Special Values of Zeta-Functions for Proper Regular Arithmetic Surfaces

Thesis by

Daniel A. Siebel

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2019
Defended November 9, 2018

Acknowledgements

First and foremost, I would like to thank my advisor, Matthias Flach, for his guidance and support throughout my time as a graduate student. I am very grateful for his help, for his many ideas and countless discussions, and for his patience and kindness which have been essential for me as well as my research.

I would also like to give thanks to my parents, my brother, and my grandparents for their endless support and care over the years.

Lastly, I thank the California Institute of Technology for going to great lengths to support their students, and its Math Department for their warm and welcoming environment and for the opportunity to pursue mathematical research.

Abstract

We explicate Flach's and Morin's special value conjectures in [8] for proper regular arithmetic surfaces $\pi : \mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ and provide explicit formulas for the conjectural vanishing orders and leading Taylor coefficients of the associated arithmetic zeta-functions. In particular, we prove compatibility with the Birch and Swinnerton-Dyer conjecture, which has so far only been known for projective smooth $\mathcal{X}$. Further, we derive a direct sum decomposition of $R\pi_*\mathbb{Z}(n)$ into motivic degree components.

Contents

Chapter 0

Introduction

Background. The Tamagawa Number Conjecture — first proposed by Bloch and Kato in [4] and then reformulated by Fontaine and Perrin-Riou in [9] — describes the vanishing order and leading Taylor coefficient (up to sign) $L^*(M,n)$ of the L-function $L(M,s)$ associated to any Chow motive M over a number field at every integer $s=n$. It vastly generalizes the Analytic Class Number Formula as well as the Birch and Swinnerton-Dyer Conjecture which could be derived as corollaries for $M=h^0(\operatorname{Spec} F)$ for any number field F and for $M=h^1(E)(1)$ for any elliptic curve E (or, more generally, any smooth projective curve) over F respectively.

Meanwhile Flach and Morin gave conjectural descriptions of the special values of arithmetic ζ-functions $\zeta(\mathscr{X},s)$ associated to any proper regular arithmetic scheme $\mathscr{X}\to\operatorname{Spec}\mathbb{Z}$ in [8]. They proved that under certain standard assumptions their conjectures are compatible with the Tamagawa Number Conjecture for projective smooth $\mathscr{X}$, in the sense that they predict the same vanishing orders and leading Taylor coefficients for the L-function associated to the Tate-twisted motive $M=h(\mathscr{X}_{\mathbb{Q}})(n)$ of the generic fiber $\mathscr{X}_{\mathbb{Q}}$ (cf. [8] Thm. 5.26).

Special Value Conjectures for arithmetic schemes. We will present their conjecture and the constructions necessary for its formulation in more detail. One may also consult Appendix A.6 for a schematic overview of the involved types of cohomology and their interconnections. Let $d=\dim\mathscr{X}$ and let n be any integer.

- Flach and Morin have constructed perfect complexes $R\Gamma_c(\mathscr{X},\mathbb{R}(n))$ and

  $$R\Gamma_{\mathrm{ar},c}(\mathscr{X},\tilde{\mathbb{R}}(n))=R\Gamma_c(\mathscr{X},\mathbb{R}(n))\oplus R\Gamma_c(\mathscr{X},\mathbb{R}(n-1))$$

  in the derived category of real vectorspaces, the former being the mapping cone of the Beilinson regulator map. Conjecturally, the vanishing orders of $L(\mathscr{X},s)$ are determined by the ranks of the resulting compact Arakelov cohomology groups:

$$\mathrm{ord}_{s=n}\zeta(\mathcal{X},s) = \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^{i,n}_{\mathrm{ar},c}(\mathcal{X},\tilde{\mathbb{R}}(n)).$$

- Further, they assume the validity of the Artin-Verdier duality conjecture, i.e., the existence of a perfect pairing

$$H^{\bullet}(\overline{\mathscr{X}},\mathbb{Z}/m(n)) \times H^{2d+1-\bullet}(\overline{\mathscr{X}},\mathbb{Z}/m(d-n)) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

  for positive integers m, to construct a perfect complex $R\Gamma_W(\overline{\mathscr{X}},\mathbb{Z}(n))$ out of the (completed) motivic cohomology $R\Gamma(\overline{\mathscr{X}},\mathbb{Z}(n))$ of Bloch's cycle complexes $\mathbb{Z}(n)$. It contains all information of both the finitely generated as well as the cofinitely generated parts of motivic cohomology. They went on to define the compactly supported Weil-étale cohomology complex $R\Gamma_{W,c}(\mathscr{X},\mathbb{Z}(n))$ via the triangle

$$R\Gamma_{W,c}(\mathscr{X},\mathbb{Z}(n)) \longrightarrow R\Gamma_W(\overline{\mathscr{X}},\mathbb{Z}(n)) \longrightarrow R\Gamma_W(\mathscr{X}_\infty,\mathbb{Z}(n)) \longrightarrow \tag{1}$$

  where the right-most complex is an integral version of Betti cohomology.

- They have shown that the above complexes fit into a distinguished triangle

$$(R\Gamma_{\mathrm{ddR}}(\mathscr{X}/\mathbb{Z})/\mathrm{Fil}^n)_{\mathbb{R}}[-1] \longrightarrow R\Gamma_{\mathrm{ar},c}(\mathscr{X},\tilde{\mathbb{R}}(n)) \longrightarrow R\Gamma_{W,c}(\mathscr{X},\mathbb{Z}(n))_{\mathbb{R}} \longrightarrow \tag{2}$$

  where $R\Gamma_{\mathrm{ddR}}(\mathscr{X}/\mathbb{Z})$ denotes the derived version of the de Rham cohomology of $\mathscr{X}$. This motivates the definition of the fundamental line

$$\Delta(\mathscr{X},n) = \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathscr{X},\mathbb{Z}(n)) \otimes \det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(\mathscr{X}/\mathbb{Z})/F^n$$

  and gives rise to its trivialization

$$\lambda_\infty(\mathscr{X},n) : \mathbb{R} \xrightarrow{\cong} \det_{\mathbb{R}} R\Gamma_{\mathrm{ar},c}(\mathscr{X},\tilde{\mathbb{R}}(n)) \xrightarrow{\cong} \Delta(\mathscr{X},n)_{\mathbb{R}}.$$

- They have defined a correction factor $C(\mathscr{X},n) \in \mathbb{Q}^\times/\{\pm 1\}$ in terms of determinants of conjecturally distinguished triangles coming from p-adic Hodge Theory. $C(\mathscr{X},n)$ is trivial for $n \leqslant 0$. The leading Taylor coefficients $\zeta^*(\mathscr{X},n)$ are now conjectured to satisfy and hence be determined by

$$\lambda_\infty(\zeta^*(\mathscr{X},n)^{-1} \cdot C(\mathscr{X},n) \cdot \mathbb{Z}) = \Delta(\mathscr{X}/\mathbb{Z},n).$$

Results and Layout. In this thesis we will explicate these conjectures for proper regular arithmetic surfaces $\mathcal{X}$ and show compatibility with the Birch and Swinnerton-Dyer conjecture. On the way we will establish decomposition results for various types of cohomology. We will interpret this as an indication for the existence of a decomposition of a hypothetical motive

$\mathit{h}(\mathcal{X})$ into motivic degree components within the framework of a yet to be developed theory of mixed motives.

In Chapter I we will prove Artin-Verdier duality for arithmetic surfaces $\mathcal{X}$ in the then remaining open case $n = 1$. We will decompose $\mathcal{X}$ into a smooth open part on which duality is easy and a collection of bad fibers on which the duality statement follows from Saito's work in [28]. The main result is Theorem 1.3. Artin-Verdier duality will be used as a computational tool in the remaining chapters.

Chapters II and III are the core of this thesis. In the second chapter we will evaluate all cohomology groups introduced before, assuming the existence of a section $s : S \to \mathcal{X}$, where S is the spectrum of the integer ring of a number field F for which there is a factorization $\pi : \mathcal{X} \to S$ of the structure map of $\mathcal{X}$. A summary of the results is given in Appendix A.5. All computations will be organized around the main result Theorem 2.11 (and its various versions throughout Section 2.2) providing a decomposition

$$R\pi_*\mathbb{Z}(n)^{\mathcal{X}} \;\simeq\; \mathbb{Z}(n)^S \oplus {}^pR^1\pi_*\mathbb{Z}(n)^{\mathcal{X}}[-1] \oplus \mathbb{Z}(n-1)^S[-2] \tag{3}$$

in the derived category of abelian sheaves on S, and hence a decomposition of the associated cohomology groups. The analogue of (3) for torsion sheaves will be proven using Verdier duality and the six functor formalism for π and s. (3) will be an extension of it incorporating Geisser's results on dualizing cycle complexes in [12]. A further insight will be the analysis of derived de Rham cohomology as given in Proposition 2.23. It will explain the occurrence of the Bloch-Kato conductor $A(\mathcal{X})$ in the special value formulas later.

In Chapter III we evaluate the trivialization factor coming from $\lambda_\infty(\mathcal{X}, n)$ (Theorem 3.13). Understanding $\lambda_\infty(\mathcal{X}, n)$ will amount to comparing the two integral structures of $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))_\mathbb{R}$ induced by the triangles (1) and (2). Note that (2) only conjecturally determines an integral lattice inside $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))_\mathbb{R}$ as follows: Flach and Morin have constructed a pairing

$$H_c^\bullet(\mathscr{X}, \mathbb{R}(n)) \times H^{2d-\bullet}(\mathscr{X}, \mathbb{Z}(d-n))_\mathbb{R} \;\longrightarrow\; \mathbb{R} \tag{4}$$

that they conjecture to be perfect and, moreover, to encode the Arakelov Intersection Pairing. This would endow the cohomology groups of the second term in (2) $R\Gamma_{\mathrm{ar},c}(\mathscr{X}, \tilde{\mathbb{R}}(n))$ with a canonical integral structure coming from $R\Gamma(\mathscr{X}, \mathbb{Z}(n))$.

Let $m = r + 2s = [F : \mathbb{Q}]$ and write g for the genus of the generic fiber X of $\mathcal{X}$. We further prove $C(\mathcal{X}, 1) = 1$ unconditionally (Theorem 3.19) and derive

$$C(\mathcal{X}, n) = ((n-1)! \cdot (n-2)!)^{m(g-1)}$$

for $n \geqslant 2$ assuming both the special value conjectures as well as the functional equation for

$\zeta(\mathcal{X}, s)$ (Theorem 3.24). We will combine this to the special value formulas

$$\zeta^*(\mathcal{X}, n) = \begin{cases} 2^{(r-l(\mathcal{X}))\epsilon_n} \prod_{1 \leqslant i \leqslant 4} \left(\#\mathrm{Tor}\, H^i(\overline{\mathcal{X}}, \mathbb{Z}(2-n))_{\mathrm{codiv}}\right)^{(-1)^i} \frac{R^{2-n}(S) R^{1-n}(S)}{R^{2-n}(\mathcal{X})} & \text{for } n \leqslant 0 \\ \frac{2^r (2\pi)^s}{(\#\mu_F)^2 \sqrt{D_F}} \cdot \frac{(\#\mathrm{Tor}\,\mathrm{Pic}^0 \mathcal{X})^2}{\#\text{Ш}(X/F) \cdot \Omega(\mathcal{X})} \cdot \frac{R(S)^2}{R(\mathcal{X})} & \text{for } n = 1 \\ \left(\frac{(n-1)! \cdot (n-2)!}{(2\pi)^{2(n-1)}}\right)^{m(g-1)} A(\mathcal{X})^{1-n} \cdot \zeta^*(\mathcal{X}, 2-n) & \text{for } n \geqslant 2 \end{cases}$$

Here the regulators $R^n(\mathcal{X})$ and $R^n(S)$ are determinants of the matrices describing the pairing (4) for $\mathcal{X}$ and S, and $\Omega(\mathcal{X})$ is the determinant of (a restriction of) the period isomorphism comparing Betti and de Rham cohomology. The case $n \geqslant 2$ requires an additional technical assumption on $\mathcal{X}$ needed to evaluate its derived de Rham cohomology in terms of $A(\mathcal{X})$. We will conclude that Flach's and Morin's conjectures for surfaces $\mathcal{X}$ and $n = 1$ are equivalent to the conjunction of the vanishing order part and the leading Taylor coefficient part of the Birch and Swinnerton-Dyer conjecture. The main result will be Theorem 3.27.

Chapter 1

Artin-Verdier duality for arithmetic surfaces

Throughout this chapter let $\mathcal{O}$ be a number ring with fraction field F. Write $S = \operatorname{Spec}\mathcal{O}$. $\mathscr{X}$ will denote a proper regular arithmetic scheme over $\mathcal{O}$ of pure dimension d − *arithmetic*, meaning that there is an integral, normal, excellent, flat map $\mathscr{X} \to \operatorname{Spec}\mathcal{O}$ of finite type with smooth generic fiber $\mathscr{X}_F$. We will write $\mathcal{X}$ for a proper regular arithmetic surface over $\mathcal{O}$. $\mathbb{Z}(n)$ will denote the motivic cycle complexes as defined in Appendix A.1. Finally, let $G_{\mathbb{R}} = \operatorname{Gal}(\mathbb{C}/\mathbb{R})$.

In this chapter we combine work by Saito with Flach's and Morin's construction of the Artin-Verdier étale topos to generalize Artin-Verdier duality to arithmetic surfaces for coefficients given by $\mathbb{Z}(1) = \mathbb{G}_m[-1]$.

1.1 The Artin-Verdier ètale topos and compact support cohomology

Classical Artin-Verdier duality gives a pairing

$$H^r(S, \mathcal{F}) \times \operatorname{Ext}^{3-r}_S(\mathcal{F}, \mathbb{G}_m) \longrightarrow H^3(S, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$$

for all constructible étale sheaves $\mathcal{F}$ on S that is in general perfect only up to 2-torsion. Duality for the 2-torsion components needs further assumptions, e.g., that the underlying number field F is totally imaginary (cf. [21] Sec. 2).

Geisser has generalized this result to arithmetic schemes $\mathscr{X}$ of any dimension (cf. [12] Thm. 7.8). Conjecturally, an analogous duality should hold for Bloch's cycle complexes replacing

$\mathcal{F}$. To treat the cases $p \neq 2$ and $p = 2$ uniformly Flach and Morin have constructed the Artin-Verdier étale topos $\overline{\mathscr{X}}_{\text{ét}}$ (cf. [8] App. A) which we will briefly review here.

The Artin-Verdier étale topos $\overline{\mathscr{X}}_{\text{ét}}$. Let $\mathcal{X}_{\text{ét}}$ denote the étale topos of $\mathscr{X}$ and write $\mathscr{X}_\infty$ for both the quotient $\mathscr{X}(\mathbb{C})/G_\mathbb{R}$ itself and its associated topos of sheaves of set $\mathrm{Sh}(\mathscr{X}(\mathbb{C})/G_\mathbb{R})$ interchangeably. The projection $\pi : \mathscr{X}(\mathbb{C}) \to \mathscr{X}_\infty$ extends to a morphism of topoi

$$\mathrm{Sh}(G_\mathbb{R}, \mathscr{X}(\mathbb{C})) \xrightarrow{\ \pi\ } \mathrm{Sh}(\mathscr{X}_\infty)$$

given by

$$\pi^*(E \to \mathscr{X}_\infty) = E \underset{\mathscr{X}_\infty}{\times} \mathscr{X}(\mathbb{C}) \to \mathscr{X}(\mathbb{C}) \qquad \text{and} \qquad (\pi_* F)(U) = F(\pi^{-1}U)^{G_\mathbb{R}},$$

where E is the étalé space associated to a sheaf in $\mathscr{X}_\infty$. Next, the functor that maps any étale covering $\mathcal{U} \to \mathscr{X}$ to the $G_\mathbb{R}$-equivariant étalé space $\mathcal{U}(\mathbb{C}) \to \mathscr{X}(\mathbb{C})$ induces a morphism of topoi $\alpha : \mathrm{Sh}(G_\mathbb{R}, \mathscr{X}(\mathbb{C})) \to \mathscr{X}_{\text{ét}}$. We now define the *Artin-Verdier étale topos* as the topos $\overline{\mathscr{X}}_{\text{ét}}$ fitting into an Open-Closed-Decomposition

$$\mathscr{X}_{\text{ét}} \xrightarrow{\ \phi\ } \overline{\mathscr{X}}_{\text{ét}} \xleftarrow{\ u_\infty\ } \mathscr{X}_\infty \tag{1.1}$$

such that $u_\infty^* \phi_* \cong \pi_* \alpha^*$. Moreover, we may write π_* as the composition

$$\mathrm{Sh}(G_\mathbb{R}, \mathscr{X}(\mathbb{C})) \xrightarrow{p_*} \mathrm{Sh}(G_\mathbb{R}, \mathscr{X}_\infty) \xrightarrow{\mathscr{H}\!om(\mathbb{Z},-)} \mathscr{X}_\infty,$$

where $(p_*\mathcal{F})(U) = \mathcal{F}(\pi^{-1}U)$ and $\mathscr{H}\!om(\mathbb{Z}, -)$ denotes the composition of the sheaf-hom functor inside $\mathrm{Sh}(G_\mathbb{R}, \mathscr{X}_\infty)$ with the forgetful functor. We summarize this in the diagram

$$\begin{array}{ccccc}
\mathscr{X}_{\text{ét}} & \xrightarrow{\ \phi\ } & \overline{\mathscr{X}}_{\text{ét}} & \xleftarrow{\ u_\infty\ } & \mathscr{X}_\infty \\
 & \nwarrow^{\alpha} & & \nearrow^{\pi} & \uparrow{\scriptstyle \mathscr{H}\!om(\mathbb{Z},-)} \\
 & & \mathrm{Sh}(G_\mathbb{R}, \mathscr{X}(\mathbb{C})) & \xrightarrow{\ p_*\ } & \mathrm{Sh}(G_\mathbb{R}, \mathscr{X}_\infty)
\end{array}$$

Next, let $P_{\geqslant 0} \to \mathbb{Z} \to 0$ be the standard resolution of the constant sheaf $\mathbb{Z}$ with trivial $G_\mathbb{R}$-action. Write Γ^* for the left-adjoint functor of the global sections functor for $G_\mathbb{R}$-equivariant sheaves on $\mathscr{X}(\mathbb{C})$. For any bounded below complex $\mathcal{A}^\bullet$ of sheaves in $\mathrm{Sh}(G_\mathbb{R}, \mathscr{X})$ one has

$$R\pi_*\mathcal{A}^\bullet \cong \int \mathscr{H}\!om(\Gamma^* P_{\geqslant 0}, p_*\mathcal{A}^\bullet).$$

Her $\int$ denotes totalization. In analogy to Tate cohomology we define the Tate analogue $R\widehat{\pi}$ of $R\pi$ by replacing $P_{\geqslant 0}$ with a complete resolution $P_\bullet$ of $\mathbb{Z}$:

$$R\widehat{\pi}_*\mathcal{A}^\bullet := \int \mathscr{H}\!om(\Gamma^* P_\bullet, p_*\mathcal{A}^\bullet).^1$$

[1] See [8] Sec. 6.4 for why $R\widehat{\pi}$ is well-defined as a functor between derived categories and for further details.

For technical reasons the analogue $\mathbb{Z}(n)^{\overline{\mathscr{X}}}$ of Bloch's cycle complexes in the derived category of abelian sheaves on $\overline{\mathscr{X}}_{\text{ét}}$ is defined via the distinguished triangle

$$\mathbb{Z}(n)^{\overline{\mathscr{X}}} \longrightarrow R\phi_*\mathbb{Z}(n)^{\mathscr{X}} \longrightarrow u_{\infty,*}\tau^{>n}R\hat{\pi}_*\alpha^*\tau^{\geqslant 0}\mathbb{Z}(n)^{\mathscr{X}} \longrightarrow .$$

However, if $\mathbb{Z}(n)^{\mathscr{X}}$ is cohomologically concentrated in degrees $\leqslant n$ then one has the more intuitive identity $\mathbb{Z}(n)^{\overline{\mathscr{X}}} = \tau^{\leqslant n}R\phi_*\mathbb{Z}(n)^{\mathscr{X}}$ (cf. [8] Prop. 6.10). The cohomology of $\mathbb{Z}(n)^{\mathscr{X}}$ and $\mathbb{Z}(n)^{\overline{\mathscr{X}}}$ only differ in 2-torsion and the precise difference will be addressed in the next chapter and in Appendix A.3.

Milne's étale cohomology with compact support. One formulation of Artin-Verdier duality uses compact support cohomology which we will briefly review here. Let $U \subset S$ be an open subscheme. Write S_∞ for all infinite places of $\mathcal{O}$ and let $S_{\text{fin}} = S\backslash U$ denote the set of all finite places of $\mathcal{O}$ not in U. For any abelian sheaf $\mathcal{F}$ on $U_{\text{ét}}$ Milne's cohomology with compact support is the cohomology of the complex $R\widehat{\Gamma}_c(U_{\text{ét}}, \mathcal{F})$ given via the distinguished triangle

$$R\widehat{\Gamma}_c(U,\mathcal{F}) \longrightarrow R\Gamma(U,\mathcal{F}) \longrightarrow \bigoplus_{v\in S_{\text{fin}}} R\Gamma(F_v, f_v^*\mathcal{F}) \oplus \bigoplus_{v\in S_\infty} R\widehat{\Gamma}(F_v, f_v^*\mathcal{F}) \longrightarrow .$$

Here $f_v : \operatorname{Spec} F_v \to U$ is the canonical embedding of the closed point v in U and $R\widehat{\Gamma}(F_v, -)$ denotes Tate cohomology of the Galois group of F_v (cf. [22] p.165ff). We extend this definition to schemes $f : \mathcal{U} \to U$ over U and abelian sheaves $\mathcal{F}$ on $\mathcal{U}_{\text{ét}}$ by setting

$$R\widehat{\Gamma}_c(\mathcal{U}_{\text{ét}}, \mathcal{F}) := R\widehat{\Gamma}_c(U_{\text{ét}}, f_*\mathcal{F}). \tag{1.2}$$

We write $\widehat{H}^i_c(\mathcal{U}_{\text{ét}}, \mathcal{F}) := H^i(R\widehat{\Gamma}_c(\mathcal{U}_{\text{ét}}, \mathcal{F}))$.

This definition covers coefficients given by $\mathbb{Z}(n)$ for $n \leqslant 1$ since these are cohomologically concentrated in one degree. It is consistent with the following definition of $R\widehat{\Gamma}_c(\mathscr{X}_{\text{ét}}, \mathbb{Z}(n))$ for general n. Let $R\hat{\phi}_!\mathbb{Z}(n)^{\mathscr{X}}$ be defined via the distinguished triangle

$$R\hat{\phi}_!\mathbb{Z}(n)^{\mathscr{X}} \longrightarrow R\phi_*\mathbb{Z}(n)^{\mathscr{X}} \longrightarrow u_{\infty,*}R\hat{\pi}_*\alpha^*\tau^{\geqslant 0}\mathbb{Z}(n)^{\mathscr{X}} \longrightarrow .$$

Define $R\widehat{\Gamma}_c(\mathscr{X}_{\text{ét}}, \mathbb{Z}(n)) := R\Gamma(\overline{\mathscr{X}}_{\text{ét}}, R\hat{\phi}_!\mathbb{Z}(n)^{\mathscr{X}})$.

1.2 The Duality Statement

Fix any integer n. The Artin-Verdier duality conjecture may be formulated as follows.

Conjecture 1.1. AV($\mathscr{X}$,n) *Let m be a positive integer. There exists a product map*

$$\mathbb{Z}(n)^{\overline{\mathscr{X}}}/m \otimes^{\mathbb{L}} \mathbb{Z}(d-n)^{\overline{\mathscr{X}}}/m \longrightarrow \mathbb{Z}(d)^{\overline{\mathscr{X}}}/m$$

in the derived category of Artin-Verdier étale sheaves on $\mathscr{X}$ such that the induced pairing

$$H^i(\overline{\mathscr{X}}_{\text{ét}}, \mathbb{Z}(n)/m) \times H^{2d+1-i}(\overline{\mathscr{X}}_{\text{ét}}, \mathbb{Z}(d-n)/m) \longrightarrow H^{2d+1}(\overline{\mathscr{X}}_{\text{ét}}, \mathbb{Z}(d)/m) \rightarrow \mathbb{Q}/\mathbb{Z}$$

is a perfect pairing of finite abelian groups integers i.

Morin has shown that this is equivalent to the following formulation using Tate cohomology (cf. [8] Thm. 6.24).

Conjecture 1.2. AV'($\mathscr{X}$,n) *For any positive integer m there exists a product map*

$$\mathbb{Z}(n)^{\mathscr{X}}/m \otimes^{\mathbb{L}} \mathbb{Z}(d-n)^{\mathscr{X}}/m \longrightarrow \mathbb{Z}(d)^{\mathscr{X}}/m$$

in the derived category of étale sheaves on $\mathscr{X}$ such that the induced pairing

$$\widehat{H}^i_c(\mathscr{X}_{\text{ét}}, \mathbb{Z}(n)/m) \times H^{2d+1-i}(\mathscr{X}_{\text{ét}}, \mathbb{Z}(d-n)/m) \longrightarrow \widehat{H}^{2d+1}_c(\mathscr{X}_{\text{ét}}, \mathbb{Z}(d)/m) \rightarrow \mathbb{Q}/\mathbb{Z}$$

is a perfect pairing of finite abelian groups for all integers i.

This conjecture is known for $n \geqslant d$ and $n \leqslant 0$ and also for all $n \in \mathbb{Z}$ if $\mathscr{X}$ is smooth over a number ring (cf. [8] Cor. 6.26, 6.27). We wish to prove AV($\mathcal{X}$,n) for proper regular arithmetic surfaces $\mathcal{X}$, i.e. for the case $d = 2$. By the above, only the case $n = 1$ remains to be shown. This will be the main result of this chapter. We formulate it for prime powers $m = p^r$.

Theorem 1.3. (AV2) *There exists a product map*

$$\mathbb{Z}(1)^{\mathcal{X}}/p^r \otimes^{\mathbb{L}} \mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow \mathbb{Z}(2)^{\mathcal{X}}/p^r \tag{1.3}$$

such that for all prime powers p^r the induced pairing of cohomology groups

$$\widehat{H}^i_c(\mathcal{X}_{\text{ét}}, \mathbb{Z}(1)/p^r) \times H^{2d+1-i}(\mathcal{X}_{\text{ét}}, \mathbb{Z}(1)/p^r) \longrightarrow \widehat{H}^5_c(\mathcal{X}_{\text{ét}}, \mathbb{Z}(2)/p^r) \rightarrow \mathbb{Z}/p^r \tag{1.4}$$

is a perfect pairing of finite p^r-torsion groups for all integers i.

Remark 1.4. The versions of Artin-Verdier duality in [8] conjecture the existence of a product map for the integral complexes

$$\mathbb{Z}(n)^{\mathscr{X}} \otimes^{\mathbb{L}} \mathbb{Z}(d-n)^{\mathscr{X}} \longrightarrow \mathbb{Z}(d)^{\mathscr{X}}$$

that induce the aforementioned product maps of torsion complexes. For the pairing of cohomology groups this does not make a difference. We will construct a product map

$$\mathbb{Z}(1)^{\mathcal{X}} \otimes^{\mathbb{L}} \mathbb{Z}(1)^{\mathcal{X}} \longrightarrow \mathbb{Z}(2)^{\mathcal{X}} \tag{1.5}$$

under the additional assumption that $\mathbb{Z}(2)^{\mathcal{X}}$ satisfies the Beilinson-Soulé conjecture. (1.3) will be constructed unconditionally.

1.3 Lichtenbaum's product map and Spiess' complex

Lichtenbaum's complex $\mathbb{Z}(2,\mathscr{X})$. Before Bloch defined the cycle complexes $\mathbb{Z}(n)^{\mathscr{X}}$ for general $n \geqslant 0$ Lichtenbaum constructed a complex $\mathbb{Z}(2,\mathscr{X})$ using K-theory and proved that it satisfies many of the axioms for arithmetic cohomology complexes as proposed by Beilinson (cf. [20]). We review its definition.

Let A be a regular noetherian ring. Write $\mathbb{A}^1_A = \operatorname{Spec} A[t]$ and $Z = \operatorname{Spec} A[t]/t(t-1)$. $a \in A$ is an *exceptional unit* if both $a, 1-a \in A^\times$. For a finite set $B \subset A$ of exceptional units let $Y_B = \operatorname{Spec} A[t]/\prod_{b\in B}(t-b)$. $(Y_B)_B$ forms an inverse system, so we may define

$$C_{n,1}(A) = \varinjlim_B K_n(\mathbb{A}^1_A \backslash Y_B, Z) \quad \text{and} \quad C_{n,2}(A) = \varinjlim_B K'_{n-1}(Y_B)$$

(cf. [20] Def. 1.5). After gluing and sheafifying the presheaves $\operatorname{Spec} A \mapsto C_{n,i}(A)$ for $i = 1,2$ we obtain abelian sheaves $\underline{C}^{\mathscr{X}}_{n,1}$ and $\underline{C}^{\mathscr{X}}_{n,2}$ on $\mathscr{X}_{\text{ét}}$.

Definition 1.5. (cf. [20] Def. 2.1) *Let $\mathbb{Z}(1,\mathscr{X})$ and $\mathbb{Z}(2,\mathscr{X})$ be the complexes in the derived category of abelian sheaves on $\mathscr{X}_{\text{ét}}$ given by*

$$\mathbb{Z}(1,\mathscr{X}) = [\overset{0}{\underline{C}^{\mathscr{X}}_{1,1}} \to \overset{1}{\underline{C}^{\mathscr{X}}_{1,2}}], \quad \textit{and} \quad \mathbb{Z}(2,\mathscr{X}) = [\overset{1}{\underline{C}^{\mathscr{X}}_{2,1}} \to \overset{2}{\underline{C}^{\mathscr{X}}_{2,2}}].$$

Lichtenbaum shows that $K_1(\mathscr{X})[-1]$ is quasi-isomorphic to $\mathbb{Z}(1,\mathscr{X})$ (cf. [20] Prop. 2.4). As the Bloch cycle complex $\mathbb{Z}(1)^{\mathscr{X}}$ is quasi-isomorphic to $K_1(\mathscr{X})[-1]$ we immediately have $\mathbb{Z}(1)^{\mathscr{X}} \cong \mathbb{Z}(1,\mathscr{X})$ in the derived category. An analogous isomorphism for $n = 2$ is only conjectured, but for arithmetic surfaces $\mathcal{X}$ partial results are known.

Spiess' complex $\mathscr{K}_{/\mathcal{X}}$ for arithmetic surfaces. Spiess constructed a complex $\mathscr{K}_{/\mathcal{X}}$ of abelian sheaves on $\mathcal{X}_{\text{ét}}$ and proved a duality of type

$$H^i(\mathcal{X},\mathcal{F}) \times \operatorname{Ext}^{6-i}(\mathcal{F},\mathscr{K}_{/\mathcal{X}}) \longrightarrow \mathbb{Q}/\mathbb{Z} \tag{1.6}$$

for any constructible étale sheaf $\mathcal{F}$ on $\mathcal{X}$ (cf. [34] Theorem 2.2.2). We review the construction of $\mathscr{K}_{/\mathcal{X}}$.

Definition 1.6. *Let*

$$\mathscr{K}_{/\mathcal{X}} = \left[\overset{1}{\bigoplus_{\xi\in\mathcal{X}^0}} (i_\xi)_*\underline{C}_{2,1}(k(\xi)) \longrightarrow \overset{2}{\bigoplus_{\xi\in\mathcal{X}^0}} (i_\xi)_*\underline{C}_{2,2}(k(\xi)) \xrightarrow{c_2} \overset{3}{\bigoplus_{\eta\in\mathcal{X}^1}} (i_\eta)_*\mathbb{G}_m \longrightarrow \overset{4}{\bigoplus_{x\in\mathcal{X}^2}} i_*\mathbb{Z}\right],$$

where c_2 arises from the composition of $C_{2,2}(k(\xi)) \to K_2(k(\xi))$ with the boundary map $\partial' : K_2(k(\xi)) \to \mathbb{G}_m$ from K-theory.

Note that there is a canonical map $\mathbb{Z}(2,\mathcal{X}) \to \mathscr{K}_{/\mathcal{X}}$. Geisser has shown a duality for $\mathbb{Z}(2)^{\mathcal{X}}$ analogous to (1.6) (cf. [12] Thm. 8.7) which suggests that $\mathscr{K}_{/\mathcal{X}}$ and $\mathbb{Z}(2)^{\mathcal{X}}$ might coincide. Zhong has found a partial answer to this conjecture.

Theorem 1.7. (Zhong, [38] Thm. 3.8) *There is a map of complexes $\mathbb{Z}(2)^{\mathcal{X}} \to \mathscr{K}_{/\mathcal{X}}$ in the derived category of etale sheaves on $\mathcal{X}$ that induces a quasi-isomorphism*

$$\tau^{\geqslant 1}\mathbb{Z}(2)^{\mathcal{X}} \xrightarrow{\simeq} \mathscr{K}_{/\mathcal{X}}.$$

In particular, [8] Conjecture 7.1 holds for arithmetic surfaces. Using Flach's and Morin's technical Lemma 7.7 in [8] we may also remove the truncation after passing to mapping cones.

Corollary 1.8. *$\mathbb{Z}(2)^{\mathcal{X}}$ is cohomologically concentrated in degrees $\leqslant 2$. Moreover, for any prime power p^r the complex $\mathbb{Z}(2)^{\mathcal{X}}/p^r$ is cohomologically concentrated in degrees $0, 1, 2$.*

Proof. Spiess remarks that $\mathscr{K}_{/\mathcal{X}}$ is concentrated in degrees $1, 2$ (cf. [34] 1.6.2.(A1)). Consequently, [8] Conjecture 7.1 holds for $\mathbb{Z}(2)^{\mathcal{X}}$ and so [8] Lemma 7.7 applies. It yields for every prime p and its associated Open-Closed-Decomposition

$$\mathcal{X}_p \xrightarrow{i} \mathcal{X} \xleftarrow{j} \mathcal{X}[1/p]$$

the distinguished triangle

$$\left(\tau^{\leqslant 1} i_*\mathbb{Z}(1)^{\mathcal{X}_p}/p\right)[-2] \longrightarrow \mathbb{Z}(2)^{\mathcal{X}}/p \longrightarrow \tau^{\leqslant 2} Rj_*\mu_p^{\otimes 2} \longrightarrow,$$

where $\mathcal{X}_p := \mathcal{X}_{\mathbb{F}_p}$. Since the left hand side is cohomologically concentrated in degree 3 it shows that $\mathscr{H}^i(\mathbb{Z}(2)^{\mathcal{X}}/p) = 0$ for all $i < 0$. □

Lichtenbaum's product map. K-theory gives product maps $K_i(A)\otimes K_j(T_0) \to K_{i+j}(T_0)$, $K_i(A) \otimes K_j(T, T_0) \to K_{i+j}(T, T_0)$ for any closed immersion $i : T_0 \hookrightarrow T$ of schemes of finite type over A. Therefore $C_{n,1}(A), C_{n,2}(A)$ have a $K_0(A)$-module structure and, moreover, there are maps $K_m(A) \otimes C_{n,1}(A) \to C_{m+n,1}(A)$. In particular, we obtain a product map

$$K_1(\mathcal{X})[-1] \otimes \mathbb{Z}(1,\mathcal{X}) \longrightarrow \mathbb{Z}(2,\mathcal{X}). \tag{1.7}$$

Recalling that $K_1(X)[-1] \cong \mathbb{Z}(1, X) \cong \mathbb{Z}(1)^X$ and that the etale localizations $(\underline{C}_{i,1}(X))_{\bar{x}} = C_{i,1}(\mathcal{O}_{X,\bar{x}})$ are flat for $i = 1, 2$ (cf. [20] Prop. 2.5) we may rewrite the left-hand side as $\mathbb{Z}(1)^{\mathcal{X}} \otimes^{\mathbb{L}} \mathbb{Z}(1)^{\mathcal{X}}$. Composition with Zhong's isomorphism yields the product map

$$\mathbb{Z}(1)^{\mathcal{X}} \otimes^{\mathbb{L}} \mathbb{Z}(1)^{\mathcal{X}} \longrightarrow \mathbb{Z}(2,\mathcal{X}) \longrightarrow \mathscr{K}_{/\mathcal{X}} \xrightarrow{\simeq} \tau^{\geqslant 1}\mathbb{Z}(2)^{\mathcal{X}}. \tag{1.8}$$

Taking mapping cones gives the pairing (1.3) as desired. When assuming Beilinson-Soulé $\tau^{\geqslant 1}\mathbb{Z}(2)^{\mathcal{X}} \simeq \mathbb{Z}(2)^{\mathcal{X}}$ the above extends to the pairing (1.5) into the full $\mathbb{Z}(2)^{\mathcal{X}}$.

1.4 A further construction of the p-torsion product map for arithmetic surfaces a la Sato

Fix a prime power p^r. We present an alternative construction of the p-torsion version of (1.3)

$$\mathbb{Z}(1)^{\mathcal{X}}/p^r \otimes^{\mathbb{L}} \mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow \mathbb{Z}(2)^{\mathcal{X}}/p^r \tag{1.9}$$

that builds on a lifting argument by Sato (cf. [29] Prop. 4.2.6). For an arithmetic surface one can drop Sato's normal crossing condition and use the explicit description of a boundary map from K-theory instead. This section should be understood as an extension of some of Sato's results and constructions in [29] to arithmetic surfaces with not necessarily semistable fibers.

An Open-Closed-Decomposition for $\mathbb{Z}(n)^{\mathcal{X}}$. Let $Z = \mathcal{X}_{\mathbb{F}_p}$ denote the special fiber of $\mathcal{X}$ over p and write $\mathcal{X}[1/p]$ for its complement. We have the Open-Closed-Decomposition

$$Z \xrightarrow{\ i\ } \mathcal{X} \xleftarrow{\ j\ } \mathcal{X}[1/p].$$

We know $\mathscr{H}^i(\mathbb{Z}(2)^{\mathcal{X}}) = 0$ for $i > 2$ (Corollary 1.8). Therefore – by Proposition A.5(ii) – $Rj_*j^*\mathbb{Z}(2)^{\mathcal{X}}/p^r = Rj_*\mu_{p^r}^{\otimes 2}$, and we have the distinguished triangles

$$i_*\mathbb{Z}^Z/p^r[-2] \longrightarrow \mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow \tau^{\leqslant 1}Rj_*\mu_{p^r} \longrightarrow, \tag{1.10}$$

$$\left(\tau^{\leqslant 1}i_*\mathbb{Z}(1)^Z/p^r\right)[-2] \longrightarrow \mathbb{Z}(2)^{\mathcal{X}}/p^r \longrightarrow \tau^{\leqslant 2}Rj_*\mu_{p^r}^{\otimes 2} \longrightarrow$$

(cf. [8] Lemma 7.7). [38] Thm. 1.1 provides the quasi-isomorphism

$$\mathbb{Z}(1)^Z/p^r \simeq \left[\bigoplus_{\eta\in Z^0}(i_\eta)_*W_r\Omega^1_{k(\eta),\log} \longrightarrow \bigoplus_{x\in Z^1}(i_x)_*W_r\Omega^0_{k(x),\log}\right][-1], \tag{1.11}$$

i.e. $\mathbb{Z}(1)^Z/p^r$ is cohomologically concentrated in degrees 1 and 2. We wish to show that

$$\mathbb{Z}(1)^{\mathcal{X}}/p^r \otimes^{\mathbb{L}} \mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow \tau^{\leqslant 1}Rj_*\mu_{p^r} \otimes^{\mathbb{L}} \tau^{\leqslant 1}Rj_*\mu_{p^r} \longrightarrow \tau^{\leqslant 2}Rj_*\mu_{p^r}^{\otimes 2}$$

lifts to a duality pairing $\mathbb{Z}(1)^{\mathcal{X}}/p^r \otimes^{\mathbb{L}} \mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow \mathbb{Z}(2)^{\mathcal{X}}/p^r$. This will follow from the picture

$$\begin{array}{ccccccc}
\left(\tau^{\leqslant 1}i_*\mathbb{Z}(1)^Z/p^r\right)[-2] & \longrightarrow & \mathbb{Z}(2)/p^r & \longrightarrow & \tau^{\leqslant 2}Rj_*\mu_{p^r}^{\otimes 2} & \longrightarrow & \left(\tau^{\leqslant 1}i_*\mathbb{Z}(1)^Z/p^r\right)[-1] \\
& \nwarrow^{\mathrm{Hom}=0} & & & \uparrow & \nearrow^{=0} & \\
& & & & \mathbb{Z}(1)^{\mathcal{X}}/p^r \otimes^{\mathbb{L}} \mathbb{Z}(1)^{\mathcal{X}}/p^r & &
\end{array} \tag{1.12}$$

which we will verify now. $\mathbb{Z}(1)^{\mathcal{X}}/p^r \otimes^{\mathbb{L}} \mathbb{Z}(1)^{\mathcal{X}}/p^r$ is concentrated in degrees $0, 1, 2$ while $(\tau^{\leqslant 1}i_*\mathbb{Z}(1)^Z/p^r)[-2]$ is concentrated in degree 3. So, there can in fact not be any non-trivial morphisms between them. Now it suffices to show the

Proposition 1.9. *The composition*

$$\mathbb{Z}(1)^{\mathcal{X}}/p^r \otimes^{\mathbb{L}} \mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow \tau^{\leqslant 2} Rj_*\mu_{p^r}^{\otimes 2} \longrightarrow \left(\tau^{\leqslant 1} i_*\mathbb{Z}(1)^Z/p^r\right)[-1]$$

on the right-hand side of (1.12) vanishes.

Proof. We use Sato's notation $M_n^r = i^* R^n j_* \mu_{p^r}^{\otimes n}$. Observe that $\mathbb{Z}(1)^{\mathcal{X}}/p^r \otimes^{\mathbb{L}} \mathbb{Z}(1)^{\mathcal{X}}/p^r \to \left(\tau^{\leqslant 1} i_*\mathbb{Z}(1)^Z/p^r\right)[-1]$ can only be non-trivial in degree 2. The map on second cohomology sheaves is supported on Z and thus can be identified with

$$\mathscr{H}^1\mathbb{Z}(1)^{\mathcal{X}}/p^r \otimes \mathscr{H}^1\mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow M_r^2 \longrightarrow \mathscr{H}^1(\mathbb{Z}(1)^Z/p^r).$$

Fix a geometric point $\bar{z}$ of Z. Then, for $A = \mathcal{O}_{\mathcal{X},\bar{z}}[1/p]$, $F = \operatorname{Frac} A$, and any $n \geqslant 1$ one has

$$(M_r^n)_{\bar{z}} = H^n_{\text{ét}}(A, \mu_{p^r}^{\otimes n}) \hookrightarrow H^n_{\text{ét}}(F, \mu_{p^r}^{\otimes n}) = K_n F/p^r.$$

The inclusion is due to Gabber as mentioned in the proof of [2], Prop. 6.1. The last equality is only needed in the easy case $n = 1, 2$ but it holds in general by the Rost-Voevodsky Theorem. Moreover, for $n = 1$ the Kummer sequence shows that $H^1_{\text{ét}}(A, \mu_{p^r}) = K_1 A/p^r$. Consequently — when writing $Z_z^0 = Z^0 \cap \overline{\{z\}}$ for the collection of all non-closed points whose closure contains z — (1.10) yields

$$(\mathscr{H}^1\mathbb{Z}(1)^{\mathcal{X}}/p^r)_{\bar{z}} = \operatorname{Ker}\left((M_r^1)_{\bar{z}} \to \bigoplus_{\eta \in Z_z^0} \mathbb{Z}/p^r\right) = \operatorname{Ker}\left(\frac{K_1 A}{p^r} \to \bigoplus_{\eta \in Z_z^0} \frac{\mathbb{Z}}{p^r}\right).$$

Also,

$$\begin{aligned}(\mathscr{H}^1\mathbb{Z}(1)^Z/p^r)_{\bar{z}} &= \operatorname{Ker}\left(\left(\bigoplus_{\eta \in Z^0} (i_\eta)_* W_r\Omega^1_{k(\eta),\log}\right)_{\bar{z}} \longrightarrow \left(\bigoplus_{x \in Z^1} (i_x)_* W_r\Omega^0_{k(x),\log}\right)_{\bar{z}}\right) \\ &= \operatorname{Ker}\left(\bigoplus_{\eta \in Z_z^0} k(\eta)^\times/p^r \longrightarrow \mathbb{Z}/p^r\right) \subset \bigoplus_{\eta \in Z_z^0} k(\eta)^\times/p^r.\end{aligned}$$

So, it suffices to show that the composition

$$\operatorname{Ker}\left(\frac{K_1 A}{p^r} \to \bigoplus_{\eta \in Z_z^0} \frac{\mathbb{Z}}{p^r}\right) \otimes \operatorname{Ker}\left(\frac{K_1 A}{p^r} \to \bigoplus_{\eta \in Z_z^0} \frac{\mathbb{Z}}{p^r}\right) \longrightarrow \frac{K_2 F}{p^r} \xrightarrow{\oplus_\eta \partial_\eta} \bigoplus_{\eta \in Z_z^0} \frac{K_1 k(\eta)}{p^r} \tag{1.13}$$

vanishes. The map $K_2F/p^r \xrightarrow{\partial_\eta} K_1 k(\eta)/p^r$ is the boundary map coming from K-theory with F being regarded as the fraction field of the DVR A_η. Let the corresponding valuation be called v_η. The composition

$$K_1 F \times K_1 F \xrightarrow{\otimes} K_2 F \xrightarrow{\partial_\eta} K_1 k(\eta)$$

admits the explicit formula

$$(a, b) \mapsto (-1)^{v_\eta(a) v_\eta(b)} \frac{a^{v_\eta(b)}}{b^{v_\eta(a)}}$$

(cf [1] Prop. 4.5(e)). In particular, the image of $a \otimes b$ under (1.13) is trivial whenever $v_\eta(a) = v_\eta(b) = 0$ for all $\eta \in \overline{\{z\}}$. Now, since the map $K_1 F \to \bigoplus_{\eta \in Z^0_{\bar{z}}} \mathbb{Z}/p^r$ is the tuple $\bigoplus_{\eta \in \overline{\{z\}}} v_\eta$ and any $a \in A$ is integral with respect to all v_η the composition (1.13) must be identically zero. □

The Gersten complex of logarithmic deRham-Witt sheaves of a variety X (which the right-hand side of (1.11) is an example of) is known to be concentrated in one degree only if X is normal-crossing. For the one-dimensional curve Z the normal-crossing condition is not necessary.

Proposition 1.10. *The boundary map*

$$\beta : \bigoplus_{\eta \in Z^0} (i_\eta)_* W_r\Omega^1_{k(\eta),\log} \longrightarrow \bigoplus_{x \in Z^1} (i_x)_* W_r\Omega^0_{k(x),\log}$$

is surjective, i.e. $\mathbb{Z}(1)^Z/p^r \simeq \mathrm{Ker}(\beta)[-1]$.

Proof. The Gersten complexes for Z and Z^{red} are identical; thus we assume Z to be reduced. The stalk of $\bigoplus_{x \in Z^1} (i_x)_* W_r\Omega^0_{k(x),\log}$ at a generic point $\eta \in Z^0$ vanishes. It therefore suffices to consider the maps on stalks at a geometric point $\bar{z} \hookrightarrow z$

$$\beta_{\bar{z}} : \bigoplus_{\eta \in \overline{\{z\}}} W_r\Omega^1_{\overline{k(\eta)},\log} \longrightarrow W_r\Omega^0_{\overline{k(x)},\log}$$

for any closed point $z \in Z$. Note that we now work over the algebraic closure $\bar{k}$ of k since we are considering etale stalks. We may assume that Z is irreducible. Let η denote its generic point. By restricting to a suitable neighborhood of z we may assume Z to be affine. We are left to show that

$$\beta_{\bar{z}} : k(\eta)^\times/p^r \longrightarrow \mathbb{Z}/p^r$$

surjects. Let $N \to Z$ be the normalization of Z and $P_1, \ldots, P_r \in N$ the preimages of z. Each local ring $\mathcal{O}_{N,P_i}$ is a DVR and hence comes with a valuation v_i. One has $\beta_{\bar{z}} = \sum_i v_i$. So, it suffices to find a rational function $f \in k(N) = k(Z)$ which has a simple zero at precisely one P_{i_0} and is non-zero at all remaining P_i. This is easy for infinite base fields.

Embed $N \subset \mathbb{A}^n_{\bar{k}}$. As $\bar{k}$ is infinite we can choose a hyperplane $H \subset \mathbb{A}^n_{\bar{k}}$ passing through some P_{i_0} that does not contain any other P_j and that also does not contain the tangent direction of Z at x. The linear function f with $\mathcal{Z}(f) = H$ will satisfy the above. □

Corollary 1.11. *One has a distinguished triangle*

$$i_*\mathbb{Z}(1)^Z/p^r[-2] \longrightarrow \mathbb{Z}(2)^{\mathcal{X}}/p^r \longrightarrow \tau^{\leqslant 2} Rj_*\mu_{p^r}^{\otimes 2} \longrightarrow$$

i.e. [8] Conjecture 7.10 holds for arithmetic surfaces and $n = 2$ *and Sato's complex* $\mathcal{I}_r(2)_{\mathcal{X}}$ *defined for normal-crossing* $\mathcal{X}$ *is quasi-isomorphic to* $\mathbb{Z}(2)^{\mathcal{X}}$ *and its construction in [29] Def. 4.2 can be carried out without the normal-crossing assumption.*

1.5 Saito's duality result and duality on the closed part

For the remainder of this chapter we let $\mathcal{X}$ be a surface over the integer ring of a local field L with perfect residue field of characteristic p. We write Z for its special fiber and $j : \mathcal{X}_L \hookrightarrow \mathcal{X}$ and $i : Z \hookrightarrow \mathcal{X}$ for the canonical open and closed embeddings. Obviously, the constructions from last section can equally be carried out for $\mathcal{X}$, i.e. we also have Spiess' and Lichtenbaum's complexes $\mathscr{K}/_{\mathcal{X}}$ and $\mathbb{Z}(2, \mathcal{X})$.

We will show (AV2) by proving dualities on the smooth, open part and the singular, closed part separately. This idea is based on Sato's proof that Artin-Verdier duality in the global and local setting are equivalent (cf. [29] Sec. 10). Duality on the closed part has essentially been shown by Saito in [28]. We will need the analogue of [28] (4-1) for $\mathcal{X}$ and will briefly sketch Sato's proof in this context.

Theorem 1.12. (AV2l) *Let $\mathcal{X}$ be a proper regular surface over the integer ring of a local field with special fiber Z. Write $H^i_Z(\mathcal{X}, \mathbb{Z}(n)) := H^i(Z, Ri^!\mathbb{Z}(n))$. The product map*

$$Ri^!\mathbb{Z}(1)^{\mathcal{X}} \otimes^{\mathbb{L}} i^*\mathbb{Z}(1)^{\mathcal{X}} \longrightarrow Ri^!\mathbb{Z}(2)^{\mathcal{X}}$$

in the derived category of abelian sheaves on $\mathcal{X}_{\text{ét}}$ induced by (1.8) induces a perfect pairing

$$H^i_Z(\mathcal{X}, \mathbb{G}_m) \times H^{4-i}(Z, i^*\mathbb{G}_m) \longrightarrow H^6_Z(\mathcal{X}, \mathbb{Z}(2)^{\mathcal{X}}) \xrightarrow{\mathrm{tr}_{\mathcal{X},Z}} \mathbb{Q}/\mathbb{Z} \tag{1.14}$$

of finitely generated abelian groups.

Proof. We will replicate the computations in [28] Section 4 using the additional simplification that L cannot have a real embedding. Note that the trace map $\mathrm{tr}_{\mathcal{X},Z}$ is Saito's trace map constructed in [28] Thm. 3.1. The duality (1.14) can be written as

$$R\Gamma_Z(\mathcal{X}, \mathbb{G}_m)^*[-4] \simeq R\Gamma(Z, i^*\mathbb{G}_m) \tag{1.15}$$

in the derived category of abelian groups, where $(-)^* = R\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$. To prove (1.15) we use the localization sequences

$$\bigoplus_{x \in Z_0} R\Gamma_x(\mathcal{X}, \mathbb{G}_m) \longrightarrow R\Gamma_Z(\mathcal{X}, \mathbb{G}_m) \longrightarrow \bigoplus_{\eta \in Z_1} R\Gamma_\eta(\mathcal{X}, \mathbb{G}_m) \longrightarrow \tag{1.16}$$

$$\bigoplus_{x \in Z_0} R\Gamma_x(Z, i^*\mathbb{G}_m) \longrightarrow R\Gamma(Z, i^*\mathbb{G}_m) \longrightarrow \bigoplus_{\eta \in Z_1} R\Gamma(\mathcal{O}_\eta, \mathbb{G}_m) \longrightarrow \tag{1.17}$$

and evaluate the enclosing complexes.

Computation of the distinguished triangle (1.16). We will now compute the terms $R\Gamma_x(X,\mathbb{G}_m)$ and $R\Gamma_\eta(X,\mathbb{G}_m)$. For $x \in Z_0$, $\eta \in Z_1$ write $D_x = \operatorname{Spec}\mathcal{O}_x \backslash x$ and K_η for the fraction field of $\mathcal{O}_\eta$. The inclusions $\{x\} \hookrightarrow \mathcal{O}_x$ and $\{\eta\} \hookrightarrow \mathcal{O}_\eta$ give rise to the distinguished triangles

$$R\Gamma_x(\mathcal{X},\mathbb{G}_m) \longrightarrow R\Gamma(\mathcal{O}_x,\mathbb{G}_m) \to R\Gamma(D_x,\mathbb{G}_m) \longrightarrow,$$

$$R\Gamma_\eta(\mathcal{X},\mathbb{G}_m) \longrightarrow R\Gamma(\mathcal{O}_\eta,\mathbb{G}_m) \to R\Gamma(K_\eta,\mathbb{G}_m) \longrightarrow .$$

Let $k(x)$, $k(\eta)$ denote the residue fields of $\mathcal{O}_x$, $\mathcal{O}_\eta$ respectively. It is well-known that $\tau^{\geqslant 1}R\Gamma(\mathcal{O}_x,\mathbb{G}_m) \simeq \tau^{\geqslant 1}R\Gamma(k(x),\mathbb{G}_m)$ and $\tau^{\geqslant 1}R\Gamma(\mathcal{O}_\eta,\mathbb{G}_m) \simeq \tau^{\geqslant 1}R\Gamma(k(\eta),\mathbb{G}_m)$. As $k(x)$ is finite, its cohomological dimension is 1. So, by Hilbert 90 and the fact that finite fields have trivial Brauer group $R\Gamma(\mathcal{O}_x,\mathbb{G}_m)$ must be concentrated in degree 0 and we have

$$R\Gamma(\mathcal{O}_x,\mathbb{G}_m) \simeq \mathcal{O}_x^\times[0].$$

Similarly, we obtain the cohomology sheaves of $R\Gamma(\mathcal{O}_\eta,\mathbb{G}_m)$ and $R\Gamma(K_\eta,\mathbb{G}_m)$. This may be written as the distinguished triangles

$$\mathcal{O}_\eta^\times \longrightarrow R\Gamma(\mathcal{O}_\eta,\mathbb{G}_m) \longrightarrow \operatorname{Br} k(\eta)[-2] \longrightarrow$$

$$K_\eta^\times \longrightarrow R\Gamma(K_\eta,\mathbb{G}_m) \longrightarrow \operatorname{Br} K_\eta[-2] \longrightarrow .$$

Indeed, we do not have 2-torsion groups in higher degrees as Saito does in his proof since K_η is an extension of the local field L, i.e. it has no real embedding.

As $\mathcal{O}_\eta$ is a DVR and the canonical map $\operatorname{Br} k(\eta) \to \operatorname{Br} K_\eta$ injects, we obtain

$$\mathbb{Z}[-1] \longrightarrow R\Gamma_\eta(\mathcal{X},\mathbb{G}_m) \longrightarrow \frac{\operatorname{Br} K_\eta}{\operatorname{Br} k(\eta)}[-3] \longrightarrow .$$

Finally, since $\mathcal{O}_x \backslash D_x \subset \mathcal{O}_x$ has codimension 2 one has $H^0(D_x,\mathbb{G}_m) = \Gamma(\mathcal{O}_x,\mathbb{G}_m) = \mathcal{O}_x^\times$ and $H^1(D_x,\mathbb{G}_m) = \operatorname{Pic}(D_x) = \operatorname{Pic}(\mathcal{O}_x) = 0$. Also, $H^2(D_x,\mathbb{G}_m) \cong H^2(\mathcal{O}_x,\mathbb{G}_m) = \operatorname{Br}(\mathcal{O}_x) = 0$ where the first isomorphism is due to Grothendieck (cf. [14], III, Thm. 6.1(b)). The duality

$$R\Gamma(D_x,\mathbb{G}_m) \simeq R\Gamma(D_x,\mathbb{G}_m)^*[-3]$$

(cf. [27]) therefore proves that

$$R\Gamma_x(\mathcal{X},\mathbb{G}_m) \simeq (\mathcal{O}_x^\times)^*[-4]$$

and the distinguished triangle (1.16) becomes

$$\begin{array}{ccccccc}
 & & & & \bigoplus_{\eta\in Z_1}\mathbb{Z}[-1] & & \\
 & & & & \downarrow & & \\
\bigoplus_{x\in Z_0}(\mathcal{O}_x^\times)^*[-4] & \longrightarrow & R\Gamma_Z(\mathcal{X},\mathbb{G}_m) & \longrightarrow & R\Gamma_\eta(\mathcal{X},\mathbb{G}_m) & \longrightarrow & \\
 & & & & \downarrow & & \\
 & & & & \bigoplus_{\eta\in Z_1}\frac{\operatorname{Br} K_\eta}{\operatorname{Br} k(\eta)}[-3] & &
\end{array}$$

Taking duals and shifting 4 terms to the right gives

$$\begin{array}{ccccc} \bigoplus_{\eta\in Z_1}\left(\frac{\operatorname{Br}K_\eta}{\operatorname{Br}k(\eta)}\right)^*[-1] & & & & \\ \downarrow & & & & \\ R\Gamma_\eta(\mathcal{X},\mathbb{G}_m)^*[-4] & \longrightarrow & R\Gamma_Z(\mathcal{X},\mathbb{G}_m)^*[-4] & \longrightarrow & \bigoplus_{x\in Z_0}\mathcal{O}_x^\times \longrightarrow \\ \downarrow & & & & \\ \bigoplus_{\eta\in Z_1}\mathbb{Q}/\mathbb{Z}[-3] & & & & \end{array} \tag{1.18}$$

Here we used finiteness of Z_1.

Computation of the distinguished triangle (1.17). To compute $R\Gamma_x(Z, i^*\mathbb{G}_m)$ we use the localization sequence

$$R\Gamma_x(Z, i^*\mathbb{G}_m) \longrightarrow R\Gamma(\mathcal{O}_x, \mathbb{G}_m) \longrightarrow \bigoplus_{v\in S_x} R\Gamma(\mathcal{O}_v, \mathbb{G}_m) \longrightarrow$$

Here $S_x = \coprod_\eta S_{x,\eta}$ and $S_{x,\eta}$ denotes the collection of all codimension 1 points of $\operatorname{Spec}\mathcal{O}_x^h$ lying over η. We write $S_\eta = \coprod_{x\in\overline{\{\eta\}}_0} S_{x,\eta}$. Note that S_η can be viewed as the collection of all finite places v of the residue field $k(\eta)$. Also, let $\mathcal{O}_v$ denote the henselization of $\mathcal{O}_\eta$ at v and $k(v)$ its residue field. We get

$$\left(\bigoplus_{v\in S_x}\mathcal{O}_v^\times\right)/\mathcal{O}_x^\times[-1] \longrightarrow R\Gamma_x(Z, i^*\mathbb{G}_m) \longrightarrow \bigoplus_{v\in S_x}\operatorname{Br}k(v)[-3] \longrightarrow .$$

So, (1.17) becomes

$$\begin{array}{ccccc} \bigoplus_{x\in Z_0}\left(\bigoplus_{v\in S_x}\mathcal{O}_v^\times\right)/\mathcal{O}_x^\times[-1] & & & & \bigoplus_{\eta\in Z_1}\mathcal{O}_\eta^\times \\ \downarrow & & & & \downarrow \\ \bigoplus_{x\in Z_0} R\Gamma_x(Z, i^*\mathbb{G}_m) & \longrightarrow & R\Gamma(Z, i^*\mathbb{G}_m) & \longrightarrow & \bigoplus_{\eta\in Z_1} R\Gamma(\mathcal{O}_\eta, \mathbb{G}_m) \longrightarrow \\ \downarrow & & & & \downarrow \\ \bigoplus_{x\in Z_0}\bigoplus_{v\in S_x}\operatorname{Br}k(v)[-3] & & & & \bigoplus_{\eta\in Z_1}\operatorname{Br}k(\eta)[-2] \end{array} \tag{1.19}$$

Fix $\eta \in Z_1$. Then $k(\eta)$ is either a number or a function field depending on whether η is a horizontal or vertical divisor of $\mathcal{X}$. So, Class Field Theory gives us the short exact sequence

$$0 \longrightarrow \operatorname{Br}k(\eta) \longrightarrow \bigoplus_{v\in S_\eta}\operatorname{Br}k(v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

and we see that (1.19) is isomorphic to the distinguished triangle

$$\bigoplus_{x\in Z_0}\left(\bigoplus_{v\in S_x}\mathcal{O}_v^\times\right)/\mathcal{O}_x^\times[-1] \oplus \bigoplus_{\eta\in Z_1}\mathbb{Q}/\mathbb{Z}[-3] \longrightarrow R\Gamma(Z, i^*\mathbb{G}_m) \longrightarrow \bigoplus_{\eta\in Z_1}\mathcal{O}_\eta^\times \longrightarrow .$$

We compare the above with the distinguished triangle (1.18). It remains to prove that the boundary maps

$$\prod_{x\in Z_0} \mathcal{O}_x^\times \quad \xrightarrow{\delta_3^*} \quad \bigoplus_{\eta\in Z_1} \left(\frac{\mathrm{Br}\,K_\eta}{\mathrm{Br}\,k(\eta)}\right)^* \tag{1.20}$$

and

$$\bigoplus_{\eta\in Z_1} \mathcal{O}_\eta^\times \xrightarrow{\delta_3'} \bigoplus_{x\in Z_0} \left(\bigoplus_{v\in S_x} \mathcal{O}_v^\times\right) / \mathcal{O}_x^\times \tag{1.21}$$

have isomorphic kernels and isomorphic cokernels. This requires some preparation.

Results from Kato's higher local class field theory. Fix $\eta \in Z_1$. Let $I \subset \mathcal{O}_\eta$ be an ideal. In analogy to the adeles and ideles of number fields we define

$$\mathbb{A}_{I,\eta} = \widetilde{\prod}'_{v\in S_\eta} \frac{K_v^\times}{1 + I\mathcal{O}_v^\times} = \left\{ (a_v)_v \in \prod_{v\in S_\eta} \frac{K_v^\times}{1+I\mathcal{O}_v^\times} \;\middle|\; \begin{array}{c} a_v \in \mathcal{O}_x^\times \text{ where } v \in S_x \\ \text{for almost all } v \end{array} \right\},$$

$$\mathbb{I}_{I,\eta} = \widetilde{\prod}'_{v\in S_\eta} \frac{\mathcal{O}_v^\times}{1 + I\mathcal{O}_v^\times} = \widetilde{\prod}'_{v\in S_\eta} (\mathcal{O}_v/I\mathcal{O}_v)^\times \;\subset\; \mathbb{A}_{I,\eta}.$$

We have diagonal embeddings of $K_\eta^\times$, $\mathcal{O}_\eta^\times$ into $\mathbb{A}_{I,\eta}$, $\mathbb{I}_{I,\eta}$ respectively. We define

$$C_{I,\eta} = \mathbb{A}_{I,\eta}/K_\eta^\times, \qquad C_{I,\eta}^0 = \mathbb{I}_{I,\eta}/\mathcal{O}_\eta^\times.$$

The existence of a perfect pairing

$$H^i(K_v, \mathbb{G}_m) \times H^{2-i}(K_v, \mathbb{G}_m) \to H^4(K_v, \mathbb{Z}(2)) \cong \mathbb{Q}/\mathbb{Z}.$$

is well-known. The induced product

$$\langle -, - \rangle_v : \mathrm{Br}\,(K_v) \times K_v^\times \longrightarrow \mathbb{Q}/\mathbb{Z}$$

has the property that for each $x \in \mathrm{Br}\,(K_v)$ the map $\langle x, - \rangle_v$ has non-vanishing kernel. $\mathrm{Br}\,\mathcal{O}_v$ can be characterized as the subgroup of $\mathrm{Br}\,K_v$ consisting of those elements x for which $\langle x, - \rangle_v$ vanishes on $\mathcal{O}_v^\times$ (cf. [29] Thm. 2.9). Consequently there is a map

$$\mathrm{Br}\,(K_v) \longrightarrow \varinjlim_{I\subset\mathcal{O}_v} \mathrm{Hom}\left(\frac{K_v^\times}{1 + I\mathcal{O}_v^\times}, \mathbb{Q}/\mathbb{Z}\right).$$

Using the embedding $\mathrm{Br}\,(K_\eta) \hookrightarrow \bigoplus_{v\in S_\eta} \mathrm{Br}\,(K_v)$ we get a map

$$\mathrm{Br}\,(K_\eta) \longrightarrow \varinjlim_{I\subset\mathcal{O}_\eta} \bigoplus_{v\in S_\eta} \mathrm{Hom}\left(\frac{K_v^\times}{1 + I\mathcal{O}_v^\times}, \mathbb{Q}/\mathbb{Z}\right) = \varinjlim_{I\subset\mathcal{O}_\eta} \mathrm{Hom}(\mathbb{A}_{\eta,I}, \mathbb{Q}/\mathbb{Z}).$$

Kato has shown that the above map factors through and surjects onto $\varinjlim_{I\subset\mathcal{O}_\eta} \mathrm{Hom}(C_{I,\eta}, \mathbb{Q}/\mathbb{Z})$ (cf. [29] Thm.2.13), i.e. we have an isomorphism

$$\mathrm{Br}\,(K_\eta) \cong \varinjlim_{I\subset\mathcal{O}_\eta} \mathrm{Hom}(C_{I,\eta}, \mathbb{Q}/\mathbb{Z}).$$

Using the characterization of $\operatorname{Br}\mathcal{O}_\eta = \operatorname{Br} k(\eta)$ from before we obtain

$$\frac{\operatorname{Br}(K_\eta)}{\operatorname{Br} k(\eta)} \cong \varinjlim_{I\subset\mathcal{O}_\eta} \operatorname{Hom}\left(C^0_{I,\eta},\, \mathbb{Q}/\mathbb{Z}\right). \tag{1.22}$$

Comparison of kernels and cokernels. Note that

$$\mathcal{O}_x^\times = \varprojlim_{I\subset\mathcal{O}_X} (\mathcal{O}_x/I\mathcal{O}_x)^\times$$

since $\mathcal{O}_x$ is a localization of $\mathcal{O}_X$. Analogous statements hold for $\mathcal{O}_\eta, \mathcal{O}_v$. So — since

$$\bigoplus_{\eta\in Z_1}\left(\frac{\operatorname{Br} K_\eta}{\operatorname{Br} k(\eta)}\right)^* = \bigoplus_{\eta\in Z_1}\varprojlim_{I\subset\mathcal{O}_\eta} C^0_{I,\eta} = \varprojlim_{I\subset\mathcal{O}_X}\bigoplus_{\eta\in Z_1}\widetilde{\prod_{v\in S_\eta}}{}'\left(\frac{\mathcal{O}_v}{I\mathcal{O}_v}\right)^\times \Big/ \left(\frac{\mathcal{O}_\eta}{I\mathcal{O}_\eta}\right)^\times$$

by (1.22) — we may rewrite the maps (1.20) and (1.21) as inverse limits $\delta_3^* = \varprojlim_{I\subset\mathcal{O}_X} \delta^*_{3,I}$ and $\delta_3' = \varprojlim_{I\subset\mathcal{O}_X} \delta'_{3,I}$ with

$$\delta^*_{3,I} : \prod_{x\in Z_0}\left(\frac{\mathcal{O}_x}{I\mathcal{O}_x}\right)^\times \longrightarrow \bigoplus_{\eta\in Z_1}\widetilde{\prod_{v\in S_\eta}}{}'\left(\frac{\mathcal{O}_v}{I\mathcal{O}_v}\right)^\times \Big/ \left(\frac{\mathcal{O}_\eta}{I\mathcal{O}_\eta}\right)^\times$$

and

$$\delta'_{3,I} : \bigoplus_{\eta\in Z_1}\left(\frac{\mathcal{O}_\eta}{I\mathcal{O}_\eta}\right)^\times \longrightarrow \bigoplus_{x\in Z_0}\left(\bigoplus_{v\in S_x}\left(\frac{\mathcal{O}_v}{I\mathcal{O}_v}\right)^\times\right) \Big/ \left(\frac{\mathcal{O}_x}{I\mathcal{O}_x}\right)^\times.$$

$\delta^*_{3,I}$ and $\delta'_{3,I}$ have isomorphic kernels and isomorphic cokernels already. Indeed, when writing

$$A = \prod_{x\in Z_0}\bigoplus_{v\in S_x}\left(\frac{\mathcal{O}_v}{I\mathcal{O}_v}\right)^\times = \bigoplus_{\eta\in Z_1}\prod_{v\in S_\eta}\left(\frac{\mathcal{O}_v}{I\mathcal{O}_v}\right)^\times, \quad A_0 = \prod_{x\in Z_0}\left(\frac{\mathcal{O}_x}{I\mathcal{O}_x}\right)^\times, \quad A_1 = \prod_{\eta\in Z_1}\left(\frac{\mathcal{O}_\eta}{I\mathcal{O}_\eta}\right)^\times$$

we have canonical embeddings $A_0, A_1 \hookrightarrow A$ and the kernels of $\delta^*_{3,I}$, $\delta'_{3,I}$ are isomorphic to $A_0\cap A_1$. Similarly, for the direct sum analogues

$$A' = \bigoplus_{x\in Z_0}\bigoplus_{v\in S_x}\left(\frac{\mathcal{O}_v}{I\mathcal{O}_v}\right)^\times = \bigoplus_{\eta\in Z_1}\bigoplus_{v\in S_\eta}\left(\frac{\mathcal{O}_v}{I\mathcal{O}_v}\right)^\times, \quad A_0' = \bigoplus_{x\in Z_0}\left(\frac{\mathcal{O}_x}{I\mathcal{O}_x}\right)^\times, \quad A_1' = \bigoplus_{\eta\in Z_1}\left(\frac{\mathcal{O}_\eta}{I\mathcal{O}_\eta}\right)^\times$$

the definition of $\widetilde{\prod}'$ ensures that the cokernels of $\delta^*_{3,I}$, $\delta'_{3,I}$ are isomorphic to $A'/(A_0'+A_1')$. This completes the proof of Theorem (AV2l). □

1.6 Duality on the open part

Theorem 1.13. (AV2s) *Fix an integer n and a prime power p^r. Let $U\subset S$ be an open subscheme and let $f:\mathcal{U}\to U$ denote a smooth scheme over U of dimension d. There is a duality in the derived category of abelian groups*

$$R\widehat{\Gamma}_c(\mathcal{U}, \mu_{p^r}^{\otimes n})[2d+1] \simeq R\Gamma(\mathcal{U}, \mu_{p^r}^{\otimes(d-n)})^*,$$

where $(-)^* = R\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ *denotes the Pontryagin-dual. In particular, for any integer* i *we have a perfect pairing*

$$\widehat{H}^i_c(\mathcal{U}, \mu_{p^r}^{\otimes n}) \times H^{2d+1-i}(\mathcal{U}, \mu_{p^r}^{\otimes (d-n)}) \quad \longrightarrow \quad \widehat{H}^{2d+1}_c(\mathcal{U}, \mu_{p^r}^{\otimes d}) \cong \mathbb{Z}/p^r.$$

Proof. We will use the structure map $f : \mathcal{U} \to U$ to reduce the above statement to usual Artin-Verdier duality for number fields. The key ingredient will be the cohomological purity result [23] XVI. Thm. 3.7, which itself is an application of the Smooth Base Change Theorem [23] XVI. Thm. 1.1. Recall that

$$\widehat{H}^i_c(U, \mathcal{F}) \times \mathrm{Ext}^{3-i}_U(\mathcal{F}, \mathbb{G}_m) \;\longrightarrow\; H^3(U, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$$

is a perfect pairing for any constructible sheaf of abelian groups $\mathcal{F}$ over U. For sheaves $\mathcal{F}$ killed by p^r this gives the quasi-isomorphism

$$R\widehat{\Gamma}_c(U, \mathcal{F})[+3] \simeq R\,\mathscr{H}\!om_U(\mathcal{F}, \mu_{p^r})^*.$$

We choose $\mathcal{F} = Rf_*(\mu_{p^r}^{\otimes n})_{\mathcal{U}}$. This yields

$$\begin{aligned}
R\widehat{\Gamma}_c(\mathcal{U}, \mu_{p^r}^{\otimes n}) &\simeq R\widehat{\Gamma}_c(U, Rf_*(\mu_{p^r}^{\otimes n})_{\mathcal{U}}) \\
&\simeq R\,\mathscr{H}\!om_U\left(Rf_*(\mu_{p^r}^{\otimes n})_{\mathcal{U}},\, \mu_{p^r}\right)^*[-3] \\
&\overset{(V)}{\simeq} R\,\mathscr{H}\!om_{\mathcal{U}}\left(\mu_{p^r}^{\otimes n},\, Rf^!(\mu_{p^r})_U\right)^*[-3] \\
&\overset{(S)}{\simeq} R\,\mathscr{H}\!om_{\mathcal{U}}\left(\mu_{p^r}^{\otimes n},\, Rf^*\mu_{p^r}^{\otimes d}[2d-2]\right)^*[-3] \\
&\simeq R\Gamma\left(\mathcal{U}, R\underline{\mathscr{H}\!om}_{\mathcal{U}}(\mu_{p^r}^{\otimes n},\, Rf^*\mu_{p^r}^{\otimes d})\right)^*[-2d-1] \\
&\simeq R\Gamma(\mathcal{U}, \mu_{p^r}^{\otimes(d-n)})^*[-2d-1].
\end{aligned} \tag{1.23}$$

For (V) we used the Verdier duality adjunction $Rf_* \vdash Rf^!$ (see also Theorem 2.3). (S) follows from Smooth Base Change applied to $f : \mathcal{U} \to U$. Note that this quasi-isomorphism is why we require $\mathcal{U}$ to be smooth. □

1.7 Proof of the global duality statement

We first rewrite (AV2l) in terms of p-torsion sheaves. We will then combine it with (AV2s) to prove global Artin-Verdier duality.

Proposition 1.14. *Keep the notation from Theorem (AV2l) and let* p^r *be a prime power. For each integer* i *there is a perfect pairing*

$$H^i_Z(\mathcal{X}, \mathbb{Z}(1)^{\mathcal{X}}/p^r) \times H^{5-i}(Z, i^*\mathbb{Z}(1)^{\mathcal{X}}/p^r) \;\longrightarrow\; H^5_Z(\mathcal{X}, \mathbb{Z}(2)^{\mathcal{X}}/p^r) \longrightarrow \mathbb{Q}/\mathbb{Z}. \tag{1.24}$$

Proof. Applying $Ri^!$ to the product map of p^r-mapping cones (1.9) gives

$$Ri^!\mathbb{Z}(1)^{\mathcal{X}}/p^r \otimes^{\mathbb{L}} i^*\mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow Ri^!\mathbb{Z}(2)^{\mathcal{X}}/p^r$$

which induces the pairing (1.24). We show that it is perfect by means of a Five Lemma argument. Consider the long exact sequence associated to $\mathbb{Z}(n)^{\mathcal{X}} \to \mathbb{Z}(n)^{\mathcal{X}} \to \mathbb{Z}(n)^{\mathcal{X}}/p^r \to$ as well as its dual under $(-)^* = R\mathrm{Hom}(-, \mathbb{Z}/p^r)$. These — together with the product maps (1.14) and (1.24) — fit into the commutative diagram

$$\begin{array}{ccccccccc}
H^i_Z(\mathcal{X},\mathbb{Z}(1)^{\mathcal{X}}) & \longrightarrow & H^i_Z(\mathcal{X},\mathbb{Z}(1)^{\mathcal{X}}) & \longrightarrow & H^i_Z(\mathcal{X},\mathbb{Z}(1)^{\mathcal{X}}/p^r) & \longrightarrow & H^{i+1}_Z(\mathcal{X},\mathbb{Z}(1)^{\mathcal{X}}) & \longrightarrow & H^{i+1}_Z(\mathcal{X},\mathbb{Z}(1)^{\mathcal{X}}) \\
\cong\downarrow{\scriptstyle (1.14)} & & \cong\downarrow{\scriptstyle (1.14)} & & \downarrow{\scriptstyle (1.24)} & & \cong\downarrow{\scriptstyle (1.14)} & & \cong\downarrow{\scriptstyle (1.14)} \\
H^{6-i}(Z,i^*\mathbb{Z}(1)^{\mathcal{X}})^* & \longrightarrow & H^{6-i}(Z,i^*\mathbb{Z}(1)^{\mathcal{X}})^* & \longrightarrow & H^{5-i}(Z,i^*\mathbb{Z}(1)^{\mathcal{X}}/p^r)^* & \longrightarrow & H^{5-i}(Z,i^*\mathbb{Z}(1)^{\mathcal{X}})^* & \longrightarrow & H^{5-i}(Z,i^*\mathbb{Z}(1)^{\mathcal{X}})^*
\end{array}$$

The outer vertical being isomorphisms is precisely the local duality result (AV2l). So, by the Five Lemma, the middle arrow represents a perfect pairing too. □

Proof of (AV2). Let $\mathcal{Z}$ be the collection of all special fibers of $\mathcal{X}$ that are singular or above p, i.e. $\mathcal{Z} = \coprod_{\mathfrak{p}\in S_{\mathrm{bad}}} Z_{\mathfrak{p}}$ where B is the (finite) union of all primes of $\mathcal{O}$ where $\mathcal{X}$ has bad reduction with all primes above p, and where $Z_{\mathfrak{p}} = \mathcal{X} \times \operatorname{Spec} k(\mathfrak{p})$. Write $\mathcal{U} = \mathcal{X}\backslash\mathcal{Z}$. The Open-Closed-Decomposition $\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{X} \overset{j}{\hookleftarrow} \mathcal{U}$ gives rise to the distinguished triangles

$$j_!\mu_{p^r} \longrightarrow \mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow i_*i^*\mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow \tag{1.25}$$

$$i_*Ri^!\mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow \mathbb{Z}(1)^{\mathcal{X}}/p^r \longrightarrow Rj_*\mu_{p^r} \longrightarrow \tag{1.26}$$

Theorem (AV2s) gives us a perfect pairing on the smooth part

$$\widehat{H}^i_c(\mathcal{U}, \mu_{p^r}) \times H^{5-i}(\mathcal{U}, \mu_{p^r}) \longrightarrow \widehat{H}^5_c(\mathcal{U}, \mu_{p^r}^{\otimes 2}) \cong \mathbb{Z}/p^r. \tag{1.27}$$

Consider the long exact sequence on compact support cohomology of the triangle (1.25) as well as the long exact sequence of (1.26). They fit into one commutative diagram as follows:

$$\begin{array}{ccccccccc}
H^{i-1}(\mathcal{Z},i^*\mathbb{Z}(1)^{\mathcal{X}}/p^r) & \longrightarrow & \widehat{H}^i_c(\mathcal{U},\mu_{p^r}) & \longrightarrow & \widehat{H}^i_c(\mathcal{X},\mathbb{Z}(1)^{\mathcal{X}}/p^r) & \longrightarrow & H^i(\mathcal{Z},i^*\mathbb{Z}(1)^{\mathcal{X}}/p^r) & \longrightarrow & \widehat{H}^{i+1}_c(\mathcal{U},\mu_{p^r}) \\
\downarrow{\scriptstyle (1.24)} & & \cong\downarrow{\scriptstyle (1.27)} & & \downarrow{\scriptstyle (1.4)} & & \downarrow{\scriptstyle (1.24)} & & \cong\downarrow{\scriptstyle (1.27)} \\
H^{6-i}_{\mathcal{Z}}(\mathcal{X},\mathbb{Z}(1)^{\mathcal{X}}/p^r)^* & \longrightarrow & H^{5-i}(\mathcal{U},\mu_{p^r})^* & \longrightarrow & H^{5-i}(\mathcal{X},\mathbb{Z}(1)^{\mathcal{X}}/p^r)^* & \longrightarrow & H^{5-i}_{\mathcal{Z}}(\mathcal{X},\mathbb{Z}(1)^{\mathcal{X}}/p^r)^* & \longrightarrow & H^{4-i}(\mathcal{U},\mu_{p^r})^*
\end{array}$$

The vertical arrows are induced by the pairings (1.24), (1.27), and (1.4) from (AV2) as indicated. We explore the map

$$H^i(\mathcal{Z}, i^*\mathbb{Z}(1)^{\mathcal{X}}/p^r) \longrightarrow H^{5-i}_{\mathcal{Z}}(\mathcal{X}, \mathbb{Z}(1)^{\mathcal{X}}/p^r)^* \tag{1.28}$$

in more detail. Write $\mathcal{X}_B = \coprod_{\mathfrak{p}} X_{\mathfrak{p}}$ where $X_{\mathfrak{p}} = X \times \operatorname{Spec} \mathcal{O}_{\mathfrak{p}}$. One may regard i as an embedding of $\mathcal{Z}$ into $\mathcal{X}_B$. The Proper Base Change Theorem then gives $H^i(\mathcal{Z}, i^*\mathbb{Z}(1)^{\mathcal{X}}/p^r) = H^i(\mathcal{X}_B, \mathbb{Z}(1)^{\mathcal{X}_B}/p^r)$ and we may write (1.28) as

$$H^i(\mathcal{X}_B, \mathbb{Z}(1)^{\mathcal{X}_B}/p^r) \longrightarrow H^{5-i}_{\mathcal{Z}}(\mathcal{X}_B, \mathbb{Z}(1)^{\mathcal{X}_B}/p^r)^*.$$

So, this map is in fact the direct sum of (1.24) for each $Z_{\mathfrak{p}} \hookrightarrow X_{\mathfrak{p}}$ with $\mathfrak{p} \in B$ and hence an isomorphism. By the Five Lemma, the middle vertical arrow must be an isomorphism too, i.e. (1.4) is a perfect pairing.

Chapter 2

Motivic decompositions of cohomology for arithmetic surfaces

Let $\mathcal{X}$ denote a regular arithmetic surface with proper structure map $\pi' : \mathcal{X} \to \operatorname{Spec}\mathbb{Z}$. Let $\mathcal{O}$ be the maximum number ring π' factors through. Unless explicitly stated otherwise, we write $S = \operatorname{Spec}\mathcal{O}$ and let $\pi : \mathcal{X} \to S$ denote the corresponding map into S. Given a point $x \in \mathcal{X}$ or $\mathfrak{p} \in S$ we write $\kappa(x)$ and $\kappa(\mathfrak{p})$ for their residue fields. We will write $\prod_{\mathfrak{p}}$ to denote an infinite product over all finite primes of $\mathcal{O}$.

Write F for the fraction field of $\mathcal{O}$. Let r and s be the number of real and complex embeddings of F respectively and write $m = r + 2s$ for its dimension over $\mathbb{Q}$. If not explicitly stated otherwise, X will denote the generic fiber $\mathcal{X}_F$ of $\mathcal{X}$. Let g be the genus of X. Also, let $n \in \mathbb{Z}$ and let ϵ_n be 0 or 1 depending on whether n is even or odd.

In this chapter we will define and partially compute the cohomology groups occuring in the conjectures [8] Conj. 5.10, 5.11 for $\mathcal{X}$. The underlying theme will be that all cohomology groups will decompose into a direct sum of h^i-parts for $i = 0, 1, 2$ — analogously to a decomposition of

$$\zeta(\mathcal{X}, s) = \prod_{i=0,1,2} {}^{p}L(H^i(\mathcal{X}), s)^{(-1)^i} = \frac{\zeta_F(s)\zeta_F(s-1)}{{}^{p}L(H^1(\mathcal{X}), s)}$$

into an alternating product of adjusted L-functions ${}^{p}L(H^i(\mathcal{X}), s)$ for $i = 0, 1, 2$. We expect this to follow in generality from the existence of a direct sum decomposition of $R\pi_*\mathbb{Z}(n)^{\overline{\mathcal{X}}}$ into *perverse* degree components ${}^{p}R^i\pi_*\mathbb{Z}(n)^{\overline{\mathcal{X}}}$:

$$R\pi_*\mathbb{Z}(n)^{\overline{\mathcal{X}}} \simeq \bigoplus_{i=0,1,2} {}^{p}R^i\pi_*\mathbb{Z}(n)^{\overline{\mathcal{X}}}[-i] \simeq \mathbb{Z}(n)^{\overline{S}} \oplus {}^{p}R^1\pi_*\mathbb{Z}(n)^{\overline{\mathcal{X}}}[-1] \oplus \mathbb{Z}(n-1)^{\overline{S}}[-2].$$

We will prove such a decomposition under the assumption that there is a section $s : S \to \mathcal{X}$ of π satisfying a further technical conjecture for higher twists $n \geqslant 2$.

2.1 L- and ζ-functions associated to arithmetic surfaces

In this section we relate $\zeta(\mathcal{X}, s)$ to the Hasse-Weil ζ-function $\zeta_{HW}(X, s)$ and review the decomposition of $\zeta(\mathcal{X}, s)$ into its 0-, 1-, and 2-part.

For any prime $\mathfrak{p}$ of $\mathcal{O}$ let $l = l_\mathfrak{p}$ denote a rational prime not divisible by $\mathfrak{p}$. Moreover, write $\mathcal{X}_{\overline{\mathfrak{p}}} := \mathcal{X}_{\overline{\kappa(\mathfrak{p})}}$. The arithmetic ζ-function associated to any proper arithmetic surface $\mathcal{X}$ is defined as

$$\begin{aligned}\zeta(\mathcal{X}, s) &= \prod_{x \in \mathcal{X}_0} \frac{1}{1 - \#\kappa(x)^{-s}} = \prod_{\mathfrak{p}} \zeta\left(\mathcal{X}_\mathfrak{p}/\kappa(\mathfrak{p}), s\right)^{-1} \\ &= \prod_{\mathfrak{p}} \prod_{m \in \mathbb{Z}} \det\left(\mathrm{id} - N\mathfrak{p}^{-s}\,\mathrm{Frob}_\mathfrak{p} \mid H^m(\mathcal{X}_{\overline{\mathfrak{p}}}, \mathbb{Q}_l\,)\right)^{(-1)^{m+1}}.\end{aligned}$$

The last equality above is a consequence of the Weil conjectures.

The Hasse-Weil ζ-function on the other hand is an object associated to the generic fiber $X = \mathcal{X}_F$ and is thus independent of the integral model $\mathcal{X}$ of X. $\zeta_{\mathrm{HW}}(X, s)$ is defined as an alternating product of Hasse-Weil L-functions which in turn only depend on the étale cohomology groups $H^m(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)$. Concretely, one defines

$$\begin{aligned}\zeta_{\mathrm{HW}}(X, s) &:= \prod_{m \in \mathbb{Z}} L(H^m(X), s)^{(-1)^{m+1}} \\ &:= \prod_{m \in \mathbb{Z}} \prod_{\mathfrak{p}} \det\left(\mathrm{id} - N\mathfrak{p}^{-s}\,\mathrm{Frob}_\mathfrak{p} \mid H^m(X \otimes_F \overline{F}, \mathbb{Q}_l\,)^{I_{\overline{\mathfrak{p}}}}\right)^{(-1)^{m+1}},\end{aligned} \tag{2.1}$$

where $I_{\overline{\mathfrak{p}}} \subset \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ denotes the inertia group of some prime $\overline{\mathfrak{p}}$ of $\overline{\mathbb{Q}}$ lying over $\mathfrak{p}$.

Note that ζ and ζ_{HW} should be thought of as associated to the scheme $\mathcal{X}$ or, equivalently, to $\mathcal{X}$ regarded as an arithmetic surface over $\mathbb{Z}$ (or its generic fiber X regarded as a curve over $\mathbb{Q}$), and *not* to arithmetic surfaces over a general number ring. However, we will make frequent use of the map $\pi : \mathcal{X} \to S$ since many objects associated to $\mathcal{X}$ such as its motivic cohomology groups can be best expressed in terms of S.

One knows that $H^m(\mathcal{X}_{\overline{\mathfrak{p}}}, \mathbb{Q}_l) \cong H^m(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)^{I_{\overline{\mathfrak{p}}}}$ for good reduction primes $\mathfrak{p}$. So, the difference between $\zeta(\mathcal{X}, s)$ and $\zeta_{HW}(X, s)$ lies in the bad reduction fibers of $\mathcal{X}$ only. Bloch worked out explicitly the difference between the above étale cohomology groups and arrived at the following result (cf. [3] Lem. 1.2 and comments).

Proposition/Definition 2.1. *Let $M_\mathfrak{p}$ denote the $\mathbb{Q}_l$-vectorspace freely generated by the irreducible components of $\mathcal{X}_{\overline{\mathfrak{p}}}$. One has $H^i(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)^{I_{\overline{\mathfrak{p}}}} = H^i(\mathcal{X}_{\overline{\mathfrak{p}}}, \mathbb{Q}_l)$ for $i = 0, 1$. For $i = 2$ there is an exact sequence of $\widehat{\mathbb{Z}}$-modules*

$$0 \longrightarrow M_\mathfrak{p}/\mathbb{Q}_l(-1) \longrightarrow H^2(\mathcal{X}_{\overline{\mathfrak{p}}}, \mathbb{Q}_l) \longrightarrow H^2(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)^{I_{\overline{\mathfrak{p}}}} \longrightarrow 0$$

and, moreover, $H^2(\mathcal{X}_{\overline{\mathfrak{p}}}, \mathbb{Q}_l) \cong M_{\mathfrak{p}}^*(-1)$. *We define*

$$\Pi(\mathcal{X}, s) := \prod_{\mathfrak{p}\ \text{bad}} \det\left(\mathrm{id} - N\mathfrak{p}^{-s}\,\mathrm{Frob}_{\mathfrak{p}} \mid M_{\mathfrak{p}}/\mathbb{Q}_l(-1)\right).$$

The short exact sequence shows

$$\zeta(\mathcal{X}, s) = \zeta_{\mathrm{HW}}(X, s) \cdot \Pi(\mathcal{X}, s)^{-1}. \tag{2.2}$$

Since $H^0(\mathcal{X}_{\overline{\mathfrak{p}}}, \mathbb{Q}_l) = \mathbb{Q}_l$ and $H^2(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_l) = \mathbb{Q}_l(-1)$ we may simplify (2.1) to

$$\zeta(\mathcal{X}, s) = \frac{\zeta_F(s)\zeta_F(s-1)}{\Pi(\mathcal{X}, s) L(H^1(X), s)} =: \frac{\zeta_F(s)\zeta_F(s-1)}{{}^{p}L(H^1(\mathcal{X}), s)}. \tag{2.3}$$

For future reference, we analyze the special values of $\Pi(\mathcal{X}, s)$.

Lemma 2.2. *For any prime* $\mathfrak{p}$ *of* F *write* $d(\mathfrak{p}) = \dim_{\mathbb{Q}_l} H^2(\mathcal{X}_{\mathfrak{p}}, \mathbb{Q}_l)$ *for the number of irreducible components* $C_1, \ldots, C_{d(\mathfrak{p})}$ *of* $\mathcal{X}_{\mathfrak{p}}$. *Further, for each* $1 \leqslant j \leqslant d(\mathfrak{p})$ *let* $n_j = n_j(\mathfrak{p})$ *be the number of irreducible components* C_j *decomposes into in* $\mathcal{X}_{\overline{\mathfrak{p}}}$.

(i) For any integer $n \neq 1$ *one has*

$$\Pi^*(\mathcal{X}, n) = \Pi(\mathcal{X}, n) = \pm \prod_{\mathfrak{p}\ \text{bad}} \left(1 - \frac{1}{N\mathfrak{p}^{n-1}}\right)^{-1} \prod_{j=1}^{d(\mathfrak{p})} \left(1 - \frac{1}{N\mathfrak{p}^{(n-1)n_j(\mathfrak{p})}}\right).$$

(ii) For $n = 1$ *one has*

$$\mathrm{ord}_{s=1}\Pi(\mathcal{X}, s) = \sum_{\mathfrak{p}\ \text{bad}} (d(\mathfrak{p}) - 1) \tag{2.4}$$

and the leading Taylor coefficient $\Pi^*(\mathcal{X}, 1)$ *equals*

$$\Pi^*(\mathcal{X}, 1) = \pm \prod_{\mathfrak{p}\ \text{bad}} (\log N\mathfrak{p})^{d(\mathfrak{p})-1} \prod_{j=1}^{d(\mathfrak{p})} n_j(\mathfrak{p}).$$

Proof. Fix a bad prime $\mathfrak{p}$ and write $d = d(\mathfrak{p})$ and $n_j = n_j(\mathfrak{p})$. Let $M_{\mathfrak{p}}^j$ denote the subspace of $M_{\mathfrak{p}}$ generated by the irreducible components of C_j in $\mathcal{X}_{\overline{\mathfrak{p}}}$. Write $P_j^{\mathfrak{p}}$ and $P^{\mathfrak{p}}$ for the characteristic polynomials for the action of $\mathrm{Frob}_{\mathfrak{p}}$ on $M_{\mathfrak{p}}^j$ and $M_{\mathfrak{p}}/\mathbb{Q}_l$ respectively. $\mathrm{Frob}_{\mathfrak{p}}^i$ cyclically permutes the components generating each $M_{\mathfrak{p}}^j$ since the $\mathrm{Frob}_{\mathfrak{p}}$-orbit of any irreducible component of C_j in $\mathcal{X}_{\overline{\mathfrak{p}}}$ must be defined over $\kappa(\mathfrak{p})$ by Galois-descent, and hence equal C_j. Therefore

$$P^{\mathfrak{p}}(T) = \frac{1}{T-1} \prod_{j=1}^{d} P_j^{\mathfrak{p}}(T) = \pm \frac{1}{T-1} \prod_{j=1}^{d} \left(T^{n_j(\mathfrak{p})} - 1\right).$$

Write $N = \dim_{\mathbb{Q}_l} H^2(\mathcal{X}_{\overline{\mathfrak{p}}}, \mathbb{Q}_l) = \sum_j n_j$. Let $s \in \mathbb{C}$ and write $x = x_{\mathfrak{p}} = N\mathfrak{p}^{s-1}$. One has

$$\det\left(1 - N\mathfrak{p}^{-s+1}\mathrm{Frob}_{\mathfrak{p}} \,\middle|\, M_{\mathfrak{p}}/\mathbb{Q}_l\right) = \frac{1}{x^{N-1}} P^{\mathfrak{p}}(x) = \pm \frac{x}{x-1} \prod_{j=1}^{d} \frac{x^{n_j(\mathfrak{p})} - 1}{x^{n_j(\mathfrak{p})}}$$
$$= \pm \left(1 - \frac{1}{x}\right)^{-1} \prod_{j=1}^{d} \left(1 - \frac{1}{x^{n_j(\mathfrak{p})}}\right).$$

Choosing $s = n$ for $n \neq 1$ proves part (i) and evaluating the vanishing order of the above at $s = 1$ yields (2.4). Now let $y_{\mathfrak{p}} = -\log N\mathfrak{p}$. Then $\frac{1}{x_{\mathfrak{p}}} = e^{y_{\mathfrak{p}}(s-1)}$ and

$$\Pi^*(\mathcal{X}, 1) = \pm \lim_{s \to 1} \prod_{\mathfrak{p} \text{ bad}} \left(\frac{1 - e^{y_{\mathfrak{p}}(s-1)}}{s-1}\right)^{-1} \prod_{j=1}^{d(\mathfrak{p})} \frac{1 - e^{y_{\mathfrak{p}} n_j(\mathfrak{p})(s-1)}}{s-1} = \pm \prod_{\mathfrak{p} \text{ bad}} \frac{1}{y_{\mathfrak{p}}} \prod_{j=1}^{d(\mathfrak{p})} y_{\mathfrak{p}} n_j(\mathfrak{p}).$$

This finishes the proof of part (ii). □

2.2 Motivic decompositions of push-forward sheaves in the presence of a section

In this section we will write $\pi : \mathscr{X} \to \mathcal{S}$ for a variety of structure maps, as specified in the proceeding paragraphs. We will derive direct sum decompositions of the kind

$$R\pi_* \mathscr{F}(n)^{\mathscr{X}} \;\simeq\; \mathscr{F}(n)^{\mathcal{S}} \oplus {}^{p}R^1\pi_*\mathscr{F}(n)^{\mathscr{X}}[-1] \oplus \mathscr{F}(n-1)^{\mathcal{S}}[-2],$$

where $\mathscr{F}$ represents locally constant sheaves with Galois twist or motivic cycle complexes on the étale sites of $\mathscr{X}$ and $\mathcal{S}$ respectively. These results will be motivated by the theory of motives and hence referred to as *motivic decompositions*. We will also write $R\pi_*\mathscr{F}(n)^{\mathscr{X}} \simeq \bigoplus_{i=0,1,2} {}^{p}R^i\pi_*\mathscr{F}(n)^{\mathscr{X}}[-i]$. The notation ${}^{p}R$ should indicate the expectation that those decompositions are reflections of a broader, yet to be developed theoretical framework which endows derived categories of motivic sheaves with *perverse t-structures*.

The proof strategies are centered on an application of Verdier duality. It will render the remaining part of the proof an exercise in the six functor formalism for π and s. We will also provide more explicit descriptions of the involved projection and inclusion morphisms when possible and elaborate on the difference between ${}^{p}R^i$ and R^i.

2.2.1 Verdier Duality and and Cohomological Purity

Verdier Duality. For any scheme X let $\mathcal{D}(X_{\text{ét}})$ denote the derived category of abelian *torsion* sheaves on $X_{\text{ét}}$. We recall the sheaf theoretic generalization of Poincaré Duality.

Theorem 2.3. (Verdier Duality, [23] Exp. XVIII, Thm 3.1.4) *Let $f : X \to Y$ be a separated, quasi-compact morphism of schemes. There is a functor $f^! : \mathcal{D}(Y_{\text{ét}}) \to \mathcal{D}(X_{\text{ét}})$ such that for all torsion sheaves $\mathcal{F}$ on $X_{\text{ét}}$ and $\mathcal{G}$ on $Y_{\text{ét}}$ one has the quasi-isomorphism*

$$Rf_* R\mathscr{H}om_X(\mathcal{F}, f^!\mathcal{G}) \simeq R\mathscr{H}om_Y(Rf_!\mathcal{F}, \mathcal{G}). \tag{2.5}$$

In other words, there is an adjunction of functors $Rf_! \vdash f^!$ between the derived categories of torsion sheaves on $X_{\text{ét}}$ and $Y_{\text{ét}}$.

$f^!$ generalizes the derived exceptional inverse image functor associated to closed immersions. However, in general $f^!$ cannot be expressed as the derived functor of any functor of sheaves. The counit tr_f of the adjunction $Rf_! \vdash f^!$ is called the *trace map*. The first part of the proof of Theorem 2.3 is the construction of the trace map. It is then used to define the map (2.5) and one subsequently shows that it is in fact a quasi-isomorphism.

If f is smooth of relative dimension d (i.e. $\dim X = \dim Y + d$), one knows that $Rf_! f^! \mathcal{F}$ is cohomologically concentrated in degrees $[0, 2d]$ for any torsion sheaf $\mathcal{F}$ on $X_{\text{ét}}$ and $\mathrm{tr}_f(\mathcal{F})$ factors through the top degree:

$$\mathrm{tr}_f(\mathcal{F}) : \; Rf_! f^! \mathcal{F} \longrightarrow R^{2d} f_! f^! \mathcal{F} \xrightarrow{\mathrm{Tr}_f} \mathcal{F}. \tag{2.6}$$

Cohomological Purity. For the rest of this section fix a prime p and let Λ denote a finite p-torsion group. For any scheme X we write Λ^X for the constant torsion sheaf with global sections equal to Λ. For any p-torsion sheaf $\mathcal{F}$ we write $\mathcal{F}(n) = \mathcal{F} \otimes \mu_{p^\infty}^{\otimes n}$. We will need the following result that had originally been conjectured by Grothendieck and that was proved in full generality by Gabber (cf. [10]).

Theorem 2.4. (Cohomological Purity) *Let $i : Z \hookrightarrow U$ be a closed immersion of regular noetherian schemes of pure codimension d. Suppose p is invertible on U. Then*

$$R^r i^! \Lambda^U \cong \begin{cases} \Lambda^Z(-d) & \text{if } r = 2d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

This result admits an extension to the functor $f^!$ between derived categories.

Proposition 2.5. *Let $f : X \to Y$ be a smooth morphism of relative dimension d. Also, let $\mathcal{F}$ be a complex of p-torsion sheaves on $X_{\text{ét}}$ and suppose p is invertible on Y. Then*

$$f^! \mathcal{F} \simeq f^* \mathcal{F}(d)[2d].$$

Let $\pi : \mathscr{X} \to \mathcal{S}$ be a projective morphism of schemes with $\mathscr{X}$ being regular and of relative dimension d over $\mathcal{S}$. Note that this includes the later most relevant case where $d = 1$ and

π is any proper morphism since projectiveness follows then from a result by Lichtenbaum (cf. [19] Thm. 2.8). We assume throughout that p is not zero in $\mathcal{O}_{\mathcal{S}}$. We then have open closed decompositions as in the diagram

$$\begin{array}{ccccc} \mathscr{X}[1/p] & \xrightarrow{j_p} & \mathscr{X} & \xleftarrow{i_p} & \mathscr{X}_p \\ \downarrow{\scriptstyle \pi[1/p]} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_p} \\ \mathcal{S}[1/p] & \xrightarrow{j'_p} & \mathcal{S} & \xleftarrow{i'_p} & \mathcal{S}_p \end{array} \tag{2.8}$$

Lemma 2.6. *In the derived category of torsion sheaves on $\mathcal{S}[1/p]_{\text{ét}}$, one has*

$$\pi[1/p]^! \Lambda^{\mathcal{S}[1/p]} \simeq \Lambda^{\mathscr{X}[1/p]}(d)[2d].$$

Proof. $\pi[1/p]$ is projective and hence factors through some $\mathbb{P}^N_{\mathcal{S}[1/p]}$ as

$$\begin{array}{ccc} \mathscr{X}[1/p] & \xrightarrow{i} & \mathbb{P}^N_{\mathcal{S}[1/p]} \\ & \searrow^{\pi} & \downarrow{\scriptstyle \Pi} \\ & & \mathcal{S}[1/p] \end{array}$$

where Π is smooth and i is a closed embedding of regular schemes. So, by Theorem 2.4 and Proposition 2.5,

$$\pi[1/p]^! \Lambda^{\mathcal{S}[1/p]} = Ri^! \Pi^! \Lambda^{\mathcal{S}[1/p]} \simeq Ri^! \Lambda^{\mathbb{P}^N_{\mathcal{S}[1/p]}}(N)[2N] \simeq \Lambda^{\mathscr{X}[1/p]}(d)[2d]. \qquad \square$$

Corollary 2.7. *Write $(-)^\vee = R\mathscr{H}\!om(-, \mathbb{Q}_p/\mathbb{Z}_p)$. For any p-torsion sheaf $\mathcal{F}$ on $\mathscr{X}[1/p]_{\text{ét}}$ one has*

$$(R\pi[1/p]_* \mathcal{F})^\vee \simeq R\pi[1/p]_* \mathcal{F}^\vee(d)[2d]. \tag{2.9}$$

Proof. This is Verdier duality for $f = \pi[1/p]$ and $\mathcal{G} = \Lambda^{\mathcal{S}[1/p]}$. Here we have used $\pi[1/p]_! = \pi[1/p]_*$ which holds since $\pi[1/p]$ is proper. $\square$

2.2.2 A motivic decomposition for push-forwards of constant torsion sheaves

To ease notation we write $\Lambda^{\mathscr{X},p} = j_{p,!}\Lambda^{\mathscr{X}[1/p]}$ and $\Lambda^{\mathcal{S},p} = j'_{p,!}\Lambda^{\mathcal{S}[1/p]}$. Note that if p is invertible on $\mathcal{S}$ then one gets the constant sheaves $\Lambda^{\mathscr{X},p} = \Lambda^{\mathscr{X}}$ and $\Lambda^{\mathcal{S},p} = \Lambda^{\mathcal{S}}$ back.

From now on, we assume $\mathcal{S}$ to be a connected at most one-dimensional regular scheme such that all closed points have perfect residue fields and such that all remaining points have residue fields of characteristic 0. Although we will need the result below only for $d = 1$ we formulate it for general d as it will make no difference for the proof.

Theorem 2.8. *Suppose $\pi : \mathscr{X} \to \mathcal{S}$ has a section $s : \mathcal{S} \to \mathscr{X}$. Then the sheaves $\Lambda^{\mathcal{S},p}(n)$ and $\Lambda^{\mathcal{S},p}(n-d)[-2d]$ split off as direct summands of the complex $R\pi_*\Lambda^{\mathscr{X},p}(n)$. If $d = 1$ we will write ${}^pR^1\pi_*\Lambda_p^{\mathscr{X}}(n)[-1]$ for the remaining summand, i.e. we will have the canonical decomposition*

$$R\pi_*\Lambda^{\mathscr{X},p}(n) \;\simeq\; \Lambda^{\mathcal{S},p}(n) \oplus {}^pR^1\pi_*\Lambda^{\mathscr{X},p}(n)[-1] \oplus \Lambda^{\mathcal{S},p}(n-1)[-2]. \qquad (2.10)$$

Proof. It is enough to prove the claim for the restriction $\pi[1/p] : \mathscr{X}[1/p] \to \mathcal{S}[1/p]$ to the open part, i.e. to show that $\Lambda^{\mathcal{S}[1/p]}(n)$ and $\Lambda^{\mathcal{S}[1/p]}(n-d)[-2d]$ split off as direct summands of $R\pi[1/p]_*\Lambda^{\mathscr{X}[1/p]}(n)$. In fact, an application of $j'_{p,!}$ will then reproduce the original claim since

$$j'_{p,!}R\pi[1/p]_*\Lambda^{\mathscr{X}[1/p]}(n) = R(j'_p\pi[1/p])_!\Lambda^{\mathscr{X}[1/p]}(n) = R\pi_!j_{p,!}\Lambda^{\mathscr{X}[1/p]}(n) = R\pi_*\Lambda^{\mathscr{X},p}(n).$$

Therefore we may assume that p is invertible on $\mathcal{S}$.

We will use the adjunctions $\pi^* \vdash \pi_*$ and $s^* \vdash s_* \vdash s^!$ to construct maps

$$\Lambda^{\mathcal{S}}(n) \xrightarrow{\varphi_0} R\pi_*\Lambda^{\mathscr{X}}(n) \xrightarrow{\psi_0} \Lambda^{\mathcal{S}}(n),$$
$$\Lambda^{\mathcal{S}}(n-d)[-2d] \xrightarrow{\varphi_{2d}} R\pi_*\Lambda^{\mathscr{X}}(n) \xrightarrow{\psi_{2d}} \Lambda^{\mathcal{S}}(n-d)[-2d] \qquad (2.11)$$

that compose as

$$\begin{pmatrix} \psi_0 \\ \psi_{2d} \end{pmatrix} \circ \begin{pmatrix} \varphi_0 & \varphi_{2d} \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix},$$

thereby proving the proposition.

The existence of φ_0, ψ_0 is a formal consequence of functoriality of the push-forward $\pi_*s_* = (\pi s)_* = \mathrm{id}$ and exactness of π^*, s^*, s_*. Using $s^*\Lambda^{\mathscr{X}} = \Lambda^{\mathcal{S}}$ and $\pi^*\Lambda^{\mathcal{S}} = \Lambda^{\mathscr{X}}$ we define

$$\varphi_0 : \Lambda^{\mathcal{S}}(n) \longrightarrow R\pi_*\pi^*\Lambda^{\mathcal{S}}(n) = R\pi_*\Lambda^{\mathscr{X}}(n),$$
$$\psi_0 : R\pi_*\Lambda^{\mathscr{X}}(n) \to R\pi_*s_*s^*\Lambda^{\mathscr{X}}(n) = s^*\Lambda^{\mathscr{X}}(n) = \Lambda^{\mathcal{S}}(n).$$

Consider the shift of the above maps for an $(d-n)$-twist by $2d$ degrees:

$$\Lambda^{\mathcal{S}}(d-n)[2d] \xrightarrow{\varphi_0[2d]} R\pi_*\Lambda^{\mathscr{X}}(d-n)[2d] \xrightarrow{\psi_0[2d]} \Lambda^{\mathcal{S}}(d-n)[2d].$$

Apply $(-)^\vee = R\mathscr{H}om_{\mathcal{S}}(-, \mathbb{Q}_p/\mathbb{Z}_p)$. Corollary 2.7 shows that one obtains

$$\Lambda^{\mathcal{S}}(n-d)[-2d] \xleftarrow{\varphi_0^\vee[-2d]} R\pi_*\Lambda^{\mathscr{X}}(n) \xleftarrow{\psi_0^\vee[-2d]} \Lambda^{\mathcal{S}}(n-d)[-2d].$$

We let $\varphi_{2d} = \psi_0^\vee[-2d]$ and $\psi_{2d} = \varphi_0^\vee[-2d]$.

$\psi_0\varphi_0 = \mathrm{id}$ is clear since both maps are adjoints of identity maps. Therefore, also $\psi_{2d}\varphi_{2d} = (\psi_0\varphi_0)^\vee[-2d] = 0$. Next, one has $\psi_{2d}\varphi_0 = 0$ for degree reasons. More precisely, φ_0 is the inclusion of the lowest degree $\Lambda^{\mathcal{S}}(n) = R^0\pi_*\Lambda^{\mathscr{X}}(n)$ into $R\pi_*\Lambda^{\mathscr{X}}(n)$ while ψ_{2d} factors through the top degree $\tau^{\geqslant 2d}R\pi_*\Lambda^{\mathscr{X}}(n) \cong R^{2d}\pi_*\Lambda^{\mathscr{X}}(n)$. Similarly, we must have $\psi_0\varphi_{2d} = 0$ since φ_{2d} maps $\Lambda^{\mathcal{S}}(n-d)[-2d]$ into the top degree of $R\pi_*\Lambda^{\mathscr{X}}(n)$ while ψ_0 is trivial in degrees > 0. This completes the proof. □

Remark 2.9. The construction of φ_0, ψ_0 did not require $\Lambda^{\mathscr{X}}$ to be torsion or p to be invertible on $\mathcal{S}$. So, the above proof more generally shows that $A^{\mathcal{S}}$ splits off as a direct summand of $R\pi_*A^{\mathscr{X}}$ for the constant sheaf $A^{\mathscr{X}}$ associated to any abelian group A. Since both $\mathscr{X}$ and $\mathcal{S}$ are connected one has $\pi_*A^{\mathscr{X}} = A^{\mathcal{S}}$ and consequently

$$R\pi_*A^{\mathscr{X}} \simeq A^{\mathcal{S}} \oplus \tau^{\geqslant 1}R\pi_*A^{\mathscr{X}}.$$

Remark 2.10. If $\pi : \mathscr{X} \to \mathcal{S}$ is smooth and proper, and $\mathcal{S}$ the spectrum of a field of characteristic unequal to p the above decomposition is well-known and the motivic components ${}^pR^i\pi_*\Lambda^{\mathscr{X}}(n)$ coincide with the cohomological components $R^i\pi_*\Lambda^{\mathscr{X}}(n)$ for degrees $i = 0, 2d$. In particular, if $d = 1$, they are identical for all degrees, i.e. one then has

$$R\pi_*\Lambda^{\mathscr{X}}(n) \;\simeq\; R^0\pi_*\Lambda^{\mathscr{X}}(n) \oplus R^1\pi_*\Lambda^{\mathscr{X}}(n)[-1] \oplus R^2\pi_*\Lambda^{\mathscr{X}}(n)[-2].$$

This also follows from direct computations when observing that (2.11) may be rewritten as

$$\Lambda^{\mathcal{S}}(n-d)[-2d] \xrightarrow{\varphi'_{2d}} R^{2d}\pi_*\Lambda^{\mathscr{X}}(n) \xrightarrow{\mathrm{Tr}_\pi} \Lambda^{\mathcal{S}}(n-d)[-2d], \tag{2.12}$$

where Tr_π is the trace map from Poincaré duality for etalé cohomology and φ'_{2d} is obtained from applying $R\pi_*$ to the adjoint of a cohomological purity isomorphism $Rs^!\Lambda^{\mathcal{S}}(n-d)[-2d] \simeq \Lambda^{\mathscr{X}}(n)$. We omit the details.

For a general proper regular arithmetic surface $\pi : \mathcal{X} \to S$ the maps in (2.12) are not necessarily isomorphisms. This suggests that the motivic decomposition of $R\pi_*\Lambda^{\mathcal{X}}(n)$ arises from the (standard) cohomological degree components $R^i\pi_*\Lambda^{\mathcal{X}}(n)$ as follows: Split $R^2\pi_*\Lambda^{\mathcal{X}}(n)$ into a component dual to $R^0\pi_*\Lambda^{\mathcal{X}}(n)$ and into another component describing the obstruction of $\mathcal{X}$ from being smooth and then regroup the latter to the motivic degree 1 part. This pattern is familiar from (2.3) and the contained definition ${}^pL(H^1(\mathcal{X}), s) := \Pi(\mathcal{X}, s)L(H^1(X), s)$ showing that the *standard* Hasse-Weil function $\zeta_{HW}(X, s)$ differs from the *motivic* function $\zeta(\mathcal{X}, s)$ only by additional terms in motivic degree 1 that are characterized entirely by the bad fibers of $\mathcal{X}$.

2.2.3 A motivic decomposition for $R\pi_*\mathbb{Z}(n)$

The fundamental insight for all following computations of motivic cohomology will be that Theorem 2.8 has an analogue for Bloch's cycle complexes. We will need a technical preparation.

For any map $f : \mathscr{X} \to \mathscr{Y}$ between arithmetic schemes and any étale neighborhood $\mathcal{V} \to \mathscr{Y}$ define

$$\mathcal{Z}^n_{\mathscr{Y}}(\mathcal{V}, j)^f := \{Z \in \mathcal{Z}^n_{\mathscr{Y}}(\mathcal{V}, j) \mid (f \times \mathrm{id}_{\Delta^j})^{-1}(Z) \text{ intersects all faces properly}\}.$$

For a section $s : \mathcal{S} \to \mathscr{X}$ of π and $n \in \mathbb{Z}$ we introduce the technical assumption

FPB(s,n) The inclusion of simplicial structures $\mathcal{Z}^n_{\mathscr{X}}(-, 2n - \bullet)^s \hookrightarrow \mathcal{Z}^n_{\mathscr{X}}(-, 2n - \bullet)$ gives rise to a quasi-isomorphism of associated derived complexes. In other words,

$$\mathrm{DK}(\mathcal{Z}^n_{\mathscr{X}}(-, 2n - \bullet)^s) \simeq \mathbb{Z}(n)^{\mathscr{X}}.$$

FPB(s,n) ensures the existence of a functorial pull-back morphism $s^*\mathbb{Z}(n)^{\mathscr{X}} \to \mathbb{Z}(n)^{\mathcal{S}}$ whose adjoint $\mathbb{Z}(n)^{\mathscr{X}} \to s_*\mathbb{Z}(n)^{\mathcal{S}}$ is given over an étale neighborhood $\mathcal{U} \to \mathscr{X}$ by the map

$$\begin{array}{ccl} \mathcal{Z}^n(\mathcal{U}, j)^s & \longrightarrow & \mathcal{Z}^n(s^{-1}\mathcal{U}, j) \\ Z & \mapsto & \begin{cases} (s \times \mathrm{id}_{\Delta^j})^{-1} Z & \begin{array}{l}\text{if for all closed points } x \in \Delta_j\text{:}\\ Z \cap (\mathscr{X} \times \{x\}) \text{ has codim 0 in } \mathscr{X} \times \{x\}\text{, or}\\ \text{is a finite union of } \textit{vertical} \text{ divisors of } \mathscr{X} \times \{x\}\end{array} \\ 0 & \text{otherwise} \end{cases} \end{array} \tag{2.13}$$

Note that for $n \leqslant 1$ one has a morphism $s^*\mathbb{Z}(n)^{\mathscr{X}} \to \mathbb{Z}(n)^{\mathcal{S}}$ that is functorial in s unconditionally since this is well-known for the sheaves $\mathbb{Q}_p/\mathbb{Z}_p(n)$, $\mathbb{Z}$, and $\mathbb{G}_m$.

The analogue of **FPB**(s,n) for morphisms between smooth varieties over fields are known (cf. [18] property 4 following Thm 1.1).

Theorem 2.11. *Suppose $\pi : \mathscr{X} \to \mathcal{S}$ is of relative dimension $d = 1$ and has a section $s : \mathcal{S} \to \mathscr{X}$. If $n \geqslant 2$ assume that s satisfies the condition* **FPB**(s,n)*. Then, for any integer n, the complexes $\mathbb{Z}(n)^{\mathcal{S}}$ and $\mathbb{Z}(n-1)^{\mathcal{S}}[-2]$ split off as direct summands of $R\pi_*\mathbb{Z}(n)^{\mathscr{X}}$. When writing ${}^pR^1\pi_*\mathbb{Z}(n)^{\mathscr{X}}[-1]$ for the remaining summand we arrive at the canonical decomposition*

$$R\pi_*\mathbb{Z}(n)^{\mathscr{X}} \simeq \mathbb{Z}(n)^{\mathcal{S}} \oplus {}^pR^1\pi_*\mathbb{Z}(n)^{\mathscr{X}}[-1] \oplus \mathbb{Z}(n-1)^{\mathcal{S}}[-2]. \tag{2.14}$$

Proof. Let $n \geqslant 1$. As before, we will prove the theorem by exhibiting maps

$$\mathbb{Z}(n)^{\mathcal{S}} \xrightarrow{\varphi_0} R\pi_*\mathbb{Z}(n)^{\mathscr{X}} \xrightarrow{\psi_0} \mathbb{Z}(n)^{\mathcal{S}},$$

$$\mathbb{Z}(n-1)^{\mathcal{S}}[-2] \xrightarrow{\varphi_2} R\pi_*\mathbb{Z}(n)^{\mathscr{X}} \xrightarrow{\psi_2} \mathbb{Z}(n-1)^{\mathcal{S}}[-2]$$

that compose as

$$\begin{pmatrix} \psi_0 \\ \psi_2 \end{pmatrix} \circ \begin{pmatrix} \varphi_0 & \varphi_2 \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Let $\Phi_0 : \pi^*\mathbb{Z}(n)^{\mathcal{S}} \to \mathbb{Z}(n)^{\mathscr{X}}$ be the flat pull-back morphism, i.e. the adjoint of the canonical morphism $\mathbb{Z}(n)^{\mathcal{S}} \to \pi_*\mathbb{Z}(n)^{\mathscr{X}}$ which, on the level of complexes over an étale neighborhood $U \to \mathcal{S}$, is given by

$$\mathscr{Z}^n(U, j) \longrightarrow \mathscr{Z}^n(\pi^{-1}U, j), \qquad Z \mapsto (\pi \times \mathrm{id}_{\Delta^j})^{-1}Z.$$

Similarly, let $\Psi_0 : s^*\mathbb{Z}(n)^{\mathscr{X}} \to \mathbb{Z}(n)^{\mathcal{S}}$ denote the pull-back morphism (2.13). By virtue of the usual adjunctions, we may now define

$$\varphi_0 : \mathbb{Z}(n)^{\mathcal{S}} \longrightarrow R\pi_*\pi^*\mathbb{Z}(n)^{\mathcal{S}} \xrightarrow{R\pi_*\Phi_0} R\pi_*\mathbb{Z}(n)^{\mathscr{X}},$$

$$\psi_0 : R\pi_*\mathbb{Z}(n)^{\mathscr{X}} \longrightarrow R\pi_*s_*s^*\mathbb{Z}(n)^{\mathscr{X}} = s^*\mathbb{Z}(n)^{\mathscr{X}} \xrightarrow{\Psi_0} \mathbb{Z}(n)^{\mathcal{S}}.$$

One sees directly that $\psi_0\varphi_0 = \mathrm{id}$, i.e. $\mathbb{Z}(n)^{\mathcal{S}}$ splits off as a direct summand of $R\pi_*\mathbb{Z}(n)^{\mathscr{X}}$.

Next, we let $\Phi_2 : s_*\mathbb{Z}(n-1)^{\mathcal{S}}[-2] \to \mathbb{Z}(n)^{\mathscr{X}}$ to be the morphism which acts on complexes as

$$\mathscr{Z}^{n-1}(s^{-1}\mathcal{U}, j) \to \mathscr{Z}^n(\mathcal{U}, j), \quad Z \mapsto (s \times \mathrm{id}_{\Delta^j})(Z).$$

Cor. 3.2 in [12] shows that Φ_2 is well-defined. Note that the adjoint of Φ_2 is a quasi-isomorphism $\mathbb{Z}(n-1)^{\mathcal{S}}[-2] \simeq Rs^!\mathbb{Z}(n)^{\mathscr{X}}$ showing cohomological purity for Bloch's cycle complexes (cf. [12] Cor. 7.2(a), Cor. 3.3(a)).

$$\varphi_2 : \mathbb{Z}(n-1)^{\mathcal{S}}[-2] = R\pi_*s_*\mathbb{Z}(n-1)^{\mathcal{S}}[-2] \xrightarrow{R\pi_*\Phi_2} R\pi_*\mathbb{Z}(n)^{\mathscr{X}}.$$

Finally, let $\Psi_2 : \pi_*\mathbb{Z}(n)^{\mathscr{X}} \to \mathbb{Z}(n-1)^{\mathcal{S}}[-2]$ be the proper push-forward map which is given on cycles by

$$\mathscr{Z}^n(\pi^{-1}U, j) \to \mathscr{Z}^{n-1}(U, j), \quad Z \mapsto \begin{cases} (\pi \times \mathrm{id}_{\Delta^j})(Z) & \begin{array}{l}\text{if for all closed points } x \in \Delta_j\text{:}\\ Z \cap (\mathscr{X} \times \{x\}) \text{ has codimension 2 in } \mathscr{X} \times \{x\}\text{, or}\\ \text{is a finite union of } \textit{horizontal} \text{ divisors of } \mathscr{X} \times \{x\}\end{array} \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

The conditions on $\mathcal{S}$ guarantee that [12] Cor. 3.2 and Cor. 7.2(b) are applicable, showing that Ψ_2 extends to a morphism

$$\psi_2 : R\pi_*\mathbb{Z}(n)^{\mathscr{X}} \longrightarrow \mathbb{Z}(n-1)^{\mathcal{S}}[-2].$$

Again it is clear that $\psi_2\varphi_2 = \mathrm{id}$. Also, from the explicit descriptions (2.13) and (2.15) we see immediately that the compositions $\psi_2\varphi_0$ and $\psi_0\varphi_2$ must be trivial. Consequently, $\mathbb{Z}(n)^{\mathcal{S}}$ and $\mathbb{Z}(n-1)^{\mathcal{S}}[-2]$ split off as distinct direct summands of $R\pi_*\mathbb{Z}(n)^{\mathscr{X}}$.

Let us now consider $\mathbb{Z}(-n)$. Recall that

$$\mathbb{Z}(-n)^{\mathcal{X}}[1] = \bigoplus_p j_{p,!}\mathbb{Q}_p/\mathbb{Z}_p(-n).$$

It suffices to show that

$$R\pi_* \bigoplus_p j_{p,!} = \bigoplus_p j'_{p,!} R\pi[1/p]_* \tag{2.16}$$

as the claim then follows from applying Theorem 2.8 to $j'_{p,!}\mathbb{Q}_p/\mathbb{Z}_p(-n)$. However, (2.16) is immediate from diagram (2.8) since $\pi_* = \pi_!$ and $\pi[1/p]_* = \pi[1/p]_!$.

Finally, for $n = 0$ the Remark 2.9 shows $R\pi_*\mathbb{Z} \simeq \mathbb{Z} \oplus \tau^{\geqslant 1} R\pi_*\mathbb{Z}$. Moreover the long exact sequence for the derived functor $R\pi_*$ associated to

$$0 \longrightarrow \mathbb{Z}^{\mathscr{X}} \longrightarrow \mathbb{Q}^{\mathscr{X}} \longrightarrow \mathbb{Q}/\mathbb{Z}^{\mathscr{X}} \longrightarrow 0$$

proves $R^1\pi_*\mathbb{Z} = 0$ and $\tau^{\geqslant 2} R\pi_*\mathbb{Z} = \tau^{\geqslant 1} R\pi_*\mathbb{Q}/\mathbb{Z}[-1]$. Indeed, one has $R^r\pi_*\mathbb{Q} = 0$ for $r > 0$ as can be seen by passing to stalks and recalling that Galois cohomology with rational coefficients vanishes. We may thus write

$$R\pi_*\mathbb{Z}^{\mathscr{X}} = \mathbb{Z}^{\mathcal{S}} \oplus \bigoplus_p \tau^{\geqslant 1} R\pi_*\mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}}[-1]. \tag{2.17}$$

For each p the Open-Closed-Decomposition (2.8) gives rise to the short exact sequence

$$0 \longrightarrow (\mathbb{Q}_p/\mathbb{Z}_p)^{\mathscr{X},p} \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}} \longrightarrow i_{p,*}\mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}_p} \longrightarrow 0.$$

Applying $R\pi_*$ yields the distinguished triangle

$$R\pi_*(\mathbb{Q}_p/\mathbb{Z}_p)_p^{\mathscr{X}} \longrightarrow R\pi_*\mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}} \longrightarrow i'_{p,*} R\pi_{p,*}\mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}_p} \longrightarrow . \tag{2.18}$$

$R\pi_{p,*}\mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}_p}$ is concentrated in degrees $0, 1$ and an analysis of the long exact sequence associated to (2.18) shows that applying $\tau^{\geqslant 1}$ preserves exactness. This yields

$$\tau^{\geqslant 1} R\pi_*\mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}} \simeq \mathrm{Cone}\left(i'_{p,*} R^1\pi_{p,*}\mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}_p}[-2] \longrightarrow \tau^{\geqslant 1} R\pi_*(\mathbb{Q}_p/\mathbb{Z}_p)^{\mathscr{X},p} \right).$$

We apply Theorem 2.8 to $\tau^{\geqslant 1} R\pi_*(\mathbb{Q}_p/\mathbb{Z}_p)^{\mathscr{X},p}$ and verify on stalks that there are no non-trivial morphisms of sheaves $i'_{p,*} R^1\pi_{p,*}\mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}_p} \to (\mathbb{Q}_p/\mathbb{Z}_p)^{\mathcal{S},p}(-1)$. Thus, we may rewrite the above as

$$\tau^{\geqslant 1} R\pi_*\mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}} \simeq \mathrm{Cone}\left(i'_{p,*} R^1\pi_{p,*}\mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}_p}[-2] \longrightarrow {}^{p}R^1\pi_*(\mathbb{Q}_p/\mathbb{Z}_p)^{\mathscr{X},p}[-1] \right) \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{\mathcal{S},p}(-1)[-2].$$

Combining this with (2.17) and making the identification

$${}^{p}R^1\pi_*\mathbb{Z}^{\mathscr{X}} := \bigoplus_p \mathrm{Cone}\left(i'_{p,*} R^1\pi_{p,*}\mathbb{Q}_p/\mathbb{Z}_p^{\mathscr{X}_p}[-2] \longrightarrow {}^{p}R^1\pi_*(\mathbb{Q}_p/\mathbb{Z}_p)^{\mathscr{X},p}[-1] \right)$$

allows us to write

$$R\pi_*\mathbb{Z}^{\mathscr{X}} \simeq \mathbb{Z}^{\mathcal{S}} \oplus {}^{p}R^1\pi_*\mathbb{Z}^{\mathscr{X}}[-1] \oplus \mathbb{Z}(-1)^{\mathcal{S}}[-2]$$

as desired. □

Remark 2.12. The proof of Theorem 2.11 uses the regularity assumption as follows. For $n \leqslant 0$ it is implicit in the use of Theorem 2.8 where in turn it is needed for the version of Verdier Duality given in Corollary 2.7. For $n \geqslant 1$ it is implicit in the use of $\mathbf{FPB}(s,n)$ since this conjecture is formulated only for regular $\mathcal{X}$. We believe $\mathbf{FPB}(s,n)$ to hold only for regular $\mathcal{X}$.

The most important instance of Theorem 2.11 is for the structure map $\pi : \mathcal{X} \to S$ of our arithmetic surface $\mathcal{X}$ as it will allow us to decompose motivic and Weil-étale motivic cohomology into degree $0, 1, 2$ components. However, it is also applicable to localizations $\pi_{\mathbb{Z}_p} : \mathcal{X}_{\mathbb{Z}_p} \to S_{\mathbb{Z}_p}$ as well as to structure maps of smooth proper curves over fields.

Note that for an elliptic surface $\pi : \mathcal{E} \to S$ one always has a section $s : S \to \mathcal{E}$. In fact, $\mathcal{E}_F$ has a rational point and $\mathcal{E}(\mathcal{O}) = \mathcal{E}_F(F)$ since $\mathcal{E}$ is proper (cf. [31] Cor. IV.4.4(a)). In the case of a general arithmetic surface $\pi : \mathcal{X} \to S$ one still has an exact triangle for $n = 1$ without assuming the existence of a section.

Let $\mathscr{P}_{\mathcal{X}/S}$ and $\mathscr{P}^0_{\mathcal{X}/S}$ denote the étale sheafifications of the functors $\mathcal{U}/S \to \mathrm{Pic}\,(\mathcal{X} \times_S \mathcal{U})$ and $\mathcal{U}/S \to \mathrm{Pic}^0(\mathcal{X} \times_S \mathcal{U})$ on S respectively.

Proposition 2.13. *One has the distinguished triangle*

$$\mathbb{Z}(1)^S \longrightarrow R\pi_*\mathbb{Z}(1)^{\mathcal{X}} \longrightarrow \mathscr{P}_{\mathcal{X}/S}[-2] \longrightarrow .$$

Proof. Clearly, $\tau^{\leqslant 1} R\pi_*\mathbb{Z}(1)^{\mathcal{X}} = (\tau^{\leqslant 0} R\pi_*\mathbb{G}_m^{\mathcal{X}})[-1] = \pi_*\mathbb{G}_m^{\mathcal{X}}[-1] = \mathbb{G}_m^S[-1]$ giving us the truncation triangle

$$\mathbb{Z}(1)^S \longrightarrow R\pi_*\mathbb{Z}(1)^{\mathcal{X}} \longrightarrow (\tau^{\geqslant 1} R\pi_*\mathbb{G}_m^{\mathcal{X}})[-1] \longrightarrow .$$

Moreover, it is well-known that

$$R^1\pi_*\mathbb{G}_m^{\mathcal{X}} \cong \mathscr{P}_{\mathcal{X}/S}. \tag{2.19}$$

It remains to show $R^i\pi_*\mathbb{G}_m^{\mathcal{X}} = 0$ for $i \geqslant 2$. Let $\overline{x} \hookrightarrow \mathcal{X}$ be a geometric point over $\mathfrak{p}$. Write $S_{\overline{\mathfrak{p}}} = \mathrm{Spec}\,\mathcal{O}_{\mathfrak{p}}^{ur}$ and $\mathcal{X}(\overline{x}) = \mathcal{X} \times_S \mathcal{O}^{sh}_{\mathcal{X},\overline{x}}$ and let $\pi_{\overline{x}}$ be the base change of π to $S_{\overline{\mathfrak{p}}}$. Then

$$\left(R^i\pi_*\mathbb{G}_m^{\mathcal{X}}\right)_{\overline{x}} = H^i(\mathcal{X}(\overline{x}), \mathbb{G}_m) = 0 \qquad \text{for } i \geqslant 2$$

by Grothendieck's result [14] Cor. 3.2 (p.98) applied to the surface $\mathcal{X}(\overline{x})$ as it is proper and flat over the spectrum $S_{\overline{\mathfrak{p}}}$ of a regular local ring. □

Corollary 2.14. *If $\pi : \mathcal{X} \to S$ has a section then ${}^pR^1\pi_*\mathbb{Z}(1) \cong \mathscr{P}^0_{\mathcal{X}/S}[-1]$ and one has the motivic decomposition*

$$R\pi_*\mathbb{Z}(1)^{\mathcal{X}} \simeq \mathbb{Z}(1)^S \oplus \mathscr{P}^0_{\mathcal{X}/S}[-1] \oplus \mathbb{Z}^S[-2].$$

2.2.4 Motivic decompositions for complex manifolds.

The analogue of Verdier duality and of Proposition 2.5 for locally compact spaces is well-known for all abelian sheaves. We use it to derive an analogue of Theorem 2.11 for the cohomology of complex manifolds with coefficients given by the locally constant $G_{\mathbb{R}}$-equivariant sheaf $\mathbb{R}(n) := (2\pi i)^n\mathbb{R}$.

Proposition 2.15. *Let $S = \{\bullet\}$ be the one-point space and let $\pi : X \to S$ be the structure map of a complex manifold X of complex dimension d. Then the sheaves $\mathbb{R}(n)^S$ and $\mathbb{R}(n-d)^S[-2d]$ split off as direct summands of $R\pi_*\mathbb{R}(n)^X$. If $d = 1$ we will write ${}^pR^1\pi_*\mathbb{R}(n)^X[-1]$ for the remaining summand, i.e. we will have the canonical decomposition*

$$\begin{aligned} R\pi_*\mathbb{R}(n)^X &\simeq \quad \mathbb{R}(n)^S \quad \oplus {}^pR^1\pi_*\mathbb{R}(n)^X[-1] \oplus \mathbb{R}(n-1)^S[-2] \\ &\simeq R^0\pi_*\mathbb{R}(n)^X \oplus R^1\pi_*\mathbb{R}(n)^X[-1] \oplus R^2\pi_*\mathbb{R}(n)^X[-2]. \end{aligned} \tag{2.20}$$

Proof. Write $(-)^\vee = R\mathscr{H}om_X(-,\mathbb{R})$. Smoothness of X implies $\pi^!\mathbb{R}^X \simeq \mathbb{R}(d)^S[2d]$. Thus, the analogue of Verdier duality yields

$$R\pi_*\mathbb{R}(d)^X[2d] \simeq R\pi_*R\mathscr{H}om_X(\mathbb{R}^X, \mathbb{R}(d)^S[2d]) \simeq R\mathscr{H}om_S(R\pi_*\mathbb{R}^X, \mathbb{R}^S) = (R\pi_*\mathbb{R}^X)^\vee.$$

Now, the proof of Theorem 2.8 holds verbatim for the structure map $\pi : X \to S$ of complex manifolds with their analytic topology when replacing (2.9) with the above duality. This yields the first line of (2.20). Equality of motivic and cohomological degree components follows analogously to Remark 2.10. □

2.2.5 The motivic picture and notation

Motivic Interpretation. The decompositions of the previous propositions are motivated by the theory of motives over a field K. We recall it here. Any smooth projective variety X over K comes with an associated motive $h(X)$, an object in a $\mathbb{Q}$-linear semi-simple abelian category $\mathrm{Mot}_{\mathbb{Q}}$. Conjecturally, $h(X)$ produces $H^*(X)$ for any Weil cohomology theory H by applying an appropriate fiber functor. One of the Standard Conjectures postulates the existence of algebraic cycles $\pi^i \subset X \times X$ that induce the projections $H^*(X) \twoheadrightarrow H^i(X)$ onto the i-th degree. It would follow that $\Delta_X = \sum_{i=0}^{2d} \pi^i$ in $C^d_\sim(X \times X)$ where $d = \dim X$. On the level of motives we would obtain the decomposition

$$h(X) = h^0(X) \oplus \cdots \oplus h^{2d}(X) \tag{2.21}$$

of $h(X)$ into its motivic degree components $h^i(X)$.

If X is a smooth projective curve over K the existence of a section, i.e. a K-rational point $x \in X(K)$ yields the above decomposition — even in the $\mathbb{Z}$-linear category of Chow motives — by setting $\pi^0 = X \times \{x\}$ and $\pi^2 = \{x\} \times X$. Indeed, on cohomology the sequence $\{x\} \hookrightarrow X \twoheadrightarrow \{x\}$ splits off $H^*(\{x\}) = H^0(X)$ and taking the transpose cycle $\pi^2 = (\pi^0)^t$ corresponds to projecting onto the Poincaré dual $H^0(X)^\vee \cong H^2(X)$. The remaining direct summand must be $h^1(X)$ and we may rewrite (2.21) as

$$h(X) = h(x) \oplus h^1(X) \oplus h(x)^\vee(-1).$$

It is expected that a similar theory holds for proper regular $\mathscr{X}$ of relative dimension 1 over more general base schemes $\mathcal{S}$. It should provide something analogous to the perverse t-structure on the derived category of l-adic sheaves on varieties over finite fields. A section $s : \mathcal{S} \to \mathscr{X}$ would then give rise to an analogous decomposition

$$h(\mathscr{X}) = {}^p h(\mathcal{S}) \oplus {}^p h^1(\mathscr{X}) \oplus {}^p h(\mathcal{S})^\vee(-1).$$

Applying the appropriate fiber functors should then reproduce (2.10) and (2.14).

Notation. Let $\mathcal{A} \in \mathcal{D}(\mathscr{X})$ be a complex in the derived category of sheaves on any fixed topology of $\mathscr{X}$. Whenever a decomposition of the kind $R\pi_*\mathcal{A} = \bigoplus_{i=0,1,2} {}^pR^i\pi_*\mathcal{A}[-i]$ holds we will call ${}^pR^i\pi_*\mathcal{A}[-i]$ the *motivic degree i component* or just shortly *h^i-component* of $R\pi_*\mathcal{A}$, and we will write

$${}^pH^i(\mathscr{X}, \mathcal{A}) := H^i(\mathscr{X}, {}^pR^i\pi_*\mathcal{A}[-i]).$$

For example, Theorem 2.11 implies

$$\begin{aligned} H^i(\mathscr{X}, \mathbb{Z}(n)) &= \bigoplus_{p=0,1,2} {}^pH^i(\mathscr{X}, \mathbb{Z}(n)) \\ &= H^i(\mathcal{S}, \mathbb{Z}(n)) \oplus {}^1H^i(\mathscr{X}, \mathbb{Z}(n)) \oplus H^{i-2}(\mathcal{S}, \mathbb{Z}(n-1)). \end{aligned}$$

2.3 Deligne Cohomology

For the remainder of this thesis we assume that $\pi : \mathcal{X} \to S$ has a section $s : S \to \mathcal{X}$ satisfying the functorial pull-back condition **FPB**(s,n) *for all integers n.*

Let X be a complex manifold and $n \geqslant 0$. Recall that for a subring $A \subset \mathbb{C}$ one defines $A(n)_\mathcal{D} = A(n)_\mathcal{D}^X$ as the bounded complex in the derived category of abelian sheaves on X given by

$$0 \to (2\pi i)^n A \to \mathcal{O}_{X/\mathbb{C}} \to \Omega_{X/\mathbb{C}} \to \ldots \to \Omega^{n-1}_{X/\mathbb{C}} \to 0$$

concentrated in degrees $[0, n]$. One defines

$$H^i_\mathcal{D}(X, A(n)) := \mathbb{H}^i(X, A(n)_\mathcal{D}).$$

For any arithmetic scheme $\mathscr{X}$, the action of $G_{\mathbb{R}}$ on $\mathscr{X}(\mathbb{C})$ carries through to $H^i_{\mathcal{D}}(\mathscr{X}(\mathbb{C}), \mathbb{R}(n))$ and we define *Deligne cohomology* to be

$$H^{i,n}_{\mathcal{D}}(\mathscr{X}) := H^i_{\mathcal{D}}(\mathscr{X}/_{\mathbb{R}}, \mathbb{R}(n)) := H^i_{\mathcal{D}}(\mathscr{X}(\mathbb{C}), \mathbb{R}(n))^{G_{\mathbb{R}}}$$

(cf. [30] §2 or [7] §1). The set of $\mathbb{C}$-points $\mathcal{X}(\mathbb{C})$ of our arithmetic surface $\mathcal{X}$ has complex dimension 1. So, we have $\Omega^i_{\mathcal{X}(\mathbb{C})/\mathbb{C}} = 0$ for $i \geqslant 2$ and Poincaré's Lemma proves $\mathbb{C} \simeq [\mathcal{O}_{\mathcal{X}(\mathbb{C})} \to \Omega_{\mathcal{X}(\mathbb{C})/\mathbb{C}}]$. Consequently

$$\mathbb{R}(n)^{\mathcal{X}(\mathbb{C})}_{\mathcal{D}} \simeq \begin{cases} \mathbb{R}(n)[0] & \text{for } n \leqslant 0 \\ \left[\mathbb{R}(1) \to \mathcal{O}_{\mathcal{X}(\mathbb{C})}\right] \simeq \mathcal{O}^{\times}_{\mathcal{X}(\mathbb{C})}/S^1[-1] & \text{for } n = 1 \\ [\mathbb{R}(n) \to \mathbb{C}] \simeq \mathbb{R}(n-1)[-1] & \text{for } n \geqslant 2. \end{cases} \tag{2.22}$$

Here the pseudo-isomorphism for $n = 1$ is given by the exponential map. Using (2.22) we reduce to singular cohomology and considering real and complex places separately gives us the table of ranks (A.12). For the computation of $H^{i,1}_{\mathcal{D}}(\mathcal{X})$ with $i \geqslant 2$ the perfect pairing

$$H^i_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}}, \mathbb{R}(n)) \times H^{3-i}_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}}, \mathbb{R}(2-n)) \;\to\; H^3_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}}, \mathbb{R}(2)) \to \mathbb{R} \tag{2.23}$$

from [8] Lemma 2.3 has been used.

Decomposition into $\mathfrak{h}^i$-components. It is easy to see that

$$\mathbb{R}(n)^{S(\mathbb{C})}_{\mathcal{D}} \simeq \begin{cases} \mathbb{R}(n)^{S(\mathbb{C})} & \text{for } n \leqslant 0 \\ \mathbb{R}(n-1)^{S(\mathbb{C})}[-1] & \text{for } n \geqslant 1. \end{cases}$$

Therefore, Proposition 2.15 shows together with (2.22) that also the Deligne complex decomposes as

$$R\pi_*\mathbb{R}(n)^{\mathcal{X}(\mathbb{C})}_{\mathcal{D}} \;\simeq\; \mathbb{R}(n)^{S(\mathbb{C})}_{\mathcal{D}} \;\oplus\; {}^pR^1\pi_*\mathbb{R}(n)^{\mathcal{X}(\mathbb{C})}_{\mathcal{D}}[-1] \oplus \mathbb{R}(n-1)^{S(\mathbb{C})}_{\mathcal{D}}[-2].$$

On cohomology we obtain

$$H^{\bullet,n}_{\mathcal{D}}(\mathcal{X}) \cong H^{\bullet,n}_{\mathcal{D}}(S) \,\oplus\, {}^1H^{\bullet,n}_{\mathcal{D}}(\mathcal{X}) \,\oplus\, H^{\bullet-2,n-1}_{\mathcal{D}}(S) \tag{2.24}$$

and the motivic degree 1 term ${}^1H^{i,n}_{\mathcal{D}}(\mathcal{X})$ equals the full $H^{i,n}_{\mathcal{D}}(\mathcal{X})$ if $i = 1, n \leqslant 0$ or $i = 2, n \geqslant 2$, and vanishes otherwise.

2.4 Étale motivic cohomology

For finitely or cofinitely generated groups we will write $G \sim H$ if G, H are isomorphic up to 2-torsion[1]. We write G_{div} and $G_{\text{codiv}} = G/G_{\text{div}}$ for the divisible and codivisible part

[1] i.e. there are homomorphisms $G \overset{\alpha}{\to} H$ and $H \overset{\beta}{\to} G$ such that kernel and cokernel of the compositions $\alpha\beta$ and $\beta\alpha$ are finite 2-torsion groups

of G. We also let $(-)^\vee = \mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ as well as $(-)^* = \mathrm{Hom}(-, \mathbb{Z})$ for abelian groups and $(-)^* = \mathrm{Hom}(-, \mathbb{R})$ for $\mathbb{R}$-vectorspaces respectively. For the remainder of this thesis we assume the validity of the following

Conjecture 2.16 ($\mathbf{L}(\mathcal{X},n)$)**.** *The groups $H^i(\mathcal{X}, \mathbb{Z}(n))$ are finitely generated for $i \leqslant 2n+1$ and vanish for sufficiently small i.*

$\mathbf{L}(\mathcal{X},1)$ is equivalent to finiteness of $\mathrm{Br}\,\mathcal{X}$ (cf. [8] Lemma 3.3 and preceding comments). Assuming $\mathbf{L}(\mathcal{X},n)$ allows us to reformulate Artin-Verdier duality as the existence of a perfect pairing of integral motivic cohomology groups (cf. [8] Prop. 3.4)

$$H^{6-i,2-n}(\overline{\mathcal{X}}) \times H^{i,n}(\overline{\mathcal{X}}) \;\longrightarrow\; \mathbb{Q}/\mathbb{Z}. \tag{2.25}$$

2.4.1 Completed motivic cohomology

Recall the Artin-Verdier étale topos $\overline{\mathcal{X}}_{\text{ét}}$ and its open closed decomposition (1.1). We write $H^{i,n}(\mathcal{X}) := H^i(\mathcal{X}_{\text{ét}}, \mathbb{Z}(n))$ and $H^{i,n}(\overline{\mathcal{X}}) := H^i(\overline{\mathcal{X}}_{\text{ét}}, \mathbb{Z}(n))$ for the motivic cohomology and for its *completed cohomology*, i.e. its cohomology with respect to the Artin-Verdier étale topos of $\mathcal{X}$. The discrepancy between these two versions of cohomology is captured by the distinguished triangle

$$\mathbb{Z}(n)^{\overline{\mathcal{X}}} \;\longrightarrow\; R\phi_*\mathbb{Z}(n) \;\longrightarrow\; u_{\infty,*}\tau^{>n}R\widehat{\pi}_*(2\pi i)^n\mathbb{Z} \;\longrightarrow . \tag{2.26}$$

(cf. [8] Cor. 6.8) In particular, for $i \leqslant n$ one has $H^{i,n}(\mathcal{X}) = H^{i,n}(\overline{\mathcal{X}})$ and for $i > n$ these cohomology groups differ only in 2-torsion. The cohomology of $u_{\infty,*}\tau^{>n}R\widehat{\pi}_*(2\pi i)^n\mathbb{Z}$ is computed in Appendix A.2. In what follows we will primarily work with completed motivic cohomology $H^{i,n}(\overline{\mathcal{X}})$ as it is this type of cohomology that will factor into the definition of Weil-étale cohomology (see Section 2.4.5).

As shown in Corollary A.12 in Appendix A.3 completed motivic cohomology also comes with a decomposition into motivic degrees

$$H^{i,n}(\overline{\mathcal{X}}) \;\cong\; H^{i,n}(\overline{S}) \oplus {}^1H^{i,n}(\overline{\mathcal{X}}) \oplus H^{i-2,n-1}(\overline{S}).$$

We will explicate this decomposition for $n = 1$.

2.4.2 The case $n = 1$

Proposition 2.17. *The groups $H^{i,1}(\overline{\mathcal{X}})$ are given by and decompose as in the table below.*

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i \geqslant 6$
$H^{i,1}(\overline{\mathcal{X}})$	$\mathcal{O}^\times$	$\mathrm{Pic}\,(\mathcal{X})$	$Ш(X/F)$	$\mathrm{Pic}\,(\mathcal{X})^\vee$	$(\mathcal{O}^\times)^\vee$	0
$H^{i,1}(\overline{S})$	$\mathcal{O}^\times$	Cl_F	0	$\mathbb{Q}/\mathbb{Z}$	0	0
${}^1H^{i,1}(\overline{\mathcal{X}})$	0	$\mathrm{Pic}^0(\mathcal{X})/\mathrm{Cl}_F$	$Ш(X/F)$	$\left(\mathrm{Pic}^0(\mathcal{X})/\mathrm{Cl}_F\right)^\vee$	0	0
$H^{i-2,0}(\overline{S})$	0	$\mathbb{Z}$	0	Cl_F	$(\mathcal{O}^\times)^\vee$	0

Proof. We use Lemma A.8(ii) to evaluate the long exact sequence on cohomology associated to the version of (2.26) for the base scheme S. In degrees 3, 4 we obtain

$$0 \to H^{3,1}(\overline{S}) \to \mathrm{Br}\,\mathcal{O} \xrightarrow{b} (\mathbb{Z}/2)^r \to \mathbb{Q}/\mathbb{Z} \to H^{4,1}(S) \to 0.$$

Since $\mathrm{Br}\,\mathcal{O} = (\mathbb{Z}/2)^{r,\Sigma=0}$ the map b must be the inclusion and we get $H^{3,1}(\overline{S}) = 0$ and $H^{4,1}(S) = \mathbb{Q}/\frac{1}{2}\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}$ as well as $H^{1,0}(S) = H^{1,0}(\overline{S}) = 0$. Therefore the motivic decomposition of $H^{3,1}(\mathcal{X}) = \mathrm{Br}\,\mathcal{X}$ is given by

$$\mathrm{Br}\,\mathcal{X} \;\cong\; \mathrm{Br}\,\mathcal{O} \oplus \frac{\mathrm{Br}\,\mathcal{X}}{\mathrm{Br}\,\mathcal{O}}.$$

So, the motivic degree 1 part of the triangle (2.26) gives

$$0 \longrightarrow {}^1H^{3,1}(\overline{\mathcal{X}}) \longrightarrow \frac{\mathrm{Br}\,\mathcal{X}}{\mathrm{Br}\,\mathcal{O}} \longrightarrow (\mathbb{Z}/2)^{{}^1\!l(\mathcal{X})}.$$

By [35] Thm. 3.1 and (1.7) the Tate-Shafarevich group $Ш(X/F)$ fits into the exact sequence

$$0 \longrightarrow Ш(X/F) \longrightarrow \frac{\mathrm{Br}\,\mathcal{X}}{\mathrm{Br}\,\mathcal{O}} \longrightarrow \prod_\sigma H^1(G_\mathbb{R}, \mathrm{Jac}X_\sigma),$$

where σ runs through all finite places of F and $X_\sigma = X \times_{F,\sigma} \mathrm{Spec}\,\mathbb{C}$. Proposition A.14 shows that the right-most terms of the above sequences are the same, so we in fact have ${}^1H^{3,1}(\overline{\mathcal{X}}) \cong Ш(X/F)$.

Up to 2-torsion the remaining entries are immediate from $\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$ and $\mathbb{Z}(0) \simeq \mathbb{Z}$ or follow from Artin-Verdier duality. The additional 2-torsion information is taken from Proposition A.15. □

2.4.3 Compact support cohomology and the perfect pairing conjecture

Fan constructs in his thesis a map between complexes $\rho : R\Gamma(\mathcal{X}_{\text{ét}}, \mathbb{Z}(n)) \to R\Gamma_\mathcal{D}(\mathcal{X}/_\mathbb{R}, \mathbb{Z}(n))$ in the derived category of abelian groups that induces the Beilinson regulator maps $H^{2n-i,n}(\mathcal{X}) \cong$

$\mathrm{CH}^n(\mathcal{X}, i) \to H_{\mathcal{D}}^{2n-i,n}(\mathcal{X})$. We define $R\Gamma_c(\mathcal{X}, \mathbb{R}(n))$ as the mapping fiber of $\rho \otimes \mathbb{R}$, i.e. we have the distinguished triangle

$$R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) \longrightarrow R\Gamma(\mathcal{X}_{\text{ét}}, \mathbb{R}(n)) \xrightarrow{\rho\otimes\mathbb{R}} R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}}, \mathbb{R}(n)) \longrightarrow \tag{2.27}$$

and write $H_c^{i,n}(\mathcal{X}) := H_c^i(\mathcal{X}, \mathbb{R}(n))$ for its cohomology groups. We have seen earlier that the motivic degrees of $R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}}, \mathbb{R}(n))$ coincide with cohomological degrees. So, $\rho \otimes \mathbb{R}$ trivially decomposes into maps between the motivic degree components of $R\Gamma_c(\mathcal{X}, \mathbb{R}(n))$ and $R\Gamma(\mathcal{X}_{\text{ét}}, \mathbb{R}(n))$ and we get a motivic decomposition of $H_c^{i,n}(\mathcal{X})$ as well:

$$H_c^{i,n}(\mathcal{X}) = H_c^{i,n}(S) \oplus {}^1H_c^{i,n}(\mathcal{X}) \oplus H_c^{i-2,n-1}(S).$$

Flach and Morin have constructed a product map

$$R\Gamma(\mathcal{X}, \mathbb{R}(n)) \otimes R\Gamma_c(\mathcal{X}, \mathbb{R}(m)) \longrightarrow R\Gamma_c(\mathcal{X}, \mathbb{R}(n+m)) \tag{2.28}$$

(cf. [8] Prop. 2.1) and have shown that under certain assumptions (cf. [8] Conj 2.9) Beilinson's conjecture (cf. [30] §3) is equivalent to

Conjecture 2.18. B($\mathcal{X}$,n) *The product map (2.28) induces for all $i, n \in \mathbb{Z}$ a perfect pairing of $\mathbb{R}$-vectorspaces*

$$H_c^{i,n}(\mathcal{X}) \times H^{4-i,2-n}(\mathcal{X})_{\mathbb{R}} \longrightarrow H_c^{4,2}(\mathcal{X}) \to \mathbb{R}. \tag{2.29}$$

Remark 2.19. **B($\mathcal{X}$,n)** is equivalent to non-degeneracy of the induced pairing

$${}^1H_c^{i,n}(\mathcal{X}) \times {}^1H^{4-i,2-n}(\mathcal{X})_{\mathbb{R}} \longrightarrow \mathbb{R}. \tag{2.30}$$

In fact, due to the decompositions (2.14) and (2.10) the conjecture $\mathbf{B}(\mathcal{X},n)$ is implied by the above together with $\mathbf{B}(S,n)$ and $\mathbf{B}(S,n-1)$. However, $\mathbf{B}(S,n)$ is known for all n. We will later see that (2.30) for $n = 1$ coincides with the *height pairing* which is known to be non-degenerate. In particular, $\mathbf{B}(\mathcal{X},1)$ is a well-known fact.

Ranks of motivic cohomology groups. From now on we assume $\mathbf{B}(\mathcal{X},n)$ to hold for all $n \in \mathbb{Z}$. We use it to compute $H^{i,n}(\mathcal{X})_{\mathbb{R}}$ and $H_c^{i,n}(\mathcal{X})$. Together with cofinite generation of the $H^{i,n}(\mathcal{X})_{\mathbb{R}}$ for $i > 2n$ one gets

$$H^{i,n}(\mathcal{X})_{\mathbb{R}} \cong \begin{cases} \mathbb{R} & \text{for } i = n = 0 \\ 0 & \text{for } n < 0 \text{ or } n = 0, i \neq 0 \end{cases}.$$

So, we can read off the ranks of $H_c^{i,n}(\mathcal{X})$ for $n \geqslant 1$ from $\mathbf{B}(\mathcal{X},n)$. Moreover, the long exact sequences associated to (2.27) for $n \neq 1$ give us

$$H_c^{i,n}(\mathcal{X}) \cong \begin{cases} H_{\mathcal{D}}^{0,0}(\mathcal{X})/H^{0,0}(\mathcal{X})_{\mathbb{R}} & \text{for } i = 1,\ n = 0 \\ H_{\mathcal{D}}^{i-1,n}(\mathcal{X}) & \text{for } n < 0 \text{ and } i \neq 1,\ n = 0 \end{cases}$$

and

$$H^{i,n}(\mathcal{X})_{\mathbb{R}} \cong \begin{cases} \mathrm{Ker}(H^{3,2}_{\mathcal{D}}(\mathcal{X}) \to H^{4,2}_c(\mathcal{X})) & \text{for } i=3,\, n=2 \\ H^{3-i,2-n}_{\mathcal{D}}(\mathcal{X})^* \cong H^{i,n}_{\mathcal{D}}(\mathcal{X}) & \text{for } n>2 \text{ or } i \neq 3,\ n=2 \end{cases}.$$

We obtain ranks as given in tables (A.15) and (A.17) in Appendix A.5. In particular,

$${}^1H^{i,n}_c(\mathcal{X}) = \begin{cases} {}^1H^{1,n}_{\mathcal{D}}(\mathcal{X}) \cong \mathbb{R}^{mg} & \text{if } i=2,\, n \leqslant 0 \\ {}^1H^{2,1}(\mathcal{X})_{\mathbb{R}} = (\mathrm{Pic}^0\mathcal{X}/\mathrm{Cl}_F)_{\mathbb{R}} & \text{if } i=2,\, n=1 \\ 0 & \text{otherwise} \end{cases}.$$

2.4.4 Torsion of motivic cohomology

Let $T^{i,n}_? = \mathrm{Tor}\, H^{i,n}(?)_{\mathrm{codiv}}$ for $? = S, \overline{S}, \mathcal{X}, \overline{\mathcal{X}}$ as well as ${}^1T^{i,n}_? = \mathrm{Tor}\, {}^1H^{i,n}(?)_{\mathrm{codiv}}$ for $? = \mathcal{X}, \overline{\mathcal{X}}$. Artin-Verdier duality gives $T^{i,n}_{\overline{S}} \cong T^{4-i,1-n}_{\overline{S}}$. Moreover, it is known that for $n \geqslant 2$, $i \neq 1,2$ one has $T^{i,n}_S \sim 0$ and even $T^{i,n}_S = 0$ for $i \leqslant 0$ (cf. [8] Section 5.8.3). In this section, we establish an analogous vanishing result for the torsion parts of the $\mathfrak{h}^1$-part of the motivic cohomology of $\mathcal{X}$.

Proposition 2.20. *Let n be any integer. One has ${}^1T^{i,n}_{\mathcal{X}} \sim {}^1T^{i,n}_{\overline{\mathcal{X}}} \sim 0$ whenever $i \neq 2,3,4$ and, moreover, ${}^1T^{i,n}_{\mathcal{X}} = 0$ for $i < 2$.*

Proof. Due to Artin-Verdier duality it suffices to consider $i < 2$. For $n < 0$ the claim is immediate from the definition $\mathbb{Z}(n) = \bigoplus_p j_{p,!}\mathbb{Q}_p/\mathbb{Z}_p(-n)$ and for $n = 0,1$ it follows from the explicit expressions $\mathbb{Z}(0) \simeq \mathbb{Z}$ and $\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$.

Let now $n \geqslant 2$ and fix a prime p. We will show that ${}^1H^i(\mathcal{X}, \mathbb{Z}_p(n)) = 0$ for $i \leqslant 0$, proving that ${}^1T^{i,n}_{\mathcal{X}}$ has trivial p-part for $i \leqslant 1$. Consider the Open-Closed-Decomposition

$$\mathcal{X}[1/p] \xrightarrow{j} \mathcal{X} \xleftarrow{i} \mathcal{X}_{\mathbb{F}_p}.$$

The proof of [8] Lemma 7.7 provides the distinguished triangle

$$i_*Ri^!\mathbb{Z}_p(n) \longrightarrow \mathbb{Z}_p(n) \longrightarrow Rj_*\mathbb{Z}_p(n) \longrightarrow \tag{2.31}$$

together with a quasi-isomorphism

$$\tau^{\leqslant n+1}\left(i_*\mathbb{Z}_p(n-1)[-2]\right) \xrightarrow{\simeq} \tau^{\leqslant n+1} i_*Ri^!\mathbb{Z}_p(n).$$

$\mathbb{Z}_p(n-1)^{\mathcal{X}_{\mathbb{F}_p}}$ is known to be cohomologically concentrated in degrees $[(n-1), 2(n-1)]$ (see [38] Thm. 1.1). Consequently,

$$\begin{aligned}\tau^{\leqslant n} R\Gamma(i_*Ri^!\mathbb{Z}_p(n)) &\simeq \tau^{\leqslant n} R\Gamma(\tau^{\leqslant n} i_*Ri^!\mathbb{Z}_p(n)) \\ &\simeq \tau^{\leqslant n} R\Gamma\left(\tau^{\leqslant n}\left(i_*\mathbb{Z}_p(n-1)[-2]\right)\right) \simeq 0.\end{aligned}$$

Therefore (2.31) implies

$$\begin{aligned}\tau^{<n} R\Gamma(\mathcal{X}, \mathbb{Z}_p(n)) &\simeq \tau^{<n} R\Gamma(\mathcal{X}, Rj_*\mathbb{Z}_p(n)) \simeq \tau^{<n} R\Gamma(\mathcal{X}, Rj_*\tau^{\leqslant n}\mathbb{Z}_p(n)) \\ &\simeq \tau^{<n} R\Gamma(\mathcal{X}, Rj_*\mu_{p^\infty}^{\otimes n}) \simeq \tau^{<n} R\Gamma\big(\mathcal{X}[\tfrac{1}{p}], \mu_{p^\infty}^{\otimes n}\big),\end{aligned} \tag{2.32}$$

where the third quasi-isomorphism is [38] Thm. 2.6. Theorem 2.8 provides the decomposition

$$R\pi[1/p]_*\mu_{p^\bullet}^{\otimes n} \;\simeq\; \mu_{p^\bullet}^{\otimes n} \oplus {}^pR^1\pi[1/p]_*\mu_{p^\bullet}^{\otimes n}[-1] \oplus \mu_{p^\bullet}^{\otimes(n-1)}[-2].$$

A direct comparison of the cohomology groups of both sides (or, alternatively, Remark 2.9) shows that ${}^pR^1\pi[1/p]_*\mu_{p^\bullet}^{\otimes n}[-1]$ is concentrated in positive degrees. So, by virtue of (2.32), one has

$$\begin{aligned}\tau^{<1}\,{}^pR^1\Gamma(\mathcal{X}, \mathbb{Z}(n)/p^\bullet) &\simeq \tau^{<1}\,{}^pR^1\Gamma(\mathcal{X}[1/p], \mu_{p^\bullet}^{\otimes n}) \\ &\simeq \tau^{<1} R\Gamma\big(\mathcal{X}[1/p], {}^pR^1\pi[1/p]_*\mu_{p^\bullet}^{\otimes n}[-1]\big) = 0,\end{aligned}$$

proving the proposition. □

2.4.5 Weil-étale cohomology

Flach's and Morin's work in [8] is founded on their insight that even in the absence of any Weil-étale topos one may construct a Weil-étale cohomology complex $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n))$ — in terms of which the special value conjectures are then formulated — utilizing Artin-Verdier duality. We recall their definitions.

Flach and Morin use perfectness of the pairing (2.25) to construct a morphism

$$\alpha_{\mathcal{X},n} : R\mathrm{Hom}(R\Gamma(\mathcal{X}, \mathbb{Q}(2-n)), \mathbb{Q}[-6]) \longrightarrow R\Gamma(\overline{\mathcal{X}}, \mathbb{Z}(n))$$

(cf. [8] Thm. 3.5) whose induced maps on cohomology $H^i(\alpha_{\mathcal{X},n})$ have image equal to the divisible part of $H^{i,n}(\overline{\mathcal{X}})$. In other words, they factor as follows:

$$H^i(\alpha_{\mathcal{X},n}) : \mathrm{Hom}_{\mathbb{Q}}(H^{6-i,2-n}(\overline{\mathcal{X}}) \otimes \mathbb{Q}, \mathbb{Q}) \twoheadrightarrow H^{i,n}(\overline{\mathcal{X}})_{\mathrm{div}} \hookrightarrow H^{i,n}(\overline{\mathcal{X}}). \tag{2.33}$$

Weil-étale cohomology $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n))$ is defined as the mapping cone of $\alpha_{\mathcal{X},n}$, i.e. one has a distinguished triangle

$$R\mathrm{Hom}(R\Gamma(\mathcal{X}, \mathbb{Q}(2-n)), \mathbb{Q}[-6]) \xrightarrow{\alpha_{\mathcal{X},n}} R\Gamma(\overline{\mathcal{X}}, \mathbb{Z}(n)) \longrightarrow R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n)) \longrightarrow.$$

We write $H_W^{i,n}(\overline{\mathcal{X}}) := H^i(R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n)))$. From the associated long exact sequence and the factoring (2.33) one easily deduces

$$H_W^{i,n}(\overline{\mathcal{X}}) \;\cong\; H^{i,n}(\overline{\mathcal{X}})_{\mathrm{codiv}} \oplus \mathrm{Hom}(H^{5-i,2-n}(\overline{\mathcal{X}}), \mathbb{Z}). \tag{2.34}$$

In particular, $\mathbf{L}(\mathcal{X},n)$ ensures that all $H^{i,n}_W(\overline{\mathcal{X}})$ are finitely generated. (2.34) gives the Weil-étale cohomology groups listed in (A.18) and for $n=1$ we get the splittings

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i\geqslant 6$
$H^{i,1}_W(\overline{\mathcal{X}})$	$\mathcal{O}^\times$	$\operatorname{Pic}\mathcal{X}$	$Ш(X/F)\oplus(\operatorname{Pic}\mathcal{X})^*$	$\operatorname{Tor}\operatorname{Pic}\mathcal{X}\oplus(\mathcal{O}^\times)^*$	μ_F	0
$H^{i,1}_W(\overline{S})$	$\mathcal{O}^\times$	Cl_F	$\mathbb{Z}$	0	0	0
${}^1H^{i,1}_W(\overline{\mathcal{X}})$	0	$\operatorname{Pic}^0\mathcal{X}/\mathrm{Cl}_F$	$Ш(X/F)\oplus(\operatorname{Pic}^0\mathcal{X})^*$	$\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}\,/\,\mathrm{Cl}_F$	0	0
$H^{i-2,0}_W(\overline{S})$	0	$\mathbb{Z}$	0	$\mathrm{Cl}_F\oplus(\mathcal{O}^\times)^*$	μ_F	0

2.5 Betti cohomology and Weil-étale cohomology with compact support

The long exact sequence induced by the regulator map splits motivic cohomology into a compactly supported part and an infinite part given by Deligne cohomology. In this section we work out the analogous decomposition for Weil-étale motivic cohomology.

Betti cohomology. Let $R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))$ be defined via the exact triangle

$$R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))\longrightarrow R\Gamma(G_\mathbb{R},\mathcal{X}(\mathbb{C}),(2\pi i)^n\mathbb{Z})\longrightarrow R\Gamma(\mathcal{X}(\mathbb{R}),\tau^{>n}R\widehat{\pi}_*(2\pi i)^n\mathbb{Z})\longrightarrow \quad (2.35)$$

(cf. [8] Def. 3.23). Since the rightmost complex is entirely 2-torsion we have

$$R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))\otimes\mathbb{R}\;\simeq\;R\Gamma(G_\mathbb{R},\mathcal{X}(\mathbb{C}),(2\pi i)^n\mathbb{Z})\otimes\mathbb{R}\;\simeq\;R\Gamma(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))^{G_\mathbb{R}}.$$

We write $H^{i,n}_{W,\infty}(\mathcal{X}):=H^i(R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n)))$. We evaluate the singular cohomology groups on the right hand side directly and get ranks as in table (A.13). The torsion groups of $H^{i,n}_{W,\infty}(\mathcal{X})$ are computed in Appendix A.2.

Lemma A.11 shows that $R\Gamma(\mathcal{X}(\mathbb{R}),\tau^{>n}R\widehat{\pi}_*(2\pi i)^n\mathbb{Z})$ and $R\Gamma(G_\mathbb{R},\mathcal{X}(\mathbb{C}),(2\pi i)^n\mathbb{Z})$ decompose into motivic degrees. Consequently, the entire triangle (2.35) decomposes and we have

$$H^{i,n}_{W,\infty}(\mathcal{X})\;\cong\;H^{i,n}_{W,\infty}(S)\,\oplus\,{}^1H^{i,n}_{W,\infty}(\mathcal{X})\,\oplus\,H^{i-2,n-1}_{W,\infty}(S).$$

Compactly supported Weil-étale cohomology. There is a canonical map $u_\infty^*:R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(n))\to R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))$ since $R\Gamma(\overline{\mathcal{X}}_{\text{ét}},\mathbb{Z}(n))$ can be regarded as the mapping fiber of the composition

$$R\Gamma(\mathcal{X}_{\text{ét}},\mathbb{Z}(n))\to R\Gamma_\mathcal{D}(\mathcal{X}/_\mathbb{R},\mathbb{Z}(n))\to R\Gamma(G_\mathbb{R},\mathcal{X}(\mathbb{C}),(2\pi i)^n\mathbb{Z})\to R\Gamma(\mathcal{X}(\mathbb{R}),\tau^{>n}R\widehat{\pi}_*(2\pi i)^n\mathbb{Z}).$$

Flach and Morin have shown that there is a unique $i_\infty^*:R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}(n))\to R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))$ making u_∞^* factor through $R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}(n))$ (cf. [8] Prop. 3.24). $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))$ is defined as the mapping fiber of i_∞^*, i.e. we have a short exact triangle

$$R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n)) \longrightarrow R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}(n)) \xrightarrow{i_\infty^*} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n)) \longrightarrow . \tag{2.36}$$

We write $H^{i,n}_{W,c}(\mathcal{X}) := H^i(R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n)))$ for its cohomology groups. As u_∞^* splits into $\hbar^i$-parts, so does i_∞^* and we obtain the usual decomposition

$$H^{i,n}_{W,c}(\mathcal{X}) \;=\; H^{i,n}_{W,c}(S) \oplus {}^1H^{i,n}_{W,c}(\mathcal{X}) \oplus H^{i-2,n-1}_{W,c}(S).$$

We will evaluate $H^{i,n}_{W,c}(\mathcal{X})$ later, as part of the computation of fundamental lines.

2.6 De Rham and derived de Rham cohomology

Algebraic de Rham cohomology. Let $\Pi : \mathscr{X} \to \mathcal{S}$ be an arithmetic scheme. Recall algebraic de Rham cohomology $R\Gamma_{\mathrm{dR}}(\mathscr{X}/\mathcal{S}) := R\Gamma(\mathscr{X},\Omega^\bullet_{\mathscr{X}/\mathcal{S}})$ and its Hodge filtration given by $R\Gamma_{\mathrm{dR}}(\mathscr{X}/\mathcal{S})/F_n := R\Gamma(\mathscr{X},\Omega^\bullet_{\mathscr{X}/\mathcal{S}}/\mathrm{Fil}_n)$ where

$$\Omega^{<n}_{\mathscr{X}/\mathcal{S}} := \Omega^\bullet_{\mathscr{X}/\mathcal{S}}/\mathrm{Fil}_n := [\mathcal{O}_{\mathscr{X}} \to \Omega_{\mathscr{X}/\mathcal{S}} \to \ldots \to \Omega^{n-1}_{\mathscr{X}/\mathcal{S}}]$$

as a complex in the derived category of sheaves of $\mathcal{O}_{\mathscr{X}}$-modules on $\mathscr{X}_{\mathrm{Zar}}$ concentrated in degrees $[0,n-1]$.

The de Rham complexes of the generic fiber X over $\mathbb{Q}$, of $\mathcal{X}_\infty$ over $\mathbb{R}$, and of $\mathcal{X}(\mathbb{C})$ over $\mathbb{C}$ are related as follows:

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})/F^n = R\Gamma_{\mathrm{dR}}(X/F)/F^n \otimes_{\mathbb{Q}} \mathbb{R} = (R\Gamma_{\mathrm{dR}}(\mathcal{X}(\mathbb{C})/\mathbb{C})/F^n)^{G_\mathbb{R}}.$$

We call $R\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})/F^n$ the *real de Rham complex* and write

$$H^{i,n}_{\mathrm{dR}}(\mathcal{X}) := H^i(R\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})/F^n) = \mathbb{H}^i(\mathcal{X}(\mathbb{C}),\Omega^{<n}_{\mathcal{X}(\mathbb{C})/\mathbb{C}})^{G_\mathbb{R}}.$$

For $n < 1$ these groups are trivial. For $n = 1$ de Rham cohomology simplifies to the cohomology of the structure sheaf $H^{i,1}_{\mathrm{dR}}(\mathcal{X}) = H^i(\mathcal{X}(\mathbb{C}),\mathcal{O}_{\mathcal{X}(\mathbb{C})})^{G_\mathbb{R}}$ which is well-understood. For $n \geqslant 2$ we use the GAGA principle to reduce to analytic cohomology. Recall that $[\mathcal{O}^{\mathrm{an}}_{\mathcal{X}(\mathbb{C})} \to \Omega^{\mathrm{an}}_{\mathcal{X}(\mathbb{C})/\mathbb{C}}]$ is a resolution of $\mathbb{C}^{\mathcal{X}(\mathbb{C})}$ (cf. [37] Lem. 8.13) so that $H^{i,2}_{\mathrm{dR}} = H^i_{\mathrm{an}}(\mathcal{X}(\mathbb{C}),\mathbb{C})^{G_\mathbb{R}}$. We infer the ranks as given in table (A.14).

We will now exhibit an integral structure for $R\Gamma_{\mathrm{dR}}(X/\mathbb{Q})/F^n$. The natural candidate arising from $R\Gamma_{\mathrm{dR}}(\mathcal{X}/\mathbb{Z})/F^n$ turns out to have undesirable properties for $n \geqslant 2$. We thus resort to derived de Rham cohomology.

Derived de Rham cohomology. Derived de Rham cohomology is a variant of the above where the alternating powers $\Omega^k_{\mathscr{X}/\mathcal{S}}$ of the Kähler differentials are replaced with derived alternating powers $L\bigwedge^k L_{\mathscr{X}/\mathcal{S}}$ of the *cotangent complex* $L_{\mathscr{X}/\mathcal{S}}$. Its construction and related notations can be found in Appendix A.4. We write

$$R\Gamma_{\mathrm{ddR}}(\mathscr{X}/\mathcal{S})/F^n := R\Gamma(\mathscr{X}_{\mathrm{Zar}}, L\Omega^\bullet_{\mathscr{X}/\mathcal{S}}/F^n) \quad \text{and} \quad H^{i,n}_{\mathrm{ddR}}(\mathscr{X}) := H^i(R\Gamma_{\mathrm{ddR}}(\mathscr{X}/\mathbb{Z})/F^n).$$

Algebraic and derived de Rham cohomology coincide for smooth schemes. In particular – when denoting the generic fiber of $\mathscr{X} \to \mathcal{S}$ by $\mathcal{X} \to \operatorname{Spec} k$ – one has $L\Omega^\bullet_{\mathscr{X}/\mathcal{S}} \otimes \mathcal{O}_{\mathcal{X}} = L\Omega^\bullet_{\mathcal{X}/k} \simeq \Omega^\bullet_{\mathcal{X}/k}$. Consequently, $R\Gamma_{\mathrm{ddR}}(\mathcal{X}/k)/F^n = R\Gamma_{\mathrm{dR}}(\mathcal{X}/k)/F^n$ attains an integral structure via

$$R\Gamma_{\mathrm{ddR}}(\mathscr{X}/\mathcal{S})/F^n \xrightarrow{-\otimes 1} R\Gamma_{\mathrm{ddR}}(\mathscr{X}/\mathcal{S})/F^n \otimes k = R\Gamma_{\mathrm{ddR}}(\mathcal{X}/k)/F^n.$$

For the remainder of this section we assume $\Pi : \mathscr{X} \to \mathcal{S}$ to be projective and regular. In particular, $\mathscr{X}$ may be any proper regular arithmetic surface or curve. Let $i : \mathscr{X} \to \mathbb{P}^N_{\mathcal{S}}$ be a closed immersion into projective space and let $\mathcal{I}$ denote the sheaf of $\mathcal{O}_{\mathbb{P}^N_{\mathcal{S}}}$-modules generated by the equations defining $\mathscr{X}$.

Lemma 2.21. *In the derived category of sheaves of $\mathcal{O}_{\mathscr{X}}$-modules one has*

$$L_{\mathscr{X}/\mathcal{S}} \simeq \Omega_{\mathscr{X}/\mathcal{S}}.$$

Proof. The morphisms of schemes $\mathscr{X} \overset{i}{\hookrightarrow} \mathbb{P}^N_{\mathcal{S}} \to \mathcal{S}$ induce a distinguished triangle of cotangent complexes

$$i^* L_{\mathbb{P}^N_{\mathcal{S}}/\mathcal{S}} \longrightarrow L_{\mathscr{X}/\mathcal{S}} \longrightarrow L_{\mathscr{X}/\mathbb{P}^N_{\mathcal{S}}} \longrightarrow$$

in the derived category $\mathcal{D}(\mathrm{Mod}_{\mathcal{O}_{\mathscr{X}}})$ of $\mathcal{O}_{\mathscr{X}}$-modules (cf. [16] Prop. 2.1.2 and 2.1.5.6). As $\mathbb{P}^N_{\mathcal{S}} \to \mathcal{S}$ is smooth we have a quasi-isomorphism $L_{\mathbb{P}^N_{\mathcal{S}}/\mathcal{S}} \simeq \Omega_{\mathbb{P}^N_{\mathcal{S}}/\mathcal{S}}$ (cf. [16] Ch. III, Prop. 3.1.2). Moreover, by [16], Prop.3.2.4(ii) one has $L_{\mathscr{X}/\mathbb{P}^N_{\mathcal{S}}} \simeq \mathcal{I}/\mathcal{I}^2[1]$. So, we have

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} i^*\Omega_{\mathbb{P}^N_{\mathcal{S}}/\mathcal{S}} \longrightarrow L_{\mathscr{X}/\mathcal{S}} \longrightarrow$$

in $\mathcal{D}(\mathrm{Mod}_{\mathcal{O}_{\mathscr{X}}})$. Here d is given by the usual Kähler differential. d has to be injective as $\mathscr{H}^{-1}(L_{\mathscr{X}/\mathcal{S}}) = 0$. In fact, by [16] Prop. 3.2.6 one has $L_{\mathscr{X}/\mathcal{S}} = [\mathcal{F} \to \mathcal{G}][+1]$ with $\mathcal{F}, \mathcal{G}$ being finitely generated and locally free. Besides, one has $L_{\mathscr{X}/\mathcal{S}} \otimes \mathcal{O}_{\mathcal{X}} = L_{\mathcal{X}/k} = \Omega_{\mathcal{X}/k}[0]$. Therefore – as $\mathcal{F}$ is torsion-free – the map $\mathcal{F} \to \mathcal{G}$ must be injective and, moreover, $L_{\mathscr{X}/\mathcal{S}} \simeq \mathscr{H}^0(L_{\mathscr{X}/\mathcal{S}})$. In particular,

$$L_{\mathscr{X}/\mathcal{S}} = [\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} i^*\Omega_{\mathbb{P}^N_{\mathcal{S}}/\mathcal{S}}][1] \simeq \Omega_{\mathscr{X}/\mathcal{S}}. \tag{2.37}$$

The last quasi-isomorphism follows from direct inspection or, alternatively, from the short exact sequence (2.1) in [3]. □

We will now compare algebraic and derived de Rham cohomology for $\pi' : \mathcal{X} \to \operatorname{Spec}\mathbb{Z}$. We begin with some preparations. The second quasi-isomorphism in (2.37) can be rewritten as the short exact sequence

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow i^*\Omega_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}} \longrightarrow \Omega_{\mathcal{X}/\mathbb{Z}} \longrightarrow 0.$$

It provides a canonical map ρ' which we regard as a two-term complex

$$\mathscr{C}' = \left[\Omega_{\mathcal{X}/\mathbb{Z}} \otimes \det_{\mathcal{O}_{\mathcal{X}}}\mathcal{I}/\mathcal{I}^2 \xrightarrow{\ \rho'\ } \det_{\mathcal{O}_{\mathcal{X}}} i^*\Omega_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}}\right][1]$$

in degrees -1 and 0 or, equivalently, a map ρ giving rise to *Bloch's complex*

$$\mathscr{C} = \left[\Omega^1_{\mathcal{X}/\mathbb{Z}} \xrightarrow{\ \rho\ } \omega_{\mathcal{X}/\mathbb{Z}}\right][1],$$

where

$$\omega_{\mathcal{X}/\mathbb{Z}} = \operatorname{Hom}\left(\det_{\mathcal{O}_{\mathcal{X}}} \mathcal{I}/\mathcal{I}^2,\ \det_{\mathcal{O}_{\mathcal{X}}} i^*\Omega_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}}\right)$$

is the canonical normal bundle of $\mathcal{X}$ (cf. [3] §2 for Bloch's treatment of $\omega_{\mathcal{X}/\mathbb{Z}}, \rho, \mathscr{C}$).

For a complex $C^\bullet$ in the derived category of abelian groups we write

$$\chi(C^\bullet) := \prod_{i\in\mathbb{Z}} \left(\#\operatorname{Tor} H^i(C^\bullet)\right)^{(-1)^i}$$

for its multiplicative Euler characteristic if it is well-defined. For a complex $\mathscr{F}^\bullet$ of abelian sheaves on $\mathcal{X}$ we write[2]

$$\chi(\mathscr{F}^\bullet) := \chi(R\Gamma(\mathcal{X}, \mathscr{F}^\bullet)) := \prod_{i\in\mathbb{Z}} \left(\#\mathbb{H}^i(\mathcal{X}, \mathscr{F}^\bullet)\right)^{(-1)^i}.$$

Recall the definition of the conductor $A(\mathcal{X}) = \chi(\Omega^\bullet_{\mathcal{X}/\mathbb{Z},\mathrm{tors}})$. We will need

Theorem 2.22. (Bloch, [3] Thm. 2.3) *One has* $A(\mathcal{X}) = \chi(\mathscr{C})$.

Proposition 2.23. *Let $\mathcal{X}$ be a regular arithmetic surface with proper structure map $\pi' : \mathcal{X} \to \operatorname{Spec}\mathbb{Z}$.*

(i) One has $\operatorname{gr}^0(L\Omega_{\mathcal{X}/\mathbb{Z}}) \simeq \mathcal{O}_{\mathcal{X}}$ and $\operatorname{gr}^1(L\Omega_{\mathcal{X}/\mathbb{Z}}) \simeq \Omega_{\mathcal{X}/\mathbb{Z}}[-1]$. Moreover, for $i \geqslant 2$ the graded piece $\operatorname{gr}^i(L\Omega_{\mathcal{X}/\mathbb{Z}})$ is locally quasi-isomorphic to $\mathscr{C}'[-2]$, i.e. for any point x of $\mathcal{X}$ there is a quasi-isomorphism of complexes of stalks

$$\operatorname{gr}^i(L\Omega_{\mathcal{X}/\mathbb{Z}})_x \simeq \mathscr{C}'_x[-2]$$

in the derived category of $\mathcal{O}_{\mathcal{X},x}$-modules. In particular, for $i \geqslant 2$ the complex $\operatorname{gr}^i L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}$ is cohomologically concentrated in degrees 1 and 2.

[2] This is an extension of Bloch's notation in [3] which he only defines for complexes with torsion cohomology. This is, e.g., the case if the base change $\mathscr{F}^\bullet \otimes \mathbb{Q}$ yields a bounded exact complex of finitely generated abelian sheaves.

(ii) Let $n \geqslant 2$. Then F^2/F^n is cohomologically concentrated in degrees 1 and 2 and there is a distinguished triangle

$$R\Gamma(\mathcal{X}, F^2/F^n) \longrightarrow R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n \longrightarrow R\Gamma(\mathcal{X}, \Omega^{\leqslant 1}_{\mathcal{X}/\mathbb{Z}}) \longrightarrow . \tag{2.38}$$

(iii) Let $n \geqslant 2$. We make the following technical assumption:

RP$(\mathcal{X})$ *All special fibers of $\mathcal{X}$ are reduced, or there is a closed immersion $i : \mathcal{X} \to \mathbb{P}^2_{\mathbb{Z}}$ into two-dimensional projective space.*

Then one has

$$\frac{\det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n}{\det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^2} = A(\mathcal{X})^{n-2}, \tag{2.39}$$

where the left hand side is understood as a quotient of lattices in the (1-dimensional) $\mathbb{Q}$-vectorspace $\det_{\mathbb{Q}} R\Gamma_{\mathrm{dR}}(X/\mathbb{Q})/F^n = \det_{\mathbb{Q}} R\Gamma_{\mathrm{dR}}(X/\mathbb{Q})/F^2$.

Proof. (i): $L\Omega^{\leqslant 1}_{\mathcal{X}/\mathbb{Z}} \simeq \Omega^{\leqslant 1}_{\mathcal{X}/\mathbb{Z}}$ is immediate from Lemma 2.21. Now, recall the identity $L_{\mathcal{X}/\mathbb{Z}} = [\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} i^*\Omega_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}}][1]$ from its proof. Note that both $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{I}/\mathcal{I}^2$ and $i^*\Omega_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}}$ are locally free of finite type. So, following Illusie we may express the derived exterior powers of $L_{\mathcal{X}/\mathbb{Z}}$ in terms of Koszul complexes.

$$\begin{aligned}
\mathrm{gr}^i L\Omega_{\mathcal{X}/\mathbb{Z}} &\simeq \left(L{\bigwedge}^i_{\mathcal{X}} L_{\mathcal{X}/\mathbb{Z}}\right)[-i] && \text{[17] Ch. VIII, (2.1.1.5) (p.277)}\\
&\simeq L\Gamma_{\mathcal{X}}(L_{\mathcal{X}/\mathbb{Z}}[-1])^i && \text{[16] I.4.3.2.1(ii), or proof of [17] VIII. Col. 2.1.2.2}\\
&\simeq \mathrm{Kos}^i(\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} i^*\Omega_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}}) && \text{[17] Ch. VIII Lem. 2.1.2.1}\\
&\simeq \left[\Gamma^i_{\mathcal{O}_{\mathcal{X}}}\mathcal{I}/\mathcal{I}^2 \to \Gamma^{i-1}_{\mathcal{O}_{\mathcal{X}}}\mathcal{I}/\mathcal{I}^2 \otimes i^*\Omega_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}} \to \ldots \to \Gamma^{i-N}_{\mathcal{O}_{\mathcal{X}}}\mathcal{I}/\mathcal{I}^2 \otimes {\bigwedge}^N i^*\Omega_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}}\right].
\end{aligned} \tag{2.40}$$

Let $i \geqslant 2$. *We treat the case $N = 2$ first.* Then there is a section f generating $\mathcal{I}$, and $i^*\Omega_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}}$ as well as $\Gamma^i\mathcal{I}/\mathcal{I}^2$ are locally free of rank 2 and rank 1 respectively. Write $\gamma_j(f)$ for the generator of $\Gamma^j\mathcal{I}/\mathcal{I}^2$. Then $\gamma_1(f) = f$. Since $\mathcal{I}/\mathcal{I}^2$ is a line bundle one has $\Gamma^j\mathcal{I}/\mathcal{I}^2 \cong (\mathcal{I}/\mathcal{I}^2)^{\otimes j}$.

Thus, the Koszul complex simplifies to

$$
\begin{aligned}
\mathrm{gr}^i L\Omega_{\mathcal{X}/\mathbb{Z}} &\simeq \Big[\Gamma^i \mathcal{I}/\mathcal{I}^2 \longrightarrow \Gamma^{i-1}\mathcal{I}/\mathcal{I}^2 \otimes i^*\Omega_{\mathbb{P}^2_{\mathbb{Z}}/\mathbb{Z}} \longrightarrow \Gamma^{i-2}\mathcal{I}/\mathcal{I}^2 \otimes {\textstyle\bigwedge^2} i^*\Omega_{\mathbb{P}^2_{\mathbb{Z}}/\mathbb{Z}}\Big] \\
&\qquad \gamma_i(f) \mapsto \gamma_{i-1}(f)\otimes df \\
&\qquad \gamma_{i-1}(f)\otimes\omega \mapsto \gamma_{i-2}(f)\otimes df \wedge \omega \\
&\simeq (\mathcal{I}/\mathcal{I}^2)^{\otimes(i-2)} \otimes \Big[(\mathcal{I}/\mathcal{I}^2)^{\otimes 2} \to \mathcal{I}/\mathcal{I}^2 \otimes i^*\Omega_{\mathbb{P}^2_{\mathbb{Z}}/\mathbb{Z}} \to {\textstyle\bigwedge^2} i^*\Omega_{\mathbb{P}^2_{\mathbb{Z}}/\mathbb{Z}}\Big] \\
&\qquad f\otimes f \mapsto f\otimes df \\
&\qquad f\otimes \omega \mapsto df\wedge\omega \\
&\simeq (\mathcal{I}/\mathcal{I}^2)^{\otimes(i-2)} \otimes \Big[\mathcal{I}/\mathcal{I}^2 \otimes \Omega_{\mathcal{X}/\mathbb{Z}} \xrightarrow{-\wedge \frac{df}{f}} {\textstyle\bigwedge^2} i^*\Omega_{\mathbb{P}^2_{\mathbb{Z}}/\mathbb{Z}}\Big][-1] \\
&\qquad f\otimes\eta \mapsto \overline{\eta}\wedge df \\
&\simeq (\mathcal{I}/\mathcal{I}^2)^{\otimes(i-2)} \otimes \Big[\Omega_{\mathcal{X}/\mathbb{Z}} \otimes \det_{\mathcal{O}_\mathcal{X}} \mathcal{I}/\mathcal{I}^2 \xrightarrow{\rho'} \det_{\mathcal{O}_\mathcal{X}} i^*\Omega_{\mathbb{P}^2_{\mathbb{Z}}/\mathbb{Z}}\Big][-1] \\
&\simeq (\mathcal{I}/\mathcal{I}^2)^{\otimes(i-2)} \otimes \mathscr{C}'[-2].
\end{aligned} \tag{2.41}
$$

Here we have used (2.37) and the fact that taking the tensor product with a line bundle is an exact operation. The claim now follows after passing to stalks. Since the claim is local the general case $N \geqslant 2$ follows from the observation that $\mathcal{X}$ may locally be described by one equation $f(u,v)=0$ as a subscheme of $\operatorname{Spec}\mathbb{Z}[\![u,v]\!]$ (cf. [3], Proof of Lemma 2.4).

(ii): We use the distinguished triangle

$$F^2/F^n \longrightarrow L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n \longrightarrow L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^2 \longrightarrow$$

and apply $R\Gamma(\mathcal{X},-)$ to obtain (2.38). Next, we consider for all $m \geqslant 2$

$$\mathrm{gr}^m L\Omega_{\mathcal{X}/\mathbb{Z}} \longrightarrow F^2/F^{m+1} \longrightarrow F^2/F^m \longrightarrow \tag{2.42}$$

and use $L\Omega^{\leqslant 1}_{\mathcal{X}/\mathbb{Z}} \simeq \Omega^{\leqslant 1}_{\mathcal{X}/\mathbb{Z}}$ to conclude inductively that all F^2/F^m are cohomologically concentrated in degrees 1 and 2. This completes (ii).

(iii): Let $i \geqslant 2$. First assume that $\mathcal{X}$ embeds into $\mathbb{P}^2_{\mathbb{Z}}$. (2.41) shows that there is a line bundle $\mathscr{L}$ on $\mathcal{X}$ such that

$$\mathrm{gr}^i L\Omega_{\mathcal{X}/\mathbb{Z}} \simeq \mathscr{L}\otimes\mathscr{C}'[-2]. \tag{2.43}$$

Therefore (2.42) gives

$$\frac{\chi(F^2/F^{i+1})}{\chi(F^2/F^i)} = \chi(\mathrm{gr}^i L\Omega_{\mathcal{X}/\mathbb{Z}}) = \chi(\mathscr{L}\otimes\mathscr{C}') = \chi(\mathscr{L}\otimes\mathscr{C}).$$

In fact, the third equality holds since ρ arises from ρ' via the adjunction between $\otimes$ and Hom. We will show the identity

$$\chi(\mathscr{L}\otimes\mathscr{C}) = \chi(\mathscr{C}) \tag{2.44}$$

later in this proof. Using (2.44) and Theorem 2.22 one obtains $\chi(F^2/F^n) = A(\mathcal{X})^{n-2}$ inductively. Consequently – by virtue of (2.38) –

$$\frac{\det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n}{\det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^2} = \frac{\det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n}{\det_{\mathbb{Z}} R\Gamma(\mathcal{X},\Omega^{\leqslant 1}_{\mathcal{X}/\mathbb{Z}})} = A(\mathcal{X})^{n-2}. \tag{2.45}$$

It remains to show (2.44). The distinguished triangle

$$\mathscr{L}\otimes\mathrm{Ker}(\rho)[1] \longrightarrow \mathscr{L}\otimes\mathscr{C} \longrightarrow \mathscr{L}\otimes\mathrm{coker}(\rho) \longrightarrow$$

proves that $\chi(\mathscr{L}\otimes\mathscr{C}) = \frac{\chi(\mathscr{L}\otimes\mathrm{coker}(\rho))}{\chi(\mathscr{L}\otimes\mathrm{Ker}(\rho))}$. The same can be said for $\chi(\mathscr{C})$. So, it suffices to prove

$$\frac{\chi(\mathscr{L}\otimes\mathrm{Ker}(\rho))}{\chi(\mathrm{Ker}(\rho))} = \frac{\chi(\mathscr{L}\otimes\mathrm{coker}(\rho))}{\chi(\mathrm{coker}(\rho))}. \tag{2.46}$$

$\mathrm{Ker}(\rho)$ and $\mathrm{coker}(\rho)$ are supported on the at most 1-dimensional subscheme $Z\subset\mathcal{X}$ of non-smooth points of $\mathcal{X}$ (since $\mathrm{Ker}(\rho)\cong\Omega_{\mathcal{X}/\mathbb{Z},\mathrm{tor}}$ and $\mathrm{coker}(\rho)$ is locally isomorphic to $\Omega^2_{\mathcal{X}/\mathbb{Z}}$). Write $Z_{\mathfrak{p}} = Z\cap\mathcal{X}_{\mathfrak{p}}$. We show (2.46) for the restrictions to each $Z_{\mathfrak{p}}$ separately. Fix a prime $\mathfrak{p}$. By abuse of notation we write $\mathscr{L}$ again for the restriction $\mathscr{L}|_{Z_{\mathfrak{p}}}$. When writing $\tilde{\chi}$ for the *additive* Euler characteristic[3] then the Riemann-Roch Theorem for non-reduced curves (cf. [36] Ex. 18.4.S) gives for any coherent sheaf $\mathscr{F}$ on $\mathcal{X}$

$$\tilde{\chi}(\mathscr{L}\otimes\mathscr{F}) - \tilde{\chi}(\mathscr{F}) = \sum_{Z_i\subset Z_{\mathfrak{p}}} \deg_{Z_i^{\mathrm{red}}}\mathscr{L}|_{Z_i^{\mathrm{red}}}\cdot\mathrm{length}_{\mathcal{O}_{\eta_i}}\mathscr{F}_{\eta_i}.$$

Here the sum is taken over all irreducible components Z_i of $Z_{\mathfrak{p}}$ and η_i denotes the generic point of Z_i. Now, (2.46) follows from applying the above formula to $\mathrm{Ker}(\rho)|_{Z_{\mathfrak{p}}}$ and $\mathrm{coker}(\rho)|_{Z_{\mathfrak{p}}}$ and using Bloch's result [3] Lemma 2.5 that

$$\mathrm{length}_{\mathcal{O}_\eta}\mathrm{Ker}(\rho)_\eta = \mathrm{length}_{\mathcal{O}_\eta}\mathrm{coker}(\rho)_\eta$$

for every codimension 1 point η of $\mathcal{X}$.

We now assume all special fibers of $\mathcal{X}$ to be reduced instead of having an embedding of $\mathcal{X}$ into $\mathbb{P}^2_{\mathbb{Z}}$. The subscheme $Z\subset\mathcal{X}$ of non-smooth points is then 0-dimensional. Write

$$\mathscr{H}^j_{\mathrm{Bl}} := \mathscr{H}^j(\mathscr{C}'[-2]) \qquad\text{and}\qquad \mathscr{H}^j_{\mathrm{ddR}} := \mathscr{H}^j(\mathrm{gr}^i L\Omega_{\mathcal{X}/\mathbb{Z}})$$

for the cohomology sheaves in degrees $j=1,2$. (2.41) shows that $\mathscr{H}^j_{\mathrm{Bl}}$ and $\mathscr{H}^j_{\mathrm{ddR}}$ have isomorphic stalks. However, since the $\mathscr{H}^j_{\mathrm{Bl}}$ are supported on the finite collection of points Z

[3] i.e. the alternating sum of the dimensions of the cohomology groups as $k(\mathfrak{p})$-vectorspaces. This means it relates to χ via $N\mathfrak{p}^{\tilde{\chi}(\mathscr{L}\otimes\mathscr{F})-\tilde{\chi}(\mathscr{F})} = \frac{\chi(\mathscr{L}\otimes\mathscr{F})}{\chi(\mathscr{F})}$.

this glues to global isomorphisms of sheaves $\mathscr{H}^j_{\mathrm{Bl}} \cong \mathscr{H}^j_{\mathrm{ddR}}$ for $j = 1, 2$. From the $(\tau^{\leqslant 1}, \tau^{\geqslant 2})$-truncation triangles for $\mathrm{gr}^i L\Omega_{\mathcal{X}/\mathbb{Z}}$ and $\mathscr{C}'[-2]$ we obtain

$$\chi(\mathrm{gr}^i L\Omega_{\mathcal{X}/\mathbb{Z}}) = \frac{\chi(\mathscr{H}^2_{\mathrm{ddR}})}{\chi(\mathscr{H}^1_{\mathrm{ddR}})} = \frac{\chi(\mathscr{H}^2_{\mathrm{Bl}})}{\chi(\mathscr{H}^1_{\mathrm{Bl}})} = \chi(\mathscr{C}') = \chi(\mathscr{C}).$$

The claim follows as in (2.45). □

Remark 2.24. We expect (2.39) to also hold without the assumption $\mathbf{RP}(\mathcal{X})$. However, it is unclear how to construct a line bundle $\mathscr{L}$ satisfying (2.43) from the local isomorphisms (2.41) without an embedding into $\mathbb{P}^2_{\mathbb{Z}}$.

The canonical bundle complex $\omega^\bullet_{\mathcal{X}/\mathbb{Z}}$. Derived de Rham cohomology endows $R\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})$ with integral structures for each $n \geqslant 2$. It will turn out to be most natural to compare them to a further integral structure coming from the complex

$$\omega^\bullet_{\mathcal{X}/\mathbb{Z}} := \left[\mathcal{O}_{\mathcal{X}} \xrightarrow{\rho\circ d} \omega_{\mathcal{X}/\mathbb{Z}}\right].$$

One indeed has $R\Gamma(\mathcal{X}, \omega^\bullet_{\mathcal{X}/\mathbb{Z}})_{\mathbb{Q}} = R\Gamma_{\mathrm{dR}}(X/\mathbb{Q})$. The advantages of using $\omega^\bullet_{\mathcal{X}/\mathbb{Z}}$ are two-fold:

(i) The cohomology of $R\Gamma(\mathcal{X}, \omega^\bullet_{\mathcal{X}/\mathbb{Z}})$ is torsion-free and its induced integral structure on $R\Gamma_{\mathrm{dR}}(X/\mathbb{Q})$ will allow for a formula for the later to be defined correction factor $C(\mathcal{X}, n)$ that does not contain the conductor $A(\mathcal{X})$ or any unspecified torsion cardinalities.

(ii) $R\Gamma(\mathcal{X}, \omega^\bullet_{\mathcal{X}/\mathbb{Z}})$ admits a motivic decomposition fitting neatly into the formalism developed in this chapter (cf. Proposition 2.30).

Proposition 2.25. *One has*

$$\frac{\det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^2}{\det_{\mathbb{Z}} R\Gamma(\mathcal{X}, \omega^\bullet_{\mathcal{X}/\mathbb{Z}})} = A(\mathcal{X}).$$

So, if $\mathbf{RP}(\mathcal{X})$ *holds, one has for any* $n \geqslant 2$

$$\frac{\det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n}{\det_{\mathbb{Z}} R\Gamma(\mathcal{X}, \omega^\bullet_{\mathcal{X}/\mathbb{Z}})} = A(\mathcal{X})^{n-1}. \tag{2.47}$$

Proof. Due to Proposition 2.23(iii) it suffices to prove the claim for $n = 2$. In this case the left hand side equals

$$\frac{\det_{\mathbb{Z}} R\Gamma(\mathcal{X}, \Omega^{\leqslant 1}_{\mathcal{X}/\mathbb{Z}})}{\det_{\mathbb{Z}} R\Gamma(\mathcal{X}, \omega^\bullet_{\mathcal{X}/\mathbb{Z}})} = \chi\left(\left[\Omega^1_{\mathcal{X}/\mathbb{Z}} \xrightarrow{\rho} \omega_{\mathcal{X}/\mathbb{Z}}\right]\right) = A(\mathcal{X}),$$

where the last equality is Theorem 2.22. □

Lemma 2.26. (cf. [3] Lemma 2.2) *Write* $(-)^* = R\mathscr{H}\!om(-, \mathbb{Z})$. *Then one has*

$$(R\pi_*\omega^\bullet_{\mathcal{X}/\mathbb{Z}})^* \simeq R\pi_*\omega^\bullet_{\mathcal{X}/\mathbb{Z}}[+2].$$

In particular, the cohomology of $R\Gamma(\mathcal{X}, \omega^\bullet_{\mathcal{X}/\mathbb{Z}})$ *is torsion-free.*

Decomposition into motivic degrees.

Proposition 2.27. *Suppose $\pi_{\mathbb{C}} : \mathcal{X}(\mathbb{C}) \to S(\mathbb{C})$ has a section $s_{\mathbb{C}} : S(\mathbb{C}) \to \mathcal{X}(\mathbb{C})$, i.e. for each real embedding σ of F one has $X_\sigma(\mathbb{R}) \neq \varnothing$. Let $n \in \mathbb{Z}$. Then $R\Gamma_{\mathrm{dR}}(S_\infty/\mathbb{R})/F^n$ and $R\Gamma_{\mathrm{dR}}(S_\infty/\mathbb{R})/F^{n-1}[-2]$ split off as direct summands of $R\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})/F^n$. When writing ${}^pR^1\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})/F^n[-1]$ for the remaining summand we have a canonical decomposition*

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})/F^n \;\simeq\; R\Gamma_{\mathrm{dR}}(S_\infty/\mathbb{R})/F^n \oplus {}^pR^1\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})/F^n[-1] \oplus R\Gamma_{\mathrm{dR}}(S_\infty/\mathbb{R})/F^{n-1}[-2]$$

in the derived category of abelian groups. Moreover, each summand on the right hand side is cohomologically concentrated in one degree only, i.e.

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})/F^n \;\simeq\; H^{0,n}_{\mathrm{dR}}(S_\infty/\mathbb{R})[0] \oplus {}^1H^{1,n}_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})[-1] \oplus H^{0,n-1}_{\mathrm{dR}}(S_\infty/\mathbb{R})[-2].$$

Proof. The claim is trivial for $n \leqslant 0$. For $n = 1$ it suffices to show that $\mathcal{O}_{S(\mathbb{C})}$ splits off as a direct summand of $R\pi_{\mathbb{C},*}\mathcal{O}_{\mathcal{X}(\mathbb{C})}$. Since $s_{\mathbb{C}}^*\mathcal{O}_{\mathcal{X}(\mathbb{C})} = \mathcal{O}_{S(\mathbb{C})}$ and $\pi_{\mathbb{C}}^*\mathcal{O}_{S(\mathbb{C})} = \mathcal{O}_{\mathcal{X}(\mathbb{C})}$ this follows verbatim as in the proof of Theorem 2.8. (Also cf. Remark 2.9). For $n = 2$ the decomposition is immediate from Proposition 2.15. Finally, a comparison with the ranks in table (A.14) shows that each complex on the right hand side is concentrated in one degree only. □

Proposition 2.28. *Suppose $\pi : \mathcal{X} \to S$ has a section $s : S \to \mathcal{X}$. Then, for any integer n, the complex $L\Omega^\bullet_{S/\mathbb{Z}}/F^n$ splits off as a direct summand of $R\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n$. When writing ${}^pR^{\geqslant 1}\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n[-1]$ for the remaining summand we have a canonical decomposition*

$$R\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n \simeq L\Omega^\bullet_{S/\mathbb{Z}}/F^n \oplus {}^pR^{\geqslant 1}\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n[-1]$$

in the derived category of $\mathcal{O}_F$-modules.

Proof. The cotangent complex formalism provides maps $\ell_\pi : \pi^*L\Omega^\bullet_{S/\mathbb{Z}}/F^n \to L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n$ and $\ell_s : s^*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n \to L\Omega^\bullet_{S/\mathbb{Z}}/F^n$ satisfying $\ell_s \circ s^*\ell_\pi = \mathrm{id}$. Consequently, the resulting maps

$$\varphi_0 : L\Omega^\bullet_{S/\mathbb{Z}}/F^n \longrightarrow R\pi_*\pi^*L\Omega^\bullet_{S/\mathbb{Z}}/F^n \xrightarrow{R\pi_*\ell_\pi} R\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n,$$

$$\psi_0 : R\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n \longrightarrow R\pi_*s_*s^*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n = s^*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n \xrightarrow{\ell_s} L\Omega^\bullet_{S/\mathbb{Z}}/F^n$$

compose to the identity on $L\Omega^\bullet_{S/\mathbb{Z}}/F^n$. So, $L\Omega^\bullet_{S/\mathbb{Z}}/F^n$ splits off as a direct summand of $R\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n$ proving the proposition. □

Remark 2.29. It is unclear whether to expect the existence of a full decomposition

$$R\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n \overset{?}{\simeq} L\Omega^\bullet_{S/\mathbb{Z}}/F^n \oplus {}^pR^1\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n[-1] \oplus L\Omega^\bullet_{S/\mathbb{Z}}/F^{n-1}[-2] \tag{2.48}$$

analogously to Theorem 2.11. In order to replicate its proof one would need a duality result of the kind

$$(R\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n)^* \simeq R\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^{n-1}[-2] \tag{2.49}$$

for a suitable duality operation $(-)^*$ that is analogues to classical Verdier duality Theorem 2.3 or Geisser's duality [12] Thm. 7.3. In any case, one has a decomposition of the integral structure determined by the canonical bundle complex.

Proposition 2.30. *Write $\omega_F = \mathrm{Hom}(\mathcal{O}_F, \mathbb{Z})$ for the different ideal of $\mathcal{O}_F$. Suppose $\pi : \mathcal{X} \to S$ has a section $s : S \to \mathcal{X}$. Then $\mathcal{O}_F$ and $\omega_F[-2]$ split off as direct summands of $R\pi_*\omega^\bullet_{\mathcal{X}/\mathbb{Z}}$. When writing ${}^pR^1\pi_*\omega^\bullet_{\mathcal{X}/\mathbb{Z}}[-1]$ for the remaining summand we have a canonical decomposition*

$$R\pi_*\omega^\bullet_{\mathcal{X}/\mathbb{Z}} \simeq \mathcal{O}_F \oplus {}^pR^1\pi_*\omega^\bullet_{\mathcal{X}/\mathbb{Z}}[-1] \oplus \omega_F[-2]$$

in the derived category of $\mathcal{O}_F$-modules.

Proof. We mimic the proof of Theorem 2.8. Define

$$\varphi_0 : \mathcal{O}_F = R^0\pi_*\omega^\bullet_{\mathcal{X}/\mathbb{Z}} \longrightarrow R\pi_*\omega^\bullet_{\mathcal{X}/\mathbb{Z}},$$

$$\psi_0 : R\pi_*\omega^\bullet_{\mathcal{X}/\mathbb{Z}} \longrightarrow R\pi_*s_*s^*\omega^\bullet_{\mathcal{X}/\mathbb{Z}} = s^*\omega^\bullet_{\mathcal{X}/\mathbb{Z}} \longrightarrow \mathcal{O}_F.$$

The composition

$$\mathcal{O}_F \xrightarrow{\varphi_0} R\pi_*\omega^\bullet_{\mathcal{X}/\mathbb{Z}} \xrightarrow{\psi_0} \mathcal{O}_F \tag{2.50}$$

is the identity. Apply $(-)^* = R\mathscr{H}\!om(-,\mathbb{Z})$ to (2.50) and then shift by -2 degrees. Due to Lemma 2.26 one obtains

$$\omega_F[-2] \xleftarrow{\varphi_0^*[-2]} R\pi_*\omega^\bullet_{\mathcal{X}/\mathbb{Z}} \xleftarrow{\psi_0^*[-2]} \omega_F[-2].$$

We let $\varphi_2 = \psi_0^*[-2]$ and $\psi_2 = \varphi_0^*[-2]$. Again, one has $\psi_2\varphi_2 = (\varphi_0\psi_0)^*[-2] = \mathrm{id}$. Furthermore, $\psi_2\varphi_0 = 0$ and $\psi_0\varphi_2 = 0$ for degree reasons. □

In the absence of a duality result of type (2.49) we will introduce ad hoc definitions to artificially force a splitting of derived de Rham cohomology on the level of determinants. This allows us to use the formalism for motivic decompositions as developed in this chapter on derived de Rham cohomology as well.

Definition 2.31. *Let n be any integer.*

(i) Let ${}^pR^0\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n := L\Omega^\bullet_{S/\mathbb{Z}}/F^n$ and ${}^pR^2\pi_*L\Omega^\bullet_{\mathcal{X}/\mathbb{Z}}/F^n := L\Omega^\bullet_{S/\mathbb{Z}}/F^{n-1}[-2]$. Write

$$\det_{\mathbb{Z}} {}^pR^1\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n := \det_{\mathbb{Z}} {}^pR^{\geqslant 1}\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n \otimes \det_{\mathbb{Z}}^{-1} R\Gamma_{\mathrm{ddR}}(S/\mathbb{Z})/F^{n-1}.$$

(ii) For $i = 0,1,2$ let

$$^{i}A(\mathcal{X}) := \frac{\det_{\mathbb{Z}} {}^pR^i\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^2}{\det_{\mathbb{Z}} {}^pR^i\Gamma(\mathcal{X}, \omega^\bullet_{\mathcal{X}/\mathbb{Z}})}.$$

(iii) For any integer n

$$t^{(n)}_{\mathrm{ddR}}(\mathcal{X}) := \chi(R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n) \quad \textit{and} \quad t^{(n)}_{\mathrm{ddR}}(S) := \chi(R\Gamma_{\mathrm{ddR}}(S/\mathbb{Z})/F^n).$$

Also, let ${}^1t^{(n)}_{\mathrm{ddR}}(\mathcal{X})$ *be defined by the equation*

$$t^{(n)}_{\mathrm{ddR}}(\mathcal{X}) := t^{(n)}_{\mathrm{ddR}}(S) \cdot {}^1t^{(n)}_{\mathrm{ddR}}(\mathcal{X})^{-1} \cdot t^{(n-1)}_{\mathrm{ddR}}(S).$$

Remark 2.32.

(i) The symbol ${}^pR^1\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n$ itself is undefined. However, Definition 2.31(i) ensures that we have the decompositions

$$\det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n = \bigotimes_{i=0,1,2} \left(\det_{\mathbb{Z}} {}^pR^i\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n\right)^{(-1)^i}. \tag{2.51}$$

as well as

$$A(\mathcal{X}) = \prod_{i=0,1,2} {}^iA(\mathcal{X})^{(-1)^i}. \tag{2.52}$$

Similarly, ${}^1t^{(n)}_{\mathrm{ddR}}(\mathcal{X})$ should be thought of as a substitute for the hypothetical Euler characteristic $\chi({}^pR^1\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n)$.

(ii) For any $n \geqslant 1$ one has $t^n_{\mathrm{ddR}}(S) = A(S)^{n-1} = (\#D_F)^{n-1}$ by [8] Prop. 5.35. Moreover, $t^{(1)}_{\mathrm{ddR}}(\mathcal{X}) = 1$ since $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathbb{Z}})$ has no torsion and $H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong H^{-1}(\mathcal{X}, \omega_{\mathcal{X}/\mathbb{Z}}) = 0$ by Serre duality.

(iii) The exact triangle $\mathcal{O}_{\mathcal{X}} \to L\Omega^{\bullet}_{\mathcal{X}/\mathbb{Z}}/F^2 \to \Omega_{\mathcal{X}/\mathbb{Z}}[-1] \to$ and the short exact sequence $0 \to \omega_F \to \mathcal{O}_F \to \Omega_{S/\mathbb{Z}} \to 0$ prove that

$${}^0A(\mathcal{X}) = \frac{\det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(S/\mathbb{Z})/F^2}{\det_{\mathbb{Z}} R\Gamma(S, \mathcal{O}_F)} = \chi(\Omega_{S/\mathbb{Z}}) = A(S),$$

$${}^2A(\mathcal{X}) = \frac{\det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(S/\mathbb{Z})/F^1}{\det_{\mathbb{Z}} R\Gamma(S, \omega_F)} = \chi(\Omega_{S/\mathbb{Z}}) = A(S).$$

Therefore, the decomposition (2.52) becomes

$$A(\mathcal{X}) = A(S) \cdot {}^1A(\mathcal{X})^{-1} \cdot A(S).$$

2.7 Completions of L- and ζ-functions

Let $H^i(\mathcal{X}(\mathbb{C}), \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}$ be the Hodge Decomposition and write $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$. Further, write $(H^{p,p})^{\pm 1}$ for the eigenspace of complex conjugation to the eigenvalue ± 1 and let $h^{p,\pm} = \dim_{\mathbb{C}}(H^{p,p})^{\pm(-1)^p}$ for integral p and $h^{p,\pm} = 0$ otherwise. Define

$$L_\infty(H^i(X), s) := \prod_{p+q=i,\ p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \cdot \prod_{p=q=\frac{i}{2}} \Gamma_{\mathbb{R}}(s-p)^{h^{p,+}} \Gamma_{\mathbb{R}}(s-p+1)^{h^{p,-}}.$$

Obviously $L_\infty(H^i(X), s)$ decomposes into a product over all infinite places of F similarly to (2.1). We analogously write

$$\zeta(\mathcal{X}_\infty, s) := \prod_{i\in\mathbb{Z}} L_\infty(H^i(X), s)^{(-1)^i} \tag{2.53}$$

and define the *completions* of $\zeta(\mathcal{X}, -)$ and $\zeta_{\mathrm{HW}}(X, -)$ to be

$$\zeta(\overline{\mathcal{X}}, s) := \zeta(\mathcal{X}_\infty, s)\zeta(\mathcal{X}, s), \quad \text{and} \quad \zeta_{\mathrm{HW}}(\overline{X}, s) := \zeta(\mathcal{X}_\infty, s)\zeta_{\mathrm{HW}}(X, s).$$

Bloch and Kato conjecture $\zeta(\overline{\mathcal{X}}, s)$ to obey a functional equation.

Conjecture 2.33 (**Functional Equation Conjecture FE**($\mathcal{X}$))**.** *For any complex s one has*

$$A(\mathcal{X})^{\frac{2-s}{2}}\zeta(\overline{\mathcal{X}}, 2-s) = \pm A(\mathcal{X})^{\frac{s}{2}}\zeta(\overline{\mathcal{X}}, s).$$

The lemma below provides special values of the completion factors $L_\infty(H^i(X), s)$. We express them in terms of the special values $\Gamma^*(n)$ of the Γ-function.

Lemma 2.34. *One has*

$$L_\infty(H^0(X), s) = \zeta(S_\infty, s) = \Gamma_{\mathbb{R}}(s)^r\Gamma_{\mathbb{C}}(s)^s \quad \textit{and} \quad L_\infty(H^2(X), s) = \zeta(S_\infty, s-1)$$

and, moreover,

	formula	$s=n$	$\operatorname{ord}_{s=n}$	leading Taylor coefficient at $s=n$
$\zeta(\mathcal{X}_\infty, s)$	$\dfrac{\Gamma_{\mathbb{C}}(s-1)^{r+s}}{\Gamma_{\mathbb{C}}(s)^{mg-s}}$	$n \leqslant 0$	$m(g-1)$	$\left(\frac{2\pi}{n-1}\right)^{r+s}\left(2(2\pi)^{-n}\Gamma^*(n)\right)^{m(1-g)}$
		$n = 1$	$-(r+s)$	$(2\pi)^{r+s}\pi^{m(g-1)}$
		$n \geqslant 2$	0	$\left(\frac{2\pi}{n-1}\right)^{r+s}\left(2(2\pi)^{-n}\Gamma^*(n)\right)^{m(1-g)}$
$L_\infty(H^1(X), s)$	$\Gamma_{\mathbb{C}}(s)^{mg}$	$n \leqslant 0$	$-mg$	$\left(2(2\pi)^{-n}\Gamma^*(n)\right)^{mg}$
		$n \geqslant 1$	0	

Proof. One checks directly that

h^{ij}	$j=0$	$j=1$
$i=0$	m	gm
$i=1$	gm	m

$h^{i,\pm}$	$+$	$-$
$i=0$	$r+s$	s
$i=1$	$r+s$	s

	$\Gamma^*_{\mathbb{C}}(n)$	$\operatorname{ord}_{s=n}\Gamma_{\mathbb{C}}$
$n \leqslant 0$	$2(2\pi)^{-n}\Gamma^*(n)$	-1
$n \geqslant 1$		0

From here the claim is straightforward. $\square$

In particular, we may read (2.53) as a decomposition of $\zeta(\mathcal{X}_\infty, s)$ into factors corresponding to both motivic and cohomological degrees $i = 0, 1, 2$. We may consequently define the completion of each perverse L-function ${}^pL(H^i(\mathcal{X}), s)$ separately:

$$ {}^pL(H^i(\overline{\mathcal{X}}), s) := L_\infty(H^i(X), s)\, {}^pL(H^i(\mathcal{X}), s). $$

Together with (2.52) Conjecture 2.33 decomposes entirely into motivic degree components and since the functional equation for $\zeta_F(s)$ is well-known we arrive at

Corollary 2.35. *Conjecture 2.33 is equivalent to*

$$ {}^1A(\mathcal{X})^{\frac{2-s}{2}}\, {}^pL(H^1(\overline{\mathcal{X}}), 2-s) = \pm\, {}^1A(\mathcal{X})^{\frac{s}{2}}\, {}^pL(H^1(\overline{\mathcal{X}}), s). $$

Chapter 3

The special value conjectures in their Weil-étale formulation

In this chapter we will define and compute the fundamental line $\Delta(\mathcal{X}, n)$ as well as the correction factor $C(\mathcal{X}, n) \in \mathbb{Q}^\times$ for arithmetic surfaces $\mathcal{X}$. $\Delta(\mathcal{X}, n)$ will be a copy of $\mathbb{Z}$ that comes with a distinguished trivialization map $\lambda_\infty(\mathcal{X}, n) : \mathbb{R} \xrightarrow{\cong} \Delta(\mathcal{X}, n) \otimes \mathbb{R}$. $\lambda_\infty(\mathcal{X}, n)$ gives rise to a unique real number up to sign $\Lambda_\infty(\mathcal{X}, n) \in \mathbb{R}^\times/\{\pm 1\}$ signifying the inverse generator of the preimage of $\Delta(\mathcal{X}, n)$. These quantities feature in the conjectures [8] Conj. 5.10 and 5.11, describing the vanishing orders and leading Taylor coefficients at all integers. We formulate them as follows.

Special Value Conjectures. *Let $\mathcal{X}$ be a proper regular arithmetic surface and let n be any integer. Then one has*

(VO)
$$\mathrm{ord}_{s=n}\zeta(\mathcal{X}, s) = \sum_{i \in \mathbb{Z}} (-1)^{i+1} \dim_{\mathbb{R}} H_c^{i,n}(\mathcal{X})$$

(TC)
$$\zeta^*(\mathcal{X}, n) = C(\mathcal{X}, n)\Lambda_\infty(\mathcal{X}, n) \tag{3.1}$$

We will explicate these conjectures using the decompositions into motivic degrees of the various cohomology groups worked out in the last chapter. In particular, for $n = 1$ the above will turn out to be equivalent to the Birch and Swinnerton-Dyer conjecture. This extends the two-dimensional case of the result [8] Thm. 5.26 — i.e. the compatibility of the above conjectures for smooth projective arithmetic surfaces $\mathcal{X}$ with the Tamagawa Number Conjecture — to (not necessarily smooth) proper regular arithmetic surfaces.

We keep the notations from Chapter 2.

3.1 Conjectural vanishing orders and compatibility with BSD

In this section we will explicate [8] Conj. 5.10 for arithmetic surfaces and show compatibility with the rank part of the Birch and Swinnerton-Dyer conjecture.

The Vanishing Order Conjecture. We will formulate the vanishing order conjecture as in [8]. Define

$$R\Gamma_{\mathrm{ar},c}(\mathcal{X},\widetilde{\mathbb{R}}(n)) = R\Gamma_c(\mathcal{X},\mathbb{R}(n)) \oplus R\Gamma_c(\mathcal{X},\mathbb{R}(n))[-1]$$

and write $H^{i,n}_{\mathrm{ar},c}(\mathcal{X}) := H^i(R\Gamma_{\mathrm{ar},c}(\mathcal{X},\widetilde{\mathbb{R}}(n)))$ for its associated cohomology groups. We wish to verify

Conjecture 3.1 (**Vanishing Order Conjecture VO**$(\mathcal{X},n)$)**.** *For all $n \in \mathbb{Z}$ one has*

$$\mathrm{ord}_{s=n}\zeta(\mathcal{X},s) = \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^{i,n}_{\mathrm{ar},c}(\mathcal{X}) = \sum_{i\in\mathbb{Z}}(-1)^{i+1}\dim_{\mathbb{R}} H^{i,n}_c(\mathcal{X}).$$

We begin with a preparational Lemma. We write $\Lambda_{\mathfrak{p}}$ for the abelian group generated by the irreducible components of $\mathcal{X}_{\mathfrak{p}}$ modulo the relation $[\mathcal{X}_{\mathfrak{p}}] = 0$, i.e. modulo the special fiber $\mathcal{X}_{\mathfrak{p}}$ interpreted as the weighted sum of its irreducible components. In other words,

$$\Lambda_{\mathfrak{p}} = \mathrm{CH}^0(\mathcal{X}_{\mathfrak{p}})/[\mathcal{X}_{\mathfrak{p}}].$$

Lemma 3.2. *We have a short exact sequence*

$$0 \longrightarrow \mathrm{Cl}_F \oplus \bigoplus_{\mathfrak{p}\text{ bad}} \Lambda_{\mathfrak{p}} \longrightarrow \mathrm{Pic}\,\mathcal{X} \longrightarrow \mathrm{Pic}\,X \longrightarrow 0. \tag{3.2}$$

Moreover, since ${}^1H^{2,1}(\mathcal{X}) \cong \frac{\mathrm{Pic}^0\mathcal{X}}{\mathrm{Cl}_F}$*, the above can be rewritten as*

$$0 \longrightarrow \bigoplus_{\mathfrak{p}\text{ bad}} \Lambda_{\mathfrak{p}} \longrightarrow {}^1H^{2,1}(\mathcal{X}) \longrightarrow \mathrm{Pic}^0 X \longrightarrow 0 \tag{3.3}$$

Proof. The theory of Chow groups provides the Localization Sequence

$$\mathrm{CH}^1(X,1) \xrightarrow{v} \bigoplus_{\mathfrak{p}} \mathrm{CH}^0(\mathcal{X}_{\mathfrak{p}}) \longrightarrow \mathrm{CH}^1(\mathcal{X}) \longrightarrow \mathrm{CH}^1(X) \longrightarrow 0.$$

We have $\mathrm{CH}^1(X,1) \cong H^0(X,\mathcal{O}_X^\times) = F^\times$ and v is the valuation map. Therefore, after taking the quotient with the image of v the above sequence becomes (3.2). Since $\mathrm{Cl}_F \oplus \bigoplus_{\mathfrak{p}\text{ bad}} \Lambda_{\mathfrak{p}}$ maps into $\mathrm{Pic}^0\mathcal{X}$ (cf. e.g. [26] Thm. (8.1.2)(i)) we also have the short exact sequence

$$0 \longrightarrow \mathrm{Cl}_F \oplus \bigoplus_{\mathfrak{p}\text{ bad}} \Lambda_{\mathfrak{p}} \longrightarrow \mathrm{Pic}^0\mathcal{X} \longrightarrow \mathrm{Pic}^0 X \longrightarrow 0.$$

For the second part of the claim take the quotient with Cl_F. □

Proposition 3.3. *The Vanishing Order Conjecture for $\mathcal{X}$ is equivalent to*

$$\mathrm{ord}_{s=n}\zeta(\mathcal{X},s) = \begin{cases} m(1-g) & \textit{for } n<0 \\ m(1-g)-1 & \textit{for } n=0 \\ r+s-1-\mathrm{rk}\ \mathrm{Pic}\,\mathcal{X} & \textit{for } n=1 \\ -1 & \textit{for } n=2 \\ 0 & \textit{for } n>2 \end{cases} \tag{3.4}$$

or to

$$\mathrm{ord}_{s=n}L(H^1(X),s) = \begin{cases} mg & \textit{for } n \leqslant 0 \\ \mathrm{rk}\ \mathrm{Pic}^0 X & \textit{for } n=1 \\ 0 & \textit{for } n \geqslant 2 \end{cases} \tag{3.5}$$

In particular, $\mathbf{VO}(\mathcal{X},1)$ *is equivalent to the vanishing order part of the Birch and Swinnerton-Dyer conjecture.*

Moreover, $\mathbf{VO}(\mathcal{X},n)$ *holds for* $n \geqslant 2$. *Also,* $\mathbf{VO}(\mathcal{X},n)$ *is compatible with the conjectural functional equation for* $L(H^1(X),s)$. *In particular, for an elliptic surface* $\mathcal{X}=\mathcal{E}$ *defined over* $\mathbb{Z}$ *(i.e.* $S=\mathrm{Spec}\,\mathbb{Z}$*) one knows* $\mathbf{VO}(\mathcal{E},n)$ *for all* $n \neq 1$.

Proof. The equivalence of (3.4) to $\mathbf{VO}(\mathcal{X},n)$ is immediate from table (A.15). We will now show equivalence to (3.5). Proposition/Definition 2.1 gives

$$\mathrm{ord}_{s=n}\zeta(\mathcal{X},s) = \mathrm{ord}_{s=n}\zeta_{\mathrm{HW}}(X,s) - \mathrm{ord}_{s=n}\Pi(\mathcal{X},s).$$

For $n \neq 1$ Lemma 2.2(i) shows $\mathrm{ord}_{s=n}\Pi(\mathcal{X},s)=0$. Hence – for $n \neq 1$ – (3.5) follows from (3.4) when using

$$\zeta_{\mathrm{HW}}(X,s) = \frac{\zeta_F(s)\zeta_F(s-1)}{L(H^1(X),s)}$$

and the well-known formula

$$\mathrm{ord}_{s=n}\zeta_F(s) = \begin{cases} \bar{\epsilon}_n r+s & \text{if } n<0 \\ r+s-1 & \text{if } n=0 \\ -1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases} \tag{3.6}$$

for the vanishing orders of ζ_F. By Lemma 2.2(ii) it remains to show

$$\mathrm{rk}\,\mathrm{Pic}\,\mathcal{X} = \mathrm{rk}\,\mathrm{Pic}\,X + \sum_{\mathfrak{p}\ \mathrm{bad}} (d(\mathfrak{p})-1)\,. \tag{3.7}$$

This in turn follows from the short exact sequence (3.3) since $d(\mathfrak{p})-1 = \mathrm{rk}\,\Lambda_{\mathfrak{p}}$.

For the second part note that $\mathrm{ord}_{s=n}L(H^1(X),s)=0$ for $n \geqslant 2$ since the infinite product expression for $L(H^1(X),s)$ converges for $\mathrm{Re}(s)>3/2$. Assuming the motivic degree 1 part of the conjectural functional equation

$${}^1A(\mathcal{X})^{\frac{s}{2}} L_\infty(H^1(X),s)\, {}^pL(H^1(\mathcal{X}),s) = {}^1A(\mathcal{X})^{\frac{2-s}{2}} L_\infty(H^1(X),2-s)\, {}^pL(H^1(\mathcal{X}),2-s) \tag{3.8}$$

and using the vanishing order part of Lemma 2.34 verifies $\mathrm{ord}_{s=n} L(H^1(X), s) = mg$ for $n \leqslant 0$. The last statement follows since (3.8) is known for X being an elliptic curve over $\mathbb{Q}$ by virtue of the Modularity Theorem. □

3.2 Integral Structures

In the next two sections we present preparatory results for the computation of the fundamental line $\Delta(\mathcal{X}/\mathbb{Z}, n)$ in Section 3.4 and introduce notation for the integral structures that are left implicit in its definition. An overview of their bases can be found in Appendix A.7.

We reserve $v|\infty$ for infinite places of F and let $j = 1, \ldots, g$. From now on we write $M_{\mathcal{B}_1,\mathcal{B}_2}(f)$ for the matrix describing a linear map f between real vectorspaces with specified bases $\mathcal{B}_1$ and $\mathcal{B}_2$.

Integral structures coming from $H^1(\mathcal{X}(\mathbb{C}), \mathbb{C})$. The ± 1-eigenspaces $H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})^{\pm}$ of the $G_{\mathbb{R}}$-action on $H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})$ induced by the $G_{\mathbb{R}}$-action on $\mathcal{X}(\mathbb{C})$ sum to a subgroup of $H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})$ of finite index

$$H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})^+ \oplus H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})^- \subset H^2(\mathcal{X}(\mathbb{C}), \mathbb{Z}). \tag{3.9}$$

Consequently, their images under the base change maps

$$H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})^{\pm} \xrightarrow{-\otimes_{\mathbb{R}} 1} H^1(\mathcal{X}(\mathbb{C}), \mathbb{R})^{\pm}$$

into the summands of the eigenspace decomposition

$$H^1(\mathcal{X}(\mathbb{C}), \mathbb{R}) = H^1(\mathcal{X}(\mathbb{C}), \mathbb{R})^+ \oplus H^1(\mathcal{X}(\mathbb{C}), \mathbb{R})^-$$

are integral lattices of maximal order. Since $H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})$ has a Hodge Decomposition after a base change to $\mathbb{C}$ both groups $H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})^{\pm}$ must have the same rank mg. In fact, Poincaré duality provides a pairing of $H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})$ with itself that restricts to a pairing between complementary eigenspaces

$$\wedge : H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})^+ \times H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})^- \longrightarrow H^2(\mathcal{X}(\mathbb{C}), \mathbb{Z}) \cong \bigoplus_v \mathbb{Z}, \tag{3.10}$$

which in turn is a direct sum of perfect pairings over all infinite places of F. Write $\mathcal{B}^+ = \{\delta_{vj}^+\}_{vj}$ for a basis of the image of $H^1(\mathcal{X}(\mathbb{C}), \mathbb{Z})^+$ in $H^1(\mathcal{X}(\mathbb{C}), \mathbb{R})^+$ and let $\mathcal{B}^- = \{\delta_{vj}^-\}_{vj} = (\mathcal{B}^+)^*$ be the basis of $H^1(\mathcal{X}(\mathbb{C}), \mathbb{R})^-$ dual to $\mathcal{B}^+$. Note that $\mathcal{B}^+ \cup \mathcal{B}^-$ is an $\mathbb{R}$-basis of $H^1(\mathcal{X}(\mathbb{C}), \mathbb{R})$ but does not necessarily generate the lattice $H^1(\mathcal{X}(\mathbb{C}),\mathbb{Z}) \subset H^1(\mathcal{X}(\mathbb{C}),\mathbb{R})$ since (3.9) is generally not an equality[1]. Next, define

$$\delta_{vj}^{+;n} = \begin{cases} (2\pi i)^n \delta_{vj}^{+} & \textit{for } n \textit{ even} \\ (2\pi i)^n \delta_{vj}^{-} & \textit{for } n \textit{ odd} \end{cases} \qquad \text{and} \qquad \delta_{vj}^{-;n} = \begin{cases} (2\pi i)^n \delta_{vj}^{+} & \textit{for } n \textit{ odd} \\ (2\pi i)^n \delta_{vj}^{-} & \textit{for } n \textit{ even} \end{cases}$$

and let $\mathcal{B}^{\pm;n} := \{\delta_{vj}^{\pm;n}\}_{vj} \subset H^1(\mathcal{X}(\mathbb{C}),\mathbb{C})$.

$\mathcal{B}^{+;n}$ is a basis of $H^{1,n}_{W,\infty}(\mathcal{X})_{\mathbb{R}} = H^1(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))^{G_{\mathbb{R}}}$ that generates the integral lattice $H^{1,n}_{W,\infty}(\mathcal{X}) \subset H^{1,n}_{W,\infty}(\mathcal{X})_{\mathbb{R}}$. Moreover, the set $\mathcal{B}^{+,n-1}$ is for $n \geqslant 2$ a basis of $H^{2,n}_{\mathcal{D}}(\mathcal{X}) = H^1(\mathcal{X}(\mathbb{C}),\mathbb{R}(n-1))^{G_{\mathbb{R}}}$ and for $n \leqslant 1$ it is a basis of $H^{1,n-1}_{\mathcal{D}}(\mathcal{X}) = H^1(\mathcal{X}(\mathbb{C}),\mathbb{R}(n-1))^{G_{\mathbb{R}}}$. Their integral structures shall be the $\mathbb{Z}$-lattices generated by $\mathcal{B}^{+;n-1}$. Further, for $n \leqslant 0$ we endow $H^{2,n}_c(\mathcal{X})$ with an integral structure via $H^{2,n}_c(\mathcal{X}) \cong H^{1,n}_{\mathcal{D}}(\mathcal{X})$. Finally, for any fixed integer n we give $H^1(\mathcal{X}(\mathbb{C}),\mathbb{C})^{G_{\mathbb{R}}}$ an integral structure via

$$H^1(\mathcal{X}(\mathbb{C}),\mathbb{C})^{G_{\mathbb{R}}} = H^1(\mathcal{X}(\mathbb{C}),\mathbb{R}(n))^{G_{\mathbb{R}}} \oplus H^1(\mathcal{X}(\mathbb{C}),\mathbb{R}(n-1))^{G_{\mathbb{R}}}$$

i.e. $\mathcal{B}^n_{\mathbb{C}} := \mathcal{B}^{+;n} \cup \mathcal{B}^{+;n-1}$ is its integral basis.

Integral structures coming from $H^2(\overline{\mathcal{X}},\mathbb{Z}(n))$ for $n \geqslant 2$. For each $n \geqslant 2$ we fix a set of generators

$$\mathcal{C}^n = \{c^n_{vj} \mid v|\infty, 1 \leqslant j \leqslant g\}$$

of the image of $H^{2,n}(\overline{\mathcal{X}})$ in $H^2(\overline{\mathcal{X}},\mathbb{Z}(n))_{\mathbb{R}}$. Since $H^{i,n}(\overline{\mathcal{X}})_{\mathbb{R}} \cong H^{i,n}_W(\overline{\mathcal{X}})_{\mathbb{R}}$ for these n, the set $\mathcal{C}^n$ also determines an integral structure on $H^{i,n}_W(\overline{\mathcal{X}})_{\mathbb{R}}$. Further, (2.34) shows that $H^{3,2-n}_W(\overline{\mathcal{X}})_{\mathbb{R}} \cong \mathrm{Hom}(H^{2,n}(\overline{\mathcal{X}}),\mathbb{R})$ for $n \geqslant 2$. We write $c^{2-n}_{vj} \in H^{3,2-n}_W(\overline{\mathcal{X}})_{\mathbb{R}}$ for the cycle class corresponding to c^2_{vj} under this isomorphism. $\mathcal{C}^{2-n} := \{c^{2-n}_{vj} \mid v|\infty, 1{\leqslant}j{\leqslant}g\}$ is a basis of the integral structure of $H^{i,n}_W(\mathcal{X})_{\mathbb{R}}$ determined by $H^{i,n}_W(\mathcal{X})$ since the duality (2.34) is a duality of integral lattices.

If there is no risk of confusion we will also use $\delta^{+;\bullet}_{vj}$ and $c^{\bullet}_{vj}$ to refer to integral elements of $H^{2,n}_c(\mathcal{X})$ for $n \leqslant 0$ and $H^{2,n}_W(\mathcal{X}), H^{3,2-n}_W(\mathcal{X})$ for $n \geqslant 2$ respectively.

Integral structures for de Rham cohomology. One has the decomposition

$$H^1_{\mathrm{dR}}(\mathcal{X}(\mathbb{C})/\mathbb{C})^{G_{\mathbb{R}}} \cong H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}})_{\mathbb{R}} \oplus H^0(\mathcal{X},\omega_{\mathcal{X}})_{\mathbb{R}}.$$

Serre duality for coherent sheaves provides a perfect pairing

$$\wedge : H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}}) \times H^0(\mathcal{X},\omega_{\mathcal{X}}) \longrightarrow H^1(\mathcal{X},\omega_{\mathcal{X}}) \cong \mathbb{Z} \tag{3.11}$$

[1]E.g. if X is an elliptic curve over $\mathbb{Q}$ so that one may write $X(\mathbb{C}) \cong \mathbb{C}/\Lambda$, the matrix B describing the base change from $H^1(\mathcal{X}(\mathbb{C}),\mathbb{Z})$ to $\mathcal{B}$ is given by $B = \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ or $B = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ respectively, depending on whether complex conjugation acts on a basis of the lattice Λ by $\left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$.

of free abelian groups. Let $\mathcal{B}^{10}_{\mathrm{dR}} = \{\omega_{vj}\}_{vj}$ be a basis of the image of $H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}}) \xrightarrow{-\otimes_{\mathbb{R}}1} H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}})_{\mathbb{R}}$ and let $\mathcal{B}^{01}_{\mathrm{dR}} := (\mathcal{B}^{10}_{\mathrm{dR}})^* := \{\eta_{vj}\}_{vj}$ be the basis of $H^0(\mathcal{X},\omega_{\mathcal{X}})$ dual to $\mathcal{B}^{10}_{\mathrm{dR}}$. We will use

$$\mathcal{B}_{\mathrm{dR}} := \mathcal{B}^{10}_{\mathrm{dR}} \cup \mathcal{B}^{01}_{\mathrm{dR}} = \{\omega_{vj},\, \eta_{vj}\}_{vj}.$$

A subtlety needs to be taken into account for the choice of bases of derived de Rham cohomology groups. Let $\widetilde{\mathcal{B}}^{n}_{\mathrm{ddR}}$ and $\widetilde{\mathcal{B}}^{(2);n}_{\mathrm{ddR}}$ be bases of the images of $H^{1,n}_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})$ and $H^{2,n}_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})$ in $H^1_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})$ and $H^2_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})$ under base change to $\mathbb{R}$. Similarly, let $\mathcal{B}^{(2)}_{\mathrm{dR}}$ be a basis of the image of $H^0(S,\omega_F) \xrightarrow{-\otimes_{\mathbb{R}}1} H^2_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})$. After splitting off the motivic degree 0 component (2.47) unravels to

$$\frac{t^{(n)}_{\mathrm{ddR}}(\mathcal{X})}{t^{(n)}_{\mathrm{ddR}}(S)} \cdot \frac{\det M_{\widetilde{\mathcal{B}}^{(2);n}_{\mathrm{ddR}},\mathcal{B}^{(2)}_{\mathrm{dR}}}\left(\mathrm{id}_{H^2(\mathcal{X}_\infty/\mathbb{R})}\right)}{\det M_{\widetilde{\mathcal{B}}^{n}_{\mathrm{ddR}},\mathcal{B}_{\mathrm{dR}}}\left(\mathrm{id}_{H^1(\mathcal{X}_\infty/\mathbb{R})}\right)} = \left(\frac{A(\mathcal{X})}{A(S)}\right)^{n-1}. \tag{3.12}$$

Due to the ad-hoc Definition 2.31(i) the motivic degree 1 and 2 parts of the quotient (2.47) can only be expressed in terms of differently defined integral bases of $H^i_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})$ for $i = 1, 2$ – which we will denote $\mathcal{B}^{n}_{\mathrm{ddR}}, \mathcal{B}^{(2),n}_{\mathrm{ddR}}$. They have to be chosen in such a way that (2.47) holds for each motivic degree component separately. Concretely, we let $\mathcal{B}^{(2);n}_{\mathrm{ddR}}$ be any basis of $H^2_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})$, satisfying

$$t^{(n-1)}_{\mathrm{ddR}}(S) \cdot \det M_{\mathcal{B}^{(2);n}_{\mathrm{ddR}},\mathcal{B}^{(2)}_{\mathrm{dR}}}\left(\mathrm{id}_{H^2(\mathcal{X}_\infty/\mathbb{R})}\right) = \frac{\det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(S/\mathbb{Z})/F^{n-1}}{\det_{\mathbb{Z}} R\Gamma(S,\omega_F)}$$

and then choose $\mathcal{B}^{n}_{\mathrm{ddR}}$ to be such a basis of $H^1_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})$ that (3.12) remains valid when replacing $\widetilde{\mathcal{B}}^{n}_{\mathrm{ddR}}, \widetilde{\mathcal{B}}^{(2),n}_{\mathrm{ddR}}$ with $\mathcal{B}^{n}_{\mathrm{ddR}}, \mathcal{B}^{(2),n}_{\mathrm{ddR}}$. Now the 1-part of (2.47) is

$${}^1t^{(n)}_{\mathrm{ddR}}(\mathcal{X}) \cdot \det M_{\mathcal{B}^{n}_{\mathrm{ddR}},\mathcal{B}_{\mathrm{dR}}}\left(\mathrm{id}_{H^1_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})}\right) = {}^1A(\mathcal{X})^{n-1}. \tag{3.13}$$

Obviously, if we had a duality result of the kind (2.48) one could choose $\mathcal{B}^{n}_{\mathrm{ddR}} = \widetilde{\mathcal{B}}^{n}_{\mathrm{ddR}}$ and $\mathcal{B}^{(2);n}_{\mathrm{dR}} = \widetilde{\mathcal{B}}^{(2);n}_{\mathrm{dR}}$. In any case, the difference will not concern us in the remainder of this thesis.

The Period Isomorphism. Let

$$\Phi : H^1(\mathcal{X}(\mathbb{C}),\mathbb{C}) \xrightarrow{\cong} H^1_{\mathrm{dR}}(\mathcal{X}(\mathbb{C})/\mathbb{C})$$

be the period isomorphism and write $\Phi^{G_{\mathbb{R}}} : H^1(\mathcal{X}(\mathbb{C}),\mathbb{C})^{G_{\mathbb{R}}} \to H^1_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})$ for its restriction to the $G_{\mathbb{R}}$-invariant part.

Lemma 3.4. *One has*

$$\det M_{\mathcal{B}^1,\mathcal{B}_{\mathrm{dR}}}(\Phi^{G_{\mathbb{R}}}) = 1.$$

Consequently, for all integers n,

$$\det M_{\mathcal{B}^n,\mathcal{B}_{\mathrm{dR}}}(\Phi^{G_{\mathbb{R}}}) = (2\pi i)^{2mg(n-1)}. \tag{3.14}$$

Proof. Most of the work towards the above identity is hidden in the definitions of $\mathcal{B}^1 = (2\pi i)\mathcal{B}^- \cup \mathcal{B}^+$ and $\mathcal{B}_{\mathrm{dR}} = \mathcal{B}_{\mathrm{dR}}^{10} \cup \mathcal{B}_{\mathrm{dR}}^{01}$ as self-dual bases. It suffices to show that

$$\det M_{\mathcal{B}^+\cup\mathcal{B}^-,\mathcal{B}_{\mathrm{dR}}^{01}\cup\mathcal{B}_{\mathrm{dR}}^{10}}(\Phi^{G_{\mathbb{R}}}) = (2\pi i)^{-mg}.$$

In other words, we need to calculate the quantity $c \in \mathbb{R}^\times/\{\pm 1\}$ for which

$$\bigwedge^{2gm}\Phi^{G_{\mathbb{R}}} : \bigwedge^{2gm} H^1(\mathcal{X}(\mathbb{C}),\mathbb{C})^{G_{\mathbb{R}}} \longrightarrow \bigwedge^{2gm} H^1_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R}) \tag{3.15}$$

acts as

$$\bigwedge_{\substack{v|\infty \\ 1\leqslant j\leqslant g}} \left(\delta_{vj}^+ \wedge \delta_{vj}^-\right) \longmapsto c\cdot \bigwedge_{\substack{v|\infty \\ 1\leqslant j\leqslant g}} (\omega_{vj}\wedge\eta_{vj}). \tag{3.16}$$

By the Poincaré and Serre dualities (3.10) and (3.11) $\delta_{vj}^+ \wedge \delta_{vj}^-$ and $\omega_{vj}\wedge\eta_{vj}$ are generators of (the v-component of) $H^2(\mathcal{X}(\mathbb{C}),\mathbb{Z})$ and $\mathbb{H}^2(\mathcal{X},\omega^\bullet_{\mathcal{X}/\mathbb{Z}})$, i.e. they correspond to classes in $H^2(\mathcal{X}(\mathbb{C}),\mathbb{C})^{G_{\mathbb{R}}}$ and $H^2_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})$ represented by a point. But it is well-known that the period isomorphism on second cohomology

$$\Phi : H^2(\mathcal{X}(\mathbb{C}),\mathbb{C})^{G_{\mathbb{R}}} \longrightarrow H^2(\mathcal{X}_\infty/\mathbb{R})$$

is just multiplication with $(2\pi i)^{-1}$ (for each v) with respect to point class bases. Consequently, one has $c = (2\pi i)^{-mg}$. □

Finally, note that the period isomorphism restricts to a map

$$\Phi^{10} : H^1(\mathcal{X}(\mathbb{C}),\mathbb{R}(1))^{G_{\mathbb{R}}} \xrightarrow{\cong} H^1(\mathcal{X},\mathcal{O}_{\mathcal{X}})_{\mathbb{R}}.$$

We define

$$\Omega(\mathcal{X}) := \det M_{\mathcal{B}^{+;1},\mathcal{B}_{\mathrm{dR}}^{10}}(\Phi^{10}).$$

The duality isomorphism $h_{B(\mathcal{X},n)}$. Let $h^{(i)}_{B(\mathcal{X},n)} : H^{i,n}(\mathcal{X})_{\mathbb{R}} \xrightarrow{\cong} H_c^{4-i,2-n}(\mathcal{X})^*$ be the isomorphism induced by the conjectural perfect pairing (2.29). It is related to the Beilinson regulator map $\rho_2 : H^{2,n}(\mathcal{X})_{\mathbb{R}} \to H^{2,n}_{\mathcal{D}}(\mathcal{X})$ as follows.

Lemma/Definition 3.5. *Let $n \geqslant 2$. Then one has*

$$\det M_{\mathcal{C}^n,\mathcal{B}^{+;2-n}}(h^{(2)}_{B(\mathcal{X},n)}) = \det M_{\mathcal{C}^n,\mathcal{B}^{+;n-1}}(\rho_2).$$

We write $R^n(\mathcal{X})$ for the above determinants and call it the n-th regulator *of $\mathcal{X}$.*

Proof. The $h^{(i)}_{B(\mathcal{X},n)}$ fit into a commutative diagram (cf. [8] Rmk. 2.6)

$$\begin{array}{ccccccccc}
\longrightarrow & H_c^{i,n}(\mathcal{X}) & \longrightarrow & H^{i,n}(\mathcal{X})_{\mathbb{R}} & \xrightarrow{\rho} & H_{\mathcal{D}}^{i,n}(\mathcal{X}) & \longrightarrow & H_c^{i+1,n}(\mathcal{X}) & \longrightarrow \\
 & \cong\downarrow (h^{(4-i)}_{B(\mathcal{X},2-n)})^* & & \cong\downarrow h^{(i)}_{B(\mathcal{X},n)} & & \cong\downarrow h^{i,n}_{\mathcal{D}} & & \cong\downarrow (h^{(3-i)}_{B(\mathcal{X},2-n)})^* & \\
\xrightarrow{\rho^*} & H^{4-i,2-n}(\mathcal{X})^*_{\mathbb{R}} & \longrightarrow & H_c^{4-i,2-n}(\mathcal{X})^* & \longrightarrow & H_{\mathcal{D}}^{3-i,2-n}(\mathcal{X})^* & \xrightarrow{\rho^*} & H^{3-i,2-n}(\mathcal{X})^*_{\mathbb{R}} & \longrightarrow
\end{array}$$

with exact rows coming from (2.27) and where $h_{\mathcal{D}}^{i,n} : H_{\mathcal{D}}^{i,n}(\mathcal{X}) \cong H_{\mathcal{D}}^{3-i,2-n}(\mathcal{X})$ is the isomorphism induced by the perfect pairing (2.23) of Deligne cohomology groups. Specializing to $i = 2$ yields the commutative square

$$\begin{array}{ccc} H^{2,n}(\mathcal{X})_{\mathbb{R}} & \xrightarrow[\cong]{\rho_2} & H_{\mathcal{D}}^{2,n}(\mathcal{X}) \\ \cong \downarrow h_{B(\mathcal{X},n)}^{(2)} & & \cong \downarrow h_{\mathcal{D}}^{2,n} \\ H_c^{2,2-n}(\mathcal{X})^* & \xrightarrow{\cong} & H_{\mathcal{D}}^{1,2-n}(\mathcal{X})^* \end{array} \tag{3.17}$$

(2.23) simplifies for $i = 2$ to the restriction of the Poincaré duality pairing of algebraic topology

$$H^1(\mathcal{X}(\mathbb{C}), \mathbb{R}(n-1)) \times H^1(\mathcal{X}(\mathbb{C}), \mathbb{R}(2-n)) \longrightarrow H^2(\mathcal{X}(\mathbb{C}), \mathbb{R}(1)).$$

to its $G_{\mathbb{R}}$-equivariant part. Since Poincaré duality also holds integrally $h_{\mathcal{D}}^{2,n}$ does not contribute to the determinant of the upper right decomposition $h_{\mathcal{D}}^{2,n} \circ \rho_2 : H^{2,n}(\mathcal{X})_{\mathbb{R}} \to H_{\mathcal{D}}^{1,2-n}(\mathcal{X})^*$ of the square. Finally, since $H_c^{2,2-n}(\mathcal{X})$ derives its integral structure from the bottom map of (3.17) the claim follows from taking determinants in (3.17). □

3.3 The Regulator $R(\mathcal{X})$

In this section we will revisit Conjecture 2.18 and introduce the additional assumption that $\mathbf{B}(\mathcal{X},n)$ specializes to the Arakelov intersection pairing. We then define the regulator $R(\mathcal{X})$ of an arithmetic surface $\mathcal{X}$ and compare it to the classical regulator $R(X)$ of the generic fiber X. The main result is Proposition 3.11.

The pairing $\mathbf{B}(\overline{\mathcal{X}},n)$. Let σ be a section of the inclusion $\tau^{\leqslant 2n-1} R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}}, \mathbb{R}(n)) \to R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}}, \mathbb{R}(n))$. We define $R\widetilde{\Gamma}(\overline{\mathcal{X}}, \mathbb{R}(n))$ as the mapping fiber of its composition $\sigma \circ \rho$ with Beilinson's regulator map. Write $\widetilde{H}^{i,n}(\overline{\mathcal{X}}) = H^i(R\widetilde{\Gamma}(\overline{\mathcal{X}}, \mathbb{Z}(n)))$. By the work of last chapter we have a decomposition into motivic degrees

$$R\widetilde{\Gamma}(\overline{\mathcal{X}}, \mathbb{R}(n)) \simeq R\widetilde{\Gamma}(\overline{S}, \mathbb{R}(n)) \oplus {}^{p}R^1\widetilde{\Gamma}(\overline{\mathcal{X}}, \mathbb{R}(n))[-1] \oplus R\widetilde{\Gamma}(\overline{S}, \mathbb{R}(n-1))[-2]. \tag{3.18}$$

$R\widetilde{\Gamma}(\overline{\mathcal{X}}, \mathbb{R}(n))$ fits into the 9-Lemma diagram [8] (29) in Section 2.3 whose associated long exact sequences show that $\widetilde{H}^{i,n}$ vanishes for $i \neq 2n$ and that one has

$$\widetilde{H}^{0,0}(\overline{\mathcal{X}}) \cong \widetilde{H}^{0,0}(\overline{S}) = H^{0,0}(S)_{\mathbb{R}} \cong \mathbb{R}, \qquad \widetilde{H}^{4,2}(\overline{\mathcal{X}}) \cong \widetilde{H}^{2,1}(\overline{S}) = H_c^{2,1}(S)_{\mathbb{R}} \cong \mathbb{R} \tag{3.19}$$

as well as the short exact sequence

$$0 \longrightarrow \operatorname{coker}(\rho_1) \longrightarrow \widetilde{H}^{2,1}(\overline{\mathcal{X}}) \longrightarrow H^{2,1}(\mathcal{X})_{\mathbb{R}} \longrightarrow 0. \tag{3.20}$$

It shows that the motivic decomposition of $\widetilde{H}^{2,1}(\overline{\mathcal{X}})$ is given by

$$\widetilde{H}^{2,1}(\overline{\mathcal{X}}) = \mathrm{coker}(\rho_1) \oplus \left(\frac{\mathrm{Pic}^0\mathcal{X}}{\mathrm{Cl}_F}\right)_{\mathbb{R}} \oplus \mathbb{R}. \tag{3.21}$$

$\widetilde{H}^{2n,n}(\overline{\mathcal{X}})$ is isomorphic to the *n-th Arakelov Chow group* $\mathrm{CH}^n(\overline{\mathcal{X}})_{\mathbb{R}}$ (cf. [8] Prop. 2.11) and the pairing $\mathbf{B}(\mathcal{X},n)$ translates into a perfect pairing

$$\widetilde{H}^{2n,n}(\overline{\mathcal{X}}) \times \widetilde{H}^{4-2n,2-n}(\overline{\mathcal{X}}) \;\longrightarrow\; \mathbb{R}. \tag{3.22}$$

Remark 3.6. Flach's and Morin's identification $\widetilde{H}^{2n,n}(\overline{\mathcal{X}}) \cong \mathrm{CH}^n(\overline{\mathcal{X}})_{\mathbb{R}}$ results from an application of the 5-Lemma and thus depends on the choice of a splitting $H^{2,1}(\mathcal{X})_{\mathbb{R}} \to \widetilde{H}^{2,1}(\overline{\mathcal{X}})$ of (3.20). However, the identifications (3.19) and the decomposition (3.21) into motivic degree components provide one such splitting, i.e. we have $\widetilde{H}^{2n,n}(\overline{\mathcal{X}}) \cong \mathrm{CH}^n(\overline{\mathcal{X}})_{\mathbb{R}}$ canonically.

From now on we will assume the following enhanced version of $\mathbf{B}(\mathcal{X},n)$.

Conjecture 3.7 ($\mathbf{B}(\overline{\mathcal{X}},n)$)**.** *Conjecture* $\mathbf{B}(\mathcal{X},n)$ *holds and the perfect pairing*

$$\mathrm{CH}^n(\overline{\mathcal{X}})_{\mathbb{R}} \times \mathrm{CH}^{2-n}(\overline{\mathcal{X}})_{\mathbb{R}} \;\longrightarrow\; \mathbb{R}$$

obtained from (3.22) via the canonical identifications $\widetilde{H}^{2n,n}(\overline{\mathcal{X}}) \cong \mathrm{CH}^n(\overline{\mathcal{X}})_{\mathbb{R}}$ *coincides with the* Arakelov Intersection Pairing $\langle -,-\rangle_{\mathrm{Ar}}$.

For arithmetic surfaces $\langle -,-\rangle_{\mathrm{Ar}}$ is only non-trivial if $n \neq 0,1,2$. Moreover, it only involves information from motivic degree 1 if $n = 1$. We now make the Arakelov pairing explicit for $n = 1$ and compare it to the intersection pairing $\langle -,-\rangle_{\cap}$ from algebraic geometry, following [15].

Proposition 3.8. (Arakelov, Hriljac)

(i) (3.21) is an orthogonal decomposition of $\widetilde{H}^{2,1}(\mathcal{X})$*, i.e. one has*

$$\widetilde{H}^{2,1}(\overline{\mathcal{X}}) = \mathrm{coker}(\rho_1) \perp \left(\frac{\mathrm{Pic}^0\mathcal{X}}{\mathrm{Cl}_F}\right)_{\mathbb{R}} \perp \mathbb{R}$$

and $\langle -,-\rangle_{\mathrm{Ar}}$ *is defined on each summand separately.* $\langle -,-\rangle_{\mathrm{Ar}}$ *is negative definite on* $(\mathrm{Pic}^0\mathcal{X}/\mathrm{Cl}_F)_{\mathbb{R}}$ *(cf. [15] Thm. 3.4, Prop. 3.3).*

(ii) Let $D, D' \in \mathrm{Pic}^0\mathcal{X}$ *be fibral divisor classes with support in the special fiber* $\mathcal{X}_{\mathfrak{p}}$*. Then*

$$\langle D, D'\rangle_{\mathrm{Ar}} = \log N\mathfrak{p} \cdot \langle D, D'\rangle_{\cap}$$

(cf. [15] def. of $(D \cdot E)_v$ *in Sec. 2).*

(iii) There is a unique linear splitting

$$\mathrm{Pic}^0 X \longrightarrow (\mathrm{Pic}^0 \mathcal{X})_{\mathbb{Q}}, \quad P \mapsto \boldsymbol{P}$$

of the natural projection $\mathrm{Pic}^0\mathcal{X} \to \mathrm{Pic}^0 X$ *such that the image is orthogonal to all fibral divisor classes in* $\mathrm{Pic}^0\mathcal{X}$ *(cf. [15] Thm. 1.3).*

(iv) Fix an isomorphism $\phi : \mathrm{Pic}^0 X \xrightarrow{\cong} \mathrm{Jac}\, X$*.* X *has a divisor such that its associated canonical height* h *satisfies for all* $P \in \mathrm{Pic}^0 X$ *(cf. [15] Thm. 3.1)*

$$\langle \boldsymbol{P}, \boldsymbol{P} \rangle_{\mathrm{Ar}} = -h(\phi(P)).$$

Definitions of $R(\mathcal{X})$, $R(X)$ and $c_{\mathfrak{p}}(X)$. Let $\mathscr{P}$ be a basis of the image of ${}^1H^{2,1}(\mathcal{X})$ in ${}^1H^{2,1}(\mathcal{X})_{\mathbb{R}} \cong (\mathrm{Pic}^0\mathcal{X}/\mathrm{Cl}_F)_{\mathbb{R}}$. $\mathscr{P}$ also defines an integral basis on $H_c^{2,1}(\mathcal{X})$ due to $H_c^{2,1}(\mathcal{X}) \cong H^{2,1}(\mathcal{X})_{\mathbb{R}}$. Let $\mathscr{P}^*$ be the basis of ${}^1H^{2,1}(\mathcal{X})_{\mathbb{R}}^*$ dual to $\mathscr{P}$. We define the *regulator* $R(\mathcal{X})$ of $\mathcal{X}$ to be

$$R(\mathcal{X}) := \det\left(\langle \mathcal{P}, \mathcal{P}' \rangle_{\mathrm{Ar}}\right)_{\mathcal{P},\mathcal{P}' \in \mathscr{P}} = \det M_{\mathscr{P},\mathscr{P}^*}\left((h^{(2)}_{B(\mathcal{X},1)})^*\right).$$

Next, fix a basis $\mathcal{P} = \{P_i\}_{1 \leqslant i \leqslant \mathrm{rk}\,\mathrm{Pic}^0 X}$ of the image of $\mathrm{Pic}^0 X$ in $(\mathrm{Pic}^0 X)_{\mathbb{R}}$. The regulator $R(X)$ of the generic fiber X equals

$$R(X) = \det\left(\langle \boldsymbol{P}, \boldsymbol{P}' \rangle_{\mathrm{Ar}}\right)_{P,P' \in \mathcal{P}}$$

since the Arakelov Intersection Pairing is by Proposition 3.8(iv) the same as the Neron-Tate height pairing on $\mathrm{Pic}^0\mathcal{X}$.

Now, fix a prime $\mathfrak{p}$. Let $\mathcal{J} \to S_{\mathfrak{p}}$ denote the Neron model of the Jacobian $J = \mathrm{Jac}\, X_{F_{\mathfrak{p}}}$ of the generic fiber of the local surface $\mathcal{X}_{\mathcal{O}_{\mathfrak{p}}}$ over $S_{\mathfrak{p}} = \mathrm{Spec}\,\mathcal{O}_{\mathfrak{p}}$. Let $\tilde{\mathcal{J}} = \mathcal{J}_{\mathfrak{p}}$ denote the special fiber of $\mathcal{J}$ and let $\tilde{\mathcal{J}}^0$ be its identity component. We also write $\mathcal{J}^0 \subset \mathcal{J}$ for the subgroup scheme with generic fiber J and special fiber equal to $\tilde{\mathcal{J}}^0$. We define

$$c_{\mathfrak{p}}(X) := \# \frac{\mathcal{J}(F_{\mathfrak{p}})}{\mathcal{J}^0(F_{\mathfrak{p}})} = \frac{\#\tilde{\mathcal{J}}(k(\mathfrak{p}))}{\#\tilde{\mathcal{J}}^0(k(\mathfrak{p}))}.$$

Decomposition of $R(\mathcal{X})$. Fix a prime $\mathfrak{p}$ of $\mathcal{O}$. Recall the notations $d(\mathfrak{p})$ and $n_j(\mathfrak{p})$ from Lemma 2.2. Also, let $\{C_j^{\mathfrak{p}}\}_{1 \leqslant j \leqslant d(\mathfrak{p})}$ be the reduced irreducible components of $\mathcal{X}_{\mathfrak{p}}$ and let $m_j(\mathfrak{p})$ be the multiplicity of $C_j^{\mathfrak{p}}$ in $\mathcal{X}_{\mathfrak{p}}$. The section $s : S \to \mathcal{X}$ provides a rational point on one component – say $C_{d(\mathfrak{p})}^{\mathfrak{p}}$ – of $\mathcal{X}_{\mathfrak{p}}$. Thus $C_{d(\mathfrak{p})}^{\mathfrak{p}}$ must be simple and cannot decompose further over any algebraic extension of $k(\mathfrak{p})$, i.e. $m_{d(\mathfrak{p})}(\mathfrak{p}) = n_{d(\mathfrak{p})}(\mathfrak{p}) = 1$. We conclude that the set of classes $\mathcal{D}^{\mathfrak{p}} := \{[C_j^{\mathfrak{p}}] \in (\Lambda_{\mathfrak{p}})_{\mathbb{R}}\}_{1 \leqslant j < d(\mathfrak{p})}$ is a basis of the image of $\Lambda_{\mathfrak{p}}$ inside $(\Lambda_{\mathfrak{p}})_{\mathbb{R}}$.

Lemma 3.9. (Raynaud; Bosch, Liu) *Fix a prime $\mathfrak{p}$ of $\mathcal{O}$. The sequence*

$$\begin{array}{ccccc} \mathrm{CH}^0(\mathcal{X}_{\mathfrak{p}}) & \xrightarrow{\alpha} & \mathrm{CH}^0(\mathcal{X}_{\mathfrak{p}}) & \xrightarrow{\beta} & \mathbb{Z} \\ C_j^{\mathfrak{p}} & \mapsto & \sum_{i=1}^{d(\mathfrak{p})} \langle C_i^{\mathfrak{p}}, C_j^{\mathfrak{p}} \rangle_{\cap} \cdot C_i^{\mathfrak{p}} & & \\ & & C_j^{\mathfrak{p}} & \mapsto & m_j(\mathfrak{p}) \end{array} \tag{3.23}$$

is a chain complex and one has

$$\# \frac{\operatorname{Ker} \beta}{\operatorname{Im} \alpha} = c_{\mathfrak{p}}(X) \cdot \prod_{j=1}^{d(\mathfrak{p})} n_j(\mathfrak{p}).$$

Proof. This is [5] Theorem 1.11 applied to the abelian variety $A = \mathcal{J}$. Indeed, the right-most term in Thm 1.11 $qd\mathbb{Z}/d'\mathbb{Z}$ vanishes since $\mathcal{X}_{\mathfrak{p}}$ has a component satisfying $m_{d(\mathfrak{p})}(\mathfrak{p}) = n_{d(\mathfrak{p})}(\mathfrak{p}) = 1$. Moreover, the geometric multiplicities e_j of $C_j^{\mathfrak{p}}$ in $C_j^{\mathfrak{p}}$ (cf. [6] Def. 9.1.3) occurring in [5] Rmk. 1.12 equal 1 since the base change of any reduced curve over a perfect field to its algebraic closure remains reduced (see e.g. [26] example (6.1.7)). □

Remark 3.10. Raynaud has shown the analogue of Lemma 3.9 for algebraically closed residue fields (cf. [26] Prop. 8.12); Bosch and Liu extended it to more general residue fields. As part of his proof Raynaud has shown that, in the case $k(\mathfrak{p}) = \overline{k(\mathfrak{p})}$, the Picard-scheme $\mathrm{Pic}^0_{\mathcal{X}_{\overline{\mathfrak{p}}}}$ is isomorphic to the *group of components* $\tilde{\mathcal{J}}/\tilde{\mathcal{J}}^0$ of $\mathcal{J}$ – a finite étale group scheme that only depends on the generic fiber X. This should serve as intuition for why $\# \frac{\operatorname{Ker} \beta}{\operatorname{Im} \alpha}$ does not depend on the special fiber $\mathcal{X}_{\mathfrak{p}}$ beyond the values of the $n_j(\mathfrak{p})$.

Proposition 3.11. *One has*

$$\begin{aligned} R(\mathcal{X}) &= \pm \frac{1}{(\#\mathrm{Tor}\,\mathrm{Pic}^0 X)^2} \left(\# \frac{\mathrm{Tor}\,\mathrm{Pic}^0 \mathcal{X}}{\mathrm{Cl}_F} \right)^2 \cdot R(X) \cdot \prod_{\mathfrak{p}} \left((\log N\mathfrak{p})^{d(\mathfrak{p})-1} \prod_{j=1}^{d(\mathfrak{p})} n_j(\mathfrak{p}) \right) c_{\mathfrak{p}}(X) \\ &= \pm \frac{1}{(\#\mathrm{Tor}\,\mathrm{Pic}^0 X)^2} \left(\# \frac{\mathrm{Tor}\,\mathrm{Pic}^0 \mathcal{X}}{\mathrm{Cl}_F} \right)^2 \cdot R(X) \cdot \Pi^*(\mathcal{X}, 1) \cdot \prod_{\mathfrak{p}} c_{\mathfrak{p}}(X). \end{aligned}$$

Proof. (3.3) gives a short exact sequence of real vectorspaces

$$0 \longrightarrow \left(\bigoplus_{\mathfrak{p}} \Lambda_{\mathfrak{p}} \right)_{\mathbb{R}} \longrightarrow \left(\frac{\mathrm{Pic}^0 \mathcal{X}}{\mathrm{Cl}_F} \right)_{\mathbb{R}} \longrightarrow \left(\mathrm{Pic}^0 X \right)_{\mathbb{R}} \longrightarrow 0. \tag{3.24}$$

Due to the splitting provided by Proposition 3.8(iii) the above sequence yields an orthogonal decomposition

$$\left(\frac{\mathrm{Pic}^0 \mathcal{X}}{\mathrm{Cl}_F} \right)_{\mathbb{R}} \cong \left(\bigoplus_{\mathfrak{p}} \Lambda_{\mathfrak{p}} \right)_{\mathbb{R}} \perp \left(\mathrm{Pic}^0 X \right)_{\mathbb{R}} \tag{3.25}$$

and we may regard $\mathscr{P}' := \mathcal{P} \cup \bigcup_{\mathfrak{p}} \mathcal{D}^{\mathfrak{p}}$ as a further ($\mathbb{R}$-)basis of ${}^1H^{2,1}(\mathcal{X})_{\mathbb{R}}$. (3.25) gives

$$\begin{aligned} R(\mathcal{X}) &= \left(\det M_{\mathscr{P},\mathscr{P}'}(\mathrm{id})\right)^2 \cdot \det\left(\langle \mathcal{P}, \mathcal{P}'\rangle_{\mathrm{Ar}}\right)_{\mathcal{P},\mathcal{P}'\in\mathscr{P}'} \\ &= \left(\det M_{\mathscr{P},\mathscr{P}'}(\mathrm{id})\right)^2 \cdot \det\left(\langle \boldsymbol{P}, \boldsymbol{P}'\rangle_{\mathrm{Ar}}\right)_{P,P'\in\mathcal{P}} \cdot \prod_{\mathfrak{p}} \det\left(\langle [C_i^{\mathfrak{p}}], [C_j^{\mathfrak{p}}]\rangle\right)_{1\leqslant i,j<d(\mathfrak{p})} \qquad (3.26) \\ &= R(X) \cdot \left(\det M_{\mathscr{P},\mathscr{P}'}(\mathrm{id})\right)^2 \cdot \prod_{\mathfrak{p}} (\log N\mathfrak{p})^{d(\mathfrak{p})-1} \det\left(\langle C_i^{\mathfrak{p}}, C_j^{\mathfrak{p}}\rangle_{\cap}\right)_{1\leqslant i,j<d(\mathfrak{p})} \end{aligned}$$

where the last equation uses Proposition 3.8(ii). We evaluate the remaining factors separately.

First, since (3.3) is an integral exact sequence, $\det M_{\mathscr{P},\mathscr{P}'}(\mathrm{id})$ measures the discrepancy in torsion between $\mathrm{Pic}^0\mathcal{X}/\mathrm{Cl}_F$ and its surrounding terms in (3.3), i.e. one has

$$\det M_{\mathscr{P},\mathscr{P}'}\left(\mathrm{id}_{\,{}^1H^{2,1}(\mathcal{X})_{\mathbb{R}}}\right) = \frac{1}{\#\mathrm{Tor}\,\mathrm{Pic}^0 X} \cdot \# \frac{\mathrm{Tor}\,\mathrm{Pic}^0\mathcal{X}}{\mathrm{Cl}_F}.$$

Finally, recall the sequence (3.23) of the previous Lemma. Since $m_{d(\mathfrak{p})} = 1$ the set $\mathcal{D}^{\mathfrak{p}}$ may also be viewed as a basis of $\mathrm{Ker}\,\beta$. Besides, α is represented by the full intersection matrix $\left(\langle C_i^{\mathfrak{p}}, C_j^{\mathfrak{p}}\rangle_{\cap}\right)_{1\leqslant i,j\leqslant d(\mathfrak{p})}$. It follows that

$$\det\left(\langle C_i^{\mathfrak{p}}, C_j^{\mathfrak{p}}\rangle_{\cap}\right)_{1\leqslant i,j<d(\mathfrak{p})} = \# \frac{\mathrm{Ker}\,\beta}{\mathrm{Im}\,\alpha}.$$

Lemma 3.9 completes the proof. □

3.4 The Fundamental Line

Consider the perfect 3×3-square from [8] Prop. 4.14.

$$\begin{array}{ccccc} R\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})/\mathrm{Fil}^n[-1] & \longrightarrow & R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}}, \mathbb{R}(n)) & \longrightarrow & R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n))_{\mathbb{R}} \\ \| & & \downarrow & & \downarrow \\ R\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})/\mathrm{Fil}^n[-1] & \longrightarrow & R\Gamma_c(\mathcal{X}, \mathbb{R}(n))[1] \oplus R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) & \longrightarrow & R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))_{\mathbb{R}}[1] \\ & & \downarrow & & \downarrow \\ & & R\Gamma(\mathcal{X}, \mathbb{R}(n))[1] \oplus R\Gamma(\mathcal{X}, \mathbb{R}(2-n))^*[-4] & \xrightarrow{\cong} & R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n))_{\mathbb{R}}[1] \end{array} \qquad (3.27)$$

The middle horizontal triangle is exact by the 9-Lemma. So, the *fundamental line*

$$\Delta(\mathcal{X}, n) := \det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) \otimes \det_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n$$

has a trivialization

$$\lambda_\infty(\mathcal{X}, n) : \mathbb{R} \xrightarrow{\cong} \Delta(\mathcal{X}, n) \otimes \mathbb{R}.$$

By the work of the previous chapter the determinants of the complexes in diagram (3.27) decompose entirely into motivic degree components in the presence of a section $s : S \to \mathcal{X}$, satisfying **FPB**(s,n). So, we may write

$$\Delta(\mathcal{X},n) = \Delta(S,n) \otimes {}^1\Delta(\mathcal{X},n)^{-1} \otimes \Delta(S,n-1)$$

as well as

$$\lambda_\infty(\mathcal{X},n) = \lambda_\infty(S,n) \otimes {}^1\lambda_\infty(\mathcal{X},n)^{-1} \otimes \lambda_\infty(S,n-1), \tag{3.28}$$

where ${}^1\Delta(\mathcal{X},n)$ and ${}^1\Lambda_\infty(\mathcal{X},n)$ denote the $\mathbb{Z}$-line and its trivialization coming from the $\hbar^1$-part of the cohomology diagram induced by (3.27). We write $\Lambda_\infty(\mathcal{X},n) \in \mathbb{R}^\times/\{\pm 1\}$ for the inverse of the generator of the inverse image of $\Delta(\mathcal{X},n)$ under $\lambda_\infty(\mathcal{X},n)$, i.e.

$$\lambda_\infty(\mathcal{X},n)(\mathbb{Z}) = \Lambda_\infty(\mathcal{X},n) \cdot \Delta(\mathcal{X},n).$$

Let ${}^1\Lambda_\infty(\mathcal{X},n)$ and $\Lambda_\infty(S,n)$ be defined analogously. (3.28) translates into

$$\Lambda_\infty(\mathcal{X},n) = \frac{\Lambda_\infty(S,n)\Lambda_\infty(S,n-1)}{{}^1\Lambda_\infty(\mathcal{X},n)}.$$

Flach and Morin have worked out $\Lambda_\infty(S,n)$ in [8].

Theorem 3.12. (cf. [8] equ. (92) following Prop. 5.33) *For $n \geqslant 1$ write*

$$R^n(S) = \mathrm{vol}\left(\mathrm{coker}\left(H^{1,n}(S) \xrightarrow{\rho_n} H^{1,n}_{\mathcal{D}}(S)\right)\right),$$

where the volume is taken with respect to the integral structure of $H^{1,n}_{\mathcal{D}}(S)$ coming from $H^{1,n}_{\mathcal{D}}(S) \cong H^0(F_\mathbb{C},(2\pi i)^{n-1}\mathbb{Z})_\mathbb{R}$. Then

(i)
$$\Lambda_\infty(S,n) = \frac{\#T^{2,1-n}_{\overline{S}}}{\#T^{1,1-n}_{\overline{S}}} \cdot R^{1-n}(S) \qquad \textit{for } n \leqslant 0$$

(ii)
$$\Lambda_\infty(S,n) = 2^{(-1)^{n-1}r}(2\pi)^{mn-r\epsilon_n-s} \cdot |D_F|^{\frac{1}{2}-n} \cdot \frac{\#T^{2,n}_{\overline{S}}}{\#T^{1,n}_{\overline{S}}} \cdot R^n(S) \qquad \textit{for } n \geqslant 1$$

The remaining part of this section will be dedicated to the proof of the below analogue of Theorem 3.12 for (the motivic degree 1 part of) arithmetic surfaces.

Theorem 3.13. *With the notations from above and the preceding sections (as well as Appendix A.2 for ${}^1l(\mathcal{X})$), one has*

(i)
$${}^1\Lambda_\infty(\mathcal{X},n) = 2^{\epsilon_n {}^1l(\mathcal{X})} \frac{\#\,{}^1T^{3,2-n}_{\overline{\mathcal{X}}}}{\#\,{}^1T^{2,2-n}_{\overline{\mathcal{X}}} \cdot \#\,{}^1T^{4,2-n}_{\overline{\mathcal{X}}}} \cdot R^{2-n}(\mathcal{X}) \qquad \textit{for } n \leqslant 0$$

(ii)
$$\begin{aligned} {}^1\Lambda(\mathcal{X},1) &= \#Ш(X/F)\left(\#\frac{\mathrm{Tor\,Pic}^0\mathcal{X}}{\mathrm{Cl}_F}\right)^{-2} R(\mathcal{X})\,\Omega(\mathcal{X}) \\ &= \frac{\#Ш(X/F)\,\Omega(\mathcal{X})\,R(X)}{(\#\mathrm{Tor\,Pic}^0 X)^2}\prod_{\mathfrak{p}}\left((\log N\mathfrak{p})^{d(\mathfrak{p})-1}\prod_{j=1}^{d(\mathfrak{p})} n_j(\mathfrak{p})\right)c_\mathfrak{p}(X) \end{aligned}$$

(iii) Suppose that the technical condition $\mathbf{RP}(\mathcal{X})$ *(or the formula (2.39)) holds. Then*

$$ {}^1\Lambda_\infty(\mathcal{X},n) = 2^{\epsilon_n {}^1l(\mathcal{X})} \frac{\#\,{}^1T^{3,n}_{\overline{\mathcal{X}}}}{\#\,{}^1T^{2,n}_{\overline{\mathcal{X}}}\cdot\#\,{}^1T^{4,n}_{\overline{\mathcal{X}}}} \cdot \left(\frac{(2\pi)^{2mg}}{{}^1A(\mathcal{X})}\right)^{n-1} \cdot R^n(\mathcal{X}) \qquad \text{for } n \geqslant 2 $$

Remark 3.14. An alternative expression for ${}^1\Lambda(\mathcal{X},1)$ in terms of $\operatorname{Br}\mathcal{X}$ instead of $\text{Ш}(X/F)$ can be obtained using the comparison sequence induced by (2.26). When defining $r-1 \leqslant c < l(\mathcal{X})$ via the equation

$$ 2^{l(\mathcal{X})-1-c} = \#\operatorname{Ker}\left(\left(\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)^{\vee} \longrightarrow {}^1H^{4,1}(\mathcal{X})\right) $$

then

$$ {}^1\Lambda(\mathcal{X},1) = 2^{-c}\#\operatorname{Br}\mathcal{X}\left(\#\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)^{-2} R(\mathcal{X})\,\Omega(\mathcal{X}). $$

Computing ${}^1\Lambda_\infty(\mathcal{X},n)$ effectively means to compare the two integral structures on $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))_{\mathbb{R}}$, one coming from the vertical and one coming from the horizontal distinguished triangle in (3.27) passing through $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))_{\mathbb{R}}$. (2.36). To do this we will provide explicit descriptions of the maps occuring in the long exact sequences associated to (the $\hbar^1$-part of) (3.27) in terms of the integral bases specified in the preceding two sections.

Contribution from 2-torsion. Before considering each case $n = 1$, $n \geqslant 2$, $n \leqslant 0$ separately we evaluate the contribution of the torsion parts of ${}^1H^{\bullet,n}_{W,\infty}(\mathcal{X})$ and ${}^1H^{\bullet,n}_{W}(\mathcal{X})$. Note that for S these torsion groups explain the occurrence of the factor $2^{(-1)^{n-1}r}$ in the formula for $\Lambda_\infty(S,n)$ with $n \geqslant 1$ since by Corollary A.16(i) and Corollary A.10

$$ \begin{aligned} \frac{\chi\left(R\Gamma_W(S,\mathbb{Z}(n))\right)}{\chi\left(R\Gamma_W(S_\infty,\mathbb{Z}(n))\right)} &= \left(2^{(-1)^n\frac{n+\epsilon_n}{2}r}\right)^{-1}\begin{cases} 2^{(-1)^n\frac{n+\epsilon_n}{2}r}\,\dfrac{\#T^{2,1-n}_{\overline{S}}}{\#T^{1,1-n}_{\overline{S}}} & \text{for } n\leqslant 0\\[2ex] 2^{(-1)^n\left(\frac{n+\epsilon_n}{2}-1\right)r}\,\dfrac{\#T^{2,n}_{\overline{S}}}{\#T^{1,n}_{\overline{S}}} & \text{for } n\geqslant 1\end{cases}\\ &= \begin{cases} \dfrac{\#T^{2,1-n}_{\overline{S}}}{\#T^{1,1-n}_{\overline{S}}} & \text{for } n\leqslant 0\\[2ex] 2^{(-1)^{n-1}r}\,\dfrac{\#T^{2,n}_{\overline{S}}}{\#T^{1,n}_{\overline{S}}} & \text{for } n\geqslant 1.\end{cases} \end{aligned} $$

We will now evaluate the $\hbar^1$-part of the analogue of the quotient above for $\mathcal{X}$. Corollary A.16(ii) and Corollary A.9 show

$$ \begin{aligned} \frac{\chi\left({}^pR^1\Gamma_W(\mathcal{X},\mathbb{Z}(n))\right)}{\chi\left({}^pR^1\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))\right)} &= \chi\left({}^pR^1\Gamma_W(\mathcal{X},\mathbb{Z}(n))\right)\\ &= \begin{cases} \#\text{Ш}(X/F)\left(\#\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)^{-2} & \text{for } n=1\\[2ex] 2^{\epsilon_n {}^1l(\mathcal{X})}\,\dfrac{\#\,{}^1T^{3,n}_{\overline{\mathcal{X}}}}{\#\,{}^1T^{2,n}_{\overline{\mathcal{X}}}\cdot\#\,{}^1T^{4,n}_{\overline{\mathcal{X}}}} & \text{for } n\neq 1.\end{cases} \end{aligned} $$

This explains the leading factors in the formulas of Theorem 3.13.

3.4.1 The Trivialization Factor for $n = 1$

The below diagram depicts the relevant parts of the long exact sequences induced by the $\hbar^1$-part of (3.27). It also displays zero-terms such as $({}^1T^{i,n}_{\mathcal{X}})_{\mathbb{R}}$ if they carry information on the involved integral structures and hence give rise to the numerical value of ${}^1\Lambda_\infty(\mathcal{X}, n)$.

$$
\begin{array}{ccccc}
\left(\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)^*_{\mathbb{R}} & & & & \\
 & & {}^1H^{1,1}_{W,\infty}(\overline{\mathcal{X}})_{\mathbb{R}} & \xrightarrow{\beta_2'} & H^1(\mathcal{X}(\mathbb{C}), \mathcal{O}_{\mathcal{X}(\mathbb{C})})^{G_{\mathbb{R}}} \\
 & & \downarrow & & \Vert \\
\left(\operatorname{Pic}^0\mathcal{X}_{\mathrm{cotor}}\right)_{\mathbb{R}} & \xrightarrow{\alpha_2} & {}^1H^{2,1}_{W,c}(\overline{\mathcal{X}})_{\mathbb{R}} & \xrightarrow{\beta_2} & H^1(\mathcal{X}(\mathbb{C}), \mathcal{O}_{\mathcal{X}(\mathbb{C})})^{G_{\mathbb{R}}} \\
\downarrow & & \downarrow & & \\
\left(\frac{\operatorname{Br}\mathcal{X}}{\operatorname{Br}\mathcal{O}}\right)^*_{\mathbb{R}} \oplus \left(\frac{\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)_{\mathbb{R}} & \xrightarrow{\cong} & \left(\frac{\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)_{\mathbb{R}} & & \\
 & & \downarrow & & \\
 & & {}^1H^{2,1}_{W,\infty}(\overline{\mathcal{X}})_{\mathbb{R}} & & \\
 & & \downarrow & & \\
\left(\operatorname{Pic}^0\mathcal{X}_{\mathrm{cotor}}\right)_{\mathbb{R}} & \xrightarrow{\alpha_3} & {}^1H^{3,1}_{W,c}(\overline{\mathcal{X}})_{\mathbb{R}} & & \\
\downarrow{\scriptstyle (h^{(2)}_{B(\mathcal{X},1)})^*} & & \downarrow & & \\
\left(\frac{\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)^*_{\mathbb{R}} \oplus \left(\frac{\operatorname{Br}\mathcal{X}}{\operatorname{Br}\mathcal{O}}\right)_{\mathbb{R}} & \longrightarrow & \left(\frac{\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)^*_{\mathbb{R}} \oplus Ш(X/F)_{\mathbb{R}} & & \\
 & & \downarrow & & \\
 & & {}^1H^{3,1}_{W,\infty}(\overline{\mathcal{X}})_{\mathbb{R}} & & \\
 & & \downarrow & & \\
 & & {}^1H^{4,1}_{W,c}(\overline{\mathcal{X}})_{\mathbb{R}} & & \\
 & & \downarrow & & \\
\left(\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)_{\mathbb{R}} & \longrightarrow & \left(\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)_{\mathbb{R}}. & &
\end{array}
$$

The quotient of the alternating product of torsion cardinalities associated to the two sequences passing through $H^{\bullet,n}_{W,c}(\mathcal{X})$ equals

$$\#\text{Ш}(X/F)\left(\#\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)^{-2} t^{(1)}_{\mathrm{ddR}}(\mathcal{X}) \;=\; \#\text{Ш}(X/F)\left(\#\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)^{-2}.$$

The integral lattices of ${}^1H^{2,1}_{W,c}(\mathcal{X})_{\mathbb{R}}$ and ${}^1H^{3,1}_{W,c}(\mathcal{X})_{\mathbb{R}}$ characterized by the vertical maps, are generated by $\mathscr{P}\cup\mathcal{B}^{+;1}$ and $\mathscr{P}^*$ respectively. α_2 acts as the identity on $\mathscr{P}$. β_2' is the period isomorphism. Therefore the trivialization factor becomes

$$\begin{aligned}
{}^1\Lambda_\infty(\mathcal{X},n) &= \#\text{Ш}(X/F)\left(\#\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)^{-2}\cdot\det M_{\mathcal{B}^{+;1},\mathcal{B}^{10}_{\mathrm{dR}}}(\beta_2')\cdot\det M_{\mathscr{P},\mathscr{P}^*}(\alpha_3)\\
&= \#\text{Ш}(X/F)\left(\#\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)^{-2}\cdot\det M_{\mathcal{B}^{+;1},\mathcal{B}^{10}_{\mathrm{dR}}}(\Phi^{10})\cdot\det M_{\mathscr{P},\mathscr{P}^*}\left((h^{(2)}_{B(\mathcal{X},1)})^*\right)\\
&= \#\text{Ш}(X/F)\left(\#\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\operatorname{Cl}_F}\right)^{-2}\cdot\Omega(\mathcal{X})\cdot R(\mathcal{X}).
\end{aligned}$$

In light of Proposition 3.11 this may be reformulated as

$${}^1\Lambda_\infty(\mathcal{X},n) = \frac{\#\text{Ш}(X/F)\,\Omega(\mathcal{X})\,R(X)}{(\#\operatorname{Tor}\operatorname{Pic}^0 X)^2}\prod_{\mathfrak{p}}\left((\log N\mathfrak{p})^{d(\mathfrak{p})-1}\prod_{j=1}^{d(\mathfrak{p})} n_j(\mathfrak{p})\right)c_{\mathfrak{p}}(X).$$

3.4.2 Trivialization Factors for $n \geqslant 2$

For $n \geqslant 2$ the $\hbar^1$-part of the diagram on cohomology induced by (3.27) becomes

$$
\begin{array}{ccccccccc}
 & & ({}^1T^{2,2-n}_{\overline{\mathcal{X}}})^*_{\mathbb{R}} & & & & & & \\
 & & & & {}^1H^{1,n}_{W,\infty}(\overline{\mathcal{X}})_{\mathbb{R}} & \xrightarrow{\beta_2'} & H^{1,n}_{\mathrm{dR}} & \xrightarrow{\gamma_2'} & H^{2,n}_{\mathcal{D}} \\
 & & & & \downarrow & & \Vert & & \\
 & & & & {}^1H^{2,n}_{W,c}(\overline{\mathcal{X}})_{\mathbb{R}} & \xrightarrow{\beta_2} & H^{1,n}_{\mathrm{dR}} & & \\
 & & & & \downarrow & & & & \\
 & & ({}^1T^{3,2-n}_{\overline{\mathcal{X}}})^*_{\mathbb{R}} \oplus (\mathbb{Z}^{mg} \oplus {}^1T^{2,n}_{\overline{\mathcal{X}}})_{\mathbb{R}} & \xrightarrow{\cong} & (\mathbb{Z}^{mg} \oplus {}^1T^{2,n}_{\overline{\mathcal{X}}})_{\mathbb{R}} & & & & \\
 & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle 0} & & & & \\
H^{1,n}_{\mathrm{dR}} & \longrightarrow & H^{2,n}_{\mathcal{D}}(\mathcal{X}) & \xrightarrow{0} & {}^1H^{2,n}_{W,\infty}(\overline{\mathcal{X}})_{\mathbb{R}} & & & & \\
\Vert & & & & \downarrow & & & & \\
H^{1,n}_{\mathrm{dR}} & & & & {}^1H^{3,n}_{W,c}(\overline{\mathcal{X}})_{\mathbb{R}} & & & & \\
 & & & & \downarrow & & & & \\
 & & ({}^1T^{4,2-n}_{\overline{\mathcal{X}}})^*_{\mathbb{R}} \oplus ({}^1T^{3,n}_{\overline{\mathcal{X}}})_{\mathbb{R}} & \xrightarrow{\cong} & ({}^1T^{3,n}_{\overline{\mathcal{X}}})_{\mathbb{R}} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & {}^1H^{3,n}_{W,\infty}(\overline{\mathcal{X}})_{\mathbb{R}} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & {}^1H^{4,n}_{W,c}(\overline{\mathcal{X}})_{\mathbb{R}} & & & & \\
 & & & & \downarrow & & & & \\
 & & ({}^1T^{4,n}_{\overline{\mathcal{X}}})_{\mathbb{R}} & \longrightarrow & ({}^1T^{4,n}_{\overline{\mathcal{X}}})_{\mathbb{R}} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & {}^1H^{4,n}_{W,\infty}(\overline{\mathcal{X}})_{\mathbb{R}}. & & & &
\end{array}
$$

The integral structure of ${}^1H^{2,n}_{W,c}(\mathcal{X})_{\mathbb{R}}$ induced by the vertical sequence is generated by $\mathcal{B}^n_c = \mathcal{B}^{+;n} \cup \mathcal{C}^n$. The diagram shows that

$$\begin{aligned}
{}^1\Lambda_\infty(\mathcal{X},n) &= 2^{\epsilon_n {}^1 l(\mathcal{X})} \frac{\#\,{}^1T^{3,n}_{\overline{\mathcal{X}}}}{\#\,{}^1T^{2,n}_{\overline{\mathcal{X}}} \cdot \#\,{}^1T^{4,n}_{\overline{\mathcal{X}}}} \cdot {}^1t^{(n)}_{\mathrm{ddR}}(\mathcal{X})^{-1} \cdot \det M_{\mathcal{B}^n_c,\mathcal{B}^n_{\mathrm{ddR}}}(\beta_2) \\
&= 2^{\epsilon_n {}^1 l(\mathcal{X})} \frac{\#\,{}^1T^{3,n}_{\overline{\mathcal{X}}}}{\#\,{}^1T^{2,n}_{\overline{\mathcal{X}}} \cdot \#\,{}^1T^{4,n}_{\overline{\mathcal{X}}}} \cdot \frac{1}{{}^1A(\mathcal{X})^{n-1}} \cdot \det M_{\mathcal{B}^n_c,\mathcal{B}_{\mathrm{dR}}}(\beta_2) && \text{by (3.13)} \\
&= 2^{\epsilon_n {}^1 l(\mathcal{X})} \frac{\#\,{}^1T^{3,n}_{\overline{\mathcal{X}}}}{\#\,{}^1T^{2,n}_{\overline{\mathcal{X}}} \cdot \#\,{}^1T^{4,n}_{\overline{\mathcal{X}}}} \cdot \left(\frac{(2\pi)^{2mg}}{{}^1A(\mathcal{X})}\right)^{n-1} \cdot \det M_{\mathcal{B}^n_c,\mathcal{B}^n}(\beta_2) && \text{by (3.14)} \\
&= 2^{\epsilon_n {}^1 l(\mathcal{X})} \frac{\#\,{}^1T^{3,n}_{\overline{\mathcal{X}}}}{\#\,{}^1T^{2,n}_{\overline{\mathcal{X}}} \cdot \#\,{}^1T^{4,n}_{\overline{\mathcal{X}}}} \cdot \left(\frac{(2\pi)^{2mg}}{{}^1A(\mathcal{X})}\right)^{n-1} \cdot \det M_{\mathcal{C}^n,\mathcal{B}^{+;n-1}}(\rho_2) && \text{by Def. 3.5} \\
&= 2^{\epsilon_n {}^1 l(\mathcal{X})} \frac{\#\,{}^1T^{3,n}_{\overline{\mathcal{X}}}}{\#\,{}^1T^{2,n}_{\overline{\mathcal{X}}} \cdot \#\,{}^1T^{4,n}_{\overline{\mathcal{X}}}} \cdot \left(\frac{(2\pi)^{2mg}}{{}^1A(\mathcal{X})}\right)^{n-1} \cdot R^n(\mathcal{X}).
\end{aligned}$$

3.4.3 Trivialization Factors for $n \leqslant 0$

Finally, for $n \leqslant 0$ the $\hbar^1$-part of (3.27) equals

$$
\begin{array}{ccc}
({}^1T^{4,2-n}_{\overline{\mathcal{X}}})^*_{\mathbb{R}} & & \\
\downarrow & & \\
H^{1,n}_{\mathcal{D}}(\mathcal{X}) & \xrightarrow{\quad \alpha_2' \quad} & {}^1H^{1,n}_{W,\infty}(\overline{\mathcal{X}})_{\mathbb{R}} \\
\downarrow{\scriptstyle\cong} & & \downarrow \\
H^{2,n}_{c}(\mathcal{X}) & \xrightarrow{\quad \alpha_2 \quad} & {}^1H^{2,n}_{W,c}(\overline{\mathcal{X}})_{\mathbb{R}} \\
\downarrow{\scriptstyle 0} & & \downarrow \\
({}^1T^{4,2-n}_{\overline{\mathcal{X}}})_{\mathbb{R}} \oplus ({}^1T^{3,2-n}_{\overline{\mathcal{X}}})^*_{\mathbb{R}} & \xrightarrow{\quad \cong \quad} & ({}^1T^{4,2-n}_{\overline{\mathcal{X}}})_{\mathbb{R}} \\
 & & \downarrow \\
 & & {}^1H^{2,n}_{W,\infty}(\overline{\mathcal{X}})_{\mathbb{R}} \\
 & & \downarrow \\
H^{2,n}_{c}(\mathcal{X}) & \xrightarrow{\quad \alpha_3 \quad} & {}^1H^{3,n}_{W,c}(\overline{\mathcal{X}})_{\mathbb{R}} \\
\downarrow{\scriptstyle (h^{(2)}_{B(\mathcal{X},2-n)})^*} & & \downarrow \\
({}^1T^{3,2-n}_{\overline{\mathcal{X}}})_{\mathbb{R}} \oplus (\mathbb{Z}^{mg} \oplus {}^1T^{2,2-n}_{\overline{\mathcal{X}}})^*_{\mathbb{R}} & \xrightarrow{\cong} & (\mathbb{Z}^{mg} \oplus {}^1T^{3,2-n}_{\overline{\mathcal{X}}})_{\mathbb{R}} \\
 & & \downarrow \\
 & & {}^1H^{3,n}_{W,\infty}(\overline{\mathcal{X}})_{\mathbb{R}} \\
 & & \downarrow \\
 & & {}^1H^{4,n}_{W,c}(\overline{\mathcal{X}})_{\mathbb{R}} \\
 & & \downarrow \\
({}^1T^{2,2-n}_{\overline{\mathcal{X}}})_{\mathbb{R}} & \xrightarrow{\quad \cong \quad} & ({}^1T^{2,2-n}_{\overline{\mathcal{X}}})_{\mathbb{R}}.
\end{array}
$$

The vertical sequence endows ${}^1H^{2,n}_{W,c}(\mathcal{X})_{\mathbb{R}}$ and ${}^1H^{3,n}_{W,c}(\mathcal{X})_{\mathbb{R}}$ with integral structures generated by $\mathcal{B}^{+;n}$ and $\mathcal{C}^{2-n}$. So, since $\det M_{\mathcal{B}^{+;n},\mathcal{B}^{+;n}}(\alpha_2) = 1$, the diagram shows

$$
{}^1\Lambda_\infty(\mathcal{X}, n) = 2^{\epsilon_n {}^1 l(\mathcal{X})} \frac{\#\, {}^1T^{3,2-n}_{\overline{\mathcal{X}}}}{\#\, {}^1T^{2,2-n}_{\overline{\mathcal{X}}} \cdot \#\, {}^1T^{4,2-n}_{\overline{\mathcal{X}}}} \cdot \det M_{\mathcal{B}^{+;n},\mathcal{C}^{2-n}}(\alpha_3).
$$

The proof of Theorem 3.13 is complete after observing that the commutative square involving α_3 shows

$$
\det M_{\mathcal{B}^{+;n},\mathcal{C}^{2-n}}(\alpha_3) = \det M_{\mathcal{B}^{+;n},\mathcal{C}^{2-n}}((h^{(2)}_{B(\mathcal{X},2-n)})^*) = R^{2-n}(\mathcal{X}).
$$

3.5 The Correction Factor

Definition of $C(\mathcal{X},n)$. Geisser has shown that the étale topology of a curve equals the *eh*-topology of the corresponding reduced curve. Therefore the definition of $C(\mathcal{X},n)$ in [8] Sec. 5.3, 5.4 simplifies for $\pi:\mathcal{X}\to S$ as follows.

Definition 3.15. *For each prime p and $n\in\mathbb{Z}$ let*

$$\chi(\mathcal{X}_{\mathbb{F}_p},\mathcal{O},n)=\sum_{\substack{0\leqslant k<n,\\ j\in\mathbb{Z}}}(-1)^{k+j}(n-k)\dim_{\mathbb{F}_p}H^j(\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p},\Omega^k_{\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p}/\mathbb{F}_p}).$$

In particular, $\chi(\mathcal{X}_{\mathbb{F}_p},\mathcal{O},n)=0$ for all p whenever $n\leqslant 0$.

Conjecture/Definition 3.16. *For any prime p and $n\in\mathbb{Z}$ one has a distinguished triangle in the derived category of $\mathbb{Q}_p$-vectorspaces*

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}_{\mathbb{Q}_p}/\mathbb{Q}_p)/F^n[-1]\longrightarrow R\Gamma(\mathcal{X}_{\mathbb{Z}_p},\mathbb{Q}_p(n))\longrightarrow R\Gamma(\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p},\mathbb{Q}_p(n))\longrightarrow. \tag{3.29}$$

In particular, there is a trivialization

$$\lambda_p(\mathcal{X},n):\left(\det_{\mathbb{Z}_p}R\Gamma(\mathcal{X}_{\mathbb{Z}_p},\mathbb{Z}_p(n))\right)_{\mathbb{Q}_p}\xrightarrow{\cong}\left(\det_{\mathbb{Z}_p}R\Gamma(\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p},\mathbb{Z}_p(n))\otimes\det^{-1}_{\mathbb{Z}_p}R\Gamma_{\mathrm{ddR}}(\mathcal{X}_{\mathbb{Z}_p}/\mathbb{Z}_p)/F^n\right)_{\mathbb{Q}_p} \tag{3.30}$$

that specifies a power $\Lambda_p(\mathcal{X},n)=\det_{\mathbb{Z}_p}\lambda_p(\mathcal{X},n)\in\mathbb{Q}_p^\times/\mathbb{Z}_p^\times$ of p. We let $c_p(\mathcal{X},n)=p^{\chi(\mathcal{X}_{\mathbb{F}_p},\mathcal{O},n)}\cdot\Lambda_p(\mathcal{X},n)$ and define the correction factor *for $\mathcal{X}$ and n to be*

$$C(\mathcal{X},n)=\prod_{p<\infty}|c_p(\mathcal{X},n)|_p.$$

The proof of [8] Prop. 5.9 shows that $c_p(\mathcal{X},n)$ is trivial unless $p\leqslant n+1$ or $\mathcal{X}_{\mathbb{F}_p}$ is a bad fiber. Thus, $C(\mathcal{X},n)$ is well-defined. Moreover, one has $C(\mathcal{X},n)=1$ for $n\leqslant 0$ (cf. [8] Prop. 5.7).

The leading Taylor coefficient conjecture. Write $\zeta^*(\mathcal{X},n)$ for the leading coefficient of the Taylor expansion of $\zeta(\mathcal{X},s)$ at $s=n$. Define $\zeta^*(\mathcal{X}_\infty,n)$ and $\zeta^*(\overline{\mathcal{X}},n)$ analogously. We can now formulate Flach's and Morin's conjectural description of $\zeta^*(\mathcal{X},n)$ (cf. [8] Conj. 5.11).

Conjecture 3.17 (Leading Taylor Coefficient Conjecture TC($\mathcal{X}$,n)). *For any integer n one has*

$$\zeta^*(\mathcal{X},n)=C(\mathcal{X},n)\Lambda_\infty(\mathcal{X},n).$$

Decomposition of $C(\mathcal{X},n)$ into motivic degrees. Let

$$^{i}\chi(\mathcal{X}_{\mathbb{F}_p},\mathcal{O},n) := (-1)^i \sum_{0\leqslant k\leqslant n} (n-k)\dim_{\mathbb{F}_p} H^{i-k}(\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p},\Omega^k_{\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p}/\mathbb{F}_p})$$

i.e. $^{i}\chi(\mathcal{X}_{\mathbb{F}_p},\mathcal{O},n)$ is the sub-summation of those summands of $\chi(\mathcal{X}_{\mathbb{F}_p},\mathcal{O},n)$ for which $j+k=i$. One has $^{i}\chi(\mathcal{X}_{\mathbb{F}_p},\mathcal{O},n)=0$ whenever $i\neq 0,1,2$ as desired. To define a decomposition $\Lambda_{\mathfrak{p}}(\mathcal{X},n)=\prod_{i=0,1,2}{}^{i}\Lambda_p(\mathcal{X},n)^{(-1)^i}$ we will proceed as in Definition 2.31 and let

$$^{p}R^0\pi_*\mathbb{Z}_p(n)^{\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p}} := R^0\pi_*\mathbb{Z}_p(n)^{\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p}} = \mathbb{Z}_p(n)^{S^{\mathrm{red}}_{\mathbb{F}_p}} \quad \text{and} \quad {}^{p}R^2\pi_*\mathbb{Z}_p(n)^{\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p}} := \mathbb{Z}_p(n-1)^{S^{\mathrm{red}}_{\mathbb{F}_p}}$$

as well as

$$\det_{\mathbb{Z}_p} {}^{p}R^1\Gamma(\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p},\mathbb{Z}_p(n)) := \det_{\mathbb{Z}_p}\tau^{\geqslant 1}R\Gamma(\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p},\mathbb{Z}_p(n)) \otimes \det^{-1}_{\mathbb{Z}_p} R\Gamma(S^{\mathrm{red}}_{\mathbb{F}_p},\mathbb{Z}_p(n-1)). \quad (3.31)$$

Again, the complex $R\Gamma(\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p},\mathbb{Z}_p(n))$ does not necessarily decompose since $\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p}$ is not necessarily smooth, i.e. the symbol $^{p}R^1\Gamma(\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p},\mathbb{Z}_p(n))$ itself is undefined. We introduce (3.31) to force a splitting on the level of determinants

$$\det_{\mathbb{Z}_p} R\Gamma(\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p},\mathbb{Z}_p(n)) = \bigotimes_{i=0,1,2}\Big(\det_{\mathbb{Z}_p} {}^{p}R^i\Gamma(\mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p},\mathbb{Z}_p(n))\Big)^{(-1)^i}.$$

Theorem 2.11 provides a motivic decomposition of $R\Gamma(\mathcal{X}_{\mathbb{Z}_p},\mathbb{Z}_p(n))$ after passing to the p-adic completion. A decomposition of $\det_{\mathbb{Z}_p} R\Gamma_{\mathrm{ddR}}(\mathcal{X}_{\mathbb{Z}_p}/\mathbb{Z}_p)/F^n$ is given in (2.51). So, every term in (3.30) decomposes and we may define $^{i}\Lambda_p(\mathcal{X},n)$ as the trivialization factor of the $\hbar^i$-component of (3.30). Finally, after defining $^{i}c_p(\mathcal{X},n)=p^{\,{}^{i}\chi(\mathcal{X}_{\mathbb{F}_p},\mathcal{O},n)}\cdot{}^{i}\Lambda_p(\mathcal{X},n)$ and

$$^{i}C(\mathcal{X},n)=\prod_{p<\infty}|{}^{i}c_p(\mathcal{X},n)|_p$$

we get the decomposition

$$C(\mathcal{X},n)=\prod_{i\in\mathbb{Z}}{}^{i}C(\mathcal{X},n)^{(-1)^i}=\frac{C(S,n)\,C(S,n-1)}{^{1}C(\mathcal{X},n)}. \quad (3.32)$$

Results for S. Flach and Morin have computed the correction factor for S assuming the validity of a conjecture from p-adic Hodge theory.

Proposition 3.18. (cf. [8] Prop. 5.33) *When assuming Conjecture $C_{EP}(\mathbb{Q}_p(n))$ in [24][App.C2] for all local fields F_v of F one has*

$$C(S,n)=\begin{cases}1 & \text{for } n\leqslant 1\\ (n-1)!^{-m} & \text{for } n\geqslant 1.\end{cases}$$

Computation of $C(\mathcal{X},1)$. Fix a rational prime p. For any complex $\mathscr{F}$ of sheaves on a scheme $\mathscr{X}$ we will use the notation

$$\widehat{R\Gamma}(\mathscr{X},\mathscr{F}) := R\Gamma\left(\mathscr{X}, \varprojlim \mathscr{F}/p^\bullet\right) = \varprojlim R\Gamma(\mathscr{X},\mathscr{F}/p^\bullet)$$

to denote p-adic completion[2].

Theorem 3.19. *For $n=1$ the triangle (3.29) exists. Moreover, $C(\mathcal{X},1)=1$.*

Proof. Write $X = \mathcal{X}_{\mathbb{Z}_p}$ and let $Z = \mathcal{X}^{\mathrm{red}}_{\mathbb{F}_p}$. Let $\iota : Z \to X$ denote the closed immersion of the reduced special fiber. Write $\mathcal{I} \subset \mathcal{O}_X$ for the ideal sheaf of Z, i.e. there is a short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0. \tag{3.33}$$

Note that $\mathcal{I}$ is the radical of $(p) = p\mathcal{O}_X$. Z is the disjoint union of fibers $Z_\mathfrak{p} = X^{\mathrm{red}}_{k(\mathfrak{p})}$ over each prime $\mathfrak{p}$ dividing p. Write $p_a^\mathfrak{p} = p_a(Z_\mathfrak{p})$ for their arithmetic genera.

For $n=1$ the triangle (3.29) becomes

$$\mathcal{D}_p(1): \quad R\Gamma(X_{\mathbb{Q}_p}, \mathcal{O}_{X_{\mathbb{Q}_p}}) \xrightarrow{\exp} \widehat{R\Gamma}(X,\mathbb{G}_m)\otimes\mathbb{Q}_p \longrightarrow \widehat{R\Gamma}(Z,\mathbb{G}_m)\otimes\mathbb{Q}_p \longrightarrow \tag{3.34}$$

after shifting by one degree since $R\Gamma(X_{\mathbb{Q}_p}, \mathcal{O}_{X_{\mathbb{Q}_p}})$ is p-adically complete already. We will show that the above triangle is exact, and moreover, compute the associated trivialization factor $\Lambda_p(\mathcal{X},1)$ by comparing it to the below distinguished triangle (3.36) of integral lattices.

Fix a power $p^\bullet$ of p. One checks on stalks that

$$1 \longrightarrow (1+\mathcal{I})/p^\bullet \longrightarrow (\mathbb{G}_m/p^\bullet)^X \longrightarrow \iota_*(\mathbb{G}_m/p^\bullet)^Z \longrightarrow 1 \tag{3.35}$$

is a short exact sequence of abelian sheaves on X. Applying the derived global sections functor and then passing to p-adic completions yields

$$\widehat{R\Gamma}(X,1+\mathcal{I}) \longrightarrow \widehat{R\Gamma}(X,\mathbb{G}_m) \longrightarrow \widehat{R\Gamma}(Z,\mathbb{G}_m) \longrightarrow. \tag{3.36}$$

The two right most complexes in (3.36) coincide with the integral structures of the two right most complexes in (3.34). So, it remains to compare the integral lattice $R\Gamma(X,\mathcal{O}_X)$ inside $R\Gamma(X_{\mathbb{Q}_p}, \mathcal{O}_{X_{\mathbb{Q}_p}})$ with $R\Gamma(X,1+\mathcal{I})$. A diagrammatic overview of this comparison as worked out in the remainder of this proof is given in Remark 3.20 below.

Write $(\mathscr{X},\mathcal{O}_\mathscr{X})$ for the formal completion of X along Z and let $\iota_\mathscr{X} : Z \to \mathscr{X}$ denote the inclusion into $\mathscr{X}$. Let $\mathcal{I}_\mathscr{X} := \mathcal{I}\mathcal{O}_\mathscr{X}$. One has $\mathcal{I}_\mathscr{X} = \varprojlim \mathcal{I}/\mathcal{I}^\bullet = \varprojlim \mathcal{I}/p^\bullet$. Moreover, the Theorem on Formal Functions gives for any $k \geqslant 0$

$$R\Gamma(X,\mathcal{I}^k) \simeq R\Gamma(\mathscr{X},\mathcal{I}^k_\mathscr{X})$$

[2] When applied to objects in a derived category $\varprojlim$ is understood to mean the homotopy limit.

since $\mathcal{I}^k$ is coherent. The transition to formal scheme theory equips us with a logarithm map $\log : 1+\mathcal{I}_{\mathscr{X}} \to \mathcal{O}_{\mathscr{X}}$ which reduces to an isomorphism $\log : 1+\mathcal{I}_{\mathscr{X}}^r \cong \mathcal{I}_{\mathscr{X}}^r$ for sufficiently large r. Consequently, for some fixed r, one has

$$R\Gamma(\mathscr{X}, \mathcal{I}_{\mathscr{X}}^r) \simeq \widehat{R\Gamma}(\mathscr{X}, 1+\mathcal{I}_{\mathscr{X}}^r).$$

Next, consider the short exact sequences

$$1 \longrightarrow 1+\mathcal{I}_{\mathscr{X}}^{k+1} \longrightarrow 1+\mathcal{I}_{\mathscr{X}}^{k} \longrightarrow (1+\mathcal{I}_{\mathscr{X}})^k/(1+\mathcal{I}_{\mathscr{X}}^{k+1}) \longrightarrow 1$$

$$0 \longrightarrow \mathcal{I}_{\mathscr{X}}^{k+1} \longrightarrow \mathcal{I}_{\mathscr{X}}^{k} \longrightarrow \mathcal{I}_{\mathscr{X}}^{k}/\mathcal{I}_{\mathscr{X}}^{k+1} \longrightarrow 0$$

for $k = 1, 2, \ldots, r-1$. One checks directly that

$$\mathcal{I}_{\mathscr{X}}^{k}/\mathcal{I}_{\mathscr{X}}^{k+1} \longrightarrow (1+\mathcal{I}_{\mathscr{X}}^{k})/(1+\mathcal{I}_{\mathscr{X}}^{k+1}), \qquad f \mapsto 1+f$$

is an isomorphism of sheaves. Therefore we obtain

$$\frac{\det_{\mathbb{Z}_p} R\Gamma(\mathscr{X}, \mathcal{I}_{\mathscr{X}})}{\det_{\mathbb{Z}_p} R\Gamma(\mathscr{X}, \mathcal{I}_{\mathscr{X}}^r)} = \frac{\det_{\mathbb{Z}_p} \widehat{R\Gamma}(\mathscr{X}, 1+\mathcal{I}_{\mathscr{X}})}{\det_{\mathbb{Z}_p} \widehat{R\Gamma}(\mathscr{X}, 1+\mathcal{I}_{\mathscr{X}}^r)},$$

where the above should be read as an equality of quotients of $\mathbb{Z}_p$-lattices inside the (1-dimensional) $\mathbb{Q}_p$-vectorspaces $\det_{\mathbb{Q}_p} R\Gamma(\mathscr{X}_{\mathbb{Q}_p}, \mathcal{I}_{\mathscr{X}_{\mathbb{Q}_p}})$ and $\det_{\mathbb{Q}_p} \widehat{R\Gamma}(\mathscr{X}_{\mathbb{Q}_p}, 1+\mathcal{I}_{\mathscr{X}_{\mathbb{Q}_p}})$ respectively. Moreover, the long exact sequence associated to (3.33) equals

$$0 \to \Gamma(\mathcal{X}, \mathcal{I}) \to \mathcal{O}_{\mathbb{Z}_p} \to \mathcal{O}_{\mathbb{F}_p}^{\mathrm{red}} \xrightarrow{0} H^1(\mathcal{X}, \mathcal{I}) \to H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \bigoplus_{\mathfrak{p}|p} k(\mathfrak{p})^{p_a^{\mathfrak{p}}} \xrightarrow{0}$$
$$\xrightarrow{0} H^2(\mathcal{X}, \mathcal{I}) \xrightarrow{\cong} H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to 0.$$

So, we obtain

$$\frac{\det_{\mathbb{Z}_p} R\Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})}{\det_{\mathbb{Z}_p} R\Gamma(\mathscr{X}, \mathcal{I}_{\mathscr{X}})} = \frac{\det_{\mathbb{Z}_p} R\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})}{\det_{\mathbb{Z}_p} R\Gamma(\mathcal{X}, \mathcal{I})} = \prod_{\mathfrak{p}|p} \frac{\# k(\mathfrak{p})^{p_a^{\mathfrak{p}}}}{\# k(\mathfrak{p})} = \prod_{\mathfrak{p}|p} N\mathfrak{p}^{(p_a^{\mathfrak{p}}-1)}. \tag{3.37}$$

It remains to relate the cohomology complexes of $(1+\mathcal{I})/p^\bullet$ and $(1+\mathcal{I}_{\mathscr{X}})/p^\bullet$. By virtue of the proper base change theorem the canonical morphisms of sheaves on Z

$$\phi_\bullet : \iota^*(1+\mathcal{I})/p^\bullet \longrightarrow \iota_{\mathscr{X}}^*(1+\mathcal{I}_{\mathscr{X}})/p^\bullet$$

induce a compatible system of maps

$$R\Gamma(\mathcal{X}, (1+\mathcal{I})/p^\bullet) = R\Gamma(Z, \iota^*(1+\mathcal{I})/p^\bullet) \xrightarrow{R\Gamma(\phi_\bullet)} R\Gamma(Z, \iota_{\mathscr{X}}^*(1+\mathcal{I}_{\mathscr{X}})/p^\bullet) = R\Gamma(\mathscr{X}, (1+\mathcal{I}_{\mathscr{X}})/p^\bullet).$$

We will show later that the induced map between p-adic completions is an isomorphism:

$$\varprojlim R\Gamma(\phi_\bullet) : \widehat{R\Gamma}(\mathcal{X}, 1+\mathcal{I}) \xrightarrow{\simeq} \widehat{R\Gamma}(\mathscr{X}, 1+\mathcal{I}_{\mathscr{X}}). \tag{3.38}$$

It will show that $\Lambda_{\mathfrak{p}}(\mathcal{X},1)$ must equal (3.37). This will finish the proof since it combines with

$$\chi(\mathcal{X}_{\mathbb{F}_p},\mathcal{O},1)=\sum_j(-1)^j\sum_{\mathfrak{p}|p}\dim_{\mathbb{F}_p}H^j(Z_{\mathfrak{p}},\mathcal{O}_{Z_{\mathfrak{p}}})$$

to the desired result

$$C(\mathcal{X},1)=p^{\chi(\mathcal{X}_{\mathbb{F}_p},\mathcal{O},1)}\cdot\Lambda_{\mathfrak{p}}(\mathcal{X},1)=\prod_j\left(\prod_{\mathfrak{p}|p}\#H^j(Z_{\mathfrak{p}},\mathcal{O}_{Z_{\mathfrak{p}}})\right)^{(-1)^j}\cdot\prod_{\mathfrak{p}|p}N\mathfrak{p}^{(p_a^{\mathfrak{p}}-1)}=1.$$

The proof of (3.38) amounts to the algebraic exercise of verifying that taking mapping cones of the multiplicative sheaf $1+\mathcal{I}$ interacts well with transitioning to the formal (i.e. p-adic) completion $\mathcal{I}_{\mathscr{X}}=\varprojlim\mathcal{I}/p^\bullet$ of the additive sheaf $\mathcal{I}$. Clearly, each $\phi_\bullet$ and hence

$$\varprojlim_n\phi_n:\varprojlim_n(1+\mathcal{I})/p^n\longrightarrow\varprojlim_n(1+\mathcal{I}_{\mathscr{X}})/p^n=\varprojlim_n\varprojlim_m(1+\mathcal{I}/p^m)/p^n$$

is injective. For surjectivity it suffices to show that for each pair of integers n,m there is an $N\geqslant n$ such that

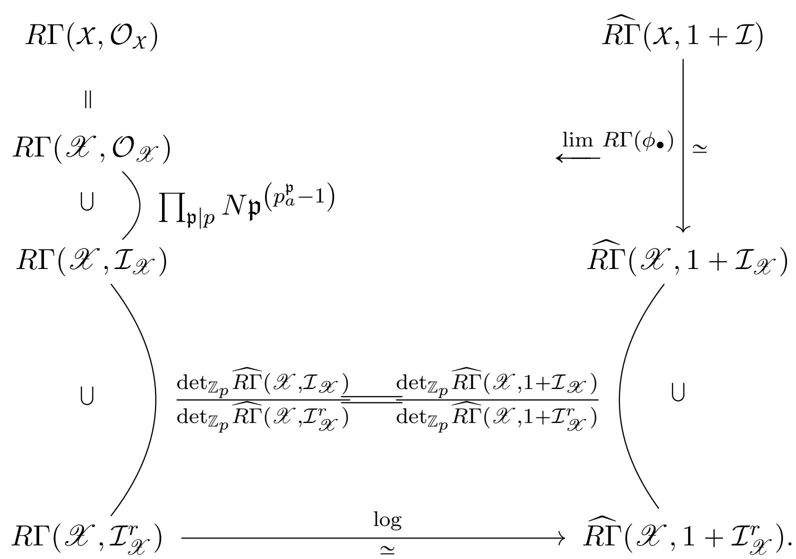
$$(1+\mathcal{I})/p^N\longrightarrow(1+\mathcal{I}/p^m)/p^n$$

surjects. To do this it is enough to exhibit a constant c such that for all $N\geqslant c$ one has

$$(1+\mathcal{I})^{p^N}\subset 1+p^{N-c}\mathcal{I}.$$

Recall that for some s one has $\mathcal{I}^s\subset(p)$. Let $f\in\mathcal{I}$ and consider $(1+f)^{p^N}=1+\sum_{k=1}^{p^N}\binom{p^N}{k}f^k$. Since $\operatorname{ord}_p\binom{p^N}{k}>N-i$ for $k<p^i$ and $f^k\in(p)^{\lfloor\frac{k}{s}\rfloor}$ one may choose $c=\max_i\left(i-\left\lfloor\frac{p^i}{s}\right\rfloor\right)$. □

Remark 3.20. The diagram below summarizes the comparison between $R\Gamma(X,\mathcal{O}_X)$ and $\widehat{R\Gamma}(X,1+\mathcal{I})$ made in the proof. Also note that the p-adic completeness of the complexes in $\mathcal{D}_p(1)$ is essential as otherwise there would be no way to relate $R\Gamma(\mathscr{X},1+\mathcal{I}_{\mathscr{X}})$ to $R\Gamma(X,1+\mathcal{I})$.



Remark 3.21. The presented proof does not use $\dim \mathcal{X} = 2$ at any point. So – when recalling the more general definition of the correction factor in [8] – the above proof shows $C(\mathscr{X}, 1) = 1$ for proper regular arithmetic schemes $\mathscr{X}$ of any dimension.

Remark 3.22. Since we already know $C(S,1) = C(S,0) = 1$ the above shows ${}^{i}C(\mathcal{X}, n) = 1$ for each $i = 0,1,2$. One has ${}^{1}\Lambda_p(\mathcal{X}, 1) = \prod_{\mathfrak{p}|p}(N\mathfrak{p})^{p_a^{\mathfrak{p}}}$ which cancels with $p^{{}^{1}\chi(\mathcal{X}_{\mathbb{F}_p}, \mathcal{O}, 1)} = \prod_{\mathfrak{p}|p}(N\mathfrak{p})^{-p_a^{\mathfrak{p}}}$.

Remark 3.23. We keep the notation of the proof and provide a more geometric version of it for X with good reduction. In this case $\mathcal{D}_p(1)$ decomposes entirely into motivic degree components and it suffices to understand its h^1-part. Write $\mathcal{J} = \mathrm{Jac}_{X/\mathbb{Z}_p}$ for the Jacobian variety of X. $\mathcal{J}$ is a projective abelian variety of relative dimension g over $\mathcal{O}_{\mathbb{Z}_p}$.

Let $\mathcal{J} \hookrightarrow \mathbb{P}^N_{\mathbb{Z}_p}$ be defined in terms of homogeneous equations in variables $T_0, \ldots, T_N$. Let $O = \operatorname{Spec} \mathcal{O}_{\mathbb{Z}_p} \hookrightarrow \mathcal{J}$ be its unit section. Choose local coordinates $X_1, \ldots, X_g$ of $\mathcal{J}$ at O, i.e. $X_1, \ldots, X_g$ is a set of generators of the maximal ideal m of the ring of regular functions $\mathcal{O}_O := (\mathcal{O}_{\mathcal{J}})_O$ at O. Since Z is smooth its Jacobian variety J is obtained from reducing $\mathcal{J}$ modulo p, i.e. $J = \mathcal{J}_{\mathbb{F}_p}$. Write $\hat{\mathcal{J}}$ and $\hat{J}$ for the formal groups of $\mathcal{J}, J$ with respect to their local coordinates $\{X_j\}_{1 \leqslant j \leqslant g}$ and $\{\overline{X}_j\}_{1 \leqslant j \leqslant g}$. The motivic degree-1-part of the H^1-groups of the long exact sequence belonging to $\mathcal{D}_p(1)$ is given by

$$0 \longrightarrow H^1(X, \mathcal{O}_X) \otimes \mathbb{Q}_p \xrightarrow{\exp} \widehat{\mathcal{J}(\mathcal{O})} \otimes \mathbb{Q}_p \longrightarrow \widehat{J(\mathcal{O}^{\mathrm{red}}_{\mathbb{F}_p})} \otimes \mathbb{Q}_p \longrightarrow 0. \tag{3.39}$$

Since $H^1(X, \mathcal{O}_X) \cong H^0(X, \omega_X)^*$ where ω_X denotes the canonical bundle of X the choice of $\{X_j\}_{1 \leqslant j \leqslant g}$ corresponds to a choice of an integral basis of $H^1(X, \mathcal{O}_X) \otimes \mathbb{Q}_p = H^1(X_{\mathbb{Q}_p}, \mathcal{O}_{X_{\mathbb{Q}_p}})$ and hence lets us identify its integral structure with $\widehat{\mathbb{G}}_a^g(\mathcal{O}_{\mathbb{Z}_p})$.

Let p be the Jacobson radical of $\mathcal{O}^{\mathrm{red}}_{\mathbb{F}_p}$, i.e. $\mathcal{O}^{\mathrm{red}}_{\mathbb{F}_p}/p = \bigoplus_{\mathfrak{p}|p} k(\mathfrak{p})$. We may assume $O = [1 : 0 : \cdots : 0]$, i.e. $T_i \in m$ for all $0 < i \leqslant N$. Then the T_i may be interpreted as power series $T_i = \hat{T}_i(X_1, \ldots, X_g) \in \widehat{\mathcal{O}_O} = \varprojlim \mathcal{O}_O/m^\bullet$ in the variables $X_1, \ldots, X_g$. This gives for any exponent $e \geqslant 1$ a morphism of abelian groups

$$\iota : \hat{\mathcal{J}}(p^e) \longrightarrow \widehat{\mathcal{J}(p^e)}, \qquad (x_j)_{1 \leqslant j \leqslant g} \mapsto \hat{T}_i(x_1, \ldots, x_g).$$

Formal Group Theory provides a logarithm map $\log : \hat{\mathcal{J}} \longrightarrow \widehat{\mathbb{G}}_a^g$ that restricts for sufficiently large $r > 0$ to an isomorphism of abelian groups

$$\log : \hat{\mathcal{J}}(p^r) \xrightarrow{\cong} \widehat{\mathbb{G}}_a^g(p^r) = \bigoplus_{1 \leqslant j \leqslant g} p^r.$$

The exponential map in (3.39) is the base change of the inverse of the above logarithm to $\mathbb{Q}_p$.

Therefore, the (non-exact) restriction of (3.39) to integral structures fits into the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathbb{G}}_a^g(\mathcal{O}) & \xrightarrow{\exp} & \widehat{\mathcal{J}(\mathcal{O}_{\mathbb{Z}_p})} & \longrightarrow & \widehat{J(\mathcal{O}^{\mathrm{red}}_{\mathbb{F}_p})} & \longrightarrow & 0 \\ & & \cup & & \uparrow{\scriptstyle \iota} & & & & \\ & & \widehat{\mathbb{G}}_a^g(p) & \xleftarrow{\log} & \hat{\mathcal{J}}(p) & & & & \\ & & \cup & & \cup & & & & \\ & & \widehat{\mathbb{G}}_a^g(p^r) & \xleftrightarrow{\cong} & \hat{\mathcal{J}}(p^r) & & & & \end{array}$$

Since, moreover, $\widehat{\mathbb{G}}_a^g(p^k)\Big/\widehat{\mathbb{G}}_a^g(p^{k+1}) \cong \hat{\mathcal{J}}(p^k)\Big/\hat{\mathcal{J}}(p^{k+1})$ for all $k \geqslant 1$, the trivialization factor associated to (3.39) does not depend on r and agrees with the trivialization factor of

$$0 \longrightarrow \hat{\mathcal{J}}(p) \xrightarrow{\iota} \widehat{\mathcal{J}(\mathcal{O}_{\mathbb{Z}_p})} \longrightarrow \widehat{J(\mathcal{O}^{\mathrm{red}}_{\mathbb{F}_p})} \longrightarrow 0 \tag{3.40}$$

up to a factor of $\#\widehat{\mathbb{G}}_a^g(\mathcal{O})/\widehat{\mathbb{G}}_a^g(p) = (\prod_{\mathfrak{p}|p} N\mathfrak{p})^g$ – which equals $\prod_{\mathfrak{p}|p}(N\mathfrak{p})^{p_a^{\mathfrak{p}}}$ since we assume Z to be smooth. So, it suffices to show that (3.40) is exact.

First, let $t = [t_0 : t_1 : \cdots : t_N] \in \widehat{\mathcal{J}(\mathcal{O}_{\mathbb{Z}_p})}$ be an integral point that is congruent to O modulo p. The $X_j = X_j(T_0 : \cdots : T_n)$ are rational functions in the T_i that vanish at O. Therefore we have $x_j := X_j(t) \in p$. We then have $\iota(x_j)_j = t$ proving exactness at the middle component. Let now $Z_n = \mathcal{X} \times \operatorname{Spec} \mathbb{Z}/p^n$. By the Theorem on Formal Functions the categories of line bundles on $\mathcal{X}$ and $\mathscr{X}$ are equivalent. So, one has $\mathcal{J}(\mathcal{O}_{\mathbb{Z}_p}) = \operatorname{Pic}^0 \mathcal{X} = \varprojlim \operatorname{Pic}^0 Z_n$. Therefore it suffices to show that each $\operatorname{Pic}^0 Z_n$ surjects onto $\mathcal{J}(\mathcal{O}_{\mathbb{F}_p}) = \operatorname{Pic}^0 Z$. This in turn follows from the long exact sequence associated to

$$1 \longrightarrow 1 + p\mathcal{O}_{Z_n} \longrightarrow \mathbb{G}_m^{Z_n} \longrightarrow \mathbb{G}_m^{Z} \longrightarrow 1$$

since $H^2(Z, 1 + p\mathcal{O}_{Z_n}) = 0$ because Z is one-dimensional. Note that exactness of (3.40) in the special case of an elliptic surface is precisely [32] Ch. VII, Prop. 2.1 & 2.2.

3.6 The Functional Equation

The correction factor for $n \geqslant 2$. For higher twists $n \geqslant 2$ the correction factor $C(\mathcal{X}, n)$ may be computed using the conjectural functional equation $\mathbf{FE}(\mathcal{X})$.

Theorem 3.24. *Assume conjectures* $\mathbf{FE}(\mathcal{X})$ *and* $\mathbf{TC}(\mathcal{X},n)$ *hold. Then for* $n \geqslant 2$ *one has*

$${}^1C(\mathcal{X}, n) = \pm \left(\frac{\Gamma^*(2-n)}{\Gamma^*(n)} \right)^{mg} = ((n-1)! \cdot (n-2)!)^{-mg}. \tag{3.41}$$

Similarly, when assuming $\mathbf{TC}(S,n)$ *one has for* $n \geqslant 1$

$$C(S, n) = (n-1)!^{-m}$$

and consequently

$$C(\mathcal{X},n) = ((n-1)!\cdot(n-2)!)^{m(g-1)}\,.$$

Proof. Let $n \geqslant 2$. Corollary 2.35 unfolds to

$$^1A(\mathcal{X})^{\frac{2-n}{2}}\,{}^pL^*_\infty(H^1(X),2-n)\,{}^1\Lambda_\infty(\mathcal{X},2-n) = \pm\,{}^1A(\mathcal{X})^{\frac{n}{2}}\,{}^pL^*_\infty(H^1(X),n)\,{}^1\Lambda_\infty(\mathcal{X},n)\,{}^1C(\mathcal{X},n).$$

By virtue of Theorem 3.13 and Lemma 2.34 this simplifies to

$$\begin{aligned}
{}^1C(\mathcal{X},n) &= \pm\,{}^1A(\mathcal{X})^{1-n}\cdot\frac{{}^pL^*_\infty(H^1(X),2-n)}{{}^pL^*_\infty(H^1(X),n)}\cdot\frac{{}^1\Lambda_\infty(\mathcal{X},2-n)}{{}^1\Lambda_\infty(\mathcal{X},n)}\\
&= \pm\,{}^1A(\mathcal{X})^{1-n}\cdot\left((2\pi)^{2(n-1)}\frac{\Gamma^*(2-n)}{\Gamma^*(n)}\right)^{mg}\left(\frac{(2\pi i)^{2mg(n-1)}}{{}^1A(\mathcal{X})^{n-1}}\right)^{-1}\\
&= \pm\left(\frac{\Gamma^*(2-n)}{\Gamma^*(n)}\right)^{mg}.
\end{aligned}$$

Similarly, for $n \geqslant 1$ we obtain from the well-known functional equation $\mathbf{FE}(S)$

$$\begin{aligned}
C(S,n) &= \pm A(S)^{\frac{1}{2}-n}\cdot\frac{\Gamma^*_{\mathbb{R}}(1-n)^r\Gamma^*_{\mathbb{C}}(1-n)^s}{\Gamma^*_{\mathbb{R}}(n)^r\Gamma^*_{\mathbb{C}}(n)^s}\cdot\frac{\Lambda_\infty(S,1-n)}{\Lambda_\infty(S,n)}\\
&= \pm(\#D_F)^{\frac{1}{2}-n}\left(\pi^{n-\frac{1}{2}}\frac{\left(\frac{1}{2}\right)^{-\epsilon_n}\Gamma^*(\frac{1-n}{2})}{\Gamma^*(\frac{n}{2})}\right)^r\left((2\pi)^{2n-1}\frac{\Gamma^*(1-n)}{\Gamma^*(n)}\right)^s\frac{1}{2^{(-1)^{n-1}r}(2\pi)^{mn-r\epsilon_n-s}(\#D_F)^{\frac{1}{2}-n}}\\
&= \pm 2^{-\left(n-\frac{1}{2}\right)r}2^{\epsilon_n r}\left(\frac{\Gamma^*(\frac{1-n}{2})}{\Gamma^*(\frac{n}{2})}\right)^r\frac{(n-1)!^{-2s}}{2^{(-1)^{n-1}r}(2\pi)^{\frac{m}{2}-r\epsilon_n-s}}\\
&= \pm 2^{\left(-n+\frac{1}{2}+\epsilon_n\right)r}(2\pi)^{-\frac{(-1)^n}{2}r}2^{(-1)^n r}\left(\frac{\Gamma^*(\frac{1-n}{2})}{\Gamma^*(\frac{n}{2})}\right)^r\frac{1}{(n-1)!^{2s}}\\
&= \pm 2^{(1-n)r}\left(\frac{2^{n-1}}{(n-1)!}\pi^{\frac{(-1)^n}{2}}\right)^r\frac{1}{(n-1)!^{2s}}\\
&= (n-1)!^{-m}.
\end{aligned}$$

Note that the factor $(1/2)^{-\epsilon_n}$ arises from the relation between leading Taylor coefficients $\Gamma^*_{\mathbb{R}}(k) = (1/2)^{\mathrm{ord}_{s=k}\Gamma_{\mathbb{R}}(s)}\,\pi^{-k/2}\Gamma^*\left(k/2\right)$. (3.32) concludes the proof. □

In particular, $C(S,n) = (n-1)!^{-m}$ also holds for $n \geqslant 1$ when assuming $\mathbf{TC}(S,n)$ instead of the conjecture $C_{\mathrm{EP}}(\mathbb{Q}_p(n))$ from p-adic Hodge Theory.

A simplified version of $\mathbf{FE}(\mathcal{X})$ for integers $n = s$. Flach and Morin provide a reformulation of $\mathbf{FE}(\mathcal{X})$ for integer arguments in terms of a newly defined quantity $x_\infty(\mathcal{X},n)\in\mathbb{R}^{>0}$, satisfying

$$x_\infty(\mathcal{X},n)^2 = \frac{\Lambda_\infty(\mathcal{X},2-n)}{\Lambda_\infty(\mathcal{X},n)}. \tag{3.42}$$

The definition of $x_\infty(\mathcal{X},n)^2$ will *not* incorporate the full fundamental line, mirroring the fact that the quotient on the right hand side is easier to evaluate than each term separately. We will review it here and use it to compute $x_\infty(\mathcal{X},n)$ independently of Theorem 3.13. Set

$$\begin{aligned}\Xi_\infty(\mathcal{X}/\mathbb{Z},n) \;:=\; &\det\nolimits_{\mathbb{Z}} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n)) \otimes \det\nolimits_{\mathbb{Z}}^{-1} R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^n \otimes \\ &\det\nolimits_{\mathbb{Z}}^{-1} R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(2-n)) \otimes \det\nolimits_{\mathbb{Z}} R\Gamma_{\mathrm{ddR}}(\mathcal{X}/\mathbb{Z})/F^{2-n}.\end{aligned} \tag{3.43}$$

This definition is made to have an isomorphism

$$\phi : \Delta(\mathcal{X}/\mathbb{Z},n)\otimes\Xi_\infty(\mathcal{X}/\mathbb{Z},n) \xrightarrow{\cong} \Delta(\mathcal{X}/\mathbb{Z},2-n).$$

Observe that the distinguished triangle

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})/F^n[-1] \longrightarrow R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}},\mathbb{R}(n)) \longrightarrow R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))_{\mathbb{R}} \longrightarrow \tag{3.44}$$

together with the duality (2.23) for Deligne cohomology gives a trivialization

$$\begin{aligned}\Xi_\infty(\mathcal{X}/\mathbb{Z},n)_{\mathbb{R}} &\simeq \det\nolimits_{\mathbb{R}} R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}},\mathbb{R}(n)) \otimes \det\nolimits_{\mathbb{R}}^{-1} R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}},\mathbb{R}(2-n)) \\ &\simeq \det\nolimits_{\mathbb{R}} R\mathrm{Hom}(R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}},\mathbb{R}(2-n)),\mathbb{R}[-3]) \otimes \det\nolimits_{\mathbb{R}}^{-1} R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}},\mathbb{R}(2-n)) \\ &\simeq \det\nolimits_{\mathbb{R}} R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}},\mathbb{R}(2-n)) \otimes \det\nolimits_{\mathbb{R}}^{-1} R\Gamma_{\mathcal{D}}(\mathcal{X}/_{\mathbb{R}},\mathbb{R}(2-n)) \\ &\simeq \mathbb{R}.\end{aligned} \tag{3.45}$$

We denote it by $\xi_\infty(\mathcal{X},n):\mathbb{R}\xrightarrow{\cong}\Xi_\infty(\mathcal{X}/\mathbb{Z},n)_{\mathbb{R}}$ and define $x_\infty^2(\mathcal{X},n)\in\mathbb{R}^{>0}$ via

$$\xi_\infty(\mathcal{X},n)(\mathbb{Z}) = x_\infty^2(\mathcal{X},n)\cdot\Xi_\infty(\mathcal{X}/\mathbb{Z},n)$$

as an equality of lattices in $\Xi_\infty(\mathcal{X}/\mathbb{Z},n)_{\mathbb{R}}$. Clearly $x_\infty(\mathcal{X},2-n) = x_\infty(\mathcal{X},n)^{-1}$. One has

Proposition 3.25. (cf. [8] Prop. 5.28, Cor. 5.30) *The diagram*

$$\begin{array}{ccc}\Delta(\mathcal{X}/\mathbb{Z},n)\otimes\Xi_\infty(\mathcal{X}/\mathbb{Z},n)\otimes\mathbb{R} & \xrightarrow{\phi\otimes\mathbb{R}} & \Delta(\mathcal{X}/\mathbb{Z},2-n)\otimes\mathbb{R} \\ \uparrow{\scriptstyle \lambda_\infty(\mathcal{X},n)\otimes\xi_\infty(\mathcal{X},n)} & & \uparrow{\scriptstyle \lambda_\infty(\mathcal{X},2-n)} \\ \mathbb{R}\otimes\mathbb{R} & = & \mathbb{R}\end{array}$$

commutes. Therefore – when assuming **TC**$(\mathcal{X},n)$ *for all integers n – the functional equation* **FE**$(\mathcal{X})$ *holds for all integers $s=n$ if and only if for all n*

$$\frac{A(\mathcal{X})^{\frac{n}{2}}\cdot\zeta^*(\mathcal{X}_\infty,n)\cdot C(\mathcal{X},n)}{x_\infty(\mathcal{X},n)} = \pm\frac{A(\mathcal{X})^{\frac{2-n}{2}}\cdot\zeta^*(\mathcal{X}_\infty,2-n)\cdot C(\mathcal{X},2-n)}{x_\infty(\mathcal{X},2-n)}. \tag{3.46}$$

Motivic decomposition of $\Xi_\infty(\mathcal{X}/\mathbb{Z},n)$. By the work of last chapter all determinants in (3.43) admit motivic decompositions. So, we obtain further decompositions

$$\Xi_\infty(\mathcal{X}/\mathbb{Z},n) = \Xi_\infty(S/\mathbb{Z},n)\otimes{}^1\Xi_\infty(\mathcal{X}/\mathbb{Z},n)^{-1}\otimes\Xi_\infty(S/\mathbb{Z},n-1)$$

and

$$x_\infty(\mathcal{X},n) = x_\infty(S,n)\cdot{}^1x_\infty(\mathcal{X},n)^{-1}\cdot x_\infty(S,n-1).$$

Proposition 3.26. *Let n be any integer. One has*

$$x_\infty^2(S,n) = 2^{(-1)^n r}(2\pi)^{r\epsilon_n+s-mn}(\# D_F)^{n-\frac{1}{2}}.$$

Moreover, when assuming the technical condition $\mathbf{RP}(\mathcal{X})$ *(or the formula (2.39)), one has*

$${}^1x_\infty^2(\mathcal{X},n) = \frac{{}^1A(\mathcal{X})^{n-1}}{(2\pi)^{2mg(n-1)}}$$

and consequently

$$x_\infty^2(\mathcal{X},n) = \Big((2\pi)^{2m(g-1)}A(\mathcal{X})\Big)^{n-1}.$$

Proof. It suffices to consider $n \geqslant 1$ since $x_\infty(S,1-n) = x_\infty(S,n)^{-1}$ and $x_\infty(\mathcal{X},2-n) = x_\infty(\mathcal{X},n)^{-1}$. The second quasi-isomorphism of (3.45) is due to Poincaré Duality which holds integrally; the third and the fourth follow directly from the determinant formalism. Therefore the trivialization factors $x_\infty^2(\mathcal{X},n)$ and $x_\infty^2(S,n)$ arise fully from a comparison of the integral structures of the complexes in (3.44) (and its analogue for S). The motivic degree 1 part of its associated long exact sequence is

$$0 \longrightarrow H^{1,n}_{W,\infty}(\mathcal{X})_\mathbb{R} \longrightarrow H^{1,n}_{\mathrm{dR}}(\mathcal{X})_\mathbb{R} \longrightarrow H^{2,n}_{\mathcal{D}}(\mathcal{X}) \longrightarrow 0,$$

which is exact integrally with respect to $\mathcal{B}^n$. Consequently, for $n \geqslant 2$,

$$\begin{aligned} {}^1x_\infty^2(\mathcal{X},n) &= {}^1t^{(n)}_{\mathrm{ddR}}(\mathcal{X}) \cdot \det M_{\mathcal{B}^n_{\mathrm{ddR}},\mathcal{B}^n}(\mathrm{id}) \\ &= \frac{1}{(2\pi)^{2mg(n-1)}} \cdot {}^1t^{(n)}_{\mathrm{ddR}}(\mathcal{X}) \cdot \det M_{\mathcal{B}^n_{\mathrm{ddR}},\mathcal{B}_{\mathrm{dR}}}(\mathrm{id}) && \text{by Lemma 3.4} \\ &= \frac{{}^1A(\mathcal{X})^{n-1}}{(2\pi)^{2mg(n-1)}} && \text{by (3.13).} \end{aligned}$$

The 2-torsion of $R\Gamma_W(\mathcal{X}_\infty,\mathbb{Z}(n))$ does not contribute due to Corollary A.9.

For S we begin noting that by Corollary A.16(i) one has for all n

$$\chi(R\Gamma_W(S_\infty,\mathbb{Z}(n))) = \prod_{i\in\mathbb{Z}} \Big(\#\mathrm{Tor}\, H^{i,n}_{W,\infty}(S)\Big)^{(-1)^i} = 2^{(-1)^n \frac{n+\epsilon_n}{2} r}.$$

The long exact sequence associated to the analogue of (3.44) for S becomes

$$0 \longrightarrow H^{0,n}_{W,\infty}(S)_\mathbb{R} \longrightarrow H^{0,n}_{\mathrm{dR}}(S)_\mathbb{R} \longrightarrow H^{1,n}_{\mathcal{D}}(S) \longrightarrow 0.$$

The implied integral structures are obvious. The outer cohomology groups contribute factors of $(2\pi i)^{-n(\bar{\epsilon}_n r+s)}$ and $(2\pi i)^{-(n-1)(\epsilon_n r+s)}$ respectively to $x_\infty^2(S,n)$. The relative determinant between the lattices $H^{0,n}_{\mathrm{ddR}}(S/\mathbb{Z})$ and $H^0(S,\mathcal{O}_F) = \mathcal{O}_F$ equals $A(S)^{n-1} = (\# D_F)^{n-1}$. A remaining factor of $\sqrt{\# D_F}$ results from the comparison of $\mathcal{O}_F$ with the integral lattice of the Minkowski space $F_\mathbb{R}$. Therefore one obtains

$$\begin{aligned} x_\infty(S,n)^2 &= \frac{2^{(-1)^{1-n}\frac{1-n+\epsilon_{1-n}}{2}r}}{2^{(-1)^n\frac{n+\epsilon_n}{2}r}} (2\pi i)^{-n(\bar{\epsilon}_n r+s)-(n-1)(\epsilon_n r+s)} (\# D_F)^{n-1}\sqrt{\# D_F} \\ &= 2^{(-1)^n r}(2\pi)^{r\epsilon_n+s-mn}(\# D_F)^{n-\frac{1}{2}} \end{aligned}$$

as claimed. Finally, we combine this to

$$x_\infty^2(\mathcal{X},n) = \frac{x_\infty(S,n)x_\infty(S,n-1)}{{}^1x_\infty(\mathcal{X},n)}$$
$$= \left(\frac{|D_F|}{(2\pi)^m}\right)^{2(n-1)} \cdot \left(\frac{(2\pi)^{2mg(n-1)}}{{}^1A(\mathcal{X})^{n-1}}\right) = \left((2\pi)^{2m(g-1)}A(\mathcal{X})\right)^{n-1}. \qquad \square$$

In particular, (3.42) holds for every motivic degree component separately. So, we can rederive the formula for the correction factor (3.41) from (3.46).

3.7 Summary of special value results

We may now combine Theorem 3.13 and Theorem 3.12 with the results on the correction factor Theorem 3.19 and Theorem 3.24 to obtain closed formulas for the leading Taylor coefficients of $\zeta(\mathcal{X},s)$. These are presented in the Theorem below.

Moreover, when combining the second equality in Theorem 3.13(ii) with Lemma 2.2(ii) one obtains

$$L^*(H^1(X),1) = \frac{{}^1\Lambda_\infty(\mathcal{X},1)\,{}^1C(\mathcal{X},1)}{\Pi^*(\mathcal{X},1)} = \frac{\#\text{Ш}(X/F)\,\Omega(\mathcal{X})\,R(X)}{(\#\operatorname{Tor}\operatorname{Pic}^0 X)^2}\prod_{\mathfrak{p}} c_{\mathfrak{p}}(X),$$

which is precisely the leading Taylor coefficient part of the Birch and Swinnerton-Dyer conjecture for the abelian variety $\operatorname{Pic}^0 X$ (cf. [35] equ. (1.5)). We may thus summarize the special value results of this chapter as follows.

Theorem 3.27. *Let the notation be as per this and the preceding chapter. We make the* standard assumptions *that* $\mathbf{L}(\mathcal{X},n)$ *and* $\mathbf{B}(\overline{\mathcal{X}},n)$ *hold for all integers* n*. Suppose* $\pi : \mathcal{X} \to S$ *has a section* $s : S \to \mathcal{X}$ *satisfying* $\mathbf{FPB}(s,n)$ *for all* $n \geqslant 2$*. Then*

(i) $\mathbf{VO}(\mathcal{X},n)$ *is equivalent to*

$$\operatorname{ord}_{s=n}\zeta(\mathcal{X},s) = \begin{cases} m(1-g) & \text{for } n<0 \\ m(1-g)-1 & \text{for } n=0 \\ r+s-1-\operatorname{rk}\operatorname{Pic}\mathcal{X} & \text{for } n=1 \\ -1 & \text{for } n=2 \\ 0 & \text{for } n>2 \end{cases}, \tag{3.47}$$

or to

$$\operatorname{ord}_{s=n}\zeta(\overline{\mathcal{X}},s) = \begin{cases} -1-\operatorname{rk}\operatorname{Pic}\mathcal{X} & \text{for } \quad n=1 \\ -1 & \text{for } |n-1|=1 \\ 0 & \text{for } |n-1|>1 \end{cases}. \tag{3.48}$$

(ii) $\mathbf{TC}(\mathcal{X},1)$ *is equivalent to*

$$\zeta^*(\mathcal{X},1) = \frac{2^r(2\pi)^s}{(\#\mu_F)^2\sqrt{\#D_F}} \cdot \frac{(\#\mathrm{Tor}\,\mathrm{Pic}^0\mathcal{X})^2}{\#\text{Ш}(X/F)\cdot\Omega(\mathcal{X})} \cdot \frac{R(S)^2}{R(\mathcal{X})}.$$

For $n \leqslant 0$, $\mathbf{TC}(\mathcal{X},n)$ *is equivalent to*

$$\zeta^*(\mathcal{X},n) = 2^{(r-l(\mathcal{X}))\epsilon_n}\frac{\#T^{2,2-n}_{\mathcal{X}}\cdot\#T^{4,2-n}_{\mathcal{X}}}{\#T^{1,2-n}_{\mathcal{X}}\cdot\#T^{3,2-n}_{\mathcal{X}}} \cdot \frac{R^{2-n}(S)R^{1-n}(S)}{R^{2-n}(\mathcal{X})}.$$

We now suppose that the technical condition $\mathbf{RP(\mathcal{X})}$ *(or the formula (2.39)) holds. When further assuming* $\mathbf{FE}(\mathcal{X})$*, the pair of conjectures* $\mathbf{TC}(\mathcal{X},n)$ *and* $\mathbf{TC}(\mathcal{X},2-n)$ *implies*

$$C(\mathcal{X},n) = \begin{cases} 1 & \textit{for } \; n \leqslant 1 \\ ((n-1)!\cdot(n-2)!)^{m(g-1)} & \textit{for } \; n \geqslant 2 \end{cases}. \tag{3.49}$$

When assuming $\mathbf{FE}(\mathcal{X})$ *and (3.49) then, for* $n \geqslant 2$, $\mathbf{TC}(\mathcal{X},n)$ *is equivalent to*

$$\zeta^*(\mathcal{X},n) = 2^{(r-l(\mathcal{X}))\epsilon_n}\frac{\#T^{2,n}_{\mathcal{X}}\#T^{4,n}_{\mathcal{X}}}{\#T^{1,n}_{\mathcal{X}}\#T^{3,n}_{\mathcal{X}}}\left(\frac{(n-1)!(n-2)!}{(2\pi)^{2(n-1)}}\right)^{m(g-1)} A(\mathcal{X})^{1-n}\frac{R^n(S)R^{n-1}(S)}{R^n(\mathcal{X})}.$$

(iii) $\mathbf{VO}(\mathcal{X},1)$ *is equivalent to the vanishing order part and* $\mathbf{TC}(\mathcal{X},1)$ *is equivalent to the leading Taylor coefficient part of the Birch and Swinnerton-Dyer conjecture for the Jacobian of the generic fiber* X.

Appendix A

Computational Material

A.1 The motivic cycle complexes $\mathbb{Z}(n)$

In this section we will review the construction of Bloch's motivic cycle complexes $\mathbb{Z}(n) = \mathbb{Z}(n)^{\mathscr{X}}$ for arithmetic schemes $\mathscr{X}$.

Simplicial structures. The *standard co-simplex* Δ is the category of finite ordinal numbers $[n] = \{0, \dots, n\}$ with order preserving maps as morphisms. A *simplicial object* A of a category $\mathcal{C}$ or a $\mathcal{C}$*-simplex* is a functor $A : \Delta^{\mathrm{op}} \to \mathcal{C}$. We write $A_n := A([n])$ and think of A as the diagram

$$\cdots A_3 \mathrel{\substack{\longrightarrow\\[-0.6ex]\longrightarrow\\[-0.6ex]\longrightarrow\\[-0.6ex]\longrightarrow}} A_2 \mathrel{\substack{\longrightarrow\\[-0.6ex]\longrightarrow\\[-0.6ex]\longrightarrow}} A_1 \rightrightarrows A_0.$$

Simplicial objects are relevant since they may be viewed as generalizations of chain complexes. Indeed, in abelian categories these two notions are equivalent (cf. [25] Thm. 2.7).

Theorem A.1. (Dold-Kan-Correspondence) *Let $\mathcal{A}$ be an abelian category and write* $\mathrm{Simp}(\mathcal{A})$ *for the category of $\mathcal{A}$-simplices and $C_{\geqslant 0}(\mathcal{A})$ for the category of chain complexes of $\mathcal{A}$ supported in non-negative degrees. There is an equivalence of categories*

$$\mathrm{DK} : \mathrm{Simp}(\mathcal{A}) \longrightarrow C_{\geqslant 0}(\mathcal{A}), \qquad A = (A_\bullet, d_\bullet^\bullet) \;\mapsto\; \mathrm{DK}(A) = (\mathrm{DK}(A)_\bullet, \partial_\bullet)$$

where $\mathrm{DK}(A)_n = \bigcap_{i=0}^{n-1} \operatorname{Ker} d_n^i$ *and* $\partial_n = (-1)^n d_n^n : \mathrm{DK}(A)_n \to \mathrm{DK}(A)_{n-1}$.

The standard co-simplex Δ may be regarded as a full subcategory of the category of arithmetic schemes by setting

$$\Delta^n = \operatorname{Spec} \frac{\mathbb{Z}\left[t_i | i \in [n]\right]}{\sum_{i \in [n]} t_i - 1} = \operatorname{Spec} \frac{\mathbb{Z}[t_0, \dots, t_n]}{\sum_{i=0}^n t_i - 1}.$$

In fact, each $\partial : [m] \to [n]$ gives rise to a canonical morphism of schemes $\Delta^m \to \Delta^n$ induced by $t_j \mapsto \sum_{i \in \partial^{-1}(j)} t_i$. A *face* of Δ^n is a subvariety defined by a set of equations of the kind $t_{i_1} = \cdots = t_{i_s} = 0$.

Arithmetic cycles. Let $\mathscr{X}$ be an arithmetic scheme of pure (Krull) dimension d.

Proposition/Definition A.2.

*(i) For any arithmetic scheme $\mathcal{U}$ and any integers $i, n \geqslant 0$ let $\Delta^i_{\mathcal{U}} = \mathcal{U} \times_{\mathbb{Z}} \Delta^i$. $\mathcal{U}^\bullet$ attains a cosimplicial structure from $\Delta^\bullet$. Next, let $\mathcal{Z}^n(\mathcal{U}, i)$ denote the abelian group freely generated by all n-*cycles *of $\Delta^i_{\mathcal{U}}$, i.e. by all irreducible subschemes of $\Delta^i_{\mathcal{U}}$ of codimension n that intersect all faces of Δ^i properly.*

The proper intersection condition ensures that the inverse image of each map $\partial : \Delta^i_{\mathcal{U}} \to \Delta^j_{\mathcal{U}}$ gives a well-defined map $\partial^{-1} : \mathcal{Z}^n(\mathcal{U}, j) \to \mathcal{Z}^n(\mathcal{U}, i)$ between cycle groups. We denote the corresponding simplex of abelian groups by $\mathcal{Z}^n(\mathcal{U}, \bullet)$.

(ii) Let $n \geqslant 0$. The presheaf $\mathcal{U} \mapsto \mathcal{Z}^n(\mathcal{U}, i)$ of abelian groups on $\mathscr{X}_{\text{ét}}$ is already sheaf. We denote it $\mathcal{Z}^n_{\mathscr{X}}(-, i)$, or just $\mathcal{Z}^n(-, i)$ if $\mathscr{X}$ is clear from context. We write $\mathcal{Z}^n_{\mathscr{X}}(-, \bullet)$ for the associated simplex of abelian sheaves. We define Bloch's cycle complex $\mathbb{Z}(n) = \mathbb{Z}(n)^{\mathscr{X}}$ *to be the chain complex of abelian sheaves on the étale site of $\mathscr{X}$ that arises from the Dold-Kan correspondence applied to the simplex $\mathcal{Z}^n_{\mathscr{X}}(-, \bullet)$ after reindexing via $\bullet \leftrightarrow 2n - \bullet$. More concisely,*

$$\mathbb{Z}(n)^{\mathscr{X}} := \mathrm{DK}\big(\mathcal{Z}^n_{\mathscr{X}}(-, 2n - \bullet) \big).$$

$\mathbb{Z}(n)^{\mathscr{X}}$ is cohomologically concentrated in degrees $\leqslant n + d$. If $d > n$ it is even concentrated in degrees $\leqslant 2n$.

(iii) For $n < 0$ one defines $\mathbb{Z}(n) := \bigoplus_p j_{p,!}(\mu_{p^\infty}^{\otimes n})[-1]$ where $j_p : \mathscr{X}[1/p] \to \mathscr{X}$ denotes the canonical open embedding.

(iv) For any n and $m > 0$ one defines $\mathbb{Z}/m(n) := \mathbb{Z}(n)/m := \mathbb{Z}(n) \otimes^{\mathbb{L}} \mathbb{Z}/m$, i.e. $\mathbb{Z}/m(n)$ is the mapping cone of the complex $\mathbb{Z}(n) \xrightarrow{m} \mathbb{Z}(n)$.

Remark A.3. The indexing in the definition of $\mathbb{Z}(n)$ agrees with Bloch's use of $\mathbb{Z}(n)$ as well as with the use in [8]. So, we will frequently call $\mathbb{Z}(n)$ *Bloch's* cycle complexes. Geisser [12] and Levine [18] use different indexing to obtain a cycle complex – here denoted $\tilde{\mathbb{Z}}(n)$ – that relates to Bloch's complex via $\tilde{\mathbb{Z}}(n)^{\mathscr{X}} = \mathbb{Z}(d - n)^{\mathscr{X}}[2d]$.

Conjecture A.4. *Let $n \geqslant 0$.*

(i) (Beilinson-Soulé) *$\mathbb{Z}(n)^{\mathscr{X}}$ is cohomologically concentrated in non-negative degrees.*

(ii) If $\mathscr{X}$ is regular then $\mathbb{Z}(n)^{\mathscr{X}}$ is cohomologically concentrated in degrees $\leqslant n$.

Geisser has shown Conjecture A.4(ii) for smooth arithmetic schemes $\mathscr{X}$ (cf. [11] Cor. 4.2). Moreover, if $n = 0, 1$, it is known for any scheme $\mathscr{X}$ as it is clear from

Proposition A.5.

(i) One has $\mathbb{Z}(0) \simeq \mathbb{Z}[0]$ and $\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$.

(ii) Let $\mathscr{X} \to \mathcal{S}$ be regular and p invertible on $\mathcal{S}$. Then, if Conjecture A.4(ii) holds true one has $\mathbb{Z}/p^r(n) \simeq \mu_{p^r}^{\otimes n}$.

Even without knowing the Beilinson-Soulé conjecture one may define the motivic cohomology complex $R\Gamma(\mathscr{X}, \mathbb{Z}(n))$ using K-injective resolutions of $\mathbb{Z}(n)$ (as defined in [33]).

A.2 $G_{\mathbb{R}}$-equivariant cohomology

Write $\mathbb{Z}(n)$ for the $G_{\mathbb{R}}$-module $(2\pi i)^n\mathbb{Z}$. For any $G_{\mathbb{R}}$-space X let Γ_X^* denote the constant sheaf functor for $G_{\mathbb{R}}$-equivariant sheaves on X. In other words, Γ_X^* is given by the adjunction $\Gamma_X^* \vdash \Gamma(X, -)$. We write $\mathbb{Z}(n)^X = \Gamma_X^*\mathbb{Z}(n)$ and omit the superscript if X is clear from context. Also, write

$$\epsilon_n = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd} \end{cases} \qquad \overline{\epsilon}_n = 1 - \epsilon_n, \quad \epsilon_{i,n} = \epsilon_{i-n} = \begin{cases} 0 & \text{if } i \equiv n \mod 2 \\ 1 & \text{if } i \not\equiv n \mod 2 \end{cases} \qquad \overline{\epsilon}_{i,n} = 1 - \epsilon_{i,n}.$$

For any infinite place v of F write $X_v = \mathcal{X}_F \times_{F,v} \operatorname{Spec}\mathbb{C}$. Real embeddings and pairs of complex conjugate embeddings of F will be denoted by σ and $\{\tau, \overline{\tau}\}$ respectively and we let $X_{\{\tau,\overline{\tau}\}} = X_\tau \amalg X_{\overline{\tau}}$. Finally, let

$$l(\sigma) := \#\ \textit{of connected components of}\ X_\sigma(\mathbb{R}) = X_\sigma^{G_{\mathbb{R}}}; \qquad \text{and} \qquad l(\mathcal{X}) := \sum_\sigma l(\sigma).$$

In this section we will establish the following computational

Lemma A.6. *One has*

(i)

$$H^i(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) = \begin{cases} 0 & \textit{for } i < 0 \\ \mathbb{Z}^{r\overline{\epsilon}_n + s} & \textit{for } i = 0 \\ \mathbb{Z}^{mg} \oplus (\mathbb{Z}/2)^{l(\mathcal{X})\epsilon_n} & \textit{for } i = 1 \\ \mathbb{Z}^{r\epsilon_n + s} \oplus (\mathbb{Z}/2)^{l(\mathcal{X})} & \textit{for } i = 2 \\ (\mathbb{Z}/2)^{l(\mathcal{X})} & \textit{for } i \geqslant 3 \end{cases}$$

(ii)

$$H^i(\mathcal{X}_\infty, \tau^{>n} R\widehat{\pi}_* \mathbb{Z}(n)) = \begin{cases} 0 & \textit{for } i \leqslant n+1 \\ (\mathbb{Z}/2)^{l(\mathcal{X})} & \textit{for } i \geqslant n+2 \end{cases}$$

(iii) One has $H^i_W(\mathcal{X}_\infty,\mathbb{Z}(n))_{\mathrm{cotor}} \cong \mathbb{Z}^{r\overline{\epsilon}_n+s}, \mathbb{Z}^{mg}, \mathbb{Z}^{r\epsilon_n+s}$ *for* $i=0,1,2$. *Moreover,*

$$H^i_W(\mathcal{X}_\infty,\mathbb{Z}(n))_{\mathrm{tor}} = \begin{cases} (\mathbb{Z}/2)^{l(\mathcal{X})} & \text{for } n \geqslant 0 \text{ and } \overline{\epsilon}_n+1 \leqslant i \leqslant n+1 \\ (\mathbb{Z}/2)^{l(\mathcal{X})} & \text{for } n<0 \text{ and } n+3 \leqslant i \leqslant \overline{\epsilon}_n+1 \\ 0 & \text{otherwise} \end{cases}$$

We begin with some preliminary remarks. Fix a real embedding σ of F and write $X = X_\sigma$. Let $\pi : X \to X/G_\mathbb{R}$ be the natural projection. Write $U = X\backslash X^{G_\mathbb{R}}$. We have a diagram of Open-Closed-Decompositions.

$$\begin{array}{ccccc} X^{G_\mathbb{R}} & \xrightarrow{i} & X & \xleftarrow{j} & U \\ \| & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ X^{G_\mathbb{R}} & \xrightarrow{i} & X/G_\mathbb{R} & \xleftarrow{j} & U/G_\mathbb{R} \end{array} \tag{A.1}$$

where, by abuse of notation, we denote the closed and open embedding on both levels by i and j. One observes directly that the analogues of proper base change

$$j^*\pi_* = \pi_* j^* \qquad \text{and} \qquad i^*\pi_* = \pi_* i^* = (-)^{G_\mathbb{R}} i^*$$

hold. Moreover, one has the following classical results.

Proposition A.7. *(cf. [13] Prop. 3.1) Let* $a = a(\sigma)$ *and* $l = l(\sigma)$ *denote the number of connected components of* U *and* $X^{G_\mathbb{R}}$ *respectively*[1]. *Then*

(i) $U/G_\mathbb{R}$ *is connected and* $X/G_\mathbb{R}$ *has Euler characteristic* $1-g$.

(ii) $0 \leqslant l \leqslant g+1$

(iii) $a = \begin{cases} 2 & \text{if } X/G_\mathbb{R} \text{ is orientable} \\ 1 & \text{otherwise} \end{cases}$

(iv) If $a=1$ *then* $l \neq g+1$. *If* $a=2$ *then* $l \neq 0$ *and* $l \equiv g+1 \bmod 2$.

Moreover, each pair (a,l) *satisfying the constraints (ii)-(iv) arise in the above way from some real smooth proper curve.*

Proof of Lemma A.6.

Computation of $R\Gamma(G_\mathbb{R},\mathcal{X}(\mathbb{C}),\mathbb{Z}(n))$. One has the decomposition

$$\begin{aligned} R\Gamma(G_\mathbb{R},\mathcal{X}(\mathbb{C}),\mathbb{Z}(n)) &= \bigoplus_\sigma R\Gamma(G_\mathbb{R},X_\sigma,\mathbb{Z}(n)) \oplus \bigoplus_{\{\tau,\overline{\tau}\}} R\Gamma(G_\mathbb{R},X_{\{\tau,\overline{\tau}\}},\mathbb{Z}(n)) \\ &= \bigoplus_\sigma R\Gamma(G_\mathbb{R},X_\sigma,\mathbb{Z}(n)) \oplus (2\pi i)^n \bigoplus_{\{\tau,\overline{\tau}\}} R\Gamma(X_{\{\tau,\overline{\tau}\}}/G_\mathbb{R},\mathbb{Z}). \end{aligned} \tag{A.2}$$

[1] Note that a and l do in fact depend on the real embedding σ as can be seen from the elliptic curves over $\mathbb{Q}[\sqrt{5}]$ given by $y^2 = x^3 \pm \sqrt{5}x - 1$.

The last equality follows since $G_{\mathbb{R}}$ acts freely on $X_{\{\tau,\overline{\tau}\}}$. Since $X_{\{\tau,\overline{\tau}\}}/G_{\mathbb{R}} \cong X_\tau$ the cohomology of $R\Gamma(X_{\{\tau,\overline{\tau}\}}/G_{\mathbb{R}}, \mathbb{Z})$ is well-understood. In particular, it has no torsion. We will now analyze the first summand of (A.2). We fix a real embedding σ and adopt the notations of the comments preceding the proof.

The diagram

$$\begin{array}{ccccc} \mathrm{Sh}_{\mathrm{Ab}}(G_{\mathbb{R}}, X) & \xrightarrow{\Gamma(X,-)} & \mathrm{Mod}_{\mathbb{Z}[G_{\mathbb{R}}]} & \xrightarrow{(-)^{G_{\mathbb{R}}}} & \mathrm{AbGrps} \\ \downarrow{\scriptstyle \pi_*} & & & \nearrow{\scriptstyle \Gamma(X/G_{\mathbb{R}},-)} & \\ \mathrm{Sh}(X/G_{\mathbb{R}}) & & & & \end{array} \tag{A.3}$$

commutes since $F(X)^{G_{\mathbb{R}}} = (\pi_* F)(X/G_{\mathbb{R}})$ for any $G_{\mathbb{R}}$-equivariant sheaf F on X. In particular,

$$R\Gamma(G_{\mathbb{R}}, X, \mathbb{Z}(n)^X) \simeq R\Gamma(X/G_{\mathbb{R}}, R\pi_*\mathbb{Z}(n)^X).$$

We will compute the right hand side.

Proposition A.7 shows that either X is the disjoint union of two copies of $X/G_{\mathbb{R}}$ or $\pi : X \to X/G_{\mathbb{R}}$ is the orientation cover of $X/G_{\mathbb{R}}$. Moreover, since $X/G_{\mathbb{R}}$ is either non-orientable or $l > 0$ one has $H_2(X/G_{\mathbb{R}}, \mathbb{Z}) = 0$. Also $X^{G_{\mathbb{R}}} = (S^1)^{\amalg l}$.

We analyze the restrictions of $R\pi_*\mathbb{Z}(n)$ to the closed and to the open part separately. One has

$$\begin{aligned} i^*\pi_*\Gamma_X^* &= (-)^{G_{\mathbb{R}}} i^*\Gamma_X^* = (-)^{G_{\mathbb{R}}}\Gamma^*_{X^{G_{\mathbb{R}}}} = \mathscr{H}\!om(\Gamma^*_{X^{G_{\mathbb{R}}}}\mathbb{Z}, \Gamma^*_{X^{G_{\mathbb{R}}}}-) \\ &= \Gamma^*_{X^{G_{\mathbb{R}}}}\mathrm{Hom}_{G_{\mathbb{R}}}(\mathbb{Z}, -) = \Gamma^*_{X^{G_{\mathbb{R}}}}(-)^{G_{\mathbb{R}}}. \end{aligned}$$

Therefore

$$i^*R\pi_*\mathbb{Z}(n)^X = R(i^*\pi_*\Gamma_X^*)\mathbb{Z}(n) = \Gamma^*_{X^{G_{\mathbb{R}}}} R\Gamma(G_{\mathbb{R}}, \mathbb{Z}(n)).$$

To compute the Galois cohomology complex we use the standard projective resolution

$$\ldots \xrightarrow{1-c} \mathbb{Z}(n)[G_{\mathbb{R}}] \xrightarrow{1+c} \mathbb{Z}(n)[G_{\mathbb{R}}] \xrightarrow{1-c} \mathbb{Z}(n)[G_{\mathbb{R}}] \xrightarrow{\deg} \mathbb{Z}(n) \longrightarrow 0,$$

where $c \in G_{\mathbb{R}}$ denotes complex conjugation. Apply $\mathrm{Hom}_{\mathbb{Z}[G_{\mathbb{R}}]}(-, \mathbb{Z}(n))$. Dropping the first term yields the complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\epsilon_n} \mathbb{Z} \xrightarrow{2\overline{\epsilon}_n} \mathbb{Z} \xrightarrow{2\epsilon_n} \ldots,$$

which is quasi-isomorphic to $\mathbb{Z}^{\overline{\epsilon}_n} \oplus \bigoplus_{k \geqslant 1} \mathbb{Z}/2[\varepsilon_n - 2k]$. In summary, we obtain

$$i^*R\pi_*\mathbb{Z}(n)^X = \Gamma^*_{X^{G_{\mathbb{R}}}}\mathbb{Z}^{\overline{\epsilon}_n} \oplus \bigoplus_{k\geqslant 1} \Gamma^*_{X^{G_{\mathbb{R}}}}\mathbb{Z}/2[\varepsilon_n - 2k]. \tag{A.4}$$

The restriction of $R\pi_*\mathbb{Z}(n)^X$ to the open part needs to be analyzed stalkwise. For $x \in X$ let $G_x \subset G_{\mathbb{R}}$ denote its stabilizer and write $\overline{x} = \pi(x) \in X/G_{\mathbb{R}}$. One has $(R\pi_*\mathbb{Z}(n))_{\overline{x}} \simeq$

$R\Gamma(G_x, \mathbb{Z}(n)_x)$ (cf. [8] Lem. 6.1). Consequently, for $x \in U$ we obtain $(R\pi_*\mathbb{Z}(n))_{\overline{x}} \simeq \mathbb{Z}$, proving that $j^*R\pi_*\mathbb{Z}(n) \simeq R\pi_*\mathbb{Z}(n)^U$ is concentrated in degree 0. Therefore all of $\tau^{\geqslant 1}R\pi_*\mathbb{Z}(n)$ is supported on $X^{G_\mathbb{R}}$. In view of (A.4) the distinguished truncation triangle for $(\tau^{\leqslant 0}, \tau^{\geqslant 1})$ equals

$$\pi_*\mathbb{Z}(n)^X \longrightarrow R\pi_*\mathbb{Z}(n)^X \longrightarrow \bigoplus_{k\geqslant 1} i_*\Gamma^*_{X^{G_\mathbb{R}}}\mathbb{Z}/2[\varepsilon_n - 2k] \longrightarrow .$$

We apply $R\Gamma(X/G_\mathbb{R}, -)$ and obtain

$$R\Gamma(X/G_\mathbb{R}, \pi_*\mathbb{Z}(n)) \longrightarrow R\Gamma(G_\mathbb{R}, X, \mathbb{Z}(n)) \longrightarrow \bigoplus_{k\geqslant 1} R\Gamma(X^{G_\mathbb{R}}, \mathbb{Z}/2)[\varepsilon_n - 2k] \longrightarrow . \qquad \text{(A.5)}$$

We make a distinction of cases to evaluate the cohomology of the left-most complex. We will arrive at

$H^i(X/G_\mathbb{R}, \pi_*\mathbb{Z}(n)^X)$		$i = 0$	$i = 1$	$i = 2$
n even	$a = 1$	$\mathbb{Z}$	$\mathbb{Z}^g$	
	$a = 2$	$\mathbb{Z}$	$\mathbb{Z}^g$	
n odd	$a = 1$		$\mathbb{Z}^g$	$\mathbb{Z}$
	$a = 2$		$\mathbb{Z}^g$	$\mathbb{Z}$

(A.6)

First, let n be even. Then $\pi_*\mathbb{Z}(n)^X = \mathbb{Z}^{X/G_\mathbb{R}}$ and the first part of table (A.6) is immediate.

Now suppose n is odd. Computing stalks shows $i^*\pi_*\mathbb{Z}(n)^X = 0$ and consequently $\pi_*\mathbb{Z}(n)^X = j_!\pi_*\mathbb{Z}(n)^U$. Therefore, the connecting morphisms of (A.5) must vanish. If $a = 2$ then $j_!\pi_*\mathbb{Z}(n)^U = j_!\mathbb{Z}^{U/G_\mathbb{R}}$. The long exact sequence associated to

$$0 \longrightarrow j_!\mathbb{Z}^{U/G_\mathbb{R}} \longrightarrow \mathbb{Z}^{X/G_\mathbb{R}} \longrightarrow i_*\mathbb{Z}^{X^{G_\mathbb{R}}} \longrightarrow 0$$

gives

$$0 = H^0(X/G_\mathbb{R}, j_!\mathbb{Z}) \to \mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}^l \to H^1(X/G_\mathbb{R}, j_!\mathbb{Z}) \to \mathbb{Z}^g \xrightarrow{\alpha} \mathbb{Z}^l \to H^2(X/G_\mathbb{R}, j_!\mathbb{Z}) \to 0.$$

Δ is the diagonal embedding. $H^1(X^{G_\mathbb{R}}, \mathbb{Z})/\mathrm{Im}(\alpha)$ is generated by a copy of S^1 in $X^{G_\mathbb{R}}$ that divides X into two components since such a loop will be trivial in the fundamental group of X and hence $X/G_\mathbb{R}$. We infer $H^1(X/G_\mathbb{R}, j_!\mathbb{Z}) \cong \mathbb{Z}^g$ and $H^2(X/G_\mathbb{R}, j_!\mathbb{Z}) \cong \mathbb{Z}$. This yields the last row of (A.6).

Let now n be odd and $a = 1$. It follows from the classification of compact surfaces with boundary that $X/G_\mathbb{R}$ is the l-times punctured connected sum of $k = g+1-l$ many projective planes. The precise value for k follows from the equality of the two expressions for the Euler characteristic $2 - k - l = 1 - g$ of $X/G_\mathbb{R}$. It is now a standard exercise to compute the Cech cohomology groups of $j_!\pi_*\mathbb{Z}(n)^U$.[2] It yields the remaining row of (A.6).

Now, (i) follows from the long exact sequence of cohomology associated to (A.5).

[2]One may choose $2k + 2l + 2 = 2(g+1)$ many simply connected open neighborhoods covering $X/G_\mathbb{R}$ such

Computation of $R\Gamma(X/G_{\mathbb{R}}, \tau^{>n}R\widehat{\pi}_*\mathbb{Z}(n))$. The analogous computation for Tate cohomology is simpler since one has on stalks

$$(R\widehat{\pi}_*\mathbb{Z}(n))_{\overline{x}} = R\widehat{\Gamma}(G_x, \mathbb{Z}(n)_x) = \begin{cases} \bigoplus_{k\in\mathbb{Z}} \mathbb{Z}/2[\varepsilon_n - 2k] & \text{if } x \in X^{G_{\mathbb{R}}} \\ 0 & \text{if } x \in U \end{cases} \tag{A.7}$$

Consequently $j^*R\widehat{\pi}_*\mathbb{Z}(n)^X = 0$ proving

$$R\widehat{\pi}_*\mathbb{Z}(n)^X = i_*i^*R\widehat{\pi}_*\mathbb{Z}(n)^X = \bigoplus_{k\in\mathbb{Z}} i_*\Gamma^*_{X^{G_{\mathbb{R}}}}\mathbb{Z}/2[\varepsilon_n - 2k] \tag{A.8}$$

and

$$\begin{aligned} R\Gamma(X/G_{\mathbb{R}}, \tau^{>n}R\widehat{\pi}_*\mathbb{Z}(n)) &= \bigoplus_{k\in\mathbb{Z},\, \epsilon_n - 2k < -n} R\Gamma(X^{G_{\mathbb{R}}}, \mathbb{Z}/2)[\epsilon_n - 2k] \\ &= \bigoplus_{k\geqslant 1} R\Gamma(X^{G_{\mathbb{R}}}, \mathbb{Z}/2)[-n-2k]. \end{aligned}$$

Part (ii) follows.

Computation of $R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n))$. Recall that $R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n))$ is defined via the distinguished triangle

$$R\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n)) \longrightarrow R\Gamma(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C}), \mathbb{Z}(n)) \longrightarrow R\Gamma(\mathcal{X}(\mathbb{R}), \tau^{>n}R\widehat{\pi}_*\mathbb{Z}(n)) \longrightarrow .$$

We decompose it analogously to (A.2) and – together with (A.5) – we get for $n \geqslant 0$

$$R\Gamma(X/G_{\mathbb{R}}, \pi_*\mathbb{Z}(n)) \longrightarrow R\Gamma_W(X/G_{\mathbb{R}}, \mathbb{Z}(n)) \longrightarrow \bigoplus_{k=1}^{\frac{n+\epsilon_n}{2}} R\Gamma(X^{G_{\mathbb{R}}}, \mathbb{Z}/2)[\epsilon_n - 2k] \longrightarrow$$

The long exact sequence on cohomology together with (A.6) proves the first case of (iii). For $n < 0$ we similarly get the distinguished triangle

$$\bigoplus_{k=1}^{-\frac{n+\epsilon_n}{2}} R\Gamma(X^{G_{\mathbb{R}}}, \mathbb{Z}/2)[-n-2k-1] \longrightarrow R\Gamma_W(X/G_{\mathbb{R}}, \mathbb{Z}(n)) \longrightarrow R\Gamma(X/G_{\mathbb{R}}, \pi_*\mathbb{Z}(n)) \longrightarrow$$

The remaining cases of (iii) follow. □

The analogous results for $S(\mathbb{C})$ are much simpler to prove. We write $i_{\mathfrak{R}} : S(\mathbb{R}) \hookrightarrow S_\infty$ and $i_{\mathfrak{C}} : S_\infty \backslash S(\mathbb{R}) \hookrightarrow S_\infty$ for the closed immersions of the collection of all real points and pairs of complex conjugate points respectively.

Lemma A.8. *One has*

that no four of them have common non-trivial intersection and such that each boundary of $X/G_{\mathbb{R}}$ intersects precisely two open neighborhoods. The resulting Cech complex has three terms and one verifies by direct computation that each of its its cohomology groups is torsion-free. We leave further details to the reader.

(i)

$$H^i(G_{\mathbb{R}}, S(\mathbb{C}), \mathbb{Z}(n)) = \begin{cases} 0 & \textit{for } i < 0 \\ \mathbb{Z}^{r\bar{\epsilon}_n + s} & \textit{for } i = 0 \\ (\mathbb{Z}/2)^{r\bar{\epsilon}_{i,n}} & \textit{for } i \geqslant 1 \end{cases}$$

(ii)

$$H^i(S_\infty, \tau^{>n} R\widehat{\pi}_* \mathbb{Z}(n)) = \begin{cases} 0 & \textit{for } i \leqslant n+1 \\ (\mathbb{Z}/2)^{r\bar{\epsilon}_{i,n}} & \textit{for } i \geqslant n+2 \end{cases}$$

(iii) One has $H^i_W(S_\infty, \mathbb{Z}(n))_{\text{cotor}} \cong \mathbb{Z}^{r\bar{\epsilon}_n + s}, 0$ *for* $i = 0, i \neq 0$. *Moreover,*

$$H^i_W(S_\infty, \mathbb{Z}(n))_{\text{tor}} = \begin{cases} (\mathbb{Z}/2)^r & \textit{for } n \geqslant 0 \textit{ and } \ \bar{\epsilon}_n + 1 \leqslant i < n+1, \ \ i \equiv n \bmod 2 \\ (\mathbb{Z}/2)^r & \textit{for } n < 0 \textit{ and } \ n+3 \leqslant i < \bar{\epsilon}_n + 1, \ \ i \not\equiv n \bmod 2 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We follow the proof of Lemma A.6. As $S_\sigma = S_F \underset{F,\sigma}{\times} \operatorname{Spec} \mathbb{C} = \operatorname{Spec} \mathbb{C}$ is just a point the analogue of (A.1) collapses to the identity of a one-point space. When combining the analogues of (A.2) and (A.4) we immediately arrive at

$$R\pi_* \mathbb{Z}(n)^{S(\mathbb{C})} = (i_{\mathfrak{C}})_* \Gamma^*_{S_\infty \setminus S(\mathbb{R})} \mathbb{Z} \oplus (i_{\mathfrak{R}})_* \left(\Gamma^*_{S(\mathbb{R})} \mathbb{Z}^{\bar{\epsilon}_n} \oplus \Gamma^*_{S(\mathbb{R})} \bigoplus_{k \geqslant 1} \mathbb{Z}/2[\epsilon_n - 2k] \right). \tag{A.9}$$

Applying $R\Gamma(S_\infty, -)$ yields

$$R\Gamma(G_{\mathbb{R}}, S(\mathbb{C}), \mathbb{Z}(n)) = \mathbb{Z}^{r\epsilon_n + s} \oplus \bigoplus_{k \geqslant 1} R\Gamma(S(\mathbb{R}), \mathbb{Z}/2)[\epsilon_n - 2k].$$

Part (i) follows. Next, mimicking the computations (A.7) and (A.8) yields

$$R\widehat{\pi}_* \mathbb{Z}(n)^{S(\mathbb{C})} \simeq (i_{\mathfrak{R}})_* \Gamma^*_{S(\mathbb{R})} \bigoplus_{k \in \mathbb{Z}} \mathbb{Z}/2[\epsilon_n - 2k]. \tag{A.10}$$

(ii) follows after truncating and applying $R\Gamma(S_\infty, -)$.

For (iii) consider the distinguished triangle defining $R\Gamma_W(S_\infty, \mathbb{Z}(n))$. For $n \geqslant 0$ one gets

$$R\Gamma_W(S_i, \mathbb{Z}(n)) \ \simeq \ \mathbb{Z}^{r\bar{\epsilon}_n + s} \oplus \bigoplus_{k=1}^{\frac{n+\epsilon_n}{2}} R\Gamma(S(\mathbb{R}), \mathbb{Z}/2)[\epsilon_n - 2k].$$

Similarly, for $n < 0$ one has

$$\bigoplus_{k=1}^{-\frac{n+\epsilon_n}{2}} R\Gamma(S(\mathbb{R}), \mathbb{Z}/2)[-n-2k-1] \longrightarrow R\Gamma_W(S_\infty, \mathbb{Z}(n)) \longrightarrow \mathbb{Z}^{r\bar{\epsilon}_n + s} \longrightarrow$$

proving the final part of the claim. □

Corollary A.9. *Write* ${}^1l(\mathcal{X}) = l(\mathcal{X}) - r$*. One has* ${}^1H^1_W(S_\infty, \mathbb{Z}(n))_{\mathrm{cotor}} \cong \mathbb{Z}^{mg}, 0$ *for* $i = 1, i \neq 1$*. Moreover,*

$$ {}^1H^i_W(\mathcal{X}_\infty, \mathbb{Z}(n))_{\mathrm{tor}} = \begin{cases} (\mathbb{Z}/2)^{{}^1l(\mathcal{X})} & \text{for } n \geqslant 0 \text{ and } \bar{\epsilon}_n + 1 \leqslant i \leqslant n+1 \\ (\mathbb{Z}/2)^{{}^1l(\mathcal{X})} & \text{for } n < 0 \text{ and } n+3 \leqslant i \leqslant \bar{\epsilon}_n + 1 \\ 0 & \text{otherwise} \end{cases} $$

In particular,

$$ \chi\left({}^pR^1\Gamma_W(\mathcal{X}_\infty, \mathbb{Z}(n))\right) = \prod_{i\in\mathbb{Z}} \left(\#\mathrm{Tor}\ {}^1H^{i,n}_{W,\infty}(\mathcal{X})\right)^{(-1)^i} = 1. $$

Corollary A.10. *For all integers n one has*

$$ \chi(R\Gamma_W(S_\infty, \mathbb{Z}(n))) = \prod_{i\in\mathbb{Z}} \left(\#\mathrm{Tor}\ H^{i,n}_{W,\infty}(S)\right)^{(-1)^i} = 2^{(-1)^n \frac{n+\epsilon_n}{2} r}. $$

A.3 Comparison between motivic and completed motivic cohomology

Recall the triangle (2.26). Building on the work done in the last section we will show that the term $\tau^{>n}R\hat{\pi}_*\mathbb{Z}(n)^{\mathcal{X}(\mathbb{C})}$ controlling the discrepancy between motivic and completed motivic cohomology allows a decomposition into motivic degrees analogous to Theorem 2.11.

Lemma A.11. *Let $\pi_\infty : \mathcal{X}_\infty \to S_\infty$ be the structure map of $\mathcal{X}_\infty$. Suppose π_∞ has a section $s_\infty : S_\infty \to \mathcal{X}_\infty$, or, equivalently, $l(\sigma) > 0$ for every real place σ. Then – in the derived category of abelian sheaves on S_∞ – the complexes $R\hat{\pi}_*\mathbb{Z}(n)^{S(\mathbb{C})}$ and $R\hat{\pi}_*\mathbb{Z}(n)^{S(\mathbb{C})}[-1]$ split off as direct summands of $R\pi_{\infty,*}R\hat{\pi}_*\mathbb{Z}(n)^{\mathcal{X}(\mathbb{C})}$.*[3] *When writing ${}^pR^1\pi_{\infty,*}R\hat{\pi}_*\mathbb{Z}(n)^{\mathcal{X}(\mathbb{C})}$ for the remaining summand we arrive at the canonical decomposition*

$$ R\pi_{\infty,*}R\hat{\pi}_*\mathbb{Z}(n)^{\mathcal{X}(\mathbb{C})} \simeq R\hat{\pi}_*\mathbb{Z}(n)^{S(\mathbb{C})} \oplus {}^pR^1\pi_{\infty,*}R\hat{\pi}_*\mathbb{Z}(n)^{\mathcal{X}(\mathbb{C})}[-1] \oplus R\hat{\pi}_*\mathbb{Z}(n-1)^{S(\mathbb{C})}[-2] $$

or, equivalently,

$$ R\pi_{\infty,*}R\hat{\pi}_*\mathbb{Z}(n) \simeq (i_{\mathfrak{R}})_*\bigoplus_{k\in\mathbb{Z}}\mathbb{Z}/2^{S(\mathbb{R})}[\epsilon_n - 2k] \oplus {}^pR^1\pi_{\infty,*}R\hat{\pi}_*\mathbb{Z}(n)[-1] \oplus (i_{\mathfrak{R}})_*\bigoplus_{k\in\mathbb{Z}}\mathbb{Z}/2^{S(\mathbb{R})}[\epsilon_n - 2k - 1] $$

Proof. The equivalence of the two decompositions is (A.10); we will prove the latter. We adopt the notations of the proof of Lemma A.6. We will also need the restrictions $\pi_\mathbb{R} = \pi_\infty|_{\mathcal{X}(\mathbb{R})}$ and $s_\mathbb{R} = s_\infty|_{S(\mathbb{R})}$. By virtue of (A.8) one has

$$ R\pi_{\infty,*}R\hat{\pi}_*\mathbb{Z}(n) = R\pi_{\infty,*}\bigoplus_{k\in\mathbb{Z}} i_*\Gamma^*_{X^{G_\mathbb{R}}}\mathbb{Z}/2 = \bigoplus_{k\in\mathbb{Z}} R\pi_{\infty,*}i_*\mathbb{Z}/2^{S(\mathbb{R})} = \bigoplus_{k\in\mathbb{Z}} (i_{\mathfrak{R}})_*R\pi_{\mathbb{R},*}\mathbb{Z}/2^{S(\mathbb{R})}. $$

[3] Beware the two different roles of the letter π. π_∞ is derived from the structure map $\mathcal{X}(\mathbb{C}) \to S(\mathbb{C})$ while $\hat{\pi}_*$ is the Tate modification of the morphism of topoi $\mathrm{Sh}(G_\mathbb{R}, \mathcal{X}(\mathbb{C})) \to \mathrm{Sh}(\mathcal{X}_\infty)$ induced by the natural projection $\mathcal{X}(\mathbb{C}) \to \mathcal{X}_\infty$.

So, it suffices to show that $\mathbb{Z}/2^{S(\mathbb{R})}$ and $\mathbb{Z}/2^{S(\mathbb{R})}[-1]$ split off as direct summands of $R\pi_{\mathbb{R},*}\mathbb{Z}/2^{S(\mathbb{R})}$. The category of constant sheaves on the finite point space $S(\mathbb{R})$ is equivalent to the category of abelian groups via the global sections functor. Therefore we have to prove a decomposition of the form

$$R\Gamma(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2) \;\simeq\; R\Gamma(S(\mathbb{R}),\mathbb{Z}/2) \oplus {}^{p}R^1\Gamma(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2) \oplus R\Gamma(S(\mathbb{R}),\mathbb{Z}/2)[-1].$$

Using the derived functor formalism as in the proof of Theorem 2.8 the identity $\pi_{\mathbb{R}}s_{\mathbb{R}} = \mathrm{id}$ shows immediately that $R\Gamma(S(\mathbb{R}),\mathbb{Z}/2)$ splits off as direct summand of $R\Gamma(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2)$ (see also Remark 2.9). Moreover, Poincaré duality gives

$$R\Gamma(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2)^\vee \simeq R\Gamma(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2)[1] \qquad \text{and} \qquad R\Gamma(S(\mathbb{R}),\mathbb{Z}/2)^\vee \simeq R\Gamma(S(\mathbb{R}),\mathbb{Z}/2).$$

Therefore $R\Gamma(S(\mathbb{R}),\mathbb{Z}/2)[-1] \;\simeq\; R\Gamma(S(\mathbb{R}),\mathbb{Z}/2)[1]^\vee$ splits off of $R\Gamma(\mathcal{X}(\mathbb{R}),\mathbb{Z}/2)$ as well. $R\Gamma(S(\mathbb{R}),\mathbb{Z}/2)$ and $R\Gamma(S(\mathbb{R}),\mathbb{Z}/2)[-1]$ must in fact be distinct direct summands for degree reasons. □

Evidently the same proof holds for any truncation of $R\widehat{\pi}_*\mathbb{Z}(n)^{\mathcal{X}(\mathbb{C})}$. Therefore, the decomposition of Theorem 2.11 extends to completed motivic cohomology.

Corollary A.12. *Suppose $\pi : \mathcal{X} \to S$ has a section $s : S \to \mathcal{X}$ satisfying* **FPB**(s,n) *for all integers $n \geqslant 2$. Then the complexes $\mathbb{Z}(n)^{\overline{S}}$ and $\mathbb{Z}(n-1)^{\overline{S}}[-2]$ split off as direct summands of $R\pi_*\mathbb{Z}(n)^{\overline{\mathcal{X}}}$ in the derived category of sheaves on the Artin-Verdier étale site $\overline{\mathcal{X}} = \overline{\mathcal{X}}_{\text{ét}}$ of $\mathcal{X}$. When writing ${}^{p}R^1\pi_*\mathbb{Z}(n)^{\overline{\mathcal{X}}}[-1]$ for the remaining summand we arrive at the canonical decomposition*

$$R\pi_*\mathbb{Z}(n)^{\overline{\mathcal{X}}} \;\simeq\; \mathbb{Z}(n)^{\overline{S}} \oplus {}^{p}R^1\pi_*\mathbb{Z}(n)^{\overline{\mathcal{X}}}[-1] \oplus \mathbb{Z}(n-1)^{\overline{S}}[-2]. \tag{A.11}$$

Remark A.13. The decomposition (2.14) of Theorem 2.11 cannot generally hold without assuming the existence of a section. Indeed, if the generic fiber $X = \mathcal{X}_F$ does not have an F-rational point then $l(\mathcal{X}) = 0$ and consequently $H^{i,n}(\overline{\mathcal{X}}) = H^{i,n}(\mathcal{X})$ for all i,n. However, we will see below that $H^{i,n}(\overline{\mathcal{X}}) = H^{i,n}(\overline{S}) = 0$ for $i \geqslant 7$, $n \geqslant 1$ while at the same time $H^{i,n}(S) = (\mathbb{Z}/2)^{r\overline{\epsilon}_{i,n}}$ for $i \geqslant 5$, $n \geqslant 1$.

If a section exists, then $l(\mathcal{X})$ can be understood in terms of the Jacobian $\mathcal{J} = \mathrm{Jac}\,\mathcal{X}_F$ of the generic fiber. In fact, one has

Proposition A.14. *Let X be a smooth proper real algebraic curve. Write $J = \mathrm{Jac}\,X(\mathbb{C})$ and let l be the number of connected components of $X(\mathbb{R})$. If $l > 0$ then*

$$H^1(G_{\mathbb{R}}, J) \cong (\mathbb{Z}/2)^{l-1}.$$

Proof. Combine [13] Prop. 1.1, Prop. 1.3, and Prop. 3.2(2). □

Computation of torsion parts. We will conclude with a summary of all information on torsion for motivic and completed motivic cohomology. Recall the notations $T_?^{i,n} = \operatorname{Tor} H^{i,n}(?)_{\mathrm{codiv}}$ for $? = S, \overline{S}, \mathcal{X}, \overline{\mathcal{X}}$ and ${}^1T_?^{i,n} = \operatorname{Tor} {}^1H^{i,n}(?)_{\mathrm{codiv}}$ for $? = \mathcal{X}, \overline{\mathcal{X}}$.

Proposition A.15. *Suppose $\pi : \mathcal{X} \to S$ has a section $s : S \to \mathcal{X}$ satisfying* $\mathbf{FPB}(s,n)$ *for all integers $n \geqslant 2$. Write ${}^1l(\mathcal{X}) = l(\mathcal{X}) - r$. Then $T_S^{i,n}$, $T_{\overline{S}}^{i,n}$ and ${}^1T_{\mathcal{X}}^{i,n}$, ${}^1T_{\overline{\mathcal{X}}}^{i,n}$ are given as in the tables below.*

$T_S^{i,n}$ / $T_{\overline{S}}^{i,n}$		$i \leqslant 0$	$i = 1$	$i = 2$	$i = 3$	$4 \leqslant i$
$n = 0$	$T_S^{i,n}$	0	0	$\sim \mathrm{Cl}_F$	$\sim \mu_F$	$(\mathbb{Z}/2)^{r\overline{\epsilon}_i}$
	$T_{\overline{S}}^{i,n}$	0	0	Cl_F	μ_F	0
$n = 1$	$T_S^{i,n}$	0	μ_F	Cl_F	$\operatorname{Br}\mathcal{O}$	$(\mathbb{Z}/2)^{r\epsilon_i}$
	$T_{\overline{S}}^{i,n}$	0	μ_F	Cl_F	0	0

$T_S^{i,n}$ / $T_{\overline{S}}^{i,n}$		$i \leqslant 0$	$i = 1$	$i = 2$	$3 \leqslant i \leqslant n+1$	$n+2 \leqslant i$
$n > 1$	$T_S^{i,n}$	0	$T_{\overline{S}}^{1,n}$	$T_{\overline{S}}^{2,n}$	$(\mathbb{Z}/2)^{r\overline{\epsilon}_{i,n}}$	$(\mathbb{Z}/2)^{r\overline{\epsilon}_{i,n}}$
	$T_{\overline{S}}^{i,n}$	0	$T_{\overline{S}}^{1,n}$	$T_{\overline{S}}^{2,n}$	$(\mathbb{Z}/2)^{r\overline{\epsilon}_{i,n}}$	0

$T_S^{i,n}$ / $T_{\overline{S}}^{i,n}$		$i \leqslant n+2$	$n+3 \leqslant i \leqslant 1$	$i = 2$	$i = 3$	$4 \leqslant i$
$n < 0$	$T_S^{i,n}$	0	0	$\sim T_{\overline{S}}^{2,1-n}$	$\sim T_{\overline{S}}^{1,1-n}$	$(\mathbb{Z}/2)^{r\epsilon_{i,n}}$
	$T_{\overline{S}}^{i,n}$	0	$(\mathbb{Z}/2)^{r\epsilon_{i,n}}$	$T_{\overline{S}}^{2,1-n}$	$T_{\overline{S}}^{1,1-n}$	0

${}^1T_{\mathcal{X}}^{i,n}$ / ${}^1T_{\overline{\mathcal{X}}}^{i,n}$		$i \leqslant 1$	$i = 2$	$i = 3$	$i = 4$	$i \geqslant 5$
$n = 0$	${}^1T_{\mathcal{X}}^{i,n}$	0	$\sim {}^1T_{\overline{\mathcal{X}}}^{4,2-n}$	$\sim {}^1T_{\overline{\mathcal{X}}}^{3,2-n}$	$\sim {}^1T_{\overline{\mathcal{X}}}^{2,2-n}$	$(\mathbb{Z}/2)^{{}^1l(\mathcal{X})}$
	${}^1T_{\overline{\mathcal{X}}}^{i,n}$	0	${}^1T_{\overline{\mathcal{X}}}^{4,2-n}$	${}^1T_{\overline{\mathcal{X}}}^{3,2-n}$	${}^1T_{\overline{\mathcal{X}}}^{2,2-n}$	0
$n = 1$	${}^1T_{\mathcal{X}}^{i,n}$	0	$\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\mathrm{Cl}_F}$	$\frac{\operatorname{Br}\mathcal{X}}{\operatorname{Br}\mathcal{O}}$	$\sim \frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\mathrm{Cl}_F}$	$(\mathbb{Z}/2)^{{}^1l(\mathcal{X})}$
	${}^1T_{\overline{\mathcal{X}}}^{i,n}$	0	$\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\mathrm{Cl}_F}$	$Ш(X/F)$	$\frac{\operatorname{Tor}\operatorname{Pic}^0\mathcal{X}}{\mathrm{Cl}_F}$	0
$n = 2$	${}^1T_{\mathcal{X}}^{i,n}$	0	${}^1T_{\overline{\mathcal{X}}}^{2,n}$	${}^1T_{\overline{\mathcal{X}}}^{3,n}$	$\sim {}^1T_{\overline{\mathcal{X}}}^{4,n}$	$(\mathbb{Z}/2)^{{}^1l(\mathcal{X})}$
	${}^1T_{\overline{\mathcal{X}}}^{i,n}$	0	${}^1T_{\overline{\mathcal{X}}}^{2,n}$	${}^1T_{\overline{\mathcal{X}}}^{3,n}$	${}^1T_{\overline{\mathcal{X}}}^{4,n}$	0

${}^1T^{i,n}_{\mathcal{X}}$ / ${}^1T^{i,n}_{\overline{\mathcal{X}}}$	$i \leqslant 1$	$i = 2,3,4$	$5 \leqslant i \leqslant n+1$	$n+2 \leqslant i$
$n > 2$	0	${}^1T^{i,n}_{\overline{\mathcal{X}}}$	$(\mathbb{Z}/2)^{{}^1l(\mathcal{X})}$	$(\mathbb{Z}/2)^{{}^1l(\mathcal{X})}$
	0	${}^1T^{i,n}_{\overline{\mathcal{X}}}$	$(\mathbb{Z}/2)^{{}^1l(\mathcal{X})}$	0

${}^1T^{i,n}_{\mathcal{X}}$ / ${}^1T^{i,n}_{\overline{\mathcal{X}}}$	$i \leqslant n+2$	$n+3 \leqslant i \leqslant 1$	$i = 2,3,4$	$5 \leqslant i$
$n < 0$	0	0	$\sim\ {}^1T^{6-i,2-n}_{\overline{\mathcal{X}}}$	$(\mathbb{Z}/2)^{{}^1l(\mathcal{X})}$
	0	$(\mathbb{Z}/2)^{{}^1l(\mathcal{X})}$	${}^1T^{6-i,2-n}_{\overline{\mathcal{X}}}$	0

In particular, $R\Gamma(\overline{S}, \mathbb{Z}(n))$ *and* ${}^pR^1\Gamma(\overline{\mathcal{X}}, \mathbb{Z}(n))$ *are perfect complexes.*

Proof. The vanishing of ${}^1T^{i,n}_{\mathcal{X}}$ and ${}^1T^{i,n}_{S}$ for $i \leqslant 1$ and $i \leqslant 0$ respectively has been established in Proposition 2.20. The torsion groups for $n = 1$ and near-central i have already been computed in in the proof of Proposition 2.17. The remaining entries are easily obtained from $\mathbb{Z}(0) \simeq \mathbb{Z}$, $\mathbb{Z}(1) \simeq \mathbb{G}_m[-1]$ and Artin-Verdier duality. Throughout, use the triangle (2.26) to translate between motivic and completed motivic cohomology. □

Corollary A.16. *One has*

(i)

$$\prod_{i\in\mathbb{Z}} \left(\#\mathrm{Tor}\, H^{i,n}(\overline{S})\right)^{(-1)^i} = \begin{cases} 2^{(-1)^n \frac{n+\epsilon_n}{2} r} \dfrac{\# T^{2,1-n}_{\overline{S}}}{\# T^{1,1-n}_{\overline{S}}} & \text{for } n \leqslant 0 \\ 2^{(-1)^n \left(\frac{n+\epsilon_n}{2} - 1\right) r} \dfrac{\# T^{2,n}_{\overline{S}}}{\# T^{1,n}_{\overline{S}}} & \text{for } n \geqslant 1 \end{cases}$$

(ii)

$$\prod_{i\in\mathbb{Z}} \left(\#\mathrm{Tor}\ {}^1H^{i,n}(\overline{\mathcal{X}})\right)^{(-1)^i} = \begin{cases} \dfrac{1}{\#\text{Ш}(X/F)} \left(\# \dfrac{\mathrm{Tor}\,\mathrm{Pic}^0 \mathcal{X}}{\mathrm{Cl}_F}\right)^2 & \text{for } n = 1 \\ 2^{-\epsilon_n {}^1l(\mathcal{X})} \dfrac{\# {}^1T^{2,n}_{\overline{\mathcal{X}}} \cdot \# {}^1T^{4,n}_{\overline{\mathcal{X}}}}{\# {}^1T^{3,n}_{\overline{\mathcal{X}}}} & \text{for } n \neq 1 \end{cases}$$

A.4 Supplementary material for derived de Rham cohomology

In this section $\pi : X \to S$ will denote any map of schemes.

Construction of the cotangent complex $L_{X/S}$. An adjunction $T \vdash U$ between categories $\mathcal{A}, \mathcal{B}$ – which we write as $\mathcal{A} \underset{U}{\overset{T}{\rightleftarrows}} \mathcal{B}$ – gives rise to a simplicial structure $(T, U)_\bullet$ on the endofunctor category $\operatorname{End} \mathcal{B}$ as follows. The adjunction data may be thought of as a pair of natural transformations

$$\alpha : \mathrm{id}_{\mathcal{A}} \longrightarrow UT \quad \text{and} \quad \beta : TU \longrightarrow \mathrm{id}_{\mathcal{B}}$$

that satisfy the compatibility relations

$$(\beta * T) \circ (T * \alpha) = \mathrm{id}_T \qquad \text{and} \qquad (U * \beta) \circ (\alpha * U) = \mathrm{id}_U.$$

We define $(T, U)_n = (TU)^{n+1}$ and let the (co)boundary maps be given by the natural transformations

$$(TU)^i * \beta * (TU)^{n-i} : (TU)^{n+1} \longrightarrow (TU)^n, \qquad (TU)^{i-1} T * \alpha * U (TU)^{n-i} : (TU)^n \longrightarrow (TU)^{n+1}.$$

Repeated application of β gives a canonical map $(T, U)_\bullet \to \mathrm{id}_{\mathcal{B}}$, i.e. a unique compatible collection of natural transformations $(T, U)_n \to \mathrm{id}_{\mathcal{B}}$.

Now, fix a scheme X and let $\mathrm{Sh}(X_{\mathrm{Zar}})$ and $\mathrm{Sh}_{\mathrm{Rings}}(X_{\mathrm{Zar}})$ denote the topoi of sheaves of sets and rings on the Zariski site X_{Zar} of X. We fix $\mathcal{O} \in \mathrm{Sh}_{\mathrm{Rings}}(X_{\mathrm{Zar}})$ and let $\mathrm{Sh}_{\mathcal{O}\text{-Alg}}(X_{\mathrm{Zar}})$ denote the topos of $\mathcal{O}$-algebras on X_{Zar}. We will apply the previous construction to the adjunction

$$\mathrm{Sh}(X_{\mathrm{Zar}}) \underset{\mathrm{Ou}}{\overset{\mathcal{O}[-]}{\rightleftarrows}} \mathrm{Sh}_{\mathcal{O}\text{-Alg}}(X_{\mathrm{Zar}}) ,$$

where the functor $\mathcal{O}[-] = \mathrm{Sym}_{\mathcal{O}} \mathcal{O}^{(-)}$ assigns to any sheaf of sets $\mathcal{F}$ on X_{Zar} the sheaf of $\mathcal{O}$-algebras $\mathcal{O}[\mathcal{F}]$ which is the sheafification of the presheaf that assigns to each Zariski-open $U \subset X$ the $\mathcal{O}(U)$-algebra freely generated by $\mathcal{F}(U)$ regarded as a set. Let $\mathcal{P}_{\mathcal{O}}(-)$ denote the resulting $\operatorname{End} \mathrm{Sh}_{\mathcal{O}\text{-Alg}}(X_{\mathrm{Zar}})$-simplex.

If X comes with a structure map $\pi : X \to S$ the above construction may be carried out for $\mathcal{O} = \pi^{-1}\mathcal{O}_S$. Applying the resulting functors to the $\pi^{-1}\mathcal{O}_S$-module $\mathcal{O}_X$ yields the $\mathrm{Sh}_{\pi^{-1}\mathcal{O}_S\text{-Alg}}(X_{\mathrm{Zar}})$-simplex $P_\bullet^{X/S} = \mathcal{P}_{\pi^{-1}\mathcal{O}_S}(\mathcal{O}_X)$. Each $\mathcal{P}_i^{X/S}$ has an algebraic de Rham resolution $\mathcal{P}_i^{X/S} \to \Omega^\bullet_{P_i^{X/S}/\pi^{-1}\mathcal{O}_S} = \Omega^\bullet_{P_i^{X/S}/\pi^{-1}\mathcal{O}_S}$. So, we obtain a complex of $\mathrm{Sh}_{\pi^{-1}\mathcal{O}_S\text{-Alg}}(X_{\mathrm{Zar}})$-simplices

$$\Omega^\bullet_{P_\bullet^{X/S}} = \begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \uparrow \\
\Rrightarrow & & \Omega^2_{P_2^{X/S}} & \Rrightarrow & \Omega^2_{P_1^{X/S}} & \rightrightarrows & \Omega^2_{P_0^{X/S}} \\
 & & \uparrow & & \uparrow & & \uparrow \\
\Rrightarrow & & \Omega_{P_2^{X/S}} & \Rrightarrow & \Omega_{P_1^{X/S}} & \rightrightarrows & \Omega_{P_0^{X/S}} \\
 & & \uparrow & & \uparrow & & \uparrow \\
\Rrightarrow & & P_2^{X/S} & \Rrightarrow & P_1^{X/S} & \rightrightarrows & P_0^{X/S}
\end{array}$$

The rows $\Omega^k_{P_\bullet^{X/S}}$ of $\Omega^\bullet_{P_\bullet^{X/S}}$ may be regarded as $P_\bullet^{X/S}$-modules and we may use the morphism $P_\bullet^{X/S} \to \mathcal{O}_X$ to define

$$\tilde{L}^k_{X/S} := \Omega^k_{P_\bullet^{X/S}} \underset{P_\bullet^{X/S}}{\otimes} \mathcal{O}_X$$

as a simplicial object of $\mathrm{Sh}_{\mathcal{O}_X\text{-Mod}}(X_{\mathrm{Zar}})$.

Write $L^k_{X/S} = \mathrm{DK}(\tilde{L}^k_{X/S})$ for the complex of $\mathcal{O}_X$-modules associated to $\tilde{L}^k_{X/S}$ via the Dold-Kan correspondence. $L_{X/S} := L^1_{X/S}$ is the *cotangent complex* of $\pi : X \to S$. One has $L^k_{X/S} = \bigwedge^k L_{X/S}$. We write

$$L\Omega^\bullet_{X/S} = \int L^\bullet_{X/S} = \int \mathrm{DK}\left(\Omega^\bullet_{P_\bullet^{X/S}} \underset{P_\bullet^{X/S}}{\otimes} \mathcal{O}_X\right)$$

for the totalization of the double complex arising from $\Omega^\bullet_{P_\bullet^{X/S}} \otimes \mathcal{O}_X$ after applying the Dold-Kan correspondence to its rows. Moreover, we define a filtration

$$F^m := \mathrm{Fil}^m L\Omega^\bullet_{X/S} := \int L^{\geqslant m}_{X/S} = \int \mathrm{DK}\left(\Omega^{\geqslant m}_{P_\bullet^{X/S}} \underset{P_\bullet^{X/S}}{\otimes} \mathcal{O}_X\right)$$

and write $L\Omega^{\leqslant m}_{X/S} := L\Omega^\bullet_{X/S}/F^m := L\Omega^\bullet_{X/S}/\mathrm{Fil}^m L\Omega^\bullet_{X/S}$ as well as $\mathrm{gr}^m L\Omega^\bullet_{X/S} = F^m/F^{m+1}$.

The de Rham conductor $A(\mathcal{X})$. The *de Rham* or *Kato conductor* of $\pi' : \mathcal{X} \to \mathbb{Z}$

$$A(\mathcal{X}) := \prod_{i \in \mathbb{Z}} \left(\#\mathbb{H}^i(\mathcal{X}, \Omega^\bullet_{\mathcal{X},\mathrm{tors}})\right)^{(-1)^i} \in \mathbb{Q}^\times$$

encapsulates the discrepancy between algebraic de Rham and derived de Rham cohomology (cf. Proposition 2.23(iii)). Since $\Omega^\bullet_{\mathcal{X},\mathrm{tors}}$ is concentrated on the non-smooth points of $\mathcal{X}$ the conductor $A(\mathcal{X})$ depends only on the bad fibers of $\mathcal{X}$. In many instances it can be computed explicitly. We give one example.

Proposition A.17. *Let $\pi' : \mathcal{X} \to \mathbb{Z}$ have semistable reduction at every prime p. Write $i_p : Z_p \hookrightarrow \mathcal{X}$ for the closed embedding of the subscheme of singular points Z_p of the special fiber $\mathcal{X}_p$ into $\mathcal{X}$. Then*

$$\Omega_{\mathcal{X}/\mathbb{Z},\mathrm{tors}} = 0 \qquad \text{and} \qquad \Omega^2_{\mathcal{X}/\mathbb{Z}} = \bigoplus_p (i_p)_* \mathbb{Z}/p.$$

In particular,

$$A(\mathcal{X}) = \prod_p p^{\# Z_p}.$$

Proof. First, observe that $Z = \bigcup_p Z_p \subset \mathcal{X}$ must be a finite collection of closed points. Let x be a singular closed point of $\mathcal{X}$ and let $\overline{x} \hookrightarrow \mathcal{X}$ be a corresponding geometric point. Semistability means that

$$\mathcal{O}_{\mathcal{X},\overline{x}} = \mathcal{O}^{sh}_{\mathcal{X},\overline{x}} \cong \left(\frac{\mathbb{Z}[u,v]}{uv-p}\right)^{sh}_{(u,v,p)},$$

i.e. étale locally at $\overline{x}$ the differentials of $\mathcal{X}$ coincide with the differentials of the scheme $\mathcal{X}_0 = \operatorname{Spec} \frac{\mathbb{Z}[u,v]}{uv-p}$ over $\mathbb{Z}$ at its singular point $O = (u, v, p)$. So, when writing $f(u, v) = uv - p$, the claim follows from the direct computations

$$\left(\Omega_{\mathcal{X}_0/\mathbb{Z}}\right)_{O,\mathrm{tors}} = \Gamma(\mathcal{X}_0, \Omega_{\mathcal{X}_0/\mathbb{Z}})_{\mathrm{tors}} = \left(\frac{\mathcal{O}_{\mathcal{X},\overline{x}}du \oplus \mathcal{O}_{\mathcal{X},\overline{x}}dv}{df}\right)_{\mathrm{tors}} = 0$$

and

$$\left(\Omega^2_{\mathcal{X}_0/\mathbb{Z}}\right)_O = \Gamma(\mathcal{X}_0, \Omega^2_{\mathcal{X}_0/\mathbb{Z}}) = \frac{\mathcal{O}_{\mathcal{X},\overline{x}}}{\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)} du \wedge dv \cong \mathbb{Z}/p.$$

□

This exercise may be repeated for singularities described by different equations f. A general formula for $A(\mathcal{X})$ — which involves Swan characters of Galois representations given by the l-adic étale cohomology of $X_{\overline{\mathbb{Q}}}$ — has been found by Bloch in [3], Prop. 1.1.

A.5 Overview of computed cohomology groups

The following tables summarize the results for the ranks of the cohomology groups associated to $\mathcal{X}$ as computed in Chapter 2 and pair them with the corresponding cohomology groups for S (see [8] Sec. 5.8 for their derivations). As indicated before, we observe in all cases a decomposition pattern as it is implied by decompositions of the kind

$$H^{i,n}_?(\mathcal{X}) = H^{i,n}_?(S) \oplus {}^1H^{i,n}_?(\mathcal{X}) \oplus H^{i-2,n-1}_?(S).$$

We also provide torsion information for motivic and Weil-étale motivic cohomology groups.

A.5.1 Rank tables

Ranks of Deligne cohomology.

$\dim_{\mathbb{R}} H^{i,n}_{\mathcal{D}}(S)$ $\dim_{\mathbb{R}} H^{i,n}_{\mathcal{D}}(\mathcal{X})$	$i = 0$	$i = 1$	$i = 2$	$i = 3$
$n \leqslant 0$	$\bar{\epsilon}_n r + s$ $\bar{\epsilon}_n r + s$	 mg	 $\epsilon_n r + s$	
$n = 1$		$r + s$ $r + s$	 $r + s$	
$n \geqslant 2$		$\epsilon_n r + s$ $\epsilon_n r + s$	 mg	 $\bar{\epsilon}_n r + s$

(A.12)

Ranks of $H^i_W(\mathcal{X}_\infty, \mathbb{Z}(n))$.

$\dim_{\mathbb{R}} H^{i,n}_{W,\infty}(S)$ $\dim_{\mathbb{R}} H^{i,n}_{W,\infty}(\mathcal{X})$	$i = 0$	$i = 1$	$i = 2$
any n	$\bar{\epsilon}_n r + s$ $\bar{\epsilon}_n r + s$	 mg	 $\epsilon_n r + s$

(A.13)

Ranks of de Rham cohomology.

$\dim_{\mathbb{R}} H^{i,n}_{\mathrm{dR}}(S)$ $\dim_{\mathbb{R}} H^{i,n}_{\mathrm{dR}}(\mathcal{X})$		$i = 0$	$i = 1$	$i = 2$
$n = 1$	$H^\bullet(F_{\mathbb{C}}, \mathbb{C})^{G_{\mathbb{R}}}$ $H^\bullet(\mathcal{X}(\mathbb{C}), \mathcal{O}_{\mathcal{X}(\mathbb{C})})^{G_{\mathbb{R}}}$	m m	 mg	
$n \geqslant 2$	$H^\bullet(F_{\mathbb{C}}, \mathbb{C})^{G_{\mathbb{R}}}$ $H^\bullet(\mathcal{X}(\mathbb{C}), \mathbb{C})^{G_{\mathbb{R}}}$	m m	 $2mg$	 m

(A.14)

Ranks of compact support cohomology.

$\dim_{\mathbb{R}} H^{i,n}_c(S)$ $\dim_{\mathbb{R}} H^{i,n}_c(\mathcal{X})$	$i = 1$	$i = 2$	$i = 3$	$i = 4$
$n < 0$	$\bar{\epsilon}_n r + s$ $\bar{\epsilon}_n r + s$	 mg	 $\epsilon_n r + s$	
$n = 0$	$r + s - 1$ $r + s - 1$	 mg	 s	
$n = 1$		1 $\mathrm{rk}\,\mathrm{Pic}\,\mathcal{X}$	 $r + s - 1$	
$n = 2$				 1

(A.15)

Ranks of Weil-étale cohomology with compact support.

$\mathrm{rk}\, H^{i,n}_{W,c}(S)$ $\mathrm{rk}\, H^{i,n}_{W,c}(\mathcal{X})$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$n<0$	$\bar{\epsilon}_n r+s$ $\bar{\epsilon}_n r+s$	$\bar{\epsilon}_n r+s$ $\bar{\epsilon}_n r+s$	 $\epsilon_n r+s$	 $\epsilon_n r+s$	
$n=0$	$r+s-1$ $r+s-1$	$r+s-1$ $mg+r+s-1$	 $mg+s$	 s	
$n=1$	$m-1$ $m-1$	 $\mathrm{rk}\,\mathrm{Pic}\,\mathcal{X}+m-1$	1 $\mathrm{rk}\,\mathrm{Pic}\,\mathcal{X}+r+s-1$	 $r+s-1$	
$n=2$	m m	 $2mg$	 $m-1$		 1
$n>2$	m m	 $2mg$	 m		

(A.16)

A.5.2 Motivic and Weil-étale motivic cohomology tables

The entries of the following tables are valid up to 2-torsion.

Motivic cohomology.

$H^{i,n}(S)$ $H^{i,n}(\mathcal{X})$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$n<0$			$T_{\overline{S}}^{2,1-n}$ $T_{\overline{S}}^{2,1-n} \oplus {}^1T_{\overline{\mathcal{X}}}^{4,2-n}$	$\mathbb{Q}/\mathbb{Z}^{r\bar{\epsilon}_n+s} \oplus T_{\overline{S}}^{1,1-n}$ $(\mathbb{Q}/\mathbb{Z})^{\bar{\epsilon}_n r+s} \oplus T_{\overline{S}}^{1,1-n} \oplus {}^1T_{\overline{\mathcal{X}}}^{3,2-n}$	$(\mathbb{Q}/\mathbb{Z})^{mg} \oplus T_{\overline{S}}^{2,2-n} \oplus {}^1T_{\overline{\mathcal{X}}}^{2,2-n}$	$(\mathbb{Q}/\mathbb{Z})^{\epsilon_n r+s} \oplus T_{\overline{S}}^{1,2-n}$	
$n=0$	$\mathbb{Z}$ $\mathbb{Z}$		Cl_F $\mathrm{Cl}_F \oplus {}^1T_{\overline{\mathcal{X}}}^{4,2}$	$(\mathcal{O}_F^\times)^\vee$ $(\mathcal{O}_F^\times)^\vee \oplus {}^1T_{\overline{\mathcal{X}}}^{3,2}$	$(\mathbb{Q}/\mathbb{Z})^{mg} \oplus T_{\overline{S}}^{2,2} \oplus {}^1T_{\overline{\mathcal{X}}}^{2,2}$	$(\mathbb{Q}/\mathbb{Z})^{s} \oplus T_{\overline{S}}^{1,2}$	
$n=1$		$\mathcal{O}_F^\times$ $\mathcal{O}_F^\times$	Cl_F $\mathrm{Pic}\,\mathcal{X}$	$\mathrm{Br}\,\mathcal{X}$	$\mathbb{Q}/\mathbb{Z}$ $(\mathrm{Pic}\,\mathcal{X})^\vee$	$(\mathcal{O}_F^\times)^\vee$	
$n=2$		$\mathbb{Z}^s \oplus T_{\overline{S}}^{1,2}$ $\mathbb{Z}^s \oplus T_{\overline{S}}^{1,2}$	$T_{\overline{S}}^{2,2}$ $\mathbb{Z}^{mg} \oplus T_{\overline{S}}^{2,2} \oplus {}^1T_{\overline{\mathcal{X}}}^{2,2}$	$\mathcal{O}_F^\times \oplus {}^1T_{\overline{\mathcal{X}}}^{3,2}$	$\mathrm{Cl}_F \oplus {}^1T_{\overline{\mathcal{X}}}^{4,2}$		$\mathbb{Q}/\mathbb{Z}$
$n\geqslant 3$		$\mathbb{Z}^{\epsilon_n r+s} \oplus T_{\overline{S}}^{1,n}$ $\mathbb{Z}^{\epsilon_n r+s} \oplus T_{\overline{S}}^{1,n}$	$T_{\overline{S}}^{2,n}$ $\mathbb{Z}^{mg} \oplus T_{\overline{S}}^{2,n} \oplus {}^1T_{\overline{\mathcal{X}}}^{2,n}$	$\mathbb{Z}^{\bar{\epsilon}_n r+s} \oplus T_{\overline{S}}^{1,n-1} \oplus {}^1T_{\overline{\mathcal{X}}}^{3,n}$	$T_{\overline{S}}^{2,n-1} \oplus {}^1T_{\overline{\mathcal{X}}}^{4,n}$		

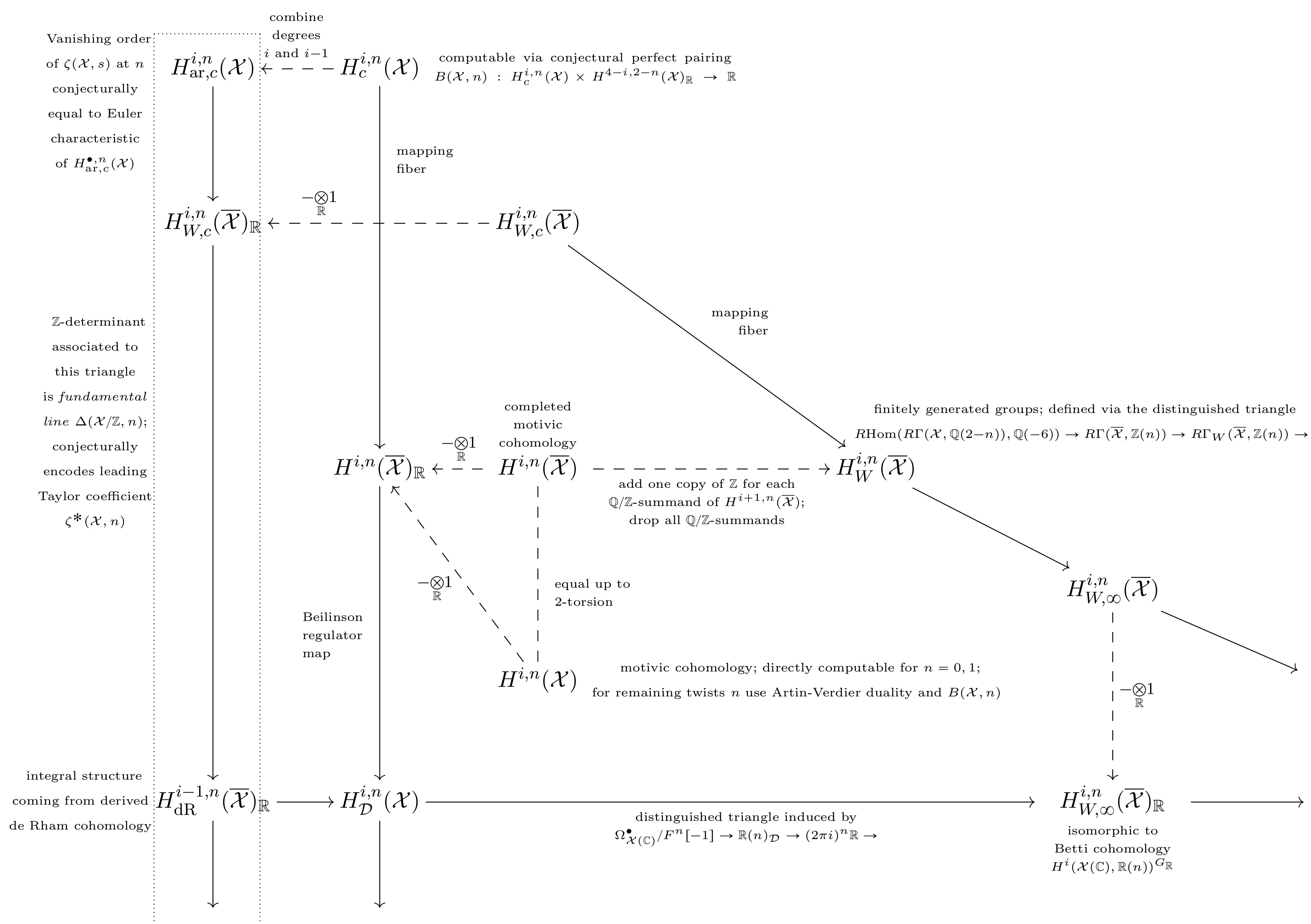
(A.17)

Weil-étale motivic cohomology.

$H_W^{i,n}(S)$ $H_W^{i,n}(\mathcal{X})$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$n<0$			$\mathbb{Z}^{r\bar{\epsilon}_n+s} \oplus T_{\overline{S}}^{2,1-n}$ $\mathbb{Z}^{r\bar{\epsilon}_n+s} \oplus T_{\overline{S}}^{2,1-n} \oplus {}^1T_{\overline{\mathcal{X}}}^{4,2-n}$	$T_{\overline{S}}^{1,1-n}$ $\mathbb{Z}^{mg} \oplus T_{\overline{S}}^{1,1-n} \oplus {}^1T_{\overline{\mathcal{X}}}^{3,2-n}$	$\mathbb{Z}^{\epsilon_n r+s} \oplus T_{\overline{S}}^{2,2-n} \oplus {}^1T_{\overline{\mathcal{X}}}^{2,2-n}$	$T_{\overline{S}}^{1,2-n}$
$n=0$	$\mathbb{Z}$ $\mathbb{Z}$		$\mathrm{Cl}_F \oplus (\mathcal{O}_F^\times)^*$ $\mathrm{Cl}_F \oplus {}^1T_{\overline{\mathcal{X}}}^{4,2} \oplus (\mathcal{O}_F^\times)^*$	μ_F $\mathbb{Z}^{mg} \oplus \mu_F \oplus {}^1T_{\overline{\mathcal{X}}}^{3,2}$	$\mathbb{Z}^s \oplus T_{\overline{S}}^{2,2} \oplus {}^1T_{\overline{\mathcal{X}}}^{2,2}$	$T_{\overline{S}}^{1,2}$
$n=1$		$\mathcal{O}_F^\times$ $\mathcal{O}_F^\times$	Cl_F $\mathrm{Pic}\,\mathcal{X}$	$\mathbb{Z}$ $\mathrm{Br}\,\mathcal{X} \oplus (\mathrm{Pic}\,\mathcal{X})^*$	$\mathrm{Tor}\,\mathrm{Pic}\,\mathcal{X} \oplus (\mathcal{O}_F^\times)^*$	μ_F
$n=2$		$\mathbb{Z}^s \oplus T_{\overline{S}}^{1,2}$ $\mathbb{Z}^s \oplus T_{\overline{S}}^{1,2}$	$T_{\overline{S}}^{2,2}$ $\mathbb{Z}^{mg} \oplus T_{\overline{S}}^{2,2} \oplus {}^1T_{\overline{\mathcal{X}}}^{2,2}$	$\mathcal{O}_F^\times \oplus {}^1T_{\overline{\mathcal{X}}}^{3,2}$	$\mathrm{Cl}_F \oplus {}^1T_{\overline{\mathcal{X}}}^{4,2}$	$\mathbb{Z}$
$n\geqslant 3$		$\mathbb{Z}^{\epsilon_n r+s} \oplus T_{\overline{S}}^{1,n}$ $\mathbb{Z}^{\epsilon_n r+s} \oplus T_{\overline{S}}^{1,n}$	$T_{\overline{S}}^{2,n}$ $\mathbb{Z}^{mg} \oplus T_{\overline{S}}^{2,n} \oplus {}^1T_{\overline{\mathcal{X}}}^{2,n}$	$\mathbb{Z}^{\bar{\epsilon}_n r+s} \oplus T_{\overline{S}}^{1,n-1} \oplus {}^1T_{\overline{\mathcal{X}}}^{3,n}$	$T_{\overline{S}}^{2,n-1} \oplus {}^1T_{\overline{\mathcal{X}}}^{4,n}$	

(A.18)

A.6 Diagram of cohomology groups



A.7 Overview of integral structures occurring in $\Delta(\mathcal{X}/\mathbb{Z}, n)$

Cohomology group	basis of integral structure	comments
$H^1(\mathcal{X}(\mathbb{C}), \mathbb{R})^+$	$\mathcal{B}^+ = \{\delta^+_{vj}\}_{vj}$	
$H^1(\mathcal{X}(\mathbb{C}), \mathbb{R})^-$	$\mathcal{B}^- = \{\delta^-_{vj}\}_{vj}$	Poincaré dual of $\mathcal{B}^+$
$H^1(\mathcal{X}(\mathbb{C}), \mathbb{R}(n))^{G_\mathbb{R}}$ for all n	$\mathcal{B}^{+;n}$	$= (2\pi i)^n \mathcal{B}^\pm$
$H^{1,n}_{W,\infty}(\mathcal{X})_\mathbb{R}$ for all n	$\mathcal{B}^{+;n}$	via $H^{1,n}_{W,\infty}(\mathcal{X})_\mathbb{R} \cong H^1(\mathcal{X}(\mathbb{C}), \mathbb{R}(n))^{G_\mathbb{R}}$
$H^{2,n}_\mathcal{D}(\mathcal{X})$ for $n \geqslant 2$	$\mathcal{B}^{+;n-1}$	via $H^{2,n}_\mathcal{D}(\mathcal{X}) \cong H^1(\mathcal{X}(\mathbb{C}), \mathbb{R}(n-1))^{G_\mathbb{R}}$
$H^{1,n}_\mathcal{D}(\mathcal{X})$ for $n \leqslant 0$	$\mathcal{B}^{+;n}$	via $H^{1,n}_\mathcal{D}(\mathcal{X}) \cong H^1(\mathcal{X}(\mathbb{C}), \mathbb{R}(n))^{G_\mathbb{R}}$
$H^{2,n}(\mathcal{X})_\mathbb{R}$ for $n \geqslant 2$	$\mathcal{C}^n = \{c^n_{vj}\}_{vj}$	
${}^1H^{2,1}(\mathcal{X})_\mathbb{R} = \left(\frac{\mathrm{Pic}^0\mathcal{X}}{\mathrm{Cl}_F}\right)_\mathbb{R}$	$\mathscr{P}$	generates image of $\frac{\mathrm{Pic}^0\mathcal{X}}{\mathrm{Cl}_F}$ inside $(\frac{\mathrm{Pic}^0\mathcal{X}}{\mathrm{Cl}_F})_\mathbb{R}$
	$\mathscr{P}' = \mathcal{P} \cup \bigcup_\mathfrak{p} D_\mathfrak{p}$	$\mathcal{P}$ basis of $(\mathrm{Pic}^0 X)_\mathbb{R}$; $D_\mathfrak{p}$ basis of $(\Lambda_\mathfrak{p})_\mathbb{R}$
${}^1H^{2,n}_W(\mathcal{X})_\mathbb{R}$ for $n \geqslant 2$	$\mathcal{C}^n$	via ${}^1H^{2,n}_W(\mathcal{X})_\mathbb{R} \cong H^{2,n}(\overline{\mathcal{X}})_\mathbb{R}$
${}^1H^{3,n}_W(\mathcal{X})_\mathbb{R}$ for $n \leqslant 0$	$\mathcal{C}^{2-n} = \{c^{2-n}_{vj}\}_{vj}$	via ${}^1H^{3,n}_W(\mathcal{X})_\mathbb{R} \cong \mathrm{Hom}(H^{2,2-n}(\overline{\mathcal{X}}), \mathbb{R})$
$H^1(\mathcal{X}, \mathcal{O}_\mathcal{X})_\mathbb{R}$	$\mathcal{B}^{10}_{\mathrm{dR}} = \{\omega_{vj}\}_{vj}$	
$H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathbb{Z}})_\mathbb{R}$	$\mathcal{B}^{01}_{\mathrm{dR}} = \{\eta_{vj}\}_{vj}$	Serre dual of $\mathcal{B}^{10}_{\mathrm{dR}}$
	$\mathcal{B}^n = \mathcal{B}^{+;n} \cup \mathcal{B}^{+;n-1}$	for any n; via $H^1(\mathcal{X}(\mathbb{C}), \mathbb{C})^{G_\mathbb{R}} = H^1(\mathcal{X}(\mathbb{C}), \mathbb{R}(n))^{G_\mathbb{R}} \oplus H^1(\mathcal{X}(\mathbb{C}), \mathbb{R}(n-1))^{G_\mathbb{R}}$
$H^1(\mathcal{X}(\mathbb{C}), \mathbb{C})^{G_\mathbb{R}} = H^1_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R})$	$\mathcal{B}_{\mathrm{dR}} = \mathcal{B}^{10}_{\mathrm{dR}} \cup \mathcal{B}^{01}_{\mathrm{dR}}$	via $H^1_{\mathrm{dR}}(\mathcal{X}_\infty/\mathbb{R}) \cong H^1(\mathcal{X}, \mathcal{O}_\mathcal{X})_\mathbb{R} \oplus H^0(\mathcal{X}, \omega_{\mathcal{X}/\mathbb{Z}})_\mathbb{R}$
	$\mathcal{B}^n_{\mathrm{ddR}}$	for $n \geqslant 2$; see Section 3.2 for definition
${}^1H^{2,n}_c(\mathcal{X})$ for $n = 1$	$\mathscr{P}$	via ${}^1H^{2,1}_c(\mathcal{X}) \cong {}^1H^{2,1}(\mathcal{X})_\mathbb{R}$
${}^1H^{2,n}_c(\mathcal{X})$ for $n \leqslant 0$	$\mathcal{B}^{+;n}$	via ${}^1H^{2,n}_c(\mathcal{X}) \cong H^{1,n}_\mathcal{D}(\mathcal{X})$

Appendix B

Bibliography

[1] H. Bass and J. Tate. The Milnor ring of a global field. In *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic (Proc. Conf., Seattle, Wash., Battelle Memorial Inst., 1972)*, pages 349–446. Lecture Notes in Math., Vol. 342. Springer, Berlin, 1973.

[2] Spencer Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61(3):267–304, 1986.

[3] Spencer Bloch. de Rham cohomology and conductors of curves. *Duke Math. J.*, 54(2):295–308, 1987.

[4] Spencer Bloch and Kazuya Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[5] Siegfried Bosch and Qing Liu. Rational points of the group of components of a Néron model. *Manuscripta Math.*, 98(3):275–293, 1999.

[6] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[7] Hélène Esnault and Eckart Viehweg. Deligne-Beilinson cohomology. In *Beilinson's conjectures on special values of L-functions*, volume 4 of *Perspect. Math.*, pages 43–91. Academic Press, Boston, MA, 1988.

[8] M. Flach and B. Morin. Weil-étale cohomology and zeta-values of proper regular arithmetic schemes, 2017.

[9] Jean-Marc Fontaine and Bernadette Perrin-Riou. Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 599–706. Amer. Math. Soc., Providence, RI, 1994.

[10] Kazuhiro Fujiwara. A proof of the absolute purity conjecture (after Gabber). In *Algebraic geometry 2000, Azumino (Hotaka)*, volume 36 of *Adv. Stud. Pure Math.*, pages 153–183. Math. Soc. Japan, Tokyo, 2002.

[11] Thomas Geisser. Motivic cohomology over Dedekind rings. *Math. Z.*, 248(4):773–794, 2004.

[12] Thomas Geisser. Duality via cycle complexes. *Ann. of Math. (2)*, 172(2):1095–1126, 2010.

[13] Benedict H. Gross and Joe Harris. Real algebraic curves. *Ann. Sci. École Norm. Sup. (4)*, 14(2):157–182, 1981.

[14] Alexander Grothendieck. Le groupe de Brauer. III. Algèbres d'Azumaya et interprétations diverses. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 88–188. North-Holland, Amsterdam, 1968.

[15] Paul Hriljac. Heights and Arakelov's intersection theory. *Amer. J. Math.*, 107(1):23–38, 1985.

[16] Luc Illusie. *Complexe cotangent et déformations. I.* Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971.

[17] Luc Illusie. *Complexe cotangent et déformations. II.* Lecture Notes in Mathematics, Vol. 283. Springer-Verlag, Berlin-New York, 1972.

[18] Marc Levine. K-theory and motivic cohomology of schemes. *http://www.math.uiuc.edu/K-theory/0336/*, 1991.

[19] Stephen Lichtenbaum. Curves over discrete valuation rings. *Amer. J. Math.*, 90:380–405, 1968.

[20] Stephen Lichtenbaum. The construction of weight-two arithmetic cohomology. *Invent. Math.*, 88(1):183–215, 1987.

[21] Barry Mazur. Notes on étale cohomology of number fields. *Ann. Sci. École Norm. Sup. (4)*, 6:521–552 (1974), 1973.

[22] J. S. Milne. *Arithmetic duality theorems*. BookSurge, LLC, Charleston, SC, second edition, 2006.

[23] J. S. Milne. *Séminaire de géométrie algébrique du Bois Marie 1962–64. Schémas en groupes (SGA 4)*. Tom I, Tom III [book review of mr2867621; mr2867622]. *Eur. Math. Soc. Newsl.*, 3(87):50–52, 2013.

[24] Bernadette Perrin-Riou. Fonctions L p-adiques des représentations p-adiques. *Astérisque*, (229):198, 1995.

[25] Arpon Raksit. The dold-kan correspondence, 2015. Available at http://web.stanford.edu/ arpon/math/files/doldkan.pdf.

[26] M. Raynaud. Spécialisation du foncteur de Picard. *Inst. Hautes Études Sci. Publ. Math.*, (38):27–76, 1970.

[27] Shuji Saito. Arithmetic on two-dimensional local rings. *Invent. Math.*, 85(2):379–414, 1986.

[28] Shuji Saito. Arithmetic theory of arithmetic surfaces. *Ann. of Math. (2)*, 129(3):547–589, 1989.

[29] Kanetomo Sato. p-adic étale Tate twists and arithmetic duality. *Ann. Sci. École Norm. Sup. (4)*, 40(4):519–588, 2007. With an appendix by Kei Hagihara.

[30] Peter Schneider. Introduction to the Beilinson conjectures. In *Beilinson's conjectures on special values of L-functions*, volume 4 of *Perspect. Math.*, pages 1–35. Academic Press, Boston, MA, 1988.

[31] Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[32] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.

[33] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.*, 65(2):121–154, 1988.

[34] Michael Spieß. Artin-Verdier duality for arithmetic surfaces. *Math. Ann.*, 305(4):705–792, 1996.

[35] John Tate. On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. In *Séminaire Bourbaki, Vol. 9*, pages Exp. No. 306, 415–440. Soc. Math. France, Paris, 1995.

[36] Ravi Vakil. Foundations of algebraic geometry, 2018. Available at http://math.stanford.edu/ vakil/216blog/FOAGjun1113public.pdf.

[37] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.

[38] Changlong Zhong. Comparison of dualizing complexes. *J. Reine Angew. Math.*, 695:1–39, 2014.