Norm Constant Holomorphic Functions on Banach Spaces

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ABSTRACT

Holomorphic functions (or maps) have been defined between Banach spaces by the use of a Taylor expansion involving Fréchet derivatives. When the Banach spaces in question coincide with $\mathfrak{L}(\mathfrak{H})$, the space of linear operators over a Hilbert space $\mathfrak{H}$, the set of holomorphic functions includes those arising from the Dunford functional calculus, but is certainly not limited to these. The holomorphic functions between Banach spaces share many of the properties of ordinary holomorphic functions from the complex plane $\mathbb{C}$ into itself. However, in many aspects they behave differently. For example, the maximum modulus theorem implies that an ordinary holomorphic function with constant modulus must be a constant function. This is no longer true even for holomorphic functions of one complex variable taking values in a Banach space. In fact, the Thorp-Whitley Theorem states that if D is a domain in $\mathbb{C}$, $\mathfrak{Y}$ a Banach space, and $F : D \to \mathfrak{Y}$ holomorphic with $\|F(\zeta)\| = 1$ on D, then F is a constant function if its range contains a complex extreme point of the unit ball of $\mathfrak{Y}$.

It is natural to ask which holomorphic functions between Banach spaces have constant norm. For the case where $F : D \subset \mathbb{C} \to \mathfrak{Y}$, the problem was solved by Globevnik, who also specialized the result to the case $F : D \subset \mathbb{C} \to \mathfrak{L}(\mathfrak{H})$. In addition, he determined under which conditions F might have constant norm in some norm equivalent to the given norm. This thesis solves the problem in the full case where F is now a holomorphic function between two Banach spaces. The following theorem analogous to Globevnik's is proved:

<u>Theorem</u>. Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces, $\mathfrak{D}$ a domain in $\mathfrak{X}$ and $F : \mathfrak{D} \to \mathfrak{Y}$

holomorphic. Then $\|F(x)\|$ will be constant for all $x \in \mathcal{D}$ if and only if

(i) The subspace $\mathcal{E}(F(x))$ is independent of $x \in \mathcal{D}$,

i.e., $\mathcal{E}(F(x)) = \mathcal{E}$ for all $x \in \mathcal{D}$,

(ii) $F(x) - F(y) \in \mathcal{E}$ for all $x, y \in \mathcal{D}$,

where for $u \in \mathcal{Y}$ the set $\mathcal{E}(u)$ is defined to be

$\mathcal{E}(u) = \{v \in \mathcal{Y} | \exists r > 0 \text{ such that } \|u + \zeta v\| \leq \|u\| \text{ for all } \zeta \in \mathbb{C}, |\zeta| \leq r\}$.

An immediate consequence is that the Thorp-Whitley Theorem also holds in this generality, that is, when F is a function between arbitrary Banach spaces.

When this theorem is applied to the case $\mathcal{X} = \mathcal{Y} = \mathcal{L}(\mathcal{H})$ a simplified criterion is obtained. The norm constant functions are precisely those annihilated by certain projection operators on $\mathcal{H}$. As a corollary to this it is shown that the only holomorphic functions arising from the Dunford calculus with constant norm are the constant functions. In contrast to the above theorem, it is also shown that any holomorphic function $F : \mathcal{D} \subset \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ with $\mathrm{Re}(F) = 0$ on $\mathcal{D}$ must be a constant function. A theorem analogous to Globevnik's for deciding when a function $F : \mathcal{D} \subset \mathcal{X} \to \mathcal{Y}$ can be norm constant under some equivalent norm is also obtained.

NOTATION

Throughout this thesis, the following notation is used:

$\mathbb{C}$	is the field of complex numbers.
$\mathfrak{X}$, $\mathfrak{Y}$	are Banach spaces.
$\mathfrak{H}$	is a Hilbert space with inner product $\langle \cdot , \cdot \rangle$.
$\mathfrak{L}(\mathfrak{H})$	is the space of bounded linear operators on $\mathfrak{H}$.
$M_n(\mathbb{C})$	is the space of $n \times n$ matrices with complex entries.
$\mathfrak{B}(x,r)$	is the open ball centered at x of radius r.
Conv(S)	is the (not necessarily closed) convex hull of the set S.
span(S)	is the linear span of the set S.
$\overline{\text{span}}(S)$	is the closed linear span of the set S.
$\sigma(T)$	is the spectrum of $T \in \mathfrak{L}(\mathfrak{H})$.
Re(T)	is the real part of $T \in \mathfrak{L}(\mathfrak{H})$, i.e., $\text{Re}(T) = \frac{1}{2}(T + T^*)$.
Im(T)	is the imaginary part of $T \in \mathfrak{L}(\mathfrak{H})$, i.e., $\text{Im}(T) = \frac{1}{2i}(T - T^*)$.
$\text{Hol}(\mathfrak{D},\mathfrak{Y})$	see the bottom of page 5.
$\mathcal{E}(z)$	see the bottom of page 12.

CONTENTS

INTRODUCTION

Banach space valued holomorphic functions of a complex variable having constant norm were first characterized by Globevnik [5] in 1975. The present work extends the study of norm constant functions to holomorphic functions from one Banach space into another. Specifically, it is shown that Globevnik's original characterization remains valid in this more general setting. A consequence of this is that a norm constant holomorphic function from one Banach space into another cannot have a complex extreme point of any ball in its range unless the function itself is constant. This generalization of the Thorp-Whitley Theorem does not appear to follow from any extension of the usual proofs. Throughout this work the main tool is the set $\mathcal{E}(z)$ which measures "how close" a point z is to being a complex extreme point.

The characterization of norm constant functions is specialized to the case where both Banach spaces are $\mathcal{L}(\mathcal{H})$, the space of linear operators over a Hilbert space $\mathcal{H}$. Here the norm constant functions are identified as those annihilated by certain projection operators. This characterization has a flavor similar to Globevnik's results for operator valued functions of a complex variable, but is essentially different since the coefficients of the power series expansion of operator valued holomorphic functions are no longer elements of $\mathcal{L}(\mathcal{H})$ in the more general setting. A consequence of the above result is that the only norm constant functions arising from the Dunford functional calculus are the constant functions. The problem of determining which holomorphic functions of operators can have constant real part is also solved. The latter result is somewhat surprising when compared to the characterization of the norm constant functions.

The organization of this thesis is as follows. The necessary background material on holomorphic functions on Banach spaces is discussed in Chapter 1. Chapter 2 summarizes the previous work concerning norm constant holomorphic functions having domains in the complex plane. It also introduces the notion of a complex extreme point and the set $\mathcal{E}(z)$. The main results of the present work are contained in Chapter 3, where the norm constant holomorphic functions on Banach spaces are characterized. The latter half of the chapter discusses which holomorphic functions can be norm constant under some norm equivalent to the original. In Chapter 4 the above results are specialized to holomorphic functions of operators. The conclusion discusses some open problems.

Chapter 1. Holomorphic Functions on Banach Spaces

This chapter presents the aspects of the theory of holomorphic functions on Banach spaces relevant to this thesis. A detailed exposition of this theory will be found in Dineen [2], Nachbin [10], and Franzoni-Vesentini [3]. The basic idea is to use the multilinear maps between Banach spaces to define polynomials; these in turn are used to define power series and then holomorphic functions are defined in terms of convergent power series.

Throughout this chapter, let $\mathfrak{X}$ be a Banach space; $\mathfrak{X}^p$ will denote the product space $\mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \times \cdots \times \mathfrak{X}$ (p-times) with the norm $\|(x_1,x_2,\ldots,x_p)\| = \max_{1 \le n \le p} \|x_n\|$. Let $\mathfrak{Y}$ also be a Banach space and let $\mathfrak{L}^p(\mathfrak{X},\mathfrak{Y})$ denote the space of all continuous p-linear maps from $\mathfrak{X}^p$ into $\mathfrak{Y}$. Endowed with the norm

$$\|A\| = \sup\{\|A(x_1,\ldots,x_p)\| \,\big|\, \|(x_1,\ldots,x_p)\| \le 1\}$$

the space $\mathfrak{L}^p(\mathfrak{X},\mathfrak{Y})$ is itself a Banach space. When $p = 1$ it is just the space of linear maps from $\mathfrak{X}$ to $\mathfrak{Y}$; set $\mathfrak{L}^0(\mathfrak{X},\mathfrak{Y}) = \mathfrak{Y}$.

Let $\hat{A} : \mathfrak{X} \to \mathfrak{Y}$ denote the map obtained by restricting $A \in \mathfrak{L}^p(\mathfrak{X},\mathfrak{Y})$ to the diagonal, i.e.

$$\hat{A}(x) = A(x,x,\ldots,x).$$

Definition. A map $P : \mathfrak{X} \to \mathfrak{Y}$ is a *p-homogeneous polynomial* if there is an $A \in \mathfrak{L}^p(\mathfrak{X},\mathfrak{Y})$ so that $P = \hat{A}$.

$\mathfrak{P}^p(\mathfrak{X},\mathfrak{Y})$ will denote the space of all p-homogeneous polynomials from $\mathfrak{X}$ to $\mathfrak{Y}$.

Definition. $P : \mathfrak{X} \to \mathfrak{Y}$ is a *polynomial* if there are $P_r \in \mathfrak{P}^r(\mathfrak{X},\mathfrak{Y})$ for $0 \le r \le d$ so that

$$P(x) = \sum_{r=0}^{d} P_r(x) \quad \text{for all} \quad x \in \mathfrak{X}.$$

This decomposition is unique. The (unique) largest integer d so that $P_d \neq 0$ is called the degree of P. The space of all polynomials from $\mathfrak{X}$ into $\mathfrak{Y}$ will be denoted by $\mathcal{P}(\mathfrak{X},\mathfrak{Y})$. For $P \in \mathcal{P}(\mathfrak{X},\mathfrak{Y})$ define

$$\|P\| = \sup\{\|P(x)\| \,\big|\, x \in \mathfrak{X}, \|x\| \leq 1\}$$

and observe that if $P \in \mathcal{P}^r(\mathfrak{X},\mathfrak{Y})$ then $\|P(x)\| \leq \|P\| \cdot \|x\|^r$.

These polynomials are now used to define power series in a manner analogous to the usual construction of a power series.

<u>Definition</u>. A *power series* F from $\mathfrak{X}$ to $\mathfrak{Y}$ is a formal expansion $F = \sum_{k=0}^{\infty} P_k$ where $P_k \in \mathcal{P}^k(\mathfrak{X},\mathfrak{Y})$.

<u>Definition</u>. The *radius of convergence* of a power series $\sum_{k=0}^{\infty} P_k$ is

$$R = \sup\{r \in \mathbb{R}^+ \,\Big|\, \sum_{k=0}^{\infty} P_k(x) \text{ converges } \textit{uniformly} \text{ on } \overline{\mathcal{B}(0,r)} \subset \mathfrak{X}\}.$$

For any $x \in \mathcal{B}(0,R)$, the power series $F = \sum_{k=0}^{\infty} P_k$ defines a function from $\mathfrak{X}$ into $\mathfrak{Y}$ given by $F(x) = \sum_{k=0}^{\infty} P_k(x)$. No confusion should arise from the fact that a power series and the function defined by it where it converges will not be distinguished in this thesis.

<u>Proposition 1.1</u> (Cauchy-Hadamard). The radius of convergence R of $\sum_{k=0}^{\infty} P_k$ is given by

$$R^{-1} = \limsup_{k\to\infty} \|P_k\|^{1/k}.$$

The completeness of $\mathfrak{Y}$ is essential in Proposition 1.1. Consider otherwise the following example. Let $\mathfrak{Y}$ be the space of all sequences $x = \{\xi_i\}_{i=1}^{\infty}$, $\xi_i \in \mathbb{C}$ having only finitely many non-zero terms, endowed with the sup norm. Consider the power series $F : \mathbb{C} \to \mathcal{Y}$, $F = \sum_{k=0}^{\infty} P_k$ given by $P_k(\zeta) = (0,\ldots,0,\underset{\text{kth place}}{\zeta^k},0,\ldots)$. Clearly $\|P_k\| = 1$ so that $R = 1$, but F fails to converge except at $\zeta = 0$.

These generalized power series have the uniqueness or analytic continuation property:

<u>Proposition 1.2</u>. If there is an $r > 0$ so that the power series $F(x) = \sum_{k=0}^{\infty} P_k(x)$ vanishes on $\mathcal{B}(0,r)$, then $P_k = 0$ for all k.

Classically, the coefficients of a power series could be obtained by repeated differentiations. This remains true in this setting. Let $d^nF(x_0)$ denote the nth Fréchet derivative of F; then $d^nF(x_0) \in \mathcal{L}^n(\mathcal{X},\mathcal{Y})$.

<u>Proposition 1.3</u>. Let $F = \sum_{k=0}^{\infty} P_k$ be a power series from $\mathcal{X}$ into $\mathcal{Y}$ with radius of convergence $R > 0$. Then

$$F : \mathcal{B}(0,R) \to \mathcal{Y} \quad \text{is infinitely Fréchet differentiable}$$

and

$$\widehat{d^nF(0)} = n!P_n.$$

Holomorphic functions are now defined in terms of convergent power series.

<u>Definition</u>. Let $\mathcal{D}$ be an open connected set in $\mathcal{X}$. A function $F : D \to \mathcal{Y}$ is *holomorphic* if for each $x_0 \in \mathcal{D}$ there is a power series $\sum_{k=0}^{\infty} P_k$ with positive radius of convergence $R(x_0)$ so that

$$F(x) = \sum_{k=0}^{\infty} P_k(x - x_0) \quad \text{for all } x \in \mathcal{B}(0,R(x_0)) \cap \mathcal{D}.$$

This power series expansion is unique by Proposition 1.2. Open connected subsets of a Banach space will be called *domains* and the space of holomorphic functions from a domain $\mathcal{D} \subset \mathcal{X}$ into $\mathcal{Y}$ will be denoted by $\mathrm{Hol}(\mathcal{D},\mathcal{Y})$. This definition of holomorphic is equivalent to, but not the same as, Hille's definition (see[8, p. 81]).

A polynomial is always holomorphic on all of $\mathcal{X}$. A power series with radius of convergence $R > 0$ defines a holomorphic function on the ball $\mathcal{B}(0,R)$. This latter fact is not entirely trivial and a proof may be found in Hille [8, p. 86].

When $\mathcal{X} = \mathcal{Y} = \mathcal{L}(\mathcal{H})$ the space of operators on a Hilbert space $\mathcal{H}$, a particularly interesting class of holomorphic functions arises from the Dunford functional calculus. Recall that if D is a domain in $\mathbb{C}$ with ∂D consisting of a finite number of closed, disjoint, rectifiable Jordan curves, then the set

$$\mathcal{D} = \{T \in \mathcal{L}(\mathcal{H}) \,\big|\, \sigma(T) \subset D\}$$

is an open set in $\mathcal{L}(\mathcal{H})$ by the upper semicontinuity of the spectrum. Let $f(\zeta)$ be a complex valued holomorphic function defined on a neighborhood of $\overline{D}$. The Dunford calculus defines

$$F(T) = \frac{1}{2\pi i}\int_{\partial D} f(\zeta)(\zeta I - T)^{-1} d\zeta \qquad \text{for } T \in \mathcal{D}.$$

To see that $F \in \mathrm{Hol}(\mathcal{D},\mathcal{L}(\mathcal{H}))$ simply expand $(\zeta I - (T + T_0))^{-1}$ in a power series and substitute into the definition of $F(T + T_0)$ to obtain that

$$F(T + T_0) = \sum_{k=0}^{\infty} P_k(T)$$

where $P_k(T) = \frac{1}{2\pi i}\int_{\partial D} f(\zeta)(\zeta I - T_0)^{-1}(T(\zeta I - T_0)^{-1})^k d\zeta$. Since $\|P_k\| \le C\|(\zeta I - T_0)^{-1}\|^{k+1}$ it follows from Proposition 1.1 that the power series has a positive radius of convergence, so $F \in \mathrm{Hol}(\mathcal{D},\mathcal{L}(\mathcal{H}))$. This class of holomorphic functions will be studied further in Chapter 4.

In view of Proposition 1.3, $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ can be expanded in the following manner about each point $x_0 \in \mathcal{D}$:

$$F(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^nF(x_0)}(x-x_0).$$

From this observation it is possible to obtain the following analogs of classical results:

Proposition 1.4 (Cauchy Formulas). Let $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$, then for any x_0, $y \in \mathcal{X}$ and $r > 0$ so that $\overline{\mathcal{B}(x_0,r)} \subset \mathcal{D}$, we have that

$$\widehat{d^nF(x_0)}(y) = n!\,\frac{\|y\|^n}{2\pi i} \int_{|\zeta|=r} \frac{1}{\zeta^{n+1}} F\left(x_0 + \frac{\zeta y}{\|y\|}\right) d\zeta.$$

Corollary 1.5 (Cauchy Inequalities). For any $x_0 \in \mathcal{D}$ and r so that $\overline{\mathcal{B}(x_0,r)} \subset \mathcal{D}$ we have that

$$\|\widehat{d^nF(x_0)}\| \leq \frac{n!}{r^n} \sup\{\|F(x)\| \,\big|\, x \in \partial\mathcal{B}(x_0,r)\}.$$

The case $n = 0$ of this corollary implies:

Proposition 1.6 (Maximum Principle). If $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ the function $H(x) = \|F(x)\|$ cannot have a maximum at $x_0 \in \mathcal{D}$ unless $H(x)$ is constant on a neighborhood of x.

Or in a more useful form:

Proposition 1.7. If $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ is bounded by M on $\partial\mathcal{B}(x_0,r) \subset \mathcal{D}$ and $\|F(x_1)\| = M$ where $x_1 \in \mathcal{B}(x_0,r)$, then $\|F(x)\|$ is constant on $\overline{\mathcal{B}(x_0,r)}$.

Now if F were complex valued, both of these Propositions could be strengthened to conclude that $F(x)$ itself is constant. This is no longer true when F takes values in an arbitrary Banach space. The reason for this is that there are nonconstant holomorphic functions that have constant norm. For example, take $\mathfrak{X} = \mathfrak{Y} = \mathfrak{L}(\mathfrak{H})$ where the Hilbert space $\mathfrak{H}$ has the orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let P_1 and P_2 be the orthogonal projections on e_1 and e_2 respectively. Then $F(T) = P_1 + P_2TP_2$ has $\|F(T)\| = 1$ for all $\|T\| \leq 1$, but $F(T)$ is not constant. There are many other examples; the characterization of holomorphic functions having constant norm is the central problem of this thesis.

Finally, it is necessary to discuss a phenomenon which occurs when the domain of a holomorphic function lies in an infinite dimensional Banach space. In this case it is possible for a function $F \in \mathrm{Hol}(\mathfrak{X},\mathfrak{Y})$ (i.e., F is "entire") to be unbounded on balls of finite radius. This is due to the noncompactness of the closed unit ball in an infinite dimensional Banach space. Loosely speaking, what happens is that there is a sequence in the unit ball which has no accumulation points. The holomorphic function can then be unbounded on this sequence without "destroying its holomorphicity." Consider the following example: Let c_0 denote the Banach space of sequences of complex numbers $x = (\xi_1,\xi_2,\ldots)$ converging to zero with the sup norm. Let $F: c_0 \to \mathbb{C}$ be defined by

$$F(x) = 1 + \xi_1 + \xi_1\xi_2 + \cdots = 1 + \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k} \xi_j\right).$$

This converges for all $x \in c_0$; to see that $F \in \mathrm{Hol}(c_0,\mathbb{C})$ let $y = (\eta_1,\eta_2,\ldots) \in c_0$ and let $x^n = (\xi_n,\xi_{n+1},\xi_{n+2},\ldots)$, then

$$F(x+y) = \sum_{k=0}^{\infty} P_k(y)$$

where
$$P_k(y) = F(x^k)\, \eta_1\eta_2 \cdots \eta_k.$$

It follows that the power series has positive radius of convergence and hence $F \in \mathrm{Hol}(c_0, \mathbb{C})$ since $x \in c_0$ was arbitrary. However note that if

$$x_k = (\overbrace{1,1,\ldots,1}^{k},0,\ldots)$$

then $\|x_k\| = 1$ but $F(x_k) = k + 1$ so F is unbounded on $\{x_k\}_{k=1}^{\infty}$.

Definition. Let $F \in \mathrm{Hol}(\mathcal{D}, \mathcal{Y})$ and $x_0 \in \mathcal{D}$, then r_b, the *radius of boundedness* of F at x_0 is

$$r_b = \sup\{r \mid \mathcal{B}(x_0, r) \subset \mathcal{D} \text{ and } F \text{ is bounded on } \overline{\mathcal{B}(x_0, s)}\ \forall s \le r\}.$$

The above example also illustrates another complication: Even though F is entire, its local representation may require several power series. In this case, the radius of convergence of the power series expansion at zero was one. These two phenomena are related by the following proposition which concludes this brief introduction to the theory of holomorphic functions on Banach spaces.

Proposition 1.8. Let $r_b(x)$ be the radius of boundedness of $F \in \mathrm{Hol}(\mathcal{D}, \mathcal{Y})$ at $x \in \mathcal{D}$, let $R(x)$ be the radius of convergence of the power series expansion of F at x, and let $d(x)$ be the distance from x to $\partial\mathcal{D}$. Then

$$r_b(x) = \min(R(x), d(x)) \quad \text{for all } x \in \mathcal{D}.$$

Chapter 2. Norm Constant Holomorphic Functions of a Complex Variable.

In this chapter previous work on the characterization of holomorphic functions with constant norm is summarized. In particular the theorem of Thorp and Whitley [11] and some of the results of Globevnik [5,6] are presented. These deal exclusively with holomorphic functions having domains in the complex plane; however they provide models for more general theorems and introduce useful tools. One of these tools in particular will play an important role in this thesis and the latter part of this chapter is dedicated to it.

The starting point for the characterization of holomorphic functions with constant norm is a theorem of E. Thorp and R. Whitley. Central to their theorem and to this thesis is the notion of a complex extreme point.

Definition. Let K be a convex subset of Banach space $\mathfrak{X}$. A point $x \in K$ is

(i) a *real extreme point* if $\{x+ty \mid -1 \leq t \leq 1\} \subset K$ implies $y=0$.

(ii) a *complex extreme point* if $\{x+\zeta y \mid 0 \leq |\zeta| \leq 1\} \subset K$ implies $y = 0$.

Geometrically, x is a real (resp. complex) extreme point of K if every real (resp. complex) "disk" centered at x sticks out of K. Every real extreme point is a complex extreme point. To see that the converse is not true consider $L_1(0,1)$ with Lebesgue measure. Every point with norm one is a complex extreme point of the closed unit ball; however, the unit ball has no real extreme points (see [11]).

Complex extreme points play a major role in the characterization of norm constant holomorphic functions, as is already suggested by the following observation. If x is not a complex extreme point of the closed unit ball of $\mathfrak{X}$ and $\|x\| = 1$, then it is possible to construct a nonconstant holomorphic function f from the unit disk in $\mathbb{C}$ into the unit ball in $\mathfrak{X}$ so that $f(0) = x$ and $\|f(\zeta)\| = 1$ for $|\zeta| < 1$. To do this, choose $y \in \mathfrak{X}$ so that $\|x+\zeta y\| \leq 1$ for $|\zeta| \leq 1$, and set $f(\zeta) = x + \zeta y$. If $\|f(\zeta_0)\| < 1$ for some $|\zeta_0| < 1$, then

$$\|x\| \leq \tfrac{1}{2}\|f(\zeta_0)\| + \tfrac{1}{2}\|x - \zeta_0 y\| < 1.$$

But $\|x\| = 1$ and so $\|f(\zeta)\| = 1$ for all $|\zeta| < 1$.

The Thorp-Whitley Theorem provides a "converse" to this example.

Theorem 2.1 (Thorp-Whitley [11]). Let $\mathfrak{Y}$ be a Banach space, D a domain in $\mathbb{C}$, and $f \in \mathrm{Hol}(D,\mathfrak{Y})$ with $\|f(\zeta)\| = 1$ for all $\zeta \in D$. If Range(f) contains a complex extreme point of the closed unit ball of $\mathfrak{Y}$, then $f(\zeta)$ is constant on D.

This theorem in particular implies that the strong maximum principle (i.e. $\|F(\zeta)\|$ has an interior maximum implies $F(\zeta)$ constant) holds only for those Banach spaces $\mathfrak{Y}$ for which every vector of norm one is a complex extreme point of the closed unit ball, as is obviously the case when $\mathfrak{Y} = \mathbb{C}$. A simple proof of Theorem 2.1 was given by Harris [7] using the following lemma, which will also find use in this thesis.

Lemma 2.2 (Harris [7]). Let ω be a complex valued function holomorphic on the open unit disk in $\mathbb{C}$ and satisfying $|\omega(\zeta)| \leq 1$ for $|\zeta| < 1$. Then

$$|\omega(0)| + \frac{1-|\zeta|}{2\,|\zeta|}\,|\omega(\zeta) - \omega(0)| \leq 1 \qquad \text{for } 0 < |\zeta| < 1.$$

A proof of Theorem 2.1 now follows quite simply by assuming that $f(0)$

is a complex extreme point and setting $\omega(\zeta) = \varphi(f(\zeta))$ where φ is any linear functional on $\mathcal{Y}$ of norm one. Then

$$\|f(0) + \lambda(f(\zeta) - f(0))\| \leq 1 \qquad \text{for } |\lambda| < \frac{1-|\zeta|}{2\,|\zeta|}, \quad 0 < |\zeta| < 1$$

since this inequality holds for all φ by Lemma 2.2. Hence $f(\zeta) = f(0)$ in a small ball, and so by Proposition 1.2 the equality holds for all $\zeta \in D$.

While little can be said of complex extreme points in an arbitrary algebra, Akemann and Russo [1] have shown that in a C^*-algebra $\mathcal{A}$ with unit the set of complex extreme points of the closed unit ball coincides with the set of real extreme points. The latter were shown by Kadison [9] to be precisely the set

$$\{a \in \mathcal{A} \mid (1 - aa^*)\,x(1 - a^*a) = 0 \text{ for all } x \in \mathcal{A}\}.$$

In particular, for the C^*-algebra $\mathcal{L}(\mathcal{H})$ of operators on a Hilbert space $\mathcal{H}$, the real (and therefore complex) extreme points of the closed unit ball are the partial isometries and their adjoints (i.e. $T \in \mathcal{L}(\mathcal{H})$ so that $T^*T = I$ or $TT^* = I$). If $\mathcal{H}$ is finite dimensional the extreme points are simply the unitary operators. The end result of this is that there will always be many nonconstant holomorphic functions with constant norm from any domain in $\mathbb{C}$ into any unitial C^*-algebra $\mathcal{A}$.

The problem of characterizing the holomorphic functions with constant norm from a domain in $\mathbb{C}$ into a Banach space $\mathcal{Y}$ was solved by Globevnik [5]. His primary tool was the following:

Definition. Let $\mathcal{Y}$ be a Banach space, and $z \in \mathcal{Y}$. The set $\mathcal{E}(z) \subset \mathcal{Y}$ is defined to be

$$\mathcal{E}(z) = \{y \in \mathcal{Y} \mid \exists r > 0 \text{ so that } \|z + \zeta y\| \leq \|z\| \text{ for } |\zeta| \leq r\}.$$

The set $\mathcal{E}(z)$ consists precisely of those $y \in \mathcal{Y}$ which "prevent" z from being a complex extreme point of the closed ball centered at zero of radius $\|z\|$. In particular, $\mathcal{E}(z) = \{0\}$ if and only if z is a complex extreme point of said ball. The set $\mathcal{E}(z)$ has many properties, foremost among them is that $\mathcal{E}(z)$ is a subspace of $\mathcal{Y}$. The properties of $\mathcal{E}(z)$ will be examined in detail in the latter half of this chapter.

Theorem 2.3 (Globevnik [5]). Let $\mathcal{Y}$ be a Banach space, D a domain in $\mathbb{C}$, and $f \in \mathrm{Hol}(D,\mathcal{Y})$. Then $\|f(\zeta)\|$ is constant on D if and only if

(i) the subspace $\mathcal{E}(f(\zeta))$ does not depend on $\zeta \in D$, i.e. $\mathcal{E}(f(\zeta)) = \mathcal{E}$ for all $\zeta \in D$.

(ii) $f(\xi) - f(\zeta) \in \mathcal{E}$ for all ξ, $\zeta \in D$.

This theorem has the following version of Theorem 2.1 as a corollary; note that the hypothesis has been slightly weakened.

Corollary 2.4 (Globevnik [5]). Let $\mathcal{Y}$, D and f be as above and suppose that $\|f(\zeta)\| = 1$ for all $\zeta \in D$. If $\mathrm{Conv}(f(D))$ contains a complex extreme point of the unit ball of $\mathcal{Y}$, then $f(\zeta)$ is constant on D.

Finally, Globevnik's specialization of Theorem 2.3 to the case where $\mathcal{Y} = \mathcal{L}(\mathcal{H})$ the space of operators on a Hilbert space $\mathcal{H}$ will be stated.

Theorem 2.5 (Globevnik [6]). Let $\mathcal{H}$ be a Hilbert space, D a domain in $\mathbb{C}$ containing zero, and $f \in \mathrm{Hol}(D,\mathcal{L}(\mathcal{H}))$ given by $f(\zeta) = \sum_{n=0}^{\infty} A_n \zeta^n$ $(\zeta \in D)$ with $\|A_0\| = 1$.

(i) If $\|f(\zeta)\| = 1$ on some neighborhood of zero then $A_nQ = A_n^*P = 0$ for $n = 1, 2, 3, \ldots$ where P and Q are the spectral projections at 1 for $A_0A_0^*$ and $A_0^*A_0$ respectively.

(ii) If there is a $\delta > 0$ so that $A_n(I - G_{1-\delta}) = A_n^*(I - E_{1-\delta}) = 0$ for $n = 1, 2, 3, \ldots$ where $\{E_\lambda\}$ and $\{G_\lambda\}$ are the spectral measures of $A_0A_0^*$ and $A_0^*A_0$ respectively and I is the identity, then there is a neighborhood of zero on which $\|F(\zeta)\|$ will be constant.

In particular, if 1 is an isolated point of $\sigma(A_0A_0^*)$ then condition (i) is both necessary and sufficient.

In the above theorems, the domain of the holomorphic function has always been an open connected subset of $\mathbb{C}$. In the following chapters, the characterization of functions with constant norm will be extended to holomorphic functions whose domains are open connected subsets of arbitrary Banach spaces. One of the principal tools will be the set $\mathcal{E}(z)$. Some properties of it will now be derived for later use. All of these are due to Globevnik [5]. The proof of the first proposition has been included; the proofs of the others have a similar flavor. In what follows, let $\mathfrak{Y}$ be a Banach space, $\mathfrak{B}$ the unit ball of $\mathfrak{Y}$ and $\mathfrak{B}'$ the unit ball of $\mathfrak{Y}'$, the dual of $\mathfrak{Y}$.

<u>Proposition 2.6</u>. Let $z \in \mathfrak{Y}$, then $y \in \mathcal{E}(z)$ if and only if a constant M exists so that

$$|\varphi(y)| \leq M(\|z\| - |\varphi(z)|) \qquad \text{for all } \varphi \in \mathfrak{B}'.$$

Proof. If $|\varphi(y)| \leq M(\|z\| - |\varphi(z)|)\ \forall\varphi \in \mathcal{B}'$,

then $|\varphi(\zeta y)| \leq \|z\| - |\varphi(z)|\ \forall\varphi \in \mathcal{B}'$, $|\zeta| \leq \frac{1}{M}$

so that $\|z+\zeta y\| \leq \|z\|$ for $|\zeta| \leq \frac{1}{M}$.

Conversely, if $r > 0$ exists so that $\|z+\zeta y\| \leq \|z\|$ for $|\zeta| \leq r$

then $|\varphi(z) + \zeta\varphi(y)| \leq \|z\|\ \forall|\zeta| \leq r$, $\varphi \in \mathcal{B}'$

so that $|\varphi(y)| \leq \frac{1}{r}(\|z\| - |\varphi(z)|)\ \forall\varphi \in \mathcal{B}'$.

Definition. Let $z \in \mathcal{Y}$ and $y \in \mathcal{E}(z)$. Define

$$\|y\|_z = \inf\{M \mid |\varphi(y)| \leq M(\|z\| - |\varphi(z)|), \forall\varphi \in \mathcal{B}'\}.$$

Proposition 2.7. Let $z \in \mathcal{Y}$, then $\mathcal{E}(z)$ is a (not necessarily closed) subspace of $\mathcal{Y}$ and $\|\cdot\|_z$ is a norm on $\mathcal{E}(z)$.

Proposition 2.8. Let $z \in \mathcal{Y}$, then $\|z + x\| \geq \|z\|\ \forall x \in \overline{\mathcal{E}(z)}$.

The last two lemmas are "stability" results for $\mathcal{E}(z)$ and are particu-arly useful.

Lemma 2.9. Let $z \in \mathcal{Y}$, $y \in \mathcal{E}(z)$ with $\|y\|_z \leq \frac{1}{2}$, then $\mathcal{E}(z+y) = \mathcal{E}(z)$.

Lemma 2.10. Let $S \subset \mathcal{Y}$ be such that every element of Conv(S) has norm one and $\mathcal{E}(z) = \mathcal{E}\ \ \forall z \in S$, then $\mathcal{E}(z) = \mathcal{E}$ for all $z \in \mathrm{Conv}(S)$.

We conclude this chapter with some examples of $\mathcal{E}(z)$.

Example 2.1. $\mathcal{Y} = M_2(\mathbb{C})$ and $z = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$, $(|\xi| < 1)$. Direct computation shows that $\mathcal{E}(z) = \{\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{C}\}$.

Example 2.2. $\mathcal{Y} = M_3(\mathbb{C})$ with the usual norm and

$$z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix} \quad \text{where} \quad \left\| \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\| \leq 1.$$

Then $$\mathcal{E}(z) = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & e & f \end{pmatrix} \middle| \; a, b, c, d, e, f \in \mathbb{C} \right\}$$

Example 2.3. Let $\mathcal{Y} = \mathcal{H}$ a Hilbert space and let $v \in \mathcal{H}$, then $\mathcal{E}(v) = \{0\}$. This is evident from the fact that

$$\|v + \zeta w\|^2 = \|v\|^2 + \|\zeta w\|^2 + 2\mathrm{Re}(\langle v, \zeta w\rangle).$$

Example 2.4. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two Hilbert spaces and set $\mathcal{Y} = \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $z = I \oplus T$ where I is the identity on $\mathcal{H}_1$ and $T \in \mathcal{L}(\mathcal{H}_2)$, $\|T\| < 1$.

Then $\mathcal{E}(z) = \{S \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2) \mid \mathcal{H}_1 \subset \ker S\}$. This follows from considering $\|(I \oplus T + \zeta S)v\|$ where $v \in \mathcal{H}_1$.

Chapter 3. Norm Constant Holomorphic Functions on Banach Spaces.

In this chapter several theorems are proved which characterize those holomorphic functions having constant norm. The first few of these theorems extend the results of Thorp-Whitley and Globevnik to holomorphic functions whose domains lie in some Banach space. The latter part of the chapter characterizes those functions which may have constant norm under some norm equivalent to the original norm.

The first and most important of these results extends Theorem 2.3 to holomorphic functions whose domain need not lie in $\mathbb{C}$.

Theorem 3.1. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, $\mathfrak{D}$ a domain in $\mathfrak{X}$ and $F \in \mathrm{Hol}(\mathfrak{D},\mathfrak{Y})$. The function $\|F(x)\|$ is constant on $\mathfrak{D}$ if and only if the following two conditions are satisfied:

(i) The subspace $\mathcal{E}(F(x))$ does not depend on $x \in \mathfrak{D}$, i.e.

$$\mathcal{E}(F(x)) = \mathcal{E} \quad \text{for all} \quad x \in \mathfrak{D}.$$

(ii) $F(x) - F(y) \in \mathcal{E}$ for all $x, y \in \mathfrak{D}$.

Proof. To show necessity assume that $\|F(x)\| = 1$ for all $x \in \mathfrak{D}$. Select $x, y \in \mathfrak{D}$ and $r < 1$ so that $\mathfrak{B}(x,r) \subset \mathfrak{D}$ and $y \in \mathfrak{B}(x,r)$. The function $h(\zeta) = F(x + \frac{(y-x)\zeta}{\|y-x\|})$ $(\zeta \in \mathbb{C})$ is holomorphic on the disk $\{|\zeta| < 1\}$ and has the property that for any linear functional φ of norm one

$$|\varphi(h(\zeta))| \leq 1 \quad \text{for} \quad |\zeta| < 1.$$

Applying Lemma 2.2 to the function $\zeta \mapsto \varphi(h(\zeta))$ yields that

$$|\varphi(h(0))| + \frac{1-|\zeta|}{2|\zeta|}|\varphi(h(\zeta)) - \varphi(h(0))| \leq 1$$

or that

$$|\varphi(h(\zeta)-h(0))| \leq \frac{2|\zeta|}{1-|\zeta|}(1-|\varphi(h(0))|) \quad \text{for} \quad |\zeta| < 1.$$

Since $\|h(0)\| = 1$, Proposition 2.6 implies that

$$h(\zeta) - h(0) \in \mathcal{E}(h(0)) \quad \text{for} \quad |\zeta| < 1.$$

Letting $\zeta = \|y - x\|$, this is equivalent to

$$F(y) - F(x) \in \mathcal{E}(F(x)). \tag{1}$$

Furthermore, if y is chosen so that $\zeta = \|y - x\| < 1/5$ then $\|F(y) - F(x)\|_{F(x)} < 1/2.$

By Lemma 2.9

$$\mathcal{E}(F(y)) = \mathcal{E}(F(x) + (F(y) - F(x))) = \mathcal{E}(F(x)).$$

Hence $\mathcal{E}(F(x))$ is constant on $\mathcal{D}$ since any $x, y \in \mathcal{D}$ can be connected by a compact arc. For the same reason, inclusion (1) above holds for any $x, y \in \mathcal{D}$.

To prove the converse, let $x, y \in \mathcal{D}$ and note that by Proposition 2.8

$$\|F(x)\| = \|F(y) + (F(x) - F(y))\| \geq \|F(y)\|$$

since $F(x) - F(y) \in \mathcal{E}(F(y))$. Because it is also true that $\mathcal{E} = \mathcal{E}(F(x))$, the above inequality remains valid if x and y are interchanged. Thus $\|F(x)\| = \|F(y)\|$ and the proof is complete.

It is worth noting that because Proposition 2.8 states that $\|x + z\| \geq \|z\|$ for all $x \in \overline{\mathcal{E}(z)}$, the following two weaker conditions (i') $\overline{\mathcal{E}(F(x))}$ does not depend on $x \in \mathfrak{D}$, i.e.

$$\overline{\mathcal{E}(F(x))} = \mathfrak{F} \text{ for all } x \in \mathfrak{D}$$

(ii') $F(x) - F(y) \in \mathfrak{F}$ for all $x, y \in \mathfrak{D}$

still imply that $\|F(x)\|$ is constant on $\mathfrak{D}$.

In the case where $\mathfrak{X} = \mathbb{C}$, Theorem 3.1 reduces to Theorem 2.3; if in addition $\mathfrak{Y} = \mathbb{C}$, the Theorem 3.1 becomes the maximum modulus theorem for complex functions since every point $\zeta \in \mathbb{C}$ has $\mathcal{E}(\zeta) = \{0\}$.

A direct consequence of Theorem 3.1 is the following extension of Theorem 2.1.

Theorem 3.2. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, $\mathfrak{D}$ a domain in $\mathfrak{X}$ and $F \in \mathrm{Hol}(\mathfrak{D},\mathfrak{Y})$ with $\|F(x)\| = 1$ for all $x \in \mathfrak{D}$. If Range(F) contains a complex extreme point of the unit ball of $\mathfrak{Y}$, then $F(x)$ is constant on $\mathfrak{D}$.

Proof. Recall that if $F(x)$ is a complex extreme point of the unit ball of $\mathfrak{Y}$, then $\mathcal{E}(F(x)) = \{0\}$. Apply Theorem 3.1.

In particular, if $\mathfrak{Y}$ is a Banach space with the property that every vector of norm one is a complex extreme point of the closed unit ball of $\mathfrak{Y}$ (for example, if $\mathfrak{Y}$ is a Hilbert space or $\mathfrak{Y} = \mathbb{C}$), then Theorem 3.2 implies that the only norm constant holomorphic functions taking values in $\mathfrak{Y}$ are constant functions.

Consider now the following examples:

Example 1. Let $\mathfrak{X} = \mathfrak{Y} = M_2(\mathbb{C})$, let $\mathfrak{D}$ be the unit ball of $M_2(\mathbb{C})$ and set

$$T = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}.$$

Define $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ by

$$F(T) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} T \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & \xi_{22}\end{pmatrix}.$$

From Example 2.1, $\mathcal{E}(F(T)) = \mathrm{span}\{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\}$ for all $T \in \mathcal{D}$, hence Theorem 3.1 implies that $\|F(T)\| = 1$ for all $T \in \mathcal{D}$ as can be verified directly. Note also that since the complex extreme points of the unit ball of $M_2(\mathbb{C})$ are the unitary matrices, the hypothesis of Theorem 3.2 fails to hold.

<u>Example 2</u>. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, $\mathcal{X} = \mathcal{L}(\mathcal{H}_1)$, $\mathcal{Y} = \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, and $\mathcal{D} = \{T \in \mathcal{X} \mid \|T\| < 1\}$. Set $\pi_k : \mathcal{L}(\mathcal{H}_k) \mapsto \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ $(k = 1, 2)$ to be the canonical injections. Let I be the identity operator on $\mathcal{H}_1$. Define $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ by $F(T) = \pi_1(I) + \pi_2(T)$. Then, by Example 2.4,

$$\mathcal{E}(F(T)) = \{S \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2) \mid \mathcal{H}_1 \subset \ker S\}$$

for all $T \in \mathcal{D}$. Again, Theorem 3.1 implies that $\|F(T)\| = 1$ on $\mathcal{D}$. Note that since the range of F does not include any partial isometries nor their adjoints, the hypothesis of Theorem 3.2 fails to hold for this example as well.

As was true in the case where $\mathcal{X} = \mathbb{C}$, the hypothesis of Theorem 3.2 can be weakened somewhat.

<u>Theorem 3.3</u>. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $\mathcal{D}$ a domain in $\mathcal{X}$ and $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ satisfying $\|F(x)\| = 1$ for all $x \in \mathcal{D}$. If $\mathrm{Conv}(F(\mathcal{D}))$ contains a complex extreme point of the unit ball of $\mathcal{Y}$, then $F(x)$ is constant on $\mathcal{D}$.

The proof of this theorem depends on a rather surprising lemma; it was originally observed by Thorp-Whitley [11] in the case $\mathfrak{X} = \mathbb{C}$.

Lemma 3.4. Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{D}$ and F be as in Theorem 3.3. Then every element in $\mathrm{Conv}(F(\mathfrak{D}))$ has norm one.

Proof. Let $y \in \mathrm{Conv}(F(\mathfrak{D}))$, then there are points $x_1, \ldots, x_n$ in $\mathfrak{D}$ and positive numbers $t_1, \ldots, t_n$ with $\sum_{j=1}^{n} t_j = 1$ so that

$$y = \sum_{j=1}^{n} t_j F(x_j).$$

Select a linear functional φ on $\mathfrak{Y}$ of norm one so that $\varphi(F(x_1)) = 1$. Now define $G \in \mathrm{Hol}(\mathfrak{D}, \mathbb{C})$ by

$$G(x) = \varphi(F(x)).$$

Then $|G(x)| \leq \|\varphi\| \cdot \|F(x)\| = 1$ for all $x \in \mathfrak{D}$.

Furthermore, $|G(x)|$ attains a maximum at $x = x_1 \in \mathfrak{D}$ so that Proposition 1.7 implies that $|G(x)| = 1$ for all $x \in \mathfrak{D}$. By Theorem 3.2, $G(x)$ itself is constant on $\mathfrak{D}$, hence

$$\varphi(y) = \sum_{j=1}^{n} t_j \, \varphi(F(x_j)) = 1.$$

Since $\|\varphi\| = 1$ it follows that $\|y\| \geq 1$. On the other hand

$$\|y\| \leq \sum_{j=1}^{n} t_j \|F(x_j)\| = 1$$

Thus $\|y\| = 1$ and the proof of the lemma is complete.

Proof of Theorem 3.3. By Theorem 3.1 and Lemma 3.4 the range of F satisfies the hypothesis of Lemma 2.10. Thus $\mathcal{E}(y) = \mathcal{E}$ for all $y \in \mathrm{Conv}(F(\mathfrak{D}))$. The fact that $\mathrm{Conv}(F(\mathfrak{D}))$ contains a complex extreme point of the closed unit ball of $\mathcal{Y}$ implies that $\mathcal{E} = \{0\}$. As before, this forces $F(x)$ to be constant on $\mathcal{D}$, completing the proof.

It is interesting to observe that if a nonconstant function $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ has constant norm on an open subset of $\mathcal{D}$, then by a straightforward application of the Hahn-Banach Theorem (combined with Theorem 3.2 and Proposition 1.2) F cannot have any zeros on $\mathcal{D}$. Furthermore, it is clear that any nonconstant $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ with a zero in $\mathcal{D}$ cannot be norm constant on any open subset of $\mathcal{D}$ for *any* norm on $\mathcal{Y}$ equivalent to the original. This suggests that if the range of F is "sufficiently" removed from zero then there should be some norm and some open subset of $\mathcal{D}$ for which F is norm constant. This was first shown to be true by Globevnik [4] in the case where $\mathcal{X} = \mathbb{C}$. The following theorem establishes the result in the general case. The basic approach of the proof remains the same.

Theorem 3.5. Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces, $\mathcal{D}$ a domain in $\mathcal{X}$ containing zero, and $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ a nonconstant function. Then

$$F(0) \notin \overline{\operatorname{span}}\{F(x) - F(0) \mid x \in \mathcal{D}\}$$

if and only if there exists an equivalent norm $|||\cdot|||$ on $\mathcal{Y}$ and an open set $\mathcal{U} \subset \mathcal{D}$ so that

$$|||F(x)||| \text{ is constant for all } x \in \mathcal{U}.$$

The proof of this theorem depends on the following two lemmas.

Lemma 3.6. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{D}$ be as above, and $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ such that

$$\|F(x)\| \equiv c > 0 \quad \text{for all } x \in \mathcal{U}$$

for some open subset $\mathcal{U}$ of $\mathcal{D}$.

Then

$$F(\mathcal{D}) \subset F(x_0) + \ker \varphi$$

where $x_0 \in \mathcal{D}$, and φ is a linear functional on $\mathcal{Y}$ of norm one such that $F(x_0) \notin \ker \varphi$.

Proof. Let $x_0 \in \mathcal{U}$. By the Hahn-Banach Theorem there is a linear functional φ of norm one such that $\varphi(F(x_0)) = c$.

Since $|\varphi(F(x))| \leq \|\varphi\| \cdot \|F(x)\| = c$ for all $x \in \mathcal{U}$ it follows from Proposition 1.7 and Theorem 3.2 that $\varphi(F(x)) = c$ for all $x \in \mathcal{U}$ and thus that $\varphi(F(x)) = c$ for all $x \in \mathcal{D}$ by Proposition 1.2. In particular this implies that $F(\mathcal{D}) \subset F(x_0) + \ker \varphi$ and $F(x_0) \notin \ker \varphi$ by construction.

Lemma 3.7. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{D}$ be as before, and $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ such that $F(\mathcal{D}) \subset H$ where H is a closed hyperplane in $\mathcal{Y}$ disjoint from zero. Then for every $x_0 \in \mathcal{D}$, there is a ball $\mathcal{B}(x_0,r)$ and a norm $|||\cdot|||$ on $\mathcal{Y}$ equivalent to the original so that

$$|||\, F(x)\, ||| = 1 \text{ for all } x \in \mathcal{B}(x_0,r).$$

Proof. Write $H = y_0 + \ker \varphi$ where φ is a linear functional on $\mathcal{Y}$ and $y_0 \notin \ker \varphi$. Let $\varphi(y_0) = c$. Given $x_0 \in \mathcal{D}$, choose a positive r less than the radius of boundedness of F at x_0, and let

$$M = \sup\{\|F(x)\| \mid x \in \mathcal{B}(x_0,r\}.$$

Define a new norm on $\mathcal{Y}$ by

$$|||\, y\, ||| = \max\{\frac{\|y\|}{M}, |\frac{\varphi(y)}{c}|\}.$$

Clearly $|||\cdot|||$ is a norm; furthermore

$$|||y||| \leq \max\left\{\frac{\|y\|}{M}, \frac{\|\varphi\|\cdot\|y\|}{|c|}\right\} = \|y\| \max\left\{\frac{1}{M}, \frac{\|\varphi\|}{|c|}\right\}$$

and

$$\|y\| \leq \max\left\{\|y\|, M\left|\frac{\varphi(y)}{c}\right|\right\} = M\cdot|||y|||$$

so that $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms. To show that $|||F(x)||| \equiv 1$ on $\mathcal{B}(x_0,r)$, first note that $F(\mathcal{D}) \subset H$ implies that

$$\left|\frac{\varphi(F(x))}{c}\right| = \left|\frac{\varphi(y_0)}{c}\right| = 1 \qquad \text{for all } x \in \mathcal{D}.$$

If in addition $x \in \mathcal{B}(x_0,r)$, then $\frac{\|F(x)\|}{M} \leq 1$ so that $|||F(x)||| = 1$ for all $x \in \mathcal{B}(x_0,r)$, as desired.

Proof of Theorem 3.5. Suppose first that

$$F(0) \notin S = \overline{\operatorname{span}}\{F(x) - F(0) \mid x \in \mathcal{D}\}.$$

Then by the Hahn-Banach Theorem it is possible to find a closed hyperplane H containing S which is disjoint from $F(0)$, i.e.

$$F(\mathcal{D}) \subset F(0) + H \quad \text{where} \quad F(0) \notin H.$$

By Lemma 3.7, around each point $x_0 \in \mathcal{D}$ there is an open ball $\mathcal{B}(x_0,r)$ and a norm $|||\cdot|||$ on $\mathcal{Y}$ equivalent to $\|\cdot\|$ so that $|||F(x)||| \equiv 1$ on $\mathcal{B}(x_0,r)$.

To prove the converse, suppose that there is an open neighborhood $\mathcal{U}$ and a norm $|||\cdot|||$ on $\mathcal{Y}$ equivalent to $\|\cdot\|$ so that

$$|||F(x)||| = C > 0 \quad \text{on } \mathcal{U}.$$

Apply Lemma 3.6 to obtain that for some $x_0 \in \mathcal{D}$

$$F(\mathcal{D}) \subset F(x_0) + H \quad \text{where} \quad F(x_0) \notin H,$$

or equivalently that

$$S = \overline{\operatorname{span}}\{F(x) - F(x_0) \mid x \in \mathcal{D}\} \subset H.$$

Notice first that $F(0) \notin S$ since

$$F(0) = F(x_0) + (F(0) - F(x_0)) \notin H$$

and second that $S = \overline{\operatorname{span}}\{F(x) - F(0) \mid x \in \mathcal{D}\}$
since $F(x) - F(0) = F(x) - F(x_0) - (F(0) - F(x_0))$.
Thus $F(0) \notin \overline{\operatorname{span}}\{F(x) - F(0) \mid x \in \mathcal{D}\}$ and the theorem is proved.

It is worth remarking at this point that Theorem 3.5 remains valid if the condition

$$F(0) \notin \overline{\operatorname{span}}\{F(x) - F(0) \mid x \in \mathcal{D}\}$$

is replaced by

$$F(x_0) \notin \overline{\operatorname{span}}\{F(x) - F(x_0) \mid x \in \mathcal{D}\}$$

where x_0 is any element of $\mathcal{D}$. This is particularly of interest when $0 \notin \mathcal{D}$.

It is quite easy to provide examples of holomorphic functions F with the property that

$$F(0) \notin \overline{\operatorname{span}}\{F(x) - F(0) \mid x \in \mathcal{D}\}.$$

In fact, Theorem 3.5 is in some sense remarkable because so many functions can be made norm constant by choosing a new norm equivalent to the old one. Because Lemma 3.7 is constructive, the new norm can actually be found explicitly if F is not very complicated. Consider the following examples.

Example 3. Let $\mathcal{X} = \mathcal{Y} = M_2(\mathbb{C})$ and let $\mathcal{D}$ be the unit ball of $M_2(\mathbb{C})$. Define $F \in \mathrm{Hol}(\mathcal{D},\mathcal{Y})$ by

$$F(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T \quad \text{for} \quad T \in \mathcal{D}.$$

Set $S = \overline{\mathrm{span}}\{F(T) - F(0) \mid T \in \mathcal{D}\} = \{\begin{pmatrix} \xi_1 & \xi_2 \\ 0 & 0 \end{pmatrix} \mid \xi_1, \xi_2 \in \mathbb{C}\}$.

It is clear that $F(0) = I \notin S$, so that F can be made norm constant. To determine this new norm $|||\cdot|||$, define a linear functional φ by

$$\varphi(T) = \langle Te_2, e_2 \rangle \quad \text{where} \quad e_2 = (0,1)$$

and $\langle \cdot , \cdot \rangle$ is the inner product on $\mathbb{C}^2$. The subspace S is annihilated by φ, so it satisfies the requirements of Lemma 3.7. Since $M = \sup\{\|F(T)\| \,\big|\, \|T\| < 1\} = 2$, the new norm is given by

$$|||T||| = \max\{\frac{\|T\|}{2}, |\langle Te_2, e_2\rangle|\}.$$

It is easy to see that $|||F(T)||| \equiv 1$ for $T \in \mathcal{D}$.

Example 4. Let $\mathcal{X} = \mathcal{Y} = \mathcal{L}(\mathcal{H})$ when $\mathcal{H}$ is a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $U \in \mathcal{L}(\mathcal{H})$ be the unilateral shift (i.e. $Ue_n = e_{n+1}$). Define $F \in \mathrm{Hol}(\mathcal{L}(\mathcal{H}),\mathcal{L}(\mathcal{H}))$ by

$$F(T) = aI + UT - TU^* \qquad (a \neq 0)$$

where I is the identity.

Let $S = \overline{\mathrm{span}}\{F(T) - F(0)\} = \overline{\mathrm{span}}\{UT - TU^* \mid T \in \mathcal{L}(\mathcal{H})\}$. The linear functional $\varphi(T) = \langle Te_1, e_1\rangle$ annihilates S and $\varphi(F(0)) = a$ so that $F(0) \notin S$. Once again the hypotheses of Theorem 3.5 are satisfied. To obtain the new norm, note that

$$M = \sup\{\|F(T)\| \,\big|\, \|T\| < 1\} \leq |a| + 2$$

so that the norm $|||\cdot|||$ is given by

$$|||\,T\,||| = \max\left\{\frac{\|T\|}{|a|+2}, \left|\frac{\langle Te_1, e_1\rangle}{a}\right|\right\}$$

Again, it is easy to verify that $|||\,F(T)\,||| \equiv 1$ for $\|T\| < 1$.

Another question that arises is the nature of the relationship between Theorem 3.1 and Theorem 3.5. Specifically, a function satisfying the conditions of Theorem 3.1 must also satisfy that of Theorem 3.5. This can be shown directly.

Proposition 3.8. Let $\mathfrak{X}$, $\mathfrak{Y}$ be Banach spaces, $\mathfrak{D}$ a domain of $\mathfrak{X}$ containing zero, and let $F \in \mathrm{Hol}(\mathfrak{D},\mathfrak{Y})$ be a nonconstant function satisfying:

(i) $\mathcal{E}(F(x)) = \mathcal{E}$ for all $x \in \mathfrak{D}$

(ii) $F(x) - F(y) \in \mathcal{E}$ for all $x, y \in \mathfrak{D}$.

Then $F(0) \notin \overline{\mathrm{span}}\{F(x) - F(0)\,|\,x \in \mathfrak{D}\}$.

Proof. Since $\mathcal{E}$ is a subspace by Proposition 2.7,

$$\mathrm{span}\{F(x) - F(0)\,|\,x \in \mathfrak{D}\} \subset \mathcal{E}$$

because of hypothesis (ii). The difficulty now is that $\mathcal{E}$ is not closed.

Since F is nonconstant, it may be assumed without loss of generality that $F(0) \neq 0$. Now recall that by Proposition 2.6, $x \in \mathcal{E} = \mathcal{E}(F(0))$ if and only if there is an M so that

$$|\varphi(x)| \leq M(\|F(0)\| - |\varphi(F(0))|)$$

for all linear functionals on $\mathcal{Y}$ of norm one. In particular, if φ is chosen so that

$$\varphi(F(0)) = \|F(0)\| \neq 0$$

then

$$\operatorname{span}\{F(x) - F(0)\} \subset \mathcal{E} \subset \ker \varphi.$$

Hence $F(0) \notin \overline{\operatorname{span}}\{F(x) - F(0) \mid x \in \mathcal{D}\}$.

Chapter 4. Norm Constant Holomorphic Functions of Operators.

In this chapter the results of the previous chapter are specialized to the case in which both Banach spaces are $\mathcal{L}(\mathcal{H})$, the space of bounded linear operators on a Hilbert space $\mathcal{H}$. In this setting it is shown that a holomorphic function will have constant norm if and only if it satisfies relatively simple algebraic conditions. Attention is also given to the class of holomorphic functions arising from the Dunford functional calculus. All functions in this class with constant norm are shown to be constant. Finally, the holomorphic functions having constant real part are determined. Actually, the results of this chapter are valid for any C^*-algebra although they are stated only for the C^*-algebra $\mathcal{L}(\mathcal{H})$.

The main result is the following analog of Theorem 3.1.

Theorem 4.1. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D}$ a domain in $\mathcal{L}(\mathcal{H})$ containing zero, and $F \in \mathrm{Hol}(\mathcal{D},\mathcal{L}(\mathcal{H}))$ with $\|F(0)\| = 1$.

(i) If there is an open neighborhood $\mathcal{U}$ of zero in $\mathcal{D}$ on which $\|F(T)\| = 1$ then for all $T \in \mathcal{D}$

$$[F(T) - F(0)]\,Q = [F(T)^* - F(0)^*]\,P = 0,$$

where P and Q are the spectral projections at 1 of $F(0)F(0)^*$ and $F(0)^*F(0)$ respectively.

(ii) If there is a $\delta > 0$ such that for all $T \in \mathcal{D}$

$$[F(T) - F(0)](1 - G_{1-\delta}) = [F(T)^* - F(0)^*](1 - E_{1-\delta}) = 0,$$

where $\{E_\lambda\}$ and $\{G_\lambda\}$ are the spectral measures of $F(0)F(0)^*$ and $F(0)^*F(0)$ respectively, then there is an open neighborhood of zero on which $\|F(T)\| = 1$.

Again, as in Theorem 2.5, if 1 is an isolated point of the spectrum of $F(0)F(0)^*$, condition (i) is both necessary and sufficient. The role of 0 in Theorem 4.1 is not essential: A translation argument shows that 0 may be replaced throughout by any $T_0 \in \mathfrak{D}$.

The first half of this theorem is a consequence of Theorem 3.1 in conjunction with the following two lemmas.

Lemma 4.2. Let $A \in \mathcal{L}(\mathcal{H})$, then $\mathcal{E}(A^*) = \mathcal{E}(A)^* = \{T^* | T \in \mathcal{E}(A)\}$. This lemma follows immediately from the fact that $\|A^* + \zeta T\| = \|A + \bar{\zeta} T^*\|$.

Lemma 4.3. Let $A \in \mathcal{L}(\mathcal{H})$ and $v \in \mathcal{H}$ so that $A^*Av = v$. Then $\mathcal{E}(A)v = \{Tv | T \in \mathcal{E}(A)\} = \{0\}$.

Proof. Let $T \in \mathcal{E}(A)$, then there is an $r > 0$ such that $\|A + \zeta T\| \leq \|A\|$ for all ζ with $|\zeta| \leq r$. But then $\|Av + \zeta Tv\| \leq \|A + \zeta T\| \cdot \|v\| = \|Av\|$ if $|\zeta| \leq r$, so that $Tv \in \mathcal{E}(Av)$. By Example 2.3 $\mathcal{E}(Av) = \{0\}$, hence $Tv = 0$.

Proof of part (i) of Theorem 4.1. Let $A = F(0)$; since $\|A\| = 1$, 1 is in the spectrum of both AA^* and A^*A. If 1 is in the point spectrum of one operator, it is in the point spectrum of the other, so there are only two cases to consider: 1 is in the point spectrum of both operators, or it is in the continuous spectrum of both. In the latter case, $P = Q = 0$ and there is nothing to prove. It is therefore assumed that 1 is an eigenvalue of both AA^* and A^*A.

Let $\mathfrak{U}$ be an open neighborhood of zero in $\mathfrak{D}$ on which $\|F(T)\| = 1$. Then Theorem 3.1 together with Lemma 4.2 implies that

$$\mathcal{E}(F(T)) = \mathcal{E} \qquad \text{for all } T \in \mathfrak{U}$$

$$F(T) - F(S) \in \mathcal{E} \quad \text{for all } T, S \in \mathfrak{U}$$

and

$$\mathcal{E}(F(T)^*) = \mathcal{E}(F(T))^* = \mathcal{E}^* \quad \text{for all} \quad T \in \mathcal{U}$$

$$F(T)^* - F(S)^* \in \mathcal{E}^* \qquad \text{for all} \quad T, S \in \mathcal{U}.$$

Let $u \in \text{Range}(P)$ and $v \in \text{Range}(Q)$, then u, v are eigenvectors of AA^* and A^*A respectively.

Application of Lemma 4.3 yields

$$\mathcal{E}v = \mathcal{E}(A)v = \{0\}$$

and

$$\mathcal{E}^*u = \mathcal{E}(A^*)u = \{0\}.$$

Hence $[F(T) - F(S)]v = [F(T)^* - F(S)^*]u = 0$ for all $T, S \in \mathcal{U}$. In particular this implies that for all $T \in \mathcal{U}$

$$[F(T) - F(0)]Q = [F(T)^* - F(0)^*]P = 0.$$

Applying Proposition 1.2 to the holomorphic functions $[F(T) - F(0)]Q$ and $P^*[F(T) - F(0)]$ it follows that both vanish identically on all of $\mathcal{D}$, which completes the proof of the first part of Theorem 3.1.

The proof of the second part of Theorem 4.1 does not appeal to Theorem 3.1. It is proved directly, proceeding along the same general lines as Globevnik's proof of Theorem 2.5 [6]. The estimates, however are somewhat more complicated. The proof relies on the following lemma which is stated without proof.

<u>Lemma 4.4</u> (Globevnik [6]). Let $T \in \mathcal{L}(\mathcal{H})$ and $\{E_\lambda\}$ and $\{G_\lambda\}$ be the spectral measures of TT^* and T^*T respectively, and let $\alpha < \beta$. If $v \in \text{Range}(G_\beta - G_\alpha)$ then $Tv \in \text{Range}(E_\beta - E_\alpha)$.

<u>Proof of part (ii) of Theorem 4.1</u>. For simplicity set $Q = 1 - G_{1-\delta}$ and $P = 1 - E_{1-\delta}$. By hypothesis there is a positive R less than the

radius of boundedness of F at zero so that for any $v \in \mathrm{Range}(Q)$ and $u \in \mathrm{Range}(P)$, $F(T)v = F(0)v$ and $F(T)^*u = F(0)^*u$ for all T with $\|T\| < R$. By Lemma 4.4 $F(0)v \in \mathrm{Range}(P)$ and thus

$$F(T)^*F(T)v = F(T)^*F(0)v = F(0)^*F(0)v$$

if $\|T\| < R$ and $v \in \mathrm{Range}(Q)$. For $w \in \mathcal{H}$, $w = v + u$ with $v = Qw$ and $u = (I-Q)w$ so that

$$\begin{aligned}\|F(T)w\|^2 &= \langle F(T)^*F(T)v,v\rangle + \langle F(T)^*F(T)v,u\rangle \\ &\quad + \langle u,F(T)^*F(T)v\rangle + \langle F(T)^*F(T)u,u\rangle \\ &= \langle F(0)^*F(0)v,v\rangle + \langle F(0)^*F(0)v,u\rangle \\ &\quad + \langle u,F(0)^*F(0)v\rangle + \langle F(T)^*F(T)u,u\rangle .\end{aligned}$$

But $\langle F(0)^*F(0)v,u\rangle = 0$ since $\mathrm{Range}(Q)$ is invariant under $F(0)^*F(0)$, so that

$$\begin{aligned}\|F(T)w\|^2 &= \langle F(0)^*F(0)v,v\rangle + \langle F(T)^*F(T)u,u\rangle \\ &\le \|F(0)^*F(0)\| \cdot \|v\|^2 + \|F(T)^*F(T)u\| \cdot \|u\| .\end{aligned}$$

To estimate $\|F(T)^*F(T)u\|$, choose an $r < R$; then $\|F(T)\|$ and $\|F(T) \pm F(0)\|$ are bounded on $\overline{\mathcal{B}(0,r)}$. Let $M(r)$ be a bound for all three on this ball. Since $u \in \mathrm{Range}(G_{1-\delta})$, $\|F(0)^*F(0)u\| \le (1-\delta)\|u\|$, so that

$$\begin{aligned}\|F(T)^*F(T)u\| &\le \|F(0)^*F(0)u\| + \|(F(T)^*F(T) - F(0)^*F(0))u\| \\ &\le (1-\delta)\|u\| + \|(F(T)^* + F(0)^*)(F(T) - F(0))u\| \\ &\quad + \|(F(T)^*F(0) - F(0)^*F(T))u\| \\ &\le (1-\delta)\|u\| + \|F(T) + F(0)\| \cdot \|F(T) - F(0)\| \cdot \|u\| \\ &\quad + \|F(T)^* - F(0)^*\| \cdot \|F(0)\| \cdot \|u\| + \|F(T) - F(0)\| \cdot \|F(0)\| \cdot \|u\| \\ &\le (1-\delta)\|u\| + M(r)\|F(T) - F(0)\| \cdot \|u\| + 2\|F(T) - F(0)\| \cdot \|u\| \\ &= (1-\delta)\|u\| + (M(r)+2)\|F(T) - F(0)\| \cdot \|u\| .\end{aligned}$$

An application of Schwarz's Lemma to the holomorphic function $F(T) - F(0)$ gives

$$\|F(T) - F(0)\| \leq \frac{M(r)}{r}\|T\|.$$

Hence

$$\|F(T)^{*}F(T)u\| \leq (1-\delta)\|u\| + (M(r)+2)\frac{M(r)}{r}\|T\| \cdot \|u\|$$
$$= (1-\delta)\|u\| + \tilde{M}(r)\|T\| \cdot \|u\|$$

and

$$\|F(T)w\|^2 \leq \|v\|^2 + (1-\delta)\|u\|^2 + \tilde{M}(r)\|T\| \cdot \|u\|^2$$
$$= \|w\|^2 + (\tilde{M}(r)\|T\| - \delta)\|u\|^2.$$

If $r_0 = \min(r, \delta/\tilde{M}(r))$, then for all T with $\|T\| \leq r_0$, $\|F(T)\| \leq 1$. Since $\|F(0)\| = 1$ Proposition 1.7 implies that $\|F(T)\| = 1$ for all T with $\|T\| \leq r_0$. This completes the proof of Theorem 4.1.

Theorem 4.1 can be used effectively to determine whether a specific function in $\mathrm{Hol}(\mathfrak{V}, \mathcal{L}(\mathcal{H}))$ has constant norm on some open ball. Furthermore, it provides a lower bound for the radius of this ball, namely

$$r_0 = \min\left(r, \frac{\delta r}{M(r)(M(r)+2)}\right),$$

where r is any positive number less than the radius of boundedness of F at zero and

$$M(r) = \sup\{\|F(T)\|, \|F(T) \pm F(0)\| \;\big|\; T \in \overline{\mathcal{B}(0,r)}\}.$$

However, this lower bound is generally quite poor as can be seen in the examples which follow.

The first example was analyzed in part in Chapter 3.

Example 1. Let $\mathcal{H} = \mathbb{C}^2$, then $\mathcal{L}(\mathcal{H}) = M_2(\mathbb{C})$. Let $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and set

$$F(T) = P_1 + P_2TP_2.$$

The spectral projections at 1 in this case are both P_1 and clearly for all T

$$[F(T) - F(0)]P_1 = [F(T)^* - F(0)^*]P_1 = 0.$$

Since 1 is an isolated point of $\sigma(F(0)^*F(0))$, Theorem 4.1 implies that $\|F(T)\| = 1$ on some ball centered at zero. Obviously this ball can be taken to have radius 1. Compare this to the lower bound given by Theorem 3.1: Here r can be taken arbitrarily large, δ arbitrarily close to 1, and $M(r) = \max(2,r)$. Thus r_0 is at most $\frac{1}{4}$.

Example 2. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$, let $U \in \mathcal{L}(\mathcal{H})$ be the unilateral shift (i.e., $Ue_n = e_{n+1}$) and let P_1 be the orthogonal projection onto $\text{span}(e_1)$. Set

$$F(T) = P_1 + UTU^*.$$

The corresponding spectral projections are both P_1 and the hypothesis of Theorem 4.1 are obviously met. Thus $\|F(T)\|$ is constant on some ball centered at zero. While the radius of this ball can be taken equal to one, r_0 is never larger than $\frac{1}{4}$, just as in the previous example.

Example 3. Let $\mathcal{H}$ and $\{e_n\}_{n=1}^{\infty}$ be as above, let D be the diagonal operator defined by

$$De_n = \left(\frac{n}{n+1}\right)^{\frac{1}{2}} e_n$$

and let P_k be the orthogonal projection onto $\text{span}(e_1,\ldots,e_k)$. Consider the holomorphic function

$$F_k(T) = D + P_kTP_k.$$

This time 1 is no longer an isolated point of $\sigma(F(0)^*F(0))$. Direct calculation shows that

$$G_{1-\delta} = E_{1-\delta} = P_m \quad \text{where} \quad m = [1/\delta].$$

So if $\delta \leq 1/(k+1)$ then $m > k$ and

$$[F_k(T) - F_k(0)](I - P_m) = [F_k(T)^* - F_k(0)^*](I - P_m) = 0.$$

Hence by Theorem 4.1, for every positive integer k, $\|F_k(T)\| = 1$ on some ball centered at zero. To estimate the radius of this ball, note that r is again arbitrary, $\delta = \frac{1}{k+1}$ and $M(r) \sim \max(r,3)$. Thus r_0 is never larger than $\frac{1}{5(k+1)}$. Again, this is far from being sharp. For example, when $k = 1$, $\|F_1(T)\| = 1$ for all $\|T\| < 1/2$.

We now consider a special class of holomorphic functions of operators, namely those functions arising from the Dunford functional calculus. It was shown in Chapter 1 that such functions arise from a complex valued function f holomorphic on the closure of a domain D in $\mathbb{C}$. For T with $\sigma(T) \subset D$, F is defined by

$$F(T) = \frac{1}{2\pi i} \int_{\partial D} f(\zeta)(\zeta I - T)^{-1} d\zeta.$$

F will be holomorphic on $\mathfrak{D} = \{T \in \mathcal{L}(\mathcal{H}) \mid \sigma(T) \subset D\}$. Such functions will be referred to as Dunford functions. If a Dunford function F has $\|F(T)\| = 1$ on $\mathfrak{D}$, then $F(T)$ will be constant on $\mathfrak{D}$ by Theorem 3.2. For if $\zeta \in D$, then $\zeta I \in \mathfrak{D}$ and $F(\zeta I) = e^{i\theta} I$, a complex extreme point of the closed unit ball. Even more is true.

<u>Proposition 4.6</u>. If $F \in \mathrm{Hol}(\mathfrak{D}, \mathcal{L}(\mathcal{H}))$ is a nonconstant Dunford function, then F cannot have constant norm under any norm equivalent to the usual norm.

Proof. First note that if

$$F(T) = \frac{1}{2\pi i}\int_{\partial D} f(\zeta)(\zeta I - T)^{-1}d\zeta$$

then $F(\xi I) = f(\xi)I$ for all $\xi I \in \mathcal{D}$. Since F is nonconstant, so is f and $\xi, \zeta \in D$ can be chosen so that $f(\xi) \neq f(\zeta)$. Then $F(\zeta I) - F(\xi I) = (f(\zeta) - f(\xi))I$ so that $F(\xi I) \in \overline{\text{span}}\{F(T) - F(\xi I) \mid T \in \mathcal{D}\}$. The result now follows by applying Theorem 3.5 to the function $G(T) = F(T + \xi I)$ which is holomorphic on a domain containing the origin and has $G(0) \in \overline{\text{span}}\{G(T) - G(0)\}$.

Another interesting class of holomorphic functions of operators is that of functions F with $\text{Im}(F(T)) \equiv 0$ $(\text{Im}(T) = \frac{1}{2i}(T - T^*))$. By analogy with the norm constant functions, it is conceivable that there would be many nonconstant functions satisfying this condition. The analogy is of course inaccurate, as the following theorem confirms.

Theorem 4.7. Let $\mathcal{D}$ be a domain in $\mathcal{H}$ and $F \in \text{Hol}(\mathcal{D}, \mathcal{L}(\mathcal{H}))$ satisfying $\text{Im}(F(T)) = 0$ for all $T \in \mathcal{D}$. Then $F(T)$ is constant on $\mathcal{D}$.

Proof. Since $\text{Im}(F(T)) = 0$ on $\mathcal{D}$, $F(T)$ is selfadjoint for all $T \in \mathcal{D}$. Define a new function $G \in \text{Hol}(\mathcal{D}, \mathcal{L}(\mathcal{H}))$ by

$$G(T) = (F(T) - iI)(F(T) + iI)^{-1}.$$

Since $F(T)$ is selfadjoint, $G(T)$ is unitary for all $T \in \mathcal{D}$. In particular, G is a norm constant function whose range includes a complex extreme point of the closed unit ball, so $G(T)$ is constant by Theorem 3.2 and hence $F(T)$ must also be constant on $\mathcal{D}$.

The conclusion of Theorem 4.7 remains valid if it is assumed that $\text{Im}(F(T)) \equiv C$ or $\text{Re}(F(T)) \equiv C$ on $\mathcal{D}$ instead of $\text{Im}(F(T)) \equiv 0$.

Theorem 4.7 is so much stronger than Theorem 4.1 or Theorem 3.1 because the analogy between holomorphic functions F with constant norm and those with $\mathrm{Im}(F(T)) \equiv 0$ is inaccurate. Requiring the imaginary part of an operator to be zero imposes a great deal of structure on the operator (i.e., it must be selfadjoint). On the other hand, requiring an operator to have norm one imposes no structure whatsoever on the operator.

A final point worth discussing is the relation of Theorem 4.1 to the results of Chapter 3. Recall that the proof of the second part of Theorem 4.1 was direct and did not appeal to any of the results of the previous chapter. Ideally, a proof of Theorem 4.1 part (ii) would use a lemma such as the following:

Lemma. Let $F \in \mathrm{Hol}(\mathfrak{D},\mathcal{L}(\mathcal{H}))$ satisfy the hypothesis of Theorem 4.1 (ii), then

$$\overline{\mathcal{E}(F(T))} = \mathfrak{J} \quad \text{for all} \quad T \in \mathfrak{D}$$

and

$$F(T) - F(S) \in \mathfrak{J} \quad \text{for all} \quad T, S \in \mathfrak{D}.$$

The proof would then be completed by invoking Theorem 3.1. Unfortunately the author has been unable to prove a lemma such as the above. However, it is a simple matter to use Theorem 3.5 to show that such a function must have constant norm in some norm equivalent to the original.

Proposition 4.5. Let $\mathfrak{D} \subset \mathcal{L}(\mathcal{H})$ be a domain containing zero and $F \in \mathrm{Hol}(\mathfrak{D},\mathcal{L}(\mathcal{H}))$ satisfying the hypothesis of Theorem 4.1 (ii), then F must have constant norm in some norm equivalent to the usual operator norm.

Proof. By Theorem 3.5, it suffices to show that

$$F(0) \notin \overline{\mathrm{span}}\{F(T) - F(0) \mid T \in \mathfrak{D}\}.$$

To do this consider the linear map from $\mathcal{L}(\mathcal{H})$ into itself given by

$$T \longmapsto T(1 - G_{1-\delta}).$$

The kernel of this map includes $F(T) - F(0)$ for all $T \in \mathfrak{D}$, but not $F(0)$ since $1 - G_{1-\delta}$ is a nonzero spectral projection for $F(0)^*F(0)$. The result now follows easily.

This proposition suggests an alternate proof of the second part of Theorem 4.1: it only remains to show that the norm in Proposition 4.5 is the usual operator norm. However, as can be seen from the construction of the norm in Lemma 3.7, this appears to require proving first that $\|F(T)\|$ is bounded by 1, so that we seem to return to the direct proof of Theorem 4.1 (ii).

Conclusion

There are several other questions related to norm constant holomorphic functions and/or maximum modulus theorems in other settings which still do not have satisfactory answers. Some of these will be briefly mentioned here.

The first involves the generalization of the Schwarz-Pick inequality. The classical result states that if ω is a holomorphic function from the complex unit disk into itself, then for all ζ, ξ in the unit disk,

$$\left|\frac{\omega(\zeta) - \omega(\xi)}{1-\overline{\omega(\xi)}\omega(\zeta)}\right| \leq \left|\frac{\zeta - \xi}{1-\overline{\xi}\zeta}\right| .$$

Nontrivial equality holds only when ω is a fractional linear transformation of the form

$$\omega(\zeta) = e^{i\theta} \frac{\zeta - \zeta_0}{1-\overline{\zeta}_0\zeta} .$$

An analog of this inequality holds in $\mathcal{L}(\mathcal{H})$. Let

$$T_B(A) = (I - BB^*)^{\frac{1}{2}}(A - B)(I - B^*A)^{-1}(I - B^*B)^{-\frac{1}{2}}.$$

Then T_B is a map of the unit ball $\mathcal{B}$ of $\mathcal{L}(\mathcal{H})$ into itself which plays the role of the fractional linear map above. If $F \in \mathrm{Hol}(\mathcal{B},\mathcal{B})$ then the following generalization of the above inequality holds for all $A, B \in \mathcal{B}$:

$$\|T_{F(B)}(F(A))\| \leq \|T_B(A)\| .$$

The problem is to determine those functions F for which nontrivial equality holds. This is equivalent to determining for which F the function

$$H(A) = A^{-1} \cdot T_{F(B)} \circ F \circ T_B(A)$$

will have constant norm. Neither Theorem 3.1 nor Theorem 4.1 provide a satisfactory answer to this question. It was this problem that originally motivated the author to study norm constant holomorphic functions.

Another unsolved problem involves a "maximum modulus theorem" in a completely different setting. Let $f \in \mathrm{Hol}(D, \mathcal{L}(\mathcal{H}))$ where D is a domain in the complex plane. Suppose that the spectrum $\sigma(f(\zeta))$ achieves its maximum on D at ζ_0 in the sense that for all $\zeta \in D$ $\sigma(f(\zeta)) \subseteq \sigma(f(\zeta_0))$. Does it follow that $\sigma(f(\zeta))$ is independent of ζ? This question was first posed by A. Brown and R. Douglas in 1966. Unfortunately none of the methods of this thesis shed any light on this interesting problem.

The usefulness of $\mathcal{E}(z)$ in dealing with problems involving complex extreme points suggests the following definition:

$$\mathfrak{F}(z) = \{y \in \mathfrak{X} \,|\, \exists\, r > 0 \text{ such that } \|z + ty\| \leq \|z\| \text{ for } -r \leq t \leq r\}.$$

The set $\mathfrak{F}(z)$ measures "how close" z is to being a real extreme point of the ball in $\mathfrak{X}$ of radius $\|z\|$. While $\mathfrak{F}(z)$ does not appear to behave as nicely as $\mathcal{E}(z)$, it is conceivable that $\mathfrak{F}(z)$ would nevertheless be a useful tool in problems dealing with real extreme points.

Finally, there remains the question brought up at the end of Chapter 4: Is it possible to give a proof of part (ii) of Theorem 4.1 which utilizes the more general theorems of Chapter 3?

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