Topics in Linear Spaces and Projective Planes

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Joel Christopher Fowler

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ABSTRACT

A linear space is an incidence structure of points and lines such that every pair of points is contained in a unique line. In the first two chapters of this thesis results are presented linking structural properties to arithmetic conditions on the number of points and lines. We provide a short new proof of Jim Totten's classification of all linear spaces for which the difference between the number of points and lines does not exceed the square root of the number of points. We extend this classification when the number of points is of a certain form. Also in these chapters we have similar classification results for more specialized finite geometrical structures such as (r,1)-designs.

The last chapter is devoted to (k,u)-arcs. A (k,u)-arc in a finite projective plane is a set of k points meeting no line of the plane in more than u points. Elementary bounds upon k can be established and we call an arc with this maximum number of points perfect. An arc not properly contained in any other is called complete. Several constructions are given for both perfect and complete arcs. The major results of this chapter concern the uniqueness of completions of a (k,u)-arc to a perfect arc.

CONTENTS

INTRODUCTION

A *finite linear space* is an incidence structure of finitely many points and lines in which two points determine exactly one line. In chapters 1 and 2 we investigate properties of linear spaces which can be deduced purely from relations upon the number of points, v, and the number of lines, b.

We say that a linear space is *embeddable in a projective plane* if it can be obtained from a projective plane by the deletion of some number of points. Lines may also be deleted if they are left with one or no points.

In chapter 1 (r,1)-designs are considered. An *(r,1)-design* is a linear space in which every point lies on exactly r lines. A theorem of Vanstone states that if $v \geq (r-1)^2$ for an (r,1)-design then it is embeddable in a projective plane of order $r - 1$. Besides providing a short proof of this result, we also prove the following stronger assertion:

Theorem A. An (r,1)-design satisfying $b - v \leq r + 1$, $r \geq 5$, is embeddable in a projective plane of order $r - 1$, for $v > 1$.

In chapter 2 linear spaces in general are considered. Before mentioning the main results we must define some particular examples. Clearly the configuration of all points upon one line satisfies the axioms for a linear space. In general we assume $b > 1$ however. A *near pencil* is a linear space in which there is one line containing $v - 1$ points and $v - 1$ lines each containing two points. An *affine plane of order* n *with a linear space at infinity* is a finite affine plane to which up to $n + 1$ new points have been added. Each new point is associated with a parallel class of the plane, a point being added to all the lines of its class. A structure of lines is imposed upon the new points such that every pair of new points lies on a unique line. In this way we obtain a linear space. For example, a projective plane can be obtained by taking an affine plane and adding at infinity the degenerate space of all points on one line.

A classic theorem of de Bruijn-Erdös states that if a linear space satisfies $b > 1$ then $b \geq v$, with equality if and only if the space is either a projective plane or a near pencil. J. Totten extended this theorem in 1976 by classifying all linear spaces satisfying $b \leq v + \sqrt{v}$. We give a new proof of this result which is considerably shorter than Totten's.

Theorem B (J. Totten). A linear space with $b > 1$ and $b \leq v + \sqrt{v}$, $n^2 \leq v < (n+1)^2$ is one of the following:

1. A near pencil.
2. Embeddable in a projective plane of order n.
3. An affine plane of order n with either a near pencil or projective plane at infinity.
4. Lin's cross, the unique linear space with $v = 6$, $b = 8$, one line each of lengths 3 and 4, and six lines of length 2.

We extend a special case of this theorem as follows:

Theorem C. A linear space in which $v = n^2 + n + 1$ and $b \leq n^2 + (2.147)n$, $b > 1$ is either a near pencil or an affine plane of order n with a (possibly degenerate) linear space at infinity.

In chapter 2 we also briefly consider the extension of these results to λ-designs. A *λ-design* is an incidence structure in which every pair of points lies on exactly λ lines. We find that a λ-design satisfying $b \leq r(r-1)/\lambda + 1$ and $r(r-2)/\lambda + 2 \leq v \leq r(r-1)/\lambda + 1$ in which no two lines meet in $> \lambda$ points is embeddable in a symmetric (v,r,λ)-design, for v sufficiently large. We close with several conjectures involving extensions of Totten's classification.

In chapter 3 we turn our attention to structures in projective planes. A (k,μ)-*arc* in a projective plane is a set of k points such that no line of the plane intersects the set in $> \mu$ points. Barlotti has shown that a (k,μ)-arc in a plane of order n must satisfy $k \leq n\mu - n + \mu$. We call an arc achieving this bound *perfect*. A (k,μ)-arc not properly contained in any (k',μ)-arc is called *complete*. We list various known properties of perfect (k,μ)-arcs and present several constructions for these. We also give a new construction for perfect $(k,2)$-arcs in some translation planes. Our main result of this chapter is the following:

Theorem D. A (k,μ)-arc in a projective plane of order n satisfying $k > n\mu - n + \mu - (n - n/\mu + 1)$ is completable in at most one way to a perfect arc. If $k = n\mu + \mu - 2n + n/\mu - 1$ then there are at most $\mu + 2$ ways to complete it to a perfect arc. Moreover, if more than one way exists then a block design on the parameters

$$b' = \frac{n}{\mu^2}\left(n - \frac{n}{\mu} + 1\right)$$

$$v' = n + 1 - n/\mu$$

$$r' = n/\mu$$

$$k' = \mu$$

$$\lambda' = 1$$

exists.

We give examples of known low order arcs which intersect in this maximum number of points.

In the last sections of this chapter we turn our attention to complete (k,μ)-arcs. We prove

Theorem E. A complete (k,μ)-arc in a plane of order n must satisfy $n \leq \frac{(k-1)(k-2)}{\mu(\mu-1)}$ for $n > \frac{\mu(\mu-1)}{2}$ and $n \leq \frac{(k-\mu+1)(k-\mu)}{\mu(\mu-1)} + \mu - 2$ for $n \leq \frac{\mu(\mu-1)}{2}$.

This extends and improves a theorem of Bruen. Equality in the case $\mu = 2$ implies the existence of a certain partial geometry. We also prove another bound for complete (k,μ)-arcs for which equality holds only for a Baer subplane.

A theorem of Segre states that in a desarguesian plane of order n, n even, a complete $(k,2)$-arc must satisfy $k = n + 2$ or $k \leq n - \sqrt{n} + 1$. We close chapter 3 by constructing low order cases of equality in the second bound using difference sets. We conjecture that our construction provides an infinite family of cases of equality.

Chapter I

(r,1)-designs

Section 1: Introduction.

We begin our investigation of linear spaces in this chapter with the study of (r,1)-designs. We first define several concepts which will be used throughout this and the following chapters.

A *finite linear space* is a finite set of points and a collection of subsets of points, called *lines*, such that every pair of points is contained in exactly one line. We will denote a given linear space by $\mathfrak{F}$, the set of lines in the space. For example

$$\mathfrak{F}_1 = \{\{1,3,5\},\{2,3,4\},\{1,4\},\{1,2\},\{4,5\},\{2,5\}\}$$

is a linear space. In general v will denote the number of points and b the number of lines in a linear space. Obviously a single line containing all points satisfies the linear space requirements, and any number of one point lines can be introduced into a linear space without violating the axioms. To avoid these degeneracies we will assume $b > 1$, and that no line contains fewer than two points, unless specifically noted.

We will use such phrases as "lies on," "passes through," "meets," and so on, in the obvious way, to denote various relationships between points and lines. The following notation will be used in connection with linear spaces. Lines will be denoted by $\ell_1, \ell_2, \ldots, \ell_b$ or in particular instances by ℓ, ℓ', or ℓ''. The *length* of a line is the number of points lying on it, denoted by $k_1, k_2, \ldots, k_b$ or simply k. Points will be variously referred to as x, y, p, or q. The number of lines containing a point is its *degree*, denoted by r_x, r_y, r_p, or r_q.

A linear space is the most general incidence structure which we will consider. We define *(r,1)-designs*, the subject of this chapter, as simply linear spaces in which every point has degree r. That is $r_x = r$ for all x.

A projective plane is a very special example of an (r,1)-design which we will encounter often. We summarize here the results on projective planes that we will use. For more details see [18], pg. 173-188 or [27], pg. 89-95. A *finite projective plane* can be defined as a finite linear space in which every pair of lines meet in exactly one point; and, to avoid degenerate configurations, we also require that there exist four points no three of which are collinear. This non-degeneracy condition is assured if all lines have at least three points and $b > 1$. Hence in the text we will not, when showing that a given linear space is a projective plane, specifically note that this condition holds.

A classic result in combinatorics states that in a finite projective plane every line contains a constant number of points, and every point lies on a constant number of lines. Moreover, these two constants are the same. In other words, calling this constant $n + 1$ (by convention and for convenience in later results), every line contains $n + 1$ points and every point lies on $n + 1$ lines. The number n is referred to as the order of the projective plane. We speak of a *projective plane of order n*. It can further be shown that a projective plane of order n contains $n^2 + n + 1$ points and $n^2 + n + 1$ lines. Thus a projective plane of order n is a linear space with $b = n^2 + n + 1$ and $v = n^2 + n + 1$. In fact, it is an $(n+1,1)$-design since every point lies on $n + 1$ lines. For example

$\mathfrak{F}_2 = \{\{1,3,5\},\{1,2,6\},\{1,4,7\},\{2,3,4\},\{4,5,6\},\{2,5,7\},\{3,6,7\}\}$

is a projective plane of order 2. Note that every pair of lines meet and each point lies on 3 lines. We will find that projective planes will occur frequently in our theorems on both (r,1)-designs and linear spaces. This is because projective planes constitute one of only two classes of linear spaces for which $b = v$. In fact, an equivalent definition of a projective plane is a linear space for which $b = v$ and there exist four points, no three collinear. We will use this alternate definition as well as that given earlier. We will return to this subject in more detail in chapter 2.

We will need the notion of linear space embeddability. We say that $\mathfrak{F}_1$ *is embeddable in* $\mathfrak{F}_2$ if by deleting some number of points from $\mathfrak{F}_2$ (and lines when they are left with 0 or 1 points) we obtain $\mathfrak{F}_1$. For the example spaces above, $\mathfrak{F}_1$ is embeddable in $\mathfrak{F}_2$ since the deletion of points 6 and 7 from $\mathfrak{F}_2$ results in $\mathfrak{F}_1$.

We note that by judiciously deleting points from a projective plane of order n we obtain many $(n+1,1)$-designs. We will find that with some restrictions on v all $(n+1,1)$-designs are obtained in this way.

Before proceeding with the main results of this section we mention a simple fact that will be used frequently in this and the next chapter. If ℓ is a line and $x \notin \ell$ then $r_x \geq |\ell|$, with equality if and only if every line through x meets ℓ. This is because the lines joining x to the points of ℓ must all be distinct. Many of our counting arguments will be based on this simple principle.

Section 2: Embeddings for v restricted.

Our first result on (r,1)-designs is the following, due to Vanstone.

Theorem 1.1. An (r,1)-design for which $v \geq (r-1)^2 - 1$, $r \geq 3$, is embeddable in a projective plane of order $r - 1$. We allow for the possibility of one point lines in this theorem.

The case $v \geq (r-1)^2$ of this theorem was first proved by Vanstone, [41]. We give here our own proof.

Proof. We first mention that what follows holds true even in the presence of one point lines. It is necessary to note this since we will be proceeding inductively, adding a point at a time, and we may introduce one point lines at some stage.

Suppose we have an (r,1)-design with $v \geq (r-1)^2 - 1$. Then no line has length $> r$, since every point has degree exactly r.

Note that if $v = r^2 - r + 1$ then for the r lines through a point, each length r or less, to cover $v = r^2 - r + 1$ points we must have every line through a point of length r. But a line of length r must be met by every other line (since all points have degree r). Thus every pair of lines meets. Hence we have a projective plane of order $r - 1$.

We now assume $(r-1)^2 - 1 = r^2 - 2r \leq v \leq r^2 - r$ and split into two cases.

Case 1: A line of length r exists. Then every line must meet a line of length r. Since every point on that line has degree r we can count the lines meeting it to obtain $b = r(r-1) + 1$. We can now compute the following.

$$\sum_{i=1}^{b} k_i = vr,$$

$$\sum_{i=1}^{b} k_i(k_i - 1) = v(v-1),$$

The first is obtained by counting pairs, (x,ℓ) such that $x \in \ell$, in two different ways. The second comes from counting triples (x,y,ℓ) such that $x \in \ell$ and $y \in \ell$. Thus

$$\sum k_i^2 = v(v-1) + vr.$$

Now first suppose $(r-1)^2 \leq v \leq r^2 - r$. Then we compute

$$\begin{aligned}\sum_{i=1}^{b} (k_i-(r-1))^2 &= \sum_{i=1}^{b} k_i^2 - 2(r-1)\sum_{i=1}^{b} k_i + (r-1)^2 \sum_{i=1}^{b} 1,\\ &= v(v-1) + vr - 2(r-1)vr + (r-1)^2(r^2 - r+1),\\ &= v(v-2r^2+3r-1) + (r-1)^2(r^2-r+1).\end{aligned}$$

This expression achieves its maximum over $(r-1)^2 \leq v \leq r^2 - r$ at either endpoint. Hence

$$\begin{aligned}\sum_{i=1}^{b} (k_i - (r-1))^2 &= (r-1)^2((r-1)^2 - 2r^2 + 3r - 1) + (r-1)^2(r^2 - r+1),\\ &= (r-1)^2 < b.\end{aligned}$$

Thus some line, ℓ, of length $r - 1$ must exist. Every point off of ℓ lies on a unique line missing ℓ (since every point has degree r and ℓ has length $r - 1$). So ℓ and the lines missing ℓ form a *parallel class*, a set of disjoint lines exhausting all points. By counting via the degrees of the points on ℓ we have that ℓ meets $(r-1)(r-1) + 1$ other lines. This leaves $r - 1$ lines disjoint from ℓ. Thus we have a total of r disjoint lines exhausting all points. We can now add a single point to all of these lines (it will thus have degree r), and proceed inductively to eventually obtain $b = v$ and thus a projective plane of order $r - 1$.

Suppose $v = (r-1)^2 - 1$ now. If a line of length $r - 1$ exists then we can proceed as above to create a parallel class and add a point. If no line length $r - 1$ exists then we have

$$0 \leq \sum_{i=1}^{b} (r-k_i)((r-2)-k_i),$$

$$= r(r-2)\sum_{i=1}^{b} 1 - 2(r-1)\sum_{i=1}^{b} k_i + \sum_{i=1}^{b} k_i^2,$$

$$= r(r-2)(r^2-r+1) - 2(r-1)vr + v(v-1) + vr,$$

$$= 0, \quad \text{with} \quad v = (r-1)^2 - 1.$$

Thus every line has length r or $r - 2$. Letting r_1 = # of lines through a point of length r and r_2 = # of lines through a point of length $r - 2$ we have

$$r_1 + r_2 = r,$$

$$(r-1)r_1 + (r-3)r_2 = v - 1 = r^2 - 2r - 1.$$

Thus $r_1 = \frac{1}{2}(r+1)$, $r_2 = \frac{1}{2}(r-1)$. Letting b_1 = # of lines of length r and b_2 = # of lines of length $r - 2$ we then have

$$b_1 + b_2 = b = r^2 - r + 1,$$

$$rb_1 + (r-2)b_2 = \sum_{i=1}^{b} k_i = vr = r(r^2 - 2r).$$

These imply $b_1 = \frac{1}{2}(r-1)(r-2)$, $b_2 = \frac{1}{2}(r^2+r)$. In this case we have precisely the (r,1)-design examined by Bose and Shrikhande in [7].

They prove that an (r,1)-design with $v = r^2 - 2r$ and $b = r^2 - r + 1$ with lines only of size r and $r - 2$, other parameters r_1, r_2, b_1, b_2 as computed above, is embeddable in a projective plane of order $r - 1$ for $r \neq 7$. The case $r = 7$ was disposed of separately by Paul de Witte in [47]. Thus this case is completed.

Case 2: No line of length r exists. Then for r lines through a point to cover $v \geq (r-1)^2 - 1$ points (lines of length $r-1$ or less) we must have either: a) $v = (r-1)^2$ and all lines through a point have length $r - 1$; or b) $v = (r-1)^2 - 1$ and each point lies on $r - 1$ lines length $r - 1$, one line of length $r - 2$.

In the first case we then have all lines of length $r - 1$. Thus

$$b(r-1) = \sum_{i=1}^{b} k_i = vr = r(r-1)^2 .$$

So $b = r^2 - r$. We can now use a line of length $r - 1$ to create a parallel class of lines (as in Case 1). Here, however, the class will only contain $r - 1$ lines, since $b = r^2 - r$ (not $r^2 - r + 1$ as previously). We adjoin a new point to these $r - 1$ lines and also include a singleton line on the new point (so it will have degree r). We can now proceed inductively.

In the second case lines can only have length $r - 1$ or $r - 2$. We count as follows.

$$\# \text{ of lines of length } r - 1 = \frac{v(r-1)}{r-1} = v = r^2 - 2r,$$

since each of v points lies on $(r-1)$ lines of length $r - 1$, while each line length $r - 1$ contains r such points. Similarly

$$\# \text{ of lines of length } r - 2 = \frac{v \cdot 1}{r-2} = r.$$

Thus $b = (r^2 - 2r) + r = r^2 - r$ and we can proceed as in the previous paragraph.

This concludes the proof of Theorem 1.1.

We mention that this theorem can be improved by the use of more complicated methods. In [23] Vanstone and McCarthy prove that an (r,1)-design with $v > (r-1)^2 - \frac{1}{4}((8r-15)^{\frac{1}{2}} - 3)$ is embeddable in a projective plane of order $r - 1$. Steven Dow in [13] has shown that an (r,1)-design satisfying $b = r^2 - r + 1$ and $v > (r-1)^2 - (2(r+2)^{\frac{1}{2}} - 6)$ is embeddable in a projective plane of order $r - 1$.

Section 3: Embeddings for $b - v$ restricted.

It is our aim in this section to establish the following.

Theorem 1.2. An $(r,1)$-design satisfying $b - v \leq r + 1$ and $r \geq 5$ is embeddable in a projective plane of order $r - 1$, for $v > 1$. Here we also allow for one point lines.

Proof. First note that if a line length r exists then all lines must meet it and thus $b = r^2 - r + 1$ as before. Then $b - v \leq r + 1$ implies $v \geq r^2 - 2r = (r-1)^2 - 1$ and we can apply Theorem 1.1 We now assume no line of length r exists.

We show that some line of length $r - 1$ exists. Let k be the maximum length of a line, ℓ. Then we have that

$$\#\text{ of lines meeting } \ell = (r-1)k + 1.$$

We count the number of lines missing ℓ as follows. Through each of the $v - k$ points off of ℓ there are $r - k$ lines missing ℓ. Each of these lines contains at most k such points (since k is the maximum length of a line). Thus

$$\#\text{ of lines missing } \ell \geq \frac{(v-k)(r-k)}{k}.$$

Hence

$$b \geq (r-1)k + 1 + \frac{(v-k)(r-k)}{k}.$$

We now manipulate this expression

$$b - \frac{v(r-k)}{k} \geq (r-1)k + 1 - (r-k),$$

$$\frac{r-k}{k}(b-v) + b\left(1 - \frac{r-k}{k}\right) \geq r(k-1) + 1.$$

Using $b - v \leq r + 1$ and simplifying further gives

$$\frac{r-k}{k}(r+1) + b\left(\frac{2k-r}{k}\right) \geq r(k-1) + 1,$$

$$b \geq \frac{rk^2+2k-r^2+k}{2k-r}$$

provided $2k - r > 0$. We can estimate v easily by examining the lengths of lines through a particular point. This yields

$$v \leq r(k-1) + 1.$$

We can now combine the estimates for b and v and use $b - v \leq r+1$ to obtain

$$r + 1 \geq b - v \geq \frac{rk^2+2k-r^2+k}{2k-r} - r(k-1) - 1.$$

Writing $k = r - \alpha$, and simplifying, eventually yields

$$0 \geq r(r(\alpha-1) - \alpha^2 - 1) + 2\alpha.$$

The assumption $2k - r > 0$ implies $r > 2\alpha$. We are attempting to show $\alpha = 1$, i.e. a line of length $r - 1$ exists. Suppose not, i. e. $\alpha \geq 2$. The above expression is increasing in r for $r > 2\alpha$ provided $2\alpha > (\alpha^2+1)/2(\alpha-1)$, which holds for $\alpha \geq 2$. Thus

$$0 \geq r(r(\alpha-1)-\alpha^2-1) + 2\alpha > 2\alpha(2\alpha(\alpha-1)-\alpha^2-1) + 2\alpha,$$

$$0 > 2\alpha^3 - 4\alpha^2,$$

a contradiction for $\alpha \geq 2$.

Thus we have that a line of length $r - 1$ exists provided $2k - r > 0$. Suppose that $2k - r \leq 0$. Then consider the following.

$$0 \leq \sum_{i=1}^{b} (k-k_i) = bk - vr,$$

$$0 \leq r(b-v) - b(r-k),$$

$$b(r-k) \leq r(b-v) \leq r(r+1),$$

$$b \leq r(r+1)/(r-k).$$

Also from the first expression

$$v \leq bk/r \leq \frac{r(r+1)}{(r-k)} \cdot \frac{k}{r} = \frac{(r+1)k}{r-k} \leq r+1,$$

since $2k - r \leq 0$ implies $(r-k) \geq k$. Thus since $b - v \leq r+1$ we have $b \leq 2r+2$. Suppose that there exist four points no three collinear. Then we can easily count that this set of points meets exactly $4(r-3) + 6 = 4r - 6$ lines. But then $b \leq 2r + 2$ implies $r \leq 4$ contrary to $r \geq 5$. If there do not exist four points no three collinear then either all points are collinear or all points but one are collinear. In either case many one point lines will be present. If all points are collinear then we can easily compute that $b = 1 + v(r-1)$, in which case $b - v \leq r + 1$ implies $v \leq 1$ (for $r \geq 5$). If all points but one are collinear we can compute $b = v + (v-1)(r-2) + (r-(v-1))$. This implies, together with $b - v \leq r + 1$, that $v \leq 1$. So we may assume $2k - r < 0$ and hence a line of length $r - 1$ exists.

We recall the bound on b derived earlier in the proof.

$$b \geq \frac{rk^2+2k-r^2-r}{2k-r}.$$

with $k = r - 1$ this becomes

$$b \geq \frac{r^3-3r^2+2r-2}{r-2} = r^2 - r - 2/(r-2).$$

Thus $b \geq r^2 - r$ for $r \geq 5$.

If $b \geq r^2 - r + 1$ then $b - v \leq r + 1$ implies $v \geq r^2 - 2r = (r-1)^2 - 1$ and we can apply Theorem 1.1.

If $b = r^2 - r$ then $v \geq r^2 - 2r - 1$. We can use a line of length $r - 1$ to create a parallel class (as in the proof of Theorem 1.1) of $r - 1$ lines. A single new point is adjoined to these $r - 1$ lines together with a singleton line on the new point. We then have an $(r,1)$-design with $v = r^2 - 2r$ and we can apply Theorem 1.1. This concludes the proof of Theorem 1.2.

Chapter II

Linear Spaces

Section 1: Introduction.

In this chapter we consider linear spaces in general. For definitions and notation used in this chapter see Chapter 1, Section 1. In addition to what is contained there we will also require several special examples of linear spaces.

A *near pencil* is a linear space in which one line contains $v - 1$ points and the remaining lines each contain two points. Thus $b = v$.

An affine plane is a type of linear space very closely related to projective planes. For more details we refer the reader to [18] pages 173-179. A *finite affine plane* is a finite linear space which satisfies "Playfair's Axiom": Given any line ℓ and any point $x \notin \ell$ there exists exactly one line through x disjoint from ℓ.

It can then be shown that every line has the same number of points, say n, and then also that every point has degree $n + 1$, $b = n^2 + n$, and $v = n^2$. We refer to this as an *affine plane of order* n. For example,

$$\begin{aligned} \mathcal{F}_1 = \{&\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,4,7\}, \\ &\{2,5,8\}, \{3,6,9\}, \{1,6,8\}, \{2,4,9\}, \\ &\{3,5,7\}, \{2,6,7\}, \{3,4,8\}, \ 1,5,9\}\} \end{aligned}$$

is an affine plane of order 3. An affine plane of order n can equivalently be defined as a linear space with $v = n^2$, $b = n^2 + n$, $k_i = n$ for all i, and $r_x = n + 1$ for all x. It is this second definition that we will use most frequently.

Playfair's Axiom gives rise to parallel classes in an affine plane. A *parallel class* is a set of disjoint lines exhausting all points. In an affine plane of order n there are n + 1 parallel classes, no two sharing a line. These parallel classes exhaust all lines. For example, $\mathfrak{F}_1$ above has the following four parallel classes.

$$\{\{1,2,3\},\ \{4,5,6\},\ \{7,8,9\}\},$$
$$\{\{1,4,7\},\ \{2,5,8\},\ \{3,6,9\}\},$$
$$\{\{1,6,8\},\ \{2,4,9\},\ \{3,5,7\}\},$$
$$\{\{2,6,7\},\ \{3,4,8\},\ \{1,5,9\}\}.$$

It should be noted that given a parallel class in a linear space a new space can be obtained by adding a single new point on the lines of the parallel class. In this way v is increased by one and b is left unchanged. This means of extending linear spaces was used in the proofs of Theorems 1.1 and 1.2.

Given an affine plane of order n we can obtain a projective plane of order n by adding n + 1 new points. We add a particular point to all the lines of one parallel class. We then place all n + 1 new points on a single new line. The new points are sometimes referred to as points at infinity, and the new line as a line at infinity. For example, given $\mathfrak{F}_1$ above we adjoin points $\infty_1, \infty_2, \infty_3, \infty_4$ at infinity to obtain

$$\begin{aligned}\mathfrak{F}_2 = \{&\{1,2,3,\infty_1\},\ \{4,5,6,\infty_1\},\ \{7,8,9,\infty_1\},\\ &\{1,4,7,\infty_2\},\ \{2,5,8,\infty_2\},\ \{3,6,9,\infty_2\},\\ &\{1,6,8,\infty_3\},\ \{2,4,9,\infty_3\},\ \{3,5,7,\infty_3\},\\ &\{2,6,7,\infty_4\}\ \ \{3,4,8,\infty_4\},\ \{1,5,9,\infty_4\},\\ &\{\infty_1,\infty_2,\infty_3,\infty_4\}\},\end{aligned}$$

a projective plane of order 3. This correspondence can be reversed, i.e., if $n + 1$ collinear points are removed from a projective plane of order n we obtain an affine plane of order n.

The next class of examples is a generalization of this process. Note that, after the points at infinity have been adjoined to their respective parallel classes (and before the line at infinity has been added), to produce a linear space we need only impose a structure of lines upon the points at infinity which guarantees that any two of these points lie on a unique line. That is, we have an *affine plane of order* n *with a linear space at infinity*. For a projective plane the linear space at infinity is the degenerate configuration of all points upon one line. In general any linear space can be placed upon the points at infinity. Also we need not adjoin a full $n + 1$ points at infinity.

To illustrate this process we adjoin three points to $\mathfrak{F}_1$ above and place a "triangle" at infinity to obtain the following linear space.

$$\begin{aligned}\mathfrak{F}_3 = \{&\{1,2,3,\infty_1\}, \{4,5,6,\infty_1\}, \{7,8,9,\infty_1\},\\ &\{1,4,7,\infty_2\}, \{2,5,8,\infty_2\}, \{3,6,9,\infty_2\},\\ &\{1,6,8,\infty_3\}, \{2,4,9,\infty_3\}, \{3,5,7,\infty_3\},\\ &\{2,6,7\}, \{3,4,8\}, \{1,5,9\},\\ &\{\infty_1,\infty_2\}, \{\infty_1,\infty_3\}, \{\infty_2,\infty_3\}\}.\end{aligned}$$

The results of sections 2 and 3 in this chapter can also be found in [14] and [16].

Section 2: Totten's classification.

The most basic theorem on linear spaces is the following due to deBruijn and Erdös [9],

Theorem 2.1 (deBruijn-Erdös, 1948). A linear space satisfies $b \geq v$ with equality only if the space is either a near pencil or projective plane.

This is the first in a series of theorems in which structural characteristics of a linear space are deduced from arithmetic relations upon b and v.

For an affine plane $(b-v)^2 = v$. It is natural to ask what linear spaces satisfy this relation. As with the deBruijn-Erdös Theorem there is a simple answer. Paul deWitte [46] proved the following:

Theorem 2.2 (deWitte 1967). A linear space satisfying $(b-v)^2 = v$ is either an affine plane of order $\sqrt{v}$ or an affine plane of order $\sqrt{v}$ with a single point at infinity from which one (non-infinite) point has been deleted.

We then ask about the linear spaces falling between the extremes of $b - v = 0$ and $b - v = \sqrt{v}$. In 1976 Jim Totten, [39] and [40], classified all linear spaces satisfying $b \leq v + \sqrt{v}$.

Theorem 2.3 (Totten 1976). A linear space satisfying $b \leq v + \sqrt{v}$ with $n^2 \leq v < (n+1)^2$ is one of the following:

1. A near pencil.
2. Embeddable in a projective plane of order n.
3. An affine plane of order n with either a near pencil or projective plane at infinity.

4. Lin's cross, the unique linear space with $v = 6$, $b = 8$, one line of length 4, one of length 3, and six of length 2.

It should be noted in regard to the second category that many linear spaces which are embeddable in a projective plane of order n do not satisfy $b \leq v + \sqrt{v}$. To have $b \leq v + \sqrt{v}$ we may delete no more than $n + 1$ points from a projective plane of order n. Up to n points may be deleted without regard to their position (if n are deleted on a line then the line is eliminated also, since it now contains only one point). We may remove $n + 1$ points provided they are either all on a line or all but one are on a line (which is then eliminated as above).

In this section we give a new proof of this classification which is much shorter than Totten's. Our approach is greatly simplified by the use of linear algebra in Lemma 2.1. We first bring together several lemmas which will be used throughout this section.

We henceforth assume throughout this section that $\mathfrak{J}$ is a linear space satisfying $b \leq v + \sqrt{v}$ and $n^2 \leq v < (n+1)^2$. Clearly then $b - v \leq n$.

<u>Lemma 2.1</u>. Let $\mathfrak{J}$ have $r_x \geq n + 1$ for all x and $k_j \leq n + 1$ for all j. Then every point x of degree $> n + 1$ lies on some line ℓ of length $n + 1$ with $r_p = n + 1$ for all $p \in \ell \backslash x$. Such a line will be called a *special* line through x.

<u>Proof</u>. We use a technique suggested to the author by R. M. Wilson (see [14] and [45]). See [17] or [22] for the matrix theory we shall employ.

Let N be the $v \times b$ incidence matrix of $\mathfrak{J}$. Then, indexing over points x,

$$NN^T = \mathrm{diag}[r_x - 1] + J = \Delta + J.$$

It is easily verified that

$$P = N^T(NN^T)^{-1}N$$

is the orthogonal projection from $\mathbb{R}^b$ onto the row space of N. By computing we also have

$$(NN^T)^{-1} = (\Delta + J)^{-1} = \Delta^{-1} - c\Delta^{-1}J\Delta^{-1},$$

where

$$c = (1 + \sum_{\text{all } x} (r_x - 1)^{-1})^{-1}.$$

So P is given by

$$P = N^T(\Delta^{-1} - c\Delta^{-1}J\Delta^{-1})N,$$

$$= N^T\Delta^{-1}N - c(\Delta^{-1}N)^T J(\Delta^{-1}N).$$

Using the notation $\alpha_S = \sum_{x \in S} \frac{1}{r_x - 1}$ for subsets S, of points, the above expression becomes

$$P = [\alpha_{\ell_i \cap \ell_j}] - c[\alpha_{\ell_i}\alpha_{\ell_j}],$$

indexing over lines ℓ_i, ℓ_j. The projection onto (row space $(N))^{\perp}$ is $Q = I - P$. Thus $\text{rank}(Q) = b - v$ and

$$Q = I - [\alpha_{\ell_i \cap \ell_j}] + c[\alpha_{\ell_i}\alpha_{\ell_j}].$$

Consider now any point x and let Q_0 be the principal submatrix of Q corresponding to the lines through x. Then on this principal submatrix

$$[\alpha_{\ell_i \cap \ell_j}] = \text{diag}[\alpha_{\ell_i} - \frac{1}{r_x - 1}] + \frac{1}{r_x - 1} J.$$

So

$$Q_0 = \text{diag}[1 - \sum_{p \in \ell_i \setminus x} \frac{1}{r_p - 1}] - \frac{1}{r_x - 1} J + c[\alpha_{\ell_i}\alpha_{\ell_j}].$$

Note that since $r_x \geq n + 1$ for all x and $k_j \leq n + 1$ for all j

we have

$$1 - \sum_{p \in \ell_i \setminus x} \frac{1}{r_p - 1} \geq 0,$$

with equality if and only if ℓ_i is special through x.

Now suppose $r_x > n + 1$ but no special lines through x exist. Then strict inequality holds above so that

$$\mathrm{diag}\left[1 - \sum_{p \in \ell_i \setminus x} \frac{1}{r_p - 1}\right]$$

is positive definite.

Then adding the positive semi-definite matrix $c[\alpha_{\ell_i} \alpha_{\ell_j}]$ still gives a positive definite matrix, hence of full rank, r_x. Subtracting the rank 1 matrix $(r_x - 1)^{-1}J$ reduces the rank by at most one. Thus $r_x - 1 \leq \mathrm{rank}(Q_0) \leq \mathrm{rank}(Q) = b - v \leq n$, contrary to $r_x > n + 1$.

Lemma 2.2. For $n \geq 3$, no lines of length $> n + 1$ exist unless $\mathfrak{F}$ is a near pencil.

Proof. We use the following result from [32]: If ℓ is a line of length k and M is the number of lines meeting ℓ (excluding ℓ) then we have

$$M \geq \frac{k^2(v-k)}{v-1}.$$

This can be proved as follows. Let $\sum$ denote the sum over lines which meet ℓ. Then

$$\sum 1 = M,$$

$$\sum (k_i - 1) = k(v - k),$$

$$\sum \binom{k_i - 1}{2} \leq \binom{v - k}{2}.$$

The second and third relations are obtained by counting the number of triples (x,y,ℓ'), $x \in \ell'$ and $y \in \ell'$, with $x \in \ell$ and $y \notin \ell$ in the first case and $x \notin \ell$ and $y \notin \ell$ in the second.

We can now estimate the variance,

$$0 \leq \sum k_i^2 - \frac{(\Sigma k_i)^2}{M} .$$

Plugging in and simplifying yields

$$M \geq \frac{k^2(v-k)}{v-1} .$$

Now, in particular, the existence of a line of length k implies

$$b \geq 1 + \frac{k^2(v-k)}{v-1} = 1 + k^2 - \frac{k^2(k-1)}{v-1} .$$

Note that this expression is monotonically increasing in k for $0 \leq k \leq \frac{2}{3}v$. Let ℓ be a line of maximum length k. We consider two cases.

Case 1: $k \leq \frac{2}{3}v$. Then if $k \geq n+2$ we have

$$b \geq 1 + (n+2)^2 - \frac{(n+2)^2(n+1)}{v-1} .$$

Also $b \leq v + n$ so that by combining and simplifying

$$v^2 - v(n^2+3n+6) + (n^3+6n^2+11n+9) \geq 0 .$$

This expression achieves its maximum on $n^2 \leq v \leq n^2 + 2n$ at $v = n^2 + 2n$ provided

$$(n^2+2n) - (\frac{n^2}{2} + \frac{3n}{2} + 3) \geq (\frac{n^2}{2} + \frac{3n}{2} + 3) - n^2 .$$

This holds if and only if $n^2 - n - 6 \geq 0$, i.e. $n \geq 3$. So, for $n \geq 3$, we have

$$(n^2+2n)^2 - (n^2+2n)(n^2+3n+6) + (n^3+6n^2+11n+9) \geq 0 .$$

This simplifies to $-2n^2 - n + 9 \geq 0$, a contradiction for $n \geq 3$.

Case 2: $k > \frac{2}{3}v$. If two points lie off ℓ then the line containing them and ℓ together meet $\geq 2(k-1)$ other lines. Thus

$$b \geq 2(k-1) + 2 > \frac{4}{3}v .$$

By using $b - v \leq n$ we then have $n > \frac{v}{3} \geq n^2/3$, a contradiction for $n \geq 3$. So at most one point of $\mathfrak{J}$ lies off ℓ. Since $b \geq 2$, $\mathfrak{J}$ must be a near pencil.

The cases $n = 1, 2$ can easily be examined by hand. The only exceptional case found is Lin's cross for $n = 2$ (a line of length $4 = n + 2$ exists).

Henceforth we assume that no line has length $> n + 1$.

Lemma 2.3. Every pair of lines of length $n + 1$ must meet.

Proof. Suppose ℓ_1 and ℓ_2 have length $n + 1$ and do not meet. Then they are both met by $(n+1)^2$ lines. Including ℓ_1 and ℓ_2 gives $b \geq n^2 + 2n + 3$. So since $b - v \leq n$, $v \geq n^2 + n + 3$. Thus since the maximum length of a line is $n + 1$, $r_x > n + 1$ for all x, contradicting Lemma 2.1.

Before proceeding further we dispose of the cases $v = n^2 + 1$ and $v = n^2$.

Lemma 2.4. If $v = n^2$ or $n^2 + 1$ then $\mathfrak{J}$ is embeddable in a projective plane of order n.

Proof. Note first that, in either case, if there are no points of degree $< n + 1$ we are done since: 1) If no points of degree $> n + 1$ exist we are done by Theorem 1.1; 2) If a point of degree $> n + 1$ exists then it lies on a special line, which must (by the degrees of the points on it) meet $\geq n^2 + n + 2$ lines (including itself), contrary to $b - v \leq n$ and $v \leq n^2 + 1$.

Thus we may assume some point x of degree $< n + 1$ exists. Then for the lines through x (necessarily of length $\leq n + 1$) to cover $v = n^2 + 1$ or n^2 points we must have $r_x = n$ and: (a) If $v = n^2 + 1$ then x lies on n lines of length $n + 1$; (b) If $v = n^2$ then x lies on $n - 1$ lines of length $n + 1$ and one line of length n. In either case all lines of length $n + 1$ must pass through x (otherwise $r_x \geq n + 1$). Hence x is the only point of degree $< n + 1$. We now split into two cases.

Case 1: $v = n^2 + 1$. Suppose some point y exists of degree $> n + 1$. Then the line, ℓ, joining x and y has length $n + 1$. It thus meets $\geq n^2 + n + 1$ other lines (including itself). But $b - v \leq n$, hence $b = n^2 + n + 1$ and all lines meet ℓ. So all points other than x and y have degree $n + 1$. Now pick some line $\ell' \neq \ell$ with $x \in \ell'$. Then $y \notin \ell'$ and ℓ' has length $n + 1$. Thus there is a line through y missing ℓ' (since $r_y > n + 1$), all of whose points then have degree $> n + 1$, contrary to x and y being the only points of degree $\neq n + 1$. Thus x is the only point not of degree $n + 1$. We can then add a singleton line on x and apply Theorem 1.1.

Case 2: $v = n^2$. Suppose y is any point of degree $> n + 1$. Then if the line joining x and y were of length $n + 1$ it would meet $\geq n^2 + n + 1$ lines (as in the previous case), contradicting $b - v \leq n$. So all points of degree $> n + 1$ lie on the unique line of length n through x. All points off this line have degree $n + 1$ and hence lie on a unique line missing this line of length n. Thus a parallel class of lines is created (as in the proof of Theorem 1.1) and a new point can be added giving $v = n^2 + 1$. Now apply the previous case.

We now assume $v \geq n^2 + 2$ and complete the proof of Theorem 2.3. Since the maximum length of a line is $n + 1$ and $v \geq n^2 + 2$ we have $r_x \geq n + 1$ for all x. So Lemma 2.1 applies. If no points of degree $> n + 1$ exist we are done by Theorem 1.1.

Henceforth we assume some point of degree $> n + 1$ exists. We will call such points *ideal points*. A line which misses some line of length $n + 1$ will be called an *ideal line*. Note that every point on an ideal line is ideal. We will find that ideal points are the points of the "spaces at infinity" in the statement of the theorem.

If there is a unique ideal point, say x, then it must lie on all lines of length $n + 1$ (otherwise $r_x > n + 1$ will give a line through x, missing the line of length $n + 1$, all of whose points will then be ideal). Thus $\mathfrak{F}$ with x deleted has $r_p = n + 1$ for all p and no lines of length $> n$, $v \geq n^2$, i.e., an affine plane. So $\mathfrak{F}$ originally had $v = n^2 + 1$ points, contrary to $v \geq n^2 + 2$. So there are at least two ideal points. By choosing a special line through one we have the existence of an ideal line through the other. Thus ideal lines exist. Also every ideal point lies on at least one ideal line.

Let ℓ_1 be an ideal line of maximal length and x of maximal degree on ℓ_1. Let ℓ_2 be a special line through x. Pick $y \in \ell_1 \backslash x$ of minimal degree, say $r_y = n + 1 + z$, $z \geq 1$.

We now count the lines meeting ℓ_1 or ℓ_2 by counting, respectively, the lines meeting $\ell_2 \backslash x$, the lines meeting x, and the lines meeting $\ell_1 \backslash x$ and missing ℓ_2. This gives

$$b \geq n^2 + r_x + (k_1 - 1)z.$$

We now estimate r_x by choosing ℓ_3 to be an ideal line of maximal length $\neq \ell_1$ through y (such a line exists since $r_y > n + 1$ and $k_2 = n + 1$) and ℓ_4 to be a special line meeting $\ell_3 \setminus y$. Then $\ell_2 \cap \ell_4 \neq \emptyset$, by Lemma 2.3, and $\ell_3 \cap \ell_2 = \emptyset$, $\ell_1 \cap \ell_4 = \emptyset$. We count the number of lines which meet ℓ_2 but not ℓ_4. Since points of $\ell_2 \setminus x$ have degree $n + 1$ and ℓ_4 has length $n + 1$, this number will be $r_x - (n+1)$. On the other hand, any point of $\ell_3 \setminus \ell_4$ has at least $k_2 - k_4 + 1$ lines through it missing ℓ_4 and meeting ℓ_2 (since it lies on ℓ_3 which meets ℓ_4 and misses ℓ_2). Thus

$$r_x - (n+1) \geq (k_2 - k_4 + 1)(k_3 - 1) = k_3 - 1.$$

So

$$r_x \geq k_3 + n.$$

Hence

$$b \geq n^2 + n + k_3 + (k_1 - 1)z.$$

We now count v by using the lines through y. The lines from y to $\ell_2 \setminus x$ have length at most $n + 1$. The other ideal lines through y, besides ℓ_1, have length at most k_3. Thus

$$v - 1 \leq n^2 + (k_1 - 1) + z(k_3 - 1).$$

Now $b - v \leq n$, so combining the two previous estimates gives, with some simplification,

$$(k_1 - k_3)(z - 1) \leq 0.$$

But ℓ_1 was chosen maximal ideal and $z \geq 1$. Thus we must have equality and either $z = 1$ or $k_1 = k_3$.

Equality implies equality in all previous estimates.

Thus:

1. All points of $\ell_1\backslash x$ have the same degree, $r_y = n + 1 + z$.
2. Every ideal line $\neq \ell_1$ through y has the same length, k_3.
3. $r_x = k_3 + n$.
4. All lines joining y to $\ell_2\backslash x$ have length $n + 1$.

Since all points of $\ell_1\backslash x$ have the same degree they are interchangeable with y in the above argument. Thus we also have:

5. Any line joining a point of $\ell_1\backslash x$ to a point of $\ell_2\backslash x$ has length $n + 1$.
6. All ideal lines through some $q \in \ell_1\backslash\{x,y\}$ have the same length, k_q (as k_3 for y). By repeating the above argument on q, $r_x = k_q + n = k_3 + n$. Hence every ideal line meeting ℓ_1 in some point other than x has length k_3.
7. All lines meet ℓ_1 or ℓ_2. Thus, since ℓ_2 is special, all ideal lines meet ℓ_1.
8. $b = n^2 + n + k_3 + z(k_1 - 1)$,

   $v = n^2 + k_1 + z(k_3 - 1)$.

Before proceeding with the cases $z = 1$ and $k_1 = k_3$ we establish the following: A line of length $n + 1$ contains at most one ideal point. To see this, let x' and y' be ideal on ℓ of length $n + 1$. Let ℓ' be special through y' and ℓ'' ideal through x', missing ℓ'. Then every line meeting ℓ' meets ℓ (since points of $\ell'\backslash y'$ have degree $n + 1$). On the other hand through every point of $\ell''\backslash x'$ there is at least one line meeting ℓ' and missing ℓ, implying ℓ'' contains fewer than 2 points.

<u>Case 1</u>: $z = 1$. Then every point of $\ell_1\backslash x$ has degree $n + 2$. Lines joining some $q \in \ell_1\backslash\{x,y\}$ to $\ell_3\backslash y$ cannot meet $\ell_2\backslash x$ (those have

length $n + 1$ and hence contain at most one ideal point, q). Thus $r_q \geq n + k_3$. Hence $k_3 = 2$ (if no such q exists then $k_1 = 2$, giving $k_3 = 2$ by maximality of k_1). Let $p = \ell_3\backslash y$. Note that $r_x = k_3 + n = n + 2$ so that x is interchangeable with any point on ℓ_1. Every ideal line meets ℓ_1 and every line meeting ℓ_1 has length $k_3 = 2$. Thus the structure of ideal points and ideal lines is that of a near pencil with vertex p. Now

$$b = (k_1 - 1)z + n^2 + n + k_3 = n^2 + n + (k_1 + 1),$$

$$v = (k_1 + 1) + n^2.$$

Thus if the near pencil is deleted we obtain an affine plane (i.e., $b = n^2 + n$, $v = n^2$, all points of degree $n + 1$).

<u>Case 2</u>: $k_1 = k_3$. Then every ideal line has length k_3 (since all meet ℓ_1). As in Case 1, the lines joining some $q \in \ell_1\backslash\{x,y\}$ to $\ell_3\backslash y$ cannot meet $\ell_2\backslash x$. Thus $r_q \geq n + k_3$. But x, chosen to be of maximal degree on ℓ_1, has $r_x = n + k_3$. Thus all points of ℓ_1 have degree $n + k_3$. Also the line joining q to any other ideal point not on ℓ_1 misses ℓ_2 and is thus interchangeable with x; all ideal lines have the same length and are thus interchangeable with ℓ_1. Thus every ideal line has length k_1 and every ideal point has degree $n + k_1$. The line joining any pair of ideal points is ideal. So the ideal points and lines form a projective plane of order $k_1 - 1$. We have

$$b = n^2 + n + (k_1 - 1)^2 + (k_1 - 1) + 1,$$

$$v = n^2 + (k_1 - 1)^2 + (k_1 - 1) + 1.$$

So the deletion of the ideal points and lines leaves an affine plane as before. This concludes the proof of Theorem 2.3.

Section 3: Further classification.

In this section we extend the classification of section 2 in the particular case of $v = n^2 + n + 1$. We prove

Theorem 2.4. A linear space satisfying $v = n^2 + n + 1$ and $b \leq n^2 + (2+c)n$ is either a projective plane of order n, an affine plane of order n with a linear space at infinity, or a near pencil, where c can be taken as .147899.

We first establish several preliminary lemmas. Henceforth let $\mathfrak{J}$ be a linear space with $v = n^2 + n + 1$ and $b \leq n^2 + (2+c)n$ for some $c < \frac{1}{2}$.

Lemma 2.5. No line of $\mathfrak{J}$ has length $> n + 1$ unless $\mathfrak{J}$ is a near pencil.

Proof. From the proof of Lemma 2.2 we have that the existence of a line of length k implies

$$b \geq 1 + \frac{k^2(v-k)}{v-1}$$

we proceed as in Lemma 2.2. Let ℓ be a line of maximal length, k. Suppose $k \geq n + 2$. We consider two cases.

Case 1: $k \leq \frac{2}{3}v$. Then since $k^2(v-k)$ is increasing for $0 \leq \ell \leq \frac{2}{3}v$ we have

$$b \geq \frac{(n+2)^2(n^2+n+1-(n+2))}{n^2+n+1-1} = n^2 + 3n + 1 - 4/n$$

a contradiction to the range of b for $n \geq 2$.

Case 2: $k > \frac{2}{3}v$. If there exist two points off of ℓ then the line through them and ℓ meet at least $2(k-2)$ other lines. Thus

$$b \geq (k-1)2 + 2 > \frac{4}{3}v = \frac{4}{3}n^2 + \frac{4}{3}n + 4/3,$$

a contradiction to the bound on b for $n \geq 1$. Thus at most one point exists off of ℓ and we have that $\mathfrak{F}$ is a near pencil, since $b > 1$.

In view of Lemma 2.5 we may assume that no line has length $> n + 1$. Using this and $v = n^2 + n + 1$ we have that every point has degree $\geq n + 1$, since fewer than $n + 1$ lines through a point (each of length $\leq n + 1$) could not cover $v = n^2 + n + 1$ points. We also have the useful fact that a point has degree $n + 1$ if and only if it lies only on lines of length $n + 1$.

Lemma 2.6. Some point of degree $n + 1$ exists.

Proof. Suppose not, then $r_x \geq n + 2$ for all x. Note that a block, ℓ, of length $n + 1$ exists since otherwise

$$bn \geq \sum_{i=1}^{b} k_i = \sum_{x} r_x \geq (n+2)v = (n+2)(n^2+n+1)$$

implying $b \geq n^2 + 3n + 2$, a contradiction. Now with $r_x \geq n + 2$ for all x we have that ℓ meets at least $(n+1)(n+1)$ other lines. Additionally any point not on ℓ lies on at least one line missing ℓ. Each such line has length at most $n + 1$. Hence at least $n^2/(n+1) = n - n/(n+1)$ lines which miss ℓ. By this $b \geq (n^2+2n+1) + n$, a contradiction.

Lemma 2.7. Every two lines of length $n + 1$ meet.

Proof. Suppose ℓ_1 and ℓ_2 have length $n + 1$ and do not meet. Then any $x \in \ell_1$ has degree $\geq n + 2$, since there are $n + 1$ lines joining x to the points of ℓ_2 and ℓ_1 is disjoint from ℓ_2. Thus through every $x \in \ell_1$ there is a line $\ell(x)$ with $x \in \ell(x)$ and $|\ell(x)| < n+1$. Every point on an $\ell(x)$ has degree $\geq n + 2$, since $\ell(x)$ has length $< n + 1$, so there are at least $|\ell(x)| - 1$ lines meeting $\ell(x)$ but missing ℓ_1. We consider two cases

Case 1: $|\ell(x)| > n/2$ for some $x \in \ell_1$. Then note that ℓ_1 and ℓ_2 together meet $(n+1)^2$ lines. So

$$b \geq (n+1)^2 + (|\ell(x)| - 1) > n^2 + 5/2\ n$$

contradicting b.

Case 2: $|\ell(x)| \leq n/2$ for every $x \in \ell_1$. Then for the lines through each such x to cover $v = n^2 + n + 1$ points we must have $r_x \geq n + 3$ for all $x \in \ell_1$. Then, by counting the lines meeting ℓ_1, we have

$$b \geq (n+1)(n+2) > n^2 + 3n,$$

a contradiction.

Note that if all lines have length $n + 1$ then every pair of lines meet. Thus $\mathfrak{F}$ is a projective plane. We henceforth assume that some line of length $< n + 1$ exists. Lines of length $n + 1$ will be called *long*. Lines of length $< n + 1$ will be called *short*.

We prove Theorem 2.4 by first showing that the number of lines of length $n + 1$ is $\geq n^2 + 1$ and then showing that this implies $\mathfrak{F}$ is an affine plane of order n with a linear space at infinity.

Let $L = \#$ of lines of length $n + 1$ and let the longest line not of length $n + 1$ be $\hat{\ell}$, of length αn, $0 < \alpha \leq 1$. Thus every line has length $n + 1$ or $\leq \alpha n$. By counting triples (x,y,ℓ) with $x \in \ell$, $y \in \ell$, $x \neq y$ we have

$$(b - L)\alpha n(\alpha n - 1) + Ln(n+1) \geq \sum_{i=1}^{b} k_i = v(v-1).$$

Using $v = n^2 + n + 1$, $b \leq n^2 + (2+c)n$ and simplifying we have, assuming $\alpha < 1$,

$$L \geq n^2 + n\left(\frac{1-2\alpha^2-c\alpha^2}{1-\alpha^2}\right) + \frac{(1+\alpha c)n+1}{n(1-\alpha^2)+1+\alpha}\ .$$

So $L \geq n^2 + 1$ for $1 - 2\alpha^2 - c\alpha^2 \geq 0$, equivalently $\alpha \leq (2+c)^{-\frac{1}{2}}$. We now take care of larger α.

Let x be a point of degree $n + 1$. Then $x \notin \hat{\ell}$. Since $\hat{\ell}$ is short there exists a line through x (necessarily of length $n + 1$) missing $\hat{\ell}$. Call this line ℓ, and the lines through x meeting $\hat{\ell}$ by $\ell_1, \ell_2, \ldots, \ell_{\alpha n}$. Consider now ℓ_1 and ℓ. Together both meet $(n+1-1)(n+1-1) + r_x = n^2 + n + 1$ lines. Through each point $y \in \hat{\ell}\backslash\ell_1$ there is at least one line meeting ℓ and missing ℓ_1 (i.e., at least one of the $n + 1$ lines from y to ℓ must miss ℓ_1, since $\hat{\ell}$ through y meets ℓ_1 and misses ℓ). Thus there are at least $\alpha n - 1$ lines meeting ℓ and missing ℓ_1. Similarly if ℓ^* is a line meeting ℓ and missing ℓ_1 there are at least $|\ell^*| - 1$ lines meeting ℓ_1 and missing ℓ. Adding these up gives

$$b \geq (n^2+n+1) + (\alpha n-1) + (|\ell^*|-1).$$

Hence $|\ell^*| \leq (1+c-\alpha)n + 1$. Thus any line meeting ℓ but missing ℓ_1 has length $\leq (1+c-\alpha)n + 1$. This same argument holds for any ℓ_i, $i = 1, \ldots, \alpha n$. Now suppose ℓ' is any line meeting ℓ. If ℓ' misses some ℓ_i, $i = 1, 2, \ldots, \alpha n$ then $|\ell'| \leq (1+c-\alpha)n + 1$ by the above. If ℓ' meets every $\ell_1, \ell_2, \ldots, \ell_{\alpha n}$ (in addition to ℓ) then $|\ell'| \geq \alpha n + 1$. Hence $|\ell'| = n + 1$, by maximality of $\hat{\ell}$.

Thus we have shown that every line meeting ℓ has length $n + 1$ or $\leq (1+c-\alpha)n + 1$. Let q be any point on ℓ and N_q = # of lines of length $n + 1$ through q other than ℓ. Then since the lines through

q must cover all $v = n^2 + n + 1$ points we have

$$\begin{pmatrix} \#\text{ of lines through } q \\ \text{of length} \le (1+c-\alpha)n+1 \end{pmatrix} \ge \frac{n^2 - nN_q}{(1+c-\alpha)n+1-1} = \frac{n - Nq}{(1+c-\alpha)} .$$

So

$$r_q - 1 \ge N_q + \frac{n - N_q}{(1+c-\alpha)} .$$

summing over $q \in \ell$ then gives

$$b - 1 \ge \sum_{q\in\ell} (r_q - 1) \ge \sum_{q\in\ell} N_q(1 - \frac{1}{1+c-\alpha}) + \frac{n}{1+c-\alpha} \sum_{q\in\ell} 1.$$

Now $\sum_{q\in\ell} N_q = L - 1$ thus

$$b - 1 \ge (L - 1)(\frac{c - \alpha}{1+c-\alpha}) + \frac{n(n+1)}{1+c-\alpha}.$$

Using $b \le n^2 + (2+c)n$ and solving for L gives

$$L \ge n^2 + n(\frac{1-(2+c)(1+c-\alpha)}{\alpha - c}) + 1/(\alpha - c).$$

Thus $L \ge n^2 + 1$ for $1 + c - \alpha \le 1/(2+c)$, i.e., $\alpha \ge (c^2 + 3c + 1)/(c+2)$. Previously $L \ge n^2 + 1$ for $\alpha \le \sqrt{\frac{1}{2+c}}$. We need only choose c so that these two ranges overlap. We can take any c such that

$$\frac{c^2+3c+1}{c+2} \le \sqrt{\frac{1}{2+c}} .$$

Equivalently, $0 \ge c^4 + 6c^3 + 11c^2 + 5c - 1$. To within six decimal places we take $c = .147899$.

We now complete the proof of Theorem 2.4 by showing that $L \ge n^2 + 1$ implies that $\mathfrak{F}$ is an affine plane with a linear space at infinity. We use Theorem 1.1, but in a dual form, that is, interchanging points and lines. The form we require reads (letting $r = n + 1$).

Theorem 1.1'. Let $S_1, \ldots, S_m$ be subsets of size $n+1$ of some set S, $|S| = n^2+n+1$, such that every pair S_i, S_j, $i \neq j$ meet in a unique point. Then if $m \geq n^2-1$ we can find subsets of S, R_1, $R_2, \ldots, R_{n^2+n+1-m}$ of size $n+1$ such that $\{S_1, S_2, \ldots, S_m, R_1, \ldots, R_{n^2+n+1-m}\}$ is a projective plane of order n.

To prove the above we simply note that with the roles of points and lines reversed, points being thought of as "containing" the lines with which they are incident, we have an $(n+1,1)$-design on $v = m \geq (n+1-1)^2-1$ "points" and $b = n^2+n+1$ "lines." We then apply Theorem 1.1 to this dual structure to obtain "points" which can be adjoined to form a projective plane. These "points" are the required sets $R_1, \ldots, R_{n^2+n+1-m}$.

To apply this to our structure, let $\ell_1, \ell_2, \ldots, \ell_{n^2+t}$ be the lines of length $n+1$ of $\mathfrak{F}$. Note that $t \leq n+1$. If $t = n+1$ we already have projective plane. Thus we may assume $1 \leq t \leq n$. The short lines of $\mathfrak{F}$ are ℓ_i, $i > n^2+t$.

By Lemma 2.7 every pair ℓ_i, ℓ_j, $1 \leq i < j \leq n^2+t$, must meet. Additionally they are size $n+1$ subsets of a set of $v = n^2+n+1$ points. By applying Theorem 1.1' we then have the existence of sets $R_1, R_2, \ldots, R_{n+1-t}$ such that $\Pi = \{\ell_1, \ell_2, \ldots, \ell_{n^2+t}, R_1, R_2, \ldots, R_{n+1-t}\}$ is a projective plane of order n.

Now consider any R_i. Because $|R_i| = n+1$ and $n+1-t < n+1$ there is some point x in R_i which lies on no other R_j, $j \neq i$. Consider any short line containing x and let y be any point on that short line. The pair $\{x,y\}$ must be covered by some line of the projective plane Π. But $\{x,y\}$ is on a short line of $\mathfrak{J}$ already so that $\{x,y\} \subsetneq \ell_j$ for $j \leq n^2 + t$. Also, by assumption, x lies on no R_j for $j \neq i$. Thus $\{x,y\} \subseteq R_i$. We have shown that any short line on x must be contained in R_i.

Note that since no point x can lie on exactly one short line of $\mathfrak{J}$ we have that every R_i, $1 \leq i \leq n + 1 - t$ contains at least two short lines.

The above argument also shows that if $t = n$, and hence R_1 is the only line added to $\ell_1, \ldots, \ell_{n^2+t}$ to produce Π, then all short lines are contained in R_1. With the removal of the points of R_1, and hence the short lines of $\mathfrak{J}$, we are left with a linear space for which $v = n^2$, $b = n^2 + n$, all lines have size n (since Π is a projective plane $\ell_1, \ldots, \ell_{n^2+t}$ intersected R_1 in a single point), and all points have degree $n + 1$. That is $\mathfrak{J}$ is an affine plane of order n with a linear space at infinity, the space at infinity being the short lines within R_1.

Thus it suffices to show that $t = n$. We proceed as follows. Note that every point in an R_i must be on some short line (since the pairs of points covered by short lines in $\mathfrak{J}$ must be covered by the lines R_i in Π, and vice versa). The short lines intersecting any R_i in at least two points induce a linear space structure on the $n + 1$ points of the R_i. Thus we have, by the deBruijn-Erdös Theorem (Theorem 2.1.) that there are at least $n + 1$ short lines meeting any given R_i in at least two points.

Now consider R_1. We have previously established that each of the remaining $n - t$ R_i contain ≥ 2 short lines. Together with the $n + 1$ short lines meeting R_1 (these will be distinct from the others since they meet in at least two points) and the long lines we have

$$b \geq (n^2 + t) + (n+1) + 2(n-t) = n^2 + 3n - t,$$

a contradiction for $t = 1$. We now suppose $2 \leq t \leq n - 1$.

Consider a particular R_i. It contains at least $(n+1) - (n-t) = t + 1$ points not on any other R_j, $j \neq i$. Let the set of such points within R_i be C_i. Thus $|C_i| \geq t + 1$. As before any short line on a point of C_i is contained entirely within R_i. We consider two cases.

Case 1: The points of C_i are covered by a single short line. Then, since every pair of R_j meet and $t < n$, there is $y \in R_i \backslash C_i$. The line joining each $x \in C_i$ to y must be short and hence is contained in R_i. Thus we obtain at least $|C_i|$ short lines contained in R_i.

Case 2: The points of C_i are covered by more than one short line. Then we can apply the deBruijn-Erdös theorem to the linear space induced by short line intersections with the points of C_i to obtain at least $|C_i|$ short lines meeting C_i. Thus there are at least $|C_i|$ short lines contained in R_i.

In either case we have at least $|C_i| \geq t + 1$ short lines contained in each R_i, $1 \leq i \leq n + 1 - t$. Including long lines we then have

$$b \geq n^2 + t + (n+1-t)(t+1) \geq n^2 + 3n - 1$$

for $2 \leq t \leq n - 1$, a contradiction. Thus $t = n$ and $\mathfrak{F}$ is an affine plane of order n with a linear space at infinity. This completes the proof of Theorem 2.4.

Section 4: Extensions to λ-spaces.

Here we consider extensions of our results on linear spaces to more general incidence structures. We give here the essential definitions and results necessary for what follows. The reader is referred to [27], pp. 96-122, [18], pp. 100-120, [11], [20], and [24] for more details.

A λ-*space* is a set of points and a collection of subsets of points called *blocks* such that every pair of points lies on exactly λ blocks. These have been variously called in the literature λ-linked designs [48], [49], pairwise balanced designs of index λ [43], and $B[K,\lambda;v]$, [20]. They are dual structures to λ-designs, [28] and [31]. We adopt the term λ-space because of its similarity to linear space and its brevity. Thus a linear space is a 1-space.

As before we will let $b = \#$ of blocks and $v = \#$ of points. Points will be denoted by x, y, p, q, etc. while blocks will be $B_1, B_2, \ldots, B_b$. Block sizes will be k_i and point degrees r_x as before. We assume that no block contains all points.

All of our previous structures have generalizations in this setting. An (r,λ)-design is a λ-space in which all points have degree r. These generalize the $(r,1)$-designs of Chapter 1. Projective planes have their counterpart as well. A (v,k,λ)-*design*, also called *symmetric* (v,k,λ)-*design* is a (k,λ)-design in which every block contains k points. We define λ-space embeddability in exactly the same way as for linear spaces.

The study of λ-spaces is complicated by the fact that we have no counterpart of the deBruijn-Erdös theorem case of equality. It can be shown that $b \geq v$ for a λ-space but no characterization of the cases of equality exists. At present only two classes of equality are known (it is conjectured that they are the only cases of equality). These are the (v,k,λ)-designs, mentioned earlier, and a class of λ-spaces obtained from (v,k,λ)-designs by manipulating the blocks in a specified way. More information on this subject can be found in [28] and [31].

Theorem 1.1 has a generalization to (r,λ)-designs. Vanstone and McCarthy (see [11] and [24]) proved the following.

Theorem 2.5. An (r,λ)-design satisfying $r(r-2)/\lambda + 1 \leq v$ for which no two blocks meet in $>\lambda$ points, and no block has size less than λ, is embeddable in a (v,r,λ)-design.

A λ-space which satisfies these conditions on block intersections and sizes is sometimes called *restricted*. This condition is quite useful for it allows one to prove the following.

Lemma 2.8. In a λ-space for which no two blocks meet in $>\lambda$ points, if x is a point and B a block such that $x \notin B$, then $r_x \geq |B|$. Equality holds if and only if every block through x meets B in exactly λ points.

Proof. We count pairs (y,B') with $y \in B \cap B'$ and $x \in B'$. On one hand we can pick $y \in B$ in any one of $|B|$ ways; there will then be λ blocks B' on x and y. On the other hand there are r_x blocks on x, each meeting B in at most λ points y. Thus

$$\lambda|B| \leq r_x\lambda.$$

The lemma then follows and the case of equality is clear from the above argument.

For linear spaces this lemma is trivial and is used many times without specific note being taken. For λ-spaces, however, it is not true without the additional assumption bounding block intersections. With this lemma we can mimic many (but not all) of the techniques used on linear spaces.

We now proceed to the main result of this section. A special case of Totten's linear space classification, Theorem 2.3, is that a non-near pencil linear space with $b \leq n^2 + n + 1$ and $v \geq n^2 + 1$ is embeddable in a projective plane of order n. We now consider the extension of this to λ-spaces with $\lambda > 1$.

We prove the following.

Theorem 2.6. A λ-space which satisfies $b \leq r(r-1)/\lambda + 1$, $r(r-2)/\lambda + 2 \leq v$, and $r \geq 4\lambda + 3$, for which no two blocks intersect in $>\lambda$ points and no block contains fewer than λ points, is embeddable in a (v,r,λ)-design.

Proof. The proof will proceed in several steps. We first show that no block has size $>r$. Let B be the largest block and p a point of smallest degree on B. First note that since no two blocks meet in $>\lambda$ points we have

$$(\#\text{ of blocks meeting } B) \geq \frac{\sum_{x \in B} (r_x - 1)}{\lambda} + 1.$$

Combining this with $b \leq \dfrac{r(r-1)}{\lambda} + 1$ then gives

$$r(r-1) \geq \sum_{x \in B} (r_x - 1).$$

Thus if $|B| \geq r + 1$ we have that $r_p \leq r - 1$. Now every block not passing through p must have size no more than r_p, by Lemma 2.8. The lines through p have size no more than $|B|$, thus

$$\sum_{i=1}^{b} k_i \le r_p(b-r_p) + r_p|B|.$$

Also every point off of B has degree at least $|B|$, while those on B have degree at least r_p. Thus

$$\sum_x r_x \ge (v-|B|)|B| + |B|r_p.$$

But $\sum_{i=1}^{b} k_i = \sum_x r_x$, by counting pairs (x,B') with $x \in B'$. We then have

$$r_p(b-r_p) + r_p|B| \ge (v-|B|)|B| + |B|r_p,$$

$$r_p(b-r_p) \ge (v-|B|)|B|.$$

The first expression achieves its maximum over $r_p \le r-1$ at $r-1$ provided $r-1 \le b/2$, which is satisfied for $r \ge 4\lambda+3$ since $b \ge v \ge r(r-2)/\lambda + 2$. For $|B| \ge r+1$, the second expression is at least $(v-(r+1))(r+1)$ provided $|B| \le v-(r+1)$. So for $|B| \le v-(r+1)$ we have

$$(r-1)\left(\frac{r(r-1)}{\lambda} - (r-1)\right) \ge \left(\frac{r(r-2)}{\lambda} + 2 - (r+1)\right)(r+1).$$

After much simplification this becomes

$$0 \ge r^2 - r(4\lambda+3) + 4\lambda,$$

a contradiction for $r \ge 4\lambda+3$. Thus to have all blocks of size $\le r$ we need only dispose of the case $|B| > v-(r+1)$. Suppose not, i.e., $|B| \ge v-r$. Then if two points lie off of B they must each have degree $\ge |B| \ge v-r$ by Lemma 2.8. They are together in λ blocks. Thus

$$b \ge 2(v-r) - \lambda.$$

Applying $b \le r(r-1)/\lambda + 1$ and $v \ge r(r-2)/\lambda + 2$ then gives $0 \ge r^2 - r(2\lambda+3) - (\lambda^2 - 3\lambda)$, a contradiction for $r \ge 4\lambda + 3$. Thus at most one point lies off of B. Since no block contains all points we have exactly one point, p, lying off of B. So $|B| = v - 1$ and $r_p \ge v - 1$. Since no two blocks meet in $>\lambda$ points we also have that all blocks besides B have size $\le \lambda + 1$. Now consider any $x \in B$. We count pairs (y,B') with $y \in B'$ and $x \in B'$ in two different ways to obtain

$$(v-1) + (r_x - 1)\lambda \ge (v-1)\lambda,$$

$$r_x \ge (v-1)\left(\frac{\lambda-1}{\lambda}\right) + 1.$$

Hence

$$\sum_x r_x \ge (v-1)\left[(v-1)\left(\frac{\lambda-1}{\lambda}\right) + 1\right] + (v-1).$$

But $\sum_x r_x = \sum_{i=1}^{b} k_i \le (b-1)(\lambda+1) + (v-1)$.

Thus

$$(b-1)(\lambda+1) + (v-1) \ge (v-1)\left[(v-1)\left(\frac{\lambda-1}{\lambda}\right) + 1\right] + (v-1),$$

which leads to a contradiction to the bounds on b and v when $\lambda > 1$. Thus no block has size $> r$.

Now consider any point x. We count the number of pairs (y,B') with $x \in B'$, $y \in B'$ in two ways to obtain

$$r_x(r-1) \ge (v-1)\lambda,$$

using the fact that no block has size $>r$. This together with $v \ge r(r-2)/\lambda + 2$ then implies $r_x \ge r$, when $\lambda > 1$. So all points have degree $\ge r$.

We now show that some block must have size r. If not then

$$b(r-1) \geq \sum_{i=1}^{b} k_i = \sum_{x} r_x \geq vr.$$

We can then apply $b \leq r(r-1)/\lambda + 1$, $v \geq r(r-2)/\lambda + 2$ and simplify to obtain $0 \geq r(\lambda - 1) + \lambda$, a contradiction.

Let B be a block of size r. Then since every point on B has degree $\geq r$ and blocks meet in at most λ points we have

$$(\# \text{ of blocks meeting } B) \geq \frac{r(r-1)}{\lambda} + 1.$$

But $b \leq r(r-1)/\lambda + 1$ by assumption. Thus we have that all blocks meet B in exactly λ points, with each point having degree r. But by the case of equality in Lemma 3.8 we also have that all points off of B have degree r exactly. Hence we have an (r,λ)-design satisfying the conditions of Theorem 2.5. Thus it is embeddable in a symmetric (v,r,λ)-design.

Section 5: Remarks.

We note here some consequences of Theorems 2.3 and 2.4 involving the possible values of b for a linear space on a fixed number, v, of points.

Suppose $\mathfrak{F}$ is a linear space on v points with $n^2 + 2 \leq v \leq n^2 + 2n$. If $b - v \leq n$ then $\mathfrak{F}$ is one of the linear spaces in Theorem 2.3. If $\mathfrak{F}$ is a near pencil then $b = v$. If $\mathfrak{F}$ is embeddable in a projective plane then $b = n^2 + n + 1$. If $\mathfrak{F}$ is an affine plane of order n with a linear space at infinity then, with the space at infinity containing α points, $v = n^2 + \alpha$ and $b = n^2 + n + \alpha$. This is because the linear space at infinity must be a near pencil or projective plane (by Theorem 2.3), for which # of points = # of lines. Thus we have shown

Corollary 2.1. A linear space on v points with $n^2 + 2 \leq v \leq n^2 + 2n$ can only have number of lines $b = v$, $b = n^2 + n + 1$, or $b \geq v + n$.

In the cases $v = n^2$ or $n^2 + 1$ we have the same result except that we must allow $b = n^2 + n$ as well, since an embeddable space on these numbers of points can have $n^2 + n$ lines.

The case $v = n^2 + n + 1$ we consider further. The above corollary gives $b = n^2 + n + 1$ or $b \geq n^2 + 2n + 1$. We can improve upon this in a special case by using Theorem 2.4.

Corollary 2.2. Let $v = n^2 + n + 1$ with $n = m^2 + m$ for some m. Then for $m \geq 6$ a linear space on v points must have $b = n^2 + n + 1$, $b = n^2 + 2n + 1$, or $b \geq n^2 + 2n + m + 1$.

Proof. Let $\mathfrak{F}$ be a linear space with $v = n^2 + n + 1 = (m^2 + m)^2 + (m^2 + m) + 1$ points and $b \leq n^2 + 2n + m$ lines, $m \geq 6$. Then $b \leq n^2 + (2.147)n$

since $(.147)n = (.147)(m^2 + m) \geq m$ for $m \geq 6$. Hence we are in the case of Theorem 2.4. If $b \neq n^2 + n + 1$ then we have an affine plane of order n with a linear space on $n + 1 = m^2 + m + 1$ points at infinity. Thus $b = n^2 + n + \alpha$, where α is the number of lines in the space at infinity. Since there are $m^2 + m + 1$ points in this space we can then apply Theorem 2.4 to obtain $\alpha = m^2 + m + 1$ or $\geq m^2 + 2m + 1$. Thus $b = n^2 + n + (m^2 + m + 1)$ or $\geq n^2 + n + (m^2 + 2m + 1)$ and the result follows.

There are several other recent results relating to questions of this kind. We cite several here, without proof.

Theorem 2.7 (Erdös, Mullin, Sós, Stinson, [15]). A linear space on v points which is not a near pencil satisfies $b \geq B(v)$ where

$$B(v) = \begin{cases} n^2 + n + 1 & \text{for } n^2 + 2 \leq v \leq n^2 + n + 1 \\ n^2 + n & \text{for } n^2 - n + 3 \leq v \leq n^2 + 1 \\ n^2 + n - 1 & \text{for } v = n^2 - n + 2. \end{cases}$$

A linear space with $b = B(v)$ on v points is embeddable in a projective plane of order n when $n^2 \leq v \leq n^2 + n + 1$, $v = n^2 - n + 2$, or $v = n^2 - \alpha$ with $\alpha^2 + \alpha(2n - 3) - (2n^2 - 2n) < 0$.

Theorem 2.8 (Stinson, deWitte [33]). A linear space with $v \geq n^2$ and $b \leq n^2 + n + 1$ is embeddable in a projective plane of order n.

Theorem 2.9 (Stinson [34]). The only finite linear space on v points and $b = n^2 + n + 2$ lines with $n^2 + 1 \leq v \leq n^2 + n + 1$ has $v = 10$, $b = 14$ (and such a space exists).

<u>Theorem 2.10</u> (Erdös, Fowler, Sós, Wilson [14]). For v sufficiently large, a linear space on b lines and v points exists for all b such that $v + v^{\frac{4}{5}} \leq b \leq \binom{v}{2} - 4$, $b = \binom{v}{2} - 2$, or $b = \binom{v}{2}$.

We close this chapter by conjecturing that, for n sufficiently large, a linear space satisfying $n^2 - n + 2 \leq v \leq n^2 + n + 1$ and $b - v \leq 2n - 2$ can only arise as an affine plane of order n, from which some points have possibly been deleted, with a (possibly degenerate) linear space at infinity.

Chapter III

(k,μ)-arcs

Section 1: Introduction.

In this chapter we restrict our attention to linear spaces which are finite projective planes. We investigate structures contained in these. In this section we define the terms which we will use. In particular instances we refer the reader to other sources for more details.

For the definition and properties of a projective plane of order n see Chapter 1, Section 1. We will denote a projective plane of order n by Π_n and its lines by $\ell_1, \ell_2, \ldots, \ell_{n^2+n+1}$, points by $p_1, p_2, \ldots, p_{n^2+n+1}$.

It should be noted that several different projective planes of a particular order n may exist. That is, the geometry of points and lines in planes of the same order may be different. We will go into more details concerning this in later sections.

A (k,μ)-*arc*, $\mu \geq 2$, in a projective plane of order n is a set of k points which meets no line of the plane in more than μ points. We will denote the set of points in a (k,μ)-arc on occasion by A and, for convenience, sometimes refer to A as a μ-*arc* or simply *arc*. A line ℓ of the plane which meets an arc in m points will be called an *m-secant*.

Note that if p is a point of a (k,μ)-arc, A, in a Π_n then the $n+1$ lines through p each meet A in at most $\mu - 1$ points other than p. Thus

$$k \leq (n+1)(\mu-1) + 1 = n\mu - n + \mu.$$

Equality holds if and only if every line of Π_n is either a 0-secant or a μ-secant. This bound and case of equality is due to Tallini-Scafati [35]. We call a (k,μ)-arc with $k = n\mu - n + \mu$ *perfect*. A perfect μ-arc can be equivalently defined as a set of points having only 0-secants and μ-secants. A (k,μ)-arc not properly contained in any (k',μ)-arc will be called *complete*. Our notation differs from the current literature in that perfect arcs here are referred to as maximal there. We feel that the term perfect is, in some sense, more descriptive of the extremal nature of $(n\mu - n + \mu,\mu)$-arcs.

A separate notation is sometimes employed in the case $\mu = 2$. In a Π_n an $(n+2,2)$-arc, n even, or an $(n+1,2)$-arc, n odd, is called an *oval* (we will see in section 2 that $(n+2,2)$-arcs do not exist for n odd). For a survey of (k,μ)-arcs we refer the reader to [2], [3], or [21].

There are several other combinatorial structures (not necessarily contained in a projective plane) that we will also need.

Later in this chapter we will encounter (b,v,r,k,λ)-designs. For more details see [18], pp. 100-120, or [27], pp. 96-116. A (b,v,r,k,λ)-*design*, also called a *balanced incomplete block design* or simply *block design*, $\lambda > 0$ and $k < v - 1$, is a set of v points and a collection of b subsets of points, called *blocks*, such that every point lies on r blocks, every block contains k points, and every pair of points lies on exactly λ blocks. It can then easily be shown that $bk = vr$ and $r(k-1) = \lambda(v-1)$. Another fundamental result on block designs is Fischer's Inequality which states that $b \geq v$ (note that the case $\lambda = 1$ is a consequence of the deBruijn-Erdös Theorem). The symmetric (v,k,λ)-designs of Chapter 2 can be equivalently defined as block designs for which $b = v$.

A *partial geometry* is a set of points and a collection of subsets of points, called *lines*, such that every point lies on r lines, every line contains k points, every pair of points is contained in at most one line, and, if ℓ is a line and $p \notin \ell$ then there are exactly α lines ℓ' such that $p \in \ell'$ and $\ell' \cap \ell \neq \emptyset$, for fixed constants r, k, and α. For more information on partial geometries see [6].

Section 2: Elementary properties.

Here we present established results on (k,μ)-arcs which will be needed later in this chapter.

The concept of duality will be quite useful. We touched on this subject briefly in Chapter 2. Given a structure of points and lines, with some notion of incidence, the *dual* of this structure is obtained by calling the lines points, the points lines, and reversing the relationship of incidence. Old points (which are now lines) contain the old lines (now points) with which they were previously incident, and old lines lie on the points they contained.

For example consider the following incidence structure with lines $\ell_1, \ell_2, \ldots, \ell_7$ and points $1, 2, \ldots, 7$.

$$\ell_1 = \{1,2,4\}, \quad \ell_2 = \{2,3,5\}, \quad \ell_3 = \{3,4,6\},$$
$$\ell_4 = \{4,5,7\}, \quad \ell_5 = \{1,5,6\}, \quad \ell_6 = \{2,6,7\},$$
$$\ell_7 = \{1,3,7\}.$$

The dual of this structure has lines $1, 2, \ldots, 7$ and points $\ell_1, \ell_2, \ldots, \ell_7$ with

$$1 = \{\ell_1,\ell_5,\ell_7\}, \quad 2 = \{\ell_1,\ell_2,\ell_6\},$$
$$3 = \{\ell_2,\ell_3,\ell_7\}, \quad 4 = \{\ell_1,\ell_3,\ell_4\},$$
$$5 = \{\ell_2,\ell_4,\ell_5\}, \quad 6 = \{\ell_3,\ell_5,\ell_6\},$$
$$7 = \{\ell_4,\ell_6,\ell_7\}.$$

We will be using duality primarily in connection with projective planes. Note that by the symmetry of the axioms of a projective plane, with respect to points and lines, the dual of Π_n is also a projective plane of order n, denoted by Π_n'. The above example is a particular case of this for $n = 2$.

Any set of lines in Π_n is a set of points in Π_n'. Given a (k,μ)-arc A in Π_n we define the *dual arc*, denoted by A', in Π_n' to be the set of points in Π_n' which correspond to lines of Π_n which do not intersect A (in Π_n).

Now suppose A is a perfect μ-arc in a Π_n. Then every line which meets A in at least one point meets it in μ. Thus if $p \notin A$ the lines through p meeting A divide A into disjoint sets of μ points. Thus # of lines through p meeting $A = |A|/\mu = n + 1 - n/\mu$, since A is perfect. Now $r_p = n + 1$ hence we have shown that the number of lines missing A through $p \notin A$ is n/μ. In other words, the dual arc A' possesses only 0-secants and (n/μ)-secants (points of A become 0-secants in Π_n'). Thus the dual of a perfect μ-arc is a perfect (n/μ)-arc, in the dual plane. We also have the necessary condition for a perfect μ-arc in a Π_n, $\mu | n$ (this shows that, as mentioned earlier, there are no $(n+2,2)$-arcs in a Π_n, for n odd). The above results are due to Cossu and can be found in [10].

We shall be concerned with arc completions. A complete (k,μ)-arc A_1 is said to be a *completion* of a (k',μ)-arc A_2 if $A_1 \supseteq A_2$. Our first theorem on completions is the following, due to Barlotti [1], concerning arcs which are one point short of perfection.

<u>Theorem 3.1</u>. For $\mu > 2$ an $(n\mu - n + \mu - 1,\mu)$-arc in a Π_n has a perfect completion. An $(n+1,2)$-arc in a Π_n, n even, has a perfect completion.

<u>Proof</u>. Suppose A is an $(n\mu - n + \mu - 1,\mu)$-arc. For any point p let a_i = # of i-secants through p. Then if $p \in A$

$$\sum_{i=1}^{\mu} a_i = n + 1,$$

$$\sum_{i=1}^{\mu} (i-1)a_i = n\mu - n + \mu - 2.$$

Combined these imply $-\mu a_1 - (\mu-1)a_2 - \cdots - 2a_{\mu-2} - a_{\mu-1} = -1$. Since $a_i \geq 0$ and an integer we then have $a_{\mu-1} = 1$, $a_i = 0$ for $i \leq \mu - 2$ and $a_\mu = n$. Thus each point of A lies on n μ-secants and one $(\mu-1)$-secant. Hence we can count # of $(\mu-1)$-secants $= |A| \cdot 1/(\mu-1) = n+1$.

Since only 0-secants, $(\mu-1)$-secants, and μ-secants exist we have that for $p \notin A$

$$(\mu-1)a_{\mu-1} + \mu a_\mu = |A| = n\mu - n + \mu - 1.$$

If $a_{\mu-1} > 0$ for every point $p \notin A$ then every point of Π_n (including those in A) lies on at least one of the $n+1$ $(\mu-1)$-secants. Because every pair of lines meets we have that for a line ℓ which is not a $(\mu-1)$-secant each $p \in \ell$ is covered by exactly one $(\mu-1)$-secant (since $|\ell| = n+1$ and there are only $n+1$ $(\mu-1)$-secants total). In other words, any point lying on a non-$(\mu-1)$-secant is covered by exactly one $(\mu-1)$-secant. Thus if p' is the intersection point of two $(\mu-1)$-secants it must lie on all $(\mu-1)$-secants. Thus p' can be added to A to product a perfect μ-arc.

Note that for an $(n+1,2)$-arc, with n even, we have every point lying on at least one 1-secant immediately, since $n+1$ is odd. We can then apply the above argument to obtain a perfect completion. Thus the second assertion is proved.

By the above we may assume that $a_{\mu-1} = 0$ for some $p \notin A$. Then for that point

$$\mu a_{\mu} = n\mu - n + \mu - 1.$$

Thus $n + 1 \equiv 0 \pmod{\mu}$. Now let ℓ be a $(\mu - 1)$-secant. As before, for any $p \in \ell \setminus A$ we have

$$(\mu - 1)a_{\mu-1} + \mu a_{\mu} = n\mu - n + \mu - 1.$$

But $n + 1 \equiv 0 \pmod{\mu}$, thus the above equation implies $a_{\mu-1} \equiv 0 \pmod{\mu}$. For each $p \in \ell \setminus A$ $a_{\mu-1} > 0$ since ℓ is a $(\mu - 1)$-secant. Hence $a_{\mu-1} \geq \mu$ for each of the $(n+1) - (\mu - 1)$ points of $\ell \setminus A$. We can then count the $(\mu - 1)$-secants by their intersections with ℓ to obtain

$$\# \text{ of } (\mu - 1)\text{-secants} \geq (n + 2 - \mu)(\mu - 1) + 1.$$

But we have already counted the number of $(\mu - 1)$-secants as $n + 1$. Hence $(n+1) \geq (n + 2 - \mu)(\mu - 1) + 1$. This simplifies to $0 \leq (\mu - 2)(\mu - (n+1))$, a contradiction for $\mu > 2$, and the result follows.

We mention a further result of this type due to B. J. Wilson. In [42] he shows that an $((n^2 - n - 4)/2, n/2)$-arc, in a Π_n, with $n \geq 8$ and even, can be completed to a perfect $(n/2)$-arc.

Section 3: Constructions.

In this section we mention several constructions for perfect arcs. To do this we will require a more detailed description of some particular kinds of projective planes.

A *desarguesian projective plane* is one in which the Axiom of Desargues holds. The exact statement of this axiom is not necessary for our discussion, more details can be found in [18], pp. 167-188. For our purposes we need only the fact that desarguesian planes can be described using vector spaces over finite fields.

Let GF(n) be the finite field of n elements, for n a prime power. Let $V_k(n)$ denote the k dimensional vector space over GF(n). A classic theorem of geometry states that a desarguesian projective plane of order n can be thought of as having as points the 1-dimensional subspaces of $V_3(n)$, and as lines the 2-dimensional subspaces of $V_3(n)$, with containment as incidence. This plane is often denoted by $PG_2(n)$. Note that this implies that a desarguesian plane can only have prime power order.

Using this description we have many ovals in desarguesian planes using the following construction, see [21] chapters 7 and 8. Let $V_3(n) = \{(x_1,x_2,x_3) : x_i \in GF(n)\}$, n a prime power. Denote points of $PG_2(n)$ by $\langle (x_1,x_2,x_3) \rangle$, not all $x_i = 0$. Let $Q(x_1,x_2,x_3)$ be a quadratic form in x_1, x_2, x_3. We say that Q is *non-singular* when no substitution $x_i = a_{i1}z_1 + a_{i2}z_2 + a_{i3}z_3$, of variables z_i for the x_i, with the matrix $[a_{ij}]_{i,j=1}^{3}$ non-singular, produces a quadratic form in fewer than three variables. For more information on quadratic forms see [17] or [22].

<u>Theorem 3.2</u>. Let $Q(x_1,x_2,x_3)$ be a non-singular quadratic form. Then $A = \{\langle(x_1,x_2,x_3)\rangle : Q(x_1,x_2,x_3) = 0, x_i \in GF(n)\}$ is an $(n+1,2)$-arc in $PG_2(n)$, n a prime power.

Using this we then immediately have ovals for n odd. Applying Theorem 3.1 gives ovals for n even. In fact all ovals for n odd in $PG_2(n)$ arise in this way (see [29]).

Another equivalent description of a desarguesian projective plane is through the affine plane embedded in it (see Chapter 2, Section 1). A desarguesian affine plane of order n (n necessarily a prime power) can be represented as follows. Points are all pairs (x,y) with x and y in $GF(n)$. Lines are all sets $L_{m,c} = \{(x,y) : y = mx + c\}$ and $L_{\infty,c} = \{(x,y) : x = c\}$, for m and c in $GF(n)$. We will denote a desarguesian affine plane of order n by $AG_2(n)$.

The set $\{L_{m,b} : b \in GF(n)\}$ is a parallel class of $AG_2(n)$. Thus the "points at infinity" that can be adjoined to parallel classes of $AG_2(n)$ to produce a projective plane of order n correspond to fixed slope values m (including the parallel class of "infinite" slope $\{L_{\infty,b} : b \in GF(n)\}$). The correspondence between projective and affine planes shows that $AG_2(n)$ is embedded in $PG_2(n)$. Any construction of arcs in the affine plane carries over into the projective plane. For our next construction, due to Denniston [12], it is more convenient to work in $AG_2(n)$.

Let $n = 2^r$ and $\mu | n$, say $\mu = 2^m$, $m < r$. We construct a perfect $(k,2^m)$-arc in $AG_2(2^r)$. We will use several properties of finite fields. For more details see [21], Chapter 1.

We first establish some facts regarding quadratic polynomials over

$GF(2^r)$. Let $f(x) = ax^2 + bx + c$ for some a, b, c in $GF(2^r)$ with $a \neq 0$ and $b \neq 0$. Then since $x \mapsto x^2$ is an automorphism of $GF(2^r)$, $f(x) - c = ax^2 + bx$ is an endomorphism of the additive group of $GF(2^r)$, with kernel $\{0, b/a\}$. Hence Range $(f(x) - c)$ is a subgroup of index 2 of the additive group of $GF(2^r)$. Each value in the range is taken on twice. Thus Range $(f(x))$ is a coset of a subgroup of index 2.

Let $\{a,h,b\} \subseteq GF(2^r)$ be such that $a, b, h \neq 0$ and $b \notin$ Range $(ax^2 + hx)$. Thus $f(x) = ax^2 + hx + b \neq 0$ for any x. Consider now $\phi(x,y) = ax^2 + hxy + by^2$. We claim that $\phi(x,y) = 0$ if and only if $x = y = 0$. One implication is obvious. Now if $\phi(x,y) = 0$ and $y = 0$ then obviously $x = 0$. If $y \neq 0$ then we have $0 = a(x/y)^2 + h(x/y) + b$, a contradiction to the choice of a, h, and b.

Let $\phi(x,y)$ be as above and H a subgroup of order 2^m of the additive group of $GF(2^r)$. Let $A = \{(x,y) : \phi(x,y) \in H\}$. We show that A is a perfect (2^m)-arc in $AG_2(2^r)$. We will show that every line meets A in either 0 or 2^m points.

For a particular line ℓ let G_ℓ be the set of values (with multiplicities) that $\phi(x,y)$ takes over the points of ℓ. We wish to show $|G_\ell \cap H| = 2^m$.

Suppose ℓ is of the form $L_{m,0}$. Then $G_\ell = \{x^2(a + hm + bm^2) : x \in GF(2^r)\}$. Since $\phi(x,y) = 0$ if and only if $x = y = 0$ we have $a + hm + bm^2 \neq 0$. Thus since $x \mapsto x^2$ is an automorphism of $GF(2^r)$, $G_\ell = GF(2^r)$ (all multiplicities one). So $|G_\ell \cap H| = 2^m$.

If $\ell = L_{\infty,0}$ then $G_\ell = \{by^2 : y \in GF(2^r)\}$ and for the same reasons as above $|G_\ell \cap H| = 2^m$.

Now let ℓ be of the form $L_{m,c}$ or $L_{x,c}$ with $c \neq 0$. Then G_ℓ is obtained by substituting $y = mx + c$ or $x = c$ in $\phi(x,y)$ and letting x (or y) range over all values in $GF(2^r)$. That is, G_ℓ is the range of a quadratic polynomial. Note that $(0,0) \notin \ell$ thus $0 \notin G_\ell$. For $\ell = L_{m,c}$ the coefficient of x^2 in f is $a + hm + bm^2$ and hence $\neq 0$. For $\ell = L_{\infty,c}$ the coefficient of y^2 is b, thus $\neq 0$.

If the coefficient of x (or y) in f is 0 then Range (f) = $GF(2^r)$, again since $x \mapsto x^2$ is an automorphism of $GF(2^r)$. But $0 \notin G$. Thus x (or y) has a non-zero coefficient, so that f is an endomorphism with non-trivial kernel and our previous discussion applies. G_ℓ = Range (f) is a coset of a subgroup of index 2, say G, of the additive group of $GF(2^r)$. Now $0 \notin G_\ell$, hence $G \neq G_\ell$ and we have $GF(2^r) = G \dot{\cup} G_\ell$. By the structure of additive subgroups of $GF(2^r)$ we have that $|G \cap H| = |H|$ or $\frac{1}{2}|H|$. If $|G \cap H| = |H|$ then $|G_\ell \cap H| = 0$. If $|G \cap H| = \frac{1}{2}|H| = 2^m$ then the other half of the elements in H are in G_ℓ, each with multiplicity 2. Thus $|G_\ell \cap H| = |H| = 2^m$.

Hence every line meets A in 0 or 2^m points. The line at infinity which we adjoin to $AG_2(2^r)$ to obtain $PG_2(2^r)$ misses A entirely. So we have

Theorem 3.3. A perfect μ-arc exists in $PG_2(n)$, n even, if and only if $\mu | n$.

We now present a new construction of ovals in a certain class of non-desarguesian projective planes called translation planes. We refer the reader to [25] for a more complete account. We need only the following description of some affine translation planes.

Let $n = p^s$ for p a prime, and $m|s$. Let $\{u_i\}_{i=1}^{p^m-1}$ be some fixed set of non-negative integers with $m|u_i$ for all i. Then taking points to be all (x,y) with $x, y \in GF(p^s)$ and lines as all

$L_{m,c} = \{(x,y) : y = mx^{p^{u_i}} + c$ with $x, y \in GF(p^s)$ and $\text{index}(m) \equiv i \pmod{p^m-1}\}$ or $L_{\infty,c} = \{(x,y) : x = c\}$ for m and c in $GF(p^s)$; we obtain an affine translation plane of order p^s. Parallel classes, as in the desarguesian affine plane, are collections of lines of fixed "slope."

We now let $\mathcal{J}$ be a translation plane of order p^s, as described above. Consider the set $B = \{(x,x^{p^a}) : x \in GF(p^s)\}$ for some fixed non-negative integer a. We examine how B intersects the lines of $\mathcal{J}$.

The lines $L_{0,c}$ intersect B in points (x,x^{p^a}) with $y = x^{p^a} = c$. Since $x \mapsto x^p$ is an automorphism of $GF(p^s)$ there will only be one such x. Thus the lines $L_{0,c}$ intersect B in one point only. Similarly the lines $L_{\infty,c}$ meet B in one point only.

The lines $L_{m,c}$, $m \neq 0, \infty$, intersect B in points (x,x^{p^a}) with $x^{p^a} = y = mx^{p^{u_i}} + c$, $\text{index}(m) \equiv i \pmod{p^m - 1}$. Because $x \mapsto x^{p^j}$ is an automorphism of $GF(p^s)$ for any j we have that $f(x) = x^{p^a} - x^{p^{u_i}}$ is an endomorphism. Hence the number of solutions of $x^{p^a} = mx^{p^{u_i}} + c$ is the same as the number of solutions to $x^{p^a} = mx^{p^{u_i}}$ when $c \in \text{Range}(f)$ and 0 if $c \notin \text{Range}(f)$. That is, besides $x = 0$, $x^{p^a-p^{u_i}} = m$. By the multiplicative structure of $GF(p^s)$ (cyclic), this equation has either 0 or $(|p^a - p^{u_i}|, p^s - 1)$ solutions, using parentheses to denote the greatest common divisor of two numbers, depending on whether $(|p^a - p^{u_i}|, p^s - 1) \mid \text{index}(m)$ or not. But $(|p^a - p^{u_i}|, p^s - 1) = (p^{|a-u_i|} - 1, p^s - 1) = p^{(|a-u_i|,s)} - 1$. Thus we have for $m \neq 0$

$$|B \cap L_{m,c}| = \begin{cases} 0 \text{ if } c \notin \text{Range}(f) \\ 1 \text{ if } c \in \text{Range}(f) \text{ but } (p^{(|a-u_i|,s)}-1) \nmid \text{index}(m) \\ p^{(|a-u_i|,s)} \text{ if } c \in \text{Range}(f) \text{ but } \\ \qquad (p^{(|a-u_i|,s)}-1) \mid \text{index}(m) \end{cases}$$

where $f(x) = x^{p^a} - mx^{p^{u_i}}$, while $|B \cap L_{\infty,c}| = |B \cap L_{0,c}| = 1$ for all c.

Now specialize to the case $p=2$ and suppose the u_i which determine $\mathcal{J}$ are such that there exists a such that $(|a-u_i|,s) = 1$ for all $i = 1, 2, \ldots, 2^m - 1$. Then we have 2^s points in $\mathcal{J}$ (affine plane of order 2^s) having only 0, 1, and 2-secants. In adjoining points at infinity to make a projective plane we can add to B the points corresponding to the two parallel classes $\{L_{\infty,c} : c \in GF(2^s)\}$ and $\{L_{0,c} : c \in GF(2^s)\}$ to produce a $(2^s+2,2)$-arc in a translation plane of order 2^s. Thus we have the following

Theorem 3.4. A translation plane of even order, described as above, for which there exists some a such that $(|a-u_i|,s) = 1$ for all i contains an oval.

We mention other results, without proof, concerning the existence of perfect arcs.

In [36] and [38] J. A. Thas constructs, for q a power of 2, perfect $(q^{2d-1} - q^d + q^{d-1}, q^{d-1})$-arcs in certain translation planes of order q^d. Also due to Thas we have

Theorem 3.5 (Thas [37]). In $PG_2(q)$, $q > 3$, there are no perfect 3-arcs.

It is conjectured that no perfect μ-arcs exist for μ odd, except for $\mu = n$ and $n+1$ in a Π_n.

Section 4: Uniqueness of completions.

We now present results on uniqueness of completions. Our first result is the following.

Theorem 3.6. Let A be a (k,μ)-arc in a Π_n. Then if $k \geq n\mu - n + \mu - (n - n/\mu)$ there is at most one way of completing A to a perfect arc.

Proof. Suppose T_1 and T_2 are both sets of points in Π_n such that $A \cup T_1$ and $A \cup T_2$ are both perfect μ-arcs. Without loss of generality assume $T_1 \cap T_2 = \emptyset$.

Pick some $p \in T$. Then, because p is external to the perfect μ-arc $A \cup T_2$, p lies on n/μ lines external to $A \cup T_2$. Each of these n/μ lines, however, must meet the perfect μ-arc $A \cup T_1$ in μ points. They are disjoint from A hence

$$|T_1| \geq 1 + (\mu - 1)(n/\mu).$$

Hence

$$|A| = (n\mu - n + \mu) - |T_1| \leq n\mu - n + \mu - (n - n/\mu + 1),$$

and the theorem follows.

The case of equality in the above theorem deserves further comment. Suppose $|A| = n\mu - n + \mu - (n - n/\mu + 1)$ and there are at least two ways to complete A to a perfect arc. Then the above proof shows that the lines through some $p \in T_1$ missing A cover all points of T_1. If we take as blocks the lines meeting T_1 in >1 point and as points the points of T_1 we then have that every pair of points (of T_1) lies in a unique block, every point of degree n/μ, and every line of length μ. This is a block design with parameters $v = n + 1 - n/\mu$, $k = \mu$, $\lambda = 1$,

$r = n/\mu$, $b = \frac{n}{\mu^2}(n - \frac{n}{\mu} + 1)$, (calculating b from the other parameters by using the relations for a block design stated in section 1). Fischer's Inequality then implies $n \geq \mu^2$. Thus the bound of Theorem 3.6 can be improved by one if $n < \mu^2$.

By using duality we can further improve the bound. Let A be a μ-arc in a Π_n, $|A| = n\mu - n + \mu - z$, for some z, with two completions, T_1 and T_2, to perfect μ-arcs, $|T_1| = |T_2| = z$, $T_1 \cap T_2 = \emptyset$. Theorem 4.6 then states that $z \geq n + 1 - n/\mu$. Consider the duals of the arcs $A \cup T_1$ and $A \cup T_2$. By the discussion in section 2, $(A \cup T_1)'$ and $(A \cup T_2)'$ will both be perfect n/μ-arcs in Π_n'. The points which are shared by these two dual arcs are the lines external to both $A \cup T_1$ and $A \cup T_2$ in Π_n. The points in $(A \cup T_1)' \backslash (A \cup T_2)'$ are those lines external to $A \cup T_1$ but meeting $A \cup T_2$. Given that there are z points in T_2, each of which lies on n/μ lines external to $A \cup T_1$, each line of which contains μ points of T_2, we have

$$|(A \cup T_1)' \backslash (A \cup T_2)'| = nz/\mu^2.$$

This immediately implies $\mu^2 | nz$. But also $(A \cup T_1)' \cap (A \cup T_2)'$ is an (n/μ)-arc in Π_n' with two different completions to perfection, $(A \cup T_1)' \backslash (A \cup T_2)'$ and $(A \cup T_2)' \backslash (A \cup T_1)'$. Thus by Theorem 3.6.

$$nz/\mu^2 = |(A \cup T_1)' \backslash (A \cup T_2)'| \geq n - n/(n/\mu) + 1,$$

$$z \geq \mu^2 - \mu^3/n + \mu^2/n.$$

Note that if $\mu^2 \leq n$ then $\mu^2 - \mu^3/n + \mu^2/n \leq n - n/\mu + 1$, while for $\mu^2 > n$, $\mu^2 - \mu^3/n + \mu^2/n > n - n/\mu + 1$. Thus this bound is only an improvement for $\mu^2 > n$.

We also note that since $(A \cup T_1)'$ is a perfect (n/μ)-arc we have

$$n\binom{n}{\mu} - n + \binom{n}{\mu} \geq |(A \cup T_1)' \backslash (A \cup T_2)'| = nz/\mu^2,$$

$$z \leq n\mu - \mu^2 + \mu.$$

Again, an improvement over the obvious bound $z \leq n\mu - n + \mu$ for $\mu^2 > n$ only. These results can be stated in terms of intersecting perfect arcs. To summarize what we have shown,

Theorem 3.7. Two perfect $(n\mu - n + \mu, \mu)$-arcs in a Π_n, $\mu | n$, which intersect in $n\mu - n + \mu - z$ points must have $\mu^2 | nz$ and

1. If $\mu^2 \leq n$ then $n\mu - n + \mu \geq z \geq n + 1 - n/\mu$. Equality on the lower bound implies the existence of a block design with parameters $v = n + 1 - n/\mu$, $k = \mu$, $\lambda = 1$, $r = n/\mu$, $b = \frac{n}{\mu^2}(n + 1 - \frac{n}{\mu})$.
2. If $\mu^2 > n$ then $\mu^2 - \mu^3/n + \mu^2/n \leq z \leq n\mu - \mu^2 + \mu$. Equality on the lower bound implying the existence of a block design with parameters $v = n + 1 - \mu$, $k = n/\mu$, $\lambda = 1$, $b = \frac{\mu^2}{n}(n + 1 - \mu)$, $r = \mu$.

For $\mu^2 > n$ this establishes a fairly limited range for the number of points in which two perfect μ-arcs can meet. In particular two μ-arcs with $\mu^2 > n$ cannot be disjoint.

To illustrate this theorem we consider the case $n = 16$. Then $\mu | n$ implies $\mu = 2, 4,$ or 8. By applying Theorem 3.7 we have

1) Two perfect 2-arcs can only meet in 0 to 9 points.

2) Two perfect 4-arcs can only meet in 0 to 37 points.

3) Two perfect 8-arcs can only meet in multiples of four from 48 to 84 points.

We now return to the extremal case of these bounds and prove the following.

Theorem 3.8. A μ-arc A with $|A| = n\mu - n + \mu - (n+1-n/\mu)$, $\mu|n$, $\mu^2 \leq n$, can be completed to a perfect arc in at most $\mu + 2$ ways.

Proof. Let $T_1, T_2, \ldots, T_m$ be completions of A to perfection. Then by Theorem 3.6 the T_i are pairwise disjoint.

We count the lines intersecting the set of points $A \cup T_1 \cup T_2 \cup \cdots \cup T_m$. By the proof of Theorem 3.6, and $|T_i| = n + 1 - n/\mu$ for all i, the line ℓ joining a point of A to a point of some T_i must contain only one point of T_i and $\mu - 1$ points of A. But then, since $A \cup T_j$ is a perfect μ-arc for all j we must have ℓ meeting T_j, $j \neq i$, in a single point. Hence through $p \in T_i$ there are $n + 1 - n/\mu$ lines, each meeting every other T_j in one point and A in $(\mu - 1)$ points. This leaves n/μ lines through p missing all T_j, $j \neq i$, and A. These are the lines of the design on the points of T_i (mentioned after Theorem 3.6). Thus, by the parameters of that design, there are $\frac{n}{\mu^2}(n+1-\frac{n}{\mu})$ lines through a T_i missing A and all T_j, $j \neq i$. This accounts for $m(n/\mu^2(n+1-n/\mu))$ lines.

There are $(n+1-n/\mu)^2$ lines joining points of T_i to points of T_j, $j \neq i$. These lines meet A in $(\mu - 1)$ points.

It remains to count the lines meeting A but no T_i. There are $n + 1 - n/\mu$ lines from any $p \in A$ to points of some T_i (and hence to all T_i). Thus there are n/μ lines through p missing all T_i, each containing μ points of A. Thus there are $n|A|/\mu = n/\mu^2(n\mu - 2n + \mu + n/\mu - 1)$ lines meeting A and missing all T_i.

So we have

$$n^2 + n + 1 = \text{total \# of lines},$$

$$\geq m(\frac{n}{\mu^2})(n+1-n/\mu) + (n+1-n/\mu)^2$$

$$+ \frac{n}{\mu^2}(n\mu - 2n + \mu + n/\mu - 1).$$

We then obtain, after some manipulation,

$$m \leq \frac{n(\mu^2+\mu-1) - (\mu^3-\mu^2-\mu)}{n(\mu-1)+\mu} = \mu + 2 + \frac{1}{\mu-1} - \frac{\mu^2(\mu^2-\mu+1)}{n(\mu-1)+\mu}.$$

Thus since $\mu \geq 2$ we have $m \leq \mu + 2$.

We mention that the $(n+1-n/\mu)^2$ lines which cut across all T_i give rise to a transversal design (see [44]) on the points of $T_1 \cup T_2 \cup \cdots \cup T_m$, and hence to a set of $m - 2$ mutually orthogonal latin squares of order $n + 1 - n/\mu$. Thus we have

Corollary 3.1. A μ-arc, A, with $|A| = n\mu - n + \mu - (n+1-n/\mu)$ with m completions to a perfect arc implies the existence of $m - 2$ mutually orthogonal latin squares of order $n + 1 - n/\mu$.

We can also dualize a μ-arc A, $\mu^2 > n$, with $|A| = n\mu - n + \mu - (\mu^2 - \mu^3/n + \mu^2/n)$, the bound of Theorem 3.7, with m completions to perfection to obtain an (n/μ)-arc A' with $|A| = n(n/\mu) - n + (\frac{n}{\mu}) - (n+1-n/(n/\mu))$ and m completions to perfection. Thus we can apply Theorem 3.8 and Corollary 3.1 to obtain

Corollary 3.2. A μ-arc A, $\mu^2 > n$, with $|A| = n\mu - n + \mu - (\mu^2 - \mu^3/n + \mu^2/n)$ has at most $n/\mu + 2$ completions to a perfect arc. The existence of m such completions implies the existence of a set of $m - 2$ mutually orthogonal latin squares of order $n + 1 - \mu$.

We mention, without proof, some further results concerning the case of equality in Theorem 3.6 for desarguesian planes. These can be proved using a technical result, Theorem 1 in Thas' [37]. Let A be a μ-arc in $PG_2(n)$ with T_1 and T_2 two completions of A to a perfect μ-arc, $|A| = n\mu - n + \mu - (n+1-n/\mu)$, $|T_1| = |T_2| = n + 1 - n/\mu$. Then A can be partitioned into sets $A = S_1 \dot{\cup} S_2 \dot{\cup} \cdots \dot{\cup} S_{\mu-1}$, $|S_i| = n + 1 - n/\mu$, with the following properties. Any line meeting two of the sets $T_1, T_2, S_1, S_2, \ldots, S_{\mu-1}$ intersects each of these $\mu + 1$ sets in exactly one point. Any line containing two points of one of these sets contains μ points of that set and no points of the others. The points of any one of these sets together with the lines meeting only that set form a block design on the parameters $b = n/\mu^2(n+1-n/\mu)$, $v = n + 1 - n/\mu$, $r = n/\mu$, $k = \mu$, $\lambda = 1$. The lines meeting all of these sets (necessarily each in one point) and the points in $T_1 \dot{\cup} T_2 \dot{\cup} S_1 \dot{\cup} \cdots \dot{\cup} S_{\mu-1}$ gives rise to a transversal design (see [44]) and hence to a set of $\mu - 1$ mutually orthogonal latin squares of order $n + 1 - n/\mu$. The above remarks imply that the union of any n of $T_1, T_2, S_1, \ldots, S_{\mu-1}$ is a perfect μ-arc in $PG_2(n)$.

We devote the remainder of this section to considering instances of arcs with at least two completions to a perfect arc. Because of the relatively small number of perfect μ-arcs known for $\mu > 2$ we have few examples of equality in the bounds of Theorem 3.7.

Consider Denniston's construction of perfect μ-arcs in $PG_2(n)$ for n even (Theorem 3.3). It can be shown, using arguments similar to those employed in the proof of the construction and facts concerning solutions of quadratic equations over finite fields (see [21], Chapter 1) that for $\phi(x,y) = ax^2 + hxy + by^2$, chosen as in Theorem 4.3, the number of solutions (x,y) of $\phi(x,y) = \alpha$ for some fixed α is $n+1$ if $\alpha \neq 0$ and 1 if $\alpha = 0$. Thus if H_1 and H_2 are two subgroups of the additive group of $GF(n)$, $|H_1| = |H_2| = \mu$, and A_1 and A_2 are the perfect μ-arcs produced in using $\emptyset(x,y)$ and Denniston's construction, we have that $|A_1 \cap A_2| = (|H_1 \cap H_2| - 1)(n+1) + 1 = |H_1 \cap H_2|(n+1) - n$ and $|A_1 \backslash A_2| = (\mu - |H_1 \cap H_2|)(n+1)$. This gives a large class of μ-arcs, $A_1 \cap A_2$, with two completions to a perfect arc. However, two additive subgroups of $GF(n)$, n even, can meet in at most half their elements. Hence $|H_1 \cap H_2| \leq \mu/2$ in the above and we cannot obtain equality in Theorem 3.7.

It is also possible to use the construction of Thas [36], mentioned near the end of Section 3. In the case $\mu^2 = n$, n even, we can obtain intersecting μ-arcs as follows. We refer directly to the construction there (not given in detail in this thesis). The reader should see [36] to follow our comments. By picking two different points p_1 and p_2 to produce μ-arcs A_1 and A_2 it can be shown that, using the properties of ovoids, if p_1 and p_2 are collinear with a point of the ovoid determining the arcs then $|A_1 \cap A_2| = \mu$, $|A_1 \backslash A_2| = \mu^3 - \mu^2$; and if p_1 and p_2 are not collinear with a point of the ovoid then $|A_1 \cap A_2| = \mu^2 - \mu$, $|A_1 \backslash A_2| = \mu^3 - 2\mu^2 + 2\mu$. The bounds of Theorem 3.7 imply $|A_1 \backslash A_2| \geq \mu^2 - \mu + 1$ so that we fall far short of the extremal case.

For $\mu = 2$ we have more success. Our bounds imply that two ovals, n even, can intersect in at most $(n+2)/2$ points. By using trial and error and Denniston's construction ovals A_1 and A_2 in $PG_2(8)$ can be found such that $|A_1 \cap A_2| = 1, 2, 3,$ and 5. We also have the duals to these.

In [19] M. Hall catalogues the ovals in $PG_2(16)$. By looking through his list we can easily find ovals intersecting in 9, 8, 7, 6, and 5 points. We can then dualize these to obtain intersecting 8-arcs.

With regard to Theorem 3.8, in $PG_2(4)$ a set of $3 = (n+2)/2$ points can be found with three different completions to an oval. This is the maximum number of completions possible since the exact bound on m in the proof of Theorem 3.8 gives $m \leq 3$ precisely for $\mu = 2$ and $n = 4$.

Section 5: Complete (k,μ)-arcs.

Here we consider small complete (k,μ)-arcs. Our first result is the following.

Theorem 3.9. A complete (k,μ)-arc in a Π_n must satisfy $n \leq \frac{(k-1)(k-2)}{\mu(\mu-1)}$, for $n > \frac{\mu(\mu-1)}{2}$, and $n \leq \frac{(k-\mu+1)(k-\mu)}{\mu(\mu-1)} + \mu - 2$ for $n \leq \frac{\mu(\mu-1)}{2}$.

This theorem improves the bound given by Bruen in [8], which is only applicable to planes of square order with $\mu \leq n$ (see also [4]). Our proof is also substantially shorter.

Proof. Let A be a complete (k,μ)-arc in a Π_n. Let ℓ be a line which meets A in a maximum number of points less than μ. Suppose $|A \cap \ell| = z < \mu$.

Since A is complete the points of $\ell \setminus A$ must be covered by at least one μ-secant. Any μ-secant passing through a point of $\ell \setminus A$ must intersect $A \setminus \ell$ in μ points. We count the maximum number of μ-secants to $A \setminus \ell$. The lines of Π_n intersecting the $k - z$ points of $A \setminus \ell$ induce a linear space on those points. Letting k_i = the length of the i-th line of this linear space and t = # of μ-secants to $A \setminus \ell$ we have, since this is a linear space on $k - z$ points,

$$t\mu(\mu-1) \leq \sum_i k_i(k_i - 1) = (k-z)(k-z-1),$$

$$t \leq \frac{(k-z)(k-z-1)}{\mu(\mu-1)}.$$

There must be sufficient μ-secants to $A \setminus \ell$ to cover the $n + 1 - z$ points of $\ell \setminus A$. Thus

$$n + 1 - z \leq \frac{(k-z)(k-z-1)}{\mu(\mu-1)},$$

$$n \leq \frac{(k-z)(k-z-1)}{\mu(\mu-1)} + (z-1).$$

Now $1 \leq z \leq \mu - 1$. The above expression achieves its maximum over this range at $z = 1$ for $k \geq \frac{\mu^2+1}{2}$ and at $z = \mu - 1$ for $k < \frac{\mu^2+1}{2}$. Thus we have

$$n \leq \frac{(k-1)(k-2)}{\mu(\mu-1)}, \quad \text{for } k \geq \frac{\mu^2+1}{2};$$

$$n \leq \frac{(k-\mu+1)(k-\mu)}{\mu(\mu-1)} + \mu - 2, \quad \text{for } k < \frac{\mu^2+1}{2}.$$

Now if $k < \frac{\mu^2+1}{2}$ the second bound holds and we then have, substitution for k,

$$n < \frac{(\frac{\mu^2+1}{2} - \mu+1)(\frac{\mu^2+1}{2} - \mu)}{\mu(\mu-1)} + \mu - 2,$$

which simplifies to $n \leq \frac{\mu(\mu-1)}{2}$. Thus if $n > \frac{\mu(\mu-1)}{2}$ we cannot have $k < \frac{\mu^2+1}{2}$ and hence the first bound (for $k \geq \frac{\mu^2+1}{2}$) holds. If $n \leq \frac{\mu(\mu-1)}{2}$ and $n > \frac{(k-\mu+1)(k-\mu)}{\mu(\mu-1)} + \mu - 2$ then necessarily $k \geq \frac{\mu^2+1}{2}$ (by our earlier bound). Hence we have

$$n > \frac{(\frac{\mu^2+1}{2} - \mu+1)(\frac{\mu^2+1}{2} - \mu)}{\mu(\mu-1)} + \mu - 2,$$

which simplifies to $n > \frac{\mu(\mu-1)}{2}$, contrary to $n \leq \frac{\mu(\mu-1)}{2}$. This establishes the two bounds in the statement of the theorem.

We mention that the derivation of the bound implies that equality in the case $n > \frac{\mu(\mu-1)}{2}$ can only hold for $\mu = 2$. We consider this case of equality in the above theorem for $\mu = 2$ (and $n > 1$). Let A be a complete $(k,2)$-arc in a Π_n with $n = \frac{(k-1)(k-2)}{2}$. Then the points of $\Pi_n \setminus A$ fall into two categories; those on no 1-secants, and those on at least one 1-secant. Those in the second category, by equality in the above argument, must lie on only one 2-secant and hence on $k-2$ 1-secants.

Consider the incidence structure, $\mathcal{P}$, with Points = points of $\Pi_n \setminus A$ on no 1-secants and Lines = 2-secants of Π_n. We easily have that # of lines in $\mathcal{P} = \binom{k}{2} = \frac{k(k-1)}{2}$ and that each point of $\mathcal{P}$ lies on $k/2$ lines of $\mathcal{P}$.

We can further compute the number of points on a line as follows. If ℓ is a 2-secant then there are $\binom{k-2}{2} = (k-2)(k-3)/2$ other secants meeting ℓ (these are generated by the $k-2$ points of A not on ℓ). Each intersection point of one of these with ℓ produces a point with >1 2-secant, hence exactly $k/2$ 2-secants (since there are only two kinds of points external to A). So the $(k-2)(k-3)/2$ 2-secants meeting ℓ intersect ℓ in bundles of $k/2 - 1$. Thus

$$(\#\text{ of points of } \mathcal{P} \text{ on } \ell) = \frac{(k-2)(k-3)/2}{k/2-1} = k-3.$$

Hence each line contains $k-3$ points. We can now count the number of points in $\mathcal{P}$. Since each point lies on $k/2$ lines, each line contains $k-3$ points, and $\mathcal{P}$ has $k(k-1)/2$ lines we have

$$(\#\text{ of points in } \mathcal{P}) = \frac{(k(k-1)/2)(k-3)}{k/2} = (k-1)(k-3).$$

Note that any pair of points of $\mathcal{P}$ lies on at most one line of $\mathcal{P}$. Suppose ℓ is a line of $\mathcal{P}$ and p a point of $\mathcal{P}$ with $p \notin \ell$. Then p lies on $k/2$ 2-secants. Now ℓ intersects A in two points, say p_1 and p_2. The lines of Π_n joining p to p_1 and p_2 must be 2-secants (since p lies only on 2-secants). The remaining $\frac{k}{2} - 2$ 2-secants through p meet ℓ (necessarily in points of $\mathcal{P}$ since they lie on at least two 2-secants, ℓ and the 2-secants through p). Thus we have shown that a point p and a line ℓ, of $\mathcal{P}$, with $p \notin \ell$ together meet $(k-4)/2$ lines of $\mathcal{P}$.

We have shown that $\mathcal{P}$ is a partial geometry (see Section 1). Thus a complete (k,2)-arc in a Π_n with $n = (k-1)(k-2)/2$ implies the existence of a partial geometry with parameters

$$\begin{aligned} \#\text{ of points} &= (k-1)(k-3), \\ \#\text{ of lines} &= k(k-1)/2, \\ \#\text{ of points on a line} &= k-3, \\ \#\text{ of lines on a point} &= k/2, \\ \alpha &= (k-4)/2. \end{aligned}$$

We cannot find examples of this since this would require a plane of non-prime power order.

We now prove one additional bound for complete (k,μ)-arcs. This is an improvement over Theorem 3.9 for values of μ close to n.

Theorem 3.10. A complete (k,μ)-arc in a Π_n must satisfy:

$$n^2 + n + 1 \leq \frac{(n+1-\mu)k(k-1)}{\mu(\mu-1)} + k.$$

Proof. Let A be a complete (k,μ)-arc in a Π_n. Then, as in the proof of Theorem 3.9, the k points of A generate at most $k(k-1)/\mu(\mu-1)$ μ-secants. Each of these μ-secants covers $n+1-\mu$ of the points of $\Pi_n \backslash A$. These μ-secants cover a maximum number of points of $\Pi_n \backslash A$ if they are disjoint outside of A. That is,

$$\begin{pmatrix} \#\text{ of points covered} \\ \text{by } \mu\text{-secants} \end{pmatrix} \leq \frac{(n+1-\mu)k(k-1)}{\mu(\mu-1)}.$$

But since A is complete all of the $n^2 + n + 1 - k$ points of $\Pi_n \backslash A$ must be covered and thus the theorem follows.

We consider the case of equality in Theorem 3.10. By the proof, a (k,μ)-arc A achieving equality must have only 0-secants, 1-secants, and

μ-secants. Also every pair of μ-secants must meet within A (otherwise they will meet outside of A creating a point of $\Pi_n \setminus A$ covered by at least two μ-secants). Thus the incidence structure of A and its μ-secants satisfies: Every pair of points determines a μ-secant and every pair of μ-secants meets (in a point of A). Thus A and its μ-secants are a subplane of order $\mu - 1$ of Π_n.

Now let ℓ be a tangent to A, $p = \ell \cap A$. Since A is a subplane of order $\mu - 1$ we have that

$$(\# \text{ of } \mu\text{-secants}) = (\mu - 1)^2 + (\mu - 1) + 1 = \mu^2 - \mu + 1.$$

Through p, μ of these will pass. The remaining $\mu^2 - 2\mu + 1$ will meet ℓ in points outside of A. Since A is complete each of these n points of $\ell \setminus A$ will be covered by at least one μ-secant, and by equality in the bound, by no more than one μ-secant. Hence

$$n = \mu^2 - 2\mu - 1 = (\mu - 1)^2.$$

Thus n is a square and $\mu = \sqrt{n} + 1$. A subplane of order $\sqrt{n}$ in a plane of order n is called a *Baer subplane*. Hence equality in Theorem 3.10 holds if and only if A is a Baer subplane.

We now turn our attention to the following result of Segre [30].

Theorem 3.11. A (k,2)-arc in $PG_2(n)$ with $k > n - \sqrt{n} + 1$ for n even, or $k > n - \sqrt{n}/4 + 7/4$ for n odd can be uniquely completed to an oval.

We do not prove this here. It requires the Hasse-Weil Theorem of algebraic geometry. The interested reader should see [30] or [21], pp. 221-240.

We construct here complete (k,2)-arcs in $PG_2(n)$ with

$k = n - \sqrt{n} + 1$, for n even. These are the largest possible, by Theorem 3.10. We will require an alternate description of $PG_2(n)$ using difference sets. For more details see [5], [26], or [18], pp. 120-166.

A desarguesian projective plane of order n can be described as follows. Let points be all elements of the group $\mathbb{Z}_{n^2+n+1}$. Lines will be all translates of a fixed set $D \subseteq \mathbb{Z}_{n^2+n+1}$, $|D| = n + 1$. D is an example of a difference set. Note that the mapping of points $p \mapsto p + i$ for any i preserves collinearity. Mappings of this type are called *collineations* of the plane.

Consider now a desarguesian projective plane of order n (necessarily a prime power) and square, presented as a difference set D in $\mathbb{Z}_{n^2+n+1}$. Note that we can factor $n^2 + n + 1 = (n+\sqrt{n}+1)(n-\sqrt{n}+1)$ since n is a square. Partition the set of points in the plane into sets A_j, $j = 0, 1, \ldots, n+\sqrt{n}$

$$A_j = \{i : i \equiv j (\mathrm{mod}\ n+\sqrt{n}+1)\} \subseteq \mathbb{Z}_{n^2+n+1}.$$

Thus $|A_j| = n - \sqrt{n} + 1$ for all j.

Now suppose $|A_j \cap D| \leq 2$ for all j. Then since lines are all translates of D we have that each A_j is an $(n-\sqrt{n}+1,2)$-arc. We claim additionally that each A_j is a complete 2-arc, for n even.

Suppose not; without loss of generality say $A_0 \cup \{s\}$ is a $(k,2)$-arc. Then by Theorem 3.11 $A_0 \cup \{s\}$ is uniquely completable to an oval. In fact A_0 itself is uniquely completable to an oval, by Theorem 3.6 and the fact that $(n+2)/2 \leq n - \sqrt{n} + 1$ for $n \geq 2$. Thus any point not on a 2-secant to A_0 is in the completion (since otherwise we would have more than one completion, one with the point in, the other with it out). But note that $p \mapsto p + (n + \sqrt{n} + 1)$ is a collineation of the plane which fixes A_0. Since s lies on no 2-secants to A_0 ($A_0 \cup \{s\}$ is a 2-arc), $s + (n + \sqrt{n} + 1)$ lies on no 2-secants to $A_0 + (n + \sqrt{n} + 1) = A_0$. In fact we then have that all points in A_i, where $s \in A_i$, lie on no 2-secants to A_0. Thus all of these points must be in the completion of A_0 to an oval. But $|A_0 \cup A_i| = 2(n - \sqrt{n} + 1) > n + 2$, a contradiction. Hence A_0 is complete.

Thus we have shown that if $|A_j \cap D| \leq 2$ for all j then every A_j is a complete $(n - \sqrt{n} + 1, 2)$-arc.

We can verify by hand, using the difference sets listed in [5], that for $n = 2^2, 3^2, 4^2, 5^2, 7^2, 8^2, 9^2$ that this procedure does produce 2-arcs. Note that this is true even for odd n. We can only prove that the 2-arcs are complete (using the above argument) for n even, however. We conjecture that in general this procedure always produces 2-arcs. In any event we have shown that complete 2-arcs exist meeting the bound of Theorem 3.11 for $n = 2^2, 4^2$, and 8^2.

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