# ON ORDER AND TOPOLOGICAL PROPERTIES OF RIESZ SPACES

Thesis by

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Στοὺς γονεῖς μου, στοὺς διδασκάλους μου καί σ'ὅσους μ' ἐβοήθησαν νὰ φθάσω μέχρις ἐδῶ.

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ABSTRACT OF

"ON ORDER AND TOPOLOGICAL PROPERTIES OF RIESZ SPACES"

by

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Chapter 1 contains a summary of results on Riesz spaces frequently used in this thesis.

Chapter 2 considers the real linear space $\mathcal{L}_b(L, M)$ of all order bounded linear transformations from a Riesz space L into a Dedekind complete Riesz space M. The order structure of the Dedekind complete Riesz space $\mathcal{L}_b(L, M)$ is studied in some detail. Dual formulas for $T(f^+)$, $T(f^-)$ and $T(|f|)$ are proved. The linear space of all extendable operators from the ideal A of L into M is denoted by $\mathcal{L}_b^e(A, M)$. Two theorems are proved:

(i) If $\theta \leqq T$ is extendable, then T has a smallest positive extension $T_m$, given by $T_m(u) = \sup\{T(v) : v \in A;\ \theta \leqq v \leqq u\}$ for all u in $L^+$.

(ii) The mapping $T \to (T^+)_m - (T^-)_m$ from $\mathcal{L}_b^e(A, M)$ into $\mathcal{L}_b(L, M)$ is a Riesz isomorphism.

Chapter 3 studies integral and normal integral transformations. Some of the theorems included in this chapter are:

(i) If $T \in \mathcal{L}_b^e(A, M)$ is a normal integral, then so is $T_m$.

(ii) If L is $\sigma$-Dedekind complete and M is super Dedekind complete, then T in $\mathcal{L}_b(L, M)$ is a normal integral if and only if $N_T = \{u \in L : |T|(|u|) = \theta\}$ is a band of L.

(iii) If L is $\sigma$-Dedekind complete and M is super Dedekind complete and if there exists a strictly positive operator for L into M, then L is super Dedekind complete.

(iv) If M admits a strictly positive linear functional which is normal, then the normal component $T_n$ of the operator $\theta \leqq T \in \mathcal{L}_b(L, M)$ is given by $T_n(u) = \inf\{\sup_\alpha T(u_\alpha) : \theta \leqq u_\alpha \uparrow u\}$ for all u in $L^+$.

Chapter 4 studies ordered topological vector spaces $(E, \tau)$ with particular emphasis on locally solid linear topological Riesz spaces. Order continuity and topological continuity are considered by introducing the properties (A, o), (A, i), (A, ii), (A, iii) and (A, iv). Some results from this chapter are:

(i) If $(L, \tau)$ is a locally solid Riesz space, then $(L, \tau)$ satisfies (A, i) iff every $\tau$-closed ideal is a $\sigma$-ideal, and $(L, \tau)$ satisfies (A, ii) iff every $\tau$-closed ideal is a band.

(ii) If $(L, \tau)$ is a metrizable locally solid Riesz space with (A, ii), then L satisfies the Egoroff property.

(iii) If $(L, \tau)$ is a metrizable locally solid Riesz space, then both (A, i) and (A, iii) hold iff (A, ii) holds. A counter example shows that this is not true for non-metrizable locally solid Riesz spaces.

The fifth and final chapter considers Hausdorff locally solid Riesz spaces $(L, \tau)$. The topological completion of $(L, \tau)$ is denoted by $(\hat{L}, \hat{\tau})$. Some results from this chapter are:

(i) $(\hat{L}, \hat{\tau})$ is a Hausdorff locally solid Riesz space with cone $\hat{L}^+ = \overline{L^+} =$ the $\hat{\tau}$-closure of $L^+$ in $\hat{L}$, containing L as a Riesz subspace.

(ii) $(\hat{L}, \hat{\tau})$ satisfies the (A, iii) property, iff $(L, \tau)$ does.

(iii) $(\hat{L}, \hat{\tau})$ satisfies the (A, ii) property, if $(L, \tau)$ does.

(iv) If $\tau$ is metrizable, then $(\hat{L}, \hat{\tau})$ satisfies the (A, i) property if $(L, \tau)$ does.

(v) If $L_\rho$ is a normed Riesz space with the (sequential) Fatou property, then $\hat{L}_{\hat{\rho}}$ has the (sequential) Fatou property.

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# CHAPTER 1

# RIESZ SPACES

## 1.1. NOTATION AND BASIC CONCEPTS

Let L be a real vector space with the zero element denoted by $\theta$. We say that L is partially ordered by $\leqq$, if $\leqq$ is an order relation of L, i.e., a reflexive, transitive and antisymmetric relation, such that

(i) $f \leqq g$ implies $f+h \leqq g+h$ for every h in L,

(ii) $\theta \leqq f$ implies $\theta \leqq \alpha f$ for every real $\alpha \geqq 0$.

The notation, $g \geqq f$ for $f \leqq g$, will also be used. The subset, $L^+ = \{f \in L: \theta \leqq f\}$, is called the (positive) cone of L.

It is easily verified that $L^+$ satisfies the following properties:

($\alpha$) $L^+ + L^+ \subseteqq L^+$

($\beta$) $\alpha L^+ \subseteqq L^+$ for all $\alpha \geqq 0$

($\gamma$) $L^+ \cap - L^+ = \{\theta\}$.

Conversely, if the subset $L^+$ of the real vector space L satisfies the above three properties ($\alpha$), ($\beta$), ($\gamma$), then the relation $\leqq$ defined by $f \leqq g$, whenever $g - f \in L^+$, partially orders L as described above.

Given two elements f, g of the (partially) ordered space L, we say that h is an upper (resp. lower) bound if $f \leqq h$ and $g \leqq h$ (resp. $h \leqq f$ and $h \leqq g$). If for every pair f, g in L the least upper bound or supremum (denoted by $f \vee g$) and the greatest lower bound or infimum (denoted by $f \wedge g$) with respect to the ordering exists in L, then L is said to be a <u>Riesz space</u> or a <u>vector lattice</u>. The notion of a Riesz space is essentially due to H. Freudenthal [4] and L. V. Kantorovich [10] and was inspired by an address of F. Riesz (see also

[25] and [26]).

In a Riesz space L we shall denote the elements of $f \vee \theta$, $(-f) \vee \theta$, $f \vee (-f)$ by $f^+$, $f^-$ and $|f|$, respectively. It is not difficult to verify that an ordered vector space is a Riesz space, if and only if, the supremum of every element of L and the zero element of L exists in L.

The elements f, g of L are called orthogonal or disjoint, if $|f| \wedge |g| = \theta$. This will be denoted by $f \perp g$.

In the next theorem we exhibit a number of simple properties of Riesz spaces.

THEOREM 1.1. *Let* L *be a Riesz space. Then we have:*

(i) $f - g \vee h = (f-g) \wedge (f-h)$, $f + g \vee h = (f+g) \vee (f+h)$, $f + g \wedge h = (f+g) \wedge (f+h)$, $\alpha(f \vee g) = (\alpha f) \vee (\alpha g)$, $\alpha(f \wedge g) = (\alpha f) \wedge (\alpha g)$ *for all* f, g *in* L *and all* $\alpha \geqq 0$,

(ii) $f + g = f \vee g + f \wedge g$, $|f| = f^+ + f^-$, $f = f^+ - f^-$, $f^+ \perp f^-$ *for all* f, g *in* L. *So, in particular* $f + g = f \vee g$ *if* f, g *in* $L^+$ and $f \perp g$,

(iii) $||f| - |g|| \leqq |f + g| \leqq |f| + |g|$ *for all* f, g *in* L,

(iv) (*Birkhoff's identity*) $|f \vee h - g \vee h| + |f \wedge h - g \wedge h| = |f - g|$ *for all* f, g, h in L,

(v) (Birkhoff's inequalities) $|f \vee h - g \vee h| \leqq |f - g|$, $|f \wedge h - g \wedge h| \leqq |f - g|$ *for all* f, g, h *in* L, *so, in particular* $|f^+ - g^+| \leqq |f - g|$ *and* $|f^- - g^-| \leqq |f - g|$, *for all* f, g *in* L,

(vi) $(f+g) \wedge h \leqq f \wedge h + g \wedge h$ *for all* f, g, h *in* $L^+$,

(vii) (*Riesz decomposition property*). *If* $u, f, g \in L^+$ *and* $\theta \leqq u \leqq f + g$, *then there are* $u_1, u_2 \in L^+$ such that $\theta \leqq u_1 \leqq f$, $\theta \leqq u_2 \leqq g$ *and* $u = u_1 + u_2$.

For a proof of the above theorem, for a number of examples of Riesz spaces and some other simple properties we refer to [18].

Note. Statement (iv) was first proved by G. Birkhoff in [1], (1$^{st}$ ed., Th. 78, p. 109). See also [8].

In the following discussion, L will denote a Riesz space. The order interval $[f, g]$ is defined to be the empty set, if $f \leqq g$ is not satisfied, and the set $\{h \in L: f \leqq h \leqq g\}$, if $f \leqq g$ is satisfied. A subset D of L is called order bounded if $D \subseteqq [f, g]$ for some interval $[f, g]$ of L. A subset D of L is bounded from above (resp. from below), if there exists f in L such that $h \leqq f$ for all h in D (resp. $f \leqq h$ for all h in D). The subset D of L has a supremum (resp. infimum) in L, denoted by sup A (resp. inf A), if A is bounded from above and the least upper bound exists in L (resp. if A is bounded from below and the upper lower bound exists in L).

A sequence $\{f_n\}$ of elements of L is called increasing if $f_1 \leqq f_2 \leqq \ldots$, and decreasing if $f_1 \geqq f_2 \geqq \ldots$ . This will be denoted by $f_n \uparrow$ or $f_n \downarrow$, respectively. If $f_n \uparrow$ and $f = \sup\{f_n\}$ exists in L, we write $f_n \uparrow f$. Similarly for a decreasing sequence. If $f_n \uparrow f$ then $f_{k_n} \uparrow f$ for every subsequence $\{f_{k_n}\}$ of $\{f_n\}$.

The indexed subset $\{f_\alpha : \alpha \in \{\alpha\}\}$ of L is said to be directed upwards or downwards, if for every pair $\alpha_1, \alpha_2 \in \{\alpha\}$ there exists $\alpha_3 \in \{\alpha\}$ such that $f_{\alpha_3} \geqq f_{\alpha_1} \vee f_{\alpha_2}$ or $f_{\alpha_3} \leqq f_{\alpha_1} \wedge f_{\alpha_2}$, respectively. If $\{f_\alpha\}$ is directed upwards we shall write $f_\alpha \uparrow$. If $f_\alpha \uparrow$ and $f = \sup\{f_\alpha\}$ exists in L, we shall write $f_\alpha \uparrow f$. Similarly if $\{f_\alpha\}$ is directed downwards. If $\{f_\alpha : \alpha \in \{\alpha\}\}$ is a directed downwards indexed set we can define a relation on $\{\alpha\}$ as follows: $\alpha_1 \geqq \alpha_2$ if $f_{\alpha_1} \leqq f_{\alpha_2}$. It is not difficult to show that this relation directs $\{\alpha\}$ in the sense of [12] (p. 65). Therefore the system

$\{f_\alpha : \alpha \in \{\alpha\}\}$ is a net according to [12], with the property $f_{\alpha_1} \leqq f_{\alpha_2}$ if $\alpha_1 \geqq \alpha_2$. Now, if $\{f_\alpha : \alpha \in \{\alpha\}\}$ is a given net we say that $\{f_\alpha\}$ is directed downwards and we write $f_\alpha \downarrow$ if $\alpha_1 \geqq \alpha_2$ implies $f_{\alpha_2} \leqq f_{\alpha_1}$ in L. It is easy to verify that the indexed set $\{f_\alpha : \alpha \in \{\alpha\}\}$ is directed downwards with respect to the previous definition. For this reason, sometimes we shall use net notation for downwards directed systems without further explanation. Similarly for upwards directed indexed sets.

The basic properties of the directed systems are summarized in the next theorem.

THEOREM 1.2. <u>For upwards indexed systems in a Riesz space L we have:</u>

(i) <u>If</u> $f_\alpha \uparrow f$, $g_\beta \uparrow g$ <u>then</u> $f_\alpha + g_\beta \uparrow_{(\alpha,\beta)} f + g$,

(ii) <u>If</u> $f_\alpha \uparrow f$ <u>then</u> $\lambda f_\alpha \uparrow \lambda f$ <u>for all</u> $\lambda \geqq 0$,

(iii) <u>If</u> $f_\alpha + g_\beta \uparrow_{(\alpha,\beta)} f + g$, $f_\alpha \uparrow f$ <u>and</u> $g_\beta \uparrow$, then $g_\beta \uparrow g$,

(iv) <u>If</u> $f_\alpha \uparrow f$, $g_\beta \uparrow g$, <u>then</u> $f_\alpha \vee g_\beta \uparrow_{(\alpha,\beta)} f \vee g$ <u>and</u> $f_\alpha \wedge g_\beta \uparrow_{(\alpha,\beta)} f \wedge g$.

A similar theorem is true for downwards directed systems.

For a proof and more details see [18], Chapter 2.

A Riesz space L is called <u>Dedekind complete</u>, if every non-empty subset of L bounded from above has a supremum, or equivalently, every non-empty subset of L bounded from below has an infimum. The Riesz space L is called <u>σ-Dedekind complete</u>, if every non-empty at most countable subset of L bounded from above has a supremum. The Riesz space L is called <u>super Dedekind complete</u>, if every non-empty bounded subset D of L has a supremum, which is also the supremum of an at most countable subset of D.

The following theorem gives some information concerning the above notions.

THEOREM 1.3. Let L be a Riesz space. Then we have:

(i) L is Dedekind complete if and only if for every indexed system $\{f_\alpha\}$ such that $\theta \leqq f_\alpha \uparrow \leqq g$, we have $f_\alpha \uparrow f$, for some f in L,

(ii) L is $\sigma$-Dedekind complete if and only if for every sequence $\{f_n\}$ such that $\theta \leqq f_n \uparrow \leqq g$, we have $f_n \uparrow f$ for some f in L,

(iii) L is super Dedekind complete if and only if for every indexed system $\{f_\alpha\}$ such that $\theta \leqq f_\alpha \uparrow \leqq g$, we have $f_\alpha \uparrow f$ for some f in L, and for some sequence $\{f_{\alpha_n}\} \subseteqq \{f_\alpha\}$ we have $f_{\alpha_n} \uparrow f$.

For a proof we refer to [18], Th. 23.2, p. 124.

A Riesz space L is called Archimedean, if the relation $\theta \leqq nu \leqq v$ for some u, v in $L^+$ and all n = 1, 2, ... implies $u = \theta$.

The next theorem characterizes the Archimedian Riesz spaces.

THEOREM 1.4. Let L be a Riesz space. Then the following statements are equivalent:

(i) L is Archimedean.

(ii) Given any directed set $\{f_\alpha\}$, $f_\alpha \uparrow \leqq f_0$ in L, and writing $G = \{g \in L : g \geqq f_\alpha$ for all $\alpha\}$, the downwards directed system $\{g - f_\alpha : g \in G, \alpha \in \{\alpha\}\}$ satisfies $g - f_\alpha \downarrow \theta$.

(iii) Given any directed set $\{f_\alpha\}$, $f_\alpha \downarrow \geqq f_0$ in L, and writing $G = \{g \in L : g \leqq f_\alpha$ for all $\alpha\}$, the downwards directed system $\{f_\alpha - g : g \in G, \alpha \in \{\alpha\}\}$ satisfies $f_\alpha - g \downarrow \theta$.

For a proof see [18], Th. 22.5, p. 115.

A vector subspace A of L is called a Riesz subspace of L if for every f, g in A we have that $f \vee g$ (taken in L) is in A. A vector subspace

A of L is called <u>an ideal</u> if $|f| \leqq |g|$ and g in A implies $f \in A$. An ideal A is called <u>a band</u>, if for every subset of A whose supremum exists in L the supremum is also in A, or equivalently, if $\theta \leqq f_\alpha \uparrow f$ in L and $\{f_\alpha\} \subseteqq A$ implies f in A.

Obviously, arbitrary intersections of ideals are ideals and arbitrary intersections of bands are bands. For every non-empty subset D of L we define the <u>orthogonal complement of D</u>, $D^d$ to be the set of all vectors of L which are orthogonal to every element of D, i.e., $D^d = \{f \in L: f \perp g \text{ for every } g \in D\}$. It is easily verified that $D^d$ is a band of L for every non-empty subset D of L.

Given a non-empty subset D of L there exists a smallest ideal containing D, namely the intersection of all ideals containing D. (The family of all ideals containing D is non-empty since L is one of them.) This smallest ideal is called the ideal generated by D, and will be denoted by $A_D$. Similarly, there is a smallest band containing D, namely the intersection of all bands containing D. (L is one band containing D, so the family of all bands containing D is non-empty.) This band is called the band generated by D and will be denoted by $\{D\}$.

The ideal generated by the element u will be denoted by $A_u$ and will be called the <u>principal ideal generated by u</u>. Obviously, $A_u = \{f \in L; |f| \leqq n |u| \text{ for some } n \in N\}$. A principal ideal is any ideal of the form $A_u$. The band generated by the element u is called the <u>principal band generated by u</u>, and will be denoted by $B_u$. A principal band is any band of the form $B_u$ for some u in L. Obviously $B_u = \{A_u\}$.

If A is an ideal of L and $\theta \leqq u \in \{A\}$, then $\theta \leqq u_\alpha \uparrow u$ for some directed system $\{u_\alpha\} \subseteqq A$. More precisely we have: given the ideal A;

$\theta \leqq u \in \{A\}$ if and only if $u = \sup\{v \in A: \theta \leqq v \leqq u\}$ (see [18], Theorem 20.2, p. 108).

The band A of L is called a projection band, if $L = A \oplus A^d$. The next theorem characterizes the projection bands.

THEOREM 1.5. *If $A \subseteqq L$ denotes a band of the Riesz space L, then the following statements are equivalent.*

(i) A *is a projection band.*

(ii) *For every* u *in* $L^+$ *the supremum of the set* $\{v \in A: \theta \leqq v \leqq u\}$ *exists in* L.

For a proof see [18], Theorem 24.5, p. 133.

A Riesz space L has the projection property (resp. the principal projection property), if every band (resp. every principal band) is a projection band. A Riesz space L has sufficiently many projections if every non-zero band contains a non-zero projection band. For the interrelation of the above concepts see [18], Theorem 25.1, p. 137.

Let A and B be two ideals of the Riesz space L. Then A is said to be order dense in B if $\{A\} \supseteqq B$. In particular, A said to be order dense in L, or simply order dense, if $\{A\} = L$. The ideal A of L is called quasi-order dense in L if $A^{dd} = L$, and A is called super order dense if for every $u \in L^+$ there exists a sequence $\{u_n\} \subseteqq A$ such that $\theta \leqq u_n \uparrow u$. Finally the ideal A is called a $\sigma$-ideal if for every countable subset of A whose supremum exists in L the supremum is also in A, or equivalently, if $\theta \leqq f_n \uparrow f$ in L and $\{f_n\} \subseteqq A$ implies $f \in A$.

An element e in $L^+$, $e \neq \theta$ is called a strong (order) unit if $A_e = L$, i.e., if for every f in L there exists an integer n such that

$f \leqq ne$. Finally, if $e \in L^+$, $e \neq \theta$, $e$ is called a <u>weak unit</u> if the band generated by $e$ is the whole $L$, i.e., $B_e = L$.

We proceed with the following theorem.

THEOREM 1.6. *Let L be a given Riesz space. Then we have:*

(i) *If L is Archimedean, in order an ideal $A \subseteqq L$ be order dense it is necessary and sufficient that* $A^d = \{\theta\}$.

(ii) *L is an Archimedian Riesz space if and only if* $\{A\} = A^{dd}$ *for every ideal* $A \subseteqq L$. *In particular, if A is a band, then* $A = A^{dd}$, *for Archimedean Riesz spaces.*

(iii) *If L is Archimedean* $A \oplus A^d$ *is an order dense ideal for every ideal A of L.*

(iv) *If L has a strong unit e, then there exists a compact Hausdorff space X such that L is Riesz isomorphic to a linear subspace of C(X). Moreover, the functions of L separate the points of X. (Two Riesz spaces L and M are called Riesz isomorphic if there exists a one-to-one linear mapping T from L onto M such that* $T(f \vee g) = T(f) \vee T(g)$ *for all f, g in L; C(X) is the Riesz space of all continuous real valued functions defined on X with ordering* $f \leqq g$ *whenever* $f(x) \leqq g(x)$ *for all* $x \in X$).

For a proof of the above theorem see [18], Theorem 22.3, p. 114. For statement (iv) see also [29].

<u>Note.</u> The necessity of statement (i) of the above theorem is due to T. Ogasawara (see [22], V. I, Ch. 2, §3, Theorem 2), and the sufficiency of (i) is due to W. A. J. Luxemburg (see [15], Note $XV_A$, Th. 48.3, p. 416). Statement (iv) is due to K. Yosida ([29]).

## 1.2. ORDER AND UNIFORM CONVERGENCE

Let L be a Riesz space and let $\{f_n\}$ be a sequence of L. We say that $\{f_n\}$ order converges to f in L, if there exists a sequence $\{g_n\}$ of L such that $|f_n - f| \leqq g_n$ for all n and $g_n \downarrow \theta$ in L. In this case we shall write $f = (o) - \lim_{n \to +\infty} f_n$ or $f_n \xrightarrow{(o)} f$.

We can verify easily that, if $f_n \xrightarrow{(o)} f$ and $f_n \xrightarrow{(o)} g$, then $f = g$. Similarly the indexed system (or a net) $\{f_\alpha\}$ order converges to f if there exists an indexed system $\{g_\alpha\}$ such that $|f_\alpha - f| \leqq g_\alpha$ for all $\alpha$ and $g_\alpha \downarrow \theta$ in L. In this case we shall write $f = (o) - \lim_\alpha f_\alpha$, or $f_\alpha \xrightarrow{(o)} f$. If $f_\alpha \xrightarrow{(o)} f$, $g_\beta \xrightarrow{(o)} g$ then $\lambda f_\alpha + \mu g_\beta \xrightarrow[(\alpha, \beta)]{(o)} \lambda f + \mu g$ for all $\lambda, \mu \in \mathbb{R}$ and $f_\alpha \vee g_\beta \xrightarrow{(o)} f \vee g$, $f_\alpha \wedge g_\beta \xrightarrow{(o)} f \wedge g$. In particular we have $|f_\alpha| \xrightarrow{(o)} |f|$, if $f_\alpha \xrightarrow{(o)} f$.

A subset V of L is called order closed whenever $\{f_n\} \subseteqq V$ and $f_n \xrightarrow{(o)} f$ implies $f \in V$. The collection of all order closed subsets of L satisfies the three axioms required for the closed sets of a certain topology in L, which is called the order topology. Unfortunately, the order topology is not a linear topology for L, in general.

The pseudo-order closure of a set $S \subseteqq L$ is defined by $S' = \{f \in L$: there exists $\{f_n\} \subseteqq S$ such that $f_n \xrightarrow{(o)} f\}$.

If clS denotes the closure of S with respect to the order topology we have $S \subseteqq S' \subseteqq (S')' \subseteqq \cdots \subseteqq \mathrm{cl}S$. It is not difficult to verify that $S' = \mathrm{cl}S$ if and only if $S' = (S')'$.

The Riesz space L has the diagonal gap property, if given any double sequence $\{f_{n_k} : n, k = 1, 2, \ldots\}$ in L, any sequence $\{f_n : n = 1, 2, \ldots\}$ in L and any $f_0$ in L such that $f_{n_k} \xrightarrow[k \to +\infty]{(o)} f_n$ for $n = 1, 2, \ldots$ and $f_n \xrightarrow{(o)} f_0$, there exists a sequence $\{f_{n_i, k(n_i)}\}$ where $n_1 < n_2 < \ldots$ such that

$f_{n_i}, k(n_i) \xrightarrow[(i\to+\infty)]{(o)} f_0$.

The Riesz space L satisfies $S' = \text{cl}S$ for all $S \subseteq L$ if and only if L has the diagonal gap property. For more details and proofs see [18], pp. 80-87.

The Riesz space L is called a <u>$K^+$-space</u> or <u>a space with the boundedness property</u>, if it has the property that a subset A of L is order bounded, i.e., $A \subseteq [f, g]$ for some f, g in L, if and only if, for every sequence $\{f_n\} \subseteq A$ and every $\{\lambda_n\} \subseteq \mathbb{R}$ such that $\lambda_n \to 0$, we have $\lambda_n f_n \xrightarrow{(o)} \theta$. For examples of $K^+$-spaces and more details we refer to [30], pp. 165-172.

The sequence $\{f_n\} \subseteq L$ is said to be <u>relatively uniformly convergent to f in L</u> (and f is called a relative uniform limit of $\{f_n\}$), if there exists some u in $L^+$ and a sequence $\{\epsilon_n\}$ of non-negative real numbers such that $|f_n - f| \leq \epsilon_n u$ for all n and $\epsilon_n \downarrow 0$ in $\mathbb{R}$, or equivalently, if for every $\epsilon > 0$, there exists $n_0(\epsilon)$ such that $|f_n - f| \leq \epsilon u$ for all $n \geq n_0(\epsilon)$. It is possible for a sequence $\{f_n\}$ to have more than one uniform limit. If L is Archimedean relative uniform convergence implies order convergence, so relative uniform limits are uniquely determined. The relative uniform convergence of an indexed system $\{f_\alpha\}$ is defined similarly. For a complete discussion for the above notions of convergence we refer the reader to [18], Chapter 2 and to [33].

It is easily verified that if $f_n \uparrow f$ then $f_n \xrightarrow{(o)} f$, and that if $f_n \uparrow$ and $f_n \xrightarrow{(o)} f$ then $f_n \uparrow f$. A similar result holds for indexed systems. For Archimedean Riesz spaces, it is also true, that if $\lambda_n \to \lambda$ (in $\mathbb{R}$) and $f_n \xrightarrow{(o)} f$ in L, that $\lambda_n f_n \xrightarrow{(o)} \lambda f$.

The sequence $\{f_n\}$ is called an <u>e-relative uniform Cauchy sequence</u> if there exists an element $e \in L^+$ such that for every $\epsilon > 0$, there exists $N_\epsilon > 0$ with $|f_n - f_m| \leqq \epsilon \cdot e$ for all $n, m \geqq N_\epsilon$. The Riesz space L is called relatively uniformly complete if for every $e \in L^+$, each e-relatively uniformly Cauchy sequence has an e-relative uniform limit. If L is Archimedean then L is uniformly complete if and only if every monotone e-uniform Cauchy sequence has an e-uniform limit ([18], Theorem 39.4, p. 253).

## 1.3. RIESZ HOMOMORPHISMS

Given two Riesz spaces L and M and a linear mapping $\pi$ from L into M we say that $\pi$ is a Riesz homomorphism if $\pi(f \vee g) = \pi(f) \vee \pi(g)$ for all f, g in L, or equivalently $\pi(f \wedge g) = \pi(f) \wedge \pi(g)$ for all f, g in L, or equivalently, $\pi(f \wedge g) = \theta$ in M, whenever $f \wedge g = \theta$ in L. The Riesz spaces L and M are called Riesz isomorphic if there exists a Riesz homomorphism $\pi$ from L onto M which is also one-to-one. In this case $\pi$ is called a Riesz isomorphism. The Riesz homomorphism $\pi$ of L into M is called a Riesz $\sigma$-homomorphism if $\pi$ preserves countable suprema, i.e., if it follows from $f = \sup\{f_n : n \in N\}$ in L, that $\pi(f) = \sup\{\pi(f_n)\}$ holds in M and $\pi$ is called a normal Riesz homomorphism, if it preserves arbitrary suprema, i.e., if it follows from $f = \sup\{f_\alpha\}$ in L, that $\pi(f) = \sup\{\pi(f_\alpha)\}$ holds in M (see [18], pp. 98-104, and chapter 9).

Given a Riesz space L and an ideal A of L the real vector space L/A is a Riesz space with the ordering $[f] \leqq [g]$, (here $[f]$ denotes the equivalence class of f) if there are two elements $f', g'$; $f' \in [f]$, $g' \in [g]$ such that $f' \leqq g'$ in L. The canonical projection $\pi: L \to L/A$, $\pi(f) = [f]$ for all $f \in L$, in a Riesz homomorphism (see [18], p. 102).

Next we prove a useful lemma.

LEMMA 1.7. Let L be a Riesz space and let $\{f_\alpha : \alpha \in \{\alpha\}\}$ be a net of L such that $f_\alpha \leqq g$ for all $\alpha \geqq \alpha_0$, for some $\alpha_0 \in \{\alpha\}$, and $f_\alpha \xrightarrow{(o)} f$. Then $f \leqq g$.

PROOF. For $\alpha \geqq \alpha_0$ we have $\theta \leqq g \vee f - g = g \vee f - g \vee f_\alpha = |g \vee f - g \vee f_\alpha| \leqq |f - f_\alpha|$ by Birkhoff's inequality. Since $f_\alpha \xrightarrow{(o)} \theta$ it follows that $|f - f_\alpha| \leqq h_\alpha \downarrow \theta$ in L, for some net $\{h_\alpha\}$ of L. So,

$$\theta \leqq g \vee f - g \leqq h_\alpha \underset{\alpha \geqq \alpha_0}{\downarrow} \theta ,$$

therefore,

$$g \vee f - g = \theta, \text{ i.e., } f \leqq g \quad . \blacksquare$$

We conclude the above discussion by introducing some notation.

If X is a non-empty set, we denote by $\mathcal{F}(X)$ the set of all finite subsets of X. The inclusion relation $\subseteqq$ directs $\mathcal{F}(X)$ in the sense of [12], p. 65. In this work, $\mathcal{F}(X)$ will be considered directed as above, by $\subseteqq$.

Note. $\mathbb{R}$ denotes the set of real numbers and N the set of natural numbers.

# CHAPTER 2

# THE SPACES $\mathcal{L}(L, M)$ AND $\mathcal{L}_b(L, M)$

## 2. 1. INTRODUCTION

Let L and M be two Riesz spaces. We shall denote by $\mathcal{L} = \mathcal{L}(L, M)$ the real linear space of all linear transformations from L into M, and by $\mathcal{L}_b = \mathcal{L}_b(L, M)$ the real subspace of all order bounded linear transformations from L into M, i. e., T is in $\mathcal{L}_b(L, M)$ if T(A) is an order bounded subset of M, whenever A is an order bounded subset of L.

A linear transformation T in $\mathcal{L}(L, M)$ is called positive, denoted by $\theta \leqq T$, whenever $\theta \leqq f \in L$, implies $\theta \leqq T(f)$ in M. We write $T_1 \leqq T_2$, $T_1, T_2 \in \mathcal{L}(L, M)$ to indicate that $\theta \leqq T_2 - T_1$. The set of all positive linear transformations of $\mathcal{L}(L, M)$ will be denoted by $\mathcal{L}^+ = \mathcal{L}^+(L, M)$. It is easy to verify that $\mathcal{L}^+(L, M) \subseteqq \mathcal{L}_b(L, M)$, and that $\mathcal{L}^+$ is a positive cone for $\mathcal{L}_b(L, M)$, and consequently for $\mathcal{L}(L, M)$. Therefore, $(\mathcal{L}_b, \mathcal{L}^+)$ is a (partially) ordered vector space. In the particular case of $M = \mathbb{R}$ we denote the linear space $\mathcal{L}_b(L, \mathbb{R})$ by $L^\sim$, i. e., $\mathcal{L}_b(L, \mathbb{R}) = L^\sim$, and we call $L^\sim$ the (order) dual of L.

LEMMA 2. 1. *Let L and M be two Riesz spaces with M Archimedean. Assume that T is an additive function from $L^+$ into $M^+$. Then, T is uniquely extendable to a positive linear transformation from L into M.*

For a proof see [30], pp. 205-206. Note that the extension is given by $T(u) = T(u^+) - T(u^-)$ for all u in L.

## 2.2. THE ORDER STRUCTURE OF $\mathcal{L}_b(L, M)$

We start with the following basic theorem dealing with the order structure of the space $\mathcal{L}_b(L, M)$.

THEOREM 2.2. (L.V. Kantorovich [9], F. Riesz [25]). Let L and M be two Riesz spaces with M Dedekind complete. Then, the (partially) ordered vector space $(\mathcal{L}_b, \mathcal{L}^+)$ is a Dedekind complete Riesz space.

For a proof we refer to [23], Proposition 2.3 on page 22. Note also that the following formulas hold:

$$\begin{aligned} T^+(f) &= \sup\{Tg : g \in L;\ \theta \leqq g \leqq f\} \\ T^-(f) &= \sup\{-Tg : g \in L;\ \theta \leqq g \leqq f\};\ f \in L^+ \\ |T|(f) &= \sup\{|Tg| : g \in L;\ |g| \leqq f\} \end{aligned} \tag{1}$$

It is also true that $T_\alpha \uparrow T$ in $\mathcal{L}_b(L, M)$ implies $T_\alpha(u) \uparrow T(u)$ for all $u$ in $L^+$.

Note. Theorem 2.2 was proved by F. Riesz in a very special case (see [25]). The general Theorem 2.2 as it is stated here was established by L. V. Kantorovich (see [9]).

REMARKS. (i) The linear mapping $|T|$ is called by W. A. J. Luxemburg and A. C. Zaanen the linear modulus of the transformation T (see [17]).

(ii) It is not difficult to verify that $T_1 \vee T_2$ and $T_1 \wedge T_2$ for $T_1, T_2 \in \mathcal{L}_b(L, M)$ are given by the formulas:

$$(T_1 \vee T_2)(u) = \sup\{T_1(u_1) + T_2(u_2) : u_1, u_2 \in L^+; u_1 + u_2 = u\}$$

$$(T_1 \wedge T_2)(u) = \inf\{T_1(u_1) + T_2(u_2) : u_1, u_2 \in L^+; u_1 + u_2 = u\}$$

for all $u \in L^+$ (see [23], p. 22).

(iii) For Riesz spaces with the principal projection property there are also some other formulas for $T^+$, $T^-$ and $|T|$ (see [17], Theorem 2.2, p. 425).

(iv) Suppose that $\{T_\alpha\}$ is a family of operators of $\mathcal{L}_b^+(L, M)$ such that the supremum of the set

$$\left\{ \sum_{i=1}^{n} T_{\alpha_i}(u) : u_i \in L^+;\ \sum_{i=1}^{n} u_i = u,\ \alpha_i \in \{\alpha\} \right\}$$

exists in M for each u in $L^+$.

For each u in $L^+$, we define S(u) as follows:

$$S(u) = \sup\left\{ \sum_{i=1}^{n} T_{\alpha_i}(u_i) : u_i \in L^+;\ \sum_{i=1}^{n} u_i = u,\ \alpha_i \in \{\alpha\} \right\} .$$

It is easy to verify that S is an additive mapping from $L^+$ into $M^+$. Hence; it is extendable, by Lemma 2.1, to all of L. It follows that S is the supremum of $\{T_\alpha : \alpha \in \{\alpha\}\}$ in $\mathcal{L}_b(L, M)$. For more details see [23], p. 21.

Our next goal is to derive some formulas which are "dual" to the formulas (1) of Theorem 2.2.

We proceed with the following lemma.

LEMMA 2.3 (Hahn-Banach). Suppose that p is a mapping from the Riesz space L into a Dedekind complete Riesz space M such that

$p(f+g) \leqq p(f) + p(g)$; $p(\lambda f) = \lambda p(f)$ for all $f, g$ in $L$ and all non-negative $\lambda$. If $T$ is a linear mapping defined on a linear subspace $A$ of $L$ with range in $M$ such that $T(f) \leqq p(f)$ for all $f \in A$, then $T$ can be extended to a linear mapping $T_1$ on $L$ into $M$ such that $T_1(f) \leqq p(f)$ for all $f$ in $L$.

See [23], p. 79.

THEOREM 2.4. Let $L$ and $M$ be two Riesz spaces with $M$ Dedekind complete. If $\theta \leqq T \in \mathcal{L}_b(L, M)$, then we have:

(i) $T(f^+) = \sup\{S(f) : S \in \mathcal{L}_b(L, M);\ \theta \leqq S \leqq T\}$

(ii) $T(f^-) = \sup\{-S(f) : S \in \mathcal{L}_b(L, M);\ \theta \leqq S \leqq T\}$

(iii) $T(|f|) = \sup\{|S(f)| : S \in \mathcal{L}_b(L, M);\ |S| \leqq T\}$

for every $f$ in $L$.

PROOF. We prove the third formula first. Assume $f$ in $L$ and define the function $T_f$ on $L^+$ by the formula $T_f(u) = \sup\{T(u \wedge n|f|) : n = 1, 2, \ldots\}$ for all $u$ in $L^+$.

It follows easily that the function defined by $p(u) = T_f(|u|)$ for all $u$ in $L$ is a positive sublinear mapping such that $p(u) \leqq T(|u|)$ for all $u$ in $L$.

Let, now, $A = \{\lambda f : \lambda \in \mathbb{R}\}$ and let $S$ be the linear mapping from $A$ into $M$ defined by $S(\lambda f) = \lambda p(f)$ for all $\lambda$ in $\mathbb{R}$. According to Lemma 2.3 there is an extension $S_1$ of $S$ to all of $L$ such that $S_1(g) \leqq p(g)$ for all $g$ in $L$. It follows easily from the last relation that $S_1 \in \mathcal{L}_b(L, M)$. Hence $T(|f|) \leqq \sup\{S(|f|) : S \in \mathcal{L}_b(L, M);\ |S| \leqq T\}$. On the other hand, $\theta \leqq |S| \leqq T$ implies that $|S(f)| \leqq |S|(|f|) \leqq T(|f|)$, so $\sup\{|S(f)| : S \in \mathcal{L}_b(L, M);\ |S| \leqq T\} \leqq T(|f|)$, hence the third formula has been proved.

For the first formula we apply the same arguments using as sublinear mapping

$$p_1(g) = \sup\{T(g^+ \wedge nf^+) : n = 1, 2, \ldots\}.$$

The second formula follows from the first by noting that $f^- = (-f)^+$. ■

Note. From the above proof we see that the above suprema are actually maxima.

The next two theorems give more information about the structure of $\mathcal{L}_b(L, M)$.

THEOREM 2.5. Let L and M be two Riesz spaces with M Dedekind complete. Suppose that $T_\alpha \xrightarrow{(o)} T$ in $\mathcal{L}_b(L, M)$. Then $T_\alpha(f) \xrightarrow{(o)} T(f)$ in M for all f in L.

PROOF. Assume that $T_\alpha \xrightarrow{(o)} T$ in $\mathcal{L}_b(L, M)$. Then there exists a net $\{S_\alpha\} \subseteq \mathcal{L}_b(L, M)$ such that $|T_\alpha - T| \leqq S_\alpha \downarrow \theta$ in $\mathcal{L}_b(L, M)$. It follows then that

$$|T_\alpha(f) - T(f)| = |(T_\alpha - T)(f)| \leqq |T_\alpha - T|(|f|) \leqq S_\alpha(|f|) \downarrow \theta$$

for every f in L, i.e., $T_\alpha(f) \xrightarrow{(o)} T(f)$ for every f in L. Note that we used the relation $\theta = \theta(|f|) = (\inf S_\alpha)(|f|) = \inf\{S_\alpha(|f|)\}$ mentioned in Theorem 2.2. ■

THEOREM 2.6. Let L and M be two Riesz spaces with M Dedekind complete. Assume further, that $\{T_\alpha\}$ is a net of $\mathcal{L}_b(L, M)$ such that $|T_\alpha| \leqq S$ for all $\alpha$ and some S in $\mathcal{L}_b(L, M)$.

If $T(f) = (o) - \lim_\alpha T_\alpha(f)$ exists in M for every f in L, then T is in $\mathcal{L}_b(L, M)$.

PROOF. It is evident that $T \in \mathcal{L}(L, M)$. We have to show only that T is order bounded. So, assume $u \in L^+$ and $v \in L$, with $\theta \leqq v \leqq u$.

It follows from $T_\alpha(v) \xrightarrow{(o)} T(v)$ that $|T_\alpha(v)| \xrightarrow{(o)} |T(v)|$. But, $|T(v)| \leqq |T_\alpha|(v) \leqq S(v) \leqq S(u)$ for all $\alpha$ implies $|T(v)| \leqq S(u)$, by Lemma 1.8. So, $|T(v)| \leqq S(u)$ for all $v$ in $L$ such that $\theta \leqq v \leqq u$, i.e., $T \in \mathcal{L}_b(L, M)$. ∎

The next theorem is a kind of converse of Theorem 2.2.

THEOREM 2.7. Let $L$ and $M$ be two Riesz spaces with $L^\sim \neq \{\theta\}$. Let $\mathcal{L}_b(L, M) = \mathcal{L}_b$ denote the real vector space of all ordered bounded linear mappings from $L$ into $M$. If the ordered vector space $(\mathcal{L}_b, \mathcal{L}^+)$ is a Dedekind complete Riesz space, then $M$ is a Dedekind complete Riesz space.

PROOF. Assume $\theta \leqq u_\alpha \uparrow \leqq u_0$ in $M$. We have to show that $u_\alpha \uparrow u$ in $M$ for some $u$ in $M$. Let $\varphi$ be a non-zero positive linear functional of $L$, and let $f_0$ in $L^+$ be such that $\varphi(f_0) = 1$. Such an $f_0$ exists since $\varphi \neq \theta$.

For $\alpha \in \{\alpha\}$ we define a linear mapping $T_\alpha$ in $\mathcal{L}_b(L,M)$ as follows:

$$T_\alpha(f) = \varphi(f)\, u_\alpha, \qquad \text{for all } f \text{ in } L.$$

It follows easily that $\theta \leqq T_\alpha \uparrow \leqq T$, where $T \in \mathcal{L}_b(L, M)$, $T(f) = \varphi(f)\, u_0$ for all $f$ in $L$. By hypothesis $\mathcal{L}_b(L, M)$ is a Dedekind complete Riesz space. Hence there exists $S$ in $\mathcal{L}_b(L, M)$ such that $T_\alpha \uparrow S$. In particular we have $T_\alpha(f_0) = \varphi(f_0)u_\alpha = u_\alpha \uparrow \leqq S(f_0)$. We show next that $S(f_0)$ is the least upper bound of the net $\{u_\alpha\}$ in $M$. Suppose that $u_\alpha \leqq w$ for all $\alpha$. Then we have $T_\alpha \leqq T_w \in \mathcal{L}_b(L, M)$ for all $\alpha$ in $\{\alpha\}$, where $T_w(f) = \varphi(f)w$, for all $f$ in $L$. Hence $S \leqq T_w$ and so, in particular, $S(f_0) \leqq T_w(f_0) = \varphi(f_0)w = w$. This shows that $u_\alpha \uparrow S(f_0)$, i.e., $M$ is Dedekind complete. ∎

Similarly we can prove the following theorem.

THEOREM 2.8. *Assume once more that* L *and* M *are two given Riesz spaces with* $L^\sim \neq \{\theta\}$. *Assume further that* $(\mathcal{L}_b, \mathcal{L}^+)$ *forms a super Dedekind complete Riesz space. Then* M *is a super Dedekind complete Riesz space.*

THEOREM 2.9. *If* L *and* M *are two given Riesz spaces with* M *Dedekind complete and with* $L^\sim \neq \{\theta\}$, *and if* $\mathcal{L}_b(L, M)$ *has a strong unit, then* M *also has a strong unit.*

PROOF. Let $\theta \leqq \varphi$ be in $L^\sim$ as in the proof of Theorem 2.7, and let $\theta \leqq T_0 \in \mathcal{L}_b(L, M)$ be a strong unit for $\mathcal{L}_b(L, M)$. Given u in M we determine $T_u$ in $\mathcal{L}_b(L, M)$ by $T_u(f) = \varphi(f)u$, for all f in L. Then we have $\theta \leqq T_u \leqq nT_0$ for some n in N, from which it follows that $\theta \leqq T_u(f_0) \leqq nT_0(f_0)$, or $\theta \leqq u \leqq nT_0(f_0)$, i.e., $T_0(f_0)$ is a strong unit for M.■

The Riesz space L is called universally complete, if every system $\{u_\alpha\}$ of mutually disjoint elements in $L^+$ has a supremum (see [18], Definition 47.3, p. 323).

THEOREM 2.10. *Let* L *be a Riesz space having a non-zero positive linear functional and let* M *be a Dedekind complete Riesz space. Assume that* $\mathcal{L}_b(L, M)$ *is universally complete. Then* M *is universally complete.*

PROOF. Let $\varphi$ be a positive linear functional of L such that $\varphi(f_0) = 1$, for some $f_0$ in $L^+$, and let $\{u_\alpha\}$ be a mutually disjoint system of $M^+$.

We show that the system $\{T_\alpha\}$, $T_\alpha(f) = \varphi(f)\cdot u_\alpha$, for all f in L, is a mutually disjoint system of elements of $\mathcal{L}_b^+(L, M)$. To this end assume that $\alpha_1 \neq \alpha_2$ and $f \in L^+$. Then we have

$$\theta \leqq (T_{\alpha_1} \wedge T_{\alpha_2})(f) \leqq T_{\alpha_1}(f) \wedge T_{\alpha_2}(f)$$

$$= (\varphi(f)\cdot u_{\alpha_2}) \wedge (\varphi(f)\cdot u_{\alpha_2}) = \varphi(f)\cdot(u_{\alpha_1} \wedge u_{\alpha_2}) = \varphi(f)\cdot\theta = \theta .$$

So, $T_{\alpha_1} \wedge T_{\alpha_2} = \theta$ in $\mathcal{L}_b(L, M)$. It follows that

$$T = \sup\{T_\alpha\} = \sup\left\{\bigvee_{\alpha\in F} T_\alpha : F \in \mathcal{F}(\{\alpha\})\right\}$$

exists in $\mathcal{L}_b(L, M)$. But from the remark (ii) following Theorem 2.2 we have that

$$\left(\bigvee_{\alpha\in F} T_\alpha\right)(f_0) = \sup\left\{\sum_{\alpha\in F} T_\alpha(f_\alpha) : f_\alpha \in L^+;\ \sum_{\alpha\in F} f_\alpha = f_0\right\}$$

$$= \sup\left\{\sum_{\alpha\in F} \varphi(f_\alpha)u_\alpha : f_\alpha \in L^+;\ \sum_{\alpha\in F} f_\alpha = f_0\right\} \leqq \sum_{\alpha\in F} u_\alpha = \bigvee_{\alpha\in F} u_\alpha .$$

The last equality holds since $\{u_\alpha\}$ is a mutually disjoint family (see Theorem 1.1 (ii)).

Since, now, it is evident that $\bigvee_{\alpha\in F} u_\alpha \leqq (\bigvee_{\alpha\in F} T_\alpha)(f_0)$ it follows that $(\bigvee_{\alpha\in F} T_\alpha)(f_0) = \bigvee_{\alpha\in F} u_\alpha$ for every F in $\mathcal{F}(\{\alpha\})$. Since $\bigvee_{\alpha\in F} T_\alpha \uparrow T$ in $\mathcal{L}_b(L,M)$, it follows from Theorem 2.2 that $\bigvee_{\alpha\in F} u_\alpha = (\bigvee_{\alpha\in F} T_\alpha)(f_0) \uparrow_F T(f_0)$ in M, i.e., $\sup\{u_\alpha\} = \sup\{\bigvee_{\alpha\in F} u_\alpha : F \in \mathcal{F}(\{\alpha\})\} = T(f_0)$ in M, and this shows that M is universally complete.■

## 2.3. EXTENSION OF ORDER BOUNDED LINEAR OPERATORS

Let L and M be two Riesz spaces with M Dedekind complete, and let A be an ideal of L. Assume that T is an order bounded linear operator from A into M. The order bounded transformation S from L into M is called an extension of T, if $S(u) = T(u)$ for all u in A, i.e., $S = T$ on A. In this case we call T an extendable transformation. It is easy to verify that if $\theta \leqq T \in \mathcal{L}_b(A, M)$ and if T is extendable, then T has a positive extension on L. Indeed, let S be an extension of T. Then, if $u \in A^+$ we have

$$S^+(u) = \sup\{S(v) : v \in L;\ \theta \leqq v \leqq u\}$$
$$= \sup\{T(v) : v \in A;\ \theta \leqq v \leqq u\} = T^+(u) = T(u)$$

i.e., $S^+$ is a positive extension of T.

More generally, if S is an extension of T then $S^+$ is an extension of $T^+$ and $S^-$ is an extension of $T^-$. In other words, T is extendable if and only if $T^+$ and $T^-$ are both extendable.

More details about extensions are included in the next theorems.

THEOREM 2.11. *Let L and M be two Riesz spaces with M Dedekind complete, and let A be an ideal of L. Then the set of all extendable order bounded transformations of A forms an ideal of* $\mathcal{L}_b(A, M)$.

PROOF. We denote by $\mathcal{L}_b^e(A, M)$ the set of all extendable order bounded transformations from A into M. Obviously, $\mathcal{L}_b^e(A, M)$ is a vector subspace of $\mathcal{L}_b(A, M)$. We prove next that $\theta \leqq S \leqq T$, and $T \in \mathcal{L}_b^e(A, M)$ implies $S \in \mathcal{L}_b^e(A, M)$, i.e., that S is extendable.

We may suppose without loss of generality, that T is defined on all of L. Then we note that

$$S(f) \leqq |S(f)| \leqq S(|f|) \leqq T(|f|) \qquad \text{for all } f \text{ in } A\ ,$$

and that the function $p: L \to M$, $p(f) = T(|f|)$, $f \in L$, satisfies the properties of the Lemma 2.3. Hence S is extendable to all of L as a linear transformation $S_1$ satisfying the relations $S_1(f) = S(f)$ for all f in A and $S_1(f) \leqq p(f) = T(|f|)$ for all f in L.

Since, $p(f) = p(-f)$ we get that $|S_1(f)| \leqq p(f) = T(|f|)$ for all f in L, and this implies that $S_1 \in \mathcal{L}_b(L, M)$. Thus S is extendable. The conclusion that $\mathcal{L}_b^e(A, M)$ is an ideal of $\mathcal{L}_b(A, M)$, now follows from the earlier observation that $T \in \mathcal{L}_b^e(A, M)$ if and only if $T^+$ and $T^-$ are both in $\mathcal{L}_b^e(A, M)$, and so, in particular $T \in \mathcal{L}_b^e(A, M)$ implies $|T| = T^+ + T^-$ in $\mathcal{L}_b^e(A. M)$. ∎

THEOREM 2.12. Let L and M be two Riesz spaces with M Dedekind complete, and let A be an ideal of L. Suppose that $\theta \leqq T \in \mathcal{L}_b(A, M)$ is an extendable transformation. Then T has a smallest positive extension $T_m$ on all of L, in the sense that for any positive extension S of T on L we have $T_m \leqq S$ in $\mathcal{L}_b(L, M)$. Moreover

$$T_m(u) = \sup\{T(v): v \in A;\ \theta \leqq v \leqq u\}$$

for all u in $L^+$.

PROOF. Let $\mathcal{E}(T) = \{S \in \mathcal{L}_b(L, M): S = T \text{ on } A \text{ and } S \geqq \theta\}$. By assumption $\mathcal{E}(T) \neq \phi$. Since $\mathcal{E}(T)$ is bounded from below by $\theta$ in $\mathcal{L}_b(L, M)$ and $\mathcal{L}_b(L, M)$ is a Dedekind complete Riesz space the infimum of $\mathcal{E}(T)$ exists in $\mathcal{L}_b(L, M)$. So, let $T_m = \inf\{S: S \in \mathcal{E}(T)\}$. Obviously, $T_m \leqq S$ for all $S \in \mathcal{E}(T)$. So, we have to show that $T_m \in \mathcal{E}(T)$. We note that, if $S_1, S_2$ are in $\mathcal{E}(T)$ then $S_1 \wedge S_2 \geqq \theta$ and for each $\theta \leqq u \in A$ we have

$$(S_1 \wedge S_2)(u) = \inf\{S_1(u_1) + S_2(u_2): u_1, u_2 \in L^+; u_1 + u_2 = u\}$$
$$= \inf\{S_1(u_1) + S_2(u_2): u_1, u_2 \in A^+; u_1 + u_2 = u\}$$
$$= \inf\{T(u_1) + T(u_2) = T(u): u_1, u_2 \in A^+; u_1 + u_2 = u\} = T(u)$$

So, $S_1 \wedge S_2 \in \mathcal{E}(T)$ and this shows that $\mathcal{E}(T)$ is directed downwards in $\mathcal{L}_b(L, M)$. It follows, now, from Theorem 2.2 that $T_m \in \mathcal{E}(T)$.

To derive the formula of the theorem we proceed as follows.

Since T is extendable, it is easy to verify that $\sup\{T(v): v \in A; \theta \leqq v \leqq u\}$ exists in M for all u in $L^+$. So, let $S(u) = \sup\{T(v): v \in A; \theta \leqq v \leqq u\}$, $u \in L^+$. It is easily verified that S is an additive mapping from $L^+$ into $M^+$. Consequently, by Lemma 2.1 S is extendable uniquely to a positive operator on L, which we shall denote also by S. Obviously S is a positive extension of T. Hence $T_m \leqq S$.

On the other hand, if U is a positive extension of T, $u \in L^+$ and $v \in A$ such that $\theta \leqq v \leqq u$ then

$$T(v) = U(v) \leqq U(u) \quad ,$$

so $S(u) = \sup\{T(v): v \in A; \theta \leqq v \leqq u\} \leqq U(u)$, or $S \leqq U$, which implies that $S \leqq T_m$. Thus $S = T_m$.■

THEOREM 2.13. Let L and M be two given Riesz spaces with M Dedekind complete and let A be an ideal of L. Then there exists a linear mapping $T \to T_m$ on $\mathcal{L}_b^e(A, M)$ into $\mathcal{L}_b(L, M)$ such that for $\theta \leqq T \in \mathcal{L}_b^e(A,M)$ the image $T_m$ is the unique smallest positive extension of T to L. The mapping is an injection and the set of all images $T_m$ is a band of $\mathcal{L}_b(L,M)$. Moreover, this mapping is a Riesz isomorphism (into).

PROOF. We shall show that the mapping $T \to T_m$ from $(\mathcal{L}_b^e(A,M))^+$ into $\mathcal{L}_b^+(L,M)$ is additive. We note first that, if $\theta \leqq T \in \mathcal{L}_b^e(A,M)$ then the set $\{T(v)\colon v \in A;\ \theta \leqq v \leqq u\}$, $u \in L^+$, is directed upwards to $T_m(u)$. So, using this and Theorem 1.2(iv) we see that, if $\theta \leqq u \in L$ and $\theta \leqq U$, $T \in \mathcal{L}_b^e(A,M)$, then we have

$$\begin{aligned}(T+U)_m(u) &= \sup\{(T+U)(v)\colon v \in A;\ \theta \leqq v \leqq u\}\\ &= \sup\{T(v)\colon v \in A;\ \theta \leqq v \leqq u\} + \sup\{U(v)\colon v \in A;\ \theta \leqq v \leqq u\}\\ &= T_m(u) + U_m(u) \text{ for all } u \text{ in } L^+, \text{ i.e., } (T+U)_m = T_m + U_m .\end{aligned}$$

Since $\mathcal{L}_b(L,M)$ is an Archimedean Riesz space, it follows from Lemma 2.1 that this mapping can be extended uniquely to a linear mapping from $\mathcal{L}_b^e(A,M)$ into $\mathcal{L}_b(L,M)$. The extension is given by $T \to T_m = (T^+)_m - (T^-)_m$ where $T^+$, $T^-$ are the positive and negative parts, respectively, of $T$ in $\mathcal{L}_b^e(A,M)$.

If, now, $T_m = \theta$, then $(T^+)_m = (T^-)_m$, so $T^+ = T^-$ on A, which implies $T = T^+ - T^- = \theta$, on A. This shows that the mapping is one-to-one.

From Theorem 2.11 it follows easily that the set of all images is a band of $\mathcal{L}_b(L,M)$.

We show that $T \to T_m$ is a Riesz isomorphism (into). Assume first $\theta \leqq T$, $U \in \mathcal{L}_b^e(A,M)$. Then it is obvious that $T_m \vee U_m$ is a positive extension of $T \vee U$, so $(T \vee U)_m \leqq T_m \vee U_m$. On the other hand, for a given positive extension S of $T \vee U$ we have $\theta \leqq T \leqq S$ and $\theta \leqq U \leqq S$ on A, so $T_m \leqq S_m \leqq S$, $U_m \leqq S_m \leqq S$ on L, thus $T_m \vee U_m \leqq S$ which implies that $T_m \vee U_m \leqq (T \vee U)_m$, so $(T \vee U)_m = T_m \vee U_m$. (Note that we used the fact: $\theta \leqq T \leqq U$ in $\mathcal{L}_b^e(A,M)$ implies $\theta \leqq T_m \leqq U_m$ in $\mathcal{L}_b(L,M)$).

Now assume $T, U \in \mathcal{L}_b^e(A, M)$. Then we can write $T = T_1 - T_2$, $U = U_1 - U_2$ with $\theta \leqq T_i, U_i \in \mathcal{L}_b^e(A, M)$, $i = 1, 2$. So, we have $\theta \leqq T + T_2 + U_2$, $U + T_2 + U_2 \in \mathcal{L}_b^e(A, M)$. It follows, now, from the above proved fact that $[(T + T_2 + U_2) \vee (U + T_2 + U_2)]_m = (T + T_2 + U_2)_m \vee (U + T_2 + U_2)_m = [T_m + (T_2 + U_2)_m] \vee [U_m + (T_2 + U_2)_m]$. So, we get $T_m \vee U_m = [T_m + (T_2 + U_2)_m] \vee [U_m + (T_2 + U_2)_m] - (T_2 + U_2)_m = [(T + T_2 + U_2) \vee (U + T_2 + U_2)]_m - (T_2 + U_2)_m = [(T + T_2 + U_2) \vee (U + T_2 + U_2) - (T_2 + U_2)]_m = (T \vee U)_m$. This completes the proof.∎

THEOREM 2.14. Let L and M be as in Theorem 2.13. If A is a projection band of L, then every T in $\mathcal{L}_b(A, M)$ is extendable. So, in this case $\mathcal{L}_b^e(A, M)$ and $\mathcal{L}_b(L, M)$ are Riesz isomorphic.

PROOF. Since $L = A \oplus A^d$ every element u in L has a unique decomposition $u = u_1 + u_2$, $u_1 \in A$, $u_2 \in A^d$. If $P_A : L \to L$ is the projection defined by $P_A(u) = u_1$, then $T \circ P_A$ is an extension of T for every T in $\mathcal{L}_b(A, M)$.

The last conclusion follows immediately from Theorem 2.13.∎

THEOREM 2.15. Let L and M be two Riesz spaces with M Dedekind complete. Suppose that $A \subseteqq L$ is a Riesz subspace of L such that for every f in L, there exists u in A such that $|f| \leqq u$. Assume further that T is a positive linear mapping from the Riesz space A into M. Then, T can be extended to a positive linear transformation from L into M.

PROOF. According to our hypothesis for given f in $L^+$ the set $\{T(u): u \in A; f \leqq u\}$ is non-empty and bounded from below by zero in M.

Therefore $\inf\{T(u): u \in A; f \leqq u\}$ exists in M. Let $p(f) = \inf\{T(u): u \in A; |f| \leqq u\}$, $f \in L$.

Then we can easily verify that p is a sublinear mapping from L into M.

It is also evident that $T(u) \leqq T(|u|) = p(|u|) = p(u)$ for all u in A. So, it follows from Lemma 2.3 that T has an extension $T_2$, on L into M, such that $T_1(f) \leqq p(f)$ for all f in L. It follows now easily that $|T_1(f)| \leqq p(f)$ for all f in L, and from this it follows easily that $T_1 \in \mathcal{L}_b(L, M)$, and that $T_1^+ = T$ on A.■

EXAMPLE 2.16. Let L and M be two Riesz spaces with $M \neq \{\theta\}$, Dedekind complete and with L having a strong unit. Let $A = \{\lambda e : \lambda \in R\}$, where e is a strong unit of L. Then A is a Riesz subspace of L satisfying the hypothesis for the Theorem 2.15. Now assume $h \in M^+$, $h \neq \theta$; define the positive linear mapping $T: A \to M$, $T(\lambda e) = \lambda h$. According to Theorem 2.15, T has a positive extension on L into M.

# CHAPTER 3

# INTEGRAL AND NORMAL INTEGRAL TRANSFORMATIONS

## 3.1. THE CONCEPT OF AN INTEGRAL

Let L be the Riesz space of all real valued, Lebesgue integrable functions defined on $[0, 1]$ with ordering $f \leqq g$ whenever $f(x) \leqq g(x)$ for all $x \in [0, 1]$. We consider the following linear functionals on L, $\varphi_1 : L \to \mathbb{R}$, $\varphi_1(f) = \int_0^1 f(x)\,dx$, $f \in L$, i.e., $\varphi_1$ is the usual Lebesgue integral, and $\varphi_2 : L \to \mathbb{R}$, $\varphi_2(f) = f(o)$, $f \in L$. We can check easily that $f_n \downarrow \theta$ in L, implies $\varphi_1(f_n) \downarrow 0$ and $\varphi_2(f_n) \downarrow 0$ in $\mathbb{R}$. Also $f_\alpha \downarrow \theta$ in L implies $\varphi_2(f_\alpha) \downarrow 0$ in $\mathbb{R}$, but not necessarily $\varphi_1(f_\alpha) \downarrow 0$ as the following example shows. Consider the system

$$\{f_\alpha : \alpha \in \mathcal{F}([0,1])\},\quad f_\alpha(x) = \begin{cases} 0 & \text{if } x \in \alpha \\ 1 & \text{if } x \notin \alpha \end{cases} ;\ x \in [0,1] \quad ,$$

then $f_\alpha \downarrow \theta$ in L, but $\varphi_1(f_\alpha) = \int_0^1 f_\alpha(x)\,dx = 1$, for all $\alpha$.

In the next definition we characterize the above properties. We have the following.

DEFINITION 3.1. <u>Let</u> L <u>and</u> M <u>be two given Riesz spaces. A transformation</u> T <u>in</u> $\mathcal{L}(L, M)$ <u>is called an integral (resp. a normal integral), if</u> $T(f_n) \xrightarrow{(o)} \theta$ <u>in</u> M <u>(resp.</u> $T(f_\alpha) \xrightarrow{(o)} \theta$ <u>in</u> M) <u>whenever</u> $f_n \xrightarrow{(o)} \theta$ <u>in</u> L <u>(resp.</u> $f_\alpha \xrightarrow{(o)} \theta$ <u>in</u> L).

It is evident that a normal integral is an integral but the converse is not always true as the above example shows. If L is Dedekind complete and M = R, there is an interesting result in [13], concerning the question "Is every integral normal?".

The following theorem follows immediately from Definition 3.1, and generalizes a result due to H. Nakano (see [21], Theorem 19.1, p. 68).

THEOREM 3.2. Let L and M be two given Riesz spaces with M having the boundedness property and with L Archimedean. Then every integral T in $\mathcal{L}(L, M)$ is order bounded, i.e., $T \in \mathcal{L}_b(L, M)$.

PROOF. Let $T \in \mathcal{L}(L, M)$ be an integral and let A be an order bounded subset of L. We prove that T(A) is an order bounded subset of M. So, let $\{T(f_n) : f_n \in A\}$ be a sequence of T(A), and let $\{\lambda_n\} \subseteqq \mathbb{R}$ be such that $\lambda_n \to 0$. Since L is Archimedean and A is order bounded in L, it follows that $\lambda_n f_n \xrightarrow{(o)} \theta$ in L, hence; since T is an integral, it follows that $\lambda_n T(f_n) = T(\lambda_n f_n) \xrightarrow{(o)} \theta$. So, from the boundedness property of M it follows that T(A) is an order bounded subset of M, i.e., $T \in \mathcal{L}_b(L, M)$.∎

The next section deals with normal integrals.

## 3.2. THE BAND OF NORMAL INTEGRALS

The next theorem characterizes the set of all normal integrals of $\mathcal{L}_b(L, M)$ and is due to T. Ogasawara (see, [22], Vol. II).

THEOREM 3.3. Let L and M be two Riesz spaces with M Dedekind complete. Then, the set of all normal integrals of $\mathcal{L}_b(L, M)$ forms a band in the Dedekind complete Riesz space $\mathcal{L}_b(L, M)$.

For a proof see [30], Theorem VIII.3.3 on page 216.

A similar argument shows that the set of all integrals of $\mathcal{L}_b(L,M)$, (M Dedekind complete) denoted by $(\mathcal{L}_b)_c = (\mathcal{L}_b(L,M))_c$ forms a band of $\mathcal{L}_b(L,M)$.

The next theorem gives a sufficient condition for an integral to be a normal integral.

THEOREM 3.4. <u>If L is a super Dedekind complete Riesz space and M is a Dedekind complete Riesz space, then every integral of</u> $\mathcal{L}_b(L,M)$ <u>is a normal integral, i.e.</u>, $(\mathcal{L}_b)_n = (\mathcal{L}_b)_c$.

PROOF. Let $T \in \mathcal{L}_b(L,M)$ be an integral. Since $T \in \mathcal{L}_b(L,M)$ is an integral if and only if $|T| \in \mathcal{L}_b(L,M)$ is an integral, according to the previous theorem, we can assume that T is positive. Now, let $U_\alpha \downarrow \theta$ in L, then $T(U_\alpha) \downarrow h \geqq \theta$ in M for some h in $M^+$. Since L is super Dedekind complete for some sequence $\{U_{\alpha_n}\} \subseteqq \{U_\alpha\}$, we have $U_{\alpha_n} \downarrow \theta$. It follows then from the integrability of T that $T(U_{\alpha_n}) \downarrow \theta$, so $\theta \geqq h$, i.e., $h = \theta$, and this shows that T is a normal integral.■

Given a Riesz space L we say that the order convergence on L is stable if $f_n \xrightarrow{(o)} \theta$ implies $\lambda_n f_n \xrightarrow{(o)} \theta$ for some sequence $\{\lambda_n\}$ of positive real numbers such that $0 \leqq \lambda_n \uparrow +\infty$ (see [33]).

The next theorem deals with the above concept.

THEOREM 3.5. <u>Let</u> L <u>and</u> M <u>be two Riesz spaces with</u> M <u>Dedekind complete. Assume that the order convergence on</u> L <u>is stable. Then every transformation</u> T <u>in</u> $\mathcal{L}_b(L,M)$ <u>is an integral, i.e.</u>, $\mathcal{L}_b(L,M) = (\mathcal{L}_b(L,M))_c$.

PROOF. Assume $T \in \mathcal{L}_b(L,M)$ and $f_n \downarrow \theta$ in L. So, in particular we have $f_n \xrightarrow{(o)} \theta$; hence there exists a sequence $\{\lambda_n\} \subseteqq \mathbb{R}$ such that $0 < \lambda_n \uparrow +\infty$ with $\lambda_n f_n \xrightarrow{(o)} \theta$.

Since $\{\lambda_n f_n\} \subseteqq L^+$ is order bounded in L (an order convergent sequence is order bounded) the sequence $\{\lambda_n T(f_n)\}$ is also order bounded in M. So, there exists g in $M^+$ such that $|\lambda_n T(f_n)| \leqq g$ for all $n = 1, 2, \ldots$, or

$$|T(f_n)| \leqq \frac{1}{\lambda_n} \cdot g \qquad \text{for} \quad n = 1, 2, \ldots \quad .$$

But M is Archimedean, since it is Dedekind complete. So, $\frac{1}{\lambda_n} g \downarrow \theta$ in M. Hence $T(f_n) \xrightarrow{(o)} \theta$ in M, i.e., $T \in (\mathcal{L}_b(L,M))_c$.■

THEOREM 3.6. <u>Let</u> L <u>and</u> M <u>be two Riesz spaces with</u> M <u>Dedekind complete, and let</u> A <u>be an ideal of</u> L. <u>Assume that</u> $T \in \mathcal{L}_b(A,M)$ <u>is a normal integral and assume further that</u> T <u>is an extendable transformation.</u>

<u>Then the minimal extension</u> $T_m$ <u>of</u> T, <u>determined by Theorem 2.12 is a normal integral.</u>

PROOF. Assume $\theta \leqq u_\alpha \uparrow u$ in L, and assume $v \in A$, $\theta \leqq v \leqq u$; then $\theta \leqq v \wedge u_\alpha \uparrow v \wedge u = v$ in L, and since $\{v \wedge u_\alpha\} \subseteqq A$ and A is an ideal of L we have also that $v \wedge u_\alpha \uparrow v$ in A. It follows, now, from the assumption that T is a normal integral on A, that $T(v \wedge u_\alpha) \uparrow T(v)$ in M (here we assume that $T \geqq \theta$, without loss of generality). So, $T_m$ is also positive. This shows that $T_m(u_\alpha) \uparrow h \leqq T_m(u)$, for some h in $M^+$, since M is Dedekind complete.

But $T(v \wedge u_\alpha) = T_m(v \wedge u_\alpha) \leqq T_m(u_\alpha) \leqq h$ in M, so $T(v) \leqq h$. Using the formula $T_m(u) = \sup\{T(v): v \in A; \theta \leqq v \leqq u\}$ provided by Theorem 2. 12 we find $T_m(u) \leqq h$, i.e., $T_m(u_\alpha) \uparrow T_m(u)$, and this shows that $T_m$ is a normal integral of $\mathcal{L}_b(L, M)$.■

The following theorem can be similarly proved.

THEOREM 3. 7. Let L, M and A be as in Theorem 3. 6 and assume $T \in \mathcal{L}_b^e(A, M)$. If T is an integral then $T_m$ is an integral.

THEOREM 3. 8. Let L and M be two Riesz spaces with M Dedekind complete and also having a strong unit. Assume that $T \in \mathcal{L}_b(L, M)$ is such that $\varphi \circ T$ is an integral of $L^\sim$ for each $\varphi \in M^\sim$. Then T is an integral of $\mathcal{L}_b(L, M)$.

PROOF. Without loss of generality we can suppose that T is positive. If $f_n \downarrow \theta$ in L, then $T(f_n) \downarrow h$ in M, for some h of M. We show that $h = \theta$. Assume that $h > \theta$. Then the Minkowski functional defined by $P(g) = \inf\{\lambda > 0: |g| \leqq \lambda e\}$, where e is a strong unit of M, is a norm since M is Archimedean, so $p(h) > 0$.

Let $K = \{\lambda h: \lambda \in \mathbb{R}\} \subseteqq M$, and $\varphi(\lambda h) = \lambda p(h)$, $\lambda \in \mathbb{R}$. Then $\varphi$ is a non-zero positive linear functional defined on K. It follows then that $\varphi$ has a positive extension on all of M (see [16], Note VI, Theorem 19. 2, p. 661). But then we have $\varphi(T(f_n)) = (\varphi \circ T)(f_n) \downarrow \theta$, and also $0 < p(h) = \varphi(h) \leqq \varphi(T(f_n)) = (\varphi \circ T)(f_n)$ for all n, a contradiction. So, T is an integral.■

The next section deals with Dedekind completions and extensions.

## 3.3. DEDEKIND COMPLETION AND EXTENSIONS

Let L and $L_1$ be two given Riesz spaces.

DEFINITION 3.9. The Riesz space $L_1$ is called a Dedekind completion of L if

(i) $L_1$ is Dedekind complete.

(ii) L is embedded in $L_1$ as a Riesz subspace (more precisely, there is a one-to-one mapping of L into a Riesz subspace $L^1$ of $L_1$ preserving the (finite) algebraic and order relations; we shall think of L and $L^1$ as identical).

(iii) For every $f_1$ in $L_1$ we have

$$f_1 = \sup\{f \in L : f \leqq f_1\} = \inf\{g \in L : f_1 \leqq g\} \quad .$$

If $L_1$ is a Dedekind completion of L, then $L_1$ is Dedekind complete, so in particular $L_1$ is Archimedean which shows that L is Archimedean.

H. Nakano has proved that a Riesz space L has a Dedekind completion if and only if L is Archimedean, and that any two Dedekind completions of L are Riesz isomorphic (see [19], Theorems 30.2 and 30.3; see also [18], Theorem 32.5, p. 191).

The following theorem tells us that $(\mathcal{L}_b(L, M)_n$ and $(\mathcal{L}_b(L_1, M))_n$ are isomorphic.

THEOREM 3.10. Let $L_1$ be the Dedekind completion of the Archimedean Riesz space L, and let M be a Dedekind complete Riesz space. Then every $\theta \leqq T \in \mathcal{L}_b(L, M)$ can be extended to a positive linear operator on $L_1$ into M and for $\theta \leqq T \in (\mathcal{L}_b(L, M))_n$ the extension is unique and normal which shows that $(\mathcal{L}_b(L, M))_n$ and $(\mathcal{L}_b(L_1, M))_n$ are Riesz isomorphic.

PROOF. Assume $\theta \leqq T \in \mathcal{L}_b(L, M)$. Since $L_1$ is the Dedekind completion of the Archimedean Riesz space L we have $u = \sup\{f \in L^+ : f \leqq u\} = \inf\{g \in L^+ : u \leqq g\}$ for all $u \in L_1^+$.

This shows in particular, that every element u of $L_1^+$ is minorized by some element of L. It follows from this observation and the fact that M is Dedekind complete that $S(u) = \inf\{T(g): g \in L^+; u \leqq g\}$ exists in M for every u in $L_1^+$.

If, now, $u_1, u_2 \in L_1^+$ and if $f_1, f_2 \in L^+$ are such that $u_1 \leqq f_1$ and $u_2 \leqq f_2$, then $u_1+u_2 \leqq f_1+f_2$, $f_1+f_2 \in L^+$. So, $S(u_1+u_2) \leqq T(f_1+f_2)$ $= T(f_1) + T(f_2)$, from which it follows easily that $S(u_1+u_2) \leqq S(u_1) + S(u_2)$. It is also evident that $S(\lambda u) = \lambda S(u)$ for all $\lambda \geqq 0$ and all u in $L_1^+$. We also note that $S(u) = T(u)$ for all u in $L^+$.

Similarly the function $V(u) = \sup\{T(f): f \in L^+; f \leqq u\}$ u in $L_1^+$ is well defined on $L_1^+$ and satisfies $V(u_1+u_2) \geqq V(u_1) + V(u_2)$, $V(\lambda u_1) = \lambda V(u_1)$ for all $u_1, u_2$ in $L_1^+$ and all $\lambda \geqq 0$. We also note that $V(u) = T(u)$, for all u in $L^+$.

Hence; the mapping $p : L_1 \to M$, $p(f) = S(|f|)$, $f \in L_1$ is a sublinear mapping satisfying

$$T(f) \leqq T(|f|) = S(|f|) = p(f) \qquad \text{for all } f \in L \quad .$$

It follows from Lemma 2.3 that T can be extended to a linear mapping $T_1$ on $L_1$ into M such that $T_1(f) \leqq p(f)$ for all f in $L_1$.

It follows now easily that $T_1^+$ is a positive extension of T to $L_1$.

In order to prove that every $\theta \leqq T \in (\mathcal{L}_b(L, M))_n$ has a unique positive normal extension, we observe that if we set $M_u = \{g \in L^+ : g \geqq u\}$ and $M_u = \{f \in L^+ : f \leqq u\}$ for any u in $L_1^+$, then the set

$M_u - m_u = \{g - f : g \in M_u, f \in m_u\}$ is directed downwards to zero in L. It follows, now, from the fact that T is a normal integral on L that $S(u) = \inf\{T(g): g \in L^+; g \geqq u\} = \sup\{T(f): f \in L^+; f \leqq u\} = V(u)$. This shows that S is additive on $L_1^+$. Hence, by Lemma 2. 1, S can be extended uniquely to all of $L_1$. It is evident that S is normal on $L_1$.

Now let $S_1$ be another positive normal extension of T. Since the set $\{f \in L^+ : f \leqq u\}$ is directed upwards to u in $L_1$ for all u in $L_1^+$ we have

$$S_1(u) = \sup\{S_1(f): f \in L^+; f \leqq u\} = \sup\{T(f): f \in L^+; f \leqq u\} = S(u) \quad ,$$

i. e., $S_1 = S$.

Hence the positive normal extension is uniquely determined. It now follows easily that $(\mathcal{L}_b(L, M))_n$ and $(\mathcal{L}_b(L_1, M))_n$ are Riesz isomorphic. ■

Some sufficient conditions for a Riesz space to satisfy one of the completeness properties are given in the next theorem.

THEOREM 3. 11. Let L and M be two Riesz spaces and let $\pi$ be a normal Riesz homomorphism from L onto M, i. e., $\pi$ is a Riesz homomorphism which is also a normal integral. Then we have

(i) If L is $\sigma$-Dedekind complete then M is $\sigma$-Dedekind complete.

(ii) If L is Dedekind complete then M is Dedekind complete.

(iii) If L is super Dedekind complete then M is super Dedekind complete.

PROOF. (i) The proof can be found in Theorem 65. 2 of [18], page 450.

(ii) A proof is given in [18] (Theorem 66. 3, p. 457). Here we give a different proof.

Let L be Dedekind complete, and let $\theta \leqq f_\alpha \uparrow \leqq f$ in M. Since $\pi$ is an onto Riesz homomorphism, for each $\alpha \in \{\alpha\}$ we have $\pi(u_\alpha) = f_\alpha$ for some $u_\alpha \in L^+$, and also $\pi(u) = f$ for some $u \in L^+$.

For each $\lambda = \{\alpha_1, \ldots, \alpha_p\} \in \mathcal{F}(\{\alpha\})$ we define the element

$$w_\lambda = \left( \bigvee_{i=1}^{p} u_{\alpha_i} \right) \wedge u \quad .$$

Obviously $\theta \leqq w_\lambda \uparrow \leqq u$. It follows from the Dedekind completeness of L that $w_\lambda \uparrow w$ for some $w \in L^+$, so $\pi(w_\lambda) \uparrow \pi(w)$ since $\pi$ is a normal integral.

But $$\pi(w_\lambda) = \left( \bigvee_{i=1}^{p} \pi(u_{\alpha_i}) \right) \wedge \pi(u) = \bigvee_{i=1}^{p} \pi(u_{\alpha_i}) \quad .$$

This shows that the sets $\{\pi(w_\lambda)\}$ and $\{\pi(u_\alpha)\}$ have the same upper bounds in M. So, $f_\alpha \uparrow \pi(w)$, and this shows that M is Dedekind complete.

(iii) By (ii) we know that M is Dedekind complete. Consider the same situation as in (ii), we have $w_{\lambda_n} \uparrow w$ for some $\{w_{\lambda_n}\} \subseteqq \{w_\lambda\}$ according to the super Dedekind completeness of L, so, $\pi(w_{\lambda_n}) \uparrow \pi(w)$. Now let $\{f_{\alpha_n}\}$ be such that $f_{\alpha_n} \uparrow$ and $\pi(w_{\lambda_n}) \leqq f_{\alpha_n} \leqq \pi(w)$ for all $n = 1, 2, \ldots$ . It follows, now, easily that $f_{\alpha_n} \uparrow \pi(w)$, and this shows that M is super Dedekind complete.■

## 3.4. A GENERALIZATION OF A THEOREM BY H. NAKANO

We begin with the following definition.

DEFINITION 3.12. Let L be a Riesz space with the following properties:

(i) L is Dedekind complete.

(ii) L is a $K^+$-space, i.e., $A \subseteqq L$ is order bounded if $\{f_n\} \subseteqq A$ and

$\lambda_n \to 0$ in $\mathbb{R}$, implies $\lambda_n f_n \xrightarrow{(o)} \theta$ in L.

(iii) A sequence $\{f_n\} \subseteq$ L (o)-converges to zero if and only if $\varphi(f_n) \to 0$ in $\mathbb{R}$ for every $\varphi \in L_n^{\sim} = (\mathcal{L}_b(L, \mathbb{R}))_n$.

We shall call every space with the above properties an R-space.

Examples of R-spaces are provided by the Riesz spaces of the form $\mathbb{R}^X$, where X is a non-empty set.

LEMMA 3.13. A subset A of a R-space L is order bounded if and only if $\varphi(A)$ is bounded in $\mathbb{R}$, for every $\varphi$ in $L_n^{\sim}$.

PROOF. It is evident that $A \subseteq L$ and A order bounded implies $\varphi(A)$ bounded in $\mathbb{R}$ for all $\varphi$ in $L_n^{\sim}$. We show that the converse also holds. To this end, assume $A \subseteq L$, and that $\varphi(A)$ is bounded in $\mathbb{R}$ for all $\varphi$ in $L_n^{\sim}$. Let $\{f_n\} \subseteq A$ and let $\{\lambda_n\} \subseteq \mathbb{R}$ be such that $\lambda_n \to o$, in $\mathbb{R}$. Then, we have $\varphi(\lambda_n f_n) = \lambda_n \varphi(f_n) \to o$, since $\{\varphi(f_n)\}$ is bounded in $\mathbb{R}$. So, according to property (iii) of the Definition 3.12 $\lambda_n f_n \xrightarrow{(o)} \theta$ in L, and so from (ii) of the same Definition we have that A is order bounded in L. ■

The next theorem is due to H. Nakano (see [19], Theorem 46.5, p. 252).

THEOREM 3.14. If L is a σ-Dedekind complete Riesz space, $\{\varphi_n\} \subseteq L^{\sim}$ and if, for every f in L, $\varphi(f) = \lim_{n \to +\infty} \varphi_n(f)$ exists and is finite, then $\varphi$ is also in $L^{\sim}$.

We generalize this theorem as follows.

THEOREM 3.15. Let L be a σ-Dedekind complete Riesz space and let M be an R-space. Suppose that $\{T_n\} \subseteq \mathcal{L}_b(L, M)$ and that

$T(f) = (o) - \lim_{n\to+\infty} T_n(f)$, exists in M for every f in L. Then T is in $\mathcal{L}_b(L, M)$.

PROOF. Let A be an order bounded subset of L, and assume $\varphi \in M_n^\sim$. Then $(\varphi o T)(f) = \varphi(T(f)) = \varphi((o) - \lim_{n\to+\infty} T_n(f)) = \lim_{n\to+\infty} \varphi(T_n(f))$ $= \lim_{n\to+\infty}(\varphi o T_n)(f)$. But $\varphi \in M_n^\sim$ and $T_n \in \mathcal{L}_b(L, M)$, implies $\varphi o T_n \in L^\sim$ for $n = 1, 2, \ldots$, so by (Nakano's) Theorem 3.14 we also have $\varphi o T \in L^\sim$, so $(\varphi o T)(A) = \varphi(T(A))$ is a bounded subset of the real line for all $\varphi \in M_n^\sim$. It follows now from Lemma 3.13 that T(A) is an order bounded subset of M, so $T \in \mathcal{L}_b(L, M)$. ∎

## 3.5. SOME PROPERTIES AND SOME CHARACTERIZATIONS OF INTEGRALS AND NORMAL INTEGRALS

We begin with the following definition.

DEFINITION 3.16. Let L and M be two Riesz spaces with M Dedekind complete. For every operator T of $\mathcal{L}_b(L, M)$ the subset $\{f \in L : |T|(|f|) = \theta\}$ is called the null ideal of T and is denoted by $N_T$.

THEOREM 3.17. Let L and M be as in Definition 3.16. Then we have:

(i) For every T in $\mathcal{L}_b(L, M)$, $N_T$ is an ideal of L and $N_T = N_{|T|} = N_{T^+} \cap N_{T^-}$.

(ii) For every T in $(\mathcal{L}_b)_n$, $N_T^d$ is the smallest band which includes both $N_{T^+}^d$ and $N_{T^-}^d$.

PROOF. Trivial. ∎

THEOREM 3.18. Let L be a $\sigma$-Dedekind complete Riesz space and let M be a Dedekind complete Riesz space. Assume $\theta \leq T \in \mathcal{L}_b(L, M)$

*is an integral. Then the operator* $\theta \leqq \hat{T} \in \mathcal{L}_b(L/N_T, M)$ *defined by* $\hat{T}([f]) = T(f)$, *is an integral.*

PROOF. It is evident that $\hat{T}$ is well defined. Since T is an integral, $N_T$ is a $\sigma$-ideal; hence the canonical projection $f \to [f]$, from L onto $L/N_T$, whose kernel is $N_T$, is a $\sigma$-homomorphism ([18], Theorem 18.11, p. 103). Now assume $[f_n] \downarrow [\theta]$ in $L/N_T$. Without loss of generality we can suppose that $f_n \downarrow \geqq \theta$. It follows from the $\sigma$-Dedekind completeness of L that $f_n \downarrow f$ in L, for some f in $L^+$. So, we have also $[f_n] \downarrow [f]$ in $L/N_T$, therefore $[f] = [\theta]$, so $f \in N_T$.

But, T being an integral implies that

$$\hat{T}([f_n]) = T(f_n) \downarrow T(f) = \theta \text{ in } M \quad ,$$

i.e., $\hat{T}$ is an integral of $\mathcal{L}_b(L/N_T, M)$. ∎

The next theorem gives a sufficient condition for $\{N_T\}$ to be a projection band.

THEOREM 3.19. *Let L be a* $\sigma$-*Dedekind complete Riesz space and let M be a super Dedekind complete Riesz space. Assume* $T \in \mathcal{L}_b(L, M)$. *Then* $N_T^d$ *is a projection band of L. It follows in particular that* $L = \{N_T\} \oplus N_T^d$.

PROOF. Since $N_T = N_{|T|}$ we can suppose that $T \geqq \theta$. Assume $u \in L^+$. We shall prove that $\sup\{v \in N_T^d : \theta \leqq v \leqq u\}$ exists in L (see Theorem 1.5).

Evidently, the set $\{v \in N_T^d : \theta \leqq v \leqq u\}$ is directed upwards in L; consequently the set $\{T(v) : \theta \leqq v \leqq u;\ v \in N_T^d\}$ is directed upwards and is bounded in M. Let $f = \sup\{T(v): \theta \leqq v \leqq u;\ v \in N_T^d\}$. The super Dedekind completeness of M implies that $T(v_n) \uparrow f$ for some sequence $\{v_n\} \subseteqq N_T^d$, $\theta \leqq v_n \leqq u$, for $n = 1, 2, \ldots$ . We can suppose that $v_n \uparrow$ in L. By the $\sigma$-Dedekind completeness of L we have that $v_n \uparrow v_o \in N_T^d$. We show that $v_o = \sup\{v \in N_T^d: \theta \leqq v \leqq u\}$. If not, there exists $v \in N_T^d$, $\theta \leqq v \leqq u$ satisfying $v_o < v_o \vee v = w \leqq u$, so $w - v_o \in N_T^d$, and $v_n \vee w - v_n = (w - v_n)^+ \geqq (w - v_o)^+ = w - v_o > \theta$, which implies that $T(v_n \vee w) - T(v_n) \geqq T(w - v_o) > \theta$.

But then it follows that $T(w - v_o) + T(v_n) \leqq T(v_n \vee w) \leqq f$ for all $n = 1, 2, \ldots$ . Hence; $T(w - v_o) + f \leqq f$, or $T(w - v_o) = \theta$, a contradiction. Since L is a $\sigma$-Dedekind complete Riesz space it is in particular Archimedean, so we have that $N_T^{dd} = \{N_T\}$ (see Theorem 1.6). So, $L = N_T^d \oplus N_T^{dd} = N_T^d \oplus \{N_T\}$.■

The next theorem characterizes the normality of an integral in some particular cases.

THEOREM 3.20. Let L be a $\sigma$-Dedekind complete Riesz space and let M be a super Dedekind complete Riesz space. The integral T of $\mathcal{L}_b(L, M)$ is a normal integral if and only if $N_T$ is a band of L.

PROOF. Assume that $\theta \leqq T$ is a normal integral of $\mathcal{L}_b(L, M)$. Let $\theta \leqq u_\alpha \uparrow u$ in L, $\{u_\alpha\} \subseteqq N_T$. Then we have $\theta = T(u_\alpha) \uparrow T(u)$, so $T(u) = \theta$, i.e., $u \in N_T$, and this shows that $N_T$ is a band of L.

Now let $\theta \leqq T$ be an integral of $\mathcal{L}_b(L, M)$ and $N_T$ be a band of L. We show that T is a normal integral of $\mathcal{L}_b(L, M)$. To this end, let $u_\alpha \downarrow \theta$ in L. We show next that $T(u_\alpha) \downarrow \theta$ in M.

Applying Theorem 3.19 we see that $L = N_T \oplus N_T^d$; therefore we may suppose that $\{u_\alpha\} \subseteq N_T^d$.

Since M is Dedekind complete and $T(u_\alpha) \downarrow \geq \theta$, we have $T(u_\alpha) \downarrow h$ in M, for some $h \in M^+$. It follows from the super Dedekind completeness of M that $T(u_{\alpha_n}) \downarrow h$ for some sequence $\{u_{\alpha_n}\} \subseteq \{u_\alpha\}$, $u_{\alpha_n} \downarrow \geq \theta$ in L. But, the σ-Dedekind completeness of L shows that $u_{\alpha_n} \downarrow u$ in L for some $u \in L^+$; hence, $T(u_{\alpha_n}) \downarrow T(u)$ in M since T is an integral of $\mathcal{L}_b(L, M)$, so $h = T(u)$. Suppose now that $h > \theta$. Then $u > \theta$. Since $u_\alpha \downarrow \theta$, it follows that for some $u_{\alpha_0}$ we must have $u_{\alpha_0} \wedge u < u$; hence, $T(u - u \wedge u_{\alpha_0}) > \theta$.

It follows, now, that $T(u \wedge u_{\alpha_0}) < T(u) = h$. But $u_{\alpha_n} \wedge u_{\alpha_0} \downarrow u \wedge u_{\alpha_0}$, so $T(u_{\alpha_n} \wedge u_{\alpha_0}) \downarrow T(u \wedge u_{\alpha_0}) < h$.

This implies that $T(u_{\alpha_{n_0}} \wedge u_{\alpha_0}) \wedge h < h$ in M for some index $n_0 \in N$. Selecting now $\alpha_1$ in $\{\alpha\}$, such that $u_{\alpha_1} \leq u_{\alpha_{n_0}} \wedge u_{\alpha_0}$ we obtain $h = T(u_{\alpha_1}) \wedge h \leq T(u_{\alpha_{n_0}} \wedge u_{\alpha_0}) \wedge h < h$, a contradiction.

This shows that $h = \theta$, and this completes the proof. ∎

Note. The proof of the necessity in the above theorem does not depend on the assumption that L is σ-Dedekind complete.

Given two Riesz spaces L and M and a linear transformation from L into M, we say that T is strictly positive if $T(f) > \theta$ in M, whenever $f > \theta$ in L.

With respect to this definition we have the following theorem.

THEOREM 3.21. *Let L and M be two given Riesz spaces with M super Dedekind complete, and let T be a strictly positive linear transformation from L into M. Then we have:*

(i) *If* $\theta \leqq u_\alpha \uparrow u$, *then* $\theta \leqq u_{\alpha_n} \uparrow u$ *for some sequence* $\{u_{\alpha_n}\} \subseteq \{u_\alpha\}$. *It follows in particular, that* L *is super Dedekind complete if it is* $\sigma$*-Dedekind complete.*

(ii) *If* T *is a strictly positive integral of* $\mathcal{L}_b(L, M)$, *then* T *is a normal integral.*

PROOF. (i) Assume $\theta \leqq u_\alpha \uparrow u$ in L. Then $T(u_\alpha) \uparrow h$ in M for some h in $M^+$. Since M is super Dedekind complete there exists a sequence $\{u_{\alpha_n}\} \subseteq \{u_\alpha\}$, $u_{\alpha_n} \uparrow$ such that $T(u_{\alpha_n}) \uparrow h$. If $u_{\alpha_n} \uparrow u$ is not valid, then there exists $u_0 < u$ such that $u_{\alpha_n} \leqq u_0$ for all $n \in N$. Then, since $u_\alpha \uparrow u$ and $u_0 < u$ we must have $u_{\alpha_0} \vee u_0 - u_0 = (u_{\alpha_0} - u_0)^+ > \theta$ for some index $\alpha_0 \in \{\alpha\}$. But then $u_{\alpha_0} \vee u_{\alpha_n} - u_{\alpha_n} \geqq (u_{\alpha_0} - u_0)^+ > \theta$ for all n. It follows then from the strict positivity of T that $T(u_{\alpha_0} \vee u_{\alpha_n}) - T(u_{\alpha_n}) \geqq T((u_{\alpha_0} - u_0)^+) > \theta$. Since $u_\alpha \uparrow u$ in L, for given n there exists an index $\beta_n \in \{\alpha\}$, such that $u_{\beta_n} \geqq u_{\alpha_0} \vee u_{\alpha_n}$; hence, $h \geqq T(u_{\beta_n}) \geqq T(u_{\alpha_n}) + T((u_{\alpha_0} - u_0)^+)$ for all n, a contradiction. This shows that $u_{\alpha_n} \uparrow u$. The last assertion follows from [14], Theorem 3.

(ii) Part (ii) is an immediate consequence of part (i).∎

COROLLARY 3.22. *Let* L *be a Dedekind complete Riesz space and let* M *be a super Dedekind complete Riesz space. Assume also that* $T \in \mathcal{L}_b(L, M)$. *Then* $N_T^d$ *is a super Dedekind complete Riesz space and if* T *is an integral, then* T *restricted to* $N_T^d$ *is a normal integral.*

PROOF. The proof follows from the previous theorem by observing that T restricted to $N_T^d$ is strictly positive and that $N_T^d$ by itself is a Dedekind complete Riesz space.∎

## 3.6. RIESZ ANNIHILATORS AND INTEGRALS

Given two Riesz spaces L and M with M Dedekind complete, the Riesz annihilator $A^o$ of a subset A of L is defined by $A^o = \{T \in \mathcal{L}_b(L, M)\}$: $T(f) = \theta$, for all $f \in A\}$. Evidently $A^o$ is a linear subspace of $\mathcal{L}_b(L, M)$.

For any subset B of $\mathcal{L}_b(L, M)$ the inverse Riesz annihilator ${}^oB$ is defined by ${}^oB = \{f \in L: T(f) = \theta$ for all $T \in B\}$. Evidently ${}^oB$ is a linear subspace of L.

We have the following theorem.

THEOREM 3.23. <u>Assume that</u> L <u>and</u> M <u>are two Riesz spaces with</u> M <u>Dedekind complete</u>. <u>Then we have:</u>

(i) <u>If</u> A <u>is an ideal of</u> L, <u>then</u> $A^o$ <u>is a band of</u> $\mathcal{L}_b(L, M)$.

(ii) <u>If</u> B <u>is an ideal of</u> $\mathcal{L}_b(L, M)$, <u>then</u> ${}^oB$ <u>is an ideal of</u> L.

(iii) <u>If</u> B <u>is an ideal of</u> $(\mathcal{L}_b)_n$, <u>then</u> ${}^oB$ <u>is a band of</u> L.

PROOF. (i) We show first that $T \in A^o$ implies $|T| \in A^o$. To this end, it suffices to show that $|T|(u) = \theta$ for every positive u in A. Now, given $\theta \leqq u \in A$, it follows from $|f| \leqq u$, that $f \in A$, and so $T(f) = \theta$, which implies by Theorem 2.2 $|T|(u) = \sup\{|T(f)|: f \in L; |f| \leqq u\} = \theta$.

Now, let $S \in \mathcal{L}_b(L, M)$, and $|S| \leqq |T|$. Then $|T| \in A^o$, and so $|S| \in A^o$ trivially. It follows then from $|S(u)| \leqq |S|(u) = \theta$, that $S(u) = \theta$ for every u in $A^+$, so $S \in A^o$. This shows that $A^o$ is an ideal of $\mathcal{L}_b(L, M)$. It remains to prove that if $\{T_\alpha\} \subseteqq A^o$ and $\theta \leqq T_\alpha \uparrow T$ in $\mathcal{L}_b(L, M)$, then $T \in A^o$. Since $T_\alpha(u) = \theta$ for every $\alpha \in \{\alpha\}$ and every $\theta \leqq u \in A$ and since $T(u) = \sup\{T_\alpha(u)\}$ by Theoren 2.2, it follows that $T(u) = \theta$ for every $u \in A^+$, so $T \in A^o$.

(ii) We prove that $f \in {}^{o}B$ implies $|f| \in {}^{o}B$. Given $T \in B^{+}$, it follows from $|S| \leqq T$, that $S \in B$, and so $S(f) = \theta$, which implies that

$$T(|f|) = \sup\{|S(f)| : S \in \mathcal{L}_b(L,M);\ |S| \leqq T\} = \theta$$

according to Theorem 2.4.

Now let $g \in L$, $f \in {}^{o}B$, and $|g| \leqq |f|$. Then $|f| \in {}^{o}B$, and so $|g| \in {}^{o}B$ trivially. It follows then from $|T(g)| \leqq |T|(|g|) = \theta$, that $T(g) = \theta$ for all $T \in B^{+}$, so $g \in {}^{o}B$. This shows that ${}^{o}B$ is an ideal of L.

(iii) Let $\theta \leqq u_\alpha \uparrow u$, $\{u_\alpha\} \subseteqq {}^{o}B$, and $B \subseteqq (\mathcal{L}_b)_n$. Then obviously $\theta = T(u_\alpha) \uparrow T(u)$ for all $\theta \leqq T \in B$, so $T(u) = \theta$, for all $\theta \leqq T \in B$. Thus, $u \in {}^{o}B$, and so ${}^{o}B$ is a band of L. ■

THEOREM 3.24. <u>Let</u> L <u>and</u> M <u>be two given Riesz spaces with</u> M <u>Dedekind complete</u>. <u>Assume that</u> ${}^{o}(M_c^{\sim}) = \{\theta\}$, <u>and that for the operator</u> $\theta \leqq T \in \mathcal{L}_b(L,M)$ <u>we have</u> $\varphi \circ T \in L_c^{\sim}$ <u>for all</u> $\varphi \in M_c^{\sim}$.

<u>Then</u> T <u>is an integral of</u> $\mathcal{L}_b(L,M)$.

PROOF. Let $f_n \downarrow \theta$ in L. Then $T(f_n) \downarrow h$ in M for some $\theta \leqq h \in M$. It follows, now, from our hypothesis that $(\varphi \circ T)(f_n) = \varphi(T(f_n)) \downarrow \varphi(h) = \theta$ for all $\theta \leqq \varphi \in M_c^{\sim}$. Since ${}^{o}(M_c^{\sim}) = \{\theta\}$, we get $h = \theta$, so $T(f_n) \downarrow \theta$. This completes the proof. ■

EXAMPLE 3.25. Let $L = C(X)$, where X is a completely regular Hausdorff topological space with the property that every point is a P-point, i.e., with the property that every intersection of a countable family of neighborhoods of the point is again a neighborhood of the point, but not a discrete point. For an example of such a space S, see [5], Problem 13P. Assume further that M is a Dedekind complete Riesz space with ${}^{o}(M_c^{\sim}) = \{\theta\}$. Then every $\theta \leqq T \in \mathcal{L}_b(L, M)$ is an integral.

To see this, we first note that $\theta \leqq \varphi \in M^{\sim}$ implies $\theta \leqq \varphi \circ T \in L^{\sim}$ and that $u_n \downarrow \theta$ in L, implies $u_n(x) \downarrow 0$ in $\mathbb{R}$ for every $x \in X$ (see [15], Note XV, Example 50.7, p. 420). But then it follows from a well-known result (see [31], Problem 150).

So, Theorem 3.24 shows that T is an integral. ■

## 3.7. THE COMPONENTS OF AN ORDER BOUNDED TRANSFORMATION

Given two Riesz spaces L and M with M Dedekind complete we denote by $(\mathcal{L}_b)_n = (\mathcal{L}_b(L,M))_n$, $(\mathcal{L}_b)_c = (\mathcal{L}_b(L,M))_c$ the bands of the normal integrals and integrals, respectively, of $\mathcal{L}_b(L,M)$.

It follows from the fact that $\mathcal{L}_b(L,M)$ is a Dedekind complete Riesz space (Th. 2.2) that

$$\mathcal{L}_b(L,M) = (\mathcal{L}_b)_n \oplus ((\mathcal{L}_b)_n)^d$$

$$\mathcal{L}_b(L,M) = (\mathcal{L}_b)_c \oplus ((\mathcal{L}_b)_c)^d$$

We shall denote the bands $((\mathcal{L}_b)_n)^d$, $((\mathcal{L}_b)_c)^d$ by $(\mathcal{L}_b)_{sn}$, $(\mathcal{L}_b)_s$, respectively, and we shall call the elements of $(\mathcal{L}_b)_{sn}$ the normal singular integrals and the elements of $(\mathcal{L}_b)_s$ the singular operators, respectively.

So, we have

$$\mathcal{L}_b(L,M) = (\mathcal{L}_b)_n \oplus (\mathcal{L}_b)_{sn} = (\mathcal{L}_b)_c \oplus (\mathcal{L}_b)_s \quad .$$

It follows, now, from the above relation and $(\mathcal{L}_b)_n \subseteqq (\mathcal{L}_b)_c$ that $(\mathcal{L}_b)_{sn} \supseteqq (\mathcal{L}_b)_s$, hence we have $(\mathcal{L}_b)_{sn} = [(\mathcal{L}_b)_{sn} \cap (\mathcal{L}_b)_s] \oplus (\mathcal{L}_b)_s$.

We denote the band $(\mathcal{L}_b)_{sn} \cap (\mathcal{L}_b)_s$ by $(\mathcal{L}_b)_{sn,c}$, and call the elements of $(\mathcal{L}_b)_{sn,c}$ singular integrals. Then we have the following decomposition for $\mathcal{L}_b(L,M)$.

$$\mathcal{L}_b(L,M) = (\mathcal{L}_b)_n \oplus (\mathcal{L}_b)_{sn,c} \oplus (\mathcal{L}_b)_s$$

i.e., every $T \in \mathcal{L}_b(L,M)$ has a unique decomposition $T = T_n + T_{sn,c} + T_s$ $(= T_c + T_s)$ where the elements on the right are in $(\mathcal{L}_b)_n$, $(\mathcal{L}_b)_{sn,c}$ and $(\mathcal{L}_b)_s$, respectively.

It is easy to see ([18], Theorem 14.4(ii)) that

$$T^+ = T_n^+ + T_{sn,c}^+ + T_s^+, \quad T^- = T_n^- + T_{sn,c}^- + T_s^-,$$

$$|T| = |T_n| + |T_{sn,c}| + |T_s|$$

are the decompositions of $T^+$, $T^-$ and $|T|$, respectively.

The operator $T_n$ is called the normal (integral) component of T, and the operator $T_c = T_n + T_{sn,c}$ is called the integral component of T.

Next we shall investigate some of the properties of the different components of T.

We start with the following lemma.

LEMMA 3.26. <u>Assume that</u> L <u>and</u> M <u>are two Riesz spaces with</u> M <u>Dedekind complete.</u> <u>We consider the following mappings from</u> $L^+$ <u>into</u> $M^+$:

(i) $T_L(u) = \inf\{(o)\text{-}\lim_{n\to+\infty} T(u_n) : \theta \leqq u_n \uparrow u\}$

(ii) $\overline{T}_L(u) = \inf\{(o)\text{-}\lim_{\alpha} T(u_\alpha) : \theta \leqq u_\alpha \uparrow u\}$

(iii) $\overline{T}(u) = \inf\{(o)\text{-}\lim_{\alpha} T(u_\alpha) : u \geqq u_\alpha \downarrow \theta\}$

<u>for every</u> $u \in L^+$, <u>where</u> $\theta \leqq T \in \mathcal{L}_b(L,M)$. <u>Then</u> $T_L$, $\overline{T}_L$ <u>and</u> $\overline{T}$ <u>are additive on</u> $L^+$.

PROOF. We give the proof for $T_L$. The proofs for $\overline{T}_L$ and $\overline{T}$ are similar.

It is evident that $T_L(u + v) \leqq T_L(u) + T_L(v)$ for all $u, v \in L^+$.

For the reverse inequality assume that $u, v \in L^+$ and $\theta \leqq w_n \uparrow u + v$. By the Riesz decomposition property (Th. 1.1(vii)) we can write $w_n = u_n + v_n$, $u_n, v_n \in L^+$ for $n = 1, 2, \ldots$, and $u_n \uparrow u$, $v_n \uparrow v$. It follows from this that $T(w_n) = T(u_n) + T(v_n)$, and so, $(o) - \lim_{n\to+\infty} T(w_n) = (o) - \lim_{n\to+\infty} T(u_n) + (o) - \lim_{n\to+\infty} T(v_n) \geqq T_L(u) + T_L(v)$. Hence, $T_L(u+v) \geqq T_L(v)$. This completes the proof. ■

The following theorem is due to W. A. J. Luxemburg and A. C. Zaanen (see [16], Note VI, p. 663, and [15], Note XV, p. 441).

THEOREM 3.27. <u>Let</u> L <u>be a Riesz space and let</u> $\theta \leqq \varphi \in L^\sim$. <u>Then for every</u> $u \in L^+$ <u>we have:</u>

(i) $\varphi_c(u) = \inf\{\lim_{n\to+\infty} \varphi(u_n) : \theta \leqq u_n \uparrow u\}$.

(ii) $\varphi_n(u) = \inf\{\lim_{\alpha} \varphi(u_\alpha) : \theta \leqq u_\alpha \uparrow u\}$.

(iii) $\varphi_{sn}(u) = \sup\{\lim_{\alpha} \varphi(u_\alpha) : u \geqq u_\alpha \downarrow \theta\}$.

We generalize this theorem as follows:

THEOREM 3.28. <u>Let</u> L <u>and</u> M <u>be two Riesz spaces with</u> M <u>Dedekind complete. Assume that</u> ${}^{o}(M_n^\sim) = \{\theta\}$ <u>or that</u> M <u>admits a strictly positive integral. If</u> $\theta \leqq T \in \mathcal{L}_b(L, M)$, <u>then for every</u> $u \in L^+$ <u>we have:</u>

(i) $T_c(u) = \inf\{(o) - \lim_{n\to+\infty} T(u_n) : \theta \leqq u_n \uparrow u\}$.

(ii) $T_n(u) = \inf\{(o) - \lim_{\alpha} T(u_\alpha) : \theta \leqq u_\alpha \uparrow u\}$.

(iii) $T_{sn}(u) = \sup\{(o) - \lim_{\alpha} T(u_\alpha) : u \geqq u_\alpha \downarrow \theta\}$.

PROOF. We verify the formula (i). A similar argument will prove (ii) and (iii).

We note first that the set $\{(o)\text{-}\lim T(u_n) : \theta \leqq u_n \uparrow u\}$ is directed downwards in M and that it is bounded by $\theta$. According to Lemma 3.26, $T_L$ is additive on $L^+$, hence by Lemma 2.1, is uniquely extendable to the whole L. We shall show next that $T_L$ is an integral of $\mathcal{L}_b(L, M)$ under the assumption ${}^o(M_n^\sim) = \{\theta\}$. (A similar argument will work in the case in which M admits a strictly positive integral. Note that in this case by Theorem 3.21 the strictly positive integral is also a normal integral.)

So, let $u_n \downarrow \theta$ in L. Then $T_L(u_n) \downarrow h \geqq \theta$ in M for some $h \in M^+$. It suffices to show that $h = \theta$.

To this end, let $\theta \leqq \varphi \in M_n^\sim$. Then we have $(\varphi o T_L)(u_n) = \varphi(T_L(u_n)) \downarrow \varphi(h)$ in $\mathbb{R}$. But, as we shall verify later, $\varphi o T_L = (\varphi o T)_c$. So, we have $(\varphi o T_L)(u_n) = (\varphi o T)_c(u_n) \downarrow \theta$ in M, hence $\varphi(h) = \theta$ for all $\theta \leqq \varphi \in M_n^\sim$, and since ${}^o(M_n^\sim) = \{\theta\}$ it follows that $h = \theta$.

To verify that $\varphi o T_L = (\varphi o T)_L$ we proceed as follows. Assume $u \in L^+$, then we have (using the earlier remarks) $(\varphi o T_L)(u) = \varphi(T_L(u)) = \varphi(\inf\{(o)\text{-}\lim_{n\to+\infty} T(u_n) : \theta \leqq u_n \uparrow u\}) = \inf\{\varphi(\{(o)\text{-}\lim_{n\to+\infty} T(u_n) : \theta \leqq u_n \uparrow u\})\} = \inf\{(o)\text{-}\lim_{n\to+\infty} (\varphi o T)(u_n) : \theta \leqq u_n \uparrow u\} = (\varphi o T)_L(u)$.

Now the relation $\theta \leqq T_L \leqq T$ implies that $T_L = (T_L)_c \leqq T_c$. On the other hand we have $T_c \leqq T$ and from this it follows that $(T_c)_L \leqq T_L$. But from the definition of $(T_c)_L$ (similarly as for $T_L$) it follows that $(T_c)_L = T_c$, thus $T_c \leqq T_L$. Combining the above two relations we get $T_L = T_c$. ∎

COROLLARY 3.29. Let L, M and T be as in Theorem 3.27. Then we have:

(i) <u>The set of all</u> $f \in L$ <u>for which there exists a sequence</u> $\theta \leqq u_n \uparrow |f|$ <u>such that</u> $T_c(|f|) = (o)\text{-}\lim_{n\to+\infty} T(u_n) = \sup\{T(u_n) : n = 1, 2, \ldots\}$ <u>is an ideal of</u> L.

(ii) <u>The set of all</u> $f \in L$ <u>for which there exists a directed system</u> $\theta \leqq u_\alpha \uparrow |f|$ <u>such that</u> $T_n(|f|) = (o)\text{-}\lim_\alpha T(u_\alpha) : \alpha \in \{\alpha\}\}$ <u>is an ideal of</u> L.

(iii) <u>The set of all</u> $f \in L$ <u>for which there exists a directed system</u> $|f| \geqq u_\alpha \downarrow \theta$, <u>such that</u> $T_{sn}(|f|) = (o)\text{-}\lim_\alpha T(u_\alpha)$ <u>is an ideal of</u> L.

PROOF. We prove (i). The proofs of (ii) and (iii) are similar. It is obvious that the set described in (i) is a Riesz subspace of L. We have only to show that if $|g| \leqq |f|$, $f, g \in L$ such that $T_c(|f|) = (o)\text{-}\lim_{n\to+\infty} T(u_n)$, for some sequence $\theta \leqq u_n \uparrow |f|$, then $T_c(|g|) = (o)\text{-}\lim_{n\to+\infty} T(v_n)$, for some sequence $\theta \leqq v_n \uparrow |g|$.

Define $v_n = u_n \wedge |g|$, then $\theta \leqq v_n \uparrow |g|$, and $T(v_n) = T(u_n - (u_n \vee |g| - |g|)) = T(u_n) - T(u_n \vee |g| - |g|)$ for $n = 1, 2, \ldots$ . Since $\theta \leqq u_n \vee |g| - |g| \uparrow |f| \vee |g| - |g| = |f| - |g|$ it follows from Theorem 3.28 that

$$T_c(|g|) \leqq (o)\text{-}\lim_{n\to+\infty} T(v_n) = (o)\text{-}\lim_{n\to+\infty} T(u_n) - (o)\text{-}\lim_{n\to+\infty} T(u_n \vee |g| - |g|) \leqq$$

$$\leqq T_c(|f|) - T_c(|f| - |g|) = T_c(|g|) ,$$

so $(o)\text{-}\lim_{n\to+\infty} T(v_n) = T(|g|)$.■

THEOREM 3.30. <u>Let</u> L, M <u>and</u> T <u>be as in Theorem 3.28.</u> <u>Then we have:</u>

(i) <u>In the formula</u> $T_c(u) = \inf\{(o)\text{-}\lim_{n\to+\infty} T(u_n) : \theta \leqq u_n \uparrow u\}$ <u>the greatest lower bound is attained for all</u> u <u>in</u> $L^+$ <u>if and only if</u> $N_{T_s}$ <u>is super order dense in</u> L.

(ii) In the formula $T_n(u) = \inf\{(o)\text{-}\lim_\alpha T(u_\alpha) : \theta \leqq u_\alpha \uparrow u\}$ the greatest lower bound is attained for all $\theta \leqq u \in L$ if and only if $N_{T_{sn}}$ is order dense in $L(T_{sn} = T_{sn,c} + T_s)$.

PROOF. Trivial. ∎

A Riesz space L is said to have the Egoroff property if, given any $u \in L^+$ and countably many sequences $\theta \leqq u_{nk} \uparrow_k u$ for $n = 1, 2, \ldots$, there exists a sequence $\theta \leqq v_m \uparrow u$, and for every pair $(m, n)$ an index $j(m, n)$ such that $v_m \leqq u_{n, j(m,n)}$ (see [18], Chapter 10).

We have the following theorem.

THEOREM 3.31. Let L, M and T be as in Theorem 3.28. Assume further that L has the Egoroff property and that M is super Dedekind complete. Then we have: $T_c(u) = \min\{(o)\text{-}\lim_{n\to+\infty} T(u_n) : \theta \leqq u_n \uparrow u\}$ for every u in $L^+$.

PROOF: According to Theorem 3.28 we have

$$T_c(u) = \inf\{(o)\text{-}\lim_{n\to+\infty} T(u_n) : \theta \leqq u_n \uparrow u\}, \quad \text{for all } u \in L^+.$$

Since M is super Dedekind complete, there exists a double sequence $\{u_{nk}\} \subseteqq L$ such that $\theta \leqq u_{nk} \uparrow_k u$ and $((o)\text{-}\lim_{k\to+\infty} T(u_{nk})) \uparrow_n T_c(u)$.

By the Egoroff property there exists a sequence $\theta \leqq v_m \uparrow u$ such that $v_m \leqq v_{n, j(m,n)}$, for given pair $(m, n)$. Hence

$$\theta \leqq T(v_n) \leqq T(u_{n, j(m,n)}) \leqq T(u_{n,k}) \quad \text{for } k \geqq j(m,n) \quad .$$

So, $\theta \leqq T(v_n) \leqq (o)\text{-}\lim_{k\to+\infty} T(u_{nk})$ for all $n = 1, 2, \ldots$ . It follows from

Lemma 1.7 that $(o)\text{-}\lim_{n\to+\infty} T(v_n) \leqq T_c(u)$. Since the converse inequality is evident we conclude that $(o)\text{-}\lim_{n\to+\infty} T(v_n) = T_c(u)$. This completes the proof. ∎

THEOREM 3.32. Let L and M be as in Theorem 3.31, and let $T \in (\mathcal{L}_b)_s$. Then the subset $N_T = \{f \in L : |T|(|f|) = \theta\}$ is an ideal with the property that for every $f \in L$ there exists a sequence $\{f_n\} \subseteq N_T$ such that $f_n^+ \uparrow f^+$ and $f_n^- \uparrow f^-$. In particular $N_T$ is a super order dense ideal of L.

PROOF. The proof is analogous to that of [14], Cor. 20.7, Note VI, p. 664. ∎

The following theorem characterizes the inverse Riesz annihilator of $(\mathcal{L}_b)_s$ in terms of a continuity property.

THEOREM 3.33. Let L and M be as in Theorem 3.28. Let $\mathcal{L}^\alpha(L,M)$ be the subset of L consisting of all $f \in L$ such that $|f| \geqq u_n \downarrow \theta$ implies $T(u_n) \xrightarrow{(o)} \theta$ in M for all $T \in \mathcal{L}_b(L,M)$. Then $\mathcal{L}^\alpha(L,M) = {}^o(\mathcal{L}_b)_s$, and hence $\mathcal{L}^\alpha(L,M)$ is an ideal of L.

PROOF. Assume first that $f \in \mathcal{L}^\alpha(L,M)$. Let $\theta \leqq T \in \mathcal{L}_b(L,M)$. For any sequence $\theta \leqq u_n \uparrow |f|$ we have $|f| \geqq |f| - u_n \downarrow \theta$, so $T(u_n) \uparrow T(|f|)$, since $f \in \mathcal{L}^\alpha(L,M)$. It follows that

$$T_c(|f|) = \inf\{(o)\text{-}\lim_{n\to+\infty} T(u_n) : \theta \leqq u_n \uparrow |f|\} = T(|f|)$$

and hence $T_s(|f|) = \theta$. In particular, if $\theta \leqq T \in (\mathcal{L}_b)_s$, then $T = T_s$, so $T(|f|) = \theta$. This shows that $|f| \in {}^o(\mathcal{L}_b)_s$ and since ${}^o(\mathcal{L}_b)_s$ is an ideal of L by Theorem 3.23, it follows that $f \in {}^{o}(\mathcal{L}_b)_s$. Hence $\mathcal{L}^{\alpha}(L, M) \subseteqq {}^{o}(\mathcal{L}_b)_s$. Conversely, let $f \in {}^{o}(\mathcal{L}_b)_s$, so $|f| \in {}^{o}(\mathcal{L}_b)_s$. Then $T(|f|) = \theta$ for all $T \in (\mathcal{L}_b)_s$, in particular for all $\theta \leqq T \in (\mathcal{L}_b)_s$.

Now let, $|f| \geqq u_n \downarrow \theta$. If $\theta \leqq T \in \mathcal{L}_b(L, M)$, then $T = T_c + T_s$. So, it follows from $T_c(u_n) \downarrow \theta$ and $T_s(u_n) = \theta$ that $T(u_n) \downarrow \theta$. But then $T(u_n) \xrightarrow{(o)} \theta$ in M for every $T \in \mathcal{L}_b(L, M)$, so $f \in \mathcal{L}^{\alpha}(L, M)$. Hence; ${}^{o}(\mathcal{L}_b)_s \subseteqq \mathcal{L}^{\alpha}(L, M)$. This completes the proof.■

THEOREM 3.34. Let L and M be as in Theorem 2.8, and let A be an ideal of L such that $\mathcal{L}_b(A, M)$ can be identified with $(\mathcal{L}_b)_c$ in the sense that

(i) the restrictions on A of different elements of $(\mathcal{L}_b)_c$ are different, and

(ii) every $T_A \in \mathcal{L}_b(A, M)$ has an extension T onto all of L such that $T \in (\mathcal{L}_b)_c$.

Then $A \subseteqq \mathcal{L}^{\alpha}(L, M)$ and $A^o = (\mathcal{L}^{\alpha}(L, M))^o = (\mathcal{L}_b)_s$.

PROOF. If $f \in A$, and $|f| \geqq u_n \downarrow \theta$, then for $\theta \leqq T \in \mathcal{L}_b(L, M)$ we have that $T_A$ (the restriction of T on A) is positive and, since, it is an integral by hypothesis, $T(u_n) \downarrow \theta$. Hence $A \subseteqq \mathcal{L}^{\alpha}(L, M)$. From this it follows easily that $(\mathcal{L}_b)_s \subseteqq [{}^{o}(\mathcal{L}_b)_s]^o = (\mathcal{L}^{\alpha}(L, M))^o \subseteqq A^o$. Using (i) and (ii) we see that $\theta \leqq T \in \mathcal{L}_b(L, M)$ and $T = \theta$ on A, imply $T = T_s$; hence $A^o \subseteqq (\mathcal{L}_b)_s$.

So, $(\mathcal{L}^{\alpha}(L, M))^o = (\mathcal{L}_b)_s = A^o$. ■

# CHAPTER 4

# ORDERED TOPOLOGICAL VECTOR SPACES AND TOPOLOGICAL RIESZ SPACES

## 4.1. THE BASIC THEORY

We recall that an ordered vector space is a real vector space E with an order relation $\leqq$ satisfying the two conditions

(i) $f \leqq g$ implies $f+h \leqq g+h$ for every $h \in E$

(ii) $\theta \leqq f$ implies $\theta \leqq \alpha f$ for every real $\alpha \geqq 0$.

The subset $K = \{f \in L : f \geqq \theta\}$ is called the (positive) cone of E with respect to the ordering $\leqq$. The cone K has the following three properties:

($\alpha$) $K + K \subseteqq K$

($\beta$) $\alpha K \subseteqq K$ for every real $\alpha \geqq 0$

($\gamma$) $K \cap - K = \{\theta\}$.

The above three properties characterize the cone K as well as the ordering $\leqq$. If a subset K of a real vector space E satisfies the properties ($\alpha$), ($\beta$) and ($\gamma$) then the relation $f \leqq g$, if $g - f \in K$ is an ordering for E satisfying the conditions (i) and (ii). The pair (E, K) where K is a cone of E will indicate an ordered vector space E with ordering induced by K.

Directed upwards and downwards systems in E are defined exactly as in the case of the Riesz spaces.

The ordered vector space (E, K) is called Archimedean if the relation $nf \leqq g$ for all $n \in N$ and some f, g in E implies $f \leqq \theta$. Note that if (E, K) is a Riesz space this definition is equivalent to that of Archimedean Riesz spaces. The element $e \in K$ is called an order (strong) unit of E, if for every element $f \in E$ there exists $\lambda \geqq 0$ (depending on f) such that $f \leqq \lambda e$. (E, K) is called σ-Dedekind complete, if $\theta \leqq f_n \uparrow \leqq f_o$ implies $f_n \uparrow f$ for some $f \in E$. Similarly (E, K) is called Dedekind complete if $\theta \leqq f_\alpha \uparrow \leqq f_o$ implies $f_\alpha \uparrow f$ for some $f \in E$. Finally, (E, K) is called super Dedekind complete, if it is Dedekind complete and $f_\alpha \uparrow f$ implies $f_{\alpha_n} \uparrow f$ for some sequence $\{f_{\alpha_n}\} \subseteqq \{f_\alpha\}$.

The cone K is called generating if $E = K - K$. If K is generating and (E, K) is σ-Dedekind complete then (E, K) is a Riesz space.

The triple $(E, \tau, K)$, where K is a cone of E, $\tau$ is a linear topology (not necessarily Hausdorff), i.e., a topology $\tau$ of E such that the operations $(f, g) \to f + g$ from $(E, \tau) \times (E, \tau)$ into $(E, \tau)$ and $(\lambda, f) \to \lambda f$ from $\mathbb{R} \times (E, \tau)$ into $(E, \tau)$, $\mathbb{R}$ with its usual topology, are both continuous, is called an ordered topological vector space.

The cone K of the ordered topological vector space $(E, \tau, K)$ is called $\tau$-normal, if there exists a neighborhood basis for the $\tau$-neighborhoods of zero consisting of full sets. A set $V \subseteqq E$ is defined to be full, if for every pair f, g in V such that $f \leqq g$ we have that $[f, g] = \{h \in E : f \leqq h \leqq g\} \subseteqq V$.

A characterization of the normal cones is given in the next theorem.

THEOREM 4.1. *Let* $(E, \tau, K)$ *be an ordered topological vector space. Then the following statements are equivalent.*

(i) K *is a* $\tau$*-normal cone.*

(ii) *For every two nets* $\{f_\alpha\}$, $\{g_\alpha\}$ *of* E *such that* $\theta \leqq f_\alpha \leqq g_\alpha$ *for all* $\alpha$ *and* $g_\alpha \xrightarrow{\tau} \theta$, *it follows that* $f_\alpha \xrightarrow{\tau} \theta$.

For a proof of Theorem 4. 1 see [23], p. 62.

The next theorem gives some information for the $\tau$-closure of K.

THEOREM 4. 2. Let $(E, \tau, K)$ be a Hausdorff ordered topological vector space with the cone K $\tau$-normal. Then the $\tau$-closure of K is a $\tau$-normal cone.

PROOF. We prove first that $\overline{K}$ is a cone. From the continuity of addition and multiplication it follows that $\overline{K} + \overline{K} \subseteqq \overline{K}$ and $\alpha\overline{K} \subseteqq \overline{K}$ for all $\alpha \geqq 0$, so, $\overline{K}$ will be a cone if $\overline{K} \cap -\overline{K} = \{\theta\}$.

To prove this, let $f \in \overline{K} \cap -\overline{K}$. Then there are two nets $\{f_\alpha\}$, $\{g_\alpha\}$ of K such that $f_\alpha \xrightarrow{\tau} f$ and $g_\alpha \xrightarrow{\tau} -f$. It follows then that $\theta \leqq f_\alpha \leqq f_\alpha + g_\alpha \xrightarrow{\tau} f - f = \theta$.

Hence from Theorem 4. 1, we get that $f_\alpha \xrightarrow{\tau} \theta$. Since $\tau$ is a Hausdorff topology we have also that $f = \theta$.

Now let $\ll$ denote the ordering induced on E by $\overline{K}$, and assume that $\theta \ll f_\alpha \ll g_\alpha \xrightarrow{\tau} \theta$.

According to Theorem 4. 1, $\overline{K}$ will be a normal cone if we show that $f_\alpha \xrightarrow{\tau} \theta$. Since $f_\alpha \in \overline{K}$ for all $\alpha$, we have $(f_\alpha + V) \cap K \neq \phi$ for all $V \in U_\theta$ = the set of all $\tau$-neighborhoods of zero, So, for each $(\alpha, V)$ there exists an element $h(\alpha, v) \in V$ such that

$$\theta \leqq f_\alpha + h(\alpha, V) \quad .$$

Similarly, $g_\alpha - f_\alpha \in \overline{K}$ for all $\alpha$, implies that for every $(\alpha, V)$ there exists an element $U(\alpha, V) \in V$ such that $\theta \leqq g_\alpha - f_\alpha + U(\alpha, V)$.

So, we have

$$\theta \leqq f_\alpha + h(\alpha, V) \leqq g_\alpha + h(\alpha, V) + U(\alpha, V) \xrightarrow[(\alpha, V)]{\tau} \theta \quad .$$

It follows, now, from Theorem 4.1 that $f_\alpha + h(\alpha, V) \xrightarrow{\tau} \theta$, from which we obtain $f_\alpha \xrightarrow{\tau} \theta$, by observing that $h(\alpha, V) \xrightarrow{\tau} \theta$. ■

Note. Theorem 4.2 is a generalization of a known theorem for Hausdorff locally convex ordered spaces (see [23], p. 63). A different proof of Theorem 4.2 was also given by M. Duhoux (see [2], p. 4, Theorem 1.1).

Normal cones have the property that the order bounded subsets of E are also topologically bounded. We have the following theorem.

THEOREM 4.3. <u>If the cone</u> K <u>of an ordered topological vector space</u> $(E, \tau, K)$ <u>is</u> $\tau$-<u>normal, then every order bounded subset of</u> E <u>is</u> $\tau$-<u>bounded</u>.

For a proof see [23], p. 62.

As a consequence we have that if K is a $\tau$-normal cone, then every $\tau$-continuous linear functional is order bounded.

The order vector space (E, K) is called a Riesz space if the least upper bound of any two elements of E exists in E.

A general discussion of the theory of Riesz spaces is given in the beginning of the present work.

If L is a Riesz space (more precisely the pair $(L, L^+)$) and if $\tau$ is a linear topology of L (not necessarily Hausdorff), i.e., a topology $\tau$ for which the algebraic operations are continuous, then $(L, \tau)$ is called a topological Riesz space.

If there exists, in addition, a basis for the $\tau$-neighborhoods of zero consisting of solid sets, (a subset of V of a Riesz space L is called a solid, if $|f| \leqq |g|$ and $g \in V$ implies $f \in V$), then the topological Riesz space $(L, \tau)$ is called a locally solid Riesz space.

Given a topological Riesz space $(L, \tau)$, the following five mappings are called the lattice operations of L.

(i) $(f, g) \to f \wedge g$, from $(L, \tau) \times (L, \tau)$ into $(L, \tau)$

(ii) $(f, g) \to f \vee g$, from $(L, \tau) \times (L, \tau)$ into $(L, \tau)$

(iii) $f \to f^+$, from $(L, \tau)$ into $(L, \tau)$

(iv) $f \to f^-$, from $(L, \tau)$ into $(L, \tau)$

(v) $f \to |f|$, from $(L, \tau)$ into $(L, \tau)$.

It is not difficult to verify that continuity of one of them implies continuity of the other four.

A characterization of the locally solid Riesz spaces is given by the following theorem.

THEOREM 4.4. *Let* L *be a Riesz space and let* $\tau$ *be a linear topology of* L, *i.e., let* $(L, \tau)$ *be a topological Riesz space. Then the following statements are equivalent.*

(i) $(L, \tau)$ *is a locally solid Riesz space.*

(ii) $L^+$ *is a* $\tau$*-normal cone and the lattice operations are continuous.*

For a proof see [23], page 104, Proposition 4.7.

## 4.2. THE PROPERTIES (A, o), (A, i), (A, ii), (A, iii), AND (A, iv)

Following W. A. J. Luxemburg and A. C. Zaanen ([16], Note X) we introduce the following conditions for an ordered topological vector space $(E, \tau, K)$:

(A, o) : $u_n \downarrow \theta$ and $\{u_n\}$ is a $\tau$-Cauchy sequence implies $u_n \xrightarrow{\tau} \theta$,

(A, i) : $u_n \downarrow \theta$ implies $u_n \xrightarrow{\tau} \theta$,

(A, ii) : $u_\alpha \downarrow \theta$ implies $u_\alpha \xrightarrow{\tau} \theta$,

(A, iii): $\theta \leqq u_n \uparrow \leqq u_o$ implies that $\{u_n\}$ is a $\tau$-Cauchy sequence, i.e., every order bounded increasing sequence in $(E, \tau)$ is a $\tau$-Cauchy sequence,

(A, iv): $\theta \leqq u_\alpha \uparrow \leqq u_o$ implies that $\{u_\alpha\}$ is a $\tau$-Cauchy net, i.e., every order bounded set in $(E, \tau)$ which is directed upwards is a $\tau$-Cauchy net.

Obviously (A, ii) implies (A, i), (A, i) implies (A, o) and (A, iv) implies (A, iii). The following examples show that many other implications do not hold in general, not even in locally solid Riesz spaces.

EXAMPLE 4.5. (i) Let L be the Riesz space of all real valued functions defined on an uncountable set X and such that for every $f \in L$ there exists a real number $f(\infty)$ such that given any $\epsilon > 0$, we have $|f(x) - f(\infty)| > \epsilon$ for finitely many x. In other words, $L = C(X_\infty)$, where $X_\infty$ is the one point compactification of the set X considered with the discrete topology. Let $\tau$ be the locally solid topology generated by the norm $\rho(f) = \sup\{|f(x)| : x \in X\}$.

Then $(L, \tau)$ satisfies (A, i), but not (A, ii) and (A, iii).

(ii) Let L be the Riesz space of all real continuous functions defined on [0, 1], i.e., $L = C_{[0,1]}$ and let $\tau$ be the locally solid topology generated by the norm $\rho(f) = \int_0^1 |f(x)|\, dx$. Then (A, iii) holds but (A, i) does not. (See [18], Exercise 18.14, p. 104.)

(iii) Let $0 < \rho < 1$ and let L be the real vector space of all real valued Lebsegue measurable functions defined on $[0, 1]$, such that $\int_0^1 |f(t)|^\rho \, dt < +\infty$.

Then L becomes a Riesz space under the ordering $f \leqq g$ whenever $f(x) \leqq g(x)$ for all $x \in [0, 1]$.

Given $n \in N$, $\delta > 0$ and $F = \{x_1, \ldots, x_n\} \subseteqq [0, 1]$ we define the set

$$W_{F, n, \delta} = \{f \in L: \int_0^1 |f(t)|^\rho \, dt < \frac{1}{n} \text{ and } |f(x_i)| < \delta \text{ for } i = 1, \ldots, n\}$$

For F varying over $\mathcal{F}([0, 1])$, n over N and $\delta$ over $(0, +\infty)$ we get a family of sets $\{W_{F, n, \delta}\}$ which is a filter basis for a neighborhood system of the origin for a uniquely determined linear topology $\tau$ of L (see [7], page 81).

Obviously each $W_{F, n, \delta}$ is a solid set. So, $(L, \tau)$ is a locally solid Riesz space.

We can verify easily the following properties:

(1) $\tau$ is a sequentially complete (but not complete) Hausdorff non-metrizable locally solid topology for L.

(2) (A, i) and (A, iii) hold, but (A, ii) does not hold. To see (2), consider the net $\{f_\alpha\}$,

$$f_\alpha(x) = \begin{cases} 0 & \text{if } x \in \alpha \\ 1 & \text{if } x \notin \alpha \end{cases}, \quad \alpha \in \mathcal{F}([0, 1]), \quad \text{for} \quad (A, ii).$$

The property (A, i) follows from Lebsegue's dominated convergence theorem and from the fact that $f_\alpha \downarrow \theta$ in L implies $f_\alpha(x) \downarrow 0$ in $\mathbb{R}$ for all $x \in [0, 1]$. Property (A, iii) is easily verified.

(3) L is $\sigma$-Dedekind complete but not Dedekind complete.

(iv) The product of the spaces in (i) and (ii) gives a locally solid Riesz space without (A, i), (A, ii) and (A, iii) (with respect to the product topology).

(v) Let L be the real vector space of all sequences which are eventually zero, i.e., f is in L if $f(k) = 0$ for all $k \geqq n_o$ for some $n_o$ (depending on f). The cone K consists of the zero element and of all sequences of L such that their last non-vanishing component is strictly positive. Then (L, K) is a non-Archimedean Riesz space.

Let $\tau$ be the linear Hausdorff topology of L generated by the norm

$$||f|| = \sum_{n=1}^{\infty} |f(n)| \quad .$$

Then the topological Riesz space $(L, \tau)$ satisfies condition (A, ii) but not (A, iii).

Note that $\tau$ is not locally solid and also that K is not $\tau$-normal. This also shows that (A, ii) does not imply normality of the cone. ■

Note. The Riesz space L of the example (v) has many interesting properties. We list some of them.

(i) The order of L is a linear order, i.e., for given f, g in L, at least one of the relations $f \leqq g$ and $g \leqq f$ is valid in L.

(ii) L is non-Archimedean (as already was mentioned).

(iii) L is relatively uniformly complete.

(iv) L has the Egoroff property.

(v) $L^{\sim} = \{\theta\}$.

(vi) $L^+$ is $\tau$-dense in L, where $\tau$ is the linear topology defined above.

We show next that (A, iii) and (A, iv) are equivalent.

LEMMA 4.6. <u>Let</u> $(E, \tau, K)$ <u>be an ordered topological vector space. Then the following statements are equivalent:</u>

(i) $(E, \tau, K)$ <u>satisfies condition</u> (A, iii).

(ii) $(E, \tau, K)$ <u>satisfies condition</u> (A, iv).

PROOF. Obviously (ii) implies (i). We show that (i) implies (ii). To this end let $\theta \leqq u_\alpha \uparrow \leqq u_o$ and assume that $\{u_\alpha\}$ is not a $\tau$-Cauchy net. This means that there exists a neighborhood U of zero such that for every $\alpha \in \{\alpha\}$ there exist $\alpha_1, \alpha_2 \geqq \alpha$ with $u_{\alpha_1} - u_{\alpha_2} \notin U$. Let now V be a circled neighborhood of zero such that $V+V \subseteqq U$. We claim that for every $\alpha \in \{\alpha\}$, there are two indices $\alpha_1$, $\alpha_2$ such that $\alpha_1 \geqq \alpha_2 \geqq \alpha$ and satisfying $u_{\alpha_1} - u_{\alpha_2} \notin V$.

Indeed, if this is not the case we must have $u_{\alpha_1} - u_{\alpha_2} \in V$ for every pair of indices $\alpha_1$, $\alpha_2$ such that $\alpha_1 \geqq \alpha_2 \geqq \alpha$. Now, for the given $\beta_1, \beta_2 \geqq \alpha$ we pick an index $\alpha_1$ such that $\alpha_1 \geqq \beta_1 \geqq \alpha$ and $\alpha_1 \geqq \beta_2 \geqq \alpha$. Then we have, $u_{\beta_1} - u_{\beta_2} = (u_{\beta_1} - u_{\alpha_1}) - (u_{\beta_2} - u_{\alpha_2}) \in V + V \subseteqq U$, a contradiction. This establishes the assertion.

Let $\alpha_1$ be an index. We pick two others $\alpha_2$, $\alpha_3$ such that $\alpha_3 \geqq \alpha_2 \geqq \alpha_1$ and $u_{\alpha_3} - u_{\alpha_2} \notin V$.

Let also $\alpha_4$, $\alpha_5$ be such that $\alpha_5 \geqq \alpha_4 \geqq \alpha_3$ and $u_{\alpha_5} - u_{\alpha_4} \notin V$, and so on.

The sequence $\{u_{\alpha_n}\}$ satisfies $\theta \leqq u_{\alpha_n} \uparrow \leqq u_o$ and it is not a $\tau$-Cauchy sequence, a contradiction.

Hence, (A, iii) implies (A, iv). ∎

Note. It is not difficult to verify that (A, iii) is also equivalent with the following statement: Every sequence $\{f_n\} \subseteqq E$ such that $\theta \leqq f_n \downarrow$ in E, is a $\tau$-Cauchy sequence.

## 4.3. TWO BASIC THEOREMS CONCERNING TOPOLOGICAL AND ORDER PROPERTIES

We begin with the following.

THEOREM 4.7. Let $(E, \tau, K)$ be an ordered topological vector space with K $\tau$-closed. Then we have

(i) E is an Archimedean ordered topological vector space.

(ii) If $f_\alpha \uparrow$ and $f_\alpha \xrightarrow{\tau} f$, then $f_\alpha \uparrow f$ in E.

Similarly for decreasing nets.

PROOF. First we note that $\tau$ is a Hausdorff topology since K is a $\tau$-closed cone.

(i) Let $nf \leqq g$, for $n = 1, 2, \ldots$ . Then we have $\theta \leqq \frac{1}{n} g - f \xrightarrow{\tau} -f$. Since K is $\tau$-closed we have $-f \in K$, or $f \leqq \theta$.

(ii) For fixed $\alpha \in \{\alpha\}$ we have $\theta \leqq f_\beta - f_\alpha$ for all $\beta \geqq \alpha$, so, since $\theta \leqq f_\beta - f_\alpha \xrightarrow[\beta \geq \alpha]{\tau} f - f_\alpha$, we see that $f - f_\alpha \geqq \theta$ for all $\alpha \in \{\alpha\}$, i.e., f is an upper bound for the net $\{f_\alpha\}$. Let now $f_\alpha \leqq g$ for all $\alpha$. Then we have $\theta \leqq g - f_\alpha \xrightarrow{\tau} g - f$, so using once more that K is $\tau$-closed we obtain $g - f \in K$, or $f \leqq g$. Hence $f_\alpha \uparrow f$. ∎

THEOREM 4.8. Every band in a Hausdorff locally solid Riesz space $(L, \tau)$ is $\tau$-closed.

PROOF. Let D be a non-empty subset of L and let $f \in \overline{D^d}$. Then there exists a net $\{f_\alpha\} \subseteq D^d$ such that $f_\alpha \xrightarrow{\tau} f$. Now if $g \in D$ it follows from Theorem 4.4 that $\theta = |f_\alpha| \wedge |g| \xrightarrow{\tau} |f| \wedge |g|$ and hence (since $\tau$ is a Hausdorff topology) $|f| \wedge |g| = \theta$, i.e., $f \in D^d$. This shows that $D^d$ is a $\tau$-closed band of L. Since every band A of L satisfies $A = A^{dd}$ (see Theorem 1.7(ii)) it follows easily that every band of L is $\tau$-closed. ∎

We recall that a Riesz space L has a countable order basis if there exists an at most countable subset of L such that the band generated by this subset is the whole space. The next corollary gives a sufficient condition for the existence of a countable order basis for Hausdorff locally solid Riesz spaces.

COROLLARY 4.9. If a Hausdorff locally solid Riesz space $(L, \tau)$ is separable then L has a countable order basis.

PROOF. Assume that $(L, \tau)$ is separable, and let $\{f_n : n = 1,2,\ldots\}$ be a $\tau$-dense subset of L. Let A be the band generated by the system $\{f_n\}$, then A is $\tau$-closed according to the previous theorem. It follows then that $A = L$. Hence A has a countable order basis. ∎

THEOREM 4.10. Let $(L, \tau)$ be a metrizable locally solid Riesz space. Then every $\sigma$-ideal is $\tau$-closed.

PROOF. Let $f_n \xrightarrow{\tau} f$ and $\{f_n\} \subseteq A$ where A is a $\sigma$-ideal of L. Since $f_n^+ \xrightarrow{\tau} f^+$ (Theorem 4.4), without loss of generality we can assume that $\{f_n\} \subseteq A \cap L^+$, $f \in A \cap L^+$.

We define the sequence

$$g_n = \left(\bigvee_{i=1}^{n} f_i\right) \wedge f \ , \qquad \text{for } n = 1,2,\ldots \ .$$

Then we have $\theta \leqq g_n \uparrow$, $\{g_n\} \subseteq A$ and $\theta \leqq f-g_n \leqq f-f_n \wedge f \leqq |f-f_n|$ for $n = 1, 2, \ldots$ . So $g_n \xrightarrow{\tau} f$.

It follows from Theorem 4.7 that $\theta \leqq g_n \uparrow f$. Since A is a $\sigma$-ideal of L it follows that $f \in A$, so $A = \overline{A}$, i.e., A is $\tau$-closed.■

The following example shows that the Hausdorff property of the topology $\tau$ cannot be removed from Theorems 4.7 and 4.8.

EXAMPLE 4.11. (i) Let L be the plane with the lexicographic order, i.e., let $L = \mathbb{R}^2$ with ordering $f = (f_1, f_2) \geqq g = (g_1, g_2)$, whenever $f_1 > g_1$ or $f_1 = g_1$ and $f_2 \geqq g_2$. Then L is a non-Archimedean Riesz space. Consider now the semi-norm $\rho$ of L defined by $\rho(f) = |f_1|$, $f = (f_1, f_2) \in L$. It is easily verified that $\theta \leqq f \leqq g$ implies $\rho(f) \leqq \rho(g)$ and that $\rho(f) = \rho(|f|)$ for all $f \in L$. So, $\rho$ defines a non-Hausdorff locally solid topology $\tau$ on L, i.e., $(L, \tau)$ is a locally solid Riesz space which, as mentioned earlier, is non-Archimedean. Consider now the sequence $f_n = (0, n)$, $n = 1, 2, \ldots$ of L. Then we have $\theta \leqq f_n \uparrow$ in L and $\rho(f_n) = 0 \to 0$, so, $f_n \xrightarrow{\tau} \theta$ in $(L, \tau)$, but the $\sup\{f_n : n \in N\}$ does not exist in L.
(ii) Consider $L = C_{[0,1]}$ with the usual order and define the Riesz semi-norm $\rho$, by $\rho(f) = \sup\{|f(x)| : x \in [0, \frac{1}{2}]\}$ for all $f \in L$. Then $\rho$ defines a locally solid topology $\tau$ for the Archimedean Riesz space L. Note that the band $B = \{f \in L : f(x) = 0 \text{ for all } x \in [\frac{1}{2}, 1]\}$ is not $\tau$-closed.■

## 4.4. SOME CHARACTERIZATIONS OF (A, i) AND (A, ii) PROPERTIES FOR LOCALLY SOLID RIESZ SPACES

The next theorem gives a characterization of the (A, i) property for locally solid Riesz spaces.

THEOREM 4.12. Let $(L, \tau)$ be a locally solid Riesz space. Then the following statements are equivalent.

(i) $(L, \tau)$ satisfies the condition (A, i).

(ii) Every $\tau$-closed ideal is a $\sigma$-ideal.

PROOF. (i)$\Rightarrow$(ii). Trivial.

(ii)$\Rightarrow$(i). Let $\theta \leqq u_n \uparrow u$ in L and let U be a solid $\tau$-neighborhood of $\theta$. We choose a solid $\tau$-neighborhood V of zero such that $V+V \subseteqq U$. Let $\alpha$ be such that $0 < \alpha < 1$ and $(1-\alpha) u \in V$.

The conclusion now follows by a way completely analogous to that of Theorem 47.3 of [15], Note XIV, p. 244. ∎

COROLLARY 4.13. Let $(L, \tau)$ be a metrizable locally solid Riesz space satisfying condition (A, i). Then an ideal A of L is $\tau$-closed iff A is a $\sigma$-ideal of L.

PROOF. This follows immediately from the above Theorem and Theorem 4.10. ∎

The next theorem generalizes a result of T. Ando and W. A. J. Luxemburg (see [15], Note XIV, p. 244), and can be proved as Theorem 4.12.

THEOREM 4.14. Let $(L, \tau)$ be a locally solid Riesz space. Then $(L, \tau)$ satisfies (A, ii) iff every $\tau$-closed ideal of L is a band.

COROLLARY 4.15. Let $(L, \tau)$ be a Hausdorff locally solid Riesz space satisfying condition (A, ii). Then an ideal A of L is $\tau$-closed if and only if A is a band of L.

PROOF. This follows immediately from the above Theorem and Theorem 4.8. ■

For Hausdorff locally convex Riesz spaces which are also locally solid there exists another characterization of the (A, ii) property. The next theorem can be found in [24], page 54.

THEOREM 4.16. *Let* $(L, \tau)$ *be a Hausdorff locally convex, locally solid Riesz space. Then the following statements are equivalent.*

(i) $(L, \tau)$ *satisfies the* (A, ii) *property.*

(ii) $(L, \tau)' \subseteq L_n^{\sim}$, *i.e., the topological dual of* $(L, \tau)$ *consists only of normal integrals.*

For locally convex, locally solid Hausdorff Riesz spaces we have also the following theorem.

THEOREM 4.17. *Let* $(L, \tau)$ *be an Archimedean locally convex, locally solid Riesz space. Then the following conditions are equivalent.*

(i) *Every* $\tau$-*closed ideal of* $L$ *is a band.*

(ii) *For every* $0 \leq \varphi \in L'$ *the null ideal* $N_\varphi$ *is a band.*

(iii) *Every order dense ideal in* $L$ *is* $\tau$-*dense.*

*In particular, it follows from Theorem 4.14 that all the above conditions are equivalent to the* (A, ii) *property.*

PROOF. The proof is analogous to that of Theorem 35.6 of [16], Note X, p. 513, and so we omit the details. ■

THEOREM 4.18. Let $(L, \tau)$ be a Hausdorff locally solid Riesz space satisfying condition (A, ii), and let M be a super Dedekind complete Riesz space. Assume further that $T \in (\mathcal{L}_b(L, M))_{sn, c}$. Then $T = \theta$, if and only if, $N_T$ is a $\tau$-closed ideal.

PROOF. Obviously if $T = \theta$, then $N_T = L$, so $N_T$ is a $\tau$-closed ideal. Let now $T \in (\mathcal{L}_b)_{sn, c}$ with $N_T$ $\tau$-closed. By Theorem 4.7(i), L is Archimedean and therefore $N_T$ is order dense in L. Using Theorem 4.14 we see that $N_T$ is also a band in L.

So, altogether we have that $L = \{N_T\} = N_T^{dd} = N_T$, i.e., $T = \theta$. ∎

## 4.5. THE EGOROFF PROPERTY

We recall that a Riesz space L has the Egoroff property if $\theta \leqq u_{nk} \uparrow_k u$, $n = 1, 2, \ldots$ implies the existence of a sequence $\{v_n\}$ such that $\theta \leqq v_n \uparrow u$, and for each pair $(m, n)$ there exists an index $j(m, n)$ such that $v_m \leqq u_{n, j(m,n)}$.

THEOREM 4.19. Let L be a Riesz space and assume that L admits a metrizable locally solid topology $\tau$, which also satisfies condition (A, ii).

Then L has the Egoroff property.

PROOF. Let $\{W_n : n \in N\}$ be a countable basis for the origin for the topology $\tau$ consisting of solid sets and such that

$$W_{n+1} + W_{n+1} \subseteqq W_n, \quad \text{for } n = 1, 2, \ldots .$$

Assume that $\theta \leqq u \in L$ and $\theta \leqq u_{nk} \uparrow_k u$ for $n = 1, 2, \ldots$ . For every pair of indices $(m, n)$ we determine an index $j(m, n)$ such that $u - u_{n, j(m,n)} \in W_{m+n}$. This can be done since $u - u_{n,k} \xrightarrow[(k)]{\tau} \theta$ according to the hypothesis. Evidently we may assume that $j(m, n)$ increases as $m$ increases. For $m$ fixed and $\alpha = \{n_1, \ldots, n_p\} \in \mathcal{F}(N)$, let

$$u_\alpha = \bigwedge_{i=1}^{p} u_{n_i, j(m, n_i)} \quad .$$

Then we have

$$\theta \leqq u - u_\alpha = u - \bigwedge_{i=1}^{p} u_{n_i, j(m,n_i)} = \bigvee_{i=1}^{p} (u - u_{n_i, j(m,n_i)})$$

$$\leqq \sum_{i=1}^{p} (u - u_{n_i, j(m,n_i)}) \in W_{m+n_1} + \ldots + W_{m+n_p} \subseteqq W_m \quad .$$

Since $W_m$ is a solid set we have

$$u - u_\alpha \in W_m, \quad \text{for all } \alpha \in \mathcal{F}(N) \quad . \tag{1}$$

Obviously the set $\{u_\alpha\}$ is directed downwards, and writing $V = \{v \in L : v \leqq u_\alpha \text{ for all } \alpha\}$ we have $u_\alpha - v \underset{(\alpha, v)}{\downarrow} \theta$, according to Theorems 1.4(iii) and 4.7(i). It follows from property (A, ii) that

$$u_\alpha - v \xrightarrow[(\alpha, v)]{\tau} \theta \quad . \tag{2}$$

Combining (1) and (2) we obtain the existence of an element $\theta \leqq w_m \in V$ such that $\theta \leqq u - w_m \in W_m + W_m \subseteqq W_{m-1}$. Now let

$$Z_\ell = \bigvee_{i=1}^{\ell} w_i \quad \text{for} \quad \ell = 1, 2, \ldots \quad .$$

Since $w_m \leqq u_{n, j(m,n)} \leqq u_{n, j(\ell, n)}$ for all n and all m = 1, . . . , $\ell$ we have $Z_\ell \leqq u_{n, j(\ell, n)}$ for all n. Furthermore $\theta \leqq Z_\ell \uparrow \leqq u$ and $\theta \leqq u - Z_\ell \leqq u - w_\ell \in W_{\ell-1}$, so $u - Z_\ell \in W_{\ell-1}$ and therefore $Z_\ell \xrightarrow{\tau} u$. It follows from Theorem 4.7(i) that $\theta \leqq Z_\ell \uparrow u$.

So the sequence $\{Z_\ell\}$ satisfies all the conditions required for the Egoroff property and this completes the proof.■

Note. The same proof works if the topology $\tau$ is not locally solid and metrizable but it is metrizable with $L^+$ $\tau$-closed and $\tau$-normal.

APPLICATION 4.20. Let $L = L_p([0,1])$, $0 < p < 1$ and let $\tau$ be the metrizable locally solid topology generated by the neighborhoods $W_\epsilon = \{f \in L : \int_0^1 |f(x)|^p dx \leqq \epsilon\}$.

It is easy to verify that L satisfies condition (A, ii) (using the fact that $L_p$ is super Dedekind complete, see [18], p. 126, Ex. 23.3(iv)).

Thus $L_p$ satisfies the Egoroff property.■

THEOREM 4.21. Let L be a Riesz space and let $(M, \tau)$ be a metrizable locally solid Riesz space satisfying condition (A, ii), which is also super Dedekind complete. If there exists a strictly positive integral from L into M, then L has the Egoroff property.

PROOF. Using the technique of Theorem 4.19 the proof can follow the same pattern as in Theorem 33.11(ii), of [16], Note X, p. 493. We omit the details.■

Note. By Theorem 4.19 M has also the Egoroff property.

The following example shows that strict positivity of T is essential in Theorem 4.21.

EXAMPLE 4.22. Let X be an uncountable set such that $L = \mathbb{R}^X$ does not have the Egoroff property (see [18], Ex. 67.6(v), p. 465 or Theorem 75.3, p. 511), and let $M = \mathbb{R}$ with its usual topology.

Assume that $x_0 \in X$ is a fixed point, and let $\pi: L \to \mathbb{R}$ be defined by $\pi(f) = f(x_0)$, for all $f \in L$.

It is easy to show that $\pi$ is a normal Riesz homomorphism from L onto M. Note in this case that $\pi$ is positive but not strictly positive. ∎

It is known that not every Riesz subspace of a Riesz space with the Egoroff property has the Egoroff property. For an example consider the space $L_1([0,1])$, i.e., the Riesz space of all equivalent classes of the Lebesgue integrable functions of [0, 1]. Then $L_1([0,1])$ has the Egoroff property but the Riesz subspace $C_{[0,1]}$ does not. (See [18], page 464, Example 67.6(iii)).

The next example gives an interesting application of Theorem 4.21.

APPLICATION 4.23. Let $(X, \Sigma, \mu)$ be a measure space with $\mu(X) < +\infty$. Assume $0 < p < +\infty$ and that L is a Riesz subspace of $L_p$ which is closed with respect to the order topology. Then L has the Egoroff property.

To see this pick q such that $0 < q < 1$ and $0 < q < p$. Since $\mu(X) < +\infty$ we have $L_p \subseteqq L_q$ (see [6], p. 196, Theorem 13.17).

Consider now the identity mapping $T: L \to L_q$, i.e., $T(f) = f$ for all $f \in L$. We show that $u_n \downarrow \theta$ in L implies $u_n \downarrow \theta$ in $L_q$. Indeed, if

$u_n \downarrow f \geqq \theta$ in $L_q$, then $u_n \geqq u_n \wedge k \downarrow f \wedge k$ in $L_p$. This shows (using our hypothesis) that $f \wedge k \in L$ ($k(x) = k$, for all $x \in X$), so $f \wedge k = \theta$ for all $k \in N$. It follows now that $f = \theta$.

The result now follows immediately applying Theorem 4.21.∎

The next theorem gives one more sufficient condition for a Riesz space L to have the Egoroff property.

THEOREM 4.24. Let L be a Riesz space containing an ideal A with $A' = L$, i.e., the pseudo order closure of A is L. Suppose further that A, considered as a Riesz space, has the Egoroff property.

Then L has the Egoroff property.

PROOF. We first show that for every $f \in L^+$, there exists a sequence $\{f_n\}$ of A such that $\theta \leqq f_n \uparrow f$. To this end, let $f \in L^+$. Since $A' = L$ we have $f_n \xrightarrow{(o)} f$ for some sequence $\{f_n\}$ of A. It follows now that $f_n^+ \xrightarrow{(o)} f$ and also that $g_n = \left( \bigvee_{k=1}^{n} f_n^+ \right) \wedge f \xrightarrow{(o)} f$, $\{g_n\} \subseteqq A^+$, so $\theta \leqq g_n \uparrow f$. In other words $A' = L$ if and only if $A_\sigma = L$ ($A_\sigma$ = the $\sigma$-ideal generated by A in L).

Assume now that $\theta \leqq u_{nk} \uparrow_k u$ in L for $n = 1, 2, \ldots$ . Let $\{g_r\}$ be a sequence of A such that $\theta \leqq g_r \uparrow u$. But then for fixed n and r we have $\theta \leqq u_{nk} \wedge g_r \uparrow_k g_r$ in A. From the Egoroff property of A we obtain a sequence $\{Z^r_\ell : \ell \in N\} \subseteqq A$ such that $\theta \leqq Z^r_\ell \uparrow_\ell g_r$ and for each pair of indices (m, n) there exists an index $j_r(m, n)$ such that $Z^r_m \leqq u_{n, j_r(m,n)} \wedge g_r \leqq u_{n, j_r(m,n)}$. Let $u_k = \bigvee_{\ell=1}^{k} \bigvee_{r=1}^{k} Z^r_\ell$, for $k = 1, 2, \ldots$ . Then it follows easily that $\theta \leqq u_k \uparrow u$ and, given a pair of indices (m, n), we determine $j_r(m, n)$ such that $Z^r_m \leqq u_{n, j_r(m,n)}$.

So, if $j(m,n) = \max\{j_r(m,n) : r = 1,\ldots,m\}$, then we have $Z^r_m \leqq u_{n,j(m,n)}$ for $r = 1,\ldots,m$ and so,

$$u_m = \bigvee_{\ell=1}^{m} \bigvee_{r=1}^{m} Z^r_\ell \leqq \bigvee_{r=1}^{m} Z^r_m \leqq u_{n,j(m,n)} .$$

This shows that the sequence $\{u_m\}$ satisfies all the properties required for the Egoroff property of L and this completes the proof.∎

COROLLARY 4.25. *Let L be a Riesz space and A an ideal of L. Then A has the Egoroff property.*

PROOF. The proof follows from the above theorem and the known result that A' is a Riesz subspace of L (actually an ideal), (see [18], p. 242, Theorem 63.1).∎

COROLLARY 4.26. *If A is an order dense ideal of a Riesz space L with the Egoroff property and if L has the diagonal gap property, then L has the Egoroff property.*

PROOF. This follows immediately from the above theorem since $\text{cl}S = S'$ for all $S \subseteqq L$ (see [18], p. 83, Theorem 16.7).∎

## 4.6. THE INTER-RELATION OF PROPERTIES (A, i), (A, ii), (A, iii) AND (A, iv).

We start with the following.

THEOREM 4.27. *Let L and M be two Riesz spaces with M Dedekind complete. Assume that there exists a metrizable locally solid topology $\tau$ on L which satisfies condition (A, ii). Then $(\mathcal{L}_b)_n = (\mathcal{L}_b)_c$, i.e., every integral of $\mathcal{L}_b(L,M)$ is a normal integral.*

PROOF. It is enough to show that $\theta \leqq u_\alpha \uparrow u$ in L implies $\theta \leqq u_{\alpha_n} \uparrow u$ for some sequence $\{u_{\alpha_n}\} \subseteqq \{u_\alpha\}$. To this end let $\{V_n : n \in N\}$ be a basis of $\tau$ for the origin in L consisting of solid sets and such that $V_{n+1} + V_{n+1} \subseteqq V_n$ for $n = 1, 2, \ldots$ . Now let $\theta \leqq u_\alpha \uparrow u$ in L. Since (A, ii) is satisfied we have that $u - u_\alpha \xrightarrow{\tau} \theta$. So for given $n \in N$ we have, for some $\alpha_n$, $\theta \leqq u - u_{\alpha_n} \in V_n$ and we can suppose that $u_{\alpha_n} \uparrow$. Now let $m \in N$. Then we have $\theta \leqq u - u_{\alpha_n} \leqq u - u_{\alpha_m} \in V_m$ for $n \geqq m$, and this shows that $u_{\alpha_n} \xrightarrow{\tau} u$. It follows now from Theorem 4. 7(ii), that $u_{\alpha_n} \uparrow u$, and this completes the proof. ∎

<u>Note.</u> The same argument can be carried out if $\tau$ is not a locally solid metrizable topology for L but it is a metrizable and $L^+$ is $\tau$-closed and $\tau$-normal.

The following example shows that metrizability of the topology $\tau$ is essential.

EXAMPLE 4. 28. Let L be the Riesz space of all real valued Lebesgue integrable functions on [0, 1], with the ordering defined by $f \leqq g$, whenever $f(x) \leqq g(x)$ for all $x \in [0, 1]$. Let $T : L \to \mathbb{R}$ denote the Lebesgue integral, i. e. , $T(f) = \int_0^1 f(x)\,dx$ for all $f \in L$. Then, T is an integral of $L^\sim$, as easily follows from the Lebesgue dominated convergence theorem (note that $f_n \downarrow \theta$ in L implies $f_n(x) \downarrow 0$ in $\mathbb{R}$ for all $x \in [0, 1]$, but T is not a normal integral (see the discussion before Definition 3. 1) ).

The family of semi-norms $(p_x)_{x \in [0,1]}$, defined by $p_x(f) = |f(x)|$ for all $f \in L$, defines a Hausdorff non-metrizable locally solid (also locally convex) topology $\tau$ on L.

We prove that $(L, \tau)$ satisfies condition (A, ii). To this end, let $\theta \leqq u_\alpha \downarrow \theta$ in L. It is easy to verify that this implies $u_\alpha(x) \downarrow 0$ in $\mathbb{R}$ for all $x \in [0,1]$, so $p_x(u_\alpha) = u_\alpha(x) \downarrow 0$ for all $x \in [0,1]$, i.e., $u_\alpha \xrightarrow{\tau} \theta$.■

Note. Theorem 4. 27 also shows that for the Riesz space L of Example 4. 28 there is no metrizable locally solid topology $\tau$ satisfying condition (A, ii).

THEOREM 4. 29. Let $(E, \tau, K)$ be a metrizable ordered topological space with K $\tau$-closed, and let $\{W_n : n \in N\}$ be a basis for the neighborhood system of the origin for $\tau$ such that $W_{n+1} + W_{n+1} \subseteqq W_n$, for $n = 1,2,\ldots$.

Then we have:

(i) If $\theta \leqq u_\alpha \uparrow$ is a $\tau$-Cauchy net, then there exists a sequence $\{u_{\alpha_n}\} \subseteqq \{u_\alpha\}$ such that $u_{\alpha_n} \uparrow$ and $u_\alpha - u_{\alpha_n} \in W_n$ for all n and all $\alpha \geqq \alpha_n$. Furthermore any upper bound of the sequence $\{u_{\alpha_n}\}$ is an upper bound for the net $\{u_\alpha\}$.

(ii) If every order bounded increasing $\tau$-Cauchy sequence has a $\tau$-limit, and $\theta \leqq u_\alpha \uparrow \leqq u_0$ is a $\tau$-Cauchy net then $u = \sup u_\alpha$ exists and the sequence $\{u_{\alpha_n}\}$ in (i) satisfies $\sup u_{\alpha_n} = u$. Furthermore $u_\alpha \xrightarrow{\tau} u$.

(iii) If E is $\sigma$-Dedekind complete, and $\theta \leqq u_\alpha \uparrow \leqq u_0$ is a $\tau$-Cauchy net, then $u = \sup u_\alpha$ exists, and the sequence $\{u_{\alpha_n}\}$ in (i) satisfies $\sup u_{\alpha_n} = u$.

PROOF. Let $n \in N$ and let $W_n$ be the corresponding neighborhood from the basis $\{W_n\}$. Since $\theta \leqq u_\alpha$ is a $\tau$-Cauchy net there exists $\alpha_n \in \{\alpha\}$ such that $u_\alpha - u_{\alpha_n} \in W_n$ for all $\alpha \geqq \alpha_n$. We can suppose $u_{\alpha_n} \uparrow$. Now let $u_{\alpha_n} \leqq v$ for all n, and let $\alpha \in \{\alpha\}$ be fixed. For given n, let $\beta_n \in \{\alpha\}$ be such that $\beta_n \geqq \alpha$ and $\beta_n \geqq \alpha_n$. Then we have

$\theta \leqq v - u_{\alpha_n} = (v - v_\alpha) + (u_\alpha - u_{\alpha_n}) \leqq (v - u_\alpha) + (u_{\beta_n} - u_{\alpha_n})$. We put $w_n = u_{\beta_n} - u_{\alpha_n} \in W_n$, $n = 1, 2, \ldots$, so $w_n \xrightarrow{\tau} \theta$ and $\theta \leqq u - u_\alpha + w_n$ for $n = 1, 2, \ldots$ . Since K is $\tau$-closed it follows that $\theta \leqq u - u_\alpha$, or $u_\alpha \leqq v$ for all $\alpha$, i.e., v is an upper bound of $\{u_\alpha\}$.

(ii) Let $\theta \leqq u_\alpha \uparrow \leqq u_0$ be a $\tau$-Cauchy net and let $\{u_{\alpha_n}\} \subseteq \{u_\alpha\}$ be the sequence determined by (i). Then we have $u_{\alpha_{n+m}} - u_{\alpha_n} \in W_n$, for $m = 1, 2, \ldots$, so $\{u_{\alpha_n}\}$ is a $\tau$-Cauchy sequence. Hence, by hypothesis $u_{\alpha_n} \xrightarrow{\tau} u$ for some u and by Theorem 4.7(ii) $u_{\alpha_n} \uparrow u$. It follows from (i) that $u_\alpha \uparrow u$. Let $n_0 \geqq n+1$ be such that $u - u_{\alpha_k} \in W_{n+1}$ for all $k \geqq n_0$. Then we have $u - u_\alpha = (u - u_{\alpha_{n_0}}) + (u_{\alpha_{n_0}} - u_\alpha) \in W_{n+1} + W_k \subseteq W_n$ for all $\alpha \geqq \alpha_{n_0}$, so $u - u_\alpha \in W_n$ for all $\alpha \geqq \alpha_{n_0}$, i.e., $u_\alpha \xrightarrow{\tau} u$.

(iii) Let $\theta \leqq u_\alpha \uparrow \leqq u_0$ be a $\tau$-Cauchy net, and let $\{u_{\alpha_n}\}$ be the sequence in part (i). It follows from the $\sigma$-Dedekind completeness of E that $\sup u_{\alpha_n}$ exists in E, so $\sup u_\alpha$ exists in E and $\sup u_\alpha = \sup u_{\alpha_n}$. ∎

We continue with the following theorem.

THEOREM 4.30. <u>Let</u> (E, $\tau$, K) <u>be a metrizable ordered topological vector space with</u> K <u>$\tau$-closed. Then the following conditions are equivalent.</u>

(i) E <u>is $\sigma$-Dedekind complete, and</u> (A, i) <u>holds.</u>

(ii) <u>Every order-bounded increasing sequence in</u> E <u>has a $\tau$-limit.</u>

(iii) E <u>is super Dedekind complete, and</u> (A, ii) <u>holds.</u>

PROOF. (i)$\Rightarrow$(ii). Let $\theta \leqq u_n \uparrow \leqq u_0$. Then $u_n \uparrow u$ for some $u \in L$. Since (A, i) holds in (E, $\tau$) it follows that $u_n \xrightarrow{\tau} u$.

(ii)$\Rightarrow$(iii). We will show first that (A, ii) holds. Let $u_\alpha \downarrow \theta$. In order to show that $u_\alpha \xrightarrow{\tau} \theta$ we may replace $\{u_\alpha\}$ by $\{u_\alpha : \alpha \geqq \alpha_0\}$ for any fixed

$\alpha_0 \in \{\alpha\}$, i.e., we can assume immediately that $u_0 \geqq u_\alpha \downarrow \theta$. Then $\theta \leqq u_0 - u_\alpha \uparrow u_0$, so $\{u_0 - u_\alpha\}$ is a $\tau$-Cauchy net by (A, iii)$\Longleftrightarrow$(A, iv) according to Lemma 4.6. But then by Theorem 4.29, part (ii), it follows immediately that $u_\alpha \xrightarrow{\tau} \theta$. We prove now that E is super Dedekind complete. To this end, let $\theta \leqq u_\alpha \uparrow \leqq u$ in E. It follows now from Lemma 4.6 that $\{u_\alpha\}$ is a $\tau$-Cauchy net. Applying Theorem 4.29, part (ii), we get easily that there exists a sequence $\{u_{\alpha_n}\} \subseteqq \{u_\alpha\}$ such that $u_{\alpha_n} \uparrow u_0 = \sup u_\alpha$.

(iii)$\Rightarrow$(i). Obvious.■

The metrizability of $\tau$ is essential as Example 4.5(iii) shows.

THEOREM 4.31. Assume that $(E, \tau, K)$ is a metrizable ordered topological vector space with K $\tau$-closed and $\tau$-normal. Consider the following statements:

(i) $(E, \tau, K)$ satisfies (A, i) and (A, iii).

(ii) $(E, \tau, K)$ satisfies (A, ii).

Then (i) implies (ii). If in addition (E, K) is a Riesz space then (ii) implies (i).

In particular, for metrizable locally solid Riesz spaces both (A, i) and (A, iii) hold if and only if (A, ii) holds.

PROOF. (i)$\Rightarrow$(ii). Let $u_\alpha \downarrow \theta$. We can suppose that $\theta \leqq u_\alpha \leqq u_0$. Then we have $\theta \leqq u_0 - u_\alpha \uparrow u_0$. It follows from Lemma 4.6 that $\{u_0 - u_\alpha\}$ is a $\tau$-Cauchy net. From Theorem 4.29 it follows that there exists an increasing sequence $\{u_0 - u_{\alpha_n}\}$ such that any upper bound of $\{u - u_\alpha\}$ is also an upper bound of $\{u_0 - u_{\alpha_n}\}$, and vice versa. Obviously $u_{\alpha_n} \downarrow$ and if

$u_{\alpha_n} \geqq w > \theta$ then $u_0 - u_{\alpha_n} \leqq u_0 - w$ and so $u - u_\alpha \leqq u_0 - w < u_0$ for all $\alpha$, a contradiction. Hence, $u_{\alpha_n} \downarrow \theta$. Since (A, i) is valid we conclude that $u_{\alpha_n} \xrightarrow{\tau} \theta$. Now let V be a full $\tau$-neighborhood of zero. Then $u_{\alpha_{n_0}} \in V$ for some $n_0$. It follows from $\theta \leqq u_\alpha \leqq u_{\alpha_{n_0}} \in V$ whenever $\alpha \geqq \alpha_{n_0}$ that $u_\alpha \in V$ if $\alpha \geqq \alpha_{n_0}$, i.e., $u_\alpha \xrightarrow{\tau} \theta$. Assume now that (E, K) is also a Riesz space. From Theorem 4.7(i) we know that E is also Archimedean. We show next that (ii)$\Rightarrow$(i).

Obviously (A, i) is satisfied. Now let $\theta \leqq u_n \uparrow \leqq u_0$ in E. Let $V = \{v \in E : v \geqq u_n, \text{ for } n = 1, 2, \dots\}$. Since E is an Archimedeam Riesz space it follows from Theorem 1.4 that $u - u_n \underset{(v,n)}{\downarrow} \theta$ and so by (A, ii), $v - v_n \xrightarrow[(v,n)]{\tau} \theta$. This implies in particular that $\{v - u_n\}$ is a $\tau$-Cauchy net. From this it follows that $\{u_n\}$ is a $\tau$-Cauchy sequence.■

COROLLARY 4.32. <u>If every order bounded increasing $\tau$-Cauchy sequence of a metrizable locally solid Riesz space (L, $\tau$) has a $\tau$-limit (in particular if (L, $\tau$) is complete) then (A, ii) and (A, iii) are equivalent. Furthermore, if in this case the equivalent conditions (A, ii) and (A, iii) hold, then L is super Dedekind complete.</u>

PROOF. By the previous theorem if (A, ii) holds then (A, iii) holds. Now assume the metrizable locally solid Riesz space (L, $\tau$) has the stated property and that (A, iii) holds. But then statements (i) and (ii) of Theorem 4.29 are satisfied and so from the same Theorem it follows that (A, ii) holds and that L is super Dedekind complete.■

Example 4.5(iii) shows that non-metrizable locally solid Riesz spaces (L, $\tau$) can satisfy (A, i) and (A, iii) but not (A, ii). Note that the space of Example 4.5(iii) is sequentially complete but not complete.

The Hausdorff property of $\tau$ is also essential for the above theorem, in other words, we cannot replace the assumptions "$\tau$ is metrizable" with "$\tau$ has a countable basis for the neighborhood system of the origin" as the following example shows. Consider the Riesz space of Example 4.5(iii) and consider the same neighborhood, $W_{F,n,\delta}$ with the restriction that $F \in \mathcal{F}(Q)$, where Q is the set of all rational numbers of [0, 1]. The collection $\{W_{F,n,\delta} : F \in \mathcal{F}(Q),\ n \in N,\ \delta > 0\}$ defines a non-Hausdorff locally solid topology $\tau$ on L, which has a countable basis for the neighborhoods of the origin. Note that both (A, i) and (A, iii) hold but (A, ii) does not.

The following theorem tells us that (A, iii) implies (A, ii) for complete ordered topological vector spaces with closed cones.

THEOREM 4.33. <u>Let</u> (E, $\tau$, K) <u>be a $\tau$-complete ordered topological vector space with</u> K <u>$\tau$-closed.</u> <u>Then</u> (A, iii) <u>implies</u> (A, ii), <u>and so, in this case,</u> (A, iii) <u>implies</u> (A, i).

PROOF. By Lemma 4.6 we know that (A, iii) is equivalent to (A, iv). Now, let $\theta \leqq u_\alpha \uparrow u$ in E. Since (A, iv) is valid, $\{u_\alpha\}$ is a $\tau$-Cauchy net, and so, by the $\tau$-completeness of E, $u_\alpha \xrightarrow{\tau} u_0$ for some $u_0 \in L$. It follows from Theorem 4.7(ii) that $u = u_0$, so $u_\alpha \xrightarrow{\tau} u$. ∎

It is not known if the condition (A, ii) implies (A, iii) in general. The next theorem shows that this is true for Archimedean topological Riesz spaces.

THEOREM 4.34. <u>Let</u> (L, $\tau$) <u>be an Archimedean topological Riesz space (for example, a Hausdorff locally solid Riesz space).</u> <u>Then</u> (A, ii) implies (A, iii), and so, in this case, (A, ii) implies both (A, i) and (A, iii).

PROOF. Assume that $(L, \tau)$ satisfies (A, ii). Let $\theta \leqq u_n \uparrow \leqq u$ in L. The set $\{v \in L : v \geqq u_n \text{ for } n = 1, 2, \ldots\}$ is non-empty and, since L has the Archimedean property, we have $v - u_n \downarrow_{(v,n)} \theta$ (see Theorem 1.4).

It follows from (A, ii) that $v - u_n \xrightarrow[(v,n)]{\tau} \theta$, so, in particular the net $\{v - u_n\}$ is a $\tau$-Cauchy net. From this it follows easily that $\{u_n\}$ is a $\tau$-Cauchy sequence.■

COROLLARY 4.35. Let $(L, \tau)$ be a Hausdorff $\tau$-complete locally solid Riesz space. Then (A, ii) holds if and only if (A, iii) holds. In particular, if (A, ii) holds, then L is Dedekind complete.

PROOF. The proof follows immediately from Theorems 4.7, 4.33 and 4.34.

If now (A, ii) holds and $\theta \leqq u_\alpha \uparrow \leqq u$ then from (A, ii) $\Leftrightarrow$ (A, iii) $\Leftrightarrow$ (A, iv) we see that $\{u_\alpha\}$ is a $\tau$-Cauchy net, hence $u_\alpha \xrightarrow{\tau} u$ for some $n \in L$. It follows from Theorem 4.7 that $\theta \leqq u_\alpha \uparrow u$, i.e., L is Dedekind complete.■

An ideal P of L is called a prime ideal if whenever $f \wedge g$ is in P implies f in P or g in P. Any ideal containing a prime ideal is necessarily prime. For more details about prime ideals we refer the reader to [18], Chapter 5.

An element $\theta < u$ of L is called an atom if $\theta \leqq w \leqq u$, $\theta \leqq v \leqq u$ and $w \wedge v = \theta$ implies $w = \theta$ or $v = \theta$. A Riesz space L is called non-atomic if L does not contain any atoms. A typical example of a non-atomic Riesz space is the space $C_{[0,1]}$. (See [18], p. 146).

THEOREM 4.36. Let $(L, \tau)$ be a Hausdorff locally solid Riesz space. Suppose that L is non-atomic and that the $\mathrm{Int}(L^+)$ is non-empty. Then $(L, \tau)$ does not satisfy the (A, ii) property.

PROOF. Assume that $(L, \tau)$ satisfies the (A, ii) property. Then it follows from Theorem 4.14 that every $\tau$-closed ideal of L is a band. Now let P be a prime ideal of L. Then $\overline{P}$ is a $\tau$-closed prime ideal of L. Hence it is a band of L. If $(\overline{P})^d \neq \{\theta\}$ then there is an element $\theta < u$ in $(\overline{P})^d$. Since L is non-atomic there are two elements $\theta < u_1 \leqq u$, $\theta < u_2 \leqq u$ with $u_1 \wedge u_2 = \theta$. But then it follows that $u_1 \in \overline{P}$ or $u_2 \in \overline{P}$, a contradiction. This shows that $(\overline{P})^d = \{\theta\}$. Since L is also Archimedean (see Theorem 4.7(i)) it follows that $\overline{P} = (\overline{P})^{dd} = L$ (see Theorem 1.7(ii)), i.e., every prime ideal of L is $\tau$-dense. We show next that every ideal of L is $\tau$-dense. To this end let A be an ideal of L such that $\overline{A} \neq L$. Let $\theta < u \in L$ and u not in $\overline{A}$. Then there exists an open $\tau$-neighborhood of zero V such that $(u+V) \cap A = \phi$. Consider the set $S = (u+V) \cap L^+$. Then S is a lower sublattice (see [18], p. 203) which does not intersect A. By the prime ideal separation theorem ([18], Theorem 33.4, p. 202. The same proof works if we replace the element $f_0$ by the lower sublattice S), it follows that there exists a prime ideal P of L with $A \subseteqq P$ and $P \cap S = \phi$. But then since $\mathrm{Int}(L^+) \neq \phi$, S contains a non-empty open set, as easily verified, and this shows that P is not $\tau$-dense in L, a contradiction to what was already proved. Hence very ideal of L is $\tau$-dense. Now let $\theta < u \in L$ and let $\theta < w_1 \leqq u$, $\theta < w_2 \leqq u$ such that $w_1 \wedge w_2 = \theta$. Consider the band generated by $w_1$, $B_{w_1}$. Then $w_2 \notin B_{w_1}$ and by Theorem 4.8 $B_{w_1}$ is $\tau$-closed. Hence $B_{w_1} = \overline{B_{w_1}} = L$, a contradiction. Thus $(L, \tau)$ does not satisfy the (A, ii) property. ∎

We close this chapter by exhibiting another characterization of the (A, ii) property for locally convex, locally solid Hausdorff Riesz spaces. The proof is completely analogous to that of [16], Theorem 36.2, Note XI, p. 577.

THEOREM 4.37. (Luxemburg-Zaanen). Let $(L, \tau)$ be a Hausdorff locally solid, locally convex Riesz space. Then the following statements are equivalent.

(i) $(L, \tau)$ satisfies (A, ii).

(ii) Every band of $L'$ is $\sigma(L', L)$-closed.

# CHAPTER 5
# THE TOPOLOGICAL COMPLETION OF A HAUSDORFF LOCALLY SOLID RIESZ SPACE

## 5.1. INTRODUCTION

It is known that every Hausdorff topological vector space $(E, \tau)$ has (up to topological and algebraic isomorphism) a unique completion $(\hat{E}, \hat{\tau})$, i.e., there is a Hausdorff complete topological vector space $(\hat{E}, \hat{\tau})$ such that $(E, \tau)$ is a dense subspace of $(\hat{E}, \hat{\tau})$ (see [27], p. 17 or [7], p. 131).

It is also known that if $\{V\}$ is a neighborhood basis for the $\tau$-neighborhoods of zero then $\{\overline{V}\}$, where $\overline{V}$ is the $\hat{\tau}$-closure of V in $\hat{E}$, is also a neighborhood basis for the $\hat{\tau}$-neighborhoods of zero in $\hat{E}$ (see [27], p. 17).

We suppose next that $(L, \tau)$ is a Hausdorff locally solid Riesz space. We shall denote the topological completion of $(L, \tau)$ by $(\hat{L}, \hat{\tau})$. Our purpose is to investigate which properties of $(L, \tau)$ are inherited also by $(\hat{L}, \hat{\tau})$.

The above problem has been investigated by W. A. J. Luxemburg ([15], Note XVI) in the case of normed Riesz spaces and by I. Kawai and M. Duhoux in the case of Hausdorff locally convex, locally solid Riesz spaces (see [11] and [2], respectively). Some questions of the general problem also were studied by D. H. Fremlin in [3].

## 5.2. THE COMPLETION SPACE $(\hat{L}, \hat{\tau})$

We start with the following important theorem.

THEOREM 5. 1. Let $(L, \tau)$ be a Hausdorff locally solid Riesz space and let $(\hat{L}, \hat{\tau})$ be the topological completion of $(L, \tau)$. If $\hat{L}^+$ denotes the $\hat{\tau}$-closure of $L^+$ in $\hat{L}$, then $\hat{L}^+$ is a cone of $\hat{L}$, and $(\hat{L}, \hat{\tau})$ with this cone is a Hausdorff locally solid Riesz space. In particular it follows that L is a Riesz subspace of $\hat{L}$.

PROOF. According to Theorem 4. 2, $\hat{L}^+$ is a $\hat{\tau}$-normal cone of $(\hat{L}, \hat{\tau})$, since $L^+$ being a $\tau$-normal cone of L implies that $L^+$ is also a $\hat{\tau}$-normal cone of $\hat{L}$.

Let now $+ : L \to \hat{L}$ be defined by $f \to f^+$. Then + is a uniformly continuous mapping from $(L, \tau)$ into the $\hat{\tau}$-complete Hausdorff topological vector space $(\hat{L}, \hat{\tau})$. Hence it can be extended uniquely to a uniformly continuous mapping p from $\hat{L}$ into $\hat{L}$ (see [12], p. 195, Theorem 26).

We show next that $p(\hat{f}) = \hat{f} \vee \theta$ in $\hat{L}$. To this end, let $\hat{f} \in \hat{L}$. Then $f_\alpha \xrightarrow{\hat{\tau}} \hat{f}$ for some net $\{f_\alpha\}$ of L. It follows from $f_\alpha^+ \geqq f_\alpha$ that $p(f_\alpha) - f_\alpha = f_\alpha^+ - f_\alpha \in L^+$, which implies $p(\hat{f}) - \hat{f} \in \overline{L^+} = \hat{L}^+$, i. e., $p(\hat{f}) \geqq \hat{f}$ in $\hat{L}$. Obviously we have also that $p(\hat{f}) \in \hat{L}^+$. Now, let $\hat{f} \leqq \hat{g}$ and $\theta \leqq \hat{g}$ in $\hat{L}$. We pick two nets $\{f_\alpha\}$, $\{g_\alpha\}$ of $L^+$ such that $f_\alpha \xrightarrow{\hat{\tau}} \hat{g} - \hat{f}$ and $g_\alpha \xrightarrow{\hat{\tau}} \hat{g}$, then we have $h_\alpha = g_\alpha - f_\alpha \xrightarrow{\hat{\tau}} \hat{f}$.

But, from the relation $\theta \leqq f_\alpha = g_\alpha - h_\alpha \in L^+$ it follows that $g_\alpha = g_\alpha^+ \geqq h_\alpha^+$, so $g_\alpha - p(h_\alpha) \in L^+$ and from this we get (by taking the $\hat{\tau}$-limits) that $\hat{g} - p(\hat{f}) \in \hat{L}^+$, i. e., $\hat{g} \geqq p(\hat{f})$ in $\hat{L}$. This shows that $p(\hat{f}) = \hat{f}^+$ in $\hat{L}$. It follows now from Theorem 4. 4 that $(\hat{L}, \hat{\tau})$ with the cone $\hat{L}^+$ is a ($\hat{\tau}$-complete) Hausdorff, locally solid Riesz space. The assertion that L is a Riesz subspace of $\hat{L}$ is obvious from the above discussion. ∎

The next theorem gives a chracterization of L in order to be an ideal of $\hat{L}$.

THEOREM 5.2. Let $(\hat{L}, \hat{\tau})$ be the topological completion of the Hausdorff locally solid Riesz space $(L, \tau)$. Then the following statements are equivalent.

(i) L is an ideal of $\hat{L}$.

(ii) Every order interval of L is $\tau$-complete.

PROOF. (i)$\Rightarrow$(ii). Since $[f,g] = f + [\theta, g-f]$ it is enough to show that every interval $[\theta, u]$, $u \in L^+$ is $\tau$-complete. So, let $\{f_\alpha\}$ be a $\tau$-Cauchy net such that $\theta \leqq f_\alpha \leqq u$ for all $\alpha \in \{\alpha\}$, u is fixed in $L^+$. Then $f_\alpha \xrightarrow{\hat{\tau}} \hat{f}$ in $\hat{L}$, for some $\hat{f}$ in $\hat{L}$. It follows that $\theta \leqq \hat{f} \leqq u$, and so, since L is an ideal of $\hat{L}$ we obtain $\hat{f}$ in L, i.e., $[\theta, u]$ is $\tau$-complete.

(ii)$\Rightarrow$(i). Assume $\theta \leqq \hat{f} \leqq u$, $\hat{f} \in \hat{L}$ and $u \in L^+$. We have to show that $\hat{f}$ is in $L^+$, since L is a Riesz subspace of $\hat{L}$. We pick a net $\{f_\alpha\} \subseteqq L^+$ such that $f_\alpha \xrightarrow{\hat{\tau}} \hat{f}$. We may suppose that $\theta \leqq f_\alpha \leqq u$ for all $\alpha$, otherwise we replace each $f_\alpha$ by $f_\alpha \wedge u$. The $\tau$-completeness of $[\theta, u]$ and the Hausdorff property of $\tau$ imply that $\hat{f}$ is in L, i.e., that L is an ideal of $\hat{L}$. ∎

Note. If L is an ideal of $\hat{L}$, then L is order dense in $\hat{L}$, i.e., $\{L\} = L$. Indeed if $\theta \leqq \hat{f} \in L^d$, then $f_\alpha \xrightarrow{\hat{\tau}} \hat{f}$ for some net $\{f_\alpha\}$ of $L^+$, therefore $\theta = f_\alpha \wedge \hat{f} \xrightarrow{\hat{\tau}} \hat{f} = \theta$, so $\hat{f} = \theta$ and hence $L^d = \{\theta\}$. Thus $\{L\} = L^{dd} = \hat{L}$.

The next theorem deals with the embedding of L into $\hat{L}$ and generalizes a result of I. Kawai (see [11], p. 296, Theorem 4.1).

THEOREM 5.3. Let $(\hat{L}, \hat{\tau})$ be the completion of a Hausdorff locally solid Riesz space $(L, \tau)$ and let $I: L \to \hat{L}$ be the embedding of $L$ into $\hat{L}$, i.e., $I(f) = f$ for all $f \in L$.

Then the following statements are equivalent.

(i) $I$ is a normal integral of $\mathcal{L}_b(L, \hat{L})$, i.e., $f_\alpha \downarrow \theta$ in $L$ implies $f_\alpha \downarrow \theta$ in $\hat{L}$, or in other words, the embedding of $L$ into $\hat{L}$, preserves arbitrary suprema and infima.

(ii) For every $\tau$-Cauchy net $\{f_\alpha\}$ of $L^+$ such that $f_\alpha \xrightarrow{(o)} \theta$ in $L$, we have that $f_\alpha \xrightarrow{\tau} \theta$.

PROOF. (i)$\Rightarrow$(ii). Let $\{f_\alpha\} \subseteq L^+$ be a $\tau$-Cauchy net such that $f_\alpha \xrightarrow{(o)} \theta$ in $L$. It follows that there exists a net $\{g_\alpha\} \subseteq L^+$ such that $\theta \leqq f_\alpha \leqq g_\alpha \downarrow \theta$ in $L$. By hypothesis we have also $g_\alpha \downarrow \theta$ in $\hat{L}$. We note that $\{f_\alpha\}$, being a $\tau$-Cauchy net, is also a $\hat{\tau}$-Cauchy net, so $f_\alpha \xrightarrow{\hat{\tau}} \hat{f}$, for some $\hat{f} \in \hat{L}$. But, for fixed $\beta \in \{\alpha\}$ and $\alpha \geqq \beta$ we have $\theta \leqq f_\alpha \leqq g_\alpha \leqq g_\beta$ which shows that $\theta \leqq \hat{f} \leqq g_\beta$ in $\hat{L}$, for all $\beta \in \{\alpha\}$. It follows then that $\hat{f} = \theta$, so $f_\alpha \xrightarrow{\hat{\tau}} \theta$, i.e., $f_\alpha \xrightarrow{\tau} \theta$.

(ii)$\Rightarrow$(i). Let $f_\alpha \downarrow \theta$ in $L$, and assume that $f_\alpha \geqq \hat{f} \geqq \theta$ for all $\alpha \in \{\alpha\}$ in $L$. We have to show that $\hat{f} = \theta$. Let $\{g_\lambda\}$ be a net of $L^+$ such that $g_\lambda \xrightarrow{\hat{\tau}} \hat{f}$. Then we have $|g_\lambda \wedge f_\alpha - \hat{f}| = |g_\lambda \wedge f_\alpha - \hat{f} \wedge f_\alpha| \leqq |g_\lambda - \hat{f}|$ for all $\alpha, \lambda$. This shows that $g_\lambda \wedge f_\alpha \xrightarrow[(\lambda,\alpha)]{\hat{\tau}} \hat{f}$. In particular we have that $\{g_\lambda \wedge f_\alpha\}$ is a $\tau$-Cauchy net of $L$. We also have $\theta \leqq g_\lambda \wedge f_\alpha \leqq f_\alpha \downarrow_{(\lambda,\alpha)} \theta$ in $L$. From our hypothesis it follows that $g_\lambda \wedge f_\alpha \xrightarrow[(\lambda,\alpha)]{\tau} \theta$. Hence $\hat{f} = \theta$ and this shows that $f_\alpha \downarrow \theta$ in $\hat{L}$. ∎

THEOREM 5.4. Let $(L, \tau)$ be a Hausdorff locally solid Riesz space. Then the following statements are equivalent.

(i) $L = \hat{L}$, *i.e.*, $L$ *is* $\tau$-*complete*.

(ii) *For every* $\hat{f}, \hat{g} \in \hat{L}$ *with* $\hat{f} < \hat{g}$, *there exists* $f \in L$ *such that* $\hat{f} \leqq f \leqq \hat{g}$.

Theorem 5.4 is a corollary of the following general theorem.

THEOREM 5.5. *Let* $L$ *be a Riesz subspace of the Archimedean Riesz space* $K$. *Then the following statements are equivalent.*

(i) $L = K$.

(ii) *For every* $f, g \in K$ *with* $f < g$, *there exists* $u \in L$ *such that* $f \leqq u \leqq g$.

PROOF. It is evident that (i)⟹(ii). We have only to show that (ii)⟹(i). To this end let $\theta < u^* \in K$. Let $A_{u^*}$ be the ideal generated by $u^*$ in $K$. Then $L_1 = L \cap A_{u^*}$ is a Riesz subspace of $A_{u^*}$ such that for every $f^*, g^*$ in $A_{u^*}$, $f^* < g^*$, there is $u \in L_1$ with $f^* \leqq u \leqq g^*$.

By the Yosida Representation Theorem (see Theorem 1.7(iv)) there exists a Hausdorff compact topological space $X$ such that $A_{u^*}$ is Riesz isomorphic to a Riesz subspace $\hat{A}_{u^*}$ of $C(X)$, with $\widehat{u^*}(x) = 1$, for all $x \in X$, and with $\hat{A}_{u^*}$ separating the points of $X$. Let $V_{x_0}$ be an open neighborhood of the point $x_0$ of $X$ and let $x$ be a point of $X$ not in $V_{x_0}$. Then there exists a function $\hat{f} \in \hat{A}_{u^*}$ such that $\hat{f}(x) = 2$ and $\hat{f}(x_0) = 0$. Without loss of generality we can suppose that $\hat{f}$ is positive, otherwise we replace $\hat{f}$ by $\hat{f}^+$. Let $V_x$ be an open neighborhood of $x$, such that $\hat{f}(y) > 1$ for all $y \in V_x$. Then the function of $\hat{A}_{u^*}$, $\hat{g} = \widehat{u^*} \wedge \hat{f}$, satisfies $\theta \leqq \hat{g} \leqq \widehat{u^*} = 1$, $\hat{g}(x_0) = 0$ and $\hat{g}(y) = 1$ for all $y \in V_x$. Since

$$X - V_{x_0} \subseteqq \bigcup_{x \in X - V_{x_0}} V_x \quad \text{and} \quad X - V_{x_0}$$

is compact, there exist neighborhoods $\{V_{x_i} : i = 1, \ldots, n\}$ such that

$$X - V_{x_0} \subseteq \bigcup_{i=1}^{n} V_{x_i} \quad .$$

Let
$$\hat{g}_0 = \bigvee_{i=1}^{n} \hat{g}_i = \widehat{\bigvee_{i=1}^{n} g_i} \quad .$$

Then $\hat{g}_0 \in \hat{A}_{u^*}$, $\hat{g}(x) = 1$ for all $x$ not in $V_{x_0}$, $\hat{g}_0(x_0) = 0$, and $\theta < \hat{g}_0 < \widehat{u_*} = 1$.

If now X consists of one point then $C(X) \cong \mathbb{R}$ and from this it follows easily that $\widehat{u^*} \in \hat{L}$ and so $u^* \in L$.

Let now X be consisting from more than one point and let $x_1, x_2$ in S be such that $x_1 \neq x_2$. We pick two open neighborhoods $V_{x_1}$, $V_{x_2}$ of $x_1$ and $x_2$, respectively, such that $V_{x_1} \cap V_{x_2} = \emptyset$. Let $\hat{f}_1$ and $\hat{f}_2$ be in $\hat{A}_{u^*}$ such that $\theta \leqq \hat{f}_1, \hat{f}_2 \leqq \hat{u}^* = 1$, $\hat{f}_1(x_1) = \hat{f}_2(x_2) = 0$, $\hat{f}_1(x) = 1$ for all $x$ not in $V_{x_1}$, $\hat{f}_2(x) = 1$ for all $x$ not in $V_{x_2}$. Since $\hat{f}_1 < \hat{u}^*$, $\hat{f}_2 < \hat{u}^*$, there are $\hat{g}_1, \hat{g}_2 \in \hat{L}$, such that $\hat{f}_1 \leqq \hat{g}_1 \leqq \hat{u}^*$, $\hat{f}_2 \leqq \hat{g}_2 \leqq \hat{u}^*$. It follows from this easily that $\widehat{g_1 \vee g_2} = \hat{g}_1 \vee \hat{g}_2 = \widehat{u^*} = 1$, so $\widehat{u^*}$ is in $\hat{L}_1$. Thus $u^* \in L_1 \subseteq L$. This shows that $L = K$. ■

A sufficient condition for some order properties of L to be inherited in $\hat{L}$ is given in the next theorem.

THEOREM 5.6. <u>Let</u> $(L, \tau)$ <u>be a Hausdorff locally solid Riesz space and assume that</u> L <u>is an ideal of</u> $\hat{L}$. <u>Then we have:</u>

(i) <u>If</u> L <u>is</u> $\sigma$-<u>Dedekind complete, then</u> $\hat{L}$ <u>is</u> $\sigma$-<u>Dedekind complete</u>.

(ii) <u>If</u> L <u>is Dedekind complete, then</u> $\hat{L}$ <u>is Dedekind complete</u>.

PROOF. The proof is similar to that of Theorem 66.5 of [15], Note XVI, p. 665. ■

## 5.3. THE PROPERTIES (A, ii) AND (A, iii) ARE PRESERVED UNDER TOPOLOGICAL COMPLETION

The next theorem deals with property (A, iii).

THEOREM 5.7. The topological completion $(\hat{L}, \hat{\tau})$ of the Hausdorff locally solid Riesz space $(L, \tau)$ satisfies condition (A, iii) if and only if $(L, \tau)$ satisfies condition (A, iii).

PROOF. Obviously $(L, \tau)$ satisfies condition (A, iii) if $(\hat{L}, \hat{\tau})$ satisfies (A, iii).

Now let $(L, \tau)$ satisfy (A, iii) and let $\theta \leqq \hat{f}_n \downarrow$ in $\hat{L}$. We have to show that $\{\hat{f}_n\}$ is a $\hat{\tau}$-Cauchy sequence. To this end let U be a solid $\hat{\tau}$-neighborhood of zero and let V be also a solid $\hat{\tau}$-neighborhood of zero such that $V + V + V \subseteqq U$.

We construct next a sequence $\{V_n\}$ of solid $\hat{\tau}$-neighborhoods of zero such that $V_1 = V$ and $V_{n+1} + V_{n+1} \subseteqq V_n$, for $n = 2, 3, \ldots$. To do this, start with $V_1 = V$. Next, pick a solid $\hat{\tau}$-neighborhood of zero $W_1$ such that $W_1 + W_2 \subseteqq V$. Let $V_2 = W_1$. Now pick a solid $\hat{\tau}$-neighborhood of zero $W_2$ such that $W_2 + W_2 \subseteqq W_1$. Let $V_3 = W_2$. Proceeding this way we construct the above sequence $\{V_n\}$.

Given $n \in N$, choose an element $g_n \in L^+$ such that $|\hat{f}_n - g_n| \in V_{n+1}$. Construct the sequence $\{f_n\}$ of $L^+$ defined by $f_n = \bigwedge_{i=1}^{n} g_i$, for $n = 1, 2, \ldots$. Then it is evident that $\theta \leqq f_n \downarrow$ in L. It follows then, from condition (A, iii) on $(L, \tau)$, that $\{f_n\}$ is a $\tau$-Cauchy sequence and so surely it is a $\hat{\tau}$-Cauchy sequence (see the note after Lemma 4.6). This shows in particular that $|f_n - f_m| \in V$ for all $n, m \geqq n_0$. We also have that

$$\hat{f}_n - f_n = \hat{f}_n - \bigwedge_{i=1}^{n} g_i = \bigvee_{i=1}^{n} (\hat{f}_n - g_i) \leqq \bigvee_{i=1}^{n} (\hat{f}_i - g_i) \leqq$$

$$\leqq \sum_{i=1}^{n} |\hat{f}_i - g_i| \in V_2 + V_3 + \ldots + V_n \subseteqq V,$$

$$f_n - \hat{f}_n \leqq g_n - \hat{f}_n \leqq \sum_{i=1}^{n} |\hat{f}_i - g_i| \in V.$$

So, $|\hat{f}_n - f_n| \leqq \sum_{i=1}^{n} |\hat{f}_i - g_i| \in V$, i.e., $|\hat{f}_n - f_n| \in V$ for all $n = 1, 2, \ldots$ .

Now, for $n, m \geqq n_0$ we have

$$|\hat{f}_n - \hat{f}_m| \leqq |\hat{f}_n - f_n| + |f_n - f_m| + |f_m - \hat{f}_m| \in V + V + V \subseteqq U .$$

Hence $|\hat{f}_n - \hat{f}_m| \in U$, for all $n, m \geqq n_0$, which shows that $\hat{f}_n - \hat{f}_m \in U$, for all $n, m \geqq n_0$, i.e., $\{\hat{f}_n\}$ is a $\hat{\tau}$-Cauchy sequence.■

Applying Corollary 4.35 we see that condition (A, ii) is satisfied in $(\hat{L}, \hat{\tau})$ if and only if (A, iii) is satisfied in $(\hat{L}, \hat{\tau})$. This observation gives rise to the following theorem.

THEOREM 5.8. <u>If the Hausdorff locally solid Riesz space</u> $(L, \tau)$ <u>satisfies condition</u> (A, ii), <u>then the topological completion</u> $(\hat{L}, \hat{\tau})$ <u>of</u> $(L, \tau)$ <u>satisfies also the condition</u> (A, ii).

PROOF. From Theorem 4.34 it follows that $(L, \tau)$ satisfies condition (A, iii) and from Theorem 5.7 we see that $(\hat{L}, \hat{\tau})$ satisfies (A, iii). It follows now from the observation preceding Theorem 5.8 that $(\hat{L}, \hat{\tau})$ satisfies (A, ii).■

The converse of Theorem 5.8 may be false as the following example shows.

EXAMPLE 5.9. Let $L = C_{[0,1]}$, and let $\tau$ be the Hausdorff locally solid topology generated by the (Riesz) norm $\rho(f) = \int_0^1 |f(x)|\,dx$. Then L satisfies (A, iii) but not (A, i) (see Example 4.5(ii)), so in particular not (A, ii). The completion $\hat{L}$ is the Riesz space $L_1([0,1]$, i.e., the Lebesgue equivalence classes with the usual ordering, with $\hat{\tau}$ generated by the (Riesz) norm $\rho([f]) = \int_0^1 |f(x)|\,dx$ and $\hat{L}^+ = \{[f] : f \geqq \theta \text{ a.e.}\}$. It is not hard to verify that $(L, \tau)$ satisfies (A, ii).■

As we have seen, condition (A, iii) in $(L, \tau)$ implies condition (A, ii) in $(\hat{L}, \hat{\tau})$, and so, using Corollary 4.35 we obtain that condition (A, iii) in $(L, \tau)$ implies Dedekind completeness in $\hat{L}$. Also, since the condition (A, iii) in $(L, \tau)$ implies (A, ii) in $(\hat{L}, \hat{\tau})$ we have always that (A, iii) in $(L, \tau)$ implies (A, i) in $(\hat{L}, \hat{\tau})$.

Next, we shall investigate under what conditions (A, i) in $(L, \tau)$ implies (A, i) in $(\hat{L}, \hat{\tau})$. Example 5.9 shows that (A, i) in $(\hat{L}, \hat{\tau})$ does not imply (A, i) in $(L, \tau)$, not even for normed Riesz spaces.

## 5.4. UPPER AND LOWER ELEMENTS IN THE METRIZABLE CASE

We shall assume next that the topology $\tau$ of the Hausdorff locally solid Riesz space $(L, \tau)$ is metrizable, i.e., the topology $\tau$ is generated by a distance, or equivalently, there exists a countable basis $\{V_n\}$ for the $\tau$-neighborhoods of zero such that $\bigcap_{n=1}^{\infty} V_n = \{\theta\}$ (see [7], Theorem 1, p. 111). We shall call $(L, \tau)$ in this case, a metrizable locally solid Riesz space.

Given a metrizable locally solid Riesz space $(L, \tau)$ we define the following subsets of $\hat{L}$:

$$U = \{\hat{f} \in \hat{L} : \exists \{f_n\} \subseteqq L^+;\ \theta \leqq f_n \uparrow \text{ and } f_n \xrightarrow{\hat{\tau}} \hat{f}\}$$

$$\Lambda = \{\hat{f} \in \hat{L} : \exists \{f_n\} \subseteqq L^+;\ \theta \leqq f_n \downarrow \text{ and } f_n \xrightarrow{\hat{\tau}} \hat{f}\}$$

We call the elements of U upper elements and the elements of $\Lambda$ lower elements.

It is easy to verify that U and $\Lambda$ are cones of $\hat{L}$ and that they are closed under the lattice operations.

Concerning the metrizable locally solid Riesz spaces we have the following Lemma.

LEMMA 5.10. Let $(L, \tau)$ be a metrizable locally solid Riesz space. Assume V is a $\tau$-neighborhood of zero of $\hat{L}$ and $\theta \leqq \hat{f} \in \hat{L}$. Then there exists an element $\hat{u} \in U$ such that $\hat{f} \leqq \hat{u}$ and $\hat{u} - \hat{f} \in V$, or in other words, every positive element of $\hat{L}$ is the $\hat{\tau}$-limit of a decreasing sequence of upper elements.

PROOF. First we pick a countable basis $\{V_n\}$ of solid $\tau$-closed neighborhoods of zero of $\hat{L}$ such that

$$V_{n+1} + V_{n+1} \subseteqq V_n, \qquad \text{for} \quad n = 1, 2, \ldots .$$

Choose a $V_k$ from $\{V_n\}$. Since $\theta \leqq \hat{f} \in \hat{L}$ we have $f_n \xrightarrow{\hat{\tau}} \hat{f}$ for some sequence $\{f_n\}$ of $L^+$. This implies that $\{f_n\}$ is a $\hat{\tau}$-Cauchy sequence. Let $\{g_n\}$ be a subsequence of $\{f_n\}$ such that $|g_{n+1} - g_n| \in V_{k+n+2}$ for $n = 1, 2, \ldots$, and let,

$$u_n = g_n + \sum_{i=1}^{n-1} |g_{i+1} - g_i| \in L^+, \quad n = 1, 2, \ldots .$$

Then $u_{n+1} - u_n = g_{n+1} - g_n + |g_{n+1} - g_n| \geqq \theta$ for $n = 1, 2, \ldots$, which shows that $\theta \leqq u_n \uparrow$ in L. Further $\{u_n\}$ is a $\hat{\tau}$-Cauchy sequence, as we can easily verify, so $u_n \xrightarrow{\hat{\tau}} \hat{u}$, for some $\hat{u} \in \hat{L}$. Since $g_n \leqq u_n$, for $n = 1, 2, \ldots$ and $g_n \xrightarrow{\hat{\tau}} \hat{f}$ in $\hat{L}$, we have also that $\hat{f} \leqq \hat{u}$.

It follows, now, from

$$|u_n - \hat{f}| \leqq |g_n - \hat{f}| + \sum_{i=1}^{n-1} |g_{i+1} - g_i| \in |g_n - \hat{f}| + \sum_{i=1}^{n-1} V_{k+2+i} \subseteqq |g_n - \hat{f}| + V_{k+1}$$

that $u_n - \hat{f} \in V_k$, for sufficiently large n. Since $V_k$ is $\hat{\tau}$-closed it follows that $\hat{u} - \hat{f} \in V_k$.

Given $n \in N$ pick an upper element $\hat{f}_n$ such that $\hat{f} \leqq \hat{f}_n$, and $\hat{f}_n - \hat{f} \in V_n$. Let $\hat{g}_n = \bigwedge_{i=1}^{n} \hat{f}_i$, $n = 1, 2, \ldots$ .

Then $\hat{g}_n$ is an upper element for all n, $\hat{f} \leqq \hat{g}_n$ for all $n = 1, 2, \ldots$, $\hat{g}_n \downarrow$ in $\hat{L}$, and $\theta \leqq \hat{g}_n - \hat{f} \leqq \hat{f}_n - \hat{f} \in V_n$, i.e., $\hat{g}_n \xrightarrow{\hat{\tau}} \hat{f}$. This completes the proof.■

The following lemma is the "dual" of Lemma 5.10 and can be proved in a similar manner as Theorem 60.6 of [15], Note XVI, p. 649.

LEMMA 5.11. Let $(L, \tau)$ be a metrizable locally solid Riesz space. Assume V is a $\hat{\tau}$-neighborhood of zero of $\hat{L}$, and $\theta \leqq \hat{f} \in \hat{L}$. Then there exists an element $\hat{u} \in \Lambda$, such that $\hat{u} \leqq \hat{f}$ and $\hat{f} - \hat{u} \in V$, or in other words, every positive element of $\hat{L}$ is the $\hat{\tau}$-limit of an increasing sequence of lower elements.

The following simple lemma will be useful later.

LEMMA 5.12. Let $(L, \tau)$ be a Hausdorff topological Riesz space with $L^+$ $\tau$-closed and $\tau$-normal (for example, a Hausdorff locally solid Riesz space). Assume that for the two nets $\{f_\alpha\}$, $\{g_\alpha\}$ of $L$ we have $\theta \leqq f_\alpha \downarrow \leqq g_\alpha \downarrow$ and $g_\alpha - f_\alpha \xrightarrow{\tau} \theta$.

Then, $f_\alpha \downarrow f$ in $L$, if and only if, $g_\alpha \downarrow f$ in $L$.

PROOF. Assume $\theta \leqq f_\alpha \downarrow \leqq g_\alpha \downarrow$, $g_\alpha - f_\alpha \xrightarrow{\tau} \theta$ and that $f_\alpha \downarrow f \geqq \theta$ in $L$. Let $g_\alpha \geqq g \geqq f$ in $L$, for all $\alpha \in \{\alpha\}$. Then, $g_\alpha - f_\alpha = (g_\alpha - f_\alpha)^+ \geqq (g - f_\alpha)^+ \geqq \theta$. Since $g_\alpha - f_\alpha \xrightarrow{\tau} \theta$ and $L^+$ is a $\tau$-normal cone it follows that $(g - f_\alpha)^+ \xrightarrow{\tau} \theta$ (see Theorem 4.1). But $(g - f_\alpha)^+ \uparrow (g - f)^+ = g - f$. So, it follows from Theorem 4.7 that $g - f = \theta$ or $g = f$, i.e., $g_\alpha \downarrow f$ in $L$.

Now, if $\theta \leqq f_\alpha \downarrow \leqq g_\alpha \downarrow$, $g_\alpha - f_\alpha \xrightarrow{\tau} \theta$ and $g_\alpha \downarrow f$ in $L$, then we have $\theta \leqq f - f \wedge f_\alpha \uparrow = f \wedge g_\alpha - f \wedge f_\alpha \leqq g_\alpha - f_\alpha \xrightarrow{\tau} \theta$. So, $f - f \wedge f_\alpha \xrightarrow{\tau} \theta$. Hence by Theorem 4.7 we obtain $\theta \leqq f - f \wedge f_\alpha \uparrow \theta$, so $f = f \wedge f_\alpha \leqq f_\alpha$ for all $\alpha \in \{\alpha\}$. Since $g_\alpha \downarrow f$ and $f_\alpha \leqq g_\alpha$ for all $\alpha$ it follows from Theorem 4.7 that $f_\alpha \downarrow f$ in $L$. ■

## 5.5. THE PROPERTIES (A, i) AND (B, i) IN THE METRISABLE CASE

We prove first that the property (A, i) is preserved under topological completion of the metrizable locally solid Riesz spaces.

THEOREM 5.13. The completion $(\hat{L}, \hat{\tau})$ of the metrizable locally solid Riesz space $(L, \tau)$ satisfies (A, i) if $(L, \tau)$ satisfies (A, i).

PROOF. Let $\{V_n\}$ be a countable basis of the neighborhood system of the origin of $\hat{L}$ consisting of solid sets and let $\hat{f}_n \downarrow \theta$ in $\hat{L}$. We shall show that $f_n \xrightarrow{\hat{\tau}} \theta$.

For a given $n \in N$, let $\hat{u}_n \in U$ be such that $\theta \leqq \hat{f}_n \leqq \hat{u}_n$ and $\hat{u}_n - \hat{f}_n \in V_n$. This is possible by Lemma 5.11. Let $\hat{w}_n = \bigwedge_{i=1}^{n} \hat{u}_i \in U$, then we have $\theta \leqq \hat{f}_n \downarrow \leqq \hat{w}_n \downarrow$ in $\hat{L}$, and $\theta \leqq \hat{w}_n - \hat{f}_n \leqq \hat{u}_n - \hat{f}_n \in V_n$, for $n = 1, 2, \ldots$, so $\hat{w}_n - \hat{f}_n \xrightarrow{\tau} \theta$.

Since $\hat{f}_n \downarrow \theta$ in $\hat{L}$, it follows from Lemma 5.12 that $\hat{w}_n \downarrow \theta$ in $\hat{L}$. This shows that we may assume that $\{\hat{f}_n\} \subseteqq U$, i.e., that $\{\hat{f}_n\}$ is a sequence of upper elements.

Now, let $V_k$ be one neighborhood from the sequence $\{V_n\}$. Given $n \in N$, choose an element $g_n \in L^+$ such that $\hat{f}_n - g_n \in V_{k+n+2}$, and $\theta \leqq g_n \leqq \hat{f}_n$. Note that this is possible since $\{\hat{f}_n\} \subseteqq U$.

Let, $f_n = \bigwedge_{i=1}^{n} g_i \in L^+$, for $n = 1, 2, \ldots$ . Then we have

$$\theta \leqq \hat{f}_n - f_n = \hat{f}_n - \bigwedge_{i=1}^{n} g_i = \bigvee_{i=1}^{n} (\hat{f}_n - g_i) \leqq \bigvee_{i=1}^{n} (\hat{f}_i - g_i) \leqq$$

$$\leqq \sum_{i=1}^{n} (\hat{f}_i - g_i) \in V_{k+3} + \ldots + V_{k+n+2} \subseteqq V_{k+1} \ .$$

So, $\hat{f}_n - f_n \in V_{k+1}$, for $n = 1, 2, \ldots$ .

Now we observe that $f_n \downarrow \theta$ in L, so $f_n \xrightarrow{\tau} \theta$ since (A, i) holds in $(L, \tau)$. This implies that $f_n \xrightarrow{\hat{\tau}} \theta$. In particular we have $f_n \in V_{k+1}$ for all $n \geqq n_0$. So, we have $\hat{f}_n = f_n + (\hat{f}_n - f_n) \in V_{k+1} + V_{k+1} \subseteqq V_k$, for $n \geqq n_0$, hence $\hat{f}_n \in V_k$, for $n \geqq n_0$, i.e., $\hat{f}_n \xrightarrow{\hat{\tau}} \theta$. ■

THEOREM 5.14. <u>Assume that</u> $(L, \tau)$ <u>and</u> $(M, \sigma)$ <u>are two Hausdorff locally solid Riesz spaces with</u> $(L, \tau)$ <u>metrizable and with</u> $(M, \sigma)$ <u>satisfying</u> (A, i).

If $T:(L,\tau)\to(M,\sigma)$ is a continuous linear operator which is also an integral, then the unique linear continuous extension $\hat{T}:(\hat{L},\hat{\tau})\to(\hat{M},\hat{\sigma})$ is also an integral.

PROOF. Without loss of generality we can assume that $\theta \leqq T$. The unique continuous linear extension $\hat{T}$, of $T$, follows from [12] (p. 195, Theorem 26), and obviously we have $\theta \leqq \hat{T}$. Let $\hat{u}_n \downarrow \theta$ in $\hat{L}$. We have to show that $\hat{T}(\hat{u}_n) \downarrow \theta$ in $\hat{M}$. We can assume that $\{\hat{u}_n\} \subseteqq U$. Indeed, as in the previous theorem we can construct a sequence $\{\hat{w}_n\} \subseteqq U$ such that $\theta \leqq \hat{u}_n \leqq \hat{w}_n$, for $n = 1,2,\ldots$ and $\hat{w}_n - \hat{u}_n \xrightarrow{\hat{\tau}} \theta$. So, $\theta \leqq \hat{T}(\hat{u}_n)\downarrow \leqq \hat{T}(\hat{w}_n)\downarrow$ for $n = 1,2,\ldots$ and $\theta \leqq \hat{T}(\hat{w}_n) - \hat{T}(\hat{u}_n) = \hat{T}(\hat{u}_n - \hat{w}_n) \xrightarrow{\hat{\sigma}} \theta$. Using Lemma 5.12 we can see that our assertion is valid. So, let $\{\hat{u}_n\} \subseteqq U$ be such that $\hat{u}_n \downarrow \theta$ in $\hat{L}$. We have to show that $\hat{T}(\hat{u}_n) \downarrow \theta$ in $\hat{M}$. Assume that $\hat{T}(\hat{u}_n) \geqq \hat{h} \geqq \theta$ for $n = 1,2,\ldots$ and some $\hat{h} \in \hat{M}$.

Let $W$ be a solid $\hat{\sigma}$-neighborhood of zero of $\hat{M}$, and let $W_1$ be another $\hat{\sigma}$-neighborhood of zero of $\hat{M}$ such that $W_1 + W_1 \subseteqq W$. We choose next a solid $\hat{\tau}$-neighborhood $V$ of zero of $\hat{L}$ such that $\hat{T}(\hat{u}) \in W_1$ for all $\hat{u} \in V$. Let $\{V_n\}$ be a basis for the neighborhood system of the origin of $\hat{L}$ consisting of solid sets and such that $V_{n+1} + V_{n+1} \subseteqq V_n$, $n = 1,2,\ldots$ and $V_1 + V_1 \subseteqq V$.

Given $n \in N$ pick an element $\theta \leqq u_n \in L^+$ such that $\theta \leqq u_n \leqq \hat{u}_n$ and $\theta \leqq \hat{u}_n - u_n \in V_n$. Let $w_n = \bigwedge_{i=1}^{n} u_i$, $n = 1,2,\ldots$ . Then obviously $w_n \downarrow \theta$ in $L$, and so, $T(w_n) \downarrow \theta$ in $M$. Hence it follows from (A, i) in $(M,\sigma)$ that $T(w_n) \xrightarrow{\sigma} \theta$. So, in particular $T(w_n) \in W_1$ for all $n \geqq n_0$.

We observe now that

$$\theta \leqq \hat{u}_n - w_n = \hat{u}_n - \bigwedge_{i=1}^{n} u_i = \bigvee_{i=1}^{n} (\hat{u}_n - u_i) \leqq \bigvee_{i=1}^{n} (\hat{u}_i - u_i) \leqq$$

$$\leqq \sum_{i=1}^{n} (\hat{u}_i - u_i) \in V_1 + \dots + V_n \subseteqq V_1 + V_1 \subseteqq V ,$$

so $\hat{u}_n - w_n \in V$ for $n = 1, 2, \dots$ . Hence $\theta \leqq \hat{h} \leqq \hat{T}(\hat{u}_n) = \hat{T}(\hat{u}_n - w_n) + T(w_n) \in W_1 + W_1 \subseteqq W$ for all $n \geqq n_0$. So, $\hat{h} \in W$ for all $\hat{\sigma}$-neighborhoods W of zero of $\hat{M}$. Since $\hat{\sigma}$ is a Hausdorff topology we have $\hat{h} = \theta$, i.e., $\hat{T}(\hat{u}_n) \downarrow \theta$. This completes the proof. ■

Next we introduce two more conditions for given topological ordered vector space $(E, \tau, K)$.

(B, i): $\theta \leqq u_n \uparrow$ in L and $\{u_n\}$ is $\tau$-bounded, then $\{u_n\}$ is a $\tau$-Cauchy sequence.

(B, ii): $\theta \leqq u_\alpha \uparrow$ in L and $\{u_\alpha\}$ is $\tau$-bounded, then $\{u_\alpha\}$ is a $\tau$-Cauchy net.

(A subset S of a topological vector space $(E, \tau)$ is called $\tau$-bounded if for every $\tau$-neighborhood V of the origin there exists a positive number $\lambda_0$ (depending on V) such that $\lambda S \subseteqq V$ for all $0 \leqq \lambda \leqq \lambda_0$, see [7], p. 108).

Using a similar argument as in Lemma 4.6 we see that (B, i) holds if and only if (B, ii) holds, i.e., (B, i) and (B, ii) are equivalent.

The following theorem says that (B, i) is preserved under the topological completion of a metrizable locally solid Riesz space.

THEOREM 5.15. _The topological completion_ $(\hat{L}, \hat{\tau})$ _of a metrizable locally solid Riesz space_ $(L, \tau)$ _satisfies_ (B, i) _if and only if_ $(L, \tau)$ _satisfies_ (B, i).

PROOF. Obviously if $(\hat{L}, \hat{\tau})$ satisfies (B, i) then $(L, \tau)$ satisfies (B, i). So, assume $(L, \tau)$ satisfies (B, i) and let $\theta \leqq \hat{f}_n \uparrow$ be a $\hat{\tau}$-bounded sequence of $\hat{L}$. We have to show that $\{\hat{f}_n\}$ is a $\hat{\tau}$-Cauchy sequence. To do this, let $\{V_n\}$ be a sequence of solid $\hat{\tau}$-neighborhoods of zero such that $V_{n+1} + V_{n+1} \subseteqq V_n$, for $n = 1, 2, \ldots$ . Pick a $V_k$ from $\{V_n\}$. Given $n > k + 2$ choose an element $g_n \in L^+$ such that $|\hat{f}_n - g_n| \in V_{n+1}$, and construct the sequence $f_n = \bigvee_{i=k+3}^{n} g_i$, for $n = k+3, k+4, \ldots$ . Then we have $\theta \leqq f_n \uparrow$ in L, and

$$f_n - \hat{f}_n = \bigvee_{i=k+3}^{n} (g_i - \hat{f}_n) \leqq \bigvee_{i=k+3}^{n} (g_i - \hat{f}_i) \leqq \sum_{i=k+3}^{n} |g_i - \hat{f}_i| \in V_{k+3} + \ldots$$

$$\ldots + V_n \subseteqq V_{k+2}, \quad \hat{f}_n - f_n \leqq \hat{f}_n - g_n \leqq \sum_{i=k+3}^{n} |g_i - \hat{f}_i| \quad ,$$

for all $n > k+2$, so,

$$|\hat{f}_n - f_n| \leqq \sum_{i=k+3}^{n} |\hat{f}_i - g_i| \in V_{k+2} \ ;$$

hence $|f_n - \hat{f}_n| \in V_{k+2}$ for $n = k+3, k+4, \ldots$ .

On the other hand we have

$$\theta \leqq \sum_{i=k+3}^{n} |\hat{f}_i - g_i| \uparrow \quad \text{and} \quad \left\{ \sum_{i=k+3}^{n} |\hat{f}_i - g_i| : n \geqq k+3 \right\}$$

is a $\hat{\tau}$-Cauchy sequence. Hence

$$\sum_{i=k+3}^{n} |\hat{f}_i - g_i| \xrightarrow[\substack{n \geqq k+3 \\ n \to +\infty}]{\hat{\tau}} \hat{f} \quad ,$$

for some $\hat{f} \in \hat{L}$. It follows from Theorem 4.7(ii) that $\sum_{i=k+3}^{n} |\hat{f}_i - g_i| \uparrow \hat{f}$, so,

$\theta \leqq |\hat{f}_n - f_n| \leqq \hat{f}$, for $n = k+3, k+4, \ldots$. Using Theorem 4.3 we see that $[\theta, \hat{f}]$ is $\hat{\tau}$-bounded, so $\{\hat{f}_n - f_n : n \geqq k+3\}$ is $\hat{\tau}$-bounded. This implies that $\{\hat{f}_n - f_n\}$ is $\hat{\tau}$-bounded.

We observe now that $f_n = \hat{f}_n - (\hat{f}_n - f_n)$ for $n = 1, 2, \ldots$ and by hypothesis, $\{\hat{f}_n\}$ is a $\hat{\tau}$-bounded sequence. Hence $\{f_n\}$ is a $\hat{\tau}$-bounded sequence of $\hat{L}$, and so surely it is a $\tau$-bounded sequence of L. Now, since $(L, \tau)$ satisfies (B, i), $\{f_n\}$ is a $\tau$-Cauchy sequence of L, and so, it is also a $\hat{\tau}$-Cauchy sequence of $\hat{L}$.

This implies, in particular, that $|f_n - f_m| \in V_{k+2}$ for all $n, m \geqq n_0 > k+3$.

Now, for $n, m \geqq n_0$ we have

$$|\hat{f}_n - \hat{f}_m| \leqq |\hat{f}_n - f_n| + |f_n - f_m| + |f_m - \hat{f}_m| \in V_{k+2} + V_{k+2} + V_{k+2} \subseteqq V_k ,$$

so, $\hat{f}_n - \hat{f}_m \in V_k$ for all $n, m \geqq n_0$, i.e., $\{\hat{f}_n\}$ is a $\hat{\tau}$-Cauchy sequence.∎

The next theorem tells us that $\hat{L}$ has the Egoroff property if $(L, \tau)$ satisfies (A, i), provided that $\tau$ is a metrizable topology.

THEOREM 5.16. Let $(L, \tau)$ be a metrizable locally solid Riesz space. Then L has the Egoroff property if $(L, \tau)$ satisfies (A, i).

PROOF. Pick a countable basis $\{V_n\}$ for the neighborhood system of the origin of L, consisting of solid, $\tau$-closed sets and such that $V_{n+1} + V_{n+1} \subseteqq V_n$ for $n = 1, 2, \ldots$. Since $(L, \tau)$ satisfies (A, i), by Theorem 5.12 $(\hat{L}, \hat{\tau})$ also satisfies (A, i).

Now let $\theta \leqq \hat{u}_{n,k} \uparrow_k \hat{u}$ in $\hat{L}$, for $n = 1, 2, \ldots$. Since $(\hat{L}, \hat{\tau})$ satisfies (A, i) it follows that for every pair of indices n, m (n, m = 1, 2, ...)

there is an index $j = j(m,n)$ such that $\theta \leqq \hat{u} - \hat{u}_{n,\, j(m,n)} \in V_{m+n}$ and we can assume that $j = j(m,n)$ $(m, n = 1, 2, \ldots)$ is increasing in the variables separately. Let

$$\hat{v}_{m,n} = \bigwedge_{k=1}^{n} \hat{u}_{k,\, j(m,k)} \quad (m, n = 1, 2, \ldots) \quad .$$

then we have

$$\theta \leqq \hat{v}_{m,n} - \hat{v}_{m,n+p} \leqq \hat{u}_{n,\, j(m,n)} - \bigwedge_{k=n}^{n+p} \hat{u}_{k,\, j(m,k)} =$$

$$= \bigvee_{k=n}^{n+p} (\hat{u}_{n,\, j(m,n)} - \hat{u}_{k,\, j(m,k)}) \leqq \bigvee_{k=n}^{n+p} (\hat{u} - \hat{u}_{k,\, j(m,k)}) \leqq$$

$$\leqq \sum_{k=n}^{n+p} (\hat{u} - u_{k,\, j(m,k)}) \in V_{m+n} + \ldots + V_{m+n+p} \subseteqq V_n \quad ,$$

for all $n = 1, 2, \ldots$ and all $p = 1, 2, \ldots$ . This shows that the sequence $\{\hat{u}_{m,n} : n = 1, 2, \ldots\}$ is a $\hat{\tau}$-Cauchy sequence for $m = 1, 2, \ldots$, so $\hat{v}_{m,n} \xrightarrow{\hat{\tau}} \hat{u}_m$, for some $\hat{u}_m \in \hat{L}$. Since $\hat{v}_{m,n} \downarrow_n$ in $\hat{L}$ and $\theta \leqq \hat{v}_{m,n} \leqq \hat{u}$ we obtain from Theorem 4.7 that $\hat{v}_{m,n} \downarrow_n \hat{u}_m$ in $\hat{L}$ and $\theta \leqq \hat{u}_m \leqq \hat{u}$.

From $\theta \leqq \hat{u} - \hat{u}_{m,n} = \bigvee_{k=1}^{n} (\hat{u} - \hat{u}_{k,\, j(m,k)}) \in V_{m+1} + \ldots + V_{m+n} \subseteqq V_m$ we obtain that $\hat{u} - \hat{u}_{m,n} \in V_m$, for $n = 1, 2, \ldots$ . Since $V_m$ is $\hat{\tau}$-closed we see that $\hat{u} - \hat{u}_m \in V_m$ for $m = 1, 2, \ldots$, so $\hat{u}_m \xrightarrow{\hat{\tau}} \hat{u}$.

Also, $\hat{v}_{m,p} \leqq \hat{v}_{m+1,p}$ for every $p$, implies $\hat{u}_m \leqq \hat{u}_{m+1}$, i.e., $\theta \leqq \hat{u}_m \uparrow$. Hence, by Theorem 4.7, $\hat{u}_m \uparrow \hat{u}$.

Furthermore, $\hat{u}_m \leqq \hat{v}_{m,n} \leqq u_{k,\, j(m,k)}$ for $k = 1, 2, \ldots, n$ and so $\hat{u}_m \leqq \hat{u}_{n,\, j(m,n)}$. This shows that $\hat{L}$ has the Egoroff property. ∎

<u>Note.</u> We observe that in the proof of Theorem 5.16 we used only the metrizability property of $\tau$ and the (A, i) property of $(\hat{L}, \hat{\tau})$.

Since (A, iii) on $(L, \tau)$ implies (A, iii) on $(\hat{L}, \hat{\tau})$ and so (A, i) on $(\hat{L}, \hat{\tau})$ we can see that if the metrizable locally solid Riesz space $(L, \tau)$ satisfies (A, iii) then $(\hat{L}, \hat{\tau})$ has the Egoroff property.

## 5.6. THE PROPERTY (A, 0) IN THE METRIZABLE CASE

We recall that the ordered topological vector space $(E, \tau, K)$ satisfies condition (A, o) whenever it follows from $u_n \downarrow \theta$ in E and $\{u_n\}$ $\tau$-Cauchy sequence that $u_n \xrightarrow{\tau} \theta$. Obviously (A, i) or (A, ii ) imply (A, o), but (A, o) does not imply necessarily (A, i). As an example consider E to be the linear space of all real sequences which are eventually constant with the usual ordering and $\tau$ the topology generated by the (Riesz) norm $||f|| = \sup\{|f(n)| : n \in N\}$.

It is also evident that if $(E, \tau, K)$ is $\tau$-complete then $(E, \tau, K)$ satisfies the property (A, o).

The following theorem gives some characterizations of the (A, o) condition for the metrizable locally solid Riesz spaces.

THEOREM 5.17. Let $(L, \tau)$ be a metrizable locally solid Riesz space. Then the following conditions are equivalent.

(i) L satisfies condition (A, o).

(ii) For every $\theta < \hat{f} \in \hat{L}$, there exists $f \in L$ such that $\theta < f \leq \hat{f}$.

(iii) For every $\theta \leq \hat{f} \in \hat{L}$, we have $\hat{f} = \sup\{f \in L : \theta \leq f \leq \hat{f}\}$.

(iv) The embedding of L into $\hat{L}$, preserves arbitrary suprema and infima, i.e., $f_\alpha \downarrow \theta$ in L, implies $f_\alpha \downarrow \theta$ in $\hat{L}$.

(v) The embedding of L into $\hat{L}$, preserves countable infima and countable suprema, i.e., $f_n \downarrow \theta$ in L, implies $f_n \downarrow \theta$ in $\hat{L}$.

Note. Condition (iii) is expressed by saying that L is strictly order dense in $\hat{L}$.

PROOF. (ii)$\Leftrightarrow$(iii). Suppose that (iii) holds, and $\theta < \hat{f} \in \hat{L}$. Since $\hat{f} = \sup\{f \in L: \theta \leqq f \leqq \hat{f}\}$ in $\hat{L}$, it follows that $0 < f \leqq \hat{f}$, for some f in L, and this shows that (ii) holds. Suppose now that (ii) holds, and $\theta \leqq \hat{f} \in \hat{L}$. Let $\hat{g} \in \hat{L}$ be such that $f \in L$; $\theta \leqq f \leqq \hat{f}$ implies $f \leqq \hat{g}$. Assume further that $\hat{g} \leqq \hat{f}$. If $\hat{g} < \hat{f}$, then since (ii) holds, we have $\theta < f \leqq \hat{f} - \hat{g} \leqq \hat{f}$ for some $\theta < f \in L$. This also implies, according to our assumptions, that $\theta < f \leqq \hat{g}$, and so $\theta < 2f \leqq \hat{f}$ which also implies $\theta \leqq 2f \leqq \hat{g}$. Proceeding inductively we see that $\theta < nf \leqq \hat{f}$ for $n = 1, 2, \ldots$ and $\theta < f$. This contradicts the fact that $\hat{L}$ is Archimedean (Theorem 4.7). So $\hat{g} = \hat{f}$ and this shows that $\hat{f} = \sup\{f \in L: \theta \leqq f \leqq \hat{f}\}$.

(iii)$\Rightarrow$(iv). Let $f_\alpha \downarrow \theta$ in L. If $\theta < \hat{f} \leqq f_\alpha$ for all $\alpha \in \{\alpha\}$ holds in $\hat{L}$, then since (ii)$\Leftrightarrow$(iii) there exists an element $\theta < f \in L$ such that $\theta < f \leqq \hat{f} \leqq f_\alpha$ for all $\alpha \in \{\alpha\}$, i.e., $\inf\{f_\alpha\} \neq \theta$ in L, a contradiction. This shows that $f_\alpha \downarrow \theta$ in $\hat{L}$.

(iv)$\Rightarrow$(v). Obvious.

(v)$\Rightarrow$(i). Let $f_n \downarrow \theta$ in L and $\{f_n\}$ be a $\tau$-Cauchy sequence. Then $f_n \xrightarrow{\hat{\tau}} \hat{f}$ in $\hat{L}$, for some $\hat{f} \in \hat{L}$, and so by Theorem 4.7 we have $f_n \downarrow \hat{f}$ in $\hat{L}$. Since (v) holds we get $\hat{f} = \theta$, i.e., $f_n \xrightarrow{\tau} \theta$.

(i)$\Rightarrow$(ii). Assume $\theta \leqq \hat{f} \in \hat{L}$. We show that if $\theta \leqq f \leqq \hat{f}$, $f \in L$ implies $f = \theta$, and this will be enough to establish (ii). To this end, let V be a solid $\hat{\tau}$-neighborhood of zero in $\hat{L}$. By Lemma 5.11 there exists $\hat{g} \in \Lambda$ (a lower element) such that, $\theta \leqq \hat{g} \leqq \hat{f}$ and $\hat{f} - \hat{g} \in V$. So, there exists a sequence $\{g_n\} \subseteqq L^+$ such that $\hat{g} \leqq g_n \downarrow$ and $g_n \xrightarrow{\hat{\tau}} \hat{g}$, so, in particular $\{g_n\}$ is a

$\tau$-Cauchy sequence. We show next that $g_n \downarrow \theta$ in L. Indeed, if $\theta \leqq g \leqq g_n$ for all n, then by Lemma 4.7 we get that $\theta \leqq g \leqq \hat{g} \leqq \hat{f}$, and so by our hypothesis $g = \theta$, i.e., $g_n \downarrow \theta$ in L. Hence, by property (A, o) of (L, $\tau$) it follows that $g_n \xrightarrow{\tau} \theta$ in L, and so $g_n \xrightarrow{\hat{\tau}} \theta$ in $\hat{L}$. This implies that $\hat{g} = \theta$. But then, $\hat{f} = \hat{f} - \theta = \hat{f} - \hat{g} \in V$, for all neighborhoods V of the origin of $\hat{L}$, and this implies that $\hat{f} = \theta$. ∎

COROLLARY 5.18. <u>If a metrizable locally solid Riesz space (L, $\tau$) satisfies $\hat{L}^+ = U$, i.e., if every positive element of $\hat{L}$ is an upper element, then (L, $\tau$) satisfies condition (A, o).</u>

PROOF. Let $\theta \leqq \hat{f} \in \hat{L}$. Then we have $\theta \leqq f_n \uparrow \hat{f}$ for some sequence $\{f_n\} \subseteq L^+$, with $f_n \xrightarrow{\hat{\tau}} \hat{f}$. So, in particular we have $\hat{f} = \sup\{f \in L: \theta \leqq f \leqq \hat{f}\}$. It follows now from Theorem 5.17 that (L, $\tau$) satisfies (A, o). ∎

Next we shall investigate under which conditions the Dedekind completeness, $\sigma$-Dedekind completeness and super Dedekind completeness of L can be carried to the topological completion. We proceed with the following lemma.

LEMMA 5.19. <u>If a metrizable locally solid Riesz space (L, $\tau$) satisfies condition (A, o) and if L is $\sigma$-Dedekind complete, then we have:</u>

(i) <u>Every interval of L is $\tau$-complete, i.e., $[\theta, u]$ is $\tau$-complete for every $u \in L^+$.</u>

(ii) <u>$\hat{L}^+ = U$, i.e., every positive element of $\hat{L}$ is an upper element.</u>

PROOF. Let $\hat{f} \in \Lambda$, i.e., let $\hat{f}$ be a lower element. This means there exists a sequence $\{f_n\} \subseteq L^+$ such that $f_n \downarrow \hat{f}$ and $f_n \xrightarrow{\hat{\tau}} \hat{f}$. So, in particular $\{f_n\}$ is a $\tau$-Cauchy sequence. Since, $\theta \leqq f_n \downarrow$ in L and L is $\sigma$-Dedekind complete we have $f_n \downarrow f \in L^+$ in L, for some f. But then it follows from Theorem 5.17 that $\hat{f} = f \in L^+$. So, $\Lambda \subseteqq L^+$, i.e., the lower elements of $\hat{L}$ are the positive elements of L.

(i) Now let $\theta \leqq \hat{f} \leqq f$ and $f \in L$. By Lemma 5.11 and the above discussion it follows that there exists a sequence $\{f_n\} \subseteqq L^+$, $f_n \uparrow$ and $f_n \xrightarrow{\hat{\tau}} \hat{f}$. But then $f_n \uparrow f_0 \leqq f$ in L, so $f_0 - f_n \downarrow \theta$ in L, and $\{f_0 - f_n\}$ is a $\tau$-Cauchy sequence. Hence, from condition (A, o) on (L, $\tau$) we see that $f_n \xrightarrow{\hat{\tau}} f$, so $\hat{f} = f \in L$. This shows that L is an ideal of L. The result of (i) now follows immediately from Theorem 5.2.

(ii) We only have to show that $\hat{f} \in \hat{L}^+$ implies $\hat{f} \in U$. This is a direct application of Lemma 5.11 and the above discussion. ∎

Next we give an application of Lemma 5.19.

We recall that a subset A of an ordered vector space (E, K) is called order complete whenever it follows from $f_\alpha \uparrow \leqq f$ in E and $\{f_\alpha\} \subseteqq A$ that $\sup f_\alpha$ exists in E and $\sup f_\alpha \in A$. A topological Riesz space (L, $\tau$) is called a <u>locally order complete</u> Reisz space if there is a neighborhood basis of zero for $\tau$ consisting of solid and order complete sets.

With respect to the above notions we have the following theorem due to H. Nakano (see [20], Theorem 4.2, and [28]).

THEOREM 5.20. <u>If</u> (L, $\tau$) <u>is a locally order complete Riesz space, then every order interval is</u> $\tau$-<u>complete</u>.

We use Lemma 5.19 to give a different proof of Theorem 5.20 in the case in which $\tau$ is metrizable. It is evident that L is Dedekind complete and so in particular it is $\sigma$-Dedekind complete. We prove next that (A, o) is satisfied in $(L, \tau)$. Indeed, if $\{f_n\}$ is a $\tau$-Cauchy sequence of L and $f_n \downarrow \theta$ in L, then we have $f_n - f_m \in V$ for all $n, m \geqq n_0$, where V is a solid and order complete neighborhood of zero. So, for fixed $n \geqq n_0$, we have $f_n - f_m \uparrow_{m \geq n_0} f_n$ in L, thus $f_n \in V$ for all $n \geqq n_0$, and this shows that $f_n \xrightarrow{\tau} \theta$, i.e., (A, o) is satisfied in $(L, \tau)$. The result now follows immediately from Lemma 5.19.

THEOREM 5.21. Let $(L, \tau)$ be a metrizable locally solid Riesz space satisfying condition (A, o).

(i) If L is $\sigma$-Dedekind complete, then $\hat{L}$ is $\sigma$-Dedekind complete.

(ii) If L is Dedekind complete, then $\hat{L}$ is Dedekind complete.

(iii) If L is super Dedekind complete, then $\hat{L}$ is super Dedekind complete.

PROOF. From Lemma 5.19(i) it follows that L is an ideal of $\hat{L}$. So, (i) and (ii) follow immediately from Theorem 5.6.

(iii) It follows from Lemma 5.19 that L is a super order dense ideal of $\hat{L}$, which by its own right is a super Dedekind complete Riesz space. Also by (i) we have that $\hat{L}$ is a $\sigma$-Dedekind complete Riesz space. The result now follows from Theorem 29.5 of [18], p. 169.■

Note. The condition (A, o) is essential for Theorem 5.21. In [15] (Note XVI, p. 665, Ex. 66.6) W. A. J. Luxemburg exhibits a super Dedekind complete normed Riesz space $L_\rho$ whose norm completion is not even $\sigma$-Dedekind complete.

## 5.7. THE PROJECTION PROPERTIES

We begin with the following lemma.

LEMMA 5.22. Let $(L, \tau)$ be a Hausdorff locally solid Riesz space. Then the closure $\overline{A}$ in $(\hat{L}, \hat{\tau})$ of the projection band A of L is a projection band of L. Moreover $\overline{A^d} = (\overline{A})^d$.

PROOF. Let A be a projection band of L. Then $L = A \oplus A^d$. We will show that $\hat{L} = \overline{A} \oplus \overline{A^d}$, where $-$ denotes the closure in $(\hat{L}, \hat{\tau})$. So, let $\theta \leqq \hat{f} \in \hat{L}$. Then there exists a net $\{f_\alpha\} \subseteqq L^+$ such that $f_\alpha \xrightarrow{\hat{\tau}} \hat{f}$. But then, since $L = A \oplus A^d$ we can write $f_\alpha = f'_\alpha + f''_\alpha$, $\theta \leqq f'_\alpha \in A$, $\theta \leqq f''_\alpha \in A^d$, for all $\alpha \in \{\alpha\}$, and so $|f_{\alpha_1} - f_{\alpha_2}| = |f'_{\alpha_1} - f'_{\alpha_2}| + |f''_{\alpha_1} - f''_{\alpha_2}|$, for all $\alpha \in \{\alpha\}$ (see [18], Theorem 14.4(i), p. 69).

The last relation shows that the nets $\{f'_\alpha\}$ and $\{f''_\alpha\}$ are two $\hat{\tau}$-Cauchy nets of $\hat{L}$. So $f'_\alpha \xrightarrow{\hat{\tau}} \hat{f}_1$ and $f''_\alpha \xrightarrow{\hat{\tau}} \hat{f}_2$ for some $\hat{f}_1, \hat{f}_2 \in \hat{L}$. So, $\hat{f} = \hat{f}_1 + \hat{f}_2 \in \overline{A} + \overline{A^d}$, i.e., $\hat{L} = \overline{A} + \overline{A^d}$. To show that $\overline{A} \cap \overline{A^d} = \{\theta\}$, let $\theta \leqq \hat{f} \in \overline{A} \cap \overline{A^d}$. Then there are two nets of $L^+$, $\{f_\alpha\} \subseteqq A$ and $\{g_\alpha\} \subseteqq A^d$ such that $f_\alpha \xrightarrow{\hat{\tau}} \hat{f}$ and $g_\alpha \xrightarrow{\hat{\tau}} \hat{f}$. Therefore $\theta = f_\alpha \wedge g_\alpha \xrightarrow{\hat{\tau}} \hat{f} \wedge \hat{f} = \hat{f}$; hence $\hat{f} = \theta$, and this shows that $\hat{L} = \overline{A} \oplus \overline{A^d}$. We show next that $\overline{A}$ and $\overline{A^d}$ are ideals of $\hat{L}$. It is clear that $\overline{A}$ is a vector subspace of $\hat{L}$. Now let $\hat{g} \in \overline{A}$. Then $f_\alpha \xrightarrow{\hat{\tau}} \hat{f}$, for some net $\{f_\alpha\} \subseteqq A$. But then $\{|f_\alpha|\} \subseteqq A$ and $|f_\alpha| \xrightarrow{\hat{\tau}} |\hat{f}|$, so $|\hat{f}| \in \overline{A}$. Now, if $\theta \leqq \hat{f} \leqq \hat{g}$ and $\hat{g} \in \overline{A}$, it follows easily that $\hat{f} \in \overline{A}$. This shows that $\overline{A}$ is an ideal of $\hat{L}$. Similarly for $\overline{A^d}$. The conclusion, now, follows from Theorem 24.1 of [18], page 131.■

We recall that the element $\theta < e$ of a Riesz space L is called a weak unit if $B_e = L$, i.e., if the band generated by e is all of L, or in other words, for Archimedean Riesz spaces, if it follows from $f \perp e$ that $f = \theta$.

The next theorem says that the completion $(\hat{L}, \hat{\tau})$ of the metrizable locally solid Riesz space $(L, \tau)$ has a weak unit under some conditions.

THEOREM 5.23. *If a metrizable locally solid Riesz space* $(L, \tau)$ *satisfies condition* $(A, o)$, *then* $(\hat{L}, \hat{\tau})$ *has a weak unit if* $(L, \tau)$ *has a weak unit.*

PROOF. Let $\theta < e \in L$ be a weak unit of L and let $\theta \leqq \hat{f} \in \hat{L}$ be such that $\hat{f} \perp e$. If $\theta < \hat{f}$, there exists, by Theorem 5.17 an element $\theta < f \in L$ such that $\theta < f \leqq \hat{f}$. But then $f \perp e$ and $f \neq \theta$, contradicting the fact that e is a weak unit of L. Hence $\hat{f} = \theta$. This shows that e is also a weak unit of $\hat{L}$. ∎

*Note.* The same result holds if L has a weak unit and it is order dense in $\hat{L}$.

From the above two results the following theorem follows immediately.

THEOREM 5.24. *If a metrizable locally solid Riesz space* $(L, \tau)$ *satisfies* $(A, o)$ *if* $\theta < u \in L$ *and if the band generated by* u, $B_u$ *is a projection band, then the closure of* $B_u$ *in* $(\hat{L}, \hat{\tau})$ *is the band generated in* $\hat{L}$ *by* u. *In particular, if* $\sup\{v \wedge nu : n = 1, 2, \ldots\}$ *exists in* L *for all* $\theta < v \in L$, *then* $\sup\{\hat{f} \wedge nu : n = 1, 2, \ldots\}$ *exists in* $\hat{L}$ *for all* $\theta < \hat{f} \in \hat{L}$, *and if* $v \in L$ *then* $\sup\{v \wedge nu : n = 1, 2, \ldots\}$ *in* L *equals* $\sup\{v \wedge nu : n = 1, 2, \ldots\}$ *in* $\hat{L}$.

PROOF. Since u is a weak unit of $B_u$ it follows from Theorem 5.23 that u is also a weak unit of $\overline{B_u}$ (the closure in $\hat{L}$ of $B_u$). Furthermore, by Lemma 5.22, $B_u$ is a $\hat{\tau}$-closed band of $(\hat{L}, \hat{\tau})$. By Theorem 4.9 every band is $\hat{\tau}$-closed, and so the smallest band containing u in $\hat{L}$

is $\overline{B_u}$. The remainder of the theorem now easily follows from Lemma 5.22 and Theorem 5.17(v).■

From the above theorem we might expect that the principal projection property and the projection property are preserved under topological completion, at least in the case when $\tau$ is metrizable and satisfies condition (A, o). Unfortunately, as it was shown by W. A. J. Luxemburg ([15], Note XVI, Example 65.6, p. 663), this is not true, and so Theorem 5.24 seems to be the best result we can get in that direction without additional assumptions.

However, as we shall show next, $\hat{L}$ has sufficiently many projections, if L has sufficiently many projections, provided (L, $\tau$) is as in Theorem 5.24.

THEOREM 5.25. Let (L, $\tau$) be a metrizable, locally solid Riesz space satisfying condition (A, o). Then $\hat{L}$ has sufficiently many projections, if L has sufficiently many projections.

PROOF. Let A be a non-zero band of $\hat{L}$, Then it follows from Theorem 5.17 that $A \cap L$ is a non-zero band of L (use part (iii) of Theorem 5.17 to show that $A \cap L \neq \{\theta\}$ and part (iv) to show that $A \cap L$ is a band of L). But then there exists a non-zero projection band B of L such that $B \subseteqq A$. It follows from Theorem 5.22 that the $\hat{\tau}$-closure $\overline{B}$ of B in ($\hat{L}$, $\hat{\tau}$) is a non-zero projection band of $\hat{L}$. So, from Theorem 4.8 we see that $\overline{B} \subseteqq A$. This shows that $\hat{L}$ has sufficiently many projections.■

The next theorem gives a condition under which L has the projection property provided $\hat{L}$ has the projection property.

THEOREM 5.26. Let $(L, \tau)$ be a Hausdorff locally solid Riesz space. Assume that L is an ideal of $\hat{L}$. Then $\hat{L}$ has the projection property if L has the projection property.

PROOF. Assume L has the projection property and let B be a band of $\hat{L}$. We shall show that $B \cap L$ is a band of L. It is obvious that $B \cap L$ is an ideal of L. Now let $\theta \leqq u_\alpha \uparrow u$ in L, for some $\{u_\alpha\} \subseteqq B \cap L$. Since L is an ideal of $\hat{L}$, it follows that $\theta \leqq u_\alpha \uparrow u$ in $\hat{L}$, and so, since B is a band of $\hat{L}$, $u \in B$, i.e., $B \cap L$ is a band of L. It follows from the projection property of L that $L = (B \cap L) \oplus (B \cap L)^d$, and so, from Theorem 5.22 we obtain $\hat{L} = \overline{B \cap L} \oplus \overline{(B \cap L)^d} = (\overline{B \cap L}) \oplus (\overline{B \cap L})^d$.

We show next that $\overline{B \cap L} = B$. From Theorem 4.8 we see easily that $\overline{B \cap L} \subset B$. Now, let $\theta \leqq \hat{f} \in B$. Then there exists sone net $\{f_\alpha\} \subseteqq L^+$ such that $\theta \leqq f_\alpha \xrightarrow{\hat{\tau}} \hat{f}$. But $\theta \leqq f_\alpha \wedge \hat{f} \xrightarrow{\hat{\tau}} \hat{f}$. Since L is an ideal of $\hat{L}$ it follows easily that $\{f_\alpha \wedge \hat{f}\} \subseteqq L$, and since B is a band of $\hat{L}$ we have also that $\{f_\alpha \wedge \hat{f}\} \subseteqq B$, i.e., $\{f_\alpha \wedge \hat{f}\} \subseteqq B \cap L$. This shows that $\hat{f} \in \overline{B \cap L}$, so, $B = \overline{B \cap L}$.

Hence $\hat{L} = B \oplus B^d$, which proves that B is a projection band of $\hat{L}$, and this completes the proof.■

## 5.8. THE FATOU PROPERTIES

We recall that a Riesz space is called a normed Riesz space if there exists a norm $\rho$ such that $\rho(u) = \rho(|u|)$ for all $u \in L$, and if $\theta \leqq f \leqq g$ in L implies $\rho(f) \leqq \rho(g)$ in $\mathbb{R}$. Every norm of L satisfying the above two properties is called a Riesz norm. We shall denote a Riesz space L with the Riesz norm $\rho$ by $L_\rho$.

Following W. A. J. Luxemburg and A. C. Zaanen ([16], Notes II and XIII) we use the following definition.

DEFINITION 5.27. (Sequential Fatou property). A Riesz norm $\rho$ is said to have the sequential Fatou property whenever $\theta \leqq u_n \uparrow u$ implies $\rho(u_n) \uparrow \rho(u)$.

(Fatou property). A Riesz norm $\rho$ is said to have the Fatou property whenever $\theta \leqq u_\alpha \uparrow u$ implies $\rho(u_\alpha) \uparrow \rho(u)$.

It is evident that the Fatou property implies the sequential Fatou property. Also, the property (A, i) implies the sequential Fatou property and the property (A, ii) implies the Fatou property.

EXAMPLE 5.28. (i) Let L be the Riesz space of all continuous functions on [0, 1], i.e., $L = C_{[0,1]}$, with the usual ordering. Let $\rho(f) = \int_0^1 |f(x)|\, dx$. Then $\rho$ is a Riesz norm for L without the sequential Fatou property as we can see by using an argument similar to that of Example 4.5(ii).

(ii) Let L be the Riesz space of all real sequences which are eventually constant, i.e., $f = \{f(n)\}$ is in L if there exists a constant c and a natural number $n_0$ (both depending on f) such that $f(n) = c$ for all $n \geqq n_0$, with the pointwise ordering. Let $\rho(f) = \sup\{|f(n)| : n \in N\}$. Then $L_\rho$ is a non-complete normed Riesz space with the (A, o) property. Note that (A, i) does not hold. An easy verification shows that $L_\rho$ satisfies the Fatou property and consequently it also satisfies the sequential Fatou property.

(iii) Let L be the Riesz space of all bounded real valued, Lebesgue measurable functions defined on $[0,1]$, with ordering $f \leqq g$ whenever $f(x) \leqq g(x)$ for all $x \in [0,1]$.

Let $\rho(f) = \int_0^1 |f(x)|\, dx + \sup\{|f(x)| : x \in [0,1]\}$. Then $\rho$ is a complete Riesz norm of L. Note that $f_\alpha \uparrow f$ in L implies $f_\alpha(x) \uparrow f(x)$ for all $x \in [0,1]$. Using this and the Lebesgue dominated convergence Theorem we see easily that $L_\rho$ satisfies the sequential Fatou property. But $L_\rho$ does not satisfy the Fatou property. To see this consider the net $\theta \leqq f_\alpha \uparrow e$, $e(x) = 1$ for all $x \in [0,1]$, $f_\alpha = \chi_\alpha$, $\alpha \in \mathcal{F}([0,1])$. Then $\rho(f_\alpha) = 1 \not\uparrow \rho(e) = 2$.

(iv) The product of the normed Riesz spaces of (ii) and (iii) gives a normed Riesz space without the Fatou property, without being norm complete but with the sequential Fatou property. ∎

We generalize Definition 5.26 to more general topologies. To do this let $(L, \tau)$ be a Hausdorff locally solid Riesz space, then we have

DEFINITION 5.29. <u>(Sequential Fatou property)</u>. $(L, \tau)$ <u>satisfies the sequential Fatou property if there exists a basis</u> $\{V\}$ <u>for the neighborhood system of the origin such that each</u> $V \in \{V\}$ <u>is solid and</u> $\theta \leqq f_n \uparrow f$ <u>in</u> L, $\{f_n\} \subseteqq V$ <u>implies</u> $f \in V$.

<u>(Fatou property)</u>. $(L, \tau)$ <u>satisfies the Fatou property if there exists a basis</u> $\{V\}$ <u>for the neighborhood system of the origin such that each</u> $V \in \{V\}$ <u>is solid and</u> $\theta \leqq f_\alpha \uparrow f$ <u>in</u> L, $\{f_\alpha\} \subseteqq V$ <u>implies</u> $f \in V$.

We shall show next that the norm completion of a normed Riesz space with the (sequential) Fatou property is also a normed Riesz space with the (sequential) Fatou property.

To do this we need the following preliminary discussion. We recall that a Riesz space L is called universally complete whenever every system $\{u_\alpha : \alpha \in \{\alpha\}\}$ of mutually disjoint elements of $L^+$ has a supremum. Given an Archimedean Riesz space L there exists a Dedekind complete and universally complete Riesz space K (unique up to a Riesz isomorphism) such that (i) K contains L as a Riesz subspace.

(ii) Every $u^*$ in $K^+$ satisfies $u^* = \sup\{u \in L : \theta \leqq u \leqq u^*\}$. (See [18], p. 338 and Theorem 50.8 on page 340.)

The following lemma is due to D. H. Fremlin (see [3], Proposition 1, p. 342).

LEMMA 5.30. Let L be an Archimedean Riesz space and let K be its universal completion. Assume that $\theta \leqq u_\alpha^* \uparrow$ in K be such that for every $\theta < u \in L$ there exists $n \in N$ such that $\sup\{u_\alpha^* \wedge nu : \alpha \in \{\alpha\}\} \neq nu$. Then $\theta \leqq u_\alpha^* \uparrow u^*$ for some $u^*$ in K.

We shall show next that the sequential Fatou property is preserved under norm completion.

THEOREM 5.31. Let $L_\rho$ be a normed Riesz space satisfying the sequential Fatou property. Then the norm completion $\hat{L}_{\hat{\rho}}$ also satisfies the sequential Fatou property.

PROOF. Let K be the universal completion of L. We define the following extended real valued function $\rho^*$ on K:

$$\rho^*(u^*) = \inf\{\lim_{n\to+\infty} \rho(u_n) : \{u_n\} \subseteqq L^+;\ u_n \uparrow \text{ and } u_n \wedge |u^*| \uparrow |u^*|\} \quad ,$$

with $\inf \emptyset = +\infty$.

Then we have:

(i) $\rho^*(u) = \rho(u)$ for all $u \in L$.

Obviously $\rho^*(u) \leqq \rho(u)$ for all $u \in L$. Now let $\{u_n\} \subseteqq L^+$, $u_n \uparrow$ and $u_n \wedge |u| \uparrow |u|$ in L. It follows from the sequential Fatou property of $L_\rho$ that $\rho(u_n \wedge |u|) \uparrow \rho(|u|) = \rho(u)$, and since $\rho(u_n \wedge |u|) \leqq \rho(u_n)$, $n = 1, 2, \ldots$, we obtain that $\rho(u) \leqq \lim_{n\to+\infty} \rho(u_n)$. Hence $\rho(u) \leqq \rho^*(u)$, i.e., $\rho(u) = \rho^*(u)$ for all $u \in L$.

(ii) $\rho(u^*) = \rho^*(|u^*|)$ for all $u^* \in K$, and $\theta \leqq u^* \leqq v^*$ implies $\rho^*(u^*) \leqq \rho^*(v^*)$.

This is an immediate consequence of the definition of $\rho^*$.

(iii) $\rho^*(u^*) \geqq 0$ for all $u^* \in K$ and $\rho^*(u^*) = 0$ implies $u^* = \theta$.

Indeed, if $\rho^*(u^*) = 0$ and $\theta < u^*$, then there exists $\theta < u \in L$ such that $\theta < u \leqq u^*$, since L is strictly order dense in K. But then it follows from (ii) and (i) that $\rho(u) = 0$, a contradiction. Hence $u^* = \theta$.

(iv) $\rho^*(u^* + v^*) \leqq \rho^*(u^*) + \rho^*(v^*)$ for all $u^*$, $v^*$ in K.

It is enough to prove the relation for $u^*$, $v^*$ in $K^+$. If one of the $\rho^*(u^*)$ and $\rho^*(v^*)$ is infinite, then the relation is obvious. So, let both $\rho^*(u^*)$ and $\rho^*(v^*)$ be finite, and let $\epsilon > 0$. Then there are two sequences $\theta \leqq u_n\uparrow$, $\theta \leqq v_n \uparrow$ of L such that $u_n \wedge u^* \uparrow u^*$, $v_n \wedge v^* \uparrow v^*$ and $\lim_{n\to+\infty} \rho(u_n) \leqq \rho^*(u^*) + \epsilon$, $\lim_{n\to+\infty} \rho(v_n) \leqq \rho^*(v^*) + \epsilon$. Since $\theta \leqq (u_n + v_n) \wedge (u^* + v^*) \uparrow u^* + v^*$, and $\rho(u_n + v_n) \leqq \rho(u_n) + \rho(v_n)$ for all $n \in N$ we obtain $\rho^*(u^* + v^*) \leqq \lim_{n\to+\infty} \rho(u_n + v_n) \leqq \lim_{n\to+\infty} \rho(u_n) + \lim_{n\to+\infty} \rho(v_n) \leqq \rho^*(u^*) + \rho^*(v^*) + 2\epsilon$ for all $\epsilon > 0$. Hence $\rho^*(u^* + v^*) \leqq \rho^*(u^*) + \rho^*(v^*)$.

(v) $\rho^*(\lambda u^*) = |\lambda|\, \rho^*(u^*)$ for all $u^* \in K$ and all $\lambda \in \mathbb{R}$.

(vi) If $\{u_n\} \subseteqq L^+$ and $\theta \leqq u_n \uparrow u^*$, then $\rho(u_n) \uparrow \rho^*(u^*)$.

It is obvious that $\rho^*(u^*) \leqq \lim_{n\to+\infty} \rho(u_n)$. Also, $\rho(u_n) = \rho^*(u_n) \leqq \rho^*(u)$; for $n = 1, 2, \ldots$. Hence, $\lim_{n\to+\infty} \rho(u_n) \leqq \rho^*(u)$, i.e., $\rho(u_n) \uparrow \rho^*(u^*)$.

(vii) Let $U = \{u^* \in K^+ : \theta \leqq u_n \uparrow u^*$, for some sequence $\{u_n\} \subseteqq L^+\}$.

Assume that $\theta \leqq u_n^* \uparrow$ in K, $\{u_n^*\} \subseteqq U$ and that $\rho^*(u_n^*) \uparrow P < +\infty$. Then there exists $u^* \in U$, $\theta \leqq u_n^* \uparrow u^*$ and $\rho^*(u^*) = P$.

Given n $(n = 1, 2, \ldots)$, let $\{u_{n,k} : k = 1, 2, \ldots\} \subseteqq L^+$ be such that $u_{n,k} \uparrow_k u_n^*$ $(n = 1, 2, \ldots)$. Define

$$w_n = \bigvee_{i=1}^{n} u_{i,n}, \quad n = 1, 2, \ldots .$$

Then $\{w_n\} \subseteqq L^+$ and $w_n \leqq u_n^*$, $n = 1, 2, \ldots$. It is obvious that $\rho(w_n) \leqq P$ for $n = 1, 2, \ldots$. Let $\theta < u \in L$. Pick $k \in N$ such that $k\rho(u) = \rho(ku) > P$. Then $\sup\{w_n \wedge ku : n = 1, 2, \ldots\} < ku$. Otherwise, if $w_n \wedge ku \uparrow ku$ it would follow from the sequential Fatou property of $L_\rho$ that $\rho(w_n \wedge ku) \uparrow \rho(ku)$ and so $\rho(ku) \leqq P$, a contradiction. It follows now from Lemma 5.30 that $\theta \leqq w_n \uparrow u^*$ for some $u^*$ in k. It follows easily now that $\theta \leqq u_n^* \uparrow u^*$, and that $u^* \in U$. From $w_n \leqq u_n^*$, $n = 1, 2, \ldots$, and from (vi) it follows also that $P = \rho^*(u^*)$.

(viii) Let $\theta \leqq u^*$, $\rho^*(u^*) < +\infty$ and let $\epsilon > 0$. Then there exists $v^* \in U$, $u^* \leqq v^*$ such that $\rho^*(v^*) \leqq \rho^*(u^*) + \epsilon$.

Indeed, let $\{u_n\} \subseteqq L^+$, $u_n \uparrow$, $u_n \wedge u^* \uparrow u^*$ be such that $\lim_{n \to +\infty} \rho(u_n) \leqq \rho^*(u^*) + \epsilon$. If $u_n \wedge ku \uparrow ku$ in L for some $u > \theta$, and all $k \in N$, then it follows easily from the sequential Fatou property of $L_\rho$ that $\rho(u) = 0$, a contradiction. It follows now from Lemma 5.30 that $u_n \uparrow v^* \in U$, and from (vi) we see that $\rho^*(v^*) \leqq \rho^*(u^*) + \epsilon$. Note also that $u^* \leqq v^*$.

(ix) Let $L_{\rho^*} = \{u^* \in K : \rho^*(u^*) < +\infty\}$. Then $L_{\rho^*}$ is a complete normed Riesz space.

Let $\{u_n^*\} \subseteqq L_{\rho^*}^+$ be such that $\sum_{n=1}^{\infty} \rho^*(u_n^*) = P < +\infty$, and let $\epsilon > 0$.

Pick $v_n^* \in U$, $u_n^* \leqq v_n^*$ such that $\rho^*(v_n^*) \leqq \rho^*(u_n^*) + \frac{\epsilon}{2^{n+1}}$, $n = 1, 2, \ldots$ .
Then we have

$$\sum_{n=1}^{\infty} \rho^*(v_n^*) \leqq P + \epsilon \quad .$$

It follows from (vii) that

$$\theta \leqq S_n^* = \sum_{k=1}^{n} v_n^* \uparrow S^*$$

and hence $\rho^*(S^*) \leqq P + \epsilon$. It follows then that $\theta \leqq \sum_{k=1}^{n} u_n^* \uparrow w^* \leqq S^*$. Thus

$$\rho^*\left(\sum_{k=1}^{\infty} u_n^*\right) = \rho^*(w^*) \leqq \rho^*(S^*) \leqq P + \epsilon$$

for all $\epsilon > 0$. Hence

$$\rho^*\left(\sum_{n=1}^{\infty} u_n^*\right) \leqq \sum_{n=1}^{\infty} \rho^*(u_n^*) \quad .$$

Theorem 26.2 of [16], Note VIII, p. 105 shows that $L_{\rho^*}$ is norm complete.

(x) The closure of $L_\rho$ in $L_{\rho^*}$, $\hat{L}_{\hat{\rho}}$, is the norm completion of $L_\rho$.

We show that $\hat{L}_{\hat{\rho}}$ satisfies the sequential Fatou property. Let $\theta \leqq u_n^* \uparrow u^*$ in $\hat{L}_{\hat{\rho}}$. Then, since $L_\rho$ is strictly order dense in K, $u_n^* \uparrow u^*$ in K. Pick an element $u_0^*$ in $\hat{L}_{\hat{\rho}}$, $u^* \leqq u_0^*$, $u_0^* \in U$ and $\rho^*(u_0^* - u^*) < \epsilon$, for given $\epsilon > 0$ (see Lemma 5.10). Similarly pick $v_n^*$ in $\hat{L}_{\hat{\rho}}$, $u_n^* \leqq v_n^*$, $\rho^*(v_n^* - u_n^*) \leqq \frac{\epsilon}{2^{n+1}}$ and $v_n^* \in U$, $n = 1, 2, \ldots$ . Obviously we can suppose that $v_n^* \leqq u_0^*$ for $n = 1, 2, \ldots$ .

Now let

$$w_n^* = \bigvee_{i=1}^{n} v_i^* , \quad n = 1, 2, \ldots \quad .$$

Then we have $\theta \leqq u_n^* \leqq w_n^* \uparrow \leqq u_0^*$, and

$$\theta \leqq w_n^* - u_n^* = \bigvee_{i=1}^{n} v_i^* - u_n^* = \bigvee_{i=1}^{n} (v_i^* - u_n^*) \leqq \bigvee_{i=1}^{n} (v_i^* - u_i^*) \leqq \sum_{i=1}^{n} (v_i^* - u_i^*) \quad .$$

Thus

$$\rho^*(w_n^* - u_n^*) \leqq \sum_{i=1}^{n} \rho^*(v_i^* - u_i^*) \leqq \sum_{i=1}^{n} \frac{\epsilon}{2^{i+1}} \leqq \epsilon \quad .$$

Let $w_n^* \uparrow u_1^* \leqq u_0^*$. Then obviously $u^* \leqq u_1^* \leqq u_0^*$. It follows then that $\rho^*(w_n^*) \leqq \rho^*(u_n^*) + \epsilon \leqq \lim_{n\to+\infty} \rho^*(u_n^*) + \epsilon$, for $n = 1, 2, \ldots$ . Thus by (vii) we have $\rho^*(u^*) \leqq \rho^*(u_1^*) = \lim_{n\to+\infty} \rho^*(w_n^*) \leqq \lim_{n\to+\infty} \rho^*(u_n^*) + \epsilon$, for all $\epsilon > 0$. Hence $\rho^*(u^*) \leqq \lim \rho^*(u_n)$, i.e., $\rho^*(u_n^*) \uparrow \rho^*(u^*)$, and this shows that $\hat{L}_{\hat{\rho}}$ satisfies the sequential Fatou property.■

The following general theorem is due to D. H. Fremlin (see [3], Theorem 1, p. 343).

THEOREM 5.32. <u>Let</u> $(L, \tau)$ <u>be a Hausdorff locally solid Riesz space satisfying the Fatou property.</u> <u>Then we have:</u>

(i) L <u>is strictly order dense in</u> $\hat{L}$.

(ii) $(\hat{L}, \hat{\tau})$ <u>satisfies the Fatou property.</u> <u>In particular, if</u> $\{V\}$ <u>is a Fatou neighborhood basis of zero for</u> $\tau$, <u>then</u> $\{\tilde{V}\}$, <u>where</u> $\tilde{V} = \{\hat{u} \in \hat{L} : u \in L$ <u>and</u> $|u| \leqq |\hat{u}|$ <u>implies</u> $u \in V\}$, <u>is a Fatou neighborhood basis of zero for</u> $\hat{\tau}$.

Given a Riesz space L and a Riesz semi-norm p on L we say that p is a Fatou semi-norm if $\theta \leqq u_\alpha \uparrow u$ in L implies $p(u_\alpha) \uparrow p(u)$.

The next theorem is based upon Theorem 5.32.

THEOREM 5.33. <u>Let</u> $(L, \tau)$ <u>be a Hausdorff locally solid, locally convex Riesz space with</u> $\tau$ <u>generated by the family of semi-norms</u> $\{p_i\}$. <u>If each</u> $p_i$ <u>is a Fatou semi-norm, then each</u> $\hat{p}_i$ <u>(the unique continuous</u>

<u>extension of</u> $p_i$ <u>to</u> $\hat{L}$) <u>is a Fatou semi-norm on</u> $\hat{L}$. <u>Moreover</u> $\hat{p}_i(\hat{u}) = \sup\{p_i(u) : u \in L;\ \theta \leqq u \leqq |\hat{u}|\}$ <u>for all</u> $\hat{u} \in \hat{L}$.

PROOF. Let $\tilde{p}_i(\hat{u}) = \sup\{p_i(u) : u \in L;\ \theta \leqq u \leqq |\hat{u}|\}$, $\hat{u} \in \hat{L}$. Then we have:

(1) $\tilde{p}_i(\hat{u}) \geqq 0$ for all $\hat{u} \in \hat{L}$.

(2) $\tilde{p}_i(\lambda\hat{u}) = |\lambda|\tilde{p}_i(\hat{u})$ for all $\hat{u} \in \hat{L}$ and all $\lambda \in \mathbb{R}$.

(3) $\tilde{p}_i(\hat{u}) = \tilde{p}_i(|\hat{u}|)$, for all $\hat{u} \in \hat{L}$, and $\theta \leqq \hat{u} \leqq \hat{v}$ implies $\tilde{p}_i(\hat{u}) \leqq \tilde{p}_i(\hat{v})$.

(4) $\tilde{p}_i(u) = p_i(u)$ for all $u \in L$.

(5) $\tilde{p}_i(\hat{u}+\hat{v}) \leqq \tilde{p}_i(\hat{u}) + \tilde{p}_i(\hat{v})$ for all $\hat{u}, \hat{v} \in \hat{L}$.

To see (5) we may suppose that $\hat{u}, \hat{v} \in \hat{L}^+$. From Theorem 5.32(i) we have that $\sup\{u+v : u, v \in L^+;\ \theta \leqq u \leqq \hat{u},\ \theta \leqq v \leqq \hat{v}\} = \hat{u} + \hat{v}$. Now let $\theta \leqq w \leqq \hat{u}+\hat{v}$, $w \in L$. Then we have $\theta \leqq (u+v) \wedge w \uparrow_{(u,v)} w$ in L. It follows from the Fatou property of $p_i$ that $p_i((u+v) \wedge w) \uparrow_{(u,v)} p_i(w)$. But $p_i((u+v) \wedge w) \leqq p_i(u+v) \leqq p_i(u) + p_i(v) \leqq \tilde{p}_i(\hat{u}) + \tilde{p}_i(\hat{v})$. Hence $p_i(w) \leqq \tilde{p}_i(\hat{u}) + \tilde{p}_i(\hat{v})$. Thus $\tilde{p}_i(\hat{u}+\hat{v}) \leqq \tilde{p}_i(\hat{u}) + \tilde{p}_i(\hat{v})$.

(6) Let $V_i = \{u \in L : p_i(u) \leqq 1\}$. Then from Theorem 5.32(ii) we get that $\tilde{V}_i = \{\hat{u} \in \hat{L} : u \in L;\ |u| \leqq |\hat{u}| \text{ implies } u \in V_i\}$. It follows easily that $\tilde{V}_i = \{\hat{u} \in \hat{L} : \tilde{p}_i(\hat{u}) \leqq 1\}$. This shows that the family of Riesz semi-norms $\{\tilde{p}_i\}$ generates $\hat{\tau}$ on $\hat{L}$. Also, (6) shows that each $\tilde{p}_i$ is a Fatou semi-norm.

Now let $\hat{p}_i$ be the unique continuous extension of $p_i$ to $\hat{L}$, and let $\hat{u} \in \hat{L}$. Then $u_\alpha \xrightarrow{\hat{\tau}} \hat{u}$ for some net $\{u_\alpha\}$ of L. It follows that $\tilde{p}(\hat{u}) = \lim \tilde{p}(u_\alpha) = \lim p(u_\alpha) = \lim \hat{p}(u_\alpha) = \hat{p}(\hat{u})$. Thus $\tilde{p} = \hat{p}$. ∎

We recall that a locally solid Riesz space $(L, \tau)$ is called locally ordered complete if it has a basis of the neighborhood system of the origin consisting of solid and order complete sets (a subset S of L is called order complete if $f_\alpha \uparrow \leqq f$ in L and $\{f_\alpha\} \subseteqq S$ imply $f_\alpha \uparrow g$ for some g, with $g \in S$).

THEOREM 5.34. <u>Let</u> $(L, \tau)$ <u>be a Hausdorff locally ordered complete Riesz space.</u> <u>Then</u> $(\hat{L}, \hat{\tau})$ <u>is also a locally ordered complete Riesz space.</u>

PROOF. It is easy to see that $(L, \tau)$ satisfies the Fatou property. Hence by Theorem 5.32 $(\hat{L}, \hat{\tau})$ satisfies also the Fatou property. The result will now follow immediately if we prove that $\hat{L}$ is a Dedekind complete Riesz space. But by Nakano's Theorem (see [23], Prop. 1.3, p. 140), it follows that all the intervals of L are $\tau$-complete, hence, it follows from Theorem 5.2 that L is an ideal of $\hat{L}$. Since L is also Dedekind complete the result follows from Theorem 5.6(ii).∎

<u>Note.</u> If $(L, \tau)$ satisfies the sequential Fatou property then $(L, \tau)$ satisfies the property (A, o).

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