A STUDY OF THE PERIODIC AND QUASI-PERIODIC SOLUTIONS OF THE DISCRETE DUFFING EQUATION

Thesis by

Mark Carter Anderson

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1986

(Submitted October 3, 1985)

ACKNOWLEDGEMENTS

The author would like to express his sincere appreciation of the assistance and consideration given by certain special people and groups of people. In particular, the guidance, enthusiasm, patience, and understanding of the thesis advisor, Prof. Thomas K. Caughey, was essential to the completion of the thesis. The stimulating academic environment within which the present work was completed was provided by the individuals that make up the Caltech community, especially by the occupants of Thomas Building. Financial support was generously provided by a fellowship from Hughes Aircraft Corporation.

A deeper, more personal expression of gratitude goes to Prof. Caughey and to my father, mother, and brother. Their support during my tenure in graduate school was vital to the achievement of a major, personal goal.

Finally, the most personal expression of gratitude goes to JoAnn Boyd Anderson, my wife. Her devotion and personal support were boundless. In addition, her mastery of the text processing software called TROFF, along with her other technical typing skills, were largely responsible for the appearance of the final form of the thesis.

ABSTRACT

The present investigation concentrates on the phenomenological and analytically quantitative study of the periodic and quasi-periodic solutions of a class of conservative, autonomous, nonlinear difference equations. In particular, an equation with a cubic nonlinearity, i.e., a form of the discrete Duffing equation, is studied. Following a simple analysis of the equilibrium solutions, the global structures of the phase portraits are illustrated phenomenologically for different values of the equation parameters. Three discrete perturbation procedures are then developed to obtain a consistent approximation for periodic and quasi-periodic solutions. These approximate solutions contain certain "small divisors" in every term other than the zero'th order term. An examination of the consequences of the vanishing of such a "small divisor" leads to a method of constructing exact periodic solutions in the form of finite Fourier series. The thesis concludes with a discussion of the quasi-periodic approximate solutions and their applicability.

CONTENTS

PAGE

CHAPTER 1

INTRODUCTION

1.1 DIFFERENCE EQUATIONS AND DYNAMICAL SYSTEMS

Difference equations may be used to model the dynamics of physical systems in a wide variety of ways. Applications of discrete models appear in the disciplines of engineering,[1,2,3] mathematical physics,[4,5] and biology,[6] just to name a few. In an applied mathematical sense, difference equations arise in three fundamental ways: as direct physical models of sampled data systems, as models of continuous equations being simulated on digital computers, and as, at least, local models of the Poincaré maps so commonly appearing in the modern geometric theory of mechanics.[4,5,7]

The use of the digital computer to simulate the behavior of physical systems is pervasive in virtually all areas of modern applied science. Any such simulation requires a discrete modeling approach at some level. The discrete modeling approach essentially always involves difference equations. Therefore, a study of the dynamics of difference equations seems quite logical. Further, since the qualitative behavior of the solutions of difference equations may be so concisely visualized in the phase space of the corresponding discrete dynamical systems, an explicit, phenomenological study of the phase portraits of a specific equation seems logically scientific.

The qualitative, geometric theory of mechanics has its roots in the pioneering works of Poincaré[8] and Birkhoff[9] on the physics and mathematics of the problem of the stability of celestial objects subject to Newtonian gravitation. Impetus for the current explosion of mathematical research was given by Kolmogorov in an address to the 1954 International Congress of Mathematicians.[3] In the thirty-odd years since then, many significant steps, e.g., Smale's work on differentiable dynamical systems,[10] have been taken toward development of a coherent, mathematically rigorous, qualitative theory. However, publications accessible to the audience of applied engineering literature have been, to say the least, rare. The present thesis may be viewed as this student's attempt to bridge such a "gap" in a way he, personally, can understand. The study of the dynamics of discrete models provides a convenient "bridge."

1.2 GOALS OF THE INVESTIGATION

The specific goals of the present thesis are threefold. These goals may be stated in the logical order of the forthcoming investigation as follows:

- Phenomenological development of a qualitative, primarily visual, understanding of the global structure of the phase portraits of a specific nonlinear difference equation;
- Explicit analytical development of the classical "Lindstedt-Poincaré type" of perturbation methods[11,12] for the study of periodic and quasi-periodic solutions of a class of nonlinear different equations; and
- Application of the perturbation theory to the study of the solutions the same, specific , nonlinear difference equation.

An equation with a cubic nonlinearity was chosen for study within the present investigation since such an equation provides a relatively simple example of a broad class of nonlinear difference equations. In particular, a second order, nonlinear difference equation of the form

$$x_{n+1} - 2x_n + x_{n-1} + kx_n + \varepsilon x_n^3 = 0 \ , \qquad (1.1)$$

will be studied. For obvious reasons, Eq. (1.1) will be referred to as the (conservative, autonomous) discrete Duffing equation.

1.3 A BRIEF OVERVIEW OF THE INVESTIGATION

The present investigation begins, in earnest, in Chapter 2. The study starts with a simple analysis of the parametric dependence of the existence and stability of equilibrium solutions of the discrete Duffing equation. Attention is then focused on a phenomenological survey of the global structure of the phase portraits of the discrete Duffing equation for various, specific values of the equation parameters, k and ε.

A development of approximate analytical methods for the study of periodic and quasi-periodic solutions of second order, nonlinear difference equations, such as the discrete Duffing equation, is presented in Chapter 3. In particular, an explicit development of the discrete analog of classical, continuous secular perturbation theory is presented.

Chapter 4 concentrates on the application of the approximate methods developed in Chapter 3 to the study of the periodic and quasi-periodic solutions of the discrete Duffing equation. A method of constructing certain exact, "closed form" periodic solutions is presented in addition to the discussion of the applications of the periodic and quasi-periodic approximate solutions. Comments connecting these explicit solutions to the qualitative, phenomenological discussion of Chapter 2 are included. A brief summary of the present study is presented in Chapter 5.

CHAPTER 2

A QUALITATIVE AND PHENOMENOLOGICAL LOOK AT THE DISCRETE DUFFING EQUATION

2.1 INTRODUCTION

The overall analysis of a given nonlinear difference equation, e.g., the discrete Duffing equation, is greatly facilitated by developing an understanding of the global behavior of the solutions. Unless exact solutions are available, the true, global phenomenology of a given difference equation can only be ascertained by numerical simulation. The (x_n, x_{n+1})-phase plane of the difference equation provides a convenient "canvas" on which the global phenomenology can be viewed. The present chapter concentrates on the generation and observational analysis of the phase portraits of the discrete Duffing equation. All of the simulations done for this investigation were performed on a microcomputer, using BASIC as the programming language.

The global analysis of the discrete Duffing equation begins with a determination and description of the equilibrium solutions in Section (2.2). A selective overview of the phenomenology of the discrete Duffing equation is presented in Section (2.3). Within Section (2.3), attention is restricted to the "physically interesting" cases, i.e., to the regions of the parameter space where periodic, quasi-periodic, and stochastic solutions appear.

2.2 EQUILIBRIUM SOLUTIONS

2.2.1 General Considerations

As mentioned in the previous section, a study of the global behavior of the solutions of a nonlinear difference equation usually begins with an analysis of the existence and character of the equilibrium solutions. An analysis of this type is performed in much the same way as the corresponding analysis of a nonlinear differential equation.

Specifically, an equilibrium analysis of the discrete Duffing equation, Eq. (1.1), begins by considering the original equation,

$$x_{n+1} - 2x_n + x_{n-1} + kx_n + \varepsilon x_n^3 = 0 \ , \tag{2.1}$$

and setting

$$x_{n+1} = x_n = x_{n-1} = x_* \quad \forall n \ . \tag{2.2}$$

The equilibrium solutions of Eq. (2.1) are determined by substituting Eq. (2.2) into Eq. (2.1) to obtain

$$kx_* + \varepsilon x_*^3 = 0 \ , \tag{2.3}$$

which yields

$$x_* = 0 \tag{2.4}$$

and

$$x_* = \pm\sqrt{\frac{-k}{\varepsilon}} \ . \tag{2.5}$$

Note that Eq. (2.5) yields real values for x_* only if k and ε are of opposite sign, i.e.,

$$sgn(k) = sgn(\varepsilon) \Rightarrow \text{one equilibrium solution} \ ,$$

and

$$sgn(k) = -\,sgn(\varepsilon) \Rightarrow \text{three equilibrium solutions} \ .$$

The general nature of the solutions in the neighborhood of the equilibrium points may be determined, in part, by slightly perturbing the solutions away from their respective equilibrium points and analyzing the respective linear approximations. This process is facilitated by taking

$$x_n = x_* + \xi_n \quad , \tag{2.6}$$

where ξ_n is a small perturbation, and substituting Eq. (2.6) into Eq. (2.1) to obtain

$$(x_* + \xi_{n+1}) - 2(x_* + \xi_n) + (x_* + \xi_{n-1}) + k(x_* + \xi_n) + \varepsilon(x_* + \xi_n)^3 = 0 \ . \tag{2.7}$$

Using Eq. (2.3), Eq. (2.7) may be reduced to

$$\xi_{n+1} - 2\xi_n + \xi_{n-1} + k\xi_n + 3\varepsilon x_*^2 \xi_n + O(|\xi_n|^2) = 0 \ . \tag{2.8}$$

Neglecting terms of order higher than one, and writing Eq. (2.8) as a first order system, results in

$$\begin{Bmatrix} \xi_{n+1} \\ \xi_n \end{Bmatrix} = \begin{bmatrix} 2 - k - 3\varepsilon x_*^2 & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \xi_n \\ \xi_{n-1} \end{Bmatrix} \ . \tag{2.9}$$

The characteristic equation for Eq. (2.9) is

$$\lambda^2 - (2 - k - 3\varepsilon x_*^2)\lambda + 1 = 0 \ ,$$

which yields

$$\lambda_\pm = \frac{1}{2}(2 - k - 3\varepsilon x_*^2) \pm \sqrt{\frac{1}{4}(2 - k - 3\varepsilon x_*^2)^2 - 1} \ . \tag{2.10}$$

The type of a given equilibrium solution, i.e., saddle, etc., may be determined using Eq. (2.10).

2.2.2 Parametric Dependence of the Global Behavior

A gross, overall picture of the global behavior may be ascertained by an analysis of the equilibrium solutions, as discussed in Section (2.2.1). In general, as illustrated by Eqs. (2.3) and (2.10), both the existence and the nature of the equilibrium solutions depend upon the parameters of the discrete Duffing equation, i.e., on ε and k. The only exception to this assertion is the existence of the equilibrium solution at the origin.

The nature of the equilibrium solution at the origin may be determined by substituting for x_* in Eq. (2.10). This yields

$$\lambda_\pm = \frac{1}{2}(2-k) \pm \sqrt{\frac{1}{4}(2-k)^2 - 1} \ .$$

The dependence of $|\lambda_\pm|$ on the parameter k is illustrated graphically in Fig. (2.1). Clearly,

$$k < 0 \Rightarrow \text{saddle point} \ ,$$

$$k > 4 \Rightarrow \text{saddle point} \ ,$$

and,

$$0 \le k \le 4 \Rightarrow \text{critical case}$$

As will be seen in Section (2.3), the origin is actually a center for $0 < k < 4$.

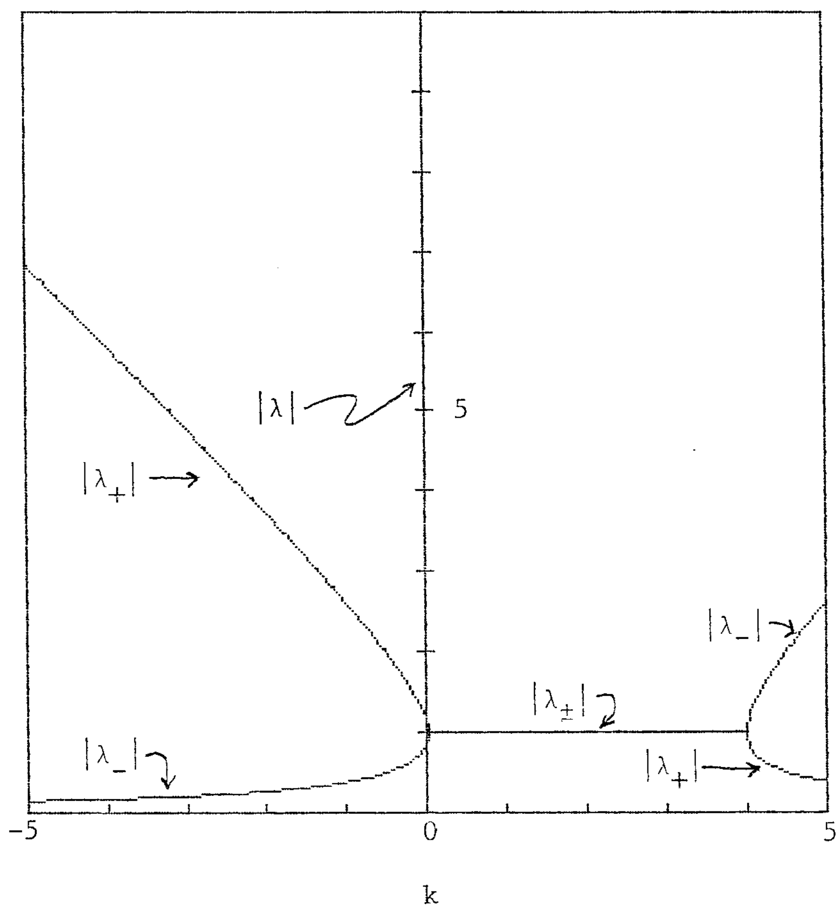
Similarly, the nature of the equilibrium points given by Eq. (2.5) may be determined. Substituting for x_* in Eq. (2.10) yields

$$\lambda_\pm = 1 + k \pm \sqrt{(1+k)^2 - 1} \ .$$

The dependence of $|\lambda_\pm|$ on k, for this case, is shown in Fig. (2.2). Obviously,

$$k > 0 \ (\varepsilon < 0) \Rightarrow \text{saddle point} \ ,$$

$$k < -2 \ (\varepsilon > 0) \Rightarrow \text{saddle point} \ ,$$



$x_* = 0 \; (|\varepsilon| > 0)$

Fig. 2.1 Parametric dependence of the absolute values of the eigenvalues, $|\lambda_{\pm}|$, of the linear approximation in the neighborhood of the equilibrium solution at the origin.

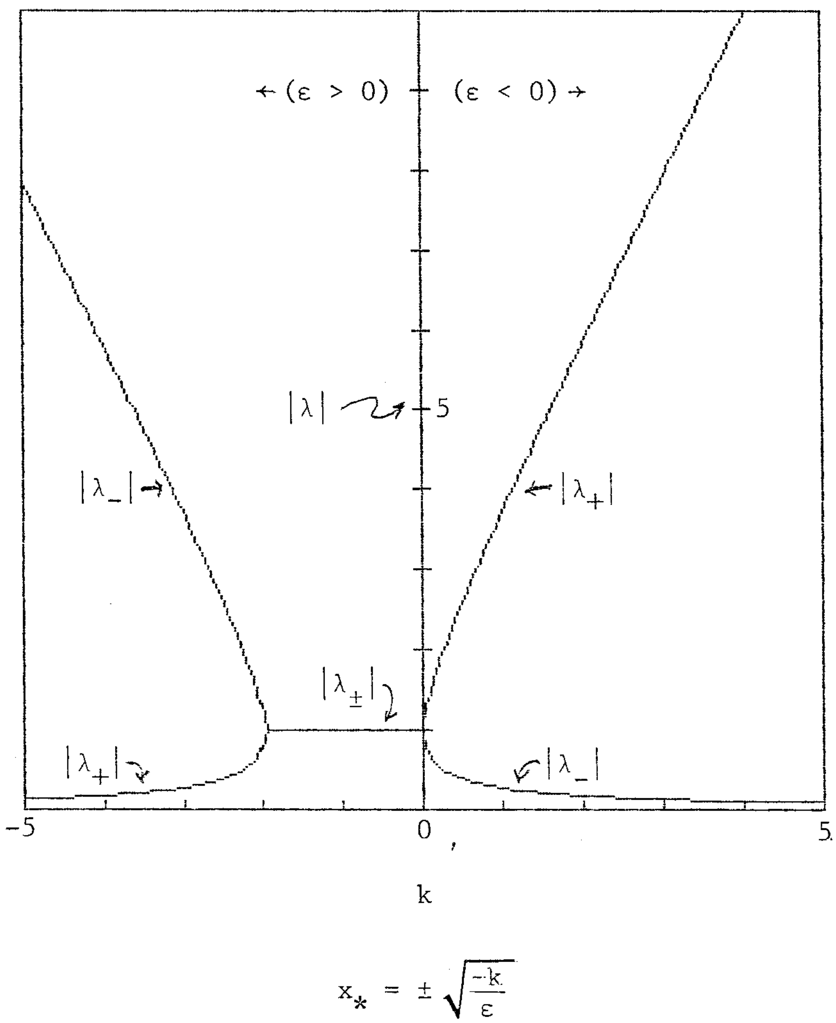


Fig. 2.2 Parametric dependence of the absolute values of the eigenvalues, $|\lambda_{\pm}|$, of the linear approximation in the neighborhood of the equilibrium solutions at $x_* = \pm\sqrt{-k/\varepsilon}$.

and

$$-2 \le k \le 0 \ (\varepsilon > 0) \Rightarrow \text{critical case}$$

Here, again, as will be seen in Section (2.3), the equilibrium point, x_*, is actually a center for $-2 < k < 0$ ($\varepsilon > 0$).

The results of the analysis, thus far, are summarized symbolically in Fig. (2.3). Obviously, except for the stable manifolds of the saddle points and the equilibrium solutions themselves, all solutions in regions I, IV, V, and VII of the parameter space are unstable. Therefore, the remainder of the present investigation will be restricted to the study of the solutions of the discrete Duffing equation occurring within regions II, III, and VI of the parameter space depicted in Fig. (2.3).

2.2.3 A Note on Degenerate Cases

The analysis of the preceding section provided a complete characterization of the equilibrium points of the discrete Duffing equation, with respect to the (k,ε)-parameter space of Fig. (2.3), except for the three vertical lines given by $k = -2(\varepsilon > 0), 0, 4$, respectively. These "lines" represent sets of parameter points at which the matrix of the linear approximation, i.e., the matrix in Eq. (2.9), yields equal eigenvalues, and hence, in general, possesses dependent eigenvectors near at least one of the equilibrium points, x_*. Of course, dependent eigenvectors indicate so-called "marginal" stability.

In particular, on the line $k = -2(\varepsilon > 0)$, the matrix possesses dependent eigenvectors at $x_* = \pm\sqrt{-k/\varepsilon}$. On the line $k = 0$, the matrix possesses dependent eigenvectors at both $x_* = 0$ and $x_* = \pm\sqrt{-k/\varepsilon}$. Finally, on the line $k = 4$, the matrix possesses dependent eigenvectors at $x_* = 0$. Qualitatively, as can be seen from Fig. (2.3), these "degeneracies" correspond to transitions between different "types" of equilibrium points or to the first appearances of equilibrium points. A specific study of such cases lies outside the scope of the present

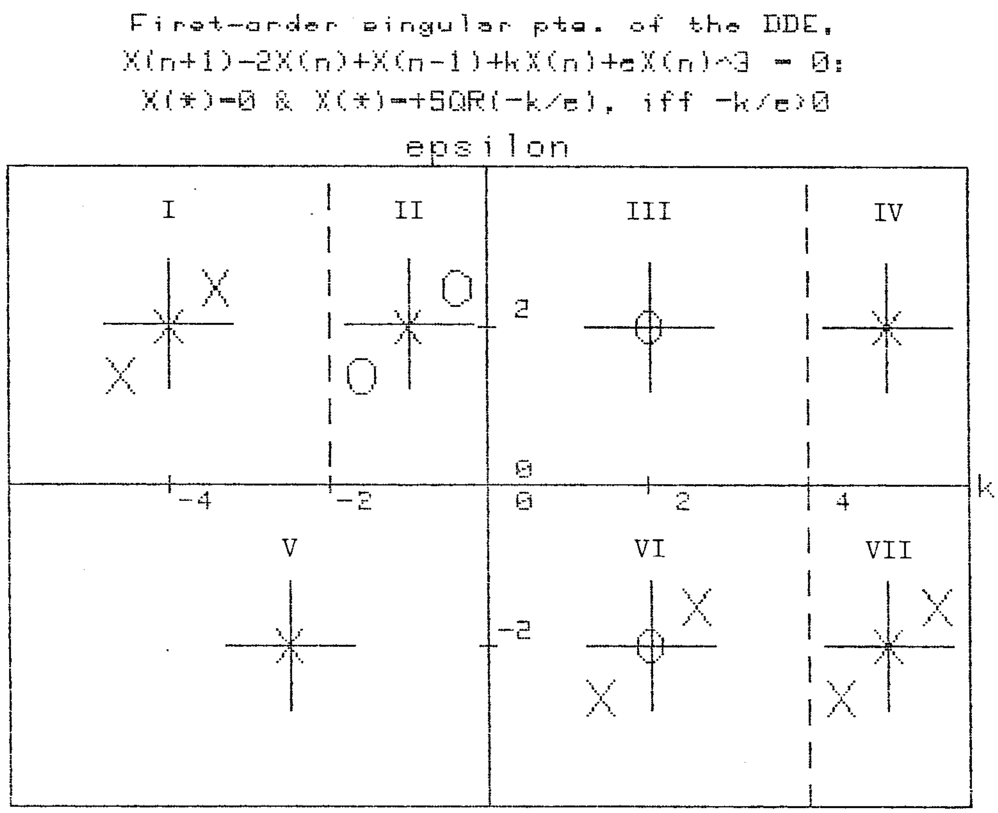


Fig. 2.3 Parametric dependence of the "macroscopic" structure of the phase portraits. Here, "X's" denote saddle points, and "O's" denote centers.

investigation.

2.3 PHENOMENOLOGY OF THE DISCRETE DUFFING EQUATION

2.3.1 Phase Portraits of the Discrete Duffing Equation

An analysis of the existence and character of equilibrium solutions, as presented in Section 2.2, provides only a "myopic" view of the phase plane structure of a nonlinear difference equation. The present section is devoted to a phenomenological look "through the looking glass" at the phase portraits of the discrete Duffing equation. In the present context, the "looking glass" is, of course, a digital computer system. Specifically, a Hewlett-Packard model 9816 microcomputer was used to generate the phase portraits. All of the programming was done in interpretive BASIC.

As will be shown in the figures to follow, the detailed structure of the phase portraits of a relatively simple, second order, nonlinear difference equation, such as the discrete Duffing equation, can be exceedingly complex. The characteristics of periodic, quasi-periodic, and so-called "stochastic" behavior are all present. Furthermore, these characteristics depend on the parameters of the specific equation under investigation. For the (conservative, autonomous) discrete Duffing equation, Eq. (2.1), the parameters are the linear "stiffness" parameter, k, and the nonlinear parameter, ε.

The specific points in the (k,ε)-parameter space, at which phase portraits are to be illustrated, are shown in Fig. (2.4). The "+"-shaped pattern of the points within each region of the parameter space was chosen so that the dependence of the phase plane structure on one of the equation parameters, with the other parameter held fixed, could be viewed. The phase portraits are presented in Figs. (2.5)-(2.19).

Keeping a picture of the familiar $(x,\dot{x})$-phase space of the (conservative, autonomous) continuous Duffing equation,

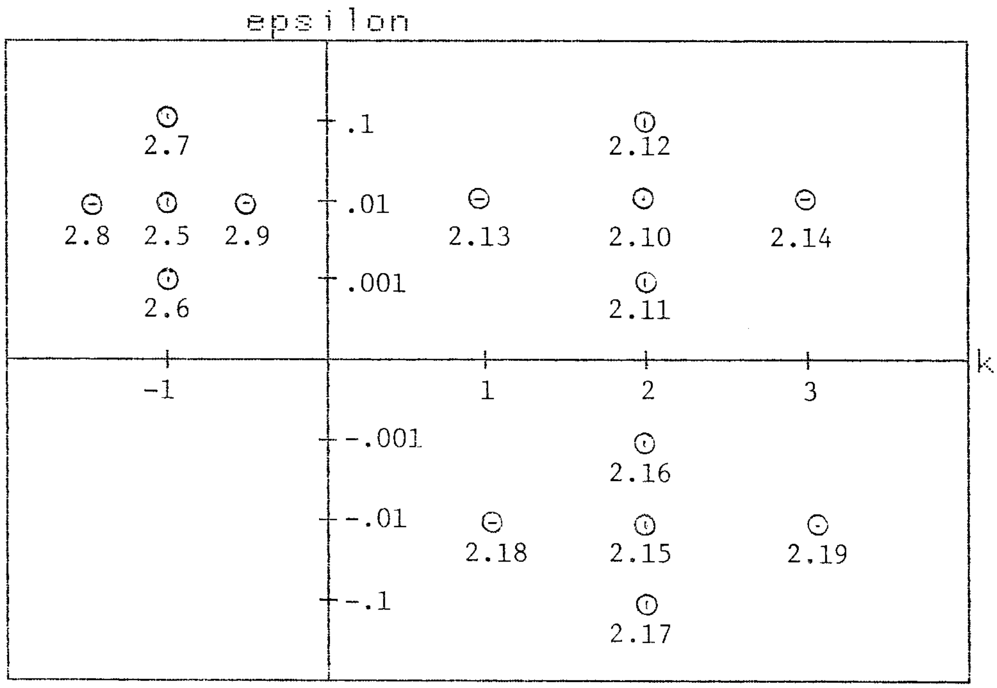


Fig. 2.4 Specific parameter space points used for the numerical simulation of phase portraits of the (conservative, autonomous) discrete Duffing equation. Each point corresponds to one of the Figs. 2.5-2.19, as shown.

$$\ddot{x} + kx + \varepsilon x^3 = 0 \ ,$$

in mind, note the general, qualitative appearance of Figs. (2.5)-(2.19). As can be seen, within a specific region of the parameter space, only the scale of the global phase portrait structure depends on ε. Further discussion of the characteristics of these phase portraits will be postponed to allow the reader to more easily cross-reference the figures and to absorb the pure, visual impact that they provide.

2.3.2 Initial Comments on Periodic and Quasi-periodic Solutions

The "regular" behavior exhibited in the neighborhood of the centers appearing in Figs. (2.5)-(2.19) is generated by periodic and quasi-periodic solutions. In particular, periodic solutions are represented by finite sets of points around the centers and quasi-periodic solutions are represented by infinite, but denumerable, sets of points "circling" the centers. These ideas will be more precisely stated in Chapters 3 and 4.

As is well known, the frequency of the periodic solutions of the (conservative, autonomous) continuous Duffing equation depends upon the amplitude of the solutions. However, the period of such solutions corresponds to only one "trip" around the center. In contrast, the period of the periodic solutions of the discrete Duffing equation may correspond to many "trips" around the center. In the same sense, the "period" of the quasi-periodic solutions of the discrete Duffing equation is infinite. Therefore, in order to facilitate a discussion of the frequency-amplitude relationship of the discrete Duffing equation, the concept of a "pseudo-period" corresponding to one "trip" around the center will prove useful.

The "pseudo-period" may be defined, heuristically, by considering the discrete independent variable, n, as "continuous" along a given trajectory. This concept is illustrated graphically in Fig. (2.20). Such a definition produces frequency-amplitude relationships of the type illustrated in Fig. (2.21). As will be seen in Chapter 4, these frequency-amplitude relationships will be useful in discussing the accuracy of the approximate methods developed in

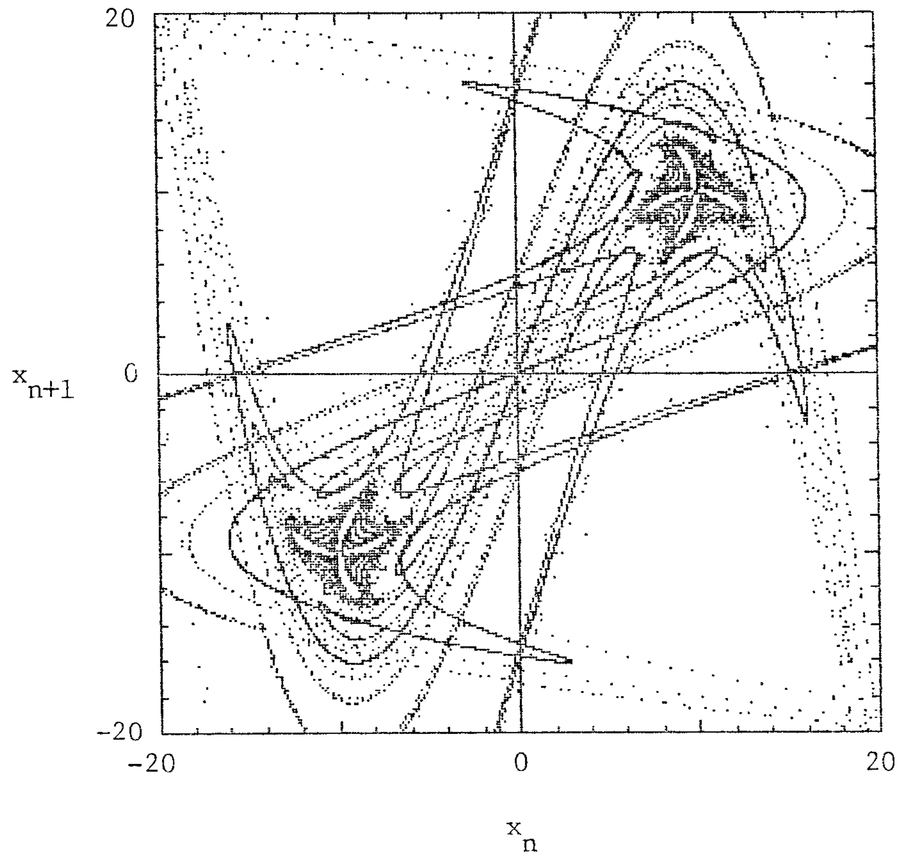


Fig. 2.5 An example of the phase portrait of the discrete Duffing equation for $k = -1$ and $\varepsilon = 0.01$.

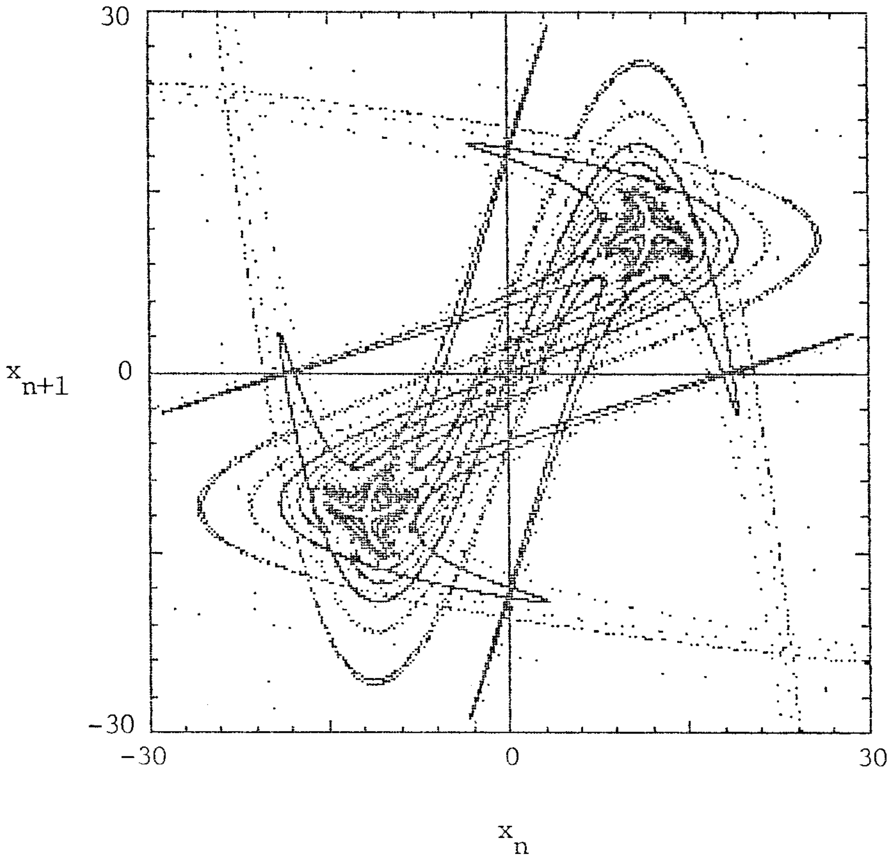


Fig. 2.6 An example of the phase portrait of the discrete Duffing equation for $k = -1$ and $\varepsilon = 0.001$.

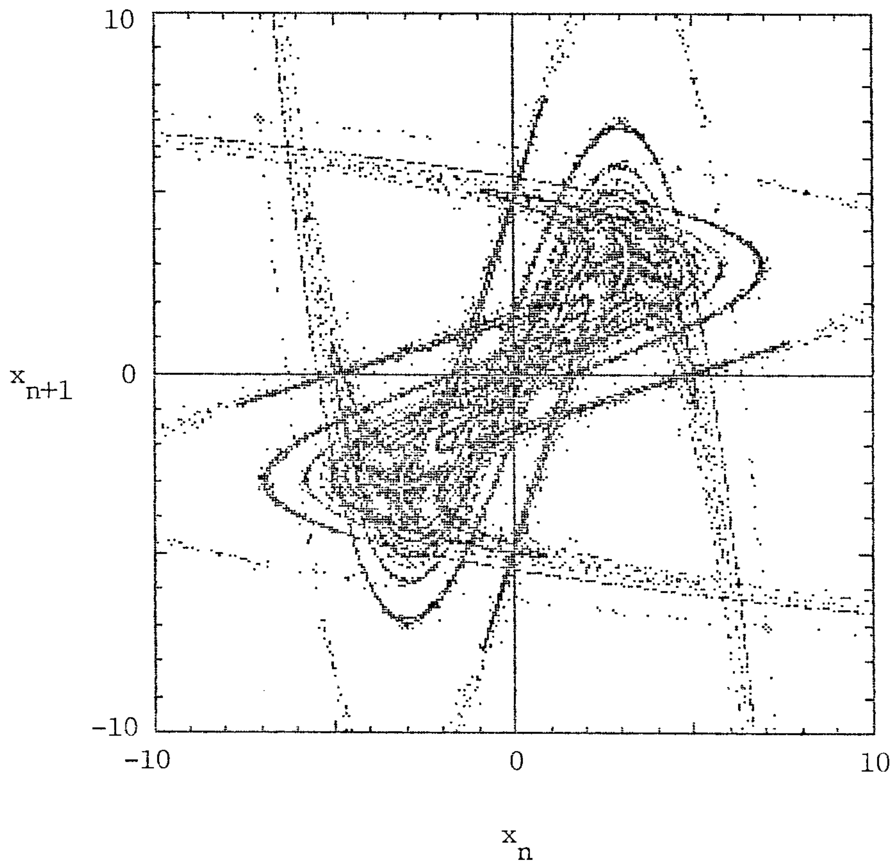


Fig. 2.7 An example of the phase portrait of the discrete Duffing equation for $k = -1$ and $\varepsilon = 0.1$.

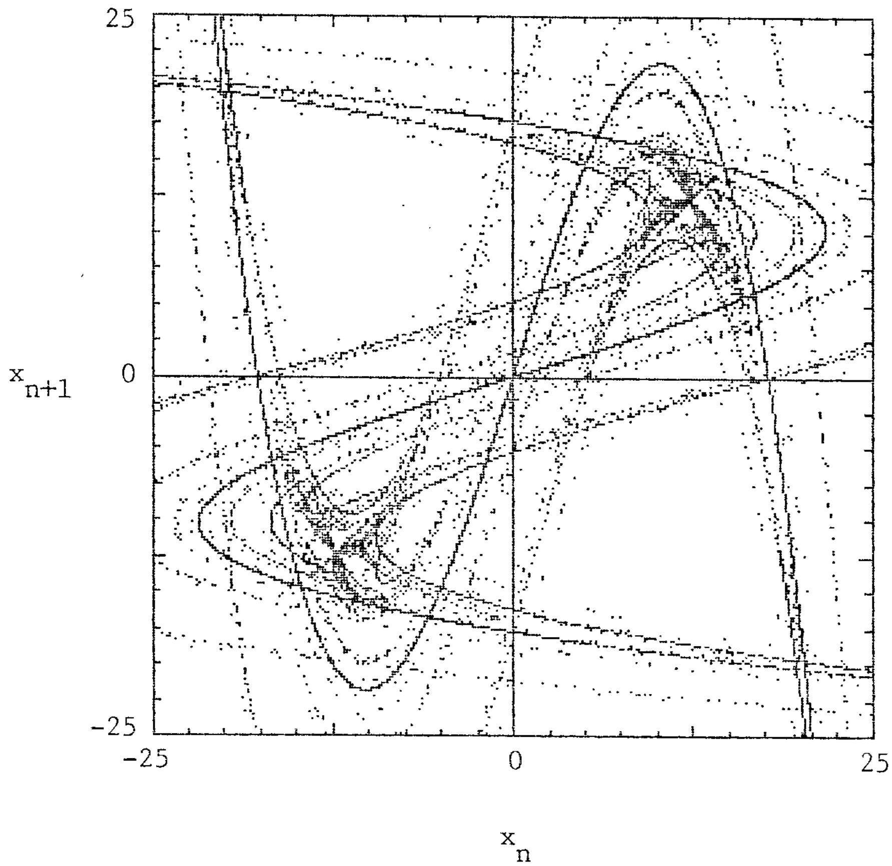


Fig. 2.6 An example of the phase portrait of the discrete Duffing equation for $k = -1.5$ and $\varepsilon = 0.01$.

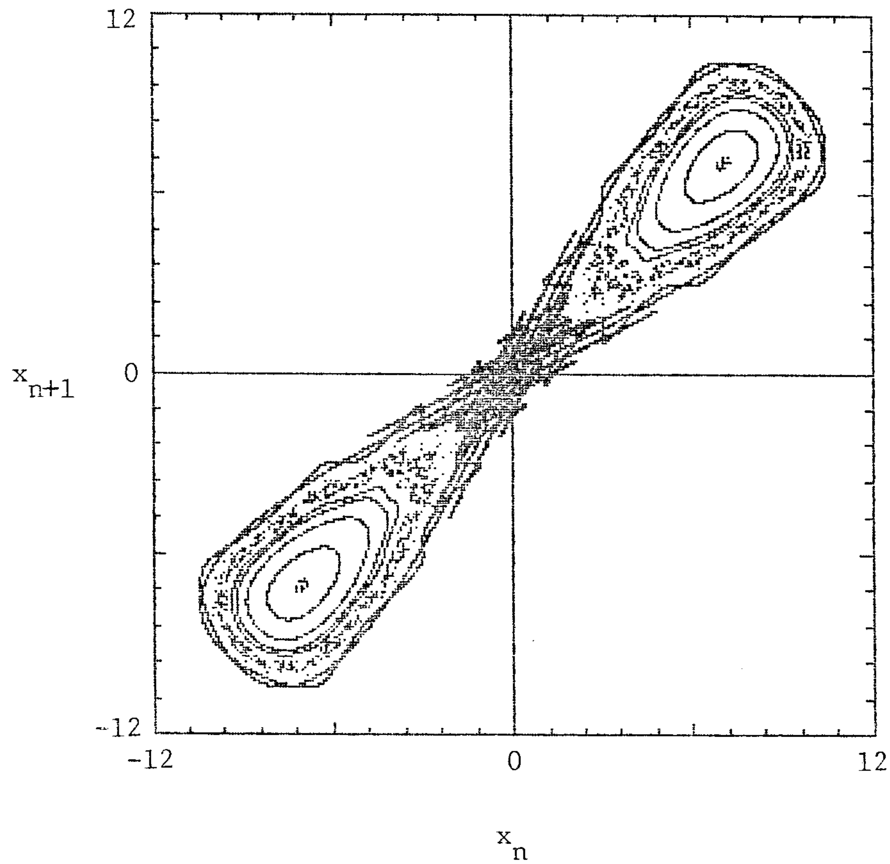


Fig. 2.9 An example of the phase portrait of the discrete Duffing equation for $k = -0.5$ and $\varepsilon = 0.01$.

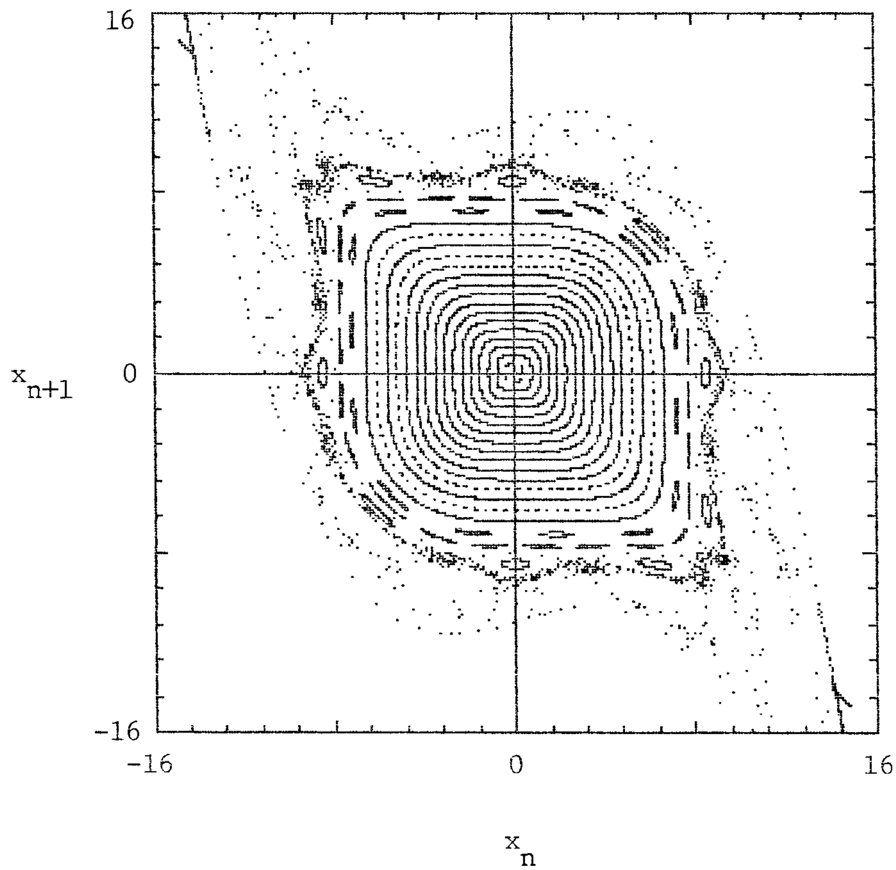


Fig. 2.10 An example of the phase portrait of the discrete Duffing equation for $k = 2$ and $\varepsilon = 0.01$.

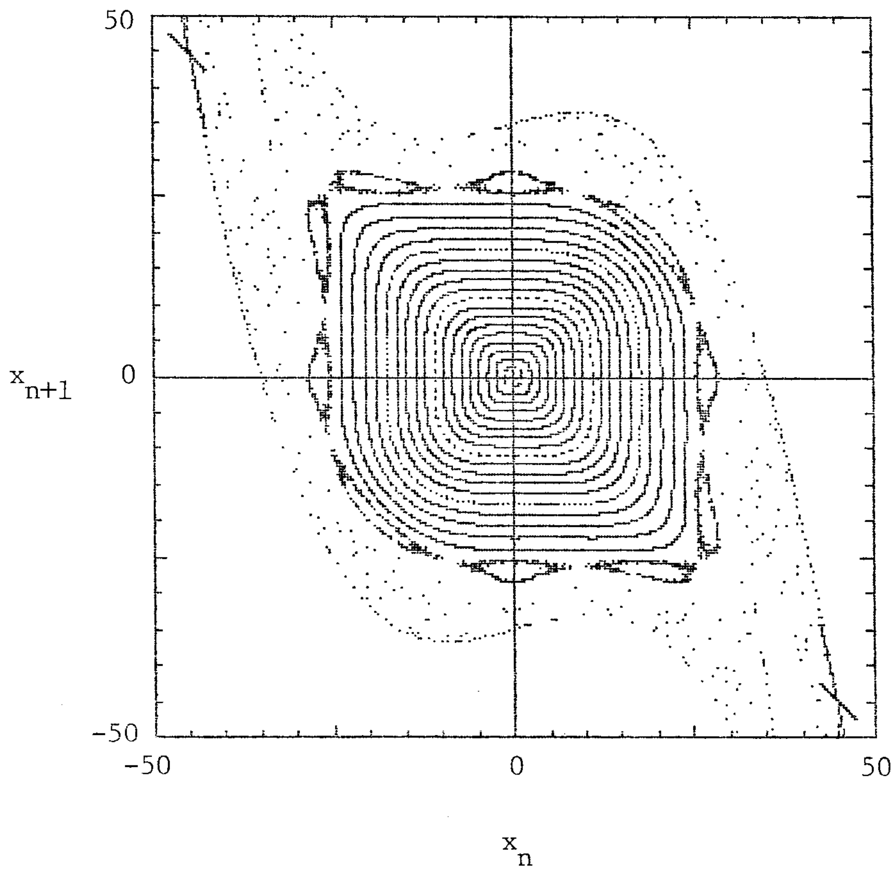


Fig. 2.11 An example of the phase portrait of the discrete Duffing equation for $k = 2$ and $\varepsilon = 0.001$.

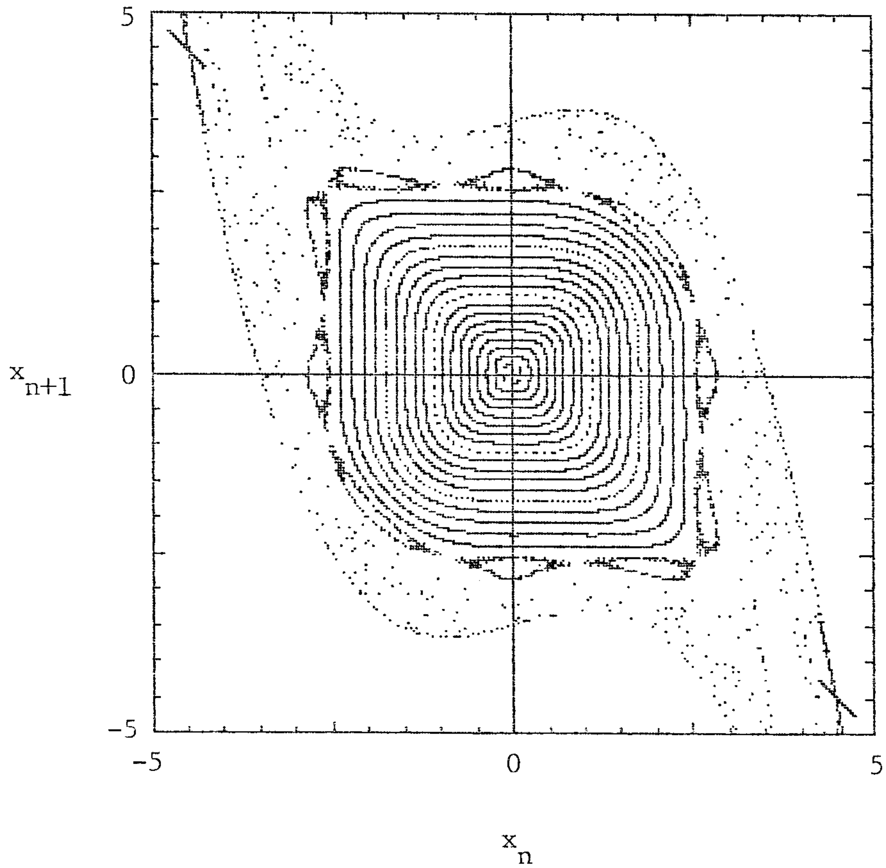


Fig. 2.12 An example of the phase portrait of the discrete Duffing equation for $k = 2$ and $\varepsilon = 0.1$.

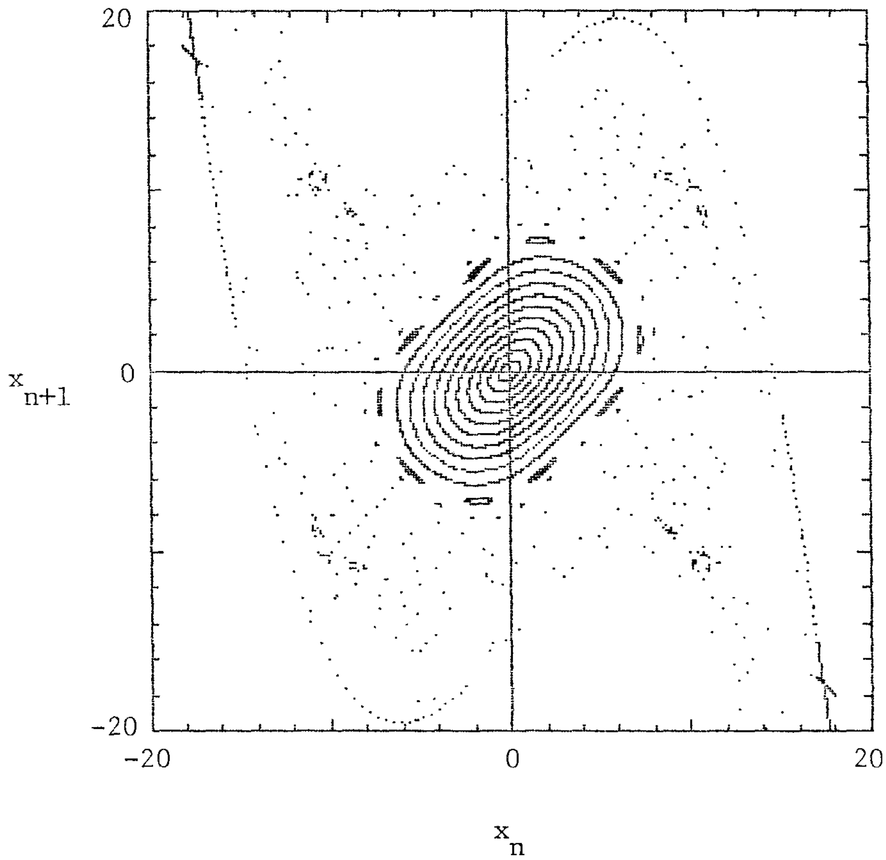


Fig. 2.13 An example of the phase portrait of the discrete Duffing equation for $k = 1$ and $\varepsilon = 0.01$.

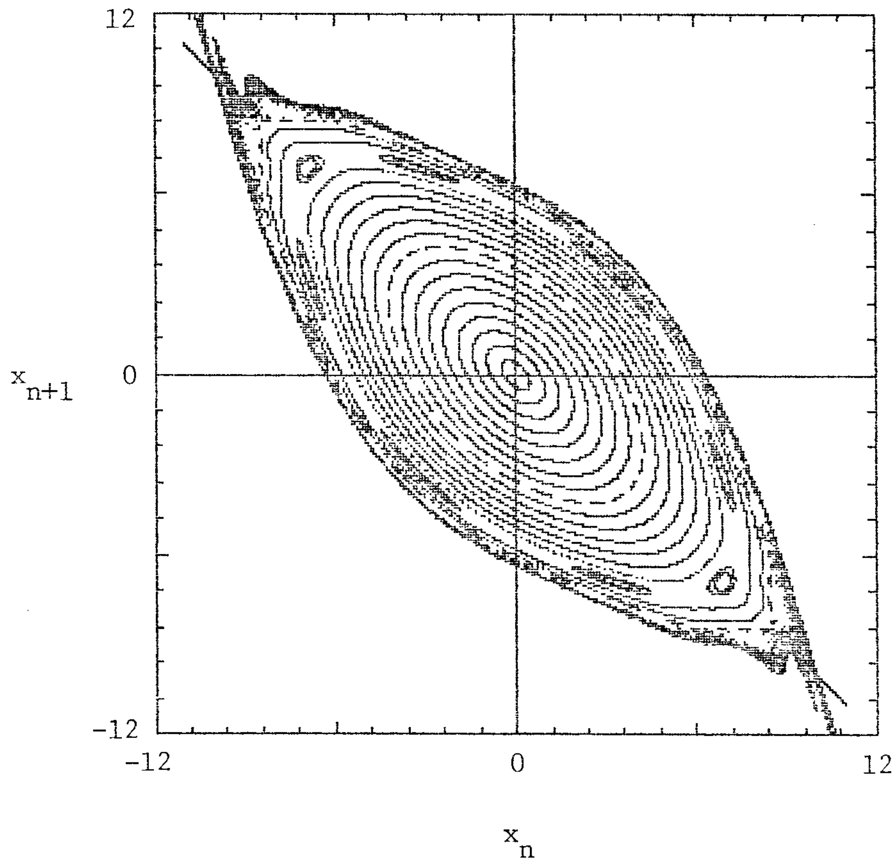


Fig. 2.14 An example of the phase portrait of the discrete Duffing equation for $k = 3$ and $\varepsilon = 0.01$.

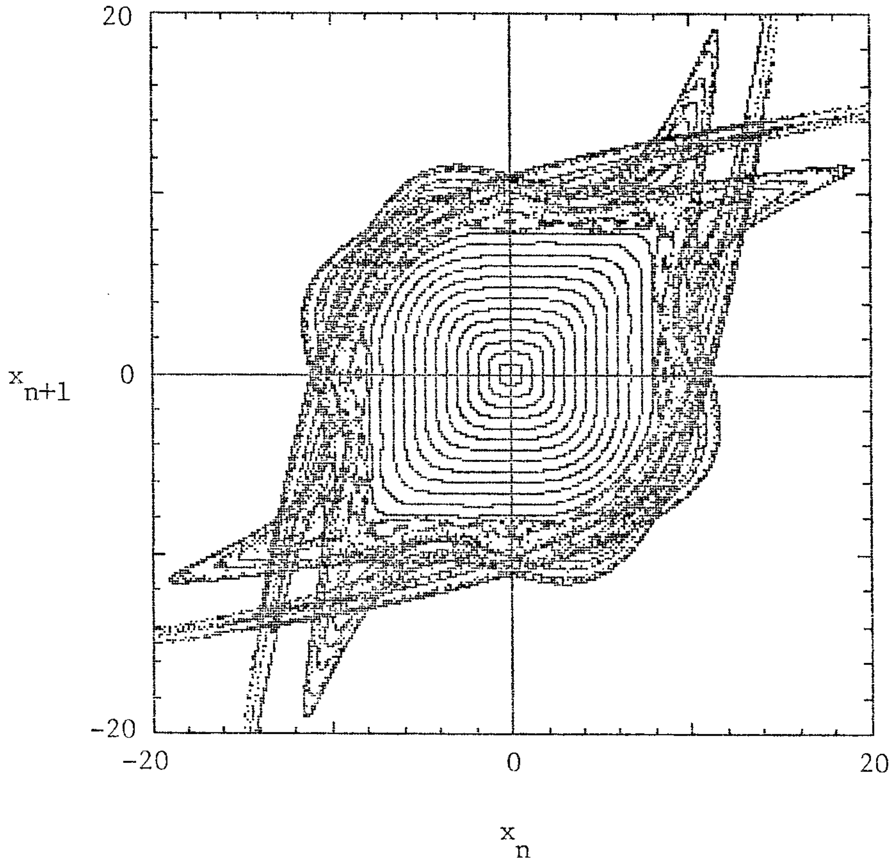


Fig. 2.15 An example of the phase portrait of the discrete Duffing equation for k = 2 and ε = −0.01.

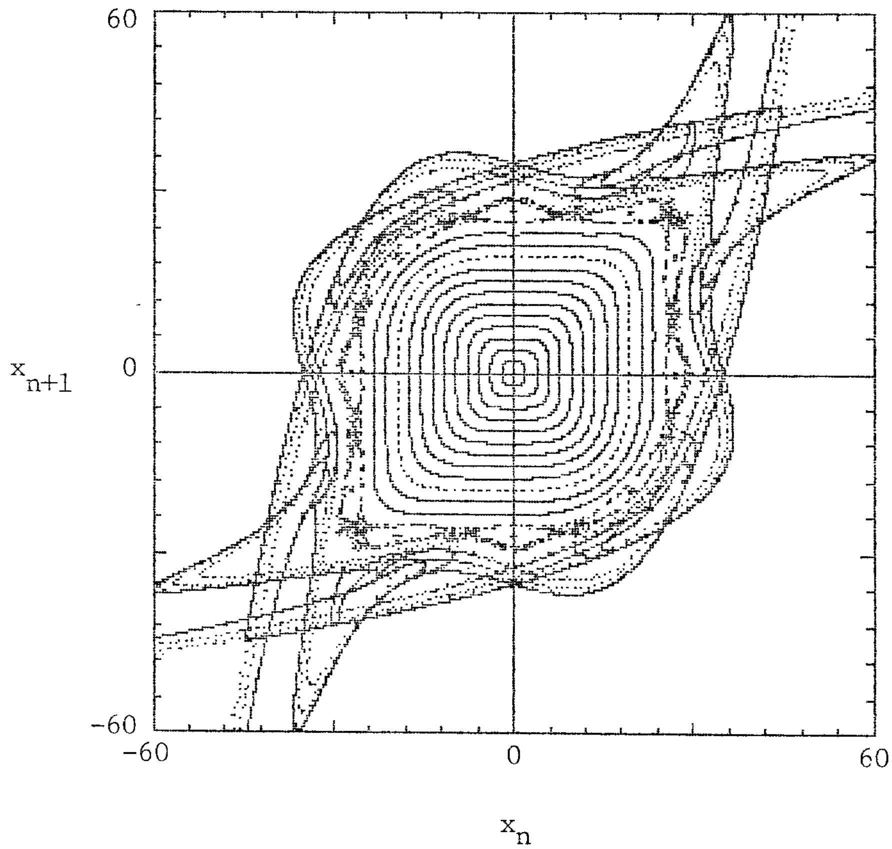


Fig. 2.16 An example of the phase portrait of the discrete Duffing equation for k = 2 and ε = −0.001.

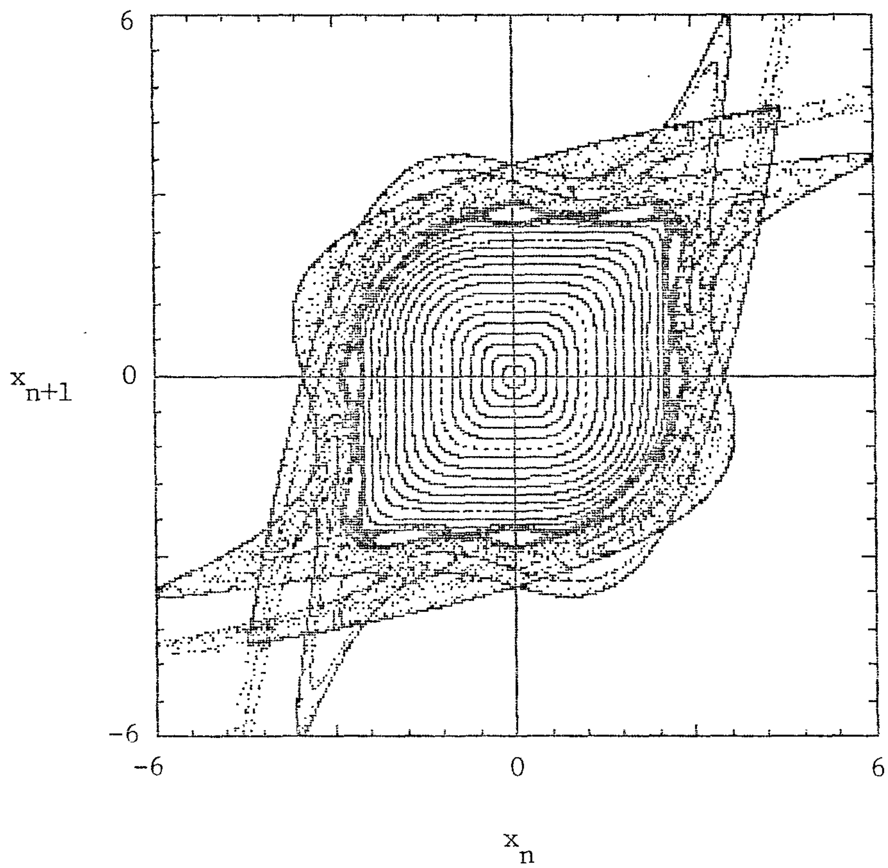


Fig. 2.17 An example of the phase portrait of the discrete Duffing equation for $k = 2$ and $\varepsilon = -0.1$.

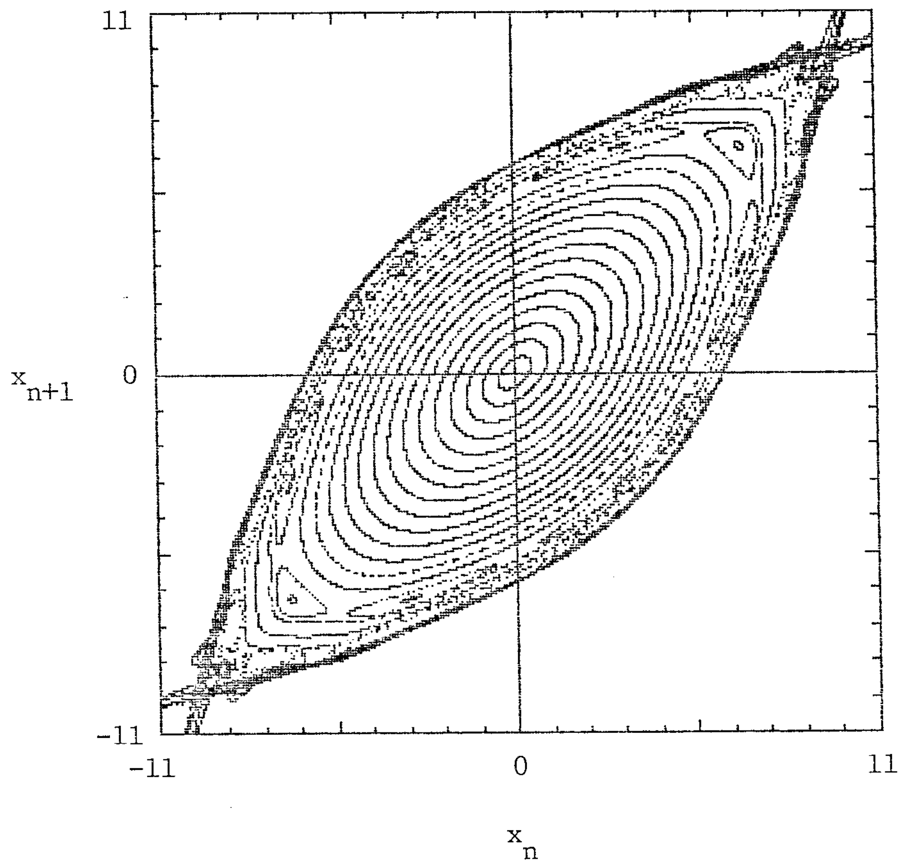


Fig. 2.18 An example of the phase portrait of the discrete Duffing equation for $k = 1$ and $\varepsilon = -0.01$.

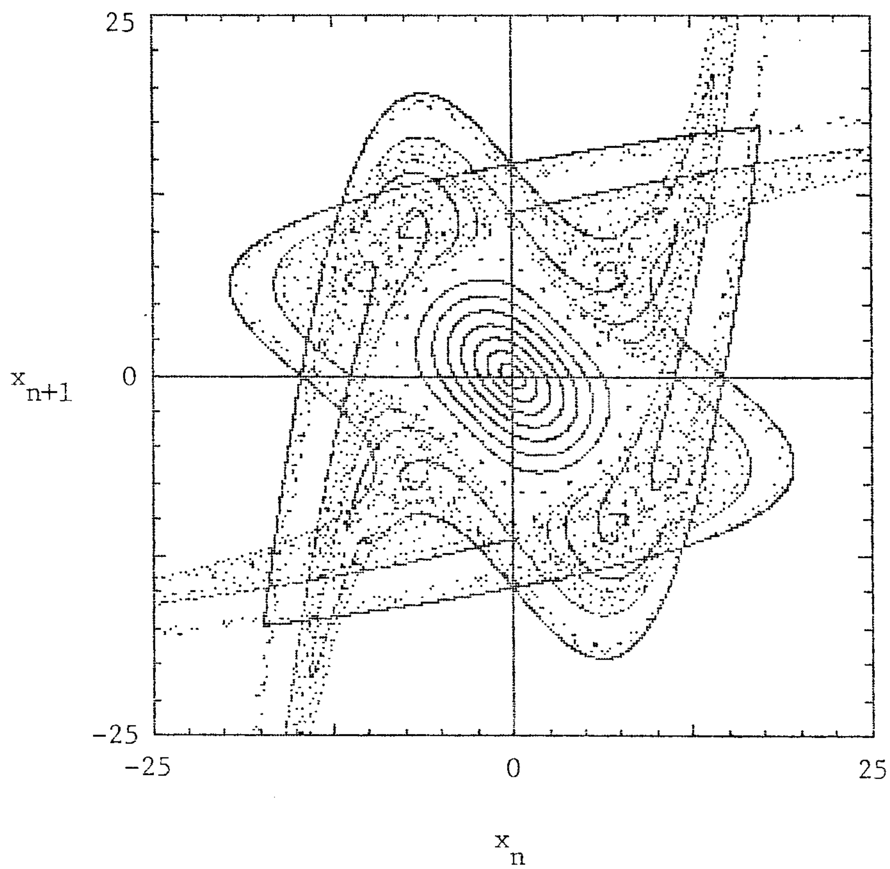


Fig. 2.19 An example of the phase portrait of the discrete Duffing equation for $k = 3$ and $\varepsilon = -0.01$.

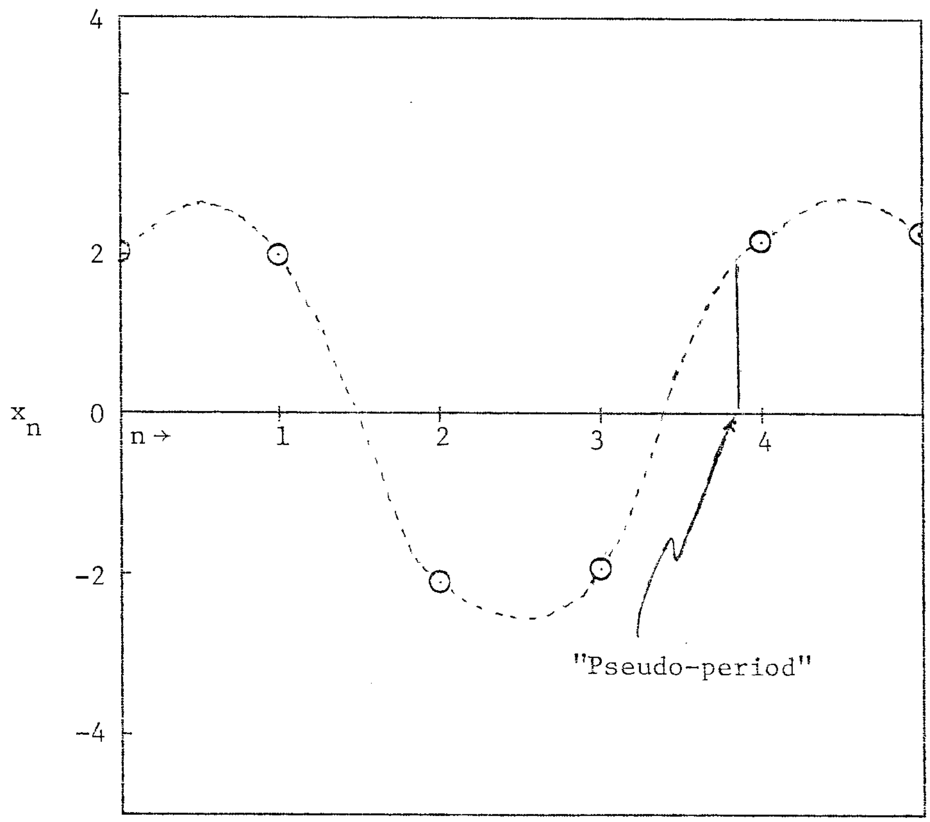


Fig. 2.20 An example illustrating the evolutionary behavior of a quasi-periodic solution, or of a periodic solution that winds around the equilibrium point many times before repeating itself. The motivation for the heuristic definition of a "pseudo-period" is clearly evident.

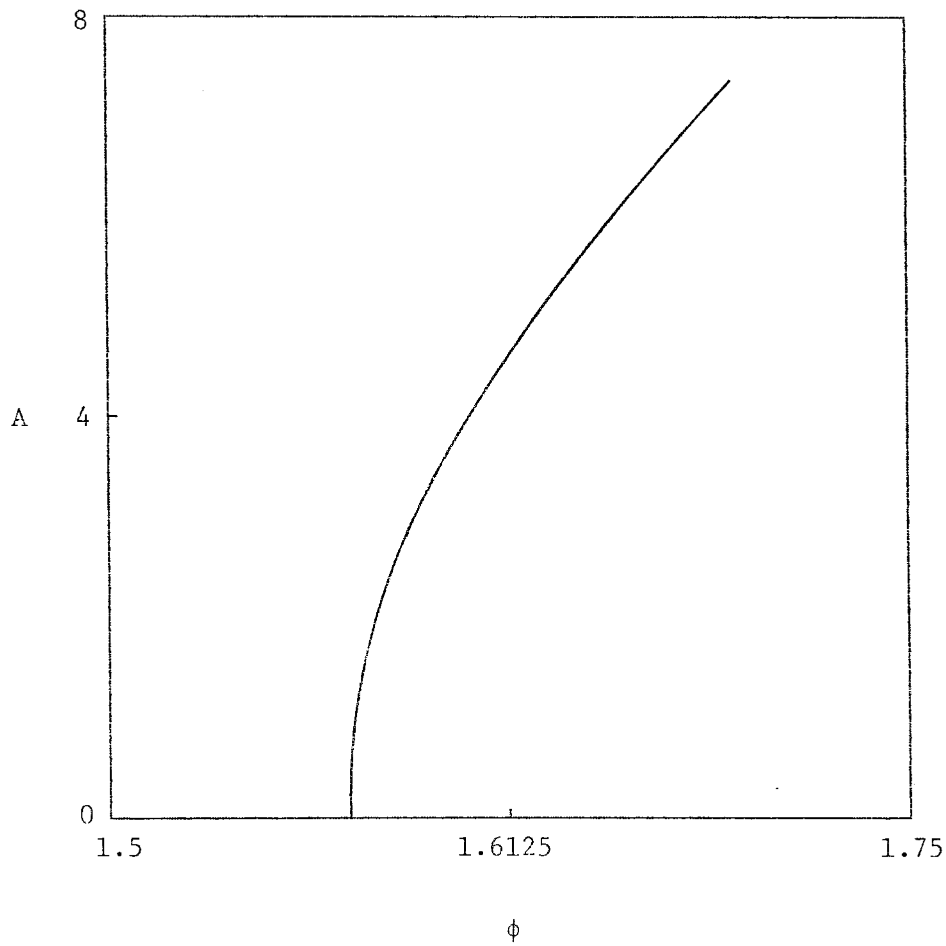


Fig. 2.21 An example of the dependence of the "pseudo-frequency" of the solutions on the amplitude of the solutions for $k = 2$ and $\varepsilon = 0.01$.

Chapter 3.

2.3.3 Some Comments on Stochastic Behavior

Certainly, one of the most obvious qualitative features of the phase portraits presented in Figs. (2.5)-(2.19) is the presence of so-called "stochastic" behavior, i.e., the seemingly irregular wandering of solutions in the neighborhood of separatrices connecting equilibrium, or periodic, saddle points. Such behavior is the result of the wild oscillations of transversally intersecting stable and unstable solutions emanating from equilibrium, or periodic, saddle points.[4,7] If the stable and unstable solutions arise from the same saddle point, the oscillations are called "homoclinic." Otherwise, they are dubbed "heteroclinic."

Stochastic behavior is not exhibited by the solutions of conservative, autonomous, second order, nonlinear differential equations. However, it is a generic attribute (i.e., persists for all $\varepsilon \neq 0$) of the phase portrait structure of a similar class of nonlinear difference equations. Phenomenological evidence of this genericity, for the equilibrium saddle point occurring in region II of the parameter space, is shown in Figs. (2.22)-(2.25).

Since the basic goal of the present thesis is the study of the periodic and quasi-periodic solutions of nonlinear difference equations, a more detailed treatment of stochastic behavior will not be presented. However, the study of stochastic behavior is the subject of much current research. The book by Lichtenberg and Lieberman[7] provides a fairly recent survey for conservative systems.

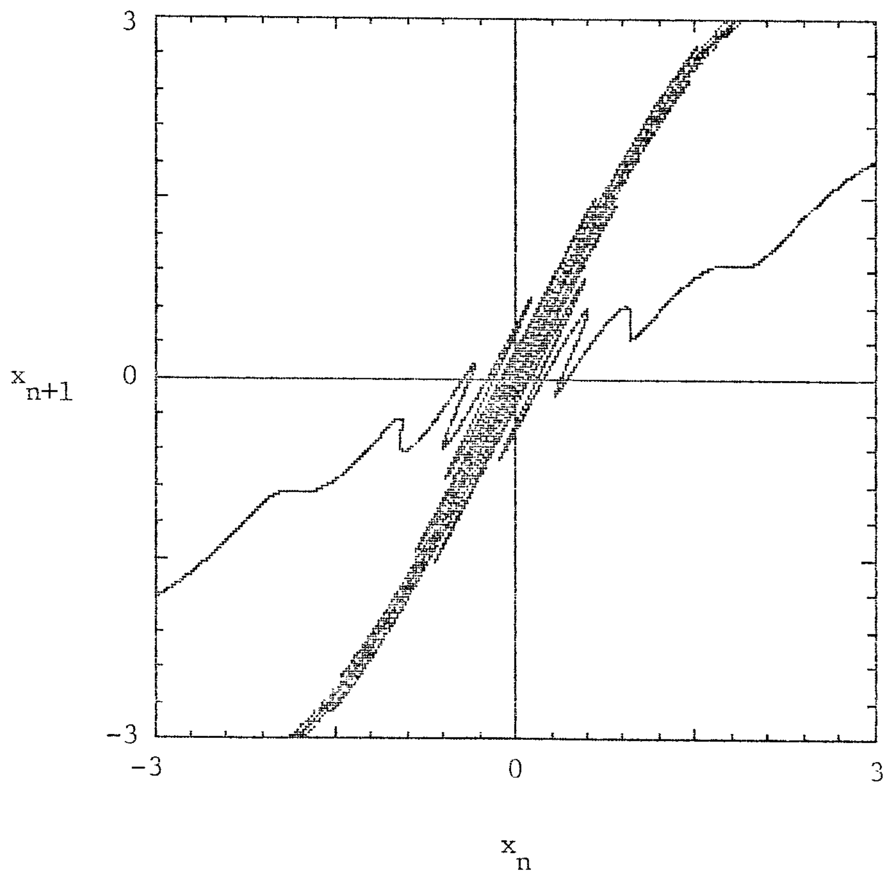


Fig. 2.22 An example of the homoclinic behavior of solutions near the equilibrium saddle point for $k = -0.5$ and $\varepsilon = 0.1$. For clarity, only the unstable manifolds are shown.

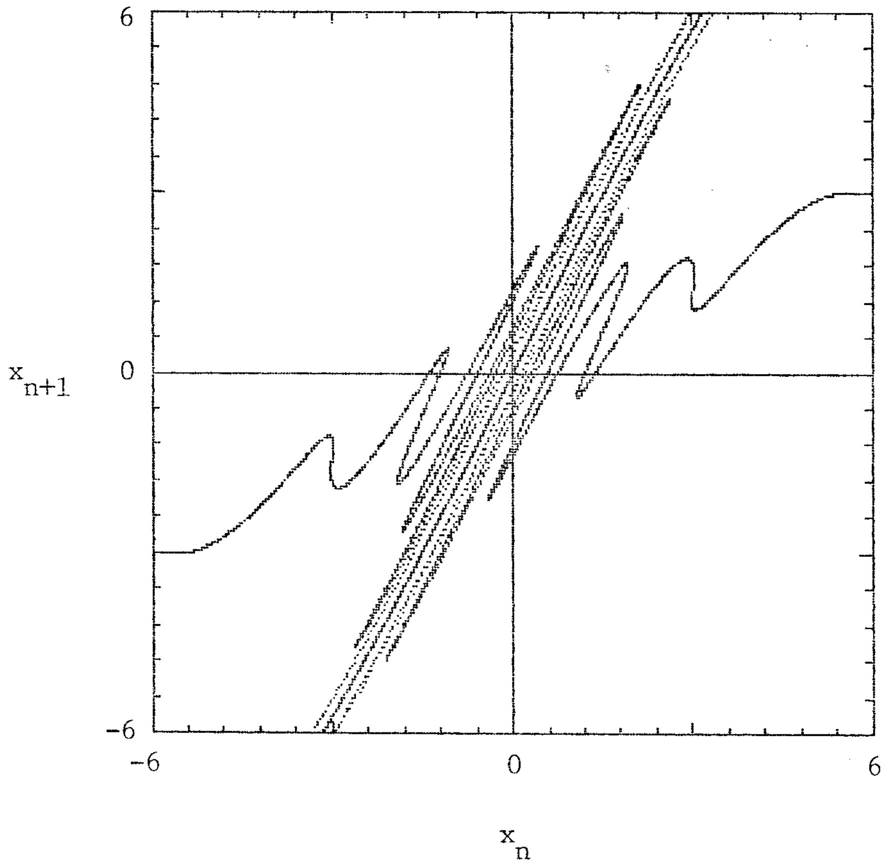


Fig. 2.23 An example of the homoclinic behavior of solutions near the equilibrium saddle point for $k = -0.5$ and $\varepsilon = 0.01$. For clarity, only the unstable manifolds are shown.

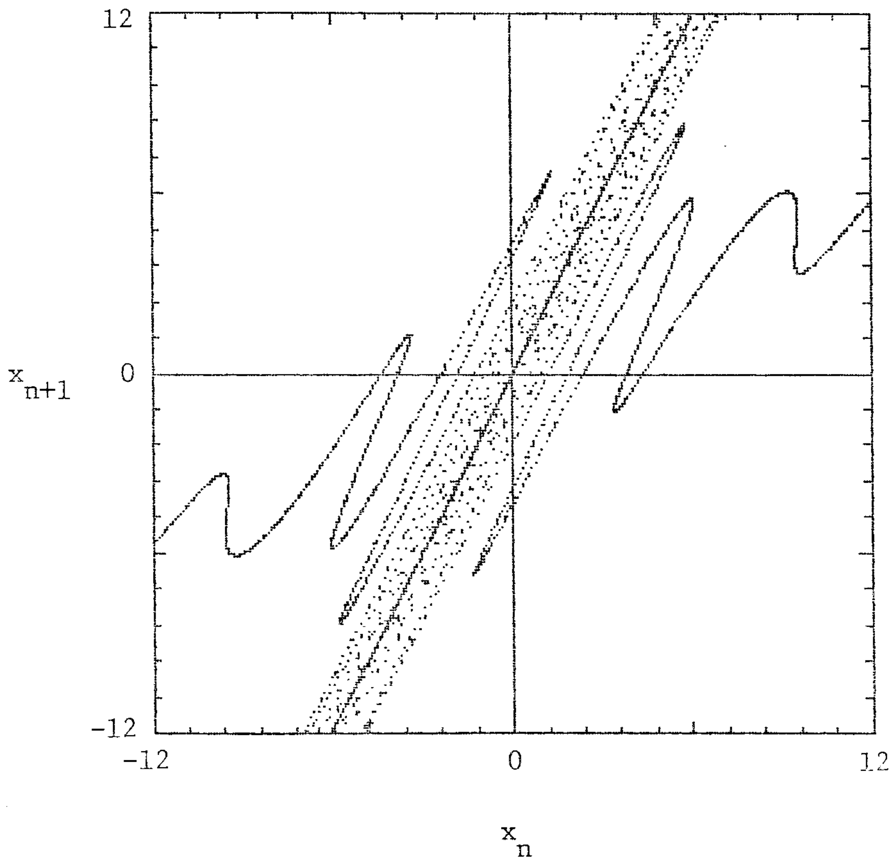


Fig. 2.24 An example of the homoclinic behavior of solutions near the equilibrium saddle point for $k = -0.5$ and $\varepsilon = 0.001$. For clarity, only the unstable manifolds are shown.

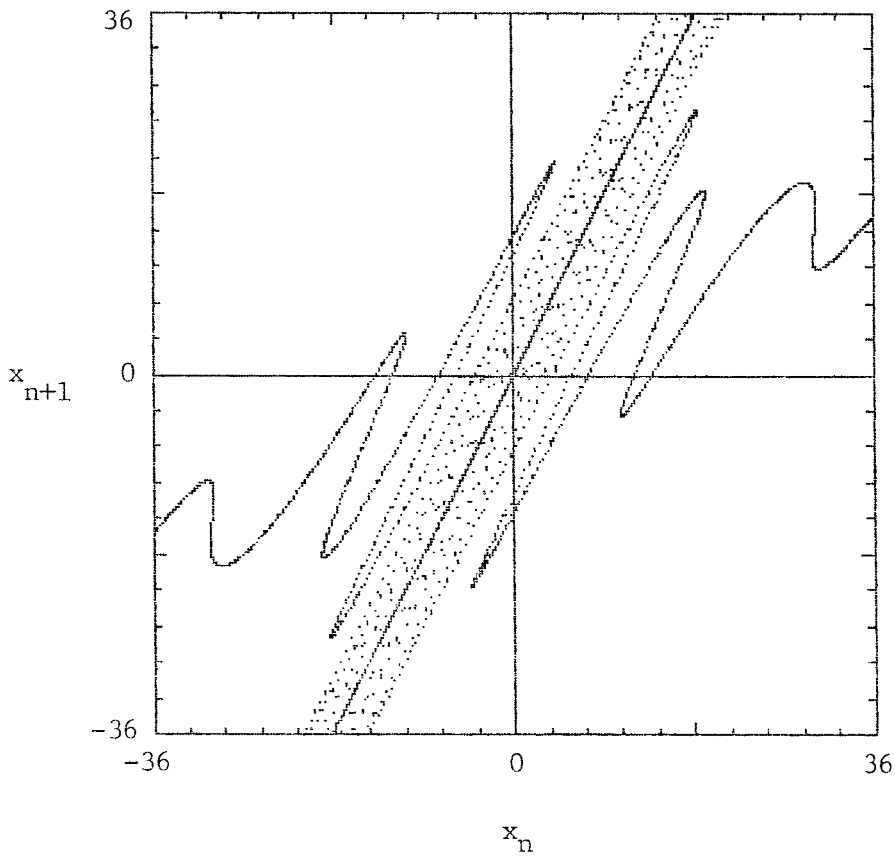


Fig. 2.25 An example of the homoclinic behavior of solutions near the equilibrium saddle point for $k = -0.5$ and $\varepsilon = 0.0001$. For clarity, only the unstable manifolds are shown.

CHAPTER 3

APPROXIMATE METHODS FOR THE STUDY OF PERIODIC AND QUASI-PERIODIC SOLUTIONS

3.1 INTRODUCTION

The utility of a mathematical model of a physical system is directly proportional to the predictive capabilities of the model. These capabilities fall into three broad categories: phenomenological (numerical), quantitative, and qualitative. In order to obtain useful quantitative information from a mathematical model, either exact analytic solutions must be available, or approximate solution techniques must be used. As illustrated in the previous chapter, even a very simple nonlinear difference equation, such as the discrete Duffing equation, can yield an incredibly complex phase portrait. Hence, in general, approximate methods will be necessary.

In physical applications, the "steady-state" solutions are of prime importance. These "steady-state" solutions may be equilibrium points, periodic solutions, or quasi-periodic solutions. Equilibrium solutions may be obtained by solving a nonlinear algebraic equation, as was discussed and carried out for the discrete Duffing equation in the previous chapter. Determination of periodic (period > 1) and quasi-periodic solutions is not as straightforward and, in practice, exact solutions are usually unobtainable. Therefore, approximate solutions are highly desirable.

Three approximate techniques for studying the periodic and quasi-periodic solutions of (autonomous) nonlinear difference equations will be developed within the present chapter. Following a preliminary discussion of periodic and quasi-periodic solutions in Section (3.2), and an illustration of the appearance of secular terms in a straightforward perturbation expansion in Section (3.3), the approximate methods will be developed in Sections (3.4), (3.5), and

(3.6). For clarity, the explicit development is restricted to regions III and VI of the parameter space of the discrete Duffing equation as shown in Fig. (2.3). The chapter closes with a comparison of the methods, as well as some comments on their application, in Section (3.7).

3.2 SOME COMMENTS ON PERIODIC AND QUASI-PERIODIC SOLUTIONS

Periodic and quasi-periodic solutions comprise all of the nontrivial, "steady-state" solutions for a given second-order, conservative, autonomous difference equation. Such an equation may be written

$$\underline{x}_{n+1} = \underline{F}(\underline{x}_n) \ . \tag{3.1}$$

Periodic solutions of Eq. (3.1), of period k, are defined via

$$\underline{x}_{n+k} = \underline{x}_n , \ k \in \mathbf{Z}^+ , k \geq 2 \tag{3.2}$$

with

$$\underline{x}_{n+l} \neq \underline{x}_n \ \forall \ l < k , l \in \mathbf{Z}^+ \ .$$

The periodicity condition given by Eq. (3.2) may be rewritten as

$$\underline{x}_n = \underline{F}^k(\underline{x}_n) \ . \tag{3.3}$$

Therefore, the determination of periodic solutions of Eq. (3.1) may be reduced to the solution of a nonlinear algebraic equation as given by Eq. (3.3). However, even for very simple forms of the map $\underline{F}$, the solution of Eq. (3.3) may be practically intractable. Hence, an efficient approximate technique for determining the period and estimating the amplitude of periodic solutions would prove useful. Later in the present chapter, a perturbation approach will be used to develop a method for the study of these periodic solutions.

Quasi-periodic solutions for Eq. (3.1) are more difficult to define rigorously. Since Eq. (3.1) is conservative and autonomous, an adequate definition may be stated as: quasi-periodic solutions of Eq. (3.1) are those solutions that remain within a closed, bounded one-dimensional submanifold of the phase space, but are not strictly periodic. Note that this definition excludes equilibrium solutions, periodic solutions, and separatrix solutions, since equilibrium and separatrix solutions may be considered strictly periodic, with periods 1 and ∞, respectively. The previous definition, however, does include the "higher-order" quasi-periodic solutions near periodic solutions. As will be seen later in the chapter, approximate expressions for quasi-periodic solutions are simply generated by a perturbation analysis.

3.3 THE DISCRETE PERTURBATION METHOD AND SECULARITY

3.3.1 A Particular Example

The formal development of a discrete perturbation technique for obtaining approximate solutions of nonlinear difference equations may best be illustrated by example. The discrete Duffing equation is an ideal example since it exhibits relative analytic simplicity, genuine practical applicability, and a history of previous theoretical work. The (conservative, autonomous) discrete Duffing equation is, again,

$$x_{n+1} - 2x_n + x_{n-1} + kx_n + \varepsilon x_n^3 = 0 \ . \tag{3.4}$$

Typically, the use of a perturbation procedure to generate approximate solutions requires knowledge of the exact solution of the unperturbed system. Note that the linear difference equation,

$$x_{n+1} - 2\cos\theta x_n + x_{n-1} = 0 \ , \tag{3.5}$$

possesses the exact solution,

$$x_n = A\cos(n\theta + \psi) \ , \tag{3.6}$$

where the amplitude, A, and the phase, ψ, depend on the initial conditions x_0 and x_1. Hence, recasting Eq. (3.4) in the form,

$$x_{n+1} - 2\cos\theta x_n + x_{n-1} + \varepsilon x_n^3 = 0 \ , \tag{3.7}$$

will be convenient for the development to follow. However, note that expressing Eq. (3.4) in the form of Eq. (3.7) seems to place a restriction on the range of the linear parameter, k, i.e.

$$k = 2(1 - \cos\theta) \Rightarrow 0 \le k \le 4 \ . \tag{3.8}$$

An examination of Fig. (2.3) reveals that the range $0 \le k \le 4$ is precisely the parameter range in which periodic and quasi-periodic solutions exist near the origin for the discrete Duffing equation. Thus, the restriction on k given by Eq. (3.8) is artificial, for the present discussion.

3.3.2 The Straightforward Expansion and Secular Terms

The most obvious, though somewhat naïve, approach to the development of a perturbation procedure is to take a known solution to the unperturbed equation and expand the desired solution in a power series in the perturbation parameter around the known solution. Specifically, set

$$x_n = x_n^0 + \varepsilon x_n^1 + \varepsilon^2 x_n^2 + \dots \ . \tag{3.9}$$

The straightforward approach is to substitute Eq. (3.9) into Eq. (3.7) and to set the coefficients of the powers of ε to zero. Thus, implementation of this procedure begins with

$$(x_{n+1}^0 + \varepsilon x_{n+1}^1 + \varepsilon^2 x_{n+1}^2 + \dots) - 2\cos\theta(x_n^0 + \varepsilon x_n^1 + \varepsilon^2 x_n^2 + \dots)$$

$$+ (x_{n-1}^0 + \varepsilon x_{n-1}^1 + \varepsilon^2 x_{n-1}^2 + \dots) + \varepsilon(x_n^0 + \varepsilon x_n^1 + \varepsilon^2 x_n^2 + \dots)^3 = 0 \ ,$$

and results in the following system of linear equations:

$$x_{n+1}^0 - 2\cos\theta x_n^0 + x_{n-1}^0 = 0 \ , \tag{3.10}$$

$$x_{n+1}^1 - 2\cos\theta x_n^1 + x_{n-1}^1 = -(x_n^0)^3 \ , \tag{3.11}$$

$$x_{n+1}^2 - 2\cos\theta x_n^2 + x_{n-1}^2 = -3(x_n^0)^2 x_n^1 \ , \tag{3.12}$$

etc.

The approximate solution generated by the straightforward approach may be obtained by solving Eq. (3.10), Eq. (3.11), Eq. (3.12), etc., in succession, and then substituting into Eq. (3.9). The general solution of Eq. (3.10) is given by Eq. (3.6) as

$$x_n^0 = A\cos(n\theta + \psi) \ . \tag{3.13}$$

To solve Eq. (3.11), first note that

$$(x_n^0)^3 = A^3\cos^3(n\theta + \psi)$$

$$= \tfrac{1}{4}A^3[3\cos(n\theta + \psi) + \cos(3n\theta + 3\psi)] \ . \tag{3.14}$$

Hence, the particular solution of Eq. (3.11) may be expressed in the form

$$x_n^1 = nB\sin(n\theta + \psi) + C\cos(3n\theta + 3\psi) \ , \tag{3.15}$$

where B and C are functions of A. Substitution of Eq. (3.15) into Eq. (3.11), use of Eq. (3.14), and use of standard trigonometric identities produces

$$B = -\frac{3A^3}{8\sin\theta} \tag{3.16}$$

and

$$C = -\frac{A^3}{8(\cos\theta - \cos 3\theta)} \ . \tag{3.17}$$

Finally, substitution of Eq. (3.13) and Eqs. (3.15)-(3.17) into Eq. (3.9) yields the first-order approximation,

$$x_n = A\cos(n\theta + \psi) - \frac{3\varepsilon A^3}{8\sin\theta} n\sin(n\theta + \psi)$$

$$- \frac{\varepsilon A^3}{8(\cos\theta - \cos 3\theta)} \sin(3\theta + 3\psi) + O(\varepsilon^2) \ . \qquad (3.18)$$

Note that the second term on the right-hand side of Eq. (3.18) contains the independent variable n, as a multiplicative factor. Hence, this term is secular, i.e., the presence of this term in the truncated perturbation series renders the expansion nonuniform as n grows large. Therefore, analogous to the situation arising in the approximate solution of nonlinear differential equations, the straightforward perturbation approach leads to the appearance of secular terms in the approximate solution. Consequently, to produce an approximate solution that is uniformly valid for all n, a method of eliminating the secular terms must be developed.

3.4 THE DISCRETE LINDSTEDT-POINCARÉ METHOD

3.4.1 The Rationale Behind the Method

As shown in Section (3.3.2), the straightforward perturbation approach fails to provide approximate solutions which are uniformly valid for all n due to the appearance of secular terms. A review of the straightforward approach reveals that secular terms arise as a result of the inhomogeneous terms in the linear difference equations for the first- and higher-order corrections, e.g., Eq. (3.11). Hence, one method of eliminating the secular terms would be to introduce additional unknowns into the expansion of the original nonlinear difference equation so that these unknowns appear in the inhomogeneous terms of the resulting system of linear difference equations. Implementation of this procedure necessitates an explicit method of introducing the additional unknowns.

A phenomenological study of the discrete Duffing equation, as discussed in Chapter 2, reveals that the pseudo-period of the quasi-periodic solutions is dependent upon the amplitude. The pseudo-period of a quasi-periodic solution, as defined in Chapter 2, can be derived by considering the independent variable, n, as a continuous variable along the solution trajectory. The dependence of the pseudo-period on the amplitude of the quasi-periodic solutions of the discrete Duffing equation is analogous to the dependence of the period on the amplitude of the periodic solutions of the (conservative, autonomous) Duffing differential equation.

Therefore, an approximate solution procedure capable of eliminating secular terms and explicitly illustrating the dependence of the pseudo-period on the amplitude of the quasi-periodic solutions would be desirable. One such procedure is outlined below.

3.4.2 Implementation of the Method

An approximate solution procedure, possessing the dual capabilities mentioned above, may be developed with reasoning analogous to that underlying the Lindstedt-Poincaré procedure for nonlinear differential equations. Specifically, the procedure may be outlined algorithmically as follows:

(i) Introduce an approximate solution frequency, say ϕ, into Eq. (3.7);

(ii) Expand the dependent variable, x_n, in a power series in the nonlinearity parameter, ε;

(iii) Expand some function of ϕ in a power series in ε, where the zero'th order term is the same function of the linear frequency, θ;

(iv) Introduce the expansions of (ii) and (iii) into Eq. (3.7); and

(v) Equate powers of ε; and solve the resulting system of equations recursively, eliminating secular terms at each step.

Implementation of this procedure may be conveniently carried out by first rewriting Eq.

(3.7) in the equivalent form,

$$x_{n+1} - 2\cos\phi x_n + x_{n-1} + 2(\cos\phi - \cos\theta)x_n + \varepsilon x_n^3 = 0 \ , \tag{3.19}$$

and then setting

$$x_n = x_n^0 + \varepsilon x_n^1 + \varepsilon^2 x_n^2 + \ldots \ , \tag{3.20}$$

and

$$\cos\phi = \cos\theta + \varepsilon a_1 + \varepsilon^2 a_2 + \ldots \ . \tag{3.21}$$

Introduction of Eqs. (3.20) and (3.21) into Eq. (3.19) yields

$$\begin{aligned}(x_{n+1}^0 &+ \varepsilon x_{n+1}^1 + \varepsilon^2 x_{n+1}^2 + \ldots) - 2\cos\phi(x_n^0 + \varepsilon x_n^1 + \varepsilon^2 x_n^2 + \ldots) \\ &+ (x_{n-1}^0 + \varepsilon x_{n-1}^1 + \varepsilon^2 x_{n-1}^2 + \ldots) + 2(\varepsilon a_1 + \varepsilon^2 a_2 + \ldots) \\ &\times (x_n^0 + \varepsilon x_n^1 + \varepsilon^2 x_n^2 + \ldots) + \varepsilon(x_n^0 + \varepsilon x_n^1 + \varepsilon^2 x_n^2 + \ldots)^3 = 0 \ .\end{aligned} \tag{3.22}$$

Algebraic manipulation of Eq. (3.22) results in

$$\begin{aligned}(x_{n+1}^0 &- 2\cos\phi x_n^0 + x_{n-1}^0) + \varepsilon[x_{n+1}^1 - 2\cos\phi x_n^1 + x_{n-1}^1 + 2a_1 x_n^0 + (x_n^0)^3] \\ &+ \varepsilon^2[x_{n+1}^2 - 2\cos\phi x_n^2 + x_{n-1}^2 + 2(a_2 x_n^0 + a_1 x_n^1) \\ &+ 3(x_n^0)^2 x_n^1] + O(\varepsilon^3) = 0 \ .\end{aligned} \tag{3.23}$$

Equation (3.23) may be satisfied iff each coefficient vanishes independently. Imposing such a requirement on Eq. (3.23) produces the recursive system of equations

$$x_{n+1}^0 - 2\cos\phi x_n^0 + x_{n-1}^0 = 0 \ , \tag{3.24}$$

$$x_{n+1}^1 - 2\cos\phi x_n^1 + x_{n-1}^1 = -2a_1 x_n^0 - (x_n^0)^3 \ , \tag{3.25}$$

$$x_{n+1}^2 - 2\cos\phi x_n^2 + x_{n-1}^2 = -2(a_2 x_n^0 + a_1 x_n^1) - 3(x_n^0)^2 x_n^1 \ , \tag{3.26}$$

etc. The equations of order k ($k = 3,4,5...$) are similar to Eqs. (3.24), (3.25), and (3.26), and may be represented generically as

$$x_{n+1}^k - 2\cos\phi x_n^k + x_{n-1}^k = f\,(x_n^0, x_n^1, \ldots, x_n^{k-1}) \ , \tag{3.27}$$

where the inhomogeneous term, $f\,(x_n^0, x_n^1, \ldots, x_n^{k-1})$, is a linear combination of terms of the first and third degrees in the trigonometric functions.

3.4.3 The First Order Approximate Solution

An approximate solution of the discrete Duffing equation, accurate to first order in ε, may be obtained by solving Eq. (3.24), choosing the constant, a_1, such that the "secular" inhomogeneity is eliminated, solving Eq. (3.25), and substituting the results into Eqs. (3.20) and (3.21). As previously noted, the solution of Eq. (3.24) is given by Eq. (3.6) as

$$x_n^0 = A\cos(n\phi + \psi) \ , \tag{3.28}$$

where A and ψ depend on the initial conditions, x_0 and x_1. Substituting Eq. (3.28) into Eq. (3.25) and using Eq. (3.14) yields

$$x_{n+1}^1 - 2\cos\phi x_n^1 + x_{n-1}^1 = -(2a_1 A + \tfrac{3}{4}A^3)\cos(n\phi + \psi)$$

$$- \tfrac{1}{4}A^3\cos(3n\phi + 3\psi) \ . \tag{3.29}$$

The "secular" inhomogeneity in Eq. (3.29) is represented by the first term on the right-hand side. As discussed in Section (3.3.2), the appearance of such a term gives rise to a particular solution of Eq. (3.29) of the form, $n\sin(n\phi + \psi)$, which leads to secular behavior. In contrast with the straightforward approach of Section (3.3), the discrete Lindstedt-Poincaré method provides a direct means of eliminating the "secular" inhomogeneity. Clearly, requiring that

$$(2a_1 A + \tfrac{3}{4}A^3)\cos(n\phi + \psi) = 0 \quad \forall\, n \quad ,$$

implies that

$$a_1 = -\tfrac{3}{8}A^2 \ . \tag{3.30}$$

Substitution of Eq. (3.30) into Eq. (3.29) transforms Eq. (3.29) into

$$x_{n+1}^1 - 2\cos\phi x_n^1 + x_{n-1}^1 = -\tfrac{1}{4}A^3\cos(3n\phi + 3\psi) \ . \tag{3.31}$$

A particular solution of Eq. (3.31) is given by

$$x_n^1 = B\cos(3n\phi + 3\psi) \ . \tag{3.32}$$

The constant, B, is determined by substituting Eq. (3.32) into Eq. (3.31), exploiting standard trigonometric identities, and requiring that the coefficients of $cos(3n\phi + 3\psi)$ cancel each other. This process leads to

$$B = \frac{A^3}{8(\cos\phi - \cos 3\phi)} \ . \tag{3.33}$$

The first order approximate solution is generated by substituting Eqs. (3.28) and (3.32) into Eq. (3.20) and using Eq. (3.33) to obtain

$$x_n = A\cos(n\phi + \psi) + \frac{\varepsilon A^3}{8(\cos\phi - \cos 3\phi)}\cos(3n\phi + 3\psi) + O(\varepsilon^2) \ . \tag{3.34}$$

The first order frequency correction is obtained by using Eq. (3.30) in Eq. (3.21) to yield

$$\cos\phi = \cos\theta - \tfrac{3}{8}\varepsilon A^2 + O(\varepsilon^2) \ . \tag{3.35}$$

The ramifications and characteristics of Eqs. (3.34) and (3.35) will be discussed in Section (3.7) and in Chapter 4.

3.4.4 The Second Order Approximate Solution

The second order corrections to Eqs. (3.34) and (3.35) may be obtained in a manner similar to the generation of the first order corrections. To begin, substitution of Eqs. (3.28), (3.30), (3.32), and (3.33) into Eq. (3.26) results in

$$x_{n+1}^2 - 2\cos\phi x_n^2 + x_{n-1}^2 = -2a_2 A\cos(n\phi+\psi)$$

$$-\frac{3A^5}{32(\cos\phi-\cos3\phi)}\cos(3n\phi+3\psi)$$

$$-\frac{3A^5}{8(\cos\phi-\cos3\phi)}\cos^2(n\phi+\psi)\cos(3n\phi+3\psi)\ . \qquad (3.36)$$

Exploiting some basic trigonometric identities yields the simplifying equation,

$$\cos^2(n\phi+\psi)\cos(3n\phi+3\psi) = \frac{1}{4}\cos(n\phi+\psi) + \frac{1}{2}\cos(3n\phi+3\psi) \qquad (3.37)$$

$$+\frac{1}{4}\cos(5n\phi+5\psi)\ .$$

Making use of Eq. (3.37) transforms Eq. (3.36) into,

$$x_{n+1}^2 - 2\cos\phi x_n^2 + x_{n-1}^2 = -\left[\ 2a_2 A + \frac{3A^5}{32(\cos\phi-\cos3\phi)}\right]\cos(n\phi+\psi)$$

$$-\frac{3A^5}{16(\cos\phi-\cos3\phi)}\cos(3n\phi+3\psi)$$

$$-\frac{3A^5}{32(\cos\phi-\cos3\phi)}\cos(5n\phi+5\psi)\ . \qquad (3.38)$$

Again, as in Eq. (3.29), the first term on the right-hand side of Eq. (3.38) represents a "secular" inhomogeneity. This inhomogeneity may be eliminated by requiring

$$\left[2a_2 A + \frac{3A^5}{32(\cos\phi - \cos 3\phi)} \right] \cos(n\phi + \psi) = 0 \quad \forall\, n \;,$$

which implies

$$a_2 = \frac{3A^4}{64(\cos\phi - \cos 3\phi)} \;. \tag{3.39}$$

Substitution of Eq. (3.39) into Eq. (3.38) gives the desired equation for the second order correction, x_n^2, i.e.,

$$x_{n+1}^2 - 2\cos\phi x_n^2 + x_{n-1}^2 = -\frac{3A^5}{16(\cos\phi - \cos 3\phi)} \cos(3n\phi + 3\psi)$$
$$- \frac{3A^5}{32(\cos\phi - \cos 3\phi)} \cos(5n\phi + 5\psi) \;. \tag{3.40}$$

A particular solution of Eq. (3.40) is given by

$$x_n^2 = C\cos(3n\phi + 3\psi) + D\cos(5n\phi + 5\psi) \;. \tag{3.41}$$

Determination of the constants, C and D, is facilitated by substitution of Eq. (3.41) into Eq. (3.40), and by trigonometric and algebraic manipulation of the resulting equation, to obtain

$$2C(\cos 3\phi - \cos\phi)\cos(3n\phi + 3\psi) + 2D(\cos 5\phi - \cos\phi)\cos(5n\phi + 5\psi)$$
$$= -\frac{3A^5}{16(\cos\phi - \cos 3\phi)} \cos(3n\phi + 3\psi)$$
$$- \frac{3A^5}{32(\cos\phi - \cos 3\phi)} \cos(5n\phi + 5\psi) \;.$$

Finally, requiring that the coefficients of $\cos(3n\phi + 3\psi)$ and $\cos(5n\phi + 5\psi)$ cancel, independently, produces

$$C = \frac{3A^5}{32(\cos\phi - \cos 3\phi)^2} \ , \tag{3.42}$$

and

$$D = \frac{3A^5}{64(\cos\phi - \cos 3\phi)(\cos\phi - \cos 5\phi)} \tag{3.43}$$

The approximate solution, accurate to the second order in the perturbation parameter, may be obtained by substituting Eqs. (3.42) and (3.43) into Eq. (3.41), and substituting this result and Eq. (3.34) into Eq. (3.20) to obtain

$$\begin{aligned} x_n = {} & A\cos(n\phi + \psi) + \varepsilon \frac{A^3}{8(\cos\phi - \cos 3\phi)} \cos(3n\phi + 3\psi) \\ & + \varepsilon^2 \left[\frac{3A^5}{32(\cos\phi - \cos 3\phi)^2} \cos(3n\phi + 3\psi) \right. \\ & \left. + \frac{3A^5}{64(\cos\phi - \cos 3\phi)(\cos\phi - \cos 5\phi)} \cos(5n\phi + 5\psi) \right] + O(\varepsilon^3) \ . \end{aligned} \tag{3.44}$$

The second order frequency correction is produced by substituting Eq. (3.39) into Eq. (3.21). Thus, using Eq. (3.35), this yields

$$\cos\phi = \cos\theta - \varepsilon \frac{3A^2}{8} - \varepsilon^2 \frac{3A^4}{64(\cos\phi - \cos 3\phi)} + O(\varepsilon^3) \ . \tag{3.45}$$

The properties and applications of Eqs. (3.44) and (3.45) will be analyzed in Section (3.7) and in Chapter 4. Higher order corrections to the approximate solution may be generated in analogous fashion.

3.5 THE DISCRETE METHOD OF RENORMALIZATION

3.5.1 The Rationale Behind the Method

The straightforward perturbation expansion, introduced in Section (3.3.2), led to the appearance of secular terms in the resulting approximate solution given by Eq. (3.18). One method of eliminating the secular terms, and, consequently, rendering the approximate solution uniformly valid for all n, was outlined in Section (3.4.2). The approach taken in Section (3.4.2) was to introduce an auxiliary expansion, Eq. (3.21), into the original nonlinear difference equation. This approach provided a means of eradicating the secular terms before the approximate solution was generated. Also, the auxiliary expansion was conveniently chosen to clearly illustrate the dependence of the nonlinear frequency, ϕ, on the amplitude of the solution.

An alternate technique for the elimination of secularity would be to introduce additional unknowns into the nonuniform approximate solution, e.g., Eq. (3.18), obtained via the straightforward expansion approach. As discussed in Section (3.4.1), a method of introducing these additional unknowns, so that the dependence of the nonlinear "frequency" (equivalently, the pseudo-period) on the amplitude is conveniently illustrated, would also be desirable. Outlined below is a technique for "renormalizing" the approximate solution, after it has been generated. This technique also explicitly displays the "frequency"-amplitude dependence.

3.5.2 Implementation of the Method

The discrete method of renormalization is implemented by first obtaining an approximate solution via the straightforward approach. The approximate solution is then rendered uniformly valid by a "renormalization" procedure. An outline of this procedure may be stated as follows:

(i) Expand the dependent variable, x_n, in a power series in the nonlinearity parameter, ε;

(ii) Introduce the expansion of (i) into Eq. (3.9);

(iii) Equate powers of ε; and solve the resulting system of equations recursively, retaining the secular terms;

(iv) Construct the (secular) approximate solution with Eq. (3.7);

(v) Expand some "convenient" function of the linear frequency, θ, in a power series in ε;

(vi) Introduce the expansion of (v) into the approximate solution of (iv); and

(vii) Eliminate any secular terms; and discard any terms of high order than the original (secular) approximate solution.

3.5.3 The First Order Approximate Solution

To obtain a first order approximate solution of the discrete Duffing equation via the discrete method of renormalization, the procedure outlined above must be followed. The first four steps of this procedure comprise, precisely, the straightforward approach previously discussed. In Section (3.3.2), this approach was applied to the discrete Duffing equation, yielding the first order (secular) approximation of Eq. (3.18). Slight algebraic manipulation of this equation produces

$$x_n = A\cos(n\theta + \psi) - \varepsilon\frac{A^3}{8}\left[\frac{3n\sin(n\theta+\psi)}{\sin\theta} - \frac{\cos(3n\theta+3\psi)}{\cos\theta - \cos3\theta}\right] + O(\varepsilon^2)\ . \tag{3.46}$$

As noted before, the secularity of the approximate solution, Eq. (3.46), renders the approximation nonuniform for large n. Hence, the approximate solution must be "renormalized." Also, the approximation given by Eq. (3.46) provides no clear means of illustrating the dependence of the nonlinear "frequency" upon the amplitude of the solution.

The renormalization of Eq. (3.46), and an explicit illustration of the nonlinear "frequency"-amplitude dependence, will be conveniently facilitated by the introduction of the expansion

$$\theta = \phi + \varepsilon b_1 + \varepsilon^2 b_2 + O(\varepsilon^3) \ . \tag{3.47}$$

Here, again, ϕ represents the nonlinear "frequency."

Implementation of the renormalization procedure is carried out by substituting Eq. (3.47) into Eq. (3.46) to obtain

$$\begin{aligned} x_n &= A\cos[n\phi + \psi + \varepsilon n b_1 + O(\varepsilon^2)] \\ &- \varepsilon \frac{A^3}{8} \left\{ \frac{3n\sin[n\phi + \psi + \varepsilon n b_1 + O(\varepsilon^2)]}{\sin[\phi + \varepsilon b_1 + O(\varepsilon^2)]} \right. \\ &\left. - \frac{\cos[3n\phi + 3\psi + 3\varepsilon n b_1 + O(\varepsilon^2)]}{\cos[\phi + \varepsilon b_1 + O(\varepsilon^2)] - \cos[3\phi + 3\varepsilon b_1 + O(\varepsilon^2)]} \right\} + O(\varepsilon^2) \ , \end{aligned} \tag{3.48}$$

for fixed n. Simplification of Eq. (3.48) may be accomplished by expanding the trigonometric functions, considering $n\phi$ as the independent variable, in Taylor series about the zero'th order parts of their respective arguments. Generically,

$$\begin{aligned} \cos[\alpha\phi + \psi + \varepsilon\alpha b_1 + O(\varepsilon^2)] &= \cos(\alpha\phi + \psi) \\ &- \varepsilon\alpha b_1 \sin(\alpha\phi + \psi) + O(\varepsilon^2) \ , \end{aligned} \tag{3.49}$$

$$\begin{aligned} \sin[\alpha\phi + \psi + \varepsilon\alpha b_1 + O(\varepsilon^2)] &= \sin(\alpha\phi + \psi) \\ &+ \varepsilon\alpha b_1 \cos(\alpha\phi + \psi) + O(\varepsilon^2) \ . \end{aligned} \tag{3.50}$$

Specifically, note that

$$\frac{1}{\sin[\phi + \varepsilon b_1 + O(\varepsilon^2)]} = \frac{1}{\sin\phi + \varepsilon b_1 \cos\phi + O(\varepsilon^2)}$$

$$= \frac{1}{\sin\phi} \left\{ \frac{1}{1 + \varepsilon \left[b_1 \left(\frac{\cos\phi}{\sin\phi} \right) + O(\varepsilon) \right]} \right\} , \qquad (3.51)$$

and that

$$\frac{1}{\cos[\phi + \varepsilon b_1 + O(\varepsilon^2)] - \cos[3\phi + 3\varepsilon b_1 + O(\varepsilon^2)]}$$

$$= \frac{1}{\cos\phi - \cos 3\phi - \varepsilon b_1 (\sin\phi - 3\sin 3\phi) + \varepsilon^2}$$

$$= \frac{1}{\cos\phi - \cos 3\phi} \left\{ \frac{1}{1 - \varepsilon \left[b_1 \left(\frac{\sin\phi - 3\sin 3\phi}{\cos\phi - \cos 3\phi} \right) + O(\varepsilon) \right]} \right\} . \qquad (3.52)$$

Assuming ε sufficiently small, i.e., small enough so that

$$|\varepsilon| < \left| \frac{1}{[\cdot]} \right|$$

in both Eqs. (3.51) and (3.52), binomial expansions may be applied to the terms in braces in both equations. Expanding the terms in braces in binomial series, transforms Eqs. (3.51) and (3.52) into

$$\frac{1}{\sin[\phi + \varepsilon b_1 + O(\varepsilon^2)]} = \frac{1}{\sin\phi} \left[1 + O(\varepsilon) \right] , \qquad (3.53)$$

and

$$\frac{1}{\cos[\phi + \varepsilon b_1 + O(\varepsilon^2)] - \cos[3\phi + \varepsilon b_1 + O(\varepsilon^2)]}$$

$$= \frac{1}{\cos\phi - \cos 3\phi}\,[1 + O(\varepsilon)] \ . \qquad (3.54)$$

Finally, substituting Eqs. (3.53) and (3.54) into Eq. (3.48), and using Eqs. (3.49) and (3.50), the first order approximate solution becomes

$$x_n = A\cos(n\phi + \psi) - \varepsilon b_1 A n \sin(n\phi + \psi) - \varepsilon\frac{A^3}{8}\left[\frac{3n\sin(n\phi + \psi)}{\sin\phi} - \frac{\cos(3n\phi + 3\psi)}{\cos\phi - \cos 3\phi}\right] + O(\varepsilon^2) \ ,$$

or

$$x_n = A\cos(n\phi + \psi) - \varepsilon A\left(b_1 + \frac{3A^3}{8\sin\phi}\right) n\sin(n\phi + \psi) + \varepsilon\frac{A^3}{8(\cos\phi - \cos 3\phi)}\cos(3n\phi + 3\psi) + O(\varepsilon^2) \ . \qquad (3.55)$$

The secularity of the approximate solution, represented by the second term on the right-hand side of Eq. (3.55), may be eliminated by proper choice of the constant, b_1. In particular, choosing

$$b_1 = -\frac{3A^2}{8\sin\phi} \ , \qquad (3.56)$$

yields the uniformly valid first order approximate solution,

$$x_n = A\cos(n\phi + \psi) + \varepsilon\frac{A^3}{8(\cos\phi - \cos 3\psi)}\cos(3n\phi + 3\psi) + O(\varepsilon^2) \ . \qquad (3.57)$$

The first order frequency correction may be obtained by substituting Eq. (3.56) into Eq. (3.47) to obtain

$$\theta = \phi - \varepsilon \frac{3A^2}{8\sin\phi} + O(\varepsilon^2) \ . \tag{3.58}$$

Taking the cosine of both sides of Eq. (3.58) results in

$$\cos\theta = \cos\left[\phi - \varepsilon \frac{3A^2}{8\sin\phi} + O(\varepsilon^2) \right] \ . \tag{3.59}$$

Applying Eq. (3.49) to Eq. (3.59), and solving for $\cos\phi$, yields, finally,

$$\cos\phi = \cos\theta - \varepsilon \frac{3A^2}{8} + O(\varepsilon^2) \ . \tag{3.60}$$

The first order approximate solution, Eq. (3.57), is exactly the same as the first order approximate solution obtained via the discrete Lindstedt-Poincaré technique, Eq. (3.34). Similarly, the first order frequency correction, represented by Eq. (3.60), is the same as before, represented by Eq. (3.35). Again, these equations will be discussed in Section (3.7) and in Chapter 4.

3.5.4 The Second Order Approximate Solution

The second and higher order approximate solutions may be generated in a fashion similar to the first order approximate solution generated above. In particular, construction of the second order approximation begins with the particular solution of Eq. (3.12) using, as before,

$$x_n^0 = A\cos(n\theta + \psi) \ , \tag{3.61}$$

and

$$x_n^1 = -\frac{A^3}{8\sin\theta} n\sin(n\theta + \psi) + \frac{A^3}{8(\cos\theta - \cos3\theta)} \cos(3n\theta + 3\psi) \ . \tag{3.62}$$

Substitution of Eqs. (3.61) and (3.62) into Eq. (3.12), exploitation of basic trigonometric identities, and a little algebra produces

$$x_{n+1}^2 - 2\cos\theta x_n^2 + x_{n-1}^2 = \frac{A^5}{16(\cos\theta - \cos3\theta)} [2\cos(n\theta + \psi)$$

$$+ \cos(3n\theta + 3\psi) + 2\cos(5n\theta + 5\psi)] - \frac{A^5 n}{32\sin\theta} [\sin(n\theta + \psi)$$

$$+ \sin(3n\theta + 3\psi)] \ . \qquad (3.63)$$

Hence, the particular solution of Eq. (3.63) is, after much algebra,

$$x_n^2 = -\frac{3A^5}{64\sin\theta} \left[\left(\frac{1}{\cos\theta - \cos3\theta} - \frac{\cos\theta}{2\sin^3\theta} \right) n\sin(n\theta + \psi) \right.$$

$$\left. + \frac{1}{\cos\theta - \cos3\theta} n\sin(3n\theta + 3\psi) + \frac{1}{2\sin\theta} n^2\cos(n\theta + \psi) \right]$$

$$+ \frac{3A^5}{32(\cos\theta - \cos3\theta)^2} \left(1 - \frac{\sin3\theta}{2\sin\theta} \right) \cos(3n\theta + 3\psi)$$

$$- \frac{3A^5}{64(\cos\theta - \cos3\theta)(\cos\theta - \cos5\theta)} (\cos5n\theta + 5\psi) \ . \qquad (3.64)$$

Therefore, the second order (secular) approximation may be constructed by substituting Eqs. (3.61), (3.62), and (3.64) into Eq. (3.9) to obtain, for fixed n,

$$x_n = A\cos(n\theta + \psi)$$

$$- \varepsilon \left[\frac{A^3}{8\sin\theta} n\sin(n\theta + \psi) - \frac{A^3}{8(\cos\theta - \cos3\theta)} \cos(3n\theta + 3\psi) \right]$$

$$- \varepsilon^2 \left\{ \frac{3A^5}{64\sin\theta} \left[\left(\frac{1}{\cos\theta - \cos3\theta} - \frac{\cos\theta}{2\sin^3\theta} \right) n\sin(n\theta + \psi) \right. \right.$$

$$+ \frac{1}{\cos\theta - \cos 3\theta} n \sin(3n\theta + 3\psi) + \frac{1}{2\sin\theta} n^2 \cos(n\theta + \psi) \Bigg]$$

$$- \frac{3A^5}{32(\cos\theta - \cos 3\theta)} \Bigg[\frac{2\sin\theta - \sin 3\theta}{2\sin\theta(\cos\theta - \cos 3\theta)} \cos(3n\theta + 3\psi)$$

$$- \frac{1}{2(\cos\theta - \cos 5\theta)} \cos(5n\theta + 5\psi) \Bigg] \Bigg] \Bigg\} + O(\varepsilon^3) \ . \qquad (3.65)$$

The uniformly valid second order approximate solution may be obtained from Eq. (3.65) via the renormalization procedure illustrated in Section (3.5.3). First, the frequency expansion, Eq. (3.47), must be substituted into Eq. (3.65). Then, all of the trigonometric factors must be expanded in Taylor series about the zero'th order parts of their respective arguments, regarding $n\phi$ as the independent variable. Binomial expansions are then used to transform the constant trigonometric terms appearing in the denominators of some of the terms of Eq. (3.65). Terms of the third and higher orders are then discarded. Finally, the constants of the frequency expansion, Eq. (3.47), are chosen so that the secular terms are obliterated. This process results in the desired second order approximate solution,

$$x_n = A\cos(n\phi + \psi) + \varepsilon \frac{A^3}{8(\cos\phi - \cos 3\phi)} \cos(3n\phi + 3\psi)$$

$$+ \varepsilon^2 \Bigg[\frac{3A^5}{32(\cos\phi - \cos 3\phi)^2} \cos(3n\phi + 3\psi)$$

$$+ \frac{3A^5}{64(\cos\phi - \cos 3\phi)(\cos\phi - \cos 5\phi)} \cos(5n\phi + 5\psi) \Bigg] + O(\varepsilon^3) \ . \qquad (3.66)$$

Proceeding as before, the "frequency"-amplitude equation becomes,

$$\cos\phi = \cos\theta - \varepsilon \frac{3A^2}{8} - \varepsilon^2 \frac{3A^4}{64(\cos\phi - \cos 3\phi)} + O(\varepsilon^3) \ . \qquad (3.67)$$

Note that Eqs. (3.66) and (3.67) are exactly the same as Eqs. (3.44) and (3.45), respectively. As previously mentioned, the utility of these equations will be discussed in Section (3.7) and in Chapter 4. Higher-order approximate solutions can be generated by the discrete method of renormalization in a similar manner.

3.6 THE DISCRETE METHOD OF DOMINANT BALANCE

3.6.1 The Rationale Behind the Method

Two approximate solution techniques have been developed thus far: the discrete Lindstedt-Poincaré method and the discrete method of renormalization. Both of these procedures begin with a generic perturbation expansion, i.e., Eq. (3.9), which assumes virtually nothing about the form of the desired approximate solution. Both methods then proceed systematically to the generation of a consistent approximate solution. However, both of these techniques, especially the discrete method of renormalization, may be cumbersome due to the amount of algebraic and trigonometric manipulations that may be required.

A slightly simpler method of generating a first order approximate solution may be developed, provided that *a priori* knowledge of the form of the first order approximation is assumed. This method, which is analogous to the method of dominant balance for differential equations, is less rigorously based than the two previous methods. However, as will be shown below, careful application of this technique results in a first order approximation that is consistent with Eqs. (3.34) and (3.57). Furthermore, the discrete method of dominant balance produces a first order nonlinear "frequency"-amplitude relationship that is consistent with Eqs. (3.35) and (3.60).

Knowledgeable consideration of the original nonlinear equation, Eq. (3.7), reveals that the zero'th order approximate solution is given by Eq. (3.6). However, since Eq. (3.7) is nonlinear, the "frequency" of the approximate solution is expected to depend upon the amplitude. Also, in view of the trigonometric identity expressed in Eq. (3.14), a correction term with a frequency three times the fundamental frequency is to be expected. Therefore, a natural choice for the form of the first order approximation is

$$x_n \approx A \cos(n\phi + \psi) + \varepsilon B \cos(3n\phi + 3\psi) \ , \tag{3.68}$$

where

$$\phi \approx \phi(A) \tag{3.69}$$

is the amplitude-dependent nonlinear "frequency." The procedure for determining the unknown constant, B, in Eq. (3.68), and for expressing the frequency-amplitude relationship, Eq. (3.69), is described below.

3.6.2 Implementation of the Method

Implementation of the discrete method of dominant balance is relatively straightforward. This procedure may be outlined as follows:

(i) Introduce the approximate solution, Eq. (3.68), into the original equation, Eq. (3.7);

(ii) Algebraically and trigonometrically manipulate the resulting equation until all trigonometric terms, with arguments of the form $\alpha n\phi + \psi$, are of the first degree;

(iii) Discard any terms of second and higher order in ε;

(iv) Discard any (non-constant) trigonometric terms with arguments different than those appearing in the approximation, Eq. (3.68); and

(v) Collect coefficients of similar terms to determine the constant, B, and the "frequency"-amplitude relationship.

3.6.3 The First Order Approximate Solution

The discrete method of dominant balance leads to a uniformly valid first order approximate solution of the discrete Duffing equation via the procedure outlined above. Substitution of Eq. (3.68) into Eq. (3.7) results in the equation

$$A\{\cos[(n+1)\phi+\psi]-2\cos\theta\cos(n\phi+\psi)+\cos[(n-1)\phi+\psi]\}$$
$$+\varepsilon B\{\cos[3(n+1)\phi+3\psi]-2\cos\theta\cos(3n\phi+3\psi)$$
$$+\cos[3(n-1)\phi+3\psi]\}+\varepsilon[A\cos(n\phi+\psi)$$
$$+\varepsilon B\cos(3n\phi+3\psi)]^3\approx 0\ . \qquad (3.70)$$

Applying simple trigonometric identities, including the identity expressed by Eq. (3.14), and algebraically manipulating Eq. (3.70) leads to

$$2A(\cos\phi-\cos\theta)\cos(n\phi+\psi)+2\varepsilon B(\cos3\phi-\cos\theta)\cos(2n\phi+\psi)$$
$$+\varepsilon\frac{A^3}{4}[3\cos(n\phi+\psi)+\cos(3n\phi+\psi)]+O(\varepsilon^2)\approx 0\ . \qquad (3.71)$$

Collecting the coefficients of similar terms in Eq. (3.17), and neglecting all terms of order higher than one, yields

$$\left[2A(\cos\phi-\cos\theta)+\varepsilon\frac{3A^3}{4}\right]\cos(n\phi+\psi)$$
$$+\left[2\varepsilon B(\cos3\phi-\cos\theta)+\varepsilon\frac{A^3}{4}\right]\cos(3n\phi+3\psi)\approx 0\ . \qquad (3.72)$$

The equations for the unknown constant, B, and the frequency-amplitude dependence are simply obtained by requiring that the respective coefficients of the two trigonometric terms in Eq. (3.72) vanish independently. That is,

$$2A\,(\cos\phi - \cos\theta) + \varepsilon\frac{3A^3}{4} = 0 \ , \tag{3.73}$$

and

$$2\varepsilon B\,(\cos 3\phi - \cos\theta) + \varepsilon\frac{A^3}{4} = 0 \ . \tag{3.74}$$

Solving for $\cos\phi$ in Eq. (3.73) yields the first order nonlinear "frequency"-amplitude relationship

$$\cos\phi \approx \cos\theta - \varepsilon\frac{3A^2}{8} \ . \tag{3.75}$$

Similarly, solving Eq. (3.74) for B yields.

$$B \approx \frac{A^3}{8(\cos\theta - \cos 3\phi)} \ . \tag{3.76}$$

Elimination of the linear "frequency," θ, from Eq. (3.76) is facilitated by introducing Eq. (3.75) into Eq. (3.76). This results in

$$B \approx \frac{A^3}{8\left[\cos\phi - \cos 3\phi + \varepsilon\dfrac{A^2}{8}\right]} \ ,$$

or

$$B \approx \frac{A^3}{8(\cos\phi - \cos 3\phi)}\left[\frac{1}{1 + \varepsilon\dfrac{A^2}{8(\cos\phi - \cos 3\phi)}}\right] \ . \tag{3.77}$$

For ε sufficiently small, i.e., for

$$|\varepsilon| < \left|\frac{8(\cos\phi - \cos 3\phi)}{A^2}\right| \ ,$$

the bracketed term in Eq. (3.77) may be expanded in a binomial series to produce

$$B \approx \frac{A^3}{8(\cos\phi - \cos 3\phi)} [1 + O(\varepsilon)] \ ,$$

or

$$B \approx \frac{A^3}{8(\cos\phi - \cos 3\phi)} \ . \tag{3.78}$$

Therefore, the first order approximate solution of the discrete Duffing equation, generated by the discrete method of dominant balance, may be written

$$x_n \approx A\cos(n\phi + \psi) + \varepsilon \frac{A^3}{8(\cos\phi - \cos 3\phi)} \cos(3n\phi + 3\psi) \ . \tag{3.79}$$

Note that Eq. (3.79) is really the same as Eqs. (3.34) and (3.57). Also, note that Eq. (3.75) is really the same as Eqs. (3.35) and (3.60). Again, these equations will be discussed in Section (3.7) and in Chapter 4.

3.6.4 Some Comments on Higher Order Approximations

In principle, approximate solutions of the second and higher orders may be constructed using the discrete method of dominant balance. However, similar to the situation arising with the method of dominant balance for nonlinear differential equations, great care must be exercised in choosing the form of the approximate solution.

In general, the assumed approximate solution of a given order will be of the form

$$x_n \approx \sum_{i=1}^{j} [B_i \cos(in\phi + \psi) + C_i \sin(in\phi + \psi)] \ , \tag{3.80}$$

where B_i and C_i are of the form

$$B_i = \sum_{k=0}^{j-1} \varepsilon^k b_k \ , \tag{3.81}$$

and

$$C_i = \sum_{k=0}^{j-1} \varepsilon^k c_k \ , \tag{3.82}$$

respectively. Improper choice of which terms to retain in Eqs. (3.80), (3.81), and (3.82) will lead to an approximate solution which will be less accurate than the one generated by the two techniques previously discussed.

In particular, for the discrete Duffing equation, a second order approximate solution, consistent with Eqs. (3.44) and (3.66), will be generated only if an approximation of the form,

$$x_n \approx A\cos(n\phi + \psi) + (\varepsilon B + \varepsilon^2 C)\cos(3n\phi + 3\psi)$$

$$+ \varepsilon^2 D\cos(5n\phi + 5\psi) \ , \tag{3.83}$$

is chosen. *A priori* knowledge of the form of the approximation, as expressed by Eq. (3.83), is generally difficult to acquire. Therefore, in practice, application of the discrete method of dominant balance should probably be restricted to the generation of first order approximate solutions. Even then, the method should only be applied to equations where the form of the approximate solution is clearly deducible.

3.7 FURTHER COMMENTS ON THE APPROXIMATE METHODS

3.7.1 An Overall Comparison

Three techniques for generating approximate solutions of nonlinear difference equations have been developed within the present chapter. All three methods are similar to methods developed for generating approximate solutions to nonlinear differential equations. However, the discrete nature of the independent variable in the present context necessitates some subtle modifications. Also, as discussed in the preceding and subsequent chapters, the discrete nature of this independent variable renders the approximate solutions quasi-periodic except for certain "special" cases.

For a given order of approximation, both the discrete Lindstedt-Poincaré method of Section (3.4), and the discrete method of renormalization, described in Section (3.5), yield exactly the same approximate solution. This equivalence is exemplified by comparing Eqs. (3.34) and (3.44) to Eqs. (3.57) and (3.66), respectively. In addition, both techniques furnish precisely the same approximation to the nonlinear "frequency"-amplitude relationship, as exemplified by comparing Eqs. (3.35) and (3.45) to Eqs. (3.60) and (3.67), respectively. However, a cursory examination of their respective applications to the discrete Duffing equation reveals that the discrete Lindstedt-Poincaré procedure is less cumbersome to apply than the discrete method of renormalization.

The third technique, the discrete method of dominant balance developed in Section (3.6), was seen to yield the same first order approximate solution, Eq. (3.79), and the same frequency-amplitude relationship, Eq. (3.75). Furthermore, application of the discrete method of dominant balance was seen to be less cumbersome than application of either of the other methods. However, successful application of the discrete method of dominant balance hinges on the correct choice of the form of the approximate solution. This requires *a priori* knowledge of the solution, which may be difficult, if not impossible, to acquire in practice. Therefore, caution must be exercised in the application of this technique.

An approximate solution technique, similar to the discrete method of dominant balance, but based on more general Fourier series approximations, may be developed. This method, which yields approximate solutions consistent with those generated within the present chapter, is similar to the method of harmonic balance for nonlinear ordinary differential equations. Approximate solutions of the second and higher orders may be produced without any *a priori* knowledge of the form of the solutions other than that they may be approximated by Fourier series. In practice, however, application of such a technique is usually very tedious and cumbersome.

An alternate approach to the generation of approximate solutions, similar to the method of slowly varying parameters for differential equations, has been developed and applied to the discrete Duffing equation. Application of this approach is little or no more tedious than application of the other methods discussed within the present chapter. However, such a technique requires *a priori* knowledge of the form of the solution. This method may also be "generalized" by following a Fourier series approach, but the generalized method is very cumbersome to apply.

3.7.2 A Note on Initial Conditions

In the development of the approximate solution techniques, the statement was made that the zero'th order amplitude, A, and the phase angle, ψ, depended upon the initial conditions, x_0 and x_1. However, the actual form of this dependence was left unspecified.

In particular, an examination of the discrete Lindstedt-Poincaré method reveals that the homogeneous solutions of the linear equations for the various order correction terms, i.e., Eqs. (3.24), (3.25), (3.26), etc., are all of the form

$$(x_n^i)_H = A_i \cos(n\phi + \psi_i)\ , \ \ i = 0,1,2,\ldots \ .$$

Hence, the total solutions are all of the form

$$x_n^i = A_i \cos(n\phi + \psi_i) + (x_n^i)_p \ , \ \ i = 0,1,2,\ldots \ , \qquad (3.84)$$

where the particular solutions, $(x_n^i)_p$ $(i = 0,1,2,\ldots)$, are the solutions that were derived in Section (3.4). Note that, for $n = 0$, Eq. (3.84) becomes

$$x_0^i = A_i \cos\psi_i + (x_0^i)_p \ , \ \ i = 0,1,2,\ldots \ ; \qquad (3.85)$$

and, for $n = 1$,

$$x_1^i = A_i \cos(\phi + \psi_i) + (x_1^i)_p \ , \ i = 0,1,2,\ldots \ . \tag{3.86}$$

To determine A and ψ as functions of the A_i and ψ_i, recall the trigonometric identity

$$\alpha\cos(\beta + \gamma) = \alpha\cos\gamma\cos\beta - \alpha\sin\gamma\sin\beta \ . \tag{3.87}$$

Denoting the sum of the fundamental "frequency" terms of the approximate solution by σ_H, application of Eq. (3.87) results in

$$\sigma_H = (A_0\cos\psi_o + \varepsilon A_1\cos\psi_1 + \ldots)\cos n\phi - (A_0\sin\psi_0 + \varepsilon A_1\sin\psi_1 + \ldots)\sin n\phi \ .$$

Applying Eq. (3.87), and a similar equation for $\alpha\sin(\beta + \gamma)$, in "reverse" yields

$$\sigma_H = A\cos(n\phi + \psi) \ ,$$

where

$$A = +\sqrt{(A_0\cos\psi_0 + \varepsilon A_1\cos\psi_1 + \ldots)^2 + (A_0\sin\psi_0 + \varepsilon A_1\sin\psi_1 + \ldots)^2} \ , \tag{3.88}$$

and

$$\psi = \tan^{-1}\left[\frac{-(A_0\sin\psi_0 + \varepsilon A_1\sin\psi_1 + \ldots)}{(A_0\cos\psi_0 + \varepsilon A_1\cos\psi_1 + \ldots)}\right] \ . \tag{3.89}$$

Two different, but equivalent, approaches can be used to determine the constants, A_i and ψ_i, in Eqs. (3.85) and (3.86). Both approaches assume that the order of the approximate solution is specified. Additionally, note that the true initial conditions, x_0 and x_1, may be expressed as

$$x_0 = x_0^0 + \varepsilon x_0^1 + \ldots \tag{3.90}$$

and

$$x_1 = x_1^0 + \varepsilon x_i + \ldots \ , \tag{3.91}$$

respectively.

One approach begins by making the assumption

$$x_0^i = x_1^i = 0 \ , \quad i = 1,2,\ldots \ .$$

The A_i and ψ_i are then determined from Eqs. (3.85) and (3.86) using Eqs. (3.88) and (3.89) and the nonlinear "frequency"-amplitude relationship. Obviously, this approach is very cumbersome.

An alternate approach is to assume that

$$A_i = \psi_i = 0 \ , \quad i = 1,2,\ldots \ .$$

This assumption transforms Eqs. (3.88) and (3.89) into

$$A = A_0 \tag{3.92}$$

and

$$\psi = \psi_0 \ , \tag{3.93}$$

respectively. The constants x_0^i and x_1^i ($i = 1,2,\ldots$) are then determined from Eqs. (3.85) and (3.86) using Eqs. (3.92) and (3.93) and the nonlinear "frequency"-amplitude relationship. The zero'th order initial conditions, x_0^0 and x_1^0, are then obtained from Eqs. (3.90) and (3.91). Finally, A_0 and ψ_0 are determined from Eqs. (3.85) and (3.86), with $i = 0$.

3.7.3 A Note on Regions of Application

The approximate solution techniques developed within the present chapter were based on an approximation of the form given by Eq. (3.9). For clarity of the discussion, this approximation was applied to the discrete Duffing equation written in the form of Eq. (3.7).

The approximate solutions subsequently generated apply directly to the periodic and quasi-periodic solutions of the discrete Duffing equation occurring around the origin. As illustrated in Chapter 2, the occurrence of such solutions corresponds to regions III and VII of the parameter space depicted in Fig. (2.3).

Periodic and quasi-periodic solutions also occur within region II of the parameter space of Fig. (2.3). The methods developed thus far also apply to this case. However, in view of Eq. (3.8), the original difference equation cannot be written in the form of Eq. (3.7). Application of the approximate techniques to region II of the parameter space requires that the origin of the phase plane be shifted to the "nonlinear center" around which solutions are desired. Specifically, taking

$$y_n = x_n - x_* \ , \tag{3.94}$$

where x_* is given by one or the other of Eqs. (2.5), substituting into Eq. (2.1), and using Eq. (2.3) leads to

$$y_{n+1} - 2(1+k)y_n + y_{n-1} + \varepsilon(3x_* y_n^2 + y_n^3) = 0 \ . \tag{3.95}$$

Application of the approximate techniques is facilitated by recasting Eq. (3.95) in the form

$$y_{n+1} - 2\cos\theta y_n + y_{n-1} + \varepsilon(3x_* y_n^2 + y_n^3) = 0 \ , \tag{3.96}$$

where

$$\cos\theta = 1 + k \ .$$

Note that this implies $-2 \le k \le 0$, precisely as desired. Using Eq. (3.96) as the new starting point, approximate solutions for y_n are generated as discussed previously. These approximate solutions are then substituted into Eq. (3.94) to give the desired approximations for x_n.

CHAPTER 4

APPLICATIONS TO THE DISCRETE DUFFING EQUATION

4.1 INTRODUCTION

The present chapter concentrates on the analysis of periodic and quasi-periodic solutions of the discrete Duffing equation using approximate solutions obtained via the perturbation techniques developed in Chapter 3. For expositional clarity, the discussion will again be restricted to regions III and VI of the parameter space of the discrete Duffing equation as shown in Fig. (2.3). A similar analysis, following the procedure outlined in Section (3.7.3), may be applied to the periodic and quasi-periodic solutions occurring within region II of the parameter space.

The general form of the approximate solution will be briefly discussed in Section (4.2). Section (4.3) begins with a discussion of how the form of the approximate solutions reveals criteria for the occurrence of periodic solutions. A method of constructing exact periodic solutions will then be discussed, followed by some comments on the stability of periodic solutions. A pair of examples of exact solutions will then be presented. The section closes with brief discussions of approximate periodic solutions and periodic solutions in general. Quasi-periodic solutions will be considered in Section (4.4).

4.2 THE GENERAL FORM OF THE APPROXIMATION

The starting point for the analysis to follow will be the general approximate solution generated by the techniques previously developed. Notational complexity will be reduced in the following discussion by defining

$$\sigma_{2i+1} = \cos\phi - \cos[(2i+1)\phi] \ , \quad i = 1,2,\ldots \ . \tag{4.1}$$

Hence, using, for example, the discrete Lindstedt-Poincaré method to generate the third order approximate solution, and substituting Eq. (4.1) into the result, yields

$$x_n = A\cos(n\phi+\psi) + \varepsilon\left[\frac{A^3}{8\sigma_3} + \frac{\varepsilon 3A^5}{32\sigma_3^2} + \frac{\varepsilon^2 A^7}{512}\left[\frac{15}{\sigma_3^3} + \frac{9}{\sigma_3^2\sigma_5}\right]\right]\cos(3n\phi+3\psi)$$

$$+\varepsilon^2\left[\frac{3A^5}{64\sigma_3\sigma_5} + \frac{\varepsilon A^7}{512}\left[\frac{21}{\sigma_3^2\sigma_5} + \frac{9}{\sigma_3\sigma_5^2}\right]\right]\cos(5n\phi+5\psi)$$

$$+\varepsilon^3\left[\frac{3A^7}{128\sigma_3^2\sigma_7}\right]\cos(7n\phi+7\psi) + O(\varepsilon^4)\ . \tag{4.2}$$

An examination of Eq. (4.2) and the lower order approximations of Chapter 3, and the procedure used to generate these approximations, indicates that the general form of the approximate solution for a given order, say j_0, can be written as

$$x_n = A\cos(n\phi+\psi) + \sum_{j=1}^{j_0} \varepsilon^j A_j \cos[(2j+1)(n\phi+\psi)] + O(\varepsilon^{j_0+1})\ . \tag{4.3}$$

The general form of the coefficient A_j, $j \in [1, j_0] \subset \mathbf{Z}^+$, may be expressed as

$$A_j = \sum_{l=1}^{j_0-j+1} c_l \varepsilon^{l-1} A^{2j+2l-1}\ , \tag{4.4}$$

where the c_l are of the form of a finite sum of terms of the form $\rho\sigma_3^{-\alpha}\sigma_5^{-\beta}\ldots\sigma_{2j_0+1}^{-\gamma}$, $\rho \in \mathbf{R}$; and $\alpha,\beta,\gamma \in \mathbf{Z}^+$. Note that every term in the summation on the right-hand side of Eq. (4.4) contains at least one divisor of the type defined in Eq. (4.1).

4.3 PERIODIC SOLUTIONS

4.3.1 Criteria for the Occurrence of Periodic Solutions

Since every term in the approximate solution, other than the zero'th order term, contains a divisor of the form of Eq. (4.1), the question of what happens when one of these divisors vanishes naturally presents itself. Clearly, if one of the divisors explicitly occurring in the approximate solution, to a given order in ε, vanishes, then the approximation is invalid.

Suppose, in particular,

$$\sigma_{2i_0+1} = 0 \ , \tag{4.5}$$

and

$$\sigma_{2i+1} \neq 0 \ \ \forall \, i < i_0 \ , \ \ i \in \mathbf{Z}^+ - \{0\} \ . \tag{4.6}$$

Thus, Eqs. (4.1) and (4.5) imply

$$\cos\phi = \cos[(2i_0 + 1)\phi] \ ,$$

and, hence,

$$(2i_0 + 1)\phi = \pm\,\phi + 2q\pi \ , \ \ q \in \mathbf{Z}^+ - \{0\} \ .$$

Solving for ϕ yields

$$\phi = \begin{cases} \dfrac{q\pi}{i_0} \\ \\ \dfrac{q\pi}{i_0 + 1} \end{cases} , q \in \mathbf{Z}^+ - \{0\} \ .$$

Imposing the condition expressed by Eq. (4.6) implies

$$\sigma_{2(i_0-1)+1} \neq 0 \ ,$$

or

$$\phi \neq \begin{cases} \dfrac{l\pi}{i_0 - 1} \\ \dfrac{l\pi}{i_0} \end{cases} , \quad l \in \mathbf{Z}^+ - \{0\} \ .$$

Therefore, Eq. (4.7a) must not be satisfied; and ϕ must be given by Eq. (4.7b). Also, examination of Eq. (4.7b), in light of Eq. (4.6), implies that q must not be an integral multiple of any of the nontrivial prime factors of $(i_0 + 1)$. Clearly, if ϕ is any given, rational multiple of π, then there exists an i_0 and a q such that Eqs. (4.5) and (4.6) are satisfied.

Note that since solutions of the discrete Duffing equation may be represented by a trigonometric series, the solutions will be strictly periodic iff there exists an integer, $P \geq 2$, such that

$$(n + P)\phi = n\phi + 2w\pi \ , \quad w \in \mathbf{Z}^+ - \{0\} \ ,$$

where w is the number of times the solution "winds" around the origin before mapping back onto itself. Thus,

$$P = \frac{2w\pi}{\phi} \ , \quad P \in \mathbf{Z}^+ - \{0,1\} \ .$$

Using Eq. (4.7b), this becomes

$$P = \frac{2w(i_0 + 1)}{q} \ , \quad P \in \mathbf{Z}^+ - \{0,1\} \ . \tag{4.8}$$

Hence, note that Eq. (4.8) implies

$$q \leq w(i_0 + 1) \ . \tag{4.9}$$

In particular, for $w = 1$, Eq. (4.9) yields

$$q \leq i_0 + 1 \ .$$

4.3.2 Construction of Exact Periodic Solutions

If Eq. (4.7b) is satisfied, an interesting thing happens to trigonometric factors of the type occurring in Eq. (4.3) for $j > i_0$. Clearly, for all $j > i_0$, j can be expressed in the form

$$j = l(i_0 + 1) + r \ , \quad l \in \mathbf{Z}^+ - \{0\} \ , \quad r \in [0, i_0] \subset \mathbf{Z}^+ \ . \tag{4.10}$$

Hence, using Eq. (4.10) and a basic trigonometric identity results in

$$\begin{aligned} \cos[(2j+1)(n\phi+\psi)] &= \cos\{[2l(i_0+1)+2r+1](n\phi+\psi)\} \\ &= \cos[(2r+1)(n\phi+\psi)]\cos[2l(i_0+1)(n\phi+\psi)] \\ &\quad - \sin[(2r+1)(n\phi+\psi)]\sin[2l(i_0+1)(n\phi+\psi)] \ . \end{aligned} \tag{4.11}$$

However, since Eq. (4.7b) is assumed to be satisfied,

$$\begin{aligned} \cos[2l(i_0+1)(n\phi+\psi)] &= \cos\left[2l(i_0+1)\left(\frac{nj\pi}{i_0+1}+\psi\right)\right] \\ &= \cos[2l(i_0+1)\psi] \ , \end{aligned} \tag{4.12}$$

and, similarly,

$$\sin[2l(i_0+1)(n\phi+\psi)] = \sin[2l(i_0+1)\psi] \ . \tag{4.13}$$

Using Eqs. (4.12) and (4.13), Eq. (4.11) becomes

$$\begin{aligned} \cos[(2j+1)(n\phi+\psi)] &= \cos[(2r+1)(n\phi+\psi)]\cos[2l(i_0+1)\psi] \\ &\quad - \sin[(2r+1)(n\phi+\psi)]\sin[2l(i_0+1)\psi] \ . \end{aligned} \tag{4.14}$$

Suppose, now, that

$$\sin[2(i_0+1)\psi]=0 \ .$$

This implies

$$\psi=\frac{s\pi}{2(i_0+1)} \ , \tag{4.15}$$

where s is restricted to be zero or q in order to avoid redundancy. Note that satisfaction of the condition expressed by Eq. (4.15) results in

$$\sin[2l(i_0+1)\psi]=\sin(ls\pi)=0 \quad \forall\, l \ , \ s\in \mathbf{Z}^+ \ , \tag{4.16}$$

and

$$\cos[2l(i_0+1)\psi]=\cos(ls\pi)=(-1)^{ls} \quad \forall\, l \ , \ s\in \mathbf{Z}^+ \ . \tag{4.17}$$

Now, assuming Eq. (4.15) is satisfied and using Eqs. (4.16) and (4.17), Eq. (4.14) becomes

$$\cos[(2j+1)(n\phi+\psi)]=(-1)^{ls}\cos[(2r+1)(n\phi+\psi)] \quad \forall\, j>i_0 \ , \tag{4.18}$$

where $r\in[0,i_0]\subset\mathbf{Z}^+$. That is, for all $j>i_0$, each trigonometric factor may be represented by one of the trigonometric factors with $r\le i_0$.

Additionally, assuming Eqs. (4.7b) and (4.15) are satisfied, a curious thing happens to the trigonometric factors for $m\le j\le i_0$, where m is the smallest number satisfying

$$m>\frac{i_0}{2} \ , \ m\in\mathbf{Z}^+ \ . \tag{4.19}$$

Clearly, in this case, j can be expressed as

$$j=i_0-r \ , \ 0\le r<\frac{i_0}{2} \ , \ r\in\mathbf{Z}^+ \ . \tag{4.20}$$

Specifically, note that Eqs. (4.20), (4.7b), and (4.15) imply

$$\cos[(2j+1)(n\phi+\psi)] = \cos\left\{ [2(i_0-r)+1] \left[\frac{nq\pi}{i_0+1} + \frac{s\pi}{2(i_0+1)} \right] \right\}$$

$$= \cos\left\{ -(2r+1) \left[\frac{nq\pi}{i_0+1} + \frac{s\pi}{2(i_0+1)} \right] + 2(i_0+1) \left[\frac{nq\pi}{i_0+1} + \frac{s\pi}{2(i_0+1)} \right] \right\}$$

$$= \cos\left\{ -(2r+1) \left[\frac{nq\pi}{i_0+1} + \frac{s\pi}{2(i_0+1)} \right] + 2nq\pi + s\pi \right\}$$

$$= (-1)^s \cos[(2r+1)(n\phi+\psi)] \,, \ \forall_j \in m \le j \le i_0 \,, \tag{4.21}$$

where r is given by Eq. (4.20) and m is the smallest number satisfying Eq. (4.19).

Therefore, satisfaction of Eqs. (4.5), (4.6), and (4.15) implies that a cosine series representation of the solutions to the discrete Duffing equation truncates with the $(2m-1)$ term, where m is the smallest number satisfying Eq. (4.19). The truncation of the series representation indicates that the periodic solutions of the discrete Duffing may be represented exactly by a finite series of the form

$$x_n = A\cos(n\phi+\psi) + \sum_{i=1}^{m-1} A_i \cos[(2i+1)(n\phi+\psi)] \,, \tag{4.22}$$

where the frequency, ϕ, and the phase angle, ψ, are given by Eqs. (4.7b) and (4.15), respectively. The coefficients, A and A_i, may be determined by substituting Eq. (4.22) into the original difference equation, Eq. (3.4). Specifically, for the case $w = 1$ in Eq. (4.8), the distinct periodic solutions obtainable from Eq. (4.22) may be identified as shown in Table I for $m = 1,\ldots,5$. Clearly, as can be seen from Table I, all $w = 1$ periodic solutions may be constructed exactly and uniquely by the method under discussion.

4.3.3 A Note on the Stability of Periodic Solutions

Note that, in general, there are two sets of values of coefficients satisfying Eq. (4.22), i.e., the two sets of values corresponding to $s = 0, q$, respectively. For example, as illustrated in Chapter 2, e.g., Fig. (2.10), if the period is even, strictly periodic solutions of the discrete Duffing equation occur as sets of alternating stable and unstable periodic points. For the conservative case presently under investigation, these points are alternating periodic centers and saddle points, respectively. Thus, for even period solutions, one value of s yields the stable periodic solution, and the other value of s yields the unstable periodic solution.

In general, the determination of the stability of a periodic solution corresponding to a specific value of s may be carried out in a manner similar to the determination of the stability of an equilibrium solution. That is, the solution is slightly perturbed away from the periodic solution, x_n^*, by taking

$$x_n = x_n^* + \xi_n \quad , \quad |\xi_n| \text{ "small"} \ . \tag{4.23}$$

Eq. (4.23) is then substituted into Eq. (3.4), using the fact that x_n^* is an exact solution, to yield

$$\xi_{n+1} - (2-k)\xi_n + \xi_{n-1} + 3\varepsilon(x_n^*)^2\xi_n + O(|\xi_n|^2) = 0 \ ,$$

or,

$$\begin{Bmatrix} \xi_{n+1} \\ \xi_n \end{Bmatrix} = \begin{bmatrix} 2-k-3\varepsilon(x_n^*)^2 & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \xi_n \\ \xi_{n-1} \end{Bmatrix} + O(|\xi_n|^2) \ . \tag{4.24}$$

Denoting the matrix occurring in Eq. (4.24) as $L_s(n)$, the stability of a specific periodic solution, x_n^*, is determined by the absolute values of the eigenvalues of

$$L_s^* = \prod_{i=0}^{P-1} L_s(P-i) \ , \tag{4.25}$$

where P is the period of the solution given by Eq. (4.8).

4.3.4 Examples of Exact Solutions

Rather than trying to prove that Eq. (4.22) furnishes an exact solution for a general value of i_0, specific examples will be considered. In particular, three examples of solutions which map back onto themselves after circling the origin once, i.e., the case where $w = 1$ in Eq. (4.8), will be treated.

Example I: $i_0 = 1, q = 1$

Examination of Eq. (4.19) reveals that $m = 1$ for this example. Hence, consider a solution of the form,

$$x_n^* = A\cos(n\phi + \psi) \ . \tag{4.26}$$

Substitution of this expression into the original form of the discrete Duffing equation, i.e., Eq. (3.4), and use of some elementary trigonometric identities results in

$$\left[2\cos\phi - 2 + k + \frac{3\varepsilon A^2}{4}\right] A\cos(n\phi + \psi)$$
$$+ \frac{\varepsilon A^3}{4}\cos(3n\phi + 3\psi) = 0 \ .$$

Use of Eq. (4.21) yields, further,

$$\left[2\cos\phi - 2 + k + \frac{3 + (-1)^s}{4}\varepsilon A^2\right] A\cos(n\phi + \psi) = 0 \ ,$$

or, simply,

$$A = + 2\sqrt{\frac{2 - k - 2\cos\phi}{\varepsilon[3 + (-1)^s]}} \tag{4.27}$$

where the exponent, s, comes from using Eq. (4.15) for the phase angle, ψ.

Table I

Distinct solutions given by Eq. (4.22) for $w = 1$ and $m = 1,\ldots,5$. The symbols at the top of the columns are defined in the text.

m	i_0	q	ϕ	P	ψ
1	1	1	$\frac{\pi}{2}$	4	$0, \frac{\pi}{4}$
		2	π	2	$0, \frac{\pi}{2}$
2	2	1	$\frac{\pi}{3}$	6	$0, \frac{\pi}{6}$
		2	$\frac{2\pi}{3}$	3	$0, \frac{\pi}{3}$
	3	1	$\frac{\pi}{4}$	8	$0, \frac{\pi}{8}$
3	4	1	$\frac{\pi}{5}$	10	$0, \frac{\pi}{10}$
		2	$\frac{2\pi}{5}$	5	$0, \frac{\pi}{5}$
	5	1	$\frac{\pi}{6}$	12	$0, \frac{\pi}{12}$
4	6	1	$\frac{\pi}{7}$	14	$0, \frac{\pi}{14}$
		2	$\frac{2\pi}{7}$	7	$0, \frac{\pi}{7}$
	7	1	$\frac{\pi}{8}$	16	$0, \frac{\pi}{16}$
5	8	1	$\frac{\pi}{9}$	18	$0, \frac{\pi}{18}$
		2	$\frac{2\pi}{9}$	9	$0, \frac{\pi}{9}$
	9	1	$\frac{\pi}{10}$	20	$0, \frac{\pi}{20}$

For the example at hand, Eqs. (4.7b), (4.8), and (4.15) yield, respectively,

$$\phi = \frac{\pi}{2} \ ,$$

and,

$$\psi = \frac{s\pi}{4} \quad ; \quad s = 0,1 \ .$$

Hence, from Eqs. (4.26) and (4.27), the periodic solution is given by

$$x_n^* = +2\sqrt{\frac{2-k}{\varepsilon[3+(-1)^s]}} \cos\left[\frac{n\pi}{2} + \frac{s\pi}{4} \right] \ . \tag{4.28}$$

Interestingly, note that $\varepsilon > 0$ implies that real, nontrivial, period four solutions, as given by Eq. (4.28), exist only for $k < 2$. Similarly, $\varepsilon < 0$ implies $k > 2$. For convenience in the remainder of the discussion of this example, suppose $\varepsilon > 0$ and, in particular,

$$k = 1 \ .$$

Now, for $s = 0$, Eq. (4.28) becomes

$$x_n^* = \frac{1}{\sqrt{\varepsilon}} \cos\left[\frac{n\pi}{2} \right] \ . \tag{4.29}$$

Hence,

$$\begin{aligned} (x_n^*)^2 &= \frac{1}{\varepsilon}\cos^2\left[\frac{n\pi}{2} \right] \\ &= \frac{1}{\varepsilon}\left[\frac{1}{2} + \frac{1}{2}\cos(n\pi) \right] \\ &= \frac{1+(-1)^n}{2\varepsilon} \ , \end{aligned}$$

so that the matrix of the linearized system in Eq. (4.24) becomes

$$L_0(n) = \begin{bmatrix} 1 - 3\left[\dfrac{1+(-1)^n}{2} \right] & -1 \\ 1 & 0 \end{bmatrix} \ .$$

Thus, using Eq. (4.25),

$$L_0^* = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} .$$

The characteristic equation of L_0^* is, therefore,

$$(\lambda^*)^2 - 14\lambda^* + 1 = 0 \ ,$$

which yields

$$\lambda_\pm^* = 7 \pm \sqrt{(7^2) - 1} \ .$$

Clearly, the eigenvalues of L_0^* "straddle" the number one on the real line. Hence, the solution given by Eq. (4.29) is a set of periodic saddle points.

Turning attention, now, to the case where $s = q = 1$ in Eq. (4.28), the solution, x_n^*, becomes

$$x_n^* = \frac{2}{\sqrt{2\varepsilon}} \cos\left[\frac{n\pi}{2} + \frac{\pi}{4} \right] . \tag{4.30}$$

Thus,

$$(x_n^*)^2 = \frac{2}{\varepsilon} \cos^2\left[\frac{n\pi}{2} + \frac{\pi}{4} \right]$$

$$= \frac{1}{\varepsilon} \left[1 + \cos\left[n\pi + \frac{\pi}{2} \right] \right]$$

$$= \frac{1}{\varepsilon} \left[1 + \sin(n\pi) \right]$$

$$= \frac{1}{\varepsilon} \ .$$

Hence, the matrix of the linearized system in Eq. (4.24) becomes

$$L_1(n) = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} .$$

Therefore,

$$L_1^* = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}^4$$

$$= \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix} ,$$

which yields the characteristic equation

$$(\lambda^*)^2 - 2\lambda^* + 1 = 0 .$$

Thus, the eigenvalues of L_1^* are

$$\lambda_\pm^* = -1 .$$

Obviously, the eigenvalues, $\lambda_\pm^*$, of L_1^* lie on the unit circle.

For the present example, with $k = 1$, the eigenvalues of L_1^* are actually equal. Hence, the stability of the periodic solution given by Eq. (4.30) is "marginal" in the sense discussed in Section (2.2.3). In reality, this "degeneracy" results from using the linearized system to determine stability. As will be shown graphically, the solution given by Eq. (4.30) is actually a set of periodic "centers," in general.

The period four solutions obtained in the present example are shown graphically in Figs. (4.1)-(4.4) for various values of ε and k. The two-"dimensionality" of one of the period four "islands," belying the results of the linearized stability analysis, is illustrated in Fig. (4.5). These figures were generated numerically from the original difference equation, i.e., Eq. (3.4). The structure of the phase plane close to the "ring" of periodic points was generated by slightly perturbing the initial values of the numerical simulation away from the

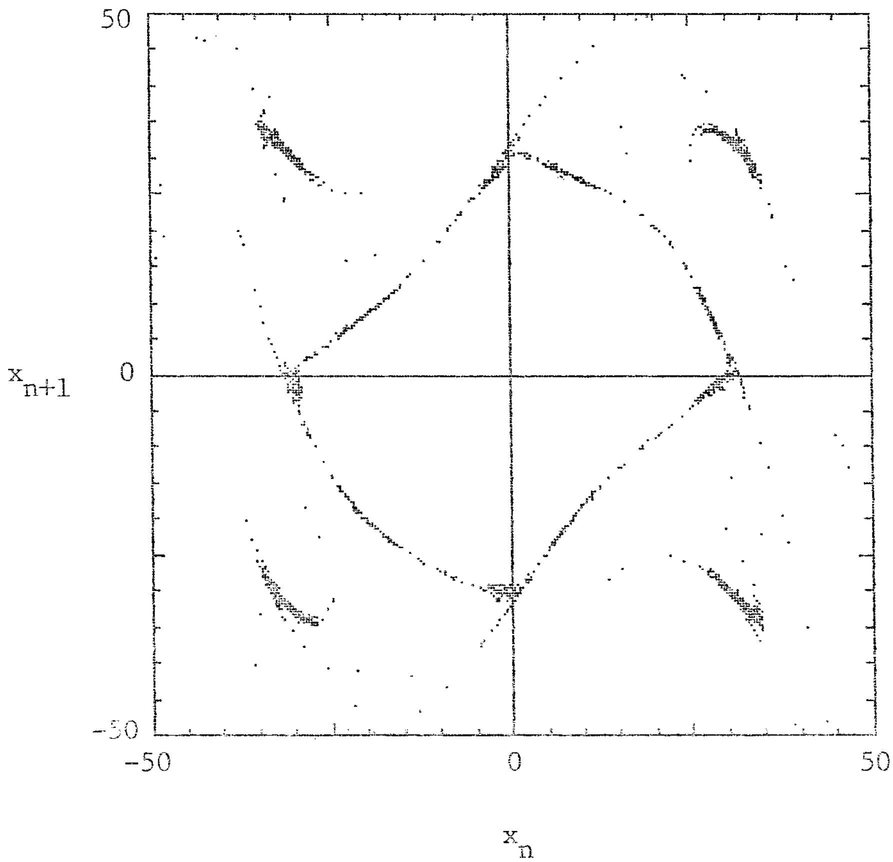


Fig. 4.1 An example of the phase plane structure near the period four solution of Example I for $k = 1$ and $\varepsilon = 0.001$.

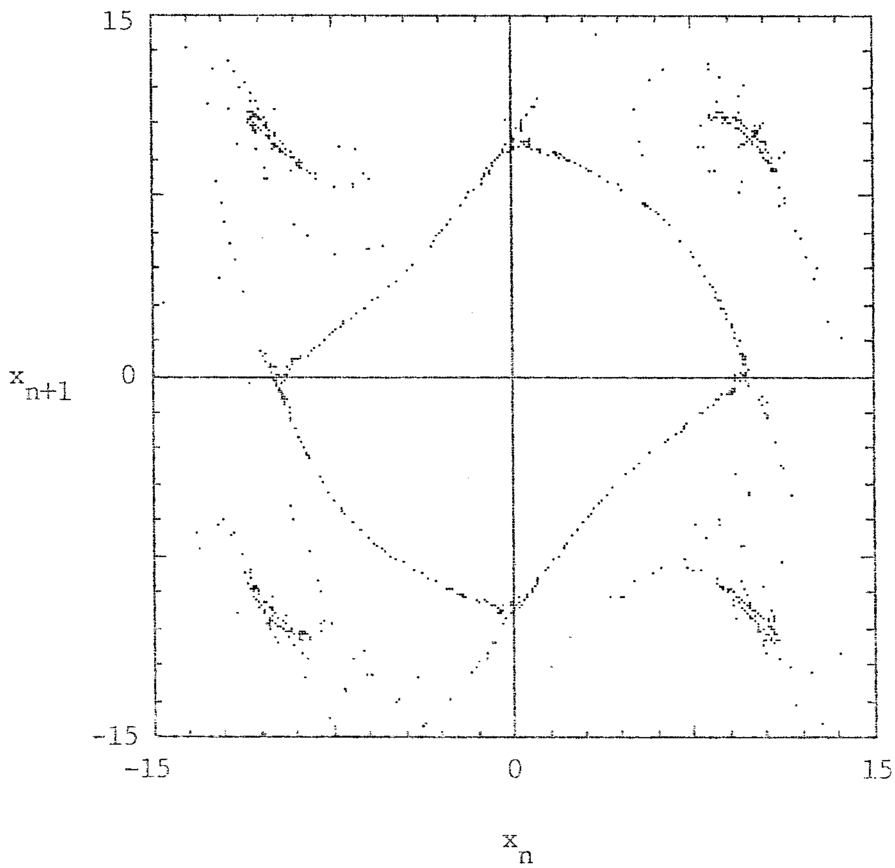


Fig. 4.2 An example of the phase plane structure near the period four solution of Example I for $k = 1$ and $\varepsilon = 0.01$.

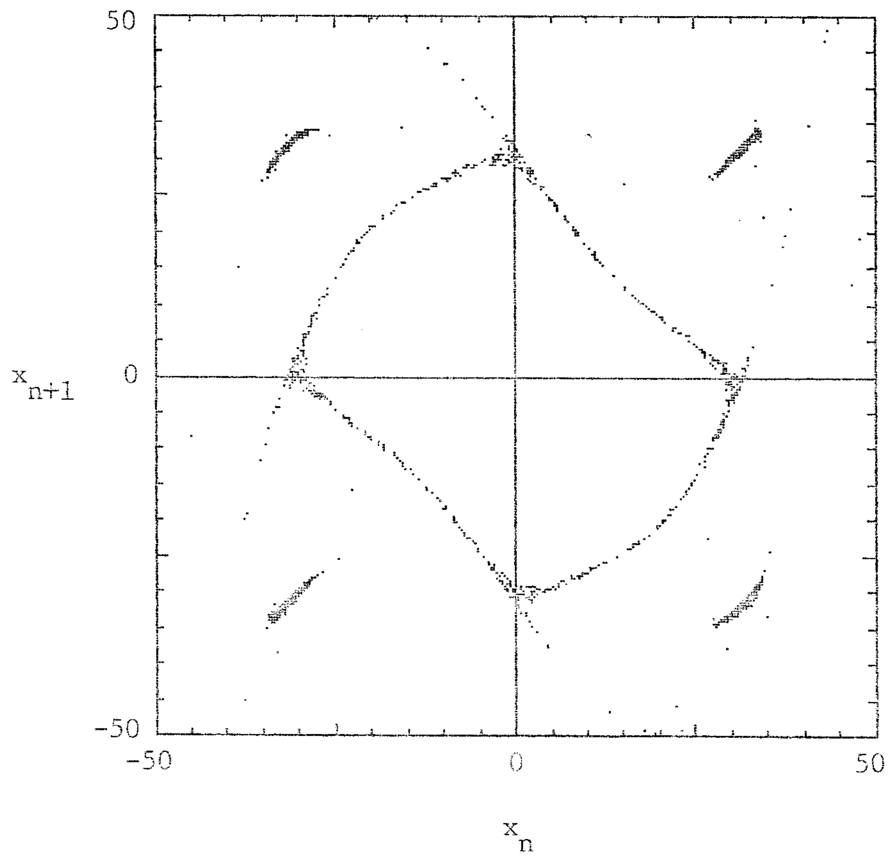


Fig. 4.3 An example of the phase plane structure near the period four solution of Example I for $k = 3$ and $\varepsilon = -0.001$.

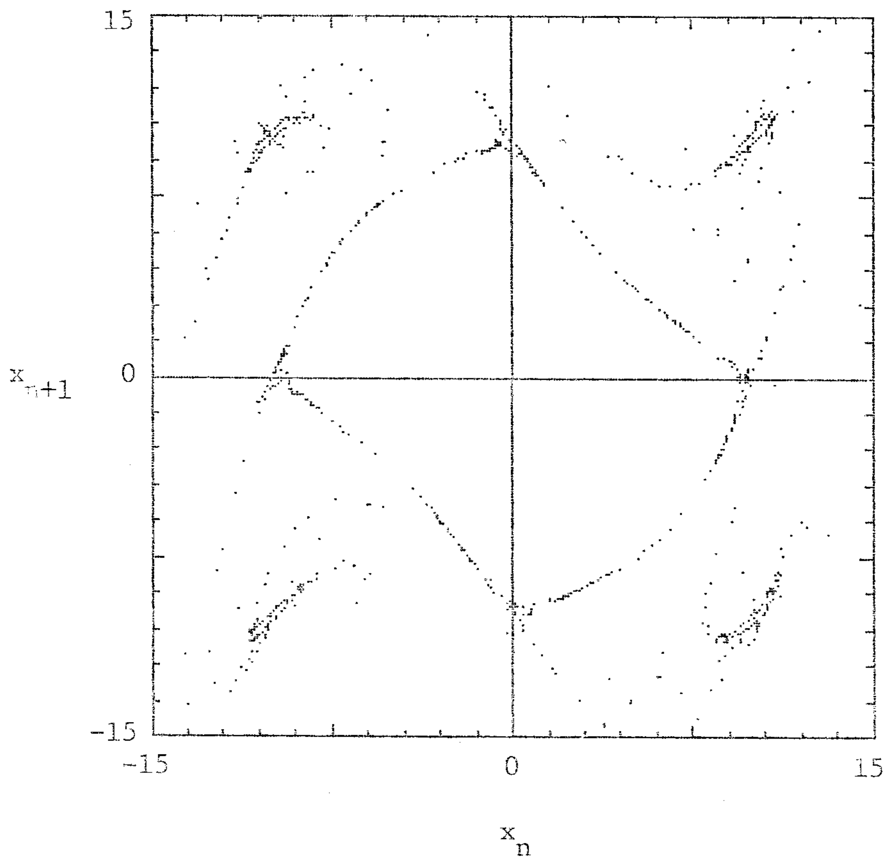


Fig. 4.4 An example of the phase plane structure near the period four solution of Example I for $k = 3$ and $\varepsilon = -0.01$.

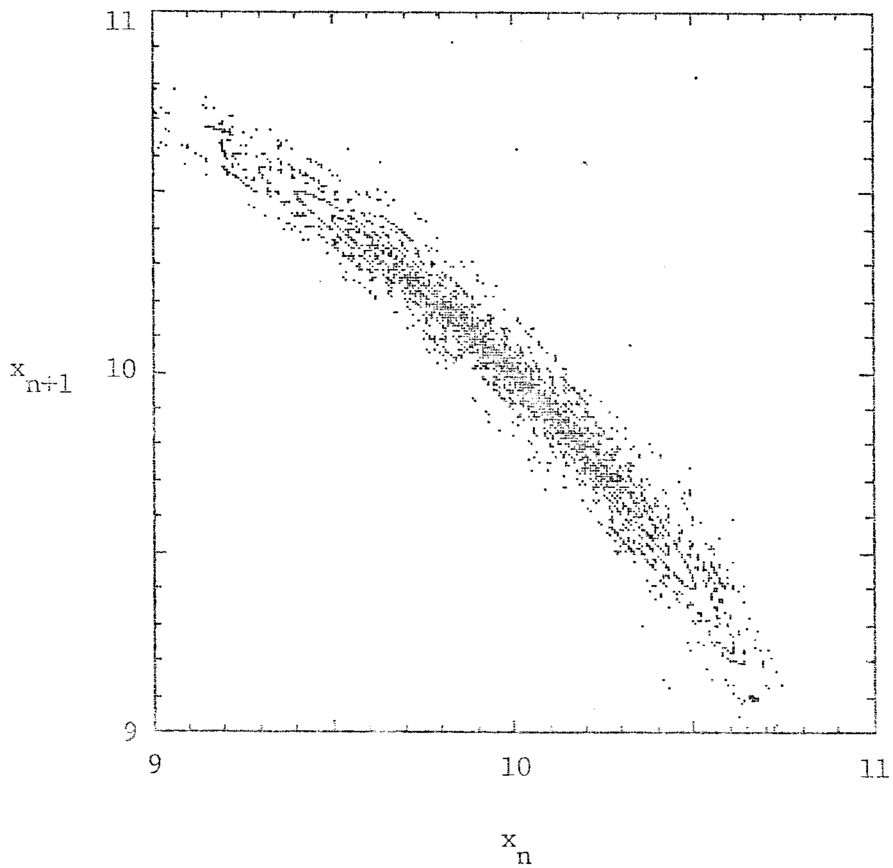


Fig. 4.5 A closeup view of one of the period four "islands" for $k = 1$ and $\varepsilon = 0.01$. Note the "two-dimensionality" of the "island."

exact periodic solutions given by Eq. (4.28).

Example II: $i_0 = 1, q = 2$

For the present situation, examination of Eq. (4.19) reveals that, again, $m = 1$. Therefore, a solution of the form of Eq. (4.26) applies, i.e.,

$$x_n^* = A\cos(n\phi + \psi) \ . \tag{4.31}$$

Following the approach illustrated in Example I and making use of Eq. (4.21) leads to Eq. (4.27), i.e.,

$$A = +2\sqrt{\frac{2-k-2\cos\phi}{\varepsilon[3+(-1)^s]}} \ . \tag{4.32}$$

For this example, Eqs. (4.7b), (4.8), and (4.15) yield, respectively,

$$\phi = \pi \ ,$$

$$P = 2 \ ,$$

and

$$\psi = \frac{s\pi}{2} \ ; \ s = 0,1 \ .$$

Hence, using Eqs. (4.26) and (4.27), the solution is given by

$$x_n^* = +2\sqrt{\frac{4-k}{\varepsilon[3+(-1)^s]}}\cos\left[n\pi + \frac{s\pi}{2}\right] \ . \tag{4.33}$$

Using a simple trigonometric identity on the cosine factor in Eq. (4.33) yields, further,

$$x_n^* = +2\sqrt{\frac{4-k}{\varepsilon[3+(-1)^s]}}(-1)^n\cos\left[\frac{s\pi}{2}\right] \ . \tag{4.34}$$

Clearly, if $\varepsilon > 0$, then the solution given by Eq. (4.34) exists $\forall\, k \in [0,4] \subset \mathbf{R}$. If $\varepsilon < 0$, then the solution given by Eq. (4.34) does not exist for any $k \in [0,4] \subset \mathbf{R}$. Therefore, for the remainder of the discussion of the present example, take $\varepsilon > 0$.

Now, note that for $s = 0$ the solution given by Eq. (4.34) becomes

$$x_n{}^* = (-1)^n \sqrt{\frac{4-k}{\varepsilon}} \ . \tag{4.35}$$

Hence,

$$(x_n{}^*)^2 = \frac{4-k}{\varepsilon} \ ,$$

so that the matrix of the linearized system in Eq. (4.24) becomes, simply,

$$L_0(n) = \begin{bmatrix} 2k - 10 & -1 \\ 1 & 0 \end{bmatrix} .$$

Therefore, the matrix, L_0^*, defined by Eq. (4.25), becomes

$$L_0^* = \begin{bmatrix} 2k - 10 & -1 \\ 1 & 0 \end{bmatrix}^2$$

$$= \begin{bmatrix} (2k-10)^2 - 1 & -(2k-10) \\ 2k - 10 & -1 \end{bmatrix} .$$

The characteristic equation of L_0^* is:

$$\lambda^2 - [(2k - 10)^2 - 2]\lambda + 1 = 0 \ ,$$

which yields the eigenvalues

$$\lambda_\pm = \frac{(2k-10)^2 - 2}{2} \pm \sqrt{\left[\frac{(2k-10)^2 - 2}{2} \right]^2 - 1} \ .$$

Clearly, $\forall\, k \in [0,4) \subset \mathbb{R}$, the eigenvalues, $\lambda_\pm$, "straddle" the number one on the real line. Therefore, the solution given by Eq. (4.35) is a set of period two saddle points.

Suppose, now, that $s = 1$ in Eqs. (4.34). The solution is just

$$x_n^* = 0 \ , \tag{4.36}$$

i.e, the trivial solution. The matrix of the linearized system in Eq. (4.24) becomes

$$L_1(n) = \begin{bmatrix} 2-k & -1 \\ 1 & 0 \end{bmatrix} \ ,$$

and, hence,

$$L_1^* = \begin{bmatrix} (2-k)^2-1 & -(2-k) \\ 2-k & -1 \end{bmatrix} \ .$$

The characteristic equation of L_1^* is

$$\lambda^2 - [(2-k)^2 - 2]\lambda + 1 = 0 \ ,$$

which produces the eigenvalues

$$\lambda_{\pm} = \frac{(2-k)^2-2}{2} \pm \sqrt{\left[\frac{(2-k)^2-2}{2}\right]^2 - 1} \ .$$

Clearly, $\forall \, k \in (0,4) \subset \mathbb{R}$, the eigenvalues, $\lambda_{\pm}$, are complex conjugates lying on the unit circle. Therefore, the solution given by Eq. (4.36) just "reflects" the stable behavior of the equilibrium point at the origin.

Examples of the period two solutions generated by Eq. (4.35) are shown in Figs. (4.6)-(4.9). Curiously enough, note that the figures indicate that the stable and unstable manifolds of the period two saddle points give rise to phase portraits that are just ninety-degree rotations of the phase portraits for $\varepsilon < 0$. However, the structure of the phase portraits for $\varepsilon < 0$ rises from the stable and unstable manifolds emanating from the equilibrium saddle points. Thus the period two solutions seem "special."

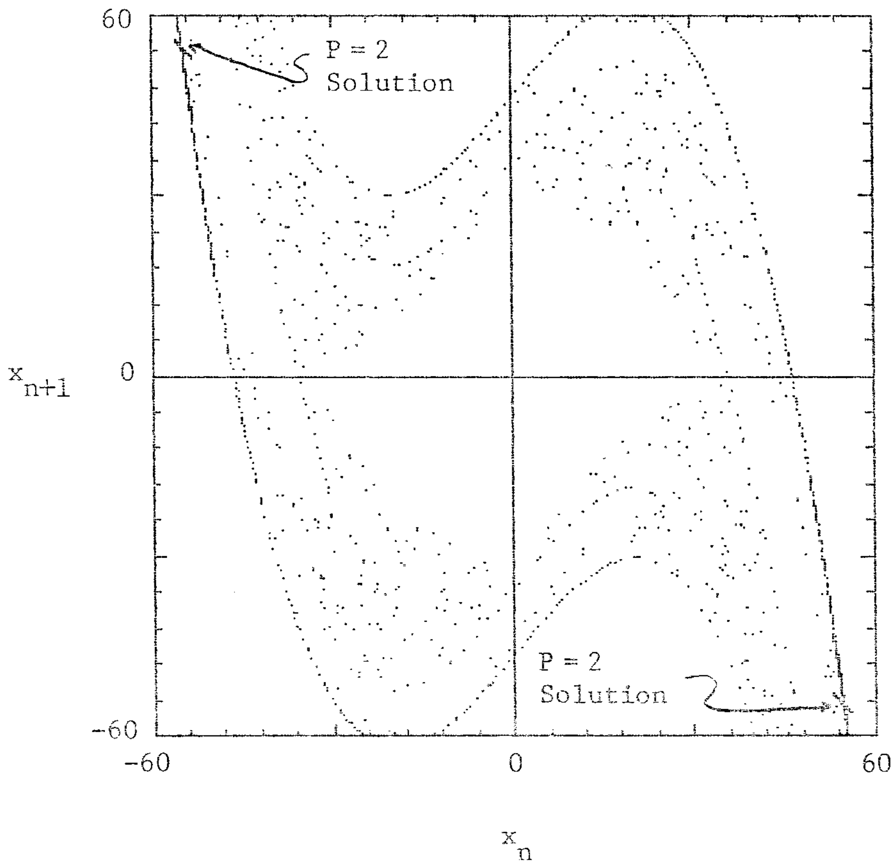


Fig. 4.6 An example of the phase plane structure showing the period two solution of Example II for k = 1 and ε = 0.001.

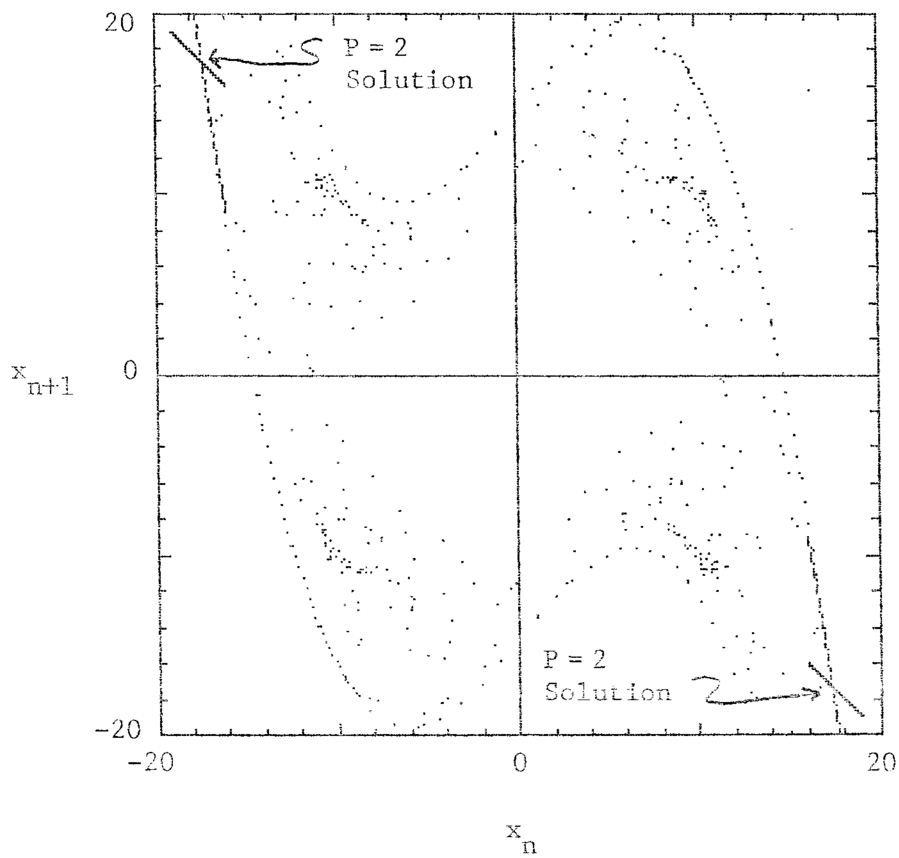


Fig. 4.7 An example of the phase plane structure showing the period two solution of Example II for $k = 1$ and $\varepsilon = 0.01$.

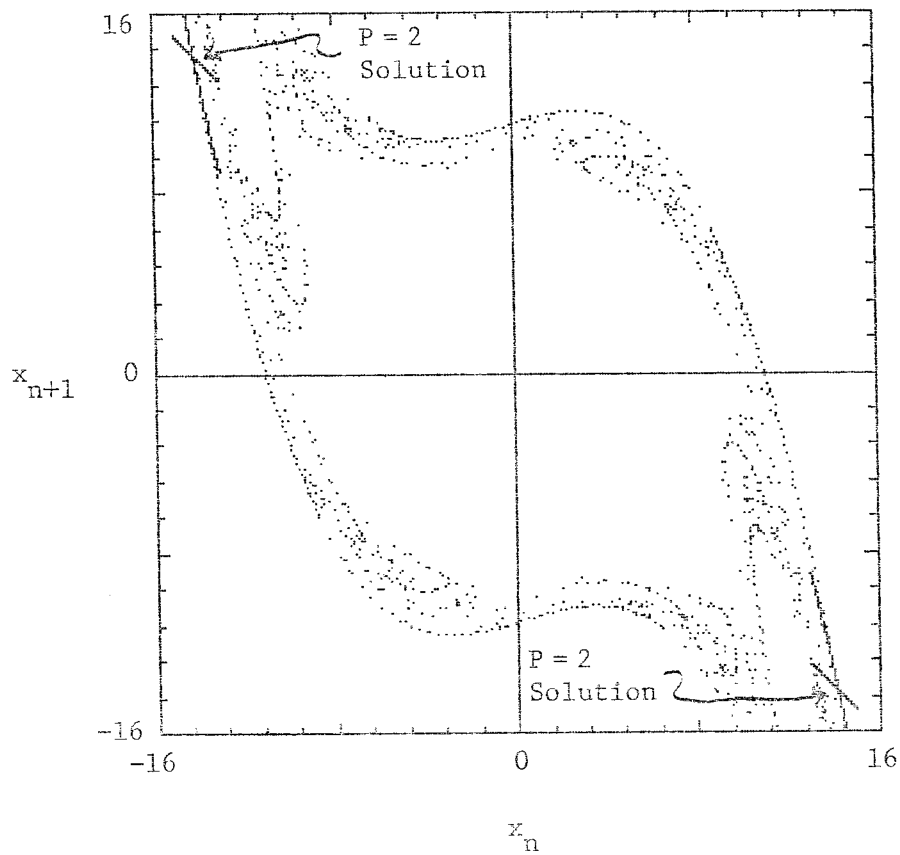


Fig. 4.8 An example of the phase plane structure showing the period two solution of Example II for $k = 2$ and $\varepsilon = 0.01$.

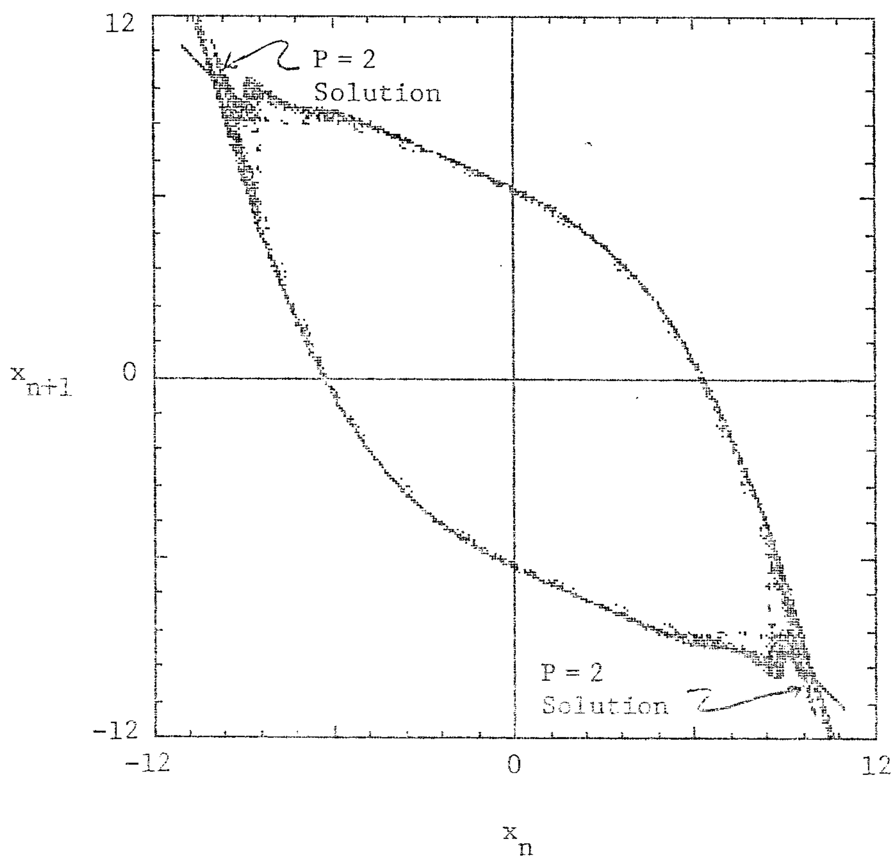


Fig. 4.9 An example of the phase plane structure showing the period two solution of Example II for $k = 3$ and $\varepsilon = 0.01$.

Example III: $i_0 = 2, q = 1$

For this example, examination of Eq. (4.19) reveals that $m = 2$. Therefore, consider a solution of the form

$$x_n^* = A\cos(n\phi + \psi) + A_1\cos(3n\phi + 3\psi) \ . \tag{4.37}$$

Substitution of Eq. (4.37) into the discrete Duffing equation, Eq. (3.4), and use of basic trigonometric identities, leads to

$$\left[(2\cos\phi - 2 + k)A + \varepsilon\left[\frac{3A^3}{4} + \frac{3A^2A_1}{4} + \frac{3AA_1^2}{4} \right] \right] \cos(n\phi + \psi)$$

$$+ \left[(2\cos3\phi - 2 + k)A_1 + \varepsilon\left[\frac{A^3}{4} + \frac{3A^3A_1}{2} + \frac{3A_1^3}{4} \right] \right] \cos(3n\phi + 3\psi)$$

$$+ \varepsilon\left[\frac{3A^2A_1}{4} + \frac{3AA_1^2}{4} \right] \cos(5n\phi + 5\psi)$$

$$+ \varepsilon\left[\frac{3AA_1^2}{4} \right] \cos(7n\phi + 7\psi) + \varepsilon\left[\frac{A_1^3}{4} \right] \cos(9n\phi + 9\psi) = 0 \ .$$

Use of Eqs. (4.7b), (4.15), (4.18), and (4.21) yields, further,

$$\left[(k-1)A + \varepsilon\left\{ \frac{3A^3}{4} + \frac{3A^2A_1}{4}[1 + (-1)^s] + \frac{3AA_1^2}{2}[1 + (-1)^s] \right\} \right] \cos(n\phi + \psi)$$

$$+ \left[(k-2)A_1 + \varepsilon\left\{ \frac{A^3}{4} + \frac{3AA_1^2}{2} + \frac{A_1^3}{4}[3 + (-1)^s] \right\} \right] \cos(3n\phi + 3\psi) = 0 \ , \tag{4.38}$$

where

$$\phi = \frac{\pi}{3}$$

and

$$\psi = \frac{s\pi}{6} \ ; \ s = 0, q \ .$$

Also, Eq. (4.8) yields, for the present example,

$$P = 6 \ .$$

Satisfaction of Eq. (4.38), for all n, implies that coefficients of the trigonometric terms must vanish independently, i.e., after minor simplification,

$$k - 1 + \frac{3\varepsilon}{4}\{A^2 + AA_1[1 + (-1)^s]\} = 0 \ , \tag{4.39}$$

and

$$(k - 2)A_1 + \frac{\varepsilon}{4}\{A^3 + 6AA_1^2 + A_1^3[3 + (-1)^s]\} = 0 \ . \tag{4.40}$$

Hence, in general, the coefficients, A and A_1, may be determined by solving Eqs. (4.39) and (4.40) simultaneously. In practice, such a solution may be obtained numerically for specific values of k and ε.

However, note that, for $s = q = 1$, Eq. (4.39) becomes, simply,

$$k - 1 + \frac{3\varepsilon}{4}(A^2) = 0 \ .$$

Thus, for $s = q = 1$,

$$A = + \frac{2}{\sqrt{3}} \sqrt{\frac{1-k}{\varepsilon}} \ .$$

Therefore, the existence of nontrivial period six solutions implies that $k < 1$ for $\varepsilon > 0$, and that $k > 1$ for $\varepsilon < 0$. Additional restrictions on k may result from solution of Eqs. (4.39) and (4.40).

Once the coefficients, A and A_1, have been obtained for a given value of s, the stability of the periodic solution, x_n^*, given by Eq. (4.37), may be investigated via the approach which was outlined in Section (4.3.3) and illustrated in Example I. In practice, however, the stability of such a solution may be more simply determined by numerical simulation. Examples of the period six solutions presently under discussion may be viewed in Figs. (4.10) and (4.11). The procedure used to obtain these figures is strictly analogous to the procedure used to obtain the figures for Example I.

4.3.5 A Note on Approximate Periodic Solutions

Clearly, the analysis of Section (4.3.1) indicates that if the frequency, ϕ, is a rational multiple of π, then there exists an integer, i_0, such that Eqs. (4.5) and (4.6) are satisfied. Hence, an exact periodic solution may be constructed in the form of Eq. (4.22), provided ε and k are such that a Fourier approximation of the type of Eq. (4.3) is applicable.

However, if i_0 is large, construction of an exact periodic solution can become tedious due to the large number of simultaneous, nonlinear algebraic equations that must be solved for the coefficients. In the event of such a circumstance, the approximate frequency-amplitude relationship to a given order in ε, e.g., Eq. (3.45), may be used to obtain a reasonably accurate estimate of the amplitude.

4.3.6 Some Closing Comments on Periodic Solutions

In summary, for $0 < k < 4$, if the frequency, ϕ, is such that Eqs. (4.5) and (4.6) are satisfied, exact periodic solutions of the discrete Duffing equation may be constructed using Eq. (4.22), where m is the smallest number satisfying Eq. (4.19). The phase angle, ψ, will be given by Eq. (4.15). The coefficients of the trigonometric terms may be determined as illustrated in the examples.

In general, however, the existence of real coefficients, i.e., the existence of true periodic solutions of the form of Eq. (4.22), places restrictions on the range of the linear "stiffness," k. Different restrictions apply to the case where $\varepsilon > 0$ than to the case where $\varepsilon < 0$.

Characterization of the phase points generated by an exact periodic solution, e.g., periodic saddle, periodic center, etc., relies on the determination of the stability of the periodic solution. The stability analysis of a given periodic solution may be carried out analytically, as outlined in Section (4.3.3), or numerically.

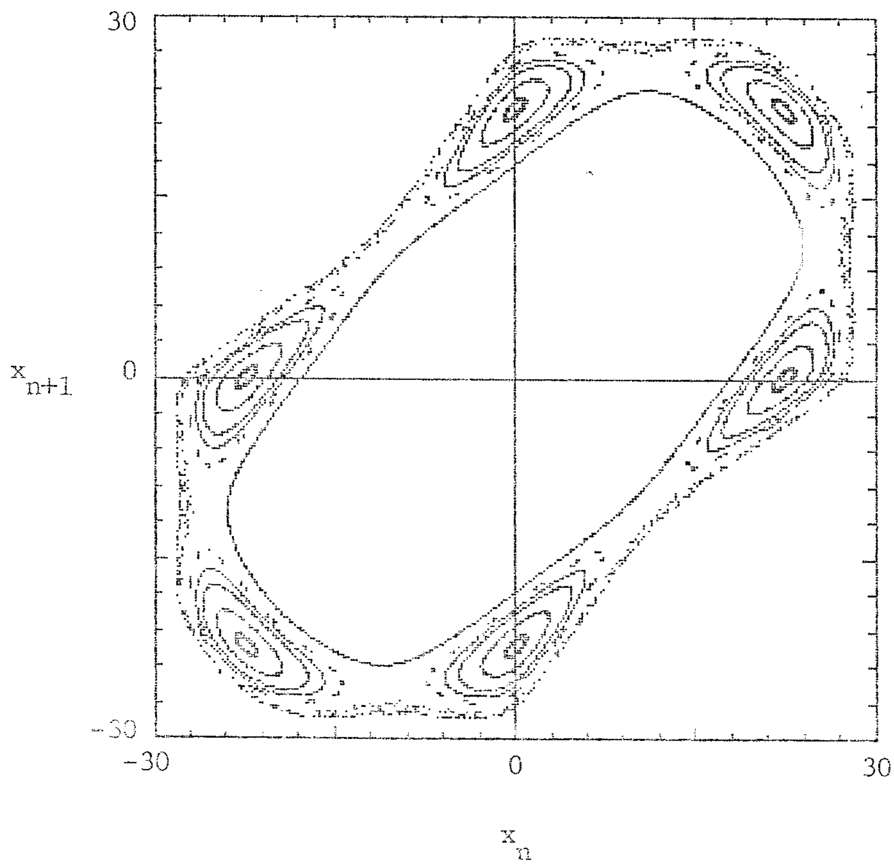


Fig. 4.10 An example of the phase plane structure near the period six solution of Example III for k = 0.5 and ε = 0.001.

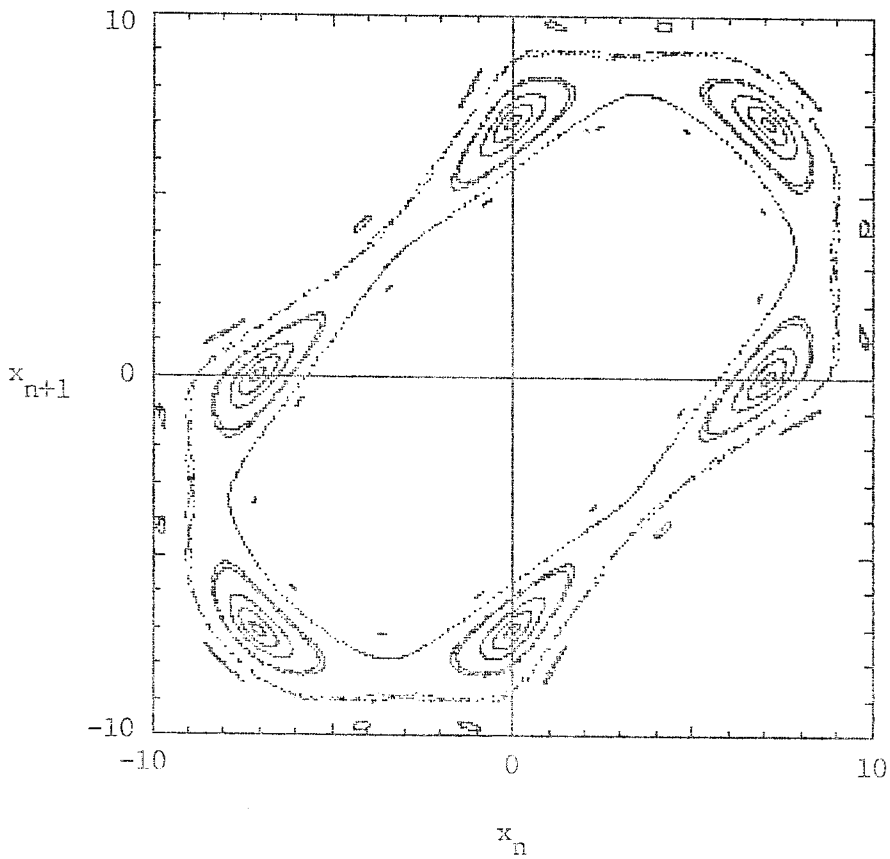


Fig. 4.11 An example of the phase plane structure near the period six solution of Example III for $k = 0.5$ and $\varepsilon = 0.01$.

Specifically, as illustrated in Chapter 2 and the examples, even period solutions which wind around the phase plane origin once before repeating themselves always occur in pairs. One solution will be a set of periodic centers and the other a set of periodic saddles. Both solutions are alternately intermingled. In this case, saddle connections occur, and the separatrices connecting the periodic saddle points exhibit generic, heteroclinic behavior. The "amount" of heteroclinic behavior increases as the absolute value of the nonlinear parameter, $|\varepsilon|$, increases. Hence, the "size" of the stable regions surrounding the periodic centers decreases as $|\varepsilon|$ increases.

The period two solutions discussed in Example II seem special because of the global ramifications of their existence and stability. Examination of the phase portraits occurring along a vertical line, say $k = 2$, in the parameter space of Fig. (2.3), indicates that the structures of the phase portraits for $\varepsilon > 0$ are just ninety-degree rotations of the structures of the phase portraits for $\varepsilon < 0$. However, for $\varepsilon < 0$, the structures of the phase portraits are essentially governed by the equilibrium points, while, for $\varepsilon > 0$, the structures are essentially governed by a pair of period two saddle points.

The characterization of odd period solutions, in general, and of even period solutions that map around the origin more than once before repeating themselves, has not been specifically attempted in the present investigation. However, the existence of such solutions is implied by the generality of the analysis of Sections (4.3.1) and (4.3.2), provided that real values of the coefficients of the exact solutions exist. Quite possibly, the phase plane symmetry exhibited by the discrete Duffing equation may significantly affect the existence and stability of such solutions.

In particular, for $w = 1$, numerical simulation indicates that odd period solutions always occur as "pairs" of pairs in the sense previously discussed for even period solutions. That is, the phase plane structure of the "ring" of periodic "islands" looks like the phase plane structure generated by an even period solution with a period twice as large as the odd period solution. However, the phase points of the odd period solution "leapfrog" from one "island" in the periodic "ring," say "island$_1$," to "island$_3$," then to "island$_5$," etc., until they return to

"island_1."

4.4 QUASI-PERIODIC SOLUTIONS

4.4.1 Approximate Quasi-periodic Solutions

If ϕ is not a rational multiple of π, then none of the divisors of the form of Eq. (4.1) vanish identically. In such a case, an exact periodic solution of the discrete Duffing equation, of the form of Eq. (4.22), does not exist. In reality, as illustrated in Chapter 2, a quasi-periodic solution may exist. The existence of such a quasi-periodic solution will, in general, depend upon the specific values of the equation parameters, ε and k. For the remainder of the present discussion, suppose that ϕ is an irrational multiple of π, and that ε and k are such that a quasi-periodic solution exists.

Suppose, additionally, that ϕ is "close" to a rational multiple of π, i.e., suppose

$$\phi = \frac{q\pi}{i_0+1} + O(|\varepsilon|^l) \ , \quad |\varepsilon| < 1 \ , \tag{4.35}$$

for some positive integers, q, i_0, and l, where q and i_0 are as defined in Section (4.3). Making use of a basic trigonometric identity, note that

$$\begin{aligned}\cos[(2i_0+1)\phi] &= \cos\left[(2i_0+1)\left(\frac{q\pi}{i_0+1}\right)\right]\cos[O(|\varepsilon|^l)] \\ &\quad - \sin\left[(2i_0+1)\left(\frac{q\pi}{i_0+1}\right)\right]\sin[O(|\varepsilon|^l)] \\ &= \cos\left\{[2(i_0+1)-1]\left(\frac{q\pi}{i_0+1}\right)\right\}\cos[O(|\varepsilon|^l)] \\ &\quad - \sin\left\{[2(i_0+1)-1]\left(\frac{q\pi}{i_0+1}\right)\right\}\sin[O(|\varepsilon|^l)] \\ &= \cos\left(\frac{q\pi}{i_0+1}\right)\cos[O(|\varepsilon|^l)]\end{aligned}$$

$$+ \sin\left[\frac{q\pi}{i_0+1}\right] \sin[O(|\varepsilon|^l)] \ , \tag{4.36}$$

and that

$$\cos\phi = \cos\left[\frac{q\pi}{i_0+1}\right] \cos[O(|\varepsilon|^l)] - \sin\left[\frac{q\pi}{i_0+1}\right] \sin[O(|\varepsilon|^l)] \ . \tag{4.37}$$

Hence, using Eqs. (4.1), (4.36), and (4.37),

$$\sigma_{2i_0+1} = -2\sin\left[\frac{q\pi}{i_0+1}\right] \sin[O(|\varepsilon|^l)] \ .$$

A Taylor series expansion of the sine function gives

$$\sin[O(|\varepsilon|^l)] \simeq O(|\varepsilon|^l) \ ,$$

and, hence,

$$\sigma_{2i_0+1} \simeq O(|\varepsilon|^l) \ .$$

Therefore, an examination of the form of the approximate solution, as discussed in Section (4.2), reveals that if ϕ is expressible in the form of Eq. (4.35), with l "large," then an approximation of the form of Eq. (4.2) is invalid. Such a situation is analogous to the "problem of small divisors" occurring for nonlinear differential equations. A method of handling the "problem of small divisors," for ε very small, will be briefly mentioned in Section (4.4.2).

Suppose, now, that ϕ is <u>not</u> close to a rational multiple of π in the sense of Eq. (4.35). Then an approximate solution of the form of Eq. (4.3) provides an asymptotic representation of the actual quasi-periodic solution. A comparison of the true and approximate quasi-periodic solutions is facilitated by use of the pseudo-period of the true (numerical) solution, as defined in Chapter 2. An example of such a comparison is shown graphically in Fig. (4.12).

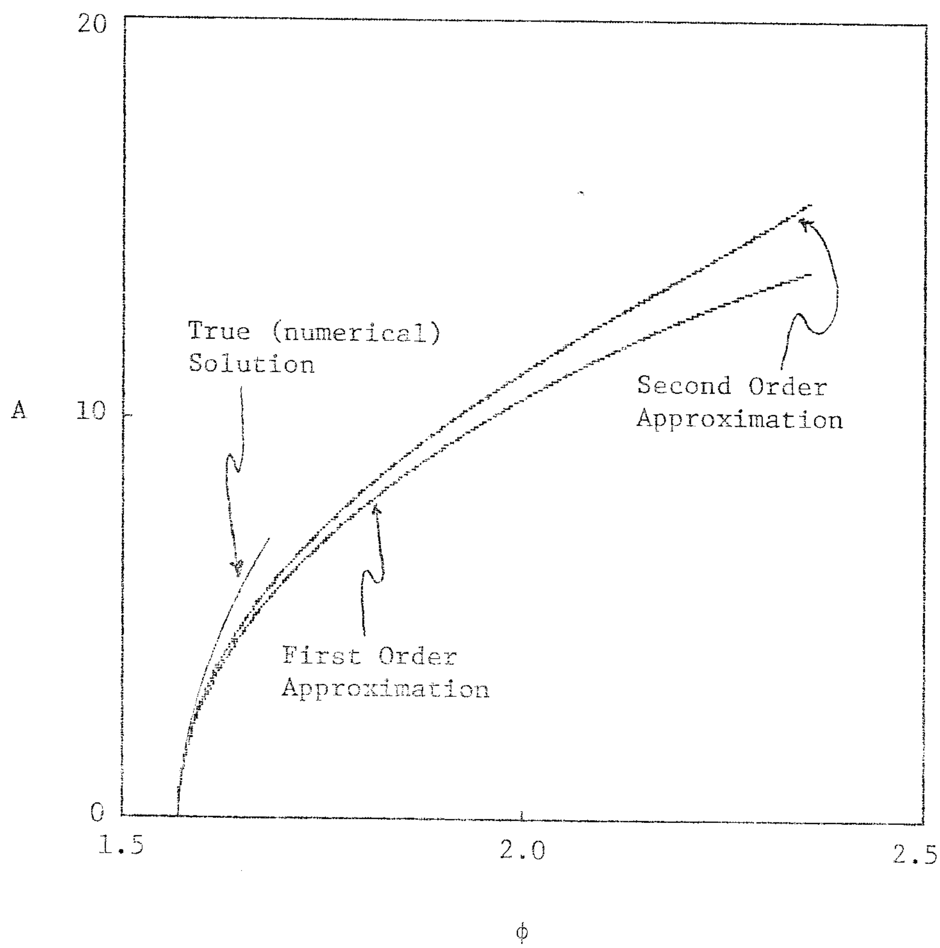


Fig. 4.12 An exemplary comparison of the frequency-amplitude relationships of the true (numerical) solutions and the approximate solutions.

Examining the phase portraits of Chapter 2, quasi-periodic solutions appear as concentric (distorted) circles around an equilibrium center and as "chains" of (distorted) circles around periodic centers. However, note that any finite digit representation of the frequency, ϕ, is obviously really a rational multiple of π. Therefore, the (distorted) circles mentioned above are actually higher order, i.e., w "large," periodic solutions that are close to the corresponding quasi-periodic solutions.

4.4.2 A Note on the Problem of Small Divisors

As previously discussed, if the frequency, ϕ, is "close" to a rational multiple of π in the sense of Eq. (4.35), then an approximation of the form of Eq. (4.3) is invalid. If, however, ε is very small, then an approximation of the amplitude may be obtained by using an extension of the discrete method of slowly varying parameters that has been developed elsewhere.

Briefly, the extension of the method is based on considering the amplitude as a function of a continuous variable, say t, and assuming, formally,

$$\varepsilon = dt \quad .$$

Subsequently, using the discrete method of slowly varying parameters leads to an approximate differential equation for the amplitude. A more explicit development will not be attempted within the present investigation since application of the extended method is restricted to the case where ε is very small.

CHAPTER 5

SUMMARY

The present investigation has focused on the phenomenological (numerical) and analytically quantitative study of the periodic and quasi-periodic solutions of conservative, autonomous, second order, nonlinear difference equations. In particular, a version of the discrete Duffing equation of the form

$$x_{n+1} - 2x_n + x_{n-1} + kx_n + \varepsilon x_n^3 = 0 \ , \tag{5.1}$$

has been studied. The periodic and quasi-periodic behavior of certain solutions of Eq. (5.1) has been investigated using numerical simulation, and approximate and exact analytical techniques. Some brief comments on the motivation for the investigation were presented in Chapter 1.

Chapter 2 concentrated on the phenomenology of the phase portraits of Eq. (5.1). Equilibrium solutions of the discrete Duffing equation were obtained by a simple analysis. The parametric dependence of the existence and stability of the equilibrium solutions was then determined in order to obtain an initial, global idea of the phase plane structure. Specifically, periodic, quasi-periodic, and separatrix solutions were observed to occur for $\varepsilon > 0$ and $-2 < k < 4$, and for $\varepsilon < 0$ and $0 < k < 4$. The remainder of the phenomenological study was then restricted to a brief, primarily graphical, discussion of the structure of the phase portraits occurring within the above-mentioned regions of the parameter space.

Chapter 3 was devoted to a development of "regular" perturbation methods for the generation of approximate solutions to a class of nonlinear difference equations. Following an initial discussion of the straightforward expansion and "secularity," three discrete perturbation techniques were derived explicitly. These techniques are strongly analogous to the

Lindstedt-Poincaré method, the method of renormalization, and the method of dominant balance, respectively, used for approximate solution of nonlinear differential equations. All of the techniques discussed apply directly to equations of the form

$$x_{n+1} - 2\cos\theta x_n + x_{n-1} + \varepsilon f(x_n) = 0 \ . \tag{5.2}$$

The discrete Lindstedt-Poincaré method begins by rewriting Eq. (5.2) in the form

$$x_{n+1} - 2\cos\phi x_n + x_{n-1} + 2(\cos\phi - \cos\theta) x_n + \varepsilon f(x_n) = 0 \ , \tag{5.3}$$

and taking

$$x_n = x_n^0 + \varepsilon x^1_n + \varepsilon^2 x_n^2 + \ldots \tag{5.4}$$

and

$$\cos\phi - \cos\theta = \varepsilon a_1 + \varepsilon^2 a_2 + \ldots \ . \tag{5.5}$$

The expansions given by Eqs. (5.4) and (5.5) are substituted into Eq. (5.3). Subsequently, requiring that the coefficient of each power of ε in the resulting equation vanish independently leads to a recursive set of linear difference equations for the x_n^i, $i = 0,1,2,\ldots$. The coefficients, a_i, $i = 1,2,\ldots$, appear in the inhomogeneous terms of these linear equations and are chosen such that any "secular" inhomogeneity is eliminated.

The discrete method of renormalization begins by considering Eq. (5.2), directly, and using the expansion of Eq. (5.4). Requiring that the coefficient of each power of ε vanishes independently results in a recursive set of linear difference equations that leads to a "secular" approximate solution. This "secularity" may be removed by substituting the expansion

$$\theta = \phi + \varepsilon b_1 + \varepsilon^2 b_2 + \ldots \ ,$$

into the approximate solution and analytically manipulating the result.

A simpler, but less generally applicable technique, called the discrete method of dominant balance, begins with a much stronger *a priori* assumption of the form of the solution. The assumed solution is substituted into Eq. (5.2), and the usual order arguments are used to determine the unknowns of the assumed approximate solution. The general use of the discrete method of dominant balance should be restricted due to the relatively detailed, *a priori* knowledge required for its correct application.

Chapter 4 concentrated on applying the methods developed in Chapter 3 to the study of the periodic and quasi-periodic solutions of the discrete Duffing equation, Eq. (5.1). To a given order, say j_0, the perturbation techniques yield a consistent approximate solution of the form

$$x_n = A\cos(n\phi + \psi) + \sum_{j=1}^{j_0} \varepsilon^j A_j \cos[(2j+1)(n\phi + \psi)] + O(\varepsilon^{j_0+1}) \ , \tag{5.6}$$

where the coefficients, A_j, $j = 1,\ldots,j_0$, are given by

$$A_j = \sum_{l=1}^{j_0-j+1} c_l \varepsilon^{l-1} A^{2j+2l-1} \ . \tag{5.7}$$

The coefficients, c_l, $l = 1,\ldots,j_0 - j + 1$, are of the form

$$c_l = \sum_i \rho_i \sigma_3^{-\gamma_i} \sigma_5^{-\beta_i} \ldots \sigma_{2j_0+1}^{-\gamma_i} \ , \tag{5.8}$$

where $\rho_i \in \mathbb{R}$ and $\alpha_i, \beta_i, \gamma_i \in \mathbf{Z}^+$, and, for a given i,

$$\gamma_i + \beta_i + \ldots + \gamma_i \geq 1 \ . \tag{5.9}$$

The "divisors" occurring in Eq. (5.8), and, hence, in Eqs. (5.7) and (5.6), are defined as

$$\sigma_{2j+1} = \cos\phi - \cos[(2j+1)\phi] \ , \quad j = 1,2,\ldots \ . \tag{5.10}$$

In view of Eq. (5.9), divisors of the form of Eq. (5.10) appear in every term in the summation on the right-hand side of Eq. (5.7), and, hence, appear in every term in the summation on the right-hand side of Eq. (5.6). Therefore, an examination of the problem of the vanishing of such divisors was crucial to an attempt to delineate the range of valid application of Eq. (5.6).

In particular, supposing, for a given i_0,

$$\sigma_{2i_0+1} = 0 \ , \tag{5.11}$$

and

$$\sigma_{2i+1} \neq 0 \quad \forall\, i \neq i_0 \ . \tag{5.12}$$

The satisfaction of these criteria lead to

$$\phi = \frac{q\pi}{i_0 + 1} \ , \ q \in \mathbf{Z}^+ \ , \ q \le w(i_0 + 1) \ , \tag{5.13}$$

where q cannot be an integral multiple of any of the nontrivial prime factors of $(i_0 + 1)$, and w is the number of times the solution wraps around the center before repeating itself.

With ϕ given by Eq. (5.13), an examination of the trigonometric factors appearing in Eq. (5.6) indicates that if $j \ge m$, where m is the smallest integer satisfying

$$m > \frac{i_0}{2} \ , \tag{5.14}$$

and if the phase angle, ψ, is taken as

$$\psi = \frac{s\pi}{2(i_o + 1)} \ , \ s = 0, q \ , \tag{5.15}$$

then the (odd) cosine series truncates with the $(m - 1)$ term. This fact implies that an <u>exact</u>

solution can be constructed using a finite number of terms, i.e.,

$$x_n = A\cos(n\phi + \psi) + \sum_{i=1}^{m-1} A_i \cos[(2i+1)(n\phi + \psi)] \ , \qquad (5.16)$$

where m, ϕ, and ψ are defined by Eqs. (5.14), (5.13), and (5.15), respectively. The coefficients, A and A_i, $i = 1,...,m-1$, may be determined by substituting Eq. (5.16) into Eq. (5.1). Solutions constructed in such a manner are clearly periodic. Three simple examples were used to illustrate the constructive procedure. As noted in the examples, for a given ε, the existence of real coefficients in Eq. (5.16) is strongly dependent on k.

If i_0, as defined above, is "large," then construction of an exact periodic solution using Eqs. (5.13)-(5.16) may be tedious. In that case, a reasonably accurate estimate of the amplitude of the periodic solution may be obtained from the approximate solution to a "lower" order via the approximate frequency-amplitude relationship, e.g., Eq. (5.5).

Clearly, in view of Eq. (5.13), if ϕ is any rational multiple of π, then there exists an i_0 such that the preceding discussion holds. If, however, ϕ is an irrational multiple of π, then none of the divisors defined in Eq. (5.10) vanishes. In that case, depending upon ε and k, the true solution may be quasi-periodic.

If, in addition, ϕ is not "close" to a rational multiple of π in the sense of Eq. (5.13) with i_0 small, then the true solution may be asymptotically approximated by Eq. (5.6). However, if ϕ is "close" to a rational multiple of π in the same sense, then the approximation is invalid. This situation corresponds to the "problem of small divisors" encountered in the regular perturbation theory of nonlinear differential equations. Solution of the "problem of small divisors" is beyond the scope of the present thesis.

Some general comments were made on the connection between the solutions generated by the techniques of Chapter 4 and the global phase portrait structures illustrated in Chapter 2. These comments arose from the stability analyses of the solutions and were conceptually verified by the corresponding numerical simulations.

REFERENCES

1. Caughey, T. K. (1979). "Nonlinear Ordinary Difference Equations," NASA/JPL publication 79-50.

2. Reinhall, P. G. (1982). "The Analysis of a Nonlinear Difference Equation Occurring in Dynamical Systems," Ph.D. Thesis, California Institute of Technology Dynamics Laboratory Report DYNL-82-2.

3. Westermo, B. D. (1978). "The Solutions of a Nonlinear Difference Equation Found in Numerical Analysis," Ph.D. Thesis, California Institute of Technology Dynamics Laboratory Report DYNL-78-01.

4. Abraham, R., and J. E. Marsden (1978). *Foundations of Mechanics*, Second Edition, Benjamin/Cummins, Reading, Mass.

5. Arnold, V. I. (1978). *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics 60, Springer-Verlag, New York.

6. Smith, M. J. (1971). *Mathematical Ideas in Biology*, Cambridge Univ. Press, Cambridge, England.

7. Lichtenberg, A. J., and M. A. Lieberman (1983). *Regular and Stochastic Motion*, Applied Mathematical Sciences 38, Springer-Verlag, New York.

8. Poincaré, H. (1957). *Les Methodes Nouvelle de la Mechanique Celeste*, Vols. I, II, III, Parish, 1892, 1893, 1899; Trans.: Dover, New York.

9. Birkhoff, G. D. (1917). "Dynamical Systems with Two Degrees of Freedom," Trans.: *Amer. Math. Soc.* **18**, 199-300.

10. Smale, S. (1967). "Differentiable Dynamical Systems," *Bull. Am. Math. Soc.* **73**, 747-817.

11. Meirovitch, L. (1970). *Methods of Analytical Dynamics*, Series in Advanced Engineering, McGraw-Hill, New York.

12. Jordan, D. W., and P. Smith (1977). *Nonlinear ordinary differential equations*, Oxford Applied Mathematics and Computing Science Series, Clarendon Press, Oxford, England.