)$. The only unusual feature in the process is developing the expressions for the source functions. Obviously, $S_{1\ell}$ is just the contribution from the boundary at $\tau_\ell = 0$, but $S_{2\ell}$ consists of the emission from $\tau_r = \tau_{ro}$ reaching region ℓ plus the emission generated within region r. From the discussion following equation (3. 20), the latter is $2J_{1r}(\theta_i, \tau_{or})$. Similarly, S_{1r} is the sum of the flux from $\tau_\ell = 0$ plus the radiation generated within region ℓ. However, care must be taken to express the functions J_1, J_2, and K, appearing in q correctly as functions of $\theta(\tau_{o\ell})$. In their definitions, (3. 18), (3. 19), and (4. 7), $\theta(\tau_{o\ell})$ is set equal to unity. This must be taken into account by multiplying the functions by θ_i^4 and transforming the variable $\theta_\ell(0)$ to $\theta_\ell(0)/\theta_i$.

2. Results for Adjacent Slabs

With the above in hand, results are readily computed. As an example, the value of θ_i as a function of τ_{or} is given in Figure 22 for $\tau_{o\ell} = 0.1$, 1.0, and 10. Since τ_{or} is the optical depth in the hotter region, it would be expected to play the dominant role in determining the temperature profile. If it is small, radiation passes directly to region ℓ. If the optical depth is large in ℓ, the flux is absorbed

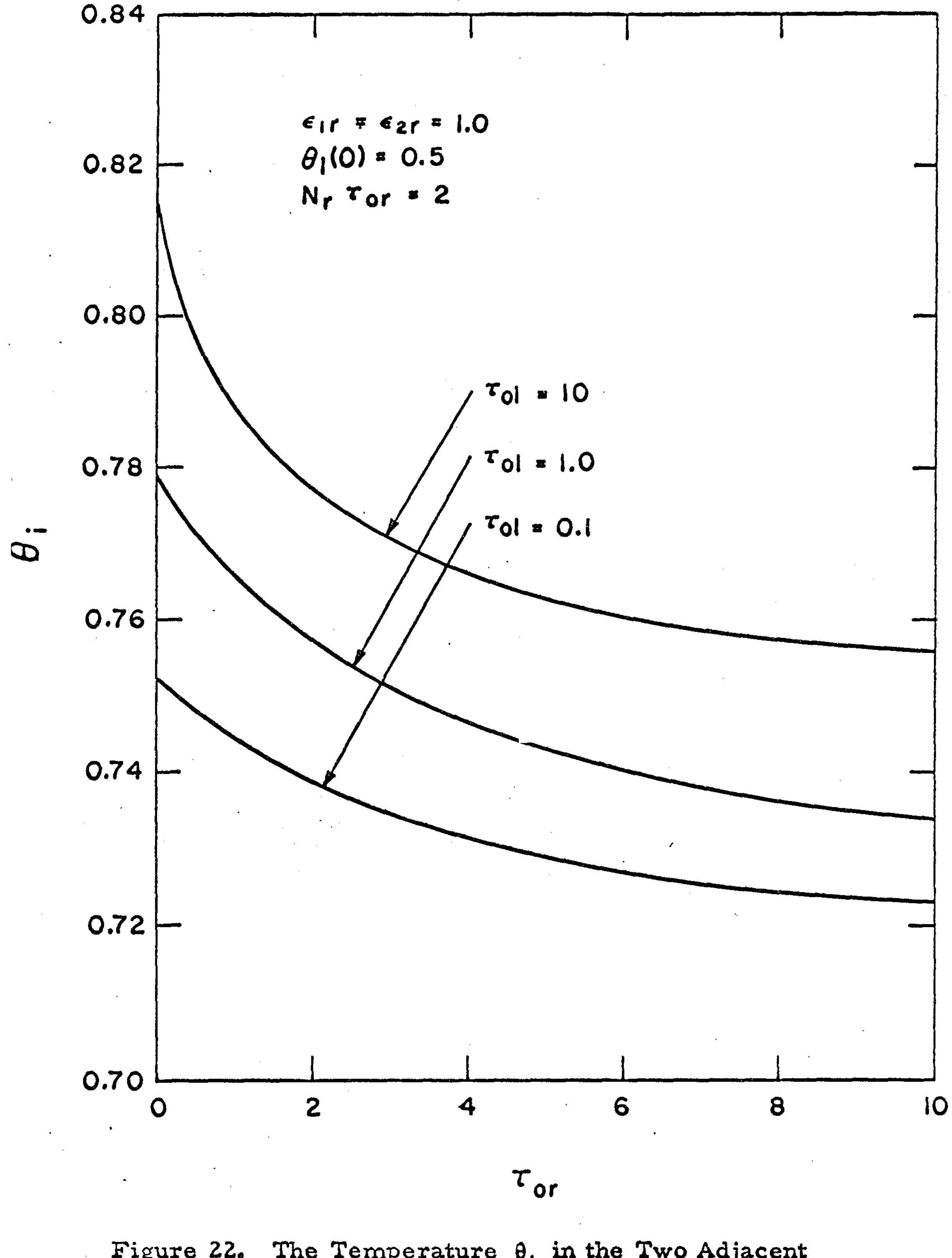

Figure 22. The Temperature θ_i in the Two Adjacent Slabs Problem.

near the interface, and is transported by conduction. As a result, θ_i becomes larger to increase the temperature gradient through ℓ. For larger τ_{or}, the radiative flux cannot pass through r as readily; θ_i decreases as more and more temperature drop is required in r. This trend is evident in the figure. Similarly, if $\tau_{o\ell} = 1.0$, and τ_{or} is very small, a large gradient is required in region ℓ. Again the gradient decreases as τ_{or} increases. If $\tau_o = 0.1$, the process reverses, and little resistance is experienced in ℓ. However, for large optical depths in r, the steep gradient required to conduct energy across the region lowers the interface temperature. Even through ℓ is optically thin, little radiant energy can be generated near the interface to contribute to the overall heat transfer rate. Thus, from the figure, θ_i is about 0.72, close to the pure conductive value of 0.75; while in the opposite case of large $\tau_{o\ell}$ and small τ_{or}, θ_i is 0.81. In terms of the heat transfer rate, in the configuration $\tau_{o\ell} << 1$ and $\tau_{or} >> 1$, q is expected to be lower than the opposite case of $\tau_{o\ell} >> 1$ and $\tau_{or} << 1$. When $\tau_{o\ell} >> 1$, there is less impedance to radiative transport since region ℓ is cool, and most of the energy is transported by conduction, even if the optical depth is decreased. In both circumstances, the heat transfer rate is small because the radiative flux cannot readily pass from one wall to the other. Results for q as a function of τ_{or} and $\tau_{o\ell} = 0.1$, 1.0, and 10 are given in Figure 23. In addition to the heat transfer property described above, the curves show the expected features of large q for both regions optically thin, with q decreasing as either $\tau_{o\ell}$ or τ_{or} increase. For both regions optically thick, q is near the pure

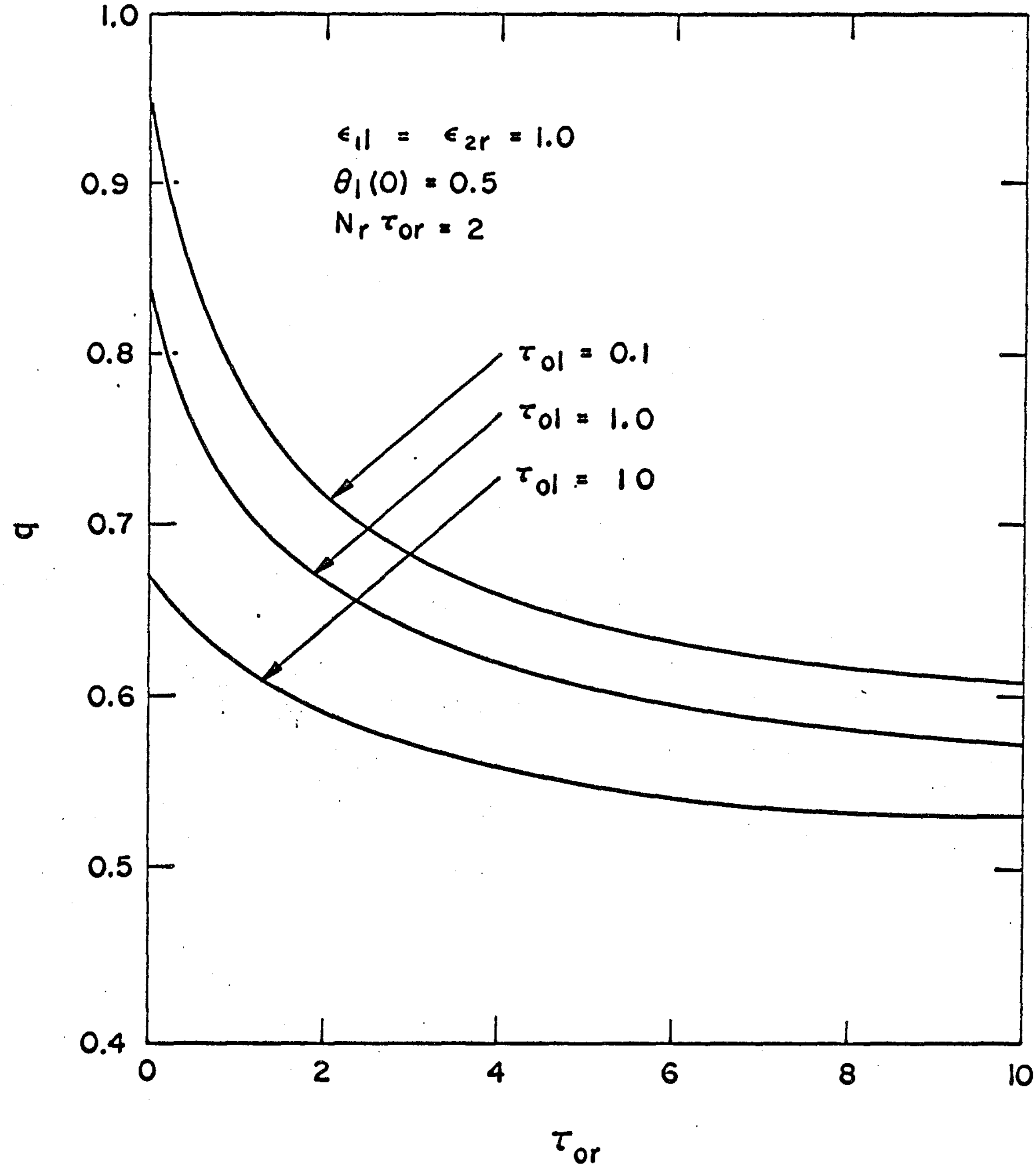

Figure 23. The Heat Transfer Rate for Two Adjacent Slabs.

conduction value of 0.5.

Another interesting comparison can be made between the results for $\tau_{o\ell} = \tau_{or}$ and a single medium having the same total optical depth and other boundary conditions. The heat transfer rate should be different in both cases. However, the difference is small. For $\tau_{o\ell} = \tau_{or} = 0.1$, q for the adjacent slabs is 0.928, while the single medium result is 0.915. The values are expected to be close, since the emission from both boundaries is already diffuse. The interface only affects the anisotropies in the radiation generated within the media, which is small since the media are optically thin. For $\tau_{o\ell} = \tau_{or} = 1.0$, the heat transfer rates are 0.711 for the adjacent slabs and 0.703 for the single medium. For greater optical depths, the difference should be smaller, since radiation would contribute little to the overall heat transfer process. In addition, the zone is very narrow where the emission reaching the interface is produced. Thus, it is nearly at a uniform temperature and the flux is practically isotropic. As a result, the boundary condition at the interface has little influence on radiative transport, and hence the heat transfer rate.

The above results show the characteristics of two adjacent slabs and how optical depth comes into play in such configurations. Further, the isotropy of the flux across an interface is shown to have little influence on the heat transfer rate. If no interface is present and the medium is optically thin, the distribution is nearly isotropic; while if the medium is optically thick, the distribution is isotropic since the flux is generated over a narrow zone where the temperature is nearly constant. Thus, in physical systems, the boundary condition relating to the isotropy of the flux is unimportant.

3. Consequences of an External Source

The case of an infinite slab with one boundary partially transparent and the other opaque merits special attention. Such configurations are found in metal oxide coatings on rocket nozzles and coatings on leading edges of hypersonic vehicles; often, an external source, in the form of combustion products or emissions from shock-heated gases, is present. The interest in this configuration stems from the two distinct ways the heat transfer rate can be calculated. The conventional approach assigns an emissivity and absorptivity to the coating and the temperature found by an energy balance equation. The second method appeals to the procedure of the previous sections.

The purpose here is not to compare directly the two approaches, but to determine which parameters must be taken into account and to find when each method is appropriate. The following assumptions are made. The coating must have isotropic and homogeneous properties independent of temperature and wavelength. The boundaries are assumed to reflect, emit, and transmit diffusely. Radiative transport must be small enough for the analysis of the previous sections to apply, and finally, steady-state conditions are postulated.

With an external source at one wall and an opaque opposite boundary, the problem is much more complex than it appears initially. The absorptivity is a characteristic of the reflective and absorptive properties of the material and hence independent of the temperature. On the other hand, the emission and radiative transport are dependent on the temperature profile, which in turn depends on the absorption at each point. Thus, the emission is not a constant but a function of specific boundary conditions, i.e., ϵ_1, t_2, $\theta(0)$, $\theta_x(\tau_o)$.

For the present analysis to hold, the ratio of radiative to conductive transport must be restricted to a range for which the approximation in the computation of the emission is suitable, since the temperature profile is approximated by θ_o, the result for pure conduction.

The importance of the various boundary conditions can best be analyzed by ignoring scattering initially. Nearly all of the phenomena occur with or without scattering, and the effects of its inclusion can be qualitatively predicted. Toward the end, however, some problems with scattering are included to show some unexpected effects.

One of the most important parameters is optical depth. It determines whether the medium appears semi-infinite or whether boundary conditions at the inner surface are important. As an example, assume the external source radiates as a black body at $\theta_x(\tau_o) = 2$ and the inner wall is at $\theta(0) = 0.5$. Under these conditions, the external radiative flux is very large, but re-radiation from the inner surface is quite small. If the outer surface is a perfect transmitter and the inner surface has an arbitrary emissivity, the optical depth, along with ϵ_1, determines the heat transfer rate.

If the medium is optically thin, radiative interactions with the internal medium can be neglected and the magnitude of ϵ_1 becomes important. At the other extreme of an optically thick medium, nearly all the radiative flux is absorbed and transported to the inner wall by conduction, and ϵ_1 does not affect the heat transfer rate. For modest optical depths, ϵ_1 is important, but part of the radiation is shielded by the absorbing medium. These effects can be seen in Figure 24, which shows q_o, the radiative term in the heat transfer rate, as a function of τ_o for $\theta(0) = 0.5$ and various inner wall emissivities. For $\epsilon_1 = 0.6$ or less, q_o reaches a maximum for τ_o between zero and one. Further, for $\tau_o > 2$, the inner wall is effectively shielded and the emissivity can be arbitrary.

Another important parameter is the transmissivity of the outer boundary, t_2. It determines the amount of the external radiation passing through the boundary and available to interact with the inner wall and the absorbing medium. Thus, its effects are quite predictable, and closely follow the previous results if the radiative influx in the previous computation is reduced in proportion to the transmissivity. Of course, some internally generated radiation is reflected at the outer boundary, but its contribution to the overall heat transfer rate is negligible compared to that of the external source (unless the transmission becomes very small).

The parameters N, $\theta(0)$, and $\theta_x(\tau_o)$ can be considered together. For $\theta_x(\tau_o)$ greater than 2, the external flux dominates the other sources of radiation since the radiative intensity goes as θ_x^4, and so the external flux is at least sixteen times that generated within the coating. Also, for the present analysis to be of value, N must be small, so conduction dominates within the medium. Further, the magnitude of $\theta(0)$ is unimportant, because the radiative flux generated at the inner boundary will be quite small.

If $\theta_x(\tau_o)$ is less than two, and especially when it is between

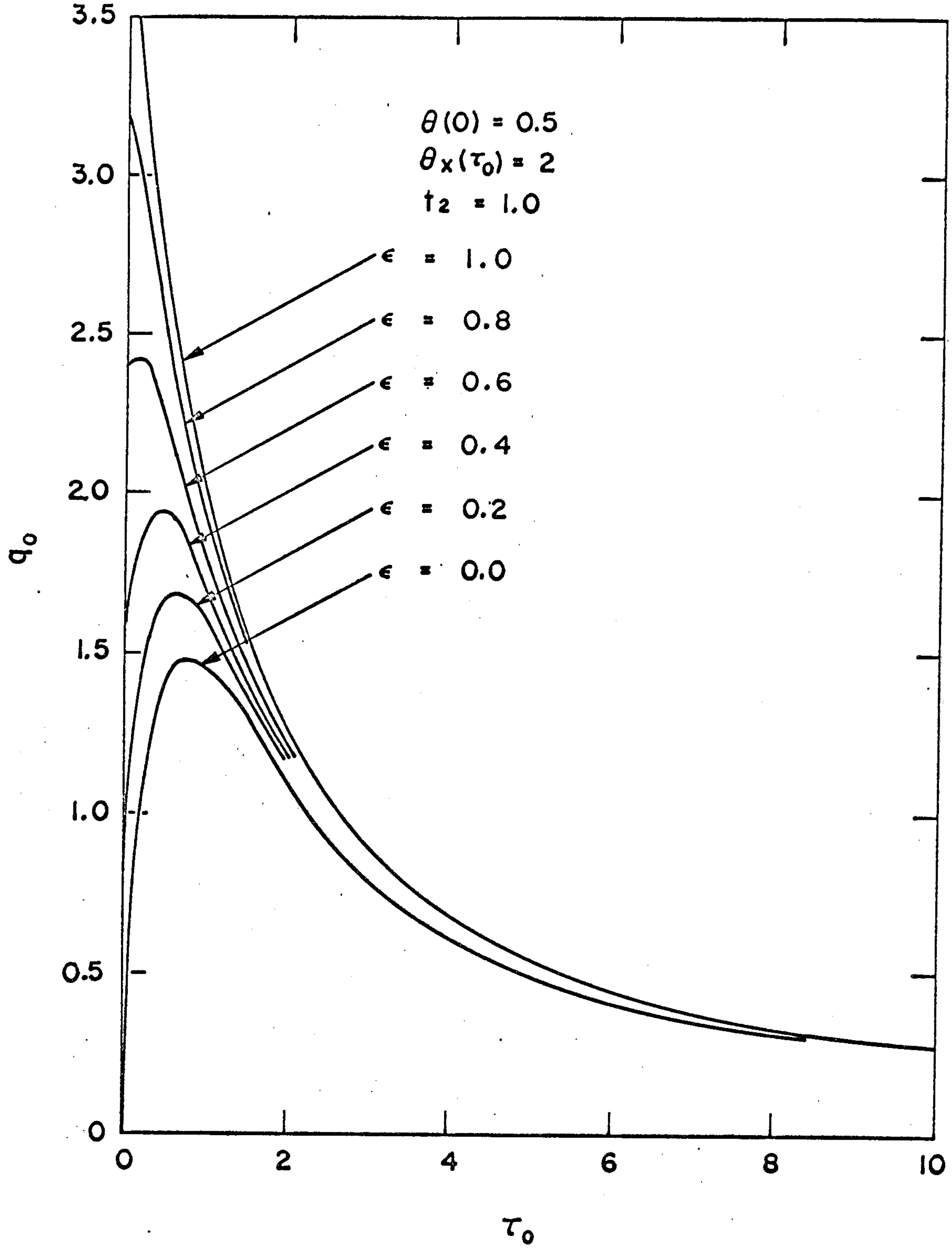

Figure 24. The Function q_o.

zero and one, the external flux is the same magnitude as the internally generated radiation. Then, of course, the size of N, $\theta(0)$, and $\theta_x(\tau_o)$ all become important, the amount dependent on their relative values and the degree of transmission, emission, and absorption.

The last parameter to consider is α, a measure of the coupling between radiation and conduction. For very small α, little absorption takes place, and the heat transfer properties approach that of the separate contributions from radiation and conduction. If it is near one, the properties of the system approach those of pure absorption.

A particular example of the effects of changes in α is shown in Figure 25, where the radiative components, $q_o + q_s(\alpha)$, of the heat transfer are given for an external source of strength $\theta_x(\tau_o) = 2$ and three optical depths, $\tau_o = 0.1$, 1.0, and 10. The other boundary conditions $\theta(0) = 0.5$, $\epsilon_1 = 1.0$, $t_2 = 1.0$ are held fixed. Even though q_o is independent of α, both components of the radiative terms are included to compare the magnitude of q_s to the entire radiative term.

Unexpectedly, for large optical depth, the heat transfer rate is greater for small α and decreases as it approaches one. This means that scattering is a more efficient transport mechanism than the concurrent processes of absorption and conduction. With the latter, most of the radiative flux is absorbed near the other boundary and must be transported by conduction. Of course, if the inner wall emissivity were much less than one, the radiative flux transported by scattering would be partially reflected. The heat transfer rate would decrease to a value less than the amount with pure absorption.

For moderate optical depths, the same reasoning holds. How-

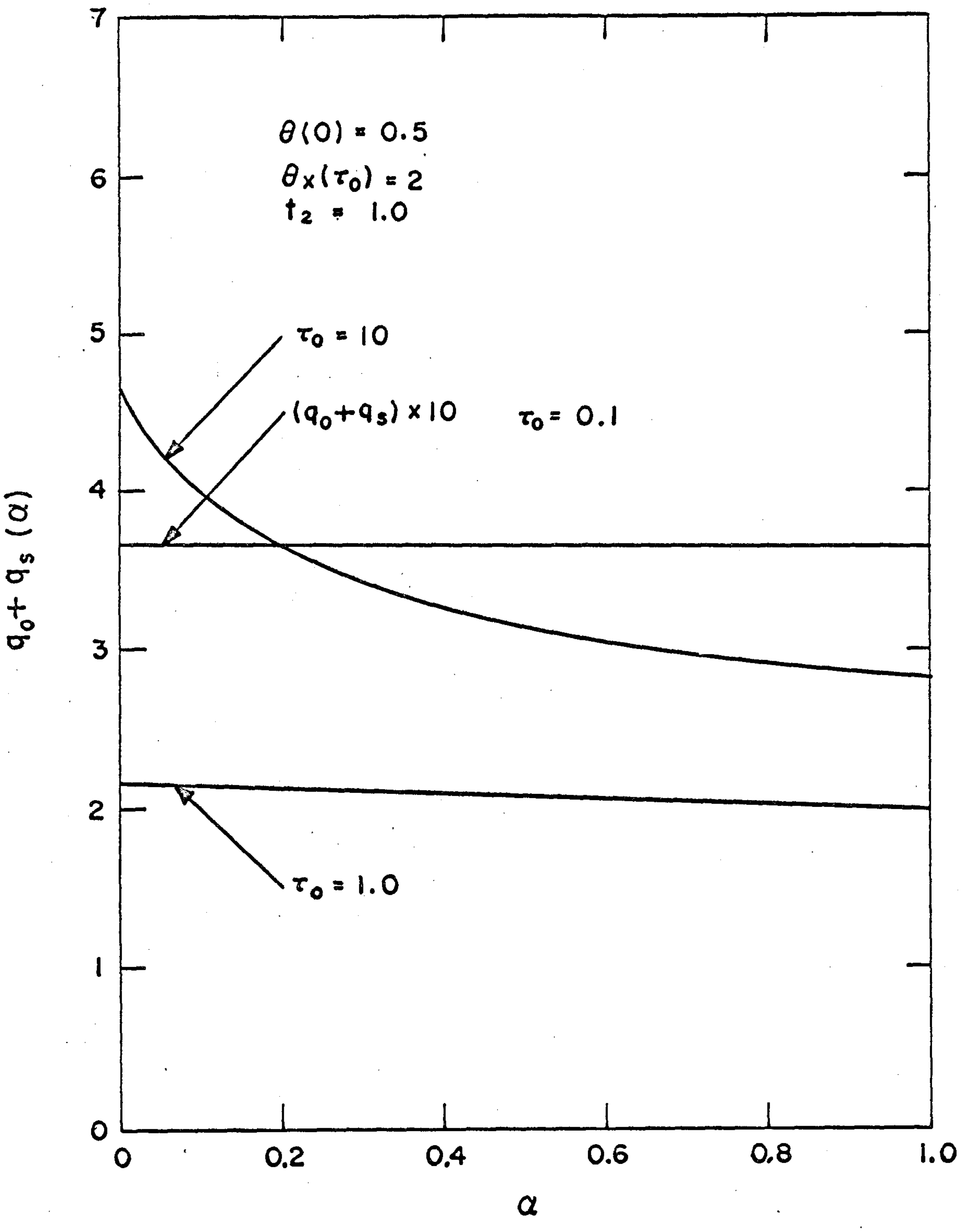

Figure 25. The Radiative Portion of the Heat Transfer Rate.

ever, the radiation is absorbed more uniformly throughout the medium. Steeper gradients exist near the inner wall, which increases the conductive heat transfer rate. The net result is little difference between the pure absorption and pure scattering cases.

For very small optical depths and $\theta_x(\tau_o) = 2$, the radiation interacts most strongly with the inner boundary and so its properties determine the heat transfer rate.

If $\theta_x(\tau_o) = 0$, the radiative flux at τ_o is directed outwards, and the radiative contribution to q can be negative. A set of results for $q_o + q_s(\alpha)$ with $\theta_x(\tau_o) = 0$ and $\epsilon_1 = 1.0$, $t_2 = 1.0$, $\tau_o = 0.1$, 1.0, and 10, and $\theta(0) = 0.5$ are shown in Figure 26. For τ_o much less than one, the heat transfer rate is insensitive to α because of the limited coupling within the medium. However, it is dependent on the inner wall emissivity if $\theta(0)$ is large enough to produce a significant radiative flux.

For moderate optical depths, q is strongly dependent on α. When scattering dominates, the only source of radiative flux is the inner wall, which is maintained at a relatively low temperature. Also, the impedance of the intervening medium cuts down the radiative transport. On the other hand, with α near one, the radiative flux is generated throughout the medium (where the temperatures, on the average, are higher) and results in higher intensities. Aside from being more intense, the flux produced near the outer walls is not appreciably attenuated while escaping.

As the optical depth becomes very large, the effects associated with a moderate thickness still exist, but shielding of inner wall emis-

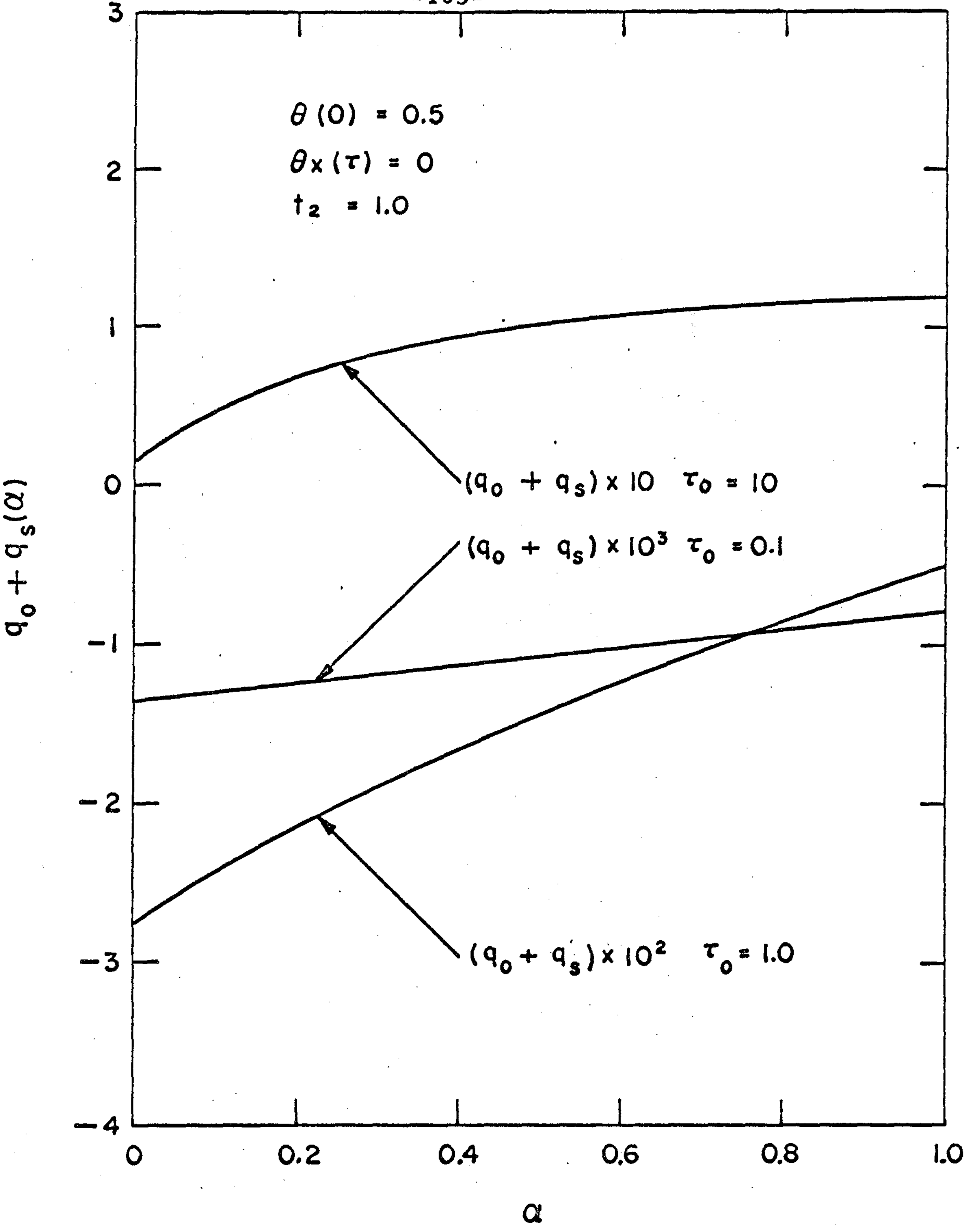

Figure 26. The Radiative Portion of the Heat Transfer Rate.

sion is so complete that for α near zero the heat transfer rate is nearly independent of N.

To summarize, the dominant radiative properties of coatings vary according to the degree of scattering and the strength of external radiative sources. For very strong sources (and for the present analysis to hold), conduction must dominate within the medium. The radiation contributes to the heat transfer rate in proportion to the amount that either directly reaches the inner wall and is absorbed, or is absorbed enough within the medium to increase conductive transport. The degree of transmission determines the radiative flux available to interact with the medium. Thus, an emissivity may be assigned to a coating irradiated by an intense source, only if the emissive properties of the inner wall are taken into account, in optically thin coatings or coatings with moderate optical depth and low values of α. For optically thick coatings, the concept of an emissivity is appropriate.

For weak external sources, optical depth becomes the dominant parameter. In optically thin media, the bulk of the radiative flux comes from the inner wall, with the amount escaping dependent on the transmissivity of the outer boundary. For an optically thick medium, both transmissivity and degree of scattering are important. Transmissivity again regulates the amount of internally generated flux which escapes through the outer boundary. The flux intensity at the outer boundary depends on the temperature where production takes place, and the impedance the radiative flux experiences traveling to the wall. As an example, a scattering medium does not generate flux within the medium but at the relatively cool inner wall. The flux then must diffuse through the coating to leave the system. On the other hand, an absorbing medium generates most of its flux in a relatively high temperature zone near the outer surface. The flux is more intense, and the emissivity is greater. For moderate optical depths, all of the above factors combine to make the process a complex function of the various parameters mentioned in connection with the optically thin and thick cases.

For weak external sources, then, the radiative transport is a complex function of wall emissivity and transmissivity, as well as the degree of scattering, the optical depth, and the actual temperature profile. The practice of assigning an emissivity to such a system is inappropriate, and an approach similar to the present is required to accurately describe its transport properties.

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APPENDIX I. Properties of the Functions $E_n(\tau)$

The function E_n, for integer n, is a generalization of the exponential integral, E_i. The conventional definition for E_i is

$$E_i(\tau) = \int_{-\infty}^{\tau} \frac{e^y}{y}\, dy \ .$$

For convenience, and because the optical depth is always positive, E_1 defined by

$$E_1(\tau) = -E_i(-\tau) \qquad \tau > 0$$

$$= \int_{\tau}^{\infty} \frac{e^{-y}}{y}\, dy \qquad \text{(I-1)}$$

is used in radiative transport.

A change of variables to $x = y/\tau$ in (I-1) reduces the integral representation of E_1 to its conventional form

$$E_1(\tau) = -E_i(-\tau) = -\int_{-\infty}^{-\tau} \frac{e^y}{y}\, dy = \int_1^{\infty} \frac{e^{-\tau x}}{x}\, dx \ .$$

With this form, E_n is defined by

$$E_n(\tau) = \int_1^{\infty} \frac{e^{-\tau x}}{x^n}\, dx \ .$$

Before discussing the properties of E_n, it is worth noting another form of the function found in the German and Russian literature, and also used on occasion by Chandrasekhar[4]:

$$E_n(\tau) = \int_1^{\infty} \frac{e^{-\tau x}}{x^n}\, dx = \int_0^1 e^{-\tau/\mu}\, \mu^{n-2} d\mu \ , \qquad \text{(I-2)}$$

which results from substituting μ^{-1} for x. This form has the advantage of being physically meaningful. If μ equals the cosine of an angle, the term μ^{n-2} is then a weight function. It corresponds to an intensity if $n = 2$, a flux for $n = 3$ and with $n = 4$ a pressure.

To continue with the properties of the exponential integrals, if E_n is integrated by parts,

$$E_n(\tau) = \int_1^\infty \frac{e^{-\tau x}}{x^n}\, dx = -\frac{1}{n-1}\left[\frac{e^{-\tau x}}{x^{n-1}}\Big|_1^\infty + \tau \int_1^\infty \frac{e^{-\tau x}}{x^{n-1}}\, dx\right] = \frac{1}{n-1}\left[e^{-\tau} - \tau E_{n-1}(\tau)\right]$$

or

$$(n-1)E_n(\tau) = e^{-\tau} - E_{n-1}(\tau) .$$

By repeated application of this process,

$$(n-1)!\, E_n(\tau) = (-\tau)^{n-1} E_1(\tau) + e^{-\tau} \sum_{\ell=0}^{n-2} (n-\ell-2)!\, (-\tau)^{\ell} .$$

Here, E_n is expressed in terms of E_1. This is advantageous, since E_1, unlike the higher order functions, is tabulated for a wide range of arguments. Two particular tables are rather extensive: The National Bureau of Standards Mathematical Tables, NBSMT, "Tables of Sine, Cosine, and Exponential Integrals," Vols. 1, 5, and 6 (1940), and the Harvard University Computation Laboratory Series, Number 21, "Tables of the Generalized Exponential-Integral Functions," (1949).

It remains to consider asymptotic forms. Expansions for E_n can be derived which show its character near $\tau = 0$ and $\tau \to \infty$. Consider E_1 written in the form (I-1)

$$E_1(\tau) = \int_\tau^\infty \frac{e^{-x}}{x}\, dx = \int_1^\infty \frac{e^{-x}}{x}\, dx + \int_\tau^1 \frac{e^{-x}}{x}\, dx \qquad \tau > 0 \quad . \qquad \text{(I-3)}$$

Integrating the last integral by parts after expanding the exponential term yields

$$\int_{\tau}^{1} \frac{e^{-x}}{x}\,dx = \int_{\tau}^{1} \frac{\sum_{n=0}^{\infty} \frac{x^n(-)^n}{n!}}{x}\,dx = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \int_{\tau}^{1} x^{n-1}dx$$

$$= -\ell n\,\tau + \sum_{n=1}^{\infty} \frac{(-)^n}{n!}\left(\frac{1}{n} - \frac{\tau^n}{n}\right) \qquad \tau > 0 \quad .$$

If this is put in (I-3) and the first term is replaced by $E_1(1)$,

$$E_1(\tau) = -\ell n\,\tau + E_1(1) + \sum_{n=1}^{\infty} \frac{(-)^n}{n!n} - \sum_{n=1}^{\infty} \frac{(-)^n \tau^n}{n!n} \quad .$$

It can be shown that

$$E_1(1) + \sum_{n=1}^{\infty} \frac{(-)^n}{n!n} = -\gamma = -.5772156 \,,$$

where γ is Euler's constant. Thus,

$$E_1(\tau) = -\gamma - \ell n\,\tau + \sum_{n=1}^{\infty} \frac{(-)^{n-1}\tau^n}{n!n} \quad . \qquad \text{(I-4)}$$

Now look at the definition of E_n and differentiate it with respect to τ.

$$\frac{d}{d\tau} E_n(\tau) = \frac{d}{d\tau} \int_1^{\infty} \frac{e^{-\tau x}}{x^n}\,dx = -\int_1^{\infty} \frac{e^{-\tau x}}{x^{n-1}}\,dx = -E_{n-1}(\tau) \quad ,$$

so that

$$E_n(\tau)' = -E_{n-1}(\tau) \quad . \qquad \text{(I-5)}$$

In addition, for $n > 1$,

$$E_n(0) = \int_1^{\infty} \frac{1}{x^n}\,dx = \frac{1}{n-1} \qquad n > 1 \quad . \qquad \text{(I-6)}$$

Thus, by integrating the expansion for $E_1(\tau)$ given in (I-4), and using the value of $E_n(0)$ to eliminate the constant of integration,

$$E_2(\tau) = 1 + \tau(\gamma-1+\ell n\tau) + \sum_{2}^{\infty} \frac{(-)^{n-1}\tau^n}{n!\,(n-1)} ,$$

$$E_3(\tau) = \tfrac{1}{2} - \tau + \frac{\tau^2}{2}(-\gamma+\frac{3}{2}-\ell n\tau) + \sum_{3}^{\infty} \frac{(-)^{n-1}\tau^n}{n!\,(n-2)} , \qquad \text{(I-7)}$$

. . .

The expansion for $E_1(\tau)$ shows that as $\tau \to 0$

$$E_1(\tau) \sim -\ell n\tau , \qquad \text{(I-8)}$$

or E_1 has a logarithmic singularity at the origin. Obviously then, from (I-5), E_2 has an infinite slope at $\tau = 0$. These properties show the rather unique form of the exponential integrals near the origin and indicate the difficulties associated with using conventional expansions, such as the method of steepest descent, to approximate them.

The above forms are useful for τ near zero. As τ approaches infinity, the functions are exponential in character. This can be seen by integrating the standard form for E_n by parts

$$E_n(\tau) = \int_1^{\infty} \frac{e^{-\tau x}}{x^n} dx = -\frac{e^{-\tau x}}{\tau x^n}\Bigg|_1^{\infty} - \int_1^{\infty} \frac{n e^{-\tau x}}{\tau x^{n+1}} dx = \frac{e^{-\tau}}{\tau} - \frac{n}{\tau}\int_1^{\infty} \frac{e^{-\tau x}}{x^{n+1}} dx .$$

Continuing, the asymptotic form for E_n is

$$E_n(\tau) \sim \frac{e^{-\tau}}{\tau}\left[1 - \frac{n}{\tau} + \ldots\right] , \qquad \tau \to \infty .$$

Notice that for all n, the dominant term is the same, so that to the first order,

$$E_n(\tau) \sim \frac{e^{-\tau}}{\tau} . \qquad \text{(I-9)}$$

Here, then, is another unusual property of E_n. For large n, all τ behave the same to the first order. This means that expansions resulting in sums of E_n are of limited value, since even for large τ all of the terms must be retained, unless, of course, some multiplicative factor in the higher term goes as a negative power of τ.

The expression for E_n shows that it goes rapidly to zero for large arguments. Thus, when these functions appear as integral kernels and the range of integration is large, local properties will dominate. As a result, the limits of integration are often unimportant. This leads to diffusion approximations for large arguments, or the "optically thick" case.

The expressions and relations developed above were taken essentially from Appendix I of Kourganoff[3]. In addition to the relations presented here, he gives detailed references as well as a broad spectrum to functions associated with the exponential integrals.

APPENDIX II. Expansion Error Analysis

The error analysis is restricted to pure absorption. In the more general problem with scattering, errors arise for two sources. Besides approximating the temperature profile by an expansion in N, a substitute kernel is used. The latter approximation introduces an entirely different type of error, which is not easily treated for arbitrary degrees of scattering. However, for pure scattering, the exact solution is available, and a comparison can be made. Grief[19] has done this for a limited number of examples using the substitute kernel results of Lick[21]. Grief compared the kernel substitution method with the Monte Carlo solution and found good agreement. With this knowledge, and an upper bound for the pure absorption error, it should be possible to determine the appropriateness of the present expansion, if not the magnitude of the actual error for arbitrary degrees of scattering.

Turning to the pure absorption problem, it was stated in the text that the temperature profile could be represented by an asymptotic expansion about N equal to zero. That is,

$$\theta(\tau, N) \sim \theta_o(\tau)+N\theta_1(\tau)+\ldots$$

with relationships between the various terms derived from the integral equation for the temperature profile (with α set equal to unity).

Assume that θ and its first k derivatives with respect to N are continuous for finite N; then, noting

$$\theta(\tau, N)-\theta(\tau, 0) = -\int_0^N \frac{\partial\theta(\tau, N-x)}{\partial x}\, dx ,$$

k integrations by parts yield

$$\theta(\tau, N)-\theta(\tau, 0) = \sum_{\ell=1}^{k} \frac{N^\ell}{\ell!} \frac{\partial \theta^\ell(\tau,x)}{\partial x^\ell}\bigg|_{x=0} + \frac{(-1)^{k+1}}{k!} \int_0^{\tau_o} x^k \frac{\partial^{k+1}\theta(\tau, N-x)}{\partial x^{k+1}} dx . \quad \text{(II-1)}$$

The remainder can be considered an error in a k-term approximation for θ. This means that if a suitable upper bound for the remainder is computed, an absolute upper bound for the error in the finite term approximation to the temperature profile can be found. It should be noted that if derivatives of all orders are continuous, and if the asymptotic series converges as $k \to \infty$, the expansion reduces to the familiar Taylor series for θ about $N = 0$. However, this approach requires assuming that $\theta(\tau, N)$ is an analytic function of N in the neighborhood of the origin.

Returning to the problem at hand, the error is estimated by computing an upper bound for the absolute value of the remainder. In the text, the first order term in the expansion for θ was retained so an estimate of the maximum magnitude of $\partial^2\theta / \partial N^2$ is required. Also, an estimate for $\partial\theta/\partial N$ is needed to compute the error in q, since only θ_o was retained in its approximation.

The remainder in (II-1) is given as a derivative of $\theta(\tau, N-x)$ integrated over the range $0 \le x \le N$. But the purpose of the approximations given in the text was to express θ as a linear function of N and to avoid the computational difficulties of solving for it for arbitrary N. Similarly, then, the remainder should be expressed as a simple function of N if it is to prove useful. Thus, an upper bound for $|\theta^{(k)}(\tau, N)|$ which is simple in form is sought. The method used requires several steps. The first is to find a form for $\max\left|\frac{\partial\theta(\tau,N)}{\partial N}\right|$ and finally $\max\left|\frac{\partial^2\theta(\tau,N)}{\partial N^2}\right|$.

Care must be taken in interpreting the meaning of the maxima of the functions used below. For convenience, the following notation is used.

$$|h(\tau, n)|$$

means the maximum of the absolute value of $h(\tau, n)$ for τ in the interval $[0, \tau_o]$ and n in $[0, N]$. Also,

$$|h(\tau, N)|$$

refers to τ in the interval $[0, \tau_o]$ for a particular value, N, of the second argument. Obviously, then,

$$|h(\tau, N)| \leq |h(\tau, n)| ,$$

since on the left hand side, the second argument is restricted to the single value N, while on the right hand side, the variable ranges from zero to N. Similarly,

$$\frac{\partial}{\partial N}|h(\tau, n)| \leq \left|\frac{\partial}{\partial n} h(\tau, n)\right| ,$$

since on the left hand side the derivative is computed only at N, while on the right hand side it is evaluated for n in the interval $[0, N]$ and then the maximum is taken. These properties will prove useful in the following pages.

From the text, θ satisfies

$$G(\theta(\tau, N)) = N[\tfrac{1}{2}G(S_1+S_2)+\tfrac{1}{2}\int_0^{\tau_o}\theta^4(t, N)G(E_3|t-\tau|)dt] \quad . \qquad \text{(II-2)}$$

For convenience, define an operator L, independent of N, by

$$L(\theta^4(\tau, N)) = \tfrac{1}{2}G(S_1(\tau, N)+S_2(\tau,N))+\tfrac{1}{2}\int_0^{\tau_o}\theta^4(t, N)G(E_3|t-\tau|)dt \,. \qquad \text{(II-3)}$$

With this notation, (II-2) can be written

$$G(\theta(\tau, N)) = NL(\theta^4(\tau, N)) \,. \qquad \text{(II-4)}$$

If this expression is differentiated with respect to N , and it is kept in mind that $\theta(0, N)$ and $\theta(\tau_o, N)$ are independent of N, it becomes

$$\frac{\partial\theta(\tau,N)}{\partial N} = L(\theta^4(\tau, N)) + N\frac{\partial}{\partial N}L(\theta^4(\tau, N)) \,. \qquad \text{(II-5)}$$

The absolute value with respect to τ_o and N of this equation satisfies

$$\left|\frac{\partial\theta(\tau, n)}{\partial n}\right| \le |L(\theta^4(\tau, n))| + N\left|\frac{\partial}{\partial n}L(\theta^4(\tau, n))\right| \,. \qquad \text{(II-6)}$$

The last term on the right hand side can be reduced further. Under quite general conditions,

$$|\theta(\tau, n)| \le 1 \qquad 0 \le \tau \le \tau_o \qquad \text{all } n \quad .$$

This is always true for opaque walls, but for very intense external radiative sources, enough radiant flux can be absorbed within the medium to cause the temperature to reach a maximum between the boundaries (i. e., some value greater than $\theta(\tau_o)$). Then the upper bound of unity would have to be replaced by θ_x, the external source temperature. In any event, the general approach given below would hold, but of course, the new bound for θ would appear in place of unity.

The last term in (II-6) is the derivative of L with respect to n . Since the operator consists, in part, of the source functions S_1 and S_2, only the portions of these terms which are integrals of θ^4, and hence dependent on N, remain. An upper bound can be computed by writing $|\partial L/\partial n|$ explicitly and separating the result into two portions:

$$|\frac{\partial}{\partial n} L(\theta^4(\tau, n))| =$$

$$\left|\tfrac{1}{2}\int_0^{\tau_o} 4\theta^3(t, n)\frac{\partial\theta(t, n)}{\partial n} \frac{\beta_1 G(E_4(\tau))}{1-\beta_1\beta_2 E_3^2(\tau_o)}[E_2(t)+\beta_2 E_3(\tau_o)E_2(\tau_o-t)]\right.$$

$$\left. + \frac{\beta_2 G(E_4(\tau_o-\tau))}{1-\beta_1\beta_2 E_3^2(\tau_o)}[E_2(\tau_o-t)+\beta_1 E_3(\tau_o)E_2(t)]+G(E_3|t-\tau|)\; dt\right|$$

$$\leq 4H(\tau_o)|\frac{\partial\theta(\tau, n)}{\partial n}| \quad , \qquad \text{(II-7)}$$

where H satisfies

$$H(\tau_o) = H_1(\tau_o) + H_2(\tau_o) , \qquad \text{(II-8)}$$

and

$$H_1(\tau_o) = \tfrac{1}{2}|G(E_4(\tau))|$$

$$\int_0^{\tau_o} \left|\frac{\beta_1[E_2(t)+\beta_2 E_3(\tau_o)E_2(\tau_o-t)]+\beta_2[E_2(\tau_o-t)+\beta_1 E_3(\tau_o)E_2(t)}{1-\beta_1\beta_2 E_3^2(\tau_o)}\right| dt$$

$$= |G(E_4(\tau))|[\frac{\beta_1+\beta_2+2\beta_1\beta_2 E_3(\tau_o)}{1-\beta_1\beta_2 E_3^2(\tau_o)}][\frac{1-2E_2(\tau_o)}{4}] \qquad \text{(II-9)}$$

$$H_2(\tau_o) = \tfrac{1}{2}\int_0^{\tau_o} \left|G(E_3|t-\tau|)\right| dt \; . \qquad \text{(II-10)}$$

In the definition of H, θ^3 has been replaced by unity. Now (II-6) can be written in terms of H :

$$|\frac{\partial\theta(\tau, n)}{\partial n}| \leq |L(\theta^4(\tau, n))| + 4NH(\tau_o)|\frac{\partial\theta(\tau, n)}{\partial n}| \quad .$$

Solving for $|\partial\theta/\partial n|$,

$$\left|\frac{\partial\theta(\tau,n)}{\partial n}\right| \le \frac{|L(\theta^4(\tau,n))|}{1-4H(\tau_o)N} \qquad N < \frac{1}{4H(\tau_o)} \quad . \qquad \text{(II-11)}$$

The limitation on N insures the right hand side of (II-11) is always finite and positive.

This procedure can be followed once again to determine an upper bound for $\left|\frac{\partial^2\theta}{\partial n^2}\right|$. If (II-5) is differentiated with respect to N ,

$$\frac{\partial^2\theta(\tau,N)}{\partial N^2} = 2\frac{\partial}{\partial N} L(\theta^4(\tau,N)) + N\frac{\partial^2}{\partial N^2} L(\theta^4(\tau,N)) \quad .$$

Again, taking the absolute value of both sides

$$\left|\frac{\partial^2\theta(\tau,n)}{\partial n^2}\right| \le 2\left|\frac{\partial}{\partial n} L(\theta^4(\tau,n))\right| + N\left|\frac{\partial^2}{\partial n^2} L(\theta^4(\tau,n))\right| \quad . \qquad \text{(II-12)}$$

The first term on the right hand side is evaluated in (II-7). The second term can be expressed in terms of H by noting that the second derivative of θ^4 with respect to n is

$$\frac{\partial^2\theta^4(\tau,n)}{\partial n^2} = 12\theta^2(\tau,n)\left(\frac{\partial\theta(\tau,n)}{\partial n}\right)^2 + 4\theta^3(\tau,n)\frac{\partial^2\theta(\tau,n)}{\partial n^2} \quad .$$

This is analogous to (II-7), so

$$\left|\frac{\partial^2}{\partial n^2} L(\theta^4(\tau,n))\right| \le 4H(\tau_o)\left[3\left|\frac{\partial\theta(\tau,n)}{\partial n}\right|^2 + \left|\frac{\partial^2\theta(\tau,n)}{\partial n^2}\right|\right] .$$

Thus, (II-12) satisfies

$$\left|\frac{\partial^2\theta(\tau,n)}{\partial n^2}\right| \le 8H(\tau_o)\left|\frac{\partial\theta(\tau,n)}{\partial n}\right|\left[1+\frac{3}{2}N\left|\frac{\partial\theta(\tau,n)}{\partial n}\right|\right] + 4H(\tau_o)N\left|\frac{\partial^2\theta(\tau,n)}{\partial n^2}\right|$$

and

$$\left|\frac{\partial^2\theta(\tau,n)}{\partial n^2}\right| \le \frac{8H(\tau_o)\left|\frac{\partial\theta(\tau,n)}{\partial n}\right|}{1-4H(\tau_o)N}\left[1+\frac{3}{2}N\left|\frac{\partial\theta(\tau,n)}{\partial n}\right|\right] \ , \ N < \frac{1}{4H(\tau_o)} . \qquad \text{(II-13)}$$

It remains to consider $|L(\theta^4)|$ in detail, since the first derivative of θ, and hence the second, are expressed in terms of this function. It is convenient to start with its definition given in (II-3). Differentiating with respect to N ,

$$\frac{\partial}{\partial N}|L(\theta^4(\tau, N))| \le |\frac{\partial}{\partial n} L(\theta^4(\tau, n))| \le 4H(\tau_o)|\frac{\partial\theta(\tau, n)}{\partial n}| \quad .$$

The last inequality follows from (II-7). But from (II-11),

$$|\frac{\partial\theta(\tau, n)}{\partial n}| \le \frac{|L(\theta^4(\tau, n)|}{1-4H(\tau_o)N} \quad ,$$

so that

$$\frac{\partial}{\partial N}|L(\theta^4(\tau, N))| \le \frac{4H(\tau_o)|L(\theta^4(\tau, n))|}{1-4H(\tau_o)N} \quad .$$

The left hand side can be replaced by

$$\frac{\partial}{\partial N}|L(\theta^4(\tau, n))| \le \frac{4H(\tau_o)|L(\theta^4(\tau, n))|}{1-4H(\tau_o)N} \quad . \qquad \text{(II-14)}$$

This holds, since if $|L(\theta^4(\tau, n))|$ is increasing with N ,

$$\frac{\partial}{\partial N}|L(\theta^4(\tau, n))| = \frac{\partial}{\partial N}|L(\theta^4(\tau, N))| \quad ,$$

while if it is constant or decreasing, the left hand side of (II-14) is zero and the inequality remains valid.

Equation (II-14) is solved by dividing by $|L(\theta^4)|$ and integrating. The constant of integration is removed by evaluating the function at N = 0. The result is

$$|L(\theta^4(\tau, n))| \le \frac{|L(\theta^4(\tau, 0))|}{1-4H(\tau_o)N} \quad . \qquad \text{(II-15)}$$

The component involving L on the right hand side is just $|\theta_1|$ as can be seen from the definition in (II-3) with θ replaced by θ_o. Thus, (II-15) can be written as

$$|L(\theta^4(\tau, n))| \le \frac{|\theta_1(\tau)|}{1-4H(\tau_o)N} \quad .$$

This can be used to place an upper bound on the derivatives of θ given in (II-11) and (II-13),

$$\left|\frac{\partial\theta(\tau,n)}{\partial n}\right| \leq \frac{|\theta_1(\tau)|}{(1-4H(\tau_o)N)^2} \tag{II-16}$$

and

$$\left|\frac{\partial^2\theta(\tau,n)}{\partial n^2}\right| \leq \frac{8H(\tau_o)|\theta_1(\tau)|}{(1-4H(\tau_o)N)^3}\left(1+\frac{3N|\theta_1(\tau)|}{1-4H(\tau_o)N}\right). \tag{II-17}$$

The last term on the right hand side of (II-17) is much less than one (for N in the range where the error bound is useful). This will be demonstrated by first neglecting the term and then later comparing its magnitude to unity. With this term ignored, (II-16) and (II-17) can be used to compute an upper bound for $|R_1|$ and $|R_2|$, the maximum possible error corresponding to the termination of the series for θ at θ_o and $\theta_o+N\theta_1$:

$$R_1(N) = \int_0^N \frac{\partial\theta(\tau,N-x)}{\partial x}\,dx\,,$$

$$|R_1(N)| \leq \int_0^N \frac{|\theta_1(\tau)|}{(1-4H(\tau_o)(N-x))^2}\,dx = \frac{|\theta_1(\tau)|}{1-4H(\tau_o)N}$$

$$N < \frac{1}{4H(\tau_o)}\,, \tag{II-18}$$

and

$$R_2(N) = \int_0^N x\,\frac{\partial^2\theta(\tau,N-x)}{\partial x^2}\,dx\,,$$

$$|R_2(N)| \leq \int_0^N \frac{4xH(\tau_o)|\theta_1(\tau)|}{(1-4H(\tau_o)(N-x))^3}\,dx = \frac{4H(\tau_o)N^2|\theta_1(\tau)|}{1-4H(\tau_o)N}$$

$$N < \frac{1}{4H(\tau_o)}\,. \tag{II-19}$$

For a moment, just consider (II-19). The error bound $|R_1|$ appears in the error analysis for q, and will be used later.

The criterion for retaining θ_1 in the expansion for θ is that $N\theta_1$ be greater than the remainder R_2. If $N|\theta_1|$ is used for the magnitude of $N\theta_1$, N must satisfy

$$R_2(N) \leq N|\theta_1(\tau)| ,$$

or, with R_2 replaced by its upper bound in (II-19),

$$\frac{4H(\tau_o)N^2|\theta_1(\tau)|}{1-4H(\tau_o)N} \leq N|\theta_1(\tau)| .$$

Then

$$\frac{N}{1-4H(\tau_o)N} \leq \frac{1}{4H(\tau_o)}$$

or

$$N \leq \frac{1}{8H(\tau_o)} .$$

Of course, for N close to $1/8H$, the error computed from (II-19) will be as large as $N\theta_1$ in the expansion for θ. In practice, the useful range for N will be further limited to values giving an acceptable error as calculated from (II-19). If conditions are such that a smaller error is expected in a particular problem, or a computation shows the actual remainder R_2 is significantly less than its upper bound, the range of N can be extended. The intention here is to place a bound on N for which the expansion is valid and indicate without a detailed calculation of $|\theta_1|$ whether the error is likely to be large.

With this result, the final form of the expansion for θ given in the text can be written

$$\theta(\tau, N) \sim \theta_o(\tau) + N\theta_1(\tau) \qquad N \leq \frac{1}{8H(\tau_o)}$$

with an error E_θ bounded by the maximum value of R_2 from (II-19),

$$E_\theta \leq \frac{4H(\tau_o)N^2|\theta_1(\tau)|}{1-4H(\tau_o)N} \quad . \qquad \text{(II-20)}$$

This form is dependent on H (which equals H_1+H_2). The two functions making up H are given in (II-9) and (II-10) and are independent of N and θ_o. However, H_1 is a function of τ_o, β_1, and β_2, while H_2 is a function of τ_o. From the definition of H_1, i.e.,

$$H_1(\tau_o) = \left[\frac{\beta_1+\beta_2+2\beta_1\beta_2E_3(\tau_o)}{1-\beta_1\beta_2E_3^2(\tau_o)}\right]\left[\frac{1-2E_3(\tau_o)}{4}\right]|G(E_4(\tau))| \quad ,$$

it can be seen that only the last term affords any computational difficulty. The value of this term is given in Table III. Also, from the definition of H_1, the maximum value will occur when β_1 and β_2 are a maximum, that is, when $\beta_1 = \beta_2 = 2$ (which corresponds to totally reflecting boundaries). In this case,

$$H_1(\tau_o) = \left[\frac{4+8E_3(\tau_o)}{1-4E_3^2(\tau_o)}\right]\left[\frac{1-2E_3(\tau_o)}{4}\right]|G(E_4(\tau))| = |G(E_4(\tau))| \; .$$

On the other hand, when β_1 and β_2 equal zero (no reflections), H_1 is identically zero.

As mentioned, the term H_2 is only dependent on τ_o,

$$H_2(\tau_o) = \tfrac{1}{2}\int_0^{\tau_o} |G(E_3|t-\tau|)|\,dt \quad .$$

It is also given in Table III. With the aid of the table, it becomes straightforward to compute H, and hence the range of usefulness for the expansion for θ. Further, once θ_1 is computed, the magnitude of the error can be found.

At the beginning of this analysis, it was stated that a complex term appearing in the expression for $|\partial^2\theta/\partial n^2|$ could be neglected. If this is to hold, the term must be much less than unity for $N \leq \frac{1}{8H(\tau_o)}$; that is,

$$\frac{N|\theta_1(\tau)|}{(1-4H(\tau_o)N)^2} \ll 1 \ ,$$

or, using the maximum value of N , $N = \frac{1}{8H(\tau_o)}$,

$$\frac{|\theta_1(\tau)|}{2H(\tau_o)} \ll 1 \ .$$

From the text, in particular Figures 3, 5, and 7, and Table III, it can be seen that this criterion is satisfied; the term generally is order of one tenth.

An error estimate, E_q , for the heat transfer rate similar to E_θ can be found. The result includes constants analogous to H_1 and H_2. However, the analysis differs slightly, since the expansion for q retains only θ_o in the expansion for θ.

In the text, q was approximated by

$$q \sim q_c + Nq_o \ . \qquad \text{(II-21)}$$

The exact solution can be written as

$$q = q_c + Nq_e(N) \ ,$$

where $q_e(N)$ is an integral over θ and a function of N. The error in (II-21) is just the difference $q_o - q_e$. It can be written explicitly in a form analogous to the right hand side of (II-7):

$$q_o - q_e(N) = \frac{1}{2}\int_0^{\tau_o} [\theta_o^4(t) - \theta^4(t,N)]\left[\left(\frac{G_1(E_4(\tau))}{1-\beta_1\beta_2 E_3(\tau_o)^2}\right)\left(\beta_1 E_2(t)[\beta_2 E_3(\tau_o)-1]\right.\right.$$

$$\left.\left. -\beta_2 E_2(\tau_o - t)[\beta_1 E_3(\tau_o)-1]\right) + G_1(E_3|t-\tau|)\right]dt\,. \qquad \text{(II-22)}$$

From (II-1),

$$\theta(\tau,N) = \theta_o(\tau) + \int_0^N \frac{\partial\theta(\tau,n)}{\partial n}\,dn = \theta_o(\tau) + R_1(N)\ .$$

The last step follows from the definition of R_1 given in (II-18). With this form,

$$|\theta^4(\tau,N) - \theta_o^4(\tau)| = |4\theta_o^3(\tau)R_1(N) + 6\theta_o^2(\tau)R_1^2(N) + \ldots|\ ,$$

or, using the upper bounds for R_1 given in (II-18), and replacing θ_o^3 by unity,

$$|\theta^4(\tau,N) - \theta_o^4(\tau)| \le \frac{4N|\theta_1(\tau)|}{1-4H(\tau_o)N} + 6\left(\frac{N|\theta_1(\tau)|}{1-4H(\tau_o)N}\right)^2 + \ldots\ . \qquad \text{(II-23)}$$

If N is sufficiently small, the higher order terms can be treated in a convenient fashion by choosing N such that

$$N \le \frac{1}{4(H(\tau_o) + 2|\theta_1(\tau)|)}\ . \qquad \text{(II-24)}$$

The ratio of the term retained in (II-23) to the first neglected is computed by substituting the form for N in (II-24)

$$\frac{6\left(\frac{N|\theta_1(\tau)|}{1-4H(\tau_o)N}\right)^2}{4\left(\frac{N|\theta_1(\tau)|}{1-4H(\tau_o)N}\right)} \le \frac{3}{16}\ .$$

Similarly, when the remaining two terms are included, the total contribution is less than 1/4, so that (II-23) satisfies

$$|\theta^4(\tau, N)-\theta_o^4(\tau)| \le (1+\frac{1}{4})\frac{4N|\theta_1(\tau)|}{1-4H(\tau_o)N} = \frac{5N|\theta_1(\tau)|}{1-4H(\tau_o)N} \,. \qquad \text{(II-25)}$$

With (II-25) substituted in (II-22), $E_q(N)$, the magnitude of the error is

$$E_q(N) = N|q_o-q_e(N)| \le \frac{5H_3(\tau_o)N^2|\theta_1(\tau)|}{1-4H(\tau_o)N} \qquad N < \frac{1}{4\left(H(\tau_o)+2|\theta_1(\tau)|\right)} \,, \qquad \text{(II-26)}$$

where H_3 is the maximum value of (II-22) with $\theta^4-\theta_o^4$ replaced by unity and the remaining integrand replaced by its absolute value:

$$H_3 = [E_4(0)-E_4(\tau_o)]\left[1-\frac{4E_3(\tau_o/2)}{1+2E_3(\tau_o)}\right]+[E_4(0)+E_4(\tau_o)]\left[1-\frac{6E_4(\tau_o/2)}{1+3E_4(\tau_o)}\right] \,.$$

Values of this function are presented in Table I. The function H is defined in (II-8) and $|\theta_1|$ is computed using (3. 20). This approximation is useful when the maximum error, $E_q(N)$, is less than Nq_o, the term retained. Approximating q_o in a form similar to that used for (q_o-q_e),

$$|q_o| \le H_3(\tau_o) \,.$$

The ratio of E_q to this approximation for Nq_o is

$$\frac{E_q(N)}{N|q_o|} \approx \frac{5N|\theta_1(\tau)|}{1-4H(\tau_o)N} \,. \qquad \text{(II-27)}$$

The maximum value of N for which the expansion is appropriate can be found by setting the left hand side of (II-27) equal to unity. The result is

$$N = \frac{1}{4(H(\tau_o)+\frac{5}{4}|\theta_1(\tau)|)} \,,$$

so the criterion in (II-24) is sufficient. Thus, the expansion

$$q \sim q_c + Nq_o \qquad N \leq \frac{1}{4(H(\tau_o)+2|\theta_1(\tau)|)} \qquad \text{(II-28)}$$

is appropriate for N satisfying the above condition, with the error bound E_q given in (II-26).

Under quite general conditions, E_q is less than one per cent of the product Nq_o, if $N\tau_o \leq 1$, and less than 10 per cent if $N\tau_o \leq 2$. However, for $\tau_o << 1$, the accuracy is greater, with an error of one per cent for $N\tau_o \leq 20$ and 10 per cent for $N\tau_o \leq 23$. These values show that E_q overestimates the errors for the cases that can be compared with references 14, 16, and 21.

In view of the magnitude of E_q compared to the actual error, its main use is to predict when the error is likely to be significant. Also, in circumstances where the problem differs significantly from the examples given in Figures 8 - 11, where comparisons of various results were made, an absolute upper limit can be placed on the error.

APPENDIX III. Reduction of the Functions of I, J_1, J_2, and K

The functions considered in this appendix are those appearing in the expressions for q_o and θ_1. From the discussion in the text, q_o, the first order expression for the radiative contribution to the heat transfer rate, is given by

$$q_o = \tfrac{1}{2} G_1(S_{10}(\tau)+S_{20}(\tau)) + \tfrac{1}{2} \int_0^{\tau_o} \theta_o^4(t) G_1(E_3|t-\tau|)dt ,$$

where

$$G_1(S_{10}(\tau)) = [E_4(\tau_o)-E_4(0)] \frac{\alpha_1+\beta_1[\alpha_2 E_3(\tau_o)+\int_0^{\tau_o}\theta_o^4 E_2(t)dt+\beta_2 E_3(\tau_o)\int_0^{\tau_o}\theta_o^4 E_2(\tau_o-t)dt]}{1-\beta_1\beta_2 E_3^2(\tau_o)}$$

$$G_2(S_{20}(\tau)) = [E_4(0)-E_4(\tau_o)] \frac{\alpha_2+\beta_2[\alpha_1 E_3(\tau_o)+\int_0^{\tau_o}\theta_o^4 E_2(\tau_o-t)dt+\beta_1 E_3(\tau_o)\int_0^{\tau_o}\theta_o^4 E_2(t)dt]}{1-\beta_1\beta_2 E_3^2(\tau_o)}$$

so that the functions involving integrals over θ_o are

$$J_1(\tau_o, \theta(0)) = \int_0^{\tau_o} \theta_o^4(t) E_2(t)dt ,$$

$$J_2(\tau_o, \theta(0)) = \int_0^{\tau_o} \theta_o^4(t) E_2(\tau_o-t)dt$$

$$K(\tau_o, \theta(0)) = \int_0^{\tau_o} \theta_o^4(t)[E_3(\tau_o-t)-E_3(t)]dt .$$

The functions J_1 and J_2 also appear in the expression for θ_1. In addition, θ_1 also contains I , which is dependent on τ;

$$I(\tau, \tau_o, \theta(0)) = \int_0^{\tau_o} \theta_o^4(t)(E_3|t-\tau|-E_3(t)-\frac{\tau}{\tau_o}[E_3(\tau_o-t)-E_3(t)])dt \quad .$$

All of the above functions are powers of t times an exponential integral. They can be integrated by parts to yield representations in terms of higher order exponential integrals times powers of τ.

Rather than proceed directly and integrate each function separately, it is convenient to define two supplemental functions L_n^k and M_n^k by

$$L_n^k(\tau) = \int_0^{\tau} t^k E_n(\tau-t)dt \quad ,$$

$$M_n^k(\tau) = \int_{\tau}^{\tau_o} t^k E_n(t-\tau)dt \quad .$$

These will prove immediately useful, since the integrals of the original functions can be represented as finite sums of them.

Consider L_n^k: integrating by parts,

$$L_n^k(\tau) = \int_0^{\tau} t^k E_n(\tau-t)dt$$

$$= t^k E_{n+1}(\tau-t)\Big|_0^{\tau} - k\int_0^{\tau} t^{k-1} E_{n+1}(\tau-t)dt$$

$$= t^k E_{n+1}(0)-k\int_0^{\tau} t^{k-1} E_{n+1}(\tau-t)dt \quad .$$

If this process is carried out k-1 more times,

$$L_n^k(\tau) = \sum_{\ell=1}^{k} \frac{(-)^{\ell}\tau^{k-\ell}k!}{(k-\ell)!} E_{n+1+\ell}(0)+(-)^k k! \int_0^{\tau} E_{n+k}(\tau-t)dt$$

$$= \sum_{\ell=0}^{k} \frac{(-)^{\ell}\tau^{k-\ell}k!}{(k-\ell!} E_{n+1+\ell}(0)+(-)^k k! E_{n+k+1}(\tau-t)\Big|_0^{\tau} \quad .$$

Note that from (I-6)

$$E_{n+1+\ell}(0) = \frac{1}{n+\ell} \quad ,$$

so that

$$L_n^k(\tau) = (-)^{k+1} k! E_{n+k+1}(\tau) + \sum_{\ell=0}^{k} \frac{(-)^{\ell} k! \, \tau^{k-\ell}}{(k-\ell)! \, (n+\ell)} \quad .$$

Similarly, M_n^k can be integrated by parts:

$$M_n^k(\tau) = \int_{\tau}^{\tau_o} t^k E_n(t-\tau)dt$$

$$= -t^k E_{n+1}(t-\tau)\Big|_{\tau}^{\tau_o} + k\int_{\tau}^{\tau_o} t^{k-1} E_{n+1}(t-\tau)dt$$

$$= -\tau_o^k E_{n+1}(\tau_o-\tau)+\tau^k E_{n+1}(0)+k\int_{\tau}^{\tau_o} t^{k-1} E_{n+1}(t-\tau)dt \quad .$$

As before, repeating the process k-1 times leads directly to

$$M_n^k(\tau) = \sum_{\ell=0}^{k-1} \frac{k!}{(k-\ell)!} [\tau^{k-\ell} E_{n+\ell+1}(0)-\tau_o^{k-\ell} E_{n+\ell+1}(\tau_o-\tau)]+k! \int_{\tau}^{\tau_o} E_{n+k}(t-\tau)dt$$

$$= \sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!} [\frac{\tau^{k-\ell}}{n+\ell} - \tau_o^{k-\ell} E_{n+\ell+1}(\tau_o-\tau)] \quad .$$

Care must be taken to note that in the last equality, the sum goes from $\ell = 0$ to $\ell = k$. Some ambiguity arises in the term with $\ell = k$ if $\tau = 0$. It must be interpreted by returning to the expression on the previous line and noting that for $\tau = 0$

$$M_n^k(0) = \frac{k!}{n+k} - \sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!} E_{n+\ell+1}(\tau_o) .$$

Now, return to the original functions and write for $\theta_o^4(t)$

$$\theta_o^4(t) = \sum_{k=0}^{4} \frac{c(k)t^k}{k!}$$

where

$$c(k) = \frac{4!}{(4-k)!} \theta_o^{4-k} \left[\frac{\theta(\tau_o)-\theta(0)}{\tau_o}\right]^k .$$

Then,

$$K(\tau_o, \theta(0)) = \int_0^{\tau_o} \theta_o^4(t)[E_3(\tau_o-t)-E_3(t)]dt$$

$$= \sum_{k=0}^{4} \frac{c(k)}{k!} \int_0^{\tau_o} t^k[E_3(\tau_o-t)-E_3(t)]dt$$

$$= \sum_{k=0}^{4} \frac{c(k)}{k!} [L_3^k(\tau_o)-M_3^k(0)]$$

$$= \sum_{k=0}^{4} c(k)\left[(-)^{k+1}E_{k+4}(\tau_o)-\frac{1}{k+3}+\sum_{\ell=0}^{k}\frac{(1+(-)^{\ell})\tau_o^{k-\ell}}{(k-\ell)!(\ell+3)}\right] .$$

Similarly,

$$J_1(\tau_o, \theta(0)) = \int_0^{\tau_o} \theta_o^4(t) E_2(t) dt$$

$$= \sum_0^4 \frac{c(k)}{k!} M_2^k(0)$$

$$= \sum_{k=0}^4 c(k) \left[\frac{1}{k+2} - \sum_{\ell=0}^{k} \frac{\tau_o^{k-\ell}}{(k-\ell)!} E_{\ell+3}(\tau_o) \right] ,$$

$$J_2(\tau_o, \theta(0)) = \int_0^{\tau_o} \theta_o^4(t) E_2(\tau_o - t) = \sum_{k=0}^4 \frac{c(k)}{k!} L_2^k(\tau_o)$$

$$= \sum_{k=0}^4 c(k) \left[(-)^{k+1} E_{k+3}(\tau) + \sum_{\ell=0}^{k} \frac{(-)^{\ell} \tau^{k-\ell}}{(k-\ell)!\,(\ell+2)} \right] ,$$

and finally,

$$I(\tau, \tau_o, \theta(0)) = \int_0^{\tau_o} \theta_o^4(t) (E|t-\tau| - E_3(t) - \frac{\tau}{\tau_o} [E_3(\tau_o - t) - E_3(t)]) dt$$

$$= \sum_{k=0}^4 \frac{c(k)}{k!} \Bigg(\int_0^{\tau} \theta_o^4(t) E_3(\tau - t) dt + \int_{\tau}^{\tau_o} \theta_o^4(t) E_3(t-\tau) dt - M_3^k(0)$$

$$- \frac{\tau}{\tau_o} [L_3^k(\tau_o) - M_3^k(0)] \Bigg)$$

$$= \sum_{k=0}^4 \frac{c(k)}{k!} \left(L_3^k(\tau) + M_3^k(\tau) - M_3^k(0) - \frac{\tau}{\tau_o} [L_3^k(\tau_o) - M_3^k(0)] \right).$$

The last expression has not been completely reduced because of the resulting algebraic complexity.

With the above expressions, I , J_1 , J_2 , and K can be tabulated as functions τ_o , $\theta(0)$, and τ in the case of I . This has been done for τ_o = 0.1 , 0.2 , 0.5 , 1.0 , 2.0 , 5.0 , and 10.0 , $\theta(0)$ = 0.1 , (0.1) , 0.9 , and τ = 0 , $(\tau_o/20)$, τ_o . The results are given in Tables II and III.

TABLE I. Values of B_o, E_3, $G_1(E_4)$, $|G(E_4)|$, H_2, and H_3.

τ_o	$B_o(\tau_o)$	$E_3(\tau_o)$	$G_1(E_4(\tau_o-\tau))$	$\lvert G(E_4(\tau))\rvert$	$H_2(\tau_o)$	$H_3(\tau_o)$
0.1	.9157	.4163	.0456	.0010	.0010	.0003
0.2	.8491	.3519	.0839	.0037	.0037	.0019
0.5	.7040	.2216	.1681	.0169	.0180	.0166
1.0	.5532	.1097	.2473	.0453	.0527	.0675
2.0	.3900	.0301	.3083	.0987	.132	.1817
5.0	(1)	.0009	.3326	.188	.315	.3109
10.0	(1)	.0000	.3333	.242	.429	.3321

(1) For $\tau_o > 3$, $B_o(\tau_o) = \frac{4/3}{\gamma+\tau_o}$, $\gamma = 1.4209$. The result is accurate to four figures.

TABLE II. Values of J_1, J_2, and K

θ_o	τ_o						
	0.1	0.2	0.5	1.0	2.0	5.0	10
				$J_1(\tau_o, \theta(0))$			
.10	.01688	.02780	.04334	.04544	.03084	.00644	.00102
.20	.01911	.03163	.05000	.05365	.03846	.01046	.00307
.30	.02198	.03661	.05888	.06507	.05001	.01834	.00848
.40	.02570	.04314	.07084	.08107	.06745	.03269	.02011
.50	.03054	.05170	.08691	.10335	.09328	.05692	.04178
.60	.03679	.06287	.10831	.13391	.13047	.09520	.07824
.70	.04482	.07730	.13647	.17507	.18249	.15246	.13518
.80	.05501	.09575	.17298	.22950	.25332	.23442	.21925
.90	.06782	.11903	.21963	.30015	.34745	.34755	.33801
				$J_2(\tau_o, \theta(0))$			
.10	.02054	.03883	.08501	.14338	.22159	.33340	.40106
.20	.02293	.04313	.09336	.15543	.23613	.34642	.41024
.30	.02591	.04845	.10352	.16972	.25276	.36050	.41982
.40	.02968	.05511	.11596	.18679	.27189	.37572	.42981
.50	.03445	.06349	.13130	.20728	.29398	.39224	.44026
.60	.04049	.07399	.15021	.23194	.31957	.41016	.45117
.70	.04808	.08713	.17346	.26159	.34927	.42966	.46257
.80	.05757	.10343	.20192	.29717	.38373	.45087	.47449
.90	.06931	.12352	.23655	.33971	.42367	.47396	.48696
				$K(\tau_o, \theta(0))$			
.10	.00117	.00410	.01892	.05092	.11047	.20607	.25925
.20	.00122	.00428	.01972	.05302	.11477	.21234	.26445
.30	.00125	.00441	.02032	.05460	.11796	.21676	.26767
.40	.00127	.00446	.02057	.05522	.11912	.21772	.26701
.50	.00125	.00440	.02025	.05434	.11709	.21313	.26000
.60	.00118	.00415	.01912	.05129	.11042	.20041	.24354
.70	.00104	.00367	.01688	.04529	.09744	.17650	.21393
.80	.00082	.00287	.01321	.03544	.07622	.13788	.16685
.90	.00048	.00168	.00772	.02072	.04455	.08054	.09738

TABLE III. Values of F_1 and I.

τ/τ_o	τ_o						
	0.1	0.2	0.5	1.0	2.0	5.0	10
				$F_1(\tau, \tau_o)$			
.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.05	.00021	.00076	.00380	.01148	.03018	.08416	.15142
.10	.00039	.00142	.00703	.02087	.05306	.13484	.21394
.15	.00055	.00199	.00975	.02845	.07015	.16469	.23733
.20	.00069	.00247	.01198	.03443	.08254	.18076	.24164
.25	.00080	.00287	.01376	.03897	.09101	.18743	.23622
.30	.00089	.00318	.01512	.04222	.09621	.18756	.22567
.35	.00096	.00341	.01607	.04429	.09864	.18307	.21237
.40	.00101	.00357	.01665	.04529	.09872	.17529	.19758
.45	.00104	.00365	.01687	.04532	.09678	.16514	.18196
.50	.00104	.00365	.01674	.04446	.09312	.15328	.16589
.55	.00102	.00359	.01630	.04277	.08796	.14016	.14955
.60	.00099	.00345	.01554	.04034	.08152	.12614	.13308
.65	.00093	.00324	.01450	.03722	.07397	.11144	.11652
.70	.00086	.00297	.01317	.03346	.06545	.09625	.09992
.75	.00076	.00263	.01157	.02912	.05609	.08087	.08329
.80	.00065	.00223	.00973	.02423	.04600	.06487	.06664
.85	.00052	.00176	.00763	.01883	.03527	.04897	.04999
.90	.00036	.00123	.00531	.01298	.02398	.03266	.03333
.95	.00019	.00065	.00276	.00669	.01221	.01647	.01666
1.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
				$I(\tau, \tau_o, \theta(0))$; $\theta(0) = -0.1$			
.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.05	.00003	.00008	.00016	-.00018	-.00221	-.00837	-.01228
.10	.00005	.00016	.00038	-.00017	-.00393	-.01614	-.02407
.15	.00008	.00025	.00065	.00006	-.00507	-.02290	-.03512
.20	.00011	.00034	.00097	.00051	-.00557	-.02864	-.04513
.25	.00014	.00044	.00136	.00120	-.00534	-.03276	-.05372
.30	.00018	.00055	.00180	.00214	-.00431	-.03531	-.06046
.35	.00021	.00066	.00229	.00333	-.00244	-.03568	-.06486
.40	.00024	.00078	.00283	.00476	.00030	-.03362	-.06637
.45	.00027	.00089	.00340	.00640	.00393	-.02885	-.06441
.50	.00030	.00101	.00399	.00822	.00837	-.02113	-.05835
.55	.00033	.00111	.00457	.01013	.01352	-.01032	-.04759
.60	.00035	.00120	.00512	.01206	.01916	-.00351	-.03158
.65	.00037	.00127	.00558	.01386	.02499	.02004	-.00992
.70	.00037	.00131	.00591	.01536	.03053	.03852	.01743
.75	.00037	.00130	.00604	.01635	.03515	.05756	.04977
.80	.00035	.00124	.00589	.01651	.03795	.07477	.08497
.85	.00031	.00110	.00535	.01549	.03776	.08630	.11794
.90	.00024	.00086	.00429	.01282	.03296	.08587	.13748
.95	.00014	.00051	.00257	.00790	.02139	.06318	.11867
1.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000

TABLE III (continued)

τ/τ_o	τ_o = 0.1	0.2	0.5	1.0	2.0	5.0	10
	$I(\tau, \tau_o, \theta(0))$; $\theta(0) = 0.2$						
.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.05	.00004	.00011	.00029	.00014	-.00162	-.00756	-.01142
.10	.00007	.00022	.00064	.00050	-.00265	-.01430	-.02206
.15	.00011	.00034	.00105	.00109	-.00302	-.01976	-.03164
.20	.00015	.00046	.00152	.00193	-.00266	-.02395	-.03982
.25	.00019	.00059	.00204	.00302	-.00153	-.02632	-.04626
.30	.00023	.00073	.00262	.00435	.00042	-.02695	-.05058
.35	.00027	.00087	.00323	.00592	.00322	-.02530	-.05235
.40	.00030	.00100	.00389	.00769	.00684	-.02121	-.05114
.45	.00034	.00114	.00455	.00962	.01125	-.01450	-.04647
.50	.00037	.00126	.00521	.01166	.01635	-.00504	-.03789
.55	.00040	.00137	.00584	.01373	.02195	.00717	-.02494
.60	.00042	.00146	.00639	.01570	.02782	.02193	-.00726
.65	.00044	.00152	.00682	.01744	.03359	.03879	.01533
.70	.00044	.00155	.00709	.01877	.03877	.05687	.04265
.75	.00043	.00152	.00710	.01943	.04268	.07467	.07382
.80	.00040	.00142	.00680	.01916	.04445	.08975	.10649
.85	.00035	.00124	.00607	.01758	.04290	.09827	.13549
.90	.00027	.00096	.00479	.01426	.03648	.09407	.14968
.95	.00015	.00056	.00282	.00863	.02314	.06718	.12458
1.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
	$I(\tau, \tau_o, \theta(0))$; $\theta(0) = 0.3$						
.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.05	.00005	.00015	.00049	.00064	-.00059	-.00577	-.00907
.10	.00010	.00030	.00103	.00152	-.00052	-.01059	-.01742
.15	.00015	.00047	.00163	.00264	.00025	-.01397	-.02455
.20	.00020	.00063	.00229	.00400	.00177	-.01598	-.03011
.25	.00024	.00080	.00299	.00559	.00408	-.01605	-.03377
.30	.00029	.00097	.00373	.00741	.00716	-.01437	-.03520
.35	.00034	.00114	.00450	.00941	.01102	-.01044	-.03407
.40	.00039	.00130	.00527	.01155	.01559	-.00422	-.03003
.45	.00043	.00145	.00604	.01379	.02079	.00439	-.02275
.50	.00046	.00158	.00676	.01604	.02647	.01541	-.01189
.55	.00049	.00170	.00741	.01820	.03241	.02871	.00283
.60	.00051	.00179	.00795	.02016	.03833	.04399	.02162
.65	.00052	.00184	.00833	.02176	.04383	.06069	.04449
.70	.00052	.00184	.00849	.02280	.04840	.07782	.07107
.75	.00050	.00178	.00836	.02305	.05135	.09382	.10031
.80	.00046	.00164	.00786	.02222	.05181	.10621	.12974
.85	.00039	.00142	.00690	.01997	.04865	.11122	.15412
.90	.00030	.00108	.00536	.01589	.04038	.10282	.16243
.95	.00017	.00062	.00311	.00945	.02507	.07141	.13068
1.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000

TABLE III (continued)

τ/τ_o	τ_o = 0.1	0.2	0.5	1.0	2.0	5.0	10
$I(\tau, \tau_o, \theta(0))$; $\theta(0) = 0.4$							
.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.05	.00007	.00021	.00077	.00140	.00108	-.00242	-.00422
.10	.00013	.00043	.00159	.00303	.00280	-.00411	-.00870
.15	.00020	.00064	.00246	.00488	.00518	-.00448	-.01231
.20	.00026	.00086	.00336	.00692	.00826	-.00358	-.01449
.25	.00032	.00107	.00429	.00915	.01201	-.00085	-.01486
.30	.00038	.00129	.00523	.01154	.01643	.00344	-.01313
.35	.00044	.00149	.00616	.01403	.02146	.00977	-.00903
.40	.00049	.00168	.00707	.01657	.02700	.01808	-.00229
.45	.00054	.00185	.00793	.01910	.03293	.02840	.00730
.50	.00058	.00199	.00871	.02151	.03907	.04064	.01998
.55	.00061	.00211	.00937	.02371	.04517	.05459	.03589
.60	.00063	.00218	.00986	.02556	.05091	.06986	.05510
.65	.00063	.00221	.01015	.02690	.05587	.08580	.07747
.70	.00062	.00218	.01016	.02754	.05953	.10135	.10251
.75	.00059	.00208	.00984	.02724	.06122	.11494	.12903
.80	.00053	.00190	.00910	.02573	.06007	.12406	.15449
.85	.00045	.00162	.00786	.02269	.05502	.12504	.17362
.90	.00034	.00122	.00601	.01773	.04466	.11204	.17558
.95	.00019	.00069	.00344	.01036	.02716	.07581	.13689
1.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
$I(\tau, \tau_o, \theta(0))$; $\theta(0) = 0.5$							
.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.05	.00009	.00030	.00118	.00253	.00366	.00322	.00440
.10	.00018	.00059	.00238	.00521	.00775	.00624	.00582
.15	.00027	.00088	.00359	.00802	.01232	.00999	.00689
.20	.00035	.00117	.00481	.01095	.01737	.01452	.00873
.25	.00043	.00144	.00602	.01396	.02289	.02047	.01191
.30	.00050	.00170	.00720	.01701	.02881	.02756	.01681
.35	.00057	.00194	.00832	.02004	.03507	.03623	.02368
.40	.00063	.00216	.00937	.02298	.04155	.04641	.03272
.45	.00068	.00235	.01032	.02576	.04810	.05805	.04408
.50	.00072	.00250	.01113	.02828	.05450	.07101	.05790
.55	.00075	.00261	.01176	.03042	.06049	.08501	.07422
.60	.00076	.00267	.01218	.03204	.06574	.09960	.09301
.65	.00076	.00267	.01232	.03299	.06984	.11408	.11402
.70	.00073	.00260	.01214	.03308	.07225	.12737	.13666
.75	.00069	.00245	.01157	.03208	.07233	.13788	.15966
.80	.00062	.00221	.01054	.02974	.06926	.14313	.18041
.85	.00052	.00185	.00897	.02576	.06202	.13960	.19372
.90	.00038	.00138	.00676	.01979	.04930	.12161	.18892
.95	.00021	.00077	.00381	.01138	.02941	.08032	.14308
1.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000

TABLE III (continued)

τ/τ_o	τ_o = 0.1	0.2	0.5	1.0	2.0	5.0	10
	$I(\tau, \tau_o, \theta(0))$; $\theta(0) = 0.6$						
.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.05	.00012	.00042	.00174	.00413	.00744	.01199	.01836
.10	.00024	.00082	.00345	.00824	.01484	.02174	.02818
.15	.00036	.00121	.00513	.01233	.02228	.03087	.03509
.20	.00046	.00157	.00675	.01636	.02979	.03975	.04137
.25	.00056	.00192	.00829	.02031	.03736	.04923	.04808
.30	.00065	.00224	.00974	.02411	.04494	.05910	.05582
.35	.00073	.00252	.01108	.02771	.05245	.06986	.06489
.40	.00080	.00277	.01227	.03104	.05976	.08145	.07552
.45	.00086	.00298	.01329	.03401	.06671	.09381	.08782
.50	.00090	.00313	.01411	.03653	.07309	.10678	.10186
.55	.00092	.00323	.01467	.03848	.07864	.12006	.11762
.60	.00093	.00326	.01496	.03973	.08304	.13317	.13501
.65	.00091	.00322	.01491	.04013	.08587	.14539	.15370
.70	.00087	.00310	.01447	.03949	.08664	.15564	.17303
.75	.00081	.00288	.01359	.03763	.08473	.16239	.19169
.80	.00072	.00256	.01221	.03429	.07937	.16318	.20708
.85	.00060	.00213	.01024	.02922	.06963	.15466	.21405
.90	.00044	.00157	.00762	.02208	.05430	.13135	.20220
.95	.00024	.00086	.00424	.01250	.03181	.08484	.14913
1.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
	$I(\tau, \tau_o, \theta(0))$; $\theta(0) = 0.7$						
.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.05	.00017	.00058	.00251	.00633	.01278	.02490	.03947
.10	.00033	.00112	.00490	.01236	.02463	.04386	.06071
.15	.00048	.00163	.00716	.01808	.03576	.05978	.07458
.20	.00061	.00210	.00927	.02347	.04626	.07369	.08542
.25	.00073	.00254	.01122	.02851	.05618	.08684	.09524
.30	.00084	.00292	.01299	.03316	.06551	.09923	.10504
.35	.00094	.00326	.01455	.03735	.07420	.11157	.11537
.40	.00101	.00354	.01588	.04102	.08214	.12386	.12651
.45	.00107	.00375	.01695	.04410	.08921	.13610	.13859
.50	.00111	.00390	.01773	.04649	.09520	.14815	.15167
.55	.00113	.00398	.01818	.04809	.09989	.15975	.16572
.60	.00113	.00397	.01827	.04878	.10297	.17043	.18058
.65	.00110	.00388	.01796	.04843	.10408	.17948	.19589
.70	.00104	.00369	.01720	.04687	.10274	.18588	.21099
.75	.00096	.00339	.01594	.04394	.09842	.18815	.22452
.80	.00084	.00298	.01413	.03942	.09040	.18388	.23392
.85	.00069	.00245	.01170	.03307	.07783	.16996	.23416
.90	.00050	.00178	.00859	.02463	.05962	.14108	.21508
.95	.00027	.00097	.00472	.01374	.03434	.08927	.15487
1.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000

TABLE III (continued)

τ/τ_o	τ_o						
	0.1	0.2	0.5	1.0	2.0	5.0	10
	$I(\tau, \tau_o, \theta(0))$; $\theta(0) = 0.8$						
.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.05	.00023	.00079	.00353	.00929	.02008	.04309	.06984
.10	.00044	.00151	.00680	.01780	.03777	.07426	.10606
.15	.00063	.00218	.00980	.02558	.05352	.09853	.12789
.20	.00080	.00278	.01252	.03263	.06759	.11804	.14303
.25	.00095	.00332	.01495	.03895	.08015	.13479	.15503
.30	.00108	.00378	.01707	.04451	.09127	.14920	.16565
.35	.00119	.00417	.01886	.04929	.10100	.16229	.17580
.40	.00128	.00448	.02032	.05323	.10928	.17426	.18595
.45	.00134	.00470	.02140	.05628	.11606	.18525	.19631
.50	.00138	.00484	.02210	.05837	.12119	.19523	.20698
.55	.00139	.00488	.02237	.05942	.12449	.20399	.21793
.60	.00137	.00482	.02220	.05934	.12571	.21114	.22900
.65	.00132	.00466	.02155	.05801	.12455	.21602	.23983
.70	.00124	.00438	.02038	.05530	.12061	.21767	.24973
.75	.00113	.00399	.01866	.05108	.11340	.21474	.25738
.80	.00098	.00347	.01633	.04516	.10230	.20485	.26028
.85	.00080	.00282	.01336	.03736	.08657	.18516	.25351
.90	.00057	.00203	.00969	.02743	.06522	.15054	.22718
.95	.00031	.00110	.00526	.01510	.03697	.09349	.16010
1.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
	$I(\tau, \tau_o, \theta(0))$; $\theta(0) = 0.9$						
.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000
.05	.00030	.00106	.00486	.01317	.02979	.06786	.11184
.10	.00058	.00202	.00925	.02487	.05498	.11480	.16716
.15	.00082	.00288	.01316	.03519	.07643	.14910	.19780
.20	.00104	.00364	.01661	.04422	.09469	.17468	.21650
.25	.00123	.00429	.01960	.05200	.11014	.19471	.22919
.30	.00138	.00485	.02211	.05855	.12304	.21028	.23879
.35	.00151	.00529	.02415	.06388	.13355	.22296	.24680
.40	.00160	.00563	.02571	.06798	.14178	.23326	.25399
.45	.00167	.00585	.02676	.07084	.14774	.24158	.26075
.50	.00170	.00596	.02731	.07243	.15141	.24805	.26726
.55	.00170	.00596	.02733	.07269	.15269	.25260	.27351
.60	.00166	.00583	.02681	.07157	.15142	.25495	.27939
.65	.00158	.00558	.02573	.06900	.14737	.25455	.28454
.70	.00147	.00520	.02406	.06488	.14025	.25051	.28830
.75	.00133	.00469	.02177	.05911	.12964	.24165	.28939
.80	.00114	.00404	.01885	.05156	.11504	.22561	.28537
.85	.00092	.00325	.01525	.04209	.09579	.19987	.27146
.90	.00065	.00232	.01094	.03050	.07105	.15946	.23808
.95	.00035	.00124	.00587	.01657	.03966	.09733	.16459
1.00	.00000	.00000	.00000	.00000	.00000	.00000	.00000