Finiteness in Supersymmetric Theories.

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To my mother and father

Abstract

A two loop calculation in the N=4 supersymmetric Yang Mills theory is performed in various dimensions. The theory is found to be two-loop finite in six dimensions or less, but infinite in seven and nine dimensions. The six-dimensional result can be explained by a formulation of the theory in terms of N=2 superfields. The divergence in seven dimensions is naively compatible with both N=2 and N=4 superfield power counting rules, but is of a form that cannot be written as an on-shell N=4 superfield integral. The hypothesized N=4 extended superfield formalism therefore either does not exist, or at least has weaker consequences than would have been expected. By analogy, four-dimensional supergravity theories are expected to be infinite at three loops.

Some general issues about the meaning of finiteness in nonrenormalizable theories are discussed. In particular, the use of field redefinitions, the generalization of wavefunction renormalizations to nonrenormalizable theories, and whether counterterms should be used in calculations in "finite" theories are studied. It is shown that theories finite to n loops can have at most simple-pole divergences at $n+1$ loops.

A method for simplifying the calculation of infinite parts of Feynman diagrams is developed. Based on the observation that counterterms are local functions, all integrals are reduced to logarithmically divergent ones with no dependence on masses or external momenta. The method is of general use, and is particularly effective for many-point Green functions at more than one loop.

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Introduction.

A minimal requirement of a theory is that, when it is used for calculations, it should give finite results for physical quantities. Quantum field theories, however, fail this simple test. This was first found in 1939 [1], when the self energy of the electron was found to be infinite in quantum electrodynamics. Since QED is a very valuable theory, ways of circumventing such divergences were sought. It was found that if the mass and charge of the electron in the lagrangian were taken, in each order of perturbation theory, to be infinite by precisely the right amount, all calculations of physical quantities yield finite answers. While this so called renormalization [2] procedure could be considered aesthetically unpleasing, it is clearly an unqualified experimental success. Furthermore, QED has been generalized to Yang-Mills theories, renormalizable gauge theories that have been successfully used to describe the strong, weak and electromagnetic interactions. All these theories have parameters that are either dimensionless or of positive mass dimension. Once these parameters are renormalized, the theories give finite results.

The renormalization idea works only if gravity is ignored, however. Einstein's theory of gravity has an interaction characterized by Newton's constant, which has negative mass dimension. Classically, the theory is both experimentally verified and theoretically beautiful. However, it does not fall into the class of renormalizable theories. As the order of perturbation theory is increased, higher powers of Newton's constant appear and, to keep the effective action dimensionless, additional coupling constants are accompanied by derivatives. Thus, successive terms in the perturbation expansion contain potentially divergent expressions, which are different at each order. This situation is commonly described by saying that such theories are "power-counting nonrenormalizable". If a procedure such as renormalization is attempted, an infinite number of different counterterm structures is needed to preserve the finiteness of the theory. In practice, these terms are suppressed by factors of the energy divided by the Planck mass, or about seventeen orders of magnitude at presently available accelerator energies. However, as a theoretical issue, when quantum effects become important, the necessity of an infinite number of renormalization parameters destroys the predictive power of the theory. In field theories containing gravity, we must therefore avoid the lure of renormalizability and demand finiteness for a successful theory.

Einstein gravity does have some success as a quantum theory. If a one-loop calculation is performed in pure Einstein gravity, it is found that the theory is finite [3]. However, as soon as gravity is coupled to matter, a divergence that cannot be eliminated by renormalization is encountered. One must thus search for a more clever way of extending the pure gravity theory.

A clue for doing this can be found from the case of renormalizable theories. It is seen there that divergences can be softened or eliminated by introducing supersymmetry [4], a symmetry between bosons and fermions. The simplest example of this is for the vacuum energy. Since supersymmetric theories have equal numbers of bosons and fermions, their free hamiltonians can be written as

$$H = \Sigma \frac{\hbar\omega}{2} (a^\dagger a + a a^\dagger + b^\dagger b + b b^\dagger) \quad ,$$

where a and b are the bosonic and fermionic annihilation operators in the Fock space. Since the bosonic oscillators commute, while the fermionic ones anticommute, the total vacuum energy seen by normal ordering H is zero. The fact that supersymmetry softens the divergences of scalar masses to logarithmic ones has made supersymmetric phenomenology popular, as quantum corrections in these theories do not give large masses to low mass scalars [5]. What is more important in this context, is that there is a class of renormalizable theories with extended supersymmetry that are completely finite [6].

With this in mind, it is natural to hope that by joining supersymmetry to gravity, one could obtain nonrenormalizable field theories that have no divergences in the perturbative expansion of their S matrix elements. Supersymmetric theories containing gravity are known as supergravity theories [7]. The supersymmetries become local, and are gauged by spin-3/2 particles known as gravitinos. Indeed supergravity theories do have improved convergence properties and are one [8] and two [9] loop finite. This is already better than the non-supersymmetric case, and could even be better than pure gravity, which may diverge at two loops. However, unlike their globally supersymmetric counterparts, there are no formal proofs of finiteness for supergravity, at all orders, and it is unclear whether or not the finiteness persists.

The purpose of this work is to gain insight into the higher-loop finiteness of these theories. The most general proofs of finiteness in renormalizable supersymmetric theories are based on formal power-counting arguments. However, for supergravity power-counting arguments can, at most, postpone the onset of divergences by a few loop orders, since the potential divergences become more severe loop by loop. The most powerful power counting arguments available have been proposed by Grisaru and Siegel [10]. In d dimensions at L loops, they would (if valid) exclude divergences for supergravity for

$$1 < L < \frac{2(N-1)}{d-2} \quad . \tag{1}$$

(Here N denotes the number of supersymmetries with respect to four dimensions.) These arguments do not apply at one loop for technical reasons, but the one loop case can be checked either by explicit calculation or by other arguments. The power-counting depends on some assumptions about the existence of superfield formalisms and these assumptions are known to be true only for $N \le 2$. The $N=2$ power counting in eq. (1) is clearly uninformative, and it is important to discover whether these rules are valid up to $N=8$, corresponding to the maximally extended supergravity theory. If this were the case, it would show that $N=8$ is at least six loop finite, which, given our present state of ignorance, would be a valuable piece of information.

The subject of this thesis is the $N=4$ supersymmetric Yang-Mills theory [11] in more than four dimensions. This is the maximally supersymmetric interacting theory not containing gravity. Because gravity calculations are extraordinarily difficult, and Yang-Mills theories are power-counting nonrenormalizable in more than four dimensions, this is a more convenient testing ground for the power counting rules than supergravity. In this case finiteness is predicted for

$$1 < L < \frac{2(N-1)}{d-4} \quad . \tag{2}$$

The rule can again be trusted only for $N \le 2$, but this is already sufficient to prove the finiteness of $N=2$ Yang-Mills theories beyond one loop in four dimensions. Most of the $N=2$ theories are infinite at one loop, but some can be found that are also one-loop finite. We shall test these rules at two loops for $N=4$ and $d>4$.

Chapter I of the thesis contains a review of supersymmetric theories and auxiliary fields, and a brief discussion of superspace and the power-counting rules. Chapter II contains a description of a new method that was developed for calculating the infinite parts of momentum integrals [12]. The method is of general applicability, and is described in the case of a φ^3 theory in six dimensions for pedagogical purposes. Chapter III addresses the meaning of finiteness in nonrenormalizable theories. In particular, field redefinitions, the generalization of wavefunction renormalizations to nonrenormalizable theories, are discussed. These issues are used in studying whether counterterms should be used in calculations in "finite" theories. It is shown that the S matrix of theories finite to n loops has at most simple-pole divergences at $n+1$ loops. Chapter IV contains the details of the calculation of the N=4 Yang-Mills theory. It is found, among other results, that N=4 Yang Mills is infinite at two loops in seven dimensions. While this is seemingly in agreement with eq. (2), the form of the divergence is incompatible with the assumptions used in its derivation. This implies that if superfield formalisms exist for N>2, they must have an unusual structure violating the assumptions of Grisaru and Siegel. A discussion of all the results and their implications appears in Chapter V. By analogy, the results suggest that all four-dimensional supergravity theories are infinite at three loops, and are thus inconsistent theories. If this is true, they could be useful only as effective low-energy descriptions of more fundamental theories, such as superstrings [13].

I. Supersymmetry and Superspace.

a) Supersymmetry.

Supersymmetry is the only symmetry that relates particles of different spins [14]. The supersymmetry generator transforms bosons into fermions and thus satisfies anticommutation relations. As could be expected from the spin statistics theorem, it is a spinor and it changes the spins of particles by $\frac{1}{2}$. The feature that characterizes the superalgebra is that the anticommutator of two supersymmetry transformations produces a spacetime translation.

Many of the features of supersymmetry can be illustrated in one of the simplest supersymmetric theories, the N = 1 Yang-Mills theory [15]. The particles of the theory are a Majorana, or real, spinor in the adjoint multiplet of a gauge group, and a Yang-Mills vector boson. As in all supersymmetric theories, there are an equal number of fermionic and bosonic degrees of freedom. Thus, a massless vector has two polarizations, and a Majorana spinor has two degrees of freedom when the Dirac equation is used. The lagrangian of the theory is

$$L = -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{i}{2}\bar{\lambda}\not{D}\lambda + \frac{1}{2}(B^a)^2 \quad , \tag{3}$$

where $F_{\mu\nu}$ is the field strength of the vector and D_μ is the gauge-covariant derivative. Here, B^a is a pseudoscalar field, which has no dynamics, and vanishes by its field equation. Such fields are called auxiliary fields. They do not represent any particles, but are introduced to enforce the equality of the Bose and Fermi degrees of freedom of the fields themselves, and not just of the particles they represent. Thus, the vector field has three components after a gauge is fixed, the spinor has four and the auxiliary field has one. The supersymmetry transformations of the fields are parametrized by an infinitesimal anticommuting Majorana spinor ε. They are

$$\delta A_\alpha = -i\bar{\varepsilon}\gamma_\alpha\lambda$$

$$\delta\lambda = \frac{1}{2}F_{\alpha\beta}\gamma_{\alpha\beta}\varepsilon + i\gamma_5 B\varepsilon$$

$$\delta B = \bar{\varepsilon}\gamma_5 \not{D}\lambda \quad . \tag{4}$$

These transformation leave the lagrangian of eq. (3) invariant up to a total derivative. Commuting two of supersymmetry transformations, labeling the generator by Q, gives

$$[\bar{\varepsilon}_1 Q, \bar{\varepsilon}_2 Q] = 2\bar{\varepsilon}_2\gamma^\mu\varepsilon_1 P_\mu \quad , \tag{5}$$

where P_μ is the translation generator, with a gauge transformation added to make it gauge covariant. Thus

$$\delta_{P_\nu} A_\mu = F_{\nu\mu} \quad , \quad \delta_{P_\nu}\lambda = D_\nu\lambda \quad , \quad \delta_{P_\nu} B = D_\nu B \quad . \tag{6}$$

Consider what happens when the auxiliary field is eliminated. This can be done either by using the equation of motion B=0, or equivalently by integrating over B in the Feynman path integral. The supersymmetry transformations of eqs (4) are modified by setting B to zero. It is clear that the modified lagrangian will be invariant under the modified supersymmetry transformations. The only change occurs in the commutator of two supersymmetry transformations on the spinor. There is now an additional term

$$i\gamma_5(\varepsilon_2\bar{\varepsilon}_1 - \varepsilon_1\bar{\varepsilon}_2)\gamma_5 \not{D}\lambda \quad . \tag{7}$$

Thus, the supersymmetry algebra closes only on the mass shell. Finding auxiliary fields is a major problem in supersymmetry, and is sometimes very difficult. It is possible to do component calculations without knowing them, but they are necessary for superfield calculations. The auxiliary field problem is discussed further in section c.

b) Superspace, Superfields and Power-Counting Rules.

Since supersymmetry is a spacetime symmetry, it is natural to attempt to understand it geometrically. This leads one to the concept of superspace [16]. Superspace has many uses and is a subject in itself. However, in this work superspace concepts will be needed only for the understanding of the theorem of Grisaru and Siegel. This section provides a brief summary of the subject.

If supersymmetry is to be realized geometrically, it is necessary to have anticommuting coordinates θ. Under a supersymmetry transformation

$$\delta\theta = \varepsilon$$

$$\delta x_\mu = 2i\bar{\varepsilon}\gamma_\mu\theta \quad . \tag{8}$$

The spacetime coordinates must transform under supersymmetry, since the anticommutator of two supersymmetries is a translation. For the rest of this section the standard practice of using SL(2,$\mathbb{C}$) notation for the Lorentz group will be followed. In this notation the supersymmetries are written as

$$\delta\theta^\alpha = \varepsilon^\alpha$$

$$\delta\bar{\theta}^{\dot\alpha} = \bar{\varepsilon}^{\dot\alpha}$$

$$\delta x^{\alpha\dot\alpha} = -\frac{i}{4}\left(\varepsilon^\alpha\bar{\theta}^{\dot\alpha} + \bar{\varepsilon}^{\dot\alpha}\theta^\alpha\right) \quad . \tag{9}$$

The indices α and $\dot{\alpha}$ run from one to two, and $\bar{\theta}$ is the complex conjugate of θ.

We now define superfields $\Phi[x,\theta,\bar{\theta}]$, which are objects that transform covariantly under supersymmetry transformations. Superfields can be formally expanded in a power series in θ and $\bar{\theta}$:

$$\Phi[x,\theta,\bar{\theta}] = \varphi + \theta^{\alpha}\psi_{\alpha} + \bar{\theta}^{\dot{\alpha}}\psi_{\dot{\alpha}} + \theta^2\chi + \cdots \tag{10}$$

Since the θ and $\bar{\theta}$'s anticommute, the power series stops after a finite number of terms (in this case nine). The component fields in the expansion of Φ can be understood as ordinary gauge, auxiliary and physical fields.

In addition to superfields, it is necessary to have the idea of derivatives. In ordinary spacetime, $\frac{\partial}{\partial x^{\mu}}$ is a covariant object. Thus, if φ is a scalar, $\partial_{\mu}\varphi$ is a vector. In superspace $\partial_{\alpha} \equiv \frac{\partial}{\partial\theta^{\alpha}}$ is not covariant, as x transforms under supersymmetry. However, it is easy to see that the derivatives

$$D_{\alpha} = \partial_{\alpha} + \frac{i}{4}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}$$

and

$$\bar{D}_{\dot{\alpha}} = \partial_{\dot{\alpha}} + \frac{i}{4}\theta^{\alpha}\partial_{\alpha\dot{\alpha}} \tag{11}$$

are invariant under supersymmetry transformations, and thus produce covariant tensors. Here $\partial_{\alpha\dot{\alpha}} \equiv \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$ denotes a spacetime derivative. The D's satisfy the algebra

$$\{D_{\alpha},D_{\beta}\} = \{\bar{D}_{\dot{\alpha}},\bar{D}_{\dot{\beta}}\} = 0$$

$$\{D_{\alpha},\bar{D}_{\dot{\alpha}}\} = \frac{i}{2}\partial_{\alpha\dot{\alpha}} \quad . \tag{12}$$

The only ingredient still needed for constructing supersymmetrically invariant quantities, is a rule for integrating over superspace. One can formally write $\int d^4x\, d^2\theta\, d^2\bar{\theta}$, but the fermionic integral must be defined. This is done using Berezin integration [17]. If X is a single Grassmann variable, $\Phi[x,X] = \varphi(x) + X\psi(x)$. Then $\int dX\Phi \equiv \psi$. This definition is the only one (up to normalization) which is invariant under translations of X. Integration over fermionic variables is thus equivalent to differentiation. The superspace integral therefore becomes

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\, \Phi = \int d^4x\, D^2 \bar{D}^2 \Phi \quad . \tag{13}$$

With this information, supersymmetric objects can be construct simply by integrating combinations of D's and superfields over superspace.

We are now in a position to write the N=1 Yang-Mills theory above in superspace [18]. Ordinary Yang-Mills is obtained by replacing ∂_μ with $\nabla_\mu \equiv \partial_\mu - A_\mu$. The procedure here is similar. The covariant derivatives ∇_α, $\nabla_{\dot{\alpha}}$ and $\nabla_{\alpha\dot{\alpha}}$ are defined. In order to preserve the structure of superspace, the algebra of eqs (12) is kept, *i.e.*

$$\{\nabla_\alpha, \nabla_\beta\} = \{\bar{\nabla}_{\dot{\alpha}}, \bar{\nabla}_{\dot{\beta}}\} = 0 \tag{14a}$$

$$\{\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}\} = \frac{i}{2} \nabla_{\alpha\dot{\alpha}} \quad . \tag{14b}$$

The second equation can be regarded merely as the definition of $\nabla_{\alpha\dot{\alpha}}$. The derivative ∇_α can be expanded as $D_\alpha - i\Gamma_\alpha$, and the potential Γ_α is then the fundamental field of the theory. However, eq. (14a) gives a constraint on Γ_α, which therefore cannot be used as an integration variable in a path integral.

Other objects can be constructed from Γ_α. For example

$$W_\alpha = \frac{i}{2}[\nabla^{\dot\alpha}, \nabla_{\alpha\dot\alpha}] \tag{15}$$

is the lowest-dimensional field strength corresponding to the connection Γ_α. It contains the spinor of the theory at $\theta = 0$. By using (super) Jacobi identities on the definition of W_α, it can be shown that it satisfies

$$\{\bar\nabla^{\dot\alpha}, W_\alpha\} = 0 \quad , \tag{16a}$$

$$\{\nabla^\alpha, W_\alpha\} = \{\bar\nabla^{\dot\alpha}, \overline{W}_{\dot\alpha}\} \equiv B \quad , \tag{16b}$$

where $\overline{W}_{\dot\alpha}$ is the complex conjugate of W_α, and the $\theta = 0$ part of the superfield B is the auxiliary field introduced in section a. The equation of motion of the theory is thus

$$B = 0 \quad \rightarrow \quad \{\nabla^\alpha, W_\alpha\} = 0 \quad . \tag{17}$$

Using these methods the supersymmetries and equations of motion can be obtained in terms of these "on-shell quantities". To quantize the theory, however, it is necessary to write Γ_α in terms of an unconstrained superfield. This means solving the constraints in eq. (14a) for ∇_α. In the case of N=1 supersymmetric Yang-Mills the solution is

$$\nabla_\alpha = \varepsilon^{-V} D_\alpha \varepsilon^{V} \quad , \tag{18}$$

where V is a complex superfield. V can now be used as a true quantum field in an action. In fact the action for N=1 Yang Mills is simply

$$S = -\frac{1}{g^2} \int d^4x \int d^2\theta \, W^\alpha(V) \, W_\alpha(V) \quad , \tag{19}$$

with W defined by eqs (15), (14b) and (18). This action is supersymmetric, even though there is no integral over $\bar{\theta}$ since W satisfies eq. (17a), it is "chiral".

The theorem of Grisaru and Siegel [11] can now be stated. *If an unconstrained lagrangian exists, all counterterms at more than one loop can be written as a complete integral over all superspace of connections and field strengths.* This means, in particular, that V can never occur, and one cannot have chiral integrals, such as $\int d^2\theta$. The theorem fails at one loop, since the quantization involves an infinite number of ghosts coupling only to background fields.

We can now see the implications of this for Yang-Mills theories and supergravity. In N-extended Yang Mills in d dimensions, the lowest dimensional counterterm possible is

$$\Delta S = (g^2)^{L-1} \int d^dx \, d^{4N}\theta \, \Gamma D^2 \Gamma \quad , \tag{20}$$

where L is the number of loops. g^2 has dimension $(4-d)$ and $\int d\theta$ has dimension $1/2$, since it acts as a derivative. As ΔS is dimensionless, it follows that

$$0 > (4-d)(\mathrm{L}-1) - d + \frac{1}{2} \cdot 4\mathrm{N} + 2 \tag{21}$$

for a counterterm to exist. This implies finiteness for $1 < \mathrm{L} < \dfrac{2(\mathrm{N}-1)}{d-4}$.

For supergravity, the lowest dimension counterterm is the superdeterminant of the supervielbein, a direct analogue of $\det V$ in ordinary gravity. Thus

$$\Delta S = (\kappa^2)^{\mathrm{L}-1} \int d^dx \, d^{4\mathrm{N}}\theta \; \mathrm{sdet} V \quad . \tag{22}$$

Here κ^2 has dimension $2-d$, and therefore

$$0 > (2-d)(\mathrm{L}-1) - d + \frac{1}{2}\cdot 4\,\mathrm{N}+0 \quad , \tag{23}$$

implying finiteness for $1 < \mathrm{L} < \frac{2(\mathrm{N}-1)}{d-2}$.

The derivation of these rules fails if no unconstrained formalism is known, since the Feynman rules cannot be constructed. This is the case for all (but one) theories with N>2. In the next section it is shown that even the auxiliary field problem is difficult to solve for N=4 Yang Mills [19]. Since superfields automatically incorporate all the physical and auxiliary fields, they cannot exist if the auxiliary fields do not. A calculation testing the power counting rules thus can provide information as to whether superfield formalisms can exist, and what form they may take.

c) Problems with Auxiliary Fields.

The auxiliary fields in the previous sections were introduced to close the supersymmetry algebra off shell. It is instructive to consider the auxiliary fields used for closing the Lorentz algebra [20]. These are used whenever a gauge theory is studied, but are generally not recognized because of the manifest covariance of the notation. The simplest such example is a free massless vector A_μ in d dimensions. It is described by the lagrangian

$$L = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \frac{1}{2} A_\mu \Box A^\mu + \frac{1}{2}(\partial\cdot A)^2 \quad . \tag{24}$$

The physical and auxiliary parts of A_μ can be untangled by going into a light-cone gauge. Defining $f^\pm \equiv (f^0 \pm f^L)/\sqrt{2}$, where L is the longitudinal direction, the metric becomes $f\cdot g = f^i g^i - f^+ g^- - f^- g^+$, with Latin indices labeling the transverse directions. If the gauge $A^+ = 0$ is used, the lagrangian can be written as

$$L = \frac{1}{2} A_i \Box A_i + \frac{1}{2} B^2 \quad , \tag{25}$$

where $B \equiv \partial \cdot A$. In this gauge it is clear that B represents an auxiliary field, while A_i represents the transverse propagating degrees of freedom of the vector.

The Lorentz transformations of these fields are obtained by taking the Lorentz transformation in the covariant theory and adding gauge transformations to preserve the gauge choice. The generators can be written as

$$J^{ij} = l^{ij} + S^{ij} \qquad J^{+-} = l^{+-} \qquad J^{+i} = l^{+i}$$

and

$$J^{-i} = l^{-i} + \frac{p_j}{p^+} S^{ji} - \frac{i}{p^+} K^i \quad , \tag{26}$$

where $l^{\alpha\beta}$ is the orbital angular momentum and S^{ij} is the transverse spin angular momentum. Since the J's represent the Lorentz algebra, S^{ij} and K^i can be shown to satisfy

$$[\, S^{ij} \,, K^k \,] = i\delta^{ik} K^j - i\delta^{jk} K^i \qquad [\, K^i \,, K^j \,] = ip^2 S^{ij} \quad . \tag{27}$$

Thus if $p^2 \neq 0$, S^{ij} and K^j together generate SO($d-1$). This is the well-known result that massive particles are classified by the little group SO($d-1$). (Massive representations occur since $p^2 \neq 0$ off shell.) In the case of the vector, the generator K^j acts as

$$\delta A^i = \delta^{ij} B$$

$$\delta B = \Box A^j \quad , \tag{28}$$

and A_i and B form the fundamental representation of SO($d-1$). If one now goes to the mass shell, $p^2 = B = 0$. The K^j generators decouple, and only SO($d-2$) remains. Therefore the massless particles form a representation of the helicity group SO($d-2$).

Finally, as in the supersymmetry example, the auxiliary field B can be set to zero by solving its equation of motion. If this is done, the K^j generators no longer exist, and even the off shell fields are only a representation of SO($d-2$). The Lorentz transformations are modified by setting B to zero. The theory is still Lorentz invariant but, in analogy with eq. (7), the commutator of two J^{-i} transformations, which should be zero in the Lorentz algebra, is proportional to p^2 and vanishes only on the mass shell. The light-cone lagrangian, with its complicated Lorentz transformations (27), is the analogue of a supersymmetric theory written in components, while the covariant lagrangian and transformations correspond to a theory in superspace. This demonstrates the simplifications that are obtained by having manifest symmetries, but shows that it is at the expense of introducing gauge and auxiliary degrees of freedom.

We now present an interesting example of a theory for which "usual" Lorentz auxiliary fields do not exist. The example exists for all $d = 4n+2$ dimensions but, to be specific, we work in six dimensions. In this case the field is $A_{\mu\nu}$†, and satisfies the field equation

†Objects with multiple indices are totally antisymmetric in those indices.

$$F_{\mu\nu\rho} = \widetilde{F}_{\mu\nu\rho} \quad , \tag{29}$$

where the field strength $F_{\mu\nu\rho}$ is defined by

$$F_{\mu\nu\rho} \equiv 3\,\partial_{[\mu}A_{\nu\rho]} \quad , \tag{30}$$

and the dual is taken with the six index Levi-Civita tensor $\varepsilon^{\lambda\mu\nu\rho\sigma\tau}$. The equation of motion is invariant under the gauge transformation

$$\delta A_{\mu\nu} \equiv 2\,\partial_{[\mu}A_{\nu]} \quad . \tag{31}$$

By taking the divergence of eq. (29), one sees that it describes a massless particle. To study the propagating modes, one can choose the on-shell momentum to be p^-, without loss of generality. Using the gauge transformation of eq. (31), the light-cone gauge $A^{i+} = 0$ can be chosen. Then eq. (29) implies that $B_j \equiv \partial_\mu A^\mu_j = 0$, and $A_{ij} = \widetilde{A}_{ij}$, where this dual is with respect to the four index transverse ε^{ijkl}.

The problem of finding the auxiliary fields for this theory, which must be solved for a covariant lagrangian to be found, is now clear. There is one physical multiplet in a self-dual representation of $SO(4)$. However, off shell the fields must be a representation of SO(5), and the anti self-dual field is automatically included. If $A^{(+)}_{ij}$ is to propagate, but $A^{(-)}_{ij}$ is to be auxiliary, the lagrangian analogous to eq. (26) would have to be

$$L = \frac{1}{8}A^{(+)}_{ij}\Box A^{(+)}_{ij} + \frac{1}{8}A^{(-)2}_{ij} + \frac{1}{2}B^2_i \quad . \tag{32}$$

Because $A^{(-)}_{ij}$ and B_i now have the same dimensionality, it is impossible to write the analogue of the K^j transformations of eq. (28). Therefore there is no quadratic action which will propagate one of these and not the other. The solution to this dilemma is to use a Lagrange multiplier term to eliminate one field [21]. The covariant lagrangian for the theory is

$$L = -\frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} + \frac{1}{4} \lambda^{\mu\nu} F^{(-)}{}_{\mu}{}^{\rho\sigma} F^{(-)}{}_{\nu\rho\sigma} \quad , \tag{33}$$

where $F^{(-)}_{\mu\nu\rho}$ is the anti-dual part of $F_{\mu\nu\rho}$, and $\lambda^{\mu\nu}$ is a symmetric traceless Lagrange multiplier field, that enforces eq. (29).

In N=4 Yang Mills, the problem is very similar. Let us consider both the on-shell and off-shell representations. The theory is a naturally written as a massless ten dimensional supersymmetric theory. On shell, the spectrum thus forms representations of the helicity group SO(8), with the supersymmetry charge Q^a a spinor of SO(8). The on-shell supersymmetry algebra is

$$\{ Q^a, Q^b \} = \delta^{ab} \quad , \tag{34}$$

where the spinor indices are real and run over eight values. Eq. (34) is simply the definition of a Clifford algebra, and if the spinor indices are regarded as the vector indices of another SO(8), the charges can be represented by γ^i matrices. The representation of these is a Dirac spinor, which can be broken into left and right handed pieces. In terms of the original SO(8), these are the dotted spinor and the vector representations and, indeed, N=4 Yang Mills contains one vector and one spinor.

Off shell, the fields form a representation of the massive little group SO(9), and the supersymmetry generator is a spinor 16 of SO(9). The algebra is

$$\{ Q^A, Q^B \} = \delta^{AB} \quad , \tag{35}$$

with A and B going from one to sixteen. This algebra is the definition of the Clifford algebra of SO(16), and its fundamental representation is the sum of the two 128 dimensional spinors of SO(16). When these are broken into SO(9) representations, they give a traceless tensor g^{IJ}, an antisymmetric tensor A_{IJK} and a "Rarita-Schwinger field" Ψ^I. These representations are familiar as the fields of eleven dimensional supergravity, which on-shell has the same algebra as eq. (35). An arbitrary representation of eq. (35) is these fields multiplied by some SO(9) representation. The result that will be important for the auxiliary field problem, is that the number of fermions in any of these representations will be a multiple of 128.

The final ingredient needed is that fermionic auxiliary fields always come in pairs. Covariantly, this is seen by noting that the lagrangian for fermionic auxiliary fields must be $\bar{\chi}\psi$, and χ and ψ cannot be equal since they must have dimensionality 3/2 or 5/2, while the lagrangian has dimension 4. Here this is seen by noting that if χ and ψ are SO(8) spinors, the K^i transformation must act on them as

$$\delta\psi = \frac{1}{2}\,\gamma^i\chi$$

$$\delta\chi = \frac{1}{2}\,\gamma^i\Box\psi \quad , \tag{36}$$

to satisfy the algebra of eq. (28). Thus ψ and χ clearly cannot be identified. Thus spinor auxiliary fields always come in pairs, and each pair has 16 degrees of freedom. However, one can now see that the counting is inconsistent. Equating the total number of fermions to the number of auxiliary and physical fermions, one needs

$$128n = 16m + 8 \quad , \tag{37}$$

but there are no (integral) solutions to this equation.

N=4 Yang Mills therefore cannot have a quadratic action in superspace, and an analogue of the Lagrange multiplier of eq. (33) is needed. Thus far this has not been found, and it is unclear what properties such a theory would have. It is this uncertainty which raises doubts about the validity of the power counting rules in these cases, motivating the calculation which follows.

II. A simple method for calculating counterterms.

Before proceeding with the calculation, it is necessary to have a method of evaluating Feynman diagrams. The difficult step in this is performing the integrations over the loop momenta of the diagrams. The integrations can be done, for example, using the method of Feynman parametrization, but it is then necessary to integrate over the parameters. These integrals are relatively easy for simple graphs, but become increasingly difficult as the number of propagators in the diagrams increases. This method is thus particularly difficult to implement for many-point functions at more than one loop, and all other available methods run into similar difficulties.

The finite parts of Green functions are complicated functions of the Mandelstam invariants and the parameters of the theory. This must be so as, in order to satisfy unitarity, they have cuts and poles in the complex plane. It is difficult to envision these functions arising as the result of a procedure much simpler than the one outlined above. On the other hand, the counterterms needed at each order of perturbation theory have a much simpler structure. They are guaranteed by general arguments to be local functions in coordinate space, and are thus merely polynomials in momentum space. This theorem is central to our method. It was rigorously proved in the case of dimensional regularization [22] in ref. [23]. The theorem does not depend on any properties of the theory, such as renormalizability, but is a statement on momentum integrals. If one is interested only in the renormalization properties of a theory, the theorem suggests that it should not be necessary to use the full Feynman parametrization procedure. This simplification is in common use at one loop. For example, if an integral is logarithmically divergent, its divergent part must be a pure number and cannot depend on any momenta. All external momenta in the propagators can therefore be set to zero, resulting in far simpler denominators in the integral. In this chapter the procedure is extended to arbitrarily divergent multiloop diagrams. In fact, the simplification is especially effective when integrals depend on many different momenta, as is the case in multiloop calculations of many-point functions.

The method described here was developed for the calculation in chapter IV of two loop four-point functions in N=4 Yang Mills theory in more than four dimensions [24]. It should also be very useful for calculations in (super) gravity theories, which are also nonrenormalizable. In this chapter the method is illustrated in the far simpler case of a massive φ^3 theory in six dimensions. This is a renormalizable theory and thus has a less divergent structure than the Yang Mills theory, but the simplifications will still be clear.

a) One Loop.

The action of the φ^3 theory in $6-\varepsilon$ dimensions is

$$S = \int d^{6-\varepsilon}x \left(\frac{1}{2}\varphi(\Box - m^2)\varphi + \frac{\lambda}{3!}\mu^{\varepsilon/2}\varphi^3 \right) \quad . \tag{1}$$

It is written in Euclidean space ($\eta_{\mu\nu} = \delta_{\mu\nu}$), to avoid the necessity of performing Wick rotations on the integrals. Here ε is the dimensional regularization parameter and μ is the dimensional regularization mass, used to keep λ dimensionless in $6-\varepsilon$ dimensions. The potential of the theory is unbounded from below, and the theory is thus ill defined, but it can nevertheless be studied in perturbation theory and is renormalizable in six dimensions, as could be expected from the dimensionlessness of λ. The vertices are simply†

†If $m \neq 0$, φ must be shifted to eliminate tadpole diagrams. This does not affect any of the results below.

 $= \frac{1}{p^2+m^2}$  $= \lambda \mu^{\varepsilon/2}$. (2)

We start by reviewing the one loop corrections to this theory. There is one propagator graph, shown in fig. 1.

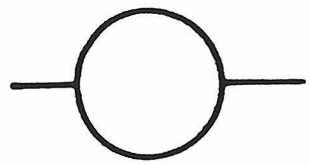

Figure 1.

The One-Loop Propagator Correction.

Its value is

$$I_2 \equiv \frac{1}{4}\,\lambda^2 \int \frac{\mu^{\varepsilon} d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\, \frac{1}{(k^2+m^2)((k+p)^2+m^2)} \quad , \tag{3}$$

where $\frac{1}{4}$ is a combinatoric factor. (Our combinatorics is chosen so that graphs are generated by the effective action.) The denominator can be simplified using the Feynman parametrization formula

$$\frac{1}{a^{\alpha} b^{\beta}} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(a x+b(1-x))^{\alpha+\beta}} \quad , \tag{4}$$

where Γ is the Euler gamma function. The momentum integral is evaluated using

$$\int \frac{d^d k}{(2\pi)}\, \frac{1}{(k^2+m^2)^n} = \frac{1}{(4\pi)^{d/2}}\, \frac{\Gamma(n-d/2)}{\Gamma(n)}\, (m^2)^{d/2-n} \quad . \tag{5}$$

Thus I_2 becomes

$$I_2 \equiv \frac{1}{4}\,\lambda^2\, \Gamma(\varepsilon/2-1)(4\pi\mu^2)^{\varepsilon/2} \int_0^1 dx\, (m^2+p^2x(1-x))^{1-\varepsilon/2} \quad , \tag{6}$$

where

$$\bar{\lambda} \equiv \frac{\lambda}{(4\pi)^{3/2}} \quad . \tag{7}$$

The Feynman parameter integral is difficult unless $m=0$. However, if one expands the expression in powers of ε, the integral can be done, giving

$$I_2 \equiv \frac{\bar{\lambda}^2}{4}\Big(\frac{p^2+6m^2}{6}\big(-\frac{2}{\varepsilon}+\gamma_e+\log\frac{m^2}{4\pi\mu^2}\big) - \frac{4}{9}p^2 - \frac{7}{3}m^2$$
$$+\frac{1}{3}(p^2+4m^2)^{3/2}\frac{1}{\sqrt{p^2}}\text{arcsinh}\big(\big(\frac{p^2}{4m^2}\big)^{1/2}\big) + O(\varepsilon) \quad . \tag{8}$$

Here γ_e is the Euler number, arising from the expansion

$$\Gamma(1+\varepsilon) = 1 - \varepsilon\gamma_e + \frac{\varepsilon^2}{2}\big(\frac{\pi^2}{6}+\gamma_e^2\big) + O(\varepsilon^3) \quad . \tag{9}$$

As promised, the finite part of I_2 is complicated, but the infinite part is a quadratic function of p and m. This information can be used to simplify the evaluation of the infinite parts of the diagrams. Consider, for example, the one loop vertex correction, which is given in fig. 2,

$$I_3 = \frac{\lambda\mu^{\varepsilon/2}}{3!}\lambda^2\int\frac{\mu^\varepsilon d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\frac{1}{((k+p)^2+m^2)((k-q)^2+m^2)(k^2+m^2)} \quad . \tag{10}$$

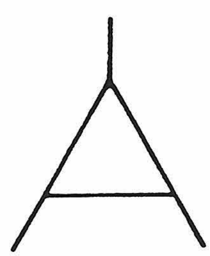

Figure 2.

The One-Loop Vertex Correction.

One factor of $\mu^{\varepsilon/2}$ has been pulled out to preserve the dimensionality of the vertex. This integral could also be done using Feynman parametrization. The finite part, however, is even more complicated than that of I_2, and involves dilogarithm functions. On the other hand, the pole part of I_3 is dimensionless , and cannot depend on p, q or m. The infinite part of I_3 can therefore be extracted from

$$I_3 = \frac{\lambda\mu^{\varepsilon/2}}{3!}\,\lambda^2 \int' \frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\,\frac{1}{(k^2)^3} \; + \; \mathit{finite} \quad . \tag{11}$$

(From this point on, finite terms will be dropped.) The prime denotes that the integral is infrared convergent and is not zero as it would formally be in dimensional regularization. This means that in eq. (10) the limit $m \to 0$ should be taken after the limit $\varepsilon \to 0$. In this case one could obtain the answer by keeping m and only letting p and q go to zero. However, in general it is easier to avoid explicit mass terms. Thus

$$\begin{aligned} \mathrm{Inf} \int' \frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\,\frac{1}{(k^2)^3} &= \lim_{m\to 0} \mathrm{Inf}\,\frac{1}{(4\pi)^{3-\varepsilon/2}}\,\frac{1}{2}\,\Gamma(\varepsilon/2)(m^2)^{-\varepsilon/2} \\ &= \lim_{m\to 0} \frac{1}{(4\pi)^3}\,\frac{1}{\varepsilon} = \frac{1}{(4\pi)^3}\,\frac{1}{\varepsilon} \quad , \end{aligned} \tag{12}$$

where Inf denotes "the infinite part of". This integral is the only one ever needed for one-loop divergence calculations. Thus, no Feynman parametrizations are needed in this case. The final result for the vertex is

$$I_3 = \frac{\lambda\mu^{\varepsilon/2}}{3!}\,\bar{\lambda}^2\,\frac{1}{\varepsilon} \quad , \tag{13}$$

where $\bar{\lambda}$ is defined in eq. (7).

Let us now calculate I_2 again, taking advantage of the polynomiality of its infinite part. In this case little is gained since I_2 contains only two propagators. However, the calculation is instructive, because I_2 is not logarithmically divergent. Since the pole part of I_2 is a polynomial of degree 2,

$$I_2 \sim (\frac{1}{2}p^\mu \frac{\partial}{\partial p_\mu} + m^2 \frac{\partial}{\partial m^2})\, I_2 \tag{14}$$

by homogeneity. In fact, since I_2 itself is homogeneous with degree $2-\varepsilon$, eq. (14) could be made exact by inserting a factor of $2/(2-\varepsilon)$. However, as will be seen when the multiloop case is considered, this is not desirable. It is better to concentrate on the pole part, for which eq. (14) is true. The diagram is now reduced to

$$I_2 \sim \frac{\lambda^2}{4}\int \frac{\mu^\varepsilon d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\Big(-\frac{m^2}{(k^2+m^2)^2((k+p)^2+m^2)}$$
$$-\frac{m^2}{(k^2+m^2)((k+p)^2+m^2)^2}-\frac{p^2}{(k^2+m^2)((k+p)^2+m^2)^2}$$
$$-\frac{k\cdot p}{(k^2+m^2)((k+p)^2+m^2)^2}\Big) \quad . \tag{15}$$

The first three integrals are now logarithmically divergent, but the fourth one is linearly divergent. The procedure is thus repeated for the linearly divergent term, using

$$I_{lin} = (p^\mu \frac{\partial}{\partial p_\mu} + 2m^2 \frac{\partial}{\partial m^2}) I_{lin} \quad . \tag{16}$$

Consequently

$$\frac{k_\nu}{(k^2+m^2)((k+p)^2+m^2)^2} \to \frac{4k_\nu k_\mu p_\mu}{(k^2+m^2)((k+p)^2+m^2)^3} \quad , \tag{17}$$

where convergent integrals have been dropped from the rhs. The infinite part of I_2 is thus

$$\frac{\lambda^2}{4}\int' \frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\left(\frac{-2m^2-p^2}{(k^2)^3}+4p_\mu p_\nu \frac{k_\mu k_\nu}{(k^2)^4}\right) \quad , \tag{18}$$

using the result in eq. (12). The second integral is a dimensionless tensor, and must therefore be proportional to $\delta_{\mu\nu}$. Since $\delta_{\mu\mu}=6-\varepsilon$, contraction with $\delta_{\mu\nu}$ shows that

$$\int' \frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\frac{k_\mu k_\nu}{(k^2)^4} = \int' \frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\frac{k^2\delta_{\mu\nu}}{(6-\varepsilon)(k^2)^4} \quad . \tag{19}$$

Thus one arrives at the final result

$$I_2 = \frac{\lambda^2}{4}\left(-2m^2-\frac{1}{3}p^2\right)\int' \frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\frac{1}{(k^2)^3}$$

$$= \frac{\hat{\lambda}^2}{4}\left(-2m^2-\frac{1}{3}p^2\right)\frac{1}{\varepsilon} \quad , \tag{20}$$

as before. This may not be the most efficient way of computing the pole part of an integral as simple as I_2, and the example is only of pedagogical value. Nonetheless, the method used is very advantageous in more complicated cases, and can be easily implemented on a computer.

The one loop renormalized action can now be seen to be

$$S_1 = S - \int d^{6-\varepsilon}x\, I_2\, \varphi^2 - \int d^{6-\varepsilon}x\, I_3\, \varphi^3$$

$$= \int d^{6-\varepsilon}x\left(\frac{1}{2}\varphi\Box\varphi\left(1-\frac{\hat{\lambda}^2}{6\varepsilon}\right)-\frac{1}{2}m^2\varphi^2\left(1-\frac{\hat{\lambda}^2}{\varepsilon}\right)+\frac{\lambda\mu^{\varepsilon/2}}{3!}\varphi^3\left(1-\frac{\hat{\lambda}^2}{\varepsilon}\right)\right) \tag{21}$$

where $\hat{\lambda}$ is defined in eq. (7).

b) Two Loops.

We can now illustrate this method for higher loop diagrams. At two loops the corrections to the propagators are given in fig. 3. Graph 2a contributes

$$I_{2a} = \frac{\lambda^4}{4}\mu^{2\varepsilon} \int \frac{\mu^\varepsilon d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}} \int \frac{\mu^\varepsilon d^{6-\varepsilon}l}{(2\pi)^{6-\varepsilon}} \frac{1}{(k^2+m^2)(l^2+m^2)}$$
$$\cdot \frac{1}{((k-l)^2+m^2)((k+p)^2+m^2)((l+p)^2+m^2)} \quad . \tag{22}$$

Figure 3.

Two-Loop Propagator Corrections.

To calculate I_{2a} using the standard techniques, it is necessary to reduce it to logarithmically divergent integrals. This can be done by using the homogeneity of I_{2a} in external momenta, as was done in eq. (14) (but keeping the ε terms). An equivalent technique is to insert the identity

$$\frac{1}{2(6-\varepsilon)}\left(\frac{\partial k_\mu}{\partial k_\mu} + \frac{\partial l_\mu}{\partial l_\mu}\right) = 1 \quad , \tag{23}$$

into the integrand, and integrate by parts. It is necessary to have logarithmically divergent integrals in order to be able to expand the integrand in powers of ε (see the appendices of ref.[25]). The resulting integrals can then be calculated, with some effort, by doing four Feynman parameter integrals. The result is

$$I_{2a} = \frac{\bar{\lambda}^4}{4}\Big(\frac{p^2+6m^2}{3}\Big(-\frac{1}{\varepsilon^2}+\frac{\gamma_e}{\varepsilon}+\frac{1}{\varepsilon}\log\frac{m^2}{4\pi\mu^2}\Big)-\frac{37}{6\varepsilon}m^2-\frac{1}{\varepsilon}p^2$$

$$+\frac{2}{3\varepsilon}(p^2+4m^2)^{3/2}\frac{1}{\sqrt{p^2}}\operatorname{arcsinh}\Big(\Big(\frac{p^2}{4m^2}\Big)^{1/2}\Big) \quad . \qquad (24)$$

Unlike the one loop case, the nonlocal functions occur even in the infinite part of I_{2a}. This is due to the well-known phenomenon of overlapping divergences. The similarity of I_{2a} and I_2, however, suggests the solution to this problem. The counterterm graphs I'_{2a} and I'_{2b} each contribute $-\frac{\bar{\lambda}^2}{\varepsilon}I_2$, since

$$\lambda_0 = \frac{\lambda\mu^{\varepsilon/2}}{3!}\Big(1-\frac{\bar{\lambda}^2}{\varepsilon}+O(\bar{\lambda}^4)\Big) \quad . \qquad (25)$$

The sum of the graphs I_{2a} and I'_{2a} thus gives

$$I_{2atot} = I_{2a}+I'_{2a} = \frac{\bar{\lambda}^4}{4}\Big(\frac{p^2+6m^2}{3\varepsilon^2}-\frac{3}{2\varepsilon}m^2-\frac{1}{9\varepsilon}p^2\Big) \quad , \qquad (26)$$

which is again a simple polynomial in p^2 and m^2. In fact even γ_e and 4π have disappeared from the total answer.

The simplicity of I_{2tot} is expected, since the two loop counterterm must be a "nice" object. However, this result is stronger, as it is true for graphs (2a) and (2b) separately. The precise theorem of ref [23] is that the infinite part of ***subtracted*** Feynman diagrams are polynomial in masses and momenta. A subtracted integral is essentially an integral minus similar ones with divergent subintegrals replaced by their infinite parts. For (2a), symbolically,

$$I_{sub} = \int dk \int dl - \int dk\ \mathrm{Inf}\Big(\int dl\Big) - \int dl\ \mathrm{Inf}\Big(\int dk\Big) \quad . \qquad (27)$$

Pictorially, one can show boxes around subdivergences as in fig. 4.

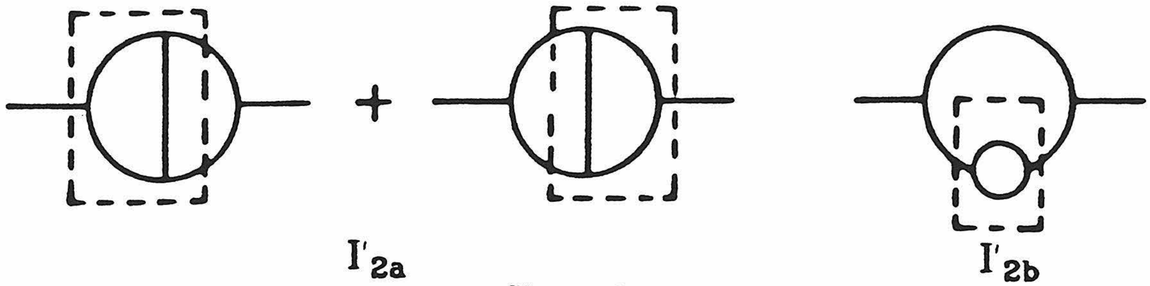

Figure 4.

Two-Loop Propagator Subtractions.

When the infinite parts of the integrals in these boxes are subtracted, the resulting divergent part is a local function. Thus $I_{2atot} = I - I_{2asub} = I_{2a} + I'_{2a}$ and $I_{2btot} = I_{2b} + I'_{2b}$ can be treated separately. In general, the sum of the subtracted diagrams is equal to the sum of diagrams plus counterterm diagrams. However, one must be careful to associate the appropriate counterterm diagrams to each graph to obtain simplifications graph by graph. If one subtracts the integrals, rather than the graphs, counterterms never need be used explicitly.

Some features of the cancellations resulting from adding I_{2a} and I'_{2a} can now be understood. The "arcsinh" term disappears, as it must in order to obtain a local answer. All factors of $\log 4\pi\mu^2$ also cancel, because there is no other dimensionful parameter that can enter in the logarithm. The cancellation of the $\log 4\pi\mu^2$ terms implies a relation between the $\frac{1}{\varepsilon^2}$ parts of I_{2a} and I'_{2a}. On dimensional grounds, from eq. (25)

$$I_{2a} = \mu^{2\varepsilon}\left(\frac{A}{\varepsilon^2} + \frac{B}{\varepsilon}\right) \simeq \frac{A}{\varepsilon^2} + \frac{B}{\varepsilon} + \frac{2A\log\mu}{\varepsilon} \quad , \tag{28}$$

while for the corresponding subtraction

$$I'_{2a} = \mu^{\varepsilon}\left(\frac{C}{\varepsilon^2} + \frac{D}{\varepsilon}\right) \simeq \frac{C}{\varepsilon^2} + \frac{D}{\varepsilon} + \frac{C\log\mu}{\varepsilon} \quad . \tag{29}$$

Thus for the logμ terms to cancel, C must equal $-2A$ and the $\frac{1}{\varepsilon^2}$ part of the counterterm diagram is thus -2 times that of the original diagram. This result is independent of the details of the diagrams or the theory. For example, since pure gravity is one loop finite, it needs no one-loop counterterms and the S matrix at two loops is at most $\frac{1}{\varepsilon}$ [26]. This is an oversimplification, because (gauge dependent) one loop Green functions can be infinite, but the argument can be made rigorous. This will be done in chapter III. At n loops, the cancellation of all the $\log^m\mu$ terms relates the $\frac{1}{\varepsilon^2}, \cdots, \frac{1}{\varepsilon^n}$ poles of the diagrams to those of the counterterm diagrams. In renormalizable theories these relations are familiar from renormalization group arguments [27]. Having recognized the great simplification that the cancellation of the overlapping divergences introduces, from now on we will restrict our attention to integrals from which subdivergences are subtracted out.

Let us now calculate the two loop vertex correction before returning to the propagator graphs. There are two graphs, shown together with their subtractions in fig. 5. Graph 3a contributes

$$I_{3a} = \frac{\lambda\mu^{\varepsilon/2}}{2}\lambda^4 \int \frac{\mu^\varepsilon d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}} \int \frac{\mu^\varepsilon d^{6-\varepsilon}l}{(2\pi)^{6-\varepsilon}} \frac{1}{(k^2+m^2)((k-l)^2+m^2)}$$
$$\cdot \frac{1}{((k+p)^2+m^2)((k-q)^2+m^2)((l+p)^2+m^2)((l-q)^2+m^2)} \quad . \tag{30}$$

It contains six propagators, and would require five Feynman parametrizations if calculated naively. However, I_{3atot} will not depend on p, q, m or factors of $(4\pi\mu^2)^\varepsilon$, and these can all be dropped. Therefore I_{3atot} simplifies to

$$I_{3atot} = \frac{1}{2}\lambda\mu^{\varepsilon/2}\lambda^4 \int' \frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}} \left(\int' \frac{d^{6-\varepsilon}l}{(2\pi)^{6-\varepsilon}} \frac{1}{l^4(k-l)^2} - \mathrm{Inf} \int' \frac{d^{6-\varepsilon}l}{(2\pi)^{6-\varepsilon}} \frac{1}{l^4(k-l)^2} \right) \tag{31}$$

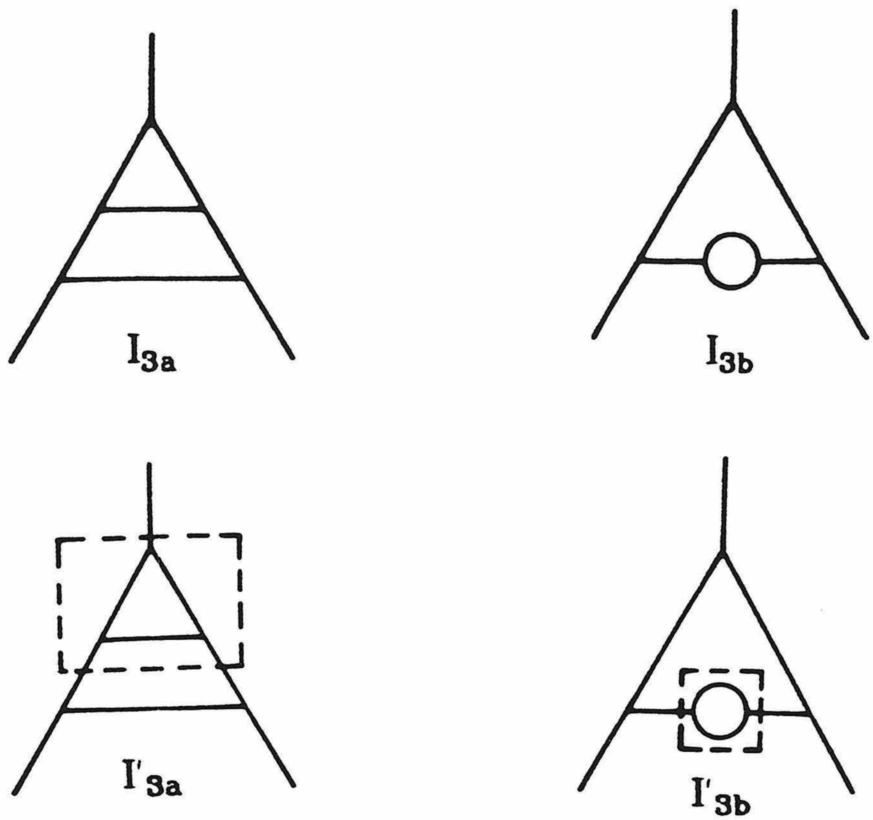

Figure 5.

Two-Loop Vertices with Subtractions.

The only subdivergence in this case comes from the l integral. The second term corresponds to the graph with the box replaced by the vertex counterterm. The l integral can be done with one Feynman parametrization, giving

$$\int' \frac{d^{6-\varepsilon}l}{(2\pi)^{6-\varepsilon}} \frac{1}{l^4(k-l)^2} = \frac{1}{(4\pi)^3}\Gamma(\varepsilon/2)\frac{\Gamma(1-\varepsilon/2)\Gamma(2-\varepsilon/2)}{\Gamma(3-\varepsilon)}(k^2)^{-\varepsilon/2} \quad , \quad (32)$$

where the integral defining the Euler beta function

$$\int_0^1 dx\, x^\alpha(1-x)^\beta = \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \quad , \quad (33)$$

has been used. The infinite part of the l integral is simply $\frac{1}{(4\pi)^3\varepsilon}$. It should be noted that, if the mass were kept for infrared regularization, deriving this result would have been more difficult, as another parameter integral would have been necessary.

The infinite part of I_{3atot} now becomes

$$\frac{1}{2}\lambda\mu^{\varepsilon/2}\lambda^2\bar{\lambda}^2\int'\frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\left[\Gamma(\varepsilon/2)\ \frac{\Gamma(1-\varepsilon/2)\Gamma(2-\varepsilon/2)}{\Gamma(3-\varepsilon)}\ \frac{1}{(k^2)^{3+\varepsilon/2}}-\frac{1}{\varepsilon}\ \frac{1}{(k^2)^3}\right] \tag{34}$$

One then arrives at the final answer using

$$\int'\frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\ \frac{1}{(k^2)^{3+\varepsilon/2}}=\frac{1}{(4\pi)^3}\ \frac{\Gamma(\varepsilon)}{\Gamma(3+\varepsilon/2)}=\frac{1}{(4\pi)^3}\ \frac{1}{2\varepsilon}\left(1+\varepsilon(-\frac{3}{4}-\gamma_e)\right) \tag{35}$$

and

$$\int'\frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}}\ \frac{1}{(k^2)^3}=\frac{1}{(4\pi)^3}\ \frac{\Gamma(\varepsilon/2)}{2}=\frac{1}{(4\pi)^3}\ \frac{1}{\varepsilon}\left(1-\frac{\varepsilon}{2}\gamma_e\right) \quad . \tag{36}$$

From these formulae it is seen that the $\frac{1}{\varepsilon^2}$ part of the subtraction is indeed twice that of the integral. Putting eqs (34-36) together gives

$$I_{3atot}=\lambda\mu^{\varepsilon/2}\left(-\frac{\bar{\lambda}^4}{4\varepsilon^2}+\frac{\bar{\lambda}^4}{16\varepsilon}\right) \quad . \tag{37}$$

The calculation of I_{3btot} is similar, and the result is

$$I_{3btot}=\lambda\mu^{\varepsilon/2}\left(\frac{\bar{\lambda}^4}{24\varepsilon^2}-\frac{7\bar{\lambda}^4}{288\varepsilon}\right) \quad . \tag{38}$$

The simplification of letting p, q and m go to zero has allowed the evaluation the vertex graph using only one fairly easy parameter integral. If the integral were done naively, not only would five parameter integrals be needed, but the resulting integral would be very complicated, with the parameters mixing in a complicated way. We stress again that the simplification comes from considering diagrams together with the corresponding subtractions, so that only an overall divergence is left.

To use these simplifications in the propagator graphs it is necessary to reduce the integrals to logarithmically divergent ones. Consider again I_{2a}. The infinite part of the subtracted integral is quadratic in p and m, so, as in one loop,

$$I_{2atot} \sim \left(\frac{1}{2}p^{\mu}\frac{\partial}{\partial p_{\mu}} + m^2\frac{\partial}{\partial m^2}\right) I_{2atot} \quad . \tag{39}$$

Since the integral and the subtraction have different degrees of homogeneity, it is seen that, if one required an exact result, a simple formula such as eq. (39) could not be used. In I_{2a}, the integrand can be reduced using

$$\frac{1}{(k^2+m^2)(l^2+m^2)((k-l)^2+m^2)((k+p)^2+m^2)((l+p)^2+m^2)}$$

$$\rightarrow \frac{2k\cdot p}{(k^2+m^2)(l^2+m^2)((k-l)^2+m^2)((k+p)^2+m^2)^2((l+p)^2+m^2)}$$

$$-\frac{4m^2}{(k^2)^3(l^2)^2(k-l)^2} - \frac{m^2}{(k^2)^2(l^2)^2((k-l)^2)^2} - \frac{2p^2}{(k^2)^3(l^2)^2(k-l)^2} \quad . \tag{40}$$

The first term above is still linearly divergent, and the procedure is repeated :

$$\frac{k_\mu}{(k^2+m^2)(l^2+m^2)((k-l)^2+m^2)((k+p)^2+m^2)^2((l+p)^2+m^2)}$$

$$\rightarrow \frac{2k_\mu l\cdot p}{(k^2)^3(l^2)^3(k-l)^2} + \frac{4k_\mu k\cdot p}{(k^2)^4(l^2)^2(k-l)^2} \quad . \tag{41}$$

As a result, the integral can be written as

$$I_{2atot} = \frac{\lambda^4}{4} \int' \frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}} \int' \frac{d^{6-\varepsilon}l}{(2\pi)^{6-\varepsilon}} \left[\frac{-4m^2-2p^2}{(k^2)^3(l^2)^2(k-l)^2} \right.$$

$$\left. + \frac{8(k\cdot p)^2}{(k^2)^4(l^2)^2(k-l)^2} - \frac{m^2}{(k^2)^2(l^2)^2(k-l)^{22}} + \frac{4k\cdot p\, l\cdot p}{(k^2)^3(l^2)^3(k-l)^2} \right] , \qquad (42)$$

where all the integrals are implicitly subtracted.

The evaluation of these integrals proceeds along lines similar to that discussed above, but there are some features which make it instructive to continue. The first integral is the same as that encountered in the vertex diagram. The second one has an additional factor of $\frac{k_\mu k_\nu}{k^2}$. The l integral and its subtraction can be done as before. The k integral is then proportional to $\delta_{\mu\nu}$, and $k_\mu k_\nu$ can be replaced by $\frac{\delta_{\mu\nu}k^2}{6-\varepsilon}$. This substitution could be done earlier, but one must be careful. If $k_\mu k_\nu$ had appeared in a subtraction integral, it must be replaced by $\frac{\delta_{\mu\nu}k^2}{6}$, since all ε's in the subtraction are set to zero.

The remaining two integrals both have no divergent subintegrals, and are thus purely of order $\frac{1}{\varepsilon}$. Once again using Feynman parametrization

$$\int' \frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}} \int' \frac{d^{6-\varepsilon}l}{(2\pi)^{6-\varepsilon}} \frac{1}{(k^2)^2(l^2)^2((k-l)^2)^2} =$$

$$\int' \frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}} \frac{1}{(k^2)^2} \left(\frac{1}{(k^2)^{1+\varepsilon/2}} (4\pi)^{-3+\varepsilon/2}\Gamma(1+\varepsilon/2) \int_0^1 dx\, (x(1-x))^{-\varepsilon/2} \right) . (43)$$

Since the integral has no $\frac{1}{\varepsilon^2}$ part, ε can be set to zero in the result of the l integral *except for the power of* k^2. (This would lead to a factor of 2 error, as can be seen from eqs (35) and (36).) The k integral is now easily evaluated, resulting in

$$\frac{1}{(4\pi)^6} \frac{1}{\varepsilon} \quad . \qquad (44)$$

In the final integration of eq. (43), it is possible to replace $k_\mu l_\nu$ with $\frac{1}{6}\,\delta_{\mu\nu} k \cdot l$ with no ambiguity, since the integral is $O(\frac{1}{\varepsilon})$. The integrand can then be written as

$$\frac{\delta_{\mu\nu}}{6}\;\frac{\frac{1}{2}(k^2+l^2-(k-l)^2)}{(k^2)^3(l^2)^3(k-l)^2} \quad . \tag{45}$$

The first two terms are the familiar integral from the vertex, while the third term factors into two one loop integrals. None of these integrals need to be subtracted, as their sum has no subdivergences. However, if one is, all must be. The result is

$$\int' \frac{d^{6-\varepsilon}k}{(2\pi)^{6-\varepsilon}} \int' \frac{d^{6-\varepsilon}l}{(2\pi)^{6-\varepsilon}}\; \frac{k_\mu l_\nu}{(k^2)^3(l^2)^3(k-l)^2} = \frac{\delta_{\mu\nu}}{48\,\varepsilon} \quad , \tag{46}$$

and substituting in eq. (42) one obtains

$$I_{2atot} = \frac{\lambda^4}{4}\left(\frac{p^2+6m^2}{3\varepsilon^2} - \frac{1}{\varepsilon}m^2 - \frac{1}{9\varepsilon}p^2 \right) \quad , \tag{47}$$

as in eq. (26). However, with this method only one simple Feynman parameter integral was needed, compared with the four nastily intertwined integrals used in the original derivation. The calculation of I_{2b} is now straightforward, yielding

$$I_{2btot} = \frac{\lambda^4}{4}\left[\frac{1}{\varepsilon^2}\left(\frac{1}{2}m^2 - \frac{1}{18}p^2\right) + \frac{1}{\varepsilon}\left(\frac{1}{24}m^2 + \frac{11}{216}p^2 \right)\right] \quad . \tag{48}$$

The final two loop renormalized action is

$$S_2 = \int d^{6-\varepsilon}x \left(\frac{1}{2}\varphi\Box\varphi\left(1 - \frac{\bar{\lambda}^2}{6\varepsilon} + \frac{5\bar{\lambda}^4}{36\varepsilon^2} - \frac{13\bar{\lambda}^4}{432\varepsilon}\right)\right.$$

$$- \frac{1}{2}m^2\varphi^2\left(1 - \frac{\bar{\lambda}^2}{\varepsilon} - \frac{5\bar{\lambda}^4}{4\varepsilon^2} + \frac{23\bar{\lambda}^4}{48\varepsilon}\right)$$

$$\left. + \frac{\lambda}{3!}\mu^{\varepsilon/2}\varphi^3\left(1 - \frac{\bar{\lambda}^2}{\varepsilon} + \frac{5\bar{\lambda}^4}{4\varepsilon^2} - \frac{11\bar{\lambda}^4}{48\varepsilon}\right)\right) \quad . \tag{49}$$

We have seen that, up to two loops in the φ^3 theory, all integrals can be reduced to fairly simple logarithmically divergent integrals with no masses or external momenta. Whereas the example is rather trivial, it should be clear that the method will work at any loop order for any theory.

c) Discussion.

We conclude this chapter with some comments for applying this method to more complicated theories. The φ^3 theory is a scalar theory with at most quadratically divergent integrals. The resulting logarithmic integrals, therefore, contained at most two vector indices. The relation $A_\mu B_\nu \propto \delta_{\mu\nu}$ could be used to eliminate these indices. In a more complicated theory one would would encounter higher rank tensors. These can be eliminated in the same way. Thus, if an integral had a numerator $k_\mu k_\nu l_\rho l_\sigma$, for example, one could use the replacement

$$k_\mu k_\nu l_\rho l_\sigma \rightarrow a\delta_{\mu\nu}\delta_{\rho\sigma} + b\left(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma}\right) \quad . \tag{50}$$

The coefficients a and b can be determined by contracting both sides with δ's in two different ways.

One important feature that has not been seen here, has to do with the indices in the subtractions. When the logarithmically divergent integrals are evaluated, the tensor $\delta_{\mu\nu}$ is generated, where (formally) the indices go from one to $(d-\varepsilon)$. If the integral occurs in a subtraction, the tensor must be replaced with $\delta_{\bar{\mu}\bar{\nu}}$, where barred indices go from one to d. This is necessary because ε is set to zero in the subtraction. If this tensor is contracted with a $d-\varepsilon$ dimensional object, the barred indices can be replaced by unbarred ones. This can be remembered by regarding ε as a positive real number. In this theory, all indices are contracted with momenta that go from one to $d-\varepsilon$, and thus barred indices are unnecessary. In fact this subtlety can be ignored in all renormalizable theories when dimensional regularization is used. This is so, because all pole parts in these theories have the same form as the original lagrangian, where indices are always contracted with momenta or fields. However, in a nonrenormalizable theory containing spinors, indices can be associated only with γ matrices, and barred indices cannot be reduced to unbarred ones.

Another point is that at more than two loops the subtractions of graphs are more complicated [28]. A three loop graph with its subtractions is shown in fig. 6. In addition, the two loop graphs subtracted in D and E must have their own one loop subdivergences subtracted (or they would not be local). Working with the subtracted graphs is equivalent to adding the one and two loop counterterm graphs, as in fig. 7. It must be noted, however, that only the counterterm from I_{3a} and not that of I_{3b} must be used in D′ and E′, as can be seen by looking at D and E. If one added both the two loop propagator diagrams together, however, one could simply add all the counterterm diagrams.

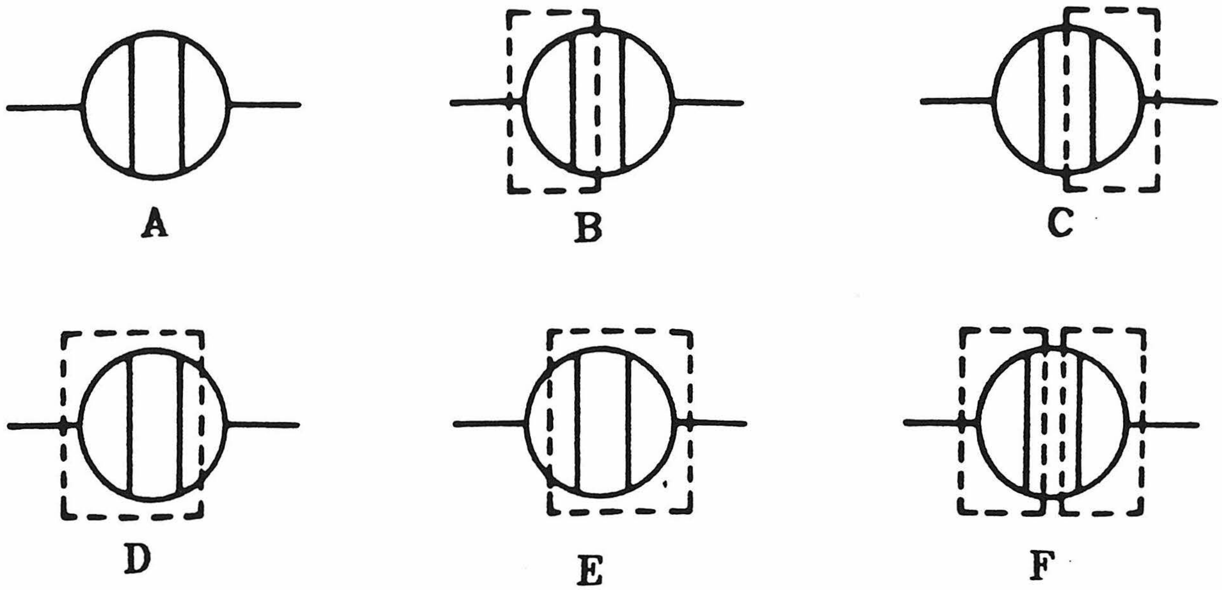

Figure 6.

A Three-Loop Propagator Graph with Subtractions.

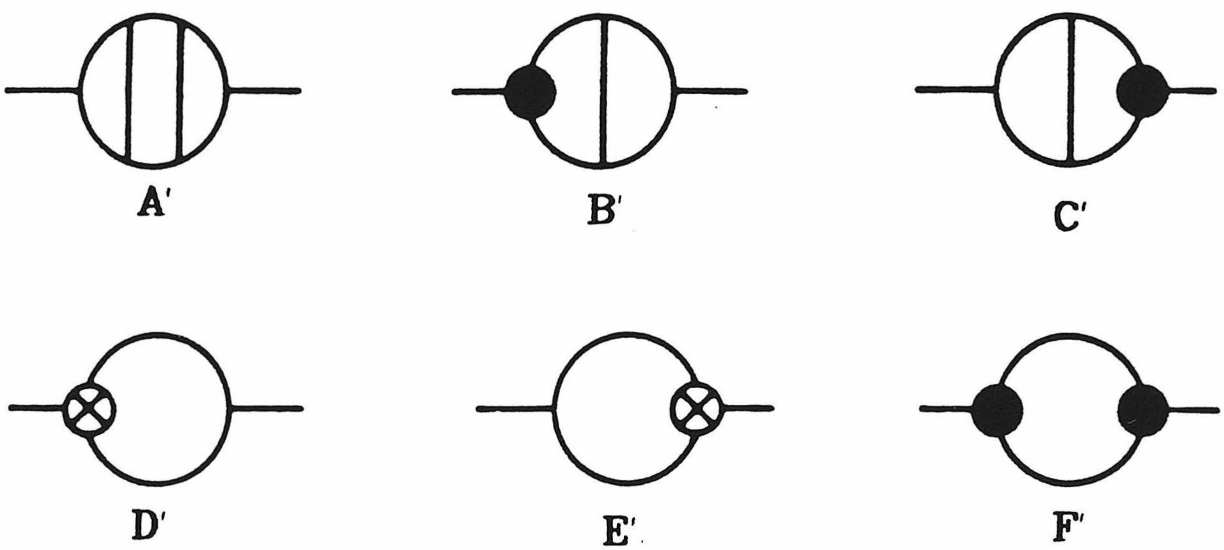

Figure 7.

A Three-Loop Propagator Graph with Counterterms.

The final comment is that at one loop the only integral required is $\int' \frac{d^{d-\varepsilon}k}{(k^2)^{d/2}}$, which can be done trivially. At two loops the integrals have the form

$$\int' d^{d-\varepsilon}k \int' d^{d-\varepsilon}l \; \frac{1}{(k^2)^a (l^2)^b (k-l)^{2c}} \quad , \tag{51}$$

with $a+b+c=d$. As the number of loops is increased, the integrals become more and more complicated. At three loops, for example, one encounters integrals of the form

$$\int' d^{d-\varepsilon}k \int' d^{d-\varepsilon}l \int' d^{d-\varepsilon}m \; \frac{1}{(k^2)^a (l^2)^b (m^2)^c (k-l)^{2e} (k-m)^{2f} (l-m)^{2g}} \; , \tag{52}$$

with $a+b+c+e+f+g=3/2d$. These integrals can sometimes be done easily using Feynman parameters, but sometimes other methods such as Gegenbauer polynomials [29] are needed. There has been much work in developing techniques for propagator integrals to many loops [29]. The method described here of differentiating graphs and setting momenta and masses to zero allows one to calculate the infinite parts of arbitrary divergent graphs with any number of external legs in any theory to n loops, using the results of massless propagator calculations at $(n-1)$ loops, and then integrating over the propagator momenta.

III. The Meaning and Properties of Finite Theories.

In asking whether or not a theory is finite, one must be careful to refer only to physical quantities, such as S matrix amplitudes. Even if a theory is physically finite, its Green functions are, in general, infinite. If the finite theory is also renormalizable, these divergences can always be removed by infinite wavefunction renormalizations of the form

$$\Phi'(x) = Z\Phi(x) \quad , \tag{1}$$

with Z a dimensionless quantity independent of the field Φ. These rescalings are unphysical and gauge dependent. This is in contrast to the renormalization of the *parameters* of the lagrangians, such as masses and coupling constants, which leads to physically meaningful quantities such as the β function. In gauge theories the redefinitions also result in a change of gauge, but this complication does not alter the discussion. In renormalizable theories finiteness can be formulated equivalently as the lack of ultraviolet divergences in S matrix elements, or by the absence of mass and charge renormalizations.

In nonrenormalizable theories, the situation is very similar. However, as there are dimensionful parameters in the theories, the rescalings of eq. (1) can be generalized to become field redefinitions

$$\Phi'(x) = f[\Phi(x)] \quad . \tag{2}$$

Here f can be nonlinear, and its only restriction is that it be a local function of Φ. In eliminating infinities, this locality will be insured by the locality of the counterterms. In gauge theories the complications due to field redefinitions involving ghosts and changes of gauge will not affect any arguments given below. We note that eq. (1) is just a special case of eq. (2), corresponding to a linear redefinition of the fields. The important feature is that, as for wavefunction renormalizations, nonlinear field redefinitions do not affect the S matrix. This well known argument goes as follows. Apart from source terms, field redefinitions can be absorbed by a change of variable in the path integral. The jacobian of this transformation is of the type $det(1+X)$, as $\varphi' = \varphi + O(\hbar)$. It can be evaluated in perturbation theory using the path integral representation

$$det(1+X) = \int d\bar{c}\, dc\, \varepsilon^{\bar{c}(1+X)c} \quad , \tag{3}$$

where c and $\bar{c}$ are anticommuting ghost fields. Since X is a local operator, the ghosts c and $\bar{c}$ have local interactions. Because the ghost propagator is simply one in momentum space, all ghost loop diagrams contain integrals of polynomials. They therefore vanish in dimensional regularization, and the jacobian is one [30]. The change of variables also alters the couplings to the sources, which become

$$J f^{-1}[\Phi'(x)] \quad , \tag{4}$$

where f^{-1} is also a local function in perturbation theory. There is thus a coupling of the source J to several fields Φ' at the same spacetime point, and the Green functions differ from those obtained by the usual $J\Phi'$ coupling. However, it is familiar from the proofs of renormalizability of Yang-Mills theories that the additional source couplings do not affect the S matrix [31]. When the sources are put on shell and the external legs are amputated for computing S matrix elements, the contributions from the nonlinear source couplings lack the necessary poles on some of the external legs, and therefore vanish. The conclusion is that finite theories in general produce redefinitions of the type (2), but these do not affect the S matrix.

In practice, when doing explicit loop calculations, the redefinition in eq. (2) appears as a power series in the coupling constants of the theory, and can be written as a power series in $\hbar$ (here not set equal to one)

$$\Phi'(x) = \Phi(x) + \hbar\,\varphi_1(x) + \hbar^2\varphi_2(x) + \cdots \tag{5}$$

It is interesting to examine the meaning of finiteness loop by loop. At one loop, inserting (5) in the action implies that the counterterm needed to make Green's functions finite in a finite theory is of the form

$$S_1[\Phi] = S[\Phi] + \hbar\,\frac{\delta S}{\delta \Phi}\varphi_1 \simeq S[\Phi'] \quad . \tag{6}$$

A theory is thus one loop finite if its counterterms vanish with the use of the classical field equations

$$\frac{\delta S}{\delta \Phi} = 0 \quad , \tag{7}$$

since they can be absorbed by a field redefinition, using eq. (6).

A well-known example of this is provided by pure Einstein quantum gravity at one loop. This was discussed by 't Hooft and Veltman [3] in a background field de Donder gauge. The Green functions are infinite, but can be made finite by introducing the counterterm

$$\Delta S = \sqrt{-g}\,\hbar(a\,R_{\mu\nu}R^{\mu\nu} + b\,R^2) \quad , \tag{8}$$

where a and b are gauge dependent coefficients. This follows simply from the power counting at one loop, general coordinate invariance and the Gauss-Bonnet identity, which implies that R^2 and $R^2_{\mu\nu}$ are the only independent general coordinate invariant scalars of dimension four. Since the equation of motion implies $R_{\mu\nu} = 0$, ΔS vanishes on shell and the theory is one loop finite.

Indeed, ΔS can be written as

$$\Delta S = \sqrt{-g}\,\hbar (R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R)(a R_{\mu\nu} - (b + \tfrac{1}{2} a) g_{\mu\nu} R) \quad , \tag{9}$$

which corresponds to the field redefinition

$$g'_{\mu\nu} = g_{\mu\nu} + \frac{4\kappa^2 \hbar}{\varepsilon}(a R_{\mu\nu} - (b + \tfrac{1}{2} a) g_{\mu\nu} R) \quad . \tag{10}$$

In fact, because the S matrix is physical, it is general coordinate invariant in any gauge and background field methods are not necessary. One loop finiteness follows simply from the fact that (8), which is the only possible correction to the *S matrix*, vanishes on shell. Another proof that the counterterm of eq. (8) is unphysical is that there is a gauge [32] (not a very convenient one for actual calculations) in which a and b are zero. In this gauge the Green functions themselves are one-loop finite, and no field redefinitions are needed.

Before asking whether a theory is finite at higher loops, one must decide whether to calculate with S or S_1, the action plus the one loop counterterms. The sensible definition of a finite theory is that the S matrix calculated from the original action must not diverge. Thus at any order, *calculating without counterterms*, the infinities of the Green functions must vanish on shell, giving a finite S matrix. In this sense, the common statement that "a theory is finite if its counterterms vanish on shell" is true. It should be noted, however, that calculations are not conveniently done in this manner. If counterterms are not used, Green functions have overlapping divergences. This is not a problem in principle but, as was shown in chapter II, it makes the evaluation of graphs very difficult.

It is thus important to see how calculations can be done using counterterms. A clue to this problem can be obtained from the renormalizable case.

There, using or not using counterterms is a question of whether or not to insert wavefunction renormalizations into Feynman diagrams. However, factors of "Z" from vertices are canceled by $1/Z$'s from propagators. Moreover, the proper definition of the S matrix involves a rescaling of the external legs, and the two procedures are clearly equivalent. In the nonrenormalizable case, one can analogously calculate either with the original action or with the action of the redefined field. This action is not just S_1, however, since additional terms from the Taylor series expansion of $S[\Phi]$ also contribute. Thus at two loops (to be definite) one can calculate with

$$S'[\Phi] = S[\Phi] + \hbar \frac{\delta S}{\delta \Phi} \varphi_1 + \frac{\hbar^2}{2} \frac{\delta^2 S}{\delta \Phi^2} {\varphi_1}^2 \quad , \tag{11}$$

rather than with $S[\Phi]$. The first correction provides the counterterms necessary for removing the divergences of Green functions at one loop. The second term is already $O(\hbar^2)$ and is thus inserted only into tree diagrams. It therefore produces no overlapping divergences, and the cancellation of the overlapping divergences by the first term is preserved. It is however necessary to include this term or one would erroneously conclude that the theory had additional divergences at two loops. This also occurs for renormalizable theories where a φ^n vertex is multiplied by Z^n, not by $(1+n(Z-1))$. If one wishes to calculate only with the original action and its counterterms (*i.e.* without the second order term in eq. (11)), a mathematically equivalent, if philosophically different, procedure is to calculate the effective action in this way and determine the counterterms. Then the *two-loop counterterm* would contain the "δ^2" term, and the theory is finite if all the counterterms fit into one and two-loop field redefinitions. This method is less suited to working with the S matrix.

To summarize, finiteness can be stated in several equivalent ways.

1) The S matrix, calculated without counterterms, is finite.

2) The counterterms of the theory combine into field redefinitions.

3) The S matrix, calculated using lower order field redefinitions to remove overlapping divergences, is finite.

We find the third definition to be the most convenient one. It can be implemented by calculating in the theory with all diagrams completely subtracted and adding the $O(\hbar^n)$ terms from the Taylor expansions of the field redefinitions at less than n loops. If the theory is finite at n loops, the remaining infinities will vanish on shell or when S matrix elements are calculated.

The equivalence of calculating with or without field redefinitions has an interesting consequence for the first physical divergence of a theory. If a theory is finite to $n-1$ loops, but infinite at n loops, the n loop divergence of the S matrix, which could *a. priori* be of the type $\frac{1}{\varepsilon} \cdots \frac{1}{\varepsilon^n}$†, is purely $\frac{1}{\varepsilon}$. In renormalizable theories, this result follows from the observation that all higher order poles can be obtained from simple poles using the renormalization group [26]. A proof of this result for pure gravity at two loops has been given by Chase [27]. He used the observation, proved in chapter II, that the $\frac{1}{\varepsilon^2}$ part of a two loop graph is $-\frac{1}{2}$ that of the corresponding counterterm graph. He then used a field redefinition to show that the gauge fixed action could be redefined back to the original Einstein-Hilbert action, and that the counterterms thus give no contribution. The gauge fixing terms and ghosts considerably complicate his proof, and the proof is, in fact, slightly flawed, as the possibility that the graviton redefinition contains ghost-ghost terms was ignored.

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†In odd dimensions $\frac{1}{\varepsilon} \cdots \frac{1}{\varepsilon^{[n/2]}}$

It is possible to evade the problems caused by gauge invariance altogether, and give a simple proof of the general result. This follows from the gauge-independence of the S matrix. If the S matrix of a theory is $n-1$ loop finite in some gauge, it will also be finite in a ghost free gauge, such as an axial or light cone gauge. In this physical gauge the theory shows no sign of its gauge invariance. If the desired result holds in non-gauge theories, it implies that the n loop S matrix has no compound poles in this gauge. Again using the gauge independence of the S matrix, this result must hold in any gauge. It is thus only necessary to prove the result for a non-gauge theory.

If a theory is finite to $n-1$ loops, one can calculate the n loop contribution to the S matrix without counterterms. It has the form

$$\mu^{n\varepsilon}\left[\frac{A_1}{\varepsilon}+\frac{A_2}{\varepsilon^2}+\cdots+\frac{A_n}{\varepsilon^n}\right] + \text{ finite} \quad , \tag{12}$$

where μ is the dimensional regularization mass introduced to preserve the dimensionality of the constants in the action. If the S matrix is calculated with field redefinitions, however, it cannot have any overlapping divergences, and thus $\log\mu$ can never appear in its divergent part. Since the two methods of calculation are equivalent, $\log\mu$ also cannot appear in the infinite part of eq. (12). Expanding

$$\mu^{n\varepsilon} = 1 + n\,\varepsilon\log\mu + \cdots \quad , \tag{13}$$

it follows that only A_1 can be non zero, giving the required result.

IV. Quantum Corrections to N=4 Yang Mills.

We now describe in some detail the various steps involved in the calculation of the ultraviolet divergences of N=4 supersymmetric Yang-Mills at two loops. The theory is formulated most simply in ten dimensions [12], where it describes the interactions of an adjoint multiplet of Majorana-Weyl (*i.e.* real left-handed) spinors with a Yang-Mills field. The action is

$$S = \int d^{10}x \left[-\frac{1}{4} F^a_{\alpha\beta} F^a_{\alpha\beta} - \frac{i}{2} \bar{\lambda}^a \gamma_\alpha (D_\alpha \lambda)^a \right] \quad , \tag{1}$$

where

$$(D_\alpha \lambda)^a \equiv \partial_\alpha \lambda^a + g f^{abc} A^b_\alpha \lambda^c \quad , \tag{2}$$

and

$$F^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + g f^{abc} A^b_\alpha A^c_\beta \quad . \tag{3}$$

Early Greek letters are used to denote ten dimensional indices. In the following, middle Greek letters will be used for d dimensional indices. The action has been Wick rotated into Euclidean space ($\eta_{\alpha\beta} = \delta_{\alpha\beta}$), to avoid the necessity of performing Wick rotations when doing loop integrals. (Rigorously these spinors do not exist in Euclidean space, but this causes no problems.) The f^{abc}'s are the structure constants of a semisimple Lie algebra, and are real and totally antisymmetric. The supersymmetry transformations of the theory are

$$\delta A^a_\alpha = -i \bar{\varepsilon} \gamma_\alpha \lambda^a$$

$$\delta \lambda^a = \frac{1}{2} F^a_{\alpha\beta} \gamma_{\alpha\beta} \varepsilon \quad , \tag{4}$$

where $\gamma_{\alpha\beta}$ is the product of γ_α and γ_β antisymmetrized with weight one. The lagrangian can be dimensionally reduced to any dimension $d<10$ simply by dropping the dependence of the fields with respect to $10-d$ spatial coordinates. The fields then split into irreducible representations of the d-dimensional Lorentz group. For example, in going from $d=10$ to $d=4$ the spinor splits into four four-dimensional Weyl spinors, and the vector splits into a four-dimensional vector and six scalars. However, while it is natural to use d dimensional indices in d dimensions, this is inconvenient for explicit calculations, because the action splits into many pieces. Therefore, we will keep the form of the action in eq. (1) for all $d\leq 10$ (changing $d^{10}x$ to $d^d x$). This simplifies the Feynman rules and also has the effect of allowing similar manipulations in all dimensions.

Since the intermediate results of a calculation diverge even in a finite theory, it is necessary to regularize the action. To preserve the gauge invariance of the regularized action, a dimensional regularization scheme should be used. Ordinary dimensional regularization does not preserve the equality of Bose and Fermi degrees of freedom, and the regularized action is thus not supersymmetric. Therefore, a supersymmetric modification of dimensional regularization, known as dimensional reduction [33], should be used. In this scheme momenta are continued to $d-\varepsilon$ dimensions, as in dimensional regularization, but fields are left d-dimensional. The notation above is very convenient for this scheme, which can be implemented simply by replacing g with $g\mu^{\varepsilon/2}$ and $d^d x$ with $d^{d-\varepsilon}x$, while leaving the ten dimensional indices on the fields untouched.

The Feynman gauge $(\partial_\alpha A_\alpha)=0$ results in the simplest vector propagator and is therefore used in the calculation. The quantum action of the theory, obtained by the usual Faddeev Popov prescription, is

$$S = \int d^{10}x \Bigg[-\frac{1}{4} F^a_{\alpha\beta} F^a_{\alpha\beta} - \frac{i}{2} \bar{\lambda}^{\,a} \gamma_\alpha (D_\alpha \lambda)^a - \frac{1}{2} (\partial_\alpha A_\alpha)^2$$

$$+ \bar{c}^{\,a} \Box c^a + g \mu^{\varepsilon/2} f^{abc} \bar{c}^{\,a} A^b_\alpha \partial_\alpha c^c \Bigg] \quad , \tag{5}$$

where c^a is the familiar ghost field. The Feynman rules are shown in fig. 1. It should be noted that, since the spinor is Majorana, the spinor propagator has no "arrow" associated with it.

We choose to calculate the four-spinor S matrix amplitude. This has two advantages. First, this case requires a total of 43 one and two-loop diagrams, to be compared with 100 for the spinor-spinor-vector-vector amplitude and 69 for the four-vector one. Secondly, four-spinor terms are superficially less divergent than terms involving vectors, even though, once gauge invariance is used, vector terms are actually less divergent than spinor terms. Multi-spinor amplitudes, however, do have the disadvantage of requiring the use of Fierz identities, relating different spinor structures.

In order to calculate this amplitude, the types of diagrams shown in fig. 2 are needed. Writing non 1PI graphs for S matrix calculations is equivalent to calculating the 1PI graphs for the effective action and using the equations of motion. In either method, three 1PI Green functions, the vector propagator, the vector-spinor-spinor vertex and the four-spinor amplitude must be calculated. From the diagrams, it can be seen that external spinors can be put on shell, but external vectors must be kept off shell since they couple to spinors. Spinor propagators corrections are proportional to $(p^2)^{(d-4)L/2}$ at L loops. Thus in four dimensions the spinor propagator is renormalized, inducing a renormalization of the external spinor legs. In more than four dimensions these corrections

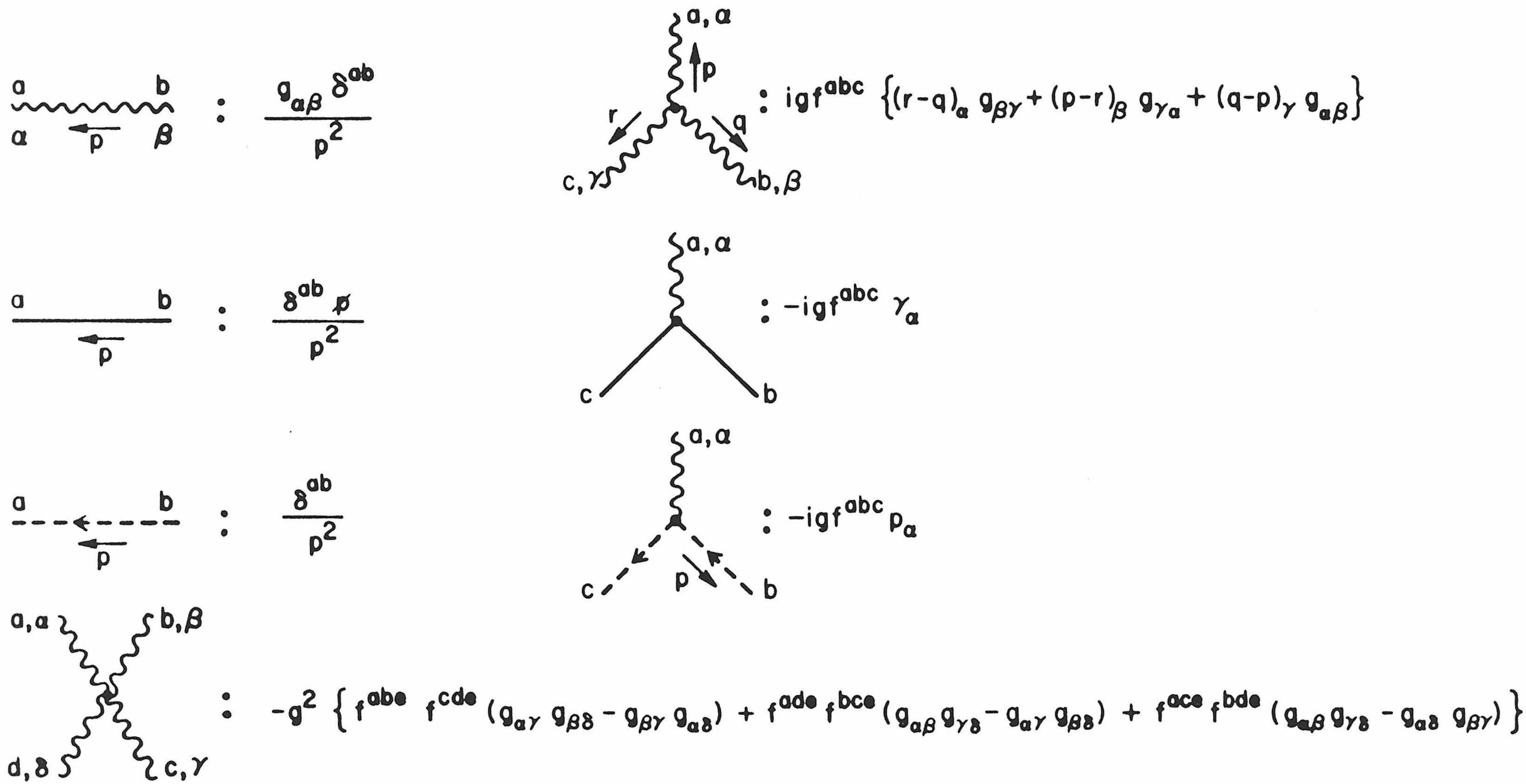


Figure 1.

The Feynman Rules for N=4 Yang Mills.

vanish when the legs are put on shell, and the spinor propagator need not be calculated.

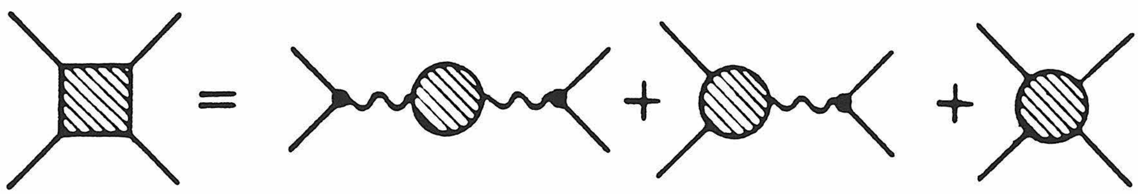


Figure 2.

The four-spinor S matrix. Circles denote 1PI graphs.

a) One Loop Graphs.

We can now study the one loop Green functions. These are both interesting in their own right, and useful as insertions in two loop diagrams. We calculate the effective action or, equivalently, the generating function for the S matrix. It is thus not necessary to permute external legs, greatly reducing the number of diagrams. Furthermore, because the spinor fields anticommute, all "minus" signs are automatically taken care of. With this method, the combinatoric factor of a graph is simply one over the dimension of the group of all symmetry transformations of the graph, allowing interchanges of both internal and external legs.

All the one-loop graphs needed are shown in fig. 3. The vector propagator corrections contribute

$$g^2 \int \frac{d^d p}{(2\pi)^d} A^a_\alpha(-p)\,(p_\alpha p_\beta - \eta_{\alpha\beta}p^2)\,A^a_\beta(p) \int \frac{\mu^\varepsilon d^{d-\varepsilon}k}{(2\pi)^{d-\varepsilon}} \frac{1}{k^2(k+p)^2} \tag{6}$$

to the effective action. This structure has already been manipulated to write it as a transverse tensor. We remind the reader that the "vector" indices in (6) run from 1 to 10, and thus also represent scalars in d dimensions. It is a non-trivial feature, that only occurs at one loop, that all d dimensional indices disappear.

The spinor propagator correction equals

$$-2g^2\bar{\lambda}^a(-p)\not{\partial}\,\lambda^a(p)\int\frac{\mu^\varepsilon d^{d-\varepsilon}k}{(2\pi)^{d-\varepsilon}}\,\frac{1}{k^2(k+p)^2} \quad , \tag{7}$$

where, from this expression on, the external momentum integrals (with their factors of $(2\pi)^d$) are omitted for brevity.

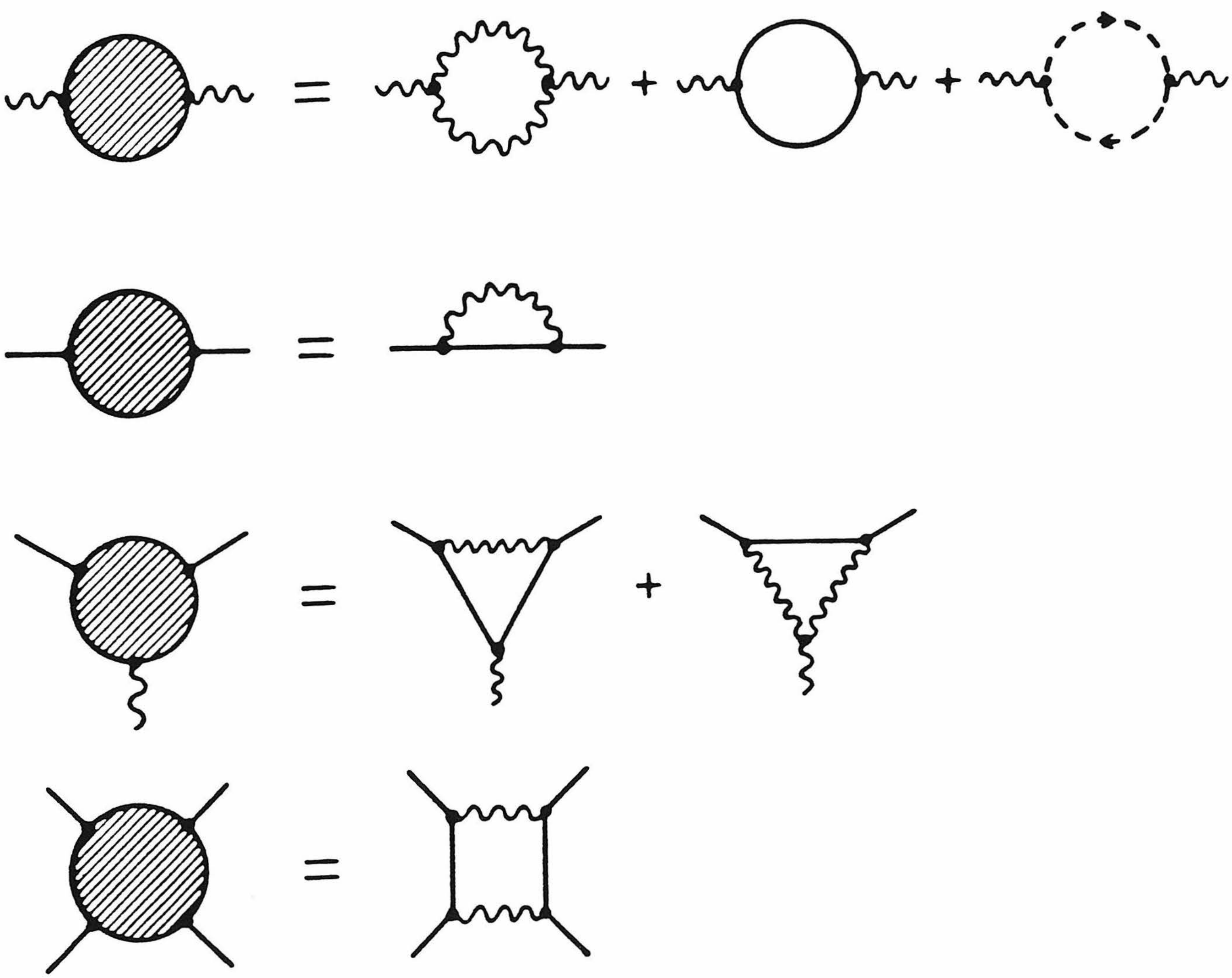


Figure 3.

The one loop graphs.

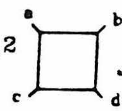
The vertex corrections give

$$\frac{i}{2} g \mu^{\varepsilon/2} g^2 f^{abc} \int \frac{\mu^{\varepsilon} d^{d-\varepsilon}k}{(2\pi)^{d-\varepsilon}} \frac{1}{k^2(k+p)^2(k-q)^2}$$

$$\cdot A^c_\alpha(-p-q)\, \bar{\lambda}^a(p) \{\gamma_\alpha [5k^2 + 6p\cdot k - 2q\cdot k - \not{k}\not{q} - 2\not{p}\not{q} - 2p\cdot q]$$

$$- 4\not{k}(p+q)_\alpha + 8\not{q}(p+k)_\alpha\}\lambda^b(q) \quad . \tag{8}$$

Finally, the four-spinor Green function equals

$$-\frac{1}{4} g^2 \mu^{\varepsilon} g^2$$  $$\int \frac{\mu^{\varepsilon} d^{d-\varepsilon}k}{(2\pi)^{d-\varepsilon}} \frac{1}{k^2(k+p)^2(k-q)^2(k-q-r)^2}$$

$$\cdot\, \bar{\lambda}^a(p)\gamma_\alpha \not{k}\, \gamma_\beta \lambda^b(q) \cdot \bar{\lambda}^c(-p-q-r)\gamma_\alpha(\not{k} - \not{q} - \not{r})\gamma_\beta \lambda^d(r) \quad . \tag{9}$$

Here the group theory factor is written diagrammatically, as explained in the appendix, using a notation first introduced by Cvitanović [34]. The indices on the group theory graphs will often be omitted if they appear in this order.

b) Pole Parts.

Extracting the pole parts from the integrals in eqs (6-8) is straightforward. First, one loop finiteness is trivial in all odd dimensions, because there are no poles at odd numbers of loops in odd dimensions. In $d=4$ the four-spinor 1PI amplitude of eq. (9) converges, whereas the amplitudes of eqs (6) and (7) give

$$-\frac{g^2}{(4\pi)^2\varepsilon}\left[4i\bar{\lambda}^a \not{\partial}\, \lambda^a - 2A^a_\alpha \Box_T A^a_\alpha + 5igf^{abc}\bar{\lambda}^a\gamma_\alpha\lambda^c A^b_\alpha\right] \quad , \tag{10}$$

where

$$\Box_T A^a_\alpha \equiv \Box\, A^a_\alpha - \partial_\alpha \partial\cdot A^a \quad . \tag{11}$$

The field equation for A_α is

$$\Box_T A^a_\alpha + \frac{i}{2} g \mu^{\varepsilon/2} f^{abc} \bar{\lambda}^b \gamma_\alpha \lambda^c + O(A^2) = 0 \quad , \tag{12}$$

while λ^a satisfies

$$(\not{D} \lambda)^a = \not{\partial} \lambda^a + g \mu^{\varepsilon/2} f^{abc} \not{A}^b \lambda^c = 0 \quad . \tag{13}$$

Substituting these results into (10), it is seen that the divergence vanishes on shell, and N=4 supersymmetric Yang-Mills theory is thus one-loop finite in four-dimensions [35]. This is, of course, well known. Here we have seen how the finiteness is recognized in terms of the on-shell effective action.

From (10), one can read off the field redefinitions (in this case simply the wavefunction renormalizations) induced by the counterterms needed to make the Green functions finite. They are

$$\delta A^a_\alpha = - \frac{2g^2}{(4\pi)^2 \varepsilon} A^a_\alpha \tag{14}$$

and

$$\delta \lambda^a = - \frac{4g^2}{(4\pi)^2 \varepsilon} \lambda^a \quad . \tag{15}$$

In six dimensions the pole parts of the effective action that contribute to the four spinor S-matrix can be written as

$$\frac{g^2}{(4\pi)^3 \varepsilon} \left[\frac{1}{3} A^a_\alpha \Box_T^2 A^a_\alpha + i g f^{abc} \frac{2}{3} \Box_T A^a_\alpha \bar{\lambda}^b \gamma_\alpha \lambda^c \right.$$

$$\left. + \frac{g^2}{4} \sum \bar{\lambda}^a \gamma_\alpha \lambda^b \cdot \bar{\lambda}^c \gamma_\alpha \lambda^d \right] \quad , \tag{16}$$

where the spinors have been put on shell. Once again, all d dimensional indices have canceled. In obtaining this expression, it was necessary to use Fierz and group theory identities, as illustrated in the appendix. For example, when multiplied by the box group theory factor,

$$\bar{\lambda}\,\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\lambda\cdot\bar{\lambda}\,\gamma^{\alpha}\gamma^{\mu}\gamma^{\beta}\lambda \rightarrow 4\;d\;\bar{\lambda}\,\gamma^{\alpha}\lambda\cdot\bar{\lambda}\,\gamma^{\alpha}\lambda \quad . \tag{17}$$

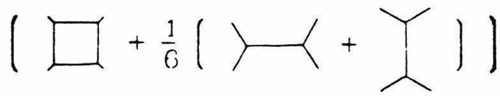
The use of the vector equation of motion now causes the divergence to vanish. Thus the divergences in the Green functions can be removed by the field redefinition

$$\delta A^{a}_{\alpha} = -\frac{g^{2}}{3\,(4\pi)^{3}\varepsilon}\left(\Box_{T} A^{a}_{\alpha} + \frac{3}{2}\,i\,g\,f^{abc}\bar{\lambda}^{\,b}\gamma_{\alpha}\lambda^{c}\right) \quad . \tag{18}$$

The redefinition is now nonlinear, as anticipated in the previous section. There are also other terms in δA^{a}_{α} which cannot be determined from this calculation, and are obtained by making other Green functions finite, as well as field redefinitions for spinors and ghosts. In general the field redefinitions mix all fields in the theory. As we have stressed, field redefinitions are gauge-dependent and unphysical, but they are useful (in principle, at least) for higher-loop calculations. In practice, the method of subtraction of subdivergences omits explicit mention of them altogether. This should be clear from the discussion that follows.

In eight dimensions even after the equation of motion is used a divergence remains [36]. This divergence is physical. After simplification using Fierz identities, symmetries of the group-theory factor and the Jacobi identity it can be written in the simple form

$$-\frac{1}{3}\bar{\lambda}^{a}\gamma_{\alpha}\partial_{\beta}\lambda^{b}\cdot\bar{\lambda}^{c}\gamma_{\alpha}\partial_{\beta}\lambda^{d}\,\frac{g^4}{(4\pi)^4\varepsilon}$$

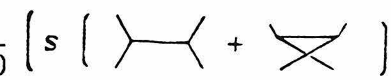


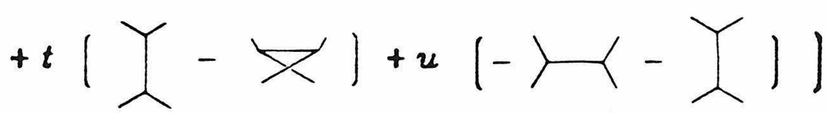
.(19)

All eight-dimensional indices have disappeared from this expression. It is important that the group theory factor is totally symmetric, as demonstrated in the appendix. This result is simply the four-spinor piece of the supersymmetric counterterm found by Green, Schwarz and Brink in their one-loop string-theory calculations [36].

The calculation can also be repeated in ten dimensions. In this case only ten dimensional indices have to be dealt with, but the integrals have a higher superficial degree of divergence, and manipulating the results is somewhat more tedious. After simplifications, the result can be written as

$$-\frac{1}{3}\bar{\lambda}^{a}\gamma_{\alpha}\partial_{\beta}\lambda^{b}\cdot\bar{\lambda}^{c}\gamma_{\alpha}\partial_{\beta}\lambda^{d}\,\frac{g^5}{(4\pi)^4\varepsilon}\,\frac{1}{360}$$





(20)

where the Mandelstam invariants are the differential operators

$$s = 2p_a p_b \quad , \quad t = 2p_a p_c \quad , \quad \text{and} \quad u = 2p_a p_d \quad . \tag{21}$$

This apparently bizarre way of writing the result is actually very convenient. The expression (20) could be simplified by using the Jacobi identity in eq. (A.5), and the relation $s+t+u = 0$, which holds for massless particles. However, we have chosen to write it in a way that the total symmetry of the second factor is manifest. The result has the same "kinematic factor" as the eight dimensional result, and is thus again part of an N=4 supersymmetric structure.

Our calculations therefore agree with the superstring result [36] that the onset of divergences at one loop occurs in eight dimensions. They also illustrate the need for nonlinear field redefinitions to make Green functions finite in theories which are finite but power-counting nonrenormalizable. Finally, they provide us with the building blocks for the two-loop calculation to be presented in the following.

c) Two Loops.

As stated in chapter III, we want to eliminate the overlapping divergences from two-loop graphs. This could be done by associating to each graph the corresponding counterterm graphs, but we shall instead use the equivalent procedure of performing subtractions on the integrals. Thus, we never explicitly mention counterterm graphs. The two-loop diagrams contributing to the vector and spinor propagator corrections, the two-loop vertex correction and the four-spinor amplitude are shown in fig. 4. The spinor propagator correction is again needed only in four dimensions. In these graphs the shaded portions denote the total one-loop propagator and vertex corrections. Using these insertions corresponds to performing a partial Schwinger-Dyson expansion. It turns out that the number of diagrams is minimized by doing this for two and three point insertions, but not for four and and higher-point ones.

Once the graphs are written down, the next problem is to extract their divergences in different dimensions. The remainder of this section will deal with technical details concerning the derivation of the results. This should clarify the remarks made at a rather more abstract level in the previous chapter. The discussion of the results and their interpretation is the subject of the next chapter.

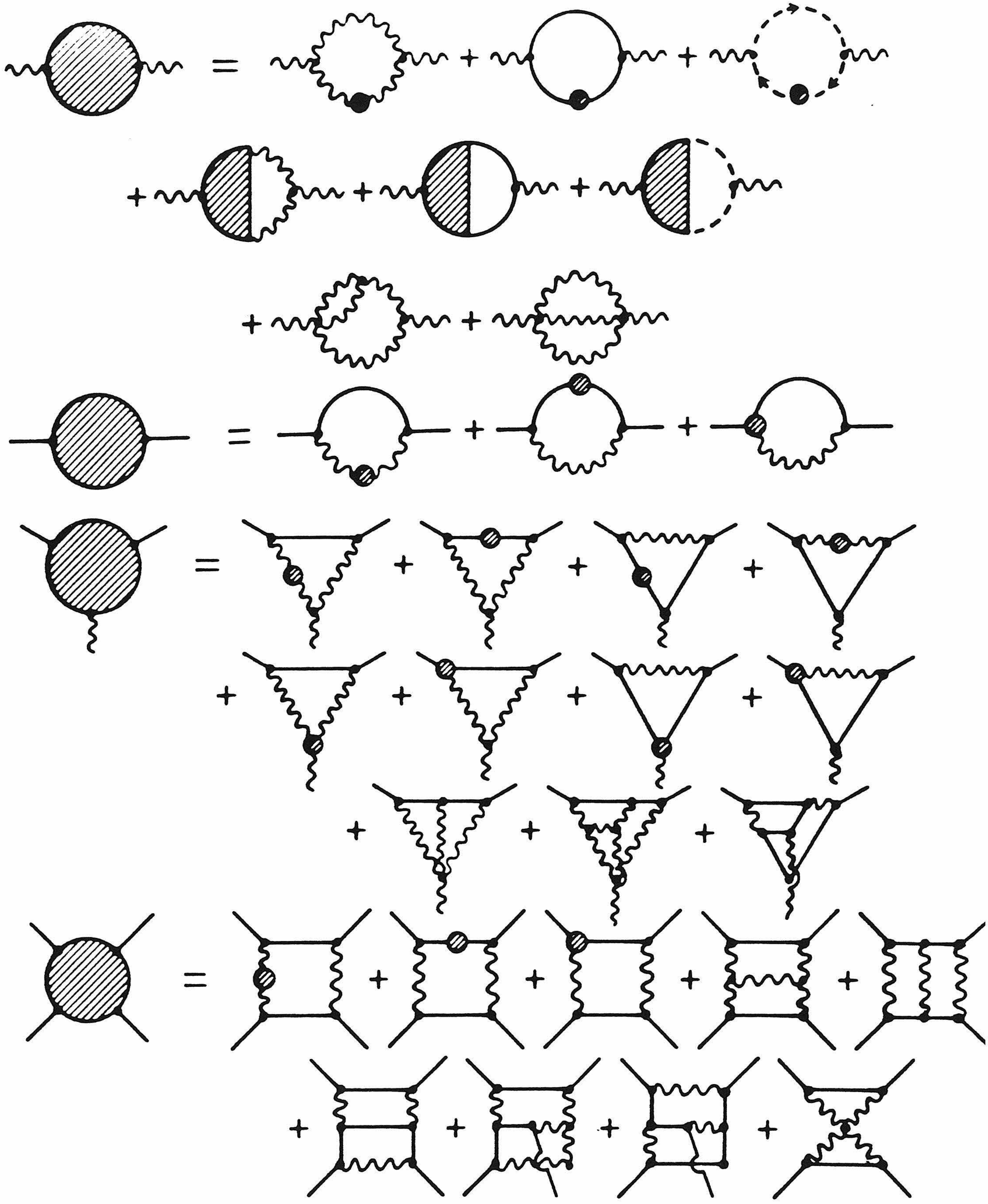


Figure 4.

Two Loop Graphs.

In four dimensions, the finiteness of N=4 Yang Mills can be seen in two equivalent ways (methods (2) and (3) of chapter III). The action plus the one and two loop vertex and propagator counterterms is

$$\frac{1}{2} A_i^a \Box A_i^a \left(1 - \frac{g^2}{(4\pi)^2}\frac{4}{\varepsilon} + \frac{g^4}{(4\pi)^4}\left(\frac{12}{\varepsilon^2} + -\frac{1}{\varepsilon}\right)\right)$$

$$-\frac{1}{4} F_{\mu\nu}^2 \left(1 - \frac{g^2}{(4\pi)^2}\frac{4}{\varepsilon} + \frac{g^4}{(4\pi)^4}\left(\frac{10}{\varepsilon^2} + \frac{1}{\varepsilon}\right)\right)$$

$$-\frac{i}{2}\bar{\lambda}^a \not{\partial} \lambda^a \left(1 - \frac{g^2}{(4\pi)^2}\frac{8}{\varepsilon} + \frac{g^4}{(4\pi)^4}\left(\frac{36}{\varepsilon^2} + \frac{6}{\varepsilon}\right)\right)$$

$$+\frac{i}{2} g f^{abc} \bar{\lambda}^a \gamma_i \lambda^b A_i^c \left(1 - \frac{g^2}{(4\pi)^2}\frac{10}{\varepsilon} + \frac{g^4}{(4\pi)^4}\left(\frac{56}{\varepsilon^2} + \frac{11}{2\varepsilon}\right)\right)$$

$$+\frac{i}{2} g f^{abc} \bar{\lambda}^a \gamma_\mu \lambda^b A_\mu^c \left(1 - \frac{g^2}{(4\pi)^2}\frac{10}{\varepsilon} + \frac{g^4}{(4\pi)^4}\left(\frac{55}{\varepsilon^2} + \frac{13}{2\varepsilon}\right)\right) \quad , \tag{22}$$

where to apply the usual methods, we have introduced the indices i and j to refer to the six scalars. Eq. (22) corresponds to the standard method of calculation. The counterterms simply induce the wavefunction renormalizations

$$A'_i = A_i\left(1 - \frac{g^2}{(4\pi)^2}\frac{2}{\varepsilon} + \frac{g^4}{(4\pi)^4}\left(\frac{4}{\varepsilon^2} - \frac{1}{2\varepsilon}\right)\right)$$

$$A'_\mu = A_\mu\left(1 - \frac{g^2}{(4\pi)^2}\frac{2}{\varepsilon} + \frac{g^4}{(4\pi)^4}\left(\frac{3}{\varepsilon^2} + \frac{1}{2\varepsilon}\right)\right)$$

$$\lambda' = \lambda\left(1 - \frac{g^2}{(4\pi)^2}\frac{4}{\varepsilon} + \frac{g^4}{(4\pi)^4}\left(\frac{10}{\varepsilon^2} + \frac{3}{\varepsilon}\right)\right) \quad , \tag{23}$$

but no charge renormalization, and the theory is thus finite [36]. If, on the other hand, we want to calculate with the field redefinitions, it is necessary to include the $\frac{\delta^2 S}{\delta\varphi^2}$ term from eq. (III.11)†. This can either be done explicitly or by adding the graphs in fig. 5.

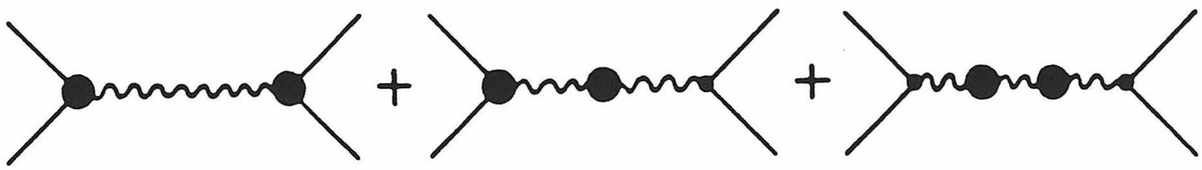


Figure 5.

Field Redefinition Contributions. Dots represent counterterms.

These give the additional contribution

$$\frac{g^4}{(4\pi)^4\varepsilon^2}\left(-2A^a_\alpha\Box_T A^a_\alpha + 8i\bar{\lambda}^a \not{\partial}\lambda^a - 16ig f^{abc}\,\bar{\lambda}^a\gamma_\alpha\lambda^b A^c_\alpha\right) \quad . \qquad (24)$$

In this way, the infinite part of the two loop effective action becomes

$$\frac{g^4}{(4\pi)^4}\left[\left(\frac{8}{\varepsilon^2}-\frac{1}{\varepsilon}\right)A^a_\alpha\Box_T A^a_\alpha - \left(\frac{2}{\varepsilon^2}-\frac{2}{\varepsilon}\right)A^a_\mu\Box_T A^a_\mu - \frac{i}{2}\left(\frac{20}{\varepsilon^2}+\frac{6}{\varepsilon}\right)\bar{\lambda}^a\not{\partial}\lambda^a\right.$$

$$\left. + \frac{i}{2}g\, f^{abc}\left(\frac{24}{\varepsilon^2}+\frac{11}{2\varepsilon}\right)(\bar{\lambda}^a\gamma_\alpha\lambda^b)A^c_\alpha + \frac{i}{2}g\, f^{abc}\left(-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\right)(\bar{\lambda}^a\gamma_\mu\lambda^b)A^c_\alpha\right] (25)$$

When the equations of motion are used, this infinity vanishes.

Next consider the $d = 5$ case. Here the extraction of pole parts is relatively simple, because no one-loop subdivergences are present. These integrals could be evaluated straightforwardly by a Feynman parametrization, but factoring out the momentum dependence is still a simpler procedure. The result for the effective action is

†As this term is purely $O(1/\varepsilon^2)$ and we know from the arguments of the previous section that the S matrix has no $1/\varepsilon^2$ part, we could ignore it altogether. However, it is kept, as it provides a useful check of the calculation.

$$\frac{g^4\pi}{(4\pi)^5\varepsilon}\Bigg[-\frac{1}{70}\,A^a_\mu\,\Box^2_{\mathrm{T}}A^a_\mu+\frac{13}{15}\,A^a_\alpha\,\Box^2_{\mathrm{T}}A^a_\alpha$$

$$+\,ig\,f^{abc}\left(-\frac{17}{420}\bar{\lambda}^a\gamma_\mu\lambda^b\Box_{\mathrm{T}}A^c_\mu+\frac{29}{40}\bar{\lambda}^a\gamma_\alpha\lambda^b\Box_{\mathrm{T}}A^c_\alpha\right)$$

$$+\,g^2\ \rotatebox{90}{$\rangle\!\!-\!\!\langle$}\left(\frac{7}{48}\bar{\lambda}^a\gamma_\alpha\lambda^b\cdot\bar{\lambda}^c\gamma_\alpha\lambda^d-\frac{1}{60}\bar{\lambda}^a\gamma_\mu\lambda^b\cdot\bar{\lambda}^c\gamma_\mu\lambda^d\right)\Bigg]\,. \qquad (26)$$

Using the equations of motion, the theory is again finite.

In six dimensions the calculation becomes more complicated. There are subtleties due both to the nonrenormalizability and to the presence of subdivergences. Moreover, the superficial degree of divergence of the diagrams increases, and the integrals of the four-spinor Green function must be differentiated up to four times to reduce them to logarithmically divergent ones. Nonetheless, the method described in Chapter II simplifies matters considerably and, after many manipulations, we arrive at the result

$$\frac{g^4}{(4\pi)^6}\Bigg[\left(-\frac{2}{3\varepsilon^2}+\frac{127}{864\varepsilon}\right)A^a_\alpha\Box^3_{\mathrm{T}}A^a_\alpha+\left(-\frac{1}{90\varepsilon^2}+\frac{1}{90\varepsilon}\right)A^a_\mu\Box^3_{\mathrm{T}}A^a_\mu$$

$$+\,ig\,f^{abc}\left(-\frac{71}{72\varepsilon^2}+\frac{275}{1728\varepsilon}\right)(\bar{\lambda}^a\gamma_\alpha\lambda^b)\Box^2_{\mathrm{T}}A^c_\alpha$$

$$+\,ig\,f^{abc}\left(\frac{1}{80\varepsilon^2}-\frac{259}{28800\varepsilon}\right)(\bar{\lambda}^a\gamma_\mu\lambda^b)\Box^2_{\mathrm{T}}A^c_\mu$$

$$+\,g^2\ \rotatebox{90}{$\rangle\!\!-\!\!\langle$}\left(-\frac{55}{144\varepsilon^2}+\frac{37}{864\varepsilon}\right)(\bar{\lambda}^a\gamma_\alpha\lambda^b)\Box(\bar{\lambda}^c\gamma_\alpha\lambda^d)$$

$$+\,g^2\ \rotatebox{90}{$\rangle\!\!-\!\!\langle$}\left(\frac{1}{288\varepsilon^2}-\frac{25}{6912\varepsilon}\right)(\bar{\lambda}^a\gamma_\mu\lambda^b)\Box(\bar{\lambda}^c\gamma_\mu\lambda^d)\Bigg] \qquad (27)$$

for the divergences from one and two-loop graphs. The algebraic manipulation program SMP [37] was used in performing the differentiations, and particularly for the γ matrix algebra needed in simplifying these results. The square of the one-loop field redefinitions from the graphs of fig. 5 give

$$-\frac{1}{18}\frac{g^4}{(4\pi)^6\varepsilon^2}\Big(A^a_\alpha \Box_T^3 A^a_\alpha + 3igf^{abc}\bar{\lambda}^a\gamma_\alpha\lambda^b\Box_T^2 A^c_\alpha$$

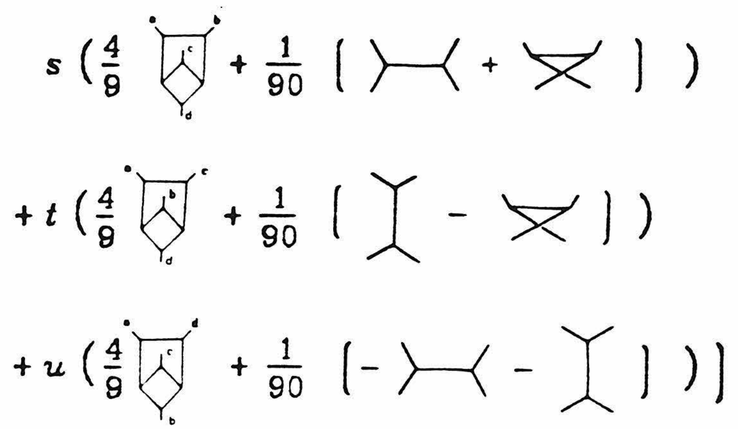
$$+\frac{g^2}{4} \Big\rangle\!\!\Big\langle (\bar{\lambda}^a\gamma_\alpha\lambda^b)\Box(\bar{\lambda}^c\gamma_\alpha\lambda^d)\Big) \quad . \qquad (28)$$

When these results are added, and the field equations are used, all $\frac{1}{\varepsilon^2}$ divergences cancel, in agreement with the general arguments presented in the previous section. The interesting result is that the $\frac{1}{\varepsilon}$ terms also cancel [24]. The the effective action has no divergences on-shell, and thus the S-matrix is finite.

Finally we describe the results in seven and nine dimensions [38]. Again, as in d=5, the calculation is conceptually rather simple, since no one-loop subdivergences occur in the integrals. The calculations are, however, very difficult in practice, due to the high superficial degree of divergence of the integrals. (The divergent part of the four point Green function in nine dimensions contains eight powers of momenta.) Because of the number of terms obtained after the differentiation, it was necessary to write specialized programs (in the C language) for their manipulation.

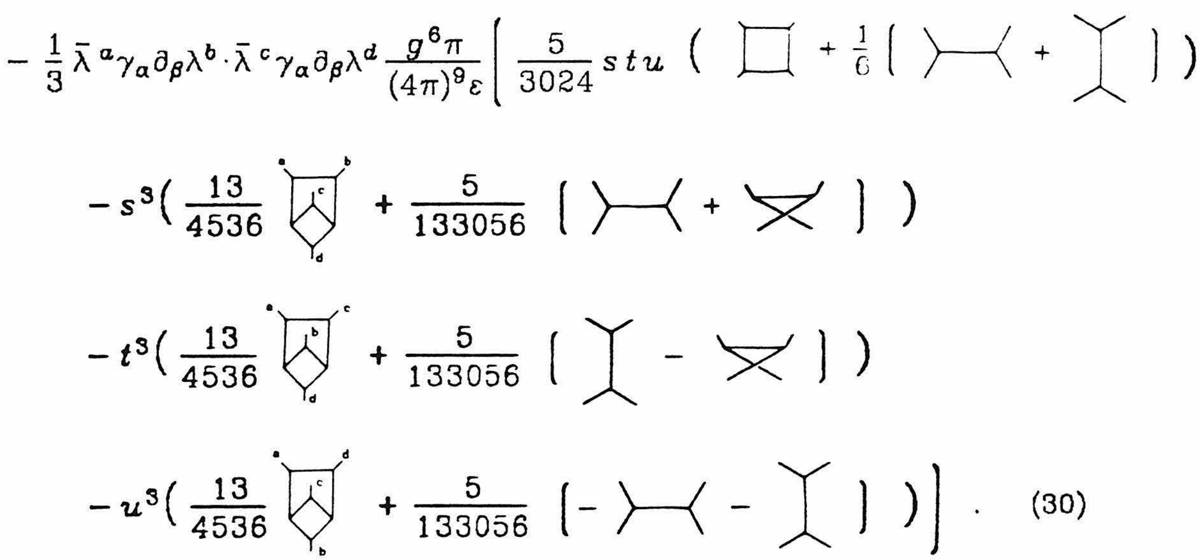
In both cases the theory is found to diverge. As in the one loop ten-dimensional result, the S matrix is the product of the kinematic factor from eq. (19), and a totally symmetric factor made from group theoretical invariants and Mandelstam variables. The seven dimensional result is

$$-\frac{1}{3}\bar{\lambda}^a\gamma_\alpha\partial_\beta\lambda^b\cdot\bar{\lambda}^c\gamma_\alpha\partial_\beta\lambda^d\frac{g^6\pi}{(4\pi)^7\varepsilon}\Big[$$



(29)

The nine dimensional result is similar, though somewhat more complicated :



(30)

In the next chapter the meaning of the results presented here, and their implications for superspace and supergravity are discussed in detail.

Appendix.

Our spinors are anticommuting and satisfy both the Majorana condition

$$\lambda = C\bar{\lambda}^{T} \tag{A.1}$$

and the Weyl condition

$$\lambda = \gamma^{11}\lambda \tag{A.2}$$

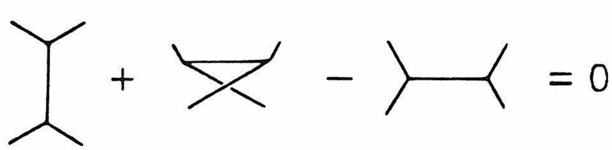
in ten dimensions. All Fierz identities can be derived from the fundamental one

$$\lambda^{a}\bar{\lambda}^{b} = -\frac{1}{16}(\bar{\lambda}^{b}\gamma_{\alpha}\lambda^{a})\gamma_{\alpha} + \frac{1}{6\cdot 16}(\bar{\lambda}^{b}\gamma_{\alpha\beta\gamma}\lambda^{a})\gamma_{\alpha\beta\gamma} - \frac{1}{32\cdot 5!}(\bar{\lambda}^{b}\gamma_{\alpha\beta\gamma\delta\varepsilon}\lambda^{a})\gamma_{\alpha\beta\gamma\delta\varepsilon} \tag{A.3}$$

For the group theory factors it is convenient to use the graphical representation introduced by Cvitanović [34]. Structure constants are represented by a trilinear vertex

$f^{abc} \rightarrow$  . (A.4)

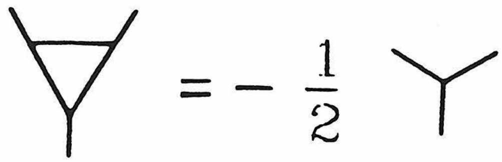
Since f^{abc} is antisymmetric, the vertex changes sign whenever two of its legs are interchanged. The Jacobi identity is a quadratic relation in structure constants, and in this notation reads

 . (A.5)

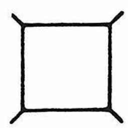
At one loop only one new group theory structure is encountered in processes with up to four particles. The one loop propagator group theory diagram gives

 , (A.6)

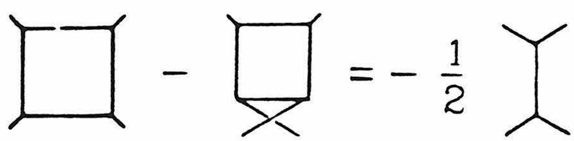
which is a definition of the quadratic Casimir, and the one loop vertex group theory diagram gives

 (A.7)

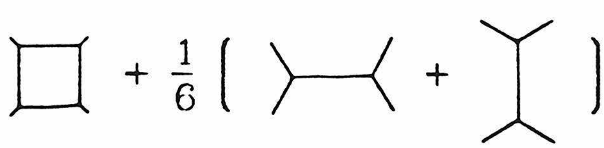
after the use of eqs (A.5) and (A.6). On the other hand, the box diagram

 (A.8)

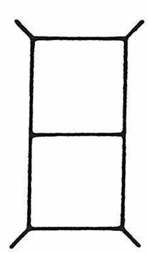
is an independent invariant tensor. It is manifestly symmetric under reflections about its diagonals and sides. Its antisymmetric part can be reduced to tree structures using

 , (A.9)

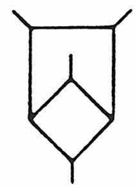
as can be seen using eqs (A.5), (A.6) and (A.7). The irreducible totally symmetric invariant tensor is

 , (A.10)

At two loops new structures emerge, but again only from diagrams with four external legs. They are represented by the double-box diagram

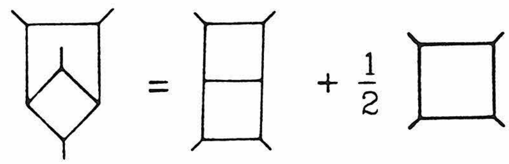
 (A.11)

and by the nonplanar diagram



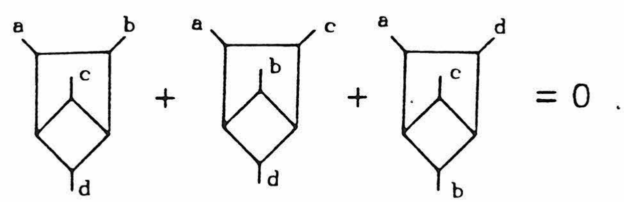
. (A.12)

These two structures are not independent. The Jacobi identity (A.5) implies



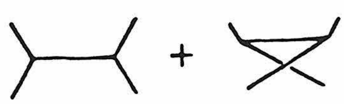
. (A.13)

The nonplanar diagram is irreducible. It is manifestly symmetric under interchanges of its two upper legs or its two lower legs, and, as can be seen from of eq. (A.13), it is symmetric under the interchange of its upper and lower legs. Finally



(A.14)

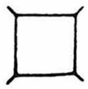
The nonplanar diagram thus has the same symmetry properties as



. (A.15)

It is crucial, however, that in general the nonplanar and tree group theory tensors are independent. (It should be appreciated how simply all these identities can be derived from the Jacobi identity (A.5) using the graphical notation. It would be very difficult to do this working directly with the products of many structure constants.)

Finally, the symmetries of the group theory factors can be combined with the Fierz identity to relate different structures, and this is of course crucial in the calculation. As an example of many such relations, consider



$$\bar{\lambda}^a(p)\gamma_\alpha\gamma_\beta\not{r}\,\lambda^b(q)\cdot\bar{\lambda}^c(-p-q-r)\gamma_\alpha\gamma_\beta\not{q}\,\lambda^d(r) \quad . \tag{A.16}$$

Since the box diagram is symmetric about its diagonals, it follows that the expression above equals the one obtained by interchanging λ^b and λ^c and relabeling $q \rightarrow -p-q-r$. Thus

$$\bar{\lambda}^a\gamma_\alpha\gamma_\beta\not{r}\,\lambda^b\cdot\bar{\lambda}^c\gamma_\alpha\gamma_\beta\not{q}\,\lambda^d \rightarrow \bar{\lambda}^a\gamma_\alpha\gamma_\beta\not{r}\,\lambda^c\cdot\bar{\lambda}^b\gamma_\alpha\gamma_\beta(-\not{p}-\not{q})\lambda^d \quad , \tag{A.17}$$

where the equation of motion of the spinor,

$$\not{r}\,\lambda^d(r) = 0 \quad , \tag{A.18}$$

has been used. Using the Fierz identity (A.3) implies

$$\bar{\lambda}^a\gamma_\alpha\gamma_\beta\not{r}\,\lambda^b\cdot\bar{\lambda}^c\gamma_\alpha\gamma_\beta\not{q}\,\lambda^d + 8\bar{\lambda}^a\not{r}\,\lambda^b\cdot\bar{\lambda}^c\not{q}\,\lambda^d - 8u\,\bar{\lambda}^a\gamma_\alpha\lambda^b\cdot\bar{\lambda}^c\gamma_\alpha\lambda^d$$

$$+ \frac{16}{3}s\,\bar{\lambda}^a\gamma_\alpha\lambda^b\cdot\bar{\lambda}^c\gamma_\alpha\lambda^d - \frac{1}{3}s\,\bar{\lambda}^a\gamma_\alpha\gamma_\beta\gamma_\gamma\lambda^b\cdot\bar{\lambda}^c\gamma_\alpha\gamma_\beta\gamma_\gamma\lambda^d = 0 \quad , \tag{A.19}$$

when multiplied by the box group theory diagram.

Performing similar manipulations on other structures, eq. (18) can be simplified into terms involving only one γ matrix. Thus, one can replace

$$\bar{\lambda}^a\gamma_\alpha\gamma_\beta\not{r}\,\lambda^b\cdot\bar{\lambda}^c\gamma_\alpha\gamma_\beta\not{q}\,\lambda^d \rightarrow \; -8\bar{\lambda}^a\not{r}\,\lambda^b\cdot\bar{\lambda}^c\not{q}\,\lambda^d$$

$$-16u\,\bar{\lambda}^a\gamma_\alpha\lambda^b\cdot\bar{\lambda}^c\gamma_\alpha\lambda^d - 8s\,\bar{\lambda}^a\gamma_\alpha\lambda^b\cdot\bar{\lambda}^c\gamma_\alpha\lambda^d \quad , \tag{A.20}$$

when the expression is multiplied by the box group theory diagram. There are many such results. They can all be derived using similar methods and will not be shown here.

V. Discussion.

The divergences found in the previous section all have similar structures. They can be written as

$$-\frac{1}{3}\bar{\lambda}^{a}\gamma_{\alpha}\partial_{\beta}\lambda^{b}\cdot\bar{\lambda}^{c}\gamma_{\alpha}\partial_{\beta}\lambda^{d} \quad , \tag{1}$$

multiplied by a totally symmetric tensor T constructed out of the structure constants f^{abc} and the Mandelstam invariants s, t and u of the four-particle process. The results of our analysis are summarized for convenience in the table below.

loops\dim	4	5	6	7	8	9	10
1	0	0	0	0	T_8	0	T_{10}
2	0	0	0	T_7	–	T_9	–

Table 1.

Divergence Structure in All Dimensions at One and Two Loops.

The calculations in eight and in ten dimensions were not done at two loops, since the theory was already found to be infinite at one loop in these cases.

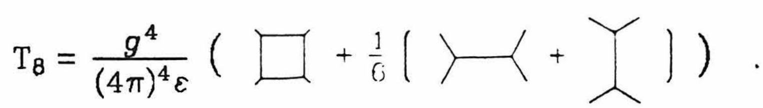
The fact that divergences take the simple form (1) was originally noticed at one loop in the string theory calculation of ref. [36], and the persistence of this result to two loops is rather fortunate. The expression (1) can be completed to give the four-point interactions including vectors by enforcing N=4 supersymmetry. The result is

$$F_{\alpha\beta}\,F_{\beta\gamma}\,F_{\gamma\delta}\,F_{\delta\alpha} - \frac{1}{4}\,F_{\alpha\beta}\,F_{\alpha\beta}\,F_{\gamma\delta}\,F_{\gamma\delta} + 2i\bar{\lambda}\,\gamma_\alpha\partial_\beta\lambda F_{\alpha\gamma}F_{\beta\gamma}$$

$$- i\bar{\lambda}\,\gamma_{\alpha\beta\gamma}\lambda\;F_{\alpha\delta}\partial_\gamma F_{\beta\delta} - \frac{1}{3}\,\bar{\lambda}\,\gamma_\alpha\partial_\beta\lambda\;\bar{\lambda}\,\gamma_\alpha\partial_\beta\lambda \quad , \qquad (2)$$

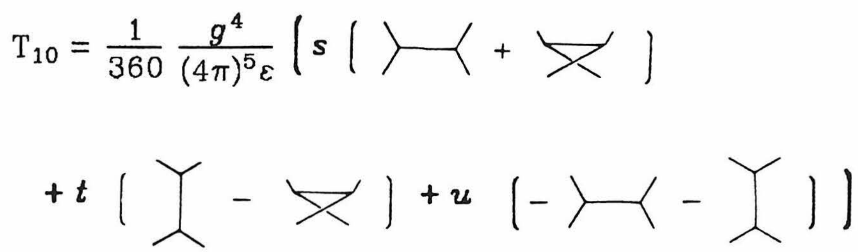
where $F_{\alpha\beta}$ is the linearized field strength of the vector. This expression is supersymmetric on-shell, *i.e.* when the the field equations are used, provided its group theory indices are contracted with a totally symmetric invariant tensor. This is somewhat tedious to prove but, once it is done, the supersymmetry of the divergences in all dimensions follows, since any totally symmetric tensor factor is invariant under supersymmetry. As the calculations were performed in a nonsupersymmetric gauge, the supersymmetry of the results is a good check.

The tensor factors in the cases where divergences are found are all different. In eight dimensions at one loop the tensor is simply



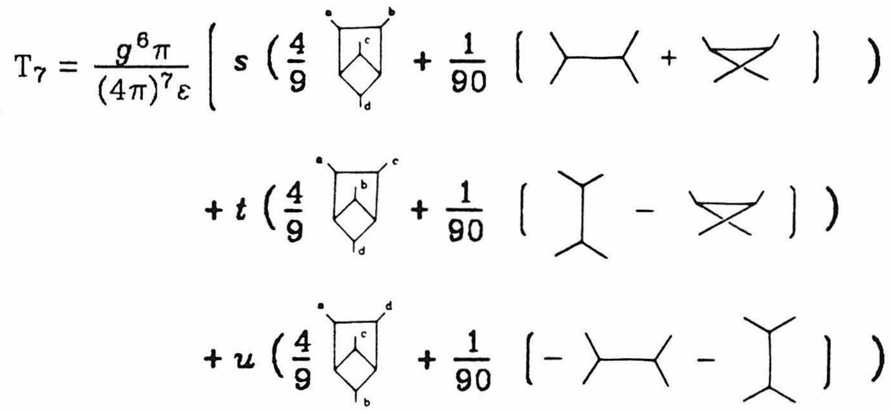
(3)

It does not contain any Mandelstam invariants, as is clear on dimensional grounds. In ten dimensions at one loop the invariant tensor does contain Mandelstam invariants. It is given by



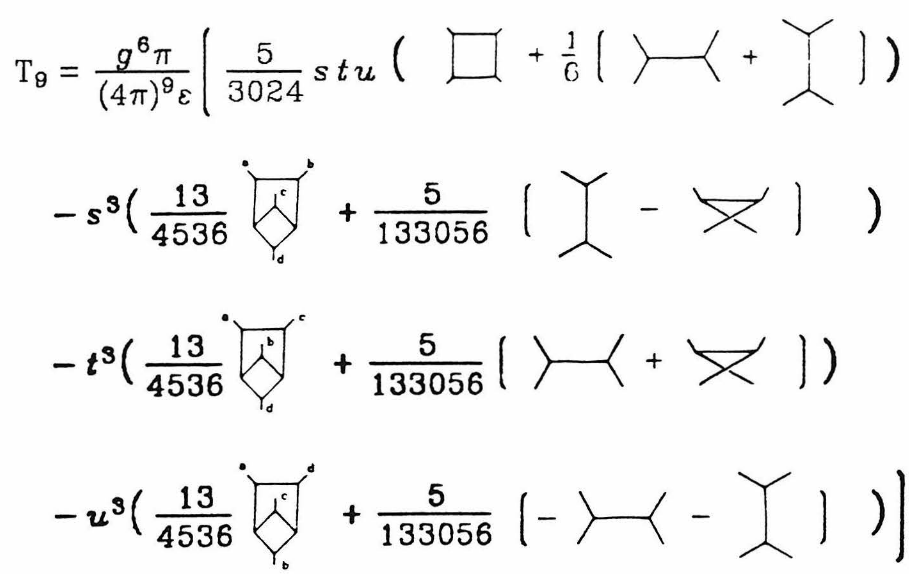
(4)

In seven dimensions at two loops the tensor is



(5)

and in nine dimensions at two loops it is



(6)

In each case the tensors that appear are the most general ones allowed on dimensional grounds. If the structure of the divergence is assumed, the only problem is to determine the coefficients. The one surprise from this point of view is that the two-loop divergence in six dimensions which could, a priori, be proportional to T_8, vanishes [24].

The expressions T_{10} and T_7 are very similar. Their only difference lies in the fact that eq. (5) contains a more complicated group theory invariant. This additional invariant cannot occur at one loop, since it has the topology of a two-loop diagram. As shown in the appendix, the tree group theory factor in eq. (5) satisfies similar identities as the nonplanar group theory factor. In general, however, it is an independent invariant. For example, in the case of SO(N), where the structure constants can be written as products of metric tensors, it can be shown that the two tensors are proportional only in the relatively trivial cases of SO(3) and SO(4). This is crucial for the interpretation of the seven dimensional result.

Let us now consider the results in the various dimensions. At one loop all theories are finite in odd dimensions. The results in even dimensions were first obtained in ref. [36] by taking the zero slope limit of the corresponding superstring theory amplitudes. The four-dimensional result has been understood by a light-cone superspace argument [39], and later by instanton methods [40] and by an application of the Adler-Bardeen theorem [41]. The result is well known, and is true to all orders. In six dimensions one-loop finiteness follows simply from supersymmetry. In that case the only dimensionally correct gauge invariant vector counterterm is of the form F^3. However, this cannot be completed into an N=4 supersymmetric invariant. Divergences do occur in eight and in ten dimensions.

At two loops in four dimensions, the methods used to show finiteness at one loop still apply. In addition, because N=2 superspace formalisms exist [42], finiteness is guaranteed by the N=2 power counting rules. Possible divergences in five dimensions at two loops have the same structure as one-loop ones in six dimensions, and vanish for the same reason. The two-loop finiteness in six dimensions is more subtle. The result found in eight dimensions at one loop, when dimensionally reduced, has the correct dimensionality to occur in six dimensions at two loops. Moreover, the superfield power counting appears to rule this out only if N=4 superfields exist. (N=4 superfields could exclude a divergence in six dimensions while allowing one in eight dimensions, because the theorem about the structure of the counterterms does not apply to one loop.)

However, Howe and Stelle [43] succeeded in explaining the six-dimensional result in terms of the existing N=2 superfield formulation. Their argument goes as follows. When N=4 Yang Mills is written in terms of N=2 superfields, the spectrum splits into two multiplets. There is a Yang-Mills multiplet and a so called hypermultiplet, or N=2 scalar multiplet. The only two-loop counterterms for the Yang-Mills multiplet, that are allowed on dimensional grounds, are of the form

$$F^{aA}\nabla_{bB}\nabla_{cC}F^{dD} \quad , \tag{7}$$

with various possible contractions of the indices. Here F is the dimension 3/2 field strength of N=2 Yang-Mills in six dimensions, which at $\theta=0$ contains the spinor of the theory†. The structure in (7) is the type of counterterm used in the derivation of the power counting rules (see eq. (I.20)). However, the superfield F satisfies the on-shell Bianchi identity

$$\nabla_{aA}F^{bA} = 0 \quad , \tag{8}$$

and its equation of motion is

$$\nabla_{AB}F^{B}_{a} = 0 \quad , \tag{9}$$

† a is an SU(2) index, and A is a spacetime spinor, or SU*(4) index.

where ∇_{AB} is a spacetime derivative. It follows that all possible contractions in eq. (7) lead to expressions that vanish on shell, and there is thus no counterterm for this sector. It is then argued that counterterms containing the hypermultiplet do not occur since the nonlinearly realized supersymmetry would relate these counterterms to the nonexistent pure vector counterterm. Thus in this case, the power counting formula can be strengthened, and N=2 superfields are sufficient to give finiteness in six dimensions.

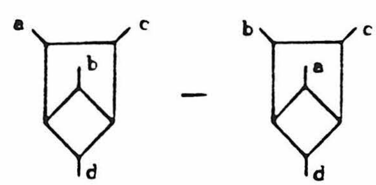
The N=2 superspace analysis does allow a counterterm at three loops [42] in six dimensions. The unique possible structure is

$$A^{\alpha\beta\gamma\delta}\int d^6x\, d^8\theta\, \varepsilon_{ABCD} F_a^{A\alpha} F_b^{B\beta} F_c^{C\gamma} F_d^{D\delta} C^{ab} C^{cd} \quad + \quad \text{more} \quad , \qquad (10)$$

where "more" denotes the hypermultiplet terms, and A is a group theory factor antisymmetric in both the $(\alpha\beta)$ and the $(\gamma\delta)$ indices, and symmetric under the interchange of the two. From the appendix of chapter IV it can be seen that, in less than three loops, such a group theory factor can be only

(11)

or



. (12)

Indeed, by integrating out the anticommuting variables, one can show that the expression in eq. (10), multiplied by the two group theory factors, reproduces the one-loop divergence in ten dimension and the two-loop divergence in seven dimensions. They both have this structure, since they have the same dimensionality as the three-loop six-dimensional result. One would expect that the three-loop six dimensional result, which is unknown, would have additional invariant group theory tensors, as the seven dimensional result had more than the ten dimensional one. It can be noted, parenthetically, that to this order, the structure of (2) is unique because of the uniqueness of eq. (10). The fact that the same kinematic factor also appears in nine dimensions has not been understood in terms of superfields. It is an open question whether this structure persists to higher orders of perturbation theory, but this appears likely.

The divergence in seven dimensions is given by eqs (2) and (5). The result, when dimensionally reduced to six dimensions, can thus be obtained from the N=2 superfield integral of eq. (10). Thus the seven-dimensional result can be written in a non-manifestly Lorentz invariant way as a N=2 superspace integral over six-dimensional superfields. (This is suggestive of an extension of six-dimensional N=2 superfields above six dimensions, as can be done with four dimensional N=1 superfields [44].) What one is really interested in, however, is attempting to write an N=4 superspace counterterm, as in eq. (I.20). The only gauge invariant candidate at the linearized level is

$$\mathrm{Tr}\int d^7x\, d^{16}\theta\, \Gamma_A W^A \quad , \tag{13}$$

where Γ_A is the spinorial connection of ten-dimensional Yang-Mills, .and W^A is the corresponding linearized on-shell field strength[†]. At the linearized level W^A is given by

[†]Here A is an SO(9,1) spinor index.

$$W^A = (\gamma_\alpha)^{AB}(\gamma^\alpha)^{CD}(D_C D_D \Gamma_B - D_B D_C \Gamma_D) \quad . \tag{14}$$

The term (13) is gauge invariant. This is not manifest, but follows immediately from the linearized on shell Bianchi identity satisfied by W^A

$$D_A\, W^A = 0 \quad . \tag{15}$$

The nonlinear generalization of (13) can be obtained using the methods in ref. [45]. The resulting expression

$$(\gamma_\alpha)^{AB}(\gamma^\alpha)^{CD}\Big(\; D_A\Gamma_B \cdot D_C\Gamma_D - D_D\Gamma_B \cdot D_C\Gamma_A$$

$$- \frac{2}{3}\, \Gamma_B[D_C\Gamma_D,\Gamma_A] + \frac{2}{3}\, \Gamma_B[D_C\Gamma_A,\Gamma_D] - \{\Gamma_B,\Gamma_C\}\cdot\{\Gamma_A,\Gamma_D\}\,\Big) \tag{16}$$

is somewhat unenlightening, but can be understood by noting that its variation is

$$2\;\delta\Gamma_A\; W^A \quad . \tag{17}$$

The important point is that the group theory factors in (16) are, at most trees. The term has a manifest N=4 supersymmetry, and thus an N=2 supersymmetry. As (2) is the unique term with N=4 supersymmetry, (13) must be proportional to it. It is even possible that the highest θ-component of the integrand in eq. (13) is a total divergence, causing the whole integral to vanish. In either case, the N=4 superspace counterterm is clearly different from the seven dimensional result, since the group theory factor in eq. (5) contains "non-planar" structures, which cannot be written as "trees". The seven dimensional result thus cannot be written as a full superspace integral of on-shell N=4 superfields.

The main conclusion of this work is that the basic ingredient of the N=4 superfield power counting, the form of the on-shell effective action, is ***explicitly violated***. Although the violation of the power-counting was discovered in N=4 Yang-Mills in higher dimensions, this demonstrates that one should not trust the power-counting of extended superspace for any theory for which the off-shell superfield formalism is unknown. The only theory for which N>2 superfields are known is simple supergravity in ten dimensions, together with its dimensional reductions [46]. However, this theory already diverges at one loop [47], and is thus somewhat uninteresting.

These results lead to the conclusion that divergences invariably appear whenever there is no good argument excluding them. There are no miracles. What are the implications for supergravity? It has been suggested that N=8 supergravity might be two-loop finite in eleven dimensions [48], even though it is already infinite in eight dimensions at one loop. This is not even sustained by the N=8 power counting. Our results suggest that this is extremely unlikely. In particular, the nine-dimensional two-loop result is a good analogue of this case, since at one loop the N=4 Yang Mills theory is infinite in eight dimensions and trivially finite in nine dimensions. If the eleven-dimensional supergravity is indeed two-loop infinite, this would undermine the use of this theory for a Kaluza-Klein approach to the unification of the fundamental interactions, as the underlying theory would be divergent. The compactification to four dimensions would not affect the divergence, as the theory would still look eleven dimensional at very small scales.

In four dimensions, all pure supergravity theories are known to be finite up to two loops. The superspace power counting arguments can be used for N=1 and N=2, but this has no further implications. Furthermore, explicit superspace counterterms for supergravity have been studied in ref. [49] and, unlike the case of six-dimensional Yang-Mills at two loops, three-loop counterterms are available for the N=2 supergravity multiplet which do not vanish on shell. It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions. If this is actually true, supergravities could not be considered to be fundamental theories. One possibility is that they could be considered to be limits of string theories at energies below the Planck mass. N=2 superstring theories in ten dimensions have been shown to be one loop finite [36], unlike their ten-dimensional field theory limits. The analysis of divergences in string theories is quite different, and one could hope that this would persist to all orders. The final word on these issues may have to await further explicit calculations.

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