OPTIMUM RANGE OF A WINGLESS ROCKET ABOUT A ROTATING EARTH

Thesis by

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SUMMARY

The motion of a wingless rocket in a vacuum about a spherical non-rotating earth describes the elliptic orbit of a material point of mass in a central field of force. The corrections for the rotation of the earth are applied to this solution to determine the effects on optimum range. Since there is no simple mathematical solution for optimizing the range, it is necessary to obtain ranges for several angles of elevation at each initial velocity and then find the optimum range by using interpolation formulas. Only the case of firing at the equator is computed although the formulas and the computational procedure outlined can be applied at any latitude.

Firing to the East gives the greatest range for a given initial velocity and firing to the West gives the least range. The angle of elevation is lowest firing East and highest firing West, although for low velocities the difference is not great and for the 5,000 mph. solution a constant angle of elevation gives ranges varying at most only one-tenth of a mile from the maximum for any direction of fire. However, for the 15,000 mph. solution all the factors are critical in obtaining the maximum range.

TABLE OF CONTENTS

SYMBOLS AND DEFINITIONS

R	= radius of the spherical earth, 3958.89 miles
v_r	= initial velocity relative to the earth
v_e	= tow speed of the earth
g_o	= non-rotational earth gravitational acceleration
μ	= $g_o R^2$
ω	= angular velocity of the earth
v_s	= spatial velocity
ψ_r	= relative angle of elevation
ψ_s	= spatial angle of elevation
β_r	= relative angle of fire, measured counterclockwise from East
β_s	= spatial angle of fire, measured clockwise from South
β_r'	= relative bearing of point of fall, measured counterclockwise from East
a	= semi-major axis of the elliptic trajectory
b	= semi-minor axis of the elliptic trajectory
ε	= eccentricity of the elliptic trajectory
η	= one-half of the angle subtended at the center of the non-rotating earth by the trajectory
t_f	= time of flight
α	= co-latitude of the point of fire
λ	= longitude of point of fire, arbitrarily chosen as $\pi/2$
ϕ	= latitude of point of fall
θ	= longitude of point of fall on non-rotating earth relative to λ (θ also used as an angle at the center of the earth as defined in Fig. 1)
$\theta-\omega t_f$	= longitude of point of fall on rotating earth relative to λ

ϑ = $\theta - \omega t_f - \frac{\pi}{2}$, longitude of point of fall on rotating earth relative to a point of fire having longitude zero

γ = 2 , the angle subtended at the center of the non-rotating earth by the trajectory

γ' = the angle subtended at the center of the rotating earth by the trajectory (Range = $\gamma' R$)

LIST OF TABLES

LIST OF FIGURES

PART I

INTRODUCTION

The motion of a wingless rocket in a vacuum about a spherical non-rotating earth describes the elliptic orbit of a material point of mass in a central field of force. The equations for such a trajectory are well defined and can be readily optimized, but when considering the effects of a rotating earth there is no simple mathematical solution for optimizing the range.

To analyze these effects it is necessary to solve the equations for various initial velocities at various angles of elevation and optimize the results by interpolation formulas.

The difficulty of carrying out more than eighty consecutive and, generally, dependent computations and arriving with the required accuracy in the answer is inherent in the problems. The computational procedure outlined in Part V and carried out in Table I for an initial velocity of 5,000 mph, firing Northeast at the equator is the end result of many attempts to shorten the computations yet preserve the accuracy. The procedure can be carried out for any latitude of the point of fire and is not restricted to the equator case.

PART II

MOTION OF A MATERIAL POINT OF MASS IN A CENTRAL FIELD OF FORCE*

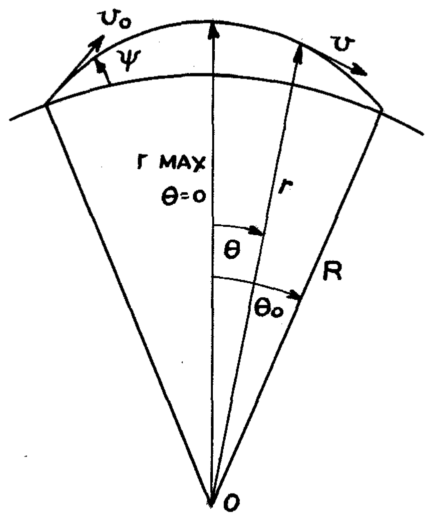

Fig. 1

Let 0 be the center of the earth having a radius R, then the attractive force for mass m is:

$$F = \frac{Km}{r^2}$$

where K is a constant and r is the distance from 0 to the mass.

$$P.E. = -\int_{\infty}^{r} \frac{Km}{r^2}(-dr) = -\frac{Km}{r},$$ if the potential energy at infinity is chosen as zero.

The Kinetic energy is:

$$K.E. = \frac{1}{2}mv^2, \text{ where } v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2$$

$$K.E. = \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + \left(\frac{rd\theta}{dt}\right)^2\right]$$

In a conservative field the total energy is constant, hence:

$$\frac{1}{2}mv_o^2 - \frac{Km}{R} = \frac{1}{2}m\left[\frac{dr}{dt}^2 + \left(\frac{rd\theta}{dt}\right)^2\right] - \frac{Km}{r} \qquad (1)$$

A further condition is that the moment of momentum remain constant about 0, hence:

$$m\left(r\frac{d\theta}{dt}\right)r = m\left(v_o \cos\psi\right)R \qquad (2)$$

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*Reference (1), page 38

Equations (1) and (2) describe the motion of m.

Putting $\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt}$ in Eq. (2)

$$\frac{dr}{dt} = \frac{Rv_o \cos\psi}{r^2} \frac{dr}{d\theta}$$

, and using this relation in Eq. (1).

$$\frac{R^2 vo^2 \cos^2\psi}{r^4} \left(\frac{dr}{d\theta}\right)^2 + \frac{R^2 vo^2 \cos^2\psi}{r^2} - \frac{2K}{r} = v_o^2 - \frac{2K}{R}$$

$$\frac{dr}{d\theta}\left(\frac{Rv_o \cos\psi}{r^2}\right) = \left[v_o^2 - \frac{2K}{R} + \frac{2K}{r} - \frac{R^2 v_o^2 \cos^2\psi}{r^2}\right]^{\frac{1}{2}}$$

Since $\frac{K}{r} = \frac{Rv_o \cos\psi}{r} \cdot \frac{K}{Rv_o \cos\psi}$

$$d\theta = \frac{-d\left[\frac{Rv_o \cos\psi}{r} - \frac{K}{Rv_o \cos\psi}\right]}{\left[\left(v_o^2 - \frac{2K}{R} + \frac{K^2}{R^2 v_o^2 \cos^2\psi}\right) - \left(\frac{Rv_o \cos\psi}{r} - \frac{K}{Rv_o \cos\psi}\right)^2\right]^{\frac{1}{2}}}$$

Integration gives:

$$\cos(\theta + \theta') = \frac{\frac{Rv_o \cos\psi}{r} - \frac{K}{Rv_o \cos\psi}}{\left[v_o^2 - \frac{2K}{R} + \frac{K^2}{R^2 v_o^2 \cos^2}\right]^{\frac{1}{2}}}$$

where θ' is the integration constant.

Solving for r,

$$r = \frac{Rv_o \cos\psi}{\frac{K}{Rv_o \cos\psi} + \left[v_o^2 - \frac{2K}{R} + \frac{K^2}{R^2 v_o^2 \cos^2\psi}\right]^{\frac{1}{2}} \cos(\theta + \theta')}$$

For r to have its maximum value $\theta = 0$ and therefore θ' must be π giving

$$r_{max} = \frac{Rv_o \cos\psi}{\frac{K}{Rv_o\cos\psi} - \left[v_o^2 - \frac{2K}{R} + \frac{K^2}{R^2 v_o^2 \cos^2\psi}\right]^{\frac{1}{2}}}$$

Now the general form for the equation of a conic section is:

$$r = \frac{p}{1 - \varepsilon\cos\theta}$$, where p and ε are constants.

Let g_o be the gravitational constant for a non-rotating earth at the surface, and putting $K = \mu^2$, then

$$\frac{\mu^2 m}{R^2} = mg_o, \quad \text{or } g_o = \frac{\mu^2}{R^2}, \text{ therefore}$$

$$r = \frac{\frac{v_o^2}{g_o}\cos^2\psi}{1 - \left[\left(\frac{v_o^2}{Rg_o}\right)^2 \cos^2\psi \left(1 - \frac{2g_oR}{v_o^2}\right) + 1\right]^{\frac{1}{2}} \cos\theta}$$

The eccentricity is then

$$\varepsilon = \left[\left(\frac{v_o^2}{Rg_o}\right)^2 \cos^2\psi\left(1 - \frac{2g_oR}{v_o^2}\right) + 1\right]^{\frac{1}{2}}$$

and if $\varepsilon < 1$, the orbit is an ellipse

if $\varepsilon = 1$, the orbit is a parabola

and if $\varepsilon > 1$, the orbit is a hyperbola.

For ε to be less than 1, then

$$\frac{2g_oR}{v_o^2} > 1 \quad \text{or} \quad v_o < \sqrt{2g_oR}$$

For $v_o = \sqrt{g_oR}$ then $\varepsilon = \sin\psi$, and if firing horizontally ($\psi = 0$) the eccentricity vanishes giving a circular orbit about the earth.

All of the following computations are for velocities less than the above "escape velocity", thus only the elliptic orbit is considered.

Using the polar form of the elliptic equation*, the half major axis, a, is:

$$a = \frac{p}{1 - \varepsilon^2}$$

$$= \frac{\frac{v_o^2}{g_o} \cos^2\psi}{1 - \left[\left(\frac{v_o^2}{Rg_o}\right)^2 \cos^2\psi \left(1 - \frac{2g_oR}{v_o^2}\right) + 1\right]}$$

$$= \frac{R}{2 - \frac{v_o^2}{g_oR}} = \frac{\mu^2}{\frac{2\mu^2}{R} - v_o^2}$$

The half minor axis, b, is:

$$b = a\sqrt{1 - \varepsilon^2}$$

$$= a\sqrt{\left(\frac{v_o^2}{g_oR}\right)^2 \cos^2\psi \left(\frac{2g_oR}{v_o^2} - 1\right)}$$

$$= \frac{av_oR\cos\psi}{\mu}\sqrt{\frac{2}{R} - \frac{v_o^2}{\mu^2}}$$

or

$$b^2 = a\left(\frac{v_oR\cos\psi}{\mu}\right)^2$$

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*Reference (2), Chap. VI, §17,19.

PART III

ELLIPTIC TRAJECTORY ABOUT A NON-ROTATING EARTH

Consider now an elliptic trajectory of a wingless rocket (i.e. point mass) in a vacuum about a spherical, non-rotating earth.

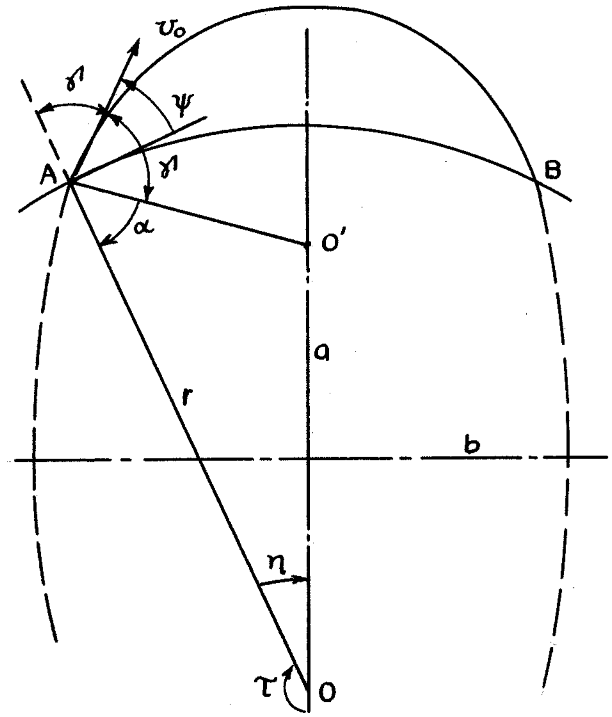

Fig. 2

The foci are 0 and 0' where 0 is the center of the earth.

$$\gamma = \frac{\pi}{2} - \psi$$

$$\alpha = \pi - 2\gamma = 2\psi$$

$$AO' = 2a - AO$$

$$= 2a - R = \ell$$

Hence: $$0'0 = \left[R^2 + \ell^2 - 2R\ell\cos 2\psi\right]^{\frac{1}{2}}$$

and: $$\sin\eta = \frac{\ell \sin 2\psi}{\left[R^2 + \ell^2 - 2R\ell\cos 2\psi\right]^{\frac{1}{2}}}$$

The arc length AB is the range such that:

$$\text{Range} = 2\eta R$$

The time of flight to traverse the distance from A to B is given by*:

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*Reference (2), Chap. VI, §19.

$$t_f = \frac{p^2}{v_o R\cos\psi} \int_{\pi-\eta}^{\pi+\eta} \frac{d\tau}{(1 + \varepsilon\cos\tau)^2}$$

$$= \frac{a}{\mu}^{3/2} \Big[u - \varepsilon \sin u \Big]_{\tau = \pi-\eta}^{\tau = \pi+\eta}$$

in which,

$$u = 2\tan^{-1}\left[\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\ \tan\frac{\tau}{2}\right]$$

PART IV

EFFECT OF ROTATING EARTH

To determine the effect of the rotating earth, the latitude and longitude of the point of fall on the non-rotating earth is first found to be* (see Fig. 3):

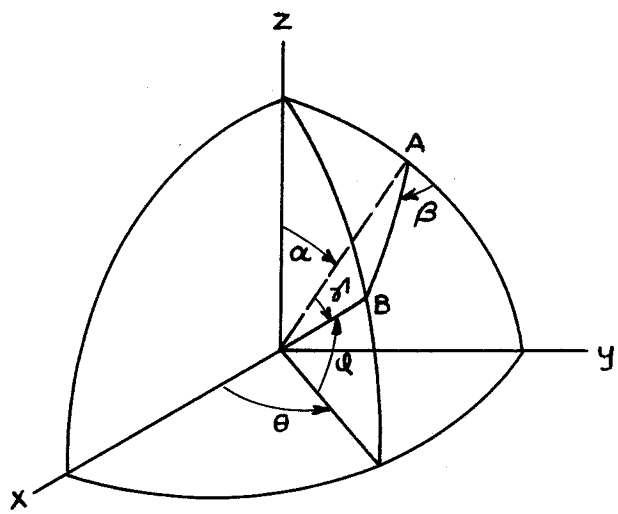

Fig. 3

Latitude:

$$\phi = \sin^{-1}(\cos\gamma\ \cos\alpha - \sin\gamma\ \cos\beta\ \sin\alpha)$$

Longitude:

$$\theta = \cos^{-1}\left[\frac{\sin\gamma\ \sin\beta}{\sin^2\gamma\sin^2\beta + (\sin\gamma\cos\beta\ \cos\alpha - \cos\gamma\sin\alpha)^2}\right]$$

The great circular arc AB is the projection of the trajectory on the non-rotating earth. Using the equations derived above with the initial velocity as the vector sum of the relative initial velocity with respect to the earth's surface and the tow speed due to the rotation of the earth yields a trajectory with respect to a non-rotating system of coordinates with the earth's center as the origin.

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*Reference (2), Chapt. VI, § 20

Since t_f time units are required to go from A to B then if ω denotes the angular velocity of the earth's rotation, the earth rotates about an angle ωt_f from West to East. On the rotating earth, the longitude of the point of fall then becomes $\theta - \omega t_f$ and the latitude is ϕ.

PART V

COMPUTATIONAL PROCEDURE FOR ROTATING EARTH

Since the radius of the earth R appears in many of the equations and the velocities are large, the calculations are considerably simplified by taking velocity units as radii of the earth per hour. Thus any place that R appears in the above equations unity replaces it.

The initial values and constants used are:

For 5,000 mph. relative initial velocity,

$v_r = 1.26298027$ Radii/hour

For 10,000 mph.

$v_r = 2.52596054$ Radii/hour

For 15,000 mph.

$v_r = 3.78894082$ Radii/hour

The tow speed of the earth at the equator is:

$v_e = 0.26251645$ Radii/hour

The non-rotational gravitational acceleration at zero height is taken as that at 45^o latitude on the actual earth and then corrected for non-rotation.

$g_o = 19.997270$ Radii/hour^2

and

$\mu^2 = g_o R^2 = 19.997270$ Radii^3/hour^2

$\mu = 4.4718307$ Radii$^{3/2}$/hour

The angular velocity of the earth is taken as

$\omega = .26251645$ Radians/hour

Columns 2 through 22 of Table I give the vector summation of relative initial velocity and tow speed of the earth. The spatial velocity, v_s, the spatial angle of elevation, ψ_s, and the spatial angle of fire, β_s, are used in the remaining calculations. (Note that β_r is measured counter-clockwise from East, and β_s is measured clockwise from South solely because of computational ease.) The equation for a using R = 1 becomes:

$$a = \frac{1}{2 - \left(\frac{v_s}{\mu}\right)^2}$$

and is computed in Columns 23 through 25. Columns 26 and 27 are required later on and give a^2 and $a^{3/2}/\mu$.

The equation for ℓ solved for in Column 28 is:

$$\ell = 2a - 1$$

Column 29 gives ℓ^2.

The equation for b solved for in Column 30 is:

$$b^2 = a\cos^2\psi_s \left(\frac{v_s^2}{\mu 2}\right)$$

The equation for ε^2 solved for in Columns 31 and 32 is:

$$\varepsilon^2 = 1 - \frac{b^2}{a^2}$$

Related functions of ε are given in Columns 33 through 36.

The equation for η solved for in Columns 37 through 43 is:

$$\eta = \sin^{-1} \frac{\ell \sin 2\psi_s}{\sqrt{\ell^2 + 1 - 2\ell\cos 2\psi_s}}$$

The equation for t_f solved for in Columns 44 through 54 is:

$$t_f = \frac{a^{3/2}}{\mu}\left[6.2831853 - 4\tan^{-1}\left\{\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\left(1.5707963 - \frac{\eta}{2}\right)\right\} + 2\varepsilon\sin\left\{2\tan^{-1}\left(1.5707963 - \frac{\eta}{2}\right)\right\}\right]$$

Columns 55 through 85 are spherical trigonometric formulas and are self-explanatory. (Note that α is the co-latitude of the point of fire).

Table II shows the interpolation procedure for optimizing the range. The Gregory-Newton interpolation formula is:

$$y = y_o + n\Delta y_o + \frac{n(n-1)}{2} \Delta^2 y_o + \ldots\ldots\ldots$$

and

$$\frac{dy}{dx} = \frac{1}{\Delta x}\left[\Delta y_o + \frac{(2n-1)}{2}\Delta^2 y_o + \ldots\ldots\ldots\right]$$

Where Δy_o is the first difference, $\Delta^2 y_o$, the second difference, etc; Δx is the constant difference between x's; $n = \frac{x-x_o}{x}$. Hence for $\frac{dy}{dx} = 0$ (maximum range),

$$n = .5 - \frac{\Delta y_o}{\Delta^2 y_o}$$

and

$$x_m = x_o + .5\Delta x - \frac{\Delta x \Delta y_o}{\Delta^2 y_o}$$

The maximum ranges and the corresponding angles of elevation thus found are shown graphically in Figs. 4, 5, 6.

The time of flight, t_f, and the azimuthal angle of bearing of the point of fall are found graphically from Figs. 7, 8, 9, and Figs. 10, 11, 12, 10a, 11a, 12a.

The pertinent results from all the computations are listed in Tables III, IV, and V, while the maximums of these Tables are listed in Table VI.

PART VI

COMPUTATIONAL PROCEDURE FOR NON-ROTATING EARTH

To obtain comparative figures for the non-rotational earth case at each velocity it is only necessary to maximize the equation previously given for η :

$$\sin \eta = \frac{\ell \sin 2\psi}{1 + \ell^2 - 2\ell \cos 2\psi}$$

for η max then:

$$\cos 2\psi = \ell$$

and

$$\sin \eta = \ell$$

where,

$$\ell = 2a - 1 = \frac{2}{2 - \frac{v_o^2}{\mu 2}} - 1$$

Table VII shows the calculations and results for the three velocities used in the non-rotational earth solution.

PART VII

DISCUSSION OF RESULTS

Figs. 13, 14, and 15 show cross plots of Figs. 4, 5, and 6 as compared with the non-rotational values of Ψ and range.

Figs. 16 and 17 give a comprehensive graphical solution to a particular firing problem. Given the range and direction of fire, then the angle of elevation, Ψ , the initial velocity, v_r, and time of flight, t_f, can be found.

Fig. 18 shows the change in deviation of the point of fall with increasing initial velocity for the case of firing Northward at the equator. The left deviation at low velocities is due to the rocket having a constant angular momentum about the axis of the earth during its flight and the trajectory remains mostly outside of a hypothetical cylinder having the earth's axis and being tangent to the earth at the equator. For the rocket to land on the same bearing as it was fired, the angular momentum would have to vary directly as the rocket's distance from the earth's axis throughout its flight. Hence portions of the trajectory outside the cylinder cause left deviation while portions inside the cylinder cause right deviation. Due to the flatness of the earth's surface for short ranges there is an almost linear increase in left deviation for initial velocities up to 6,000 mph. As the velocity increases, the angle of elevation decreases so that less and less of the trajectory is outside of the cylinder. At an initial velocity of 11,600 mph the trajectory is such that the left deviation portion cancels the right deviation portion. At higher velocities the deviation increases rapidly to the right.

A similar analysis can be applied to the range effect and to cases of firing at latitudes other than the equator.

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(2) "Ballistics of the Future", J. M. J. Kooy and J. W. H. Uytenbogaart, 1946, McGraw-Hill Book Company, New York and London.

(3) Tables of Sines and Cosines, National Bureau of Standards, 1940.

(4) Table of Arc Tan X, National Bureau of Standards, 1942.

TABLE I *

Rotating Earth Solution for Range at ψ_r = .650, .725, and .800, v = 5000 mph, Firing Northeast ($\beta_r = \pi/4$)

1	2	3	4	5
			sin(3)	cos(3)
Row	β_r	ψ_r	$\sin\psi_r$	$\cos\psi_r$
2a	$\pi/4$	.65000	.60518641	.79608380
b	$\pi/4$	.72500	.66313544	.74849942
c	$\pi/4$	.80000	.71735609	.69670671

6	7	8	9	10
	(6)x(4)	(6)x(5)		sin(2)
v_r	$v_r \sin\psi_r$	$v_r \cos\psi_r$	v_e	$\sin\beta_r$
1.26298027	.76433850	1.00543813	.26251645	.70710670
1.26298027	.83752698	.94534000	.26251645	.70710670
1.26298027	.90600659	.87992683	.26251645	.70710670

11	12	13	14	15
cos (2)	(8) x (11) + (9)	$(12)^2$	$((8)\text{x}(10))^2$	$\sqrt{(13)+(14)}$
$\cos\beta_r$				$v_s \cos\psi_s$
.70710670	.97346849	.9476409	.5054528	1.2054434
.70710670	.93097269	.8667101	.4468338	1.1460996
.70710670	.88471861	.7827269	.3871355	1.0816019

16	17	18	19	20
$(7)^2$	(16) + (13) + (14)	$\sqrt{(17)}$	(7) ÷ (18)	(15) ÷ (18)
	v_s^2	v_s	$\sin\psi_s$	$\cos\psi_s$
.5842133	2.0373073	1.4273427	.5354975	.8445368
.7014514	2.0149956	1.4195054	.5900132	.8073936
.8208479	1.9907106	1.4109254	.6421364	.7665904

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*References (3) and (4) were used in these computations

20A	21	22	23	24
$(20)^2$	2 x (19) x (20)	2 x (20A) - 1	$\frac{(17)}{19.997270}$	2 - (23)
$\cos^2 \psi_s$	$\sin 2\psi_s$	$\cos 2\psi_s$	v_s^2/μ^2	1/a
.7132424	.9044947	.4264848	.1018793	1.8981207
.6518844	.9527458	.3037688	.1007635	1.8992365
.5876608	.9845112	.1753216	.0995491	1.9004509

25	26	27	28	29
1/ (24)	$(25)^2$	$\frac{\sqrt{(25)} \times (25)}{4.4718307}$	2 x (25) - 1	$(28)^2$
a	a^2	$\frac{a^{3/2}}{\mu}$	ℓ	ℓ^2
.5268369	.2775571	.08551236	.0536738	.0028809
.5265274	.2772311	.08543702	.0530548	.0028148
.5261909	.2768769	.08535513	.0523818	.0027439

30	31	32	33	34
(20A) x (25) x (23)	(30) / (26)	1 - (31)	$\sqrt{(32)}$	1 - (33)
b^2	b^2/a^2	ε^2	ε	$1 - \varepsilon$
.0382824	.1379262	.8620738	.9284793	.0715207
.0345856	.1247537	.8752463	.9355460	.0644540
.0307827	.1111783	.8888217	.9427734	.0572266

35	36	37	38	39
1 + (33)	$\sqrt{\frac{(34)}{(35)}}$	(28) x (21)	(29) + 1	2 x (28) x (22)
$1 + \varepsilon$	$\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$	$\ell \sin 2\psi_s$	$\ell^2 + 1$	$2\ell \cos 2\psi_s$
1.9284793	.1925788	.0485477	1.0028809	.0457821
1.9355460	.1824834	.0505477	1.0028148	.0322328
1.9427734	.1716278	.0515705	1.0027439	.0183673

40	41	42	43	44
(38) - (39)	$\sqrt{(40)}$	(37) / (41)	$\sin^{-1}$ (42)	(43) / 2
		$\sin \eta$	η	$\eta/2$
.9570988	.9783143	.0496238	.0496442	.0248221
.9705820	.9851812	.0513080	.0513305	.0256653
.9843766	.9921575	.0519781	.0520015	.0260008

45	46	47	48	49
1.5707963 - (44)	tan (45)	(36) x (46)	$\tan^{-1}$ (47)	2 x (48)
$\tau/2$	$\tan \tau/2$			u_1
1.5459742	40.27839	7.7567640	1.4425838	2.8851676
1.5451310	38.95455	7.1085587	1.4310379	2.8620758
1.5447955	38.45164	6.5993704	1.4204108	2.8408216

50	51	52	53	54
6.2831853 -4 x (48)	sin (49)	2 x (33) x (51)	(50) + (52)	(27) x (53)
$u_2 - u_1$	$\sin u_1$	$2 \varepsilon \sin u_1$		t_f
.5128501	.2536241	.4709695	.9838196	.0841287
.5590337	.2758912	.5162178	1.0752515	.0918663
.6015421	.2962567	.5586059	1.1601480	.0990246

55	56*	58	59	60
.26251645 x (54)	2 x (43)		sin (56)	cos (56)
ωt_f	γ	α	$\sin \gamma$	$\cos \gamma$
.0220852	.0992884	$\pi/2$	.0991254	.9950749
.0241164	.1026610	$\pi/2$	.1024808	.9947349
.0259956	.1040030	$\pi/2$	.1038156	.9945966

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* No column 57

61	62	63	64	65
$-\frac{(12)}{(15)}$	$\sqrt{(14)}$ $-\overline{(15)}$	sin (58)	cos (58)	(60) x (64)
$\sin\beta_s$	$\cos\beta_s$	sin α	cos α	cosγcosα
-.8075605	-.5897847	1.000	0	0
-.8122966	-.5832446	1.000	0	0
-.8179707	-.5752599	1.000	0	0

66	67	68	69	70
(59) x (62) x (63)	(65) - (66)	$\sin^{-1}$ (67)	cos (68)	(59) x (61)
sinγcos β_s x sinα	sin φ	φ	cos φ	sinγsin β_s
-.0584626	.0584626	.0584959	.9982895	-.0800498
-.0597714	.0597714	.0598071	.9982120	-.0832448
-.0597210	.0597210	.0597566	.9982150	-.0849181

71	72	73	74	75
(70) / (69)	$\cos^{-1}$ (71)	(72) - (55)	(73) - $\frac{\pi}{2}$	cos (74)
cos θ	θ	θ - ωt_f	ϑ	cos ϑ
-.0801870	1.6510695	1.6289843	.0581879	.9983075
-.0833939	1.6542872	1.6301708	.0593744	.9982377
-.0850699	1.6559692	1.6299736	.0591773	.9982495

76	77	78	79	80
(69) x (75)	$\cos^{-1}$ (76)	sin (77)	3958.89 x (77)	(67) / (78)
cosγ'	γ'	sinγ'	Range	sin β_r'
.9965999	.0824854	.0823918	326.551	.7095682
.9964529	.0842512	.0841516	333.541	.7102824
.9964676	.0840758	.0839768	332.847	.7111607

81	82*	83*	84*	85*
$\sin^{-1}$ (80)	tan (68)	sin (74)	(82)/(83)	$\tan^{-1}$ (84)
β_r'	$\tan \phi$	$\sin \vartheta$	$\tan \beta_r'$	β_r'
.7888852 .7898993 .7911466				

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* Cols. 82 through 85 must be used in place of Cols. 80 and 81 to obtain β_r' accurately when $\pi/4 < \beta_r < 3\pi/4$.

TABLE II

Optimum Range Solution for Results of Table I

v_r = 5000 mph. $\beta_r = \pi/4$

For x_o = .80, ΔX = -.075

1	2	3	4	5
ψ_r	Range	1st Diffs.	2nd Diff.	(3)/(4)
x	y	Δy_o	$\Delta^2 y_o$	$\Delta y_o / \Delta^2 y_o$
.800 .725 .650	332.847 333.541 326.551	0.694 -6.990	-7.684	-.0903

6	7	8	9	10
.075 x(5)	.7625 +(6)	.5 -(5)	$\frac{(8) - 1}{2}$	(8) x (9)
	x_m	n	$\frac{n-1}{2}$	$\frac{n(n-1)}{2}$
-.0068	.7557	.5903	-.2049	-.1210

11	12	13	14	
(2)	(8)x(3)	(10) x(4)	(11)+ (12) + (13)	
y_o	$n\Delta y_o$	$\frac{n(n-1)}{2}\Delta^2 y_o$	y_m	
332.847	.410	.930	334.187	

TABLE III

Range, Time of Flight, and Point of Fall Bearing for Various Angles of Elevation at v = 5000 mph.

v_r = 1.26298027 (5000 mph)

Row	β_r	ψ_r	t_f	Range	β_r'
1a	0	.65000	.0848425	328.715	0
b	0	.72500	.0925949	335.402	0
c	0	.80000	.0997507	334.325	0
2a	$\pi/4$	.65000	.0841287	326.551	.7889
b	$\pi/4$	.72500	.0918663	333.541	.7899
c	$\pi/4$	.80000	.0990246	332.847	.7911
3a	$\pi/2$	.65000	.0824536	321.448	1.5757
b	$\pi/2$	.72500	.0902379	329.433	1.5771
c	$\pi/2$	.80000	.0973135	329.324	1.5788
4a	$3\pi/4$	.65000	.0808777	316.532	2.3597
b	$3\pi/4$	.72500	.0885033	324.821	2.3606
c	$3\pi/4$	.80000	.0956602	325.859	2.3618
5a	π	.65000	.0801944	314.493	π
b	π	.72500	.0878358	323.059	π
c	π	.80000	.0949921	324.448	π

TABLE IV

Range, Time of Flight and Point of Fall Bearing for Various Angles of Elevation at v = 10, 000 mph.

v_r = 2.52596054 (10,000 mph.)

Row	β_r	ψ_r	t_f	Range	β_r'
1a	0	.57500	.2063123	1564.249	0
b	0	.65000	.2256390	1588.199	0
c	0	.72500	.2430204	1573.430	0
2a	$\pi/4$	.57500	.2016607	1537.220	.7883
b	$\pi/4$	.65000	.2209331	1565.315	.7900
c	$\pi/4$	.72500	.2383689	1555.475	.7923
3a	$\pi/2$	.65000	.2102985	1512.581	1.5776
b	$\pi/2$	.72500	.2277953	1513.545	1.5806
c	$\pi/2$	.80000	.2437099	1481.100	1.5845
4a	$3\pi/4$	.65000	.2005900	1463.115	2.3611
b	$3\pi/4$	.72500	.2180675	1473.502	2.3632
c	$3\pi/4$	.80000	.2341679	1452.095	2.3658
5a	π	.65000	.1968131	1443.505	π
b	π	.72500	.2142643	1457.437	π
c	π	.80000	.2304174	1440.010	π

TABLE V

Range, Time of Flight, and Point of Fall Bearing for Various Angles of Elevation at v = 15,000 mph.

v_r = 3.78894082 (15,000 mph.)

Row	β_r	ψ_r	t_f	Range	β_r'
1a	0	.35000	.4704944	5403.947	0
b	0	.42500	.5156391	5420.436	0
c	0	.50000	.5526277	5290.940	0
2a	$\pi/4$	.35000	.4368645	5105.443	.7548
b	$\pi/4$	.42500	.4836113	5186.277	.7531
c	$\pi/4$	.50000	.5224184	5111.081	.7543
3a	$\pi/2$	.42500	.4367317	4676.989	1.5415
b	$\pi/2$	.50000	.4593001	4717.839	1.5396
c	$\pi/2$	.57500	.4951522	4643.161	1.5425
4a	$3\pi/4$	.50000	.4076910	4362.052	2.3406
b	$3\pi/4$	.57500	.4448764	4368.866	2.3420
c	$3\pi/4$	.65000	.4783027	4284.959	2.3449
5a	π	.50000	.3890890	4225.707	π
b	π	.57500	.4264796	4260.205	π
c	π	.65000	.4603740	4204.210	π

TABLE VI

Rotating Earth Optimum Range, Time of Flight, and Point of Fall Bearing for v = 5000, 10,000 and 15,000 mph.

v = 5000 mph.

β_r	ψ_r	Range	β_r'	t_f
0	.7521	335.908	0	.0948
$\pi/4$	.7557	334.187	.7906	.0944
$\pi/2$	.7635	330.391	1.5780	.0934
$3\pi/4$	.7732	326.321	2.3614	.0923
π	.7770	324.785	π	.0919

v = 10,000 mph.

β_r	ψ_r	Range	β_r'	t_f
0	.6589	1588.472	0	.2277
$\pi/4$	.6680	1566.415	.7912	.2253
$\pi/2$	.6897	1517.254	1.5786	.2199
$3\pi/4$	.7120	1473.978	2.3629	.2151
π	.7208	1457.487	π	.2133

v = 15,000 mph.

β_r	ψ_r	Range	β_r'	t_f
0	.3960	5431.199	0	.4995
$\pi/4$	.4264	5186.305	.7534	.4835
$\pi/2$	.4890	4719.079	1.5403	.4540
$3\pi/4$	.5431	4377.057	2.3364	.4302
π	.5661	4260.840	π	.4226

TABLE VII

Non-rotating Earth Solution for Optimum Range

1	2	3	4	5
	①^2/19.997270	2 - ②	1 /③	2 x④ - 1
v_o	v_o^2 / μ^2	1/a	a	$\ell = \sin \eta$
1.26298027	.07976685	1.92023315	.52077010	.04154020
2.52596054	.31906739	1.68093261	.59490785	.18981570
3.78894082	.71790162	1.28209838	.77997135	.55994270

6	7	8	9
$\cos^{-1}$ ⑤	⑥ / ②	$\sin^{-1}$ ⑤	⑧x7917.78
2ψ	ψ	η	$2\eta R$
1.52924417	.76462209	.04155216	329.001
1.37982188	.68991094	.19097443	1512.094
.97647960	.48823980	.59431671	4705.669

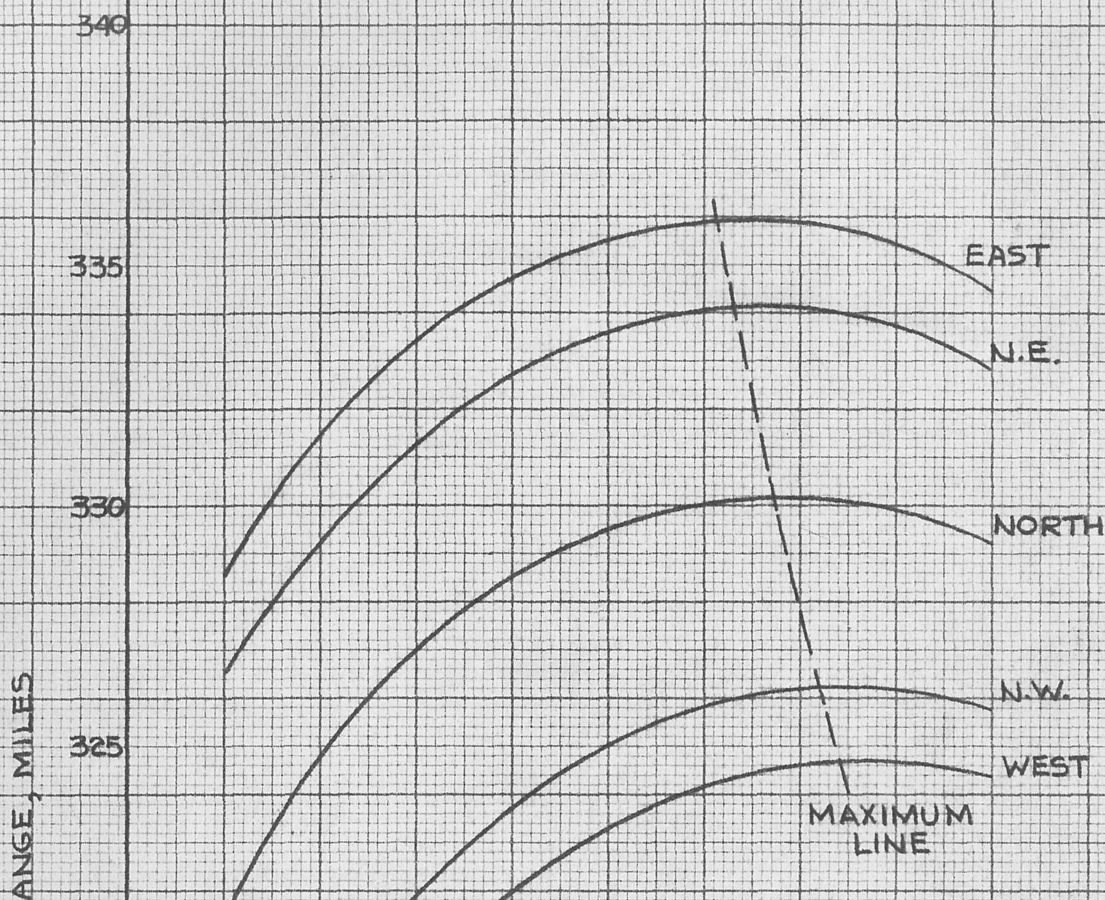

FIG. 4

RANGE vs. ANGLE OF ELEVATION FOR v = 5000 m.p.h.

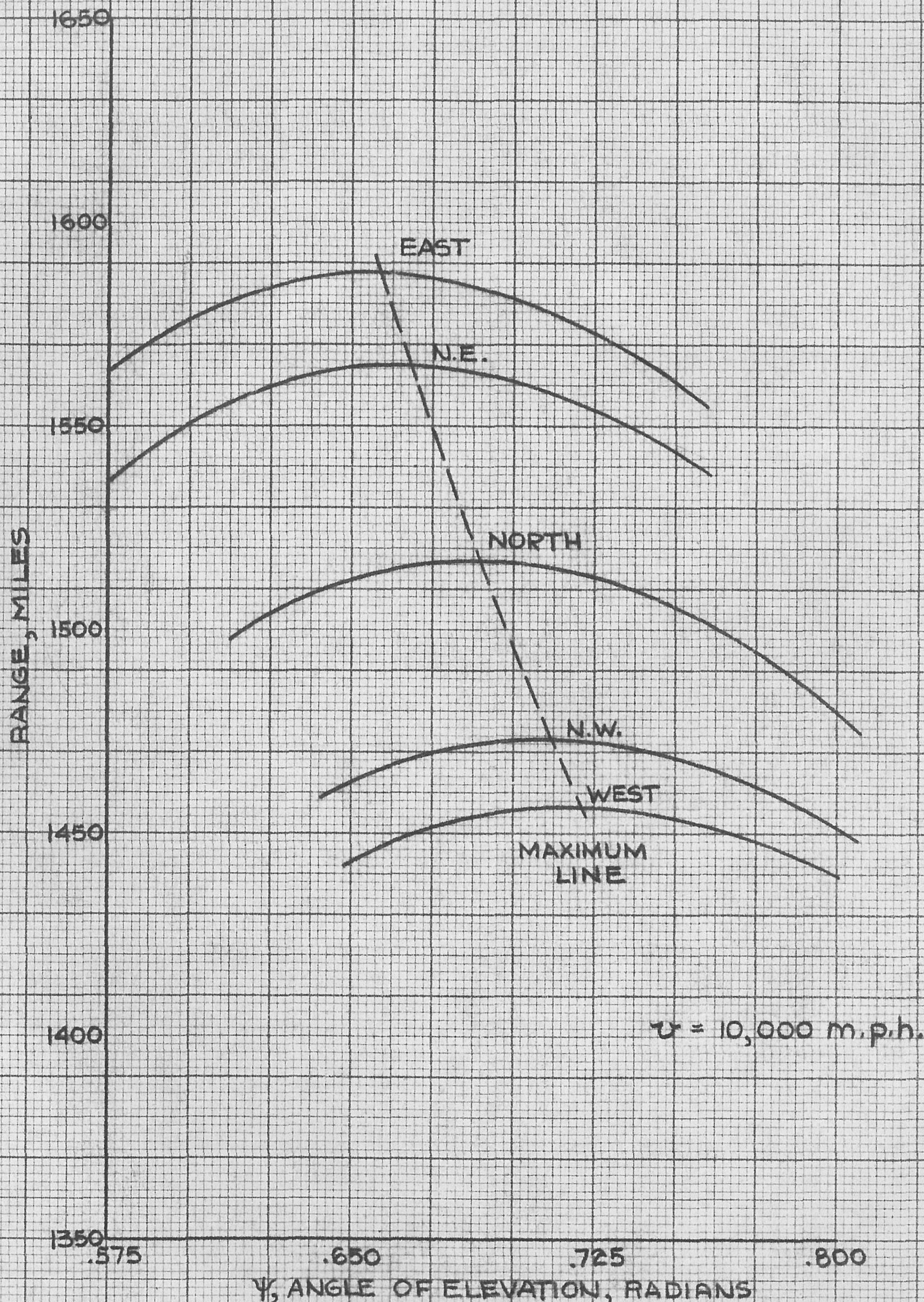

FIG. 5

RANGE VS ANGLE OF ELEVATION FOR

v = 10,000 m.p.h.

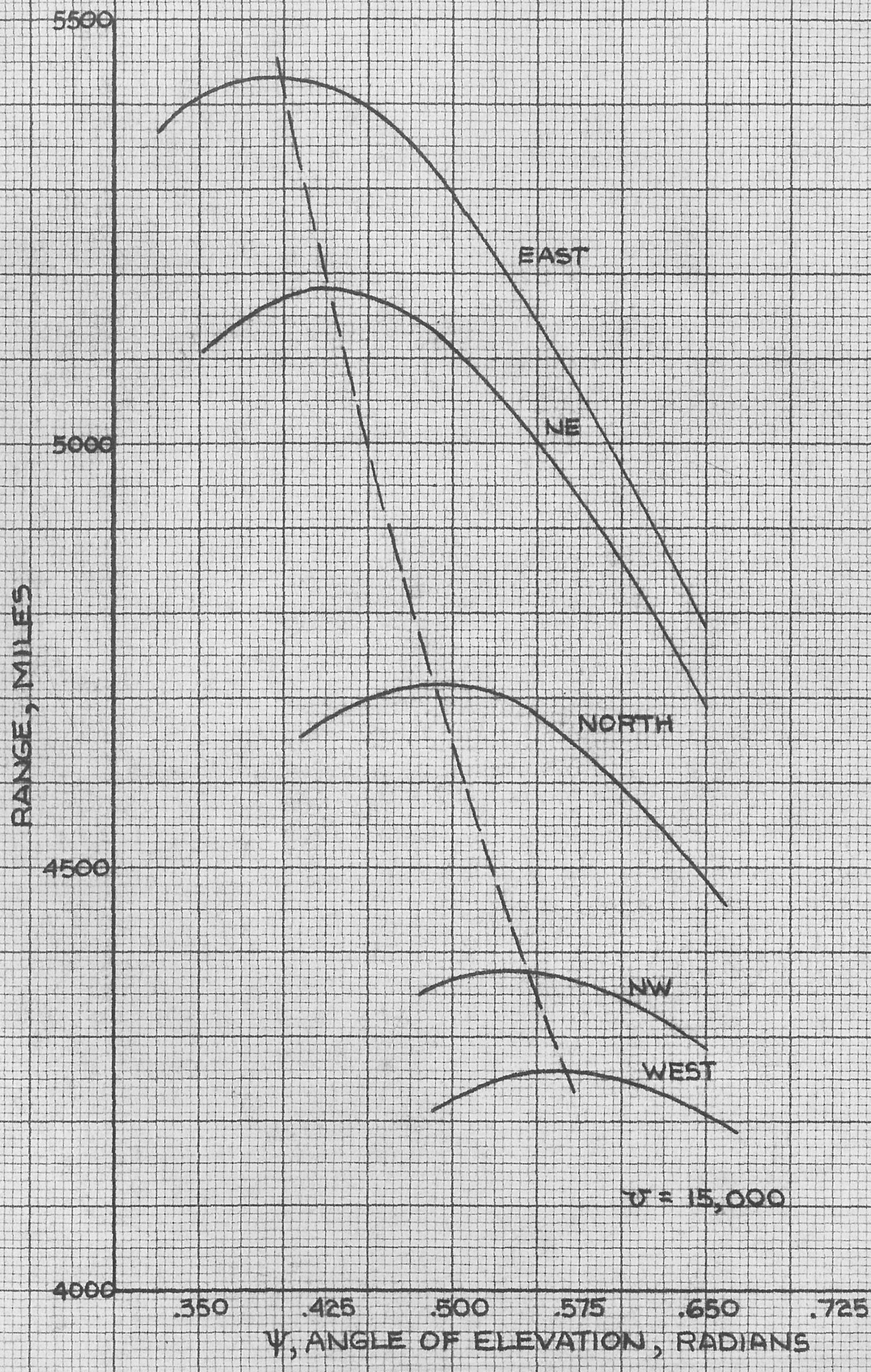

FIG. 6

RANGE VS ANGLE OF ELEVATION FOR

v = 15,000 m.p.h.

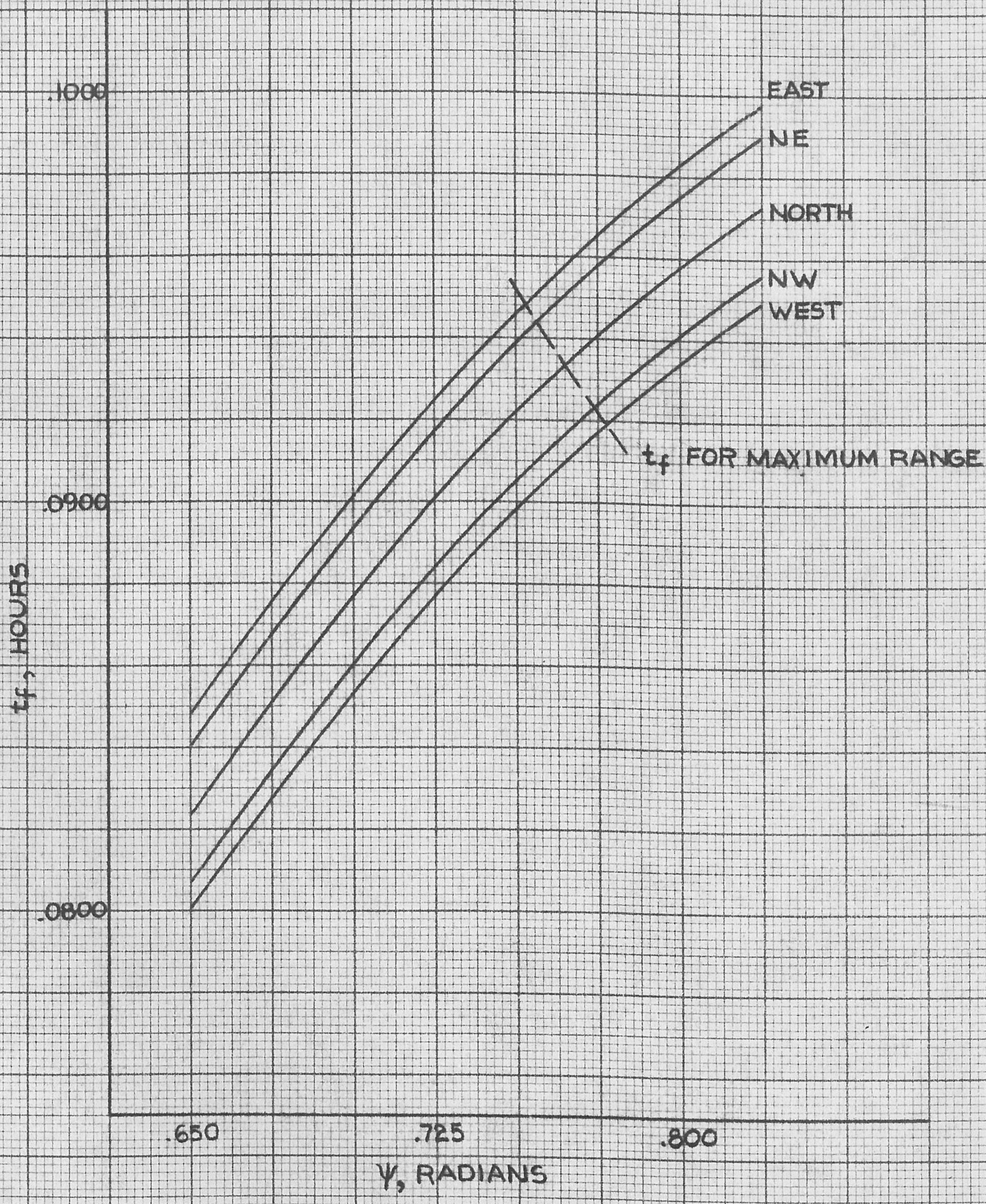

FIG. 7

TIME OF FLIGHT vs ANGLE OF ELEVATION

FOR v = 5000 mph

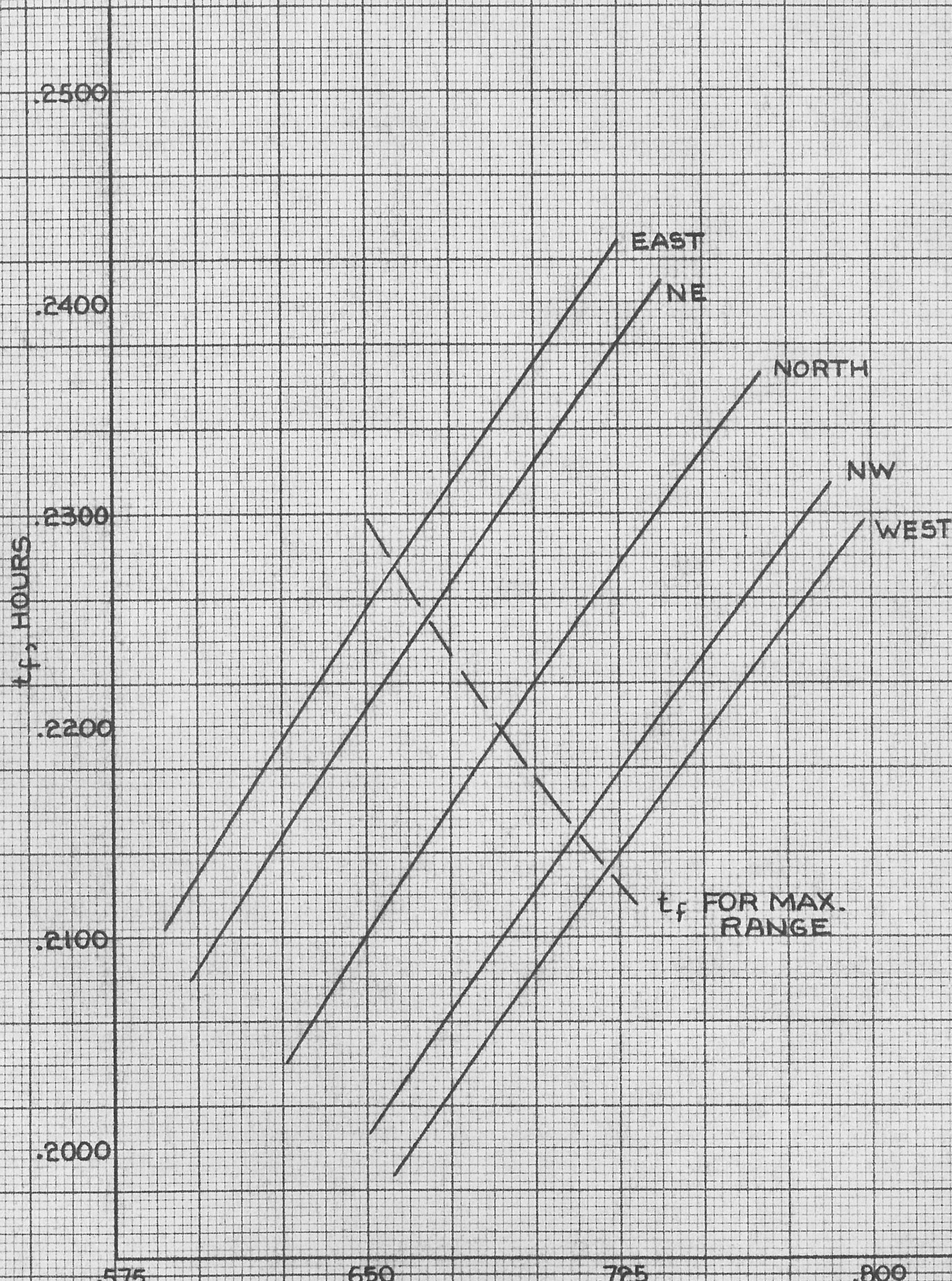

FIG. 8

TIME OF FLIGHT vs ANGLE OF ELEVATION

FOR v = 10,000 m.p.h.

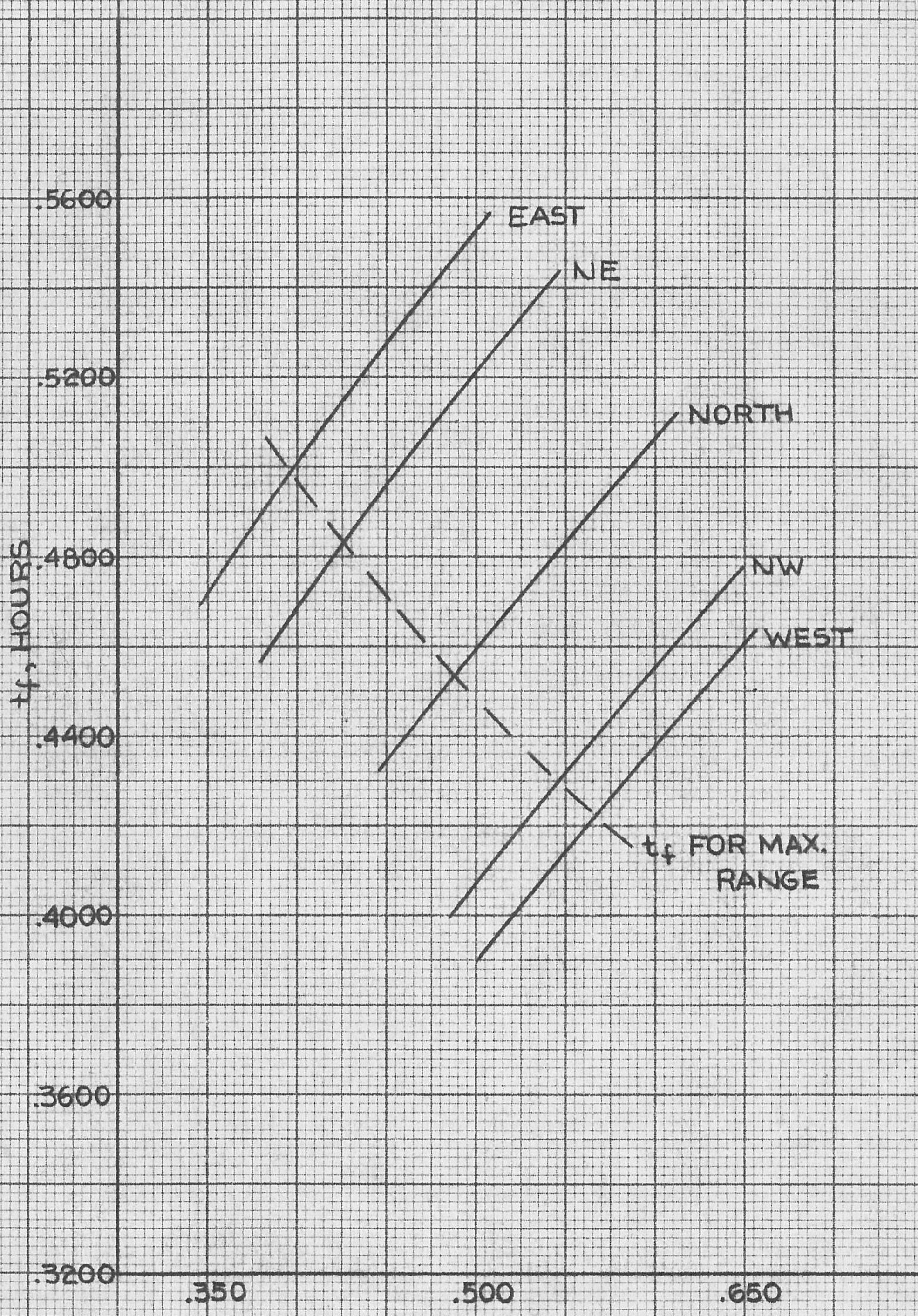

FIG. 9

TIME OF FLIGHT vs ANGLE OF ELEVATION FOR v = 15,000 m.p.h.

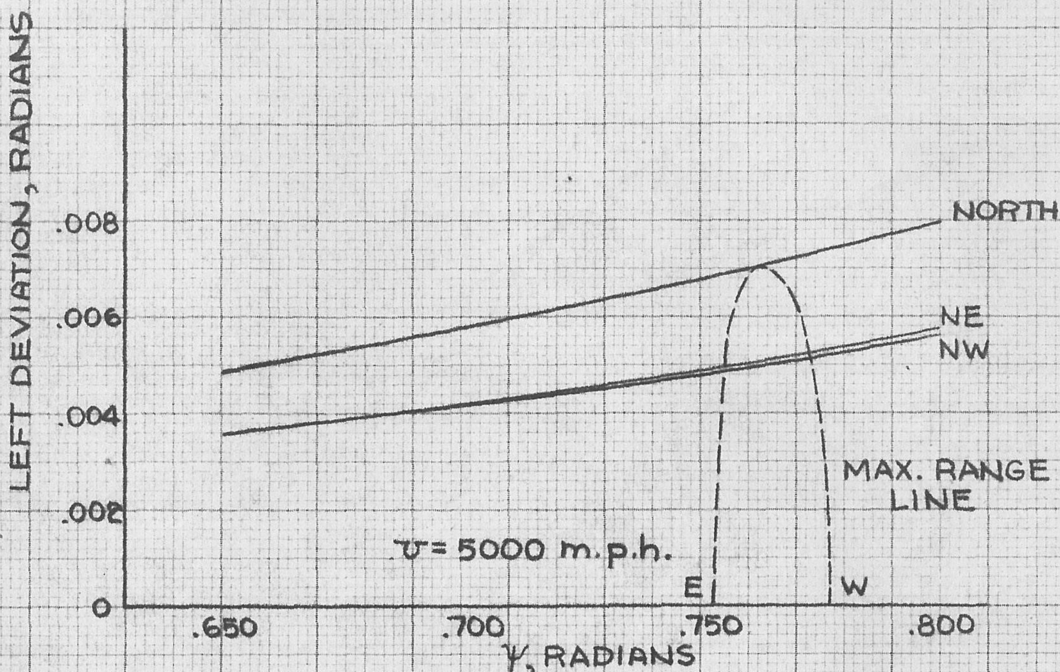

FIG. 10
DEVIATION OF POINT OF FALL FROM FIRING LINE FOR v = 5000 m.p.h.

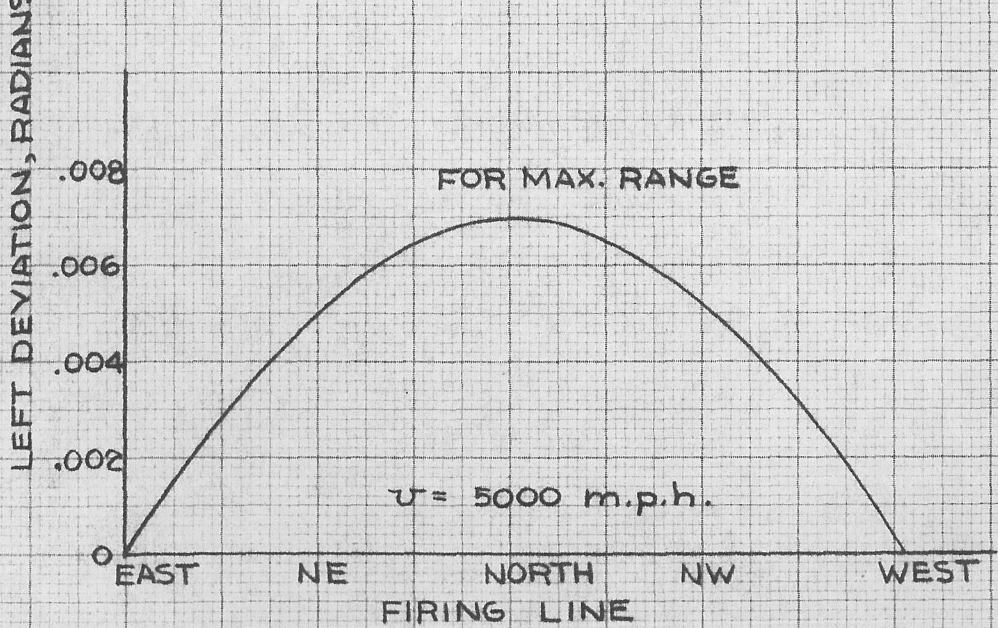

FIG. 10a
DEVIATION VS FIRING LINE FOR v = 5000 m.p.h.

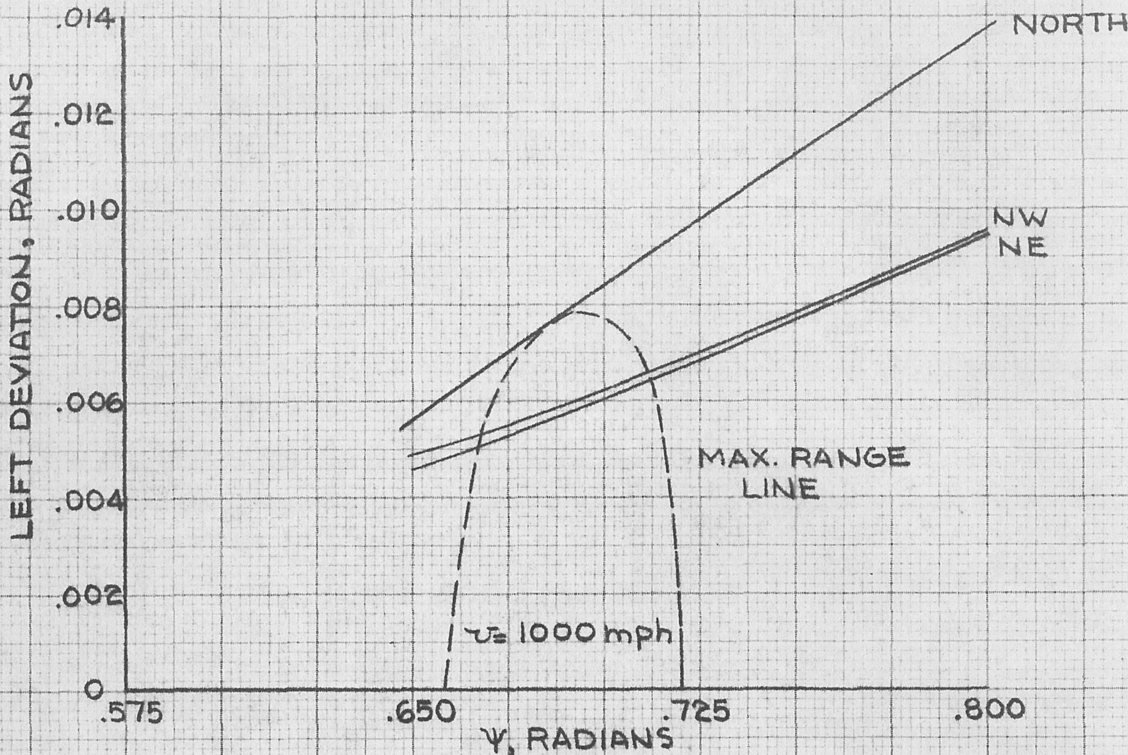

FIG. 11
DEVIATION OF POINT OF FALL FROM FIRING LINE FOR $v = 10{,}000$ m.p.h.

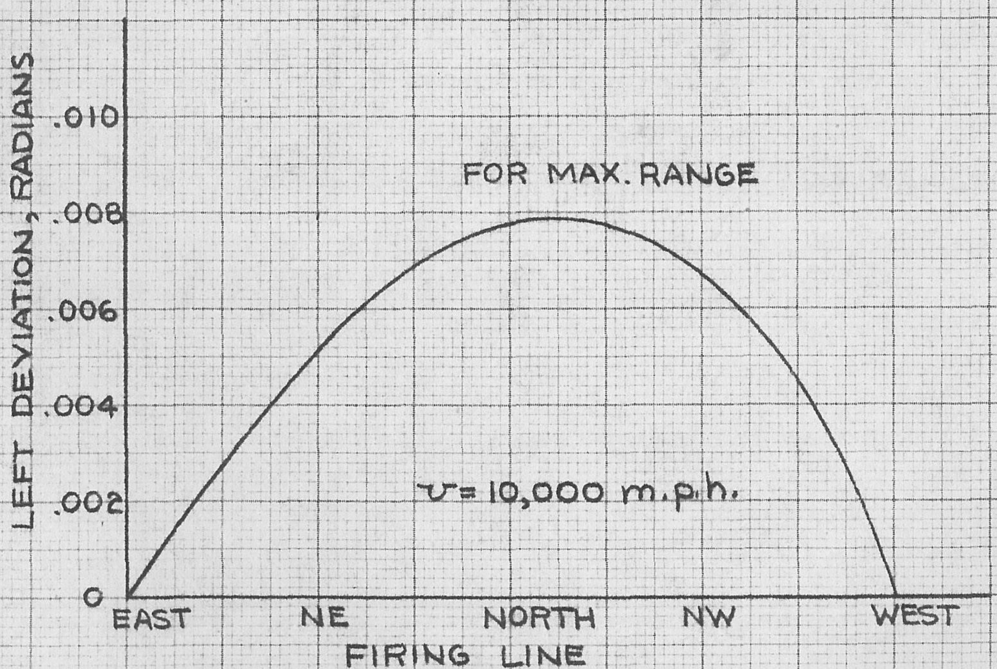

FIG. 11a
DEVIATION vs FIRING LINE FOR $v = 10{,}000$ m.p.h.

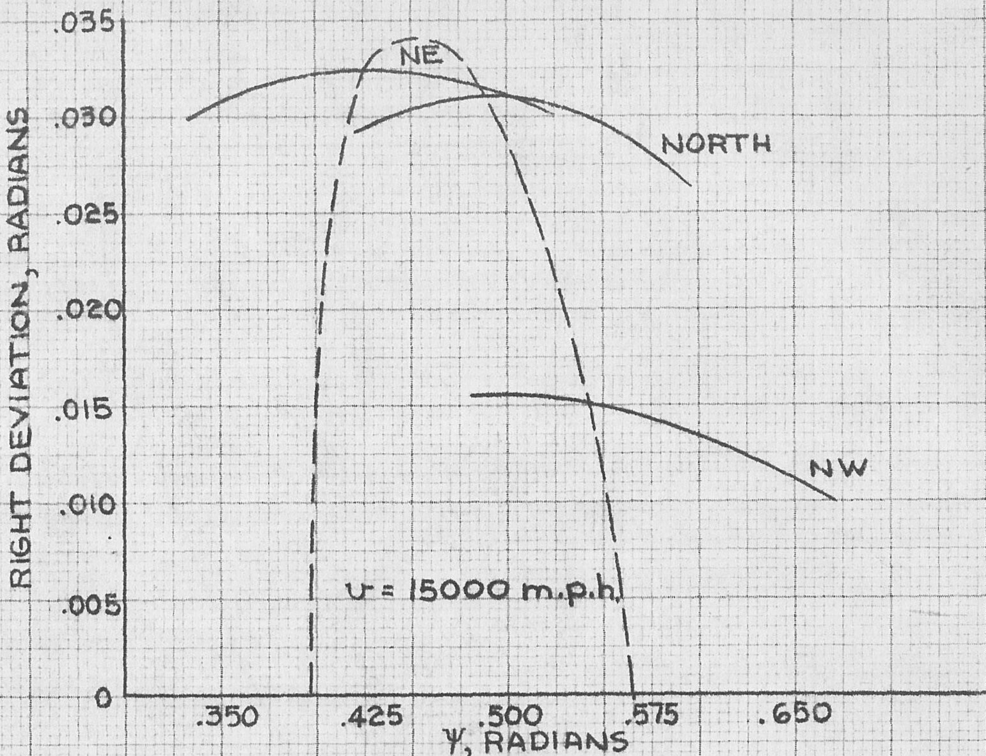

FIG 12

DEVIATION OF POINT OF FALL FROM FIRING LINE FOR v = 15,000 m.p.h.

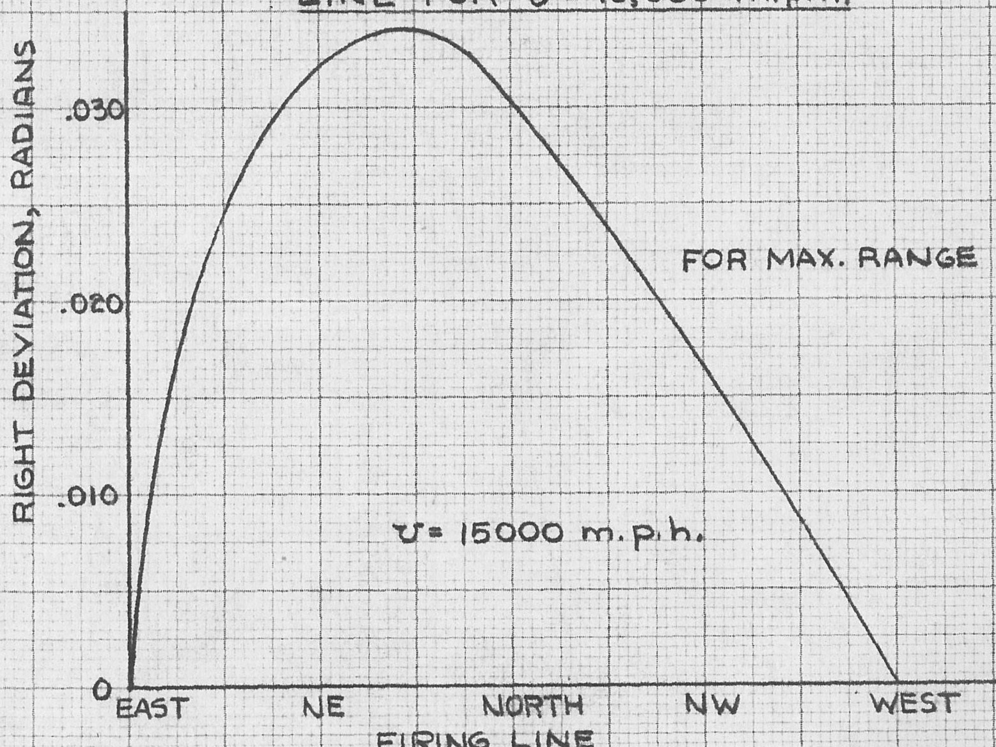

FIG 12a

DEVIATION vs FIRING LINE FOR v = 15,000 m.p.h.

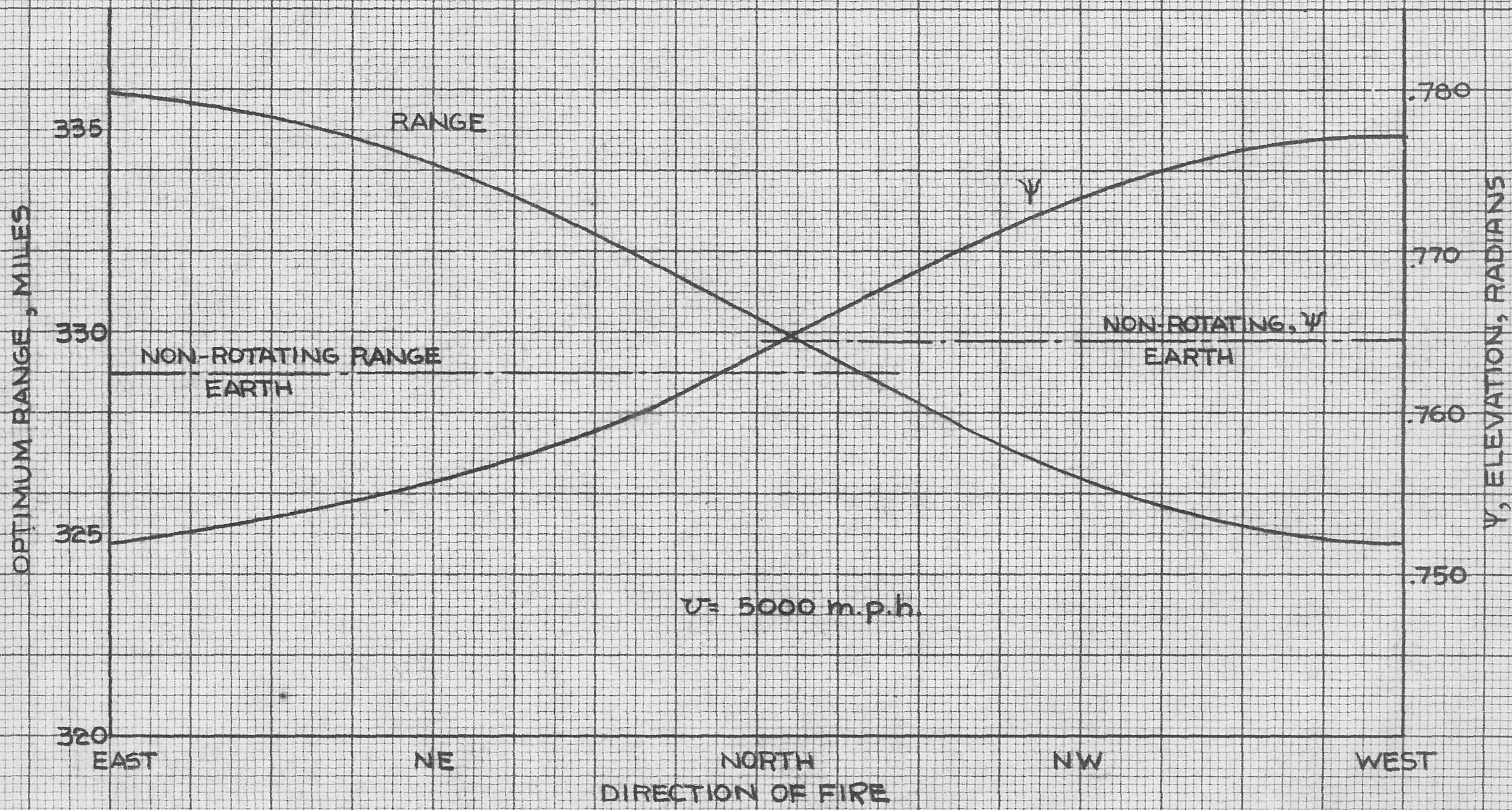

FIG. 13

OPTIMUM RANGE AND ANGLE OF ELEVATION vs DIRECTION OF FIRE FOR v = 5000 m.p.h.

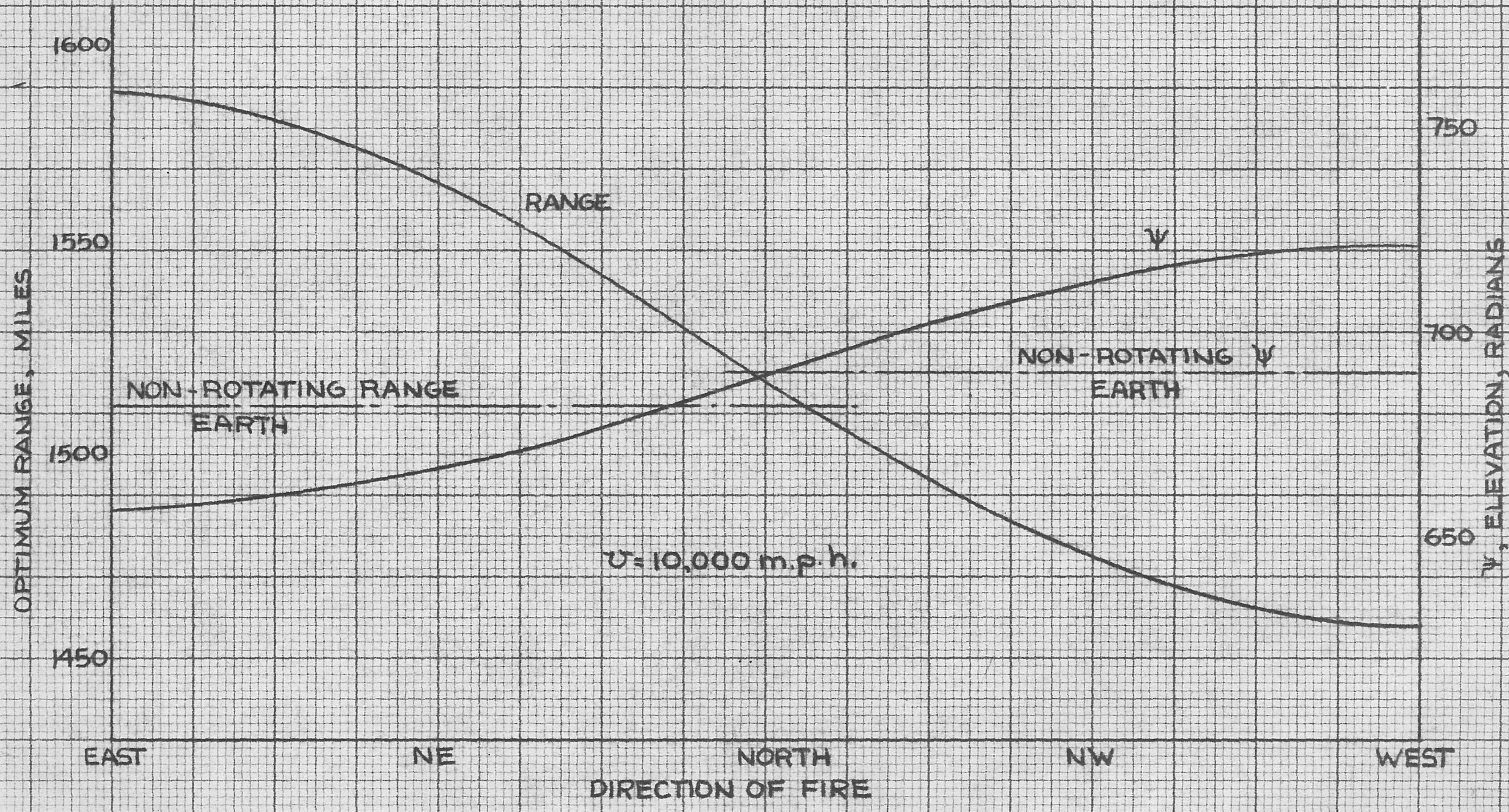

FIG. 14

OPTIMUM RANGE AND ANGLE OF ELEVATION VS DIRECTION OF FIRE

FOR v = 10,000 m.p.h.

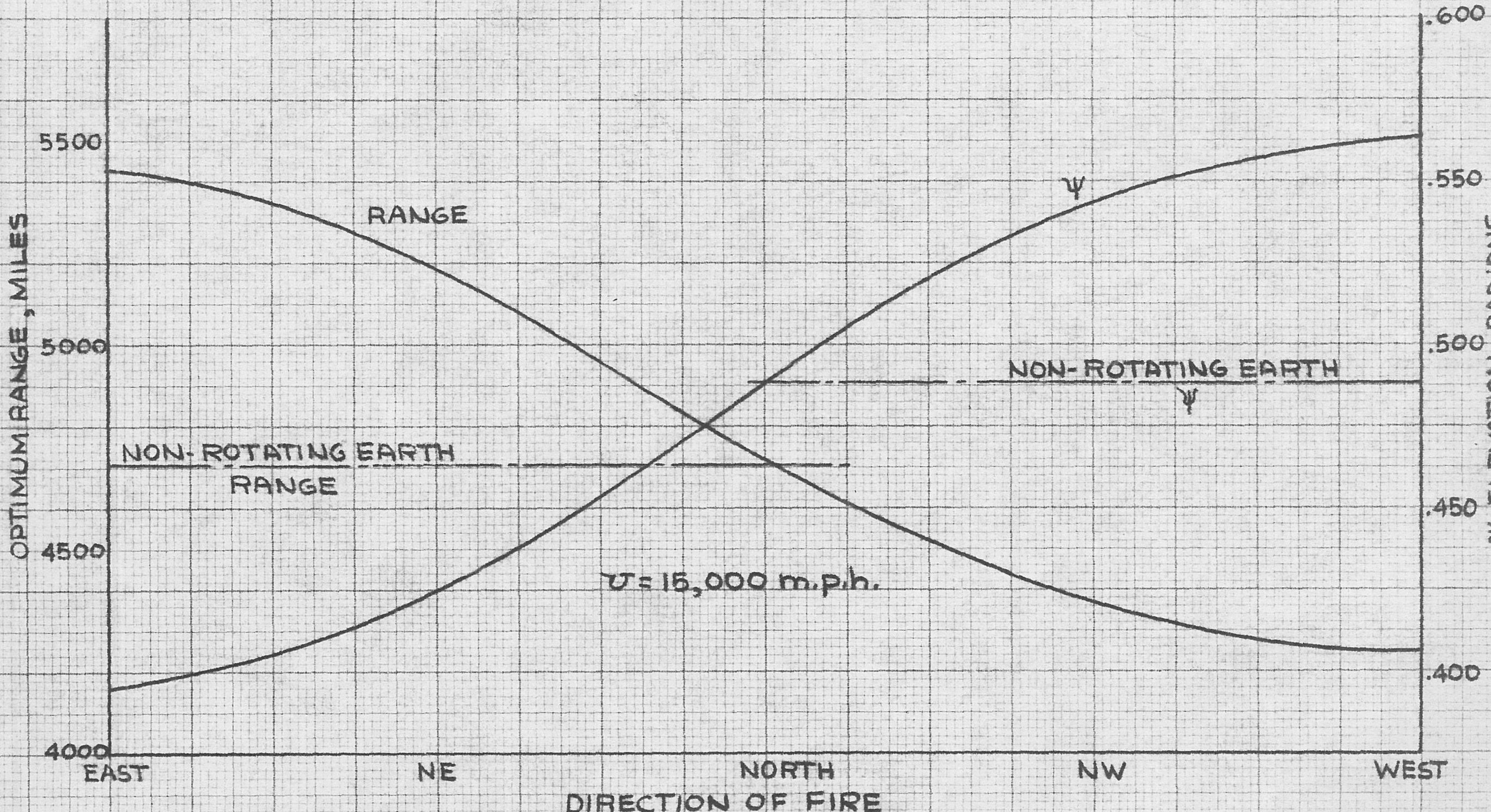

FIG. 15

OPTIMUM RANGE AND ANGLE OF ELEVATION vs. DIRECTION OF FIRE

FOR $v = 15{,}000$ m.p.h.

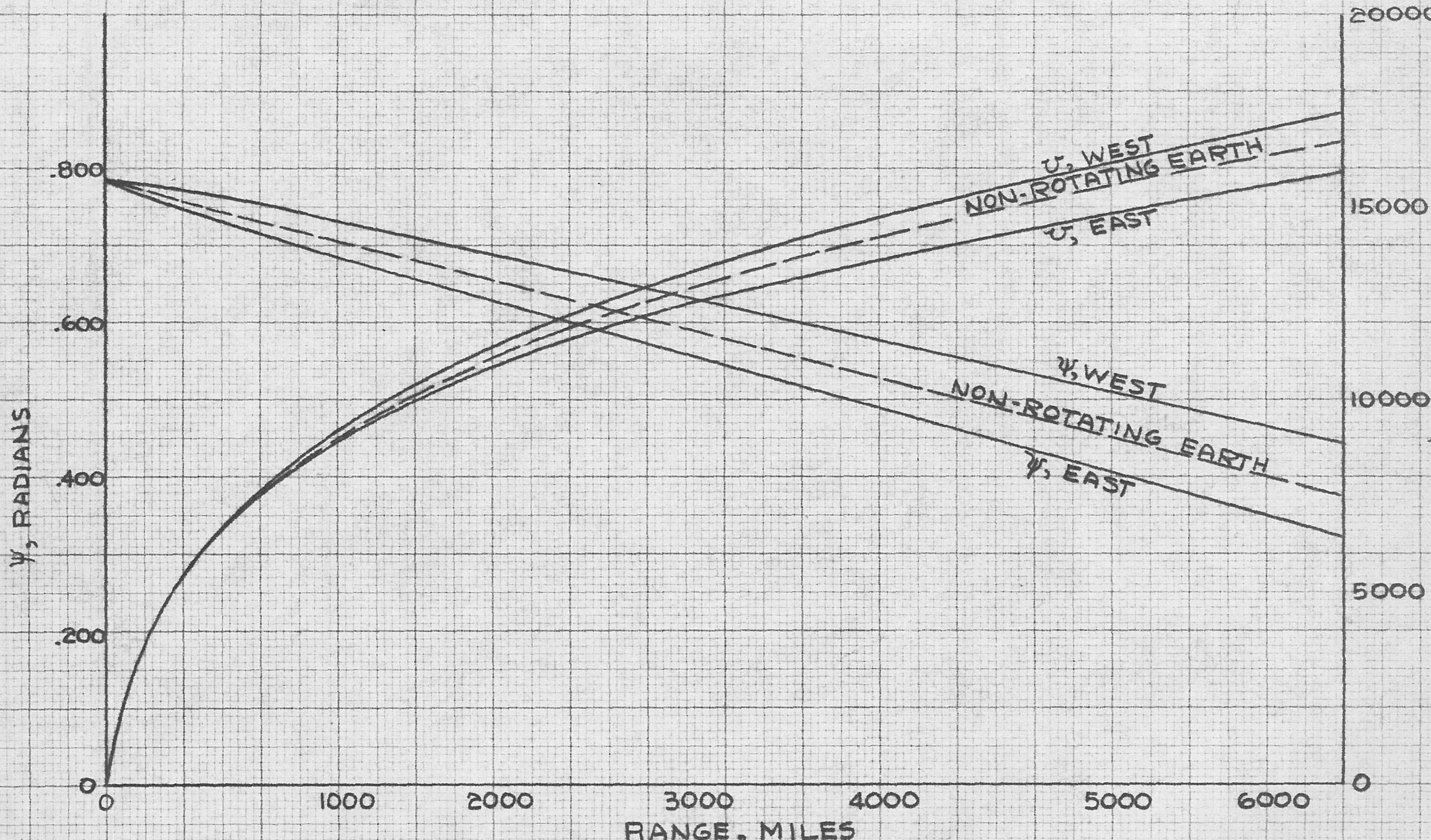

FIG. 16

OPTIMUM RANGE vs. ANGLE OF ELEVATION AND INITIAL VELOCITY

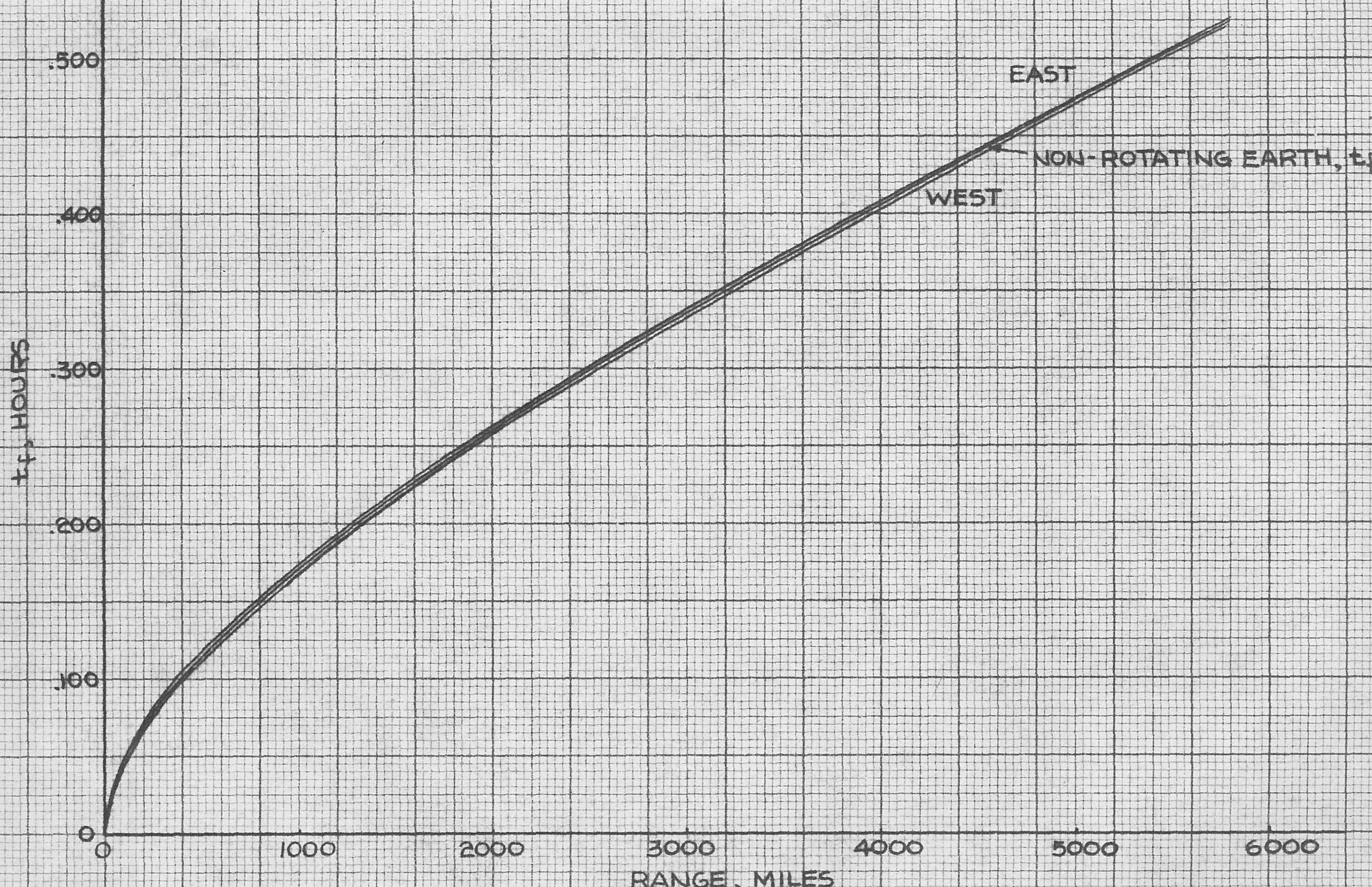

FIG. 17

TIME OF FLIGHT vs OPTIMUM RANGE

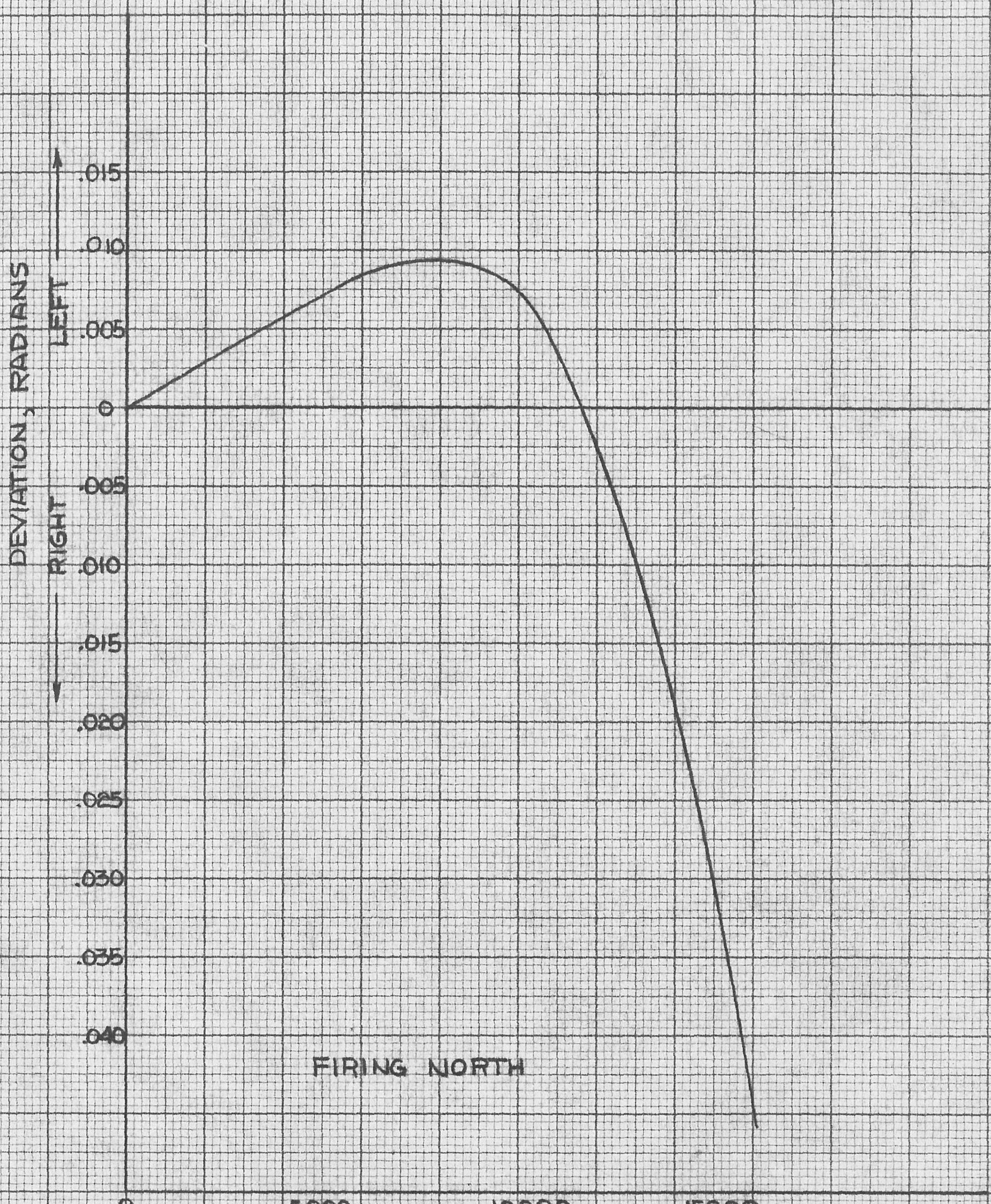

FIG. 18
DEVIATION vs INITIAL VELOCITY