AN INVESTIGATION OF FORCED FLEXURAL TORSIONAL OSCILLATIONS OF A WING AND THE PHENOMENON OF FLUTTER

Thesis

by

Lieut. A. McB. Jackson, Jr., U. S. Navy

and

Lieut. E. W. Parish, Jr., U. S. Navy

In Partial Fulfillment of the

Requirements for the Degree of Master of Science

in Aeronautics

California Institute of Technology

Pasadena, California

1939

ACKNOWLEDGEMENT

The authors wish to express their deep appreciation to those who have aided in this investigation. They are especially grateful to Mr. A. E. Lombard, Jr., without whose assistance and continued guidance this investigation would not have been carried out. Dr. Th. von Kármán has kindly given advice and helpful suggestions throughout the work.

The authors are also very grateful to the Douglas Aircraft Company and to its Chief Engineer, Mr. A. E. Raymond for furnishing specific data pertaining to various airplane wings. They wish further to express appreciation for the cooperation and interest displayed by the engineers of the Research Department of the Douglas Aircraft Company.

ABSTRACT

In this thesis the tortional-flexural response of a two-dimensional airfoil to forced oscillations of various frequencies and at various airspeeds is investigated. The airfoil chosen has characteristics which are typical of modern American transport wings, and the speeds cover the range from zero airspeed up through the speeds for torsional-flexural flutter and torsional divergence.

In the latter part of this thesis, curves are plotted showing the effects which changes in the assumed wing parameters have on the torsional-flexural flutter speed and the torsional divergence speed.

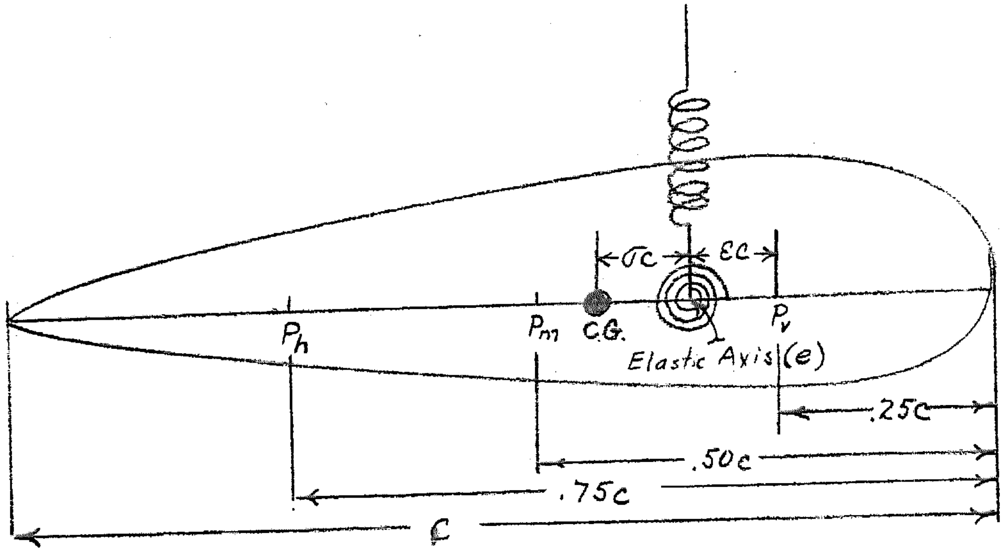
INTRODUCTION

A study of the phenomenon of flutter of an airfoil in an airstream leads at once to the conclusion that it is very complicated in nature and consequently practically impossible, or at least very difficult, to visualize clearly. The equations of motion of an airfoil oscillating in an airstream have been derived by a number of investigators; and the conditions at flutter, or at the limit of stability of the wing, have been investigated. In this thesis, the conditions which exist at velocities and frequencies other than flutter are investigated in hopes that such an investigation will shed additional light on the nature of flutter and how it develops.

The type of flutter selected for this investigation is the torsional-flexural type or one which involves the bending and twisting of the wing. Such a system is one of two degrees of freedom. The main difference between the oscillating wing and the usual type of oscillating system that we are accustomed to consider is that the wing in itself is not a conservative system, for under the proper conditions it will absorb energy from the airstream, and it is this fact that makes flutter possible. The flexing and twisting of the wing in the airstream cause the aerodynamic forces and moments on the wing to change. These changes in turn cause the wing to flex and twist. In general, the magnitude and phase relation of these changes are functions of the velocity and frequency. The majority of the time, the net effect of these forces and moments is to damp the oscillation, but under certain conditions their phase relations may be such that they have a negative damping effect and cause the oscillations to build up. This condition we call flutter.

Most of the theoretical investigations of flutter have been two dimensional, that is, the variations of the parameters along the wing span have been neglected or at the best averaged. For this reason, and because the parameters involved are difficult to determine, it has been suggested that it may be possible to determine the flutter speed of an actual plane by plotting curves of the natural frequency of the wing in bending and torsion against airspeed and then extrapolating these curves to intersection under the supposition that the speed at which the two natural frequencies coincide is the flutter speed. The nature of the curves must be known in order to perform this extrapolation and will be investigated in this thesis. The natural frequencies could be determined in flight by means of a forcing oscillator and pick-up or by means of a pick-up alone allowing the accidental "air bumps" to set the wing in vibration.

NOTATION



c = chord of wing

ε = distance of the elastic axis behind the quarter chord point in fraction of chord.

σ = distance of the center of gravity or inertia axis behind the elastic axis expressed as a fraction of the chord.

e = elastic axis, i.e. the point at which an applied force will produce no rotation of the wing.

k = spring constant of the wing in bending of flexure (force/unit vertical displacement).

T/ϕ = torsional constant of the wing (torque/radian).

$r_e = \frac{T/\phi}{k}$ or the "elastic radius", ϕ = angular deflection of the wing.

y_j = vertical displacement of any point j of the wing, positive up.

ν = circular frequency of oscillation (radians/sec).

$\nu_1 = \sqrt{\frac{k}{m_f}}$ natural frequency of the wing in bending neglecting the effect of the apparent mass of the air.

m_F = mass of wing per unit span.

m_L = "apparent mass of air", i.e. the mass of the circumscribed cylinder of air one unit in length.

v = forward velocity or airspeed.

$V = \frac{v}{\nu c}$ = reduced velocity.

$\overline{P}$ = A-iB = a complex function of the reduced velocity which enters into the consideration of the effect of the wake on a wing in steady state oscillations.

$(\ )_s$ = static response, that is, the response at $\nu = 0$ and $v = 0$.

$\overline{F}_j$ = vertical oscillatory force applied at the point j, positive up.

$\overline{M}_j$ = moment of the applied force about the point j, positive nosing up.

ρ = mass density of air.

P_v = .25 chord point.

P_m = .50 chord point.

P_h = .75 chord point.

L = lift force

I = moment of inertia.

EQUATIONS FOR TORSIONAL-FLEXURAL OSCILLATIONS

The case to be considered is a two-dimensional one where the airfoil is assumed to have uniform characteristics along the span and to be infinitely long.

The aerodynamic forces and moments on an airfoil in steady state oscillation are as follows:

(1) The forces and moments which arise from the fact that a certain amount of air around the wing also must be accelerated when the wing is accelerated. The "apparent" mass of air involved when the wing is accelerated perpendicular to its chord is equal to $m_L = \frac{\pi \rho c^2}{4}$ or the circumscribed cylinder of air. The "apparent" moment of inertia of air which is the amount involved in angular accelerations of the wing is equal to $I_L = \frac{\pi \rho c^4}{128}$.

(2) The changes in the lift and moment due to vertical velocity, the angular velocity, and the angular rotation of the wing considering the motion as quasi-steady, that is, the motion is considered slow enough to allow the circulation or lift to follow the changes.

(3) The forces and moments which arise from the effect of the vortices shed by a wing in steady state oscillation. This classification is that suggested by Kármán and Sears in reference (1).

The above considerations result in the following forces and moments on an airfoil in steady state oscillations:

At 25 per cent of the chord

$$L = \pi \rho c v^2 \left(\phi - \frac{\dot{y}_h}{v}\right) \bar{P}$$

At 50 per cent of the chord

$$L = \frac{\pi \rho c^2}{4} \left(\ddot{y}_h + \frac{\ddot{\phi} c}{4} \right)$$

At 75 per cent of the chord

$$L = \frac{\pi \rho c^2}{4} v \dot{\phi}$$

Moment couple

$$M = \frac{\pi \rho c^4}{128} \ddot{\phi}$$

where ($\dot{}$) and ($\ddot{}$) are the first and second time derivatives. These relations have been derived by Lombard in reference (2).

In addition to the aerodynamic forces, the following elastic and inertia forces will act:

(1) The moments and forces arising from the deflection of the elastic axis and the rotation about the elastic axis.

$$F = -ky_e \qquad \text{and} \qquad M = -k\eta^2 c^2 \phi$$

(2) The forces and moments arising from the inertia and moment of inertia of the wing's mass.

$$F = -m_F \ddot{y}_{CG} \qquad \text{and} \qquad M = -I \ddot{\phi}$$

Using these forces and moments, Lombard in reference (2) has set up the equations of equilibrium for a two-dimensional wing in steady state oscillation. The derivation assumes no structural damping and small oscillations, the latter condition allowing second order and higher terms to be neglected in certain instances. The equations are as follows:

$$(1a) \quad +\frac{\overline{F}}{m_F} = \overline{y}_h \left[+\nu_1^2 - \left(1+\frac{1}{\mu}\right)\nu^2 + \frac{4i}{\mu} \overline{P} \nu \frac{v}{c} \right]$$

$$+ \overline{\phi} c \left[+\nu_1^2 \left(\frac{1}{2} - \varepsilon\right) + \left(\varepsilon + \sigma - \frac{1}{2} - \frac{1}{4\mu}\right)\nu^2 - \frac{4}{\mu} \overline{P} \frac{v^2}{c^2} - \frac{i}{\mu} \frac{\nu v}{c} \right]$$

$$(1b)\quad +\frac{\overline{M}_{.25c}}{cm_F} = \overline{y}_h\left[-\nu_1^2\varepsilon + \left(\varepsilon + \sigma + \frac{1}{4\mu}\right)\nu^2\right] + \overline{\phi}_c\left[-\nu_1^2\left\{\varepsilon\left(\frac{1}{2}-\varepsilon\right)-\eta^2\right\} + \frac{i\nu}{2\mu}\frac{v}{c} + \nu^2\left\{-i_F^2 + (\varepsilon+\sigma)\left(\frac{1}{2}-\varepsilon-\sigma\right) + \frac{1}{32\mu}\right\}\right]$$

In the above equations the barred quantities are vectors rotating in the complex plane at an angular velocity ν. The real parts of these vectors represent the actual quantity and their relative angular positions determine their phase relations. This use of the complex plane is common practice in many fields of engineering, especially in electrical engineering, and it is particularly adapted to vibration problems. A thorough explanation of its use is given in "Mechanical Vibrations" by J. P. Den Hartog.

To determine the limit of stability or flutter conditions the applied forces and moments are set equal to zero.

The equations then have the form:

$$0 = \overline{y}_h\,\overline{A}_{11} + \overline{\phi}_c\,\overline{A}_{12}$$

$$0 = \overline{y}_h\,\overline{A}_{21} + \overline{\phi}_c\,\overline{A}_{22}$$

In order that there be a solution other than 0,0 the determinant of the coefficients of $\overline{y}$ and $\overline{\phi}_c$ must equal zero, and since this involves complex quantities, we obtain two equations to determine V and ν from the condition that both the reals and imaginaries must equal zero. Since $\overline{F}$ cannot be expressed explicitly, the equations cannot be solved explicitly, but Kassner and Fingado in reference (3) have provided curves for the solution of the equations.

This solution gives only the conditions at flutter speed and it is desirable to investigate the conditions which exist at other speeds and frequencies.

Because of the number of parameters which enter into the equilibrium equations, it was necessary to assume certain numerical values for all of them except the velocity of flight and the frequency of vibration. In choosing these numerical values an attempt was made to select ones which were typical of modern American transport airplanes. The values selected were as follows:

1. Elastic axis at 35 per cent of the chord, i.e. $\varepsilon = 0.10$

2. Center of gravity of wing at 40 per cent of chord, i.e.,

$$\varepsilon + \sigma = 0.15$$

3. Radius of gyration of wing $= \sqrt{0.0625c^2}$ i.e. $i_p^2 = 0.0625$

4. Bending frequency at zero airspeed, neglecting the apparent mass of the air

$$= 300 \text{ cycles/min} = 10\pi \text{ rad/sec}$$

5. Elastic radius $= \sqrt{0.50}$ i.e., $\eta^2 = 0.50$

6. Ratio of the mass of wing per unit length to the apparent mass of the air $\mu = 6.00$

7. Wing chord = 7.5 ft.

8. Wing weight = 2.7 lbs/ft

Then by Kassner Fingado Chart flutter occurs at:

$\nu = 57.2$ rad/sec

$v/c = 73.2$ 1/sec corresponding to 374 mph.

$V = 1.28$

Returning now to the equations of equilibrium (1a) & (1b) and substituting the wing parameters, we obtain:

$$(2a)\quad + \frac{\overline{F}}{m_F} = \overline{y}_h \Big[\overbrace{+ 987 + 0.667\, i\nu \tfrac{v}{c}\,\overline{P} - 1.167\,\nu^2}^{\overline{A}_{11}}\Big] + \overline{\phi c}\,\Big[\overbrace{+395 - 0.667\,\tfrac{v^2}{c^2}\,\overline{P} - 0.167\, i\nu \tfrac{v}{c} - 0.3917\,\nu^2}^{\overline{A}_{12}}\Big]$$

$$(2b)\quad + \frac{\overline{M}_{.25c}}{cm_F} = \overline{y}_h \Big[\overbrace{-98.7 + 0.1917\,\nu^2}^{\overline{A}_{21}}\Big] + \overline{\phi c}\,\Big[\overbrace{+ 454 - 0.0048\,\nu^2 + 0.0833\, i\nu \tfrac{v}{c}}^{\overline{A}_{22}}\Big]$$

Note: This notation of $\overline{A}_{11}$, $\overline{A}_{12}$, etc. does not agree exactly with the notation of $\overline{A}_{11}$, $\overline{A}_{12}$, etc. in Lombard's report.

DETERMINATION OF THE NORMAL MODES AT v=0

Setting v = 0 in equations (2) we obtain:

$$(3a)\quad + \frac{\overline{F}}{m_F} = \overline{y}_h \left[+ 987 - 1.167\,\nu^2\right] + \overline{\phi c}\left[+ 395 - 0.3917\,\nu^2\right]$$

$$(3b)\quad + \frac{\overline{M}_{.25c}}{cm_F} = \overline{y}_h \left[- 98.7 + 0.1917\,\nu^2\right] + \overline{\phi c}\left[+ 454 - 0.0048\,\nu^2\right]$$

To determine the natural frequencies of vibration we set the determinant of the coefficients of $\overline{y}$ and $\overline{\phi c}$ to zero and obtain the frequency equation:

$$\nu^4 - 8030\,\nu^2 + 6{,}030{,}000 = 0$$

the roots of which are:

$\nu = 29.0$ rad/sec

$\nu = 84.8$ rad/sec

Now using equation (3b) and substituting $\nu = 29.0$, we obtain

$$0 = \overline{y}_h\left[- 98.7 + 161\right] + \overline{\phi c}\left[+ 454 - 4.0\right]$$

$$\overline{\phi c} = -.138\,\overline{y}_h$$

This equation is satisfied if the wing oscillates with a nodal point 6.5 chord lengths ahead of the leading edge.

Again using equation (3b) and substituting $\nu = 84.8$, we obtain

$$0 = \bar{y}_h\left[-98.7 + 1378\right] + \bar{\phi}c\left[+ 454 - 34.5\right]$$

$$\bar{\phi}c = -3.05\ \bar{y}_h$$

This equation indicates a nodal point at 42.2 per cent of the chord behind the leading edge.

If an oscillating force is applied at the nodal point of one normal mode, the resulting vibrations will be in the other normal mode.

We will now shift the flexural coordinate of our equations from $\bar{y}_h$ to $\bar{y}_{.422c}$. Then from the geometry of the set-up, we obtain

$$\bar{y}_h = \bar{y}_{.422} - .328\ \bar{\phi}c$$

which when substituted in equations (2) results in:

(4a) $$\frac{+\bar{F}}{m_F} = \bar{y}_{.422}\left[\bar{A}_{11}\right] + \bar{\phi}c\left[\bar{A}_{12} - .328\bar{A}_{11}\right]$$

(4b) $$+\frac{\bar{M}_{.25c}}{c\,m_F} = \bar{y}_{.422}\left[\bar{A}_{21}\right] + \bar{\phi}c\left[\bar{A}_{22} - .328\ \bar{A}_{21}\right]$$

Further transformation will place these equations in a more useful form. Noting that

$$\bar{M}_{-6.5c} = \bar{M}_{.25c} - \bar{F} \times 6.75c$$

and substituting (4a) and (4b) in this, we obtain:

$$\frac{+\bar{M}_{-6.5c}}{c\,m_F} = \bar{y}_{.422}\left[+\bar{A}_{21} - 6.75\bar{A}_{11}\right] + \bar{\phi}c\left[\bar{A}_{22} - .328\ \bar{A}_{21} - 6.75\bar{A}_{12} + 2.22\bar{A}_{11}\right]$$

or

(5a) $$\frac{+\bar{M}_{-6.5c}}{6.75\,m_F c} = \bar{y}_{.422}\left[-\bar{A}_{11} + .148\bar{A}_{21}\right] + \bar{\phi}c\left[.328\bar{A}_{11} - \bar{A}_{12} - .0486\bar{A}_{21} + .148\bar{A}_{22}\right]$$

Likewise;

$\bar{M}_{.422c} = \bar{M}_{.25c} + \bar{F} \times .172c$ which when (4a) and (4b) are substituted in, results in

(5b) $$+\frac{\bar{M}_{.422c}}{c\,m_F} = \bar{y}_{.422}\left[\bar{A}_{21} + .172\ \bar{A}_{11}\right] + \bar{\phi}c\left[-.0564\bar{A}_{11} + .172\ \bar{A}_{12} - .328\ \bar{A}_{21} + \bar{A}_{22}\right]$$

Noting that

$$\frac{6.92\bar{F}_{.422c}}{6.75\, m_F} = -\frac{\bar{M}_{-6.5c}}{6.75c m_F}$$

and $\dfrac{6.92\bar{F}_{-6.5c}}{m_F} = \dfrac{\bar{M}_{.422c}}{c m_F}$, equations (5) then reduce to the following

when the values of the A's from (2) are introduced.

$$\text{(6a)} \quad \frac{1.025\bar{F}_{.422c}}{m_F} = \bar{y}_{.422}\Big[\overbrace{+1002 + 0.667 i \nu \bar{P} v/c - 1.195\nu^2}^{\bar{B}_{11}}\Big] + \bar{\phi}c\Big[\overbrace{-0.219 i \bar{P}\nu v/c - 0.179 i \nu v/c - 0.667\bar{P}v^2/c^2}^{\bar{B}_{12}}\Big]$$

$$\text{(6b)} \quad \frac{6.92\bar{F}_{-6.5c}}{m_F} = \bar{y}_{.422}\Big[\overbrace{+71.1 + 0.1147 i \nu \bar{P} v/c - .0090\nu^2}^{\bar{B}_{21}}\Big] + \bar{\phi}c\Big[\overbrace{499 + .0547 i \nu v/c - 0.0376 i \nu \bar{P} v/c - .1147\bar{P}v^2/c^2 - .0692\nu^2}^{B_{22}}\Big]$$

Equations (6) are not actually in the normal form at zero velocity, because the normal coordinates are $y_{.422c}$ and $y_{-6.5c}$, but since the latter point does not lie on the wing, and since it is desirable to retain ϕ as a coordinate in order to better visualise the motion, the equations have been placed in this form.

The equations of motion are now such that we can obtain the two principal modes of oscillation at zero airspeed by setting either $F_{.422c}$ or $F_{-6.5c}$ equal to zero, thus applying the oscillating force at a nodal point. The resulting oscillations will be about the opposite nodal point.

MODE I

To produce what we shall call Mode 1, we apply an oscillating force of the form $F'\cos\nu t$, or in the complex plane of the form $F'e^{i\nu t}$, at the .422 chord point, and in order to express the amplitude of response in a form which is independent of the magnitude of the applied force, we divide by the static deflection which this applied force would produce if the airspeed (v) and the frequency (ν) were zero.

Equations (6) then become

(7a) $$\frac{1.025\,\bar{F}_{.422c}}{m_F} = F' e^{i\nu t} = \bar{y}_{.422}\left[\bar{B}_{11}\right] + \bar{\phi}c\left[\bar{B}_{12}\right]$$

(7b) $$\frac{6.92\,\bar{F}_{-6.5c}}{m_F} = 0 = \bar{y}_{.422}\left[\bar{B}_{21}\right] + \bar{\phi}c\left[\bar{B}_{22}\right]$$

(7c) $$\bar{y}_{.422} = +\frac{\bar{B}_{22}}{\Delta} F' e^{i\nu t} \qquad \frac{\bar{y}_{.422}}{(\bar{y}_{.422})_0} = \frac{\bar{B}_{22}/\Delta}{(\bar{B}_{22}/\Delta)_0} e^{i\nu t}$$

(7d) $$\bar{\phi}c = -\frac{\bar{B}_{11}}{\Delta} F' e^{i\nu t} \qquad \frac{\bar{\phi}c}{(\bar{\phi}c)_0} = \frac{\bar{B}_{21}/\Delta}{(\bar{B}_{21}/\Delta)_0} e^{i\nu t}$$

(7e) $$\frac{\phi c}{y_{.422}} = -\frac{\bar{B}_{21}}{\bar{B}_{22}}$$

where $$\Delta = \begin{vmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{vmatrix} = \bar{B}_{11}\bar{B}_{22} - \bar{B}_{12}\bar{B}_{21}$$

and $(\ )_0$ denotes the static deflection.

MODE 2

In a similar manner we can produce Mode 2 by applying the oscillating force at the -6.5 chord point.

(8a) $$\frac{1.025\,\bar{F}_{.422c}}{m_F} = 0 = \bar{y}_{.422}\left[\bar{B}_{11}\right] + \bar{\phi}c\left[\bar{B}_{12}\right]$$

(8b) $$\frac{6.92\,\bar{F}_{-6.5c}}{m_F} = F'e^{i\nu t} = \bar{y}_{.422}\left[\bar{B}_{21}\right] + \phi c\left[\bar{B}_{22}\right]$$

(8c) $$\bar{y}_{.422} = -\frac{\bar{B}_{12}}{\Delta}F'e^{i\nu t} \qquad \frac{\bar{y}_{.422}}{(\bar{\phi}c)_0} = \frac{\bar{B}_{12}/\Delta}{(\bar{B}_{11}/\Delta)_0}e^{i\nu t}$$

(8d) $$\bar{\phi}c = +\frac{\bar{B}_{11}}{\Delta}F'e^{i\nu t} \qquad \frac{\bar{\phi}c}{(\bar{\phi}c)_0} = \frac{\bar{B}_{11}/\Delta}{(\bar{B}_{11}/\Delta)_0}e^{i\nu t}$$

(8e) $$\frac{\bar{y}_{.422}}{\bar{\phi}c} = -\frac{\bar{B}_{12}}{\bar{B}_{11}}$$

where Δ and $(\;)_0$ have the same significance as in Mode 1.

In equation (8c), the flexural response has been divided by the static angular response because the static flexural response of the .422c point is zero.

RESPONSE OF THE WING AT VARIOUS VELOCITIES AND FREQUENCIES

The above equations enable us to obtain response curves for the wing at any velocity. The procedure followed in obtaining the response curves in this thesis was first to choose a value of v/c. This was substituted in equations (7) and (8). Values of ν from 0 to 120 in steps of 10 were then assumed and the value $\bar{\bar{F}}$ determined from the corresponding $V = v/\nu c$. This enabled us to calculate the $\bar{B}$'s and hence the deflections. The calculations are, of course, in the complex form. The deflections plotted are the maximum deflections; that is, the modulus of $\bar{\bar{y}}$ and $\bar{\bar{\phi}}$.

TORSIONAL DIVERGENCE

Torsional divergence will occur in the wing when the rate of change of moment of the aerodynamic forces with respect to a change in the angle of attack exceeds the rate of change of the elastic restoring moment with respect to a change in angle.

In order to determine the velocity at which this divergence occurs, we must determine the velocity at which Δ of our equations of motion becomes equal to zero at zero frequency, since this is the frequency at which divergence occurs.

Then noting that at $\nu = 0$,

$$V = \frac{v}{\nu c} = \infty \text{ and } \overline{P} = 1, \text{ the } \Delta \text{ of equation (1a) and (1b)}$$

becomes

$$\begin{vmatrix} [\nu_1^2] & \left[+\nu_1^2(1/2 - \varepsilon) - \frac{4}{\mu}\frac{v^2}{c^2}\right] \\ [-\nu_1^2 \varepsilon] & \left[-\nu_1^2\left\{\varepsilon(1/2 - \varepsilon) - \eta^2\right\}\right] \end{vmatrix} = 0$$

from which we obtain

$$v_d/c = \frac{\nu_1 \eta}{2}\sqrt{\frac{\mu}{\varepsilon}} \quad \text{at divergence}$$

Substituting our parameters

$$v_d/c = \frac{10\pi \times .707}{2}\sqrt{\frac{6}{.1}} = 86 \text{ rad/sec}$$

$$v_d = 86 \times 7.5 = 645 \text{ ft/sec} = 440 \text{ m.p.h.}$$

Response curves for this velocity have been plotted.

RESPONSE OF THE WING WITH ONE DEGREE OF FREEDOM

We can obtain an equation for the total applied force on the wing by adding the force at .422c to the force at -6.5c. Then by using equations (6), we obtain

$$\frac{\bar{F}_{TOTAL}}{m_F} = \frac{\bar{F}_{.422c}}{m_F} + \frac{\bar{F}_{-6.5c}}{m_F}$$

(9a) $$\frac{\bar{F}_{TOTAL}}{m_F} = \bar{y}_{.422}\overbrace{\left[\frac{\bar{B}_{11}}{1.025} + \frac{\bar{B}_{21}}{6.92}\right]}^{\bar{C}_{11}} + \bar{\Phi}c\overbrace{\left[\frac{\bar{B}_{12}}{1.025} + \frac{\bar{B}_{22}}{6.92}\right]}^{\bar{C}_{12}}$$

If we now set $\bar{\phi}c$ equal to zero, we will obtain an equation for the motion of a wing with infinite torsional rigidity. By setting $\bar{\phi}c = 0$, we imply that we are applying the force so that its moment varies in such a manner that the moment equation (6a) or (6b) is also satisfied for $\bar{\phi}c = 0$.

Equation (9a) is equivalent to equation (4a), but as the B's have already been calculated, it is the more convenient form.

Now again applying an oscillating force of the form $F' \cos \nu t$ or $F'e^{i\nu t}$ in the complex plane and dividing the response amplitude by the static response, we obtain

(9b) $$\frac{\bar{F}_{TOTAL}}{m_F} = F'e^{i\nu t} = \bar{y}_{.422}\left[\bar{C}_{11}\right] + 0\left[\bar{C}_{12}\right]$$

(9c) $$\bar{y}_{.422} = \frac{F'e^{i\nu t}}{\bar{C}_{11}}$$

(9d) $$\frac{\bar{y}_{.422}}{(\bar{y}_{.422})_0} = \frac{(\bar{C}_{11})_0}{\bar{C}_{11}} e^{i\nu t}$$ (Flexure only)

Returning now to the equation (6b) and setting $\bar{y}_{.422}$ equal to zero, we obtain the equation of motion of the wing with the .422 chord point fixed.

(10a) $$\frac{6.92\,\bar{F}_{-6.5c}}{m_F} = \frac{\bar{M}_{.422c}}{c\,m_F} = 0\left[\bar{B}_{21}\right] + \bar{\phi}c\left[\bar{B}_{22}\right]$$

It is also implied that we are applying a force at the .422 chord point of such variable magnitude that equation (6a) is satisfied for the condition $\bar{y}_{.422} = 0$.

Then applying an oscillating moment of the form $M'\cos\nu t$, or $M'e^{i\nu t}$ in the complex plane, and dividing the response by the static response, we obtain:

(10b) $$\bar{\phi}c = \frac{M'e^{i\nu t}}{\bar{B}_{22}}$$

(10c) $$\frac{\bar{\phi}c}{(\bar{\phi}c)_0} = \frac{(\bar{B}_{22})_0}{\bar{B}_{22}}\,e^{i\nu t}$$ (.422 chord point fixed)

Response curves for the above two modes of oscillations have been plotted. The amplitudes plotted are the maximum values or modulus.

DISCUSSION OF THE RESPONSE CURVES

The response curves for one degree of freedom as shown in Figures 1 and 2 are typical families of damped oscillations with the damping increasing with airspeed. The torsional response curves show the effect of the torsional divergence phenomenon near zero frequency with the response becoming large at 332 m.p.h. which is slightly below the divergence speed of the wing with the .422 chord point fixed. At speeds above this, the deflection is actually opposite to the applied force which means that a force must be applied to keep the wing from diverging.

The response curves given in Figures 3 to 11 for Mode 1, or the mode which at zero airspeed is predominately flexural, indicate that natural frequency of flexure which occurs at zero airspeed is damped out as the airspeed increases. The damping is even greater in the two degrees of freedom than in the one degree of freedom case discussed above. It should be remembered in examining these curves that as soon as the airspeed is other than zero, the modes are no longer normal modes, and we are obtaining some cross responses. The peaks on the right of the curves of response of $y_{.422c}$ to a force at the .422 chord point are of this nature since they arise from the ϕc or torsional response.

The curves of Mode 2 or the torsional mode at zero airspeed, as given in Figures 12 to 19, indicate that the torsional oscillations are increasingly damped with increasing airspeed and that the frequency of maximum response moves to the left or decreases. However, as the flutter speed is approached, the damping becomes less and the amplitude builds up. The curves for 440 m.p.h. indicate torsional divergence occuring at zero frequency.

As stated in the introduction, one of the purposes of this investigation was to determine the possibility of predicting the speed at which flutter would occur by plotting the natural torsional and flexural frequencies at various airspeeds as determined by flight tests. If the frequency at which the maximum response occurs at any given airspeed is called the natural frequency and these frequencies plotted against airspeed, the curves will be those which would be obtained by flight tests and are shown in Figure 21. It can be seen at once that flutter does not occur where the torsion and flexural curves intersect, but that the flexural mode has been damped out, as was pointed out before, and that flutter occurs at a point on the torsional curve without warning other than a building up of the amplitude of response. This method of attempting to predict flutter appears to be impractical.

Figure 1

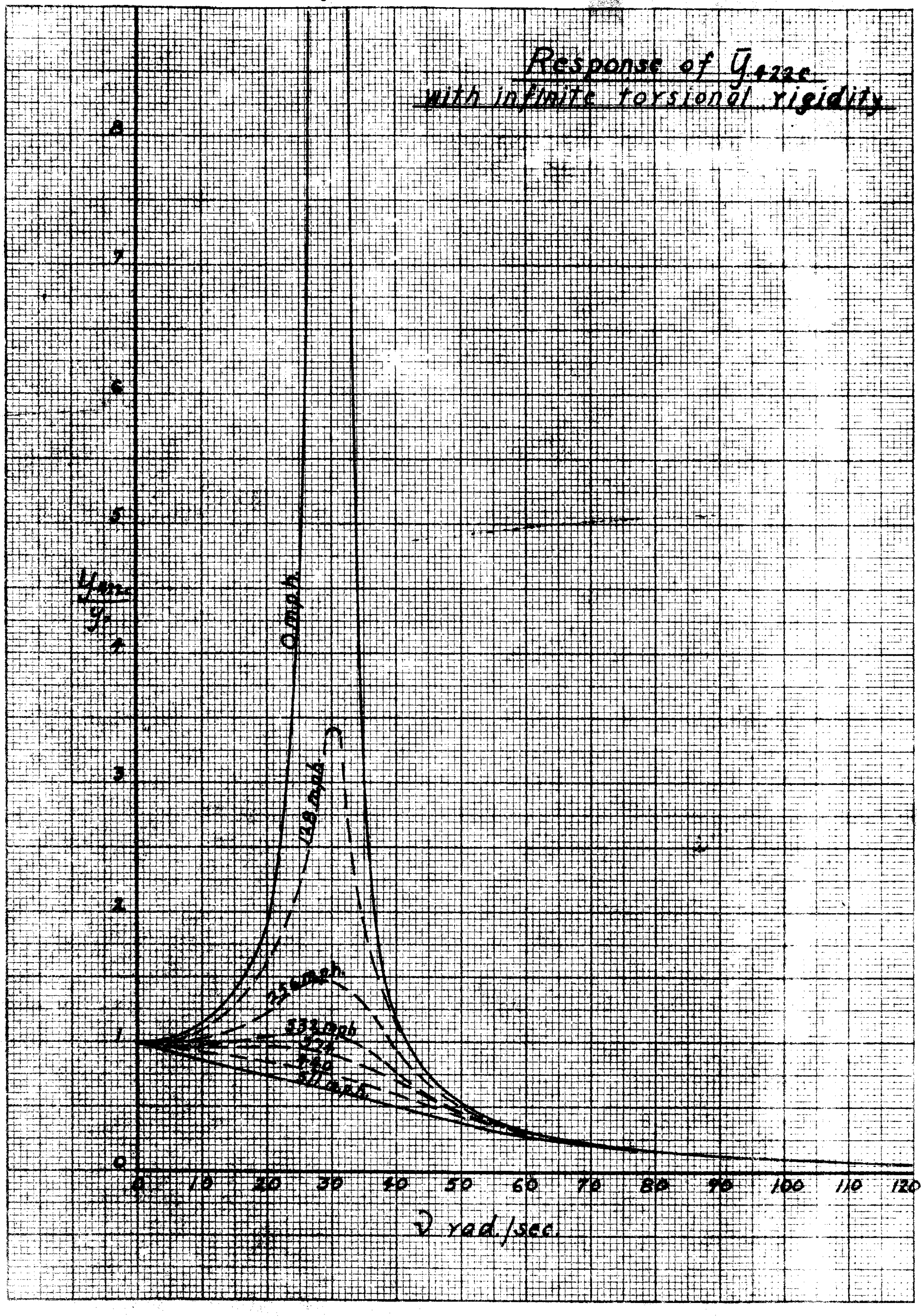
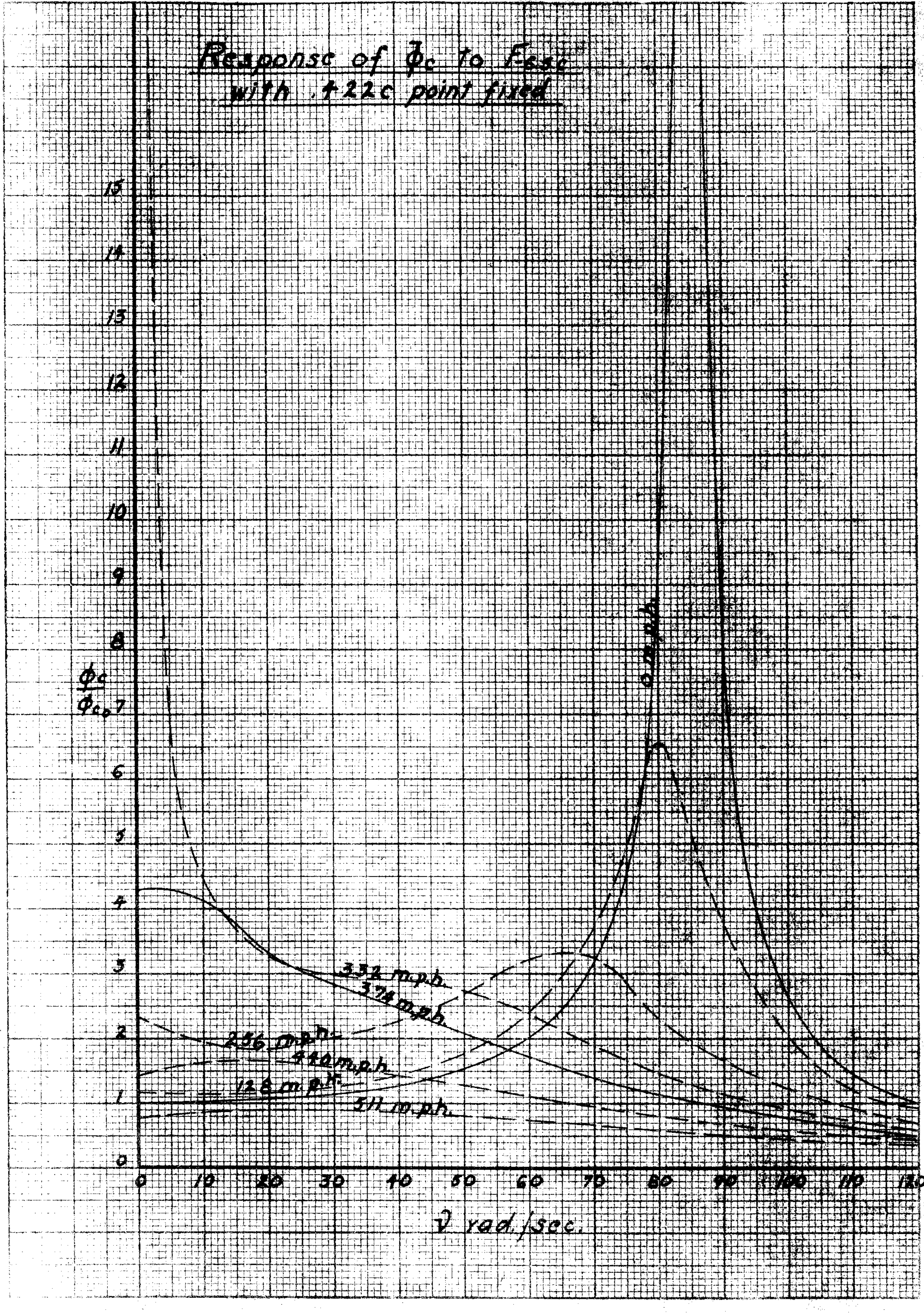

Response of $\bar{y}_{422c}$
with infinite torsional rigidity
0 m.p.h.
148 m.p.h.
256 m.p.h.
332 m.p.h.
$\frac{y_{422c}}{y_0}$
ν rad./sec.
0
1
2
3
4
5
6
7
8
10
20
30
40
50
60
70
80
90
100
110
120

Figure 2



Response of ϕ_c to F_{ssc}
with .422c point fixed
0 m.p.h.
332 m.p.h.
374 m.p.h.
256 m.p.h.
410 m.p.h.
128 m.p.h.
511 m.p.h.
$\frac{\phi_c}{\phi_{c_0}}$
ν rad./sec.
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
10
20
30
40
50
60
70
80
90
100
110
120

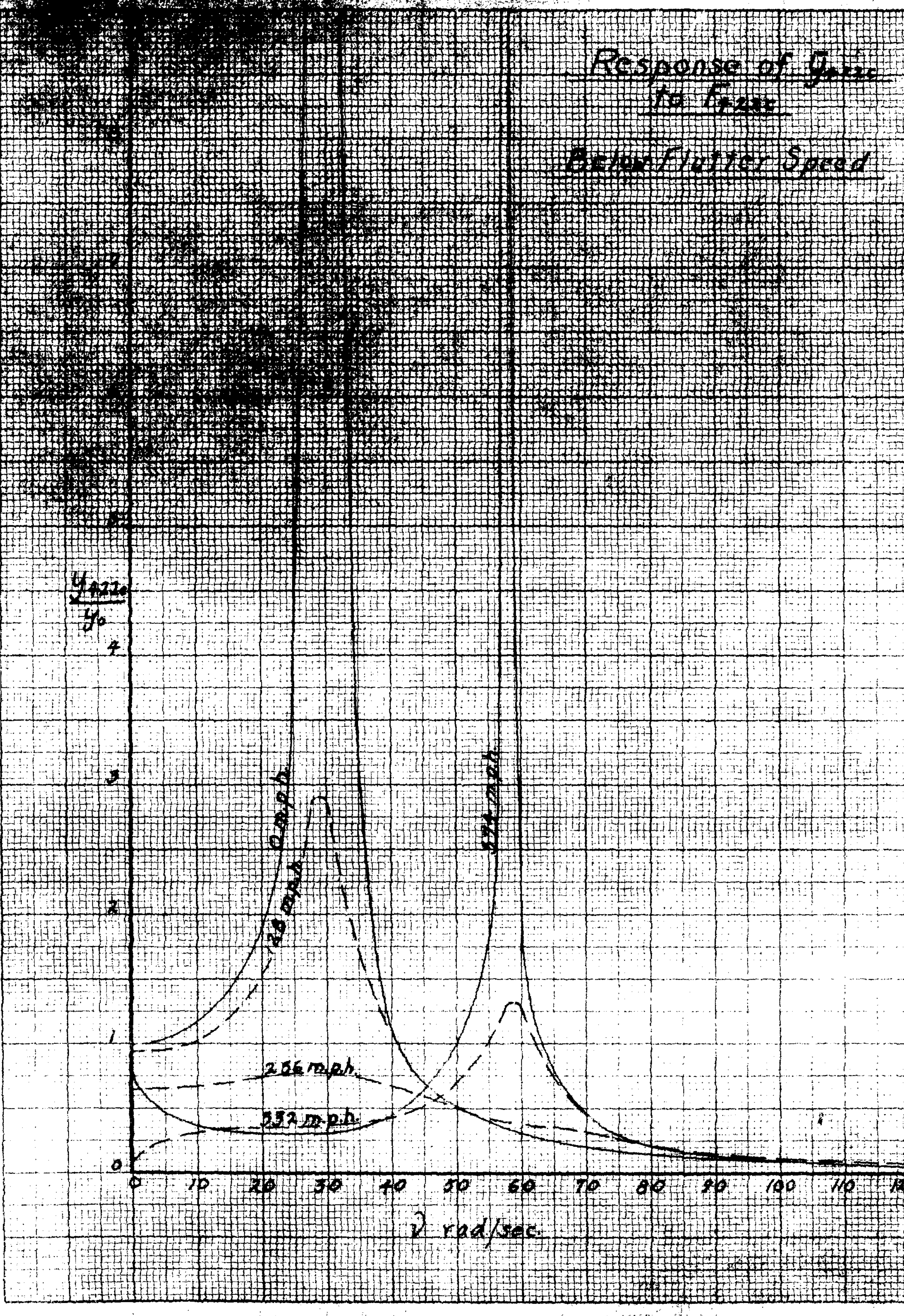

Response of y_{422c} to F_{422c}
Below Flutter Speed
$\frac{y_{422c}}{y_0}$
0
1
2
3
4
0 m.p.h.
46 m.p.h.
256 m.p.h.
332 m.p.h.
374 m.p.h.
0
10
20
30
40
50
60
70
80
90
100
110
120
ν rad/sec

Figure 4

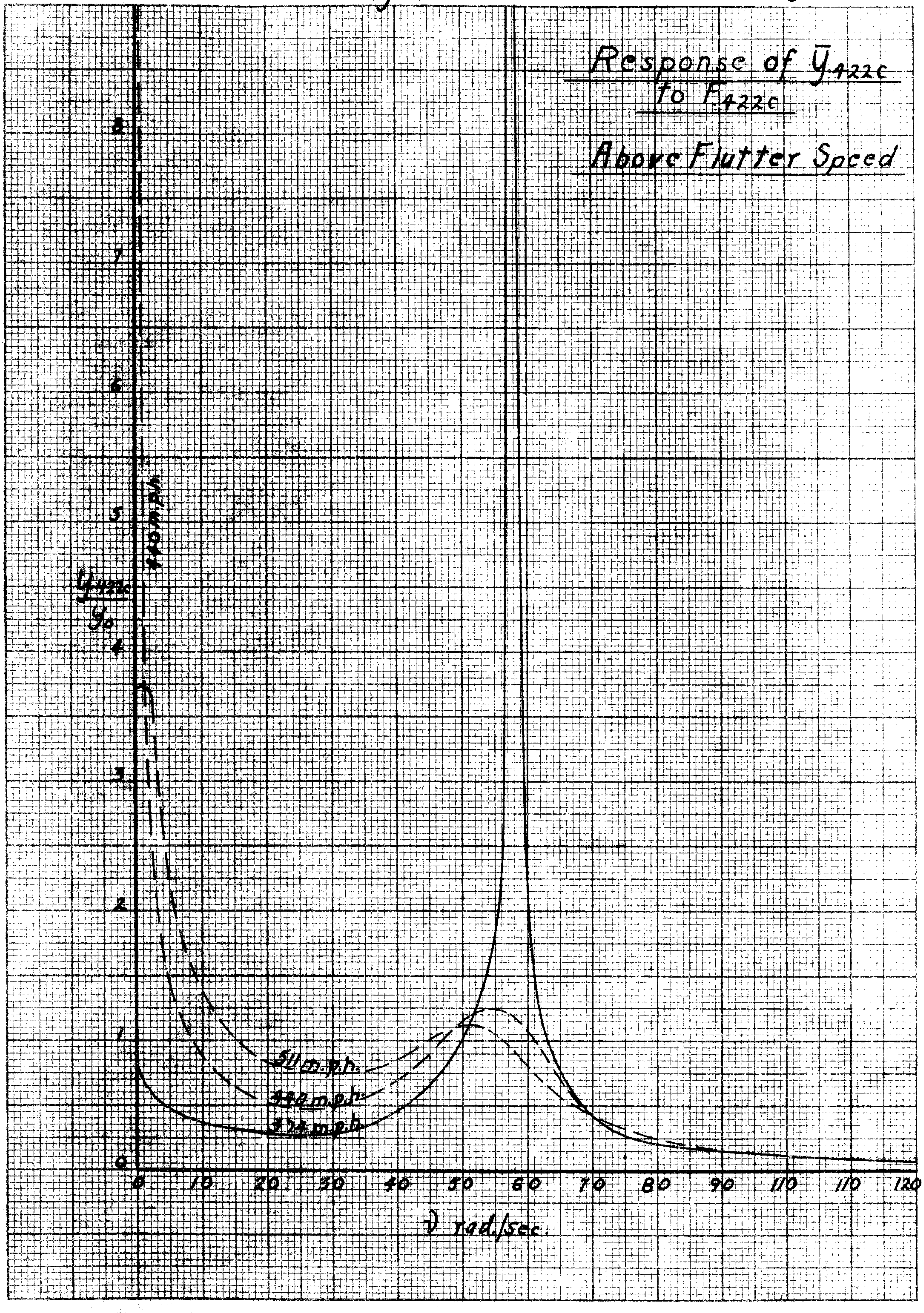

Response of ȳ4222c to F4222c
Above Flutter Speed
511 m.p.h.
440 m.p.h.
374 m.p.h.
440 m.p.h.
ν rad./sec.

Figure 5

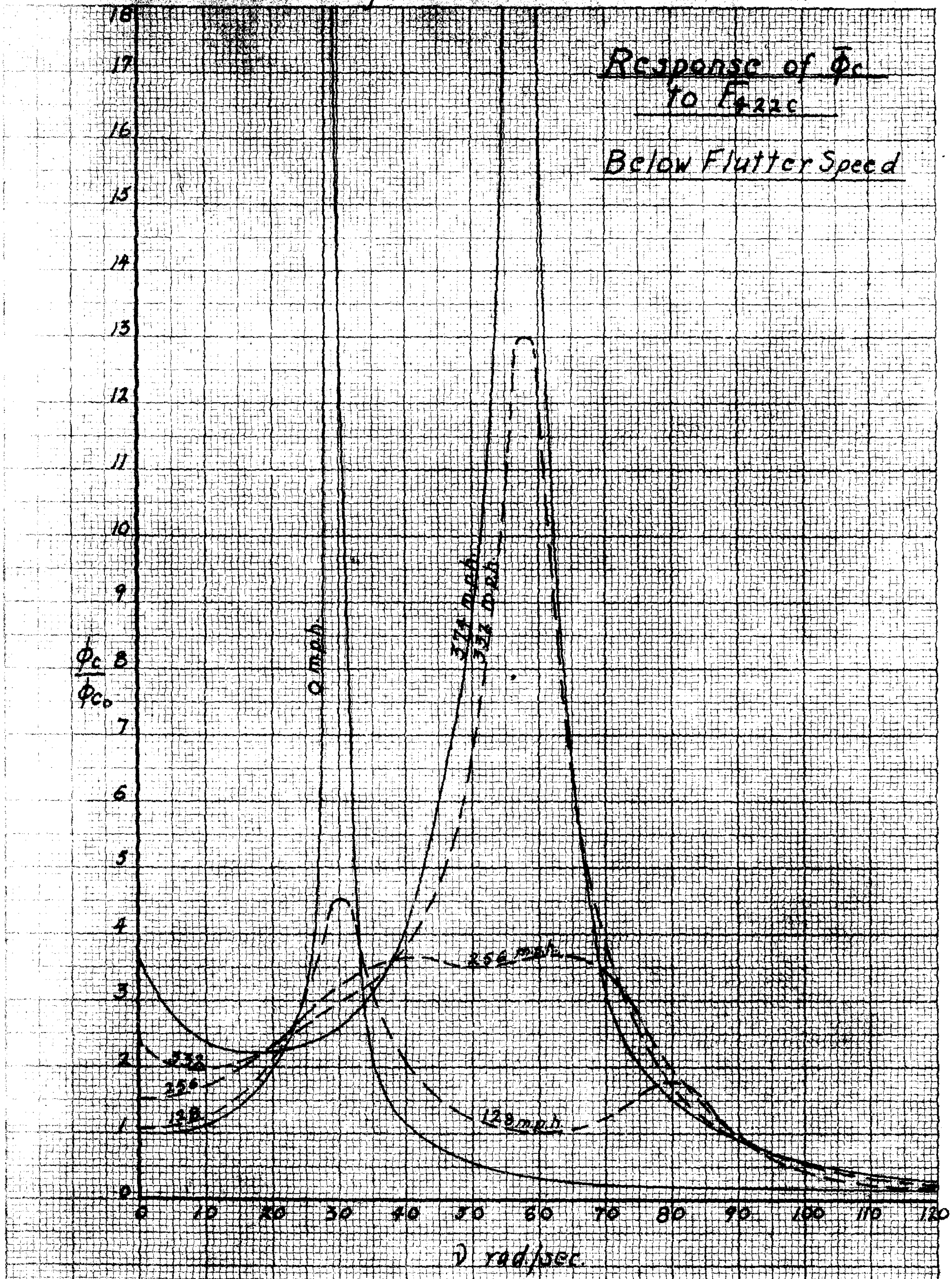

Response of $\bar{\Phi}_c$
to $F_{\phi 22c}$
Below Flutter Speed
ϕ_c / ϕ_{c_0}
0 m.p.h.
374 m.p.h.
332 m.p.h.
256 m.p.h.
128 m.p.h.
332
256
128
ν rad/sec.
0
10
20
30
40
50
60
70
80
90
100
110
120
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18

Figure 6

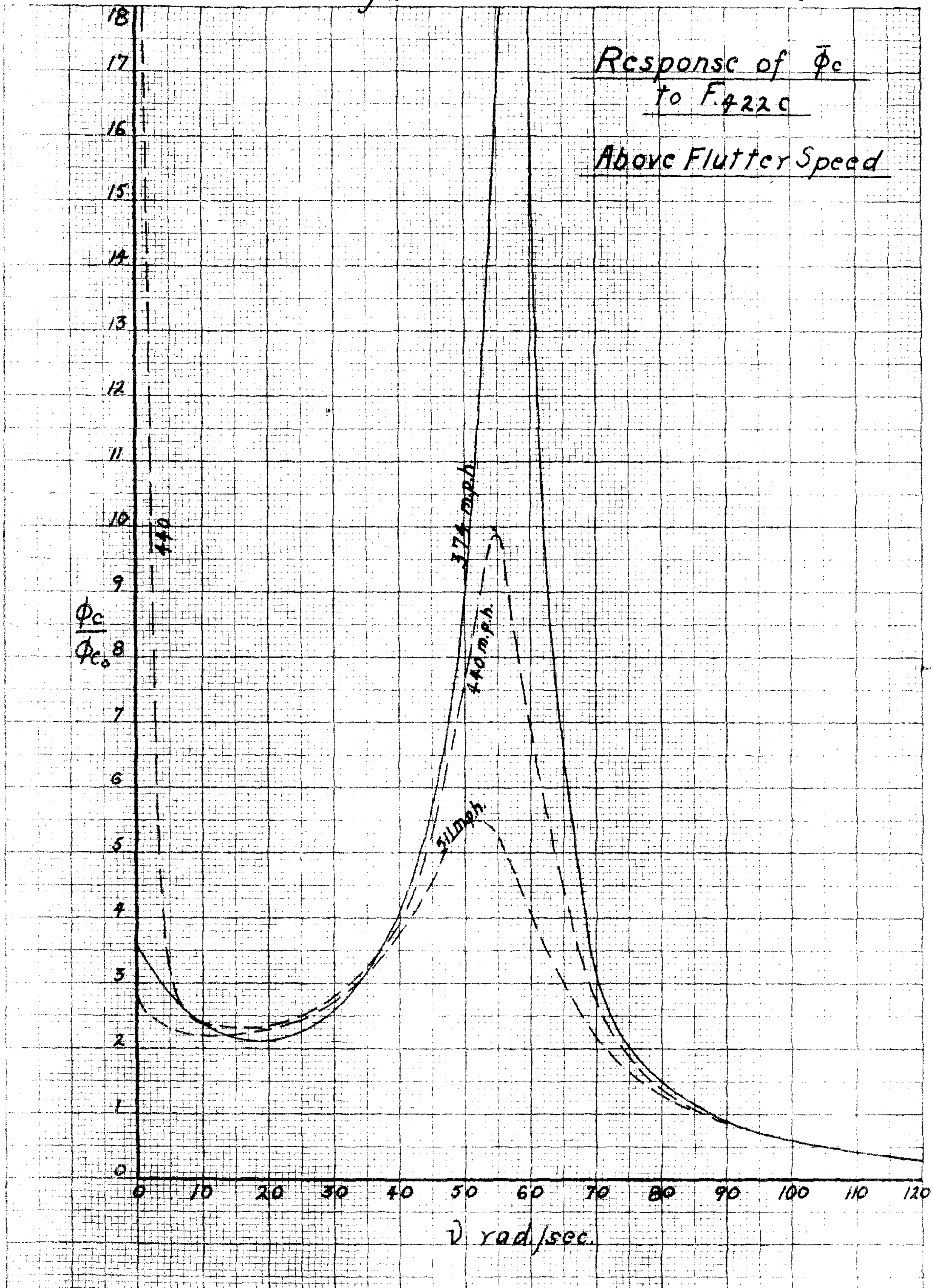

Response of $\bar{\Phi}_c$ to F_{422c}
Above Flutter Speed
$\frac{\Phi_c}{\Phi_{c_0}}$
ν rad/sec.
440
374 m.p.h.
440 m.p.h.
511 mph.
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
10
20
30
40
50
60
70
80
90
100
110
120

Figure 7

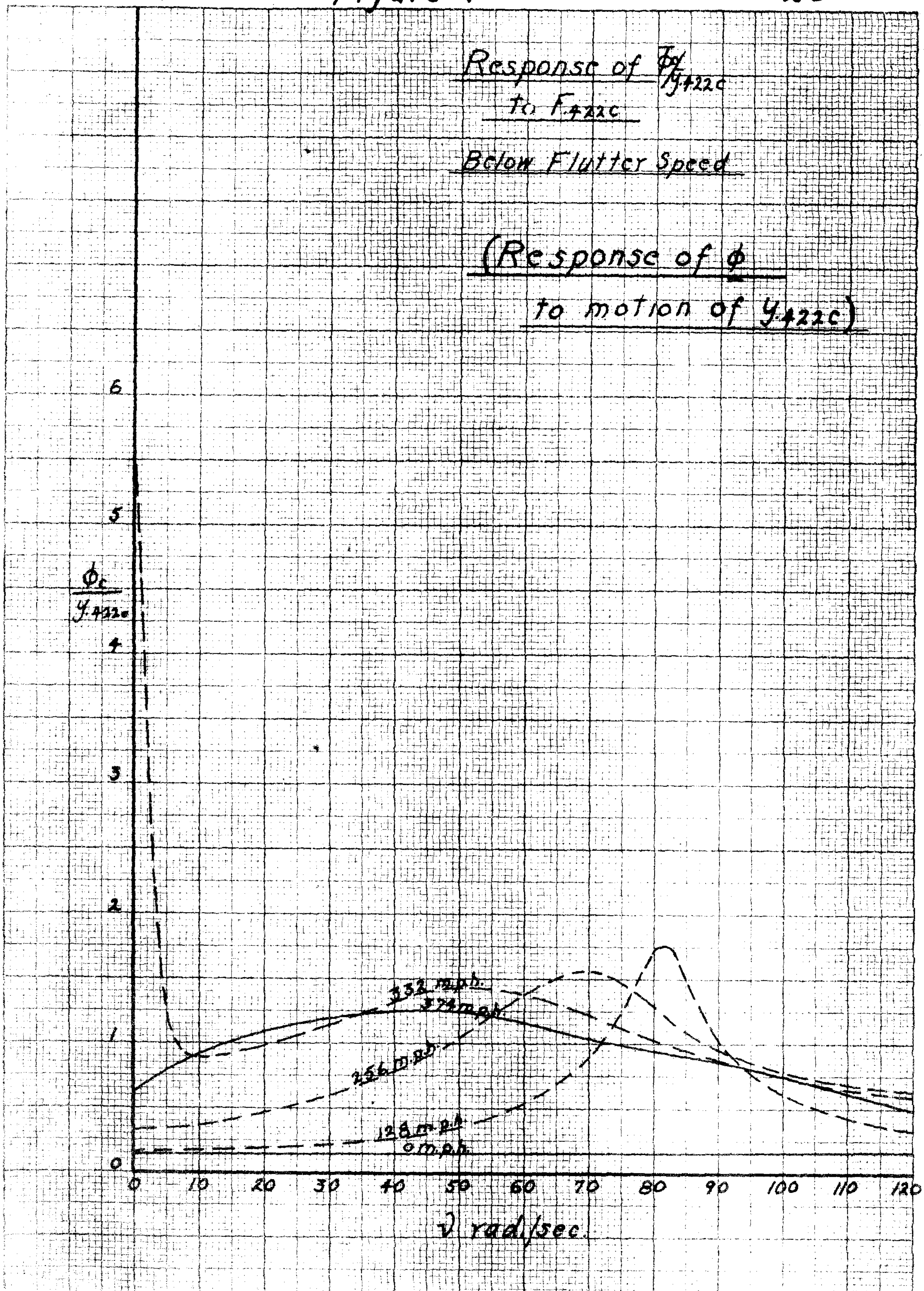
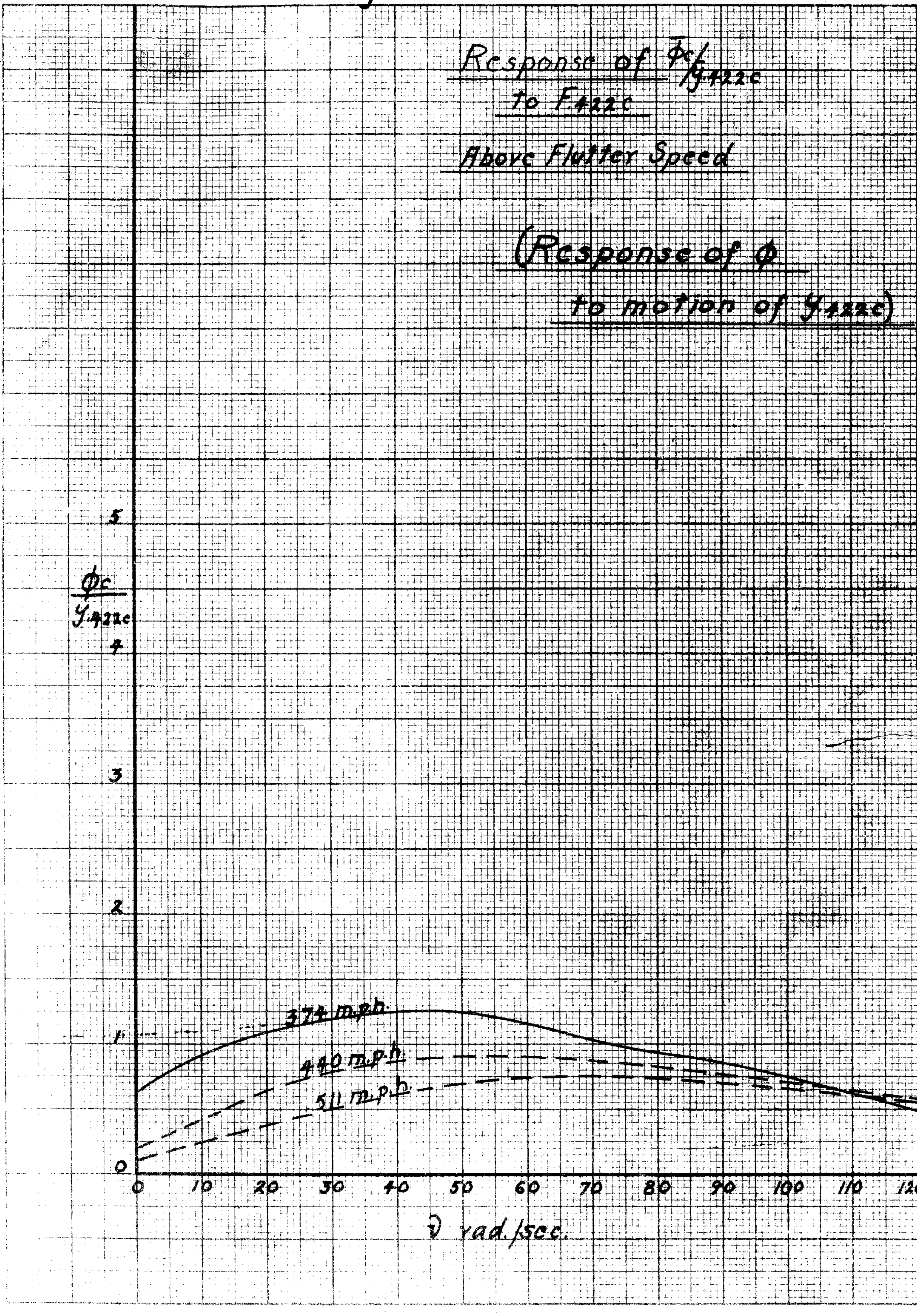

Response of Φ/y.422c
to F.422c
Below Flutter Speed
(Response of φ
to motion of y.422c)
φc / y.422c
0
1
2
3
4
5
6
332 mph.
374 mph.
256 mph.
128 mph.
0 mph.
0
10
20
30
40
50
60
70
80
90
100
110
120
ν rad/sec.

Figure 8


Response of $\Phi_c / y_{.422c}$
to $F_{.422c}$
Above Flutter Speed
(Response of ϕ
to motion of $y_{.422c}$)
$\phi_c / y_{.422c}$
0
1
2
3
4
5
0
10
20
30
40
50
60
70
80
90
100
110
120
ν rad./sec.
374 m.p.h.
440 m.p.h.
511 m.p.h.

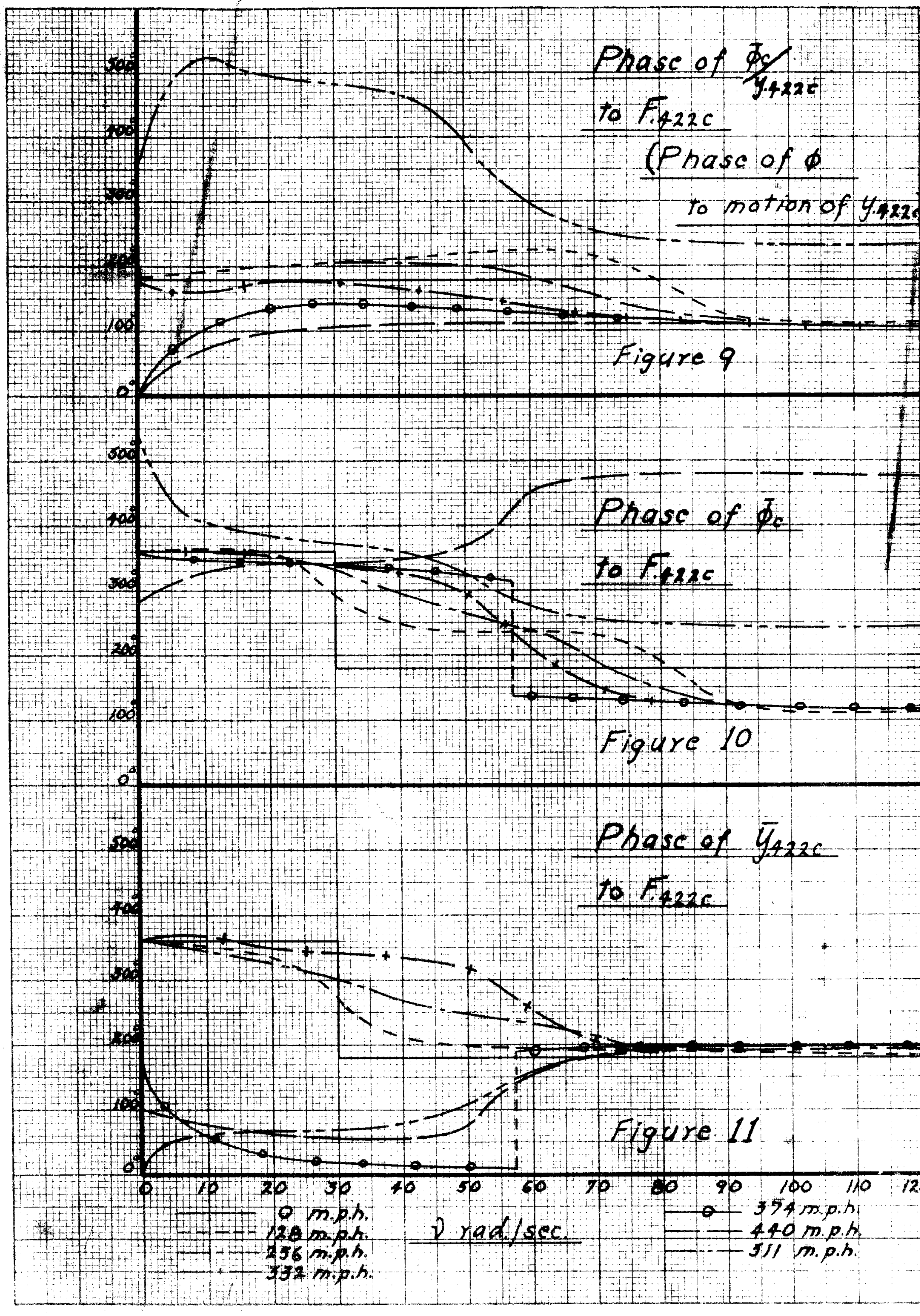


Figure 9

Figure 10

Figure 11

Figure 12

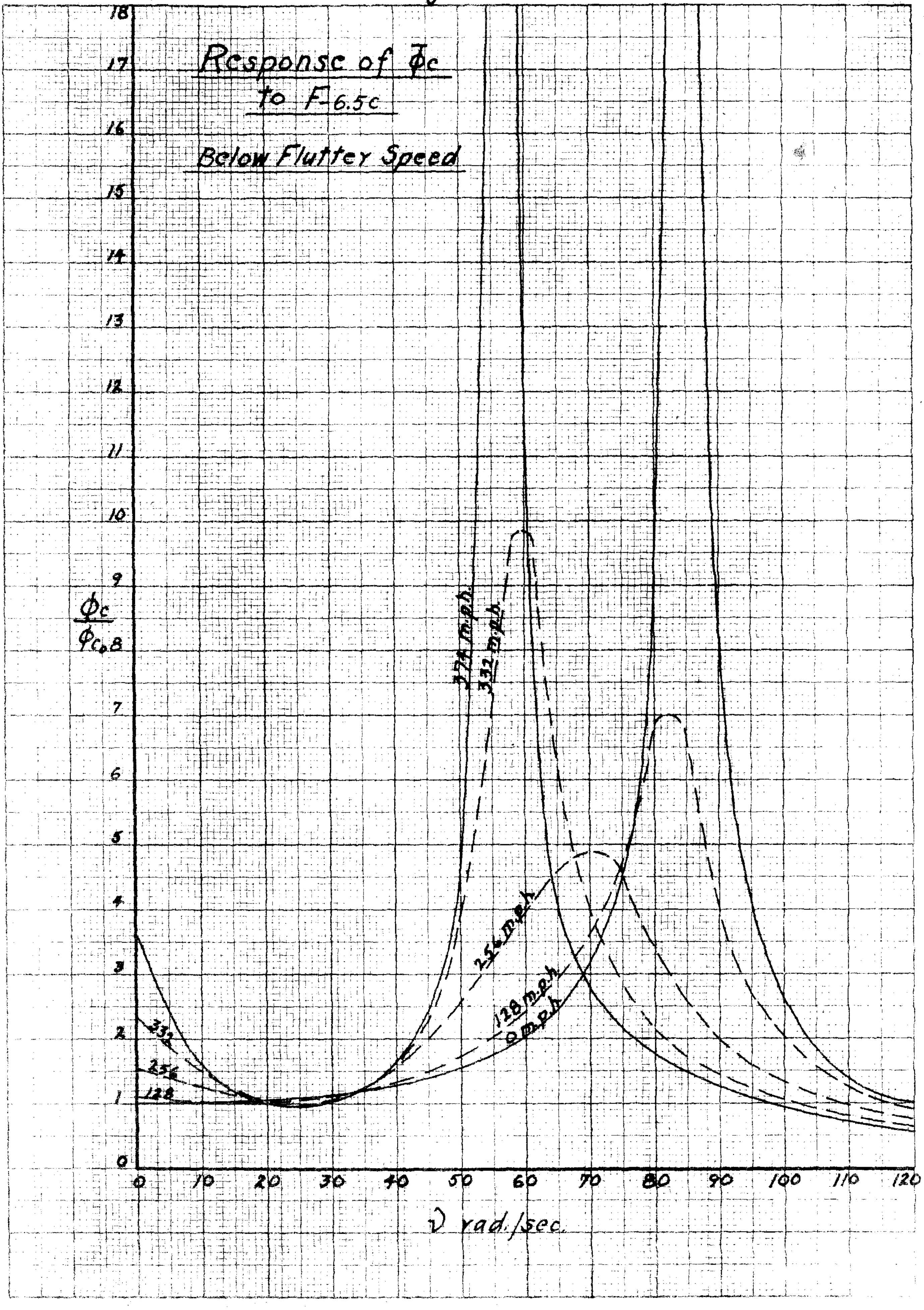

Response of $\bar{\Phi}_c$ to $F_{6.5c}$
Below Flutter Speed
$\frac{\phi_c}{\phi_{c_0}}$
374 m.p.h.
332 m.p.h.
256 m.p.h.
128 m.p.h.
0 m.p.h.
332
256
128
ν rad./sec.
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
0
10
20
30
40
50
60
70
80
90
100
110
120

Figure 13

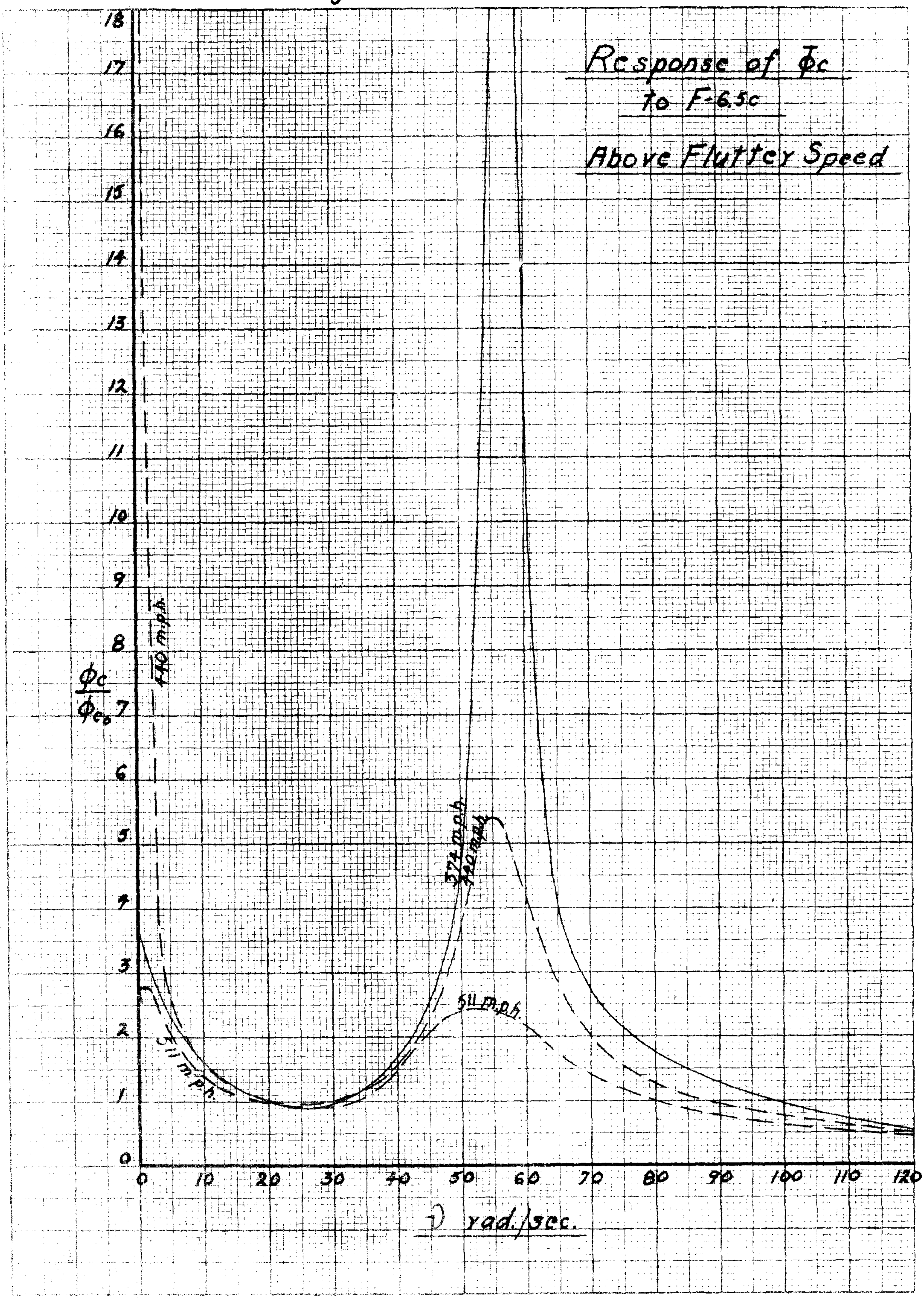

Response of Φ_c
to F-6.5c
Above Flutter Speed
$\frac{\phi_c}{\phi_{c_0}}$
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
440 m.p.h.
511 m.p.h.
374 m.p.h.
420 m.p.h.
511 m.p.h.
0
10
20
30
40
50
60
70
80
90
100
110
120
ν rad./sec.

Figure 14

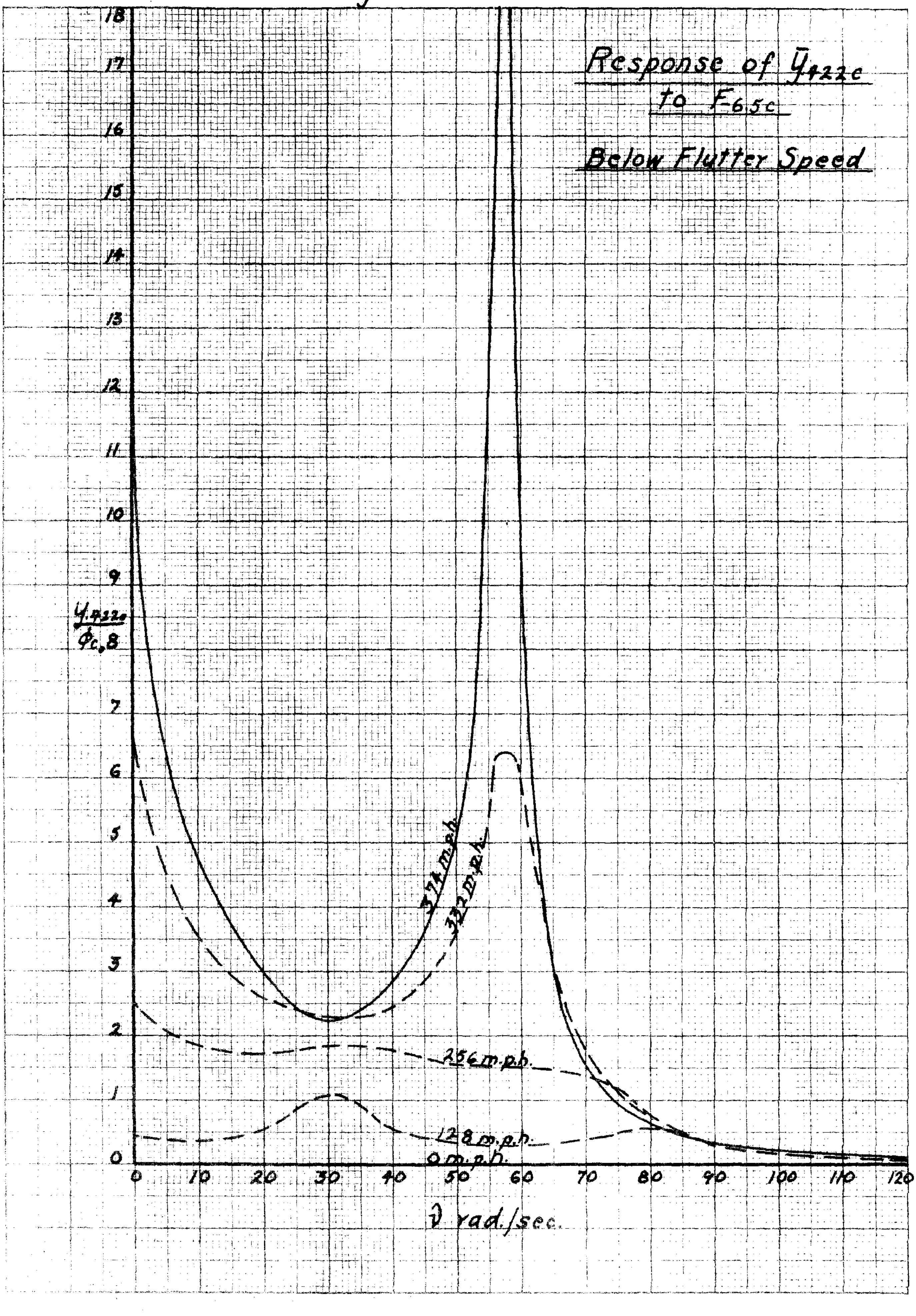

Response of $\bar{y}_{422c}$ to F_{65c}
Below Flutter Speed
$\frac{y_{422c}}{\phi_{c_0}}$
374 m.p.h.
333 m.p.h.
256 m.p.h.
128 m.p.h.
0 m.p.h.
ϑ rad./sec.

Figure 15

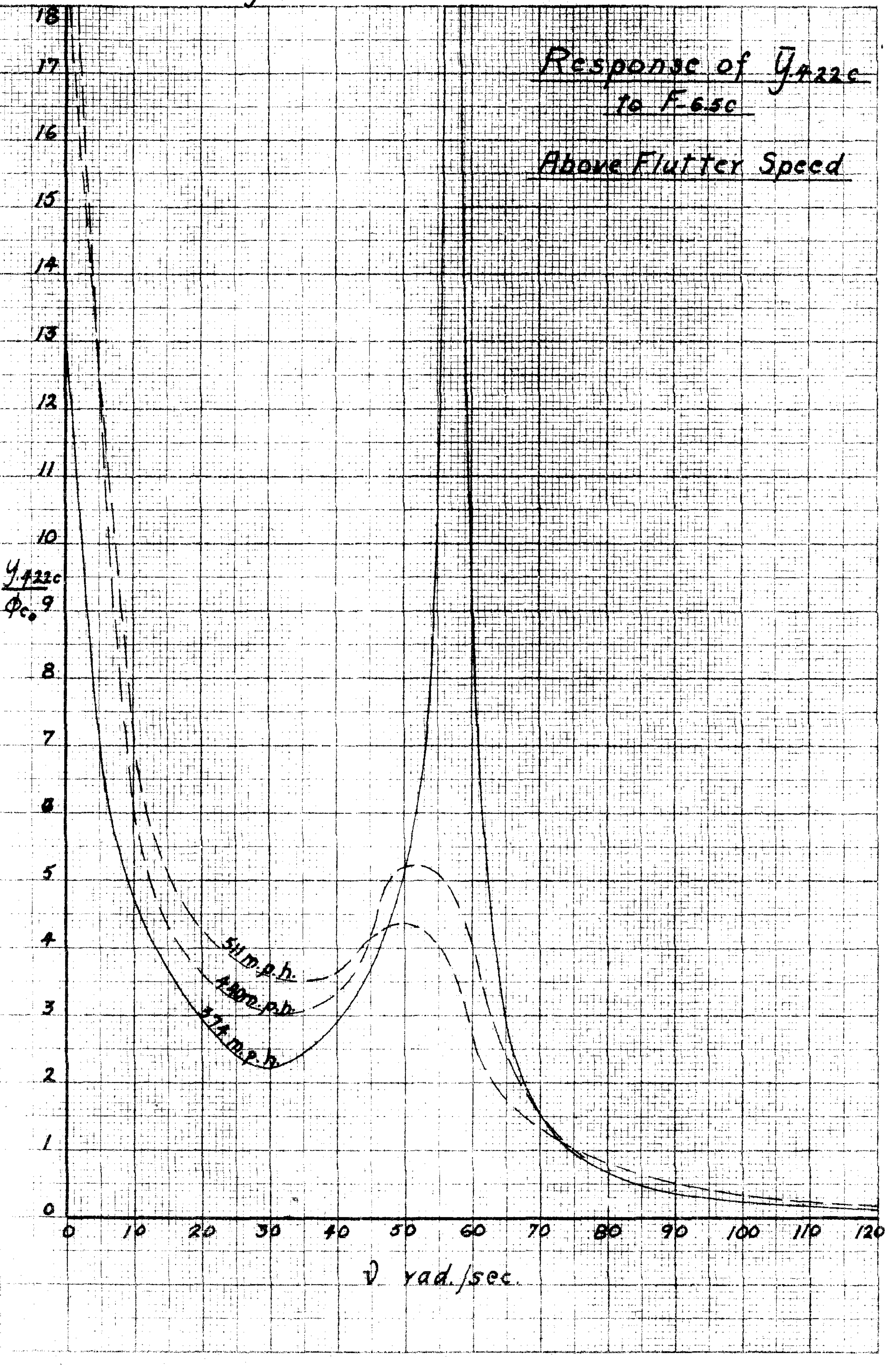
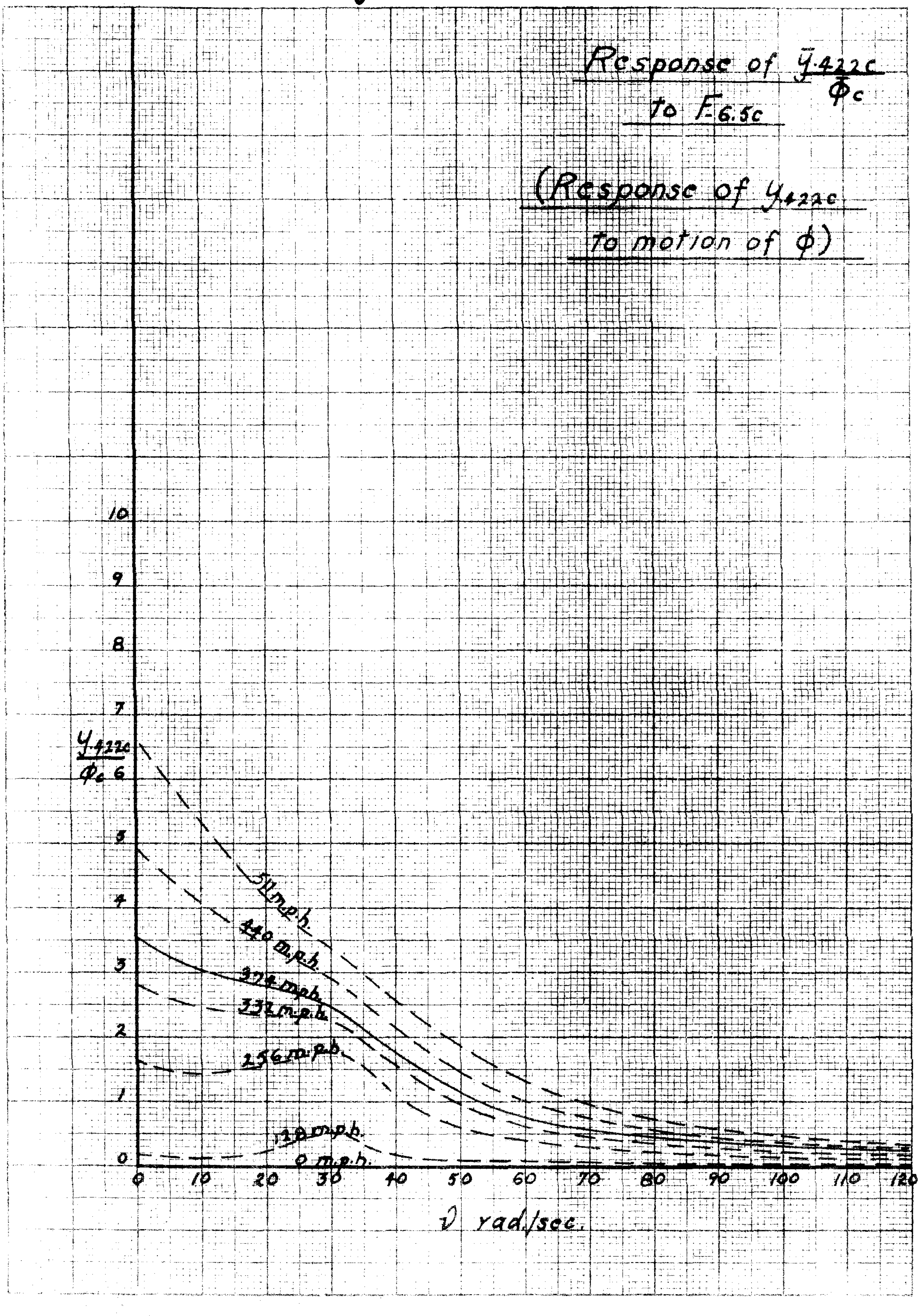

Response of $\bar{y}_{.422c}$ to $F_{.65c}$
Above Flutter Speed
$\frac{y_{.422c}}{\phi_{c_0}}$
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
0
10
20
30
40
50
60
70
80
90
100
110
120
ν rad./sec.
511 m.p.h.
480 m.p.h.
374 m.p.h.

Figure 16


Response of $\bar{y}_{4.22c}$ / Φ_c to $F_{6.5c}$
(Response of y_{422c} to motion of ϕ)
$\frac{y_{422c}}{\Phi_c}$
512 m.p.h.
440 m.p.h.
374 m.p.h.
332 m.p.h.
256 m.p.h.
128 m.p.h.
0 m.p.h.
0 1 2 3 4 5 6 7 8 9 10
0 10 20 30 40 50 60 70 80 90 100 110 120
ν rad/sec.

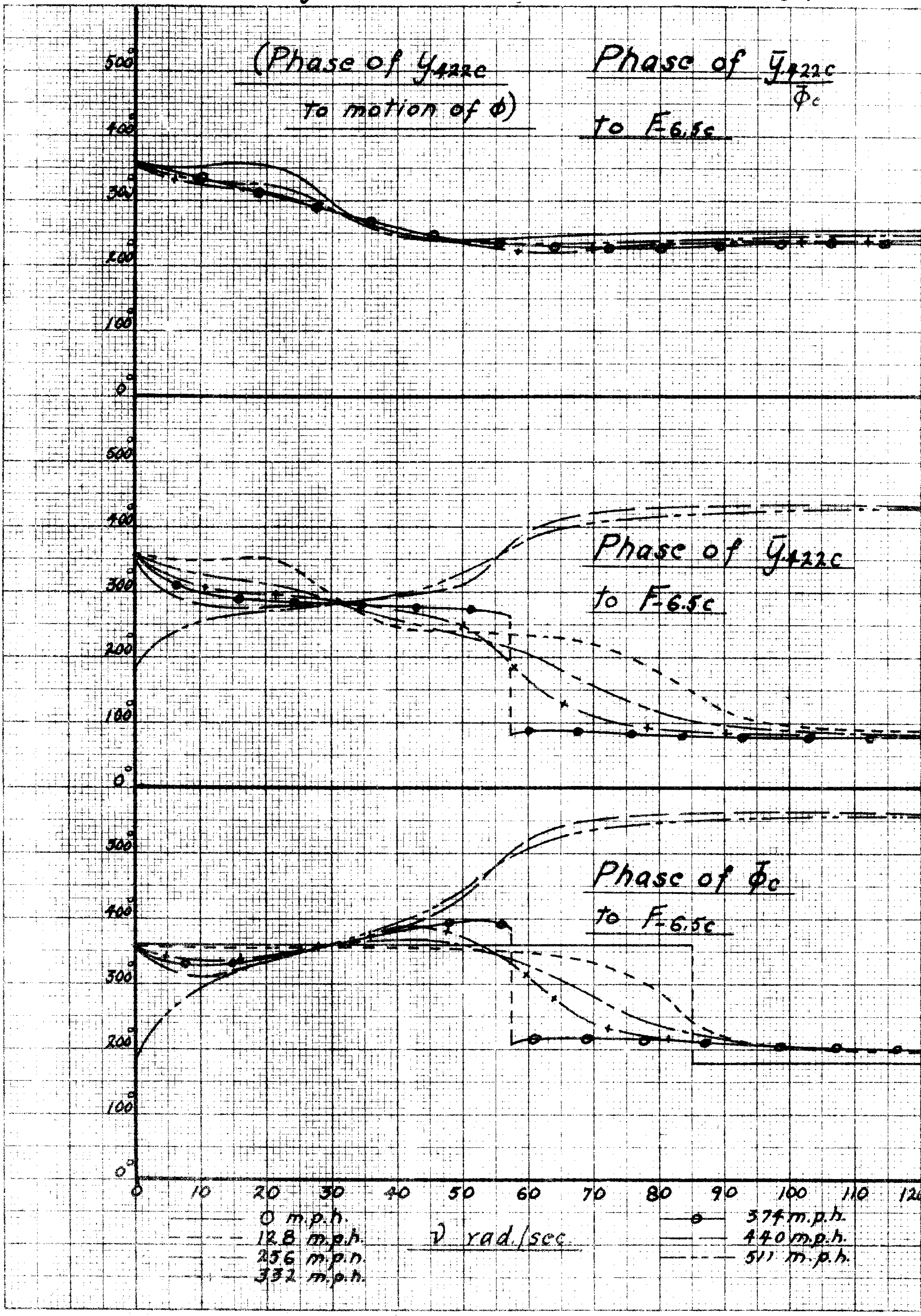
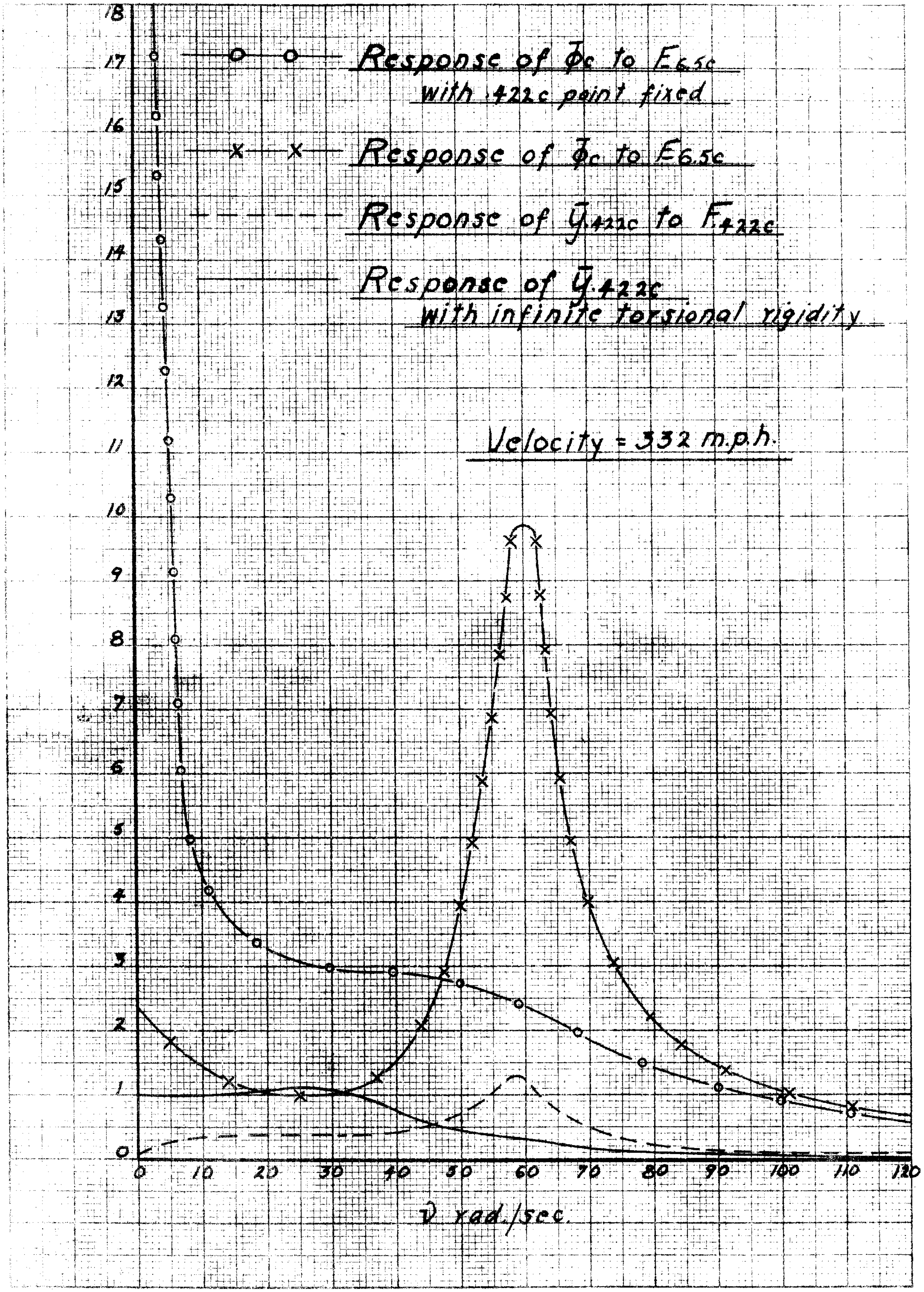

(Phase of y_{422c}
to motion of ϕ)
Phase of $\bar{y}_{422c}/\bar{\Phi}_c$
to $F_{6.5c}$
Phase of $\bar{y}_{422c}$
to $F_{6.5c}$
Phase of $\bar{\Phi}_c$
to $F_{6.5c}$
500°
400°
300°
200°
100°
0°
0
10
20
30
40
50
60
70
80
90
100
110
120
0 m.p.h.
128 m.p.h.
256 m.p.h.
332 m.p.h.
ν rad./sec
374 m.p.h.
440 m.p.h.
511 m.p.h.

Figure 20


Response of $\bar{\Phi}_c$ to E_{65c} with .422c point fixed
Response of $\bar{\Phi}_c$ to E_{65c}
Response of $\bar{y}_{.422c}$ to $F_{.422c}$
Response of $\bar{y}_{.422c}$ with infinite torsional rigidity
Velocity = 332 m.p.h.
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
0
10
20
30
40
50
60
70
80
90
100
110
120
ν rad./sec.

Variation of frequency for maximum amplitude with airspeed.

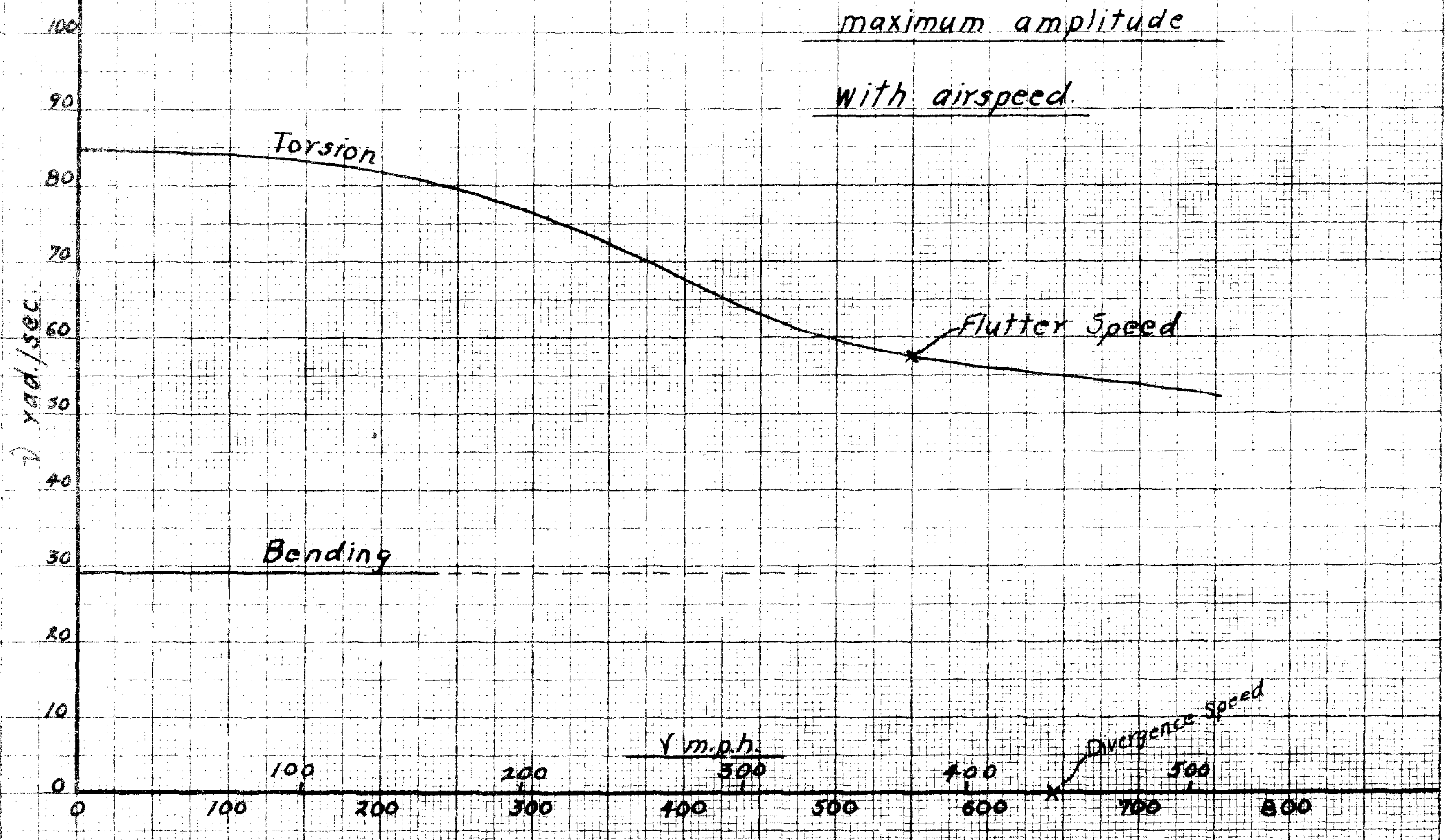


Figure 21

EFFECT OF CHANGES IN THE VARIOUS WING PARAMETERS

In order to obtain some idea of the importance of the various parameters and to investigate the effects which changes in these parameters will have on the torsional flexural flutter speed and the torsional divergence speed, one parameter at a time has been varied while holding the remaining wing parameters at the values chosen for the response investigation. By use of the Kassner and Fingado chart of reference (3) for flutter speed and the previously derived equation for torsional divergence speed, curves have been plotted showing their variation and are given in Figures 22 to 29. The circled points indicate the basic wing.

The torsional divergence being non-dynamic in nature is rather easy to visualize. Obviously the bending frequency and inertia axis have nothing to do with torsional divergence hence the torsional divergence speed remains constant as they are varied. Increasing the wing weight while holding the torsional frequency constant or increasing the torsional frequency while holding i and m_p, i.e. the moment of inertia, constant means that the torsional stiffness and hence the divergence speed is increasing. Moving the elastic axis back on the wing in effect gives the aerodynamic forces, which can be considered as acting at the quarter chord, a longer moment arm and hence decreased the divergence speed. Of course, the elastic axis at the quarter chord point would result in an infinite divergence speed. An increase in altitude decreased the aerodynamic forces on the wing at a given speed and thus increases the divergence speed.

The reasons behind the changes in the flutter speeds are very complex and an explanation will not be attempted. However, the following pertinent points should be noted:

1. The flexural frequency or stiffness is apparently of minor importance.
2. The flutter speed increases with increased torsional frequency in nearly a linear relation.
3. Moving the elastic axis to the rear increases the flutter speed.
4. The position of the inertia axis has a marked effect on the flutter speed, a forward position being favorable.
5. From the curve showing the effect of changes in the radius of gyration of wing, it should be noted that even at zero radius which corresponds to a high torsional frequency the flutter speed is still quite nominal. From this we can conclude that the major effect of increased torsional frequency comes from an increase in torsional stiffness.
6. Changes in wing weight have a peculiar effect. Since the frequencies have been held constant part of the effect can be attributed to the fact that the stiffnesses must change as the weight is varied.
7. The flutter speed increases with altitude.
8. It should be remembered that a definite wing has been considered and that only one parameter has been varied at a time. If another wing were considered, or if two or more parameters were varied at one time the changes in flutter speed and

even their character might well be different. It will be noted that the effect of moving the elastic and inertia axes simultaneously is not the sum of the effects of moving them individually.

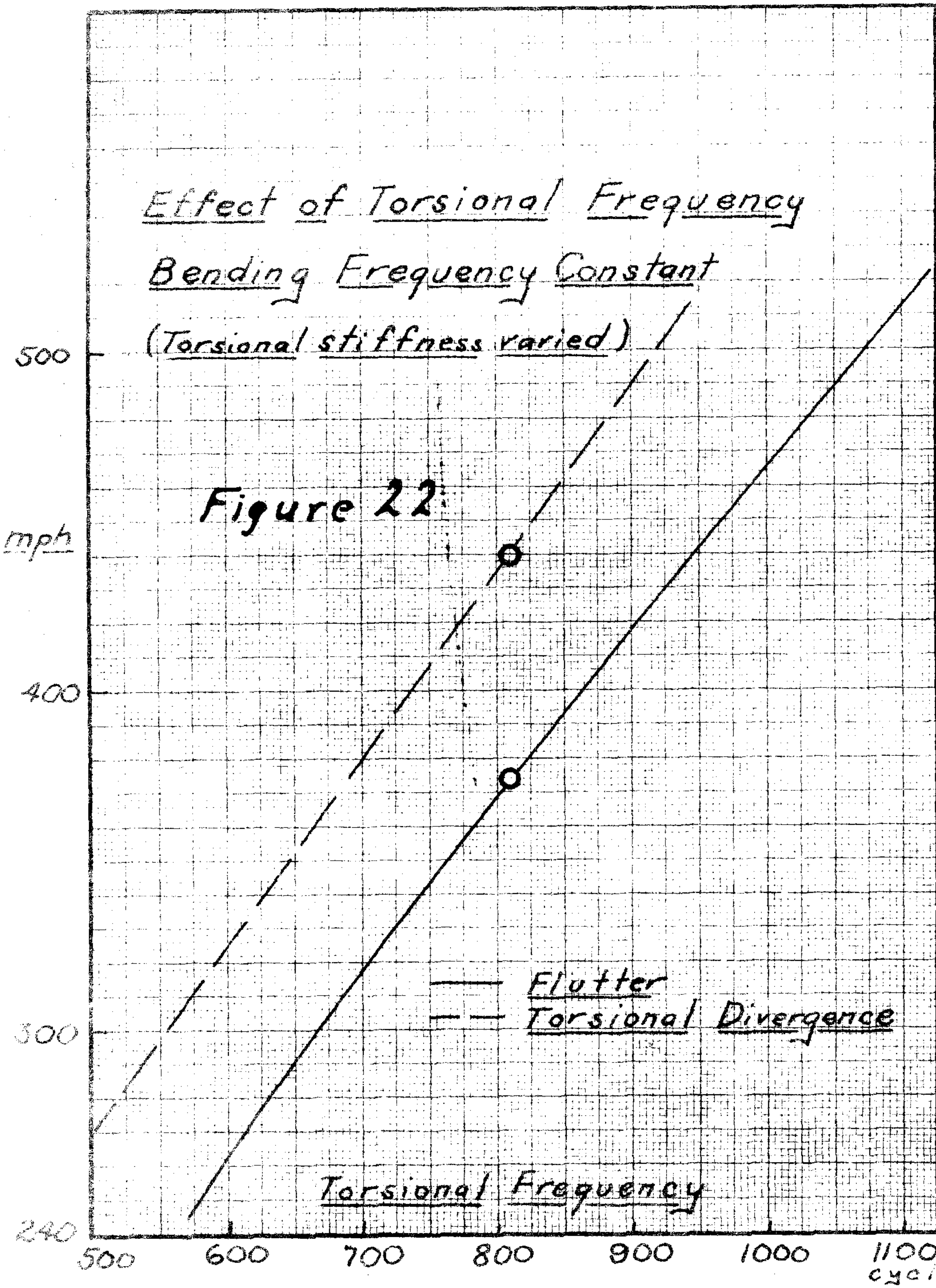


Figure 22

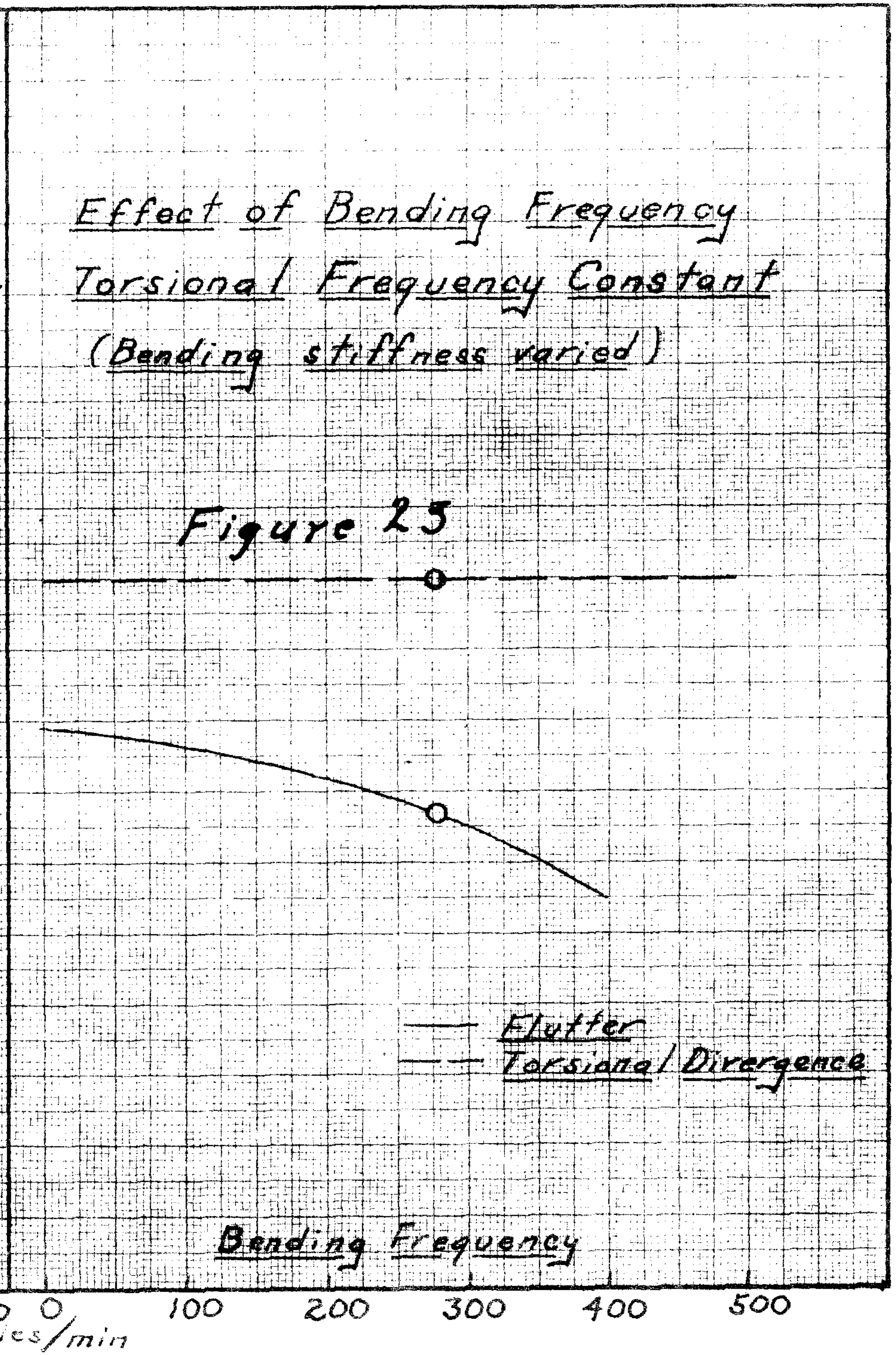


Figure 23

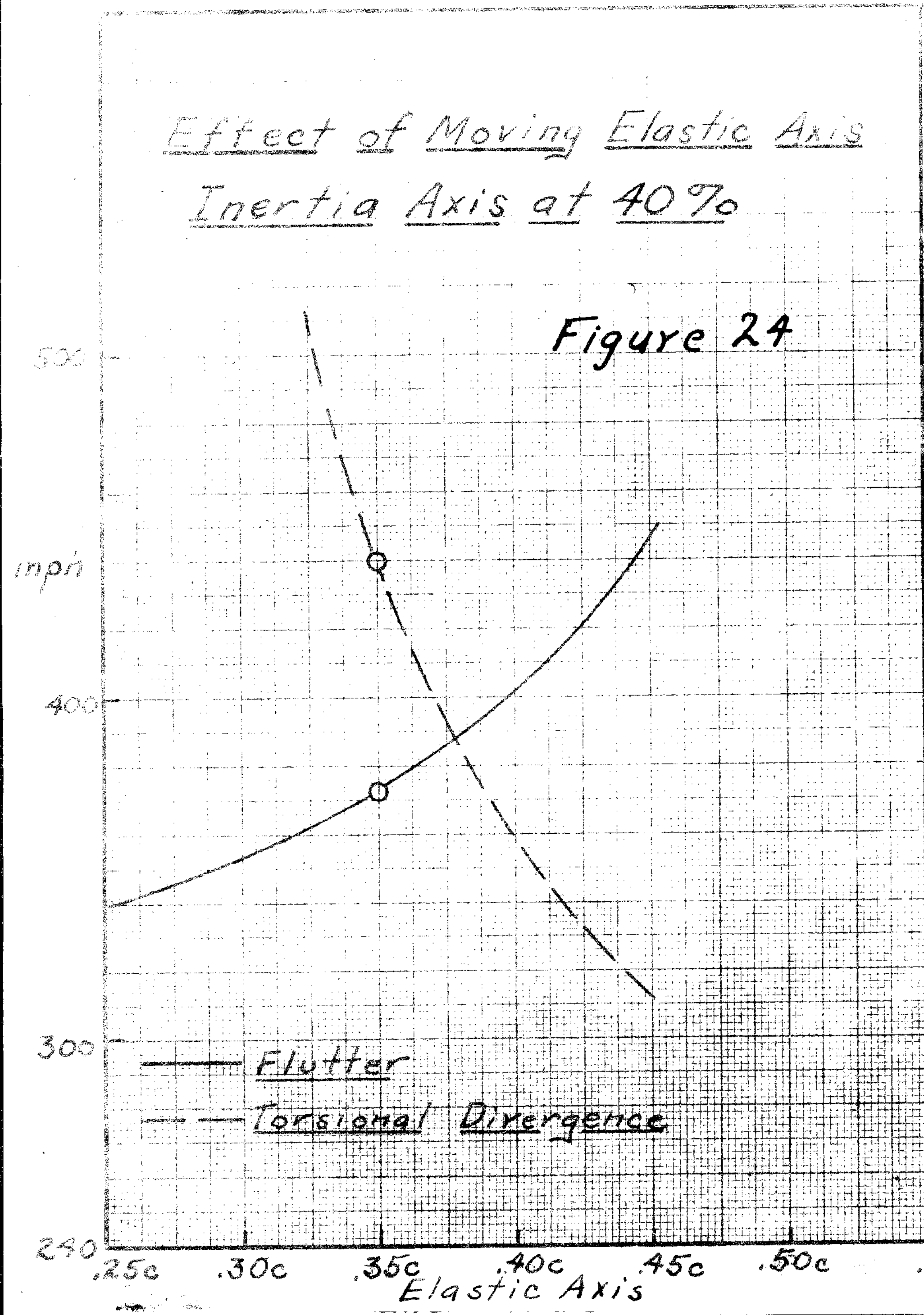


Figure 24

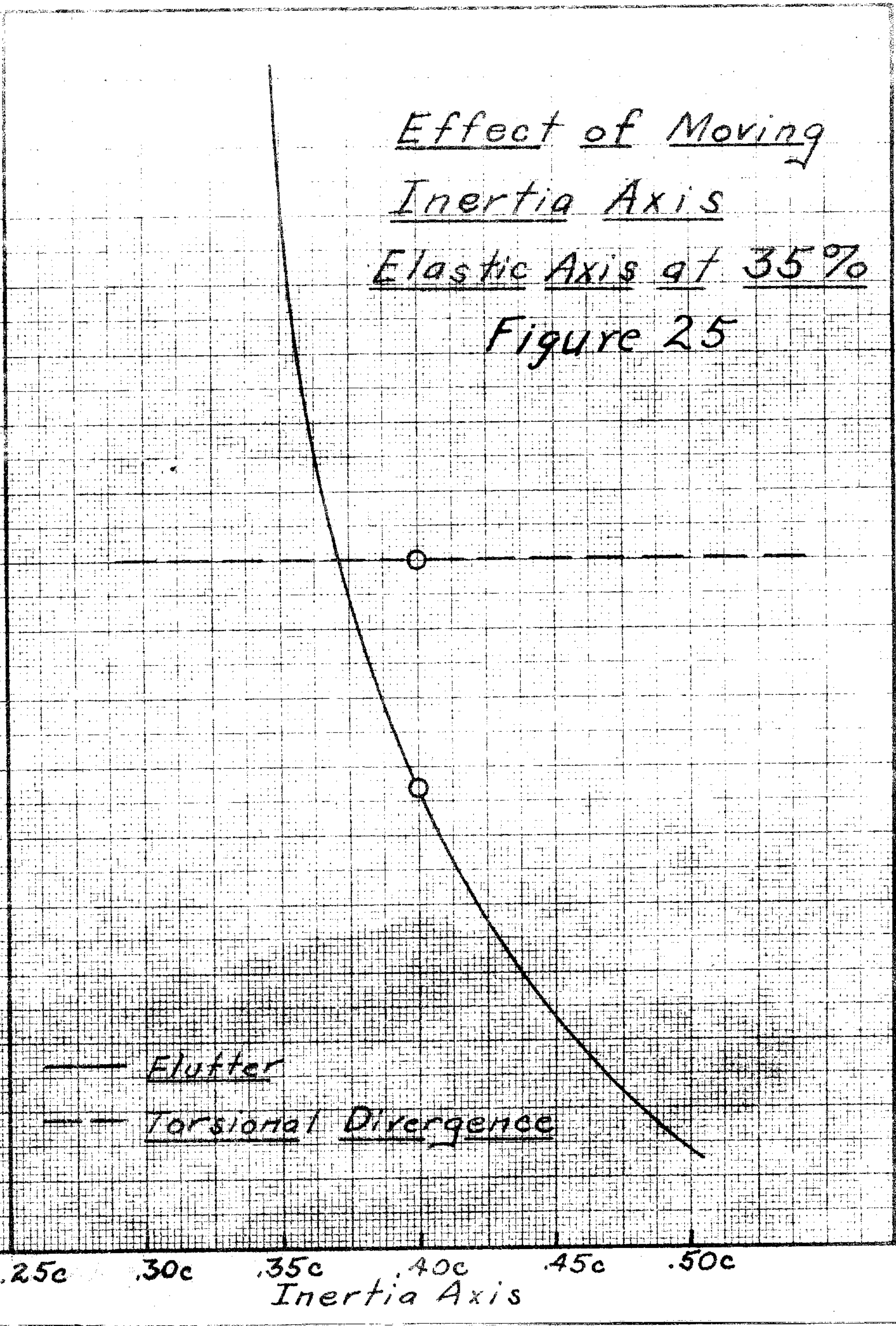


Figure 25

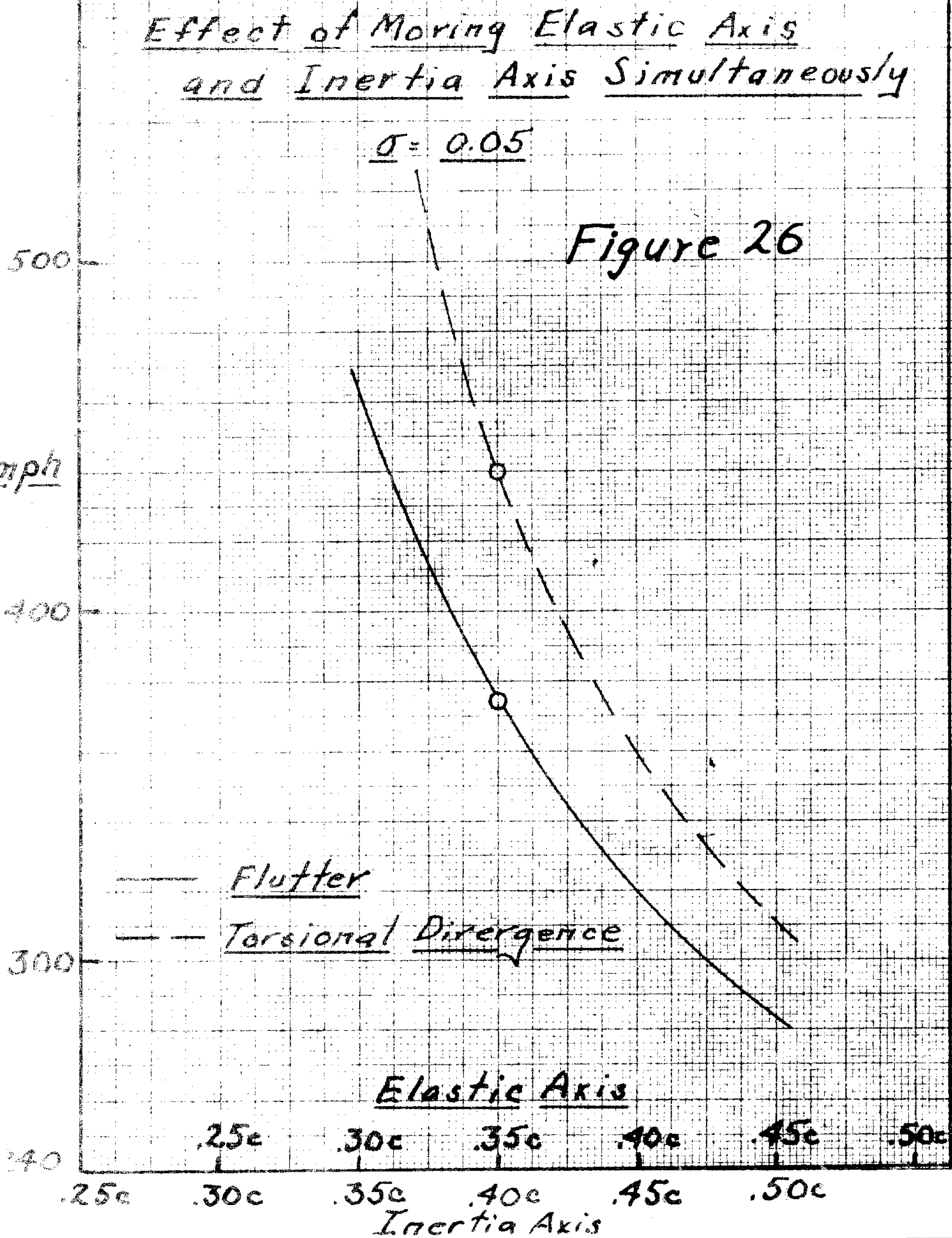


Figure 26

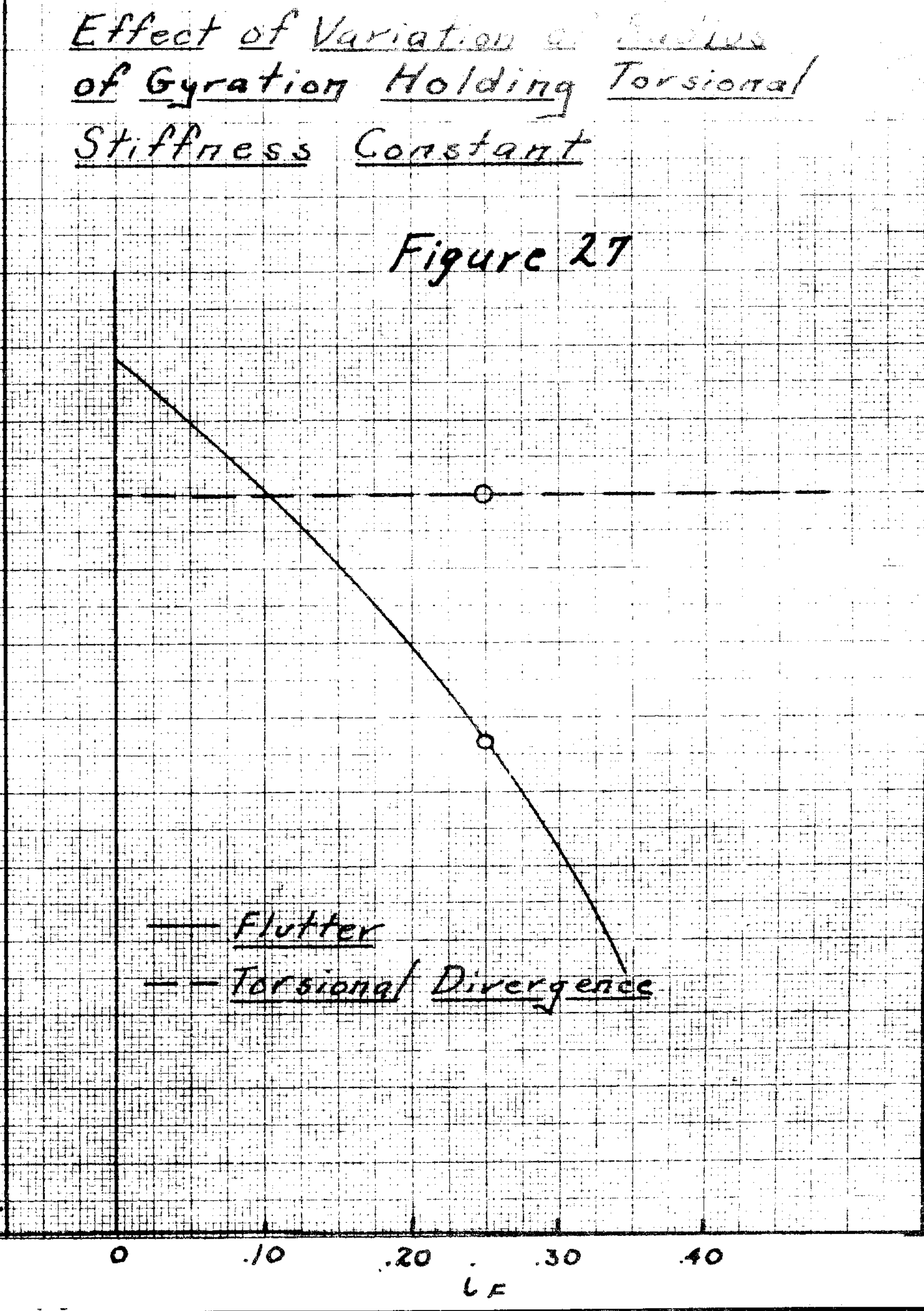


Figure 27

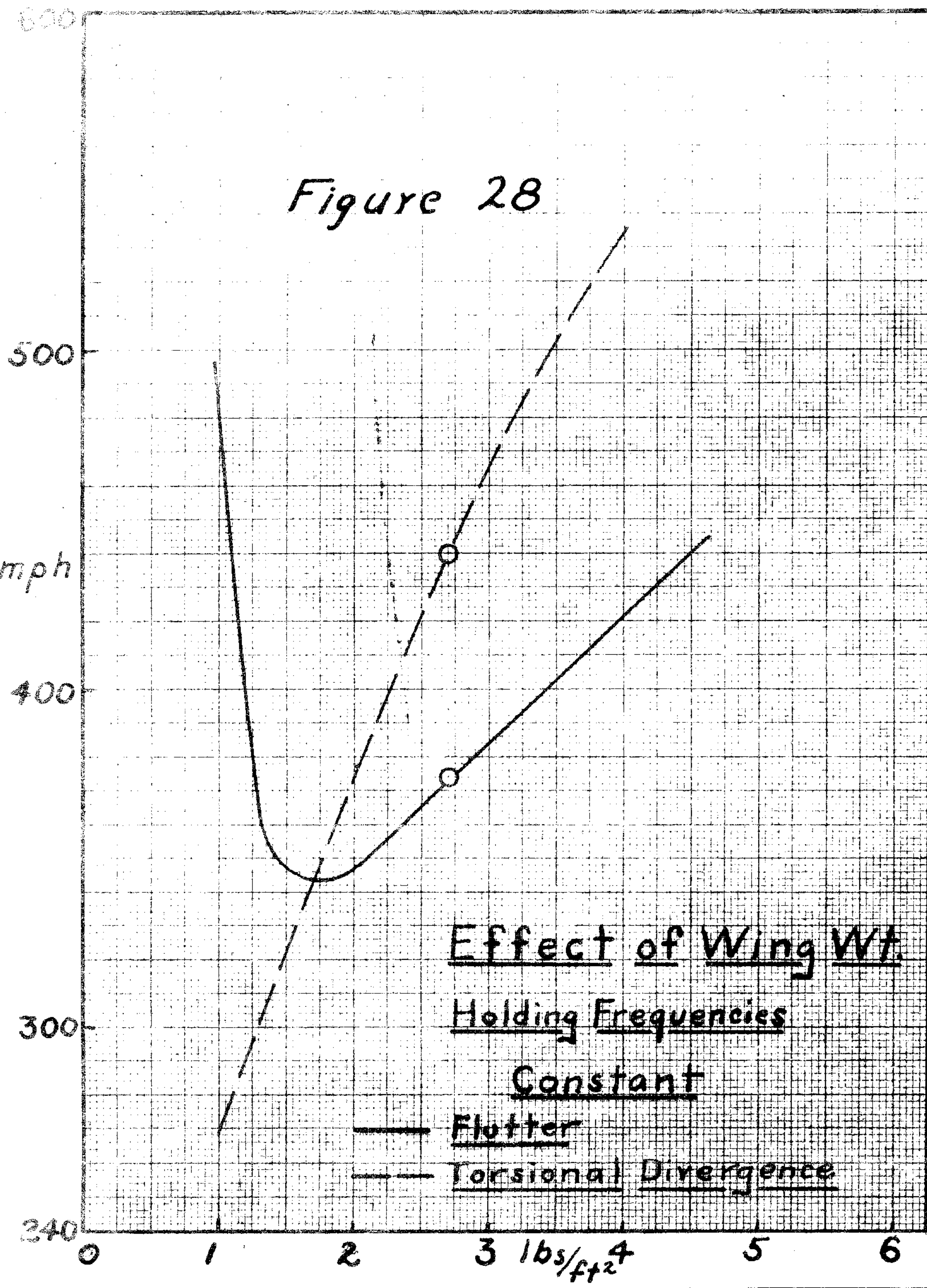


Figure 28
600
500
mph
400
300
240
0
1
2
3 lbs/ft²
4
5
6
Effect of Wing Wt.
Holding Frequencies
Constant
Flutter
Torsional Divergence

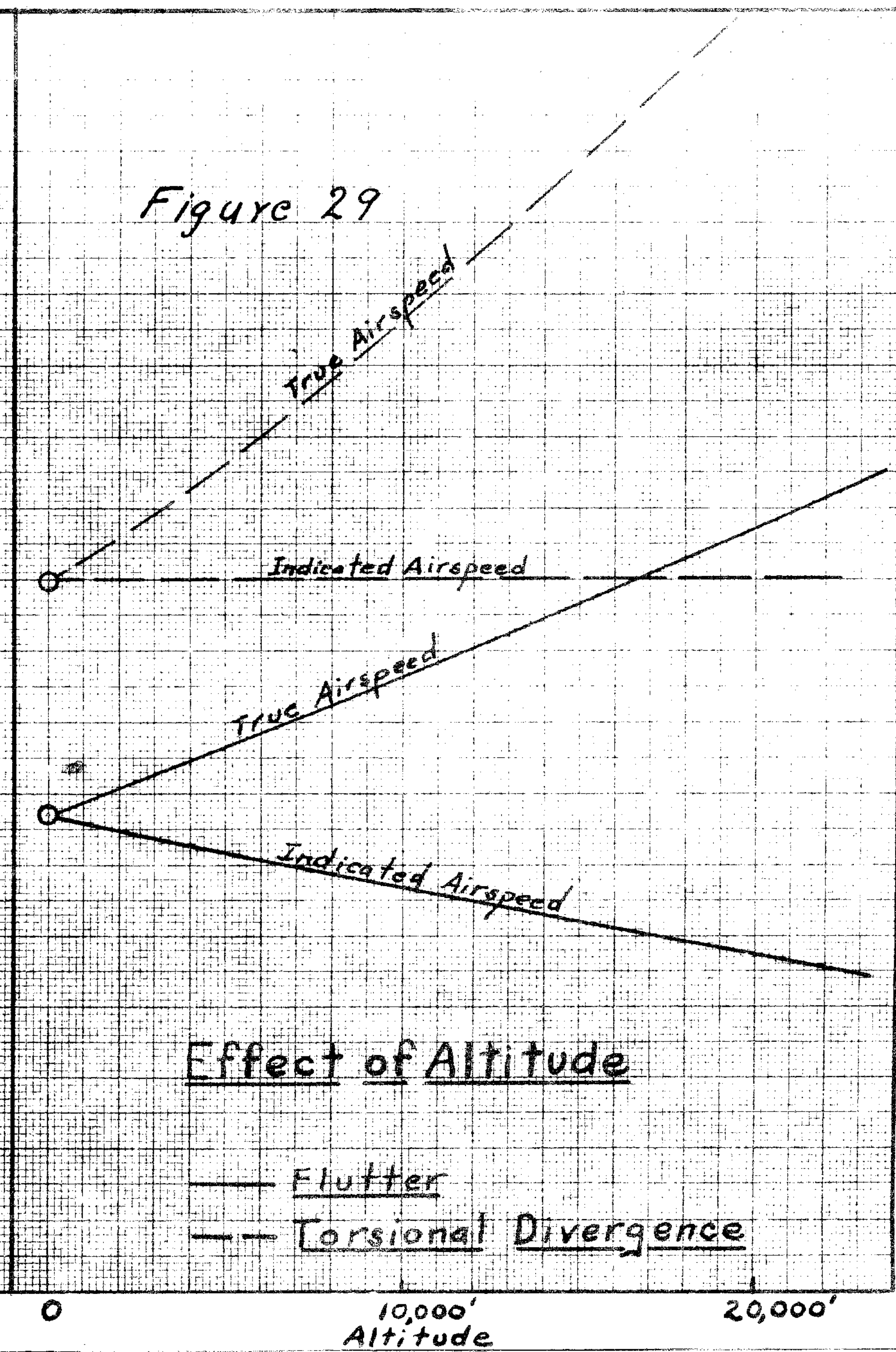
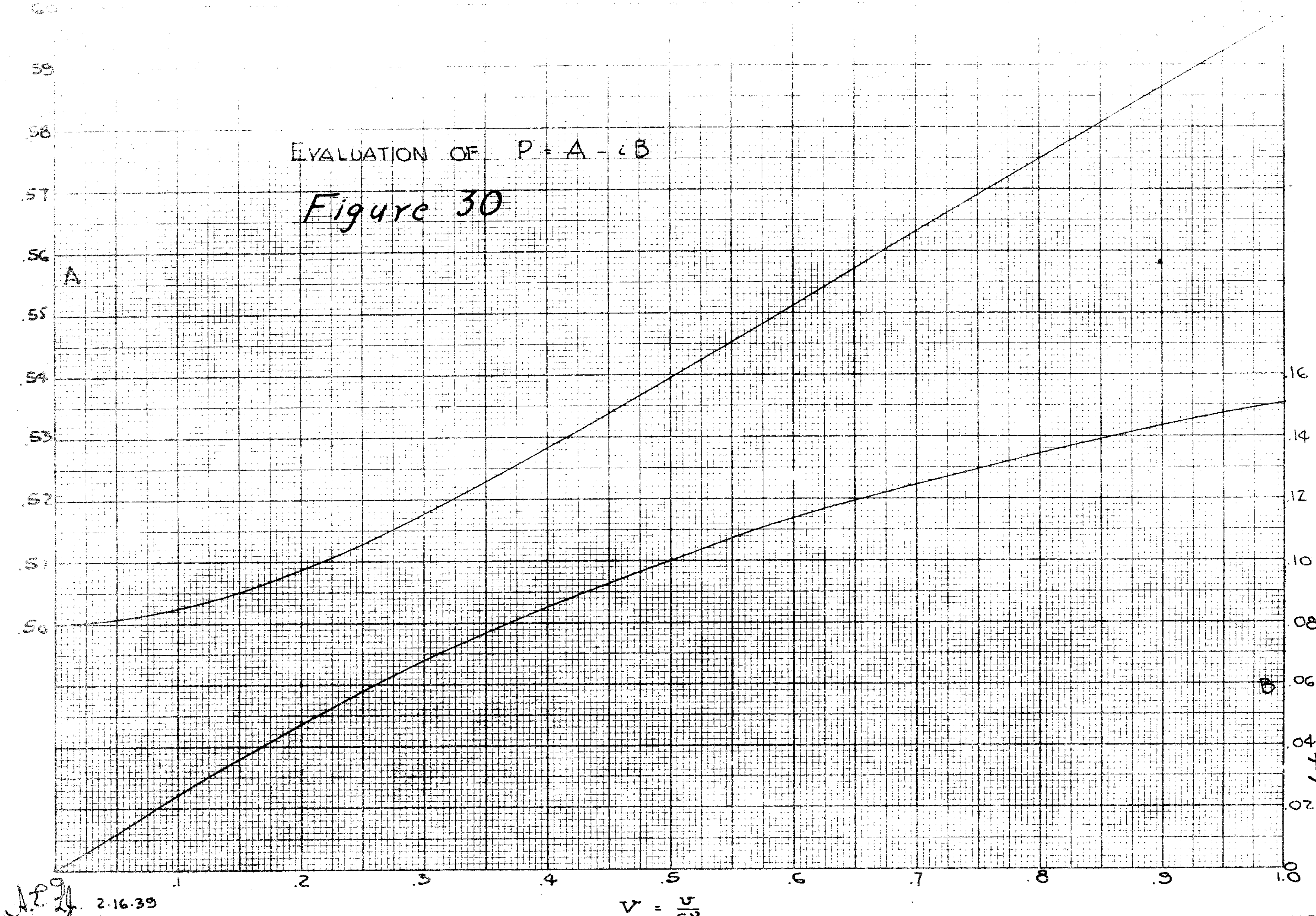
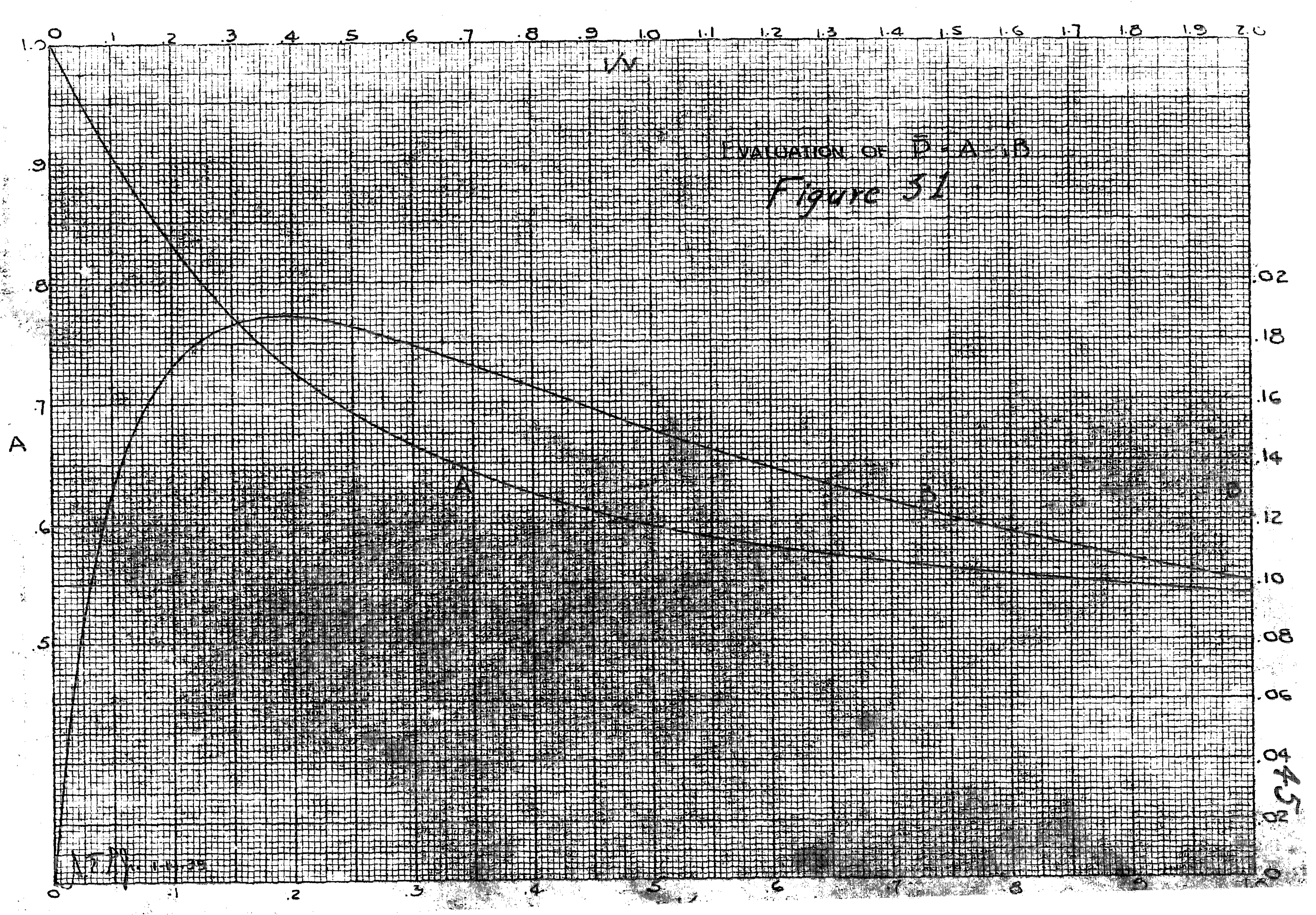


Figure 29
True Airspeed
Indicated Airspeed
True Airspeed
Indicated Airspeed
Effect of Altitude
Flutter
Torsional Divergence
0
10,000'
20,000'
Altitude



EVALUATION OF P = A - iB
Figure 30
A
B
.50
.51
.52
.53
.54
.55
.56
.57
.58
.59
.02
.04
.06
.08
.10
.12
.14
.16
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
V = v/cv
2.16.39



45

REFERENCES

1. Von Karman, Th. and Sears, W.R.

AIRFOIL THEORY FOR NON-UNIFORM MOTION

2. Lombard, A.E.

AN INVESTIGATION OF THE CONDITIONS FOR THE OCCURENCE OF FLUTTER IN AIRCRAFT AND THE DEVELOPMENT OF CRITERIA FOR THE PREDICTION AND ELIMINATION OF SUCH FLUTTER

3. Kassner, R. and Fingado, H.

DAS EBENE PROBLEM DER FLÜGELSCHWINGUNG