Limit Theorems for Classical Spin Systems with an Abelian Discrete Symmetry

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Nestor Caticha Alfonso

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Também te vi chorar... Também sofreste

A dor de ver secarem pela estrada

As fontes da esperança... E não cedeste!...

Manuel Bandeira

"Estrela da Vida Inteira"

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I cannot explain in how many ways and how much my brother Ariel has helped me. I cannot express what he already knows.

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And then Rita..., a constant source of encouragement, companionship and love.

to Rita

to my parents

and yes, to Bruno.

Abstract

Classical spin models with a discrete abelian symmetry ($\mathbf{Z}_p$)are studied and compared to analogous models with a continuous ($\mathbf{O}(2)$) symmetry.

The dependence on p (the number of states) of some quantities, e.g., the pressure and correlation functions, is studied. For high p, under fairly general conditions, the pressure of the $\mathbf{Z}_p$ invariant model converges exponentially, in p, to that of the $\mathbf{O}(2)$ model. Results of a similar nature, although obtained under more restrictive conditions, are presented for a class of expectation values.

Several different methods of proving Mermin- and Wagner-like results are reviewed and it is suggested that these methods are not sufficiently powerful to be used in obtaining upper bounds on the magnetization temperature of the two dimensional $\mathbf{Z}_p$ model. A rigorous lower bound is obtained using a Peierls-Chessboard method.

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I.1 Introduction

The study of lattice systems serves as a tool for determining the applicability of the general theory of Statistical Mechanics. It provides a ground where mathematically rigorous statements can be made about infinitely large systems. Examples of general statements would be proofs of the existence of thermodynamic functions in the infinite volume limit and existence of equilibrium states. More specific results would be proofs of the existence or nonexistence of phase transitions, results concerning the invariance or non invariance of the equilibrium states under certain symmetry transformation of the Hamiltonian, etc.. The fact that this type of result can be obtained for nontrivial lattice systems justifies the use of Statistical Mechanics in the study of real physical systems, even though the underlying dynamics of the latter may be much more complicated and therefore general results may not be available. The agreement between its predictions and experiment is, however, the main practical justification.

The mathematical results are also of importance in the study of Quantum Field theories due to the "remarkable" similarity between the Euclidean space Green's function generating functionals and partition functions.

Historically, the subject originated with the proposal in 1920 by W. Lenz and the study by the E. Ising (in one dimension) [1] of what is known as the Ising model in order to explain ferromagnetism as a collective phenomenon. In one dimension, this model with nearest neighbor interaction is rather trivial, can be solved exactly and shows no phase transition and zero spontaneous magnetization. Ising's conjecture that this would be the case for higher dimensions was proved to be wrong in 1936 by R. Peierls [2] who showed, through the use of a very clever argument (see Chapter III), that for sufficiently low temperatures the spontaneous magnetization was nonzero in two dimensions. The argument was completed by R. B. Griffiths [3] in 1964, and by Dobrushin [15] who showed that the spontaneous magnetization was zero at sufficiently high temperatures, establishing the existence of a phase transition. Meanwhile, the existence had been proved by L. Onsager [4] in 1944 by explicitly solving the model in two dimensions -- an exact calculation of the free energy in zero external field -- and finding the celebrated logarithmic divergence of the specific heat.

A large number of models have been since proposed, allowing for different interactions, more states, different symmetries and lattices.

It is of great interest to understand which features of a given lattice Hamiltonian are indeed responsible for the general thermodynamic behavior of the system and which are responsible for specific details. To obtain this kind of information, it is necessary to consider a general formulation of the problem, in order to simultaneously treat classes of interaction which share a given property. Many works have contributed to the formalism of lattice gases and spin systems, initiated in the 1960s by Gallavoti, Miracle-Sole, Ruelle, Robinson and Lanford (Wightman [5]).

We will mainly study classical spin systems with a discrete Z_p symmetry, (where p is the number of allowed states of a spin). The limit of $p \to \infty$ is particularly interesting since the resulting system has a continuous (O(2)) symmetry. In two dimensions the spontaneous breaking of this continuous symmetry is forbidden. We study various proofs of this result in Chapter II, trying to understand if these methods can be applied to the Z_p system to obtain upper bounds on the temperature, above which, there is no spontaneous symmetry breaking.

Chapters III and IV deal with the discrete symmetry system. After reviewing results [6,7] about the phase structure in two dimensions in Section III.1, we turn to the new results in this work. In section III.2 we use the Peierls-Chessboard method to obtain a rigorous lower bound on the magnetization temperature of the two dimensional $\mathbf{Z}_p$ system. This method provides, besides the rigorous lower bound result, an intuition on the mechanism driving the phase transition, and based on it, an heuristic argument is presented in Section III.4, indicating that Chapter II's methods of proving the absence of symmetry breaking for the continuous system cannot be generalized to obtain rigorous upper bounds on the magnetization temperature of the $\mathbf{Z}_p$ model. The reason can be trailed back to the weak treatment of entropy contributions.

In Chapter IV we analyze the convergence of certain quantities, especially the pressure, in the discrete systems to their analogue in the continuous case. It is proven that the convergence is exponentially fast in the number of states p for any fixed temperature and p sufficiently large. The result is general in the sense that it is valid for a large class of Hamiltonians and any lattice dimension. We obtain similar but weaker results for some type of correlation functions at high temperatures.

I.2 Formalism

First let us define the problem. We use the notation of Simon [8]. Consider a Z^ν lattice and at each site i associate a spin variable s_i which takes values on a compact metric space Ω. Denote by $P(Z^\nu)$ the set of all subsets of Z^ν and $P_f(Z^\nu)$ the set of finite subsets. An assignment of values of s_i for all $i \in \Lambda$, $\Lambda \in P(Z^\nu)$ is called a configuration and Ω^Λ is used to denote the set of configurations in Λ.

C_Λ will denote the set of real continuous functions on Ω^Λ. We define an interaction as the assignment of a function $\Phi(X) \in C_{Z^\nu}$ to each $X \in P_f(Z^\nu)$. We will only be interested in translation invariant interactions, such that $\Phi(X+i) = \tau_i \Phi(X)$ for $i \in Z^\nu$, and τ_i the translation mapping from $C_X \rightarrow C_{X+i}$. The interactions that will be physically interesting will have restrictions on the growth of $\Phi(X)$ with $|X|$. Therefore, we introduce the following norms for $\Phi \in C_\Lambda$

$$\|\Phi\|_\infty = sup_{s \in \Omega^\Lambda} |\Phi(s)| \qquad (1a)$$

$$|\!|\!|\Phi|\!|\!| = \sum_{0 \in X} |X|^{-1} \|\Phi\|_\infty \qquad (1b)$$

$$\|\Phi\|_1 = \sum_{0 \in X} \|\Phi\|_\infty \ . \qquad (1c)$$

where $X \in P_f(Z^\nu)$, $|X|$ is the number of sites in X and the sum is over all those X which contain the site with all coordinates zero.

Using the norms of eq(1) we define B, the set of interactions Φ with $|\!|\!|\Phi|\!|\!|$ finite, and B_1, the set of Φ with $\|\Phi\|_1$ finite. B and B_1 are Banach spaces in these norms and $B_1 \subset B$. Another interesting space B_0 is obtained by considering the finite range interactions, for which $\Phi(X) = 0$ if $|X| > R$, then $B_0 \subset B_1$.

The free boundary condition Hamiltonian $H_\Lambda(\Phi)$ of a system in $\Lambda \in P_f(Z^\nu)$ associated to an interaction Φ is defined by

$$H_\Lambda(\Phi)=\sum_{X\subset\Lambda}\Phi(X) \quad . \tag{2}$$

To introduce the statistical aspects of the problem we introduce a measure $d\mu_{0i}$ on $\Omega_{\{i\}}$ called the *a priori* measure. μ_{0i} will always be normalized to one

$$\int_\Omega d\mu_{0i}\equiv\mu_0(\Omega)=1 \ . \tag{3}$$

We now introduce the partition function Z_Λ at temperature $T=1/\beta$

$$Z_\Lambda(\Phi)=\int e^{-\beta H_\Lambda(\Phi)}\prod_{i\in\Lambda}d\mu_{0i} \ , \tag{4}$$

the finite volume pressure $p_\Lambda(\Phi)$

$$p_\Lambda(\Phi)=\frac{1}{|\Lambda|}\ln Z_\Lambda(\Phi) \quad , \tag{5}$$

and the expectation values of observables, i.e. functions $f\in C_\Lambda$

$$<f>_{H,\Lambda}=\frac{1}{Z_\Lambda(\Phi)}\int f\, e^{-\beta H(\Phi)}\prod_{i\in\Lambda}d\mu_{0i} \quad . \tag{6}$$

This defines the canonical ensemble and we will call a state of the finite physical system an expectation value functional on the functions $f\in C_\Lambda$. We are interested in the equilibrium states of a physical system which can be determined by all the expectation values of functions of an isolated system when we let the time approach infinity. Typically, the equilibrium state of a physical system will consist of at least one macroscopically homogeneous region, which is called a phase and is described by giving the values of a small set of parameters, (e.g., temperature, external magnetic field, pressure, etc.). The structure of the phases will depend on the values of these parameters and any change in the phase structure is believed to be associated with nonanalytical dependence of the equilibrium state on these parameters. To obtain nonanalytical behavior of the partition function or the pressure as defined by equations (4) and (5) we must consider the thermodynamic limit, which is obtained by letting the volume of Λ tend to infinity in a suitable manner as defined below. The first rigorous treatment of this was given by Van Hove [9]. We will describe this limit procedure although the simpler procedure of letting Λ be a hypercube of volume a^{ν} centered at 0 and letting $a \to \infty$ will suffice in most of the problems treated here.

We say that a sequence of sets Λ_n tends to infinity in Van Hove's sense if for any integer a and $\partial^a \Lambda_n = \{i \in Z^{\nu} /$ cube C of side a centered at i , $C \cap \Lambda_n \neq \phi$ and $C \cap \Lambda_n^c \neq \phi\}$,

$$\frac{|\partial^a \Lambda_n|}{|\Lambda_n|} \to 0 \text{ as } n \to \infty \quad .$$

For further developments concerning the infinite volume limit, see Fisher [10] and Fisher and Ruelle [11].

One might argue that the experiments are done with finite size samples and therefore one might expect that transitions are not sharp but extended over an interval of temperature which depends on sample size. However, for samples which are much larger than the correlation lengths, the approximation of infinite volume should be appropriate. Furthermore, in these cases, due to the calculational difficulties arising from considering large but finite volumes, many questions can be better treated by considering the infinite volume limit. We will not deal here with finite size effects. Although sometimes working in a finite volume, we will take $\Lambda \to \infty$ at the end of the calculations. Experimentally (see e.g.Ahlers [12]), the finite size will only affect measurements which have a resolution in $|t| = |T - T_c| / T_c$ of the order of 10^{-9}. Before that, for the best solid materials, sample inhomogeneities will come into play since they smooth the transitions over a range of about 10^{-4}. For liquid-gas transitions there is a gravitational effect which also acts in a range of the order of $t = 10^{-4}$, due to the density difference between the two phases. For a theoretical treatment of finite size effects see V.Privman and M.Fisher [13] and references therein.

With this in mind, we define the pressure as the limit of eq(5)

$$p(\Phi)=\lim_{\Lambda\to\infty} p_\Lambda(\Phi) \quad . \tag{7}$$

Whether the limit exists depends on the interaction Φ. It can be proved (see e.g.Simon [8])that if $\Phi\in B$ then the limit exists. An interesting case of interactions outside B for which the limit exists is the Coulomb interaction.

It should be noted that calling p the pressure is terminology borrowed from the grand canonical ensemble and that the pressure as defined above is minus the free energy per site times β.

The interesting point to make about the pressure is that by considering arbitrary interactions, the thermal averages or expectation values of functions can be obtained by taking directional derivatives, i.e., by looking at the tangent functionals to the pressure. For example, for $A\in C^X$ define the interaction Ψ_A^X to be the translate of A

$$\Psi_A^X(X+i)=\tau_i A \quad \text{for} \quad i\in Z^\nu$$

$$\Psi_A^X(Y)=0 \quad \text{if } Y \textit{ is not a translate of } X \; .$$

Then

$$\frac{1}{|\Lambda|} < \sum_{i+X\subset\Lambda} \tau_i A > \; = \; \frac{-1}{\beta}\frac{d}{dt} p_\Lambda(\Phi+t\,\Psi_A^X) \quad . \tag{8}$$

This procedure may be used to define translation invariant equilibrium states of the infinite system for interactions on the B space (see e.g., [14]). This is equivalent to an alternative definition using the Variational Principle, which says that the invariant equilibrium states minimize the free energy per site. This is directly related to the fact that the pressure is convex in the interactions, which is easily seen using the convexity of exp(.) and Holder's inequality:

$$p_\Lambda(t\Phi+(1-t)\Psi) \leq tp_\Lambda(\Phi)+(1-t)p_\Lambda(\Psi) \, . \tag{9}$$

This can be extended to the infinite volume for Φ and $\Psi \in B$.

We now introduce a piece of notation, the concept of boundary conditions and yet another definition of equilibrium states, which is useful even for noninvariant states, in terms of the so called DLR equations. DLR stands for Dobrushin-Lanford-Ruelle [15,16].

Given Λ' and $\Lambda, \Lambda\subset\Lambda', t\in\Omega^{\Lambda'|\Lambda}, s\in\Omega^\Lambda$ then let $s\times t$ denote

$$\begin{aligned}(s\times t)_\alpha &= s_\alpha \quad \text{if } \alpha\in\Lambda \\ &= t_\alpha \quad \text{if } \alpha\in\Lambda'|\Lambda \quad .\end{aligned} \tag{10}$$

Consider now the Hamiltonian

$$H_\Lambda(s/t) = \sum_{X\cap\Lambda\neq\phi} \Phi(X)(s\times t) \, . \tag{11}$$

The partition function $Z_\Lambda(s/t)$ obtained by summing over all configurations on Ω^Λ keeping a configuration of $t\in\Omega^{\Lambda'|\Lambda}$ fixed is the partition function with boundary conditions $t\in\Omega^{\Lambda'|\Lambda}$.

The DLR equations are suggested by comparing different configurations in Λ with the same boundary conditions.

If $\Phi\subset B_0$ (finite range) we take Λ' to include all sites i for which there is an X such that $i\in X$ and $X\cap\Lambda\neq\phi$. Then the conditional probabilities of configurations s_1 and $s_2\in\Omega^\Lambda$, given $t\in\Omega^{\Lambda'|\Lambda}$ fixed are related by

$$P(s_1/t) = P(s_2/t)\, e^{\beta(H_\Lambda(s_2/t)-H_\Lambda(s_1/t))} \quad . \tag{12}$$

$P(s_i/t)$ are conditional probability densities if Ω is not discrete. Then for a state $\rho_\Lambda{}'$ and $f\in C_\Lambda{}'$ we can associate a measure $\rho_\Lambda{}'(s,dt)$ on $\Omega^{\Lambda'|\Lambda}$ by

$$\rho_{\Lambda'}(f) \equiv \langle f \rangle_{\Lambda'} = \int_{\Omega^\Lambda} \int_{\Omega^{\Lambda' \setminus \Lambda}} f(s \times t) \rho_{\Lambda'}(s, dt) d\mu(s) \tag{13a}$$

($d\mu(s) = \prod_{\alpha \in \Lambda} d\mu_\alpha$) such that

$$\rho_{\Lambda'}(s_1, dt) = \rho_{\Lambda'}(s_2, dt)\, e^{\beta(H_\Lambda(s_2/t) - H_\Lambda(s_1/t))} \quad . \tag{13b}$$

The important point is that a similar formula holds for $\Lambda' = Z^\nu$ and $\Phi \subset B$, therefore defining properties of the states of the infinite system. It can be proved that invariant states satisfying these equations also satisfy the variational principle and are thus also equilibrium states. In many practical cases it is better to work with the DLR states since one can then make statements which hold in the thermodynamic limit.

Another way, equivalent to the DLR equations, of defining the equilibrium states is through a set of inequalities proposed by Fannes, Vanheuverzwijn and Verbeure [17].

Suppose there is a group G of transformations $\Omega \to \Omega$ such that for all $s, t \in \Omega$ there is a $U \in G$ with the property $Us = t$ (U acts transitively on Ω). Denote by Q the set of maps from C_{Z^ν} onto C_{Z^ν} which satisfy

(i) the measure $d\mu$ is invariant under U.

(ii) $U = \prod_{i \in \Lambda} U_i$, where $U_i \in G_i$ and G_i is a copy of G at site i, Λ is bounded.

Note that U is invertible and $(Us)_\alpha = s_\alpha$ if α is not in Λ.

For any $U \in Q$ define $\tilde{U}$ on C_{Z^ν} by

$$(\tilde{U} f)(s) = f(U^{-1} s)$$

for $f \in C_{Z^\nu}$, $s \in \Omega^{Z^\nu}$.

For $H \in B$ denote by

$$\tilde{U} H - H = \lim_{\Lambda' \to \infty} \tilde{U} H_{\Lambda'} - H_{\Lambda'} \ .$$

Then we have

Theorem [17]

For all $f \in C_{Z^\nu}$, $f \geq 0, f \neq 0$ and $U \in Q$ a state ρ satisfies

$$\rho(f) \leq \rho(\tilde{U}f) \exp \beta \frac{\rho((\tilde{U}^{-1}H - H)f)}{\rho(f)} \tag{14}$$

if and only if ρ satisfies the DLR equations.

Remarks :

1) DLR imply eq(14) under the weaker conditions on Q that : (i) $d\mu$ is invariant under U, (ii) U is invertible and, (iii) $(Us)_\alpha = s_\alpha$ if α is not in Λ.

2) For the converse to be true we need the stronger conditions which are satisfied by all current models.

3) In most applications we only need the weaker result

$$\rho(f) \leq \rho(\tilde{U}f) \exp \|\tilde{U}^{-1}H - H\| , \tag{15}$$

a direct consequence of Jensen's inequality.

4) The inequalities are statements about the infinite volume system.

5) We only prove the first part since we will not use the second.For a proof that the inequalities imply the DLR equations see the original reference [17].

Proof

DLR imply the inequalities. Since $f \geq 0, f \neq 0$ and $\rho(f) > 0$

$$\rho(\tilde{U}f) = \int_{\Omega^\Lambda} \int_{\Omega^{\Lambda 1 \Lambda}} f(U^{-1}(s \times t))\rho(s, dt) d\mu(s) \ .$$

Using condition (1iii): $f(U^{-1}(s \times t) = f((U^{-1}s) \times t)$, change variables $s \to Us$, then

$$\rho(\tilde{U}f) = \int_{\Omega^\Lambda} \int_{\Omega^{\Lambda 1 \Lambda}} f(s \times t)\rho(Us, dt)\ d\mu(s)$$

since $d\mu$ is invariant. Using eq(13b) for infinite volume

$$\rho(\tilde{U}f) = \int_{\Omega^\Lambda}\int_{\Omega^{\Lambda|\Lambda}} f(s\times t)\,\rho(s,dt)\; e^{\beta H_\Lambda(s/t)-\beta H_\Lambda(Us/t)} d\mu(s) \;.$$

Multiply and divide the rhs by $\rho(f)$, and using that

$$d\nu = \frac{f(s\times t)\rho(s,dt)d\mu(s)}{\rho(f)}$$

is a probability measure ,and the convexity of the exponential, we can apply Jensen's inequality to

$$\rho(\tilde{U}f) = \rho(f) \int_{\Omega^\Lambda}\int_{\Omega^{\Lambda|\Lambda}} e^{\beta H(s/t) - \beta H(Us/t)} d\nu$$

to obtain

$$\rho(\tilde{U}f) \geq \rho(f)\; \exp\beta\int_{\Omega^\Lambda}\int_{\Omega^{\Lambda|\Lambda}} \frac{f(s\times t)[H(s/t)-H(Us/t)]\rho(s,dt)d\mu(s)}{\rho(f)}$$

$$\rho(\tilde{U}f) \geq \rho(f)\; \exp\; \frac{\rho(f\,\beta(H(s/t)-H(Us/t)))}{\rho(f)}$$

$$\rho(\tilde{U}f) \geq \rho(f)\; \exp\; \beta\frac{\rho(f\,(H-\tilde{U}^{-1}H))}{\rho(f)}$$

which is equation (14).

These inequalities are very useful in applications if a bound of the type

$$\exp\,\beta\frac{\rho(f\,(H-\tilde{U}^{-1}H))}{\rho(f)} \leq K$$

can be obtained, where K is independent of the domain of f, in view of the fact that if two extremal DLR states are mutually absolutely continuous they are equal, proving that the states are invariant under the transformation U, (S.Sakai [18,19], J.Bricmont, J.Lebowitz and C.Pfister [20]).

II. Continuous Symmetry

The absence of spontaneous symmetry breaking in the one dimensional nearest neighbor interaction Ising model led to the conjecture, by Ising, that the same would happen for the higher dimensional cases. This was proved to be wrong by Peierls, by specifically obtaining a lower bound for the critical temperature. It also happens that there is spontaneous symmetry breaking in one dimension if the interactions are allowed to be of long range (decaying as $1/r^2$ or slower), as shown first by Dyson for the hierarchical models. For systems with continuous symmetries, a similar effect occurs in two dimensions. The word similar is used here in the sense that if the interactions decay sufficiently fast, in two dimensions there is no spontaneous symmetry breaking. In higher dimensional lattices, even those models with short range interactions exhibit spontaneous symmetry breaking for non-zero temperatures. The reason for this phenomenon may be traced to the existence of excitations of arbitrarily small energies which disorder the system at any finite temperature. A similar result, known as the Coleman theorem [21], is that continuous symmetries cannot be spontaneously broken in two-dimensional (continuum) field theories, since if they could, massless Goldstone bosons would appear, which is not allowed by strong infrared divergences.

Several results concerning this type of phenomenon have appeared in the literature. The first rigorous results in this subject were obtained by Mermin and Wagner [22] and Mermin [23] who showed that the spontaneous magnetization was zero for some quantum Heisenberg ferromagnets and for classical systems, respectively, and also by Hohenberg [24] who showed that certain Bose gases do not undergo a Bose condensation in two dimensions. A deeper result was obtained by Dobrushin and Shlosman [25], showing that every equilibrium state is invariant under the continuous symmetry, for classical systems with finite range. This result has been extended [26,33] to include some long range interactions, with even optimal results for some classes of classical spin systems. The first results concerning the invariance of states were obtained by Garrison, Wong and Morisson [52].

In this chapter we will look at some of the proofs of what has been called the Mermin and Wagner phenomenon. We will not concentrate on obtaining optimal results concerning the range of interactions but, we will study the mechanisms behind the methods in order to see if they can be generalized to study systems with discrete symmetries. In Chapter III an argument will be presented against this possible generalization.

II.2. Classical Bogoliubov Inequalities and Local Ward Identities

The first rigorous proofs of the absence of symmetry breaking for certain classical models in two dimensions by Mermin used the classical analogue of the Bogoliubov inequality for quantum systems. Driessler, Landau and Fernando Perez [27] have obtained general results concerning this inequality, in which they emphasize the fact that it is related to the existence of a one-parameter group of transformations leaving the measure invariant. This symmetry leads to a set of identities, in much the same manner as gauge invariance leads to the Ward identities (or Slavnov Taylor identities) in gauge theories, and are thus called local Ward identities.

Consider a one parameter group of transformations that acts on Ω

$$U(t):\Omega\to\Omega \ .$$

For $A \in C_{\mathbb{Z}^2}$ define A' by

$$A' = \frac{d}{dt}\Big[U(t)A\Big]_{t=0} \ . \tag{1}$$

Driessler *et al.* prove the following :

Proposition 1

$$<H'>=0 \tag{2}$$

$$<A'>=\beta<H'A> \ . \tag{3}$$

Proof. $<A>=\int d\mu e^{-\beta H}A$ ($e^{-\beta H}d\mu$ is normalized : $\int d\mu e^{-\beta H}=1$)

Since the measure is invariant

$$1=\int d\mu e^{-\beta H}=\int d\mu e^{-\beta(U(t)H)}$$

then

$$0 = \ln\int d\mu e^{-\beta(U(t)H)} = \frac{d}{dt}\ln\int d\mu e^{-\beta(U(t)H)}\Big|_{t=0} =$$

$$= \int d\mu(-\beta H\,')e^{-\beta H} = <H\,'>,$$

proving Equation (2).

For (3):

$$<U(t)A> = \int d\mu e^{-\beta U(t)^{-1}H}A$$

$$\frac{d}{dt}<U(t)A>\Big|_{t=0} = <A'> = \beta\int d\mu(-\frac{d}{dt}(U(t)^{-1}H)e^{-\beta U(t)^{-1}H}A\Big|_{t=0}$$

$$<A'> = \beta<H\,'A> \quad .$$

The classical Bogoliubov inequality can now be obtained by using Schwartz inequality $|<AB>|^2 \leq <|A|^2><|B|^2>$, and equation (3)

$$|<A'>|^2 = \beta^2|<H\,'A>|^2 \leq \beta^2<A^2><H\,'^2> \tag{4}$$

Using the notation $H'' = (H\,')'$ and eq.(3) for $A = H\,'$

$<H''> = \beta<H\,'H\,'>$ obtaining

$$|<A'>|^2 \leq \beta<A^2><H''> \tag{5}$$

The importance of eq. (5) in applications is derived from the fact that $<A'>$ is bounded by a product of two factors, one of them independent of A. If $<H''>$ can be proved to be zero for a suitable transformation U, then the state will be invariant under U. It is this separation that allows one to obtain information about a general state.

The Ward identities are also useful in proving the existence of critical lengths in some classical models in dimension ≥ 3 and obtaining upper bounds (mean field) for critical temperatures [27]. Actually these problems provided the motivation for their study. For mean field upper bounds on the critical temperatures using the Ward identities see also B.Simon [28].

II.3 The Mermin and Wagner Theorem

The initial proofs of the Mermin and Wagner theorem showed that as the external magnetic field h is reduced to zero the spontaneous magnetization reduces logarithmically in h (like $(|\ln h|)^{-1/2}$).

It is of more interest to obtain results concerning the invariance of the states under global $O(2)$ transformations.

Let A be as in the previous section and U and $\tilde{U}$ be given by

$$U(t) = \prod_{i \in Z^2} u_t(i) \tag{1.a}$$

$$\tilde{U}(t) = \prod_{i \in Z^2} u_{f(i)t}(i) \tag{1.b}$$

where $f(i): Z^2 \to R$ will be chosen later; $u_t(i)$ is a single spin transformation.

The idea is to choose $f(i)$ such that inside a bounded region Λ_0, $U = \tilde{U}$ and far away from Λ_0, $\tilde{U} = 1$. Then one proves that $<\tilde{H}''>$ can be made arbitrarily small, which implies, through (5), that $<A>$ is invariant under U for $A \in \Omega^{\Lambda_0}$. Since Λ_0 is arbitrary it then follows that the state is invariant.

We will just treat the case of nearest neighbor interactions (Simon [8]). The general case has been studied by Klein *et al.* [29] and by Bonato *et al.* [30], who obtain the best possible conditions on the interaction range and also study clustering properties using the same approach.

Consider

$$\tilde{U} = \prod_{i \varepsilon \Lambda} u_{f(i)t} \tag{2}$$

where $u_{f(i)t}$ acts on the angle θ_i as $u_{f(i)t}\theta_i = \theta_i + tf(i)$.

Then

$$\tilde{U}H = -\sum_{\langle ij \rangle} \cos(\theta_i - \theta_j + t(f(i) - f(j)) \tag{3}$$

and

$$\frac{d^2}{dt^2}\tilde{U}H|_{t=0} = \sum_{ij}(f(i)-f(j))^2\cos(\theta_i-\theta_j)$$

$$|<\frac{d^2}{dt^2}\tilde{U}H>|_{t=0} \le \sum_{ij}(f(i)-f(j))^2 \tag{4}$$

by eq.(5) of the previous section

$$<\frac{d}{dt}\tilde{U}A|_{t=0}>^2 \le \beta<A^2>\sum_{ij}(f(i)-f(j))^2 \ . \tag{5}$$

Remembering that f has to satisfy

$$f(i) = \begin{cases} 1 & i \in \Lambda_0 \\ 0 & |i| >> diam\,\Lambda_0 \end{cases} \tag{6}$$

we consider $g_\varepsilon(x): R^2 \to R$ such that

$$g_\varepsilon(x) = \begin{cases} 1 & |x| \le 1 \\ |x|^{-\varepsilon} & |x| \ge 1. \end{cases} \tag{7}$$

Note that

$$\int \left|\nabla g_\varepsilon(x)\right|^2 d^2x = \pi\varepsilon \tag{8}$$

and

$$\int \left|\nabla g_\varepsilon(x)\right|^2 = \lim_{n\to\infty}\sum_{ij}\frac{1}{n^2}\left[\left(g_\varepsilon(in^{-1})-g_\varepsilon(jn^{-1})\right)n\right]^2$$

$$= \lim_{n\to\infty}\sum_{ij}\left[g_\varepsilon(in^{-1})-g_\varepsilon(jn^{-1})\right]^2 \ . \tag{9}$$

Therefore, if $f(i)=g_\varepsilon(in^{-1})$ is chosen, $<\frac{d^2}{dt^2}\tilde{U}H|_{t=0}>$ can be made arbitrarily small and the state proved to be invariant since Λ_0 is arbitrary. As mentioned before, this method can be used to study clustering properties.

However, it "only" obtains logarithmic decay lower bounds [50], when we know that in some models (McBryan and Spencer [31]) the correlation function decays like a power. See Section III.2. for more comments on this.

II-4 Energy Entropy Inequalities - A

The idea behind the method described here (Simon & Sokal [32]) proving that the spontaneous magnetization for the plane rotor model is zero, is a very nice formalization of arguments concerned with the balance of entropy and energy that go back to Herring and Kittel [53]. It also draws from the concept of relative entropy introduced by Araki [54]. Heuristically, one initially considers a magnetized phase, represented by a configuration with all spins parallel. One can decrease the free energy by doing the following: Flip all spins inside a block of side L and surround the block with layers of spins rotated very slowly. For example, rotate the first layer by $\pi[1-(1/L)]$, the next by $\pi[1-(2/L)]$ and so on, so that after L layers, the spins are left unchanged. This transformation resembles that performed in the previous section. The energy is increased by $[1-cos(2\pi/L)]L^2 \approx$ constant. The entropy gain is related to the number of places where we could choose the block $\approx \ln(\Lambda/L^2)$($\Lambda \approx$ volume of the system). For large volumes the entropy gain dominates the energy shift and the free energy is decreased for any temperature T > 0. Thus the initial magnetized phase cannot be an equilibrium state.

Needless to say, this is not at all rigorous, since the argument is cavalier about notions of phase and entropy. Also, it is not clear why it does not work for more than two dimensions, since the requirement would be to dominate an energy $\approx L^{d-2}$ with an entropy $\approx ln(\Lambda/L^d)$, which can be done if Λ is large enough.

The method is as follows: First compare the entropy S_{mix} of a statistical mixture of n states, each obtained by a suitable transformation of the original state, and the entropy of the original state S_0. Simon and Sokal obtain that (see Theorem 1 below):

$$S_{mix} - S_0 \leq -(\text{Energy shift})$$

Then the entropy of the mixture is compared with the average of the entropies S_i of the mixture components. If the mixture components were totally disjoint one would have

$$S_{mix} = (1/n)\sum_i S_i + ln(n)$$

For not totally disjoint but "almost disjoint" components (see Theorem 2) Simon and Sokal prove that

$$S_{mix} \geq (1/n)\sum_i S_i + ln(n) - constant$$

If the transformations performed leave the *a priori* measure invariant, then $(1/n)\sum_i S_i = S_0$. For large n, the two inequalities are contradictory since the energy shift can be shown to be finite and one concludes that the components are not "almost disjoint". The important point is that if one assumes the spontaneous magnetization (infinite volume) to be non zero it can be proved that the components are almost disjoint in the precise sense of theorem 2, conditions (i) and (ii). Therefore, the magnetization has to vanish.

The notation is as follows:

Λ is the volume of the lattice.

$$g = \frac{1}{Z_\Lambda} e^{-BH_\Lambda} , \; f = \frac{1}{n}\sum_{i=1}^{n} f_i \, , \; f_i = T_i \, g$$

$d\mu_0$ the *a priori* measure (a probability measure).

$d\nu_i = f_i \, d\mu_0$ (probability measure, if T_i leaves $d\mu_0$ invariant).

T_i are transformations which rotate the spins as suggested by the heuristic argument. See theorem 2.

Theorem 1

Let $S(f)=-\int f \ln f \, d\mu_0$, then $S(f)+\int f \ln g \, d\mu_0 \leq 0$.

Proof

This is a direct consequence of Jensen's inequality and the convexity of - log

$$S(f)+\int f \ln g \, d\mu_0 = \int \ln(g/f) f \, d\mu_0 \leq \ln(\int g \, d\mu_0) = 0 \quad .$$

Note that $\int f \ln g \, d\mu_0$ can be rewritten as

$$\int f \ln g \, d\mu_0 = \int g \ln g \, d\mu_0 + \int (f-g) \ln g \, d\mu_0 = -S(g)-\Delta E$$

where ΔE is the "energy shift"

$$\Delta E = -\int (f-g) \ln g \, d\mu = \frac{-1}{n} \sum_i \int g \, (\ln \tilde{f}_i - \ln g) \, d\mu$$

where $\tilde{f}_i$ is obtained from g by transforming with T_i^{-1}.

Therefore Theorem 1 expresses the first inequality

$$S(f)-S(g) \leq -\Delta E \quad . \tag{1}$$

The second inequality is obtained from the following theorem.

Theorem 2

If the components are almost disjoint in the following sense:

There exists a $c \geq 0$ such for each i there is a set A_i (A_i^c the complement of A_i such that $A_i \cup A_i^c = \Omega^\Lambda$, the internal space) with

(i) $\int_{A_i^c} d\nu_i = \nu_i(A_i^c) \leq \frac{c}{n}$

(ii) $\sum_{j\neq i} \int_{A_i} d\nu_j = \sum_{j\neq i} \nu_j(A_i) \leq c$

then

$$S(f) - \frac{1}{n}\sum S(f_i) \geq \ln n - \Delta c^{1/2}. \qquad (2)$$

Proof

We just follow the proof by Simon and Sokal:

$$S(f) = -\frac{1}{n}\sum S(f_i) - \ln n =$$

$$= \frac{1}{n}\{-\sum_{i=1}^{n} \int f_i \ln[\frac{\sum f_i}{f_i}]\, d\mu_0\} =$$

$$= -\frac{2}{n}\{\sum_{i=1}^{n} \int d\nu_i \ln(\frac{\sum_j f_j}{f_j})^{1/2}\}$$

by Jensen's inequality:

$$\geq -\frac{2}{n}\sum_{i=1}^{n} \ln\int d\mu_0\, (f_i^2 + \sum_{j\neq i} f_i f_j)^{1/2}$$

using $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$

$$\geq -\frac{2}{n}\sum_{i=1}^{n} ln[\, 1 + \int d\mu_0\, (\sum_{j\neq i} f_i f_j)^{1/2}] \;,$$

using $-\ln(1+x) > -x$

$$\geq -\frac{2}{n}\sum_{i=1}^{n} 1 + \int d\mu_0\, (\sum_{j\neq i} f_i f_j)^{1/2}$$

$$\geq \frac{-2}{n}\sum_{i=1}^{n}[\int_{A_i} d\mu_0 f_i^{1/2}(\sum_{j\neq i} f_j)^{1/2} + \int_{A_i^c} d\mu_0 f_i^{1/2}(\sum_{j\neq i} f_j)^{1/2}] \;,$$

using Schwarz inequality: $\int fg \leq (\sqrt{\int f^2})(\sqrt{\int g^2})$

$$\geq \frac{-2}{n}\sum_{i=1}^{n}\{\nu_i(A_i)^{1/2}[\sum_{j\neq i}\nu_j(A_i)]^{1/2} + \nu_i(A_i^c)^{1/2}[\sum_{j\neq i}\nu_j(A_i^c)]^{1/2}\} \;,$$

using conditions (i) and (ii) finally

$$\geq \frac{-2}{n} n \{1.c^{1/2}+(c/n)^{1/2}(n-1)^{1/2}\} \geq -4c^{1/2}$$

is obtained, proving Theorem 2 .

Theorems 1 and 2 have made no assumption about the form of the transformations T_i or how conditions (i) and (ii) can be satisfied. The subsets A_i and A_i^c have not been specified yet.

These conditions can be obtained, using a Chebyshev inequality under the assumption that the infinite volume spontaneous magnetization is not zero. But first, a comment about the range of interactions is necessary.

The restriction that the energy shift be finite constrains the range of the interaction. Simon and Sokal prove the following theorem.

Theorem 3

The spontaneous magnetization is zero for a plane rotor with Hamiltonian $H=-\sum_{ij} J(i-j)\cos(\vartheta_i-\vartheta_j)$, $J(i-j)\geq 0$ and

$$\sum_{i\neq 0} |i|^2 J(i) < \infty \tag{3}$$

This constraint roughly states that the couplings decay as $r^{-4-\varepsilon}$, $\varepsilon>0$ or faster. Pfister [33] proves a stronger result, which for this case is:

Theorem 4

All states are O(2) invariant if there exist a constant C > 0 and an integer p so that

$$\sum_{|i|\leq L} |i|^2 J(i) < CF_p(L) \tag{4}$$

where $F_p(L)$ diverges at most like $ln(L)\ln_2(L)\cdots\ln_p(L)$ for large L and $\ln_k(.)=\ln\ln_{k-1}(.)$ (See [33] for a proof).

This constraint is such that the theorem is valid if the coupling behaves like $r^{-4}\ln_2 r \cdots \ln_p r$, for large distances, a behavior which seems critical in view of the fact that the theorem is false if J(r) $\approx r^{-4}\ln_2 r \cdots \ln_{p-1} r(\ln_p r)^{1+\varepsilon}$. See comment 4 of ref.[33] . Pfister's method is related to the one presented in the next section. However, his main contribution lies in the way the energy shifts are bounded, an idea which could be used in the present context.

Now let us return to the problem. Consider a volume Λ of size $(2l+1).(2l+1)$ centered at the origin with all spins outside Λ pointing in the 1- direction. The transformations T_i will be such that they rotate the spins in a $3k \times 3k$ block $\tilde{B}_i$, flipping the spins inside a block B_i (concentric to $\tilde{B}_i$) of $k \times k$ spins and slowly rotating those spins outside B_i: rotate by $\pi[1-(1/k)]$ the first layer of spins outside B_i, by $\pi[1-(2/k)]$ the next, and so on until the last layer, the border of $\tilde{B}_i$, is rotated by π/k. We have the liberty of choosing l, k, and n as fit.

The concept of "almost disjoint", as stated by condition (i) and (ii) is as follows:

Choose A_i as the subset of Ω^Λ such that

$$A_i\colon \sum_{\alpha\varepsilon B_i} \sigma_\alpha^{(1)} > 0$$

and (5)

$$A_i^c\colon \sum_{\alpha\varepsilon B_i} \sigma_\alpha^{(1)} \leq 0$$

$\sigma_\alpha^{(1)}$ is the component in the "1-direction" of the spin $\vec{\sigma}_\alpha$. With this choice

$$\nu(A_i^c) = \textit{Probability that} \sum_{\alpha\in B_i} \sigma_\alpha^{(1)} \leq 0 = P_{+,\Lambda}(\sum_{\alpha\in B_i} \sigma_\alpha^{(1)})$$

$$\nu(A_i^c) = \frac{1}{Z_\Lambda} \int_{\sum\sigma_\alpha^{(1)}\leq 0} e^{-\beta H} d\mu_0 \ .$$

Denoting the states by $< . >_{+,\lambda}$, call

$$a = <\sum_{B_i} \sigma_\alpha^{(1)}>_{+,\lambda} ,$$

and

$$F = \sum_{B_i} \sigma^{(1)} - a \quad .$$

Choose $b < a$ then

$$P_{+,\Lambda}(\sum_{B_i} \sigma_\alpha^{(1)} \leq b) = \int_{\sum \sigma^{(1)} \leq b} \frac{e^{-\beta H}}{Z_\Lambda} d\mu_0$$
$$= \frac{1}{(b-a)^2} \int_{\sum \sigma^{(1)} \leq b} (b-a)^2 \frac{e^{-\beta H}}{Z_\Lambda} d\mu_0$$
$$\leq \frac{1}{(b-a)^2} \int_{\sum \sigma^{(1)} \leq b} (\sum_{B_i} \sigma_\alpha^{(1)} - a)^2 \frac{e^{-\beta H}}{Z_\Lambda} d\mu 0$$

since $(x-a)^2$ is minimum when x=b for $x \leq b$. By adding positive contributions, we can extend the integral to Ω^Λ, for b=0

$$P(\sum_B \sigma^1 \leq 0) \leq \frac{1}{a^2} <F^2>_{+,\lambda} \quad .$$

It is easy to understand that $<\sum \sigma^{(1)}>_{+,\lambda}$ decreases when the interactions are decreased, which is what effectively happens if we increase the volume Λ. This, in fact, can be rigorously proved by the correlation inequalities of Griffiths [34], Kelly and Sherman [35] and Ginibre [36]. Thus, by a GKS inequality:

$$a \geq mk^2$$

where $m = <\sigma^1>_{+,\infty}$ is the spontaneous magnetization.

Assume $m \neq 0$, then

$$P_{+,\Lambda}(\sum_{B_i} \sigma_\alpha^i \leq 0) \leq \frac{1}{m^2 k^4} <F^2>_{+,\Lambda} \tag{6}$$

and is beginning to look like condition (i), provided $<F^2>$ is well behaved. It can be proved that

$$\frac{1}{k^4}<F^2>_{+,\Lambda}=\frac{1}{k^4}\sum_{\alpha,\gamma\in B_i}[<\sigma_\alpha\sigma_\gamma>_{+,\Lambda}-<\sigma_\alpha>_{+,\Lambda}<\sigma_\gamma>_{+,\Lambda}] \tag{7}$$

can be made arbitrarily small when $l\to\infty$ since $<.>_{+,\infty}$ is ergodic. It satisfies some decomposition properties, see, e.g., Israel [14].

Now given n choose k and l (sufficiently large) so that

$$P_{+,\Lambda}(\sum_{\alpha\in B_i}\sigma_\alpha^{(1)}\leq 0)\leq\frac{1}{n} \tag{8}$$

for all $1\leq i\leq n$. l is chosen so that (7) can be made small and the $\tilde{B}_i$ blocks do not intersect.

If (8) is satisfied we can apply Theorem 2. Combining with Theorem 1 (remember that T_i leave the measure $d\mu_0$ invariant).

$$\ln(n) - \Delta C^{1/2}\leq S(f) - S(g)\leq -\Delta E \ . \tag{9}$$

The next step is to bound the energy shift ΔE

$$\Delta E = \int (g-f)\, lng\ d\mu$$

$$\Delta E=\frac{1}{n}\sum\Delta E_i,$$

$$\Delta E_i=\frac{1}{Z}\int e^{-\beta H}(H-T_i^{-1}H) \ .$$

The only contributions to $H-T_i^{-1}H$ come from the interactions of spins inside $\tilde{B}_i$ with

a) spins outside Λ

b) spins inside Λ.

By taking Λ sufficiently large (a) can be made arbitrarily small.

T_i^{-1} can be written as a product of single spin rotations

$$T_i^{-1} = \prod_{\alpha \in \tilde{B}_i} R_{i\alpha} \ .$$

Then the contribution from (b) will be of the form

$$\Delta E_i = - \sum_{<\alpha,\gamma> \in \Lambda} J(\alpha-\gamma)[<R_{i\alpha}\vec{\sigma}_\alpha \cdot R_{i\gamma}\vec{\sigma}_\gamma>_{+,\Lambda} - <\vec{\sigma}_\alpha \cdot \vec{\sigma}_\gamma>_{+,\Lambda}]$$

$$\Delta E_i = \sum_{<\alpha,\gamma> \in \Lambda} J(\alpha-\gamma)[<\cos(\theta_\alpha-\theta_\gamma+\varphi_\alpha-\varphi_\gamma)> - <\cos(\theta_\alpha-\theta_\gamma)>]$$

$$|\Delta E_i| \le \sum J(\alpha-\gamma) | \cos(\varphi_\alpha-\varphi_\gamma) - 1 | \le \sum J(\alpha-\gamma) \frac{(\varphi_\alpha-\varphi_\gamma)^2}{2} \ .$$

Where φ_α is given by

$$\varphi = \pi(1-\frac{r_\alpha}{k})$$

$$r_\alpha = \begin{cases} 0, & \alpha=\alpha_B \in B \\ \|\alpha\| - \frac{k}{2}, & \alpha=\tilde{\alpha} \in \tilde{B} | B \ , \ \|\alpha\| = \max\{\alpha_x, \alpha_y\} \\ k, & \alpha=\bar{\alpha} \in \Lambda | \tilde{B} \end{cases}$$

then

$$\Delta E_i \le \frac{\pi^2}{2k^2} \sum J(\alpha-\gamma)(r_\alpha-r_\gamma)^2 \ .$$

Write $\sum_{\alpha,\gamma}$ as

$$\sum_{\alpha\gamma} J(\alpha-\gamma)(r_\alpha-r_\gamma)^2 = \sum_{\alpha_B\gamma_B} + \sum_{\alpha_B\tilde{\gamma}} + \sum_{\alpha_B\bar{\gamma}} + \sum_{\tilde{\alpha}\tilde{\gamma}} + \sum_{\tilde{\alpha}\bar{\gamma}} + \sum_{\bar{\alpha}\bar{\gamma}} \ .$$

The first and last terms vanish and each of the other four terms can be bounded by

$$\sum_{\alpha\gamma} \leq Ck^2 \sum_{\alpha} J(\alpha)|\alpha|^2 \ .$$

Therefore, there exists a $K<\infty$ such that

$$|\Delta E_i| < K$$

and (9) is contradictory if n is sufficiently large, proving that the magnetization vanishes for any non-zero temperature.

Finally, we note that the most interesting use of this method is the proof, modulo a slight technical assumption, of the so called Thouless effect [37], that the magnetization of the one-dimensional $1/r^2$ Ising model is discontinuous at the critical temperature [32]. The energy shift diverges logarithmically and still can be controlled by the entropy. That infinite energy shifts can be controlled using this method seems to be a characteristic of one-dimensional lattices. For a comment on this see Chapter III.

II-5 Energy Entropy Inequalities-B

In this section we present yet another method to prove the absence of spontaneous symmetry breaking in two dimensions. The argument consists of a very simple application of the powerful energy entropy inequalities of Fannes, Vanheuverzwijn and Verbeure [17], (see section I.2) and a trick already used in Section II.3 to control the second order energy shifts.

The result to be proved is the following.

Theorem

The states of the (nearest neighbor interaction) plane rotor model are $O(2)$ invariant for any temperature $T>0$.

Remarks

1. Pfister, [33], controls the second order energy shifts for long range interactions. See Theorem 4 of Section II.4 for constraints on the decay of the interactions.

2. In the previous section the magnetization was proved to be null. That method can also be used to prove the invariance of the states under group transformations, [32].

Proof : Let g be an even function of each of its arguments θ_i which lie inside a block Λ_0 of $n \times n$ sites.

Let $\tilde{U}$ be the transformation defined in section II.3, that acts on the Hamiltonian

$$H=-\sum_{\substack{<i,j> \\ |i-j|=1}} \cos(\theta_i-\theta_j) \tag{1}$$

as

$$\tilde{U}^{-1}H=-\sum\cos(\theta_i-\theta_j-t(f(i)-f(j))). \tag{2}$$

Now, following section I.2 eq.14, the energy entropy inequality relates the expectation values of g and $\tilde{U}^{-1}$

$$\rho(\tilde{U}g) \geq \rho(g) \exp \beta \frac{\rho(g(H-\tilde{U}^{-1}H)}{\rho(g)} \tag{3}$$

and from equations (1) and (2)

$$\Delta H = H - \tilde{U}^{-1}H =$$

$$= -\sum \cos(\theta_i - \theta_j)[1 - \cos t(f(i) - f(j))] +$$

$$+ \sum \sin(\theta_i - \theta_j) \sin t(f(i) - f(j))$$

so that in $\rho(g\Delta H)$ we have integrals of even times even and even times odd functions. Only the first term (even times even) survives (cosine), but this term can be made as small as we want by choosing f appropriately, as already seen in Section II.3. Therefore $\rho(\tilde{U}g)$ and $\rho(g)$ are proved to be equal. By letting n, and therefore Λ_0, go to ∞, the states are proved to be invariant under global rotations.

III.1 Discrete Symmetry: The Z_p Model

In order to compare the Ising model and the plane rotor models and understand their differences, it is interesting to consider a class of models which interpolate between them. So, allow the spins to point to any of the vertices of a regular p-sided polygon. By maintaining the functional form of the interactions and by appropriately normalizing the *a priori* measures, we can expect that in the $p=2$ case and in the limit of p infinite, we reobtain the Ising and the plane rotor models respectively. We will be interested in how certain quantities, such as the pressure and correlation functions , depend on the number of states p.

These models are invariant under simultaneous rotation of the spins by $2\pi/p$, the $\mathbf{Z_p}$ group and are therefore called $\mathbf{Z_p}$ models. They are also known as the clock or vector Potts model, introduced in the fifties by Domb and Potts [38] and have proved to be interesting also in their own right, since there seems to be a relationship between the two-dimensional classical model and the four-dimensional $\mathbf{Z_p}$ lattice gauge theories, which could be attributed to the similar manner they behave under duality transformations. These gauge theories may be relevant to the problem of color confinement in Quantum Chromodynamics ('t Hooft [39], Polyakov [40]).

The interest in this model is not only theoretical, because it seems to be related to problems of thin films of noble gases in crystalline substrates [41] and problems related to melting in two dimensions [42].

III.2 Phase Structure in Two Dimensions

We will now discuss some results concerning the phase diagram in two dimensions [6], [7], [43]. The Kosterlitz Thouless (KT) phase transition of the plane rotor model , [44], [45] can be understood heuristically in terms of the topological excitations, called vortices. At high temperatures the main contributions to the partition function is due to configurations with unbound vortices. The vortex can be associated with configurations where the spins in a region wind up around a central position in a whirlpool manner. At low temperatures the vortices bind in pairs of opposite handness. The differences between the two phases are shown in the clustering properties. At high temperatures the correlation function decays, as usual, exponentially, while the decay is as an inverse power at low temperatures. The existence of this transition was proved rigorously by Frohlich and Spencer [46] who developed a formalism that might be considered to be the first rigorous version of the Renormalization Group (other rigorous versions of the RG , for example the treatment by Eckmann and Collet [47] of the Hierarchical model, are restricted to very specific models).

For sufficiently high p (and T), one expects that the discrete system will behave like the plane rotors and that it even might exhibit a KT transition. At sufficiently low temperatures, however, the discreteness of the system will cause spontaneous symmetry breaking. So, with hindsight, one expects the $\mathbf{Z_p}$ model to have at least three different phases , for p high enough:broken symmetry phase with long range order at low temperatures; short range ordered or "topologically" ordered phase , with algebraic decay of correlations;and a disordered phase with exponential decay of correlations at high temperatures.

Elitzur, Pearson and Shigemitsu [7] proposed this phase structure for the nearest neighbor (nn) cosine and Villain Z_p models [48]. They proved that for the Villain model the three region-picture is consistent with (i)the existence of the KT transition in the plane rotors Villain model, (ii)self duality and (iii)correlation inequalities (comparing the correlations of the discrete and continuous systems of the original as well as the dual variables). The result is not complete for the cosine interactions , since this model is not self-dual, and also since self-duality does not exclude the possibility of a richer phase structure.

Their results indicate that the temperature of onset of magnetization decreases as $1/p^2$.

Also in [46], Frohlich and Spencer rigorously prove the existence of an intermediate phase with a power law behavior, without explicitly giving the p dependence of the magnetization temperature.

Monte Carlo calculations (see for example , Alcaraz et al. [49]) also suggest a $1/p^2$ dependence of the magnetization temperature.

We will now apply the Peierls-Chessboard argument to obtain a lower bound on this temperature. We obtain a $1/p^2$ behavior for this lower bound that is, there exists a constant K' such that for $T<K'/p^2$ there are multiple phases.

III.3 Lower Bound - Reflection Positivity and the Peierls-Chessboard Argument

The beauty of the Peierls argument lies not only in the simplicity of the proofs of existence of phase transitions, but also in the fact that it gives an intuitive explanation of the mechanism that drives the transition.

Consider the nearest neighbor (n.n.) ferromagnetic Ising model in two dimensions with plus boundary conditions . For any configuration draw unit length lines across the bonds that join two n.n. spins in different states. One readily sees that each configuration will be associated with a set of closed lines called contours. See Figure 4. The probability of having a contour of length $|\gamma|$ can be bounded by the product of two terms: $Ce^{-\beta|\gamma|}$, which is an energy contribution; and the number of such contours which can be bounded by $|\gamma|^2 \exp C|\gamma|$, an entropy contribution. The expectation value of a spin depends on the balance of these two terms and can be shown to be non zero for β sufficiently large.

This choice of contours is well suited for the Ising model but for other models, as is the case for the $\mathbf{Z}_p$ model, this may not be appropriate. Bounding the probability of these contours, requires in these cases more sophisticated methods, such as reflection positivity plus chessboard estimates. For the $\mathbf{Z}_p$ model one can think of separating spins in different states by contours of different type, depending on the angle of separation of the neighboring spins. This is cumbersome and it is not clear how to proceed . There are many possible choices of contours. For example, contours that separate spins pointing in a given fixed direction from the rest (see Figure 5) or separate spins pointing down from those pointing up, where up and down are subsets of the internal space. The problem that arises is that now there is not a one to one mapping between sets of contours and configurations. This means one has to take into account some additional entropy factors. The problem arises in the estimative of the probability of the contours, and is handled by the method of chessboard estimates.

We will now introduce the notion of Reflection Positivity. The origin of such a notion comes from the positivity of the inner product in Hilbert spaces in Quantum Field Theories translated to the Euclidean sector, and is originally due to Osterwalder and Schrader [56]. It was originally used in relation to the study of phase transitions by Glimm, Jaffe and Spencer [57] . Further developments appear in Frolich and Lieb [58], Frolich, Simon and Spencer [59], Dyson, Lieb and Simon [60], and Frolich, Israel, Lieb and Simon [61].

First,we will define and prove Reflection Positivity (RP) for the $\mathbf{Z}_p$ model.

Consider Λ, a $2^N \times 2^N$ square, let $< \, . \, >_0$ denote the noninteracting system expectations and impose periodic boundary conditions. Let λ be any vertical (or horizontal) line that passes between sites.

Let F be a real function of the variables to the right of λ and $\theta_\lambda F$ the "reflected" function, i.e. , if

$$F = f\,(\sigma_{ix,iy})$$

then

$$\theta_\lambda F = f\,(\sigma_{-(ix-\lambda x) Mod 2^N, iy})$$

λx is the coordinate of the vertical line λ.

The measure will be said to be Reflection Positive if the following is satisfied :

$$< F\theta_\lambda F > \geq 0 \,. \qquad \text{(RP)}$$

It is obvious that

$$(< F >_0)^2 \geq 0$$

leading to

$$0 \le (< F >_0)^2 = <F>_0 <\theta_\lambda F>_0 = < F\theta_\lambda F>_0 \quad . \tag{1}$$

Therefore, we have RP for the noninteracting measure.

The nearest neighbor interaction $\mathbf{Z}_p$ Hamiltonian with periodic BC can be written for any λx as

$$H = H_1 + \theta_\lambda H_1 - \sum_{iy} \sigma_{\lambda x + ½, iy} \cdot \sigma_{\lambda x - ½, iy} - \sum_{iy} \sigma_{\lambda x + 2^{N-1} + ½, iy} \cdot \sigma_{\lambda x + 2_{N-1} - ½, iy}$$

or symbolically :

$$H = H_1 + \theta_\lambda H_1 - h\theta_\lambda h - h'\theta_\lambda h' \ .$$

(The minus signs are essential.)

Let $< . >$ denote the usual expectation values :

$$< . > = \frac{< . \exp -\beta H >_0}{< . >_0} \ .$$

Then

$$< F\theta_\lambda F > = \frac{1}{\mathbf{Z}} <(Fe^{-\beta H_1})\theta_\lambda (Fe^{-\beta H_1}) e^{\beta h \theta h} e^{\beta h' \theta h'} >$$

expand the last two exponentials and use eq.(1) for each term to obtain :

$$< F\theta_\lambda F > \ge 0 \ . \tag{2}$$

This proves reflection positivity for the interacting measure.

Combining this result with the proof of the Schwarz inequality, one obtains

$$< F\theta G >^2 \le < F\theta F > < G\theta G > \tag{3}$$

which is a result of great technical utility.

Consider $|\Lambda|$ positive functions f_i of only one variable σ. Equation (3) is useful to study the following form :

$$< \prod_{i\in\Lambda} f_i(\sigma_i) > . \tag{4}$$

Choose a λ plane. This separates Λ into two parts Λ_L and Λ_R , (4) can be written as :

$$< \prod_{i\in\Lambda_R} f_i(\sigma_i)\theta_\lambda \prod_{j\in\Lambda_L} f_j(\sigma_{\theta j}) >$$

where θj is the site obtained by reflecting j .From (3)

$$< \prod_{i\in\Lambda} f_i(\sigma_i) >\le$$

$$< \prod_{i\in\Lambda_R} f_i(\sigma_i) \prod_{i'\in\Lambda_R} f_{i'}(\sigma_{\theta i'}) >^{1/2} < \prod_{j\in\Lambda_L} f_j(\sigma_j) \prod_{j'\in\Lambda_L} f_{j'}(\sigma_{\theta j'}) >^{1/2}$$

this procedure can be repeated for all vertical planes λ and then again for the horizontal ones, to obtain

$$< \prod_{i\in\Lambda} f_i(\sigma_i) >\le \prod_{i\in\Lambda} < \prod_{j\in\Lambda} f_i(\sigma_j) >^{1/|\Lambda|} . \tag{5}$$

The f 's above were functions of a single site. In applications we will need a more general result [61]. Separate the sites into two types, Λ_e and Λ_o, depending on whether the x coordinate is even or odd. Let the site i be of the even type and j its nearest neighbor to the right. Consider the following expectation :

$$< \prod_{i\in\Lambda_e} f_i(\sigma_i) f_j(\sigma_j) > .$$

Using similar arguments to those that led to Eq.(5) we can obtain

$$< \prod_{i\in\Lambda_e} f_i(\sigma_i) f_j(\sigma_j) > \le \prod_{i\in\Lambda_e} A(i) \tag{6}$$

where $A(i)$ is the following

$$A(i) = < \prod_{\alpha-\eta \, a \, \Lambda_e \, bond} f_u(\sigma_\alpha) f_v(\sigma_\eta) >^{2/|\Lambda|}$$

where u (resp v) is i (resp j) or j (resp i) depending on the x coordinate of α , for ($n=0,\pm 1,\pm 2, \cdots$)

$$i_x - \alpha_x = 4n, 4n-1$$

then $u=i, v=j$, and if

$$i_x - \alpha_x = 4n-2, 4n-3$$

then $u=j$ and $v=i$.

The separation done here is one of four possible. One could take the odd components, or look at vertical bonds. From this, three formulae equivalent to (6) result.

The functions f will now be tailored to represent contours separating regions of spins which point in different directions. The contours can be chosen to separate regions of spins pointing, for instance, in the zero-direction from others pointing elsewhere. Another possibility is to consider contours that separate regions with spins "up" from regions with spins "down". A spin will be said to be up if it points in the direction n, with $0 \le n < p/2$ and down if $p/2 \le n \le p-1$.

Let χ_i^n be the characteristic function of the event

" σ_i points in the n^{th} direction "

and χ_i^+ , χ_i^- the characteristic functions for σ_i being up or down, respectively.

It will turn out that the use of the second type of contour has technical advantages in proving the existence of a phase transition. So, let γ be a contour of the second type and call $|\gamma|$ its length. The probability of γ occurring is

$$P(\gamma) = < \prod_{ij} \chi_i^+ \chi_j^- > . \tag{7}$$

The product is taken over all bonds ij that cross the contour, with i inside and j outside γ . As above, four types of bonds can be distinguished, two horizontal types, with the left site having x coordinate even (h.e.) or odd (h.o.), and two vertical, with the lower site's x coordinate even (v.e.) or odd (v.o.). Choose one of these types, making sure that the number of bonds is larger than or equal to $|\gamma|/2$. We can suppose, without any loss, that this condition is satisfied by the h.e. type. "Throw away" the rest by substituting their projectors, in eq.(7), by one. This gives the bound:

$$P(\gamma)=< \prod_{ij h.e.} \chi_i^+ \chi_j^- > \tag{8}$$

where the product is now over only one type. Using Eq.(6)

$$P(\gamma) \leq \prod_{ij h.e. \in \gamma} A_{ij} \tag{9}$$

where A is given by

$$A_{ij} = < \prod_{all\ bonds\ \mu\nu\ h.e.\ in\ \Lambda} \chi_\mu^u \chi_\nu^v >^{2/|\Lambda|} \tag{10}$$

where now (u,v) is $(+,-)$ or $(-,+)$ depending on the x coordinate of i :

$$\text{if } ix - \mu x = 4n \ or \ 4n-1 \ then \ (u,v)=(+,-) \ ,$$

$$\text{if } ix - \mu x = 4n-2 \ or \ 4n-3 \ then \ (u,v)=(-,+) \ .$$

However, due to the periodic boundary conditions, all the A's are equal. Since there are at least $|\gamma|/4$ such A's (or h.e. bonds), then

$$P(\gamma) \leq A^{|\gamma|/4} \ . \tag{11}$$

The important thing that has been accomplished is that we can now estimate A using simple thermodynamic arguments. The structure of A is similar to that of a partition function with the following constraints: two columns of spins pointing "up" alternate with two columns of spins pointing "down".

$$A^{|\Lambda|/2}=\frac{<\prod \chi_\mu^i \chi_\nu^j e^{-\beta H}>_0}{\mathbf{Z}} \tag{12}$$

The bound

$$A^{|\Lambda|/2}\leq\frac{e^{\frac{3}{2}\beta|\Lambda|}}{\mathbf{Z}}\Big[\sum_{n=0}^{[p/2]-1}\sum_{m=[p/2]}^{p-1} e^{\beta\cos\frac{2\pi}{p}(m-n)}\Big]^{\frac{1}{2}|\Lambda|}$$

or

$$A^{|\Lambda|/2}\leq\Big[\frac{e^{2\beta|\Lambda|}}{\sum e^{-\beta H}}\Big]\Big[\sum_{n=0}^{[p/2]-1}\sum_{m=[p/2]}^{p-1} e^{-\beta(1-\cos\frac{2\pi}{p}(m-n))}\Big]^{\frac{1}{2}|\Lambda|} \tag{13}$$

is obtained by substituting for all the interactions of "like" spins (both pointing either up or down) its least possible value, -1. The interactions between "unlike" spins are kept, and summed over. Notice that the entropy contributions are still considered, since the number of configurations is kept the same. Here $[p/2]=p/2$ for p even, and $[p/2]=(p+1)/2$ for p odd.

Now let $\beta=\beta_0 p^2$. We will use the following bound. One can find a constant c such that $1-\cos x>cx^2$ for $|x|<\pi$. Also there is a function $C(\beta_0)$ such that

$$\sum_{n=0}^{[p/2]-1}\sum_{m=[p/2]}^{p-1} e^{-\beta(1-\cos\frac{2\pi}{p}(m-n))}\leq 2\sum_{k=1}^{[p/2]} k e^{-4\beta_0\pi^2 k^2}$$
$$\leq e^{-C(\beta_0)} \tag{14}$$

$C(\beta_0)$ goes to ∞ as β_0 goes to ∞, which means, given K one can find a K' such that for $\beta_0>K'$ one has $C(\beta_0)>K$.

The probability of the contour $\mathrm{P}(\gamma)$, therefore has the following bound :

$$\mathrm{P}(\gamma)\leq\exp[-\frac{|\gamma|}{4}C(\beta_0)]\ . \tag{15}$$

We now turn to the problem of proving the existence of a phase transition. Since we have worked all along with periodic boundary conditions, we cannot calculate a magnetization, because by symmetry the expected value of any spin component is zero. However, we will be able to prove the existence of multiple phases, by proving that under certain conditions the state is not ergodic. That is, it fails to satisfy certain decomposition properties, and therefore it is not a pure state. See for instance Israel [14]. For a pure state the following is true :

$$\lim_{|\Lambda|\to\infty}|\Lambda|^{-2}\sum_{i,j\in|\Lambda|}[<\chi_i^+\chi_j^->-<\chi_i^+><\chi_j^->]=0 \tag{16}$$

By symmetry, all directions of the spin are equally probable

$$<\chi_i^n>=\frac{1}{p}$$

and so

$$<\chi_i^+>=\frac{1}{p}\times[p/2]$$

$$<\chi_j^->=\frac{1}{p}\times(p-[p/2])$$

which is,

$$<\chi_i^+> = \begin{cases}1/2 & \text{for } p \text{ } even\\ (p+1)/2p & \text{for } p \text{ } odd\end{cases} \tag{17a}$$

$$<\chi_j^-> = \begin{cases}1/2 & \text{for } p \text{ } even\\ (p-1)/2p & \text{for } p \text{ } odd\end{cases} \tag{17b}$$

Therefore, if we prove that $<\chi_i^+\chi_j^->$ is less than $1/4$, for p even, or less than $(p^2-1)/4p^2$ for p odd, for any (i, j) Eq.(16) will fail to be satisfied.

The only configurations that will contribute to $<\chi_i^+\chi_j^->$ are those which have a contour that either is wrapped around i or j or Λ :

$$<\chi_i^+\chi_j^->=\sum_{i\in\gamma}P(\gamma)+\sum_{j\in\gamma}P(\gamma)+\sum_{\gamma \text{ around } \Lambda}P(\gamma) \tag{18}$$

from (15)

$$\leq \sum_{|\gamma|=4}^{\infty} [\# \text{ of contours of length } |\gamma|] e^{-|\gamma| C(\beta_0)} \quad .$$

The number of contours $N(|\gamma|)$ is bounded by a product of two contributions,

$$N(|\gamma|) \leq (|\gamma|+1)^2 \times e^{c|\gamma|}$$

the first term coming from a translational degree of freedom of the contour, and the second from the three possible choices one has when constructing a contour step by step, $c < 3$.

$$<\chi_i^+ \chi_j^-> \leq \sum_{|\gamma|=4}^{\infty} (|\gamma|+1)^2 \times e^{c|\gamma|} e^{-|\gamma| C(\beta_0)/4} \ .$$

Note that for $C(\beta_0) > c$ the right hand side is summable, and that there is a K, such that for $C(\beta_0) \geq K > c$ it is less than $1/4$ (for even p) or $(p^2-1)/4p^2$ (if p is odd). This means that there are multiple phases for

$$\beta_0 > K'$$

that is

$$\beta > K' p^2 \quad or \quad T < \frac{K'^{-1}}{p^2}$$

III.4 Upper Bounds: An Heuristic Argument

The proposed phase structure of the two dimensional Z_p models [7] shows an ordered phase for temperatures below $T_c(p)$, which decreases to zero as p tends to infinity, as should be expected from the Mermin and Wagner theorem. This leads one to think that a decreasing (going to zero when p tends to infinity) upper bound for $T_c(p)$, could be obtained if the same methods used to prove the Mermin and Wagner theorem, are applied to the Z_p model. For example, if in the energy-entropy methods, the first order contribution to the energy shift ceased to be zero below a certain temperature it would prove that the magnetization is zero above such a temperature. These procedures, however, seem inappropriate to treat the discrete symmetry system, when studied more carefully. Roughly, the reasons for this are the following. The main problem that appears in the generalization to models with discrete symmetries, of the methods based in the energy-entropy inequalities, lies in the fact that the first-order energy shifts are always finite.

The methods that use the Bogoliubov inequality need, implicitly, the existence of a one parameter, twice differentiable, family of symmetry transformations. A generalization of the Ward identities to discrete symmetries is possible. However, the term that contains the Hamiltonian, in the Bogoliubov inequality, cannot be made arbitrarily small, in the discrete symmetry case, for reasons that resemble those that make the first order energy shifts that occur in the energy-entropy inequalities methods at least finite.

McBryan and Spencer [31] prove the absence of symmetry breaking in the two dimensional plane rotor model in a completely unrelated (technically) manner. Their method consists of obtaining an upper bound for the two point correlation function that decays like an inverse power.This bound is obtained by deforming the contours of integration of the angular variables into the complex plane and using some properties of the two dimensional lattice massless propagator. The Z_p invariant model is related to the plane rotor model plus symmetry breaking fields as seen in the work of Jose *et al* [6]. It can be seen that the McBryan and Spencer technique cannot be applied to this model with continuous variables.The reason for this is that the massive, instead of the massless lattice propagator, now appears, and this does not have the appropriate decay properties.

In the absence of a rigorous method to obtain upper bounds on the magnetization temperature $T_c(p)$, we present an heuristic argument which claims the $T_c(p)$ to go to zero at least as fast as $1/p$. The motivation for this comes from looking back at the Peierls argument of the previous section.

For the Z_p model (or Ising) the upper bound on the number of contours of length $|\gamma|$, enclosing a site i , has roughly two contributions.

$$N(|\gamma|) \le \left[c_1 |\gamma|^2\right] \times \left[e^{c_2 |\gamma|}\right]$$

The first term comes from a translational degree of freedom of a contour of area $= |\gamma|^2$. We will call this contribution to the entropy, for reference, as translational entropy. The second term is obtained by considering that at each step of the construction of a contour one has at least three choices of how to proceed to the next step, thus a $\sim exp(|\gamma| ln 3)$ upper bound. This is clearly an upper bound since no restriction to close the contour has been made. However, there is also a lower bound on this number since otherwise the high temperature regime would be ordered. The contribution from this term to the entropy will be called shape entropy. While there is no clear-cut difference between these two contributions a rough distinction can be made. If our space, instead of a lattice, were a continuum these two could be understood as infrared (translation) and ultraviolet (shape) contributions to phase space.

One of the appealing facts about the Peierls argument is that it gives an idea of the mechanism driving the phase transition. The system is magnetized at low temperatures because the energy dominates. At higher temperatures the shape entropy dominates the energy and the system is disordered. The translational entropy alone would not be enough to disorder the system.

For systems with a continuous symmetry, (as seen in Chapter II), flipping the spins inside fixed regions, suitably surrounded by layers of slowly rotated spins, is enough to prove, (as in the Simon and Sokal method), that the magnetization is always zero, provided the number of regions where one could have done the rotation is of the order of the area of the region. It is then only necessary to include entropy contributions that could be called translational. In no place was it necessary to allow the shapes of the regions to vary. The translational entropy is enough to prevent spontaneous symmetry breaking. It is therefore reasonable to expect that these methods, when applied to systems with discrete symmetries, will not be strong enough to obtain the upper bound on $T_c(p)$ since they do not include the shape entropy. Note that in one dimension this differentiation of the entropy into two types is senseless and the energy entropy inequalities can be applied, for example, to study the Thouless effect in the one-dimensional Ising model with $1/r^2$ interaction, [37] and [32] .

The methods based on the Bogoliubov inequality do not take into account the many possible shapes of the region where the transformation can be done and therefore only consider the translational entropy. Since this method can be used to obtain best possible results concerning the range of interactions (Bonato et al [30]) one is led to think that if the system does not have spontaneous symmetry breaking for any $T>0$, then translational entropy alone is sufficient. Clustering bounds obtained by this method characteristically show only logarithmic decay for the correlations instead of the power law behavior obtained in some specific models by other methods (McBryan and Spencer [81]). This weak result could probably be traced back to the soft treatment of the entropy. To conclude, we present the following argument.

Consider a magnetized phase with all spins in the $[\frac{p}{2}]$ state ($[\frac{p}{2}]$ is the largest integer less or equal to $\frac{p}{2}$) and also consider a square block B of side $N_B = N + [\frac{p}{2}] - 2$. Rotate the spins in the outer row of B, by $\frac{2\pi}{p}$ (the smallest possible change) clockwise; in the next row, second closest to the boundary of B, rotate the spins by $2\frac{2\pi}{p}$ and repeat this until the row that the spins are rotated by $\frac{2\pi}{p}([\frac{p}{2}]-2)$. Inside the square of side N concentric to B, choose $L > kN_B$ and rotate all spins outside a closed contour of length L by $\frac{2\pi}{p}([\frac{p}{2}]-1)$ and by $\frac{2\pi}{p}[\frac{p}{2}]$ those inside.

The energy increase due to this rotation can be bounded by

$$\Delta E \le \frac{c_1}{p^2}\sum L_n$$

where L_n is the perimeter of a square in the n^{th} step. The factor $1/p^2$ is due to the change in energy of each nearest neighbor pair of spins $=1-\cos\frac{2\pi}{p}$. There are of the order of $p/2$ steps and each L_n is bounded by $L_n < c'L$ so

$$\Delta < \frac{c_1}{p^2}(\frac{p}{2})c'_0 L = \frac{c_2 L}{p} .$$

One can think of the many different possible shapes of the contour of length L as contributing to the entropy of the system, since all these possible configurations have the same energy (# of contours of length L). This has a lower bound

$$\Delta S > \ln e^{c_3 L}$$

then the change in free energy is

$$\Delta F = \Delta E - \Delta S < \frac{c_2 L}{p} - Tc_3 L$$

which is negative and thus the original phase unstable if

$$T > \frac{c_4}{p} .$$

Note one could improve this bound by considering the entropy generated by allowing the outer squares to change shapes, perhaps even to the optimal result $T > c/p^2$ which is believed to be the correct behavior (see [7,49]).

IV.1 Exponential Bound - A Limit Theorem

In describing the phase structure of the **Zp** model in two dimensions, it was said, in Chapter 3, that for high temperature and large p its behavior should be similar to that of the plane rotor model. The idea behind that comment is that the "energy fluctuations" per unit spin would be larger than the finite excitation energy of a discrete spin, under those conditions.

Our intention, in this section, is to study this loose statement, trying to understand how certain quantities depend on the number of states p. From these comments, one might think that the resemblance would appear only at high temperatures. One could also, naively expect the existence of an expansion for certain thermodynamic functions around the continuous symmetry theory, in inverse powers of the number of states. It is of interest, therefore to have the following result, proved below in Theorem 1. For any fixed temperature, the pressure of the Z_p systems converge exponentially fast in the number of states, to that of the $O(2)$ model. The proof presented below is remarkably simple, and avoids cumbersome perturbation expansions by looking at properties of the *a priori* integration measures. It is due to this simplicity that a quite general result, with respect to the range of interactions and the lattice dimension, can be obtained

A related result, concerning the convergence of correlation functions, is also presented (Theorem 2.). The result is that this convergence is also exponential in the number of states, provided certain conditions concerning the Holder continuity of the correlations (see definition 2) are satisfied. These conditions are likely to hold at high temperatures.

IV.2 Convergence of the Pressure

Consider a $\mathbf{Z}_p$ invariant model in a volume $\Lambda \subset Z^\nu$ with Hamiltonian

$$H_\Lambda(\Phi)=\sum_{X\subset\Lambda} \Phi_X(\theta) \tag{1}$$

periodic in each of its variables θ_i , and a Z_p invariant *a priori* measure

$$d\nu_p=\prod_{i\in\Lambda} d\mu_p(\theta_i) \tag{2}$$

where

$$d\mu_p(\theta)=\frac{d\theta}{p}\sum_{n=0}^{p-1}\delta(\theta-2\frac{\pi}{p}n) \quad . \tag{3}$$

Call $d\nu$ the $O(2)$ invariant measure

$$d\nu=\prod_{i\in\Lambda} d\mu_i(\theta_i) \tag{4}$$

where

$$d\mu_i=d\theta_i/\,2\pi \quad . \tag{5}$$

Introduce the partition functions

$$Z_\Lambda^{\mathbf{Z}_p}=\int d\nu_p\, e^{-\beta H_\Lambda} \tag{6a}$$

$$Z_\Lambda^{O(2)}=\int d\nu e^{-\beta H_\Lambda} \tag{6b}$$

and the pressures

$$P^{\mathbf{Z}_p,O(2)}=\lim_{\Lambda\to\infty} P_\Lambda^{\mathbf{Z}_p,O(2)}$$

where

$$P_\Lambda^{\mathbf{Z}_p,O(2)}=\frac{1}{\Lambda}\ln Z_\Lambda^{\mathbf{Z}_p,O(2)} \quad . \tag{7}$$

Denote by $<\,.\,>_\Lambda^{\mathbf{Z}_p,O(2)}$ the expectation values.

We will prove the following result:

Theorem (1):

If:

$$\Phi_X \in B_1 = \{ \Phi / \|\Phi\|_1 = \sum_{0 \in X} sup.|\Phi_X|, \ is\ finite \}$$

Φ_X is periodic in each of $\theta_i \in R$, and

Φ_X is analytic for $|\mathrm{Im}\theta_i| \leq A$, and there is $0 < g(\alpha)$ such that

$$|\mathrm{Re}\Phi_X(\theta + i\alpha, \vec{\theta})| \leq |\Phi_X(\theta, \vec{\theta})| g(\alpha) \ ,$$

then there is a bounded function $C(\beta)$, and a constant $a, 0 < a \leq A$ such that, for fixed β,

$$1 - \frac{2C(\beta)e^{-pa}}{1-e^{-pa}} \leq exp(P^{Z_p}(\beta) - P^{O(2)}(\beta)) \leq 1 + \frac{2C(\beta)e^{-pa}}{1-e^{-pa}} \tag{8}$$

with $C(\beta)$ given by $C(\beta) = \exp[\beta\|\Phi\|_1(1+g(a))]$ and $g(a)$ is a monotonic increasing function of a (grows exponentially in a).

Remarks

1) The difference of the pressures is exponentially small in p, the number of states for fixed β and sufficiently large p, since in that case the left hand side of eq.(8) is positive and therefore

$$\ln(1-\frac{2C(\beta)e^{-pa}}{1-e^{-pa}})\leq P^{Z_p}(\beta)-P^{O(2)}(\beta)\leq\ln(1+\frac{2C(\beta)e^{-pa}}{1-e^{-pa}})$$

using the expansion $ln(1+x)=x+\ldots$ the difference is seen to be exponential.

2) The theorem implies there is no " 1 over number of states " expansion for the pressure of the Z_p system around the O(2) system.

3) In one dimension, for the nearest neighbor cosine interaction plane rotors, the pressures can be calculated explicitly and seen to converge exponentially in the number of states.

4) For the Villain model in two dimensions, heuristical arguments about high temperatures seems to suggest that a gaussian convergence can occur ($\exp -p^2$).

5) The result is independent of the dimension ν of the lattice.

6) The theorem can be extended to lattice Z_p and U (1) gauge theories.

7) This theorem improves results (see Simon [8] chapter 2) using Bishop-de Leew order that prove that the convergence is at least as fast as $1/p^2$

8) The theorem is a consequence of the fact that for periodic functions Riemann sums converge exponentially to the Riemann integral in the number of points.

9) If there are only n-body interactions, n finite, the last condition is

satisfied automatically, given that $\Phi_X \in B_1$.

We need the following well known preliminary result concerning the exponential decay of the Fourier coefficients of a periodic function.

Proposition 1

Let $f(\theta): C \to \mathbf{R}$ be a periodic ($f(\theta+2\pi)=f(\theta)$) positive function, analytic for $|\mathrm{Im}(\theta)|<A$.

Call

$$\hat{f}_k=\frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{ik\theta}d\theta \ . \tag{9}$$

Then for any a, $A>a>0$ there is a positive constant (which depends on a but not on k) C_1 such that

$$-C_1\hat{f}_0 e^{-ka} \leq \hat{f}_k \leq +C_1\hat{f}_0 e^{-ka} \tag{10}$$

Proof of Proposition 1:

Consider a rectangular contour B in the complex θ-plane, with vertices at $0, 2\pi, 2\pi+ia, ia$. Since f is analytic inside and on the contour

$$\int_B f(\theta)e^{ik\theta}\frac{d\theta}{2\pi}=0 \quad .$$

(Integration is performed counterclockwise.)

Then

$$0=\int_0^{2\pi} f(\theta)e^{ik\theta}d\theta/2\pi+\int_{2\pi}^{2\pi+ia} f(\theta)e^{ik\theta}d\theta/2\pi$$

$$+\int_{2\pi+ia}^{ia} f(\theta)e^{ik\theta}d\theta/2\pi+\int_{ia}^{0} f(\theta)e^{ik\theta}d\theta/2\pi\ .$$

Since $f(\theta)e^{ik\theta}$ is periodic, the second and fourth integral cancel, leaving

$$\hat{f}_k=\int_0^{2\pi} f(\theta)e^{ik\theta}d\theta/2\pi=\int_{ia}^{2\pi+ia} f(\theta)e^{ik\theta}d\theta/2\pi\ .$$

Changing $\theta\to\theta+ia$, the right- hand side can be written as

$$\hat{f}_k=e^{-ka}\int_0^{2\pi} f(\theta+ia)e^{ik\theta}d\theta/2\pi\ ,$$

and then

$$-C_1\hat{f}_0e^{-ka}\leq\hat{f}_k\leq C_1\hat{f}_0e^{-ka}\ ,$$

where

$$C_1 = \frac{\int_0^{2\pi} |f(\theta + ia)| d\theta}{\int_0^{2\pi} f(\theta) d\theta} . \tag{11}$$

Since $\int f \, d\theta \leq \int |f| d\theta$, this completes the proof of Proposition 1.

We now prove the following result which shows the exponential convergence of Riemann sums to the Riemman integral for periodic functions.

Proposition 2:

Let f , C_1 and a be as above and $d\mu_p$ and $d\nu$ defined by eqs. (3) and (5). Then

$$\int d\mu f(\theta)[1 - \frac{2C_1 e^{-pa}}{1-e^{-pa}}] \leq \int d\mu_p f(\theta) \leq [1 + \frac{2C_1 e^{-pa}}{1-e^{-pa}}] \int d\mu f(\theta) . \tag{12}$$

Proof of Proposition 2:

First we need the following result:

$$\int d\mu_p f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}_{kp} \tag{13}$$

which comes from

$$\int d\mu_p f(\theta) = \frac{1}{p}\sum_{n=0}^{p-1}\int_0^{2\pi} d\theta\ \delta(\theta - 2\pi\frac{n}{p}) f(\theta)$$

$$= \frac{1}{p}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} d\theta\,\delta(\theta - 2\pi\frac{n}{p}) f(\theta)\ .$$

Using

$$\frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikp\theta} = \frac{1}{p}\sum_{n=-\infty}^{\infty}\delta(\theta - \frac{2\pi n}{p})\ , \tag{14}$$

$$\int d\mu_p f(\theta) = \frac{1}{2\pi}\int_0^{2\pi} e^{ikp\theta} f(\theta) d\theta$$

$$= \sum_{k=-\infty}^{\infty}\hat{f}_{kp}$$

$$= \hat{f}_0 + \sum_{k=1}^{\infty}(\hat{f}_{kp} + \hat{f}_{-kp})\ .$$

Using Proposition 1 this can be bounded by

$$\hat{f}_0 - 2C_1\hat{f}_0\sum_{k=1}^{\infty} e^{-kpa} \le \int d\mu_p\ f(\theta) \le \hat{f}_0 + 2C_1\hat{f}_0\sum_{k=1}^{\infty} e^{-kpa}$$

using

$$\sum_{k=1}^{\infty} e^{-kpa} = \frac{e^{-pa}}{1-e^{-pa}} \quad \text{and,} \ \hat{f}_0 = \int \frac{d\theta}{2\pi} f(\theta) \ ,$$

we obtain

$$(1-\frac{2C_1 e^{-pa}}{1-e^{-pa}}) \int d\mu f(\theta) \leq \int d\mu_p f(\theta)$$
$$\leq (1+\frac{2C_1 e^{-pa}}{1-e^{-pa}}) \int d\mu f(\theta) \ .$$

This completes the proof of Proposition 2. We now turn to the proof of Theorem 1.

Proof of Theorem 1.

Assume there are only n-body interactions with n finite. Consider the Boltzmann factor $\exp -\beta H_\Lambda$ and a site i. Integrate the variable θ_i.

Since $\exp -\beta H_\Lambda$ satisfies the conditions of Propositions 1 and 2 we have

$$[1-\frac{2C(\tilde{\theta})e^{-pa}}{1-e^{-pa}}] \int d\mu_i e^{-\beta H_\Lambda(\theta_i,\tilde{\theta})}$$
$$\leq \int d\mu_{pi} e^{-\beta H_\Lambda(\theta_i,\tilde{\theta})}$$
$$\leq [1+\frac{2C(\tilde{\theta})e^{-pa}}{1-e^{-pa}}] \int d\mu_i e^{-\beta H_\Lambda(\theta_i,\tilde{\theta})} \ . \tag{15}$$

The Hamiltonian is written as $H(\theta_i,\tilde{\theta})$, where $\tilde{\theta}$ represents all the other variables. We now need a bound on $C(\tilde{\theta})$ independent of $\tilde{\theta}$

$$C(\tilde{\theta})=\frac{\int_0^{2\pi}|e^{-\beta H(\theta_i+ia,\tilde{\theta})}|d\theta_i}{\int_0^{2\pi}e^{-\beta H(\theta_i,\tilde{\theta})}d\theta_i} .$$

So consider

$$|e^{-\beta H(\theta_i+ia,\tilde{\theta})}|-e^{-\beta H(\theta_i,\tilde{\theta})}$$

$$=e^{-\beta H(\theta_i,\tilde{\theta})}[e^{-\beta(\mathrm{Re}H(\theta_i+ia,\tilde{\theta}))-H(\theta_i,\tilde{\theta})}-1] .$$

Write H as

$$H(\theta,\tilde{\theta})=\sum_{\substack{i\in X\\ X\subset\Lambda}}\Phi_X(\theta,\tilde{\theta}) .$$

Since by hypothesis $\Phi_X(\theta,\tilde{\theta})$ is analytic for $|\mathrm{Im}(\theta)|<A$; for $a<A$, and for $\theta,\tilde{\theta}$ real we have the bound :

$$|\mathrm{Re}\Phi_X(\theta+ia,\tilde{\theta})|\leq|\Phi_X(\theta,\tilde{\theta})|g(a)$$

where $g(a)$ is a positive bounded function for $a<A$. If there are infinite body interactions this property will be imposed as a requirement. Then

$$e^{-\beta[\mathrm{Re}H(\theta+ia,\tilde{\theta})-H(\theta,\tilde{\theta})]}=e^{-\beta[\mathrm{Re}\sum_{i\subset X}\Phi_X(\theta+ia,\tilde{\theta})-\sum_{i_X}\Phi_X(\theta,\tilde{\theta})]}$$

$$\leq e^{\beta\sum_{i\subset X}|\Phi_X|(1+g(a))}$$

$$\leq e^{\beta\|\Phi\|_1(1+g(a))}$$

where $\|\Phi\|_1 = \sum_{X\subset Z^\nu} \|\Phi\|$ (infinite volume), which for $\Phi\subset B_1$ is finite and independent of θ and $\tilde{\theta}$. Therefore $C(\tilde{\theta})$ can be bounded by:

$$C(\tilde{\theta})-1=\frac{\int e^{-\beta H(\theta+ia,\tilde{\theta})}d\theta-\int e^{-\beta H(\theta,\tilde{\theta})}d\theta}{\int e^{-\beta H(\theta,\tilde{\theta})}}$$

$$\leq e^{\beta\|\Phi\|_1(1+g(a))}-1$$

and

$$C(\tilde{\theta})\leq e^{\|\Phi\|_1(1+g(a))}\equiv C_\beta \quad .$$

Therefore Equation (15) can be rewritten as

$$[1-\frac{2C_\beta e^{-pa}}{1-e^{-pa}}]\int d\mu_i\, e^{-\beta H(\theta_i,\tilde{\theta})}$$

$$\leq\int d\mu_i e^{-\beta H(\theta_i,\tilde{\theta})}$$

$$\leq[1+\frac{2C_\beta e^{-pa}}{1-e^{-pa}}]\int d\mu_i e^{-\beta H(\theta_i,\tilde{\theta})} \ . \tag{16}$$

We now note that if two functions $f\,,g\in C^\Lambda$ are such that

$$f(\tilde{\theta})\geq g(\tilde{\theta}) \text{for } any\ \tilde{\theta}\in\Omega^\Lambda$$

then

$$\int d\mu_i f\geq\int d\mu_i g \text{ and } \int d\mu_{pi} f\geq\int d\mu_{pi} g \,.$$

Call

$$K^\pm=1\pm\frac{2C_\beta e^{-pa}}{1-e^{-pa}}$$

then for $\beta<\beta_0 p-ln\,2$ where

$$\beta_0\geq\frac{a}{\|\Phi\|_1(1+g(a))}$$

K^- is positive.

We now choose another site, j . Integrate w.r.t. $d\mu_j$ and multiply by K^- the right inequality of Eq. (16) to get

$$K^{-2}\int d\mu_i d\mu_j e^{-\beta H} \leq K^- \int d\mu_{pi} \int d\mu j e^{-\beta H} \leq$$
$$\int d\mu_{pi} \int d\mu_{pj} e^{-\beta H} .$$

Integrating w.r.t. $d\mu_j$ and multiplying by K^+ the left inequality of Eq. (16), we obtain

$$\int d\mu_{pi} \int d\mu_{pj} e^{-\beta H} \leq K^+ \int d\mu_{pi} d\mu_j e^{-\beta H}$$
$$\leq K^{+2} \int d\mu_i \int d\mu_j e^{-\beta H} .$$

Repeating this procedure, of integrating w.r.t. the continuous measure and bounding the result with the integral w.r.t. the $\mathbf{Z}_p$ and a $K^{\pm}$ factor for every site in Λ, we get, by definition of the finite volume partition function

$$K^{-|\Lambda|} Z_\Lambda^{O(2)} \leq Z_\Lambda^{\mathbf{Z}_p} \leq K^{+|\Lambda|} Z_\Lambda^{O(2)} . \tag{17}$$

Since $\log(\ .\ )$ is monotone increasing, we can take the logarithm and divide by the volume $|\Lambda|$ to obtain

$$\ln K^- + P_\Lambda^{O(2)} \leq P_\Lambda^{\mathbf{Z}_p} \leq \ln K^+ + P_\Lambda^{O(2)} . \tag{18}$$

or, using the values of $K^{\pm}$

$$ln\left(1-\frac{2C_{\beta}e^{-pa}}{1-e^{-pa}}\right) \leq \quad P_{\Lambda}^{\mathbf{Z}_p} - P_{\Lambda}^{O(2)} \leq \ln\left(1+\frac{2C_{\beta}e^{-pa}}{1-e^{-pa}}\right) \quad . \qquad (18')$$

Since the bound of $C(\tilde{\theta})$ was obtained even for the limit of infinite volume, Equation (19) is valid in that limit, which completes the proof of Theorem 1.

IV.3 Convergence of the Correlations

We now turn to the expectation values of functions in a translationally invariant state.

Consider changing the Hamiltonian by adding a function $A \in C_X, X \subset P_f(Z^\nu)$ and all its translates in Λ

$$H_\Lambda(\Phi_X + t\Psi_X^A) = \sum_{X\subset\Lambda} (\Phi_X + t\Psi_X^A) = H_\Lambda(\Phi) + t \sum_{(i+X)\subset\Lambda} \tau_i A \qquad 20$$

then

$$\frac{d}{dt} P_\Lambda(\Phi + t\Psi_X^A)|_{t=0} = -\frac{\beta}{|\Lambda|} < \sum_{(i+X)\subset\Lambda} \tau_i A >_{\Lambda,\Phi} \qquad (21)$$

where $< . >_{\Lambda,\Phi}$ denotes expectation values with respect to the original measure.Note that for translationally invariant states, the limit $\Lambda \to \infty$ exists, and

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} < \sum_{(i+X)\subset\Lambda} \tau_i A >_{\Lambda,\Phi} = < A >_\Phi \qquad (22)$$

The problem we want to consider is the following :

Let V be a linear vector space. Given a sequence F_n of functions from V $\to R$ that converges pointwise to F such that for any n, $|F_n - F| < Ce^{-an}$, we want to know under which conditions

a)The derivatives $DF_n \to DF$, and

b)The convergence is also exponential.

To answer this we need the following (see, e.g., Simon [8])

Definition 1. Let F be a real valued function on a vector space V . F is convex if and only if

$$F(\theta x + (1-\theta)y) \leq \theta F(x) + (1-\theta)F(y) \qquad (23)$$

for all $x, y \in$ V and $0 \leq \theta \leq 1$, and also

Proposition 3 (*idem*)

Let F be a convex function on R then at every x

$$(D^{\pm}F)(x)=\lim_{y\downarrow 0}\frac{[F(x\pm y)-F(x)]}{\pm y}$$

exits, and for every $x\leq w$

$$D^{-}F(x)\leq D^{+}F(x)\leq D^{-}F(w)\leq D^{+}F(w) \tag{24}$$

and also, for all but countable x

$$(D^{-}F)(x)=(D^{+}F)(x) \quad . \tag{25}$$

Proposition 4 (*idem*)

Let $\{F_n\}$ be a sequence of convex functions on R that converge pointwise to F, then

(i) F is convex.

(ii) Convergence is uniform in compact subsets.

(iii) For any fixed x

$$(D^{-}F)(x)\leq\lim(D^{-}F_n)(x)\leq\lim(D^{+}F_n)(x)\leq(D^{+}F)(x) \quad .$$

If F is differentiable at x then

$$(DF)(x)=\lim_{n\to\infty}(D^{+}F_n)(x) \quad .$$

We now introduce the notion of Holder continuity

Definition 2. A function $h:R\to R$ is said to be Holder continuous of order γ in a region D if there exist positive constants M and γ such that:

$$|h(y)-h(x)|< M|x-y|^{\gamma} \tag{26}$$

for all $x, y \in D$

Proposition 5

Let F_n be a sequence of convex functions that converge to F. Let C and a be positive constants such that

(i) $|F_n(z)-F(z)|<Ce^{-na}$ for any z

(ii) $F_n(z)$ and $F(z)$ are differentiable in a neighborhood of x .

(iii) $DF_n(x)$ are Holder continuous of order γ

Then

$$|DF_n(x)-DF(x)| < (2C+M)e^{-n\frac{a\gamma}{1+\gamma}} \tag{27}$$

Proof of Proposition 2. We want to bound quantities of the type

$$|DF_n(x)-DF(x)| \ .$$

First of all, consider ($y > x$, $0 \leq \theta \leq 1$)

$$D^+F_n(x)=\lim_{\theta\uparrow 1}\frac{F_n(\theta x+(1-\theta)y)-F(x)}{(1-\theta)(y-x)}$$

then by convexity

$$D^+F_n(x)\leq\frac{F_n(y)-F_n(x)}{y-x} \tag{28}$$

and similarly

$$D^-F_n(y)=\lim_{\theta\downarrow 0}\frac{F_n(y)-F_n(\theta x+(1-\theta)y)}{\theta(y-x)}\geq\frac{F_n(y)-F(x)}{y-x} \quad . \tag{29}$$

Similar formulas hold for $D^{\pm}F$ since F is convex by proposition 4. If F and F_n are differentiable at x

$$D\pm F=DF, \ \ D\pm F_n=DF_n \ \ .$$

Suppose $DF_n(x)-DF(y)\geq 0$ then

$$D^+F_n(x)-D^-F(y)\leq\frac{F_n(y)-F_n(x)-F(y)+F(x)}{y-x}$$
$$\leq\frac{|F_n(y)-F(y)|+|F_n(x)-F(x)|}{y-x} \quad .$$

Consider a sequence $y_m=x+e^{-mA}$, m and A to be chosen later. Using condition (i) of the proposition

$$D^+F_n(x)-D^-F(y_m)\leq 2Ce^{-na+mA} \quad . \tag{30}$$

If $DF_n-DF(y)$ were negative consider instead $D^+F(y)-D^-F_n(y)$ to obtain

$$D^+F(y_m)-D^-F_n(x)\leq 2Ce^{-na+mA} \quad . \tag{31}$$

Since the F 's are assumed differentiable in a neighborhood of x, we obtain from Equations 30 and 31

$$|DF(y_m)-DF_n(x)|\leq 2C\, e^{-na+mA} \quad . \tag{32}$$

Note that if no further assumptions are made about DF the bound cannot be used for $y_m\to x(m\to\infty)$. Here enters the Holder continuity, condition (iii) (see Eq. 26), for if

$$|DF(y)-DF(x)|<M|y-x|^\gamma \tag{33}$$

then

$$|DF(x)-DF_n(x)|=|DF(x)-DF(y_m)+DF(y_m)-DF_n(x)|$$

$$\leq|DF(x)-DF(y_m)|+|DF(y_m)-DF_n(x)| \, .$$

From Equations 32 and 33 we obtain

$$|DF(x)-DF_n(x)| \leq Me^{-ma\gamma}+2Ce^{-na+mA} \quad . \tag{34}$$

We can now choose m and A. The best choices are

$$m=n \quad \text{and} \quad A=\frac{a}{1+\gamma} \quad . \tag{35}$$

Then

$$|DF(x)-DF_n(x)| \leq (M+2C)e^{-na\frac{\gamma}{1+\gamma}}$$

completing the proof of Proposition 5.

Note that Holder continuity on $DF(x)$ implies in Holder continuity on the $DF_n(x)$, for under conditions (i) through (iii)

$$|DF_n(y)-DF_n(x)| \leq |DF_n(y)-DF(x)|+|DF(x)-DF(y)|+|DF(y)-DF_n(x)|$$

$$\leq 2Ce^{-na+mA}+Me^{-m\gamma A}+2Ce^{-na+mA}$$

and for $y=y_m$ as above

$$|DF_n(y_m)-DF_n(x)| \leq (4C+M)|x-y_m|^{\gamma} \quad .$$

With this in mind we can now prove the following theorem.

Theorem 2. If $H_\Lambda(\Phi_X+t\Psi_X^A)$ satisfies the conditions of Theorem 1, $P_\Lambda^{\mathbf{Zp}}(\Phi+t\Psi)$ and $P_\Lambda^{\varrho(2)}(\Phi+t\Psi)$ are differentiable w.r.t. t in a neighborhood of $t=0$, and $\frac{d}{dt}P_\Lambda^{\mathbf{Zp}}(\Phi+t\Psi)$ or $\frac{d}{dt}P_\Lambda^{\varrho(2)}(\Phi+t\Psi)$ are Holder continuous, and the number of states p is sufficiently large such that the difference between the pressures is exponentially small, then there are positive constants a', C_β', M and γ such that

$$|< A >_\Phi^{\mathbf{Zp}}-< A >_\Phi^{\varrho(2)}| \leq \left(\frac{2C_\beta'+M}{\beta}\right)e^{-pa'\frac{\gamma}{1+\gamma}}$$

Proof :

The proof follows immediately from Proposition 5 once we note that:

1. $P_{\Lambda}^{Z^p}(\Phi+t\Psi)$ are convex.

2. Since a bound uniform in $|\Lambda|$ can be obtained using Proposition 5, the bound is valid in the limit $\Lambda\to\infty$.

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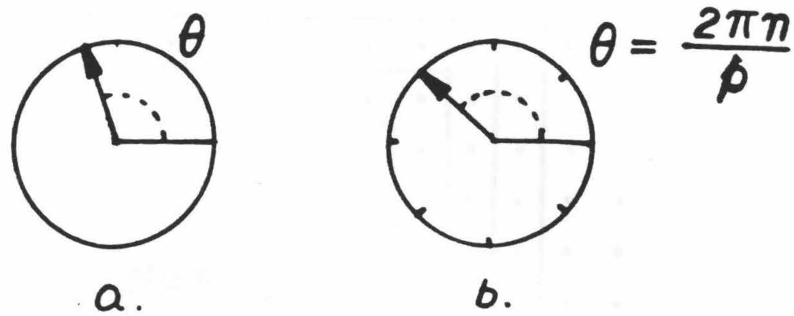

Fig 1

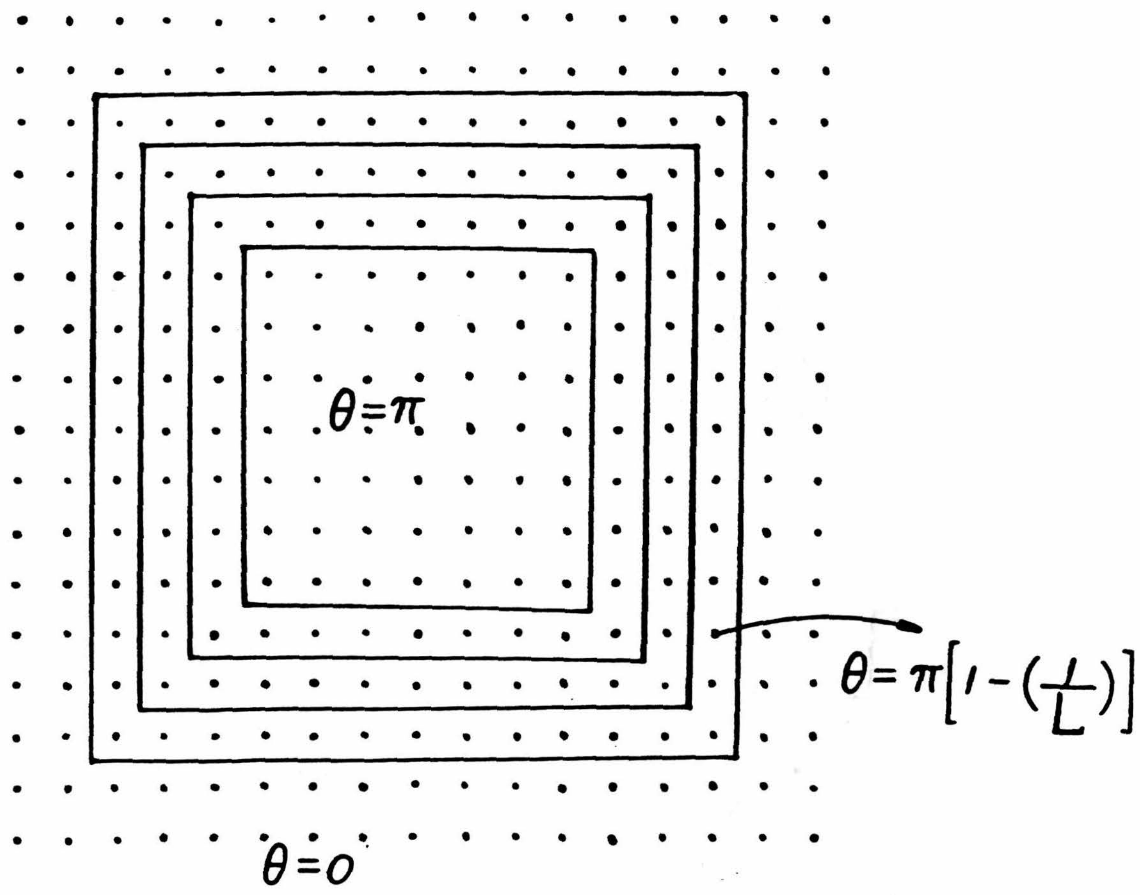

Fig 2

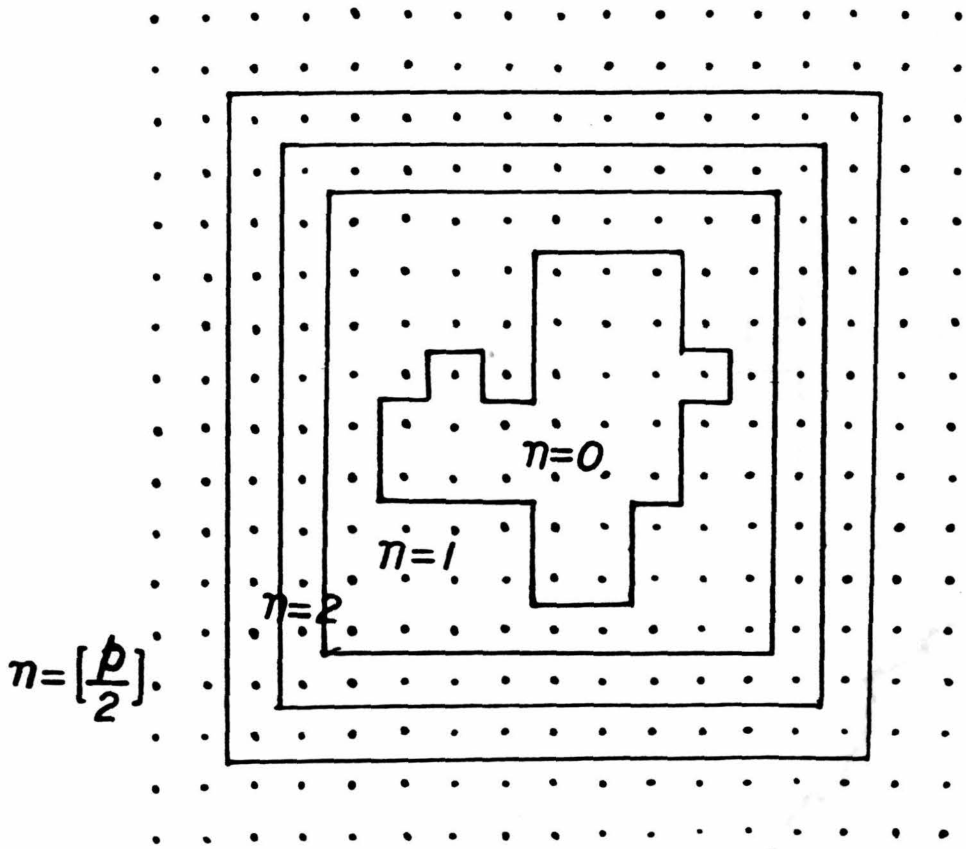

Fig 3

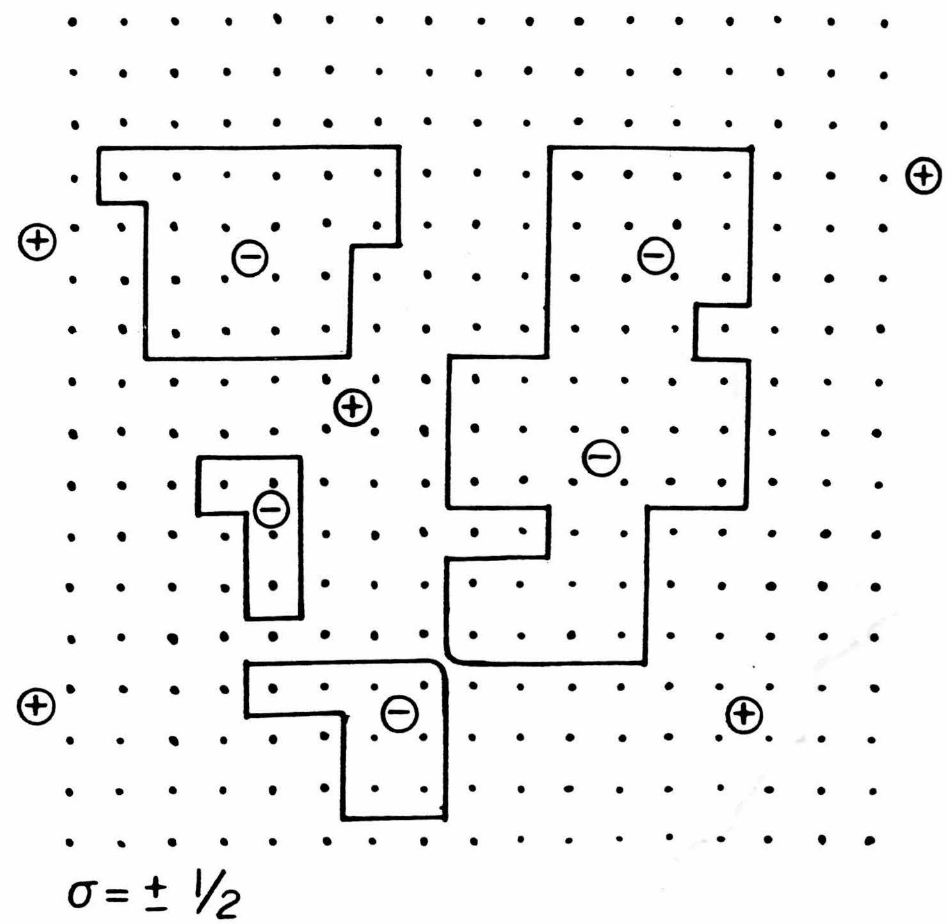

Fig 4

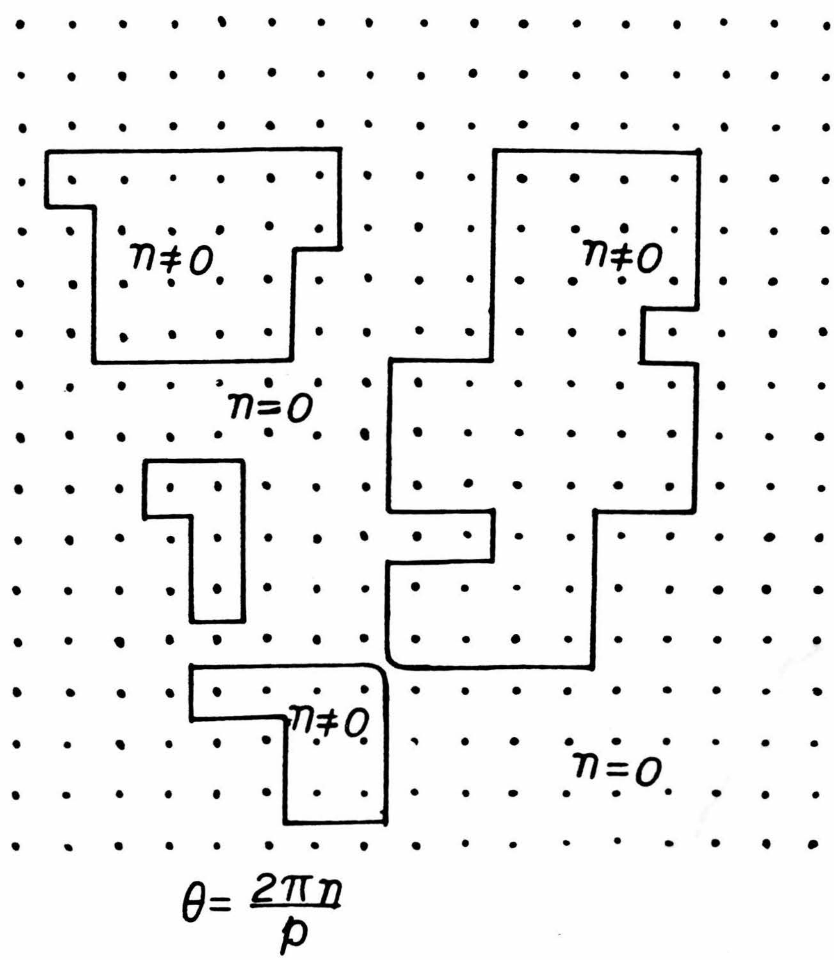

Fig 5