I. Idempotent Multipliers of H^1 on the Circle

II. A Mean Oscillation Inequality for Rearrangements

Thesis by

Ivo Klemes

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in Mathematics

California Institute of Technology

Pasadena, California

1985

Submitted 10 May 1985

Acknowledgements

I am most grateful to my advisor, Thomas Wolff, for his help, encouragement, and for keeping me on the right track. I also thank him for his constant willingness to spare time for discussions.

I would like to thank Brent Smith for providing key references, and for his course "Harmonic Number Theory," from which I gained useful inisghts for thinking about 1-norms of exponential sums. I acknowledge with great pleasure the inspiration for mathematics I gained from Arnold E. Ross of the Ohio State University, through his excellent program in number theory in the summer of 1978.

I also would like to thank Charles DePrima and W.A.J. Luxemburg for their encouragement and for their very enjoyable courses in complex and functional analysis.

Words cannot express how much I appreciate the expert typing of this thesis done by Frances Williams.

I. Idempotent Multipliers of H^1 on the Circle

II. A Mean Oscillation Inequality for Rearrangements

by Ivo Klemes

Department of Mathematics

California Institute of Technology

Abstract

$H^1(T)$ is the space of integrable functions f on the circle T such that the Fourier coefficients $\hat{f}(n)$ vanish for negative integers n. A multiplier is by definition a map m of H^1 to itself such that the Fourier transform diagonalizes m. Let $\hat{m}(n)$ denote the diagonal coefficients of m for nonnegative n. Then m is called idempotent if each coefficient is zero or one.

Theorem: If m is idempotent, then the set of n for which $\hat{m}(n) = 1$ is a finite Boolean combination of sets of nonnegative integers of the following three types: finite sets, arithmetic sequences, and lacunary sequences.

By definition, a sequence is lacunary if there is a real number $q > 1$ such that each term of the sequence is at least as large as q times the preceding term. The theorem implies a classification of the projections in H^1 which commute with translations, or, what is equivalent on the circle (but not on the line), of the closed, translation invariant subspaces which are complemented in H^1. In the course of the proof, a lower bound is obtained on the operator

norm of a multiplier whose coefficients are 0 or greater than 1 in magnitude. This bound implies that the number of nonzero coefficients in disjoint intervals of the same length is the same, up to some factor depending on the norm of m, provided that both intervals are shorter than their distance from 0.

Part II is unrelated to Part I. There it is proved that a general expression measuring the oscillation of a function on an interval is minimized by the decreasing rearrangement of the function. A special case of this expression is the BMO norm for functions of bounded mean oscillation.

TABLE OF CONTENTS

Page

Acknowledgements .... (i)

Abstract.. (ii)

Introduction 1

Notation 3

Part I 4

1. Statement and context of the result 4

2. Proof of theorem 1 8

3. Some consequences and remarks 24

References 31

Part II 33

References 39

INTRODUCTION

In a paper 1½ pages long written in 1933, Paley [9] proved the inequality

$$\left(\sum_{k=1}^{\infty} |a_{n_k}|^2\right)^{\frac{1}{2}} \leq c(q)\int_0^{2\pi} |f(\theta)|\, d\theta,$$

where f is an integrable complex valued function on the circle $T = \mathbb{R}/2\pi\mathbb{Z}$ having Fourier series

$$f \sim \sum_{n=0}^{\infty} a_n e^{in\theta}, \quad (f \in H^1)$$

and where $\{n_k\}_{k=1}^{\infty}$ is q-lacunary; $q > 1$ and

$$n_{k+1} \geq qn_k \quad k = 1, 2, \cdots .$$

Paley's inequality implies the existence of a square-integrable function g with Fourier series

$$g \sim \sum_{k=1}^{\infty} a_{n_k} e^{in_k\theta},$$

and also the boundedness of the projection $f \mapsto g$ from H^1 into H^2, for a fixed $\{n_k\}$.

In 1953, Helson [4] characterized the subsets $E \subset \mathbb{Z}$ having the property that

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta} \in L^1 \text{ implies } \sum_{n \in E} a_n e^{in\theta} \in L^1.$$

Helson's theorem is that this holds if and only if

$$E = \bigcup_{i=1}^{N} (\alpha_i \mathbb{Z} + \beta_i) \setminus F$$

for some $N \geq 0$, α_i, $\beta_i \in \mathbb{Z}$, and a finite set $F \subset \mathbb{Z}$. The question of what happens when H^1 replaces L^1 in Helson's theorem is the main subject of this thesis. The result (Theorem 1) is that E (a subset of the nonnegative integers) is obtained by union and complementation from finitely many sets of two kinds. These are the sets in Helson's theorem (restricted to nonnegative integers) and the lacunary sequences appearing in Paley's inequality. The proof also consists of two corresponding parts. Two slightly different proofs of the second part, concerning lacunarity, are given. One of these is a new lower bound on the norm of a multiplier having coefficients either 0 or greater than 1 in magnitude (lemma 5). Both versions depend on the inequality of McGehee, Pigno, Smith [8] (see Theorem 2), combined with some counting arguments, and an upper bound on the L^1 norm of certain sums of Fejér kernels (lemma 3). The latter lemma is probably well-known.

By the H^1-BMO duality, the property

$$\sum_{n=0}^{\infty} a_n e^{in\theta} \in H^1 \Rightarrow \sum_{n\in E} a_n e^{in\theta} \in H^1$$

is equivalent to the property

$$\sum_{n=0}^{\infty} a_n e^{in\theta} \in \text{BMO} \Rightarrow \sum_{n\in E} a_n e^{in\theta} \in \text{BMO}.$$

It turns out that the proof of Theorem 1 can be modified, using the full McGehee, Pigno, Smith result on one-sided interpolation, to prove the same conclusion about E assuming only the property

$$\sum_{n=0}^{\infty} a_n e^{in\theta} \in H^{\infty} \Rightarrow \sum_{n\in E} a_n e^{in\theta} \in \text{BMO}.$$

These modifications are considered in section 3.

Part II is completely disjoint from Part I. It is a paper about an inequality concerning the nonincreasing rearrangement f^* of a real function f on [0,1]. A special case of this result is that $\|f^*\|_{\text{BMO}} \leq \|f\|_{\text{BMO}}$.

Notation

$T = \mathbb{R}/2\pi\mathbb{Z}$, the circle

$H^1(T)$: the space of complex valued functions f on the circle T such that f is Lebesgue integrable and $\hat{f}(n) = 0$, $n < 0$, where

$$\hat{f}(n) = \int_0^{2\pi} e^{-in\theta} f(\theta) d\theta/2\pi.$$

H^1 is a Banach space with the norm

$$\|f\|_1 = \int_0^{2\pi} |f(\theta)| \, d\theta/2\pi .$$

$H^\infty(T)$: the space of complex valued functions f on the circle T such that ess sup $|f| = \|f\|_\infty < \infty$ and $\hat{f}(n) = 0$, $n < 0$. H^∞ is a Banach space with the norm $\|f\|_\infty$.

$M(T)$: the space of Borel measures on T

For $\mu \in M(T)$, $\|\mu\|$ denotes the total mass of μ and also equals:

$$\sup \{ \|f * \mu\|_1 : f \in L^1(T), \ \|f\|_1 \leq 1 \} .$$

$BMO(T)$: the space of complex valued integrable functions f on T such that the supremum over intervals $I \subset T$ of

$$\frac{1}{|I|} \int_I |f - f_I|$$

is finite, where $f_I = \frac{1}{|I|} \int_I f$.

$BMOA(T)$: the subspace of $BMO(T)$ of functions f such that $\hat{f}(n) = 0$, $n < 0$ (analytic BMO).

PART I

1. Statement and context of the result.

1.1 Definitions

A multiplier of $H^1(T)$ is a map $m : H^1 \to H^1$ such that for some sequence $\{c_n\}_{n=0}^{\infty}$ in $\mathbb{C}$ and for all $f \in H^1$, $n \geq 0$,

$$\widehat{m(f)}(n) = c_n \hat{f}(n). \tag{1}$$

Here $\hat{g}(n)$ is the nth Fourier coefficient of g:

$$\hat{g}(n) = \int_0^{2\pi} e^{-in\theta} g(\theta)d\theta/2\pi.$$

A multiplier of $L^1(T)$ is, similarly, a map $m : L^1 \to L^1$ such that (1) holds for some two-sided sequence $\{c_n\}_{n\in\mathbb{Z}}$ for all $f \in L^1$ and $n \in \mathbb{Z}$. I will use the notation

$$\hat{m}(n) = c_n$$

to indicate the correspondence between m and $\{c_n\}$. When the domain of m is H^1, the domain of $\hat{m}$ is $\{n \in \mathbb{Z}, n \geq 0\} = \mathbb{Z}_{\geq 0}$ by definition. Also, $\text{supp}\ \hat{m} = \{n : \hat{m}(n) \neq 0\}$ denotes the support of $\hat{m}$.

An idempotent multiplier is one such that $m \circ m = m$ (a projection). In this case it is clear that $\hat{m}(n) = 0$ or 1 for each n.

1.2 Background on idempotent measures

Let m_1, m_2 be idempotent multipliers of H^1 and e the identity map of H^1. Then

$$e,\ e-m_1,\ m_1 \circ m_2,\ m_1 + m_2 - m_1 \circ m_2$$

are idempotent multipliers having coefficient sequences

$$\hat{e} \equiv 1,\ \hat{e}-\hat{m}_1,\ \hat{m}_1\hat{m}_2,\ \hat{m}_1 + \hat{m}_2 - \hat{m}_1\hat{m}_2,$$

with supports

$$\mathbb{Z}_{\geq 0},\ \mathbb{Z}_{\geq 0} \setminus \text{supp } \hat{m}_1,\ \text{supp } \hat{m}_1 \cap \text{supp } \hat{m}_2,\ \text{supp } \hat{m}_1 \cup \text{supp } \hat{m}_2\ ,$$

respectively. The same holds for idempotent multipliers of L^1 except that $\mathbb{Z}$ replaces $\mathbb{Z}_{\geq 0}$. In both cases it is clear that the collection of all supports, $\{\text{supp } \hat{m},\ m$ is an idempotent multiplier$\}$ is closed under finite union, finite intersection, and complementation (this is called a Boolean ring).

In terms of multipliers, Helson's theorem reads as follows:

Helson's theorem [4]. Let $E \subset \mathbb{Z}$. There exists an idempotent multiplier $m : L^1 \to L^1$ with supp $\hat{m} = E$ if and only if E is in the ring of subsets of $\mathbb{Z}$ generated by the sets $\alpha\mathbb{Z} + \beta$, $\alpha, \beta \in \mathbb{Z}$.

The sets $\alpha\mathbb{Z} + \beta$ are just cosets of additive subgroups of $\mathbb{Z}$, and the ring they generate is called the coset ring of $\mathbb{Z}$. Yet another equivalent statement of the theorem is that a sequence $\hat{m}$ of 0's and 1's defines an L^1 multiplier if and only if for some integer $p \geq 1$ we have $\hat{m}(n+p) = \hat{m}(n)$ for all but finitely many $n \in \mathbb{Z}$.

When $\alpha \neq 0$, the idempotent m with supp $\hat{m} = \alpha\mathbb{Z} + \beta$ has the explicit form

$$\pi(f)(\theta) = \frac{1}{|\alpha|} \sum_{k=0}^{|\alpha|-1} e^{2\pi i\beta k/|\alpha|}\ f(\theta - 2\pi k/|\alpha|).$$

This is more often written as the convolution

$$m(f) = \mu * f\ ,\quad f \in L^1(T) \tag{2}$$

where μ is the discrete measure on T with masses $\frac{1}{|\alpha|}\, e^{2\pi i\beta k/|\alpha|}$ at the points $2\pi k/|\alpha|$ in T, $k = 0, \ldots, |\alpha| - 1$. In fact, it is well-known that (2) is a 1-1 correspondence between all multipliers

$m : L^1(T) \to L^1(T)$ and all Borel measures μ on T. If m and μ satisfy (2), then

$$\hat{m}(n) = \hat{\mu}(n), \ n \in \mathbb{Z}$$

where $\hat{\mu}$ is defined by

$$\hat{\mu}(n) = \int_T e^{-in\theta} d\mu(\theta).$$

If m is idempotent, then $\mu * \mu = \mu$, and such μ are called idempotent measures. Helson's theorem is usually stated as a characterization of idempotent measures in terms of supp $\hat{\mu}$. More complete information on idempotent measures, on arbitrary locally compact Abelian groups, may be found in Chapter 1 of [3].

1.3 Result on H^1

The theorem to be proved states:

Let $E \subset \mathbb{Z}_{\geq 0}$. There exists an idempotent multiplier $m : H^1 \to H^1$ with supp $\hat{m} = E$ if and only if E is in the ring of subsets of $\mathbb{Z}_{\geq 0}$ generated by lacunary sequences and the sets $(\alpha\mathbb{Z} + \beta) \cap \mathbb{Z}_{\geq 0}$, $\alpha, \beta \in \mathbb{Z}$.

Paley's inequality implies that, for each lacunary sequence E, there exists an idempotent multiplier $m : H^1 \to H^2 \subset H^1$ with supp $\hat{m} = E$. Also, if $m : L^1 \to L^1$ is an idempotent multiplier, then so is the restriction m_0 of m to H^1, and supp $\hat{m}_0 = (\text{supp } \hat{m}) \cap \mathbb{Z}_{\geq 0}$. Hence the "if" part of the theorem is a consequence of the "easy" part of Helson's theorem, and Paley's inequality. The main result of this thesis is the "only if" part:

Theorem 1: Let $m : H^1 \to H^1$ be an idempotent multiplier. Then supp $\hat{m}$ is in the ring of subsets of $\mathbb{Z}_{\geq 0}$ generated by the arithmetic sequences, finite sets, and lacunary sequences.

The proof will use the fact that a multiplier is necessarily a bounded linear transformation. This general property of multipliers of a commutative Banach algebra may be found in [7]. It is a direct consequence of the closed graph theorem and the uniqueness of Fourier coefficients.

Definition $\|m\| = \sup\{\|m(f)\|_1 : \|f\|_1 = \int_0^{2\pi} |f(\theta)|\, d\theta/2\pi \leq 1,\ f \in H^1\}$

Before continuing, I will give an excuse for use of the multiplier terminology. Helson [5] actually later proved a stronger version of his theorem as follows (the semi idempotent theorem):

Let $\mu \in M(T)$. If $\hat{\mu}(n) = 0$ or 1 for all $n \geq 0$, then $(\text{supp}\ \hat{\mu}) \cap \mathbb{Z}_{\geq 0}$ differs by a finite set from a periodic sequence in $\mathbb{Z}_{\geq 0}$.

This implies that all idempotents $m : H^1 \to H^1$ of the form $m(f) = \mu * f$, $\mu \in M(T)$ are also of the form $m(f) = \nu * f$ where $\nu \in M(T)$ and $\nu * \nu = \nu$. As a corollary, when supp $\hat{m}$ is lacunary, we have an object which exists only as a multiplier and cannot be a measure.

I would finally like to isolate two main ingredients of the proof of Theorem 1 (for the proof see the next section). Helson [5] considered the weak* limit points of the set of measures $\{e^{-in\theta} d\mu(\theta) : n \geq 0\}$ to get useful information about μ. The same idea occurs in [3], where a proof of Cohen's idempotent theorem is presented. (The generalization of Helson's to locally compact abelian groups). I use a variation of this; considering weak* limit points of $\{e^{-in\theta} m(e^{in\theta} K_n(\theta)) : n \geq 0\}$ where m is an idempotent H^1 multiplier and $K_n(\theta)$ is Fejér's kernel.

Next, the argument depends on:

Theorem 2 (McGehee, Pigno, Smith [8]):

$$\int_0^{2\pi} \left| \sum_{k=1}^{N} a_k e^{in_k\theta} \right| d\theta \geq c \sum_{k=1}^{N} \frac{|a_k|}{k}$$

where $n_1 < \cdots < n_N$ are integers, N is a natural number, $\{a_k\} \subset \mathbb{C}$, and $c > 0$ is an absolute constant.

This result resolved the Littlewood conjecture

$$\int_0^{2\pi} \left| \sum_{k=1}^{N} e^{in_k\theta} \right| d\theta \geq c \log N.$$

Historically, Cohen's theorem is also related to Littlewood's conjecture, through the fact that Cohen [1] started by obtaining the lower bound $c(\log N/\log\log N)^{1/8}$ and then used the method of proof in his proof of the idempotent theorem. The McGehee, Pigno, Smith inequality will give certain lower bounds on $\|m\|$ when $|\hat{m}(n)| \geq 1$ or $\hat{m}(n) = 0$ on an interval of integers n (lemmas 2,5).

2. Proof of Theorem 1.

2.1

Lemma 1. For each idempotent multiplier $m : H^1 \to H^1$ there is an idempotent measure $\mu \in M(T)$ such that the multiplier of H^1 defined by

$$m_0(f) = m(f) - \mu * f, \quad f \in H^1$$

satisfies the condition:

For all integers $x \geq 0$ there is an integer $g \geq 0$ such that $[g, g + x] \cap \operatorname{supp} \hat{m}_0 = \emptyset$. Briefly, supp $\hat{m}_0$ has arbitrarily large gaps. (3)

m_0 may not be idempotent, but satisfies $\hat{m}_0(n) \in \{-1, 0, 1\}$ for all n. Also note that supp $\hat{\mu}$ is in the coset ring (by Helson's theorem) and that

$$\text{supp } \hat{m} = (\text{supp } \hat{m}_0) \,\Delta\, (\mathbb{Z}_{\geq 0} \cap \text{supp } \hat{\mu}).$$

Hence, if Theorem 1 is to be true, it must be that supp $\hat{m}_0$ is either finite or a finite union of lacunary sequences (and the theorem would also follow if this were so). Proving the latter will be the next step, but first the proof of the lemma:

Proof of lemma 1:

For each $n \geq 0$ let K_n denote the Fejér kernel

$$K_n(\theta) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ij\theta}, \quad \theta \in T.$$

Recall that $K_n \geq 0$ and

$$\| K_n \|_1 = \int_0^{2\pi} K_n(\theta) d\theta/2\pi = 1$$

for all n. Fix an idempotent multiplier $m: H^1 \to H^1$. Since the function $e^{in\theta}K_n(\theta)$ is in H^1, we may define functions $g_n(\theta)$ by

$$g_n(\theta) = e^{-in\theta} m(e^{in\theta} K_n(\theta)) \quad n = 0, 1, \ldots$$

Then $\|g_n\|_1 \leq \|m\| \, \|K_n\|_1 = \|m\|$ for all n; hence the sequence $\{g_n(\theta)d\theta/2\pi\}$ has a weak* limit point ν in M(T). This implies that, for some increasing sequence $\{n_k\}_{k=1}^{\infty}$ and for all $\ell \in \mathbb{Z}$,

$$\lim_{k\to\infty} \hat{g}_{n_k}(\ell) = \hat{\nu}(\ell).$$

Note that for $|\ell| \leq n$ we have

$$\hat{g}_n(\ell) = \hat{K}_n(\ell)\hat{m}(n+\ell) = \left(1 - \frac{|\ell|}{n+1}\right)\hat{m}(n+\ell).$$

Now for fixed $\ell \in \mathbb{Z}$ we eventually have $|\ell| \leq n_k$ so that

$$\lim_{k\to\infty} \hat{g}_{n_k}(\ell) = \lim_{k\to\infty} \left(1 - \frac{|\ell|}{n_k+1}\right)\hat{m}(n_k+\ell) = \lim_{k\to\infty} \hat{m}(n_k+\ell).$$

Since $\hat{m}(n) \in \{0, 1\}$, this limit is 0 or 1; hence ν is idempotent. By Helson's theorem there exist $p \geq 1$ and $t \geq 0$ such that

$$\hat{\nu}(\ell+p) = \hat{\nu}(\ell), \quad |\ell| \geq t.$$

Consider the remainders of $\{n_k\}$ modulo p. There must be some r, $0 \leq r \leq p - 1$ such that $n_k \equiv r \bmod p$ for infinitely many n_k. Letting

$$d\mu(\theta) = e^{ir\theta}d\nu(\theta)$$

satisfies the lemma, as will be verified:

Clearly $\hat{\mu}(n) = \hat{\nu}(n-r)$ and μ is idempotent. Let $x \geq 0$ be given. For fixed ℓ, $\hat{\nu}(\ell) = \hat{m}(n_k+\ell)$ eventually, and thus for sufficiently large k we have

$$\hat{\nu}(\ell) = \hat{m}(n_k+\ell), \quad \ell = t, t+1, \ldots, t+x. \tag{4}$$

By the definition of r, there is also some $n_k \equiv r \bmod p$, $n_k \geq r$ such that (4) holds. Then

$$\hat{\nu}(\ell) = \hat{\nu}(\ell+n_k-r) = \hat{\mu}(n_k+\ell), \quad \ell = t, t+1, \ldots, t+x;$$

hence $\hat{m}_0(n) = \hat{m}(n) - \hat{\mu}(n) = 0$ for all $n \in [n_k + t, n_k + t + x]$, so we can take $g = n_k + t$.

2.2 Lacunarity and Fejér's Kernel

It is easy to see that a set E of nonnegative integers is either finite or a finite union of lacunary sequences if and only if for some $c > 0$ and $0 < \alpha < \beta$ we have

$$|[\alpha n, \beta n] \cap E| \leq c \qquad n = 0, 1, 2, \ldots$$

(Here $|A|$ is the cardinality of the set A).

Rudin [12] used the following argument to show that supp $\hat{m}$ must be of this form if $m : H^1 \to H^2$ is an idempotent multiplier. First, by the closed graph theorem

$$B \equiv \sup\{\|m(f)\|_2 : \|f\|_1 \leq 1, f \in H^1\} < \infty.$$

Next, considering $f(\theta) = e^{i3n\theta}K_{3n}(\theta)$ where K is Fejér's kernel, we get by Bessel's inequality that

$$\begin{aligned} B^2 \geq \|m(f)\|_2^2 &\geq \sum_{j=-n}^{n} |m(3n+j)\hat{f}(3n+j)|^2 \\ &= \sum_{j=-n}^{n} \left| \hat{m}(3n+j)\left(1 - \frac{j}{3n+1}\right)\right|^2 \\ &\geq \frac{4}{9} \sum_{j=-n}^{n} \hat{m}(3n+j) \\ &= \frac{4}{9} |[2n,4n] \cap \text{supp } \hat{m}| . \end{aligned}$$

Similarly, for H^1 multipliers, and using Theorem 2 instead of Bessel's inequality, we get the following lemma:

<u>Lemma 2.</u> Let $m : H^1 \to H^1$ be a multiplier such that

$$|\hat{m}(n)| \geq 1 \text{ for all } n \in \text{supp } \hat{m}.$$

Let $a,\ y \in \mathbb{Z},\ a \geq y \geq 1$ and define

$$A = |\,[a,\ a+y) \cap \operatorname{supp} \hat{m}\,|$$

$$B = |\,[a+y,\ a+2y) \cap \operatorname{supp} \hat{m}\,|$$

then

$$\| m \| \geq \frac{c}{2}\ \left| \log \left(\frac{1+A}{1+B}\right) \right|$$

where c is the absolute constant in Theorem 2.

Proof: Define $V \in H^1$ by

$$V(\theta) = (e^{ia\theta} + e^{i(a+y)\theta})\ K_{y-1}(\theta).$$

Then $\| V \|_1 \leq 2$ and

$$\hat{V}(j) = 1, \qquad a \leq j \leq a+y$$

$$\hat{V}(j) = 0, \qquad j \geq a + 2y.$$

Therefore

$$|\widehat{m(V)}(j)| = |\hat{m}(j)|\ |\hat{V}(j)| = |\hat{m}(j)| \geq 1, \quad j \in [a,\ a+y] \cap \operatorname{supp} \hat{m}$$

$$\operatorname{supp} \widehat{m(V)} \subset [0,\ a+2y) \cap \operatorname{supp} \hat{m}\ .$$

Now estimate $\| m(V) \|_1$ using Theorem 2, but enumerating supp $\widehat{m(V)}$ from right to left, say

$$\operatorname{supp} \widehat{m(V)} = \{n_1 > n_2 > \cdots > n_N\}.$$

By definition, $n_{B+1}, n_{B+2}, \ldots, n_{B+A} \in [a,\ a+y) \cap \operatorname{supp} \hat{m}$;hence

$$\| m(V) \|_1 \geq c \sum_{k=1}^{N} |\widehat{m(V)}(n_k)|/k$$

$$\geq c \sum_{k=B+1}^{B+A} \frac{1}{k} \geq c\ \log\left(\frac{1+B+A}{1+B}\right) \geq c\ \log\left(\frac{1+A}{1+B}\right).$$

This gives

$$\|m\| \geq \|m(V)\|_1 / \|V\|_1 > \frac{c}{2} \log\left(\frac{1+A}{1+B}\right).$$

Similarly, considering

$$W(\theta) = (e^{i(a+y)\theta} + e^{i(a+2y)\theta})K_{y-1}(\theta)$$

and numbering supp $\widehat{m(W)}$ from left to right gives

$$\| m \| \geq \frac{c}{2} \log\left(\frac{1+B}{1+A}\right).$$

The method of lemma 2 gives more general estimates if we use more general combinations of Fejér kernels, but only if we estimate the norm of these combinations more carefully, as in the next lemma.

<u>Lemma 3.</u> Let $y > 0$, $a_1 < a_2 < \cdots < a_N$ be integers satisfying

$$a_{k+1} - a_k \geq y + 1, \quad k = 1, 2, \ldots, N - 1.$$

Let $c_1, c_2, \ldots, c_N \in \mathbb{C}$; then

$$\int_0^{2\pi} \left| \sum_{k=1}^{N} c_k e^{ia_k\theta} K_y(\theta) \right| d\theta/2\pi \leq \left(\sum_{k=1}^{N} |c_k|^2 \right)^{\frac{1}{2}}.$$

<u>Proof</u>: Since $K_y \geq 0$, the Cauchy-Schwartz inequality gives

$$\left(\int_0^{2\pi} \left| \sum_{k=1}^{N} c_k e^{ia_k\theta} K_y(\theta) \right| d\theta/2\pi \right)^2 = \left(\int_0^{2\pi} \left| \sum_{k=1}^{N} c_k e^{ia_k\theta} \right| \sqrt{K_y(\theta)} \sqrt{K_y(\theta)}\, d\theta/2\pi \right)^2$$

$$\leq \int_0^{2\pi} \left(\sum_{k=1}^{N} c_k e^{ia_k\theta} \right) \left(\sum_{\ell=1}^{N} \bar{c}_\ell e^{-ia_\ell\theta} \right) K_y(\theta) d\theta/2\pi.$$

Since

$$K_y(\theta) = \sum_{j=-y}^{y} \left(1 - \frac{|j|}{y+1}\right) e^{ij\theta},$$

and from $|j| \leq y$, $a_{k+1} - a_k \geq y + 1$ follows

$$a_k - a_\ell + j = 0 \Leftrightarrow k = \ell, j = 0,$$

we see that the last integral equals $\sum_{k=1}^{N} |c_k|^2$.

Only the special case $c_1 = c_2 = \ldots = c_N = 1$ will be needed. Some discussion concerning the significance of the general case is given in section 3.

2.3 First proof of Theorem 1

Let m_0 be the multiplier in the conclusion of lemma 1. It remains to show that supp $\hat{m}_0$ is finite or is a finite union of lacunary sequences. It suffices to show

$$\sup_{y \in \mathbb{N}} |[3y, 6y) \cap \text{supp } \hat{m}_0| < \infty$$

under the assumptions (3) and $|\hat{m}_0(n)| \geq 1$, $n \in$ supp $\hat{m}_0$. By lemma 2, there is a constant $\rho > 1$ depending only on $\|m_0\|$ such that

$$\frac{1}{\rho} B \leq A \leq \rho B \text{ whenever } \max(A,B) \geq \rho \tag{5}$$

where, as in lemma 2,

$$A = |[a, a + x) \cap \text{supp } \hat{m}_0|,$$

$$B = |[a + x, a + 2x) \cap \text{supp } \hat{m}_0|,$$

$$a \geq x \geq 1, \ a, x \in \mathbb{Z}.$$

The constant ρ may be chosen from $\mathbb{Z}$ for convenience.

Now suppose

$$3\rho \leq |[3y, 6y) \cap \operatorname{supp} \hat{m}_0| \equiv s \text{ for some } y \in \mathbb{N}.$$

Define $N \geq 1$ by

$$3\rho^N \leq s < 3\rho^{N+1}.$$

I claim there is a sequence of integers $3y \leq x_1 < x_2 < \cdots < x_N$ satisfying

(i) $x_{k+1} - x_k \geq 3y$, $k = 1, 2, \ldots, N-1$,

(ii) $|[x_k, x_k + 3y) \cap \operatorname{supp} \hat{m}_0| = 3\rho^{N-k+1}$, $k = 1, 2, \ldots, N$.

The claim follows easily from the gap condition (3) and the uniformity condition (5): There exists $g > 3y$ such that $|[g, g + 3y) \cap \operatorname{supp} \hat{m}_0| = 0$. But $s \geq 3\rho^N$, so there exists x_1, $3y \leq x_1 < g$, such that

$$|[x_1, x_1 + 3y) \cap \operatorname{supp} \hat{m}_0| = 3\rho^N.$$

If $N \geq 2$, continue by noting that for

$$A = |[x_1, x_1 + 3y) \cap \operatorname{supp} \hat{m}_0| = 3\rho^N,$$

$$B = |[x_1 + 3y, x_1 + 6y) \cap \operatorname{supp} \hat{m}_0|,$$

condition (5) implies $B \geq 3\rho^{N-1}$. Again some $g > x_1 + 3y$ such that $|[g, g + 3y) \cap \operatorname{supp} \hat{m}_0| = 0$ is available; hence

$$|[x_2, x_2 + 3y) \cap \operatorname{supp} \hat{m}_0| = 3\rho^{N-1}$$

for some $x_2 \geq x_1 + 3y$. Continuing this way gives the required sequence.

By property (ii) and the uniformity (5) it follows that

$$|[x_k + y, x_k + 2y) \cap \operatorname{supp} \hat{m}_0| \geq \rho^{N-k} \tag{6}$$

for each k. Finally, define $f \in H^1$ by

$$f(\theta) = \sum_{k=1}^{N} \left(e^{i(x_k+y)\theta} + e^{i(x_k+2y)\theta}\right)K_{y-1}(\theta) .$$

By lemma 3, $\|f\|_1 \leq \sqrt{2N}$. Now estimate $\|m_0(f)\|_1$ as in the proof of lemma 2 by enumerating supp $\widehat{m_0(f)}$ backwards, say

$$\begin{aligned} \text{supp } \widehat{m_0(f)} &= \{n_T < n_{T-1} < \cdots < n_2 < n_1\} \\ &= \bigcup_{k=1}^{N} (x_k, x_k + 3y) \cap \text{supp } \hat{m}_0. \end{aligned}$$

Note that for $n_\ell \in (x_k, x_k + 3y)$ we have

$$\ell \leq 3\rho + 3\rho^2 + \cdots + 3\rho^{N-k+1} \leq 3\rho^{N-k+2} . \tag{7}$$

Also note that $\hat{f} = 1$ on $[x_k + y, x_k + 2y]$. Thus by Theorem 2:

$$\begin{aligned} \|m_0(f)\|_1 &\geq c \sum_{\ell=1}^{T} |\widehat{m_0(f)}(n_\ell)| / \ell \\ &\geq c \sum_{k=1}^{N} \sum_{n_\ell \in [x_k+y, x_k+2y)} |\hat{m}_0(n_\ell)| / \ell \\ &\geq c \sum_{k=1}^{N} \rho^{N-k}/3\rho^{N-k+2} \qquad \text{by (6) and (7),} \\ &= c\, N/3\rho^2 . \end{aligned}$$

Hence $\|m_0\| \geq \|m_0(f)\|_1 / \|f\|_1 \geq c\sqrt{N}/(3\sqrt{2}\rho^2)$. But $3\rho^{N+1} > s$, so s is bounded by constants not depending on y, as was to be shown.

This concludes the first proof of Theorem 1. The second proof will give a better estimate on the number $s = |[3y, 6y) \cap \text{supp } \hat{m}_0|$ above. The argument above resulted in

$$s \leq 3\rho^{N+1} \leq 3\rho^{\gamma\rho^4\|m_0\|^2}$$

where γ is some absolute positive constant. If ρ were independent of $\|m_0\|$, this estimate would be

$$s \leq e^{a\|m_0\|^2} \tag{8}$$

for some constant a. But, in fact, lemma 2 only gives $\rho \leq e^{c\|m_0\|}$. Nevertheless, (8) is true. The proof involves a more careful counting argument which yields a sequence analogous to $\{x_k\}$ above, without the use of lemma 2, as will be seen below.

2.4 Second proof of Theorem 1.

Lemma 4. Let $E \subset \mathbb{Z}$, $a, b, y \in \mathbb{Z}$ and

$$y > 0, \quad b \geq a + y.$$

Define
$$A = |[a, a + y) \cap E|,$$
$$B = |[b, b + y) \cap E|.$$

Suppose that $A > B$. Then there is a finite sequence of integers $a - y \leq x_N < x_{N-1} < \cdots < x_2 < x_1 \leq b - y$ with the following properties:

(i) $x_k - x_{k+1} \geq y, \quad k = 1, 2, \ldots, N-1$

(ii) Define $F = \bigcup_{k=1}^{N} [x_k - y, x_k + 2y)$.

Define the function M on $\mathbb{Z}$ by

$$M(n) = |F \cap E \cap [n, \infty)| .$$

Then M satisfies

1) $M(x_1 + y) \leq 2B$, $M(x_1) - M(x_1 + y) > B$

$$\left.\begin{array}{l} \text{2) } M(x_{k+1} + y) - M(x_k) \leq 2M(x_k) \\ \text{3) } M(x_{k+1}) - M(x_{k+1} + y) > M(x_k) \end{array}\right\} \quad k = 1, 2, \ldots, N-1$$

4) $M(x_N) \geq A/2$, $M(x_{N-1}) < A/2$.

Remark: Let $\tilde{F} = \bigcup_{k=1}^{N} [x_k, x_k + y) \cap E$. Condition (ii) is set up to imply that

$$\sum_{n \in \tilde{F}} \frac{1}{M(n)} > c_1 \log\left(\frac{1+A}{1+B}\right) + c_2$$

and also
$$N \leq c_3 \log\left(\frac{1+A}{1+B}\right) + c_4,$$

for some absolute constants $c_1 > 0$, $c_3 > 0$, c_2, c_4. This will be proved later, in the course of lemma 5.

Proof of lemma 4: Let x_1 be the largest integer in $(-\infty, b - y]$ satisfying $|[x_1, x_1 + y) \cap E| > B$. Such an integer exists because $a \in (-\infty, -b - y]$ and $|[a, a + y) \cap E| = A > B$. This also implies $a \leq x_1 \leq b - y$. Define inductively a finite or infinite sequence $x_1 > x_2 > \ldots$ as follows:

Suppose $x_1 > \cdots > x_k$, $k \geq 1$, have been defined. Let x_{k+1} be the largest integer in $(-\infty, x_k - y]$ satisfying:

$$|[x_{k+1}, x_{k+1} + y) \cap E| > \left|\left([x_k, x_k + 2y) \cup \bigcup_{\ell=1}^{k-1} [x_\ell - y, x_\ell + 2y)\right) \cap E\right|, \tag{9}$$

if such an integer exists; if not, then stop.

The sequence having been defined, let N be the least index such that

$$\left|\left([x_N, x_N + 2y) \cup \bigcup_{\ell=1}^{N-1} [x_\ell - y, x_\ell + 2y)\right) \cap E\right| \geq A/2.$$

We now check that N is well-defined and that (i) and (ii) hold. Let K be the largest index such that x_K exists and $x_K \geq a - y$. K exists since $x_1 \geq a > a - y$ and since there are finitely many x_k in $[a - y, x_1]$. We claim that

$$|([x_K, x_K + 2y) \cup \bigcup_{\ell=1}^{K-1} [x_\ell - y, x_\ell + 2y)) \cap E| \geq A/2 . \qquad (10)$$

There are three cases: 1) $a + y < x_K$. Then by definition of x_K, condition (9) fails for each x_{K+1} in $[a - y, x_K - y] \subset [a - y, a]$. In particular it fails for $x_{K+1} = a$, and this yields (10) since $|[a, a + y) \cap E| = A \geq A/2$. 2) $a \leq x_K < a + y$. Then either $|[a, x_K) \cap E| \geq A/2$ or $|[x_K, a + y)| \geq A/2$; hence either $|[x_K - y, x_K) \cap E| \geq A/2$ or $|[x_K, x_K + y) \cap E| \geq A/2$. But (9) must fail for $x_{K+1} = x_K - y \geq a - y$; thus both of the latter possibilities imply (10). 3) $a - y \leq x_K < a$. Then $|[x_K, x_K + 2y) \cap E| \geq |[a, a + y) \cap E| = A$, so (10) is clear.

The existence of K implies that N is well-defined and also that $x_N \geq a - y$. Now (i) is true by definition. As for (ii):

1) $M(x_1) - M(x_1 + y) = |[x_1, x_1 + y) \cap E| > B$ by definition. Next, we have $M(x_1 + y) = |[x_1 + y, x_1 + 2y) \cap E|$. Suppose $x_1 + y \leq b - y$. Then by definition of x_1 we must have $|[x_1 + y, x_1 + 2y) \cap E| \leq B$. Now suppose $b - y < x_1 + y$. Then

$$\begin{aligned} |[x_1 + y, x_1 + 2y) \cap E| &= |[x_1 + y, b) \cap E| + |[b, x_1 + 2y) \cap E| \\ &\leq |[x_1 + y, b) \cap E| + |[b, b + y) \cap E| \\ &= |[x_1 + y, b) \cap E| + B. \end{aligned}$$

If $x_1 = b - y$, then $[x_1 + y, b) \cap E = |\emptyset| = 0$. If $x_1 \leq b - y - 1$, then $B \geq |[b - y, b) \cap E|$ by definition of x_1, and $|[b - y, b) \cap E| \geq |[x_1 + y, b) \cap E|$ since $b - y < x_1 + y$. Thus $M(x_1 + y) \leq 2B$ in all cases.

2) Suppose $x_{k+1} = x_k - y$. Then $M(x_{k+1} + y) - M(x_k) = 0$. Suppose $x_{k+1} < x_k - y$. Then by (9) we have

$$|[x, x + y) \cap E| \leq M(x_k) \tag{11}$$

for all $x \in (x_{k+1}, x_k - y]$. If $x_k - y < x_{k+1} + y$, then

$$M(x_{k+1} + y) - M(x_k) \leq M(x_k - y) - M(x_k)$$
$$= |[x_k - y, x_k) \cap E| \leq M(x_k).$$

If $x_{k+1} + y \leq x_k - y$, then

$$M(x_{k+1} + y) - M(x_k) = |([x_{k+1} + y, x_{k+1} + 2y) \cup [x_k - y, x_k)) \cap E|$$
$$\leq |[x_{k+1} + y, x_{k+1} + 2y) \cap E| + |[x_k - y, x_k) \cap E|$$
$$\leq 2M(x_k) \text{ by (11).}$$

3) This is just (9).

4) This is the definition of x_N.

Combined with the McGehee Pigno Smith inequality, lemma 4 implies:

Lemma 5. There is a constant $\delta > 0$ such that any multiplier $m : H^1 \to H^1$ with $|\hat{m}(n)| \geq 1$, $n \in \operatorname{supp} \hat{m}$ satisfies:

$$\|m\| \geq \delta \left|\log\left(\frac{1+A}{1+B}\right)\right|^{\frac{1}{2}},$$

where
$$A = |[a, a + y) \cap \operatorname{supp} \hat{m}|,$$
$$B = |[b, b + y) \cap \operatorname{supp} \hat{m}|,$$

and $a, b, y \in \mathbb{Z}$ are any integers satisfying $a \geq 2y > 0$, $b \geq a + y$.

Proof: For the proof we assume $A > B$ since an entirely symmetric argument works for $A < B$. Let $E = \operatorname{supp} \hat{m}$ and consider the sequence $\{x_k\}_{k=1}^{N}$ obtained by applying lemma 4 to the data E, a, b, y. The function f defined by

$$f(\theta) = \sum_{k=1}^{N} (e^{ix_k\theta} + e^{i(x_k+y)\theta})K_{y-1}(\theta)$$

is in H^1 since $x_1 > \cdots > x_N \geq a - y \geq y$, and we have $\|f\|_1 \leq 2\sqrt{N}$ by lemma 3 and the condition $x_k - x_{k+1} \geq y$.

The function m(f) satisfies

$$\operatorname{supp} \widehat{m(f)} = \operatorname{supp} \hat{m} \cap \operatorname{supp} \hat{f}$$
$$\subset \bigcup_{k=1}^{N} [x_k - y, x_k + 2y) \cap E = F$$

and
$$|\widehat{m(f)}(n)| = |\hat{m}(n)\hat{f}(n)|$$
$$\geq 1 \text{ for } n \in \bigcup_{k=1}^{N} [x_k, x_k + y) \cap E = \tilde{F}.$$

By Theorem 2 and the definition of M(n) in lemma 4,

$$\|m(f)\|_1 \geq c \sum_{n\in F} |\widehat{m(f)}(n)|/M(n)$$
$$\geq c \sum_{n\in\tilde{F}} 1/M(n).$$

We will now prove the remark following lemma 4: We have

$$\sum_{n\in\tilde{F}} 1/M(n) = \sum_{k=1}^{N} \sum_{t=M(x_k+y)+1}^{M(x_k)} 1/t \; .$$

This is almost like $\sum_{t=B+1}^{M(x_N)} 1/t$ except for some missing pieces which are small. In fact:

$$\sum_{t=B+1}^{M(x_1+y)} 1/t \leq (M(x_1+y)-B)/B \leq 1 \text{ by ii(1)}$$

and

$$\sum_{t=M(x_1+y)+1}^{M(x_1)} 1/t \geq \sum_{t=M(x_1+y)+1}^{M(x_1+y)+B} 1/t \geq \frac{B}{M(x_1+y)+B} \geq \frac{1}{3}$$

(again by ii(1)). Therefore

$$\sum_{t=M(x_1+y)+1}^{M(x_1)} 1/t \geq \frac{1}{4} \sum_{t=B+1}^{M(x_1)} 1/t \ . \qquad (12)$$

Similarly, for k = 1, 2, ..., N-1 we have

$$\sum_{t=M(x_k)+1}^{M(x_{k+1}+y)} 1/t \leq (M(x_{k+1} + y) - M(x_k))/M(x_k) \leq 2 \text{ by ii(2)}$$

and

$$\sum_{t=M(x_{k+1}+y)+1}^{M(x_{k+1})} 1/t \geq \sum_{t=M(x_{k+1}+y)+1}^{M(x_{k+1}+y)+M(x_k)} 1/t \geq \frac{M(x_k)}{M(x_{k+1}+y)+M(x_k)} \geq \frac{1}{4}$$

(by ii(3) and ii(2)). Therefore

$$\sum_{t=M(x_{k+1}+y)+1}^{M(x_{k+1})} 1/t \geq \frac{1}{9} \sum_{t=M(x_k)+1}^{M(x_{k+1})} 1/t \ . \qquad (13)$$

Putting together (12) and (13) we have

$$\begin{aligned} \sum_{n \in \tilde{F}} 1/M(n) &\geq \frac{1}{9} \sum_{t=B+1}^{M(x_N)} 1/t \\ &\geq \frac{1}{9} \log \left(\frac{1+M(x_N)}{1+B} \right) \\ &\geq \frac{1}{9} \log \left(\frac{1+A/2}{1+B} \right) \quad \text{by ii(4)}. \end{aligned}$$

On the other hand,

$$M(x_1) > B + M(x_1 + y) \geq B \qquad \text{by ii(1)}$$

$$M(x_{k+1}) > M(x_k) + M(x_{k+1} + y) \qquad \text{by ii(3)}$$

$$\geq 2M(x_k) \qquad \text{since } x_{k+1} + y \leq x_k,$$

imply $M(x_{N-1}) \geq 2^{N-2}M(x_1) \geq 2^{N-2}(1+B)$, or $$N \le \frac{\log\left(\frac{M(x_{N-1})}{1+B}\right)}{\log 2} + 2$$

$$< \log\left(\frac{1+A/2}{1+B}\right)/\log 2 + 2 \quad \text{by ii(4).}$$

We therefore have

$$\|f\|_1 \le 2\left(\log\left(\frac{1+A/2}{1+B}\right)/\log 2 + 2\right)^{\frac{1}{2}},$$

$$\|m(f)\|_1 \ge \frac{c}{9}\log\left(\frac{1+A/2}{1+B}\right),$$

and this implies

$$\|m\| \ge \delta\left(\log\left(\frac{1+A}{1+B}\right)\right)^{\frac{1}{2}}$$

for some constant $\delta > 0$.

This concludes the proof of lemma 5. The second proof of Theorem 1 is the following:

Let m_0 be the multiplier in the conclusion of lemma 1. (Thus supp $\hat{m}_0$ has arbitrarily large gaps and $|\hat{m}_0(n)| \ge 1$ for $n \in$ supp $\hat{m}_0$). Let $y > 0$ be an integer, let $a = 2y$, and let $b \ge a + y$ be such that $[b, b + y)$ is contained in a gap; i.e.,

$$B = |[b, b + y) \cap \text{supp } \hat{m}_0| = 0.$$

By lemma 5 we have

$$A = |[2y, 3y) \cap \text{supp } \hat{m}_0| \le c_1 e^{c_2\|m_0\|^2}$$

and this holds for all y. Thus supp $\hat{m}_0$ is finite or a finite union of lacunary sequences, as was to be shown.

I do not know whether the exponent $\frac{1}{2}$ is sharp in lemma 5, even for the case $B = 0$. On the other hand, the exponent 1 in lemma 2 is clearly sharp for the case $B = 0$, for the same reason that the McGehee, Pigno, Smith inequality is sharp; the multiplier determined by

$$\hat{m}(n) = \begin{cases} 1 & n \in [0,A] \\ 0 & \text{otherwise} \end{cases}$$

has norm the order of log A. One may speculate that lemmas 2 and 5 are far from being sharp when A and B are both large, since it may be that a lower bound in terms of $|A-B|$, rather than A/B, exists.

3. Some consequences and remarks.

Recall that in the proof of Theorem 1, an important step was identifying a sequence of disjoint intervals.

$$[x_k, x_k + 3y) \qquad k = 1, 2, \ldots, N$$

$$(\text{where } 0 < 3y \leq x_1 < x_2 < \cdots < x_N)$$

having the property that

$$|[x_k, x_k + 3y) \cap \operatorname{supp} \hat{m}| = \rho^{N-k+1}$$

for some number $\rho > 1$. This made it possible to conclude that $\|m\|$ was large if N was large, regardless of how far x_N was from x_1. An analogy may be drawn between this situation and a certain gap theorem of Y. Mayer, as proved by J. Fournier in [2]. I will state this result since it, in fact, motivated my approach above:

Let

$$0 \leq n_0 \leq m_1 < n_1 \leq m_2 < n_2 \leq \cdots \leq m_k < n_k \leq \cdots$$

be integers such that

$$n_{k+1} - m_{k+1} \geq \varepsilon n_k \ , \ k = 1, 2, \ldots$$

for some positive number ε. Let μ be a Borel measure such that $\hat{\mu}(n) = 0$ for $n_k < n < m_k$, $k = 1, 2, \ldots$

Then

$$\sum_{k=1}^{\infty} |\hat{\mu}(m_k)|^2 \leq c\|\mu\|^2$$

for some absolute constant c.

Here are some of the analogies:

1) The density of supp $\hat{\mu}$ in $[m_k, n_k)$ is decreasing at least geometrically. The density of supp $\hat{m}$ in $[x_k, x_k + 3y)$ is decreasing geometrically.

2) There is no restriction from above on how far m_{k+1} is from n_k. There is no restriction from above on how far x_{k+1} is from x_k.

3) Suppose only finitely many (m_k, n_k) are considered and $\hat{\mu}(m_k) = 1$, $k = 1, 2, \ldots, N$. Then the estimate is $\|\mu\| \geq c\sqrt{N}$, the same as in the proof of Theorem 1 for $\|m\|$.

Fournier proves the above gap theorem by a special case of his general method for constructing bounded functions with prescribed Fourier coefficients. The result is that a function $G \in L^\infty$ exists such that

$$\hat{G}(m_k) = \hat{\mu}(m_k) \qquad k = 1, 2, \ldots$$

$$\text{supp } \hat{G} \subset \bigcup_{k=1}^{\infty} [m_k, n_k)$$

$$\|G\|_\infty^2 \leq c \sum_{k=1} |\hat{\mu}(m_k)|^2.$$

Similarly, the McGehee, Pigno, Smith inequality is a consequence of another L^∞ construction; if $\ell_1 > \ell_2 > \ldots > \ell_T \geq 0$ are given integers, and $\gamma_1, \gamma_2, \ldots \gamma_T$ are on the unit circle, then a function $\phi \in L^\infty$ exists such that

$$\mathrm{Re}(\hat{\phi}(\ell_k)\gamma_k) \geq \frac{1}{k}, \; k = 1, 2, \ldots, T$$

$$\mathrm{supp}\, \hat{\phi} \subset [\ell_T, \infty),$$

$$\|\phi\|_\infty \leq c, \text{ an absolute constant.}$$

Now observe that in both constructions the functions are in fact analytic; $G, \phi \in H^\infty$. By taking into account this additional property of ϕ, the proof of Theorem 1, and of the case $A > B$ of lemma 5 can be modified to yield the same conclusions from the weaker assumption that m is a Fourier multiplier of the space H^∞ into BMOA, the space of analytic functions of bounded mean oscillation. This is a weaker assumption because, by the H^1-BMOA duality, the property

$$m : H^1 \to H^1$$

is equivlent to

$$m : \mathrm{BMOA} \to \mathrm{BMOA}$$

for Fourier multipliers m. I will now discuss the changes needed to make the proofs work under the assumption that $m : H^\infty \to \mathrm{BMOA}$. Let

$$\|m\| = \sup\{\|m(\phi)\|_{\mathrm{BMOA}} : \phi \in H^\infty, \; \|\phi\|_\infty \leq 1\}.$$

Lemma 1: The only change is that we define g_n by

$$g_n(\theta) = e^{-i3n\theta} m(e^{i3n\theta} K_n(\theta)) \quad n = 0, 1, \ldots$$

We show that $\|g_n\|_1$, $n = 1, 2, \ldots$, are bounded by duality:

Let

$$V_n(\theta) = (e^{i2n\theta} + e^{i4n\theta})K_{2n}(\theta).$$

Then

$$\hat{V}_n(j) = 1 \quad 2n \leq j \leq 4n,$$

$$\text{supp } \hat{V}_n \subset [0, 6n),$$

$$\|V_n\|_1 \leq 2.$$

Let $f \in L^\infty, |f| \leq 1$. Then the convolution $V_n * f \in H^\infty$ satisfies $|V_n * f| \leq 2$ and

$$\int_0^{2\pi} e^{i3n\theta} g_n(\theta)\overline{f(\theta)}d\theta/2\pi$$

$$= \int_0^{2\pi} m(e^{i3n\theta}K_n(\theta))\overline{f(\theta)}d\theta/2\pi$$

$$= \int_0^{2\pi} m(e^{i3n\theta}K_n(\theta))\overline{(V_n*f)}(\theta)d\theta/2\pi$$

$$= \int_0^{2\pi} e^{i3n\theta}K_n(\theta)\ \overline{m(V_n*f)(\theta)}\ d\theta/2\pi \quad (\text{since } \hat{m}=0 \text{ or } 1).$$

By the H^1-BMOA duality, the magnitude of the last integral is bounded by

$$c\,\|e^{i3n\theta}K_n(\theta)\|_1\|\ m(V_n*f)\|_{BMOA}$$

$$\leq c\,\|m\|\ \|V_n*f\|_\infty \leq 2\,c\,\|m\|.$$

But the supremum of the first integral over $\|f\|_\infty \leq 1$ is just $\|g_n\|_1$; hence $\|g_n\|_1 \leq 2c\|m\|$ for all n.

I thank T. Wolff for suggesting the use of the function V_n in the above calculation.

Lemma 2 and the first proof of Theorem 1: Consider, for instance, the set $\{n_T < n_{T-1} < \ldots < n_2 < n_1\}$ in the proof. Let $\phi \in H^\infty$ be the McGehee, Pigno, Smith construction for this set, with $\gamma_\ell = \overline{\operatorname{sgn} \hat{m}(n_\ell)}$; i.e.,

$$\operatorname{Re}\left(\hat{\phi}(n_\ell) \cdot \overline{\operatorname{sgn}(\hat{m}(n_\ell))}\right) \geq 1/\ell$$

$$\operatorname{supp} \hat{\phi} \subset [n_T, \infty)$$

$$\|\phi\|_\infty \leq c.$$

We then show that $\|m(\phi)\|_{BMOA}$ is large, by duality: Integrating against the same function $f \in H^1$ used in the proof gives

$$\sqrt{2N} \, \|m(\phi)\|_{BMOA} \geq \|f\|_1 \, \|m(\phi)\|_{BMOA}$$

$$\geq \left| c \operatorname{Re} \int_0^{2\pi} m(\phi)\overline{f} \, d\theta/2\pi \right|$$

$$\geq c \sum_{\ell=1}^{T} |\hat{m}(n_\ell)| \hat{f}(n_\ell)/\ell$$

$$\geq c \, N/3\rho^2 \quad \text{as in the proof.}$$

So the final estimate, $\|m\| \geq c\sqrt{N}/(3\sqrt{2}\rho^2)$, has the same form as before. Notice that we can get $\phi \in H^\infty$, instead of merely L^∞, because of the backwards enumeration of the set $\{n_\ell\}$. In general, the set may be very spread-out, so this is not possible in the other direction. For this reason, lemma 5 does not carry over completely, but does work for the case $A > B$, $m : H^\infty \to BMO$, $|\hat{m}(n)| \geq 1$, $n \in \operatorname{supp} \hat{m}$.

As a final remark, I will note the connection between lemma 2 and the following theorem of C. Fefferman, which I quote from Smith [13].

Let $a_k \geq 0$, $k = 0, 1, 2, \ldots$. Then the following two conditions are equivalent:

1) $\sum_{k=0}^{\infty} \eta_k a_k e^{ik\theta} \in$ BMOA for all sequences $\{\eta_k\}$ such that $|\eta_k| = 1$, $\eta_k \in \mathbb{C}$, $k = 0, 1, 2, \ldots$.

2) $\sup_{n \in \mathbb{Z}^+} \sum_{j=1}^{\infty} \left(\sum_{k=jn}^{(j+1)n-1} a_k \right)^2 < \infty.$

The connection is that lemma 2 and the H^1-BMOA duality immediately give the implication 1) $\Rightarrow$ 2). In fact, the case $\eta_k = 1$ implies

$$\phi(\theta) = \sum_{k=0}^{\infty} a_k e^{ik\theta} \in \text{BMO}.$$

Now fix $y \in \mathbb{Z}^+$ and define

$$f(\theta) = \sum_{j=1}^{\infty} c_j (e^{ijy\theta} + e^{i(j+1)y\theta}) K_{y-1}(\theta)$$

where
$$c_j = \sum_{k=jy}^{(j+1)y-1} a_k .$$

By a direct calculation it can be seen that

$$\int_0^{2\pi} \phi(\theta)\overline{f(\theta)}\, d\theta/2\pi = \sum_{j=1}^{\infty} c_j \sum_{t=-y+1}^{y-1} (a_{jy+t} + a_{(j+1)y+t})\left(1-\frac{|t|}{y}\right)$$

$$\geq \sum_{j=1}^{\infty} c_j \sum_{k=jy}^{jy+y-1} a_k$$

$$= \sum_{j=1}^{\infty} c_j^2 .$$

On the other hand, the integral is bounded by

$$c\,\|\phi\|_{BMOA}\,\|f\|_1 \leq c\,\|\phi\|_{BMOA} \cdot 2\left(\sum_{j=1}^{\infty} c_j^2\right)^{\frac{1}{2}}$$

where we have used lemma 2 to estimate $\|f\|_1$. We conclude

$$\sum_{j=1}^{\infty}\left(\sum_{k=jy}^{(j+1)y-1} a_k\right)^2 \leq c\,\|\phi\|^2_{BMOA} < \infty$$

for all $y \in \mathbb{Z}^+$, and so 2) follows.

REFERENCES

[1] Cohen, P. J., On a conjecture of Littlewood and idempotent measures Amer. J. Math. 82(1960), 191-212.

[2] Fournier, J. J. F., On a theorem of Paley and the Littlewood conjecture, Arkiv for Mathematik 17(1979), 199-216.

[3] Graham, C. C., McGehee, O. C., Essays in commutative harmonic analysis. Springer Verlag, New York 1979.

[4] Helson, H., Note on harmonic functions, Proc. Amer. Math. Soc., 4(1953), 686-691.

[5] Helson, H., On a theorem of Szego, Proc. Amer. Math. Soc. 6(1955), 235-242.

[6] Katznelson, Y., An Introduction to Harmonic Analysis, 2nd ed. Dover, New York 1976.

[7] Larsen, R., An Introduction to the Theory of Multipliers, Springer-Verlag, New York 1971.

[8] McGehee, O. C., Pigno, L., Smith, B., Hardy's inequality and the L^1 norm of exponential sums, Ann. of Math. 113(1981), 613-618.

[9] Paley, R. E. A. C., On the lacunary coefficients of power series, Ann. of Math. 34(1933), 615-616.

[10] Pigno, L., Smith, B., Quantitative behaviour of the norms of an analytic measure, Proc. Amer. Math. Soc. 86, no. 4(1982) 581-585.

[11] Pigno, L., Smith, B., A Littlewood-Paley inequality for analytic measures, Arkiv for Matematik 20, no. 2, (1982) 271-274.

[12] Rudin, W., Remarks on a theorem of Paley, J. London Math. Soc. 32(1957), 307-311.

[13] Smith, B., Two trigonometric designs: one-sided Riesz products and Littlewood products, ISNM 64 reprint, E. F. Beckenbach, W. Walter ed., Birkhauser Verlag Basel 1983, 141-148.

PART II

The purpose of this paper is to prove an extremal property of the decreasing rerrangement of a function on an interval. As a consequence, we obtain the sharp constant c = 1 in the inequality $\|f^*\|_{BMO} \leq c\|f\|_{BMO}$, thus refining a result contained in Theorem 3.1 of [1].

Let $I \subset \mathbb{R}$ be a bounded nondegenerate interval. For $f \in L^1(I)$ let $f^* \in L^1(I)$ denote the decreasing rearrangement of f. For a subset $E \subset I$ with $|E| > 0$, denote the average of f over E by $f_E \equiv (1/|E|)\int_E f(t)dt$. The letters F,G will be reserved for functions $F, G : [0,\infty) \to [0,\infty)$ with the special properties that $F(0) = 0$, $F(\lambda)/\lambda$ is increasing for $\lambda > 0$ and $G(\lambda)/\lambda$ is decreasing for $\lambda > 0$. (We note for clarity that the only condition on G(0) is $G(0) \geq 0$.)

With the above notation the main result is:

Theorem. Let $f \in L^1(I)$ and let $\alpha \in \mathbb{R}$ such that $f_K = \alpha$ for at least one interval $K \subset I$. Suppose $F \circ |f - \alpha|$, $G \circ |f - \alpha| \in L^1(I)$ where F,G have the above properties. Then

$$(1) \qquad \sup_J \frac{\int_J F \circ |f^* - \alpha|}{\int_J G \circ |f^* - \alpha|} \leq \sup_K \frac{\int_K F \circ |f - \alpha|}{\int_K G \circ |f - \alpha|}$$

where the supremum on the left is over all intervals $J \subset I$ with $f^*_J = \alpha$, the supremum on the right is over all intervals $K \subset I$ such that $f_K = \alpha$, and where by definition $\frac{0}{0} = 0$, $\frac{x}{0} = \infty$ for $x > 0$.

Before proving the theorem we apply it to $f \in BMO(I)$. For $1 \leq p < \infty$, taking $F(\lambda) = \lambda^p$, $G(\lambda) = 1$, and taking the supremum over all α on both sides of (1) gives:

Corollary. For $f \in BMO(I)$ and $1 \leq p < \infty$ we have

$$\sup_{J \subset I} \frac{1}{|J|} \int_J |f^* - f^*_J|^p \leq \sup_{J \subset I} \frac{1}{|J|} \int_J |f - f_J|^p$$

where on both sides the supremum is over all intervals $J \subset I$ such that $|J| > 0$.

The proof of the theorem may be divided into two lemmas. The first is a refinement of the well-known rising sun lemma and its proof is omitted (for the original rising sun lemma see [3] or [2], p. 293).

Lemma A. Let $f \in L^1(I)$ where I is a bounded interval, let $\alpha \in \mathbb{R}$, and suppose $f_I \leq \alpha$. Then there is a finite or countable set $\mathcal{L}$ of pairwise disjoint subintervals of I such that $f_L = \alpha$ for each $L \in \mathcal{L}$ and $f \leq \alpha$ almost everywhere on $I \setminus \cup\mathcal{L}$.

The second lemma contains the main computations.

Lemma B. Let $f \in L^1(I)$ where I is a bounded interval. Suppose $J \subset I$ is an interval with $|J| > 0$ and $E \subset I$ is a set with $|E| > 0$, such that

(i) $f^*_J = f_E \equiv \alpha$ and (ii) $\int_J |f^* - \alpha| \leq \int_E |f - \alpha|$.

Assume that $F \circ |f - \alpha|$, $G \circ |f - \alpha| \in L^1(I)$ where F,G are the functions in the theorem Then

$$\frac{\int_J F \circ |f^* - \alpha|}{\int_J G \circ |f^* - \alpha|} \leq \frac{\int_E F \circ |f - \alpha|}{\int_E G \circ |f - \alpha|} \tag{2}$$

where by definition $\frac{0}{0} = 0$ and $\frac{x}{0} = \infty$ for $x > 0$.

Proof of Lemma B. If $\int_J |f^* - \alpha| = 0$, then $\int_J F \circ |f^* - \alpha| = 0$ since $F(0) = 0$, and this gives (2). We now assume $\int_J |f^* - \alpha| > 0$. Define the functions m, M, n, N for $\lambda \geq 0$ by

$$m(\lambda) = \int_{\{t:f^*(t)-\alpha>\lambda\}\cap J} f^*(t) - \alpha dt \quad , \quad M(\lambda) = \int_{\{t:f(t)-\alpha>\lambda\}\cap E} f(t) - \alpha dt \ ,$$

$$n(\lambda) = \int_{\{t:\alpha-f^*(t)>\lambda\}\cap J} \alpha - f^*(t) dt \quad , \quad N(\lambda) = \int_{\{t:\alpha-f(t)>\lambda\}\cap E} \alpha - f(t) dt \ .$$

Letting H = F or G, the integrals in (2) are given by

$$\int_J H\circ |f^*-\alpha| = -\int_{(0,\infty)} H(\lambda)/\lambda dm(\lambda) + H(0)\,|\{t : f^*(t) = \alpha\}\cap J| - \int_{(0,\infty)} H(\lambda)/\lambda dn(\lambda)$$

$$\int_E H\circ |f - \alpha| = -\int_{(0,\infty)} H(\lambda)/\lambda dM(\lambda) + H(0)\,|\{t : f\ (t) = \alpha\}\cap E| - \int_{(0,\infty)} H(\lambda)/\lambda dN(\lambda).$$

We claim that

(F1) $-\int_{(0,\infty)} F(\lambda)/\lambda dm(\lambda)/m(0) \leq -\int_{(0,\infty)} F(\lambda)/\lambda dM(\lambda)/M(0)$,

(F2) $-\int_{(0,\infty)} F(\lambda)/\lambda dn(\lambda)/n(0) \leq -\int_{(0,\infty)} F(\lambda)/\lambda dN(\lambda)/N(0)$,

(G1) $-\int_{(0,\infty)} G(\lambda)/\lambda dm(\lambda)/m(0) \geq -\int_{(0,\infty)} G(\lambda)/\lambda dM(\lambda)/M(0)$,

(G2) $-\int_{(0,\infty)} G(\lambda)/\lambda dn(\lambda)/n(0) \geq -\int_{(0,\infty)} G(\lambda)/\lambda dN(\lambda)/N(0)$,

(G3) $G(0)\,|\{t : f^*(t) = \alpha\}\cap J|/m(0) \geq G(0)|\{t : f(t) = \alpha\}\cap E|/M(0)$.

Assuming the claim for the moment, we derive (2) as follows: Hypotheses (i) and (ii) of the lemma imply $m(0) = n(0) = \frac{1}{2}\int_J |f^* - \alpha| \leq \frac{1}{2}\int_E |f - \alpha| = N(0) = M(0)$. In particular, $m(0) = n(0) \neq 0$ and $M(0) = N(0) \neq 0$. Recalling that $F(0) = 0$, we obtain (2) by dividing the sum of (F1) and (F2) by the sum of (G1), (G2), and (G3).

We now prove the claim. The easiest inequality is (G3): We note that $\{t : f^*(t) = \alpha\} \subset J$ since $f^*_J = \alpha$, $\int_J |f^* - \alpha| > 0$, and f^* is decreasing. Therefore $|\{t : f^*(t) = \alpha\} \cap J| = |\{t : f^*(t) = \alpha\}| = |\{t : f(t) = \alpha\}| \geq |\{t : f(t) = \alpha\} \cap E|$. Since $G(0) \geq 0$ and $m(0) \leq M(0)$, (G3) follows.

The other inequalities will follow by integration by parts, once we establish

$$\text{(iiia)} \quad \frac{m(\lambda)}{m(0)} \leq \frac{M(\lambda)}{M(0)} \quad \text{and} \quad \text{(iiib)} \quad \frac{n(\lambda)}{n(0)} \leq \frac{N(\lambda)}{N(0)} \quad \text{for } \lambda \geq 0.$$

To prove (iiia) write $J = [a,b]$ where $a < b$ and set $\lambda_1 = f^*(a+0) - \alpha$. (We include the possibility that $\lambda_1 = +\infty$.) For $0 \leq \lambda < \lambda_1$ we have

$$m(0) - m(\lambda) = \int_{\{t:0<f^*(t)-\alpha\leq\lambda\}\cap J} f^*(t) - \alpha\, dt = \int_{\{t:0<f^*(t)-\alpha\leq\lambda\}} f^*(t) - \alpha\, dt$$

$$= \int_{\{t:0<f(t)-\alpha\leq\lambda\}} f(t) - \alpha dt \geq \int_{\{t:0<f(t)-\alpha\leq\lambda\}\cap E} f(t) - \alpha dt =$$

$$= M(0) - M(\lambda).$$

Recalling that $m(0) \leq M(0)$ we obtain $\frac{m(0)-m(\lambda)}{m(0)} \geq \frac{M(0)-M(\lambda)}{M(0)}$ so that $m(\lambda)/m(0) \leq M(\lambda)/M(0)$. For $\lambda \geq \lambda_1$ we have $m(\lambda) = 0$, so (iiia) is proved. The proof of (iiib) is similar and we omit it.

To prove (F1) we may assume without loss of generality that $F(\lambda)/\lambda \equiv \phi(\lambda)$ is a *bounded* increasing function (by the monotone convergence theorem). Since $m(0+0) = m(0)$ and $m(\lambda) \to 0$ as $\lambda \to \infty$, integration by parts yields

$$-\int_{(0,\infty)} \phi(\lambda) dm(\lambda)/m(0) = \phi(0+0) + \int_{(0,\infty)} m(\lambda)/m(0) d\phi(\lambda).$$

Similarly,

$$-\int_{(0,\infty)}\phi(\lambda)dM(\lambda)/M(0) = \phi(0+0) + \int_{(0,\infty)}M(\lambda)/M(0)d\phi(\lambda).$$

In view of (iiia) and the fact that $d\phi \geq 0$, we obtain (F1) by comparing the last two equations. The proofs of (F2), (G1), and (G2) are similar. This concludes the proof of lemma B.

<u>Proof of the Theorem</u>. We first note that both sides of inequality (1) are invariant under the transformation $f \mapsto -f$, $\alpha \mapsto -\alpha$, so that we may assume $f_I \leq \alpha$. Let $\mathcal{L}$ be the set of intervals given by lemma A and set $E = \cup\mathcal{L}$. Let $J \subset I$ be any interval such that $f_J^* = \alpha$. We check that $\int_E f - \alpha = \sum_{L\in\mathcal{L}} \int_L f - \alpha = 0$ so that $f_E = \alpha$. Also,

$$\int_E |f-\alpha| = \sum_{L\in\mathcal{L}} \int_L |f-\alpha| = \sum_{L\in\mathcal{L}} 2\int_{\{t:f(t)>\alpha\}\cap L} f-\alpha =$$

$$= 2\int_{\{t:f(t)>\alpha\}\cap E} f-\alpha = 2\int_{\{t:f(t)>\alpha\}} f-\alpha = 2\int_{\{t:f^*(t)>\alpha\}} f^*-\alpha$$

$$\geq 2\int_{\{t:f^*(t)>\alpha\}\cap J} f^*-\alpha = \int_J |f^*-\alpha|.$$

Therefore lemma B applies and we obtain inequality (2). If $\int_E G\circ|f-\alpha| > 0$ then

$$\frac{\int_E F\circ|f-\alpha|}{\int_E G\circ|f-\alpha|} = \frac{\sum_{L\in\mathcal{L}}\int_L F\circ|f-\alpha|}{\sum_{L\in\mathcal{L}}\int_L G\circ|f-\alpha|} \leq \sup_{L\in\mathcal{L}} \frac{\int_L F\circ|f-\alpha|}{\int_L G\circ|f-\alpha|},$$

and the latter does not exceed the right-hand side of (1). If $\int_E G\circ|f-\alpha| = 0$ and $\int_E F\circ|f-\alpha| = 0$, then by our convention $\int_E F\circ|f-\alpha| / \int_E G\circ|f-\alpha| = 0 \leq$ R.H.S. of (1). If $\int_E G\circ|f-\alpha| = 0$ and $\int_E F\circ|f-\alpha| > 0$, then $\int_L F\circ|f-\alpha| > 0$ for some $L\in\mathcal{L}$, whereas

$\int_L G \circ |f - \alpha| = 0$, so that the right-hand side of (1) is $+\infty$ and we again have $\int_E F \circ |f - \alpha| / \int_E G \circ |f - \alpha| \leq$ R.H.S. of (1). Hence in all cases we have by (2) that $\int_J F \circ |f^* - \alpha| / \int_J G \circ |f^* - \alpha| \leq$ R.H.S. of (1). Taking the supremum over all admissible J yields (1).

We now consider functions of bounded mean oscillation on the circle $T \equiv \mathbb{R}/2\pi\mathbb{Z}$. For $f \in BMO(T)$ define

$$\|f\| = \sup_J \frac{1}{|J|} \int_J |f - f_J|,$$

where J ranges over all intervals in T. Let $f^\#$ denote the symmetric decreasing rearrangement of f. Making use of the corollary, it can be shown that $\|f^\#\| \leq 2\|f\|$. However, there are functions f such that $\|f^\#\| > \|f\|$. We give an example, omitting the computations.

Define $f(0) = f(1) = f(-1) = 0$, $f(\frac{1}{3}) = f(-\frac{1}{3}) = 1$ and interpolate linearly for the remaining θ $[-\pi, \pi]$. Then $f^\#$ is the piecewise linear function with corners (-1,0), (0,1), (1,0), and it can be shown that $\|f^\#\| > \|f\|$,

The failure of the inequality $\|f^\#\| \leq \|f\|$ is due to the fact that the supremum $\|f^\#\|$ may be achieved for an interval J on which $f^\#$ is not monotone. This makes it possible to construct an (equimeasurable) "perturbation" f for which $\|f\| < \|f^\#\|$ (as in the above example).

We conclude with two problems.:

1) For $f \in BMO(T)$, are there any equimeasurable rearrangements g for which $\|g\|$ is minimal, and if so, describe them.

2) What is the best constant c such that $\|f^\#\| \leq c\|f\|$ for all $f \in BMO(T)$?

Acknowledgements. The author wishes to thank Professor T. H. Wolff for suggesting the problems in this paper and pointing out the relevance of the rising sun lemma. The basic covering strategy was motivated by the proof of Theorem 3.1 in [1].

REFERENCES

[1] Bennett, C., DeVore, R. A., Sharpley, R., Weak-L^∞ and BMO., Ann. of Math. 113 (1981), 601-611.

[2] Hardy, G. H., Littlewood, J. E. Pólya, G., Inequalities, Cambridge University Press, 1967.

[3] Riesz, F., Sur un théorème de maximum de MM. Hardy et Littlewood., Journ. L.M.S. 7 (1932), 10-13.